MAT-004B - DIFFERENTIAL EQUATIONS

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MAT-004B - Differential Equations (Reed)

William F. Trench

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Detailed Licensing

Licensing

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CHAPTER OVERVIEW

Chapter 1: Introduction

In this chapter, we begin our study of differential equations.

1.1: Applications Leading to Differential Equations

1.2: Basic Concepts

1.2E: Basic Concepts (Exercises)

1.3: Direction Fields for First Order Equations

1.3E: Direction Fields for First Order Equations (Exercises)

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1.1: Applications Leading to Differential Equations

In order to apply mathematical methods to a physical or "real life" problem, we must formulate the problem in mathematical terms; that is, we must construct a mathematical model for the problem. Many physical problems concern relationships between changing quantities. Since rates of change are represented mathematically by derivatives, mathematical models often involve equations relating an unknown function and one or more of its derivatives. Such equations are differential equations. They are the subject of this book.

Much of calculus is devoted to learning mathematical techniques that are applied in later courses in mathematics and the sciences; you wouldn't have time to learn much calculus if you insisted on seeing a specific application of every topic covered in the course. Similarly, much of this book is devoted to methods that can be applied in later courses. Only a relatively small part of the book is devoted to the derivation of specific differential equations from mathematical models, or relating the differential equations that we study to specific applications. In this section we mention a few such applications. The mathematical model for an applied problem is almost always simpler than the actual situation being studied, since simplifying assumptions are usually required to obtain a mathematical problem that can be solved. For example, in modeling the motion of a falling object, we might neglect air resistance and the gravitational pull of celestial bodies other than Earth, or in modeling population growth we might assume that the population grows continuously rather than in discrete steps.

A good mathematical model has two important properties:

- It's sufficiently simple so that the mathematical problem can be solved.
- It represents the actual situation sufficiently well so that the solution to the mathematical problem predicts the outcome of the real problem to within a useful degree of accuracy. If results predicted by the model don't agree with physical observations,the underlying assumptions of the model must be revised until satisfactory agreement is obtained.

We will now give examples of mathematical models involving differential equations. We willreturn to these problems at the appropriate times, as we learn how to solve the various types of differential equations that occur in the models. All the examples in this section deal with functions of time, which we denote by t. If y is a function of t, y' denotes the derivative of y with respect to t; thus,

$$y' = \frac{dy}{dt}.$$

Population Growth and Decay

Although the number of members of a population (people in a given country, bacteria in a laboratory culture, wildflowers in a forest, etc.) at any given time t is necessarily an integer, models that use differential equations to describe the growth and decay of populations usually rest on the simplifying assumption that the number of members of the population can be regarded as a differentiable function $P = P(t)$. In most models it is assumed that the differential equation takes the form

$$P' = a(P)P \tag{1.1.1}$$

where a is a continuous function of P that represents the rate of change of population per unit time per individual. In the **Malthusian model**, it is assumed that $a(P)$ is a constant, so Equation 1.1.1 becomes

$$P' = aP. \tag{1.1.2}$$

(When you see a name in blue italics, just click on it for information about the person.) This model assumes that the numbers of births and deaths per unit time are both proportional to the population. The constants of proportionality are the birth rate (births per unit time per individual) and the death rate (deaths per unit time per individual); a is the birth rate minus the death rate. You learned in calculus that if c is any constant then

$$P = ce^{at} \tag{1.1.3}$$

satisfies Equation 1.1.2, so Equation 1.1.2 has infinitely many solutions. To select the solution of the specific problem that we are considering, we must know the population P_0 at an initial time, say $t = 0$. Setting $t = 0$ in Equation 1.1.3 yields $c = P(0) = P_0$, so the applicable solution is

$$P(t) = P_0 e^{at}.$$

This implies that

$$\lim_{t\to\infty} P(t) = \begin{cases} \infty & \text{if } a > 0, \\ 0 & \text{if } a < 0; \end{cases}$$

that is, the population approaches infinity if the birth rate exceeds the death rate, or zero if the death rate exceeds the birth rate.

To see the limitations of the Malthusian model, suppose we are modeling the population of a country, starting from a time $t = 0$ when the birth rate exceeds the death rate (so $a > 0$), and the country's resources in terms of space, food supply, and other necessities of life can support the existing population. Then the prediction $P = P_0 e^{at}$ may be reasonably accurate as long as it remains within limits that the country's resources can support. However, the model must inevitably lose validity when the prediction exceeds these limits. (If nothing else, eventually there will not be enough space for the predicted population!) This flaw in the Malthusian model suggests the need for a model that accounts for limitations of space and resources that tend to oppose the rate of population growth as the population increases.

Perhaps the most famous model of this kind is the **Verhulst model**, where Equation 1.1.2 is replaced by

$$P' = aP(1 - \alpha P), \tag{1.1.4}$$

where α is a positive constant. As long as P is small compared to $1/\alpha$, the ratio P'/P is approximately equal to a. Therefore the growth is approximately exponential; however, as P increases, the ratio P'/P decreases as opposing factors become significant.

Equation 1.1.4 is the **logistic equation**. You will learn how to solve it in Section 1.2. (See **Exercise 2.2.28**.) The solution is

$$P = \frac{P_0}{\alpha P_0 + (1 - \alpha P_0)e^{-at}},$$

where $P_0 = P(0) > 0$. Therefore $\lim_{t\to\infty} P(t) = 1/\alpha$, independent of P_0.

Figure 1.1.1 shows typical graphs of P versus t for various values of P_0.

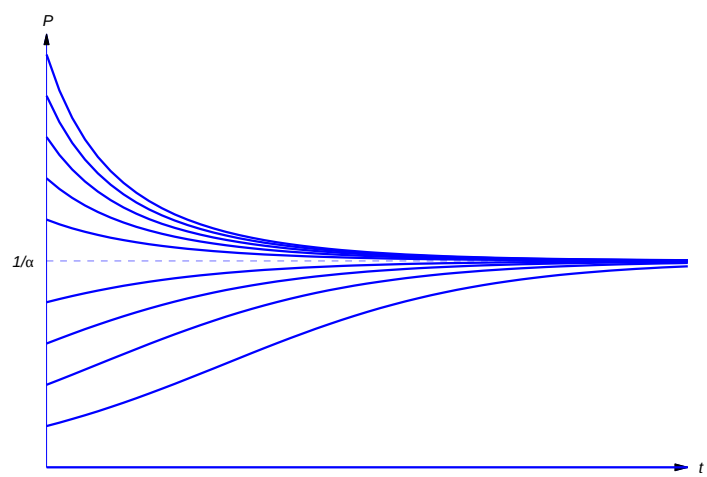

Figure 1.1.1 : Solutions of the logistic equation

Newton's Law of Cooling

According to **Newton's law of cooling**, the temperature of a body changes at a rate proportional to the difference between the temperature of the body and the temperature of the surrounding medium. Thus, if T_m is the temperature of the medium and $T = T(t)$ is the temperature of the body at time t, then

$$T' = -k(T - T_m) \tag{1.1.5}$$

where k is a positive constant and the minus sign indicates; that the temperature of the body increases with time if it is less than the temperature of the medium, or decreases if it is greater. We will see in Section 4.2 that if T_m is constant then the solution of Equation 1.1.5 is

$$T = T_m + (T_0 - T_m)e^{-kt} \tag{1.1.6}$$

where T_0 is the temperature of the body when $t=0$. Therefore

$$\lim_{t\to\infty} T(t) = T_m$$

independent of T_0 (Common sense suggests this. Why?). Figure 1.1.2 shows typical graphs of T versus t for various values of T_0.

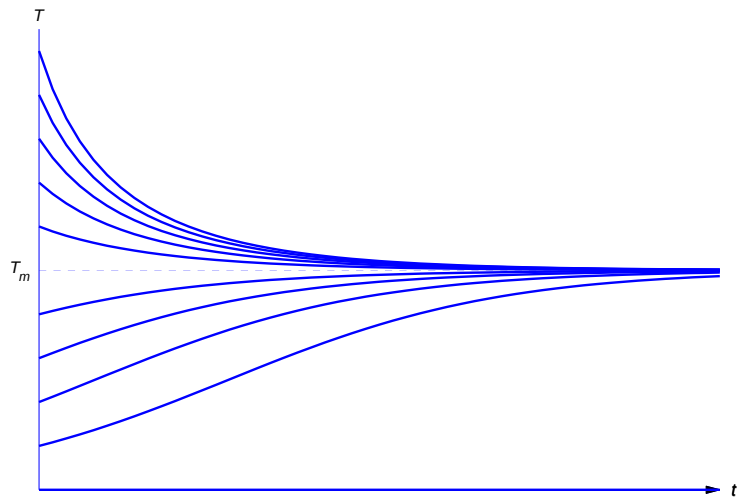

Figure 1.1.2 : Temperature according to Newton's Law of Cooling

Assuming that the medium remains at constant temperature seems reasonable if we are considering a cup of coffee cooling in a room, but not if we are cooling a huge cauldron of molten metal in the same room. The difference between the two situations is that the heat lost by the coffee isn't likely to raise the temperature of the room appreciably, but the heat lost by the cooling metal is. In this second situation we must use a model that accounts for the heat exchanged between the object and the medium. Let $T=T(t)$ and $T_m=T_m(t)$ be the temperatures of the object and the medium respectively, and let T_0 and T_m0 be their initial values. Again, we assume that T and Tm are related by Equation 1.1.5. We also assume that the change in heat of the object as its temperature changes from T_0 to T is $a(T-T_0)$ and the change in heat of the medium as its temperature changes from T_{m0} to T_m is $a_m(T_m-T_{m0})$, where a and am are positive constants depending upon the masses and thermal properties of the object and medium respectively. If we assume that the total heat of the in the object and the medium remains constant (that is, energy is conserved), then

$$a(T-T_0)+a_m(T_m-T_{m0})=0.$$

Solving this for T_m and substituting the result into Equation 1.1.6 yields the differential equation

$$T' = -k\left(1+\frac{a}{a_m}\right)T + k\left(T_{m0}+\frac{a}{a_m}T_0\right)$$

for the temperature of the object. After learning to solve linear first order equations, you'll be able to show (**Exercise 4.2.17**) that

$$T = \frac{aT_0+a_mT_{m0}}{a+a_m} + \frac{a_m\left(T_0-T_{m0}\right)}{a+a_m}e^{-k(1+a/a_m)t}$$

Glucose Absorption by the Body

Glucose is absorbed by the body at a rate proportional to the amount of glucose present in the blood stream. Let λ denote the (positive) constant of proportionality. Suppose there are G_0 units of glucose in the bloodstream when $t=0$, and let $G=G(t)$ be the number of units in the bloodstream at time $t>0$. Then, since the glucose being absorbed by the body is leaving the bloodstream, G satisfies the equation

$$G' = -\lambda G. \tag{1.1.7}$$

From calculus you know that if c is any constant then

$$G = ce^{-\lambda t} \tag{1.1.8}$$

satisfies Equation (1.1.7), so Equation 1.1.7 has infinitely many solutions. Setting $t=0$ in Equation 1.1.8 and requiring that $G(0)=G_0$ yields $c=G_0$, so

$$G(t)=G_0e^{-\lambda t}.$$

Now let's complicate matters by injecting glucose intravenously at a constant rate of r units of glucose per unit of time. Then the rate of change of the amount of glucose in the bloodstream per unit time is

$$G'=-\lambda G+r \tag{1.1.9}$$

where the first term on the right is due to the absorption of the glucose by the body and the second term is due to the injection. After you've studied Section 2.1, you'll be able to show that the solution of Equation 1.1.9 that satisfies $G(0)=G_0$ is

$$G=\frac{r}{\lambda}+\left(G_0-\frac{r}{\lambda}\right)e^{-\lambda t}$$

Graphs of this function are similar to those in Figure 1.1.2 . (Why?)

Spread of Epidemics

One model for the spread of epidemics assumes that the number of people infected changes at a rate proportional to the product of the number of people already infected and the number of people who are susceptible, but not yet infected. Therefore, if S denotes the total population of susceptible people and $I=I(t)$ denotes the number of infected people at time t, then $S-I$ is the number of people who are susceptible, but not yet infected. Thus,

$$I'=rI(S-I)$$

, where r is a positive constant. Assuming that $I(0)=I_0$, the solution of this equation is

$$I=\frac{SI_0}{I_0+(S-I_0)e^{-rSt}}$$

(**Exercise 2.2.29**). Graphs of this function are similar to those in Figure 1.1.1. (Why?) Since $\lim_{t\to\infty}I(t)=S$, this model predicts that all the susceptible people eventually become infected.

Newton's Second Law of Motion

According to Newton's second law of motion, the instantaneous acceleration a of an object with constant mass m is related to the force F acting on the object by the equation $F=ma$. For simplicity, let's assume that $m=1$ and the motion of the object is along a vertical line. Let y be the displacement of the object from some reference point on Earth's surface, measured positive upward. In many applications, there are three kinds of forces that may act on the object:

a. A force such as gravity that depends only on the position y, which we write as $-p(y)$, where $p(y)>0$ if $y\geq 0$.
b. A force such as atmospheric resistance that depends on the position and velocity of the object, which we write as $-q(y,y')y'$, where q is a nonnegative function and we've put y' "outside" to indicate that the resistive force is always in the direction opposite to the velocity.
c. A force $f=f(t)$, exerted from an external source (such as a towline from a helicopter) that depends only on t.

In this case, Newton's second law implies that

$$y''=-q(y,y')y'-p(y)+f(t),$$

which is usually rewritten as

$$y''+q(y,y')y'+p(y)=f(t).$$

Since the second (and no higher) order derivative of y occurs in this equation, we say that it is a second order differential equation.

Interacting Species: Competition

Let $P=P(t)$ and $Q=Q(t)$ be the populations of two species at time t, and assume that each population would grow exponentially if the other did not exist; that is, in the absence of competition we would have

$$P'=aP \quad \text{and} \quad Q'=bQ, \tag{1.1.10}$$

where a and b are positive constants. One way to model the effect of competition is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so Equation 1.1.10 is replaced by

$$\begin{aligned} P' &= aP - \alpha Q \\ Q' &= -\beta P + bQ, \end{aligned}$$

where α and β are positive constants. (Since negative population doesn't make sense, this system works only while P and Q are both positive.) Now suppose $P(0) = P_0 > 0$ and $Q(0) = Q_0 > 0$. It can be shown (**Exercise 10.4.42**) that there's a positive constant ρ such that if (P_0, Q_0) is above the line L through the origin with slope ρ, then the species with population P becomes extinct in finite time, but if (P_0, Q_0) is below L, the species with population Q becomes extinct in finite time. Figure 1.1.3 illustrates this. The curves shown there are given parametrically by $P = P(t), Q = Q(t),\ t > 0$. The arrows indicate direction along the curves with increasing t.

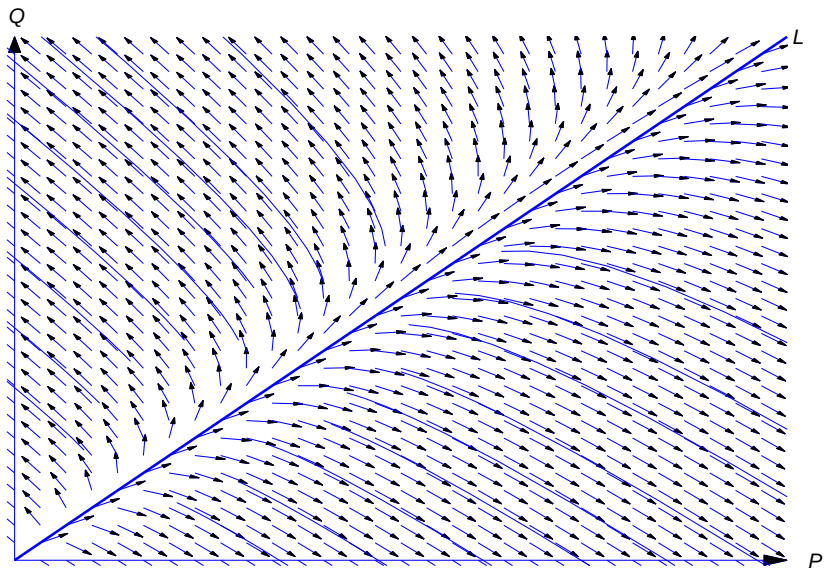

Figure 1.1.3 : Populations of competing species

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1.2: Basic Concepts

A **differential equation** is an equation that contains one or more derivatives of an unknown function. The **order** of a differential equation is the order of the highest derivative that it contains. A differential equation is an **ordinary differential equation** if it involves an unknown function of only one variable, or a **partial differential equation** if it involves partial derivatives of a function of more than one variable. For now we'll consider only ordinary differential equations, and we'll just call them **differential equations.**

> Throughout this text, all variables and constants are real unless it is stated otherwise. We'll usually use x for the independent variable unless the independent variable is time; then we'll use t.

The simplest differential equations are first order equations of the form

$$\frac{dy}{dx} = f(x)$$

or equivalently

$$y' = f(x),$$

where f is a known function of x. We already know from calculus how to find functions that satisfy this kind of equation. For example, if

$$y' = x^3,$$

then

$$y = \int x^3\,dx = \frac{x^4}{4} + c,$$

where c is an arbitrary constant. If $n > 1$ we can find functions y that satisfy equations of the form

$$y^{(n)} = f(x) \tag{1.2.1}$$

by repeated integration. Again, this is a calculus problem.

Except for illustrative purposes in this section, there's no need to consider differential equations like Equation 1.2.1. We'll usually consider differential equations that can be written as

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \tag{1.2.2}$$

where at least one of the functions y, y', …, $y^{(n-1)}$ actually appears on the right. Here are some examples:

$$\begin{aligned}
\tfrac{dy}{dx} - x^2 &= 0 && \text{(first order)},\\
\tfrac{dy}{dx} + 2xy^2 &= -2 && \text{(first order)},\\
\tfrac{d^2y}{dx^2} + 2\tfrac{dy}{dx} + y &= 2x && \text{(second order)},\\
xy''' + y^2 &= \sin x && \text{(third order)},\\
y^{(n)} + xy' + 3y &= x && \text{(n-th order)}.
\end{aligned}$$

Although none of these equations is written as in Equation 1.2.2, all of them **can** be written in this form:

$$\begin{aligned} y' &= x^2, \\ y' &= -2-2xy^2, \\ y'' &= 2x-2y'-y, \\ y''' &= \frac{\sin x - y^2}{x}, \\ y^{(n)} &= x - xy' - 3y. \end{aligned}$$

Solutions of Differential Equations

A **solution** of a differential equation is a function that satisfies the differential equation on some open interval; thus, y is a solution of Equation 1.2.2 if y is n times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$$

for all x in some open interval (a, b). In this case, we also say that y **is a solution of Equation 1.2.2 on (a, b)**. Functions that satisfy a differential equation at isolated points are not interesting. For example, $y = x^2$ satisfies

$$xy' + x^2 = 3x$$

if and only if $x = 0$ or $x = 1$, but it is not a solution of this differential equation because it does not satisfy the equation on an open interval.

The graph of a solution of a differential equation is a **solution curve**. More generally, a curve C is said to be an **integral curve** of a differential equation if every function $y = y(x)$ whose graph is a segment of C is a solution of the differential equation. Thus, any solution curve of a differential equation is an integral curve, but an integral curve need not be a solution curve.

Example 1.2.1

If a is any positive constant, the circle

$$x^2 + y^2 = a^2 \tag{1.2.3}$$

is an integral curve of

$$y' = -\frac{x}{y}. \tag{1.2.4}$$

Solution

To see this, note that the only functions whose graphs are segments of Equation 1.2.3 are

$$y_1 = \sqrt{a^2 - x^2} \quad \text{and} \quad y_2 = -\sqrt{a^2 - x^2}.$$

We leave it to you to verify that these functions both satisfy Equation 1.2.4 on the open interval $(-a, a)$. However, Equation 1.2.3 is not a solution curve of Equation 1.2.4, since it is not the graph of a function.

Example 1.2.2

Verify that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{1.2.5}$$

is a solution of

$$xy' + y = x^2 \tag{1.2.6}$$

on $(0, \infty)$ and on $(-\infty, 0)$.

Solution

Substituting Equation 1.2.5 and

$$y' = \frac{2x}{3} - \frac{1}{x^2}$$

into Equation 1.2.6 yields

$$xy'(x) + y(x) = x\left(\frac{2x}{3} - \frac{1}{x^2}\right) + \left(\frac{x^2}{3} + \frac{1}{x}\right) = x^2$$

for all $x \neq 0$. Therefore y is a solution of Equation 1.2.6 on $(-\infty, 0)$ and $(0, \infty)$. However, y isn't a solution of the differential equation on any open interval that contains $x = 0$, since y is not defined at $x = 0$.

Figure 1.2.2 shows the graph of Equation 1.2.5. The part of the graph of Equation 1.2.5 on $(0, \infty)$ is a solution curve of Equation 1.2.6, as is the part of the graph on $(-\infty, 0)$.

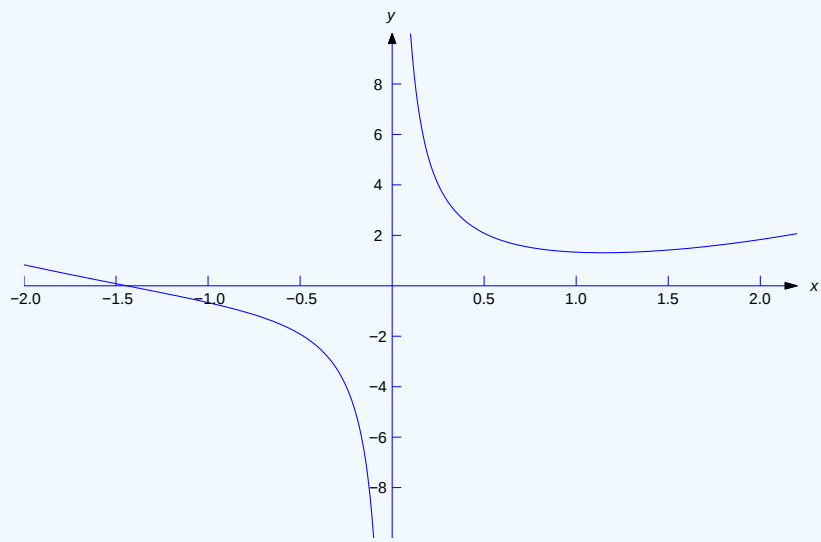

Figure 1.2.2 : $y = \frac{x^2}{3} + \frac{1}{x}$

Example 1.2.3

Show that if c_1 and c_2 are constants then

$$y = (c_1 + c_2 x)e^{-x} + 2x - 4 \tag{1.2.7}$$

is a solution of

$$y'' + 2y' + y = 2x \tag{1.2.8}$$

on $(-\infty, \infty)$.

Solution

Differentiating Equation 1.2.7 twice yields

$$y' = -(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2$$

and

$$y'' = (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x},$$

so

$$\begin{aligned} y'' + 2y' + y &= (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x} + 2\left[-(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2\right] + (c_1 + c_2 x)e^{-x} + 2x - 4 \\ &= (1 - 2 + 1)(c_1 + c_2 x)e^{-x} + (-2 + 2)c_2 e^{-x} + 4 + 2x - 4 \\ &= 2x \end{aligned}$$

for all values of x. Therefore y is a solution of Equation 1.2.8 on $(-\infty, \infty)$.

Example 1.2.4

Find all solutions of

$$y^{(n)} = e^{2x}. \tag{1.2.9}$$

Solution

Integrating Equation 1.2.9 yields

$$y^{(n-1)} = \frac{e^{2x}}{2} + k_1,$$

where k_1 is a constant. If $n \geq 2$, integrating again yields

$$y^{(n-2)} = \frac{e^{2x}}{4} + k_1 x + k_2.$$

If $n \geq 3$, repeatedly integrating yields

$$y = \frac{e^{2x}}{2^n} + k_1 \frac{x^{n-1}}{(n-1)!} + k_2 \frac{x^{n-2}}{(n-2)!} + \cdots + k_n, \tag{1.2.10}$$

where k_1, k_2, …, k_n are constants. This shows that every solution of Equation 1.2.9 has the form Equation 1.2.10 for some choice of the constants k_1, k_2, …, k_n. On the other hand, differentiating Equation 1.2.10 n times shows that if k_1, k_2, …, k_n are arbitrary constants, then the function y in Equation 1.2.10 satisfies Equation 1.2.9.

Since the constants k_1, k_2, …, k_n in Equation 1.2.10 are arbitrary, so are the constants

$$\frac{k_1}{(n-1)!}, \frac{k_2}{(n-2)!}, \cdots, k_n.$$

Therefore Example 1.2.4 actually shows that all solutions of Equation 1.2.9 can be written as

$$y = \frac{e^{2x}}{2^n} + c_1 + c_2 x + \cdots + c_n x^{n-1},$$

where we renamed the arbitrary constants in Equation 1.2.10 to obtain a simpler formula. As a general rule, arbitrary constants appearing in solutions of differential equations should be simplified if possible. You'll see examples of this throughout the text.

Initial Value Problems

In Example 1.2.4 we saw that the differential equation $y^{(n)} = e^{2x}$ has an infinite family of solutions that depend upon the n arbitrary constants c_1, c_2, …, c_n. In the absence of additional conditions, there's no reason to prefer one solution of a differential equation over another. However, we'll often be interested in finding a solution of a differential equation that satisfies one or more specific conditions. The next example illustrates this.

Example 1.2.5

Find a solution of

$$y' = x^3$$

such that $y(1) = 2$.

Solution

At the beginning of this section we saw that the solutions of $y' = x^3$ are

$$y = \frac{x^4}{4} + c.$$

To determine a value of c such that $y(1) = 2$, we set $x = 1$ and $y = 2$ here to obtain

$$2 = y(1) = \frac{1}{4} + c$$

so

$$c = \frac{7}{4}.$$

Therefore the required solution is

$$y = \frac{x^4 + 7}{4}.$$

Figure 1.2.2 shows the graph of this solution. Note that imposing the condition $y(1) = 2$ is equivalent to requiring the graph of y to pass through the point $(1, 2)$.

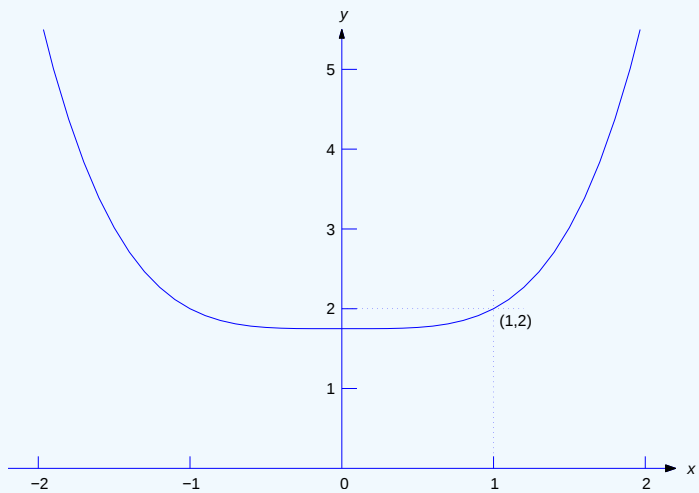

Figure 1.2.2 : $y = \frac{x^2+7}{4}$

We can rewrite the problem considered in Example 1.2.5 more briefly as

$$y' = x^3, \quad y(1) = 2.$$

We call this an **initial value problem**. The requirement $y(1) = 2$ is an **initial condition**. Initial value problems can also be posed for higher order differential equations. For example,

$$y'' - 2y' + 3y = e^x, \quad y(0) = 1, \quad y'(0) = 2 \tag{1.2.11}$$

is an initial value problem for a second order differential equation where y and y' are required to have specified values at $x = 0$. In general, an initial value problem for an n-th order differential equation requires y and its first $n-1$ derivatives to have specified values at some point x_0. These requirements are the **initial conditions**.We'll denote an initial value problem for a differential equation by writing the initial conditions after the equation, as in Equation 1.2.11. For example, we would write an initial value problem for Equation 1.2.2 as

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}),\ y(x_0) = k_0,\ y'(x_0) = k_1,\ \ldots,\ y^{(n-1)} = k_{n-1}. \tag{1.2.12}$$

Consistent with our earlier definition of a solution of the differential equation in Equation 1.2.12, we say that y is a solution of the initial value problem Equation 1.2.12 if y is n times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \ldots, y^{(n-1)}(x))$$

for all x in some open interval (a, b) that contains x_0, and y satisfies the initial conditions in Equation 1.2.12. The largest open interval that contains x_0 on which y is defined and satisfies the differential equation is the **interval of validity** of y.

Example 1.2.6

In Example 1.2.5 we saw that

$$y = \frac{x^4+7}{4} \tag{1.2.13}$$

is a solution of the initial value problem

$$y' = x^3, \quad y(1) = 2.$$

Since the function in Equation 1.2.13 is defined for all x, the interval of validity of this solution is $(-\infty, \infty)$.

Example 1.2.7

In Example 1.2.2 we verified that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{1.2.14}$$

is a solution of

$$xy' + y = x^2$$

on $(0, \infty)$ and on $(-\infty, 0)$. By evaluating Equation 1.2.14 at $x = \pm 1$, you can see that Equation 1.2.14 is a solution of the initial value problems

$$xy' + y = x^2, \quad y(1) = \frac{4}{3} \tag{1.2.15}$$

and

$$xy' + y = x^2, \quad y(-1) = -\frac{2}{3}. \tag{1.2.16}$$

The interval of validity of Equation 1.2.14 as a solution of Equation 1.2.15 is $(0, \infty)$, since this is the largest interval that contains $x_0 = 1$ on which Equation 1.2.14 is defined. Similarly, the interval of validity of Equation 1.2.14 as a solution of Equation 1.2.16 is $(-\infty, 0)$, since this is the largest interval that contains $x_0 = -1$ on which Equation 1.2.14 is defined.

Free Fall Under Constant Gravity

The term **initial value problem** originated in problems of motion where the independent variable is t (representing elapsed time), and the initial conditions are the position and velocity of an object at the initial (starting) time of an experiment.

Example 1.2.8

An object falls under the influence of gravity near Earth's surface, where it can be assumed that the magnitude of the acceleration due to gravity is a constant g.

a. Construct a mathematical model for the motion of the object in the form of an initial value problem for a second order differential equation, assuming that the altitude and velocity of the object at time $t = 0$ are known. Assume that gravity is the only force acting on the object.
b. Solve the initial value problem derived above to obtain the altitude as a function of time.

Solution a

Let $y(t)$ be the altitude of the object at time t. Since the acceleration of the object has constant magnitude g and is in the downward (negative) direction, y satisfies the second order equation

$$y'' = -g,$$

where the prime now indicates differentiation with respect to t. If y_0 and v_0 denote the altitude and velocity when $t = 0$, then y is a solution of the initial value problem

$$y'' = -g, \quad y(0) = y_0, \quad y'(0) = v_0. \tag{1.2.17}$$

Solution b

Integrating Equation 1.2.17 twice yields

$$\begin{aligned} y' &= -gt + c_1, \\ y &= -\frac{gt^2}{2} + c_1 t + c_2. \end{aligned}$$

Imposing the initial conditions $y(0) = y_0$ and $y'(0) = v_0$ in these two equations shows that $c_1 = v_0$ and $c_2 = y_0$. Therefore the solution of the initial value problem Equation 1.2.17 is

$$y = -\frac{gt^2}{2} + v_0 t + y_0.$$

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1.2E: Basic Concepts (Exercises)

1. Find the order of the equation.

a. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}\ \frac{d^3y}{dx^3}+x=0$

b. $y''-3y'+2y=x^7$

c. $y'-y^7=0$

d. $y''y-(y')^2=2$

2. Verify that the function is a solution of the differential equation on some interval, for any choice of the arbitrary constants appearing in the function.

a. $y=\frac{x^2}{3}+\frac{c}{x};\quad xy'+y=x^2$

b. $y=\frac{1}{2}+ce^{-x^2};\quad y'+2xy=x$

c. $y=\tan\left(\frac{x^3}{3}+c\right);\quad y'=x^2(1+y^2)$

d. $y=x^{-1/2}(c_1\sin x+c_2\cos x)+4x+8\ ;x^2y''+xy'+\left(x^2-\frac{1}{4}\right)y=4x^3+8x^2+3x-2$

3. Find all solutions of the equation.

a. $y'=-x$

b. $y''=x\cos x$

c. $y'''=7e^{4x}$

4. Solve the initial value problem.

a. $y'=-xe^x,\quad y(0)=1$

b. $y'=x\sin(x^2),\quad y\left(\sqrt{\frac{\pi}{2}}\right)=1$

c. $y'''=2+\sin(2x),\quad y(0)=1,\quad y'(0)=-6,\quad y''(0)=3$

5. Verify that the function is a solution of the initial value problem.

a. $y=\frac{2}{x-2};\quad y'=\frac{-y(y+1)}{x},\quad y(1)=-2$

b. $y=\frac{x^2}{3}+x-1;\quad y''=\frac{x^2-xy'+y+1}{x^2},\quad y(1)=\frac{1}{3},\quad y'(1)=\frac{5}{3}$

6. Suppose an object is launched from a point 320 feet above the earth with an initial velocity of 128 ft/sec upward, and the only force acting on it thereafter is gravity. Take $g=32$ ft/sec^2.

a. Find the highest altitude attained by the object.
b. Determine how long it takes for the object to fall to the ground.

7. Verify that

$$y=\begin{cases} e^x-1, & x\geq 0,\\ 1-e^{-x}, & x<0,\end{cases}$$

is a solution of

$$y' = |y| + 1$$

on $(-\infty, \infty)$.

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1.3: Direction Fields for First Order Equations

It's impossible to find explicit formulas for solutions of some differential equations. Even if there are such formulas, they may be so complicated that they're useless. In this case we may resort to graphical or numerical methods to get some idea of how the solutions of the given equation behave.

In Section 2.3 we'll take up the question of existence of solutions of a first order equation

$$y' = f(x, y). \tag{1.3.1}$$

In this section we'll simply assume that Equation 1.3.1 has solutions and discuss a graphical method for approximating them. In Chapter 3 we discuss numerical methods for obtaining approximate solutions of Equation 1.3.1. Recall that a solution of Equation 1.3.1 is a function $y = y(x)$ such that

$$y'(x) = f(x, y(x))$$

for all values of x in some interval, and an integral curve is either the graph of a solution or is made up of segments that are graphs of solutions. Therefore, not being able to solve Equation 1.3.1 is equivalent to not knowing the equations of integral curves of Equation 1.3.1. However, it is easy to calculate the slopes of these curves. To be specific, the slope of an integral curve of Equation 1.3.1 through a given point (x_0, y_0) is given by the number $f(x_0, y_0)$. This is the basis of **the method of direction fields**.

If f is defined on a set R, we can construct a **direction field** for Equation 1.3.1 in R by drawing a short line segment through each point (x, y) in R with slope $f(x, y)$. Of course, as a practical matter, we can't actually draw line segments through **every** point in R; rather, we must select a finite set of points in R. For example, suppose f is defined on the closed rectangular region

$$R : \{a \le x \le b, c \le y \le d\}.$$

Let

$$a = x_0 < x_1 < \cdots < x_m = b$$

be equally spaced points in $[a, b]$ and

$$c = y_0 < y_1 < \cdots < y_n = d$$

form a **rectangular grid** (Figure 1.3.1). Through each point in the grid we draw a short line segment with slope $f(x_i, y_j)$. The result is an approximation to a direction field for Equation 1.3.1 in R. If the grid points are sufficiently numerous and close together, we can draw approximate integral curves of Equation 1.3.1 by drawing curves through points in the grid tangent to the line segments associated with the points in the grid.

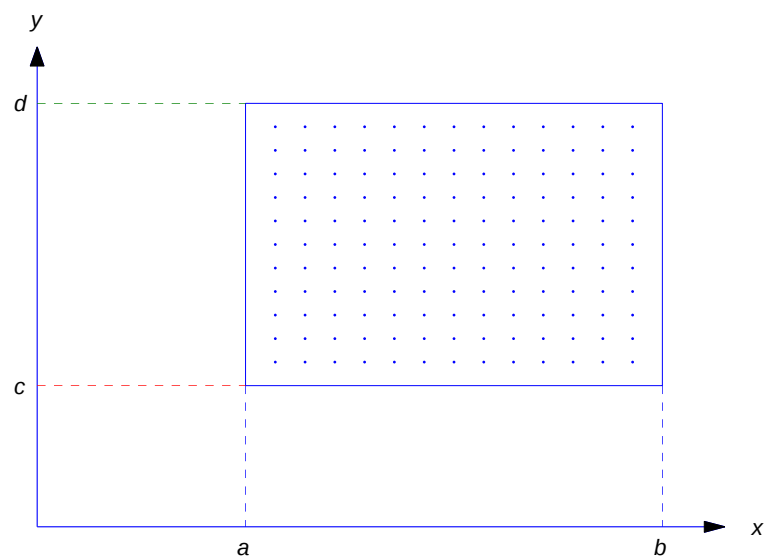

Figure 1.3.1 : A rectangular grid

Unfortunately, approximating a direction field and graphing integral curves in this way is too tedious to be done effectively by hand. However, there is software for doing this. As you'll see, the combination of direction fields and integral curves gives useful

insights into the behavior of the solutions of the differential equation even if we can't obtain exact solutions.

We'll study numerical methods for solving a single first order equation Equation 1.3.1 in Chapter 3. These methods can be used to plot solution curves of Equation 1.3.1 in a rectangular region R **if f is continuous on** R. Figures 1.3.2 , 1.3.3 , and 1.3.4 show direction fields and solution curves for the differential equations:

- $y' = \dfrac{x^2 - y^2}{1 + x^2 + y^2}$,
- $y' = 1 + xy^2$, and
- $y' = \dfrac{x - y}{1 + x^2}$.

which are all of the form Equation 1.3.1 with f continuous for all (x, y).

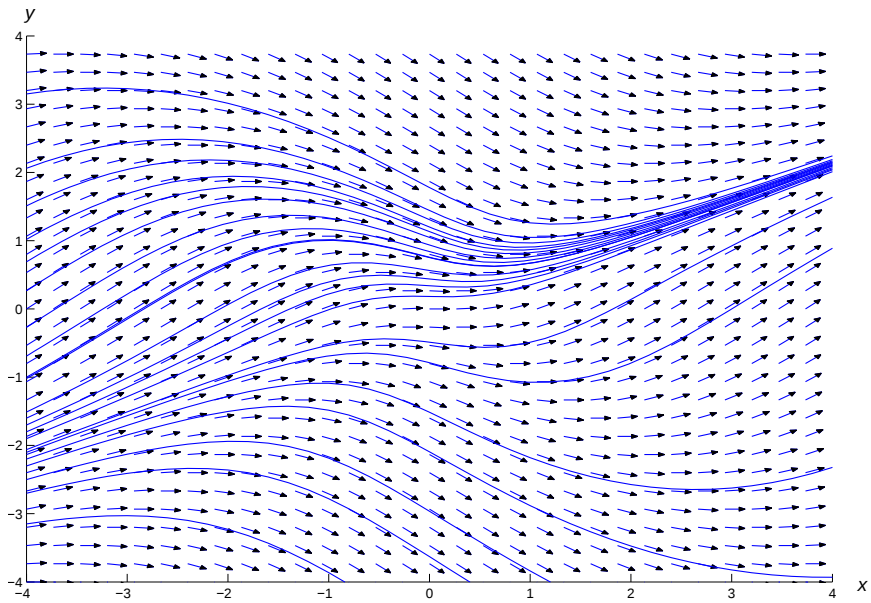

Figure 1.3.2 : A direction and integral curves for $y' = \frac{x^2 - y^2}{1 + x^2 + y^2}$.

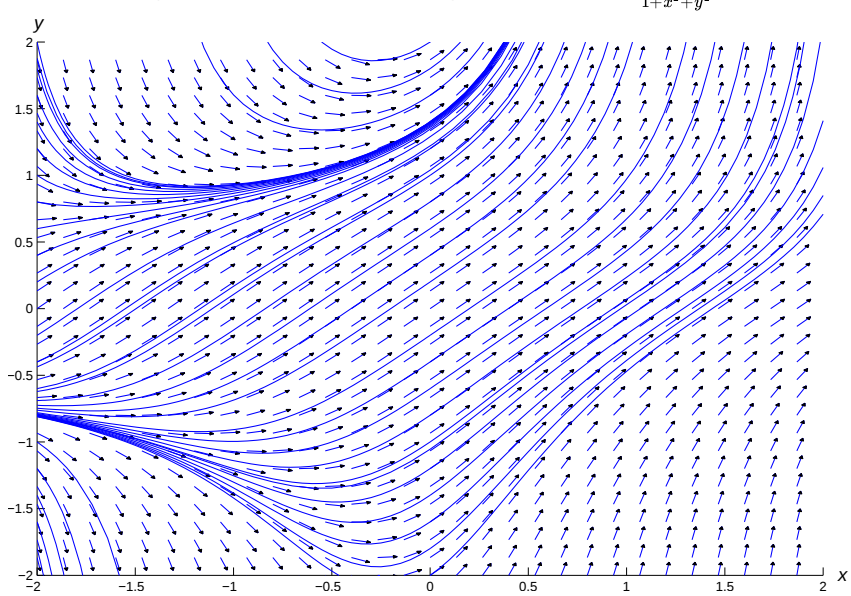

Figure 1.3.3 : A direction and integral curves for $y' = 1 + xy^2$.

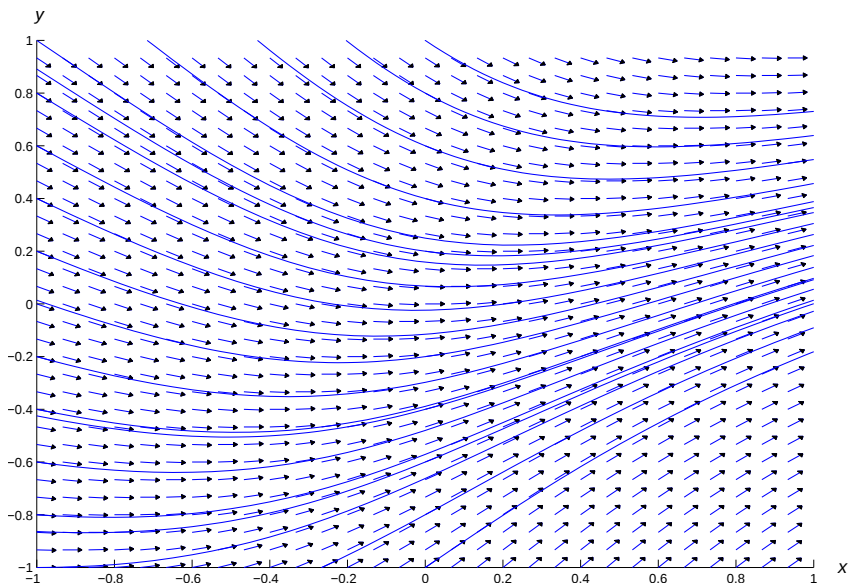

Figure 1.3.4 : A direction and integral curves for $y' = \frac{x-y}{1+x^2}$.

The methods of Chapter 3 will not work for the equation

$$y' = -x/y \tag{1.3.2}$$

if R contains part of the x-axis, since $f(x,y) = -x/y$ is undefined when $y = 0$. Similarly, they will not work for the equation

$$y' = \frac{x^2}{1 - x^2 - y^2} \tag{1.3.3}$$

if R contains any part of the unit circle $x^2 + y^2 = 1$, because the right side of Equation 1.3.3 is undefined if $x^2 + y^2 = 1$. However, Equation 1.3.2 and Equation 1.3.3 can written as

$$y' = \frac{A(x,y)}{B(x,y)} \tag{1.3.4}$$

where A and B are continuous on any rectangle R. Because of this, some differential equation software is based on numerically solving pairs of equations of the form

$$\frac{dx}{dt} = B(x,y), \quad \frac{dy}{dt} = A(x,y) \tag{1.3.5}$$

where x and y are regarded as functions of a parameter t. If $x = x(t)$ and $y = y(t)$ satisfy these equations, then

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{A(x,y)}{B(x,y)},$$

so $y = y(x)$ satisfies Equation 1.3.4.

Equations 1.3.2 and 1.3.3 can be reformulated as in Equation 1.3.4 with

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x$$

and

$$\frac{dx}{dt} = 1 - x^2 - y^2, \quad \frac{dy}{dt} = x^2,$$

respectively. Even if f is continuous and otherwise "nice" throughout R, your software may require you to reformulate the equation $y' = f(x,y)$ as

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = f(x,y),$$

which is of the form Equation 1.3.5 with $A(x,y)=f(x,y)$ and $B(x,y)=1$.

Figure 1.3.5 shows a direction field and some integral curves for Equation 1.3.2. As we saw in Example [example:1.2.1} and will verify again in Section 2.2, the integral curves of Equation 1.3.2 are circles centered at the origin.

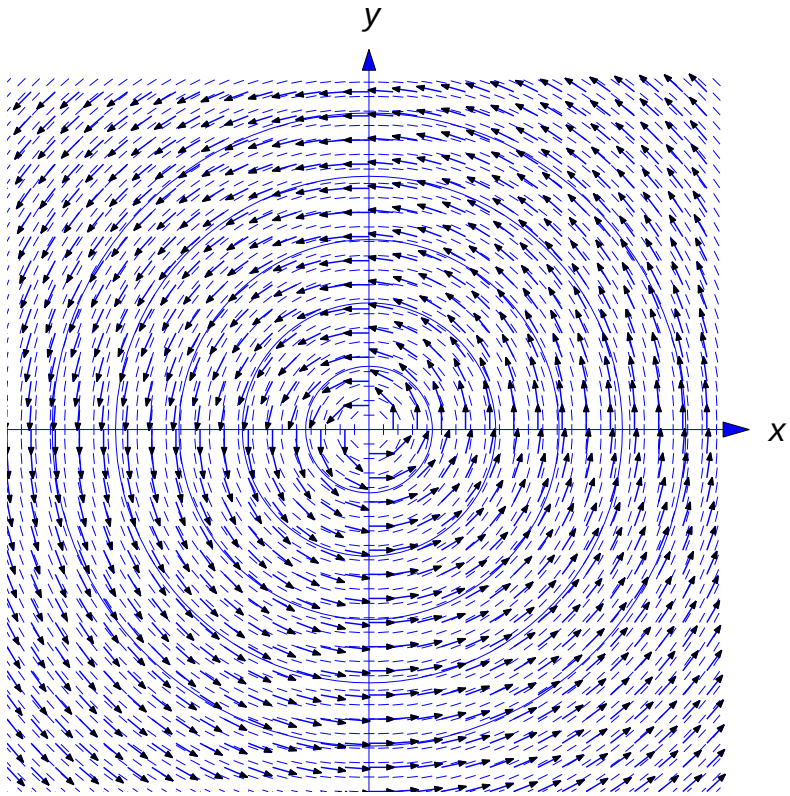

Figure 1.3.5 : A direction field and integral curves for $y'=-\frac{x}{y}$

Figure 1.3.6 shows a direction field and some integral curves for Equation 1.3.3. The integral curves near the top and bottom are solution curves. However, the integral curves near the middle are more complicated.

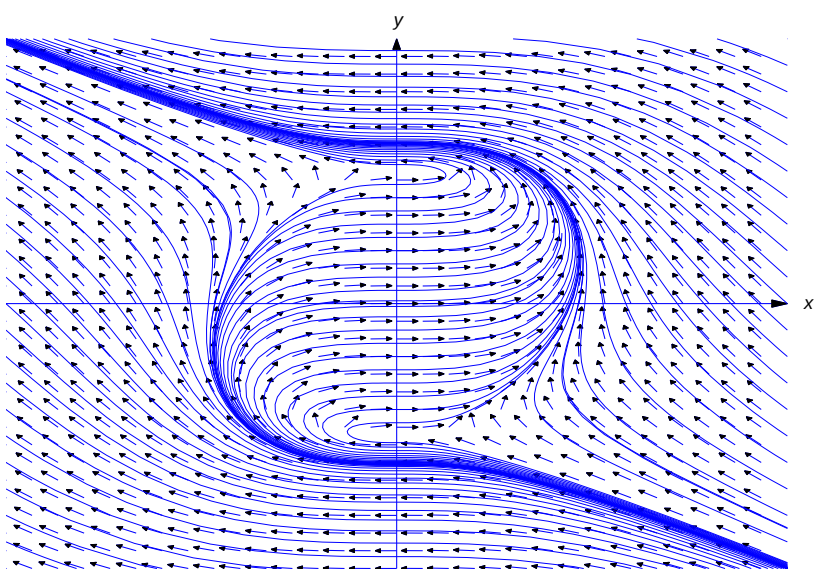

Figure 1.3.6 : A direction field and integral curves for $y'=\dfrac{x^2}{1-x^2-y^2}$.

For example, Figure 1.3.7 shows the integral curve through the origin. The vertices of the dashed rectangle are on the circle $x^2+y^2=1$ ($a\approx .846$, $b\approx .533$), where all integral curves of Equation 1.3.3 have infinite slope. There are three solution curves of Equation 1.3.3 on the integral curve in the figure: the segment above the level $y=b$ is the graph of a solution on $(-\infty,a)$, the

segment below the level $y=-b$ is the graph of a solution on $(-a,\infty)$, and the segment between these two levels is the graph of a solution on $(-a,a)$.

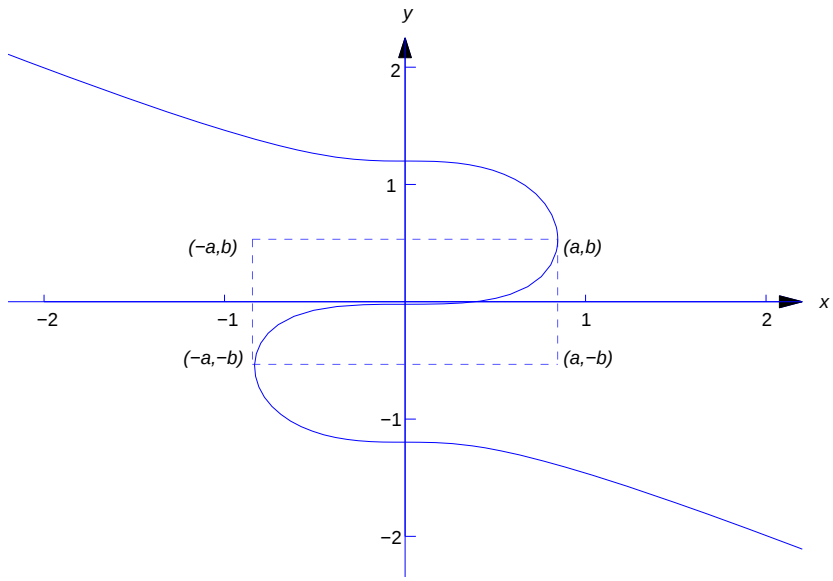

Figure 1.3.7 : An integral curve for $y'=\dfrac{x^2}{1-x^2-y^2}$.

Using Technology

As you study from this book, you'll often be asked to use computer software and graphics. Exercises with this intent are marked as (computer or calculator required), (computer and/or graphics required), or (laboratory work requiring software and/or graphics). Often you may not completely understand how the software does what it does. This is similar to the situation most people are in when they drive automobiles or watch television, and it doesn't decrease the value of using modern technology as an aid to learning. Just be careful that you use the technology as a supplement to thought rather than a substitute for it.

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1.3E: Direction Fields for First Order Equations (Exercises)

Exercises for Section 1.3

In Exercises 1–11 a direction field is drawn for the given equation. Sketch some integral curves.

1.

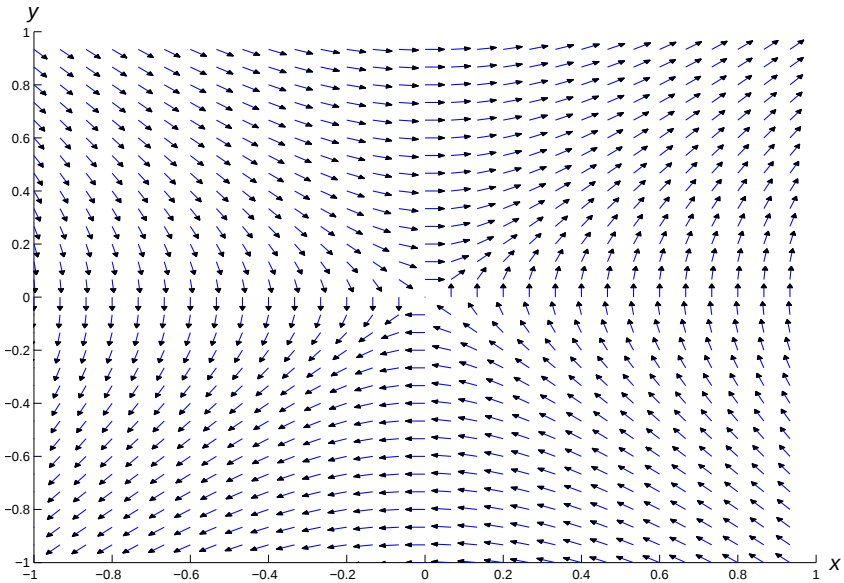

Figure 1.3E.1 : A direction field for $y' = \frac{x}{y}$

2.

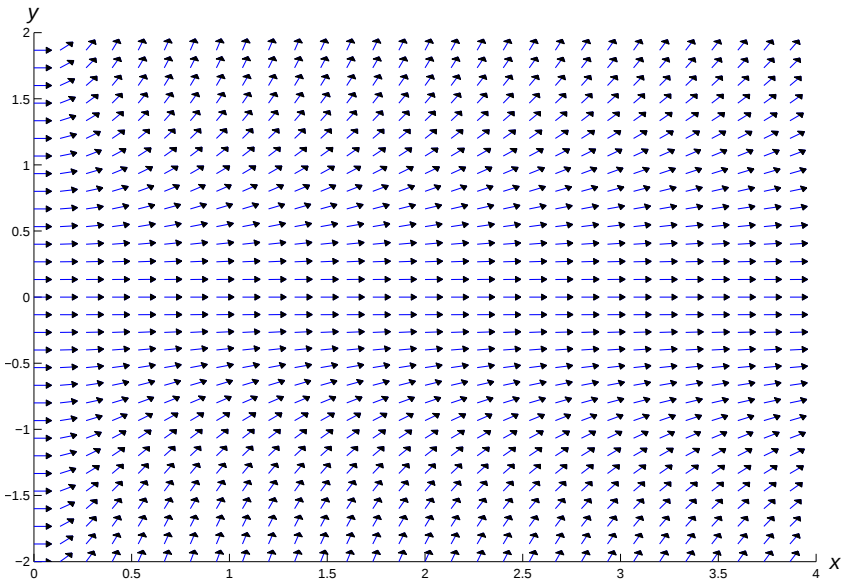

Figure 1.3E.2 : A direction field for $y' = \frac{2xy^2}{1+x^2}$

3.

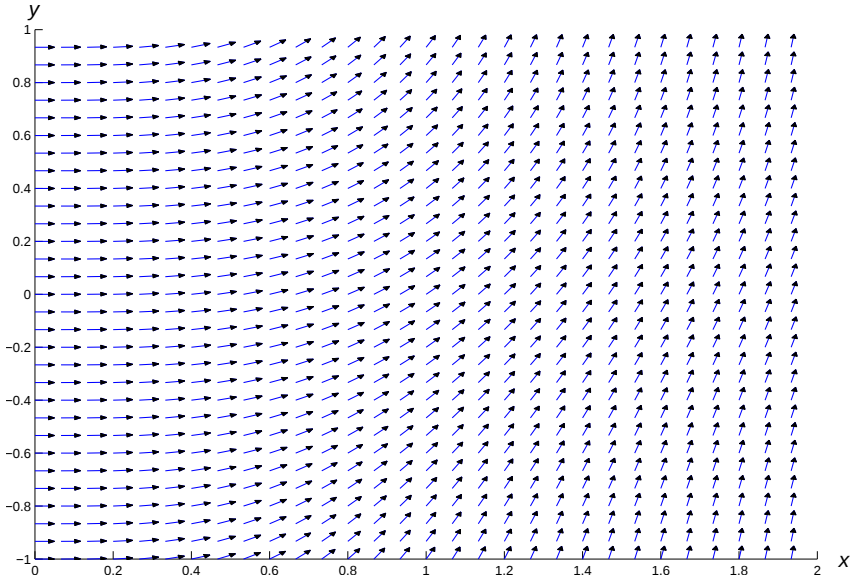

Figure 1.3E.3 : A direction field for $y' = x^2(1+y^2)$

4.

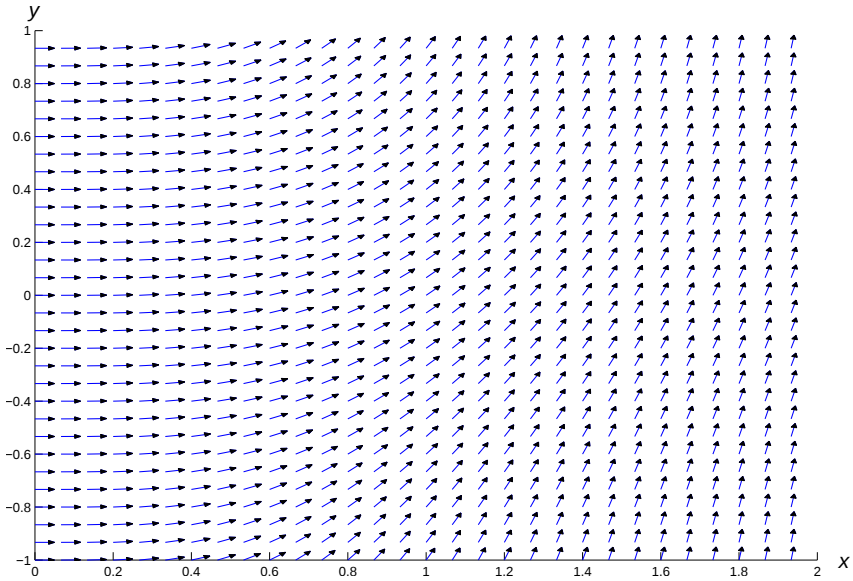

Figure 1.3E.4 : A direction field for $y' = \frac{1}{1+x^2+y^2}$

5.

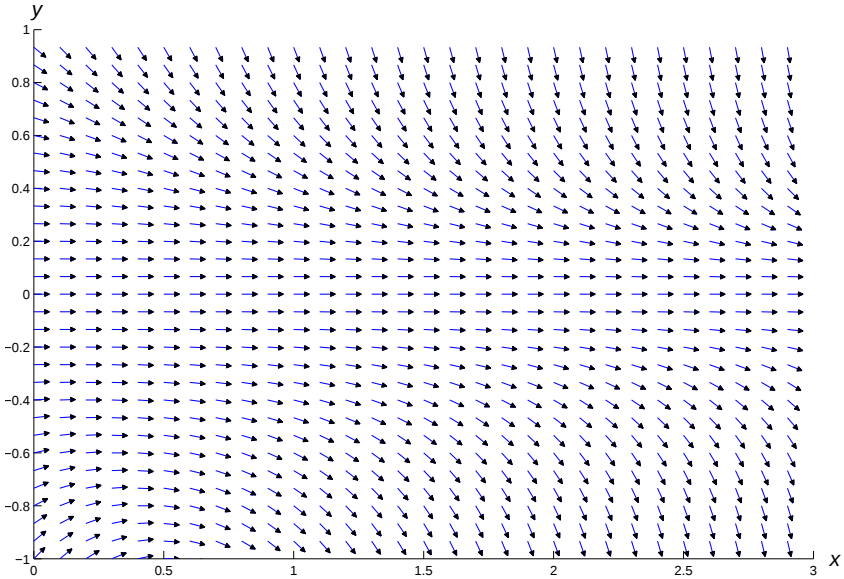

Figure 1.3E.5 : A direction field for $y' = -(2xy^2 + y^3)$

6.

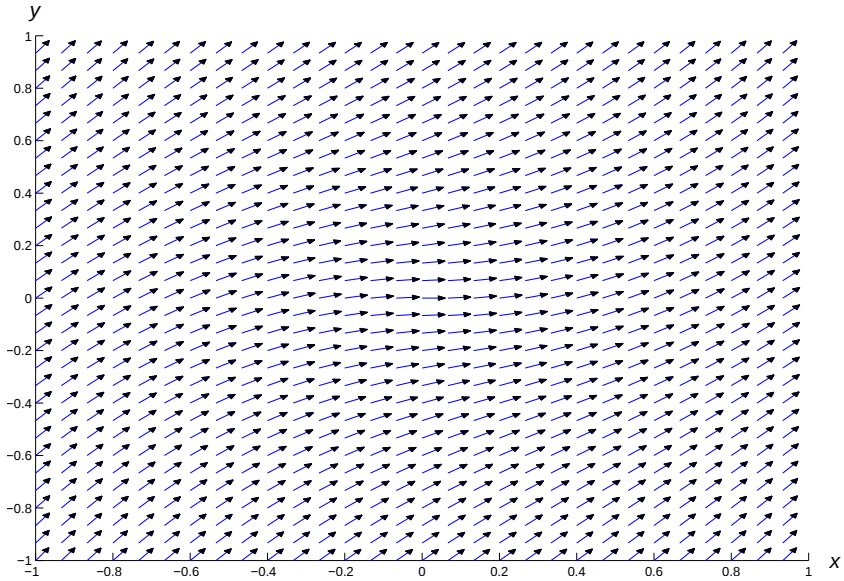

Figure 1.3E.6 : A direction field for $y' = (x^2 + y^2)^{1/2}$

7.

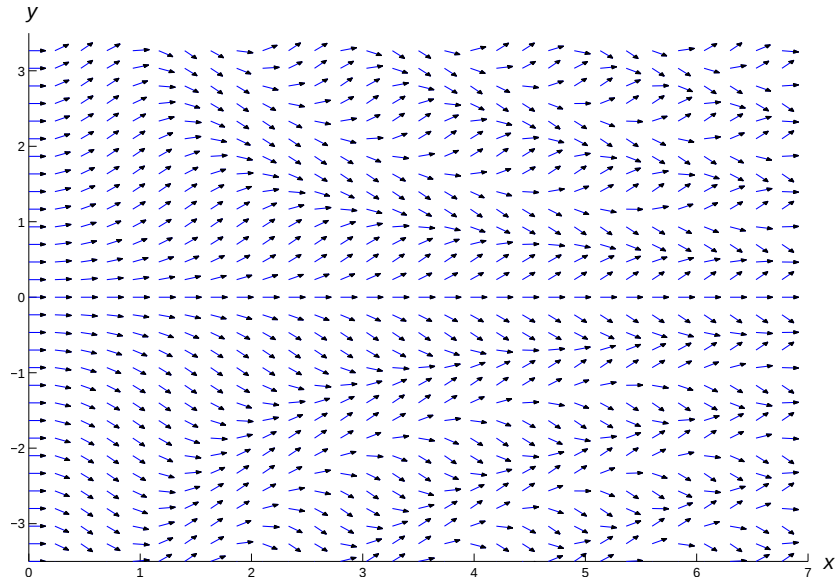

Figure 1.3E.7 : A direction field for $y' = \sin xy$

8.

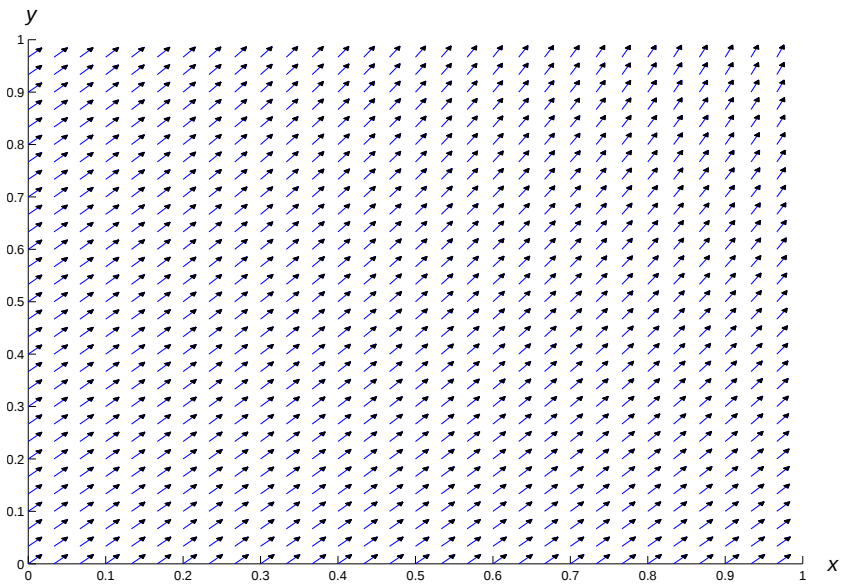

Figure 1.3E.8 : A direction field for $y' = e^{xy}$

9.

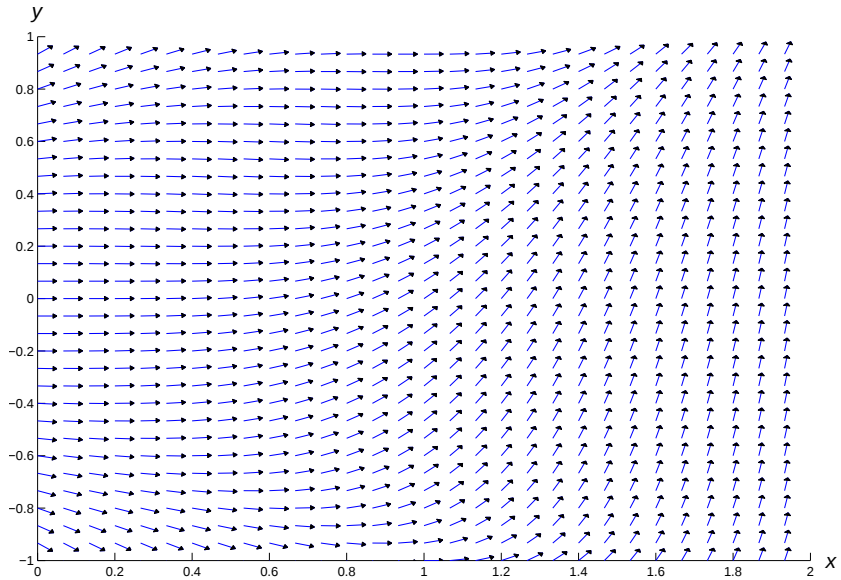

Figure 1.3E.9 : A direction field for $y' = (x - y^2)(x^2 - y)$

10.

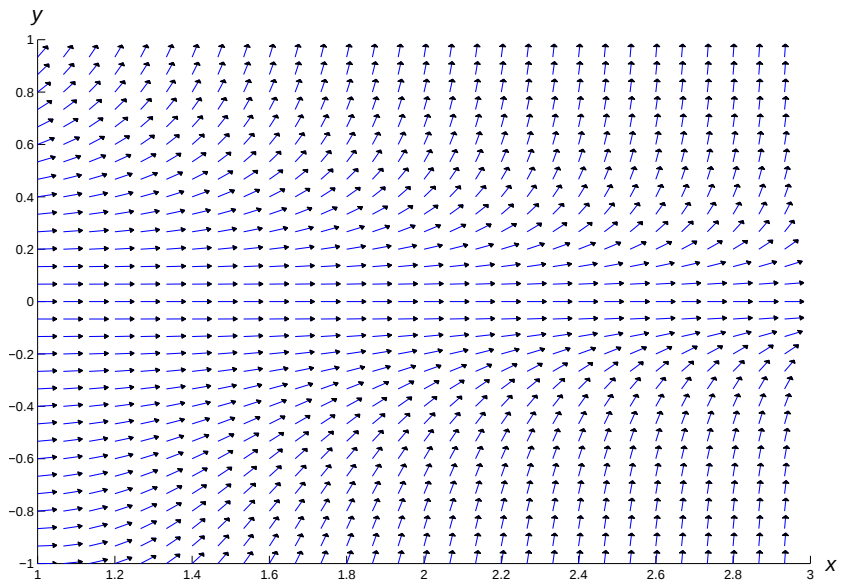

Figure 1.3E.10 : A direction field for $y' = x^3y^2 + xy^3$

11.

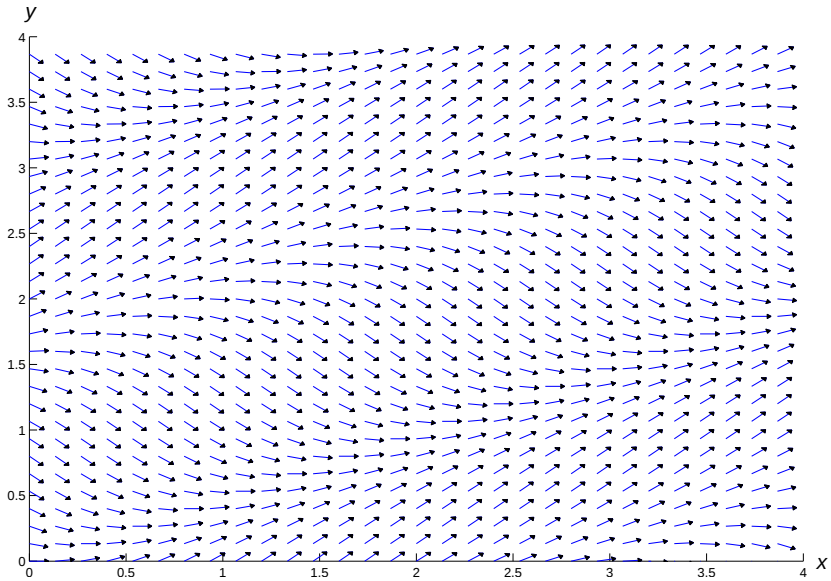

Figure 1.3E.11 : A direction field for $y' = \sin(x - 2y)$

In Exercises 12 - 22 construct a direction field and plot some integral curves in the indicated rectangular region.

12. $y' = y(y-1); \quad \{-1 \le x \le 2,\ -2 \le y \le 2\}$

13. $y' = 2 - 3xy; \quad \{-1 \le x \le 4,\ -4 \le y \le 4\}$

14. $y' = xy(y-1); \quad \{-2 \le x \le 2,\ -4 \le y \le 4\}$

15. $y' = 3x + y; \quad \{-2 \le x \le 2,\ 0 \le y \le 4\}$

16. $y' = y - x^3; \quad \{-2 \le x \le 2,\ -2 \le y \le 2\}$

17. $y' = 1 - x^2 - y^2; \quad \{-2 \le x \le 2,\ -2 \le y \le 2\}$

18. $y' = x(y^2 - 1); \quad \{-3 \le x \le 3,\ -3 \le y \le 2\}$

19. $y' = \frac{x}{y(y^2-1)}; \quad \{-2 \le x \le 2,\ -2 \le y \le 2\}$

20. $y' = \frac{xy^2}{y-1}; \quad \{-2 \le x \le 2,\ -1 \le y \le 4\}$

21. $y' = \frac{x(y^2-1)}{y}; \quad \{-1 \le x \le 1,\ -2 \le y \le 2\}$

22. $y' = -\frac{x^2+y^2}{1-x^2-y^2}; \quad \{-2 \le x \le 2,\ -2 \le y \le 2\}$

23. By suitably renaming the constants and dependent variables in the equations

$$T' = -k(T - T_m) \tag{A}$$

and

$$G' = -\lambda G + r \tag{B}$$

discussed in Section 1.2 in connection with Newton's law of cooling and absorption of glucose in the body, we can write both as

$$y' = -ay + b, \tag{C}$$

where a is a positive constant and b is an arbitrary constant. Thus, (A) is of the form (C) with $y = T$, $a = k$, and $b = kT_m$, and (B) is of the form (C) with $y = G$, $a = \lambda$, and $b = r$. We'll encounter equations of the form (C) in many other applications in Chapter 2.

Choose a positive a and an arbitrary b. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T,\ c \le y \le d\}$$

of the ty-plane. Vary T, c, and d until you discover a common property of all the solutions of (C). Repeat this experiment with various choices of a and b until you can state this property precisely in terms of a and b.

24. By suitably renaming the constants and dependent variables in the equations

$$P' = aP(1 - \alpha P) \tag{A}$$

and

$$I' = rI(S - I) \tag{B}$$

discussed in Section 1.1 in connection with Verhulst's population model and the spread of an epidemic, we can write both in the form

$$y' = ay - by^2, \tag{C}$$

where a and b are positive constants. Thus, (A) is of the form (C) with $y = P$, $a = a$, and $b = a\alpha$, and (B) is of the form (C) with $y = I$, $a = rS$, and $b = r$. In Chapter 2 we'll encounter equations of the form (C) in other applications..

Choose positive numbers a and b. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T,\ 0 \le y \le d\}$$

of the ty-plane. Vary T and d until you discover a common property of all solutions of (C) with $y(0) > 0$. Repeat this experiment with various choices of a and b until you can state this property precisely in terms of a and b.

Choose positive numbers a and b. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T,\ c \le y \le 0\}$$

of the ty-plane. Vary a, b, T and c until you discover a common property of all solutions of (C) with $y(0) < 0$.

You can verify your results later by doing Exercise 2.2.27.

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CHAPTER OVERVIEW

Chapter 2: First Order Equations

In this chapter, we study first-order differential equations for which there are general methods of solution.

2.1: Separable Equations

2.1E: Separable Equations (Exercises)

2.2: Linear First Order Equations

2.2E: Linear First Order Equations (Exercises)

2.3: Existence and Uniqueness of Solutions of Nonlinear Equations

2.3E: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises)

2.4: Transformation of Nonlinear Equations into Separable Equations

2.4E: Transformation of Nonlinear Equations into Separable Equations (Exercises)

2.5: Exact Equations

2.5E: Exact Equations (Exercises)

2.6: Integrating Factors

2.6E: Integrating Factors (Exercises)

2.7: Euler's Method

2.7E: Euler’s Method (Exercises)

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2.1: Separable Equations

A first order differential equation is *separable* if it can be written as

$$h(y)y' = g(x), \tag{2.1.1}$$

where the left side is a product of y' and a function of y and the right side is a function of x. Rewriting a separable differential equation in this form is called *separation of variables.* In Section 2.1, we used separation of variables to solve homogeneous linear equations. In this section we'll apply this method to nonlinear equations.

To see how to solve Equation 2.1.1, let's first assume that y is a solution. Let $G(x)$ and $H(y)$ be antiderivatives of $g(x)$ and $h(y)$; that is,

$$H'(y) = h(y) \quad \text{and} \quad G'(x) = g(x). \tag{2.1.2}$$

Then, from the chain rule,

$$\frac{d}{dx}H(y(x)) = H'(y(x))y'(x) = h(y)y'(x).$$

Therefore Equation 2.1.1 is equivalent to

$$\frac{d}{dx}H(y(x)) = \frac{d}{dx}G(x).$$

Integrating both sides of this equation and combining the constants of integration yields

$$H(y(x)) = G(x) + c. \tag{2.1.3}$$

Although we derived this equation on the assumption that y is a solution of Equation 2.1.1, we can now view it differently: Any differentiable function y that satisfies Equation 2.1.3 for some constant c is a solution of Equation 2.1.1. To see this, we differentiate both sides of Equation 2.1.3, using the chain rule on the left, to obtain

$$H'(y(x))y'(x) = G'(x),$$

which is equivalent to

$$h(y(x))y'(x) = g(x)$$

because of Equation 2.1.2.

In conclusion, to solve Equation 2.1.1 it suffices to find functions $G = G(x)$ and $H = H(y)$ that satisfy Equation 2.1.2. Then any differentiable function $y = y(x)$ that satisfies Equation 2.1.3 is a solution of Equation 2.1.1.

Example 2.1.1

Solve the equation

$$y' = x(1 + y^2).$$

Solution

Separating variables yields

$$\frac{y'}{1 + y^2} = x.$$

Integrating yields

$$\tan^{-1} y = \frac{x^2}{2} + c$$

Therefore

$$y = \tan\left(\frac{x^2}{2} + c\right).$$

Example 2.1.2

a. Solve the equation

$$y' = -\frac{x}{y}. \tag{2.1.4}$$

b. Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = 1. \tag{2.1.5}$$

c. Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = -2. \tag{2.1.6}$$

Solution a

Separating variables in Equation 2.1.4 yields

$$yy' = -x.$$

Integrating yields

$$\frac{y^2}{2} = -\frac{x^2}{2} + c, \quad \text{or equivalently} \quad x^2 + y^2 = 2c.$$

The last equation shows that c must be positive if y is to be a solution of Equation 2.1.4 on an open interval. Therefore we let $2c = a^2$ (with $a > 0$) and rewrite the last equation as

$$x^2 + y^2 = a^2. \tag{2.1.7}$$

This equation has two differentiable solutions for y in terms of x:

$$y = \sqrt{a^2 - x^2}, \quad -a < x < a, \tag{2.1.8}$$

and

$$y = -\sqrt{a^2 - x^2}, \quad -a < x < a. \tag{2.1.9}$$

The solution curves defined by Equation 2.1.8 are semicircles above the x-axis and those defined by Equation 2.1.9 are semicircles below the x-axis (Figure 2.1.1).

Solution b

The solution of Equation 2.1.5 is positive when $x = 1$; hence, it is of the form Equation 2.1.8. Substituting $x = 1$ and $y = 1$ into Equation 2.1.7 to satisfy the initial condition yields $a^2 = 2$; hence, the solution of Equation 2.1.5 is

$$y = \sqrt{2 - x^2}, \quad -\sqrt{2} < x < \sqrt{2}.$$

Solution c

The solution of Equation 2.1.6 is negative when $x = 1$ and is therefore of the form Equation 2.1.9. Substituting $x = 1$ and $y = -2$ into Equation 2.1.7 to satisfy the initial condition yields $a^2 = 5$. Hence, the solution of Equation 2.1.6 is

$$y = -\sqrt{5 - x^2}, \quad -\sqrt{5} < x < \sqrt{5}.$$

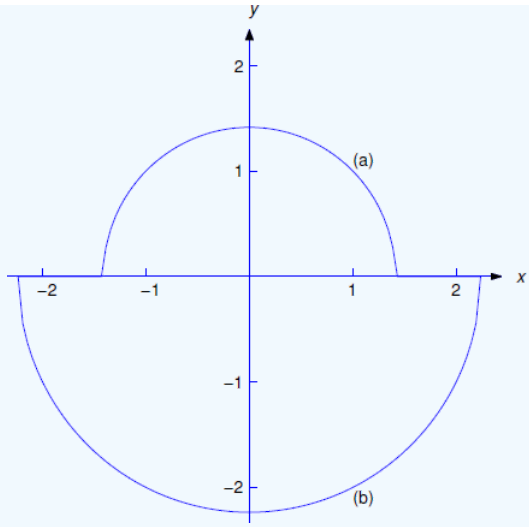

Figure 2.1.1: **(a)** $y=\sqrt{2-x^2}$, $-\sqrt{2}<x<\sqrt{2}$; **(b)** $y=-\sqrt{5-x^2}$, $-\sqrt{5}<x<\sqrt{5}$

Implicit Solutions of Separable Equations

In Examples $2.1.1$ and $2.1.2$ we were able to solve the equation $H(y)=G(x)+c$ to obtain explicit formulas for solutions of the given separable differential equations. As we'll see in the next example, this isn't always possible. In this situation we must broaden our definition of a solution of a separable equation. The next theorem provides the basis for this modification. We omit the proof, which requires a result from advanced calculus called as the *implicit function theorem*.

Theorem 2.1.1: implicit function theorem

Suppose $g=g(x)$ is continous on (a,b) and $h=h(y)$ are continuous on (c,d). Let G be an antiderivative of g on (a,b) and let H be an antiderivative of h on (c,d). Let x_0 be an arbitrary point in (a,b), let y_0 be a point in (c,d) such that $h(y_0)\neq 0$, and define

$$c=H(y_0)-G(x_0). \tag{2.1.10}$$

Then there's a function $y=y(x)$ defined on some open interval (a_1,b_1), where $a\leq a_1<x_0<b_1\leq b$, such that $y(x_0)=y_0$ and

$$H(y)=G(x)+c \tag{2.1.11}$$

for $a_1<x<b_1$. Therefore y is a solution of the initial value problem

$$h(y)y'=g(x), \quad y(x_0)=x_0. \tag{2.1.12}$$

It's convenient to say that Equation 2.1.11 with c arbitrary is an *implicit solution* of $h(y)y'=g(x)$. Curves defined by Equation 2.1.11 are integral curves of $h(y)y'=g(x)$. If c satisfies Equation 2.1.10, we'll say that Equation 2.1.11 is an *implicit solution of the initial value problem* Equation 2.1.12. However, keep these points in mind:

- For some choices of c there may not be any differentiable functions y that satisfy Equation 2.1.11.
- The function y in Equation 2.1.11 (not Equation 2.1.11 itself) is a solution of $h(y)y'=g(x)$.

Example 2.1.3

a. Find implicit solutions of

$$y'=\frac{2x+1}{5y^4+1}. \tag{2.1.13}$$

b. Find an implicit solution of

$$y'=\frac{2x+1}{5y^4+1}, \quad y(2)=1. \tag{2.1.14}$$

Solution a

Separating variables yields

$$(5y^4+1)y' = 2x+1.$$

Integrating yields the implicit solution

$$y^5+y = x^2+x+c. \tag{2.1.15}$$

of Equation 2.1.13.

Solution b

Imposing the initial condition $y(2)=1$ in Equation 2.1.15 yields $1+1=4+2+c$, so $c=-4$. Therefore

$$y^5+y = x^2+x-4$$

is an implicit solution of the initial value problem Equation 2.1.14. Although more than one differentiable function $y=y(x)$ satisfies Equation 2.1.13 near $x=1$, it can be shown that there is only one such function that satisfies the initial condition $y(1)=2$. Figure 2.1.2 shows a direction field and some integral curves for Equation 2.1.13.

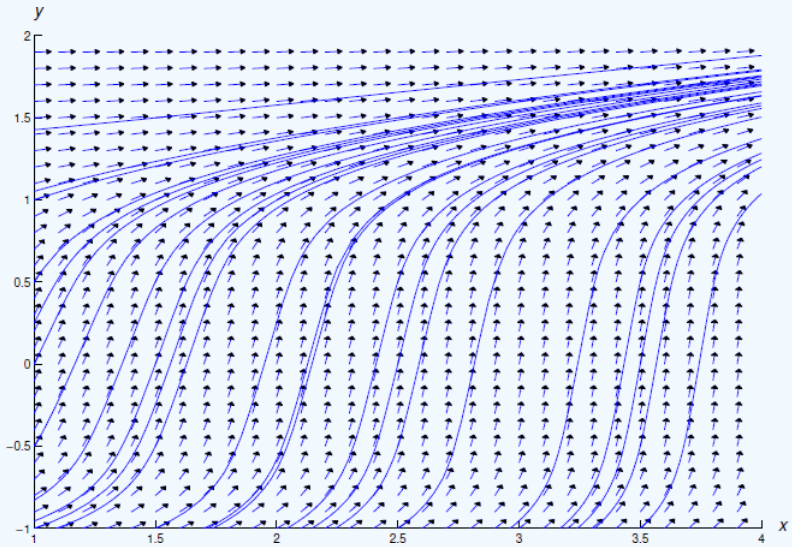

Figure 2.1.2: A direction field and integral curves for $y' = \frac{2x+1}{5y^4+1}$

Constant Solutions of Separable Equations

An equation of the form

$$y' = g(x)p(y)$$

is separable, since it can be rewritten as

$$\frac{1}{p(y)}y' = g(x).$$

However, the division by $p(y)$ is not legitimate if $p(y)=0$ for some values of y. The next two examples show how to deal with this problem.

Example 2.1.4

Find all solutions of

$$y' = 2xy^2. \tag{2.1.16}$$

Solution

Here we must divide by $p(y)=y^2$ to separate variables. This isn't legitimate if y is a solution of Equation 2.1.16 that equals zero for some value of x. One such solution can be found by inspection: $y\equiv 0$. Now suppose y is a solution of Equation 2.1.16 that isn't identically zero. Since y is continuous there must be an interval on which y is never zero. Since division by y^2 is legitimate for x in this interval, we can separate variables in Equation 2.1.16 to obtain

$$\frac{y'}{y^2}=2x.$$

Integrating this yields

$$-\frac{1}{y}=x^2+c,$$

which is equivalent to

$$y=-\frac{1}{x^2+c}. \tag{2.1.17}$$

We've now shown that if y is a solution of Equation 2.1.16 that is not identically zero, then y must be of the form Equation 2.1.17. By substituting Equation 2.1.17 into Equation 2.1.16, you can verify that Equation 2.1.17 is a solution of Equation 2.1.16. Thus, solutions of Equation 2.1.16 are $y\equiv 0$ and the functions of the form Equation 2.1.17. Note that the solution $y\equiv 0$ isn't of the form Equation 2.1.17 for any value of c.

Figure 2.1.3 shows a direction field and some integral curves for Equation 2.1.16

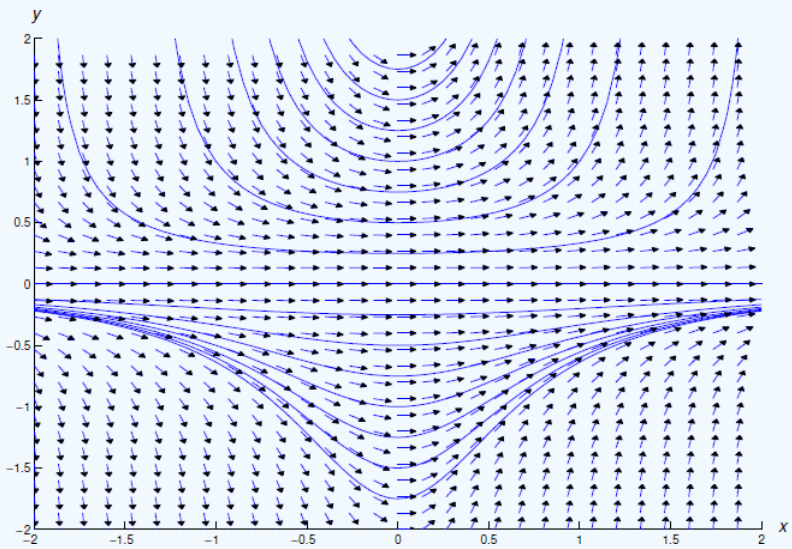

Figure 2.1.3: A direction field and integral curves for $y'=2xy^2$

Example 2.1.5

Find all solutions of

$$y'=\frac{1}{2}x(1-y^2). \tag{2.1.18}$$

Here we must divide by $p(y)=1-y^2$ to separate variables. This isn't legitimate if y is a solution of Equation 2.1.18 that equals ± 1 for some value of x. Two such solutions can be found by inspection: $y\equiv 1$ and $y\equiv -1$. Now suppose y is a solution of Equation 2.1.18 such that $1-y^2$ isn't identically zero. Since $1-y^2$ is continuous there must be an interval on which $1-y^2$ is never zero. Since division by $1-y^2$ is legitimate for x in this interval, we can separate variables in Equation 2.1.18 to obtain

$$\frac{2y'}{y^2-1}=-x.$$

A partial fraction expansion on the left yields

$$\left[\frac{1}{y-1}-\frac{1}{y+1}\right]y'=-x,$$

and integrating yields

$$\ln\left|\frac{y-1}{y+1}\right|=-\frac{x^2}{2}+k;$$

hence,

$$\left|\frac{y-1}{y+1}\right|=e^k e^{-x^2/2}.$$

Since $y(x)\neq\pm1$ for x on the interval under discussion, the quantity $(y-1)/(y+1)$ cannot change sign in this interval. Therefore we can rewrite the last equation as

$$\frac{y-1}{y+1}=ce^{-x^2/2},$$

where $c=\pm e^k$, depending upon the sign of $(y-1)/(y+1)$ on the interval. Solving for y yields

$$y=\frac{1+ce^{-x^2/2}}{1-ce^{-x^2/2}}. \tag{2.1.19}$$

We've now shown that if y is a solution of Equation 2.1.18 that is not identically equal to ±1, then y must be as in Equation 2.1.19. By substituting Equation 2.1.19 into Equation 2.1.18 you can verify that Equation 2.1.19 is a solution of Equation 2.1.18. Thus, the solutions of Equation 2.1.18 are $y\equiv1$, $y\equiv-1$ and the functions of the form Equation 2.1.19. Note that the constant solution $y\equiv1$ can be obtained from this formula by taking $c=0$; however, the other constant solution, $y\equiv-1$, cannot be obtained in this way.

Figure 2.1.4 shows a direction field and some integrals for Equation 2.1.18.

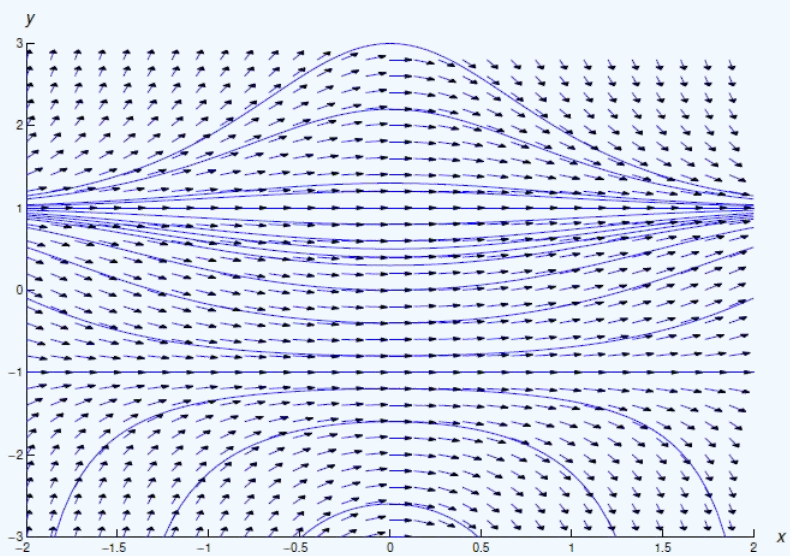

Figure 2.1.4: A direction field and integral curves for $y'=\frac{x(1-y^2)}{2}$

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2.1E: Separable Equations (Exercises)

Q2.1.1

In *Exercises 2.1.1-2.1.6* find all solutions.

1. $y' = \frac{3x^2+2x+1}{y-2}$ &

2. $(\sin x)(\sin y) + (\cos y)y' = 0$

3. $xy' + y^2 + y = 0$ &

4. $y' \ln|y| + x^2 y = 0$

5. $(3y^3 + 3y\cos y + 1)y' + \frac{(2x+1)y}{1+x^2} = 0$

6. $x^2 yy' = (y^2-1)^{3/2}$

Q2.1.2

In *Exercises 2.1.7-2.1.10* find all solutions. Also, plot a direction field and some integral curves on the indicated rectangular region.

7. $y' = x^2(1+y^2);\ \{-1 \le x \le 1,\ -1 \le y \le 1\}$

8. $y'(1+x^2) + xy = 0;\ \{-2 \le x \le 2,\ -1 \le y \le 1\}$

9. $y' = (x-1)(y-1)(y-2);\ \{-2 \le x \le 2,\ -3 \le y \le 3\}$

10. $(y-1)^2 y' = 2x+3;\ \{-2 \le x \le 2,\ -2 \le y \le 5\}$

Q2.1.3

In *Exercises 2.1.11* and *2.1.12* solve the initial value problem.

11. $y' = \frac{x^2+3x+2}{y-2}, \quad y(1) = 4$

12. $y' + x(y^2+y) = 0, \quad y(2) = 1$

Q2.1.4

In *Exercises 2.1.13-2.1.16* solve the initial value problem and graph the solution.

13. $(3y^2+4y)y' + 2x + \cos x = 0, \quad y(0) = 1$

14. $y' + \frac{(y+1)(y-1)(y-2)}{x+1} = 0, \quad y(1) = 0$

15. $y' + 2x(y+1) = 0, \quad y(0) = 2$

16. $y' = 2xy(1+y^2), \quad y(0) = 1$

Q2.1.5

In *Exercises 2.1.17-2.1.23* solve the initial value problem and find the interval of validity of the solution.

17. $y'(x^2+2) + 4x(y^2+2y+1) = 0, \quad y(1) = -1$

18. $y' = -2x(y^2-3y+2), \quad y(0) = 3$

19. $y' = \frac{2x}{1+2y}, \quad y(2) = 0$ &

20. $y' = 2y - y^2, \quad y(0) = 1$

21. $x + yy' = 0, \quad y(3) = -4$

22. $y' + x^2(y+1)(y-2)^2 = 0, \quad y(4) = 2$

23. $(x+1)(x-2)y' + y = 0, \quad y(1) = -3$

Q2.1.6

24. Solve $y' = \frac{(1+y^2)}{(1+x^2)}$ explicitly.

25. Solve $y'\sqrt{1-x^2} + \sqrt{1-y^2} = 0$ explicitly.

26. Solve $y' = \frac{\cos x}{\sin y}, \quad y(\pi) = \frac{\pi}{2}$ explicitly.

27. Solve the initial value problem

$$y' = ay - by^2, \quad y(0) = y_0. \tag{2.1E.1}$$

Discuss the behavior of the solution if a $y_0 \geq 0$; b $y_0 < 0$.

28. The population $P = P(t)$ of a species satisfies the logistic equation

$$P' = aP(1 - \alpha P) \tag{2.1E.2}$$

and $P(0) = P_0 > 0$. Find P for $t > 0$, and find $\lim_{t\to\infty} P(t)$.

29. An epidemic spreads through a population at a rate proportional to the product of the number of people already infected and the number of people susceptible, but not yet infected. Therefore, if S denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time t, then

$$I' = rI(S - I), \tag{2.1E.3}$$

where r is a positive constant. Assuming that $I(0) = I_0$, find $I(t)$ for $t > 0$, and show that $\lim_{t\to\infty} I(t) = S$.

30. The result of *Exercise 2.2.29* is discouraging: if any susceptible member of the group is initially infected, then in the long run all susceptible members are infected! On a more hopeful note, suppose the disease spreads according to the model of *Exercise 2.2.29*, but there's a medication that cures the infected population at a rate proportional to the number of infected individuals. Now the equation for the number of infected individuals becomes

$$I' = rI(S - I) - qI \tag{A}$$

where q is a positive constant.

a. Choose r and S positive. By plotting direction fields and solutions of (A) on suitable rectangular grids

$$R = \{0 \leq t \leq T,\ 0 \leq I \leq d\} \tag{2.1E.4}$$

in the (t, I)-plane, verify that if I is any solution of (A) such that $I(0) > 0$, then $\lim_{t\to\infty} I(t) = S - q/r$ if $q < rS$ and $\lim_{t\to\infty} I(t) = 0$ if $q \geq rS$.

b. To verify the experimental results of (a), use separation of variables to solve (A) with initial condition $I(0) = I_0 > 0$, and find $\lim_{t\to\infty} I(t)$.

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2.2: Linear First Order Equations

A first order differential equation is said to be **linear** if it can be written as

$$y' + p(x)y = f(x). \tag{2.2.1}$$

A first order differential equation that cannot be written like this is **nonlinear**. We say that Equation 2.2.1 is **homogeneous** if $f \equiv 0$; otherwise it is **nonhomogeneous**. Since $y \equiv 0$ is obviously a solution of the homgeneous equation

$$y' + p(x)y = 0,$$

we call it the **trivial solution**. Any other solution is **nontrivial**.

Example 2.2.1

The first order equations

$$\begin{aligned} x^2y' + 3y &= x^2 \\ xy' - 8x^2y &= \sin x \\ xy' + (\ln x)y &= 0 \\ y' &= x^2y - 2 \end{aligned}$$

are not in the form in Equation 2.2.1, but they are linear, since they can be rewritten as

$$\begin{aligned} y' + \frac{3}{x^2}y &= 1 \\ y' - 8xy &= \frac{\sin x}{x} \\ y' + \frac{\ln x}{x}y &= 0 \\ y' - x^2y &= -2 \end{aligned}$$

Example 2.2.2

Here are some nonlinear first order equations:

$$\begin{aligned} xy' + 3y^2 &= 2x && \text{(because } y \text{ is squared)} \\ yy' &= 3 && \text{(because of the product } yy') \\ y' + xe^y &= 12 && \text{(because of } e^y) \end{aligned}$$

General Solution of a Linear First Order Equation

To motivate a definition that we'll need, consider the simple linear first order equation

$$y' = \frac{1}{x^2}. \tag{2.2.2}$$

From calculus we know that y satisfies this equation if and only if

$$y = -\frac{1}{x} + c, \tag{2.2.3}$$

where c is an arbitrary constant. We call c a **parameter** and say that Equation 2.2.3 defines a **one–parameter family** of functions. For each real number c, the function defined by Equation 2.2.3 is a solution of Equation 2.2.2 on $(-\infty, 0)$ and $(0, \infty)$; moreover, every solution of Equation 2.2.2 on either of these intervals is of the form Equation 2.2.3 for some choice of c. We say that Equation 2.2.3 is **the general solution** of Equation 2.2.2.

We'll see that a similar situation occurs in connection with any first order linear equation

$$y' + p(x)y = f(x); \tag{2.2.4}$$

that is, if p and f are continuous on some open interval (a, b) then there's a unique formula $y = y(x, c)$ analogous to Equation 2.2.3 that involves x and a parameter c and has the these properties:

- For each fixed value of c, the resulting function of x is a solution of Equation 2.2.4 on (a, b).
- If y is a solution of Equation 2.2.4 on (a, b), then y can be obtained from the formula by choosing c appropriately.

We'll call $y = y(x, c)$ the **general solution** of Equation 2.2.4.

When this has been established, it will follow that an equation of the form

$$P_0(x)y' + P_1(x)y = F(x) \tag{2.2.5}$$

has a general solution on any open interval (a, b) on which P_0, P_1, and F are all continuous and P_0 has no zeros, since in this case we can rewrite Equation 2.2.5 in the form Equation 2.2.4 with $p = P_1/P_0$ and $f = F/P_0$, which are both continuous on (a, b).

To avoid awkward wording in examples and exercises, we will not specify the interval (a, b) when we ask for the general solution of a specific linear first order equation. Let's agree that this always means that we want the general solution on every open interval on which p and f are continuous if the equation is of the form Equation 2.2.4, or on which P_0, P_1, and F are continuous and P_0 has no zeros, if the equation is of the form Equation 2.2.5. We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0, P_1, and F are all continuous on an open interval (a, b), but P_0 **does** have a zero in (a, b), then Equation 2.2.5 may fail to have a general solution on (a, b) in the sense just defined. Since this isn't a major point that needs to be developed in depth, we will not discuss it further; however, see **Exercise 2.1.44** for an example.

Homogeneous Linear First Order Equations

We begin with the problem of finding the general solution of a homogeneous linear first order equation. The next example recalls a familiar result from calculus.

Example 2.2.3

Let a be a constant.

a. Find the general solution of

$$y' - ay = 0. \tag{2.2.6}$$

b. Solve the initial value problem

$$y' - ay = 0, \quad y(x_0) = y_0.$$

Solution a

(a) You already know from calculus that if c is any constant, then $y = ce^{ax}$ satisfies Equation 2.2.6. However, let's pretend you've forgotten this, and use this problem to illustrate a general method for solving a homogeneous linear first order equation.

We know that Equation 2.2.6 has the trivial solution $y \equiv 0$. Now suppose y is a nontrivial solution of Equation 2.2.6. Then, since a differentiable function must be continuous, there must be some open interval I on which y has no zeros. We rewrite Equation 2.2.6 as

$$\frac{y'}{y} = a$$

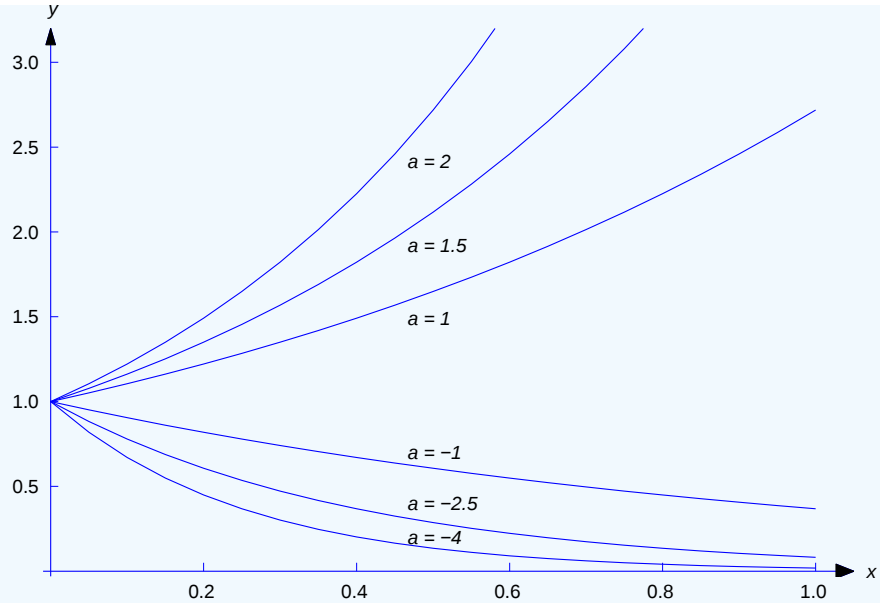

Figure 2.2.1 : Solutions of $y' - ay = 0$, $y(0) = 1$

for x in I. Integrating this shows that

$$\ln|y| = ax + k, \quad \text{so} \quad |y| = e^k e^{ax},$$

where k is an arbitrary constant. Since e^{ax} can never equal zero, y has no zeros, so y is either always positive or always negative. Therefore we can rewrite y as

$$y = ce^{ax} \tag{2.2.7}$$

where

$$c = \begin{cases} e^k & \text{if } y > 0, \\ -e^k & \text{if } y < 0. \end{cases}$$

This shows that every nontrivial solution of Equation 2.2.6 is of the form $y = ce^{ax}$ for some nonzero constant c. Since setting $c = 0$ yields the trivial solution, **all** solutions of Equation 2.2.6 are of the form Equation 2.2.7. Conversely, Equation 2.2.7 is a solution of Equation 2.2.6 for every choice of c, since differentiating Equation 2.2.7 yields $y' = ace^{ax} = ay$.

Solution b

Imposing the initial condition $y(x_0) = y_0$ yields $y_0 = ce^{ax_0}$, so $c = y_0 e^{-ax_0}$ and

$$y = y_0 e^{-ax_0} e^{ax} = y_0 e^{a(x-x_0)}.$$

Figure 2.2.1 show the graphs of this function with $x_0 = 0$, $y_0 = 1$, and various values of a.

Example 2.2.4

a. Find the general solution of

$$xy' + y = 0. \tag{2.2.8}$$

b. Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3. \tag{2.2.9}$$

Solution a

We rewrite Equation 2.2.8 as

$$y' + \frac{1}{x} y = 0, \tag{2.2.10}$$

where x is restricted to either $(-\infty, 0)$ or $(0, \infty)$. If y is a nontrivial solution of Equation 2.2.10, there must be some open interval I on which y has no zeros. We can rewrite Equation 2.2.10 as

$$\frac{y'}{y} = -\frac{1}{x}$$

for x in I. Integrating shows that

$$\ln|y| = -\ln|x| + k, \quad so \quad |y| = \frac{e^k}{|x|}.$$

Since a function that satisfies the last equation can't change sign on either $(-\infty, 0)$ or $(0, \infty)$, we can rewrite this result more simply as

$$y = \frac{c}{x} \tag{2.2.11}$$

where

$$c = \begin{cases} e^k & \text{if} y > 0, \\ -e^k & \text{if} y < 0. \end{cases}$$

We have now shown that every solution of Equation 2.2.10 is given by Equation 2.2.11 for some choice of c. (Even though we assumed that y was nontrivial to derive Equation 2.2.11, we can get the trivial solution by setting $c = 0$ in Equation 2.2.11.) Conversely, any function of the form Equation 2.2.11 is a solution of Equation 2.2.10, since differentiating Equation 2.2.11 yields

$$y' = -\frac{c}{x^2},$$

and substituting this and Equation 2.2.11 into Equation 2.2.10 yields

$$\begin{aligned} y' + \frac{1}{x}y = & \quad -\frac{c}{x^2} + \frac{1}{x}\frac{c}{x} \\ = & \quad -\frac{c}{x^2} + \frac{c}{x^2} = 0. \end{aligned}$$

Figure 2.2.2 shows the graphs of some solutions corresponding to various values of c

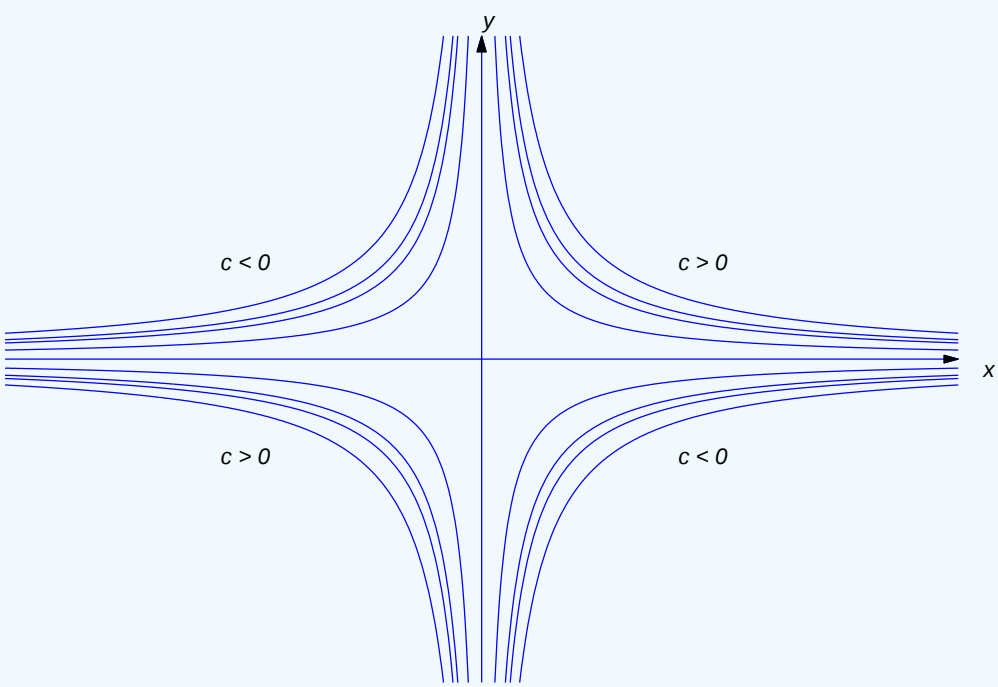

Figure 2.2.2 : Solutions of $xy' + y = 0$ on $(0, \infty)$ and $(-\infty, 0)$

Solution b

Imposing the initial condition $y(1) = 3$ in Equation 2.2.11 yields $c = 3$. Therefore the solution of Equation 2.2.9 is

$$y = \frac{3}{x}.$$

The interval of validity of this solution is $(0, \infty)$.

The results in Examples $2.2.3a$ and $2.2.4b$ are special cases of the next theorem.

Theorem 2.2.1

If p is continuous on (a, b), then the general solution of the homogeneous equation

$$y' + p(x)y = 0 \tag{2.2.12}$$

on (a, b) is

$$y = ce^{-P(x)},$$

where

$$P(x) = \int p(x)\, dx \tag{2.2.13}$$

is any antiderivative of p on (a, b); that is ,

$$P'(x) = p(x), \quad a < x < b \tag{2.2.14}$$

Proof

If $y = ce^{-P(x)}$, differentiating y and using Equation 2.2.14 shows that

$$y' = -P'(x)ce^{-P(x)} = -p(x)ce^{-P(x)} = -p(x)y,$$

so $y' + p(x)y = 0$; that is, y is a solution of Equation 2.2.12, for any choice of c.

Now we'll show that any solution of Equation 2.2.12 can be written as $y = ce^{-P(x)}$ for some constant c. The trivial solution can be written this way, with $c = 0$. Now suppose y is a nontrivial solution. Then there's an open subinterval I of (a, b) on which y has no zeros. We can rewrite Equation 2.2.12 as

$$\frac{y'}{y} = -p(x) \tag{2.2.15}$$

for x in I. Integrating Equation 2.2.15 and recalling Equation 2.2.13 yields

$$\ln|y| = -P(x) + k,$$

where k is a constant. This implies that

$$|y| = e^k e^{-P(x)}.$$

Since P is defined for all x in (a, b) and an exponential can never equal zero, we can take $I = (a, b)$, so y has zeros on (a, b) (a, b), so we can rewrite the last equation as $y = ce^{-P(x)}$, where

$$c = \begin{cases} e^k & \text{if } y > 0 \text{ on } (a, b), \\ -e^k & \text{if } y < 0 \text{ on } (a, b). \end{cases}$$

REMARK: Rewriting a first order differential equation so that one side depends only on y and y' and the other depends only on x is called separation of variables. We did this in Examples 2.2.3 and 2.2.4 , and in rewriting Equation 2.2.12 and Equation 2.2.15. We will apply this method to nonlinear equations in Section 2.2.

Linear Nonhomogeneous First Order Equations

We'll now solve the nonhomogeneous equation

$$y' + p(x)y = f(x). \tag{2.2.16}$$

When considering this equation we call

$$y' + p(x)y = 0$$

the **complementary equation**.

We'll find solutions of Equation 2.2.16 in the form $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation and u is to be determined. This method of using a solution of the complementary equation to obtain solutions of a nonhomogeneous equation is a special case of a method called **variation of parameters**, which you'll encounter several times in this book. (Obviously, u can't be constant, since if it were, the left side of Equation 2.2.16 would be zero. Recognizing this, the early users of this method viewed u as a "parameter" that varies; hence, the name "variation of parameters.")

If

$$y = uy_1, \quad \text{then} \quad y' = u'y_1 + uy_1'.$$

Substituting these expressions for y and y' into Equation 2.2.16 yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x),$$

which reduces to

$$u'y_1 = f(x), \tag{2.2.17}$$

since y_1 is a solution of the complementary equation; that is,

$$y_1' + p(x)y_1 = 0.$$

In the proof of Theorem 2.2.1 we saw that y_1 has no zeros on an interval where p is continuous. Therefore we can divide Equation 2.2.17 through by y_1 to obtain

$$u' = f(x)/y_1(x).$$

We can integrate this (introducing a constant of integration), and multiply the result by y_1 to get the general solution of Equation 2.2.16. Before turning to the formal proof of this claim, let's consider some examples.

Example 2.2.5

Find the general solution of

$$y' + 2y = x^3e^{-2x}. \tag{2.2.18}$$

By applying a of Example 2.1.3 with $a = -2$, we see that $y_1 = e^{-2x}$ is a solution of the complementary equation $y' + 2y = 0$. Therefore we seek solutions of Equation 2.2.18 in the form $y = ue^{-2x}$, so that

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y' + 2y = u'e^{-2x} - 2ue^{-2x} + 2ue^{-2x} = u'e^{-2x}. \tag{2.2.19}$$

Therefore y is a solution of Equation 2.2.18 if and only if

$$u'e^{-2x} = x^3e^{-2x} \quad \text{or, equivalently,} \quad u' = x^3.$$

Therefore

$$u = \frac{x^4}{4} + c,$$

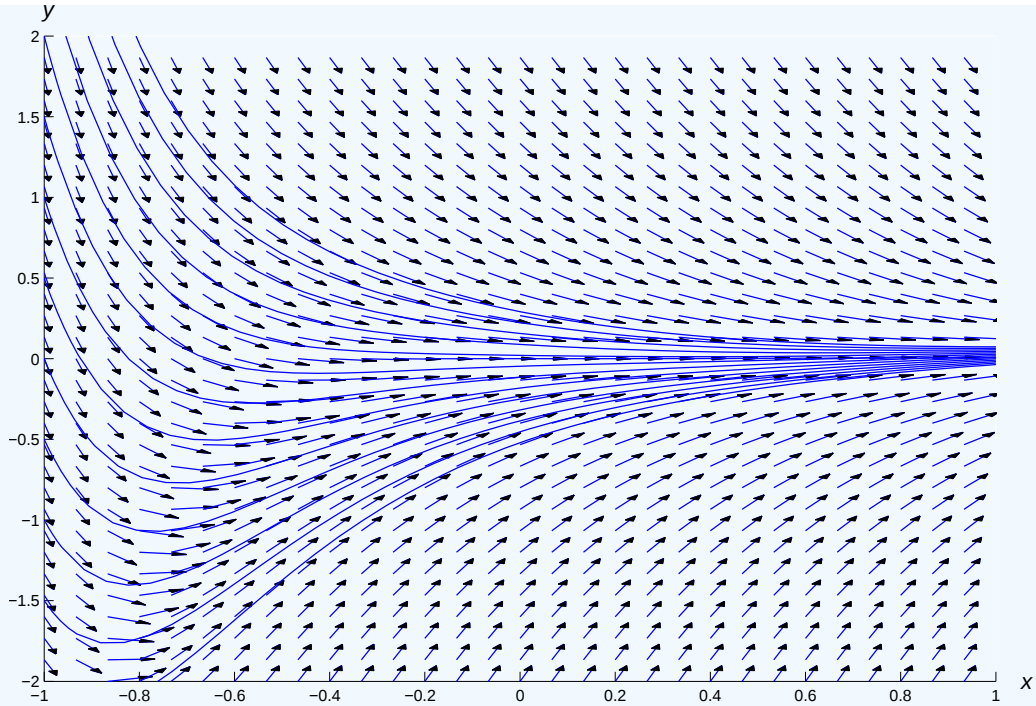

Figure 2.2.3 : A direction field and integral curves for $y' + 2y = x^2 e^{-2x}$

and

$$y = ue^{-2x} = e^{-2x}\left(\frac{x^4}{4} + c\right)$$

is the general solution of Equation 2.2.18.

Figure 2.2.3 shows a direction field and some integral curves for Equation 2.2.18.

Example 2.2.6

Find the general solution

$$y' + (\cot x)y = x \csc x. \tag{2.2.20}$$

Solve the initial value problem

$$y' + (\cot x)y = x \csc x, \quad y(\pi/2) = 1. \tag{2.2.21}$$

Solution

Here $p(x) = \cot x$ and $f(x) = x \csc x$ are both continuous except at the points $x = r\pi$, where r is an integer. Therefore we seek solutions of Equation 2.2.20 on the intervals $(r\pi, (r+1)\pi)$. We need a nontrival solution y_1 of the complementary equation; thus, y_1 must satisfy $y_1' + (\cot x)y_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = -\cot x = -\frac{\cos x}{\sin x}. \tag{2.2.22}$$

Integrating this yields

$$\ln|y_1| = -\ln|\sin x|,$$

where we take the constant of integration to be zero since we need only **one** function that satisfies Equation 2.2.22. Clearly $y_1 = 1/\sin x$ is a suitable choice. Therefore we seek solutions of Equation 2.2.20 in the form

$$y = \frac{u}{\sin x},$$

so that

$$y' = \frac{u'}{\sin x} - \frac{u\cos x}{\sin^2 x} \tag{2.2.23}$$

and

$$\begin{aligned} y' + (\cot x)y &= \tfrac{u'}{\sin x} - \tfrac{u\cos x}{\sin^2 x} + \tfrac{u\cot x}{\sin x} \\ &= \tfrac{u'}{\sin x} - \tfrac{u\cos x}{\sin^2 x} + \tfrac{u\cos x}{\sin^2 x} \\ &= \tfrac{u'}{\sin x}. \end{aligned} \tag{2.2.24}$$

Therefore y is a solution of Equation 2.2.20 if and only if

$$u'/\sin x = x\csc x = x/\sin x \quad \text{or, equivalently,} \quad u' = x.$$

Integrating this yields

$$u = \frac{x^2}{2} + c, \quad \text{and} \quad y = \frac{u}{\sin x} = \frac{x^2}{2\sin x} + \frac{c}{\sin x}. \tag{2.2.25}$$

is the general solution of Equation 2.2.20 on every interval $(r\pi, (r+1)\pi)$ ($r =$integer).

b. Imposing the initial condition $y(\pi/2) = 1$ in Equation 2.2.25 yields

$$1 = \frac{\pi^2}{8} + c \text{ or} c = 1 - \frac{\pi^2}{8}.$$

Thus,

$$y = \frac{x^2}{2\sin x} + \frac{(1-\pi^2/8)}{\sin x}$$

is a solution of Equation 2.2.20. The interval of validity of this solution is $(0, \pi)$; Figure 2.1.4 shows its graph.

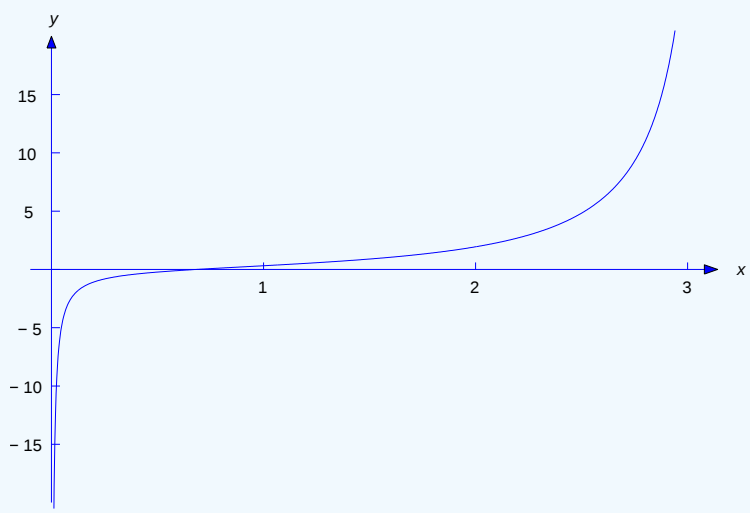

Figure 2.2.4 : Solution of $y' + (\cot x)y = x\csc x, y(\pi/2) = 1$

REMARK: It wasn't necessary to do the computations 2.2.23 and 2.2.24 in Example 2.2.6 , since we showed in the discussion preceding Example 2.1.5 that if $y = uy_1$ where $y_1' + p(x)y_1 = 0$, then $y' + p(x)y = u'y_1$. We did these computations so you would see this happen in this specific example. We recommend that you include these "unnecesary" computations in doing exercises, until you're confident that you really understand the method. After that, omit them.

We summarize the method of variation of parameters for solving

$$y' + p(x)y = f(x) \tag{2.2.26}$$

as follows:

a. Find a function y_1 such that

$$\frac{y_1'}{y_1} = -p(x).$$

For convenience, take the constant of integration to be zero.

b. Write

$$y = uy_1 \tag{2.2.27}$$

to remind yourself of what you're doing.

c. Write $u'y_1 = f$ and solve for u'; thus, $u' = f/y_1$.

d. Integrate u' to obtain u, with an arbitrary constant of integration.

e. Substitute u into Equation 2.2.27 to obtain y.

To solve an equation written as

$$P_0(x)y' + P_1(x)y = F(x),$$

we recommend that you divide through by $P_0(x)$ to obtain an equation of the form Equation 2.2.26 and then follow this procedure.

Solutions in Integral Form

Sometimes the integrals that arise in solving a linear first order equation can't be evaluated in terms of elementary functions. In this case the solution must be left in terms of an integral.

Example 2.2.7

Find the general solution of

$$y' - 2xy = 1.$$

Solve the initial value problem

$$y' - 2xy = 1, \quad y(0) = y_0. \tag{2.2.28}$$

Solution

a. To apply variation of parameters, we need a nontrivial solution y_1 of the complementary equation; thus, $y_1' - 2xy_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = 2x.$$

Integrating this and taking the constant of integration to be zero yields

$$\ln|y_1| = x^2, \quad \text{so} \quad |y_1| = e^{x^2}.$$

We choose $y_1 = e^{x^2}$ and seek solutions of Equation 2.2.28 in the form $y = ue^{x^2}$, where

$$u'e^{x^2} = 1, \quad \text{so} \quad u' = e^{-x^2}.$$

Therefore

$$u = c + \int e^{-x^2}\,dx,$$

but we can't simplify the integral on the right because there's no elementary function with derivative equal to e^{-x^2}. Therefore the best available form for the general solution of Equation 2.2.28 is

$$y = ue^{x^2} = e^{x^2}\left(c + \int e^{-x^2}\,dx\right). \tag{2.2.29}$$

b. Since the initial condition in Equation 2.2.28 is imposed at $x_0 = 0$, it is convenient to rewrite Equation 2.2.29 as

$$y = e^{x^2}\left(c + \int_0^x e^{-t^2}\,dt\right), \quad \text{since} \quad \int_0^0 e^{-t^2}\,dt = 0$$

Setting $x = 0$ and $y = y_0$ here shows that $c = y_0$. Therefore the solution of the initial value problem is

$$y = e^{x^2}\left(y_0 + \int_0^x e^{-t^2}\,dt\right). \tag{2.2.30}$$

For a given value of y_0 and each fixed x, the integral on the right can be evaluated by numerical methods. An alternate procedure is to apply the numerical integration procedures discussed in Chapter 3 directly to the initial value problem Equation 2.2.28. Figure 2.1.5 shows graphs of of Equation 2.2.30 for several values of y_0.

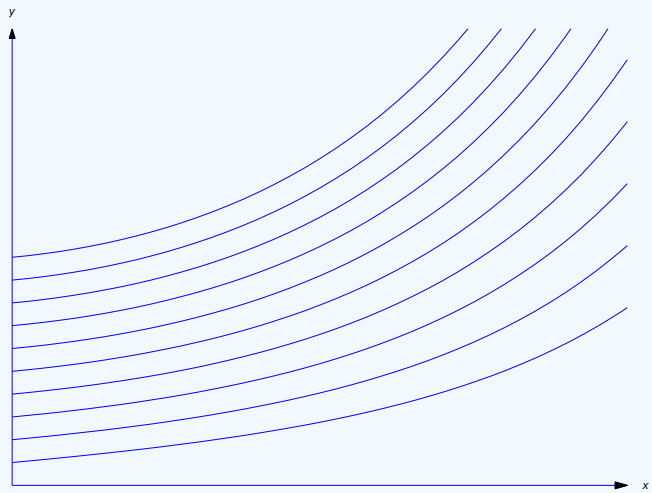

Figure 2.2.5 : Solution of $y' - 2xy = 1, y(0) = y_0$

An Existence and Uniqueness Theorem

The method of variation of parameters leads to this theorem.

Theorem 2.2.2 Existence and Uniqueness Theorem

Suppose p and f are continuous on an open interval (a, b), and let y_1 be any nontrivial solution of the complementary equation

$$y' + p(x)y = 0$$

on (a, b). Then:

a. The general solution of the nonhomogeneous equation

$$y' + p(x)y = f(x) \tag{2.2.31}$$

on (a, b) is

$$y = y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right). \tag{2.2.32}$$

b. If x_0 is an arbitrary point in (a, b) and y_0 is an arbitrary real number , then the initial value problem

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has the unique solution

$$y = y_1(x)\left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^x \frac{f(t)}{y_1(t)}\,dt\right)$$

on (a, b).

Proof

(a) To show that Equation 2.2.32 is the general solution of Equation 2.2.31 on (a,b), we must prove that:

i. If c is any constant, the function y in Equation 2.2.32 is a solution of Equation 2.2.31 on (a,b).
ii. If y is a solution of Equation 2.2.31 on (a,b) then y is of the form Equation 2.2.32 for some constant c.

To prove (i), we first observe that any function of the form Equation 2.2.32 is defined on (a,b), since p and f are continuous on (a,b). Differentiating Equation 2.2.32 yields

$$y' = y_1'(x)\left(c + \int f(x)/y_1(x)\,dx\right) + f(x).$$

Since $y_1' = -p(x)y_1$, this and Equation 2.2.32 imply that

$$\begin{aligned} y' &= -p(x)y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right) + f(x) \\ &= -p(x)y(x) + f(x), \end{aligned}$$

which implies that y is a solution of Equation 2.2.31.

To prove (ii), suppose y is a solution of Equation 2.2.31 on (a,b). From the proof of Theorem 2.1.1, we know that y_1 has no zeros on (a,b), so the function $u = y/y_1$ is defined on (a,b). Moreover, since

$$y' = -py + f \quad \text{and} \quad y_1' = -py_1,$$

$$\begin{aligned} u' &= \frac{y_1 y' - y_1' y}{y_1^2} \\ &= \frac{y_1(-py+f) - (-py_1)y}{y_1^2} = \frac{f}{y_1}. \end{aligned}$$

Integrating $u' = f/y_1$ yields

$$u = \left(c + \int f(x)/y_1(x)\,dx\right),$$

which implies Equation 2.2.32 since $y = uy_1$.

(b) We've proved (a), where $\int f(x)/y_1(x)\,dx$ in Equation 2.2.32 is an arbitrary antiderivative of f/y_1. Now it is convenient to choose the antiderivative that equals zero when $x = x_0$, and write the general solution of Equation 2.2.31 as

$$y = y_1(x)\left(c + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt\right).$$

Since

$$y(x_0) = y_1(x_0)\left(c + \int_{x_0}^{x_0} \frac{f(t)}{y_1(t)}\,dt\right) = cy_1(x_0),$$

we see that $y(x_0) = y_0$ if and only if $c = y_0/y_1(x_0)$.

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2.2E: Linear First Order Equations (Exercises)

Q2.2.1

In *Exercises 2.2.1-2.2.5* find the general solution.

1. $y' + ay = 0$ (a=constant)

2. $y' + 3x^2 y = 0$

3. $xy' + (\ln x)y = 0$

4. $xy' + 3y = 0$

5. $x^2 y' + y = 0$

Q2.2.2

In *Exercises 2.2.6-2.2.11* solve the initial value problem.

6. $y' + \left(\frac{1+x}{x}\right) y = 0, \quad y(1) = 1$

7. $xy' + \left(1 + \frac{1}{\ln x}\right) y = 0, \quad y(e) = 1$

8. $xy' + (1 + x \cot x)y = 0, \quad y\left(\frac{\pi}{2}\right) = 2$

9. $y' - \left(\frac{2x}{1+x^2}\right) y = 0, \quad y(0) = 2$

10. $y' + \frac{k}{x} y = 0, \quad y(1) = 3 \quad (k = \text{constant})$

11. $y' + (\tan kx)y = 0, \quad y(0) = 2 \quad (k = \text{constant})$

Q2.2.3

In *Exercises 2.2.12-2.2.15* find the general solution. Also, plot a direction field and some integral curves on the rectangular region $\{-2 \le x \le 2, -2 \le y \le 2\}$.

12. $y' + 3y = 1$

13. $y' + \left(\frac{1}{x} - 1\right) y = -\frac{2}{x}$

14. $y' + 2xy = xe^{-x^2}$

15. $y' + \frac{2x}{1+x^2} y = \frac{e^{-x}}{1+x^2}$

Q2.2.4

In *Exercises 2.2.16-2.2.24* find the general solution.

16. $y' + \frac{1}{x} y = \frac{7}{x^2} + 3$

17. $y' + \frac{4}{x-1} y = \frac{1}{(x-1)^5} + \frac{\sin x}{(x-1)^4}$

18. $xy' + (1 + 2x^2)y = x^3 e^{-x^2}$

19. $xy' + 2y = \frac{2}{x^2} + 1$

20. $y' + (\tan x)y = \cos x$

21. $(1 + x)y' + 2y = \frac{\sin x}{1+x}$

22. $(x - 2)(x - 1)y' - (4x - 3)y = (x - 2)^3$

23. $y' + (2 \sin x \cos x)y = e^{-\sin^2 x}$

24. $x^2 y' + 3xy = e^x$

Q2.2.5

In *Exercises 2.2.25-2.2.29* solve the initial value problem and sketch the graph of the solution.

25. $y' + 7y = e^{3x}, \quad y(0) = 0$

26. $(1 + x^2)y' + 4xy = \frac{2}{1+x^2}, \quad y(0) = 1$

27. $xy' + 3y = \frac{2}{x(1+x^2)}, \quad y(-1) = 0$

28. $y' + (\cot x)y = \cos x, \quad y\left(\frac{\pi}{2}\right) = 1$

29. $y' + \frac{1}{x}y = \frac{2}{x^2} + 1, \quad y(-1) = 0$

Q2.2.6

In *Exercises 2.2.30-2.2.37*, solve the initial value problem.

30. $(x-1)y' + 3y = \frac{1}{(x-1)^3} + \frac{\sin x}{(x-1)^2}, \quad y(0) = 1$

31. $xy' + 2y = 8x^2, \quad y(1) = 3$

32. $xy' - 2y = -x^2, \quad y(1) = 1$

33. $y' + 2xy = x, \quad y(0) = 3$

34. $(x-1)y' + 3y = \frac{1+(x-1)\sec^2 x}{(x-1)^3}, \quad y(0) = -1$

35. $(x+2)y' + 4y = \frac{1+2x^2}{x(x+2)^3}, \quad y(-1) = 2$

36. $(x^2-1)y' - 2xy = x(x^2-1), \quad y(0) = 4$

37. $(x^2-5)y' - 2xy = -2x(x^2-5), \quad y(2) = 7$

Q2.2.7

In *Exercises 2.2.38-2.2.42* solve the initial value problem and leave the answer in a form involving a definite integral. (You can solve these problems numerically by methods discussed in Chapter 3.)

38. $y' + 2xy = x^2, \quad y(0) = 3$

39. $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2$

40. $y' + y = \frac{e^{-x}\tan x}{x}, \quad y(1) = 0$

41. $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1$

42. $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2$

43. Experiments indicate that glucose is absorbed by the body at a rate proportional to the amount of glucose present in the bloodstream. Let λ denote the (positive) constant of proportionality. Now suppose glucose is injected into a patient's bloodstream at a constant rate of r units per unit of time. Let $G = G(t)$ be the number of units in the patient's bloodstream at time $t > 0$. Then

$$G' = -\lambda G + r, \tag{2.2E.1}$$

where the first term on the right is due to the absorption of the glucose by the patient's body and the second term is due to the injection. Determine G for $t > 0$, given that $G(0) = G_0$. Also, find $\lim_{t\to\infty} G(t)$.

44.

(a) Plot a direction field and some integral curves for

$$xy' - 2y = -1 \tag{A}$$

on the rectangular region $\{-1 \le x \le 1, -.5 \le y \le 1.5\}$. What do all the integral curves have in common?

(b) Show that the general solution of (A) on $(-\infty, 0)$ and $(0, \infty)$ is

$$y=\frac{1}{2}+cx^2. \tag{2.2E.2}$$

(c) Show that y is a solution of (A) on $(-\infty,\infty)$ if and only if

$$y=\begin{cases}\frac{1}{2}+c_1x^2, & x\geq 0,\\ \frac{1}{2}+c_2x^2, & x<0,\end{cases} \tag{2.2E.3}$$

where c_1 and c_2 are arbitrary constants.

(d) Conclude from c that all solutions of (A) on $(-\infty,\infty)$ are solutions of the initial value problem

$$xy'-2y=-1, \quad y(0)=\frac{1}{2}. \tag{2.2E.4}$$

(e) Use (b) to show that if $x_0\neq 0$ and y_0 is arbitrary, then the initial value problem

$$xy'-2y=-1, \quad y(x_0)=y_0 \tag{2.2E.5}$$

has infinitely many solutions on ($-\infty,\infty$). Explain why this doesn't contradict Theorem 2.1.1.

45. Assume that all functions in this exercise are defined on a common interval (a,b).

(a) Prove: If y_1 and y_2 are solutions of

$$y'+p(x)y=f_1(x) \tag{2.2E.6}$$

and

$$y'+p(x)y=f_2(x) \tag{2.2E.7}$$

respectively, and c_1 and c_2 are constants, then $y=c_1y_1+c_2y_2$ is a solution of

$$y'+p(x)y=c_1f_1(x)+c_2f_2(x). \tag{2.2E.8}$$

(This is the *principle of superposition.)*

(b) Use (a) to show that if y_1 and y_2 are solutions of the nonhomogeneous equation

$$y'+p(x)y=f(x), \quad \text{(A)} \tag{2.2E.9}$$

then y_1-y_2 is a solution of the homogeneous equation

$$y'+p(x)y=0. \quad \text{(B)} \tag{2.2E.10}$$

(c) Use (a) to show that if y_1 is a solution of (A) and y_2 is a solution of (B), then y_1+y_2 is a solution of (A).

46. Some nonlinear equations can be transformed into linear equations by changing the dependent variable. Show that if

$$g'(y)y'+p(x)g(y)=f(x) \tag{2.2E.11}$$

where y is a function of x and g is a function of y, then the new dependent variable $z=g(y)$ satisfies the linear equation

$$z'+p(x)z=f(x). \tag{2.2E.12}$$

47. Solve by the method discussed in Exercise 46.

(a) $(\sec^2 y)y'-3\tan y=-1$

(b) $e^{y^2}\left(2yy'+\frac{2}{x}\right)=\frac{1}{x^2}$

(c) $\frac{xy'}{y}+2\ln y=4x^2$

(d) $\frac{y'}{(1+y)^2}-\frac{1}{x(1+y)}=-\frac{3}{x^2}$

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2.3: Existence and Uniqueness of Solutions of Nonlinear Equations

Although there are methods for solving some nonlinear equations, it is impossible to find useful formulas for the solutions of most. Whether we are looking for exact solutions or numerical approximations, it is useful to know conditions that imply the existence and uniqueness of solutions of initial value problems for nonlinear equations. In this section we state such a condition and illustrate it with examples.

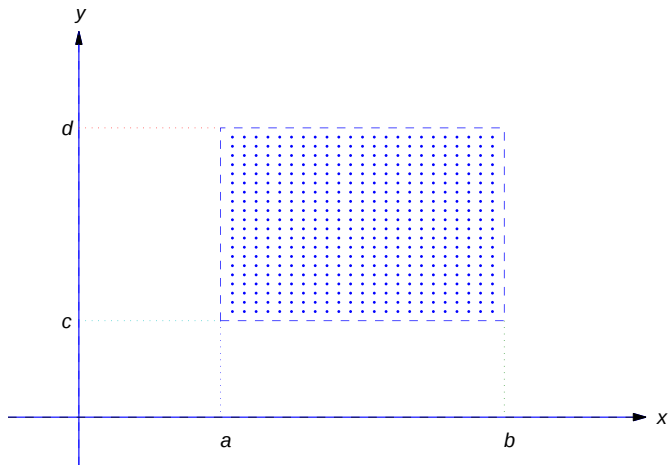

Figure 2.3.1 : An open rectangle

Some terminology: an **open rectangle R** is a set of points (x, y) such that

$$a < x < b \quad \text{and} \quad c < y < d$$

(Figure 2.3.1). We'll denote this set by $R: \{a < x < b, c < y < d\}$. "Open" means that the boundary rectangle (indicated by the dashed lines in Figure 2.3.1) is not included in R.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for first order nonlinear differential equations. We omit the proof, which is beyond the scope of this book.

Theorem 2.3.1 : existence and uniqueness

a. If f is continuous on an open rectangle

$$R: \{a < x < b, c < y < d\}$$

that contains (x_0, y_0) then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2.3.1}$$

has at least one solution on some open subinterval of (a, b) that contains x_0.

b. If both f and f_y are continuous on R then Equation 2.3.1 has a unique solution on some open subinterval of (a, b) that contains x_0

It's important to understand exactly what Theorem 2.3.1 says.

- (a) is an **existence theorem**. It guarantees that a solution exists on some open interval that contains x_0, but provides no information on how to find the solution, or to determine the open interval on which it exists. Moreover, (a) provides no information on the number of solutions that Equation 2.3.1 may have. It leaves open the possibility that Equation 2.3.1 may have two or more solutions that differ for values of x arbitrarily close to x_0. We will see in Example 2.3.6 that this can happen.
- (b) is a **uniqueness theorem**. It guarantees that Equation 2.3.1 has a unique solution on some open interval (a,b) that contains x_0. However, if $(a, b) \neq (-\infty, \infty)$, Equation 2.3.1 may have more than one solution on a larger interval that contains (a, b). For example, it may happen that $b < \infty$ and all solutions have the same values on (a, b), but two solutions y_1 and y_2 are defined on some interval (a, b_1) with $b_1 > b$, and have different values for $b < x < b_1$; thus, the graphs of the y_1 and y_2 "branch off" in different directions at $x = b$. (See Example 2.3.7 and Figure 2.3.3). In this case, continuity implies that $y_1(b) = y_2(b)$ (call their common value y), and y_1 and y_2 are both solutions of the initial value problem

$$y = f(x, y), \quad y(b) = \bar{y} \tag{2.3.2}$$

that differ on every open interval that contains b. Therefore f or f_y must have a discontinuity at some point in each open rectangle that contains (b, y), since if this were not so, 2.3.2 would have a unique solution on some open interval that contains b. We leave it to you to give a similar analysis of the case where $a > -\infty$.

Example 2.3.1

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0. \tag{2.3.3}$$

Since

$$f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{2y(1 + 2x^2)}{(1 + x^2 + y^2)^2}$$

are continuous for all (x, y), Theorem 2.3.1 implies that if (x_0, y_0) is arbitrary, then Equation 2.3.3 has a unique solution on some open interval that contains x_0.

Example 2.3.2

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{x^2 + y^2}, \quad y(x_0) = y_0. \tag{2.3.4}$$

Here

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{4x^2 y}{(x^2 + y^2)^2}$$

are continuous everywhere except at $(0, 0)$. If $(x_0, y_0) \neq (0, 0)$, there's an open rectangle R that contains (x_0, y_0) that does not contain $(0, 0)$. Since f and f_y are continuous on R, Theorem 2.3.1 implies that if $(x_0, y_0) \neq (0, 0)$ then Equation 2.3.4 has a unique solution on some open interval that contains x_0.

Example 2.3.3

Consider the initial value problem

$$y' = \frac{x + y}{x - y}, \quad y(x_0) = y_0. \tag{2.3.5}$$

Here

$$f(x, y) = \frac{x + y}{x - y} \quad \text{and} \quad f_y(x, y) = \frac{2x}{(x - y)^2}$$

are continuous everywhere except on the line $y = x$. If $y_0 \neq x_0$, there's an open rectangle R that contains (x_0, y_0) that does not intersect the line $y = x$. Since f and f_y are continuous on R, Theorem 2.3.1 implies that if $y_0 \neq x_0$, Equation 2.3.5 has a unique solution on some open interval that contains x_0.

Example 2.3.4

In Example 2.2.4, we saw that the solutions of

$$y' = 2xy^2 \tag{2.3.6}$$

are

$$y \equiv 0 \quad \text{and} \quad y = -\frac{1}{x^2+c},$$

where c is an arbitrary constant. In particular, this implies that no solution of Equation 2.3.6 other than $y \equiv 0$ can equal zero for any value of x. Show that Theorem $2.3.1b$ implies this.

We'll obtain a contradiction by assuming that Equation 2.3.6 has a solution y_1 that equals zero for some value of x, but is not identically zero. If y_1 has this property, there's a point x_0 such that $y_1(x_0) = 0$, but $y_1(x) \neq 0$ for some value of x in every open interval that contains x_0. This means that the initial value problem

$$y' = 2xy^2, \quad y(x_0) = 0 \tag{2.3.7}$$

has two solutions $y \equiv 0$ and $y = y_1$ that differ for some value of x on every open interval that contains x_0. This contradicts Theorem 2.3.1 (b), since in Equation 2.3.6 the functions

$$f(x,y) = 2xy^2 \quad \text{and} \quad f_y(x,y) = 4xy.$$

are both continuous for all (x,y), which implies that Equation 2.3.7 has a unique solution on some open interval that contains x_0.

Example 2.3.5

Consider the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(x_0) = y_0. \tag{2.3.8}$$

a. For what points (x_0, y_0) does Theorem $2.3.1a$ imply that Equation 2.3.8 has a solution?
b. For what points (x_0, y_0) does Theorem $2.3.1b$ imply that Equation 2.3.8 has a unique solution on some open interval that contains x_0?

Solution a

Since

$$f(x,y) = \frac{10}{3}xy^{2/5}$$

is continuous for all (x,y), Theorem 2.3.1 implies that Equation 2.3.8 has a solution for every (x_0, y_0).

Solution b

Here

$$f_y(x,y) = \frac{4}{3}xy^{-3/5}$$

is continuous for all (x,y) with $y \neq 0$. Therefore, if $y_0 \neq 0$ there's an open rectangle on which both f and f_y are continuous, and Theorem 2.3.1 implies that Equation 2.3.8 has a unique solution on some open interval that contains x_0.

If $y = 0$ then $f_y(x,y)$ is undefined, and therefore discontinuous; hence, Theorem 2.3.1 does not apply to Equation 2.3.8 if $y_0 = 0$.

Example 2.3.6

Example 2.3.5 leaves open the possibility that the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 0 \tag{2.3.9}$$

has more than one solution on every open interval that contains $x_0 = 0$. Show that this is true.

Solution

By inspection, $y \equiv 0$ is a solution of the differential equation

$$y' = \frac{10}{3}xy^{2/5}. \tag{2.3.10}$$

Since $y \equiv 0$ satisfies the initial condition $y(0) = 0$, it is a solution of Equation 2.3.9.

Now suppose y is a solution of Equation 2.3.10 that is not identically zero. Separating variables in Equation 2.3.10 yields

$$y^{-2/5}y' = \frac{10}{3}x$$

on any open interval where y has no zeros. Integrating this and rewriting the arbitrary constant as $5c/3$ yields

$$\frac{5}{3}y^{3/5} = \frac{5}{3}(x^2 + c).$$

Therefore

$$y = (x^2 + c)^{5/3}. \tag{2.3.11}$$

Since we divided by y to separate variables in Equation 2.3.10, our derivation of Equation 2.3.11 is legitimate only on open intervals where y has no zeros. However, Equation 2.3.11 actually defines y for all x, and differentiating Equation 2.3.11 shows that

$$y' = \frac{10}{3}x(x^2 + c)^{2/3} = \frac{10}{3}xy^{2/5}, \ -\infty < x < \infty$$

Therefore Equation 2.3.11 satisfies Equation 2.3.10 on $(-\infty, \infty)$ even if $c \le 0$, so that $y(\sqrt{|c|}) = y(-\sqrt{|c|}) = 0$. In particular, taking $c = 0$ in Equation 2.3.11 yields

$$y = x^{10/3}$$

as a second solution of Equation 2.3.9. Both solutions are defined on $(-\infty, \infty)$, and they differ on every open interval that contains $x_0 = 0$ (Figure 2.3.2). In fact, there are **four** distinct solutions of Equation 2.3.9 defined on $(-\infty, \infty)$ that differ from each other on every open interval that contains $x_0 = 0$. Can you identify the other two?

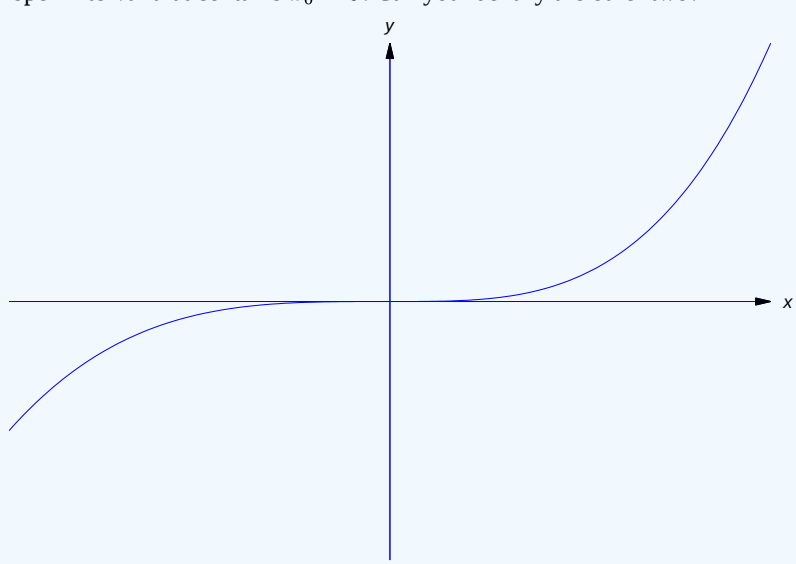

Figure 2.3.2 : Two solutions ($y = 0$ and $y = x^{1/2}$) of Equation 2.3.9 that differ on every interval containing $x_0 = 0$

Example 2.3.7

From Example 2.3.5 , the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1 \tag{2.3.12}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find a solution and determine the largest open interval (a, b) on which it is unique.

Solution

Let y be any solution of Equation 2.3.12. Because of the initial condition $y(0) = -1$ and the continuity of y, there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form Equation 2.3.11. Setting $x = 0$ and $y = -1$ in Equation 2.3.11 yields $c = -1$, so

$$y = (x^2 - 1)^{5/3} \tag{2.3.13}$$

for x in I. Therefore every solution of Equation 2.3.12 differs from zero and is given by Equation 2.3.13 on $(-1, 1)$; that is, Equation 2.3.13 is the unique solution of Equation 2.3.12 on $(-1, 1)$. This is the largest open interval on which Equation 2.3.12 has a unique solution. To see this, note that Equation 2.3.13 is a solution of Equation 2.3.12 on $(-\infty, \infty)$. From **Exercise 2.2.15**, there are infinitely many other solutions of Equation 2.3.12 that differ from Equation 2.3.13 on every open interval larger than $(-1, 1)$. One such solution is

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 \le x \le 1, \\ 0, & |x| > 1. \end{cases}$$

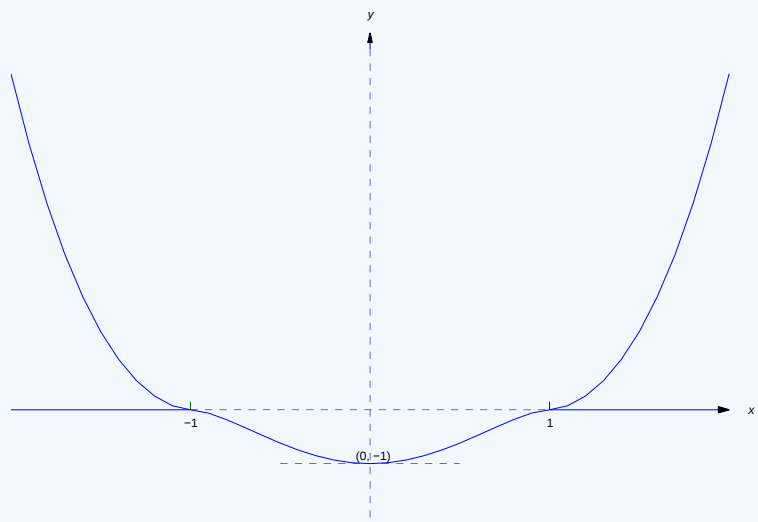

Figure 2.3.3 : Two solutions of Equation 2.3.12 on (−1,1) that coincide on (−1, 1), but on no larger open interval. (right)

Example 2.3.8

From Example 2.3.5), the initial value problem

$$y' = \frac{10}{3} x y^{2/5}, \quad y(0) = 1 \tag{2.3.14}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it is unique.

Solution

Let y be any solution of Equation 2.3.14. Because of the initial condition $y(0) = 1$ and the continuity of y, there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form Equation 2.3.11. Setting $x = 0$ and $y = 1$ in Equation 2.3.11 yields $c = 1$, so

$$y = (x^2 + 1)^{5/3} \tag{2.3.15}$$

for x in I. Therefore every solution of Equation 2.3.14 differs from zero and is given by Equation 2.3.15 on $(-\infty, \infty)$; that is, Equation 2.3.15 is the unique solution of Equation 2.3.14 on $(-\infty, \infty)$. Figure 2.3.4) shows the graph of this solution.

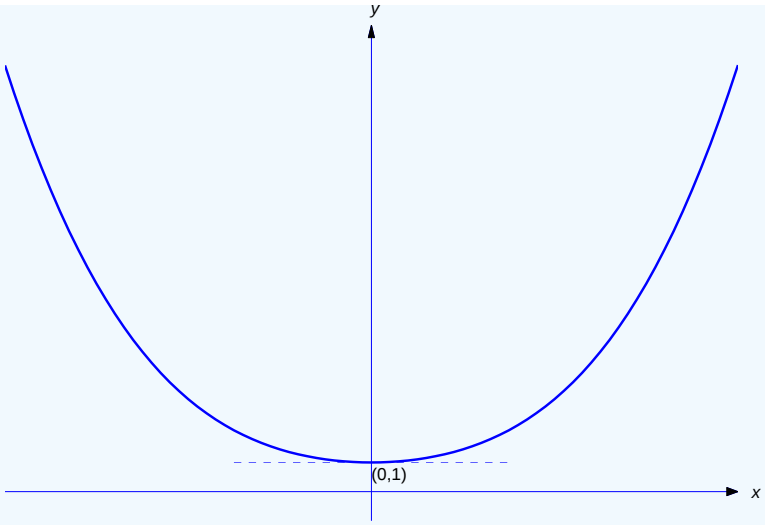

Figure 2.3.4 : The unique solution of Equation 2.3.14.

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2.3E: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises)

Q2.3.1

In **Exercises 2.3.1-2.3.13**, find all (x_0, y_0) for which Theorem 2.3.1 implies that the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ has (a) a solution and (b) a unique solution on some open interval that contains x_0.

1. $y' = \frac{x^2+y^2}{\sin x}$
2. $y' = \frac{e^x+y}{x^2+y^2}$
3. $y' = \tan xy$
4. $y' = \frac{x^2+y^2}{\ln xy}$
5. $y' = (x^2+y^2)y^{1/3}$
6. $y' = 2xy$
7. $y' = \ln(1+x^2+y^2)$
8. $y' = \frac{2x+3y}{x-4y}$
9. $y' = (x^2+y^2)^{1/2}$
10. $y' = x(y^2-1)^{2/3}$
11. $y' = (x^2+y^2)^2$
12. $y' = (x+y)^{1/2}$
13. $y' = \frac{\tan y}{x-1}$

Q2.3.2

14. Apply Theorem 2.3.1 to the initial value problem

$$y' + p(x)y = q(x), \quad y(x_0) = y_0$$

for a linear equation, and compare the conclusions that can be drawn from it to those that follow from Theorem 2.1.2.

15.

a. Verify that the function

$$y = \begin{cases} (x^2-1)^{5/3}, & -1 < x < 1, \\ 0, & |x| \geq 1, \end{cases}$$

is a solution of the initial value problem

$$y' = \frac{10}{3} x y^{2/5}, \quad y(0) = -1$$

on $(-\infty, \infty)$. HINT: You'll need the definition

$$y'(\overline{x}) = \lim_{x \to \overline{x}} \frac{y(x) - y(\overline{x})}{x - \overline{x}}$$

to verify that y satisfies the differential equation at $\overline{x} = \pm 1$.

b. Verify that if $\epsilon_i = 0$ or 1 for $i = 1, 2$ and $a, b > 1$, then the function

$$y=\begin{cases}\epsilon_1(x^2-a^2)^{5/3}, & -\infty<x<-a,\\ 0, & -a\le x\le -1,\\ (x^2-1)^{5/3}, & -1<x<1,\\ 0, & 1\le x\le b,\\ \epsilon_2(x^2-b^2)^{5/3}, & b<x<\infty,\end{cases}$$

is a solution of the initial value problem of a on $(-\infty,\infty)$.

16. Use the ideas developed in **Exercise 2.3.15** to find infinitely many solutions of the initial value problem

$$y'=y^{2/5},\quad y(0)=1$$

on $(-\infty,\infty)$.

17. Consider the initial value problem

$$y'=3x(y-1)^{1/3},\quad y(x_0)=y_0. \tag{A}$$

a. For what points (x_0,y_0) does Theorem 2.3.1 imply that (A) has a solution?
b. For what points (x_0,y_0) does Theorem 2.3.1 imply that (A) has a unique solution on some open interval that contains x_0?

18. Find nine solutions of the initial value problem

$$y'=3x(y-1)^{1/3},\quad y(0)=1$$

that are all defined on $(-\infty,\infty)$ and differ from each other for values of x in every open interval that contains $x_0=0$.

19. From Theorem 2.3.1, the initial value problem

$$y'=3x(y-1)^{1/3},\quad y(0)=9$$

has a unique solution on an open interval that contains $x_0=0$. Find the solution and determine the largest open interval on which it is unique.

20.

a. From Theorem 2.3.1, the initial value problem

$$y'=3x(y-1)^{1/3},\quad y(3)=-7 \tag{A}$$

has a unique solution on some open interval that contains $x_0=3$. Determine the largest such open interval, and find the solution on this interval.
b. Find infinitely many solutions of (A), all defined on $(-\infty,\infty)$.

21. Prove:

a. If

$$f(x,y_0)=0,\quad a<x<b, \tag{A}$$

andx_0 is in (a,b), then $y\equiv y_0$ is a solution of

$$y'=f(x,y),\quad y(x_0)=y_0$$

on (a,b).
b. If f and f_y are continuous on an open rectangle that contains (x_0,y_0) and (A) holds, no solution of $y'=f(x,y)$ other than $y\equiv y_0$ can equal y_0 at any point in (a,b).

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2.4: Transformation of Nonlinear Equations into Separable Equations

In Section 2.1, we found that the solutions of a linear nonhomogeneous equation

$$y' + p(x)y = f(x)$$

are of the form $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation

$$y' + p(x)y = 0 \tag{2.4.1}$$

and u is a solution of

$$u'y_1(x) = f(x).$$

Note that this last equation is separable, since it can be rewritten as

$$u' = \frac{f(x)}{y_1(x)}.$$

In this section we'll consider nonlinear differential equations that are not separable to begin with, but can be solved in a similar fashion by writing their solutions in the form $y = uy_1$, where y_1 is a suitably chosen known function and u satisfies a separable equation. We'll say in this case that we *transformed* the given equation into a separable equation.

Bernoulli Equations

A *Bernoulli equation* is an equation of the form

$$y' + p(x)y = f(x)y^r, \tag{2.4.2}$$

where r can be any real number other than 0 or 1. (Note that Equation 2.4.2 is linear if and only if $r = 0$ or $r = 1$.) We can transform Equation 2.4.2 into a separable equation by variation of parameters: if y_1 is a nontrivial solution of Equation 2.4.1, substituting $y = uy_1$ into Equation 2.4.2 yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x)(uy_1)^r,$$

which is equivalent to the separable equation

$$u'y_1(x) = f(x)(y_1(x))^r u^r \quad \text{or} \quad \frac{u'}{u^r} = f(x)(y_1(x))^{r-1},$$

since $y_1' + p(x)y_1 = 0$.

Example 2.4.1:

Solve the Bernoulli equation

$$y' - y = xy^2. \tag{2.4.3}$$

Since $y_1 = e^x$ is a solution of $y' - y = 0$, we look for solutions of Equation 2.4.3 in the form $y = ue^x$, where

$$u'e^x = xu^2e^{2x} \quad \text{or equivalently} \quad u' = xu^2e^x.$$

Separating variables yields

$$\frac{u'}{u^2} = xe^x,$$

and integrating yields

$$-\frac{1}{u} = (x-1)e^x + c.$$

Hence,

$$u = -\frac{1}{(x-1)e^x + c}$$

and

$$y = -\frac{1}{x - 1 + ce^{-x}}.$$

Figure 2.4.1 shows direction field and some integral curves of Equation 2.4.3.

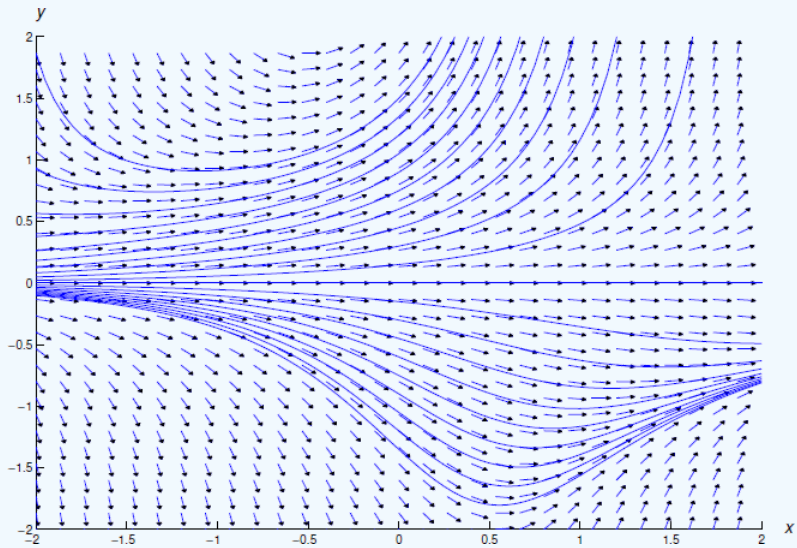

Figure 2.4.1: A direction field and integral curves for $y' - y = xy^2$

Other Nonlinear Equations That Can be Transformed Into Separable Equations

We've seen that the nonlinear Bernoulli equation can be transformed into a separable equation by the substitution $y = uy_1$ if y_1 is suitably chosen. Now let's discover a sufficient condition for a nonlinear first order differential equation

$$y' = f(x, y) \tag{2.4.4}$$

to be transformable into a separable equation in the same way. Substituting $y = uy_1$ into Equation 2.4.4 yields

$$u'y_1(x) + uy_1'(x) = f(x, uy_1(x)),$$

which is equivalent to

$$u'y_1(x) = f(x, uy_1(x)) - uy_1'(x). \tag{2.4.5}$$

If

$$f(x, uy_1(x)) = q(u)y_1'(x)$$

for some function q, then Equation 2.4.5 becomes

$$u'y_1(x) = (q(u) - u)y_1'(x), \tag{2.4.6}$$

which is separable. After checking for constant solutions $u \equiv u_0$ such that $q(u_0) = u_0$, we can separate variables to obtain

$$\frac{u'}{q(u) - u} = \frac{y_1'(x)}{y_1(x)}.$$

Homogeneous Nonlinear Equations

In the text we will consider only the most widely studied class of equations for which the method of the preceding paragraph works. Other types of equations appear in Exercises 2.4.44- 2.4.51.

The differential equation Equation 2.4.4 is said to be *homogeneous* if x and y occur in f in such a way that $f(x,y)$ depends only on the ratio y/x; that is, Equation 2.4.4 can be written as

$$y' = q(y/x), \tag{2.4.7}$$

where $q = q(u)$ is a function of a single variable. For example,

$$y' = \frac{y + xe^{-y/x}}{x} = \frac{y}{x} + e^{-y/x}$$

and

$$y' = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$$

are of the form Equation 2.4.7, with

$$q(u) = u + e^{-u} \quad \text{and} \quad q(u) = u^2 + u - 1,$$

respectively. The general method discussed above can be applied to Equation 2.4.7 with $y_1 = x$ (and therefore $y_1' = 1$). Thus, substituting $y = ux$ in Equation 2.4.7 yields

$$u'x + u = q(u),$$

and separation of variables (after checking for constant solutions $u \equiv u_0$ such that $q(u_0) = u_0$) yields

$$\frac{u'}{q(u) - u} = \frac{1}{x}.$$

Before turning to examples, we point out something that you may've have already noticed: the definition of *homogeneous equation* given here is not the same as the definition given in Section 2.1, where we said that a linear equation of the form

$$y' + p(x)y = 0$$

is homogeneous. We make no apology for this inconsistency, since we didn't create it historically, *homogeneous* has been used in these two inconsistent ways. The one having to do with linear equations is the most important. This is the only section of the book where the meaning defined here will apply.

Since y/x is in general undefined if $x = 0$, we'll consider solutions of nonhomogeneous equations only on open intervals that do not contain the point $x = 0$.

Example 2.4.2

Solve

$$y' = \frac{y + xe^{-y/x}}{x}. \tag{2.4.8}$$

Substituting $y = ux$ into Equation 2.4.8 yields

$$u'x + u = \frac{ux + xe^{-ux/x}}{x} = u + e^{-u}.$$

Simplifying and separating variables yields

$$e^u u' = \frac{1}{x}.$$

Integrating yields $e^u = \ln|x| + c$. Therefore $u = \ln(\ln|x| + c)$ and $y = ux = x\ln(\ln|x| + c)$.

Figure 2.4.2 shows a direction field and integral curves for Equation 2.4.8.

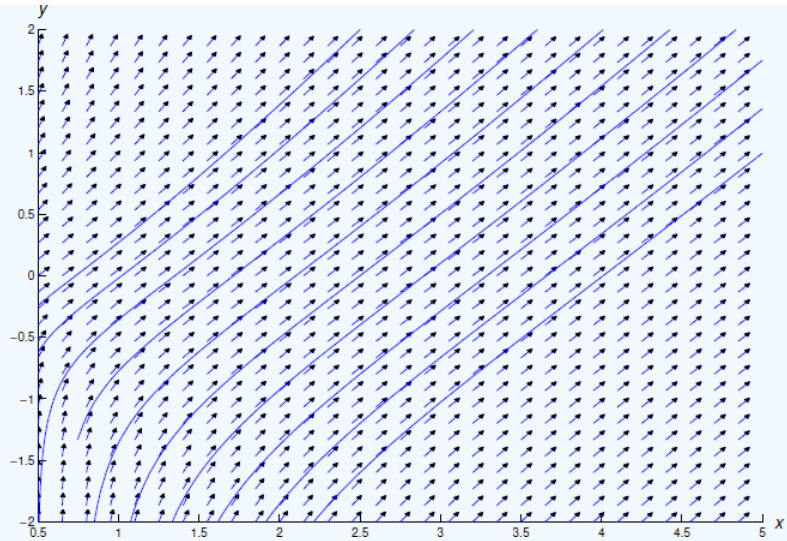

Figure 2.4.2: A direction field and some integral curves for $y' = \dfrac{y + xe^{-y/x}}{x}$

Example 2.4.3

a. Solve

$$x^2 y' = y^2 + xy - x^2. \tag{2.4.9}$$

b. Solve the initial value problem

$$x^2 y' = y^2 + xy - x^2, \quad y(1) = 2. \tag{2.4.10}$$

Solution a

We first find solutions of Equation 2.4.9 on open intervals that don't contain $x = 0$. We can rewrite Equation 2.4.9 as

$$y' = \frac{y^2 + xy - x^2}{x^2}$$

for x in any such interval. Substituting $y = ux$ yields

$$u'x + u = \frac{(ux)^2 + x(ux) - x^2}{x^2} = u^2 + u - 1,$$

so

$$u'x = u^2 - 1. \tag{2.4.11}$$

By inspection this equation has the constant solutions $u \equiv 1$ and $u \equiv -1$. Therefore $y = x$ and $y = -x$ are solutions of Equation 2.4.9. If u is a solution of Equation 2.4.11 that does not assume the values ± 1 on some interval, separating variables yields

$$\frac{u'}{u^2 - 1} = \frac{1}{x},$$

or, after a partial fraction expansion,

$$\frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right] u' = \frac{1}{x}.$$

Multiplying by 2 and integrating yields

$$\ln\left|\frac{u-1}{u+1}\right| = 2\ln|x| + k,$$

or

$$\left|\frac{u-1}{u+1}\right| = e^k x^2,$$

which holds if

$$\frac{u-1}{u+1} = cx^2 \tag{2.4.12}$$

where c is an arbitrary constant. Solving for u yields

$$u = \frac{1+cx^2}{1-cx^2}.$$

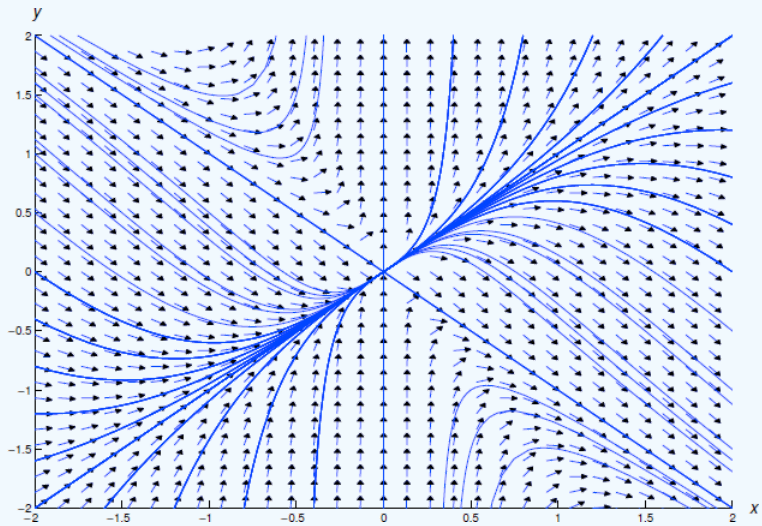

Figure 2.4.3: A direction field and integral curves for $x^2y' = y^2 + xy - x^2$

Therefore

$$y = ux = \frac{x(1+cx^2)}{1-cx^2} \tag{2.4.13}$$

is a solution of Equation 2.4.10 for any choice of the constant c. Setting $c = 0$ in Equation 2.4.13 yields the solution $y = x$. However, the solution $y = -x$ can't be obtained from Equation 2.4.13. Thus, the solutions of Equation 2.4.9 on intervals that don't contain $x = 0$ are $y = -x$ and functions of the form Equation 2.4.13.

The situation is more complicated if $x = 0$ is the open interval. First, note that $y = -x$ satisfies Equation 2.4.9 on $(-\infty, \infty)$. If c_1 and c_2 are arbitrary constants, the function

$$y = \begin{cases} \frac{x(1+c_1x^2)}{1-c_1x^2}, & a < x < 0 \\ \frac{x(1+c_2x^2}{1-c_2x^2}, & 0 \le x < b \end{cases} \tag{2.4.14}$$

is a solution of Equation 2.4.9 on (a, b), where

$$a = \begin{cases} -\frac{1}{\sqrt{c_1}} & \text{if } c_1 > 0, \\ -\infty & \text{if } c_1 \le 0, \end{cases} \quad \text{and} \quad b = \begin{cases} \frac{1}{\sqrt{c_2}} & \text{if } c_2 > 0, \\ \infty & \text{if } c_2 \le 0. \end{cases}$$

We leave it to you to verify this. To do so, note that if y is any function of the form Equation 2.4.13 then $y(0) = 0$ and $y'(0) = 1$.

Figure 2.4.3 shows a direction field and some integral curves for Equation 2.4.9.

Solution b

We could obtain c by imposing the initial condition $y(1)=2$ in Equation 2.4.13, and then solving for c. However, it is easier to use Equation 2.4.12. Since $u=y/x$, the initial condition $y(1)=2$ implies that $u(1)=2$. Substituting this into Equation 2.4.12 yields $c=1/3$. Hence, the solution of Equation 2.4.10 is

$$y=\frac{x(1+x^2/3)}{1-x^2/3}.$$

The interval of validity of this solution is $(-\sqrt{3},\sqrt{3})$. However, the largest interval on which Equation 2.4.10 has a unique solution is $(0,\sqrt{3})$. To see this, note from Equation 2.4.14 that any function of the form

$$y=\begin{cases}\frac{x(1+cx^2)}{1-cx^2}, & a<x\le 0\\ \frac{x(1+x^2/3)}{1-x^2/3} & 0\le x<\sqrt{3}\end{cases} \tag{2.4.15}$$

is a solution of Equation 2.4.10 on $(a,\sqrt{3})$, where $a=-1/\sqrt{c}$ if $c>0$ or $a=-\infty$ if $c\le 0$. Why does this not contradict Theorem 2.3.1?

Figure 2.4.4 shows several solutions of the initial value problem Equation 2.4.10. Note that these solutions coincide on $(0,\sqrt{3})$.

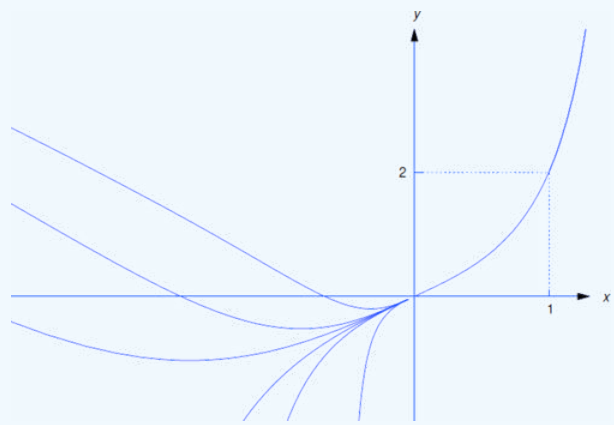

Figure 2.4.4: Solutions of $x^2y'=y^2+xy-x^2$, $y(1)=2$

In the last two examples we were able to solve the given equations explicitly. However, this is not always possible, as you'll see in the exercises.

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2.4E: Transformation of Nonlinear Equations into Separable Equations (Exercises)

Q2.4.1

In *Exercises 2.4.1-2.4.4* solve the given Bernoulli equation.

1. $y' + y = y^2$

2. $7xy' - 2y = -\frac{x^2}{y^6}$

3. $x^2y' + 2y = 2e^{1/x}y^{1/2}$

4. $(1+x^2)y' + 2xy = \frac{1}{(1+x^2)y}$

Q2.4.2

In *Exercises 2.4.5* and *2.4.6* find all solutions. Also, plot a direction field and some integral curves on the indicated rectangular region.

5. $y' - xy = x^3y^3; \quad \{-3 \le x \le 3, 2 \le y \ge 2\}$

6. $y' - \frac{1+x}{3x}y = y^4; \quad \{-2 \le x \le 2, -2 \le y \le 2\}$

Q2.4.3

In *Exercises 2.4.7-2.4.11* solve the initial value problem.

7. $y' - 2y = xy^3, \quad y(0) = 2\sqrt{2}$

8. $y' - xy = xy^{3/2}, \quad y(1) = 4$

9. $xy' + y = x^4y^4, \quad y(1) = 1/2$

10. $y' - 2y = 2y^{1/2}, \quad y(0) = 1$

11. $y' - 4y = \frac{48x}{y^2}, \quad y(0) = 1$

Q2.4.4

In *Exercises 2.4.12* and *2.4.13* solve the initial value problem and graph the solution.

12. $x^2y' + 2xy = y^3, \quad y(1) = 1/\sqrt{2}$

13. $y' - y = xy^{1/2}, \quad y(0) = 4$

Q2.4.5

14. You may have noticed that the logistic equation

$$P' = aP(1 - \alpha P) \tag{2.4E.1}$$

from Verhulst's model for population growth can be written in Bernoulli form as

$$P' - aP = -a\alpha P^2. \tag{2.4E.2}$$

This isn't particularly interesting, since the logistic equation is separable, and therefore solvable by the method studied in Section 2.2. So let's consider a more complicated model, where a is a positive constant and α is a positive continuous function of t on $[0, \infty)$. The equation for this model is

$$P' - aP = -a\alpha(t)P^2, \tag{2.4E.3}$$

a non-separable Bernoulli equation.

a. Assuming that $P(0) = P_0 > 0$, find P for $t > 0$.
b. Verify that your result reduces to the known results for the Malthusian model where $\alpha = 0$, and the Verhulst model where α is a nonzero constant.

c. Assuming that

$$\lim_{t\to\infty} e^{-at}\int_0^t \alpha(\tau)e^{a\tau}\,d\tau = L \tag{2.4E.4}$$

exists (finite or infinite), find $\lim_{t\to\infty} P(t)$.

Q2.4.6

In *Exercises 2.4.15-2.4.18* solve the equation explicitly.

15. $y' = \frac{y+x}{x}$

16. $y' = \frac{y^2+2xy}{x^2}$

17. $xy^3y' = y^4 + x^4$

18. $y' = \frac{y}{x} + \sec\frac{y}{x}$

Q2.4.7

In *Exercises 2.4.19-2.4.21* solve the equation explicitly. Also, plot a direction field and some integral curves on the indicated rectangular region.

19. $x^2y' = xy + x^2 + y^2; \quad \{-8 \le x \le 8, -8 \le y \le 8\}$

20. $xyy' = x^2 + 2y^2; \quad \{-4 \le x \le 4, -4 \le y \le 4\}$

21. $y' = \frac{2y^2+x^2e^{-(y/x)^2}}{2xy}; \quad \{-8 \le x \le 8, -8 \le y \le 8\}$

Q2.4.8

In *Exercises 2.4.22-2.4.27* solve the initial value problem.

22. $y' = \frac{xy+y^2}{x^2}, \quad y(-1) = 2$

23. $y' = \frac{x^3+y^3}{xy^2}, \quad y(1) = 3$

24. $xyy' + x^2 + y^2 = 0, \quad y(1) = 2$

25. $y' = \frac{y^2-3xy-5x^2}{x^2}, \quad y(1) = -1$

26. $x^2y' = 2x^2 + y^2 + 4xy, \quad y(1) = 1$

27. $xyy' = 3x^2 + 4y^2, \quad y(1) = \sqrt{3}$

Q2.4.9

In *Exercises 2.4.28-2.4.34* solve the given homogeneous equation implicitly.

28. $y' = \frac{x+y}{x-y}$

29. $(y'x - y)(\ln|y| - \ln|x|) = x$

30. $y' = \frac{y^3+2xy^2+x^2y+x^3}{x(y+x)^2}$

31. $y' = \frac{x+2y}{2x+y}$

32. $y' = \frac{y}{y-2x}$

33. $y' = \frac{xy^2+2y^3}{x^3+x^2y+xy^2}$

34. $y' = \frac{x^3+x^2y+3y^3}{x^3+3xy^2}$

Q2.4.10

35.

a. Find a solution of the initial value problem

$$x^2y' = y^2 + xy - 4x^2, \quad y(-1) = 0 \tag{A}$$

on the interval $(-\infty, 0)$. Verify that this solution is actually valid on $(-\infty, \infty)$.

b. Use Theorem 2.3.1 to show that (A) has a unique solution on $(-\infty, 0)$.

c. Plot a direction field for the differential equation in (A) on a square

$$\{-r \le x \le r, -r \le y \le r\}, \tag{2.4E.5}$$

where r is any positive number. Graph the solution you obtained in (a) on this field.

d. Graph other solutions of (A) that are defined on $(-\infty, \infty)$.

e. Graph other solutions of (A) that are defined only on intervals of the form $(-\infty, a)$, where is a finite positive number.

36.

a. Solve the equation

$$xyy' = x^2 - xy + y^2 \tag{A}$$

implicitly.

b. Plot a direction field for (A) on a square

$$\{0 \le x \le r, 0 \le y \le r\} \tag{2.4E.6}$$

where r is any positive number.

c. Let K be a positive integer. (You may have to try several choices for K.) Graph solutions of the initial value problems

$$xyy' = x^2 - xy + y^2, \quad y(r/2) = \frac{kr}{K}, \tag{2.4E.7}$$

for $k = 1$, 2, …, K. Based on your observations, find conditions on the positive numbers x_0 and y_0 such that the initial value problem

$$xyy' = x^2 - xy + y^2, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution (i) on $(0, \infty)$ or (ii) only on an interval (a, ∞), where $a > 0$?

d. What can you say about the graph of the solution of (B) as $x \to \infty$? (Again, assume that $x_0 > 0$ and $y_0 > 0$.)

37.

a. Solve the equation

$$y' = \frac{2y^2 - xy + 2x^2}{xy + 2x^2} \tag{A}$$

implicitly.

b. Plot a direction field for (A) on a square

$$\{-r \le x \le r, -r \le y \le r\} \tag{2.4E.8}$$

where r is any positive number. By graphing solutions of (A), determine necessary and sufficient conditions on (x_0, y_0) such that (A) has a solution on (i) $(-\infty, 0)$ or (ii) $(0, \infty)$ such that $y(x_0) = y_0$.

38. Follow the instructions of *Exercise 2.4.37* for the equation

$$y' = \frac{xy + x^2 + y^2}{xy}. \tag{2.4E.9}$$

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2.5: Exact Equations

In this section it is convenient to write first order differential equations in the form

$$M(x,y)\,dx+N(x,y)\,dy=0. \tag{2.5.1}$$

This equation can be interpreted as

$$M(x,y)+N(x,y)\,\frac{dy}{dx}=0, \tag{2.5.2}$$

where x is the independent variable and y is the dependent variable, or as

$$M(x,y)\,\frac{dx}{dy}+N(x,y)=0, \tag{2.5.3}$$

where y is the independent variable and x is the dependent variable. Since the solutions of Equation 2.5.2 and Equation 2.5.3 will often have to be left in implicit form we will say that $F(x,y)=c$ is an implicit solution of Equation 2.5.1 if every differentiable function $y=y(x)$ that satisfies $F(x,y)=c$ is a solution of Equation 2.5.2 and every differentiable function $x=x(y)$ that satisfies $F(x,y)=c$ is a solution of Equation 2.5.3

Here are some examples:

Table 2.5.1: Examples of Exact Differential Equations in three forms

Equation 2.5.1	Equation 2.5.2	Equation 2.5.3
$3x^2y^2\,dx+2x^3y\,dy=0$	$3x^2y^2+2x^3y\,\frac{dy}{dx}=0$	$3x^2y^2\,\frac{dx}{dy}+2x^3y=0$
$(x^2+y^2)\,dx+2xy\,dy=0$	$(x^2+y^2)+2xy\,\frac{dy}{dx}=0$	$(x^2+y^2)\,\frac{dx}{dy}+2xy=0$
$3y\sin x\,dx-2xy\cos x\,dy=0$	$3y\sin x-2xy\cos x\,\frac{dy}{dx}=0$	$3y\sin x\,\frac{dx}{dy}-2xy\cos x=0$

Note that a separable equation can be written as Equation 2.5.1 as

$$M(x)\,dx+N(y)\,dy=0.$$

we will develop a method for solving Equation 2.5.1 under appropriate assumptions on M and N. This method is an extension of the method of separation of variables. Before stating it we consider an example.

Example 2.5.1

Show that

$$x^4y^3+x^2y^5+2xy=c \tag{2.5.4}$$

is an implicit solution of

$$(4x^3y^3+2xy^5+2y)\,dx+(3x^4y^2+5x^2y^4+2x)\,dy=0. \tag{2.5.5}$$

Solution

Regarding y as a function of x and differentiating Equation 2.5.4 implicitly with respect to x yields

$$(4x^3y^3+2xy^5+2y)+(3x^4y^2+5x^2y^4+2x)\,\frac{dy}{dx}=0.$$

Similarly, regarding x as a function of y and differentiating Equation 2.5.4 implicitly with respect to y yields

$$(4x^3y^3+2xy^5+2y)\frac{dx}{dy}+(3x^4y^2+5x^2y^4+2x)=0.$$

Therefore Equation 2.5.4 is an implicit solution of Equation 2.5.5 in either of its two possible interpretations.

You may think Example 2.5.1 is pointless, since concocting a differential equation that has a given implicit solution is not particularly interesting. However, it illustrates the next important theorem, which we will prove by using implicit differentiation, as in Example 2.5.1.

Theorem 2.5.1

If $F = F(x, y)$ has continuous partial derivatives F_x and F_y, then

$$F(x, y) = c \tag{2.5.6}$$

(with c as a constant) is an implicit solution of the differential equation

$$F_x(x, y)\, dx + F_y(x, y)\, dy = 0. \tag{2.5.7}$$

Proof

Regarding y as a function of x and differentiating Equation 2.5.6implicitly with respect to x yields

$$F_x(x, y) + F_y(x, y)\, \frac{dy}{dx} = 0.$$

On the other hand, regarding x as a function of y and differentiating Equation 2.5.6implicitly with respect to y yields

$$F_x(x, y)\, \frac{dx}{dy} + F_y(x, y) = 0.$$

Thus, Equation 2.5.6is an implicit solution of Equation 2.5.7in either of its two possible interpretations.

We will say that the equation

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{2.5.8}$$

is **exact** on an an open rectangle R if there's a function $F = F(x, y)$ such F_x and F_y are continuous, and

$$F_x(x, y) = M(x, y) \quad \text{and} \quad F_y(x, y) = N(x, y) \tag{2.5.9}$$

for all (x, y) in R. This usage of "exact" is related to its usage in calculus, where the expression

$$F_x(x, y)\, dx + F_y(x, y)\, dy$$

(obtained by substituting Equation 2.5.9 into the left side of Equation 2.5.8) is the **exact differential of** F.

Example 2.5.1 shows that it is easy to solve Equation 2.5.8 if it is exact **and** we know a function F that satisfies Equation 2.5.9. The important questions are:

- Question 1. Given an equation Equation 2.5.8, how can we determine whether it is exact?
- Question 2. If Equation 2.5.8 is exact, how do we find a function F satisfying Equation 2.5.9?

To discover the answer to Question 1, assume that there's a function F that satisfies Equation 2.5.9 on some open rectangle R, and in addition that F has continuous mixed partial derivatives F_{xy} and F_{yx}. Then a theorem from calculus implies that

$$F_{xy} = F_{yx}. \tag{2.5.10}$$

If $F_x = M$ and $F_y = N$, differentiating the first of these equations with respect to y and the second with respect to x yields

$$F_{xy} = M_y \quad \text{and} \quad F_{yx} = N_x. \tag{2.5.11}$$

From Equation 2.5.10 and Equation 2.5.11, we conclude that a necessary condition for exactness is that $M_y = N_x$. This motivates the next theorem, which we state without proof.

Theorem 2.5.2: The Exactness Condition

Suppose M and N are continuous and have continuous partial derivatives M_y and N_x on an open rectangle R. Then

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is exact on R if and only if

$$M_y(x,y) = N_x(x,y) \tag{2.5.12}$$

for all (x,y) in R..

To help you remember the exactness condition, observe that the coefficients of dx and dy are differentiated in Equation 2.5.12 with respect to the "opposite" variables; that is, the coefficient of dx is differentiated with respect to y, while the coefficient of dy is differentiated with respect to x.

Example 2.5.2

Show that the equation

$$3x^2y\,dx + 4x^3\,dy = 0$$

is not exact on any open rectangle.

Solution

Here

$$M(x,y) = 3x^2y \quad \text{and} \quad N(x,y) = 4x^3$$

so

$$M_y(x,y) = 3x^2 \quad \text{and} N_x(x,y) = 12x^2.$$

Therefore $M_y = N_x$ on the line $x = 0$, but not on any open rectangle, so there's no function F such that $F_x(x,y) = M(x,y)$ and $F_y(x,y) = N(x,y)$ for all (x,y) on any open rectangle.

The next example illustrates two possible methods for finding a function F that satisfies the condition $F_x = M$ and $F_y = N$ if $M\,dx + N\,dy = 0$ is exact.

Example 2.5.3

Solve

$$(4x^3y^3 + 3x^2)\,dx + (3x^4y^2 + 6y^2)\,dy = 0. \tag{2.5.13}$$

Solution (Method 1)

Here

$$M(x,y) = 4x^3y^3 + 3x^2, \quad N(x,y) = 3x^4y^2 + 6y^2,$$

and

$$M_y(x,y) = N_x(x,y) = 12x^3y^2$$

for all (x,y). Therefore Theorem 2.5.2 implies that there's a function F such that

$$F_x(x,y) = M(x,y) = 4x^3y^3 + 3x^2 \tag{2.5.14}$$

and

$$F_y(x,y) = N(x,y) = 3x^4y^2 + 6y^2 \tag{2.5.15}$$

for all (x,y). To find F, we integrate Equation 2.5.14 with respect to x to obtain

$$F(x,y) = x^4y^3 + x^3 + \phi(y), \tag{2.5.16}$$

where $\phi(y)$ is the "constant" of integration. (Here ϕ is "constant" in that it is independent of x, the variable of integration.) If ϕ is any differentiable function of y then F satisfies Equation 2.5.14. To determine ϕ so that F also satisfies Equation 2.5.15, assume that ϕ is differentiable and differentiate F with respect to y. This yields

$$F_y(x,y) = 3x^4y^2 + \phi'(y).$$

Comparing this with Equation 2.5.15 shows that

$$\phi'(y) = 6y^2.$$

We integrate this with respect to y and take the constant of integration to be zero because we are interested only in finding **some** F that satisfies Equation 2.5.14 and Equation 2.5.15. This yields

$$\phi(y) = 2y^3.$$

Substituting this into Equation 2.5.16 yields

$$F(x,y) = x^4y^3 + x^3 + 2y^3. \tag{2.5.17}$$

Now Theorem 2.5.1 implies that

$$x^4y^3 + x^3 + 2y^3 = c$$

is an implicit solution of Equation 2.5.13. Solving this for y yields the explicit solution

$$y = \left(\frac{c - x^3}{2 + x^4}\right)^{1/3}.$$

Solution (Method 2)

Instead of first integrating Equation 2.5.14 with respect to x, we could begin by integrating Equation 2.5.15 with respect to y to obtain

$$F(x,y) = x^4y^3 + 2y^3 + \psi(x), \tag{2.5.18}$$

where ψ is an arbitrary function of x. To determine ψ, we assume that ψ is differentiable and differentiate F with respect to x, which yields

$$F_x(x,y) = 4x^3y^3 + \psi'(x).$$

Comparing this with Equation 2.5.14 shows that

$$\psi'(x) = 3x^2.$$

Integrating this and again taking the constant of integration to be zero yields

$$\psi(x) = x^3.$$

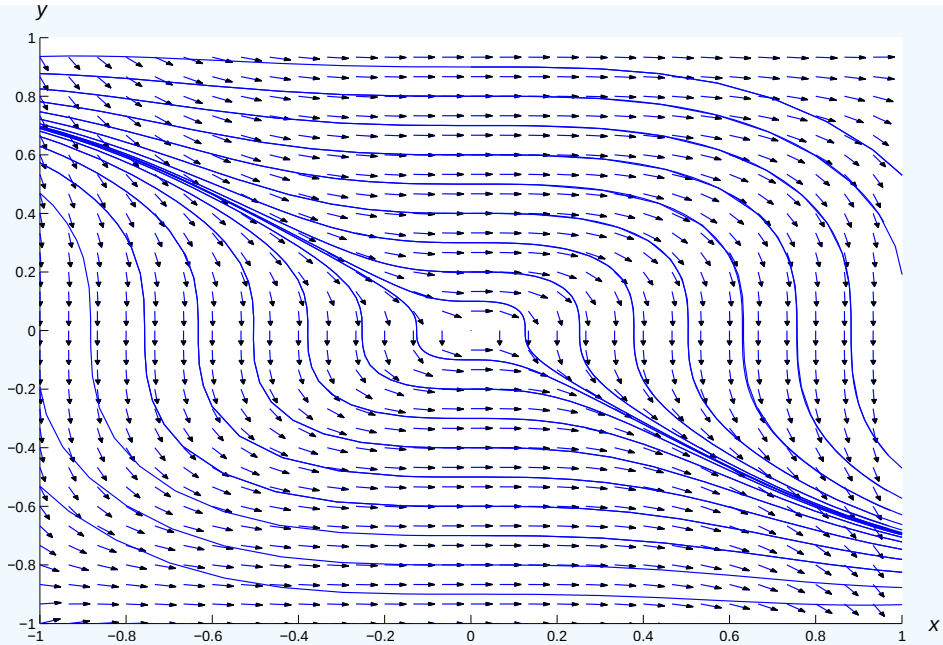

Figure 2.5.1: A direction field and integral curves for $(4x^3y^3+3x^2)\,dx+(3x^4y^2+6y^2)\,dy=0$

Substituting this into Equation 2.5.18 yields Equation 2.5.17.

Figure 2.5.1 shows a direction field and some integral curves of Equation 2.5.13.

Here's a summary of the procedure used in Method 1 of this example. You should summarize procedure used in Method 2.

HOWTO: Procedure For Solving An Exact Equation (Method 1)

- **Step 1.** Check that the equation
$$M(x,y)\,dx+N(x,y)\,dy=0 \tag{2.5.19}$$
satisfies the exactness condition $M_y=N_x$. If not, don't go further with this procedure.
- **Step 2.** Integrate
$$\frac{\partial F(x,y)}{\partial x}=M(x,y)$$
with respect to x to obtain
$$F(x,y)=G(x,y)+\phi(y), \tag{2.5.20}$$
where G is an antiderivative of M with respect to x, and ϕ is an unknown function of y.
- **Step 3.** Differentiate Equation 2.5.20 with respect to y to obtain
$$\frac{\partial F(x,y)}{\partial y}=\frac{\partial G(x,y)}{\partial y}+\phi'(y).$$
- **Step 4.** Equate the right side of this equation to N and solve for ϕ'; thus,
$$\frac{\partial G(x,y)}{\partial y}+\phi'(y)=N(x,y), \quad \text{so} \quad \phi'(y)=N(x,y)-\frac{\partial G(x,y)}{\partial y}.$$
- **Step 5.** Integrate ϕ' with respect to y, taking the constant of integration to be zero, and substitute the result in Equation 2.5.20 to obtain $F(x,y)$.
- **Step 6.** Set $F(x,y)=c$ to obtain an implicit solution of Equation 2.5.19. If possible, solve for y explicitly as a function of x.

It's a common mistake to omit Step 6 in the procedure above. However, it is important to include this step, since F isn't itself a solution of Equation 2.5.19. Many equations can be conveniently solved by either of the two methods used in Example 2.5.3. However, sometimes the integration required in one approach is more difficult than in the other. In such cases we choose the approach that requires the easier integration.

Example 2.5.4

Solve the equation

$$\left(ye^{xy}\tan x + e^{xy}\sec^2 x\right)dx + xe^{xy}\tan x\,dy = 0 \tag{2.5.21}$$

Solution

We leave it to you to check that $M_y = N_x$ on any open rectangle where $\tan x$ and $\sec x$ are defined. Here we must find a function F such that

$$F_x(x,y) = ye^{xy}\tan x + e^{xy}\sec^2 x \tag{2.5.22}$$

and

$$F_y(x,y) = xe^{xy}\tan x. \tag{2.5.23}$$

It's difficult to integrate Equation 2.5.22 with respect to x, but easy to integrate Equation 2.5.23 with respect to y. This yields

$$F(x,y) = e^{xy}\tan x + \psi(x). \tag{2.5.24}$$

Differentiating this with respect to x yields

$$F_x(x,y) = ye^{xy}\tan x + e^{xy}\sec^2 x + \psi'(x).$$

Comparing this with Equation 2.5.22 shows that $\psi'(x) = 0$. Hence, ψ is a constant, which we can take to be zero in Equation 2.5.24, and

$$e^{xy}\tan x = c,$$

is an implicit solution of Equation 2.5.21.

Attempting to apply our procedure to an differential equation that is not exact will lead to failure in Step 4, since the function

$$N - \frac{\partial G}{\partial y}$$

will not be independent of x if $M_y \neq N_x$, and therefore cannot be the derivative of a function of y alone. Example 2.5.5 illustrates this.

Example 2.5.5

Verify that the equation

$$3x^2y^2\,dx + 6x^3y\,dy = 0 \tag{2.5.25}$$

is not exact, and show that the procedure for solving exact equations fails when applied to Equation 2.5.25.

Solution

Here

$$M_y(x,y) = 6x^2y \quad \text{and} \quad N_x(x,y) = 18x^2y,$$

so Equation 2.5.25 is not exact. Nevertheless, let us try to find a function F such that

$$F_x(x,y) = 3x^2y^2 \tag{2.5.26}$$

and

$$F_y(x,y) = 6x^3y. \tag{2.5.27}$$

Integrating Equation 2.5.26 with respect to x yields

$$F(x,y)=x^3y^2+\phi(y),$$

and differentiating this with respect to y yields

$$F_y(x,y)=2x^3y+\phi'(y).$$

For this equation to be consistent with Equation 2.5.27,

$$6x^3y=2x^3y+\phi'(y),$$

or

$$\phi'(y)=4x^3y.$$

This is a contradiction, since ϕ' must be independent of x. Therefore the procedure fails.

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2.5E: Exact Equations (Exercises)

Q2.5.1

In *Exercises 2.5.1-2.5.17* determine which equations are exact and solve them.

1. $6x^2y^2\,dx+4x^3y\,dy=0$
2. $(3y\cos x+4xe^x+2x^2e^x)\,dx+(3\sin x+3)\,dy=0$
3. $14x^2y^3\,dx+21x^2y^2\,dy=0$
4. $(2x-2y^2)\,dx+(12y^2-4xy)\,dy=0$
5. $(x+y)^2\,dx+(x+y)^2\,dy=0$
6. $(4x+7y)\,dx+(3x+4y)\,dy=0$
7. $(-2y^2\sin x+3y^3-2x)\,dx+(4y\cos x+9xy^2)\,dy=0$
8. $(2x+y)\,dx+(2y+2x)\,dy=0$
9. $(3x^2+2xy+4y^2)\,dx+(x^2+8xy+18y)\,dy=0$
10. $(2x^2+8xy+y^2)\,dx+(2x^2+xy^3/3)\,dy=0$
11. $\left(\frac{1}{x}+2x\right)\,dx+\left(\frac{1}{y}+2y\right)\,dy=0$
12. $(y\sin xy+xy^2\cos xy)\,dx+(x\sin xy+xy^2\cos xy)\,dy=0$
13. $\frac{x\,dx}{(x^2+y^2)^{3/2}}+\frac{y\,dy}{(x^2+y^2)^{3/2}}=0$
14. $\left(e^x(x^2y^2+2xy^2)+6x\right)\,dx+(2x^2ye^x+2)\,dy=0$
15. $\left(x^2e^{x^2+y}(2x^2+3)+4x\right)\,dx+(x^3e^{x^2+y}-12y^2)\,dy=0$
16. $\left(e^{xy}(x^4y+4x^3)+3y\right)\,dx+(x^5e^{xy}+3x)\,dy=0$
17. $(3x^2\cos xy-x^3y\sin xy+4x)\,dx+(8y-x^4\sin xy)\,dy=0$

Q2.5.2

In *Exercises 2.5.18-2.5.22* solve the initial value problem.

18. $(4x^3y^2-6x^2y-2x-3)\,dx+(2x^4y-2x^3)\,dy=0,\quad y(1)=3$
19. $(-4y\cos x+4\sin x\cos x+\sec^2 x)\,dx+(4y-4\sin x)\,dy=0,\quad y(\pi/4)=0$
20. $(y^3-1)e^x\,dx+3y^2(e^x+1)\,dy=0,\quad y(0)=0$
21. $(\sin x-y\sin x-2\cos x)\,dx+\cos x\,dy=0,\quad y(0)=1$
22. $(2x-1)(y-1)\,dx+(x+2)(x-3)\,dy=0,\quad y(1)=-1$

Q2.5.3

23. Solve the exact equation

$$(7x+4y)\,dx+(4x+3y)\,dy=0.$$

Plot a direction field and some integral curves for this equation on the rectangle

$$\{-1\le x\le 1,-1\le y\le 1\}.$$

24. Solve the exact equation

$$e^x(x^4y^2+4x^3y^2+1)\,dx+(2x^4ye^x+2y)\,dy=0.$$

Plot a direction field and some integral curves for this equation on the rectangle

$$\{-2 \le x \le 2, -1 \le y \le 1\}.$$

25. Plot a direction field and some integral curves for the exact equation

$$(x^3y^4 + x)\,dx + (x^4y^3 + y)\,dy = 0$$

on the rectangle $\{-1 \le x \le 1, -1 \le y \le 1\}$. (See *Exercise 2.5.37(a)*).

26. Plot a direction field and some integral curves for the exact equation

$$(3x^2 + 2y)\,dx + (2y + 2x)\,dy = 0$$

on the rectangle $\{-2 \le x \le 2, -2 \le y \le 2\}$. (See E*xercise 2.5.37(b)*).

27.

a. Solve the exact equation

$$(x^3y^4 + 2x)\,dx + (x^4y^3 + 3y)\,dy = 0 \tag{A}$$

implicitly.

b. For what choices of (x_0, y_0) does Theorem 2.3.1 imply that the initial value problem

$$(x^3y^4 + 2x)\,dx + (x^4y^3 + 3y)\,dy = 0, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution on an open interval (a, b) that contains x_0?

c. Plot a direction field and some integral curves for (A) on a rectangular region centered at the origin. What is the interval of validity of the solution of (B)?

28.

a. Solve the exact equation

$$(x^2 + y^2)\,dx + 2xy\,dy = 0 \tag{A}$$

implicitly.

b. For what choices of (x_0, y_0) does Theorem 2.3.1 imply that the initial value problem

$$(x^2 + y^2)\,dx + 2xy\,dy = 0, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution $y = y(x)$ on some open interval (a, b) that contains x_0?

c. Plot a direction field and some integral curves for (A). From the plot determine, the interval (a, b) of b, the monotonicity properties (if any) of the solution of (B), and $\lim_{x\to a+} y(x)$ and $\lim_{x\to b-} y(x)$.

29. Find all functions M such that the equation is exact.

a. $M(x, y)\,dx + (x^2 - y^2)\,dy = 0$
b. $M(x, y)\,dx + 2xy \sin x \cos y\,dy = 0$
c. $M(x, y)\,dx + (e^x - e^y \sin x)\,dy = 0$

30. Find all functions N such that the equation is exact.

a. $(x^3y^2 + 2xy + 3y^2)\,dx + N(x, y)\,dy = 0$
b. $(\ln xy + 2y \sin x)\,dx + N(x, y)\,dy = 0$
c. $(x \sin x + y \sin y)\,dx + N(x, y)\,dy = 0$

31. Suppose M, N, and their partial derivatives are continuous on an open rectangle R, and G is an antiderivative of M with respect to x; that is,

$$\frac{\partial G}{\partial x} = M.$$

Show that if $M_y \ne N_x$ in R then the function

$$N - \frac{\partial G}{\partial y}$$

is not independent of x.

32. Prove: If the equations $M_1\,dx+N_1\,dy=0$ and $M_2\,dx+N_2\,dy=0$ are exact on an open rectangle R, so is the equation

$$(M_1+M_2)\,dx+(N_1+N_2)\,dy=0.$$

33. Find conditions on the constants A, B, C, and D such that the equation

$$(Ax+By)\,dx+(Cx+Dy)\,dy=0$$

is exact.

34. Find conditions on the constants A, B, C, D, E, and F such that the equation

$$(Ax^2+Bxy+Cy^2)\,dx+(Dx^2+Exy+Fy^2)\,dy=0$$

is exact.

35. Suppose M and N are continuous and have continuous partial derivatives M_y and N_x that satisfy the exactness condition $M_y=N_x$ on an open rectangle R. Show that if (x,y) is in R and

$$F(x,y)=\int_{x_0}^{x}M(s,y_0)\,ds+\int_{y_0}^{y}N(x,t)\,dt,$$

then $F_x=M$ and $F_y=N$.

36. Under the assumptions of *Exercise 2.5.35*, show that

$$F(x,y)=\int_{y_0}^{y}N(x_0,s)\,ds+\int_{x_0}^{x}M(t,y)\,dt.$$

37. Use the method suggested by *Exercise 2.5.35*, with $(x_0,y_0)=(0,0)$, to solve the these exact equations:

a. $(x^3y^4+x)\,dx+(x^4y^3+y)\,dy=0$
b. $(x^2+y^2)\,dx+2xy\,dy=0$
c. $(3x^2+2y)\,dx+(2y+2x)\,dy=0$

38. Solve the initial value problem

$$y'+\frac{2}{x}y=-\frac{2xy}{x^2+2x^2y+1},\quad y(1)=-2.$$

39. Solve the initial value problem

$$y'-\frac{3}{x}y=\frac{2x^4(4x^3-3y)}{3x^5+3x^3+2y},\quad y(1)=1.$$

40. Solve the initial value problem

$$y'+2xy=-e^{-x^2}\left(\frac{3x+2ye^{x^2}}{2x+3ye^{x^2}}\right),\quad y(0)=-1.$$

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2.6: Integrating Factors

In the previous section on Exact Equations we saw that if M, N, M_y and N_x are continuous and $M_y = N_x$ on an open rectangle R then

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.6.1}$$

is exact on R. Sometimes an equation that isn't exact can be made exact by multiplying it by an appropriate function. For example,

$$(3x + 2y^2)\,dx + 2xy\,dy = 0 \tag{2.6.2}$$

is not exact, since $M_y(x,y) = 4y \neq N_x(x,y) = 2y$ in Equation 2.6.2. However, multiplying Equation 2.6.2 by x yields

$$(3x^2 + 2xy^2)\,dx + 2x^2y\,dy = 0, \tag{2.6.3}$$

which is exact, since $M_y(x,y) = N_x(x,y) = 4xy$ in Equation 2.6.3. Solving Equation 2.6.3 by the procedure given in Section 2.5 yields the implicit solution

$$x^3 + x^2y^2 = c.$$

A function $\mu = \mu(x,y)$ is an **integrating factor** for Equation 2.6.1 if

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0 \tag{2.6.4}$$

is exact. If we know an integrating factor μ for Equation 2.6.1, we can solve the exact equation Equation 2.6.4 by the method of Section 2.5. It would be nice if we could say that Equation 2.6.1 and Equation 2.6.4 always have the same solutions, but this isn't so. For example, a solution $y = y(x)$ of Equation 2.6.4 such that $\mu(x, y(x)) = 0$ on some interval $a < x < b$ could fail to be a solution of 2.6.1 (**Exercise 2.6.1**), while Equation 2.6.1 may have a solution $y = y(x)$ such that $\mu(x, y(x))$ isn't even defined (**Exercise 2.6.2**). Similar comments apply if y is the independent variable and x is the dependent variable in Equation 2.6.1 and Equation 2.6.4. However, if $\mu(x,y)$ is defined and nonzero for all (x,y), Equation 2.6.1 and Equation 2.6.4 are equivalent; that is, they have the same solutions.

Finding Integrating Factors

By applying Theorem 2.5.2 (with M and N replaced by μM and μN), we see that Equation 2.6.4 is exact on an open rectangle R if μM, μN, $(\mu M)_y$, and $(\mu N)_x$ are continuous and

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

or, equivalently

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

on R. It's better to rewrite the last equation as

$$\mu(M_y - N_x) = \mu_x N - \mu_y M, \tag{2.6.5}$$

which reduces to the known result for exact equations; that is, if $M_y = N_x$ then Equation 2.6.5 holds with $\mu = 1$, so Equation 2.6.1 is exact.

You may think Equation 2.6.5 is of little value, since it involves **partial** derivatives of the unknown integrating factor μ, and we haven't studied methods for solving such equations. However, we'll now show that Equation 2.6.5 is useful if we restrict our search to integrating factors that are products of a function of x and a function of y; that is, $\mu(x,y) = P(x)Q(y)$. We're not saying that **every** equation $M\,dx + N\,dy = 0$ has an integrating factor of this form; rather, we are saying that **some** equations have such integrating factors.We'llnow develop a way to determine whether a given equation has such an integrating factor, and a method for finding the integrating factor in this case.

If $\mu(x,y) = P(x)Q(y)$, then $\mu_x(x,y) = P'(x)Q(y)$ and $\mu_y(x,y) = P(x)Q'(y)$, so Equation 2.6.5 becomes

$$P(x)Q(y)(M_y - N_x) = P'(x)Q(y)N - P(x)Q'(y)M, \tag{2.6.6}$$

or, after dividing through by $P(x)Q(y)$,

$$M_y - N_x = \frac{P'(x)}{P(x)}N - \frac{Q'(y)}{Q(y)}M. \tag{2.6.7}$$

Now let

$$p(x) = \frac{P'(x)}{P(x)} \quad \text{and} \quad q(y) = \frac{Q'(y)}{Q(y)},$$

so Equation 2.6.7 becomes

$$M_y - N_x = p(x)N - q(y)M. \tag{2.6.8}$$

We obtained Equation 2.6.8 by **assuming** that $M\,dx + N\,dy = 0$ has an integrating factor $\mu(x,y) = P(x)Q(y)$. However, we can now view Equation 2.6.7 differently: If there are functions $p = p(x)$ and $q = q(y)$ that satisfy Equation 2.6.8 and we define

$$P(x) = \pm e^{\int p(x)\,dx} \quad \text{and} \quad Q(y) = \pm e^{\int q(y)\,dy}, \tag{2.6.9}$$

then reversing the steps that led from Equation 2.6.6 to Equation 2.6.8 shows that $\mu(x,y) = P(x)Q(y)$ is an integrating factor for $M\,dx + N\,dy = 0$. In using this result, we take the constants of integration in Equation 2.6.9 to be zero and choose the signs conveniently so the integrating factor has the simplest form.

There's no simple general method for ascertaining whether functions $p = p(x)$ and $q = q(y)$ satisfying Equation 2.6.8 exist. However, the next theorem gives simple sufficient conditions for the given equation to have an integrating factor that depends on only one of the independent variables x and y, and for finding an integrating factor in this case.

Theorem 2.6.1

Let M, N, M_y, and N_x be continuous on an open rectangle R. Then:

a. If $(M_y - N_x)/N$ is independent of y on R and we define

$$p(x) = \frac{M_y - N_x}{N}$$

then

$$\mu(x) = \pm e^{\int p(x)\,dx} \tag{2.6.10}$$

is an integrating factor for

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.6.11}$$

on R.

b. If $(N_x - M_y)/M$ is independent of x on R and we define

$$q(y) = \frac{N_x - M_y}{M},$$

then

$$\mu(y) = \pm e^{\int q(y)\,dy} \tag{2.6.12}$$

is an integrating factor for Equation 2.6.11 on R.

Proof

(a) If $(M_y - N_x)/N$ is independent of y, then Equation 2.6.8 holds with $p = (M_y - N_x)/N$ and $q \equiv 0$. Therefore

$$P(x) = \pm e^{\int p(x)\,dx} \quad \text{and} \quad Q(y) = \pm e^{\int q(y)\,dy} = \pm e^0 = \pm 1,$$

so Equation 2.6.10 is an integrating factor for Equation 2.6.11 on R.

(b) If $(N_x - M_y)/M$ is independent of x then eqrefeq:2.6.8 holds with $p \equiv 0$ and $q = (N_x - M_y)/M$, and a similar argument shows that Equation 2.6.12 is an integrating factor for Equation 2.6.11 on R.

The next two examples show how to apply Theorem 2.6.1 .

Example 2.6.1

Find an integrating factor for the equation

$$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + (3x^2y^2 + 4y)\,dy = 0 \tag{2.6.13}$$

and solve the equation.

Solution

In Equation 2.6.13

$$M = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x,\ N = 3x^2y^2 + 4y,$$

and

$$M_y - N_x = (6xy^2 - 6x^3y^2 - 8xy) - 6xy^2 = -6x^3y^2 - 8xy,$$

so Equation 2.6.13 isn't exact. However,

$$\frac{M_y - N_x}{N} = -\frac{6x^3y^2 + 8xy}{3x^2y^2 + 4y} = -2x$$

is independent of y, so Theorem 2.6.1 (a) applies with $p(x) = -2x$. Since

$$\int p(x)\,dx = -\int 2x\,dx = -x^2,$$

$\mu(x) = e^{-x^2}$ is an integrating factor. Multiplying Equation 2.6.13 by μ yields the exact equation

$$e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + e^{-x^2}(3x^2y^2 + 4y)\,dy = 0. \tag{2.6.14}$$

To solve this equation, we must find a function F such that

$$F_x(x, y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) \tag{2.6.15}$$

and

$$F_y(x, y) = e^{-x^2}(3x^2y^2 + 4y). \tag{2.6.16}$$

Integrating Equation 2.6.16 with respect to y yields

$$F(x, y) = e^{-x^2}(x^2y^3 + 2y^2) + \psi(x). \tag{2.6.17}$$

Differentiating this with respect to x yields

$$F_x(x, y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2) + \psi'(x).$$

Comparing this with Equation 2.6.15 shows that $\psi'(x) = 2xe^{-x^2}$; therefore, we can let $\psi(x) = -e^{-x^2}$ in Equation 2.6.17 and conclude that

$$e^{-x^2}\left(y^2(x^2y + 2) - 1\right) = c$$

is an implicit solution of Equation 2.6.14. It is also an implicit solution of Equation 2.6.13.

Figure 2.6.1 shows a direction field and some integral curves for Equation 2.6.13

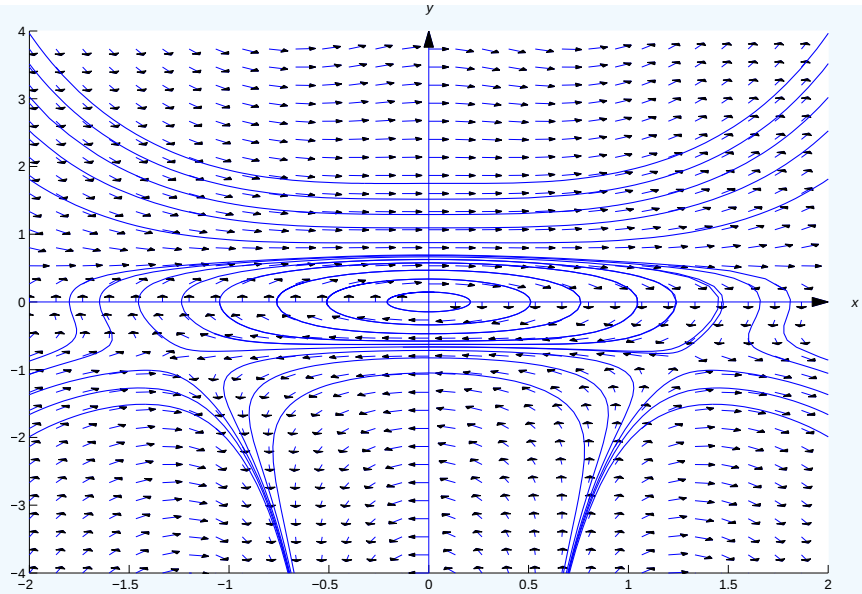

Figure 2.6.1 : A direction field and integral curves for $(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + (3x^2y^2 + 4y)\,dy = 0$

Example 2.6.2

Find an integrating factor for

$$2xy^3dx + (3x^2y^2 + x^2y^3 + 1)dy = 0 \tag{2.6.18}$$

and solve the equation.

Solution

In Equation 2.6.18,

$$M = 2xy^3, \quad N = 3x^2y^2 + x^2y^3 + 1,$$

and

$$M_y - N_x = 6x^2 - (6xy^2 + 2xy^3) = -2xy^3,$$

so Equation 2.6.18 isn't exact. Moreover,

$$\frac{M_y - N_x}{N} = -\frac{2xy^3}{3x^2y^2 + x^2y^2 + 1}$$

is not independent of y, so Theorem 2.6.1(a) does not apply. However, Theorem 2.6.1(b) does apply, since

$$\frac{N_x - M_y}{M} = \frac{2xy^3}{2xy^3} = 1$$

is not independent of x, so we can take $q(y) = 1$. Since

$$\int q(y)dy = \int dy = y,$$

$\mu(y) = e^y$ is an integrating factor. Multiplying Equation 2.6.18 by μ yields the exact equation.

$$2xy^3e^ydx + (3x^2y^2 + x^2y^3 + 1)e^ydy = 0. \tag{2.6.19}$$

To solve this equation, we must find a function F such that

$$F_x(x, y) = 2xy^3e^y \tag{2.6.20}$$

and

$$F_y(x,y) = (3x^2y^2 + x^2y^3 + 1)e^y. \tag{2.6.21}$$

Integrating Equation 2.6.20 with respect to x yields

$$F(x,y) = x^2y^3e^y + \phi(y) \tag{2.6.22}$$

Differentiating this with respect to y yields

$$F_y = (3x^2y^2 + x^2y^3)e^y + \phi'(y)$$

and comparing this with Equation 2.6.21 shows that φ 0 (y) = e y . Therefore we set φ(y) = e y in Equation 2.6.22 and conclude that

$$(x^2y^3 + 1)e^y = c$$

is an implicit solution of 2.6.19. It is also an implicit solution of 2.6.18. Figure 2.6.2 shows a direction field and some integral curves for 2.6.18.

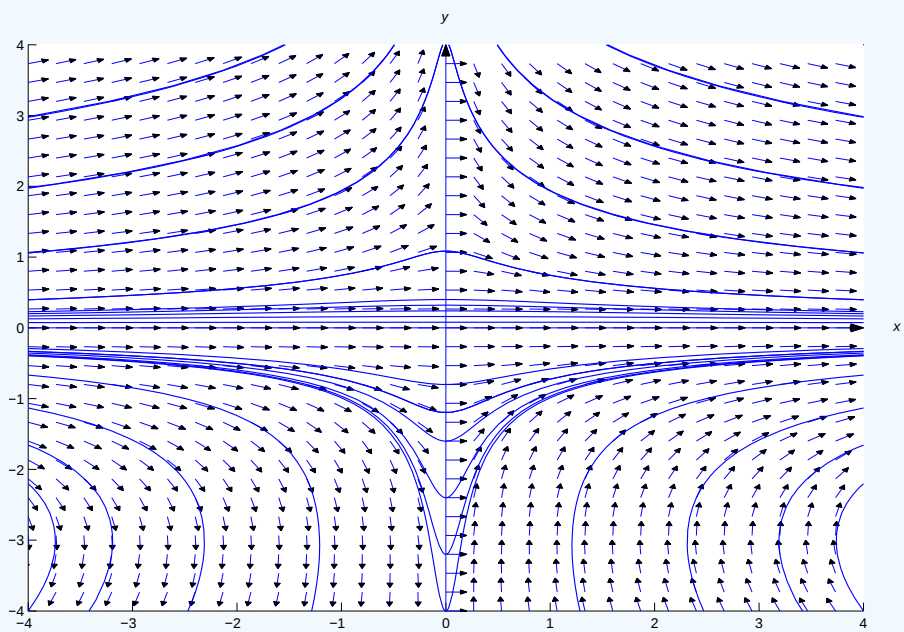

Figure 2.6.2 : A direction field and integral curves for $(2xy^3e^y dx + (3x^2y^2 + x^2y^3 + 1)e^y dy = 0$

Theorem 2.6.1 does not apply in the next example, but the more general argument that led to Theorem 2.6.1 provides an integrating factor.

Example 2.6.3

Find an integrating factor for

$$(3xy + 6y^2)dx + (2x^2 + 9xy)dy = 0 \tag{2.6.23}$$

and solve the equation.

Solution

In Equation 2.6.23

$$M = 3xy + 6y^2, \quad N = 2x^2 + 9xy,$$

and

$$M_y - N_x = (3x + 12y) - (4x + 9y) = -x + 3y.$$

Therefore

$$\frac{M_y - N_x}{M} = \frac{-x+3y}{3xy+6y^2} \quad \text{and} \quad \frac{N_x - M_y}{N} = \frac{x-3y}{2x^2+9xy}$$

so Theorem 2.6.1 does not apply. Following the more general argument that led to Theorem 2.6.1 , we look for functions $p = p(x)$ and $q = q(y)$ such that

$$M_y - N_x = p(x)N - q(y)M;$$

that is,

$$-x+3y = p(x)(2x^2+9xy) - q(y)(3xy+6y^2).$$

Since the left side contains only first degree terms in x and y, we rewrite this equation as

$$xp(x)(2x+9y) - yq(y)(3x+6y) = -x+3y.$$

This will be an identity if

$$xp(x) = A \quad \text{and} yq(y) = B, \tag{2.6.24}$$

where A and B are constants such that

$$-x+3y = A(2x+9y) - B(3x+6y),$$

or, equivalently,

$$-x+3y = (2A-3B)x + (9A-6B)y.$$

Equating the coefficients of x and y on both sides shows that the last equation holds for all (x, y) if

$$\begin{aligned} 2A - 3B &= -1 \\ 9A - 6B &= 3 \end{aligned}$$

which has the solution A = 1, B = 1. Therefore Equation 2.6.24 implies that

$$p(x) = \frac{1}{x} \quad \text{and} \quad q(y) = \frac{1}{y}.$$

Since

$$\int p(x)dx = \ln|x| \quad \text{and} \quad \int q(y)dy = \ln|y|,$$

we can let $P(x) = x$ and $Q(y) = y$; hence, $\mu(x, y) = xy$ is an integrating factor. Multiplying Equation 2.6.23 by μ yields the exact equation

$$(3x^2y^2 + 6xy^3)dx + (2x^3y + 9x^2y^2)dy = 0.$$

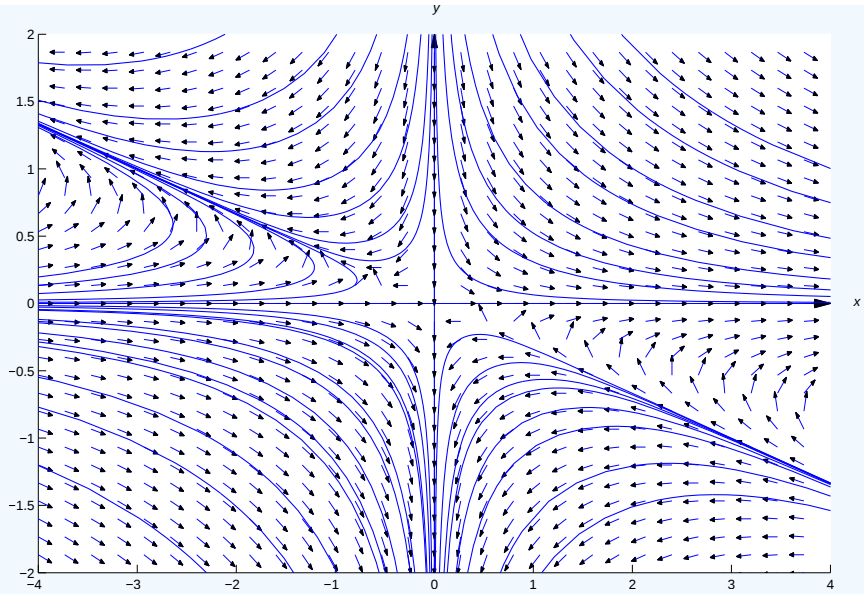

Figure 2.6.3 : A direction field and integral curves for $(3xy+6y^2)dx+(2x^2+9xy)dy=0$

We leave it to you to use the method of Section 2.5 to show that this equation has the implicit solution

$$x^3y^2+3x^2y^3=c. \tag{2.6.25}$$

This is also an implicit solution of Equation 2.6.23. Since x ≡ 0 and y ≡ 0 satisfy Equation 2.6.25, you should check to see that x ≡ 0 and y ≡ 0 are also solutions of Equation 2.6.23. (Why is it necesary to check this?) Figure 2.6.3 shows a direction field and integral curves for Equation 2.6.23. See **Exercise 2.6.28** for a general discussion of equations like Equation 2.6.23.

Example 2.6.4

The separable equation

$$-ydx+(x+x^6)dy=0 \tag{2.6.26}$$

can be converted to the exact equation

$$-\frac{dx}{x+x^6}+\frac{dy}{y}=0 \tag{2.6.27}$$

by multiplying through by the integrating factor

$$\mu(x,y)=\frac{1}{y(x+x^6)}.$$

However, to solve Equation 2.6.27 by the method of Section 2.5 we would have to evaluate the nasty integral

$$\int\frac{dx}{x+x^6}.$$

Instead, we solve Equation 2.6.26 explicitly for y by finding an integrating factor of the form $\mu(x,y)=x^ay^b$.

Solution

In Equation 2.6.26

$$M=-y,\ N=x+x^6,$$

and

$$M_y-N_x=-1-(1+6x^5)=-2-6x^5.$$

We look for functions $p=p(x)$ and $q=q(y)$ such that

$$M_y - N_x = p(x)N - q(y)M;$$

that is,

$$-2 - 6x^5 = p(x)(x + x^6) + q(y)y. \tag{2.6.28}$$

The right side will contain the term $-6x^5$ if $p(x) = -6/x$. Then Equation 2.6.28 becomes

$$-2 - 6x^5 = -6 - 6x^5 + q(y)y,$$

so $q(y) = 4/y$. Since

$$\int p(x)\,dx = -\int \frac{6}{x}\,dx = -6\ln|x| = \ln\frac{1}{x^6},$$

and

$$\int q(y)\,dy = \int \frac{4}{y}\,dy = 4\ln|y| = \ln y^4,$$

we can take $P(x) = x^{-6}$ and $Q(y) = y^4$, which yields the integrating factor $\mu(x, y) = x^{-6}y^4$. Multiplying Equation 2.6.26 by μ yields the exact equation

$$-\frac{y^5}{x^6}\,dx + \left(\frac{y^4}{x^5} + y^4\right)\,dy = 0.$$

We leave it to you to use the method of the Section 2.5 to show that this equation has the implicit solution

$$\left(\frac{y}{x}\right)^5 + y^5 = k.$$

Solving for y yields

$$y = k^{1/5}x(1 + x^5)^{-1/5},$$

which we rewrite as

$$y = cx(1 + x^5)^{-1/5}$$

by renaming the arbitrary constant. This is also a solution of Equation 2.6.26.

Figure 2.6.4 shows a direction field and some integral curves for Equation 2.6.26.

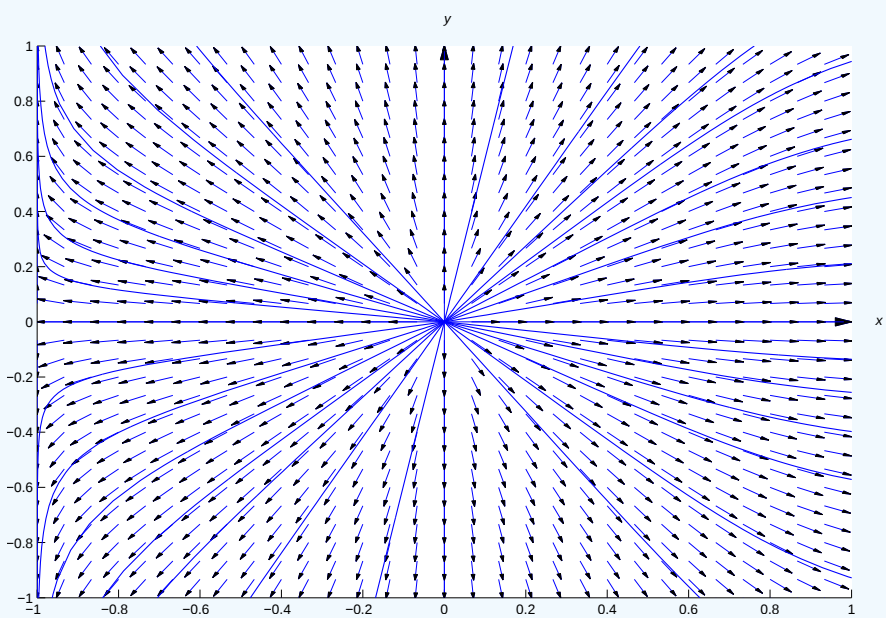

Figure 2.6.4 : A direction field and integral curves for $-ydx + (x + x^6)dy = 0$

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2.6E: Integrating Factors (Exercises)

Q2.6.1

1.

a. Verify that $\mu(x,y)=y$ is an integrating factor for

$$ydx+\left(2x+\frac{1}{y}\right)dy=0 \tag{A}$$

on any open rectangle that does not intersect the x axis or, equivalently, that

$$y^2dx+(2xy+1)dy=0 \tag{B}$$

is exact on any such rectangle.

b. Verify that $y\equiv 0$ is a solution of (B), but not of (A).

c. Show that

$$y(xy+1)=c \tag{C}$$

is an implicit solution of (B), and explain why every differentiable function $y=y(x)$ other than $y\equiv 0$ that satisfies (C) is also a solution of (A).

2.

a. Verify that $\mu(x,y)=1/(x-y)^2$ is an integrating factor for

$$-y^2dx+x^2dy=0 \tag{A}$$

on any open rectangle that does not intersect the line $y=x$ or, equivalently, that

$$-\frac{y^2}{(x-y)^2}dx+\frac{x^2}{(x-y)^2}dy=0 \tag{B}$$

is exact on any such rectangle.

b. Use Theorem 2.2.1 to show that

$$\frac{xy}{(x-y)}=c \tag{C}$$

is an implicit solution of (B), and explain why it is also an implicit solution of (A)

c. Verify that $y=x$ is a solution of (A), even though it can't be obtained from (C).

Q2.6.2

In *Exercises 2.6.3-2.6.16* find an integrating factor; that is a function of only one variable, and solve the given equation.

3. $ydx-xdy=0$

4. $3x^2ydx+2x^3dy=0$

5. $2y^3dx+3y^2dy=0$

6. $(5xy+2y+5)dx+2xdy=0$

7. $(xy+x+2y+1)\,dx+(x+1)\,dy=0$

8. $(27xy^2+8y^3)\,dx+(18x^2y+12xy^2)\,dy=0$

9. $(6xy^2+2y)\,dx+(12x^2y+6x+3)\,dy=0$

10. $y^2\,dx+\left(xy^2+3xy+\frac{1}{y}\right)\,dy=0$

11. $(12x^3y+24x^2y^2)\,dx+(9x^4+32x^3y+4y)\,dy=0$

12. $(x^2y+4xy+2y)\,dx+(x^2+x)\,dy=0$

13. $-y\,dx+(x^4-x)\,dy=0$

14. $\cos x\cos y\,dx+(\sin x\cos y-\sin x\sin y+y)\,dy=0$

15. $(2xy+y^2)\,dx+(2xy+x^2-2x^2y^2-2xy^3)\,dy=0$

16. $y\sin y\,dx+x(\sin y-y\cos y)\,dy=0$

Q2.6.3

In *Exercises 2.6.17-2.6.23* find an integrating factor of the form $\mu(x,y)=P(x)Q(y)$ and solve the given equation.

17. $y(1+5\ln|x|)\,dx+4x\ln|x|\,dy=0$

18. $(\alpha y+\gamma xy)\,dx+(\beta x+\delta xy)\,dy=0$

19. $(3x^2y^3-y^2+y)\,dx+(-xy+2x)\,dy=0$

20. $2y\,dx+3(x^2+x^2y^3)\,dy=0$

21. $(a\cos xy-y\sin xy)\,dx+(b\cos xy-x\sin xy)\,dy=0$

22. $x^4y^4\,dx+x^5y^3\,dy=0$

23. $y(x\cos x+2\sin x)\,dx+x(y+1)\sin x\,dy=0$

Q2.6.4

In *Exercises 2.6.24-2.6.27* find an integrating factor and solve the equation. Plot a direction field and some integral curves for the equation in the indicated rectangular region.

24. $(x^4y^3+y)\,dx+(x^5y^2-x)\,dy=0;\quad\{-1\le x\le 1,-1\le y\le 1\}$

25. $(3xy+2y^2+y)\,dx+(x^2+2xy+x+2y)\,dy=0;\quad\{-2\le x\le 2,-2\le y\le 2\}$

26. $(12xy+6y^3)\,dx+(9x^2+10xy^2)\,dy=0;\quad\{-2\le x\le 2,-2\le y\le 2\}$

27. $(3x^2y^2+2y)\,dx+2x\,dy=0;\quad\{-4\le x\le 4,-4\le y\le 4\}$

Q2.6.5

28. Suppose a, b, c, and d are constants such that $ad-bc\neq 0$, and let m and n be arbitrary real numbers. Show that

$$(ax^my+by^{n+1})\,dx+(cx^{m+1}+dxy^n)\,dy=0 \tag{2.6E.1}$$

has an integrating factor $\mu(x,y)=x^\alpha y^\beta$.

29. Suppose M, N, M_x, and N_y are continuous for all (x,y), and $\mu=\mu(x,y)$ is an integrating factor for

$$M(x,y)dx+N(x,y)dy=0. \tag{A}$$

Assume that μ_x and μ_y are continuous for all (x,y), and suppose $y=y(x)$ is a differentiable function such that $\mu(x,y(x))=0$ and $\mu_x(x,y(x))\neq 0$ for all x in some interval I. Show that y is a solution of (A) on I.

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2.7: Euler's Method

If an initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{2.7.1}$$

cannot be solved analytically, it is necessary to resort to numerical methods to obtain useful approximations to a solution of Equation 2.7.1. we will consider such methods in this chapter.

We're interested in computing approximate values of the solution of Equation 2.7.1 at equally spaced points x_0, x_1, …, $x_n = b$ in an interval $[x_0, b]$. Thus,

$$x_i = x_0 + ih, \quad i = 0, 1, \ldots, n,$$

where

$$h = \frac{b - x_0}{n}.$$

we will denote the approximate values of the solution at these points by y_0, y_1, …, y_n; thus, y_i is an approximation to $y(x_i)$. we will call

$$e_i = y(x_i) - y_i$$

the *error at the* ith step. Because of the initial condition $y(x_0) = y_0$, we will always have $e_0 = 0$. However, in general $e_i \neq 0$ if $i > 0$.

We encounter two sources of error in applying a numerical method to solve an initial value problem:

- The formulas defining the method are based on some sort of approximation. Errors due to the inaccuracy of the approximation are called *truncation errors*.
- Computers do arithmetic with a fixed number of digits, and therefore make errors in evaluating the formulas defining the numerical methods. Errors due to the computer's inability to do exact arithmetic are called *roundoff errors.*

Since a careful analysis of roundoff error is beyond the scope of this book, we will consider only truncation errors.

Euler's Method

The simplest numerical method for solving Equation 2.7.1 is *Euler's method*. This method is so crude that it is seldom used in practice; however, its simplicity makes it useful for illustrative purposes. Euler's method is based on the assumption that the tangent line to the integral curve of Equation 2.7.1 at $(x_i, y(x_i))$ approximates the integral curve over the interval $[x_i, x_{i+1}]$. Since the slope of the integral curve of Equation 2.7.1 at $(x_i, y(x_i))$ is $y'(x_i) = f(x_i, y(x_i))$, the equation of the tangent line to the integral curve at $(x_i, y(x_i))$ is

$$y = y(x_i) + f(x_i, y(x_i))(x - x_i). \tag{2.7.2}$$

Setting $x = x_{i+1} = x_i + h$ in Equation 2.7.2 yields

$$y_{i+1} = y(x_i) + hf(x_i, y(x_i)) \tag{2.7.3}$$

as an approximation to $y(x_{i+1})$. Since $y(x_0) = y_0$ is known, we can use Equation 2.7.3 with $i = 0$ to compute

$$y_1 = y_0 + hf(x_0, y_0).$$

However, setting $i = 1$ in Equation 2.7.3 yields

$$y_2 = y(x_1) + hf(x_1, y(x_1)),$$

which isn't useful, since we *don't know* $y(x_1)$. Therefore we replace $y(x_1)$ by its approximate value y_1 and redefine

$$y_2 = y_1 + hf(x_1, y_1).$$

Having computed y_2, we can compute

$$y_3 = y_2 + hf(x_2, y_2).$$

In general, Euler's method starts with the known value $y(x_0) = y_0$ and computes $y_1, y_2, \ldots, y_n$ successively by with the formula

$$y_{i+1} = y_i + hf(x_i, y_i), \quad 0 \le i \le n-1. \tag{2.7.4}$$

The next example illustrates the computational procedure indicated in Euler's method.

Example 2.7.1

Use Euler's method with $h = 0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1 \tag{2.7.5}$$

at $x = 0.1, 0.2, 0.3$

Solution

We rewrite Equation 2.7.5 as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form Equation 2.7.1, with

$$f(x, y) = -2y + x^3 e^{-2x},\ x_0 = 0,\ \text{and}\, y_0 = 1.$$

Euler's method yields

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + (0.1)f(0, 1) = 1 + (0.1)(-2) = 0.8, \\ y_2 &= y_1 + hf(x_1, y_1) \\ &= 0.8 + (0.1)f(0.1, 0.8) = 0.8 + (0.1)\left(-2(0.8) + (0.1)^3 e^{-0.2}\right) = 0.640081873, \\ y_3 &= y_2 + hf(x_2, y_2) \\ &= 0.640081873 + (0.1)\left(-2(0.640081873) + (0.2)^3 e^{-0.4}\right) = 0.512601754. \end{aligned}$$

We've written the details of these computations to ensure that you understand the procedure. However, in the rest of the examples as well as the exercises in this chapter, we will assume that you can use a programmable calculator or a computer to carry out the necessary computations.

Examples Illustrating The Error in Euler's Method

Example 2.7.2

Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0$, 0.1, 0.2, 0.3, …, 1.0. Compare these approximate values with the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4), \tag{2.7.6}$$

which can be obtained by the method of Section 2.1. (Verify.)

Table 2.7.1 shows the values of the exact solution Equation 2.7.6 at the specified points, and the approximate values of the solution at these points obtained by Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$. In examining this table, keep in mind that the approximate values in the column corresponding to $h = 0.05$ are actually the results of 20 steps with Euler's method. We haven't listed the estimates of the solution obtained for $x = 0.05$, 0.15, …, since there's nothing to compare them with in the column corresponding to $h = 0.1$. Similarly, the approximate values in the column corresponding to $h = 0.025$ are actually the results of 40 steps with Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact

x	$h=0.1$	$h=0.05$	$h=0.025$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.810005655	0.814518349	0.818751221
0.2	0.640081873	0.656266437	0.663635953	0.670588174
0.3	0.512601754	0.532290981	0.541339495	0.549922980
0.4	0.411563195	0.432887056	0.442774766	0.452204669
0.5	0.332126261	0.353785015	0.363915597	0.373627557
0.6	0.270299502	0.291404256	0.301359885	0.310952904
0.7	0.222745397	0.242707257	0.252202935	0.261398947
0.8	0.186654593	0.205105754	0.213956311	0.222570721
0.9	0.159660776	0.176396883	0.184492463	0.192412038
1.0	0.139778910	0.154715925	0.162003293	0.169169104

Table* $2.7.1$: Numerical solution of* $y'+2y=x^3e^{-2x},\ y(0)=1$ *, by Euler's method.*

You can see from Table $2.7.1$ that decreasing the step size improves the accuracy of Euler's method. For example,

$$y_{exact}(1)-y_{approx}(1)\approx\begin{cases}0.0293\text{with}h=0.1,\\ 0.0144 \text{ with } h=0.05,\\ 0.0071 \text{ with } h=0.025.\end{cases}$$

Based on this scanty evidence, you might guess that the error in approximating the exact solution at a *fixed value of* x by Euler's method is roughly halved when the step size is halved. You can find more evidence to support this conjecture by examining Table $2.7.2$, which lists the approximate values of $y_{exact}-y_{approx}$ at $x=0.1, 0.2, \ldots, 1.0$.

x	$h=0.1$	$h=0.05$	$h=0.0.25$
0.1	0.0187	0.0087	0.0042
0.2	0.0305	0.0143	0.0069
0.3	0.0373	0.0176	0.0085
0.4	0.0406	0.0193	0.0094
0.5	0.0415	0.0198	0.0097
0.6	0.0406	0.0195	0.0095
0.7	0.0386	0.0186	0.0091
0.8	0.0359	0.0174	0.0086
0.9	0.0327	0.0160	0.0079
1.0	0.0293	0.0144	0.0071

***Table* $2.7.2$*:* *Errors in approximate solutions of* $y'+2y=x^3e^{-2x},\ y(0)=1$ *, obtained by Euler's method.*

Example $2.7.3$

Tables $2.7.2$ and $2.7.4$ show analogous results for the nonlinear initial value problem

$$y'=-2y^2+xy+x^2,\ y(0)=1, \tag{2.7.7}$$

except in this case we cannot solve Equation 2.7.7 exactly. The results in the "Exact" column were obtained by using a more accurate numerical method known as the *Runge-Kutta* method with a small step size. They are exact to eight decimal places.

x	$h=0.1$	$h=0.05$	$h=0.025$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.821375000	0.829977007	0.837584494
0.2	0.681000000	0.707795377	0.719226253	0.729641890
0.3	0.605867800	0.633776590	0.646115227	0.657580377
0.4	0.559628676	0.587454526	0.600045701	0.611901791
0.5	0.535376972	0.562906169	0.575556391	0.587575491
0.6	0.529820120	0.557143535	0.569824171	0.581942225
0.7	0.541467455	0.568716935	0.581435423	0.593629526
0.8	0.569732776	0.596951988	0.609684903	0.621907458
0.9	0.614392311	0.641457729	0.654110862	0.666250842
1.0	0.675192037	0.701764495	0.714151626	0.726015790

***Table 2.7.3**: Numerical solution of $y' = -2y^2 + xy + x^2,\ y(0) = 1$, by Euler's method.*

Since we think it is important in evaluating the accuracy of the numerical methods that we will be studying in this chapter, we often include a column listing values of the exact solution of the initial value problem, even if the directions in the example or exercise don't specifically call for it. If quotation marks are included in the heading, the values were obtained by applying the Runge-Kutta method in a way that's explained in Section 3.3. If quotation marks are not included, the values were obtained from a known formula for the solution. In either case, the values are exact to eight places to the right of the decimal point.

x	$h=0.1$	$h=0.05$	$h=0.025$
0.1	0.0376	0.0162	0.0076
0.2	0.0486	0.0218	0.0104
0.3	0.0517	0.0238	0.0115
0.4	0.0523	0.0244	0.0119
0.5	0.0522	0.0247	0.0121
0.6	0.0521	0.0248	0.0121
0.7	0.0522	0.0249	0.0122
0.8	0.0522	0.0250	0.0122
0.9	0.0519	0.0248	0.0121
1.0	0.0508	0.0243	0.0119

***Table 2.7.4:** Errors in approximate solutions of $y' = -2y^2 + xy + x^2,\ y(0) = 1$, obtained by Euler's method.*

Truncation Error in Euler's Method

Consistent with the results indicated in Tables 2.7.1 - 2.7.4, we will now show that under reasonable assumptions on f there's a constant K such that the error in approximating the solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0,$$

at a given point $b > x_0$ by Euler's method with step size $h = (b - x_0)/n$ satisfies the inequality

$$|y(b) - y_n| \le Kh,$$

where K is a constant independent of n.

There are two sources of error (not counting roundoff) in Euler's method:

1. The error committed in approximating the integral curve by the tangent line Equation 2.7.2 over the interval $[x_i, x_{i+1}]$.
2. The error committed in replacing $y(x_i)$ by y_i in Equation 2.7.2 and using Equation 2.7.4 rather than Equation 2.7.2 to compute y_{i+1}.

Euler's method assumes that y_{i+1} defined in Equation 2.7.2 is an approximation to $y(x_{i+1})$. We call the error in this approximation the *local truncation error at the* ith step, and denote it by T_i; thus,

$$T_i = y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i)). \tag{2.7.8}$$

we will now use *Taylor's theorem* to estimate T_i, assuming for simplicity that f, f_x, and f_y are continuous and bounded for all (x, y). Then y'' exists and is bounded on $[x_0, b]$. To see this, we differentiate

$$y'(x) = f(x, y(x))$$

to obtain

$$\begin{aligned} y''(x) = & \quad f_x(x, y(x)) + f_y(x, y(x))y'(x) \\ = & \quad f_x(x, y(x)) + f_y(x, y(x))f(x, y(x)). \end{aligned}$$

Since we assumed that f, f_x and f_y are bounded, there's a constant M such that

$$|f_x(x, y(x)) + f_y(x, y(x))y'(x)| \le M \quad x_0 < x < b$$

which implies that

$$|y''(x)| \le M, \quad x_0 < x < b \tag{2.7.9}$$

Since $x_{i+1} = x_i + h$, Taylor's theorem implies that

$$y(x_{i+1}) = y(x_i) + hy'(x_i) + \frac{h^2}{2}y''(\tilde{x}_i),$$

where $\tilde{x}_i$ is some number between x_i and x_{i+1}. Since $y'(x_i) = f(x_i, y(x_i))$ this can be written as

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + \frac{h^2}{2}y''(\tilde{x}_i),$$

or, equivalently,

$$y(x_{i+1}) - y(x_i) - hf(x_i, y(x_i)) = \frac{h^2}{2}y''(\tilde{x}_i).$$

Comparing this with Equation 2.7.8 shows that

$$T_i = \frac{h^2}{2}y''(\tilde{x}_i).$$

Recalling Equation 2.7.9, we can establish the bound

$$|T_i| \le \frac{Mh^2}{2}, \quad 1 \le i \le n. \tag{2.7.10}$$

Although it may be difficult to determine the constant M, what is important is that there's an M such that Equation 2.7.10 holds. We say that the local truncation error of Euler's method is *of order* h^2, which we write as $O(h^2)$.

Note that the magnitude of the local truncation error in Euler's method is determined by the second derivative y'' of the solution of the initial value problem. Therefore the local truncation error will be larger where $|y''|$ is large, or smaller where $|y''|$ is small.

Since the local truncation error for Euler's method is $O(h^2)$, it is reasonable to expect that halving h reduces the local truncation error by a factor of 4. This is true, but halving the step size also requires twice as many steps to approximate the solution at a given point. To analyze the overall effect of truncation error in Euler's method, it is useful to derive an equation relating the errors

$$e_{i+1} = y(x_{i+1}) - y_{i+1} \quad \text{and} \quad e_i = y(x_i) - y_i.$$

To this end, recall that

$$y(x_{i+1}) = y(x_i) + hf(x_i, y(x_i)) + T_i \tag{2.7.11}$$

and

$$y_{i+1} = y_i + hf(x_i, y_i). \tag{2.7.12}$$

Subtracting Equation 2.7.12 from Equation 2.7.11 yields

$$e_{i+1} = e_i + h\left[f(x_i, y(x_i)) - f(x_i, y_i)\right] + T_i. \tag{2.7.13}$$

The last term on the right is the local truncation error at the ith step. The other terms reflect the way errors made at *previous steps* affect e_{i+1}. Since $|T_i| \le Mh^2/2$, we see from Equation 2.7.13 that

$$|e_{i+1}| \le |e_i| + h|f(x_i, y(x_i)) - f(x_i, y_i)| + \frac{Mh^2}{2}. \tag{2.7.14}$$

Since we assumed that f_y is continuous and bounded, the mean value theorem implies that

$$f(x_i, y(x_i)) - f(x_i, y_i) = f_y(x_i, y_i^*)(y(x_i) - y_i) = f_y(x_i, y_i^*)e_i,$$

where y_i^* is between y_i and $y(x_i)$. Therefore

$$|f(x_i, y(x_i)) - f(x_i, y_i)| \le R|e_i|$$

for some constant R. From this and Equation 2.7.14,

$$|e_{i+1}| \le (1 + Rh)|e_i| + \frac{Mh^2}{2}, \quad 0 \le i \le n-1. \tag{2.7.15}$$

For convenience, let $C = 1 + Rh$. Since $e_0 = y(x_0) - y_0 = 0$, applying Equation 2.7.15 repeatedly yields

$$\begin{aligned}
|e_1| &\le \frac{Mh^2}{2} \\
|e_2| &\le C|e_1| + \frac{Mh^2}{2} \le (1 + C)\frac{Mh^2}{2} \\
|e_3| &\le C|e_2| + \frac{Mh^2}{2} \le (1 + C + C^2)\frac{Mh^2}{2} \\
&\vdots \\
|e_n| &\le C|e_{n-1}| + \frac{Mh^2}{2} \le (1 + C + \cdots + C^{n-1})\frac{Mh^2}{2}.
\end{aligned} \tag{2.7.16}$$

Recalling the formula for the sum of a geometric series, we see that

$$1 + C + \cdots + C^{n-1} = \frac{1 - C^n}{1 - C} = \frac{(1 + Rh)^n - 1}{Rh}$$

(since $C = 1 + Rh$). From this and Equation 2.7.16,

$$|y(b) - y_n| = |e_n| \le \frac{(1 + Rh)^n - 1}{R}\frac{Mh}{2}. \tag{2.7.17}$$

Since Taylor's theorem implies that

$$1 + Rh < e^{rh}$$

(verify),

$$(1+Rh)^n < e^{nRh} = e^{R(b-x_0)} \quad (\text{since } nh = b - x_0).$$

This and Equation 2.7.17 imply that

$$|y(b) - y_n| \le Kh, \tag{2.7.18}$$

with

$$K = M\frac{e^{R(b-x_0)} - 1}{2R}.$$

Because of Equation 2.7.18 we say that the *global truncation error of Euler's method is of order* h, which we write as $O(h)$.

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2.7E: Euler's Method (Exercises)

Q3.1.1

You may want to save the results of these exercises, since we will revisit in the next two sections. In *Exercises 3.1.1-3.1.5* use Euler's method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$, where x_0 is the point where the initial condition is imposed and $i = 1, 2, 3$. The purpose of these exercises is to familiarize you with the computational procedure of Euler's method.

1. $y' = 2x^2 + 3y^2 - 2, \quad y(2) = 1; \quad h = 0.05$

2. $y' = y + \sqrt{x^2 + y^2}, \quad y(0) = 1; \quad h = 0.1$

3. $y' + 3y = x^2 - 3xy + y^2, \quad y(0) = 2; \quad h = 0.05$

4. $y' = \frac{1+x}{1-y^2}, \quad y(2) = 3; \quad h = 0.1$

5. $y' + x^2 y = \sin xy, \quad y(1) = \pi; \quad h = 0.2$

Q3.1.2

6. Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 3y = 7e^{4x}, \quad y(0) = 2 \tag{2.7E.1}$$

at $x = 0, 0.1, 0.2, 0.3, \ldots, 1.0$. Compare these approximate values with the values of the exact solution $y = e^{4x} + e^{-3x}$, which can be obtained by the method of Section 2.1. Present your results in a table like *Table 3.1.1*.

7. Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + \frac{2}{x}y = \frac{3}{x^3} + 1, \quad y(1) = 1 \tag{2.7E.2}$$

at $x = 1.0, 1.1, 1.2, 1.3, \ldots, 2.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{1}{3x^2}(9\ln x + x^3 + 2), \tag{2.7E.3}$$

which can be obtained by the method of Section 2.1. Present your results in a table like *Table 3.1.1*.

8. Use Euler's method with step sizes $h = 0.05$, $h = 0.025$, and $h = 0.0125$ to find approximate values of the solution of the initial value problem

$$y' = \frac{y^2 + xy - x^2}{x^2}, \quad y(1) = 2 \tag{2.7E.4}$$

at $x = 1.0, 1.05, 1.10, 1.15, \ldots, 1.5$. Compare these approximate values with the values of the exact solution

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3} \tag{2.7E.5}$$

obtained in Example [example:2.4.3}. Present your results in a table like *Table 3.1.1*.

9. In Example [example:2.2.3} it was shown that

$$y^5 + y = x^2 + x - 4 \tag{2.7E.6}$$

is an implicit solution of the initial value problem

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \tag{A}$$

Use Euler's method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of (A) at $x=2.0$, 2.1, 2.2, 2.3, …, 3.0. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x,y)=y^5+y-x^2-x+4 \tag{2.7E.7}$$

for each value of (x,y) appearing in the first table.

10. You can see from *Example 2.5.1* that

$$x^4y^3+x^2y^5+2xy=4 \tag{2.7E.8}$$

is an implicit solution of the initial value problem

$$y'=-\frac{4x^3y^3+2xy^5+2y}{3x^4y^2+5x^2y^4+2x}, \quad y(1)=1. \tag{A}$$

Use Euler's method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of (A) at $x=1.0$, 1.1, 1.2, 1.3, …, 2.0. Present your results in tabular form. To check the error in these approximate values, construct another table of values of the residual

$$R(x,y)=x^4y^3+x^2y^5+2xy-4 \tag{2.7E.9}$$

for each value of (x,y) appearing in the first table.

11. Use Euler's method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of the initial value problem

$$(3y^2+4y)y'+2x+\cos x=0, \quad y(0)=1; \quad \text{(Exercise 2.2.13)} \tag{2.7E.10}$$

at $x=0$, 0.1, 0.2, 0.3, …, 1.0.

12. Use Euler's method with step sizes $h=0.1$, $h=0.05$, and $h=0.025$ to find approximate values of the solution of the initial value problem

$$y'+\frac{(y+1)(y-1)(y-2)}{x+1}=0, \quad y(1)=0 \quad \text{(Exercise 2.2.14)} \tag{2.7E.11}$$

at $x=1.0$, 1.1, 1.2, 1.3, …, 2.0.

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CHAPTER OVERVIEW

Chapter 3: Applications of First Order Equations

In this chapter, we consider applications of first order differential equations.

3.1: Growth and Decay

3.1E: Growth and Decay (Exercises)

3.2: Elementary Mechanics

3.2E: Elementary Mechanics (Exercises)

3.3: Autonomous Second Order Equations

3.3E: Autonomous Second Order Equations (Exercises)

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3.1: Growth and Decay

This section begins with a discussion of exponential growth and decay, which you have probably already seen in calculus. We consider applications to radioactive decay, carbon dating, and compound interest. We also consider more complicated problems where the rate of change of a quantity is in part proportional to the magnitude of the quantity, but is also influenced by other other factors for example, a radioactive substance is manufactured at a certain rate, but decays at a rate proportional to its mass, or a saver makes regular deposits in a savings account that draws compound interest.

Since the applications in this section deal with functions of time, we'll denote the independent variable by t. If Q is a function of t, Q' will denote the derivative of Q with respect to t; thus,

$$Q' = \frac{dQ}{dt}.$$

Exponential Growth and Decay

One of the most common mathematical models for a physical process is the **exponential model**, where it is assumed that the rate of change of a quantity Q is proportional to Q; thus

$$Q' = aQ, \tag{3.1.1}$$

where a is the constant of proportionality.

From Example 2.1.3, the general solution of Equation 3.1.1 is

$$Q = ce^{at}$$

and the solution of the initial value problem

$$Q' = aQ, \quad Q(t_0) = Q_0$$

is

$$Q = Q_0 e^{a(t-t_0)}. \tag{3.1.2}$$

Since the solutions of $Q' = aQ$ are exponential functions, we say that a quantity Q that satisfies this equation **grows exponentially** if $a > 0$, or **decays exponentially** if $a < 0$ (Figure 4.1.1).

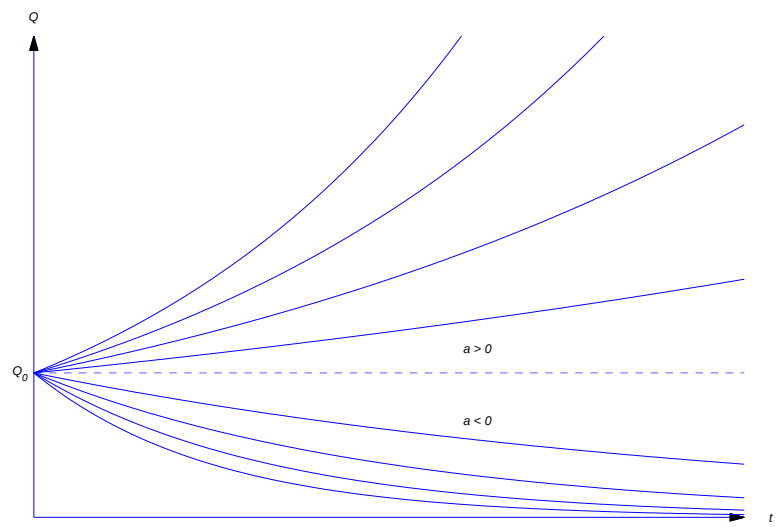

Figure 4.1.1 : Exponential growth and decay

Radioactive Decay

Experimental evidence shows that radioactive material decays at a rate proportional to the mass of the material present. According to this model the mass $Q(t)$ of a radioactive material present at time t satisfies Equation 3.1.1, where a is a negative constant

whose value for any given material must be determined by experimental observation. For simplicity, we'll replace the negative constant a by $-k$, where k is a positive number that we'll call the **decay constant** of the material. Thus, Equation 3.1.1 becomes

$$Q' = -kQ.$$

If the mass of the material present at $t = t_0$ is Q_0, the mass present at time t is the solution of

$$Q' = -kQ, \quad Q(t_0) = Q_0.$$

From Equation 3.1.2 with $a = -k$, the solution of this initial value problem is

$$Q = Q_0 e^{-k(t-t_0)}. \tag{3.1.3}$$

The **half–life** τ of a radioactive material is defined to be the time required for half of its mass to decay; that is, if $Q(t_0) = Q_0$, then

$$Q(\tau + t_0) = \frac{Q_0}{2}. \tag{3.1.4}$$

From Equation 3.1.3 with $t = \tau + t_0$, Equation 3.1.4 is equivalent to

$$Q_0 e^{-k\tau} = \frac{Q_0}{2},$$

so

$$e^{-k\tau} = \frac{1}{2}.$$

Taking logarithms yields

$$-k\tau = \ln\frac{1}{2} = -\ln 2,$$

so the half-life is

$$\tau = \frac{1}{k}\ln 2. \tag{3.1.5}$$

(Figure 4.1.2). The half-life is independent of t_0 and Q_0, since it is determined by the properties of material, not by the amount of the material present at any particular time.

Example 4.1.1

A radioactive substance has a half-life of 1620 years.

a. If its mass is now 4 g (grams), how much will be left 810 years from now?
b. Find the time t_1 when 1.5 g of the substance remain.

Solution a

From Equation 3.1.3 with $t_0 = 0$ and $Q_0 = 4$,

$$Q = 4e^{-kt}, \tag{3.1.6}$$

where we determine k from Equation 3.1.5, with τ= 1620 years:

$$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{1620}.$$

Substituting this in Equation 3.1.6 yields

$$Q = 4e^{-(t\ln 2)/1620}. \tag{3.1.7}$$

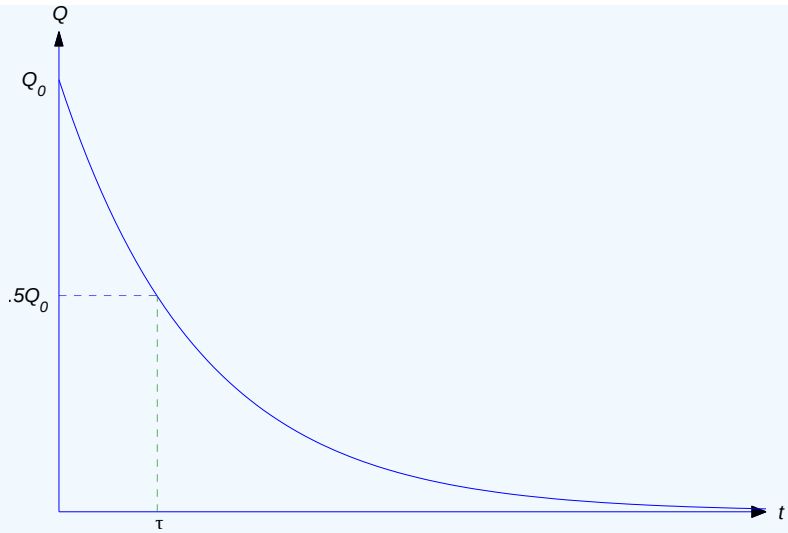

Figure 4.1.2 : Half-life of a radioactive substance

Therefore the mass left after 810 years will be

$$\begin{aligned} Q(810) &= 4e^{-(810\ln 2)/1620} = 4e^{-(\ln 2)/2} \\ &= 2\sqrt{2}\text{ g}. \end{aligned}$$

Solution b

Setting $t = t_1$ in Equation 3.1.7 and requiring that $Q(t_1) = 1.5$ yields

$$\frac{3}{2} = 4e^{(-t_1 \ln 2)/1620}.$$

Dividing by 4 and taking logarithms yields

$$\ln\frac{3}{8} = -\frac{t_1 \ln 2}{1620}.$$

Since $\ln 3/8 = -\ln 8/3$,

$$t_1 = 1620\frac{\ln 8/3}{\ln 2} \approx 2292.4 \text{ years}.$$

Interest Compounded Continuously

Suppose we deposit an amount of money Q_0 in an interest-bearing account and make no further deposits or withdrawals for t years, during which the account bears interest at a constant annual rate r. To calculate the value of the account at the end of t years, we need one more piece of information: how the interest is added to the account, or—as the bankers say—how it is **compounded**. If the interest is compounded annually, the value of the account is multiplied by $1 + r$ at the end of each year. This means that after t years the value of the account is

$$Q(t) = Q_0(1 + r)^t.$$

If interest is compounded semiannually, the value of the account is multiplied by $(1 + r/2)$ every 6 months. Since this occurs twice annually, the value of the account after t years is

$$Q(t) = Q_0\left(1 + \frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded n times per year, the value of the account is multiplied n times per year by $(1 + r/n)$; therefore, the value of the account after t years is

$$Q(t) = Q_0\left(1 + \frac{r}{n}\right)^{nt}. \tag{3.1.8}$$

Thus, increasing the frequency of compounding increases the value of the account after a fixed period of time. Table 4.1.1 shows the effect of increasing the number of compoundings over $t=5$ years on an initial deposit of $Q_0=100$ (dollars), at an annual interest rate of 6%.

Table 4.1.1 : The effect of compound interest

n (number of compoundings per year)	$\$100\left(1+\frac{.06}{n}\right)^{5n}$ (value in dollars after 5 years)
1	\$133.82
2	\$134.39
4	\$134.68
8	\$134.83
364	\$134.98

You can see from Table 4.1.1 that the value of the account after 5 years is an increasing function of n. Now suppose the maximum allowable rate of interest on savings accounts is restricted by law, but the time intervals between successive compoundings isn't; then competing banks can attract savers by compounding often. The ultimate step in this direction is to **compound continuously**, by which we mean that $n\to\infty$ in Equation 3.1.8. Since we know from calculus that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n=e^r,$$

this yields

$$\begin{aligned} Q(t) &= \lim_{n\to\infty} Q_0\left(1+\tfrac{r}{n}\right)^{nt}=Q_0\left[\lim_{n\to\infty}\left(1+\tfrac{r}{n}\right)^n\right]^t \\ &= Q_0e^{rt}. \end{aligned}$$

Observe that $Q=Q_0e^{rt}$ is the solution of the initial value problem

$$Q'=rQ,\quad Q(0)=Q_0;$$

that is, with continuous compounding the value of the account grows exponentially.

Example 4.1.2

If \$150 is deposited in a bank that pays $5\frac{1}{2}\%$ annual interest compounded continuously, the value of the account after t years is

$$Q(t)=150e^{.055t}$$

dollars. (Note that it is necessary to write the interest rate as a decimal; thus, $r=.055$.) Therefore, after $t=10$ years the value of the account is

$$Q(10)=150e^{.55}\approx\$259.99.$$

Example 4.1.3

We wish to accumulate \$10,000 in 10 years by making a single deposit in a savings account bearing $5\frac{1}{2}\%$ annual interest compounded continuously. How much must we deposit in the account?

Solution

The value of the account at time t is

$$Q(t)=Q_0e^{.055t}. \tag{3.1.9}$$

Since we want $Q(10)$ to be \$10,000, the initial deposit Q_0 must satisfy the equation

$$10000=Q_0e^{.55}, \tag{3.1.10}$$

obtained by setting $t = 10$ and $Q(10) = 10000$ in Equation 3.1.9. Solving Equation 3.1.10 for Q_0 yields

$$Q_0 = 10000e^{-.55} \approx \$5769.50.$$

Mixed Growth and Decay

Example 4.1.4

A radioactive substance with decay constant k is produced at a constant rate of a units of mass per unit time.

a. Assuming that $Q(0) = Q_0$, find the mass $Q(t)$ of the substance present at time t.
b. Find $\lim_{t\to\infty} Q(t)$.

Solution a:

Here

$$Q' = \text{ rate of increase of } Q - \text{ rate of decrease of } Q.$$

The rate of increase is the constant a. Since Q is radioactive with decay constant k, the rate of decrease is kQ. Therefore

$$Q' = a - kQ.$$

This is a linear first order differential equation. Rewriting it and imposing the initial condition shows that Q is the solution of the initial value problem

$$Q' + kQ = a, \quad Q(0) = Q_0. \tag{3.1.11}$$

Since e^{-kt} is a solution of the complementary equation, the solutions of Equation 3.1.11 are of the form $Q = ue^{-kt}$, where $u'e^{-kt} = a$, so $u' = ae^{kt}$. Hence,

$$u = \frac{a}{k}e^{kt} + c$$

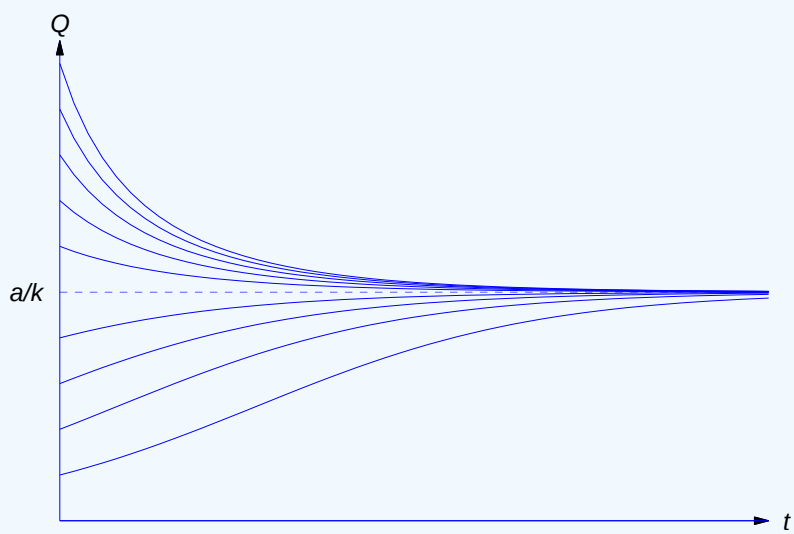

Figure 4.1.3 : $Q(t)$ approaches the steady state value $\frac{a}{k}$ as $t \to \infty$

and

$$Q = ue^{-kt} = \frac{a}{k} + ce^{-kt}.$$

Since $Q(0) = Q_0$, setting $t = 0$ here yields

$$Q_0 = \frac{a}{k} + c \quad \text{or} \quad c = Q_0 - \frac{a}{k}.$$

Therefore

$$Q = \frac{a}{k} + \left(Q_0 - \frac{a}{k}\right) e^{-kt}. \tag{3.1.12}$$

b. Since $k > 0$, $\lim_{t\to\infty} e^{-kt} = 0$, so from Equation 3.1.12

$$\lim_{t\to\infty} Q(t) = \frac{a}{k}.$$

This limit depends only on a and k, and not on Q_0. We say that a/k is the **steady state** value of Q. From Equation 3.1.12 we also see that Q approaches its steady state value from above if $Q_0 > a/k$, or from below if $Q_0 < a/k$. If $Q_0 = a/k$, then Q remains constant (Figure 4.1.3).

Carbon Dating

The fact that Q approaches a steady state value in the situation discussed in Example 4 underlies the method of **carbon dating**, devised by the American chemist and Nobel Prize Winner **W.S. Libby**.

Carbon 12 is stable, but carbon-14, which is produced by cosmic bombardment of nitrogen in the upper atmosphere, is radioactive with a half-life of about 5570 years. Libby assumed that the quantity of carbon-12 in the atmosphere has been constant throughout time, and that the quantity of radioactive carbon-14 achieved its steady state value long ago as a result of its creation and decomposition over millions of years. These assumptions led Libby to conclude that the ratio of carbon-14 to carbon-12 has been nearly constant for a long time. This constant, which we denote by R, has been determined experimentally.

Living cells absorb both carbon-12 and carbon-14 in the proportion in which they are present in the environment. Therefore the ratio of carbon-14 to carbon-12 in a living cell is always R. However, when the cell dies it ceases to absorb carbon, and the ratio of carbon-14 to carbon-12 decreases exponentially as the radioactive carbon-14 decays. This is the basis for the method of carbon dating, as illustrated in the next example.

Example 4.1.5

An archaeologist investigating the site of an ancient village finds a burial ground where the amount of carbon-14 present in individual remains is between 42 and 44% of the amount present in live individuals. Estimate the age of the village and the length of time for which it survived.

Solution

Let $Q = Q(t)$ be the quantity of carbon-14 in an individual set of remains t years after death, and let Q_0 be the quantity that would be present in live individuals. Since carbon-14 decays exponentially with half-life 5570 years, its decay constant is

$$k = \frac{\ln 2}{5570}.$$

Therefore

$$Q = Q_0 e^{-t(\ln 2)/5570}$$

if we choose our time scale so that $t_0 = 0$ is the time of death. If we know the present value of Q we can solve this equation for t, the number of years since death occurred. This yields

$$t = -5570 \frac{\ln Q/Q_0}{\ln 2}.$$

It is given that $Q = 0.42Q_0$ in the remains of individuals who died first. Therefore these deaths occurred about

$$t_1 = -5570 \frac{\ln .42}{\ln 2} \approx 6971$$

years ago. For the most recent deaths, $Q = .44Q_0$; hence, these deaths occurred about

$$t_2 = -5570 \frac{\ln .44}{\ln 2} \approx 6597$$

years ago. Therefore it is reasonable to conclude that the village was founded about 7000 years ago, and lasted for about 400 years.

A Savings Program

Example 4.1.6

A person opens a savings account with an initial deposit of \$1000 and subsequently deposits \$50 per week. Find the value $Q(t)$ of the account at time $t > 0$, assuming that the bank pays 6% interest compounded continuously.

Solution

Observe that Q isn't continuous, since there are 52 discrete deposits per year of \$50 each. To construct a mathematical model for this problem in the form of a differential equation, we make the simplifying assumption that the deposits are made continuously at a rate of \$2600 per year. This is essential, since solutions of differential equations are continuous functions. With this assumption, Q increases continuously at the rate

$$Q' = 2600 + 0.06Q$$

and therefore Q satisfies the differential equation

$$Q' - .06Q = 2600. \tag{3.1.13}$$

(Of course, we must recognize that the solution of this equation is an approximation to the true value of Q at any given time. We'll discuss this further below.) Since $e^{.06t}$ is a solution of the complementary equation, the solutions of Equation 3.1.13 are of the form $Q = ue^{.06t}$, where $u'e^{.06t} = 2600$. Hence, $u' = 2600e^{-.06t}$,

$$u = -\frac{2600}{.06}e^{-0.06t} + c$$

and

$$Q = ue^{.06t} = -\frac{2600}{.06} + ce^{.06t}. \tag{3.1.14}$$

Setting $t = 0$ and $Q = 1000$ here yields

$$c = 1000 + \frac{2600}{0.06},$$

and substituting this into Equation 3.1.14 yields

$$Q = 1000e^{.06t} + \frac{2600}{.06}(e^{.06t} - 1) \tag{3.1.15}$$

where the first term is the value due to the initial deposit and the second is due to the subsequent weekly deposits.

Mathematical models must be tested for validity by comparing predictions based on them with the actual outcome of experiments. Example 6 is unusual in that we can compute the exact value of the account at any specified time and compare it with the approximate value predicted by Equation 3.1.15 (See **Exercise 4.1.21**). The following table gives a comparison for a ten year period. Each exact answer corresponds to the time of the year-end deposit, and each year is assumed to have exactly 52 weeks.

Table 4.1.3

Year	Approximate Value of Q (Example 4.1.6)	Exact Value of P (Exercise 4.1.21)	Error $Q - P$	Percentage Error $(Q - P)/P$
1	\$3741.42	\$3739.87	\$1.55	.0413
2	6652.36	6649.17	3.19	.0479
3	9743.30	9738.37	4.93	.0506
4	13,025.38	13,018.60	6.78	.0521
5	16,510.41	16,501.66	8.75	.0530

Year	Approximate Value of Q (Example 4.1.6)	Exact Value of P (Exercise 4.1.21)	Error $Q-P$	Percentage Error $(Q-P)/P$
6	$20,210.94$	$20,200.11$	10.83	$.0536$
7	$24,140.30$	$24,127.25$	13.05	$.0541$
8	$28,312.63$	$28,297.23$	15.40	$.0544$
9	$32,742.97$	$32,725.07$	17.90	$.0547$
10	$37,447.27$	$37,426.72$	20.55	$.0549$

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3.1E: Growth and Decay (Exercises)

Q4.1.1

1. The half-life of a radioactive substance is 3200 years. Find the quantity $Q(t)$ of the substance left at time $t > 0$ if $Q(0) = 20$ g.

2. The half-life of a radioactive substance is 2 days. Find the time required for a given amount of the material to decay to 1/10 of its original mass.

3. A radioactive material loses 25% of its mass in 10 minutes. What is its half-life?

4. A tree contains a known percentage p_0 of a radioactive substance with half-life τ. When the tree dies the substance decays and isn't replaced. If the percentage of the substance in the fossilized remains of such a tree is found to be p_1, how long has the tree been dead?

5. If t_p and t_q are the times required for a radioactive material to decay to $1/p$ and $1/q$ times its original mass (respectively), how are t_p and t_q related?

6. Find the decay constant k for a radioactive substance, given that the mass of the substance is Q_1 at time t_1 and Q_2 at time t_2.

7. A process creates a radioactive substance at the rate of 2 g/hr and the substance decays at a rate proportional to its mass, with constant of proportionality $k = .1(\text{hr})^{-1}$. If $Q(t)$ is the mass of the substance at time t, find $\lim_{t\to\infty} Q(t)$.

8. A bank pays interest continuously at the rate of 6%. How long does it take for a deposit of Q_0 to grow in value to $2Q_0$?

9. At what rate of interest, compounded continuously, will a bank deposit double in value in 8 years?

10. A savings account pays 5% per annum interest compounded continuously. The initial deposit is Q_0 dollars. Assume that there are no subsequent withdrawals or deposits.

a. How long will it take for the value of the account to triple?
b. What is Q_0 if the value of the account after 10 years is \$100,000 dollars?

11. A candymaker makes 500 pounds of candy per week, while his large family eats the candy at a rate equal to $Q(t)/10$ pounds per week, where $Q(t)$ is the amount of candy present at time t.

a. Find $Q(t)$ for $t > 0$ if the candymaker has 250 pounds of candy at $t = 0$.
b. Find $\lim_{t\to\infty} Q(t)$.

12. Suppose a substance decays at a yearly rate equal to half the square of the mass of the substance present. If we start with 50 g of the substance, how long will it be until only 25 g remain?

13. A super bread dough increases in volume at a rate proportional to the volume V present. If V increases by a factor of 10 in 2 hours and $V(0) = V_0$, find V at any time t. How long will it take for V to increase to $100V_0$?

14. A radioactive substance decays at a rate proportional to the amount present, and half the original quantity Q_0 is left after 1500 years. In how many years would the original amount be reduced to $3Q_0/4$? How much will be left after 2000 years?

15. A wizard creates gold continuously at the rate of 1 ounce per hour, but an assistant steals it continuously at the rate of 5% of however much is there per hour. Let $W(t)$ be the number of ounces that the wizard has at time t. Find $W(t)$ and $\lim_{t\to\infty} W(t)$ if $W(0) = 1$.

16. A process creates a radioactive substance at the rate of 1 g/hr, and the substance decays at an hourly rate equal to 1/10 of the mass present (expressed in grams). Assuming that there are initially 20 g, find the mass $S(t)$ of the substance present at time t, and find $\lim_{t\to\infty} S(t)$.

17. A tank is empty at $t = 0$. Water is added to the tank at the rate of 10 gal/min, but it leaks out at a rate (in gallons per minute) equal to the number of gallons in the tank. What is the smallest capacity the tank can have if this process is to continue forever?

18. A person deposits \$25,000 in a bank that pays 5% per year interest, compounded continuously. The person continuously withdraws from the account at the rate of \$750 per year. Find $V(t)$, the value of the account at time t after the initial deposit.

19. A person has a fortune that grows at rate proportional to the square root of its worth. Find the worth W of the fortune as a function of t if it was \$1 million 6 months ago and is \$4 million today.

20. Let $p = p(t)$ be the quantity of a product present at time t. The product is manufactured continuously at a rate proportional to p, with proportionality constant 1/2, and it is consumed continuously at a rate proportional to p^2, with proportionality constant 1/8. Find $p(t)$ if $p(0) = 100$.

21.

a. In the situation of Example 4.1.6 find the exact value P(t) of the person's account after t years, where t is an integer. Assume that each year has exactly 52 weeks, and include the year-end deposit in the computation.

HINT: At time t the initial $1000 has been on deposit for t years. There have been $52t$ deposits of $50 each. The first $50 has been on deposit for $t - 1/52$ years, the second for $t - 2/52$ years ... in general, the **j** th $50 has been on deposit for $t - j/52$ years ($1 \leq j \leq 52t$). Find the present value of each $50 deposit assuming 6% interest compounded continuously, and use the formula

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} (x \neq 1)$$

to find their total value.

b. Let

$$p(t) = \frac{Q(t) - P(t)}{P(t)}$$

be the relative error after t years. Find

$$p(\infty) = \lim_{t \to \infty} p(t).$$

22. A homebuyer borrows P_0 dollars at an annual interest rate r, agreeing to repay the loan with equal monthly payments of M dollars per month over N years.

a. Derive a differential equation for the loan principal (amount that the homebuyer owes) $P(t)$ at time $t > 0$, making the simplifying assumption that the homebuyer repays the loan continuously rather than in discrete steps. (See Example 4.1.6.)

b. Solve the equation derived in (a).

c. Use the result of (b) to determine an approximate value for M assuming that each year has exactly 12 months of equal length.

d. It can be shown that the exact value of M is given by

$$M = \frac{rP_0}{12} \left(1 - (1 + r/12)^{-12N}\right)^{-1}.$$

Compare the value of M obtained from the answer in (c) to the exact value if (i) $P_0 = \$50,000$, \(r=7{1\over2}\)%, $N = 20$ (ii) $P_0 = \$150,000$, $r = 9.0\%$, $N = 30$.

23. Assume that the homebuyer of **Exercise 4.1.22** elects to repay the loan continuously at the rate of αM dollars per month, where α is a constant greater than 1. (This is called **accelerated payment**.)

a. Determine the time $T(\alpha)$ when the loan will be paid off and the amount $S(\alpha)$ that the homebuyer will save.
b. Suppose $P_0 = \$50,000$, $r = 8\%$, and $N = 15$. Compute the savings realized by accelerated payments with $\alpha = 1.05, 1.10$, and 1.15.

24. A benefactor wishes to establish a trust fund to pay a researcher's salary for T years. The salary is to start at S_0 dollars per year and increase at a fractional rate of a per year. Find the amount of money P_0 that the benefactor must deposit in a trust fund paying interest at a rate r per year. Assume that the researcher's salary is paid continuously, the interest is compounded continuously, and the salary increases are granted continuously.

25. A radioactive substance with decay constant k is produced at the rate of

$$\frac{at}{1 + btQ(t)}$$

units of mass per unit time, where a and b are positive constants and $Q(t)$ is the mass of the substance present at time t; thus, the rate of production is small at the start and tends to slow when Q is large.

a. Set up a differential equation for Q.

b. Choose your own positive values for a, b, k, and $Q_0 = Q(0)$. Use a numerical method to discover what happens to $Q(t)$ as $t \to \infty$. (Be precise, expressing your conclusions in terms of a, b, k. However, no proof is required.)

26. Follow the instructions of **Exercise 4.1.25**, assuming that the substance is produced at the rate of $at/(1+bt(Q(t))^2)$ units of mass per unit of time.

27. Follow the instructions of **Exercise 4.1.25**, assuming that the substance is produced at the rate of $at/(1+bt)$ units of mass per unit of time.

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3.2: Elementary Mechanics

Newton's Second Law of Motion

In this section we consider an object with constant mass m moving along a line under a force F. Let $y = y(t)$ be the displacement of the object from a reference point on the line at time t, and let $v = v(t)$ and $a = a(t)$ be the velocity and acceleration of the object at time t. Thus, $v = y'$ and $a = v' = y''$, where the prime denotes differentiation with respect to t. Newton's second law of motion asserts that the force F and the acceleration a are related by the equation

$$F = ma. \tag{3.2.1}$$

Note: Units

In applications there are three main sets of units in use for length, mass, force, and time: the cgs, mks, and British systems. All three use the second as the unit of time. Table 4.3.1 shows the other units. Consistent with Equation 3.2.1, the unit of force in each system is defined to be the force required to impart an acceleration of (one unit of length)$/s^2$ to one unit of mass.

Table 4.3.1

Set	Length	Force	Mass
cgs	centimeter (cm)	dyne (d)	gram (g)
mks	meter (m)	newton (N)	kilogram (kg)
British	foot (ft)	pound (lb)	slug (sl)

If we assume that Earth is a perfect sphere with constant mass density, Newton's law of gravitation (discussed later in this section) asserts that the force exerted on an object by Earth's gravitational field is proportional to the mass of the object and inversely proportional to the square of its distance from the center of Earth. However, if the object remains sufficiently close to Earth's surface, we may assume that the gravitational force is constant and equal to its value at the surface. The magnitude of this force is mg, where g is called the **acceleration due to gravity**. (To be completely accurate, g should be called the **magnitude of the acceleration due to gravity at Earth's surface**.) This quantity has been determined experimentally. Approximate values of g are

$$\begin{aligned} g &= 980 \text{ cm/s}^2 && \text{(cgs)} \\ g &= 9.8 \text{ m/s}^2 && \text{(mks)} \\ g &= 32 \text{ ft/s}^2 && \text{(British).} \end{aligned}$$

In general, the force F in Equation 3.2.1 may depend upon t, y, and y'. Since $a = y''$, Equation 3.2.1 can be written in the form

$$my'' = F(t, y, y'), \tag{3.2.2}$$

which is a second order equation. We'll consider this equation with restrictions on F later; however, since Chapter 2 dealt only with first order equations, we consider here only problems in which Equation 3.2.2 can be recast as a first order equation. This is possible if F does not depend on y, so Equation 3.2.2 is of the form

$$my'' = F(t, y').$$

Letting $v = y'$ and $v' = y''$ yields a first order equation for v:

$$mv' = F(t, v). \tag{3.2.3}$$

Solving this equation yields v as a function of t. If we know $y(t_0)$ for some time t_0, we can integrate v to obtain y as a function of t.

Equations of the form Equation 3.2.3 occur in problems involving motion through a resisting medium.

Motion Through a Resisting Medium Under Constant Gravitational Force

Now we consider an object moving vertically in some medium. We assume that the only forces acting on the object are gravity and resistance from the medium. We also assume that the motion takes place close to Earth's surface and take the upward direction to be positive, so the gravitational force can be assumed to have the constant value $-mg$. We'll see that, under reasonable assumptions on the resisting force, the velocity approaches a limit as $t \to \infty$. We call this limit the **terminal velocity**.

Example 4.3.1

An object with mass m moves under constant gravitational force through a medium that exerts a resistance with magnitude proportional to the speed of the object. (Recall that the speed of an object is $|v|$, the absolute value of its velocity v.) Find the velocity of the object as a function of t, and find the terminal velocity. Assume that the initial velocity is v_0.

Solution

The total force acting on the object is

$$F = -mg + F_1, \tag{3.2.4}$$

where $-mg$ is the force due to gravity and F_1 is the resisting force of the medium, which has magnitude $k|v|$, where k is a positive constant. If the object is moving downward ($v \leq 0$), the resisting force is upward (Figure 4.3.1a), so

$$F_1 = k|v| = k(-v) = -kv.$$

On the other hand, if the object is moving upward ($v \geq 0$), the resisting force is downward (Figure 4.3.1b , so

$$F_1 = -k|v| = -kv.$$

Thus, Equation 3.2.4 can be written as

$$F = -mg - kv, \tag{3.2.5}$$

regardless of the sign of the velocity.

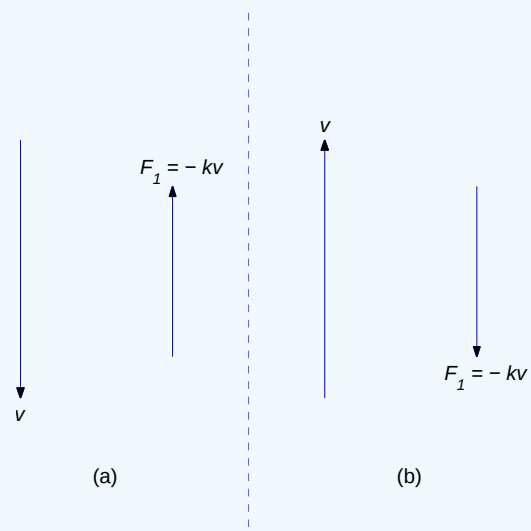

Figure 4.3.1 : Resistive forces

From Newton's second law of motion,

$$F = ma = mv',$$

so Equation 3.2.5 yields

$$mv' = -mg - kv,$$

or

$$v' + \frac{k}{m}v = -g. \tag{3.2.6}$$

Since $e^{-kt/m}$ is a solution of the complementary equation, the solutions of Equation 3.2.6 are of the form $v = ue^{-kt/m}$, where $u'e^{-kt/m} = -g$, so $u' = -ge^{kt/m}$. Hence,

$$u = -\frac{mg}{k}e^{kt/m} + c,$$

so

$$v = ue^{-kt/m} = -\frac{mg}{k} + ce^{-kt/m}. \tag{3.2.7}$$

Since $v(0) = v_0$,

$$v_0 = -\frac{mg}{k} + c,$$

so

$$c = v_0 + \frac{mg}{k}$$

and Equation 3.2.7 becomes

$$v = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}.$$

Letting $t \to \infty$ here shows that the terminal velocity is

$$\lim_{t\to\infty} v(t) = -\frac{mg}{k},$$

which is independent of the initial velocity v_0 (Figure 4.3.2).

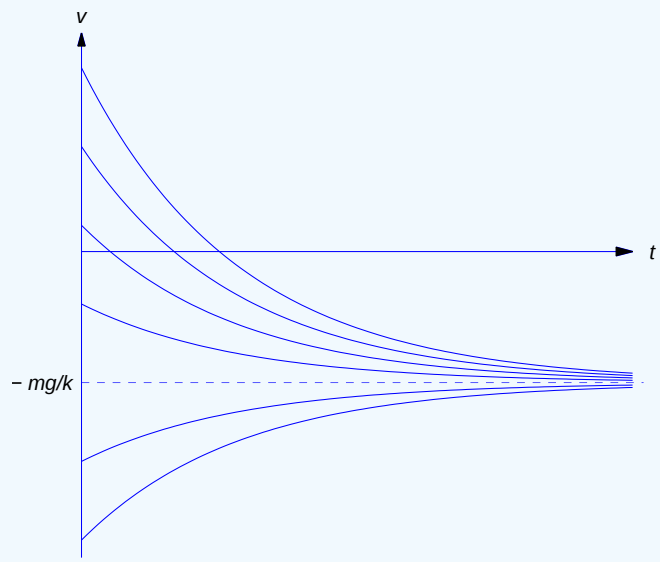

Figure 4.3.2 : Solutions of $mv' = -mg - kv$

Example 4.3.2 : Terminal Velocity

A 960-lb object is given an initial upward velocity of 60 ft/s near the surface of Earth. The atmosphere resists the motion with a force of 3 lb for each ft/s of speed. Assuming that the only other force acting on the object is constant gravity, find its velocity v as a function of t, and find its terminal velocity.

Solution

Since $mg = 960$ and $g = 32$, $m = 960/32 = 30$. The atmospheric resistance is $-3v$ lb if v is expressed in feet per second. Therefore

$$30v' = -960 - 3v,$$

which we rewrite as

$$v' + \frac{1}{10}v = -32.$$

Since $e^{-t/10}$ is a solution of the complementary equation, the solutions of this equation are of the form $v = ue^{-t/10}$, where $u'e^{-t/10} = -32$, so $u' = -32e^{t/10}$. Hence,

$$u = -320e^{t/10} + c,$$

so

$$v = ue^{-t/10} = -320 + ce^{-t/10}. \tag{3.2.8}$$

The initial velocity is 60 ft/s in the upward (positive) direction; hence, $v_0 = 60$. Substituting $t = 0$ and $v = 60$ in Equation 3.2.8 yields

$$60 = -320 + c,$$

so $c = 380$, and Equation 3.2.8 becomes

$$v = -320 + 380e^{-t/10} \text{ ft/s}$$

The terminal velocity is

$$\lim_{t\to\infty} v(t) = -320 \text{ ft/s}.$$

Example 4.3.3

A 10 kg mass is given an initial velocity $v_0 \le 0$ near Earth's surface. The only forces acting on it are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the resistance is 8 N if the speed is 2 m/s, find the velocity of the object as a function of t, and find the terminal velocity.

Solution

Since the object is falling, the resistance is in the upward (positive) direction. Hence,

$$mv' = -mg + kv^2, \tag{3.2.9}$$

where k is a constant. Since the magnitude of the resistance is 8 N when $v = 2$ m/s,

$$k(2^2) = 8,$$

so $k = 2$ N-s^2/m^2. Since $m = 10$ and $g = 9.8$, Equation 3.2.9 becomes

$$10v' = -98 + 2v^2 = 2(v^2 - 49). \tag{3.2.10}$$

If $v_0 = -7$, then $v \equiv -7$ for all $t \ge 0$. If $v_0 \ne -7$, we separate variables to obtain

$$\frac{1}{v^2 - 49}v' = \frac{1}{5}, \tag{3.2.11}$$

which is convenient for the required partial fraction expansion

$$\frac{1}{v^2 - 49} = \frac{1}{(v-7)(v+7)} = \frac{1}{14}\left[\frac{1}{v-7} - \frac{1}{v+7}\right]. \tag{3.2.12}$$

Substituting Equation 3.2.12 into Equation 3.2.11 yields

$$\frac{1}{14}\left[\frac{1}{v-7} - \frac{1}{v+7}\right]v' = \frac{1}{5},$$

so

$$\left[\frac{1}{v-7}-\frac{1}{v+7}\right]v'=\frac{14}{5}.$$

Integrating this yields

$$\ln|v-7|-\ln|v+7|=14t/5+k.$$

Therefore

$$\left|\frac{v-7}{v+7}\right|=e^k e^{14t/5}.$$

Since Theorem 2.3.1 implies that $(v-7)/(v+7)$ cannot change sign (why?), we can rewrite the last equation as

$$\frac{v-7}{v+7}=ce^{14t/5}, \tag{3.2.13}$$

which is an implicit solution of Equation 3.2.10. Solving this for v yields

$$v=-7\frac{c+e^{-14t/5}}{c-e^{-14t/5}}. \tag{3.2.14}$$

Since $v(0)=v_0$, it Equation 3.2.13 implies that

$$c=\frac{v_0-7}{v_0+7}.$$

Substituting this into Equation 3.2.14 and simplifying yields

$$v=-7\frac{v_0(1+e^{-14t/5})-7(1-e^{-14t/5})}{v_0(1-e^{-14t/5})-7(1+e^{-14t/5})}.$$

Since $v_0\le 0$, v is defined and negative for all $t>0$. The terminal velocity is

$$\lim_{t\to\infty}v(t)=-7\text{ m/s},$$

independent of v_0. More generally, it can be shown (**Exercise 4.3.11**) that if v is any solution of Equation 3.2.9 such that $v_0\le 0$ then

$$\lim_{t\to\infty}v(t)=-\sqrt{\frac{mg}{k}}.$$

This is demonstrated in Figure 4.3.3 .

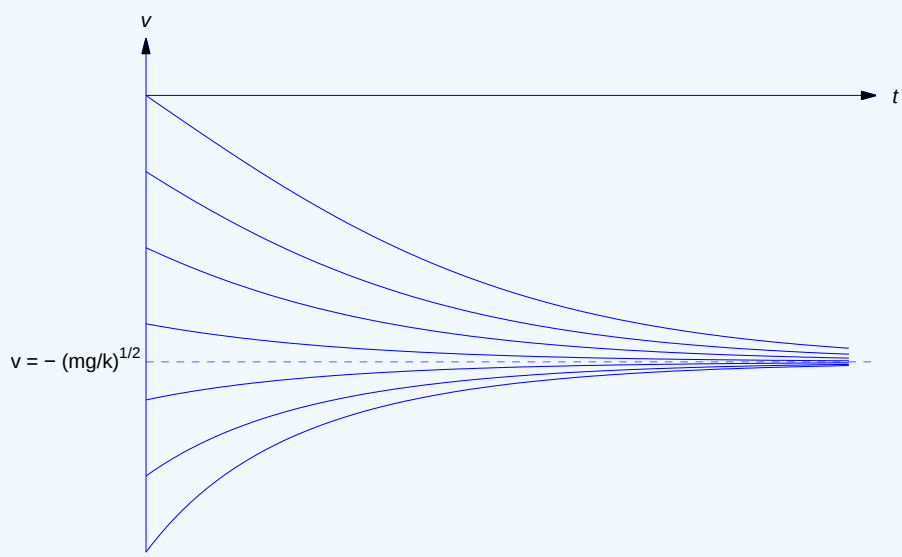

Figure 4.3.3 : Solutions of $mv'=-mg+kv^2,\ v(0)=v_0\le 0$

Example 4.3.4

A 10-kg mass is launched vertically upward from Earth's surface with an initial velocity of v_0 m/s. The only forces acting on the mass are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the atmospheric resistance is 8 N if the speed is 2 m/s, find the time T required for the mass to reach maximum altitude.

Solution

The mass will climb while $v > 0$ and reach its maximum altitude when $v = 0$. Therefore $v > 0$ for $0 \le t < T$ and $v(T) = 0$; therefore, we replace Equation 3.2.10 by

$$10v' = -98 - 2v^2. \tag{3.2.15}$$

Separating variables yields

$$\frac{5}{v^2 + 49} v' = -1,$$

and integrating this yields

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + c.$$

(Recall that $\tan^{-1} u$ is the number θ such that $-\pi/2 < \theta < \pi/2$ and $\tan\theta = u$.) Since $v(0) = v_0$,

$$c = \frac{5}{7} \tan^{-1} \frac{v_0}{7},$$

so v is defined implicitly by

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + \frac{5}{7} \tan^{-1} \frac{v_0}{7}, \quad 0 \le t \le T. \tag{3.2.16}$$

Solving this for v yields

$$v = 7 \tan\left(-\frac{7t}{5} + \tan^{-1} \frac{v_0}{7}\right). \tag{3.2.17}$$

Using the identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

with $A = \tan^{-1}(v_0/7)$ and $B = 7t/5$, and noting that $\tan(\tan^{-1}\theta) = \theta$, we can simplify Equation 3.2.17 to

$$v = 7 \frac{v_0 - 7\tan(7t/5)}{7 + v_0 \tan(7t/5)}.$$

Since $v(T) = 0$ and $\tan^{-1}(0) = 0$, Equation 3.2.16 implies that

$$-T + \frac{5}{7} \tan^{-1} \frac{v_0}{7} = 0.$$

Therefore

$$T = \frac{5}{7} \tan^{-1} \frac{v_0}{7}.$$

Since $\tan^{-1}(v_0/7) < \pi/2$ for all v_0, the time required for the mass to reach its maximum altitude is less than

$$\frac{5\pi}{14} \approx 1.122 \text{ s}$$

regardless of the initial velocity. Figure 4.3.4 shows graphs of v over $[0, T]$ for various values of v_0.

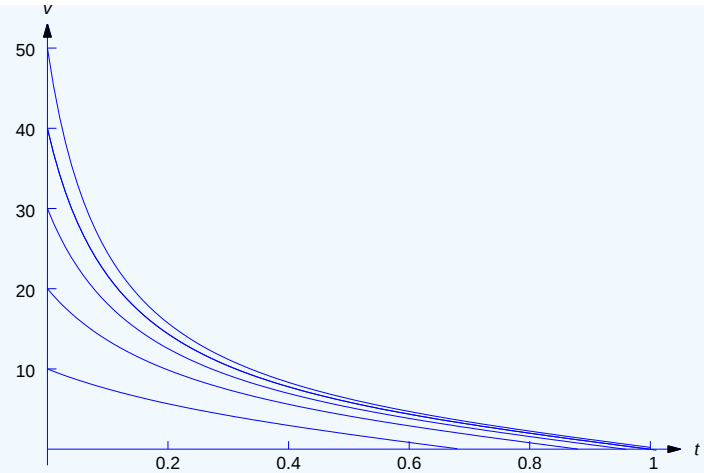

Figure 4.3.4 : Solutions of 4.3.15 for various $v_0 > 0$

Escape Velocity

Suppose a space vehicle is launched vertically and its fuel is exhausted when the vehicle reaches an altitude h above Earth, where h is sufficiently large so that resistance due to Earth's atmosphere can be neglected. Let $t = 0$ be the time when burnout occurs. Assuming that the gravitational forces of all other celestial bodies can be neglected, the motion of the vehicle for $t > 0$ is that of an object with constant mass m under the influence of Earth's gravitational force, which we now assume to vary inversely with the square of the distance from Earth's center; thus, if we take the upward direction to be positive then gravitational force on the vehicle at an altitude y above Earth is

$$F = -\frac{K}{(y+R)^2}, \tag{3.2.18}$$

where R is Earth's radius (Figure 4.3.5).

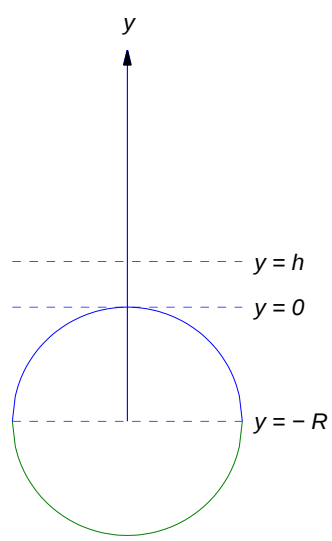

Figure 4.3.5 : Escape velocity

Since $F = -mg$ when $y = 0$, setting $y = 0$ in Equation 3.2.18 yields

$$-mg = -\frac{K}{R^2};$$

therefore $K = mgR^2$ and Equation 3.2.18 can be written more specifically as

$$F = -\frac{mgR^2}{(y+R)^2}. \tag{3.2.19}$$

From Newton's second law of motion,

$$F = m\frac{d^2y}{dt^2},$$

so Equation 3.2.19 implies that

$$\frac{d^2y}{dt^2} = -\frac{gR^2}{(y+R)^2}. \tag{3.2.20}$$

We'll show that there's a number v_e, called the **escape velocity**, with these properties:

1. If $v_0 \geq v_e$ then $v(t) > 0$ for all $t > 0$, and the vehicle continues to climb for all $t > 0$; that is, it "escapes" Earth. (Is it really so obvious that $\lim_{t\to\infty} y(t) = \infty$ in this case? For a proof, see **Exercise 4.3.20**.)
2. If $v_0 < v_e$ then $v(t)$ decreases to zero and becomes negative. Therefore the vehicle attains a maximum altitude y_m and falls back to Earth.

Since Equation 3.2.20 is second order, we cannot solve it by methods discussed so far. However, we are concerned with v rather than y, and v is easier to find. Since $v = y'$ the chain rule implies that

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Substituting this into Equation 3.2.20 yields the first order separable equation

$$v\frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}. \tag{3.2.21}$$

When $t = 0$, the velocity is v_0 and the altitude is h. Therefore we can obtain v as a function of y by solving the initial value problem

$$v\frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}, \quad v(h) = v_0.$$

Integrating Equation 3.2.21 with respect to y yields

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + c. \tag{3.2.22}$$

Since $v(h) = v_0$,

$$c = \frac{v_0^2}{2} - \frac{gR^2}{h+R},$$

so Equation 3.2.22 becomes

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right). \tag{3.2.23}$$

If

$$v_0 \geq \left(\frac{2gR^2}{h+R}\right)^{1/2},$$

the parenthetical expression in Equation 3.2.23 is nonnegative, so $v(y) > 0$ for $y > h$. This proves that there's an escape velocity v_e. We'll now prove that

$$v_e = \left(\frac{2gR^2}{h+R}\right)^{1/2}$$

by showing that the vehicle falls back to Earth if

$$v_0 < \left(\frac{2gR^2}{h+R}\right)^{1/2}. \tag{3.2.24}$$

If Equation 3.2.24 holds then the parenthetical expression in Equation 3.2.23 is negative and the vehicle will attain a maximum altitude $y_m > h$ that satisfies the equation

$$0 = \frac{gR^2}{y_m + R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right).$$

The velocity will be zero at the maximum altitude, and the object will then fall to Earth under the influence of gravity.

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3.2E: Elementary Mechanics (Exercises)

Except where directed otherwise, assume that the magnitude of the gravitational force on an object with mass m is constant and equal to mg. In exercises involving vertical motion take the upward direction to be positive.

Q4.3.1

1. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to his speed, with $k = 2.5$ lb-s/ft. Assuming that he starts from rest, find his velocity as a function of time and find his terminal velocity.

2. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to her speed, with constant of proportionality k. Find k, given that her terminal velocity is -16 ft/s, and then find her velocity v as a function of t. Assume that she starts from rest.

3. A boat weighs $64,000$ lb. Its propellor produces a constant thrust of $50,000$ lb and the water exerts a resistive force with magnitude proportional to the speed, with $k = 2000$ lb-s/ft. Assuming that the boat starts from rest, find its velocity as a function of time, and find its terminal velocity.

4. A constant horizontal force of 10 N pushes a 20 kg-mass through a medium that resists its motion with $.5$ N for every m/s of speed. The initial velocity of the mass is 7 m/s in the direction opposite to the direction of the applied force. Find the velocity of the mass for $t > 0$.

5. A stone weighing $1/2$ lb is thrown upward from an initial height of 5 ft with an initial speed of 32 ft/s. Air resistance is proportional to speed, with $k = 1/128$ lb-s/ft. Find the maximum height attained by the stone.

6. A 3200-lb car is moving at 64 ft/s down a 30-degree grade when it runs out of fuel. Find its velocity after that if friction exerts a resistive force with magnitude proportional to the square of the speed, with $k = 1$ lb-s$^2/$ft^2 . Also find its terminal velocity.

7. A 96 lb weight is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the speed. Find its velocity as a function of time if its terminal velocity is -128 ft/s.

8. An object with mass m moves vertically through a medium that exerts a resistive force with magnitude proportional to the speed. Let $y = y(t)$ be the altitude of the object at time t, with $y(0) = y_0$. Use the results of Example 4.3.1 to show that

$$y(t) = y_0 + \frac{m}{k}(v_0 - v - gt).$$

9. An object with mass m is launched vertically upward with initial velocity v_0 from Earth's surface ($y_0 = 0$) in a medium that exerts a resistive force with magnitude proportional to the speed. Find the time T when the object attains its maximum altitude y_m. Then use the result of **Exercise 4.3.8** to find y_m.

10. An object weighing 256 lb is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the square of the speed. The magnitude of the resisting force is 1 lb when $|v| = 4$ ft$/$s. Find v for $t > 0$, and find its terminal velocity.

11. An object with mass m is given an initial velocity $v_0 \le 0$ in a medium that exerts a resistive force with magnitude proportional to the square of the speed. Find the velocity of the object for $t > 0$, and find its terminal velocity.

12. An object with mass m is launched vertically upward with initial velocity v_0 in a medium that exerts a resistive force with magnitude proportional to the square of the speed.

a. Find the time T when the object reaches its maximum altitude.
b. Use the result of **Exercise 4.3.11** to find the velocity of the object for $t > T$.

13. An object with mass m is given an initial velocity $v_0 \le 0$ in a medium that exerts a resistive force of the form $a|v|/(1 + |v|)$, where a is positive constant.

a. Set up a differential equation for the speed of the object.
b. Use your favorite numerical method to solve the equation you found in (a), to convince yourself that there's a unique number a_0 such that $\lim_{t\to\infty} s(t) = \infty$ if $a \le a_0$ and $\lim_{t\to\infty} s(t)$ exists (finite) if $a > a_0$. (We say that a_0 is the **bifurcation value** of a.) Try to find a_0 and $\lim_{t\to\infty} s(t)$ in the case where $a > a_0$.

14. An object of mass m falls in a medium that exerts a resistive force $f = f(s)$, where $s = |v|$ is the speed of the object. Assume that $f(0) = 0$ and f is strictly increasing and differentiable on $(0, \infty)$.

a. Write a differential equation for the speed $s = s(t)$ of the object. Take it as given that all solutions of this equation with $s(0) \geq 0$ are defined for all $t > 0$ (which makes good sense on physical grounds).
b. Show that if $\lim_{s\to\infty} f(s) \leq mg$ then $\lim_{t\to\infty} s(t) = \infty$.
c. Show that if $\lim_{s\to\infty} f(s) > mg$ then $\lim_{t\to\infty} s(t) = s_T$ (terminal speed), where $f(s_T) = mg$..

15. A 100-g mass with initial velocity $v_0 \leq 0$ falls in a medium that exerts a resistive force proportional to the fourth power of the speed. The resistance is .1 N if the speed is 3 m/s.

a. Set up the initial value problem for the velocity v of the mass for $t > 0$.
b. Use **Exercise 4.3.14** (c) to determine the terminal velocity of the object.
c. To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 1]$ (seconds) of the initial value problem of (a) , with initial values $v_0 = 0$, -2, -4, …, -12. Present your results in graphical form similar to Figure 4.3.3.

16. A 64-lb object with initial velocity $v_0 \leq 0$ falls through a dense fluid that exerts a resistive force proportional to the square root of the speed. The resistance is 64 lb if the speed is 16 ft/s.

a. Set up the initial value problem for the velocity v of the mass for $t > 0$.
b. Use **Exercise 4.3.14** (c) to determine the terminal velocity of the object.
c. To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 4]$ (seconds) of the initial value problem of (a), with initial values $v_0 = 0$, -5, -10, …, -30. Present your results in graphical form similar to Figure 4.3.3.

Q4.3.2

In **Exercises 4.3.17-4.3.20**, assume that the force due to gravity is given by Newton's law of gravitation. Take the upward direction to be positive.

17. A space probe is to be launched from a space station 200 miles above Earth. Determine its escape velocity in miles/s. Take Earth's radius to be 3960 miles.

18. A space vehicle is to be launched from the moon, which has a radius of about 1080 miles. The acceleration due to gravity at the surface of the moon is about 5.31 ft/s^2. Find the escape velocity in miles/s.

19.

a. Show that (Equation 4.3.27) can be rewritten as

$$v^2 = \frac{h-y}{y+R} v_e^2 + v_0^2.$$

b. Show that if $v_0 = \rho v_e$ with $0 \leq \rho < 1$, then the maximum altitude y_m attained by the space vehicle is

$$y_m = \frac{h + R\rho^2}{1-\rho^2}.$$

c. By requiring that $v(y_m) = 0$, use (Equation 4.3.26) to deduce that if $v_0 < v_e$ then

$$|v| = v_e \left[\frac{(1-\rho^2)(y_m - y)}{y+R} \right]^{1/2},$$

where y_m and ρ are as defined in (b) and $y \geq h$.
d. Deduce from (c) that if $v < v_e$, the vehicle takes equal times to climb from $y = h$ to $y = y_m$ and to fall back from $y = y_m$ to $y = h$.

20. In the situation considered in the discussion of escape velocity, show that $\lim_{t\to\infty} y(t) = \infty$ if $v(t) > 0$ for all $t > 0$.

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3.3: Autonomous Second Order Equations

A second order differential equation that can be written as

$$y'' = F(y, y') \tag{3.3.1}$$

where F is independent of t, is said to be **autonomous**. An autonomous second order equation can be converted into a first order equation relating $v = y'$ and y. If we let $v = y'$, Equation 3.3.1 becomes

$$v' = F(y, v). \tag{3.3.2}$$

Since

$$v' = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}, \tag{3.3.3}$$

Equation 3.3.2 can be rewritten as

$$v\frac{dv}{dy} = F(y, v). \tag{3.3.4}$$

The integral curves of Equation 3.3.4 can be plotted in the (y, v) plane, which is called the **Poincaré phase plane** of Equation 3.3.1. If y is a solution of Equation 3.3.1 then $y = y(t), v = y'(t)$ is a parametric equation for an integral curve of Equation 3.3.4. we will call these integral curves **trajectories** of Equation 3.3.1, and we will call Equation 3.3.4 the **phase plane equivalent** of Equation 3.3.1.

In this section we will consider autonomous equations that can be written as

$$y'' + q(y, y')y' + p(y) = 0. \tag{3.3.5}$$

Equations of this form often arise in applications of Newton's second law of motion. For example, suppose y is the displacement of a moving object with mass m. It's reasonable to think of two types of time-independent forces acting on the object. One type - such as gravity - depends only on position. We could write such a force as $-mp(y)$. The second type - such as atmospheric resistance or friction - may depend on position and velocity. (Forces that depend on velocity are called **damping** forces.) We write this force as $-mq(y, y')y'$, where $q(y, y')$ is usually a positive function and we've put the factor y' outside to make it explicit that the force is in the direction opposing the motion. In this case Newton's, second law of motion leads to Equation 3.3.5.

The phase plane equivalent of Equation 3.3.5 is

$$v\frac{dv}{dy} + q(y, v)v + p(y) = 0. \tag{3.3.6}$$

Some statements that we will be making about the properties of Equation 3.3.5 and Equation 3.3.6 are intuitively reasonable, but difficult to prove. Therefore our presentation in this section will be informal: we will just say things without proof, all of which are true if we assume that $p = p(y)$ is continuously differentiable for all y and $q = q(y, v)$ is continuously differentiable for all (y, v). We begin with the following statements:

Properties

- **Statement 1:** If y_0 and v_0 are arbitrary real numbers then Equation 3.3.5 has a unique solution on $(-\infty, \infty)$ such that $y(0) = y_0$ and $y'(0) = v_0$.
- **Statement 2:** If $y = y(t)$ is a solution of Equation 3.3.5 and τ is any constant then $y_1 = y(t - \tau)$ is also a solution of Equation 3.3.5, and y and y_1 have the same trajectory.
- **Statement 3:** If two solutions y and y_1 of Equation 3.3.5 have the same trajectory then $y_1(t) = y(t - \tau)$ for some constant τ.
- **Statement 4:** Distinct trajectories of Equation 3.3.5 cannot intersect; that is, if two trajectories of Equation 3.3.5 intersect, they are identical.
- **Statement 5:** If the trajectory of a solution of Equation 3.3.5 is a closed curve then $(y(t), v(t))$ traverses the trajectory in a finite time T, and the solution is periodic with period T; that is, $y(t + T) = y(t)$ for all t in $(-\infty, \infty)$.

If $\overline{y}$ is a constant such that $p(\overline{y})=0$ then $y\equiv\overline{y}$ is a constant solution of Equation 3.3.5. We say that $\overline{y}$ is an **equilibrium** of Equation 3.3.5 and $(\overline{y},0)$ is a **critical point** of the phase plane equivalent equation Equation 3.3.6. We say that the equilibrium and the critical point are **stable** if, for any given $\epsilon>0$ **no matter how small**, there's a $\delta>0$, **sufficiently small**, such that if

$$\sqrt{(y_0-\overline{y})^2+v_0^2}<\delta$$

then the solution of the initial value problem

$$y''+q(y,y')y'+p(y)=0,\quad y(0)=y_0,\quad y'(0)=v_0$$

satisfies the inequality

$$\sqrt{(y(t)-\overline{y})^2+(v(t))^2}<\epsilon$$

for all $t>0$. Figure 4.4.1 illustrates the geometrical interpretation of this definition in the Poincaré phase plane: if (y_0,v_0) is in the smaller shaded circle (with radius δ), then $(y(t),v(t))$ must be in in the larger circle (with radius ϵ) for all $t>0$.

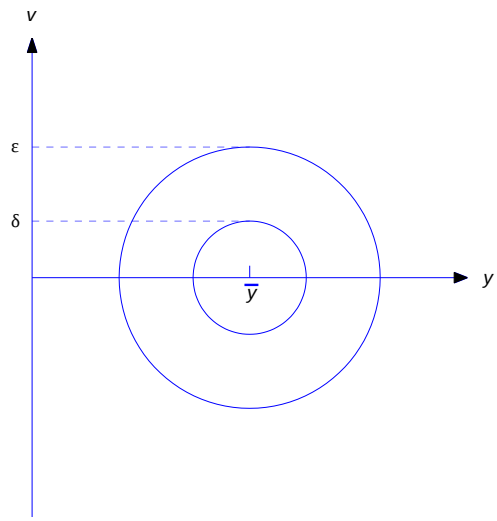


Figure 4.4.1 : Stability: if (y_0,v_0) is in the smaller circle then $(y(t),v(t))$is in the larger circle for all $t>0$

If an equilibrium and the associated critical point are not stable, we say they are **unstable**. To see if you really understand what **stable** means, try to give a direct definition of **unstable** (**Exercise 4.4.22**). we will illustrate both definitions in the following examples.

The Undamped Case

We will begin with the case where $q\equiv 0$, so Equation 3.3.5 reduces to

$$y''+p(y)=0. \tag{3.3.7}$$

We say that this equation - as well as any physical situation that it may model - is **undamped**. The phase plane equivalent of Equation 3.3.7 is the separable equation

$$v\frac{dv}{dy}+p(y)=0.$$

Integrating this yields

$$\frac{v^2}{2}+P(y)=c, \tag{3.3.8}$$

where c is a constant of integration and $P(y)=\int p(y)\,dy$ is an antiderivative of p.

If Equation 3.3.7 is the equation of motion of an object of mass m, then $mv^2/2$ is the kinetic energy and $mP(y)$ is the potential energy of the object; thus, Equation 3.3.8 says that the total energy of the object remains constant, or is **conserved**. In particular, if

a trajectory passes through a given point (y_0, v_0) then

$$c = \frac{v_0^2}{2} + P(y_0).$$

Example 4.4.1 : The Undamped Spring - Mass System

Consider an object with mass m suspended from a spring and moving vertically. Let y be the displacement of the object from the position it occupies when suspended at rest from the spring (Figure 4.4.2).

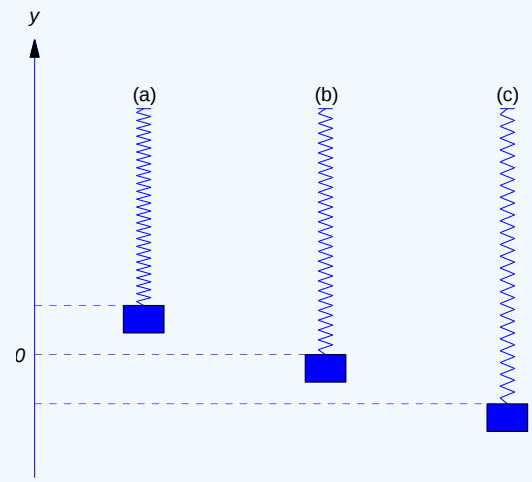


Figure 4.4.2 : (a) $y > 0$ (b) $y = 0$ (c) $y < 0$.

Assume that if the length of the spring is changed by an amount ΔL (positive or negative), then the spring exerts an opposing force with magnitude $k|\Delta L|$, where k is a positive constant. In Section 6.1, it will be shown that if the mass of the spring is negligible compared to m and no other forces act on the object then Newton's second law of motion implies that

$$my'' = -ky, \tag{3.3.9}$$

which can be written in the form Equation 3.3.7 with $p(y) = ky/m$. This equation can be solved easily by a method that we will study in Section 5.2, but that method isn't available here. Instead, we will consider the phase plane equivalent of Equation 3.3.9.

From Equation 3.3.3, we can rewrite Equation 3.3.9 as the separable equation

$$mv\frac{dv}{dy} = -ky.$$

Integrating this yields

$$\frac{mv^2}{2} = -\frac{ky^2}{2} + c,$$

which implies that

$$mv^2 + ky^2 = \rho \tag{3.3.10}$$

($\rho = 2c$). This defines an ellipse in the Poincaré phase plane (Figure 4.4.3).

We can identify ρ by setting $t = 0$ in Equation 3.3.10; thus, $\rho = mv_0^2 + ky_0^2$, where $y_0 = y(0)$ and $v_0 = v(0)$. To determine the maximum and minimum values of y we set $v = 0$ in Equation 3.3.10; thus,

$$y_{\max} = R \quad \text{and} \quad y_{\min} = -R, \quad \text{with} R = \sqrt{\frac{\rho}{k}}. \tag{3.3.11}$$

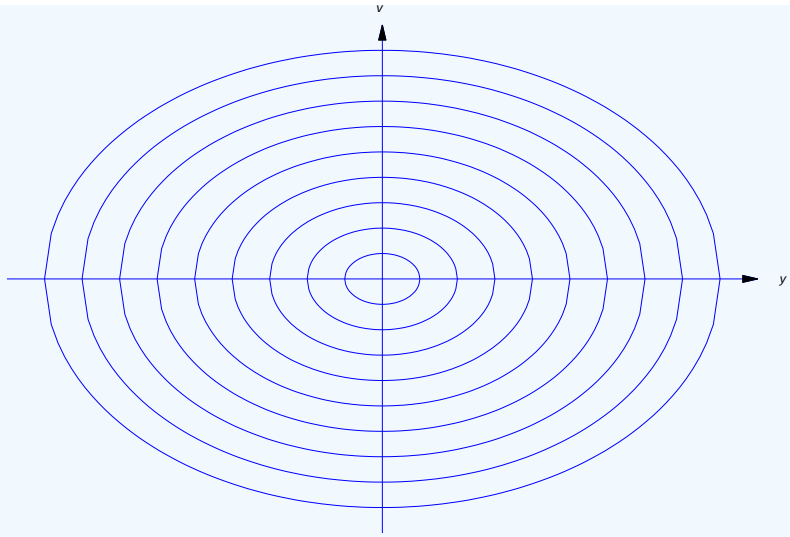


Figure 4.4.3 : Trajectories of $my'' + ky = 0$

Equation 3.3.9 has exactly one equilibrium, $\bar{y} = 0$, and it is stable. You can see intuitively why this is so: if the object is displaced in either direction from equilibrium, the spring tries to bring it back.

In this case we can find y explicitly as a function of t. (Don't expect this to happen in more complicated problems!) If $v > 0$ on an interval I, Equation 3.3.10 implies that

$$\frac{dy}{dt} = v = \sqrt{\frac{\rho - ky^2}{m}}$$

on I. This is equivalent to

$$\frac{\sqrt{k}}{\sqrt{\rho - ky^2}} \frac{dy}{dt} = \omega_0, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{3.3.12}$$

Since

$$\begin{aligned} \int \frac{\sqrt{k}\, dy}{\sqrt{\rho - ky^2}} &= \sin^{-1}\left(\sqrt{\frac{k}{\rho}}\, y\right) + c \\ &= \sin^{-1}\left(\frac{y}{R}\right) + c \end{aligned}$$

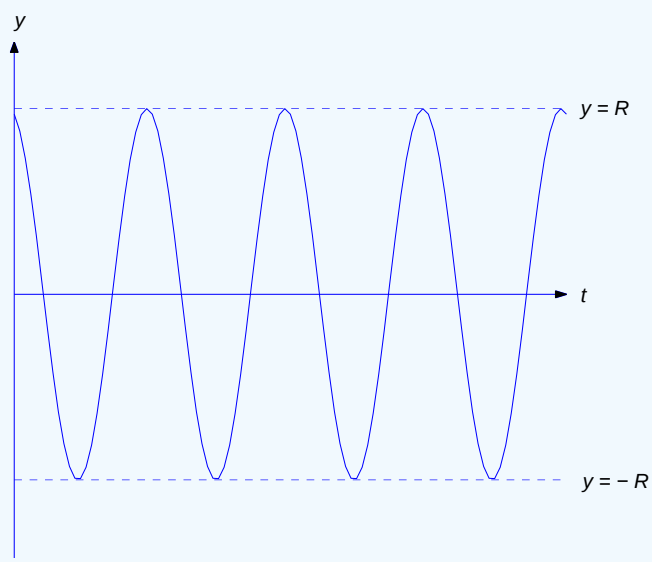


Figure 4.4.4 : $y = R\sin(\omega_0 t + \phi)$

(see Equation 3.3.11), Equation 3.3.12 implies that that there's a constant ϕ such that

$$\sin^{-1}\left(\frac{y}{R}\right) = \omega_0 t + \phi$$

or

$$y = R\sin(\omega_0 t + \phi)$$

for all t in I. Although we obtained this function by assuming that $v > 0$, you can easily verify that y satisfies Equation 3.3.9 for all values of t. Thus, the displacement varies periodically between $-R$ and R, with period $T = 2\pi/\omega_0$ (Figure 4.4.4). (If you've taken a course in elementary mechanics you may recognize this as **simple harmonic motion**.)

Example 4.4.2 : The Undamped Pendulum

Now we consider the motion of a pendulum with mass m, attached to the end of a weightless rod with length L that rotates on a frictionless axle (Figure 4.4.5). We assume that there's no air resistance.

Let y be the angle measured from the rest position (vertically downward) of the pendulum, as shown in Figure 4.4.5 .

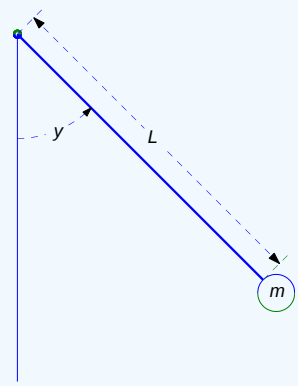


Figure 4.4.5 : The undamped pendulum

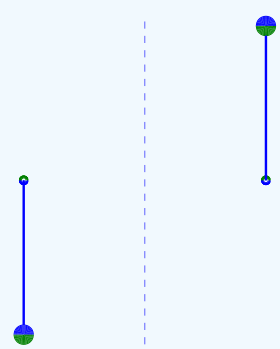
Newton's second law of motion says that the product of m and the tangential acceleration equals the tangential component of the gravitational force; therefore, from Figure 4.4.5 ,

$$mLy'' = -mg\sin y,$$

or

$$y'' = -\frac{g}{L}\sin y. \tag{3.3.13}$$

Since $\sin n\pi = 0$ if n is any integer, Equation 3.3.13 has infinitely many equilibria $\overline{y}_n = n\pi$. If n is even, the mass is directly below the axle (Figure $3.3.6a$) and gravity opposes any deviation from the equilibrium.



(a) Stable equilibrium (b) Unstable equilibrium

Figure 4.4.6 : (a) Stable equilibrium (b) Unstable equilibrium

However, if n is odd, the mass is directly above the axle (Figure 3.3.6b) and gravity increases any deviation from the equilibrium. Therefore we conclude on physical grounds that $\overline{y}_{2m} = 2m\pi$ is stable and $\overline{y}_{2m+1} = (2m+1)\pi$ is unstable.

The phase plane equivalent of Equation 3.3.13 is

$$v\frac{dv}{dy} = -\frac{g}{L}\sin y,$$

where $v = y'$ is the angular velocity of the pendulum. Integrating this yields

$$\frac{v^2}{2} = \frac{g}{L}\cos y + c. \tag{3.3.14}$$

If $v = v_0$ when $y = 0$, then

$$c = \frac{v_0^2}{2} - \frac{g}{L},$$

so Equation 3.3.14 becomes

$$\begin{aligned}\frac{v^2}{2} &= \frac{v_0^2}{2} - \frac{g}{L}(1-\cos y)\\ &= \frac{v_0^2}{2} - \frac{2g}{L}\sin^2\frac{y}{2},\end{aligned}$$

which is equivalent to

$$v^2 = v_0^2 - v_c^2\sin^2\frac{y}{2}, \tag{3.3.15}$$

where

$$v_c = 2\sqrt{\frac{g}{L}}.$$

The curves defined by Equation 3.3.15 are the trajectories of Equation 3.3.13. They are periodic with period 2π in y, which isn't surprising, since if $y = y(t)$ is a solution of Equation 3.3.13 then so is $y_n = y(t) + 2n\pi$ for any integer n. Figure 4.4.7 shows trajectories over the interval $[-\pi, \pi]$.

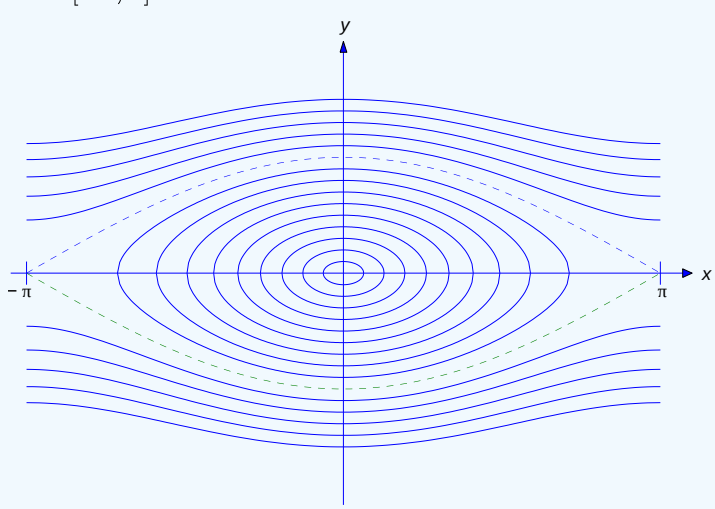


Figure 4.4.7 : Trajectories of the undamped pendulum

From Equation 3.3.15, you can see that if $|v_0| > v_c$ then v is nonzero for all t, which means that the object whirls in the same direction forever, as in Figure 4.4.8 . The trajectories associated with this whirling motion are above the upper dashed curve and below the lower dashed curve in Figure 4.4.7 . You can also see from Equation 3.3.15 that if $0 < |v_0| < v_c$, then $v = 0$ when $y = \pm y_{max}$, where

$$y_{\max} = 2\sin^{-1}(|v_0|/v_c).$$

In this case the pendulum oscillates periodically between $-y_{\max}$ and $y_{\max}$, as shown in Figure 4.4.9 . The trajectories associated with this kind of motion are the ovals between the dashed curves in Figure 4.4.7 .

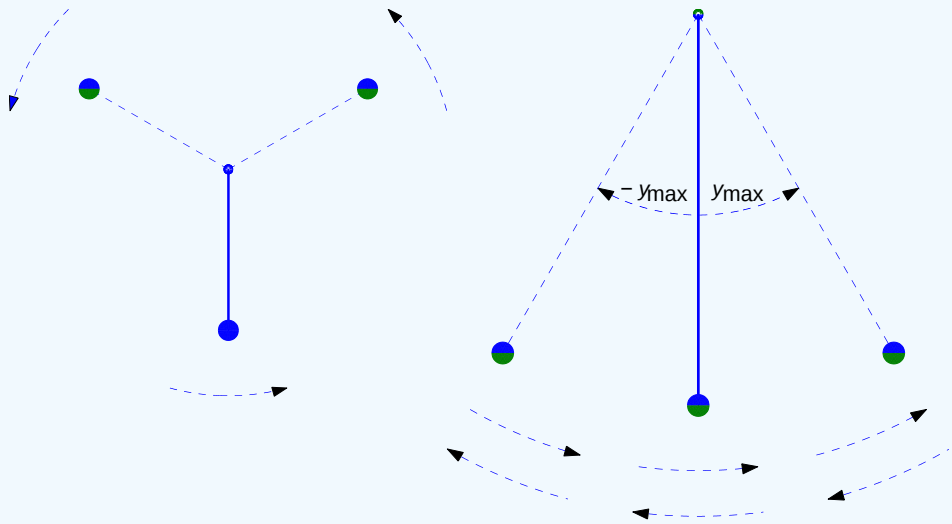


Figure 4.4.8 (left) and 4.4.9 (right): (left) The whirling undamped pendulum and (b) The oscillating undamped pendulum

It can be shown (see **Exercise 4.4.21** for a partial proof) that the period of the oscillation is

$$T = 8\int_0^{\pi/2} \frac{d\theta}{\sqrt{v_c^2 - v_0^2 \sin^2\theta}}. \tag{3.3.16}$$

Although this integral cannot be evaluated in terms of familiar elementary functions, you can see that it is finite if $|v_0| < v_c$.>

The dashed curves in Figure 4.4.7 contain four trajectories. The critical points $(\pi, 0)$ and $(-\pi, 0)$ are the trajectories of the unstable equilibrium solutions $\overline{y} = \pm\pi$. The upper dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0,\ v(t_0) = v_c$. If y is any solution with this trajectory then

$$\lim_{t\to\infty} y(t) = \pi \quad \text{and} \quad \lim_{t\to-\infty} y(t) = -\pi.$$

The lower dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0,\ v(t_0) = -v_c$. If y is any solution with this trajectory then

$$\lim_{t\to\infty} y(t) = -\pi \quad \text{and} \quad \lim_{t\to-\infty} y(t) = \pi.$$

Consistent with this, the integral Equation 3.3.16 diverges to ∞ if $v_0 = \pm v_c$. (**Exercise 4.4.21**).

Since the dashed curves separate trajectories of whirling solutions from trajectories of oscillating solutions, each of these curves is called a **separatrix**.

In general, if Equation 3.3.7 has both stable and unstable equilibria then the separatrices are the curves given by Equation 3.3.8 that pass through unstable critical points. Thus, if $(\overline{y}, 0)$ is an unstable critical point, then

$$\frac{v^2}{2} + P(y) = P(\overline{y}) \tag{3.3.17}$$

defines a separatrix passing through $(\overline{y}, 0)$.

Stability and Instability Conditions for y'' + p(y) = 0

It can be shown (**Exercise 4.4.23**) that an equilibrium $\overline{y}$ of an undamped equation

$$y'' + p(y) = 0 \tag{3.3.18}$$

is stable if there's an open interval (a, b) containing $\overline{y}$ such that

$$p(y) < 0 \text{ if } a < y < \overline{y} \text{ and } p(y) > 0 \text{ if } \overline{y} < y < b \tag{3.3.19}$$

If we regard $p(y)$ as a force acting on a unit mass, Equation 3.3.19 means that the force resists all sufficiently small displacements from $\bar{y}$.

We've already seen examples illustrating this principle. The equation Equation 3.3.9 for the undamped spring-mass system is of the form Equation 3.3.18 with $p(y)=ky/m$, which has only the stable equilibrium $\bar{y}=0$. In this case Equation 3.3.19 holds with $a=-\infty$ and $b=\infty$. The equation Equation 3.3.13 for the undamped pendulum is of the form Equation 3.3.18 with $p(y)=(g/L)\sin y$. We've seen that $\bar{y}=2m\pi$ is a stable equilibrium if m is an integer. In this case

$$p(y)=\sin y<0 \text{ if } (2m-1)\pi<y<2m\pi$$

and

$$p(y)>0 \text{ if } 2m\pi<y<(2m+1)\pi$$

It can also be shown (**Exercise 4.4.24**) that $\bar{y}$ is unstable if there's a $b>\bar{y}$ such that

$$p(y)<0 \text{ if } \bar{y}<y<b \tag{3.3.20}$$

or an $a<\bar{y}$ such that

$$p(y)>0 \text{ if } a<y<\bar{y} \tag{3.3.21}$$

If we regard $p(y)$ as a force acting on a unit mass, Equation 3.3.20 means that the force tends to increase all sufficiently small positive displacements from $\bar{y}$, while Equation 3.3.21 means that the force tends to increase the magnitude of all sufficiently small negative displacements from $\bar{y}$.

The undamped pendulum also illustrates this principle. We've seen that $\bar{y}=(2m+1)\pi$ is an unstable equilibrium if m is an integer. In this case

$$\sin y<0 \text{ if } (2m+1)\pi<y<(2m+2)\pi$$

so Equation 3.3.20 holds with $b=(2m+2)\pi$, and

$$\sin y>0 \text{ if } 2m\pi<y<(2m+1)\pi$$

so Equation 3.3.21 holds with $a=2m\pi$.

Example 4.4.3

The equation

$$y''+y(y-1)=0 \tag{3.3.22}$$

is of the form Equation 3.3.18 with $p(y)=y(y-1)$. Therefore $\bar{y}=0$ and $\bar{y}=1$ are the equilibria of Equation 3.3.22. Since

$$\begin{aligned} y(y-1) &>0 \qquad \text{if } y<0 \text{ or } y>1 \\ &<0 \qquad \text{if } 0<y<1 \end{aligned}$$

$\bar{y}=0$ is unstable and $\bar{y}=1$ is stable.

The phase plane equivalent of Equation 3.3.22 is the separable equation

$$v\frac{dv}{dy}+y(y-1)=0.$$

Integrating yields

$$\frac{v^2}{2}+\frac{y^3}{3}-\frac{y^2}{2}=C,$$

which we rewrite as

$$v^2+\frac{1}{3}y^2(2y-3)=c \tag{3.3.23}$$

after renaming the constant of integration. These are the trajectories of Equation 3.3.22. If y is any solution of Equation 3.3.22, the point $(y(t), v(t))$ moves along the trajectory of y in the direction of increasing y in the upper half plane ($v = y' > 0$), or in the direction of decreasing y in the lower half plane ($v = y' < 0$).

Figure 4.4.10 shows typical trajectories. The dashed curve through the critical point $(0, 0)$, obtained by setting $c = 0$ in Equation 3.3.23, separates the y-v plane into regions that contain different kinds of trajectories; again, we call this curve a **separatrix**. Trajectories in the region bounded by the closed loop (b) are closed curves, so solutions associated with them are periodic. Solutions associated with other trajectories are not periodic. If y is any such solution with trajectory not on the separatrix, then

$$\begin{aligned} \lim_{t\to\infty} y(t) &= -\infty, & \lim_{t\to-\infty} y(t) &= -\infty, \\ \lim_{t\to\infty} v(t) &= -\infty, & \lim_{t\to-\infty} v(t) &= \infty. \end{aligned}$$

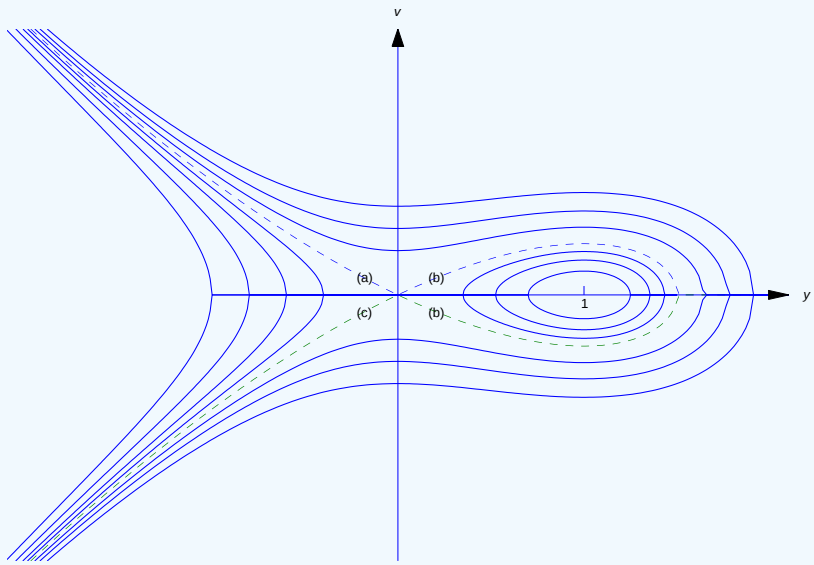


Figure 4.4.10 : Trajectories of $y'' + y(y-1) = 0$

The separatrix contains four trajectories of Equation 3.3.22. One is the point $(0, 0)$, the trajectory of the equilibrium $\bar{y} = 0$. Since distinct trajectories cannot intersect, the segments of the separatrix marked (a), b, and (c)– which don't include $(0, 0)$ – are distinct trajectories, none of which can be traversed in finite time. Solutions with these trajectories have the following asymptotic behavior:

$$\begin{aligned} \lim_{t\to\infty} y(t) &= 0, & \lim_{t\to-\infty} y(t) &= -\infty, & & \\ \lim_{t\to\infty} v(t) &= 0, & \lim_{t\to-\infty} v(t) &= \infty & & (\text{on (a)}) \\ \lim_{t\to\infty} y(t) &= 0, & \lim_{t\to-\infty} y(t) &= 0, & & \\ \lim_{t\to\infty} v(t) &= 0, & \lim_{t\to-\infty} v(t) &= 0 & & (\text{on (b)}) \\ \lim_{t\to\infty} y(t) &= -\infty, & \lim_{t\to-\infty} y(t) &= 0, & & \\ \lim_{t\to\infty} v(t) &= -\infty, & \lim_{t\to-\infty} v(t) &= 0 & & (\text{on (c)}). \end{aligned}$$

The Damped Case

The phase plane equivalent of the damped autonomous equation

$$y'' + q(y, y')y' + p(y) = 0 \tag{3.3.24}$$

is

$$v\frac{dv}{dy}+q(y,v)v+p(y)=0.$$

This equation isn't separable, so we cannot solve it for v in terms of y, as we did in the undamped case, and conservation of energy doesn't hold. (For example, energy expended in overcoming friction is lost.) However, we can study the qualitative behavior of its solutions by rewriting it as

$$\frac{dv}{dy}=-q(y,v)-\frac{p(y)}{v} \tag{3.3.25}$$

and considering the direction fields for this equation.

In the following examples we will also be showing computer generated trajectories of this equation, obtained by numerical methods. The exercises call for similar computations. The methods discussed in Chapter 3 are not suitable for this task, since $p(y)/v$ in Equation 3.3.25 is undefined on the y axis of the Poincaré phase plane. Therefore we are forced to apply numerical methods briefly discussed in Section 10.1 to the system

$$\begin{aligned} y' &= v \\ v' &= -q(y,v)v-p(y), \end{aligned}$$

which is equivalent to Equation 3.3.24 in the sense defined in Section 10.1. Fortunately, most differential equation software packages enable you to do this painlessly.

In the text we will confine ourselves to the case where q is constant, so Equation 3.3.24 and Equation 3.3.25 reduce to

$$y''+cy'+p(y)=0 \tag{3.3.26}$$

and

$$\frac{dv}{dy}=-c-\frac{p(y)}{v}.$$

(we will consider more general equations in the exercises.) The constant c is called the **damping constant**. In situations where Equation 3.3.26 is the equation of motion of an object, c is positive; however, there are situations where c may be negative.

The Damped Spring-Mass System

Earlier we considered the spring - mass system under the assumption that the only forces acting on the object were gravity and the spring's resistance to changes in its length. Now we will assume that some mechanism (for example, friction in the spring or atmospheric resistance) opposes the motion of the object with a force proportional to its velocity. In Section 6.1 it will be shown that in this case Newton's second law of motion implies that

$$my''+cy'+ky=0, \tag{3.3.27}$$

where $c>0$ is the **damping constant**. Again, this equation can be solved easily by a method that we will study in Section 5.2, but that method isn't available here. Instead, we will consider its phase plane equivalent, which can be written in the form Equation 3.3.25 as

$$\frac{dv}{dy}=-\frac{c}{m}-\frac{ky}{mv}. \tag{3.3.28}$$

(A minor note: the c in Equation 3.3.26 actually corresponds to c/m in this equation.) Figure 4.4.11 shows a typical direction field for an equation of this form. Recalling that motion along a trajectory must be in the direction of increasing y in the upper half plane ($v>0$) and in the direction of decreasing y in the lower half plane ($v<0$), you can infer that all trajectories approach the origin in clockwise fashion.

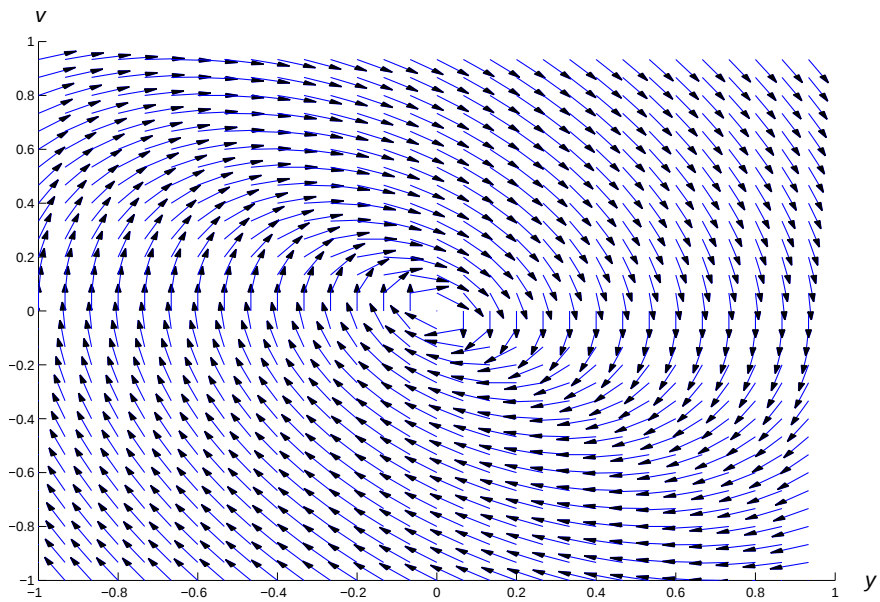


Figure 4.4.11 : A typical direction field for $my'' + cy' + ky = 0$ with $0 < c < c_1$

To confirm this, Figure 4.4.12 shows the same direction field with some trajectories filled in. All the trajectories shown there correspond to solutions of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0, \quad y'(0) = v_0,$$

where

$$mv_0^2 + ky_0^2 = \rho \quad (\text{a positive constant});$$

thus, if there were no damping ($c = 0$), all the solutions would have the same dashed elliptic trajectory, shown in Figure 4.4.14 .

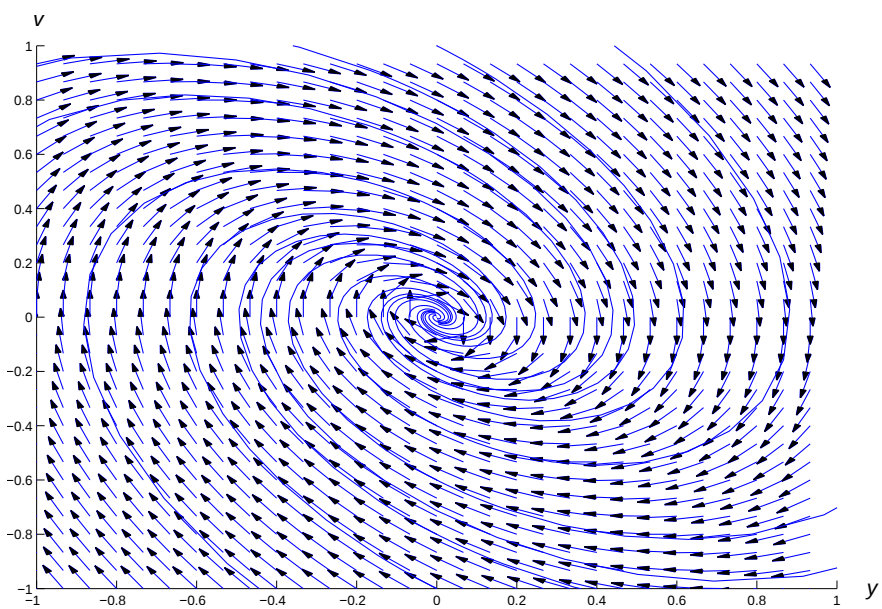


Figure 4.4.11 with some trajectories added

Solutions corresponding to the trajectories in Figure 4.4.12 cross the y-axis infinitely many times. The corresponding solutions are said to be **oscillatory** (Figure 4.4.13)

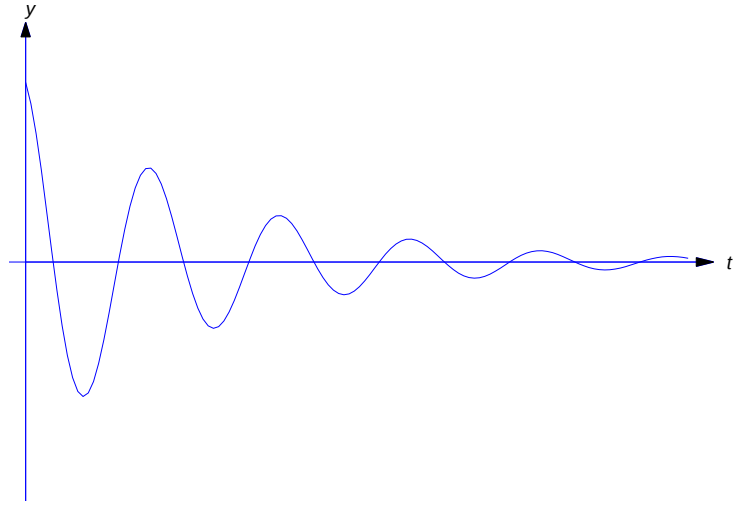


Figure 4.4.13 : An oscillatory solution of $my'' + cy' + ky = 0$

It is shown in Section 6.2 that there's a number c_1 such that if $0 \le c < c_1$ then all solutions of (4.4.27) are oscillatory, while if $c \ge c_1$, no solutions of (4.4.27) have this property. (In fact, no solution not identically zero can have more than two zeros in this case.) Figure 4.4.14 shows a direction field and some integral curves for (4.4.28) in this case.

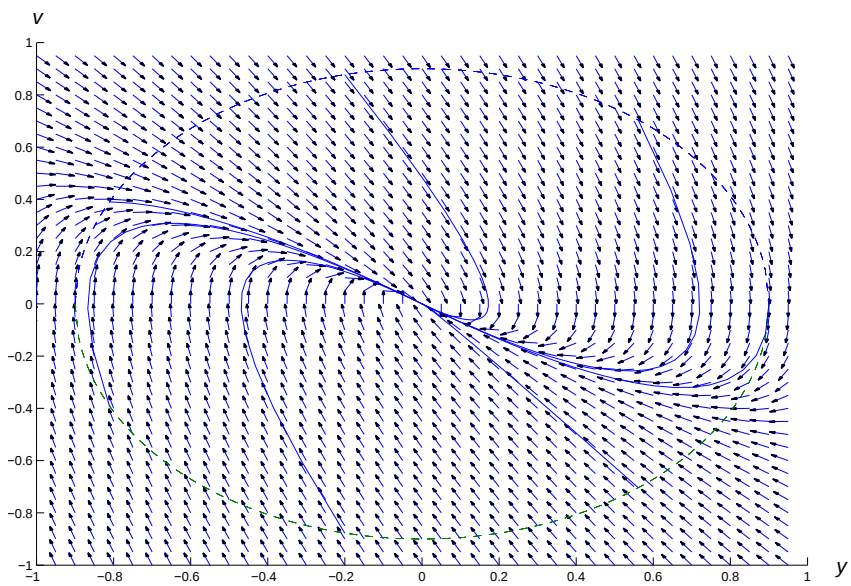


Figure 4.4.14 : A typical direction field for $my'' + cy' + ky = 0$ with $c > c_1$

Example 4.4.4

Now we return to the pendulum. If we assume that some mechanism (for example, friction in the axle or atmospheric resistance) opposes the motion of the pendulum with a force proportional to its angular velocity, Newton's second law of motion implies that

$$mLy'' = -cy' - mg\sin y, \tag{3.3.29}$$

where $c > 0$ is the damping constant. (Again, a minor note: the c in Equation 3.3.26 actually corresponds to c/mL in this equation.) To plot a direction field for Equation 3.3.29 we write its phase plane equivalent as

$$\frac{dv}{dy} = -\frac{c}{mL} - \frac{g}{Lv}\sin y.$$

Figure 4.4.15 shows trajectories of four solutions of Equation 3.3.29, all satisfying $y(0) = 0$. For each $m = 0,\ 1,\ 2,\ 3$, imparting the initial velocity $v(0) = v_m$ causes the pendulum to make m complete revolutions and then settle into decaying oscillation about the stable equilibrium $\bar{y} = 2m\pi$.

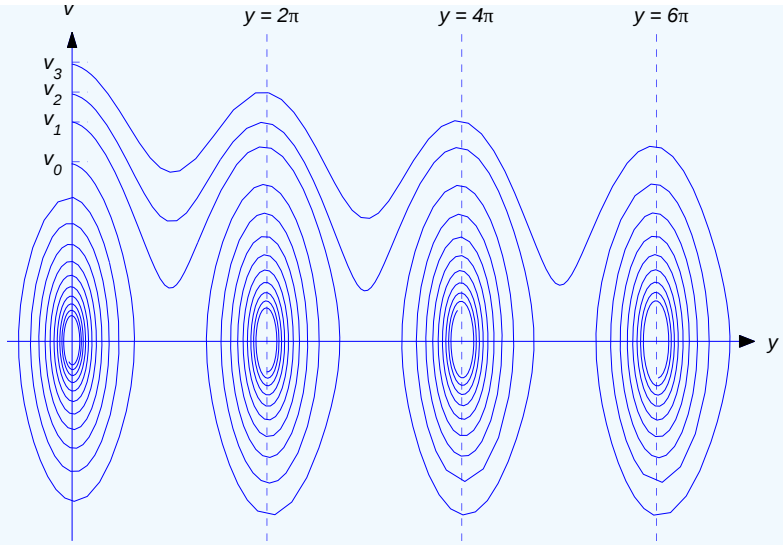


Figure 4.4.15 : Four trajectories of the damped pendulum.

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3.3E: Autonomous Second Order Equations (Exercises)

Q4.4.1

In **Exercises 4.4.1-4.4.4** find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and plot some trajectories.

1. $y''+y^3=0$
2. $y''+y^2=0$
3. $y''+y|y|=0$
4. $y''+ye^{-y}=0$

Q4.4.2

In **Exercises 4.4.5–4.4.8** find the equations of the trajectories of the given undamped equation. Identify the equilibrium solutions, determine whether they are stable or unstable, and find the equations of the separatrices (that is, the curves through the unstable equilibria). Plot the separatrices and some trajectories in each of the regions of Poincaré plane determined by them.

5. $y''-y^3+4y=0$
6. $y''+y^3-4y=0$
7. $y''+y(y^2-1)(y^2-4)=0$
8. $y''+y(y-2)(y-1)(y+2)=0$

Q4.4.3

In **Exercises 4.4.9–4.4.12** plot some trajectories of the given equation for various values (positive, negative, zero) of the parameter a. Find the equilibria of the equation and classify them as stable or unstable. Explain why the phase plane plots corresponding to positive and negative values of a differ so markedly. Can you think of a reason why zero deserves to be called the **critical value** of a?

9. $y''+y^2-a=0$
10. $y''+y^3-ay=0$
11. $y''-y^3+ay=0$
12. $y''+y-ay^3=0$

Q4.4.4

In **Exercises 4.4.13-4.4.18** plot trajectories of the given equation for $c=0$ and small nonzero (positive and negative) values of c to observe the effects of damping.

13. $y''+cy'+y^3=0$
14. $y''+cy'-y=0$
15. $y''+cy'+y^3=0$
16. $y''+cy'+y^2=0$
17. $y''+cy'+y|y|=0$
18. $y''+y(y-1)+cy=0$

Q4.4.5

19. The **van der Pol equation**

$$y''-\mu(1-y^2)y'+y=0, \tag{A}$$

where μ is a positive constant and y is electrical current (Section 6.3), arises in the study of an electrical circuit whose resistive properties depend upon the current. The damping term $-\mu(1-y^2)y'$ works to reduce $|y|$ if $|y|<1$ or to increase $|y|$ if $|y|>1$. It can be shown that van der Pol's equation has exactly one closed trajectory, which is called a **limit cycle**. Trajectories inside the limit cycle spiral outward to it, while trajectories outside the limit cycle spiral inward to it (Figure [figure:4.4.16}). Use your favorite differential equations software to verify this for $\mu=.5, 1.1.5, 2$. Use a grid with $-4<y<4$ and $-4<v<4$.

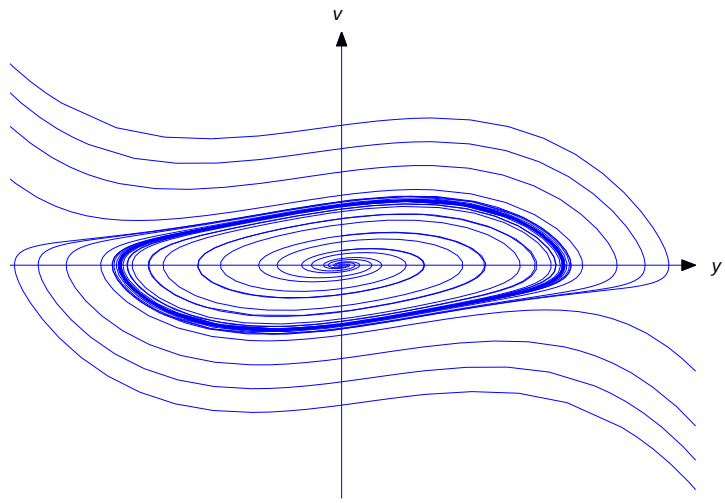

Figure 4.4E.16 : Trajectories of van der Pol's euqation

20. **Rayleigh's equation**,

$$y''-\mu(1-(y')^2/3)y'+y=0$$

also has a limit cycle. Follow the directions of **Exercise 4.4.19** for this equation.

21. In connection with Equation 4.4.16, suppose $y(0)=0$ and $y'(0)=v_0$, where $0<v_0<v_c$.

a. Let T_1 be the time required for y to increase from zero to $y_{\max}=2\sin^{-1}(v_0/v_c)$. Show that

$$\frac{dy}{dt}=\sqrt{v_0^2-v_c^2\sin^2 y/2}, \quad 0\le t<T_1. \tag{A}$$

b. Separate variables in (A) and show that

$$T_1=\int_0^{y_{\max}}\frac{du}{\sqrt{v_0^2-v_c^2\sin^2 u/2}} \tag{B}$$

c. Substitute $\sin u/2=(v_0/v_c)\sin\theta$ in (B) to obtain

$$T_1=2\int_0^{\pi/2}\frac{d\theta}{\sqrt{v_c^2-v_0^2\sin^2\theta}}. \tag{C}$$

d. Conclude from symmetry that the time required for $(y(t),v(t))$ to traverse the trajectory

$$v^2=v_0^2-v_c^2\sin^2 y/2$$

is $T=4T_1$, and that consequently $y(t+T)=y(t)$ and $v(t+T)=v(t)$; that is, the oscillation is periodic with period T.

e. Show that if $v_0=v_c$, the integral in (C) is improper and diverges to ∞. Conclude from this that $y(t)<\pi$ for all t and $\lim_{t\to\infty}y(t)=\pi$.

22. Give a direct definition of an unstable equilibrium of $y''+p(y)=0$.

23. Let p be continuous for all y and $p(0)=0$. Suppose there's a positive number ρ such that $p(y)>0$ if $0<y\le\rho$ and $p(y)<0$ if $-\rho\le y<0$. For $0<r\le\rho$ let

$$\alpha(r)=\min\left\{\int_0^r p(x)\,dx,\ \int_{-r}^0|p(x)|\,dx\right\}\backslash\text{quad and }\backslash\text{quad}\beta(r)=\max\left\{\int_0^r p(x)\,dx,\ \int_{-r}^0|p(x)|\,dx\right\}.$$

Let y be the solution of the initial value problem

$$y''+p(y)=0, \quad y(0)=v_0, \quad y'(0)=v_0,$$

and define $c(y_0,v_0)=v_0^2+2\int_0^{y_0}p(x)\,dx$.

a. Show that

$$0<c(y_0,v_0)<v_0^2+2\beta(|y_0|)\backslash \text{quad if } \backslash \text{quad} 0<|y_0|\leq \rho.$$

b. Show that

$$v^2+2\int_0^y p(x)\,dx=c(y_0,v_0), \quad t>0.$$

c. Conclude from (b) that if $c(y_0,v_0)<2\alpha(r)$ then $|y|<r,\ t>0$.
d. Given $\epsilon>0$, let $\delta>0$ be chosen so that

$$\delta^2+2\beta(\delta)<\max\left\{\epsilon^2/2, 2\alpha(\epsilon/\sqrt{2})\right\}.$$

Show that if $\sqrt{y_0^2+v_0^2}<\delta$ then $\sqrt{y^2+v^2}<\epsilon$ for $t>0$, which implies that $\overline{y}=0$ is a stable equilibrium of $y''+p(y)=0$.

e. Now let p be continuous for all y and $p(\overline{y})=0$, where $\overline{y}$ is not necessarily zero. Suppose there's a positive number ρ such that $p(y)>0$ if $\overline{y}<y\leq\overline{y}+\rho$ and $p(y)<0$ if $\overline{y}-\rho\leq y<\overline{y}$. Show that $\overline{y}$ is a stable equilibrium of $y''+p(y)=0$.

24. Let p be continuous for all y.

a. Suppose $p(0)=0$ and there's a positive number ρ such that $p(y)<0$ if $0<y\leq\rho$. Let ϵ be any number such that $0<\epsilon<\rho$. Show that if y is the solution of the initial value problem

$$y''+p(y)=0, \quad y(0)=y_0, \quad y'(0)=0$$

with $0<y_0<\epsilon$, then $y(t)\geq\epsilon$ for some $t>0$. Conclude that $\overline{y}=0$ is an unstable equilibrium of $y''+p(y)=0$.

b. Now let $p(\overline{y})=0$, where $\overline{y}$ isn't necessarily zero. Suppose there's a positive number ρ such that $p(y)<0$ if $\overline{y}<y\leq\overline{y}+\rho$. Show that $\overline{y}$ is an unstable equilibrium of $y''+p(y)=0$.
c. Modify your proofs of (a) and (b) to show that if there's a positive number ρ such that $p(y)>0$ if $\overline{y}-\rho\leq y<\overline{y}$, then $\overline{y}$ is an unstable equilibrium of $y''+p(y)=0$.

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CHAPTER OVERVIEW

Chapter 4: Linear Second Order Equations

In this Chapter, we study a particularly important class of second order equations. Because of their many applications in science and engineering, second order differential equation have historically been the most thoroughly studied class of differential equations. Research on the theory of second order differential equations continues to the present day. This chapter is devoted to second order equations that can be written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x).$$

Such equations are said to be *linear*. As in the case of first order linear equations, this differential equation is said to be *homogeneous* if $F \equiv 0$, or *nonhomogeneous* if $F \not\equiv 0$.

4.1: Homogeneous Linear Equations

4.1E: Homogeneous Linear Equations (Exercises)

4.2: Introduction to Linear Higher Order Equations

4.2E: Introduction to Linear Higher Order Equations (Exercises)

4.3: Constant Coefficient Homogeneous Equations

4.3E: Constant Coefficient Homogeneous Equations (Exercises)

4.4: Higher Order Constant Coefficient Homogeneous Equations

4.4E: Higher Order Constant Coefficient Homogeneous Equations (Exercises)

4.5: Nonhomgeneous Linear Equations

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4.6: The Method of Undetermined Coefficients I

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4.8: Undetermined Coefficients for Higher Order Equations

4.8E: Undetermined Coefficients for Higher Order Equations (Exercises)

4.9: Reduction of Order

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4.11: Variation of Parameters for Higher Order Equations

4.11E: Variation of Parameters for Higher Order Equations (Exercises)

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4.1: Homogeneous Linear Equations

A second order differential equation is said to be **linear** if it can be written as

$$y''+p(x)y'+q(x)y=f(x). \tag{4.1.1}$$

We call the function f on the right a **forcing function**, since in physical applications it is often related to a force acting on some system modeled by the differential equation. We say that Equation 4.1.1 is **homogeneous** if $f\equiv 0$ or **nonhomogeneous** if $f\not\equiv 0$. Since these definitions are like the corresponding definitions in Section 2.1 for the linear first order equation

$$y'+p(x)y=f(x), \tag{4.1.2}$$

it is natural to expect similarities between methods of solving Equation 4.1.1 and Equation 4.1.2. However, solving Equation 4.1.1 is more difficult than solving Equation 4.1.2. For example, while Theorem 5.1.1 gives a formula for the general solution of Equation 4.1.2 in the case where $f\equiv 0$ and Theorem 5.1.2 gives a formula for the case where $f\not\equiv 0$, there are no formulas for the general solution of Equation 4.1.1 in either case. Therefore we must be content to solve linear second order equations of special forms.

In Section 2.1 we considered the homogeneous equation $y'+p(x)y=0$ first, and then used a nontrivial solution of this equation to find the general solution of the nonhomogeneous equation $y'+p(x)y=f(x)$. Although the progression from the homogeneous to the nonhomogeneous case isn't that simple for the linear second order equation, it is still necessary to solve the homogeneous equation

$$y''+p(x)y'+q(x)y=0 \tag{4.1.3}$$

in order to solve the nonhomogeneous equation Equation 4.1.1. This section is devoted to Equation 4.1.3.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for Equation 4.1.3. We omit the proof.

Theorem 5.1.1

Suppose p and q are continuous on an open interval (a,b), let x_0 be any point in (a,b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem

$$y''+p(x)y'+q(x)y=0,\ y(x_0)=k_0,\ y'(x_0)=k_1$$

has a unique solution on (a,b).

Since $y\equiv 0$ is obviously a solution of Equation 4.1.3 we call it the **trivial** solution. Any other solution is **nontrivial**. Under the assumptions of Theorem 5.1.1 , the only solution of the initial value problem

$$y''+p(x)y'+q(x)y=0,\ y(x_0)=0,\ y'(x_0)=0$$

on (a,b) is the trivial solution (**Exercise 5.1.24**).

The next three examples illustrate concepts that we'll develop later in this section. You shouldn't be concerned with how to **find** the given solutions of the equations in these examples. This will be explained in later sections.

Example 5.1.1

The coefficients of y' and y in

$$y''-y=0 \tag{4.1.4}$$

are the constant functions $p\equiv 0$ and $q\equiv -1$, which are continuous on $(-\infty,\infty)$. Therefore Theorem 5.1.1 implies that every initial value problem for Equation 4.1.4 has a unique solution on $(-\infty,\infty)$.

a. Verify that $y_1=e^x$ and $y_2=e^{-x}$ are solutions of Equation 4.1.4 on $(-\infty,\infty)$.
b. Verify that if c_1 and c_2 are arbitrary constants, $y=c_1e^x+c_2e^{-x}$ is a solution of Equation 4.1.4 on $(-\infty,\infty)$.
c. Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{4.1.5}$$

Solution:

a. If $y_1 = e^x$ then $y_1' = e^x$ and $y_1'' = e^x = y_1$, so $y_1'' - y_1 = 0$. If $y_2 = e^{-x}$, then $y_2' = -e^{-x}$ and $y_2'' = e^{-x} = y_2$, so $y_2'' - y_2 = 0$.

b. If

$$y = c_1 e^x + c_2 e^{-x} \tag{4.1.6}$$

then

$$y' = c_1 e^x - c_2 e^{-x} \tag{4.1.7}$$

and

$$y'' = c_1 e^x + c_2 e^{-x},$$

so

$$\begin{aligned} y'' - y &= (c_1 e^x + c_2 e^{-x}) - (c_1 e^x + c_2 e^{-x}) \\ &= c_1(e^x - e^x) + c_2(e^{-x} - e^{-x}) = 0 \end{aligned}$$

for all x. Therefore $y = c_1 e^x + c_2 e^{-x}$ is a solution of Equation 4.1.4 on $(-\infty, \infty)$.

c.

We can solve Equation 4.1.5 by choosing c_1 and c_2 in Equation 4.1.6 so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in Equation 4.1.6 and Equation 4.1.7 shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3. \end{aligned}$$

Solving these equations yields $c_1 = 2$ and $c_2 = -1$. Therefore $y = 2e^x - e^{-x}$ is the unique solution of Equation 4.1.5 on $(-\infty, \infty)$.

Example 5.1.2

Let ω be a positive constant. The coefficients of y' and y in

$$y'' + \omega^2 y = 0 \tag{4.1.8}$$

are the constant functions $p \equiv 0$ and $q \equiv \omega^2$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 5.1.1 implies that every initial value problem for Equation 4.1.8 has a unique solution on $(-\infty, \infty)$.

a. Verify that $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$ are solutions of Equation 4.1.8 on $(-\infty, \infty)$.
b. Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 \cos \omega x + c_2 \sin \omega x$ is a solution of Equation 4.1.8 on $(-\infty, \infty)$.
c. Solve the initial value problem

$$y'' + \omega^2 y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{4.1.9}$$

Solution:

a. If $y_1 = \cos \omega x$ then $y_1' = -\omega \sin \omega x$ and $y_1'' = -\omega^2 \cos \omega x = -\omega^2 y_1$, so $y_1'' + \omega^2 y_1 = 0$. If $y_2 = \sin \omega x$ then, $y_2' = \omega \cos \omega x$ and $y_2'' = -\omega^2 \sin \omega x = -\omega^2 y_2$, so $y_2'' + \omega^2 y_2 = 0$.

b. If

$$y = c_1 \cos \omega x + c_2 \sin \omega x \tag{4.1.10}$$

then

$$y' = \omega(-c_1 \sin \omega x + c_2 \cos \omega x) \tag{4.1.11}$$

and

$$y'' = -\omega^2(c_1 \cos \omega x + c_2 \sin \omega x),$$

so

$$\begin{aligned} y'' + \omega^2 y &= -\omega^2(c_1 \cos \omega x + c_2 \sin \omega x) + \omega^2(c_1 \cos \omega x + c_2 \sin \omega x) \\ &= c_1\omega^2(-\cos \omega x + \cos \omega x) + c_2\omega^2(-\sin \omega x + \sin \omega x) = 0 \end{aligned}$$

for all x. Therefore $y = c_1 \cos \omega x + c_2 \sin \omega x$ is a solution of Equation 4.1.8 on $(-\infty, \infty)$.

c. To solve Equation 4.1.9, we must choosing c_1 and c_2 in Equation 4.1.10 so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in Equation 4.1.10 and Equation 4.1.11 shows that $c_1 = 1$ and $c_2 = 3/\omega$. Therefore

$$y = \cos \omega x + \frac{3}{\omega} \sin \omega x$$

is the unique solution of Equation 4.1.9 on $(-\infty, \infty)$.

Theorem 5.1.1 implies that if k_0 and k_1 are arbitrary real numbers then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{4.1.12}$$

has a unique solution on an interval (a, b) that contains x_0, provided that P_0, P_1, and P_2 are continuous and P_0 has no zeros on (a, b). To see this, we rewrite the differential equation in Equation 4.1.12 as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem 5.1.1 with $p = P_1/P_0$ and $q = P_2/P_0$.

Example 5.1.3

The equation

$$x^2y'' + xy' - 4y = 0 \tag{4.1.13}$$

has the form of the differential equation in Equation 4.1.12, with $P_0(x) = x^2$, $P_1(x) = x$, and $P_2(x) = -4$, which are are all continuous on $(-\infty, \infty)$. However, since $P(0) = 0$ we must consider solutions of Equation 4.1.13 on $(-\infty, 0)$ and $(0, \infty)$. Since P_0 has no zeros on these intervals, Theorem 5.1.1 implies that the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(0, \infty)$ if $x_0 > 0$, or on $(-\infty, 0)$ if $x_0 < 0$.

a. Verify that $y_1 = x^2$ is a solution of Equation 4.1.13 on $(-\infty, \infty)$ and $y_2 = 1/x^2$ is a solution of Equation 4.1.13 on $(-\infty, 0)$ and $(0, \infty)$.
b. Verify that if c_1 and c_2 are any constants then $y = c_1x^2 + c_2/x^2$ is a solution of Equation 4.1.13 on $(-\infty, 0)$ and $(0, \infty)$.
c. Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 0. \tag{4.1.14}$$

d. Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 0. \tag{4.1.15}$$

Solution:

a. If $y_1 = x^2$ then $y_1' = 2x$ and $y_1'' = 2$, so

$$x^2y_1'' + xy_1' - 4y_1 = x^2(2) + x(2x) - 4x^2 = 0$$

for x in $(-\infty, \infty)$. If $y_2 = 1/x^2$, then $y_2' = -2/x^3$ and $y_2'' = 6/x^4$, so

$$x^2y_2'' + xy_2' - 4y_2 = x^2\left(\frac{6}{x^4}\right) - x\left(\frac{2}{x^3}\right) - \frac{4}{x^2} = 0$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

b. If

$$y = c_1 x^2 + \frac{c_2}{x^2} \tag{4.1.16}$$

then

$$y' = 2c_1 x - \frac{2c_2}{x^3} \tag{4.1.17}$$

and

$$y'' = 2c_1 + \frac{6c_2}{x^4},$$

so

$$\begin{aligned} x^2 y'' + xy' - 4y &= x^2\left(2c_1 + \frac{6c_2}{x^4}\right) + x\left(2c_1 x - \frac{2c_2}{x^3}\right) - 4\left(c_1 x^2 + \frac{c_2}{x^2}\right) \\ &= c_1(2x^2 + 2x^2 - 4x^2) + c_2\left(\frac{6}{x^2} - \frac{2}{x^2} - \frac{4}{x^2}\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0 \end{aligned}$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

c. To solve Equation 4.1.14, we choose c_1 and c_2 in Equation 4.1.16 so that $y(1) = 2$ and $y'(1) = 0$. Setting $x = 1$ in Equation 4.1.16 and Equation 4.1.17 shows that this is equivalent to

$$\begin{aligned} c_1 + \ \ c_2 &= 2 \\ 2c_1 - 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1 = 1$ and $c_2 = 1$. Therefore $y = x^2 + 1/x^2$ is the unique solution of Equation 4.1.14 on $(0, \infty)$.

d. We can solve Equation 4.1.15 by choosing c_1 and c_2 in Equation 4.1.16 so that $y(-1) = 2$ and $y'(-1) = 0$. Setting $x = -1$ in Equation 4.1.16 and Equation 4.1.17 shows that this is equivalent to

$$\begin{aligned} c_1 + \ \ c_2 &= 2 \\ -2c_1 + 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1 = 1$ and $c_2 = 1$. Therefore $y = x^2 + 1/x^2$ is the unique solution of Equation 4.1.15 on $(-\infty, 0)$.

Although the **formulas** for the solutions of Equation 4.1.14 and Equation 4.1.15 are both $y = x^2 + 1/x^2$, you should not conclude that these two initial value problems have the same solution. Remember that a solution of an initial value problem is defined **on an interval that contains the initial point**; therefore, the solution of Equation 4.1.14 is $y = x^2 + 1/x^2$ **on the interval** $(0, \infty)$, which contains the initial point $x_0 = 1$, while the solution of Equation 4.1.15 is $y = x^2 + 1/x^2$ **on the interval** $(-\infty, 0)$, which contains the initial point $x_0 = -1$.

The General Solution of a Homogeneous Linear Second Order Equation

If y_1 and y_2 are defined on an interval (a, b) and c_1 and c_2 are constants, then

$$y = c_1 y_1 + c_2 y_2$$

is a **linear combination of y_1** and y_2. For example, $y = 2\cos x + 7\sin x$ is a linear combination of $y_1 = \cos x$ and $y_2 = \sin x$, with $c_1 = 2$ and $c_2 = 7$.

The next theorem states a fact that we've already verified in Examples 5.1.1 , 5.1.2 , 5.1.3 .

Theorem 5.1.2

If y_1 and y_2 are solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{4.1.18}$$

on (a, b), then any linear combination

$$y = c_1 y_1 + c_2 y_2 \tag{4.1.19}$$

of y_1 and y_2 is also a solution of (4.1.18) on (a, b).

Proof

If

$$y = c_1 y_1 + c_2 y_2$$

then

$$y' = c_1 y_1' + c_2 y_2' \quad \text{and} \quad y'' = c_1 y_1'' + c_2 y_2''.$$

Therefore

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (c_1 y_1'' + c_2 y_2'') + p(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1 \left(y_1'' + p(x)y_1' + q(x)y_1\right) + c_2 \left(y_2'' + p(x)y_2' + q(x)y_2\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0, \end{aligned}$$

since y_1 and y_2 are solutions of Equation 4.1.18

We say that $\{y_1, y_2\}$ is a **fundamental set of solutions of** (4.1.18) on (a, b) if every solution of Equation 4.1.18 on (a, b) can be written as a linear combination of y_1 and y_2 as in Equation 4.1.19. In this case we say that Equation 4.1.19 is **general solution of** (4.1.18) on (a, b).

Linear Independence

We need a way to determine whether a given set $\{y_1, y_2\}$ of solutions of Equation 4.1.18 is a fundamental set. The next definition will enable us to state necessary and sufficient conditions for this.

We say that two functions y_1 and y_2 defined on an interval (a, b) are **linearly independent on** (a, b) if neither is a constant multiple of the other on (a, b). (In particular, this means that neither can be the trivial solution of Equation 4.1.18, since, for example, if $y_1 \equiv 0$ we could write $y_1 = 0y_2$.) We'll also say that the set $\{y_1, y_2\}$ **is linearly independent on** (a, b).

Theorem 5.1.3

Suppose p and q are continuous on (a, b). Then a set $\{y_1, y_2\}$ of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{4.1.20}$$

on (a, b) is a fundamental set if and only if $\{y_1, y_2\}$ is linearly independent on (a, b).

Proof

We'll present the proof of Theorem 5.1.3 in steps worth regarding as theorems in their own right. However, let's first interpret Theorem 5.1.3 in terms of Examples 5.1.1 , 5.1.2 , 5.1.3 .

Example 5.1.4

Since $e^x / e^{-x} = e^{2x}$ is nonconstant, Theorem 5.1.3 implies that $y = c_1 e^x + c_2 e^{-x}$ is the general solution of $y'' - y = 0$ on $(-\infty, \infty)$.

Since $\cos\omega x/\sin\omega x=\cot\omega x$ is nonconstant, Theorem 5.1.3 implies that $y=c_1\cos\omega x+c_2\sin\omega x$ is the general solution of $y''+\omega^2y=0$ on $(-\infty,\infty)$.

Since $x^2/x^{-2}=x^4$ is nonconstant, Theorem 5.1.3 implies that $y=c_1x^2+c_2/x^2$ is the general solution of $x^2y''+xy'-4y=0$ on $(-\infty,0)$ and $(0,\infty)$.

The Wronskian and Abel's Formula

To motivate a result that we need in order to prove Theorem 5.1.3 , let's see what is required to prove that $\{y_1,y_2\}$ is a fundamental set of solutions of Equation 4.1.20 on (a,b). Let x_0 be an arbitrary point in (a,b), and suppose y is an arbitrary solution of Equation 4.1.20 on (a,b). Then y is the unique solution of the initial value problem

$$y''+p(x)y'+q(x)y=0,\quad y(x_0)=k_0,\quad y'(x_0)=k_1; \tag{4.1.21}$$

that is, k_0 and k_1 are the numbers obtained by evaluating y and y' at x_0. Moreover, k_0 and k_1 can be any real numbers, since Theorem 5.1.1 implies that Equation 4.1.21 has a solution no matter how k_0 and k_1 are chosen. Therefore $\{y_1,y_2\}$ is a fundamental set of solutions of Equation 4.1.20 on (a,b) if and only if it is possible to write the solution of an arbitrary initial value problem Equation 4.1.21 as $y=c_1y_1+c_2y_2$. This is equivalent to requiring that the system

$$\begin{aligned} c_1y_1(x_0)+c_2y_2(x_0) &= k_0\\ c_1y_1'(x_0)+c_2y_2'(x_0) &= k_1 \end{aligned} \tag{4.1.22}$$

has a solution (c_1,c_2) for every choice of (k_0,k_1). Let's try to solve Equation 4.1.22.

Multiplying the first equation in Equation 4.1.22 by $y_2'(x_0)$ and the second by $y_2(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_2'(x_0)+c_2y_2(x_0)y_2'(x_0) &= y_2'(x_0)k_0\\ c_1y_1'(x_0)y_2(x_0)+c_2y_2'(x_0)y_2(x_0) &= y_2(x_0)k_1, \end{aligned}$$

and subtracting the second equation here from the first yields

$$\left(y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)\right)c_1=y_2'(x_0)k_0-y_2(x_0)k_1. \tag{4.1.23}$$

Multiplying the first equation in Equation 4.1.22 by $y_1'(x_0)$ and the second by $y_1(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_1'(x_0)+c_2y_2(x_0)y_1'(x_0) &= y_1'(x_0)k_0\\ c_1y_1'(x_0)y_1(x_0)+c_2y_2'(x_0)y_1(x_0) &= y_1(x_0)k_1, \end{aligned}$$

and subtracting the first equation here from the second yields

$$\left(y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)\right)c_2=y_1(x_0)k_1-y_1'(x_0)k_0. \tag{4.1.24}$$

If

$$y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)=0,$$

it is impossible to satisfy Equation 4.1.23 and Equation 4.1.24 (and therefore Equation 4.1.22) unless k_0 and k_1 happen to satisfy

$$\begin{aligned} y_1(x_0)k_1-y_1'(x_0)k_0 &= 0\\ y_2'(x_0)k_0-y_2(x_0)k_1 &= 0. \end{aligned}$$

On the other hand, if

$$y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)\neq 0 \tag{4.1.25}$$

we can divide Equation 4.1.23 and Equation 4.1.24 through by the quantity on the left to obtain

$$\begin{aligned} c_1 &= \frac{y_2'(x_0)k_0-y_2(x_0)k_1}{y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)}\\ c_2 &= \frac{y_1(x_0)k_1-y_1'(x_0)k_0}{y_1(x_0)y_2'(x_0)-y_1'(x_0)y_2(x_0)}, \end{aligned} \tag{4.1.26}$$

no matter how k_0 and k_1 are chosen. This motivates us to consider conditions on y_1 and y_2 that imply Equation 4.1.25.

Theorem 5.1.4

Suppose p and q are continuous on (a,b), let y_1 and y_2 be solutions of

$$y''+p(x)y'+q(x)y=0 \tag{4.1.27}$$

on (a,b), and define

$$W=y_1y_2'-y_1'y_2. \tag{4.1.28}$$

Let x_0 be any point in (a,b). Then

$$W(x)=W(x_0)e^{-\int_{x_0}^{x}p(t)\,dt}, \quad a<x<b \tag{4.1.29}$$

Therefore either W has no zeros in (a,b) or $W\equiv 0$ on (a,b).

Proof

Differentiating Equation 4.1.28yields

$$W'=y_1'y_2'+y_1y_2''-y_1'y_2'-y_1''y_2=y_1y_2''-y_1''y_2. \tag{4.1.30}$$

Since y_1 and y_2 both satisfy Equation 4.1.27,

$$y_1''=-py_1'-qy_1 \quad \text{and} \quad y_2''=-py_2'-qy_2.$$

Substituting these into Equation 4.1.30yields

$$\begin{aligned} W' &= -y_1\left(py_2'+qy_2\right)+y_2\left(py_1'+qy_1\right) \\ &= -p(y_1y_2'-y_2y_1')-q(y_1y_2-y_2y_1) \\ &= -p(y_1y_2'-y_2y_1')=-pW. \end{aligned}$$

Therefore $W'+p(x)W=0$; that is, W is the solution of the initial value problem

$$y'+p(x)y=0, \quad y(x_0)=W(x_0).$$

We leave it to you to verify by separation of variables that this implies Equation 4.1.29. If $W(x_0)\neq 0$, Equation 4.1.29 implies that W has no zeros in (a,b), since an exponential is never zero. On the other hand, if $W(x_0)=0$, Equation 4.1.29implies that $W(x)=0$ for all x in (a,b).

The function W defined in Equation 4.1.28is the **Wronskian of $\{y_1,y_2\}$**. Formula Equation 4.1.29is **Abel's formula**.

The Wronskian of $\{y_1,y_2\}$ is usually written as the determinant

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

The expressions in Equation 4.1.26for c_1 and c_2 can be written in terms of determinants as

$$c_1=\frac{1}{W(x_0)}\begin{vmatrix} k_0 & y_2(x_0) \\ k_1 & y_2'(x_0) \end{vmatrix} \quad \text{and} \quad c_2=\frac{1}{W(x_0)}\begin{vmatrix} y_1(x_0) & k_0 \\ y_1'(x_0) & k_1 \end{vmatrix}.$$

If you've taken linear algebra you may recognize this as **Cramer's rule**.

Example 5.1.5

Verify Abel's formula for the following differential equations and the corresponding solutions, from Examples 5.1.1 , 5.1.2 , 5.1.3 .

a. $y''-y=0; \quad y_1=e^x,\ y_2=e^{-x}$
b. $y''+\omega^2y=0; \quad y_1=\cos\omega x,\ y_2=\sin\omega x$
c. $x^2y''+xy'-4y=0; \quad y_1=x^2,\ y_2=1/x^2$

Solution:

a. Since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$W(x)=\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}=e^x(-e^{-x})-e^xe^{-x}=-2$$

for all x.

b. Again, since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$\begin{aligned} W(x) &= \begin{vmatrix} \cos\omega x & \sin\omega x \\ -\omega\sin\omega x & \omega\cos\omega x \end{vmatrix} \\ &= \cos\omega x(\omega\cos\omega x)-(-\omega\sin\omega x)\sin\omega x \\ &= \omega(\cos^2\omega x+\sin^2\omega x)=\omega \end{aligned}$$

for all x.

c. Computing the Wronskian of $y_1=x^2$ and $y_2=1/x^2$ directly yields

$$W=\begin{vmatrix} x^2 & 1/x^2 \\ 2x & -2/x^3 \end{vmatrix}=x^2\left(-\frac{2}{x^3}\right)-2x\left(\frac{1}{x^2}\right)=-\frac{4}{x}. \tag{4.1.31}$$

To verify Abel's formula we rewrite the differential equation as

$$y''+\frac{1}{x}y'-\frac{4}{x^2}y=0$$

to see that $p(x)=1/x$. If x_0 and x are either both in $(-\infty,0)$ or both in $(0,\infty)$ then

$$\int_{x_0}^{x}p(t)\,dt=\int_{x_0}^{x}\frac{dt}{t}=\ln\left(\frac{x}{x_0}\right),$$

so Abel's formula becomes

$$\begin{aligned} W(x) &= W(x_0)e^{-\ln(x/x_0)}=W(x_0)\frac{x_0}{x} \\ &= -\left(\frac{4}{x_0}\right)\left(\frac{x_0}{x}\right) \quad \text{from}(4.1.31) \\ &= -\frac{4}{x}, \end{aligned}$$

which is consistent with Equation 4.1.31.

The next theorem will enable us to complete the proof of Theorem 5.1.3 .

Theorem 5.1.5

Suppose p and q are continuous on an open interval (a,b), let y_1 and y_2 be solutions of

$$y''+p(x)y'+q(x)y=0 \tag{4.1.32}$$

on (a,b), and let $W=y_1y_2'-y_1'y_2$. Then y_1 and y_2 are linearly independent on (a,b) if and only if W has no zeros on (a,b).

Proof

We first show that if $W(x_0)=0$ for some x_0 in (a,b), then y_1 and y_2 are linearly dependent on (a,b). Let I be a subinterval of (a,b) on which y_1 has no zeros. (If there's no such subinterval, $y_1 \equiv 0$ on (a,b), so y_1 and y_2 are linearly independent, and we are finished with this part of the proof.) Then y_2/y_1 is defined on I, and

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{W}{y_1^2}. \tag{4.1.33}$$

However, if $W(x_0) = 0$, Theorem 5.1.4 implies that $W \equiv 0$ on (a, b). Therefore Equation 4.1.33 implies that $(y_2/y_1)' \equiv 0$, so $y_2/y_1 = c$ (constant) on I. This shows that $y_2(x) = cy_1(x)$ for all x in I. However, we want to show that $y_2 = cy_1(x)$ for all x in (a, b). Let $Y = y_2 - cy_1$. Then Y is a solution of Equation 4.1.32on (a, b) such that $Y \equiv 0$ on I, and therefore $Y' \equiv 0$ on I. Consequently, if x_0 is chosen arbitrarily in I then Y is a solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0,$$

which implies that $Y \equiv 0$ on (a, b), by the paragraph following Theorem 5.1.1 . (See also **Exercise 5.1.24**). Hence, $y_2 - cy_1 \equiv 0$ on (a, b), which implies that y_1 and y_2 are not linearly independent on (a, b).

Now suppose W has no zeros on (a, b). Then y_1 can't be identically zero on (a, b) (why not?), and therefore there is a subinterval I of (a, b) on which y_1 has no zeros. Since Equation 4.1.33implies that y_2/y_1 is nonconstant on I, y_2 isn't a constant multiple of y_1 on (a, b). A similar argument shows that y_1 isn't a constant multiple of y_2 on (a, b), since

$$\left(\frac{y_1}{y_2}\right)' = \frac{y_1' y_2 - y_1 y_2'}{y_2^2} = -\frac{W}{y_2^2}$$

on any subinterval of (a, b) where y_2 has no zeros.

We can now complete the proof of Theorem 5.1.3 . From Theorem 5.1.5 , two solutions y_1 and y_2 of Equation 4.1.32 are linearly independent on (a, b) if and only if W has no zeros on (a, b). From Theorem 5.1.4 and the motivating comments preceding it, $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 4.1.32 if and only if W has no zeros on (a, b). Therefore $\{y_1, y_2\}$ is a fundamental set for Equation 4.1.32on (a, b) if and only if $\{y_1, y_2\}$ is linearly independent on (a, b).

The next theorem summarizes the relationships among the concepts discussed in this section.

Theorem 5.1.6

Suppose p and q are continuous on an open interval (a, b) and let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{4.1.34}$$

on (a, b). Then the following statements are equivalent; that is, they are either all true or all false.

a. The general solution of (4.1.34) on (a, b) is $y = c_1 y_1 + c_2 y_2$.
b. $\{y_1, y_2\}$ is a fundamental set of solutions of (4.1.34) on (a, b).
c. $\{y_1, y_2\}$ is linearly independent on (a, b).
d. The Wronskian of $\{y_1, y_2\}$ is nonzero at some point in (a, b).
e. The Wronskian of $\{y_1, y_2\}$ is nonzero at all points in (a, b).

We can apply this theorem to an equation written as

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

on an interval (a, b) where P_0, P_1, and P_2 are continuous and P_0 has no zeros.dd proof here and it will automatically be hidden

Theorem 5.1.7

Suppose c is in (a, b) and α and β are real numbers, not both zero. Under the assumptions of Theorem 5.1.7 , suppose y_1 and y_2 are solutions of Equation 4.1.34such that

$$\alpha y_1(c) + \beta y_1'(c) = 0 \quad \text{and} \quad \alpha y_2(c) + \beta y_2'(c) = 0. \tag{4.1.35}$$

Then $\{y_1, y_2\}$ isn't linearly independent on (a, b).

Proof

Since α and β are not both zero, Equation 4.1.35implies that

$$\begin{vmatrix} y_1(c) & y_1'(c) \\ y_2(c) & y_2'(c) \end{vmatrix} = 0, \quad \text{so} \quad \begin{vmatrix} y_1(c) & y_2(c) \\ y_1'(c) & y_2'(c) \end{vmatrix} = 0$$

and Theorem 5.1.6 implies the stated conclusion.

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4.1E: Homogeneous Linear Equations (Exercises)

Q4.1.1

1.

a. Verify that $y_1 = e^{2x}$ and $y_2 = e^{5x}$ are solutions of

$$y'' - 7y' + 10y = 0 \tag{A}$$

on $(-\infty, \infty)$.

b. Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 e^{2x} + c_2 e^{5x}$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 1.$$

d. Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

2.

a. Verify that $y_1 = e^x \cos x$ and $y_2 = e^x \sin x$ are solutions of

$$y'' - 2y' + 2y = 0 \tag{A}$$

on $(-\infty, \infty)$.

b. Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 e^x \cos x + c_2 e^x \sin x$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = -2.$$

d. Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

3.

a. Verify that $y_1 = e^x$ and $y_2 = xe^x$ are solutions of

$$y'' - 2y' + y = 0 \tag{A}$$

on $(-\infty, \infty)$.

b. Verify that if c_1 and c_2 are arbitrary constants then $y = e^x(c_1 + c_2 x)$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 7, \quad y'(0) = 4.$$

d. Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

4.

a. Verify that $y_1 = 1/(x-1)$ and $y_2 = 1/(x+1)$ are solutions of

$$(x^2 - 1)y'' + 4xy' + 2y = 0 \tag{A}$$

on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. What is the general solution of (A) on each of these intervals?

b. Solve the initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = -5, \quad y'(0) = 1.$$

What is the interval of validity of the solution?

c. Graph the solution of the initial value problem.

d. Verify Abel's formula for y_1 and y_2, with $x_0 = 0$.

5. Compute the Wronskians of the given sets of functions.

a. $\{1, e^x\}$
b. $\{e^x, e^x \sin x\}$
c. $\{x+1, x^2+2\}$
d. $\{x^{1/2}, x^{-1/3}\}$
e. $\{\frac{\sin x}{x}, \frac{\cos x}{x}\}$
f. $\{x \ln|x|, x^2 \ln|x|\}$
g. $\{e^x \cos\sqrt{x}, e^x \sin\sqrt{x}\}$

6. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$y'' + 3(x^2+1)y' - 2y = 0,$$

given that $W(\pi) = 0$.

7. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0,$$

given that $W(0) = 1$. (This is *Legendre's equation*.)

8. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

given that $W(1) = 1$. (This is *Bessel's equation*.)

9. (This exercise shows that if you know one nontrivial solution of $y'' + p(x)y' + q(x)y = 0$, you can use Abel's formula to find another.)

Suppose p and q are continuous and y_1 is a solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

that has no zeros on (a, b). Let $P(x) = \int p(x)\,dx$ be any antiderivative of p on (a, b).

a. Show that if K is an arbitrary nonzero constant and y_2 satisfies

$$y_1y_2' - y_1'y_2 = Ke^{-P(x)} \tag{B}$$

on (a, b), then y_2 also satisfies (A) on (a, b), and $\{y_1, y_2\}$ is a fundamental set of solutions on (A) on (a, b).

b. Conclude from (a) that if $y_2 = uy_1$ where $u' = K\frac{e^{-P(x)}}{y_1^2(x)}$, then $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on (a, b).

Q4.1.2

In *Exercises 4.1.10-4.1.23* use the method suggested by *Exercise 4.1.9* to find a second solution y_2 that isn't a constant multiple of the solution y_1. Choose K conveniently to simplify y_2.

10. $y'' - 2y' - 3y = 0$; $y_1 = e^{3x}$

11. $y'' - 6y' + 9y = 0$; $y_1 = e^{3x}$

12. $y'' - 2ay' + a^2y = 0$ ($a =$ constant); $y_1 = e^{ax}$

13. $x^2y'' + xy' - y = 0$; $y_1 = x$

14. $x^2y'' - xy' + y = 0$; $y_1 = x$

15. $x^2y'' - (2a-1)xy' + a^2y = 0$ ($a =$ nonzero constant); $x > 0$; $y_1 = x^a$

16. $4x^2y'' - 4xy' + (3 - 16x^2)y = 0$; $y_1 = x^{1/2}e^{2x}$

17. $(x-1)y'' - xy' + y = 0$; $y_1 = e^x$

18. $x^2y'' - 2xy' + (x^2+2)y = 0$; $y_1 = x\cos x$

19. $4x^2(\sin x)y''-4x(x\cos x+\sin x)y'+(2x\cos x+3\sin x)y=0$; $y_1=x^{1/2}$

20. $(3x-1)y''-(3x+2)y'-(6x-8)y=0$; $y_1=e^{2x}$

21. $(x^2-4)y''+4xy'+2y=0$; $y_1=\frac{1}{x-2}$

22. $(2x+1)xy''-2(2x^2-1)y'-4(x+1)y=0$;$y_1=\frac{1}{x}$

23. $(x^2-2x)y''+(2-x^2)y'+(2x-2)y=0$;$y_1=e^x$

Q4.1.3

24. Suppose p and q are continuous on an open interval (a,b) and let x_0 be in (a,b). Use Theorem 4.1.1 to show that the only solution of the initial value problem

$$y''+p(x)y'+q(x)y=0, \quad y(x_0)=0, \quad y'(x_0)=0$$

on (a,b) is the trivial solution $y\equiv 0$.

25. Suppose P_0, P_1, and P_2 are continuous on (a,b) and let x_0 be in (a,b). Show that if either of the following statements is true then $P_0(x)=0$ for some x in (a,b).

a. The initial value problem

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0, \quad y(x_0)=k_0, \quad y'(x_0)=k_1$$

has more than one solution on (a,b).

b. The initial value problem

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0, \quad y(x_0)=0, \quad y'(x_0)=0$$

has a nontrivial solution on (a,b).

26. Suppose p and q are continuous on (a,b) and y_1 and y_2 are solutions of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

on (a,b). Let

$$z_1=\alpha y_1+\beta y_2 \quad \text{and} \quad z_2=\gamma y_1+\delta y_2,$$

where α, β, γ, and δ are constants. Show that if $\{z_1,z_2\}$ is a fundamental set of solutions of (A) on (a,b) then so is $\{y_1,y_2\}$.

27. Suppose p and q are continuous on (a,b) and $\{y_1,y_2\}$ is a fundamental set of solutions of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

on (a,b). Let

$$z_1=\alpha y_1+\beta y_2 \quad \text{and} \quad z_2=\gamma y_1+\delta y_2,$$

where α,β,γ, and δ are constants. Show that $\{z_1,z_2\}$ is a fundamental set of solutions of (A) on (a,b) if and only if $\alpha\gamma-\beta\delta\neq 0$.

28. Suppose y_1 is differentiable on an interval (a,b) and $y_2=ky_1$, where k is a constant. Show that the Wronskian of $\{y_1,y_2\}$ is identically zero on (a,b).

29. Let

$$y_1=x^3 \quad \text{and} \quad y_2=\begin{cases} x^3, & x\geq 0, \\ -x^3, & x<0. \end{cases}$$

a. Show that the Wronskian of $\{y_1,y_2\}$ is defined and identically zero on $(-\infty,\infty)$.
b. Suppose $a<0<b$. Show that $\{y_1,y_2\}$ is linearly independent on (a,b).
c. Use *Exercise 4.1.25b* to show that these results don't contradict Theorem 4.1.5, because neither y_1 nor y_2 can be a solution of an equation

$$y''+p(x)y'+q(x)y=0$$

on (a,b) if p and q are continuous on (a,b).

30. Suppose p and q are continuous on (a,b) and $\{y_1,y_2\}$ is a set of solutions of

$$y''+p(x)y'+q(x)y=0$$

on (a,b) such that either $y_1(x_0)=y_2(x_0)=0$ or $y_1'(x_0)=y_2'(x_0)=0$ for some x_0 in (a,b). Show that $\{y_1,y_2\}$ is linearly dependent on (a,b).

31. Suppose p and q are continuous on (a,b) and $\{y_1,y_2\}$ is a fundamental set of solutions of

$$y''+p(x)y'+q(x)y=0$$

on (a,b). Show that if $y_1(x_1)=y_1(x_2)=0$, where $a<x_1<x_2<b$, then $y_2(x)=0$ for some x in (x_1,x_2).

32. Suppose p and q are continuous on (a,b) and every solution of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

on (a,b) can be written as a linear combination of the twice differentiable functions $\{y_1,y_2\}$. Use Theorem 4.1.1 to show that y_1 and y_2 are themselves solutions of (A) on (a,b).

33. Suppose p_1, p_2, q_1, and q_2 are continuous on (a,b) and the equations

$$y''+p_1(x)y'+q_1(x)y=0 \quad \text{and} \quad y''+p_2(x)y'+q_2(x)y=0$$

have the same solutions on (a,b). Show that $p_1=p_2$ and $q_1=q_2$ on (a,b).

34. (For this exercise you have to know about 3×3 determinants.) Show that if y_1 and y_2 are twice continuously differentiable on (a,b) and the Wronskian W of $\{y_1,y_2\}$ has no zeros in (a,b) then the equation

$$\frac{1}{W}\begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix}=0$$

can be written as

$$y''+p(x)y'+q(x)y=0, \tag{A}$$

where p and q are continuous on (a,b) and $\{y_1,y_2\}$ is a fundamental set of solutions of (A) on (a,b).

35. Find a linear homogeneous equation for which the given functions form a fundamental set of solutions on some interval.

a. $e^x\cos 2x, e^x\sin 2x$
b. x, e^{2x}
c. $x, x\ln x$
d. $\cos(\ln x), \sin(\ln x)$
e. $\cosh x, \sinh x$
f. x^2-1, x^2+1

36. Suppose p and q are continuous on (a,b) and $\{y_1,y_2\}$ is a fundamental set of solutions of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

on (a,b). Show that if y is a solution of (A) on (a,b), there's exactly one way to choose c_1 and c_2 so that $y=c_1y_1+c_2y_2$ on (a,b).

37. Suppose p and q are continuous on (a,b) and x_0 is in (a,b). Let y_1 and y_2 be the solutions of

$$y''+p(x)y'+q(x)y=0 \tag{A}$$

such that

$$y_1(x_0)=1, \quad y_1'(x_0)=0 \quad \text{and} \quad y_2(x_0)=0,\ y_2'(x_0)=1.$$

(Theorem 4.1.1 implies that each of these initial value problems has a unique solution on (a,b).)

a. Show that $\{y_1, y_2\}$ is linearly independent on (a, b).
b. Show that an arbitrary solution y of (A) on (a, b) can be written as $y = y(x_0)y_1 + y'(x_0)y_2$.
c. Express the solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2.

38. Find solutions y_1 and y_2 of the equation $y'' = 0$ that satisfy the initial conditions

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Then write the solution of the initial value problem

$$y'' = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

as a linear combination of y_1 and y_2.

39. Let x_0 be an arbitrary real number. Given (Example 4.1.1) that e^x and e^{-x} are solutions of $y'' - y = 0$, find solutions y_1 and y_2 of $y'' - y = 0$ such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \; y_2'(x_0) = 1.$$

Then use *Exercise 4.1.37 (c)* to write the solution of the initial value problem

$$y'' - y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2.

40. Let x_0 be an arbitrary real number. Given (Example 4.1.2) that $\cos \omega x$ and $\sin \omega x$ are solutions of $y'' + \omega^2 y = 0$, find solutions of $y'' + \omega^2 y = 0$ such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \; y_2'(x_0) = 1.$$

Then write the solution of the initial value problem

$$y'' + \omega^2 y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2. Use the identities

$$\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ \sin(A+B) &= \sin A \cos B + \cos A \sin B \end{aligned}$$

to simplify your expressions for y_1, y_2, and y.

41. Recall from *Exercise 4.1.4* that $1/(x-1)$ and $1/(x+1)$ are solutions of

$$(x^2 - 1)y'' + 4xy' + 2y = 0 \tag{A}$$

on $(-1, 1)$. Find solutions of (A) such that

$$y_1(0) = 1, \quad y_1'(0) = 0 \quad \text{and} \quad y_2(0) = 0, \; y_2'(0) = 1.$$

Then use *Exercise 4.1.37 (c)* to write the solution of initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

as a linear combination of y_1 and y_2.

42.

a. Verify that $y_1 = x^2$ and $y_2 = x^3$ satisfy

$$x^2 y'' - 4xy' + 6y = 0 \tag{A}$$

on $(-\infty, \infty)$ and that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(-\infty, 0)$ and $(0, \infty)$.
b. Let a_1, a_2, b_1, and b_2 be constants. Show that

$$y=\begin{cases} a_1x^2+a_2x^3, & x\ge 0,\\ b_1x^2+b_2x^3, & x<0\end{cases}$$

is a solution of (A) on $(-\infty,\infty)$ if and only if $a_1=b_1$. From this, justify the statement that y is a solution of (A) on $(-\infty,\infty)$ if and only if

$$y=\begin{cases} c_1x^2+c_2x^3, & x\ge 0,\\ c_1x^2+c_3x^3, & x<0,\end{cases}$$

where c_1, c_2, and c_3 are arbitrary constants.

c. For what values of k_0 and k_1 does the initial value problem

$$x^2y''-4xy'+6y=0,\quad y(0)=k_0,\quad y'(0)=k_1$$

have a solution? What are the solutions?

d. Show that if $x_0\ne 0$ and k_0,k_1 are arbitrary constants, the initial value problem

$$x^2y''-4xy'+6y=0,\quad y(x_0)=k_0,\quad y'(x_0)=k_1 \tag{B}$$

has infinitely many solutions on $(-\infty,\infty)$. On what interval does (B) have a unique solution?

43.

a. Verify that $y_1=x$ and $y_2=x^2$ satisfy

$$x^2y''-2xy'+2y=0 \tag{A}$$

on $(-\infty,\infty)$ and that $\{y_1,y_2\}$ is a fundamental set of solutions of (A) on $(-\infty,0)$ and $(0,\infty)$.

b. Let a_1, a_2, b_1, and b_2 be constants. Show that

$$y=\begin{cases} a_1x+a_2x^2, & x\ge 0,\\ b_1x+b_2x^2, & x<0\end{cases}$$

is a solution of (A) on $(-\infty,\infty)$ if and only if $a_1=b_1$ and $a_2=b_2$. From this, justify the statement that the general solution of (A) on $(-\infty,\infty)$ is $y=c_1x+c_2x^2$, where c_1 and c_2 are arbitrary constants.

c. For what values of k_0 and k_1 does the initial value problem

$$x^2y''-2xy'+2y=0,\quad y(0)=k_0,\quad y'(0)=k_1$$

have a solution? What are the solutions?

d. Show that if $x_0\ne 0$ and k_0,k_1 are arbitrary constants then the initial value problem

$$x^2y''-2xy'+2y=0,\quad y(x_0)=k_0,\quad y'(x_0)=k_1$$

has a unique solution on $(-\infty,\infty)$.

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4.2: Introduction to Linear Higher Order Equations

An nth order differential equation is said to be **linear** if it can be written in the form

$$y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=f(x). \tag{4.2.1}$$

We considered equations of this form with $n=1$ in Section 2.1 and with $n=2$ in Chapter 5. In this chapter n is an arbitrary positive integer. In this section we sketch the general theory of linear nth order equations. Since this theory has already been discussed for $n=2$ in Sections 5.1 and 5.3, we'll omit proofs.

For convenience, we consider linear differential equations written as

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=F(x), \tag{4.2.2}$$

which can be rewritten as Equation 4.2.1 on any interval on which P_0 has no zeros, with $p_1=P_1/P_0$, ..., $p_n=P_n/P_0$ and $f=F/P_0$. For simplicity, throughout this chapter we'll abbreviate the left side of Equation 4.2.2 by Ly; that is,

$$Ly=P_0y^{(n)}+P_1y^{(n-1)}+\cdots+P_ny.$$

We say that the equation $Ly=F$ is **normal** on (a,b) if P_0, P_1, ..., P_n and F are continuous on (a,b) and P_0 has no zeros on (a,b). If this is so then $Ly=F$ can be written as Equation 4.2.1 with p_1, ..., p_n and f continuous on (a,b).

The next theorem is analogous to Theorem 5.3.1.

Theorem 9.1.1

Suppose $Ly=F$ is normal on (a,b), let x_0 be a point in (a,b), and let k_0, k_1, ..., k_{n-1} be arbitrary real numbers. Then the initial value problem

$$Ly=F,\quad y(x_0)=k_0,\quad y'(x_0)=k_1,\ldots,\quad y^{(n-1)}(x_0)=k_{n-1}$$

has a unique solution on (a,b).

Homogeneous Equations

Equation 4.2.2 is said to be **homogeneous** if $F\equiv 0$ and **nonhomogeneous** otherwise. Since $y\equiv 0$ is obviously a solution of $Ly=0$, we call it the **trivial** solution. Any other solution is **nontrivial**.

If y_1, y_2, ..., y_n are defined on (a,b) and c_1, c_2, ..., c_n are constants, then

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n \tag{4.2.3}$$

is a **linear combination** of $\{y_1,y_2\ldots,y_n\}$. It's easy to show that if y_1, y_2, ..., y_n are solutions of $Ly=0$ on (a,b), then so is any linear combination of $\{y_1,y_2,\ldots,y_n\}$. (See the proof of Theorem 5.1.2.) We say that $\{y_1,y_2,\ldots,y_n\}$ is a **fundamental set of solutions of $Ly=0$** on (a,b) if every solution of $Ly=0$ on (a,b) can be written as a linear combination of $\{y_1,y_2,\ldots,y_n\}$, as in Equation 4.2.3. In this case we say that Equation 4.2.3 is the **general solution of $Ly=0$** on (a,b).

It can be shown (**Exercises 9.1.14** and **9.1.15**) that if the equation $Ly=0$ is normal on (a,b) then it has infinitely many fundamental sets of solutions on (a,b). The next definition will help to identify fundamental sets of solutions of $Ly=0$.

We say that $\{y_1,y_2,\ldots,y_n\}$ is **linearly independent** on (a,b) if the only constants c_1, c_2, ..., c_n such that

$$c_1y_1(x)+c_2y_2(x)+\cdots+c_ny_n(x)=0,\quad a<x<b, \tag{4.2.4}$$

are $c_1=c_2=\cdots=c_n=0$. If Equation 4.2.4 holds for some set of constants c_1, c_2, ..., c_n that are not all zero, then $\{y_1,y_2,\ldots,y_n\}$ is **linearly dependent** on (a,b)

The next theorem is analogous to Theorem 5.1.3.

Theorem 9.1.1

If $Ly=0$ is normal on (a,b), then a set $\{y_1,y_2,\ldots,y_n\}$ of n solutions of $Ly=0$ on (a,b) is a fundamental set if and only if it is linearly independent on (a,b).

Example 9.1.1

The equation

$$x^3y''' - x^2y'' - 2xy' + 6y = 0 \tag{4.2.5}$$

is normal and has the solutions $y_1 = x^2$, $y_2 = x^3$, and $y_3 = 1/x$ on $(-\infty, 0)$ and $(0, \infty)$. Show that $\{y_1, y_2, y_3\}$ is linearly independent on $(-\infty, 0)$ and $(0, \infty)$. Then find the general solution of Equation 4.2.5 on $(-\infty, 0)$ and $(0, \infty)$.

Solution

Suppose

$$c_1x^2 + c_2x^3 + \frac{c_3}{x} = 0 \tag{4.2.6}$$

on $(0, \infty)$. We must show that $c_1 = c_2 = c_3 = 0$. Differentiating Equation 4.2.6 twice yields the system

$$\begin{aligned} c_1x^2 + c_2x^3 + \frac{c_3}{x} &= 0 \\ 2c_1x + 3c_2x^2 - \frac{c_3}{x^2} &= 0 \\ 2c_1 + 6c_2x + \frac{2c_3}{x^3} &= 0. \end{aligned} \tag{4.2.7}$$

If Equation 4.2.7 holds for all x in $(0, \infty)$, then it certainly holds at $x = 1$; therefore,

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 + 6c_2 + 2c_3 &= 0. \end{aligned} \tag{4.2.8}$$

By solving this system directly, you can verify that it has only the trivial solution $c_1 = c_2 = c_3 = 0$; however, for our purposes it is more useful to recall from linear algebra that a homogeneous linear system of n equations in n unknowns has only the trivial solution if its determinant is nonzero. Since the determinant of Equation 4.2.8 is

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -3 \\ 2 & 4 & 0 \end{vmatrix} = 12,$$

it follows that Equation 4.2.8 has only the trivial solution, so $\{y_1, y_2, y_3\}$ is linearly independent on $(0, \infty)$. Now Theorem 9.1.2 implies that

$$y = c_1x^2 + c_2x^3 + \frac{c_3}{x}$$

is the general solution of Equation 4.2.5 on $(0, \infty)$. To see that this is also true on $(-\infty, 0)$, assume that Equation 4.2.6 holds on $(-\infty, 0)$. Setting $x = -1$ in Equation 4.2.7 yields

$$\begin{aligned} c_1 - c_2 - c_3 &= 0 \\ -2c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 - 6c_2 - 2c_3 &= 0. \end{aligned}$$

Since the determinant of this system is

$$\begin{vmatrix} 1 & -1 & -1 \\ -2 & 3 & -1 \\ 2 & -6 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 2 & -4 & 0 \end{vmatrix} = -12,$$

it follows that $c_1 = c_2 = c_3 = 0$; that is, $\{y_1, y_2, y_3\}$ is linearly independent on $(-\infty, 0)$.

Example 9.1.2

The equation

$$y^{(4)}+y'''-7y''-y'+6y=0 \tag{4.2.9}$$

is normal and has the solutions $y_1=e^x$, $y_2=e^{-x}$, $y_3=e^{2x}$, and $y_4=e^{-3x}$ on $(-\infty,\infty)$. (Verify.) Show that $\{y_1,y_2,y_3,y_4\}$ is linearly independent on $(-\infty,\infty)$. Then find the general solution of Equation 4.2.9.

Solution

Suppose c_1, c_2, c_3, and c_4 are constants such that

$$c_1e^x+c_2e^{-x}+c_3e^{2x}+c_4e^{-3x}=0 \tag{4.2.10}$$

for all x. We must show that $c_1=c_2=c_3=c_4=0$. Differentiating Equation 4.2.10 three times yields the system

$$\begin{aligned} c_1e^x+c_2e^{-x}+\ \ c_3e^{2x}+\ \ \ c_4e^{-3x} &= 0\\ c_1e^x-c_2e^{-x}+2c_3e^{2x}-\ 3c_4e^{-3x} &= 0\\ c_1e^x+c_2e^{-x}+4c_3e^{2x}+\ 9c_4e^{-3x} &= 0\\ c_1e^x-c_2e^{-x}+8c_3e^{2x}-27c_4e^{-3x} &= 0. \end{aligned} \tag{4.2.11}$$

If Equation 4.2.11 holds for all x, then it certainly holds for $x=0$. Therefore

$$\begin{aligned} c_1+c_2+\ \ c_3+\ \ \ c_4 &= 0\\ c_1-c_2+2c_3-\ 3c_4 &= 0\\ c_1+c_2+4c_3+\ 9c_4 &= 0\\ c_1-c_2+8c_3-27c_4 &= 0. \end{aligned}$$

The determinant of this system is

$$\begin{vmatrix} 1 & 1 & 1 & 1\\ 1 & -1 & 2 & -3\\ 1 & 1 & 4 & 9\\ 1 & -1 & 8 & -27 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1\\ 0 & -2 & 1 & -4\\ 0 & 0 & 3 & 8\\ 0 & -2 & 7 & -28 \end{vmatrix} = \begin{vmatrix} -2 & 1 & -4\\ 0 & 3 & 8\\ -2 & 7 & -28 \end{vmatrix} \tag{4.2.12}$$

$$[5pt] \quad = \begin{vmatrix} -2 & 1 & -4\\ 0 & 3 & 8\\ 0 & 6 & -24 \end{vmatrix} = -2\begin{vmatrix} 3 & 8\\ 6 & -24 \end{vmatrix} = 240,$$

so the system has only the trivial solution $c_1=c_2=c_3=c_4=0$. Now Theorem 9.1.2 implies that

$$y=c_1e^x+c_2e^{-x}+c_3e^{2x}+c_4e^{-3x}$$

is the general solution of Equation 4.2.9.

The Wronskian

We can use the method used in Examples 9.1.1 and 9.1.2 to test n solutions $\{y_1,y_2,\ldots,y_n\}$ of any nth order equation $Ly=0$ for linear independence on an interval (a,b) on which the equation is normal. Thus, if c_1, c_2,…, c_n are constants such that

$$c_1y_1+c_2y_2+\cdots+c_ny_n=0, \quad a<b,$$

then differentiating $n-1$ times leads to the $n\times n$ system of equations

$$\begin{aligned} c_1y_1(x)+c_2y_2(x)+ \quad &\cdots \quad +c_ny_n(x)=0 \\ c_1y_1'(x)+c_2y_2'(x)+ \quad &\cdots \quad +c_ny_n'(x)=0 \\ &\vdots \\ c_1y_1^{(n-1)}(x)+c_2y_2^{(n-1)}(x)+ \quad &\cdots \quad +c_ny_n^{(n-1)}(x)=0 \end{aligned} \tag{4.2.13}$$

for c_1, c_2, ..., c_n. For a fixed x, the determinant of this system is

$$W(x)=\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}.$$

We call this determinant the **Wronskian** of $\{y_1,y_2,\ldots,y_n\}$. If $W(x)\neq 0$ for some x in (a,b) then the system Equation 4.2.13 has only the trivial solution $c_1=c_2=\cdots=c_n=0$, and Theorem 9.1.2 implies that

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n$$

is the general solution of $Ly=0$ on (a,b).

The next theorem generalizes Theorem 5.1.4. The proof is sketched in **Exercises 9.1.17-9.1.20**.

Theorem 9.1.3 Abel's Formula

Suppose the homogeneous linear nth order equation

$$P_0(x)y^{(n)}+P_1(x)y^{n-1}+\cdots+P_n(x)y=0 \tag{4.2.14}$$

is normal on (a,b), let y_1, y_2, ..., y_n be solutions of Equation 4.2.14 on (a,b), and let x_0 be in (a,b). Then the Wronskian of $\{y_1,y_2,\ldots,y_n\}$ is given by

$$W(x)=W(x_0)\exp\left\{-\int_{x_0}^{x}\frac{P_1(t)}{P_0(t)}\,dt\right\}, \quad a<b. \tag{4.2.15}$$

Therefore, either W has no zeros in (a,b) or $W\equiv 0$ on (a,b).

Formula Equation 4.2.15 is **Abel's formula**.

The next theorem is analogous to Theorem 5.1.6.

Theorem 9.1.4

Suppose $Ly=0$ is normal on (a,b) and let y_1, y_2, ..., y_n be n solutions of $Ly=0$ on (a,b). Then the following statements are equivalent; that is, they are either all true or all false:

a. The general solution of $Ly=0$ on (a,b) is $y=c_1y_1+c_2y_2+\cdots+c_ny_n$.
b. $\{y_1,y_2,\ldots,y_n\}$ is a fundamental set of solutions of $Ly=0$ on (a,b).
c. $\{y_1,y_2,\ldots,y_n\}$ is linearly independent on (a,b).
d. The Wronskian of $\{y_1,y_2,\ldots,y_n\}$ is nonzero at some point in (a,b).
e. The Wronskian of $\{y_1,y_2,\ldots,y_n\}$ is nonzero at all points in (a,b).

Example 9.1.3

In Example 9.1.1 we saw that the solutions $y_1=x^2$, $y_2=x^3$, and $y_3=1/x$ of

$$x^3y'''-x^2y''-2xy'+6y=0$$

are linearly independent on $(-\infty, 0)$ and $(0, \infty)$. Calculate the Wronskian of $\{y_1, y_2, y_3\}$.

Solution

If $x \neq 0$, then

$$W(x) = \begin{vmatrix} x^2 & x^3 & \frac{1}{x} \\ 2x & 3x^2 & -\frac{1}{x^2} \\ 2 & 6x & \frac{2}{x^3} \end{vmatrix} = 2x^3 \begin{vmatrix} 1 & x & \frac{1}{x^3} \\ 2 & 3x & -\frac{1}{x^3} \\ 1 & 3x & \frac{1}{x^3} \end{vmatrix}$$

where we factored x^2, x, and 2 out of the first, second, and third rows of $W(x)$, respectively. Adding the second row of the last determinant to the first and third rows yields

$$W(x) = 2x^3 \begin{vmatrix} 3 & 4x & 0 \\ 2 & 3x & -\frac{1}{x^3} \\ 3 & 6x & 0 \end{vmatrix} = 2x^3 \left(\frac{1}{x^3}\right) \begin{vmatrix} 3 & 4x \\ 3 & 6x \end{vmatrix} = 12x$$

Therefore $W(x) \neq 0$ on $(-\infty, 0)$ and $(0, \infty)$.

Example 9.1.4

In Example 9.1.2 we saw that the solutions $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{2x}$, and $y_4 = e^{-3x}$ of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

are linearly independent on every open interval. Calculate the Wronskian of $\{y_1, y_2, y_3, y_4\}$.

Solution

For all x,

$$W(x) = \begin{vmatrix} e^x & e^{-x} & e^{2x} & e^{-3x} \\ e^x & -e^{-x} & 2e^{2x} & -3e^{-3x} \\ e^x & e^{-x} & 4e^{2x} & 9e^{-3x} \\ e^x & -e^{-x} & 8e^{2x} & -27e^{-3x} \end{vmatrix}.$$

Factoring the exponential common factor from each row yields

$$W(x) = e^{-x} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{vmatrix} = 240e^{-x},$$

from Equation 4.2.12.

Note

Under the assumptions of Theorem 9.1.4 , it isn't necessary to obtain a formula for $W(x)$. Just evaluate $W(x)$ at a convenient point in (a, b), as we did in Examples 9.1.1 and 9.1.2 .

Theorem 9.1.5

Suppose c is in (a, b) and α_1, α_2, …, are real numbers, not all zero. Under the assumptions of Theorem 10.3.3, suppose y_1 and y_2 are solutions of Equation 5.1.35 such that

$$\alpha y_i(c)+y_i'(c)+\cdots+y_i^{(n-1)}(c)=0, \quad 1\le i\le n. \tag{4.2.16}$$

Then $\{y_1, y_2, \ldots y_n\}$ isn't linearly independent on (a, b).

Proof

Since α_1, α_2, …, α_n are not all zero, Equation 4.2.14implies that

$$\begin{vmatrix} y_1(c) & y_1'(c) & \cdots & y_1^{(n-1)}(c) \\ y_2(c) & y_2'(c) & \cdots & y_2^{(n-1)}(c) \\ \vdots & \vdots & \ddots & \vdots \\ y_n(c) & y_n'(c) & \cdots & y_n^{(n-1)}(c) \end{vmatrix}=0,$$

so

$$\begin{vmatrix} y_1(c) & y_2(c) & \cdots & y_n(c) \\ y_1'(c) & y_2'(c) & \cdots & y_n'(c) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(c) & y_2^{(n-1)}(c)(c) & \cdots & y_n^{(n-1)}(c)(c) \end{vmatrix}=0$$

and Theorem 9.1.4 implies the stated conclusion.

General Solution of a Nonhomogeneous Equation

The next theorem is analogous to Theorem 5.3.2. It shows how to find the general solution of $Ly=F$ if we know a particular solution of $Ly=F$ and a fundamental set of solutions of the **complementary equation** $Ly=0$.

Theorem 9.1.6

The probabilities assigned to events by a distribution function on a sample space are given by.

Proof

Suppose $Ly=F$ is normal on (a, b). Let y_p be a particular solution of $Ly=F$ on (a, b), and let $\{y_1, y_2, \ldots, y_n\}$ be a fundamental set of solutions of the complementary equation $Ly=0$ on (a, b). Then y is a solution of $Ly=F$ on (a, b) if and only if

$$y=y_p+c_1y_1+c_2y_2+\cdots+c_ny_n,$$

where $c_1, c_2, \ldots, c_n$ are constants.

The next theorem is analogous to Theorem 5.3.2.

Theorem 9.1.7 The Principle of Superposition

Suppose for each $i=1$, 2, …, r, the function y_{p_i} is a particular solution of $Ly=F_i$ on (a, b). Then

$$y_p=y_{p_1}+y_{p_2}+\cdots+y_{p_r}$$

is a particular solution of

$$Ly=F_1(x)+F_2(x)+\cdots+F_r(x)$$

on (a, b).

We'll apply Theorems 9.1.6 and 9.1.7 throughout the rest of this chapter.

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4.2E: Introduction to Linear Higher Order Equations (Exercises)

Q4.2.1

1. Verify that the given function is the solution of the initial value problem.

a. $x^3y''' - 3x^2y'' + 6xy' - 6y = \dfrac{-24}{x}, \quad y(-1) = 0 \ , y'(-1) = 0, \quad y''(-1) = 0 \ ; y = -6x - 8x^2 - 3x^3 + \frac{1}{x}$

b. $y''' - \dfrac{1}{x}y'' - y' + \dfrac{1}{x}y = \dfrac{x^2-4}{x^4}, \quad y(1) = \dfrac{3}{2}, \quad y'(1) = \dfrac{1}{2}, y''(1) = 1 \quad ; y = x + \dfrac{1}{2x}$

c. $xy''' - y'' - xy' + y = x^2, \quad y(1) = 2, \quad y'(1) = 5, \quad y''(1) = -1 \quad ; y = -x^2 - 2 + 2e^{(x-1)} - e^{-(x-1)} + 4x$

d. $4x^3y''' + 4x^2y'' - 5xy' + 2y = 30x^2, \quad y(1) = 5, \quad y'(1) = \dfrac{17}{2} \quad ; y''(1) = \dfrac{63}{4}; \quad y = 2x^2 \ln x - x^{1/2} + 2x^{-1/2} + 4x^2$

e. $x^4y^{(4)} - 4x^3y''' + 12x^2y'' - 24xy' + 24y = 6x^4, \quad y(1) = -2 \quad ; y'(1) = -9, \quad y''(1) = -27, \quad y'''(1) = -52 \ ;$
$y = x^4 \ln x + x - 2x^2 + 3x^3 - 4x^4$

f. $xy^{(4)} - y''' - 4xy'' + 4y' = 96x^2, \quad y(1) = -5, \quad y'(1) = -24 \ ;$
$y''(1) = -36; \quad y'''(1) = -48; \quad y = 9 - 12x + 6x^2 - 8x^3$

2. Solve the initial value problem

$$x^3y''' - x^2y'' - 2xy' + 6y = 0, \quad y(-1) = -4, \quad y'(-1) = -14, \quad y''(-1) = -20.$$

HINT: *See Example 9.1.1.*

3. Solve the initial value problem

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0, \quad y(0) = 5, \quad y'(0) = -6, \quad y''(0) = 10, \quad y'''(0) - 36.$$

HINT: *See Example 9.1.2.*

4. Find solutions y_1, y_2, …, y_n of the equation $y^{(n)} = 0$ that satisfy the initial conditions

$$y_i^{(j)}(x_0) = \begin{cases} 0, & j \neq i-1, \\ 1, & j = i-1, \end{cases} \quad 1 \leq i \leq n.$$

5.

a. Verify that the function

$$y = c_1x^3 + c_2x^2 + \frac{c_3}{x}$$

satisfies

$$x^3y''' - x^2y'' - 2xy' + 6y = 0 \tag{A}$$

if c_1, c_2, and c_3 are constants.

b. Use (a) to find solutions y_1, y_2, and y_3 of (A) such that

$$\begin{aligned} y_1(1) &= 1, \quad y_1'(1) = 0, \quad y_1''(1) = 0 \\ y_2(1) &= 0, \quad y_2'(1) = 1, \quad y_2''(1) = 0 \\ y_3(1) &= 0, \quad y_3'(1) = 0, \quad y_3''(1) = 1. \end{aligned}$$

c. Use (b) to find the solution of (A) such that

$$y(1) = k_0, \quad y'(1) = k_1, \quad y''(1) = k_2.$$

6. Verify that the given functions are solutions of the given equation, and show that they form a fundamental set of solutions of the equation on any interval on which the equation is normal.

a. $y''' + y'' - y' - y = 0; \quad \{e^x, \ e^{-x}, \ xe^{-x}\}$

b. $y''' - 3y'' + 7y' - 5y = 0; \quad \{e^x, \ e^x \cos 2x, \ e^x \sin 2x\}$.

c. $xy''' - y'' - xy' + y = 0; \quad \{e^x,\ e^{-x},\ x\}$
d. $x^2y''' + 2xy'' - (x^2+2)y = 0; \quad \{e^x/x,\ e^{-x}/x,\ 1\}$
e. $(x^2-2x+2)y''' - x^2y'' + 2xy' - 2y = 0; \quad \{x,\ x^2,\ e^x\}$
f. $(2x-1)y^{(4)} - 4xy''' + (5-2x)y'' + 4xy' - 4y = 0; \quad \{x,\ e^x,\ e^{-x}, e^{2x}\}$
g. $xy^{(4)} - y''' - 4xy' + 4y' = 0; \quad \{1, x^2,\ e^{2x},\ e^{-2x}\}$

7. Find the Wronskian W of a set of three solutions of

$$y''' + 2xy'' + e^x y' - y = 0,$$

given that $W(0) = 2$.

8. Find the Wronskian W of a set of four solutions of

$$y^{(4)} + (\tan x)y''' + x^2y'' + 2xy = 0,$$

given that $W(\pi/4) = K$.

9.

a. Evaluate the Wronskian $W\ \{e^x,\ xe^x,\ x^2e^x\}$. Evaluate $W(0)$.
b. Verify that y_1, y_2, and y_3 satisfy

$$y''' - 3y'' + 3y' - y = 0. \tag{A}$$

c. Use $W(0)$ from (a) and Abel's formula to calculate $W(x)$.
d. What is the general solution of (A)?

10. Compute the Wronskian of the given set of functions.

a. $\{1,\ e^x,\ e^{-x}\}$
b. $\{e^x,\ e^x \sin x,\ e^x \cos x\}$
c. $\{2,\ x+1,\ x^2+2\}$
d. $x,\ x\ln x,\ 1/x\}$
e. $\{1,\ x,\ \frac{x^2}{2!},\ \frac{x^3}{3!}, \cdots,\ \frac{x^n}{n!}\}$
f. $\{e^x,\ e^{-x},\ x\}$
g. $\{e^x/x,\ e^{-x}/x,\ 1\}$
h. $\{x,\ x^2,\ e^x\}$
i. $\{x,\ x^3,\ 1/x,\ 1/x^2\}$
j. $\{e^x,\ e^{-x},\ x,\ e^{2x}\}$
k. $\{e^{2x},\ e^{-2x},\ 1,\ x^2\}$

Q4.2.2

11. Show that if the Wronskian of the n-times continuously differentiable functions $\{y_1, y_2, \ldots, y_n\}$ has no zeros in (a, b), then the differential equation obtained by expanding the determinant

$$\begin{vmatrix} y & y_1 & y_2 & \cdots & y_n \\ y' & y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y^{(n)} & y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = 0,$$

in cofactors of its first column is normal and has $\{y_1, y_2, \ldots, y_n\}$ as a fundamental set of solutions on (a, b).

12. Use the method suggested by *Exercise 4.2.11* to find a linear homogeneous equation such that the given set of functions is a fundamental set of solutions on intervals on which the Wronskian of the set has no zeros.

a. $\{x,\ x^2-1,\ x^2+1\}$
b. $\{e^x,\ e^{-x},\ x\}$
c. $\{e^x,\ xe^{-x},\ 1\}$

d. $\{x,\ x^2,\ e^x\}$
e. $\{x,\ x^2,\ 1/x\}$
f. $\{x+1,\ e^x,\ e^{3x}\}$
g. $\{x,\ x^3,\ 1/x,\ 1/x^2\}$
h. $\{x,\ x\ln x,\ 1/x,\ x^2\}$
i. $\{e^x,\ e^{-x},\ x,\ e^{2x}\}$
j. $\{e^{2x},\ e^{-2x},\ 1,\ x^2\}$

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4.3: Constant Coefficient Homogeneous Equations

This section deals with homogeneous equations of the special form

$$ay'' + by' + cy = 0, \tag{4.3.1}$$

where a, b, and c are constant ($a \neq 0$). When you've completed this section you'll know everything there is to know about solving such equations.

If a, b, and c are real constants and $a \neq 0$, then

$$ay'' + by' + cy = F(x)$$

is said to be a **constant coefficient equation**. In this section we consider the homogeneous constant coefficient equation

$$ay'' + by' + cy = 0. \tag{4.3.2}$$

As we'll see, all solutions of Equation 4.3.2 are defined on $(-\infty, \infty)$. This being the case, we'll omit references to the interval on which solutions are defined, or on which a given set of solutions is a fundamental set, etc., since the interval will always be $(-\infty, \infty)$.

The key to solving Equation 4.3.2 is that if $y = e^{rx}$ where r is a constant then the left side of Equation 4.3.2 is a multiple of e^{rx}; thus, if $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$, so

$$ay'' + by' + cy = ar^2 e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx}. \tag{4.3.3}$$

The quadratic polynomial

$$p(r) = ar^2 + br + c$$

is the **characteristic polynomial** of Equation 4.3.2, and $p(r) = 0$ is the **characteristic equation**. From Equation 4.3.3 we can see that $y = e^{rx}$ is a solution of Equation 4.3.2 if and only if $p(r) = 0$.

The roots of the characteristic equation are given by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{4.3.4}$$

Consider Three Cases

- Case 1: $b^2 - 4ac > 0$, so the characteristic equation has two distinct real roots.
- Case 2: $b^2 - 4ac = 0$, so the characteristic equation has a repeated real root.
- Case 3: $b^2 - 4ac < 0$, so the characteristic equation has complex roots.

In each case we'll start with an example.

Case 1: Distinct Real Roots

Example 5.2.1

a. Find the general solution of

$$y'' + 6y' + 5y = 0. \tag{4.3.5}$$

b. Solve the initial value problem

$$y'' + 6y' + 5y = 0, \quad y(0) = 3,\ y'(0) = -1. \tag{4.3.6}$$

Solution

a. The characteristic polynomial of Equation 4.3.5 is

$$p(r) = r^2 + 6r + 5 = (r+1)(r+5).$$

Since $p(-1)=p(-5)=0$, $y_1=e^{-x}$ and $y_2=e^{-5x}$ are solutions of Equation 4.3.5. Since $y_2/y_1=e^{-4x}$ is nonconstant, Theorem 5.1.6 implies that the general solution of Equation 4.3.5 is

$$y=c_1e^{-x}+c_2e^{-5x}. \tag{4.3.7}$$

b. We must determine c_1 and c_2 in Equation 4.3.7 so that y satisfies the initial conditions in Equation 4.3.6. Differentiating Equation 4.3.7 yields

$$y'=-c_1e^{-x}-5c_2e^{-5x}. \tag{4.3.8}$$

Imposing the initial conditions $y(0)=3,\ y'(0)=-1$ in Equation 4.3.7 and Equation 4.3.8 yields

$$\begin{aligned} c_1+\ c_2 &= 3 \\ -c_1-5c_2 &= -1. \end{aligned}$$

The solution of this system is $c_1=7/2, c_2=-1/2$. Therefore the solution of Equation 4.3.6 is

$$y=\frac{7}{2}e^{-x}-\frac{1}{2}e^{-5x}.$$

Figure 5.2.1 is a graph of this solution.

If the characteristic equation has arbitrary distinct real roots r_1 and r_2, then $y_1=e^{r_1x}$ and $y_2=e^{r_2x}$ are solutions of $ay''+by'+cy=0$. Since $y_2/y_1=e^{(r_2-r_1)x}$ is nonconstant, Theorem 5.1.6 implies that $\{y_1,y_2\}$ is a fundamental set of solutions of $ay''+by'+cy=0$.

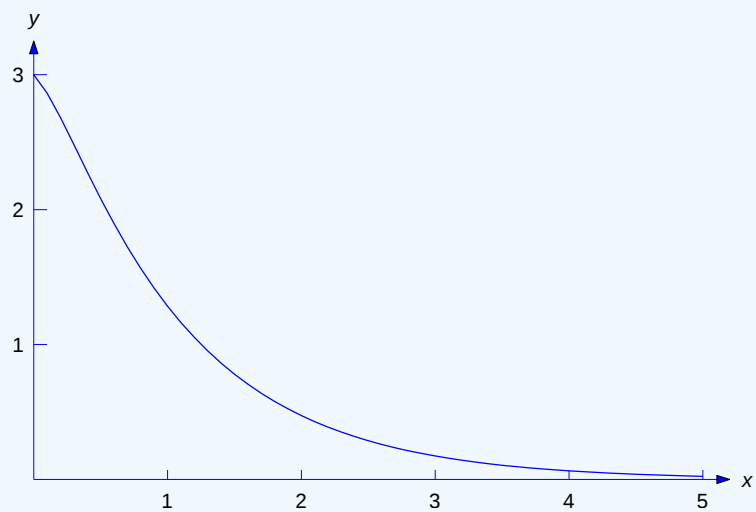

Figure 5.2.1 : $y=\frac{7}{2}e^{-x}-\frac{1}{2}e^{-5x}$

Case 2: A Repeated Real Root

Example 5.2.2

a. Find the general solution of

$$y''+6y'+9y=0. \tag{4.3.9}$$

b. Solve the initial value problem

$$y''+6y'+9y=0,\quad y(0)=3,\ y'(0)=-1. \tag{4.3.10}$$

Solution

a. The characteristic polynomial of Equation 4.3.9 is

$$p(r)=r^2+6r+9=(r+3)^2,$$

so the characteristic equation has the repeated real root $r_1 = -3$. Therefore $y_1 = e^{-3x}$ is a solution of Equation 4.3.9. Since the characteristic equation has no other roots, Equation 4.3.9 has no other solutions of the form e^{rx}. We look for solutions of the form $y = uy_1 = ue^{-3x}$, where u is a function that we'll now determine. (This should remind you of the method of variation of parameters used in Section 2.1 to solve the nonhomogeneous equation $y' + p(x)y = f(x)$, given a solution y_1 of the complementary equation $y' + p(x)y = 0$. It's also a special case of a method called **reduction of order** that we'll study in Section 5.6. For other ways to obtain a second solution of Equation 4.3.9 that's not a multiple of e^{-3x}, see **Exercises 5.1.9, 5.1.12,** and **5.1.33**.

If $y = ue^{-3x}$, then

$$y' = u'e^{-3x} - 3ue^{-3x} \quad \text{and} \quad y'' = u''e^{-3x} - 6u'e^{-3x} + 9ue^{-3x},$$

so

$$\begin{aligned} y'' + 6y' + 9y &= e^{-3x}\left[(u'' - 6u' + 9u) + 6(u' - 3u) + 9u\right] \\ &= e^{-3x}\left[u'' - (6-6)u' + (9 - 18 + 9)u\right] = u''e^{-3x}. \end{aligned}$$

Therefore $y = ue^{-3x}$ is a solution of Equation 4.3.9 if and only if $u'' = 0$, which is equivalent to $u = c_1 + c_2 x$, where c_1 and c_2 are constants. Therefore any function of the form

$$y = e^{-3x}(c_1 + c_2 x) \tag{4.3.11}$$

is a solution of Equation 4.3.9. Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-3x}$ that we already knew. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = xe^{-3x}$. Since $y_2/y_1 = x$ is nonconstant, Theorem 5.1.6 implies that $\{y_1, y_2\}$ is fundamental set of solutions of Equation 4.3.9, and Equation 4.3.11 is the general solution.

b. Differentiating Equation 4.3.11 yields

$$y' = -3e^{-3x}(c_1 + c_2 x) + c_2 e^{-3x}. \tag{4.3.12}$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ in Equation 4.3.11 and Equation 4.3.12 yields $c_1 = 3$ and $-3c_1 + c_2 = -1$, so $c_2 = 8$. Therefore the solution of Equation 4.3.10 is

$$y = e^{-3x}(3 + 8x).$$

Figure 5.2.2 is a graph of this solution.

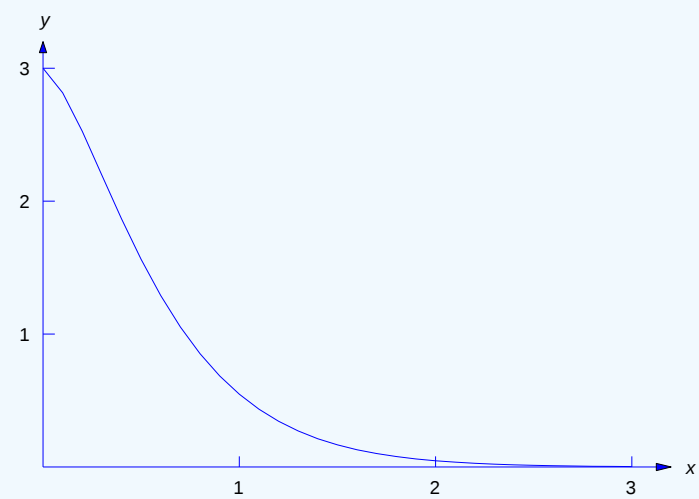

Figure 5.2.2 : $y = e^{-3x}(3 + 8x)$

If the characteristic equation of $ay'' + by' + cy = 0$ has an arbitrary repeated root r_1, the characteristic polynomial must be

$$p(r) = a(r - r_1)^2 = a(r^2 - 2r_1 r + r_1^2).$$

Therefore

$$ar^2 + br + c = ar^2 - (2ar_1)r + ar_1^2,$$

which implies that $b=-2ar_1$ and $c=ar_1^2$. Therefore $ay''+by'+cy=0$ can be written as $a(y''-2r_1y'+r_1^2y)=0$. Since $a\neq 0$ this equation has the same solutions as

$$y''-2r_1y'+r_1^2y=0. \tag{4.3.13}$$

Since $p(r_1)=0$, t $y_1=e^{r_1x}$ is a solution of $ay''+by'+cy=0$, and therefore of Equation 4.3.13. Proceeding as in Example 5.2.1 , we look for other solutions of Equation 4.3.13 of the form $y=ue^{r_1x}$; then

$$y'=u'e^{r_1x}+rue^{r_1x} \quad \text{and} \quad y''=u''e^{r_1x}+2r_1u'e^{r_1x}+r_1^2ue^{r_1x},$$

so

$$\begin{aligned} y''-2r_1y'+r_1^2y &= e^{rx}\left[(u''+2r_1u'+r_1^2u)-2r_1(u'+r_1u)+r_1^2u\right] \\ &= e^{r_1x}\left[u''+(2r_1-2r_1)u'+(r_1^2-2r_1^2+r_1^2)u\right]=u''e^{r_1x}. \end{aligned}$$

Therefore $y=ue^{r_1x}$ is a solution of Equation 4.3.13 if and only if $u''=0$, which is equivalent to $u=c_1+c_2x$, where c_1 and c_2 are constants. Hence, any function of the form

$$y=e^{r_1x}(c_1+c_2x) \tag{4.3.14}$$

is a solution of Equation 4.3.13. Letting $c_1=1$ and $c_2=0$ here yields the solution $y_1=e^{r_1x}$ that we already knew. Letting $c_1=0$ and $c_2=1$ yields the second solution $y_2=xe^{r_1x}$. Since $y_2/y_1=x$ is nonconstant, Theorem 5.1.6 implies that $\{y_1,y_2\}$ is a fundamental set of solutions of Equation 4.3.13, and Equation 4.3.14 is the general solution.

Case 3: Complex Conjugate Roots

Example 5.2.3

a. Find the general solution of

$$y''+4y'+13y=0. \tag{4.3.15}$$

b. Solve the initial value problem

$$y''+4y'+13y=0, \quad y(0)=2,\ y'(0)=-3. \tag{4.3.16}$$

Solution

a. The characteristic polynomial of Equation 4.3.15 is

$$p(r)=r^2+4r+13=r^2+4r+4+9=(r+2)^2+9.$$

The roots of the characteristic equation are $r_1=-2+3i$ and $r_2=-2-3i$. By analogy with Case 1, it is reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions of Equation 4.3.15. This is true (see **Exercise 5.2.34**); however, there are difficulties here, since you are probably not familiar with exponential functions with complex arguments, and even if you are, it is inconvenient to work with them, since they are complex–valued. We'll take a simpler approach, which we motivate as follows: the exponential notation suggests that

$$e^{(-2+3i)x}=e^{-2x}e^{3ix} \quad \text{and} \quad e^{(-2-3i)x}=e^{-2x}e^{-3ix},$$

so even though we haven't defined e^{3ix} and e^{-3ix}, it is reasonable to expect that every linear combination of $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ can be written as $y=ue^{-2x}$, where u depends upon x. To determine u, we note that if $y=ue^{-2x}$ then

$$y'=u'e^{-2x}-2ue^{-2x} \quad \text{and} \quad y''=u''e^{-2x}-4u'e^{-2x}+4ue^{-2x},$$

so

$$\begin{aligned} y''+4y'+13y &= e^{-2x}\left[(u''-4u'+4u)+4(u'-2u)+13u\right] \\ &= e^{-2x}\left[u''-(4-4)u'+(4-8+13)u\right]=e^{-2x}(u''+9u). \end{aligned}$$

Therefore $y=ue^{-2x}$ is a solution of Equation 4.3.15 if and only if

$$u''+9u=0.$$

From Example 5.1.2, the general solution of this equation is

$$u = c_1 \cos 3x + c_2 \sin 3x.$$

Therefore any function of the form

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) \tag{4.3.17}$$

is a solution of Equation 4.3.15. Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-2x} \cos 3x$. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = e^{-2x} \sin 3x$. Since $y_2/y_1 = \tan 3x$ is nonconstant, Theorem 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 4.3.15, and Equation 4.3.17 is the general solution.

b. Imposing the condition $y(0) = 2$ in Equation 4.3.17 shows that $c_1 = 2$. Differentiating Equation 4.3.17 yields

$$y' = -2e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + 3e^{-2x}(-c_1 \sin 3x + c_2 \cos 3x),$$

and imposing the initial condition $y'(0) = -3$ here yields $-3 = -2c_1 + 3c_2 = -4 + 3c_2$, so $c_2 = 1/3$. Therefore the solution of Equation 4.3.16 is

$$y = e^{-2x}(2 \cos 3x + \frac{1}{3} \sin 3x).$$

Figure 5.2.3 is a graph of this function.

Now suppose the characteristic equation of $ay'' + by' + cy = 0$ has arbitrary complex roots; thus, $b^2 - 4ac < 0$ and, from Equation 4.3.4, the roots are

$$r_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a}, \quad r_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a},$$

which we rewrite as

$$r_1 = \lambda + i\omega, \quad r_2 = \lambda - i\omega, \tag{4.3.18}$$

with

$$\lambda = -\frac{b}{2a}, \quad \omega = \frac{\sqrt{4ac - b^2}}{2a}.$$

Don't memorize these formulas. Just remember that r_1 and r_2 are of the form Equation 4.3.18, where λ is an arbitrary real number and ω is positive; λ and ω are the **real** and **imaginary parts**, respectively, of r_1. Similarly, λ and $-\omega$ are the real and imaginary parts of r_2. We say that r_1 and r_2 are **complex conjugates**,

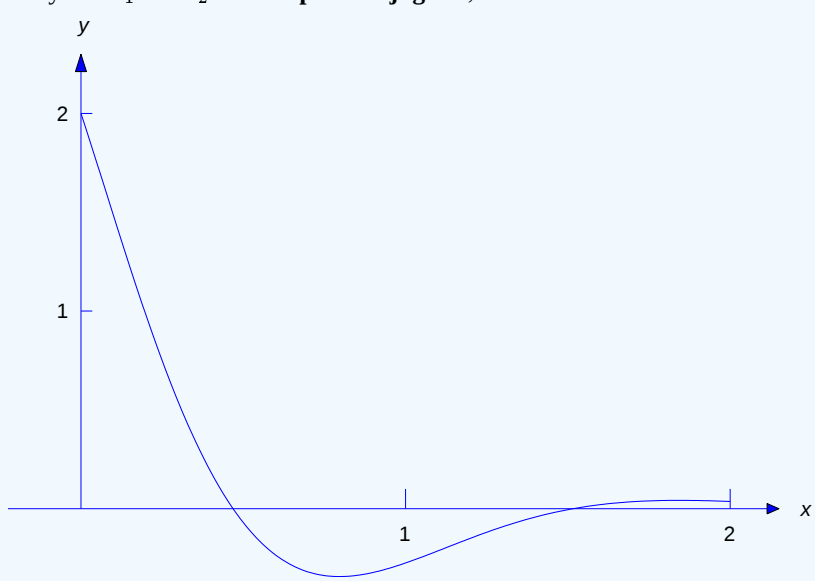

Figure 5.2.3 : $y = e^{-2x}(2 \cos 3x + \frac{1}{3} \sin 3x)$

which means that they have the same real part and their imaginary parts have the same absolute values, but opposite signs.

As in Example 5.2.3 , it is reasonable to to expect that the solutions of $ay''+by'+cy=0$ are linear combinations of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$. Again, the exponential notation suggests that

$$e^{(\lambda+i\omega)x}=e^{\lambda x}e^{i\omega x} \quad \text{and} \quad e^{(\lambda-i\omega)x}=e^{\lambda x}e^{-i\omega x},$$

so even though we haven't defined $e^{i\omega x}$ and $e^{-i\omega x}$, it is reasonable to expect that every linear combination of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$ can be written as $y=ue^{\lambda x}$, where u depends upon x. To determine u we first observe that since $r_1=\lambda+i\omega$ and $r_2=\lambda-i\omega$ are the roots of the characteristic equation, p must be of the form

$$\begin{aligned} p(r) &= a(r-r_1)(r-r_2) \\ &= a(r-\lambda-i\omega)(r-\lambda+i\omega) \\ &= a\left[(r-\lambda)^2+\omega^2\right] \\ &= a(r^2-2\lambda r+\lambda^2+\omega^2). \end{aligned}$$

Therefore $ay''+by'+cy=0$ can be written as

$$a\left[y''-2\lambda y'+(\lambda^2+\omega^2)y\right]=0.$$

Since $a\neq 0$ this equation has the same solutions as

$$y''-2\lambda y'+(\lambda^2+\omega^2)y=0. \tag{4.3.19}$$

To determine u we note that if $y=ue^{\lambda x}$ then

$$y'=u'e^{\lambda x}+\lambda ue^{\lambda x} \quad \text{and} \quad y''=u''e^{\lambda x}+2\lambda u'e^{\lambda x}+\lambda^2 ue^{\lambda x}.$$

Substituting these expressions into Equation 4.3.19 and dropping the common factor $e^{\lambda x}$ yields

$$(u''+2\lambda u'+\lambda^2 u)-2\lambda(u'+\lambda u)+(\lambda^2+\omega^2)u=0,$$

which simplifies to

$$u''+\omega^2 u=0.$$

From Example 5.1.2, the general solution of this equation is

$$u=c_1\cos\omega x+c_2\sin\omega x.$$

Therefore any function of the form

$$y=e^{\lambda x}(c_1\cos\omega x+c_2\sin\omega x) \tag{4.3.20}$$

is a solution of Equation 4.3.19. Letting $c_1=1$ and $c_2=0$ here yields the solution $y_1=e^{\lambda x}\cos\omega x$. Letting $c_1=0$ and $c_2=1$ yields a second solution $y_2=e^{\lambda x}\sin\omega x$. Since $y_2/y_1=\tan\omega x$ is nonconstant, so Theorem 5.1.6 implies that $\{y_1,y_2\}$ is a fundamental set of solutions of Equation 4.3.19, and Equation 4.3.20 is the general solution.

Summary

The next theorem summarizes the results of this section.

Theorem 5.2.1

Let $p(r)=ar^2+br+c$ be the characteristic polynomial of

$$ay''+by'+cy=0. \tag{4.3.21}$$

Then:

a. If $p(r)=0$ has distinct real roots r_1 and r_2, then the general solution of Equation 4.3.21 is

$$y=c_1e^{r_1x}+c_2e^{r_2x}.$$

b. If $p(r)=0$ has a repeated root r_1, then the general solution of Equation 4.3.21 is

$$y=e^{r_1 x}(c_1+c_2 x).$$

c. If $p(r)=0$ has complex conjugate roots $r_1=\lambda+i\omega$ and $r_2=\lambda-i\omega$ (where $\omega>0$), then the general solution of Equation 4.3.21 is

$$y=e^{\lambda x}(c_1\cos\omega x+c_2\sin\omega x).$$

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4.3E: Constant Coefficient Homogeneous Equations (Exercises)

Q4.3.1

In *Exercises 4.3.1-4.3.12* find the general solution.

1. $y''+5y'-6y=0$
2. $y''-4y'+5y=0$
3. $y''+8y'+7y=0$
4. $y''-4y'+4y=0$
5. $y''+2y'+10y=0$
6. $y''+6y'+10y=0$
7. $y''-8y'+16y=0$
8. $y''+y'=0$
9. $y''-2y'+3y=0$
10. $y''+6y'+13y=0$
11. $4y''+4y'+10y=0$
12. $10y''-3y'-y=0$

Q4.3.2

In *Exercises 4.3.13-4.3.17* solve the initial value problem.

13. $y''+14y'+50y=0, \quad y(0)=2, \quad y'(0)=-17$
14. $6y''-y'-y=0, \quad y(0)=10, \quad y'(0)=0$
15. $6y''+y'-y=0, \quad y(0)=-1, \quad y'(0)=3$
16. $4y''-4y'-3y=0, \quad y(0)=\frac{13}{12}, \quad y'(0)=\frac{23}{24}$
17. $4y''-12y'+9y=0, \quad y(0)=3, \quad y'(0)=\frac{5}{2}$

Q4.3.3

In *Exercises 4.3.18-4.3.21* solve the initial value problem and graph the solution.

18. $y''+7y'+12y=0, \quad y(0)=-1, \quad y'(0)=0$
19. $y''-6y'+9y=0, \quad y(0)=0, \quad y'(0)=2$
20. $36y''-12y'+y=0, \quad y(0)=3, \quad y'(0)=\frac{5}{2}$
21. $y''+4y'+10y=0, \quad y(0)=3, \quad y'(0)=-2$
22. $y''+4y=0, \quad y(\pi/12)=1, \quad y'(\pi/12)=0$

Q4.3.4

23.

a. Suppose y is a solution of the constant coefficient homogeneous equation

$$ay''+by'+cy=0. \tag{A}$$

Let $z(x)=y(x-x_0)$, where x_0 is an arbitrary real number. Show that

$$az''+bz'+cz=0.$$

b. Let $z_1(x) = y_1(x - x_0)$ and $z_2(x) = y_2(x - x_0)$, where $\{y_1, y_2\}$ is a fundamental set of solutions of (A). Show that $\{z_1, z_2\}$ is also a fundamental set of solutions of (A).

c. The statement of Theorem 5.2.1 is convenient for solving an initial value problem

$$ay'' + by' + cy = 0, \quad y(0) = k_0, \quad y'(0) = k_1,$$

where the initial conditions are imposed at $x_0 = 0$. However, if the initial value problem is

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1, \tag{B}$$

where $x_0 \neq 0$, then determining the constants in

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad y = e^{r_1 x}(c_1 + c_2 x), \text{ or } y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x)$$

(whichever is applicable) is more complicated. Use (b) to restate Theorem 5.2.1 in a form more convenient for solving (B).

Q4.3.5

In *Exercises 5.2.23-5.2.28* use a method suggested by *Exercise 5.2.22* to solve the initial value problem.

24. $y'' + 3y' + 2y = 0, \quad y(1) = -1, \quad y'(1) = 4$

25. $y'' - 6y' - 7y = 0, \quad y(2) = -\frac{1}{3}, \quad y'(2) = -5$

26. $y'' - 14y' + 49y = 0, \quad y(1) = 2, \quad y'(1) = 11$

27. $9y'' + 6y' + y = 0, \quad y(2) = 2, \quad y'(2) = -\frac{14}{3}$

28. $9y'' + 4y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$

29. $y'' + 3y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -1$

Q4.3.6

30. Prove: If the characteristic equation of

$$ay'' + by' + cy = 0 \tag{A}$$

has a repeated negative root or two roots with negative real parts, then every solution of (A) approaches zero as $x \to \infty$.

31. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has distinct real roots r_1 and r_2. Use a method suggested by *Exercise 5.2.22* to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

32. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has a repeated real root r_1. Use a method suggested by *Exercise 5.2.22* to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

33. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has complex conjugate roots $\lambda \pm i\omega$. Use a method suggested by *Exercise 5.2.22* to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

34. Suppose the characteristic equation of

$$ay'' + by' + cy = 0 \tag{A}$$

has a repeated real root r_1. Temporarily, think of e^{rx} as a function of two real variables x and r.

a. Show that

$$a\frac{\partial^2}{\partial^2 x}(e^{rx}) + b\frac{\partial}{\partial x}(e^{rx}) + ce^{rx} = a(r - r_1)^2 e^{rx}. \tag{B}$$

b. Differentiate (B) with respect to r to obtain

$$a\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial^2 x}(e^{rx})\right)+b\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}(e^{rx})\right)+c(xe^{rx})=[2+(r-r_1)x]a(r-r_1)e^{rx}. \tag{C}$$

c. Reverse the orders of the partial differentiations in the first two terms on the left side of (C) to obtain

$$a\frac{\partial^2}{\partial x^2}(xe^{rx})+b\frac{\partial}{\partial x}(xe^{rx})+c(xe^{rx})=[2+(r-r_1)x]a(r-r_1)e^{rx}. \tag{D}$$

d. Set $r=r_1$ in (B) and (D) to see that $y_1=e^{r_1x}$ and $y_2=xe^{r_1x}$ are solutions of (A)

35. In calculus you learned that e^u, $\cos u$, and $\sin u$ can be represented by the infinite series

$$e^u=\sum_{n=0}^{\infty}\frac{u^n}{n!}=1+\frac{u}{1!}+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots+\frac{u^n}{n!}+\cdots \tag{A}$$

$$\cos u=\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n}}{(2n)!}=1-\frac{u^2}{2!}+\frac{u^4}{4!}+\cdots+(-1)^n\frac{u^{2n}}{(2n)!}+\cdots, \tag{B}$$

and

$$\sin u=\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n+1}}{(2n+1)!}=u-\frac{u^3}{3!}+\frac{u^5}{5!}+\cdots+(-1)^n\frac{u^{2n+1}}{(2n+1)!}+\cdots \tag{C}$$

for all real values of u. Even though you have previously considered (A) only for real values of u, we can set $u=i\theta$, where θ is real, to obtain

$$e^{i\theta}=\sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}. \tag{D}$$

Given the proper background in the theory of infinite series with complex terms, it can be shown that the series in (D) converges for all real θ.

a. Recalling that $i^2=-1$, write enough terms of the sequence $\{i^n\}$ to convince yourself that the sequence is repetitive:

$$1,i,-1,-i,1,i,-1,-i,1,i,-1,-i,1,i,-1,-i,\cdots.$$

Use this to group the terms in (D) as

$$\begin{aligned}e^{i\theta}&=\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{4}+\cdots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\cdots\right)\\&=\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!}+i\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!}.\end{aligned}$$

By comparing this result with (B) and (C), conclude that

$$e^{i\theta}=\cos\theta+i\sin\theta. \tag{E}$$

This is *Euler's identity*.

b. Starting from

$$e^{i\theta_1}e^{i\theta_2}=(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2),$$

collect the real part (the terms not multiplied by i) and the imaginary part (the terms multiplied by i) on the right, and use the trigonometric identities

$$\begin{aligned}\cos(\theta_1+\theta_2)&=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\\\sin(\theta_1+\theta_2)&=\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2\end{aligned}$$

to verify that

$$e^{i(\theta_1+\theta_2)}=e^{i\theta_1}e^{i\theta_2},$$

as you would expect from the use of the exponential notation $e^{i\theta}$.

c. If α and β are real numbers, define

$$e^{\alpha+i\beta}=e^{\alpha}e^{i\beta}=e^{\alpha}(\cos\beta+i\sin\beta). \tag{F}$$

Show that if $z_1=\alpha_1+i\beta_1$ and $z_2=\alpha_2+i\beta_2$ then

$$e^{z_1+z_2}=e^{z_1}e^{z_2}.$$

d. Let a, b, and c be real numbers, with $a\neq 0$. Let $z=u+iv$ where u and v are real-valued functions of x. Then we say that z is a solution of

$$ay''+by'+cy=0 \tag{G}$$

if u and v are both solutions of (G). Use Theorem 5.2.1 (c) to verify that if the characteristic equation of (G) has complex conjugate roots $\lambda\pm i\omega$ then $z_1=e^{(\lambda+i\omega)x}$ and $z_2=e^{(\lambda-i\omega)x}$ are both solutions of (G).

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4.4: Higher Order Constant Coefficient Homogeneous Equations

If a_0, a_1, …, a_n are constants and $a_0 \neq 0$, then

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = F(x)$$

is said to be a **constant coefficient equation**. In this section we consider the homogeneous constant coefficient equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0. \tag{4.4.1}$$

Since Equation 4.4.1 is normal on $(-\infty, \infty)$, the theorems in Section 9.1 all apply with $(a, b) = (-\infty, \infty)$.

As in Section 5.2, we call

$$p(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_n \tag{4.4.2}$$

the **characteristic polynomial** of Equation 4.4.1. We saw in Section 5.2 that when $n = 2$ the solutions of Equation 4.4.1 are determined by the zeros of the characteristic polynomial. This is also true when $n > 2$, but the situation is more complicated in this case. Consequently, we take a different approach.

If k is a positive integer, let D^k stand for the k-th derivative operator; that is

$$D^k y = y^{(k)}.$$

If

$$q(r) = b_0 r^m + b_1 r^{m-1} + \cdots + b_m$$

is an arbitrary polynomial, define the operator

$$q(D) = b_0 D^m + b_1 D^{m-1} + \cdots + b_m$$

such that

$$q(D)y = (b_0 D^m + b_1 D^{m-1} + \cdots + b_m)y = b_0 y^{(m)} + b_1 y^{(m-1)} + \cdots + b_m y$$

whenever y is a function with m derivatives. We call $q(D)$ a **polynomial operator**.

With p as in Equation 4.4.2,

$$p(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_n,$$

so Equation 4.4.1 can be written as $p(D)y = 0$. If r is a constant then

$$\begin{aligned} p(D)e^{rx} &= \left(a_0 D^n e^{rx} + a_1 D^{n-1} e^{rx} + \cdots + a_n e^{rx}\right) \\ &= (a_0 r^n + a_1 r^{n-1} + \cdots + a_n)e^{rx}; \end{aligned}$$

that is

$$p(D)(e^{rx}) = p(r)e^{rx}.$$

This shows that $y = e^{rx}$ is a solution of Equation 4.4.1 if $p(r) = 0$. In the simplest case, where p has n distinct real zeros r_1, r_2,…, r_n, this argument yields n solutions

$$y_1 = e^{r_1 x}, \quad y_2 = e^{r_2 x}, \ldots, \quad y_n = e^{r_n x}.$$

It can be shown (**Exercise 9.2.39**) that the Wronskian of $\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\}$ is nonzero if r_1, r_2, …, r_n are distinct; hence, $\{e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}\}$ is a fundamental set of solutions of $p(D)y = 0$ in this case.

Example 9.2.1

a. Find the general solution of

$$y''' - 6y'' + 11y' - 6y = 0. \tag{4.4.3}$$

b. and solve the initial value problem

$$y''' - 6y'' + 11y' - 6y = 0, \quad y(0) = 4, \quad y'(0) = 5, \quad y''(0) = 9. \tag{4.4.4}$$

Solution a

The characteristic polynomial of Equation 4.4.3 is

$$p(r) = r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3).$$

Therefore $\{e^x, e^{2x}, e^{3x}\}$ is a set of solutions of Equation 4.4.3. It is a fundamental set, since its Wronskian is

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x} \neq 0.$$

Therefore the general solution of Equation 4.4.3 is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}. \tag{4.4.5}$$

Solution b

We must determine c_1, c_2 and c_3 in Equation 4.4.5 so that y satisfies the initial conditions in Equation 4.4.4. Differentiating Equation 4.4.5 twice yields

$$\begin{aligned} y' &= c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x} \\ y'' &= c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}. \end{aligned} \tag{4.4.6}$$

Setting $x = 0$ in Equation 4.4.5 and Equation 4.4.6 and imposing the initial conditions yields

$$\begin{aligned} c_1 + \ \ c_2 + \ \ c_3 &= 4 \\ c_1 + 2c_2 + 3c_3 &= 5 \\ c_1 + 4c_2 + 9c_3 &= 9. \end{aligned}$$

The solution of this system is $c_1 = 4$, $c_2 = -1$, $c_3 = 1$. Therefore the solution of Equation 4.4.4 is

$$y = 4e^x - e^{2x} + e^{3x}$$

(Figure 9.2.1).

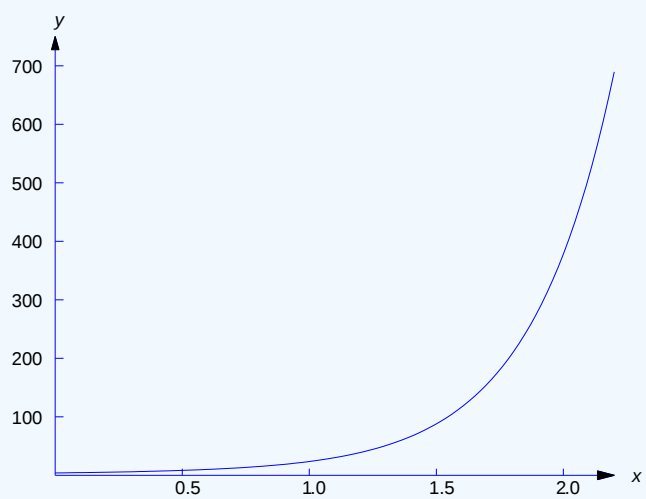

Figure 9.2.1 : $y = 4e^x - e^{2x} + e^{3x}$

Now we consider the case where the characteristic polynomial Equation 4.4.2 does not have n distinct real zeros. For this purpose it is useful to define what we mean by a factorization of a polynomial operator. We begin with an example.

Example 9.2.2

Consider the polynomial

$$p(r) = r^3 - r^2 + r - 1$$

and the associated polynomial operator

$$p(D) = D^3 - D^2 + D - 1.$$

Since $p(r)$ can be factored as

$$p(r) = (r-1)(r^2+1) = (r^2+1)(r-1),$$

it is reasonable to expect that p(D) can be factored as

$$p(D) = (D-1)(D^2+1) = (D^2+1)(D-1). \tag{4.4.7}$$

However, before we can make this assertion we must **define** what we mean by saying that two operators are equal, and what we mean by the products of operators in Equation 4.4.7. We say that two operators are equal if they apply to the same functions and always produce the same result. The definitions of the products in Equation 4.4.7 is this: if y is any three-times differentiable function then

a. $(D-1)(D^2+1)y$ is the function obtained by first applying D^2+1 to y and then applying $D-1$ to the resulting function
b. $(D^2+1)(D-1)y$ is the function obtained by first applying $D-1$ to y and then applying D^2+1 to the resulting function.

From (a):

$$\begin{aligned}(D-1)(D^2+1)y &= (D-1)[(D^2+1)y] \\ &= (D-1)(y''+y) = D(y''+y) - (y''+y) \\ &= (y'''+y') - (y''+y) \\ &= y''' - y'' + y' - y = (D^3 - D^2 + D - 1)y.\end{aligned} \tag{4.4.8}$$

This implies that

$$(D-1)(D^2+1) = (D^3 - D^2 + D - 1).$$

From (b):

$$\begin{aligned}(D^2+1)(D-1)y &= (D^2+1)[(D-1)y] \\ &= (D^2+1)(y'-y) = D^2(y'-y) + (y'-y) \\ &= (y'''-y'') + (y'-y) \\ &= y''' - y'' + y' - y = (D^3 - D^2 + D - 1)y,\end{aligned} \tag{4.4.9}$$

$$(D^2+1)(D-1) = (D^3 - D^2 + D - 1),$$

which completes the justification of Equation 4.4.7.

Example 9.2.3

Use the result of Example 9.2.2 to find the general solution of

$$y''' - y'' + y' - y = 0. \tag{4.4.10}$$

Solution

From Equation 4.4.8, we can rewrite Equation 4.4.10 as

$$(D-1)(D^2+1)y = 0,$$

which implies that any solution of $(D^2+1)y=0$ is a solution of Equation 4.4.10. Therefore $y_1=\cos x$ and $y_2=\sin x$ are solutions of Equation 4.4.10.

From Equation 4.4.9, we can rewrite Equation 4.4.10 as

$$(D^2+1)(D-1)y=0,$$

which implies that any solution of $(D-1)y=0$ is a solution of Equation 4.4.10. Therefore $y_3=e^x$ is solution of Equation 4.4.10.

The Wronskian of $\{e^x,\cos x,\sin x\}$ is

$$W(x)=\begin{vmatrix}\cos x & \sin x & e^x\\ -\sin x & \cos x & e^x\\ -\cos x & -\sin x & e^x\end{vmatrix}.$$

Since

$$W(0)=\begin{vmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & 0 & 1\end{vmatrix}=2,$$

$\{\cos x,\sin x,e^x\}$ is linearly independent and

$$y=c_1\cos x+c_2\sin x+c_3e^x$$

is the general solution of Equation 4.4.10.

Example 9.2.4

Find the general solution of

$$y^{(4)}-16y=0. \tag{4.4.11}$$

Solution

The characteristic polynomial of Equation 4.4.11 is

$$\begin{aligned}p(r)&=r^4-16\\&=(r^2-4)(r^2+4)\\&=(r-2)(r+2)(r^2+4).\end{aligned}$$

By arguments similar to those used in Examples 9.2.2 and 9.2.4 , it can be shown that Equation 4.4.11 can be written as

$$(D^2+4)(D+2)(D-2)y=0$$

or

$$(D^2+4)(D-2)(D+2)y=0$$

or

$$(D-2)(D+2)(D^2+4)y=0.$$

Therefore y is a solution of Equation 4.4.11 if it is a solution of any of the three equations

$$(D-2)y=0,\quad (D+2)y=0,\quad (D^2+4)y=0.$$

Hence, $\{e^{2x},e^{-2x},\cos 2x,\sin 2x\}$ is a set of solutions of Equation 4.4.11. The Wronskian of this set is

$$W(x)=\begin{vmatrix} e^{2x} & e^{-2x} & \cos 2x & \sin 2x \\ 2e^{2x} & -2e^{-2x} & -2\sin 2x & 2\cos 2x \\ 4e^{2x} & 4e^{-2x} & -4\cos 2x & -4\sin 2x \\ 8e^{2x} & -8e^{-2x} & 8\sin 2x & -8\cos 2x \end{vmatrix}.$$

Since

$$W(0)=\begin{vmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 4 & 4 & -4 & 0 \\ 8 & -8 & 0 & -8 \end{vmatrix}=-512,$$

$\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$is linearly independent, and

$$y_1=c_1e^{2x}+c_2e^{-2x}+c_3\cos 2x+c_4\sin 2x$$

is the general solution of Equation 4.4.11.

It is known from algebra that every polynomial

$$p(r)=a_0r^n+a_1r^{n-1}+\cdots+a_n$$

with real coefficients can be factored as

$$p(r)=a_0p_1(r)p_2(r)\cdots p_k(r),$$

where no pair of the polynomials p_1, p_2, …, p_k has a commom factor and each is either of the form

$$p_j(r)=(r-r_j)^{m_j}, \tag{4.4.12}$$

where r_j is real and m_j is a positive integer, or

$$p_j(r)=\left[(r-\lambda_j)^2+\omega_j^2\right]^{m_j}, \tag{4.4.13}$$

where λ_j and ω_j are real, $\omega_j\neq 0$, and m_j is a positive integer. If Equation 4.4.12 holds then r_j is a real zero of p, while if Equation 4.4.13 holds then $\lambda+i\omega$ and $\lambda-i\omega$ are complex conjugate zeros of p. In either case, m_j is the **multiplicity** of the zero(s).

By arguments similar to those used in our examples, it can be shown that

$$p(D)=a_0p_1(D)p_2(D)\cdots p_k(D) \tag{4.4.14}$$

and that the order of the factors on the right can be chosen arbitrarily. Therefore, if $p_j(D)y=0$ for some j then $p(D)y=0$. To see this, we simply rewrite Equation 4.4.14 so that $p_j(D)$ is applied first. Therefore the problem of finding solutions of $p(D)y=0$ with p as in Equation 4.4.14reduces to finding solutions of each of these equations

$$p_j(D)y=0, \quad 1\leq j\leq k,$$

where p_j is a power of a first degree term or of an irreducible quadratic. To find a fundamental set of solutions $\{y_1, y_2, \ldots, y_n\}$ of $p(D)y=0$, we find fundamental set of solutions of each of the equations and take $\{y_1, y_2, \ldots, y_n\}$ to be the set of all functions in these separate fundamental sets. In **Exercise 9.2.40** we sketch the proof that $\{y_1, y_2, \ldots, y_n\}$ is linearly independent, and therefore a fundamental set of solutions of $p(D)y=0$.

To apply this procedure to general homogeneous constant coefficient equations, we must be able to find fundamental sets of solutions of equations of the form

$$(D-a)^my=0$$

and

$$\left[(D-\lambda)^2+\omega^2\right]^m y=0,$$

where m is an arbitrary positive integer. The next two theorems show how to do this.

Theorem 9.2.1

If m is a positive integer, then

$$\{e^{ax}, xe^{ax}, \ldots, x^{m-1}e^{ax}\} \tag{4.4.15}$$

is a fundamental set of solutions of

$$(D-a)^m y=0. \tag{4.4.16}$$

Proof

We'll show that if

$$f(x)=c_1+c_2x+\cdots+c_m x^{m-1}$$

is an arbitrary polynomial of degree $\leq m-1$, then $y=e^{ax}f$ is a solution of Equation 4.4.16. First note that if g is any differentiable function then

$$(D-a)e^{ax}g=De^{ax}g-ae^{ax}g=ae^{ax}g+e^{ax}g'-ae^{ax}g,$$

so

$$(D-a)e^{ax}g=e^{ax}g'. \tag{4.4.17}$$

Therefore

$$\begin{aligned}
(D-a)e^{ax}f &= e^{ax}f' && (\text{from (4.4.17) with } g=f)\\
(D-a)^2e^{ax}f &= (D-a)e^{ax}f'=e^{ax}f'' && (\text{from (4.4.17) with } g=f')\\
(D-a)^3e^{ax}f &= (D-a)e^{ax}f''=e^{ax}f''' && (\text{from (4.4.17) with } g=f'')\\
&\vdots\\
(D-a)^me^{ax}f &= (D-a)e^{ax}f^{(m-1)}=e^{ax}f^{(m)} && (\text{from (4.4.17) with } g=f^{(m-1)}).
\end{aligned}$$

Since $f^{(m)}=0$, the last equation implies that $y=e^{ax}f$ is a solution of Equation 4.4.16 if f is any polynomial of degree $\leq m-1$. In particular, each function in Equation 4.4.15 is a solution of Equation 4.4.16. To see that Equation 4.4.15 is linearly independent (and therefore a fundamental set of solutions of Equation 4.4.16), note that if

$$c_1e^{ax}+c_2xe^{ax}+c\cdots+c_{m-1}x^{m-1}e^{ax}=0$$

for all x in some interval (a,b), then

$$c_1+c_2x+c\cdots+c_{m-1}x^{m-1}=0$$

for all x in (a,b). However, we know from algebra that if this polynomial has more than $m-1$ zeros then $c_1=c_2=\cdots=c_n=0$.

Example 9.2.5

Find the general solution of

$$y'''+3y''+3y'+y=0. \tag{4.4.18}$$

Solution

The characteristic polynomial of Equation 4.4.18 is

$$p(r)=r^3+3r^2+3r+1=(r+1)^3.$$

Therefore Equation 4.4.18 can be written as

$$(D+1)^3 y = 0,$$

so Theorem 9.2.1 implies that the general solution of Equation 4.4.18 is

$$y = e^{-x}(c_1 + c_2 x + c_3 x^2).$$

The proof of the next theorem is sketched in **Exercise 9.2.41**.

Theorem 9.2.2

If $\omega \neq 0$ and m is a positive integer, then

$$\begin{aligned}\{e^{\lambda x}\cos\omega x, xe^{\lambda x}\cos\omega x, \quad &\ldots, x^{m-1}e^{\lambda x}\cos\omega x,\\ e^{\lambda x}\sin\omega x, xe^{\lambda x}\sin\omega x, \quad &\ldots, x^{m-1}e^{\lambda x}\sin\omega x\}\end{aligned}$$

is a fundamental set of solutions of

$$[(D-\lambda)^2 + \omega^2]^m y = 0.$$

Example 9.2.6

Find the general solution of

$$(D^2 + 4D + 13)^3 y = 0. \tag{4.4.19}$$

Solution

The characteristic polynomial of Equation 4.4.19 is

$$p(r) = (r^2 + 4r + 13)^3 = \left((r+2)^2 + 9\right)^3.$$

Therefore Equation 4.4.19 can be be written as

$$[(D+2)^2 + 9]^3 y = 0,$$

so Theorem 9.2.2 implies that the general solution of Equation 4.4.19 is

$$y = (a_1 + a_2 x + a_3 x^2)e^{-2x}\cos 3x + (b_1 + b_2 x + b_3 x^2)e^{-2x}\sin 3x.$$

Example 9.2.7

Find the general solution of

$$y^{(4)} + 4y''' + 6y'' + 4y' = 0. \tag{4.4.20}$$

Solution

The characteristic polynomial of Equation 4.4.20 is

$$\begin{aligned}p(r) &= r^4 + 4r^3 + 6r^2 + 4r\\ &= r(r^3 + 4r^2 + 6r + 4)\\ &= r(r+2)(r^2 + 2r + 2)\\ &= r(r+2)[(r+1)^2 + 1].\end{aligned}$$

Therefore Equation 4.4.20 can be written as

$$[(D+1)^2 + 1](D+2)Dy = 0.$$

Fundamental sets of solutions of

$$\left[(D+1)^2+1\right] y=0, \quad (D+2)y=0, \quad \text{and} \quad Dy=0.$$

are given by

$$\{e^{-x}\cos x, e^{-x}\sin x\}, \quad \{e^{-2x}\}, \quad \text{and} \quad \{1\},$$

respectively. Therefore the general solution of Equation 4.4.20 is

$$y=e^{-x}(c_1\cos x+c_2\sin x)+c_3e^{-2x}+c_4.$$

Example 9.2.8

Find a fundamental set of solutions of

$$[(D+1)^2+1]^2(D-1)^3(D+1)D^2y=0. \tag{4.4.21}$$

Solution

A fundamental set of solutions of Equation 4.4.21 can be obtained by combining fundamental sets of solutions of

- $\left[(D+1)^2+1\right]^2 y=0$
- $(D-1)^3y=0$
- $(D+1)y=0$
- $D^2y=0$

Fundamental sets of solutions of these equations are given by

- $\{e^{-x}\cos x, xe^{-x}\cos x, e^{-x}\sin x, xe^{-x}\sin x\}$
- $\left\{e^x, xe^x, x^2e^x\right\}$
- $\{e^{-x}\}$,
- $\{1, x\}$

respectively. These ten functions form a fundamental set of solutions of Equation 4.4.21.

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4.4E: Higher Order Constant Coefficient Homogeneous Equations (Exercises)

Q4.4.1

In *Exercises 4.4.1-4.4.14* find the general solution.

1. $y''' - 3y'' + 3y' - y = 0$
2. $y^{(4)} + 8y'' - 9y = 0$
3. $y''' - y'' + 16y' - 16y = 0$
4. $2y''' + 3y'' - 2y' - 3y = 0$
5. $y''' + 5y'' + 9y' + 5y = 0$
6. $4y''' - 8y'' + 5y' - y = 0$
7. $27y''' + 27y'' + 9y' + y = 0$
8. $y^{(4)} + y'' = 0$
9. $y^{(4)} - 16y = 0$
10. $y^{(4)} + 12y'' + 36y = 0$
11. $16y^{(4)} - 72y'' + 81y = 0$
12. $6y^{(4)} + 5y''' + 7y'' + 5y' + y = 0$
13. $4y^{(4)} + 12y''' + 3y'' - 13y' - 6y = 0$
14. $y^{(4)} - 4y''' + 7y'' - 6y' + 2y = 0$

Q4.4.2

In *Exercises 4.4.15-4.4.27* solve the initial value problem. Graph the solution for *Exercises 4.4.17-4.4.19* and 4.4.27.

15. $y''' - 2y'' + 4y' - 8y = 0, \quad y(0) = 2, \quad y'(0) = -2, \; y''(0) = 0$
16. $y''' + 3y'' - y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = 14, \quad y''(0) = -40$
17. $y''' - y'' - y' + y = 0, \quad y(0) = -2, \quad y'(0) = 9, \quad y''(0) = 4$
18. $y''' - 2y' - 4y = 0, \quad y(0) = 6, \quad y'(0) = 3, \quad y''(0) = 22$
19. $3y''' - y'' - 7y' + 5y = 0, \quad y(0) = \frac{14}{5}, \quad y'(0) = 0, \quad y''(0) = 10$
20. $y''' - 6y'' + 12y' - 8y = 0, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = -4$
21. $2y''' - 11y'' + 12y' + 9y = 0, \quad y(0) = 6, \quad y'(0) = 3, \quad y''(0) = 13$
22. $8y''' - 4y'' - 2y' + y = 0, \quad y(0) = 4, \quad y'(0) = -3, \quad y''(0) = -1$
23. $y^{(4)} - 16y = 0, \quad y(0) = 2, \; y'(0) = 2, \; y''(0) = -2, \; y'''(0) = 0$
24. $y^{(4)} - 6y''' + 7y'' + 6y' - 8y = 0, \quad y(0) = -2, \quad y'(0) = -8, \quad y''(0) = -14, \quad y'''(0) = -62$
25. $4y^{(4)} - 13y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 3, \quad y''(0) = 1, \quad y'''(0) = 3$
26. $y^{(4)} + 2y''' - 2y'' - 8y' - 8y = 0, \quad y(0) = 5, \quad y'(0) = -2, \quad y''(0) = 6, \quad y'''(0) = 8$
27. $4y^{(4)} + 8y''' + 19y'' + 32y' + 12y = 0, \quad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -\frac{7}{2}, \quad y'''(0) = \frac{31}{4}$

Q4.4.3

28. Find a fundamental set of solutions of the given equation.

   a. $y''' - y = 0$
   b. $y''' + y = 0$
   c. $y^{(4)} + 64y = 0$

d. $y^{(6)} - y = 0$
e. $y^{(6)} + 64y = 0$

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4.5: Nonhomgeneous Linear Equations

We'll now consider the nonhomogeneous linear second order equation

$$y''+p(x)y'+q(x)y=f(x), \tag{4.5.1}$$

where the forcing function f isn't identically zero. The next theorem, an extension of Theorem 5.1.1, gives sufficient conditions for existence and uniqueness of solutions of initial value problems for Equation 4.5.1. We omit the proof, which is beyond the scope of this book.

Theorem 5.3.1 : Uniqueness

Suppose p, , q and f are continuous on an open interval (a,b), let x_0 be any point in (a,b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem

$$y''+p(x)y'+q(x)y=f(x), \quad y(x_0)=k_0, \quad y'(x_0)=k_1$$

has a unique solution on (a,b).

To find the general solution of Equation 4.5.1 on an interval (a,b) where p, q, and f are continuous, it is necessary to find the general solution of the associated homogeneous equation

$$y''+p(x)y'+q(x)y=0 \tag{4.5.2}$$

on (a,b). We call Equation 4.5.2 the **complementary equation** for Equation 4.5.1.

The next theorem shows how to find the general solution of Equation 4.5.1 if we know one solution y_p of Equation 4.5.1 and a fundamental set of solutions of Equation 4.5.2. We call y_p a **particular solution** of Equation 4.5.1 ; it can be any solution that we can find, one way or another.

Theorem 5.3.2

Suppose p, q, and f are continuous on (a,b). Let y_p be a particular solution of

$$y''+p(x)y'+q(x)y=f(x) \tag{4.5.3}$$

on (a,b), and let $\{y_1,y_2\}$ be a fundamental set of solutions of the complementary equation

$$y''+p(x)y'+q(x)y=0 \tag{4.5.4}$$

on (a,b). Then y is a solution of (4.5.3) on (a,b) if and only if

$$y=y_p+c_1y_1+c_2y_2, \tag{4.5.5}$$

where c_1 and c_2 are constants.

Proof

We first show that y in Equation 4.5.5 is a solution of Equation 4.5.3 for any choice of the constants c_1 and c_2. Differentiating Equation 4.5.5twice yields

$$y'=y_p'+c_1y_1'+c_2y_2' \quad \text{and} \quad y''=y_p''+c_1y_1''+c_2y_2'',$$

so

$$\begin{aligned} y''+p(x)y'+q(x)y &= (y_p''+c_1y_1''+c_2y_2'')+p(x)(y_p'+c_1y_1'+c_2y_2')+q(x)(y_p+c_1y_1+c_2y_2) \\ &= (y_p''+p(x)y_p'+q(x)y_p)+c_1(y_1''+p(x)y_1'+q(x)y_1)+c_2(y_2''+p(x)y_2'+q(x)y_2) \\ &= f+c_1\cdot 0+c_2\cdot 0=f, \end{aligned}$$

since y_p satisfies Equation 4.5.3and y_1 and y_2 satisfy Equation 4.5.4

Now we'll show that every solution of Equation 4.5.3has the form Equation 4.5.5for some choice of the constants c_1 and c_2. Suppose y is a solution of Equation 4.5.3 We'll show that $y-y_p$ is a solution of Equation 4.5.4, and therefore of the form $y-y_p=c_1y_1+c_2y_2$, which implies Equation 4.5.5 To see this, we compute

$$\begin{aligned}(y-y_p)''+p(x)(y-y_p)'+q(x)(y-y_p) &= (y''-y_p'')+p(x)(y'-y_p')+q(x)(y-y_p)\\ &= (y''+p(x)y'+q(x)y)-(y_p''+p(x)y_p'+q(x)y_p)\\ &= f(x)-f(x)=0,\end{aligned}$$

since y and y_p both satisfy Equation 4.5.3

We say that Equation 4.5.5is the **general solution of** (4.5.3) on (a,b).

If P_0, P_1, and F are continuous and P_0 has no zeros on (a,b), then Theorem 5.3.2 implies that the general solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F(x) \tag{4.5.6}$$

on (a,b) is $y=y_p+c_1y_1+c_2y_2$, where y_p is a particular solution of Equation 4.5.6 on (a,b) and $\{y_1,y_2\}$ is a fundamental set of solutions of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0$$

on (a,b). To see this, we rewrite Equation 4.5.6 as

$$y''+\frac{P_1(x)}{P_0(x)}y'+\frac{P_2(x)}{P_0(x)}y=\frac{F(x)}{P_0(x)}$$

and apply Theorem 5.3.2 with $p=P_1/P_0$, $q=P_2/P_0$, and $f=F/P_0$.

To avoid awkward wording in examples and exercises, we will not specify the interval (a,b) when we ask for the general solution of a specific linear second order equation, or for a fundamental set of solutions of a homogeneous linear second order equation. Let's agree that this always means that we want the general solution (or a fundamental set of solutions, as the case may be) on every open interval on which p, q, and f are continuous if the equation is of the form Equation 4.5.3, or on which P_0, P_1, P_2, and F are continuous and P_0 has no zeros, if the equation is of the form Equation 4.5.6. We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0, P_1, P_2, and F are all continuous on an open interval (a,b), but P_0 **does** have a zero in (a,b), then Equation 4.5.6 may fail to have a general solution on (a,b) in the sense just defined. **Exercises 5.1.42-5.1.44** illustrate this point for a homogeneous equation.

In this section we to limit ourselves to applications of Theorem 5.3.2 where we can guess at the form of the particular solution.

Example 5.3.1

a. Find the general solution of

$$y''+y=1. \tag{4.5.7}$$

b. Solve the initial value problem

$$y''+y=1,\quad y(0)=2,\quad y'(0)=7. \tag{4.5.8}$$

Solution

a. We can apply Theorem 5.3.2 with $(a,b)=(-\infty,\infty)$, since the functions $p\equiv 0$, $q\equiv 1$, and $f\equiv 1$ in Equation 4.5.7 are continuous on $(-\infty,\infty)$. By inspection we see that $y_p\equiv 1$ is a particular solution of Equation 4.5.7. Since $y_1=\cos x$ and $y_2=\sin x$ form a fundamental set of solutions of the complementary equation $y''+y=0$, the general solution of Equation 4.5.7 is

$$y=1+c_1\cos x+c_2\sin x. \tag{4.5.9}$$

b. Imposing the initial condition $y(0)=2$ in Equation 4.5.9 yields $2=1+c_1$, so $c_1=1$. Differentiating Equation 4.5.9 yields

$$y' = -c_1 \sin x + c_2 \cos x.$$

Imposing the initial condition $y'(0)=7$ here yields $c_2=7$, so the solution of Equation 4.5.8 is

$$y = 1 + \cos x + 7 \sin x.$$

Figure 5.3.1 is a graph of this function.

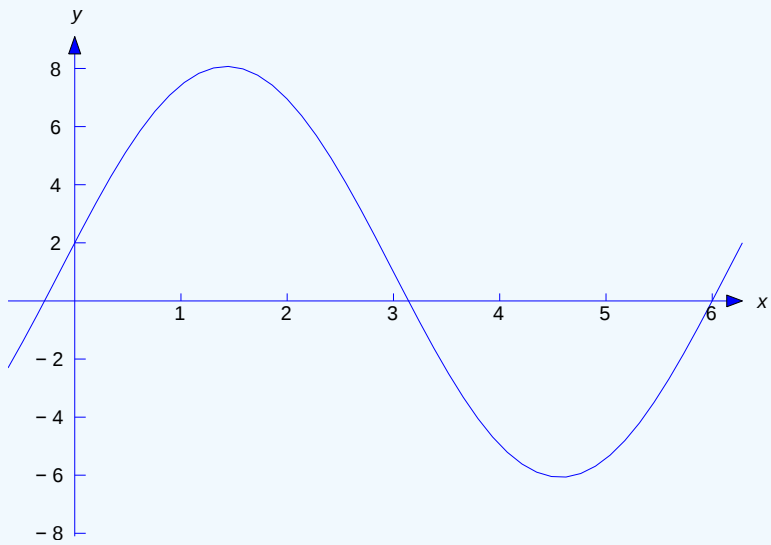

Figure 5.3.1 : $y = 1 + \cos x + 7 \sin x$

Example 5.3.2

a. Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \tag{4.5.10}$$

b. Solve the initial value problem

$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2, \quad y'(0) = 1. \tag{4.5.11}$$

Solution

a. The characteristic polynomial of the complementary equation

$$y'' - 2y' + y = 0$$

is $r^2 - 2r + 1 = (r-1)^2$, so $y_1 = e^x$ and $y_2 = xe^x$ form a fundamental set of solutions of the complementary equation. To guess a form for a particular solution of Equation 4.5.10, we note that substituting a second degree polynomial $y_p = A + Bx + Cx^2$ into the left side of Equation 4.5.10 will produce another second degree polynomial with coefficients that depend upon A, B, and C. The trick is to choose A, B, and C so the polynomials on the two sides of Equation 4.5.10 have the same coefficients; thus, if

$$y_p = A + Bx + Cx^2 \quad \text{then} \quad y_p' = B + 2Cx \quad \text{and} \quad y_p'' = 2C,$$

so

$$\begin{aligned} y_p'' - 2y_p' + y_p &= 2C - 2(B + 2Cx) + (A + Bx + Cx^2) \\ &= (2C - 2B + A) + (-4C + B)x + Cx^2 = -3 - x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} C &= 1 \\ B-4C &= -1 \\ A-2B+2C &= -3, \end{aligned}$$

so $C=1$, $B=-1+4C=3$, and $A=-3-2C+2B=1$. Therefore $y_p=1+3x+x^2$ is a particular solution of Equation 4.5.10 and Theorem 5.3.2 implies that

$$y=1+3x+x^2+e^x(c_1+c_2x) \tag{4.5.12}$$

is the general solution of Equation 4.5.10.

b. Imposing the initial condition $y(0)=-2$ in Equation 4.5.12 yields $-2=1+c_1$, so $c_1=-3$. Differentiating Equation 4.5.12 yields

$$y'=3+2x+e^x(c_1+c_2x)+c_2e^x,$$

and imposing the initial condition $y'(0)=1$ here yields $1=3+c_1+c_2$, so $c_2=1$. Therefore the solution of Equation 4.5.11 is

$$y=1+3x+x^2-e^x(3-x).$$

Figure 5.3.2 is a graph of this solution.

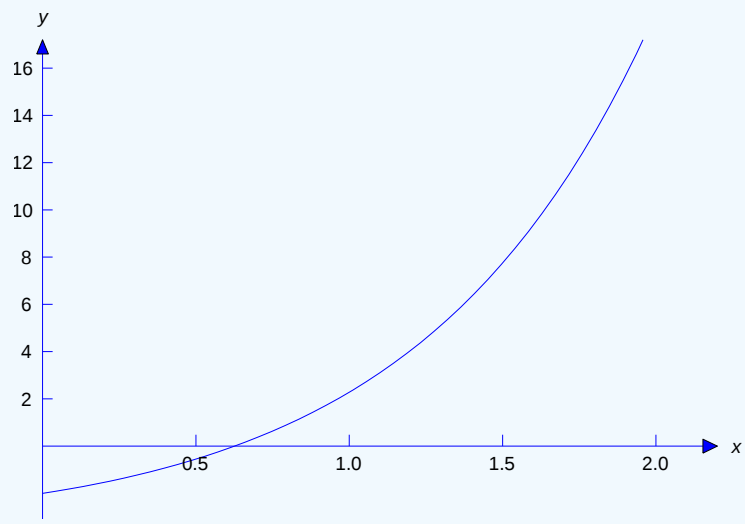

Figure 5.3.2 : $y=1+3x+x^2-e^x(3-x)$

Example 5.3.3

Find the general solution of

$$x^2y''+xy'-4y=2x^4 \tag{4.5.13}$$

on $(-\infty,0)$ and $(0,\infty)$.

Solution

In Example 5.1.3, we verified that $y_1=x^2$ and $y_2=1/x^2$ form a fundamental set of solutions of the complementary equation

$$x^2y''+xy'-4y=0$$

on $(-\infty,0)$ and $(0,\infty)$. To find a particular solution of Equation 4.5.13, we note that if $y_p=Ax^4$, where A is a constant then both sides of Equation 4.5.13 will be constant multiples of x^4 and we may be able to choose A so the two sides are equal. This is true in this example, since if $y_p=Ax^4$ then

$$x^2y_p''+xy_p'-4y_p=x^2(12Ax^2)+x(4Ax^3)-4Ax^4=12Ax^4=2x^4$$

if $A=1/6$; therefore, $y_p=x^4/6$ is a particular solution of Equation 4.5.13 on $(-\infty,\infty)$. Theorem 5.3.2 implies that the general solution of Equation 4.5.13 on $(-\infty,0)$ and $(0,\infty)$ is

$$y=\frac{x^4}{6}+c_1x^2+\frac{c_2}{x^2}.$$

The Principle of Superposition

The next theorem enables us to break a nonhomogeous equation into simpler parts, find a particular solution for each part, and then combine their solutions to obtain a particular solution of the original problem.

Theorem 5.3.3 the principle of superposition

Suppose y_{p_1} is a particular solution of

$$y''+p(x)y'+q(x)y=f_1(x)$$

on (a,b) and y_{p_2} is a particular solution of

$$y''+p(x)y'+q(x)y=f_2(x)$$

on (a,b). Then

$$y_p=y_{p_1}+y_{p_2}$$

is a particular solution of

$$y''+p(x)y'+q(x)y=f_1(x)+f_2(x)$$

on (a,b).

Proof

If $y_p=y_{p_1}+y_{p_2}$ then

$$\begin{aligned} y_p''+p(x)y_p'+q(x)y_p &= (y_{p_1}+y_{p_2})''+p(x)(y_{p_1}+y_{p_2})'+q(x)(y_{p_1}+y_{p_2}) \\ &= \left(y_{p_1}''+p(x)y_{p_1}'+q(x)y_{p_1}\right)+\left(y_{p_2}''+p(x)y_{p_2}'+q(x)y_{p_2}\right) \\ &= f_1(x)+f_2(x). \end{aligned}$$

It's easy to generalize Theorem 5.3.3 to the equation

$$y''+p(x)y'+q(x)y=f(x) \tag{4.5.14}$$

where

$$f=f_1+f_2+\cdots+f_k;$$

thus, if y_{p_i} is a particular solution of

$$y''+p(x)y'+q(x)y=f_i(x)$$

on (a,b) for $i=1,2,\ldots,k$, then $y_{p_1}+y_{p_2}+\cdots+y_{p_k}$ is a particular solution of Equation 4.5.14 on (a,b). Moreover, by a proof similar to the proof of Theorem 5.3.3 we can formulate the principle of superposition in terms of a linear equation written in the form

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F(x)$$

that is, if y_{p_1} is a particular solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F_1(x)$$

on (a,b) and y_{p_2} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_2(x)$$

on (a, b), then $y_{p_1} + y_{p_2}$ is a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x) + F_2(x)$$

on (a, b).

Example 5.3.4

The function $y_{p_1} = x^4/15$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 \tag{4.5.15}$$

on $(-\infty, \infty)$ and $y_{p_2} = x^2/3$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 4x^2 \tag{4.5.16}$$

on $(-\infty, \infty)$. Use the principle of superposition to find a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 + 4x^2 \tag{4.5.17}$$

on $(-\infty, \infty)$.

Solution

The right side $F(x) = 2x^4 + 4x^2$ in Equation 4.5.17 is the sum of the right sides

$$F_1(x) = 2x^4 \quad \text{and} \quad F_2(x) = 4x^2.$$

in Equation 4.5.15 and Equation 4.5.16. Therefore the principle of superposition implies that

$$y_p = y_{p_1} + y_{p_2} = \frac{x^4}{15} + \frac{x^2}{3}$$

is a particular solution of Equation 4.5.17.

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4.5E: Nonhomgeneous Linear Equations (Exercises)

Q4.5.1

In *Exercises 4.5.1-4.5.12* find a particular solution. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

1. $y''+5y'-6y=22+18x-18x^2$
2. $y''-4y'+5y=1+5x$
3. $y''+8y'+7y=-8-x+24x^2+7x^3$
4. $y''-4y'+4y=2+8x-4x^2$
5. $y''+2y'+10y=4+26x+6x^2+10x^3, \quad y(0)=2, \quad y'(0)=9$
6. $y''+6y'+10y=22+20x, \quad y(0)=2,\ y'(0)=-2$
7. $y''+5y'-6y=6e^{3x}$
8. $y''-4y'+5y=e^{2x}$
9. $y''+8y'+7y=10e^{-2x}, \quad y(0)=-2,\ y'(0)=10$
10. $y''-4y'+4y=e^{x}, \quad y(0)=2, \quad y'(0)=0$
11. $y''+2y'+10y=e^{x/2}$
12. $y''+6y'+10y=e^{-3x}$

Q4.5.2

13. Show that

$$y''+y'=1+2x+x^2; \tag{A}$$

will not yield a particular solution of the form $y_p=A+Bx+Cx^2$, where A, B, and C are constants.

14. Show that

$$y''-7y'+12y=5e^{4x}; \tag{A}$$

will not yield a particular solution of the form $y_p=Ae^{4x}$.

15. Prove: If α and M are constants and $M\neq 0$ then constant coefficient equation

$$ay''+by'+cy=Me^{\alpha x} \tag{4.5E.1}$$

has a particular solution $y_p=Ae^{\alpha x}$ ($A=$ constant) if and only if $e^{\alpha x}$ isn't a solution of the complementary equation.

Q4.5.3

In *Exercises 4.5.16-4.5.21* find a particular solution. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

16. $y''-8y'+16y=23\cos x-7\sin x$
17. $y''+y'=-8\cos 2x+6\sin 2x$
18. $y''-2y'+3y=-6\cos 3x+6\sin 3x$
19. $y''+6y'+13y=18\cos x+6\sin x$
20. $y''+7y'+12y=-2\cos 2x+36\sin 2x, \quad y(0)=-3, \quad y'(0)=3$
21. $y''-6y'+9y=18\cos 3x+18\sin 3x, \quad y(0)=2, \quad y'(0)=2$

Q4.5.4

In *Exercises 4.5.22-5.3.27* use the principal of superposition to find a particular solution. Then find the general solution.

22. $y''+5y'-6y=22+18x-18x^2+6e^{3x}$

23. $y''-4y'+5y=1+5x+e^{2x}$

24. $y''+8y'+7y=-8-x+24x^2+7x^3+10e^{-2x}$

25. $y''-4y'+4y=2+8x-4x^2+e^x$

26. $y''+2y'+10y=4+26x+6x^2+10x^3+e^{x/2}$

27. $y''+6y'+10y=22+20x+e^{-3x}$

Q4.5.5

28. Prove: If y_{p_1} is a particular solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F_1(x) \tag{4.5E.2}$$

on (a,b) and y_{p_2} is a particular solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F_2(x) \tag{4.5E.3}$$

on (a,b), then $y_p=y_{p_1}+y_{p_2}$ is a solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F_1(x)+F_2(x) \tag{4.5E.4}$$

on (a,b).

29. Suppose p, q, and f are continuous on (a,b). Let y_1, y_2, and y_p be twice differentiable on (a,b), such that $y=c_1y_1+c_2y_2+y_p$ is a solution of

$$y''+p(x)y'+q(x)y=f \tag{4.5E.5}$$

on (a,b) for every choice of the constants c_1,c_2. Show that y_1 and y_2 are solutions of the complementary equation on (a,b).

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4.6: The Method of Undetermined Coefficients I

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x} G(x), \tag{4.6.1}$$

where α is a constant and G is a polynomial.

From Theorem 5.3.2, the general solution of Equation 4.6.1 is $y = y_p + c_1 y_1 + c_2 y_2$, where y_p is a particular solution of Equation 4.6.1 and $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation

$$ay'' + by' + cy = 0.$$

In Section 5.2 we showed how to find $\{y_1, y_2\}$. In this section we'll show how to find y_p. The procedure that we'll use is called **the method of undetermined coefficients**. Our first example is similar to **Exercises 5.3.16-5.3.21**.

Example 5.4.1

Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{4.6.2}$$

Then find the general solution.

Solution

Substituting $y_p = Ae^{2x}$ for y in Equation 4.6.2 will produce a constant multiple of Ae^{2x} on the left side of Equation 4.6.2, so it may be possible to choose A so that y_p is a solution of Equation 4.6.2. Let's try it; if $y_p = Ae^{2x}$ then

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 2Ae^{2x} = 4e^{2x}$$

if $A = 2$. Therefore $y_p = 2e^{2x}$ is a particular solution of Equation 4.6.2. To find the general solution, we note that the characteristic polynomial of the complementary equation

$$y'' - 7y' + 12y = 0 \tag{4.6.3}$$

is $p(r) = r^2 - 7r + 12 = (r-3)(r-4)$, so $\{e^{3x}, e^{4x}\}$ is a fundamental set of solutions of Equation 4.6.3. Therefore the general solution of Equation 4.6.2 is

$$y = 2e^{2x} + c_1 e^{3x} + c_2 e^{4x}.$$

Example 5.4.2

Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{4.6.4}$$

Then find the general solution.

Solution

Fresh from our success in finding a particular solution of Equation 4.6.2 — where we chose $y_p = Ae^{2x}$ because the right side of Equation 4.6.2 is a constant multiple of e^{2x} — it may seem reasonable to try $y_p = Ae^{4x}$ as a particular solution of Equation 4.6.4. However, this will not work, since we saw in Example 5.4.1 that e^{4x} is a solution of the complementary equation Equation 4.6.3, so substituting $y_p = Ae^{4x}$ into the left side of Equation 4.6.4) produces zero on the left, no matter how we choose$A$. To discover a suitable form for y_p, we use the same approach that we used in Section 5.2 to find a second solution of

$$ay'' + by' + cy = 0$$

in the case where the characteristic equation has a repeated real root: we look for solutions of Equation 4.6.4 in the form $y = ue^{4x}$, where u is a function to be determined. Substituting

$$y = ue^{4x}, \quad y' = u'e^{4x} + 4ue^{4x}, \quad \text{and} \quad y'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x} \tag{4.6.5}$$

into Equation 4.6.4 and canceling the common factor e^{4x} yields

$$(u''+8u'+16u)-7(u'+4u)+12u=5,$$

or

$$u''+u'=5.$$

By inspection we see that $u_p=5x$ is a particular solution of this equation, so \(y_p=5xe^{4x}\) is a particular solution of Equation 4.6.4. Therefore

$$y=5xe^{4x}+c_1e^{3x}+c_2e^{4x}$$

is the general solution.

Example 5.4.3

Find a particular solution of

$$y''-8y'+16y=2e^{4x}. \tag{4.6.6}$$

Solution

Since the characteristic polynomial of the complementary equation

$$y''-8y'+16y=0 \tag{4.6.7}$$

is $p(r)=r^2-8r+16=(r-4)^2$, both $y_1=e^{4x}$ and $y_2=xe^{4x}$ are solutions of Equation 4.6.7. Therefore Equation 4.6.6) does not have a solution of the form $y_p=Ae^{4x}$ or $y_p=Axe^{4x}$. As in Example 5.4.2 , we look for solutions of Equation 4.6.6 in the form $y=ue^{4x}$, where u is a function to be determined. Substituting from Equation 4.6.5 into Equation 4.6.6 and canceling the common factor e^{4x} yields

$$(u''+8u'+16u)-8(u'+4u)+16u=2,$$

or

$$u''=2.$$

Integrating twice and taking the constants of integration to be zero shows that $u_p=x^2$ is a particular solution of this equation, so $y_p=x^2e^{4x}$ is a particular solution of Equation 4.6.4 Therefore

$$y=e^{4x}(x^2+c_1+c_2x)$$

is the general solution.

The preceding examples illustrate the following facts concerning the form of a particular solution y_p of a constant coefficent equation

$$ay''+by'+cy=ke^{\alpha x},$$

where k is a nonzero constant:

a. If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay''+by'+cy=0, \tag{4.6.8}$$

then $y_p=Ae^{\alpha x}$, where A is a constant. (See Example 5.4.1).

b. If $e^{\alpha x}$ is a solution of Equation 4.6.8 but $xe^{\alpha x}$ is not, then $y_p=Axe^{\alpha x}$, where A is a constant. (See Example 5.4.2 .)

c. If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of Equation 4.6.8, then $y_p=Ax^2e^{\alpha x}$, where A is a constant. (See Example 5.4.3 .)

See **Exercise 5.4.30** for the proofs of these facts.

In all three cases you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p''+by_p'+cy_p=ke^{\alpha x},$$

and solve for the constant A, as we did in Example 5.4.1 . (See **Exercises 5.4.31-5.4.33**.) However, if the equation is

$$ay'' + by' + cy = ke^{\alpha x} G(x),$$

where G is a polynomial of degree greater than zero, we recommend that you use the substitution $y = ue^{\alpha x}$ as we did in Examples 5.4.2 and 5.4.3 . The equation for u will turn out to be

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x), \tag{4.6.9}$$

where $p(r) = ar^2 + br + c$ is the characteristic polynomial of the complementary equation and $p'(r) = 2ar + b$ (**Exercise 5.4.30**); however, you shouldn't memorize this since it is easy to derive the equation for u in any particular case. Note, however, that if $e^{\alpha x}$ is a solution of the complementary equation then $p(\alpha) = 0$, so Equation 4.6.9 reduces to

$$au'' + p'(\alpha)u' = G(x),$$

while if both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation then $p(r) = a(r-\alpha)^2$ and $p'(r) = 2a(r-\alpha)$, so $p(\alpha) = p'(\alpha) = 0$ and Equation 4.6.9) reduces to

$$au'' = G(x).$$

Example 5.4.4

Find a particular solution of

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2). \tag{4.6.10}$$

Solution

Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and} y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into Equation 4.6.10) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 3(u' + 3u) + 2u = -1 + 2x + x^2,$$

or

$$u'' + 3u' + 2u = -1 + 2x + x^2. \tag{4.6.11}$$

As in Example 5.3.2, in order to guess a form for a particular solution of Equation 4.6.11), we note that substituting a second degree polynomial $u_p = A + Bx + Cx^2$ for u in the left side of Equation 4.6.11) produces another second degree polynomial with coefficients that depend upon A, B, and C; thus,

$$\text{if} \quad u_p = A + Bx + Cx^2 \quad \text{then} \quad u_p' = B + 2Cx \quad \text{and} \quad u_p'' = 2C.$$

If u_p is to satisfy Equation 4.6.11), we must have

$$\begin{aligned} u_p'' + 3u_p' + 2u_p &= 2C + 3(B + 2Cx) + 2(A + Bx + Cx^2) \\ &= (2C + 3B + 2A) + (6C + 2B)x + 2Cx^2 = -1 + 2x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} 2C &= 1 \\ 2B + 6C &= 2 \\ 2A + 3B + 2C &= -1. \end{aligned}$$

Solving these equations for C, B, and A (in that order) yields $C = 1/2$, $B = -1/2$, $A = -1/4$. Therefore

$$u_p = -\frac{1}{4}(1 + 2x - 2x^2)$$

is a particular solution of Equation 4.6.11, and

$$y_p = u_p e^{3x} = -\frac{e^{3x}}{4}(1+2x-2x^2)$$

is a particular solution of Equation 4.6.10.

Example 5.4.5

Find a particular solution of

$$y'' - 4y' + 3y = e^{3x}(6+8x+12x^2). \tag{4.6.12}$$

Solution

Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and } y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into Equation 4.6.12) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 4(u' + 3u) + 3u = 6 + 8x + 12x^2,$$

or

$$u'' + 2u' = 6 + 8x + 12x^2. \tag{4.6.13}$$

There's no u term in this equation, since e^{3x} is a solution of the complementary equation for Equation 4.6.12). (See **Exercise 5.4.30**.) Therefore Equation 4.6.13) does not have a particular solution of the form $u_p = A + Bx + Cx^2$ that we used successfully in Example 5.4.4 , since with this choice of u_p,

$$u_p'' + 2u_p' = 2C + (B + 2Cx)$$

can't contain the last term ($12x^2$) on the right side of Equation 4.6.13). Instead, let's try $u_p = Ax + Bx^2 + Cx^3$ on the grounds that

$$u_p' = A + 2Bx + 3Cx^2 \quad \text{and} \quad u_p'' = 2B + 6Cx$$

together contain all the powers of x that appear on the right side of Equation 4.6.13).

Substituting these expressions in place of u' and u'' in Equation 4.6.13) yields

$$(2B + 6Cx) + 2(A + 2Bx + 3Cx^2) = (2B + 2A) + (6C + 4B)x + 6Cx^2 = 6 + 8x + 12x^2.$$

Comparing coefficients of like powers of x on the two sides of the last equality shows that u_p satisfies Equation 4.6.13) if

$$\begin{aligned} 6C &= 12 \\ 4B + 6C &= 8 \\ 2A + 2B &= 6. \end{aligned}$$

Solving these equations successively yields $C = 2$, $B = -1$, and $A = 4$. Therefore

$$u_p = x(4 - x + 2x^2)$$

is a particular solution of Equation 4.6.13), and

$$y_p = u_p e^{3x} = xe^{3x}(4 - x + 2x^2)$$

is a particular solution of Equation 4.6.12).

Example 5.4.6

Find a particular solution of

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2). \tag{4.6.14}$$

Solution

Substituting

$$y = ue^{-x/2}, \quad y' = u'e^{-x/2} - \frac{1}{2}ue^{-x/2}, \quad \text{and} \quad y'' = u''e^{-x/2} - u'e^{-x/2} + \frac{1}{4}ue^{-x/2}$$

into Equation 4.6.14) and canceling $e^{-x/2}$ yields

$$4\left(u'' - u' + \frac{u}{4}\right) + 4\left(u' - \frac{u}{2}\right) + u = 4u'' = -8 + 48x + 144x^2,$$

or

$$u'' = -2 + 12x + 36x^2, \tag{4.6.15}$$

which does not contain u or u' because $e^{-x/2}$ and $xe^{-x/2}$ are both solutions of the complementary equation. (See **Exercise 5.4.30**.) To obtain a particular solution of Equation 4.6.15) we integrate twice, taking the constants of integration to be zero; thus,

$$u_p' = -2x + 6x^2 + 12x^3 \quad \text{and} \quad u_p = -x^2 + 2x^3 + 3x^4 = x^2(-1 + 2x + 3x^2).$$

Therefore

$$y_p = u_p e^{-x/2} = x^2 e^{-x/2}(-1 + 2x + 3x^2)$$

is a particular solution of Equation 4.6.14).

Summary

The preceding examples illustrate the following facts concerning particular solutions of a constant coefficent equation of the form

$$ay'' + by' + cy = e^{\alpha x}G(x),$$

where G is a polynomial (see **Exercise 5.4.30**):

a. If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay'' + by' + cy = 0, \tag{4.6.16}$$

then $y_p = e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 5.4.4).

b. If $e^{\alpha x}$ is a solution of Equation 4.6.16but $xe^{\alpha x}$ is not, then $y_p = xe^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 5.4.5 .)

c. If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of Equation 4.6.16, then $y_p = x^2e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 5.4.6 .)

In all three cases, you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p'' + by_p' + cy_p = e^{\alpha x}G(x),$$

and solve for the coefficients of the polynomial Q. However, if you try this you will see that the computations are more tedious than those that you encounter by making the substitution $y = ue^{\alpha x}$ and finding a particular solution of the resulting equation for u. (See **Exercises 5.4.34-5.4.36**.) In Case (a) the equation for u will be of the form

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x),$$

with a particular solution of the form $u_p = Q(x)$, a polynomial of the same degree as G, whose coefficients can be found by the method used in Example 5.4.4 . In Case (b) the equation for u will be of the form

$$au'' + p'(\alpha)u' = G(x)$$

(no u term on the left), with a particular solution of the form $u_p = xQ(x)$, where Q is a polynomial of the same degree as G whose coefficents can be found by the method used in Example 5.4.5 . In Case (c), the equation for u will be of the form

$$au'' = G(x)$$

with a particular solution of the form $u_p = x^2Q(x)$ that can be obtained by integrating $G(x)/a$ twice and taking the constants of integration to be zero, as in Example 5.4.6 .

Using the Principle of Superposition

The next example shows how to combine the method of undetermined coefficients and Theorem 5.3.3, the principle of superposition.

Example 5.4.7

Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}. \tag{4.6.17}$$

Solution

In Example 5.4.1 we found that $y_{p_1} = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x},$$

and in Example 5.4.2 we found that $y_{p_2} = 5xe^{4x}$ is a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

Therefore the principle of superposition implies that $y_p = 2e^{2x} + 5xe^{4x}$ is a particular solution of Equation 4.6.17).

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4.6E: The Method of Undetermined Coefficients I (Exercises)

Q4.6.1

In *Exercises 4.6.1-4.6.14* find a particular solution.

1. $y''-3y'+2y=e^{3x}(1+x)$
2. $y''-6y'+5y=e^{-3x}(35-8x)$
3. $y''-2y'-3y=e^{x}(-8+3x)$
4. $y''+2y'+y=e^{2x}(-7-15x+9x^2)$
5. $y''+4y=e^{-x}(7-4x+5x^2)$
6. $y''-y'-2y=e^{x}(9+2x-4x^2)$
7. $y''-4y'-5y=-6xe^{-x}$
8. $y''-3y'+2y=e^{x}(3-4x)$
9. $y''+y'-12y=e^{3x}(-6+7x)$
10. $2y''-3y'-2y=e^{2x}(-6+10x)$
11. $y''+2y'+y=e^{-x}(2+3x)$
12. $y''-2y'+y=e^{x}(1-6x)$
13. $y''-4y'+4y=e^{2x}(1-3x+6x^2)$
14. $9y''+6y'+y=e^{-x/3}(2-4x+4x^2)$

Q4.6.2

In *Exercises 4.6.15-4.6.19* find the general solution.

15. $y''-3y'+2y=e^{3x}(1+x)$
16. $y''-6y'+8y=e^{x}(11-6x)$
17. $y''+6y'+9y=e^{2x}(3-5x)$
18. $y''+2y'-3y=-16xe^{x}$
19. $y''-2y'+y=e^{x}(2-12x)$

Q4.6.3

In *Exercises 4.6.20-4.6.23* solve the initial value problem and plot the solution.

20. $y''-4y'-5y=9e^{2x}(1+x), \quad y(0)=0, \quad y'(0)=-10$
21. $y''+3y'-4y=e^{2x}(7+6x), \quad y(0)=2, \quad y'(0)=8$
22. $y''+4y'+3y=-e^{-x}(2+8x), \quad y(0)=1, \quad y'(0)=2$
23. $y''-3y'-10y=7e^{-2x}, \quad y(0)=1, \quad y'(0)=-17$

Q4.6.4

In *Exercises 4.6.24-4.6.29* use the principle of superposition to find a particular solution.

24. $y''+y'+y=xe^{x}+e^{-x}(1+2x)$
25. $y''-7y'+12y=-e^{x}(17-42x)-e^{3x}$
26. $y''-8y'+16y=6xe^{4x}+2+16x+16x^2$
27. $y''-3y'+2y=-e^{2x}(3+4x)-e^{x}$

28. $y'' - 2y' + 2y = e^x(1+x) + e^{-x}(2 - 8x + 5x^2)$

29. $y'' + y = e^{-x}(2 - 4x + 2x^2) + e^{3x}(8 - 12x - 10x^2)$

Q4.6.5

30. Suppose $\alpha \neq 0$ and k is a positive integer. In most calculus books integrals like $\int x^k e^{\alpha x}\, dx$ are evaluated by integrating by parts k times. This exercise presents another method. Let

$$y = \int e^{\alpha x} P(x)\, dx$$

with

$$P(x) = p_0 + p_1 x + \cdots + p_k x^k$$

(where $p_k \neq 0$).

a. Show that $y = e^{\alpha x} u$, where

$$u' + \alpha u = P(x). \tag{A}$$

b. Show that (A) has a particular solution of the form

$$u_p = A_0 + A_1 x + \cdots + A_k x^k,$$

where A_k, A_{k-1}, …, A_0 can be computed successively by equating coefficients of $x^k, x^{k-1}, \ldots, 1$ on both sides of the equation

$$u_p' + \alpha u_p = P(x).$$

c. Conclude that

$$\int e^{\alpha x} P(x)\, dx = \left(A_0 + A_1 x + \cdots + A_k x^k\right) e^{\alpha x} + c,$$

where c is a constant of integration.

31. Use the method of *Exercise 4.6.30* to evaluate the integral.

a. $\int e^x(4+x)dx$
b. $\int e^{-x}(-1+x^2)dx$
c. $\int x^3 e^{-2x} dx$
d. $\int e^x(1+x)^2 dx$
e. $\int e^{3x}(-14+30x+27x^2)dx$
f. $\int e^{-x}(1+6x^2-14x^3+3x^4)dx$

32. Use the method suggested in *Exercise 4.6.30* to evaluate $\int x^k e^{\alpha x}\, dx$, where k is an arbitrary positive integer and $\alpha \neq 0$.

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4.7: The Method of Undetermined Coefficients II

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{4.7.1}$$

where λ and ω are real numbers, $\omega \neq 0$, and P and Q are polynomials. We want to find a particular solution of Equation 4.7.1. As in Section 5.4, the procedure that we will use is called *the method of undetermined coefficients*.

Forcing Functions Without Exponential Factors

We begin with the case where $\lambda = 0$ in Equation 4.7.1; thus, we we want to find a particular solution of

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x, \tag{4.7.2}$$

where P and Q are polynomials.

Differentiating $x^r\cos\omega x$ and $x^r\sin\omega x$ yields

$$\frac{d}{dx}x^r\cos\omega x = -\omega x^r\sin\omega x + rx^{r-1}\cos\omega x$$

and

$$\frac{d}{dx}x^r\sin\omega x = \quad \omega x^r\cos\omega x + rx^{r-1}\sin\omega x.$$

This implies that if

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x$$

where A and B are polynomials, then

$$ay_p'' + by_p' + cy_p = F(x)\cos\omega x + G(x)\sin\omega x,$$

where F and G are polynomials with coefficients that can be expressed in terms of the coefficients of A and B. This suggests that we try to choose A and B so that $F = P$ and $G = Q$, respectively. Then y_p will be a particular solution of Equation 4.7.2. The next theorem tells us how to choose the proper form for y_p. For the proof see *Exercise 5.5.37*.

Theorem 4.7.1

Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x, \tag{4.7.3}$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \text{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

provided that $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation. The solutions of

$$a(y'' + \omega^2 y) = P(x)\cos\omega x + Q(x)\sin\omega x$$

for which $\cos\omega x$ and $\sin\omega x$ are solutions of the complementary equation are of the form of Equation 4.7.3, where

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \text{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}.$$

Theorem 4.7.1 implies the next theorem:

Theorem 4.7.1

Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{4.7.4}$$

has a particular solution

$$y_p = e^{\lambda x}\left(A(x)\cos\omega x + B(x)\sin\omega x\right), \tag{A}$$

where

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k \quad \text{and} \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k, \tag{4.7.5}$$

provided that $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are not solutions of the complementary equation. The equation

$$a\left[y'' - 2\lambda y' + (\lambda^2 + \omega^2)y\right] = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{4.7.6}$$

(for which $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are solutions of the complementary equation) has a particular solution of the form (A), where

$$A(x) = A_0 x + A_1 x^2 + \cdots + A_k x^{k+1} \quad \text{and} \quad B(x) = B_0 x + B_1 x^2 + \cdots + B_k x^{k+1}. \tag{4.7.7}$$

Example 4.7.1

Find a particular solution of

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x. \tag{4.7.8}$$

Solution

In Equation 4.7.8 the coefficients of $\cos 2x$ and $\sin 2x$ are both zero degree polynomials (constants). Therefore Theorem 4.7.1 implies that Equation 4.7.8 has a particular solution

$$y_p = A\cos 2x + B\sin 2x.$$

Since

$$y_p' = -2A\sin 2x + 2B\cos 2x \quad \text{and} \quad y_p'' = -4(A\cos 2x + B\sin 2x),$$

replacing y by y_p in Equation 4.7.8 yields

$$\begin{aligned} y_p'' - 2y_p' + y_p &= -4(A\cos 2x + B\sin 2x) - 4(-A\sin 2x + B\cos 2x) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (A\cos 2x + B\sin 2x) \\ &= (-3A - 4B)\cos 2x + (4A - 3B)\sin 2x. \end{aligned}$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ here with the corresponding coefficients on the right side of Equation 4.7.8 shows that y_p is a solution of Equation 4.7.8 if

$$\begin{aligned} -3A - 4B &= 5 \\ 4A - 3B &= 10. \end{aligned}$$

Solving these equations yields $A = 1$, $B = -2$. Therefore

$$y_p = \cos 2x - 2\sin 2x$$

is a particular solution of Equation 4.7.8.

Example 4.7.2

Find a particular solution of

$$y'' + 4y = 8\cos 2x + 12\sin 2x. \tag{4.7.9}$$

Solution

The procedure used in Example 4.7.1 doesn't work here; substituting $y_p = A\cos 2x + B\sin 2x$ for y in Equation 4.7.9 yields

$$y_p'' + 4y_p = -4(A\cos 2x + B\sin 2x) + 4(A\cos 2x + B\sin 2x) = 0$$

for any choice of A and B, since $\cos 2x$ and $\sin 2x$ are both solutions of the complementary equation for Equation 4.7.9. We're dealing with the second case mentioned in Theorem 4.7.1, and should therefore try a particular solution of the form

$$y_p = x(A\cos 2x + B\sin 2x). \tag{4.7.10}$$

Then

$$\begin{aligned} y_p' &= A\cos 2x + B\sin 2x + 2x(-A\sin 2x + B\cos 2x) \\ \text{and} y_p'' &= -4A\sin 2x + 4B\cos 2x - 4x(A\cos 2x + B\sin 2x) \\ &= -4A\sin 2x + 4B\cos 2x - 4y_p \text{ (see (4.7.10))}, \end{aligned}$$

so

$$y_p'' + 4y_p = -4A\sin 2x + 4B\cos 2x.$$

Therefore y_p is a solution of Equation 4.7.9 if

$$-4A\sin 2x + 4B\cos 2x = 8\cos 2x + 12\sin 2x,$$

which holds if $A = -3$ and $B = 2$. Therefore

$$y_p = -x(3\cos 2x - 2\sin 2x)$$

is a particular solution of Equation 4.7.9.

Example 4.7.3

Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x. \tag{4.7.11}$$

Solution

The coefficients of $\cos x$ and $\sin x$ in Equation 4.7.11 are polynomials of degree one and zero, respectively. Therefore Theorem 4.7.1 tells us to look for a particular solution of Equation 4.7.11 of the form

$$y_p = (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x. \tag{4.7.12}$$

Then

$$y_p' = (A_1 + B_0 + B_1 x)\cos x + (B_1 - A_0 - A_1 x)\sin x \tag{4.7.13}$$

and

$$y_p'' = (2B_1 - A_0 - A_1 x)\cos x - (2A_1 + B_0 + B_1 x)\sin x, \tag{4.7.14}$$

so

$$y_p'' + 3y_p' + 2y_p \quad = [A_0 + 3A_1 + 3B_0 + 2B_1 + (A_1 + 3B_1)x]\cos x + [B_0 + 3B_1 - 3A_0 - 2A_1 + (B_1 - 3A_1)x]\sin x. \tag{4.7.15}$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in Equation 4.7.11 shows that y_p is a solution of Equation 4.7.11 if

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + B_0 - 2A_1 + 3B_1 &= 10. \end{aligned}$$

Solving the first two equations yields $A_1 = 2$, $B_1 = 6$. Substituting these into the last two equations yields

$$\begin{aligned} A_0 + 3B_0 &= 16 - 3A_1 - 2B_1 = -2 \\ -3A_0 + B_0 &= 10 + 2A_1 - 3B_1 = -4. \end{aligned}$$

Solving these equations yields $A_0 = 1$, $B_0 = -1$. Substituting $A_0 = 1$, $A_1 = 2$, $B_0 = -1$, $B_1 = 6$ into Equation 4.7.12 shows that

$$y_p = (1 + 2x)\cos x - (1 - 6x)\sin x$$

is a particular solution of Equation 4.7.11.

A Useful Observation

In Equations 4.7.13, 4.7.14, and 4.7.15 the polynomials multiplying $\sin x$ can be obtained by replacing A_0, A_1, B_0, and B_1 by $B_0, B_1, -A_0$, and $-A_1$, respectively, in the polynomials mutiplying $\cos x$. An analogous result applies in general, as follows (*Exercise 5.5.36*).

Theorem 4.7.2

If

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x,$$

where $A(x)$ and $B(x)$ are polynomials with coefficients A_0 …, A_k and B_0, …, B_k, then the polynomials multiplying $\sin\omega x$ in

$$y_p', \quad y_p'', \quad ay_p'' + by_p' + cy_p \quad \text{and} \quad y_p'' + \omega^2 y_p$$

can be obtained by replacing A_0, …, A_k by B_0, …, B_k and B_0, …, B_k by $-A_0$, …, $-A_k$ in the corresponding polynomials multiplying $\cos\omega x$.

We will not use this theorem in our examples, but we recommend that you use it to check your manipulations when you work the exercises.

Example 4.7.4

Find a particular solution of

$$y'' + y = (8 - 4x)\cos x - (8 + 8x)\sin x. \tag{4.7.16}$$

Solution

According to Theorem 4.7.1, we should look for a particular solution of the form

$$y_p = (A_0 x + A_1 x^2)\cos x + (B_0 x + B_1 x^2)\sin x, \tag{4.7.17}$$

since $\cos x$ and $\sin x$ are solutions of the complementary equation. However, let's try

$$y_p = (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x \tag{4.7.18}$$

first, so you can see why it doesn't work. From Equation 4.7.14,

$$y_p'' = (2B_1 - A_0 - A_1 x)\cos x - (2A_1 + B_0 + B_1 x)\sin x,$$

which together with Equation 4.7.18implies that

$$y_p'' + y_p = 2B_1 \cos x - 2A_1 \sin x.$$

Since the right side of this equation does not contain $x\cos x$ or $x\sin x$, Equation 4.7.18can't satisfy Equation 4.7.16no matter how we choose A_0, A_1, B_0, and B_1.

Now let y_p be as in Equation 4.7.17. Then

$$\begin{aligned} y_p' &= \left[A_0 + (2A_1 + B_0)x + B_1 x^2\right]\cos x \\ &\quad + \left[B_0 + (2B_1 - A_0)x - A_1 x^2\right]\sin x \end{aligned}$$

and

$$\begin{aligned} y_p'' &= \left[2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1 x^2\right]\cos x \\ &\quad + \left[2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1 x^2\right]\sin x, \end{aligned}$$

so

$$y_p'' + y_p = (2A_1 + 2B_0 + 4B_1 x)\cos x + (2B_1 - 2A_0 - 4A_1 x)\sin x.$$

Comparing the coefficients of $\cos x$ and $\sin x$ here with the corresponding coefficients in Equation 4.7.16shows that y_p is a solution of Equation 4.7.16if

$$\begin{aligned} 4B_1 &= -4 \\ -4A_1 &= -8 \\ 2B_0 + 2A_1 &= 8 \\ -2A_0 + 2B_1 &= -8. \end{aligned}$$

The solution of this system is $A_1 = 2$, $B_1 = -1$, $A_0 = 3$, $B_0 = 2$. Therefore

$$y_p = x\left[(3 + 2x)\cos x + (2 - x)\sin x\right]$$

is a particular solution of Equation 4.7.16.

Forcing Functions with Exponential Factors

To find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{4.7.19}$$

when $\lambda \neq 0$, we recall from Section 5.4 that substituting $y = ue^{\lambda x}$ into Equation 4.7.19will produce a constant coefficient equation for u with the forcing function $P(x)\cos\omega x + Q(x)\sin\omega x$. We can find a particular solution u_p of this equation by the procedure that we used in Examples 4.7.1-4.7.4. Then $y_p = u_p e^{\lambda x}$ is a particular solution of Equation 4.7.19.

Example 4.7.5

Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right]. \tag{4.7.20}$$

Let $y = ue^{-2x}$. Then

$$\begin{aligned} y''-3y'+2y &= e^{-2x}\left[(u''-4u'+4u)-3(u'-2u)+2u\right] \\ &= e^{-2x}(u''-7u'+12u) \\ &= e^{-2x}\left[2\cos 3x-(34-150x)\sin 3x\right] \end{aligned}$$

if

$$u''-7u'+12u=2\cos 3x-(34-150x)\sin 3x. \tag{4.7.21}$$

Since $\cos 3x$ and $\sin 3x$ aren't solutions of the complementary equation

$$u''-7u'+12u=0,$$

Theorem 4.7.1tells us to look for a particular solution of Equation 4.7.21of the form

$$u_p=(A_0+A_1x)\cos 3x+(B_0+B_1x)\sin 3x. \tag{4.7.22}$$

Then

$$\begin{aligned} u_p' &= (A_1+3B_0+3B_1x)\cos 3x+(B_1-3A_0-3A_1x)\sin 3x \\ \text{and} \qquad u_p'' &= (-9A_0+6B_1-9A_1x)\cos 3x-(9B_0+6A_1+9B_1x)\sin 3x, \end{aligned}$$

so

$$\begin{aligned} u_p''-7u_p'+12u_p &= [3A_0-21B_0-7A_1+6B_1+(3A_1-21B_1)x]\cos 3x \\ &\quad +[21A_0+3B_0-6A_1-7B_1+(21A_1+3B_1)x]\sin 3x. \end{aligned}$$

Comparing the coefficients of $x\cos 3x$, $x\sin 3x$, $\cos 3x$, and $\sin 3x$ here with the corresponding coefficients on the right side of Equation 4.7.21shows that u_p is a solution of Equation 4.7.21if

$$\begin{aligned} 3A_1-21B_1 &= 0 \\ 21A_1+\ 3B_1 &= 150 \\ 3A_0-21B_0-7A_1+\ 6B_1 &= 2 \\ 21A_0+\ 3B_0-6A_1-\ 7B_1 &= -34. \end{aligned} \tag{4.7.23}$$

Solving the first two equations yields $A_1=7$, $B_1=1$. Substituting these values into the last two equations of Equation 4.7.23yields

$$\begin{aligned} 3A_0-21B_0 &= 2+7A_1-6B_1=45 \\ 21A_0+\ 3B_0 &= -34+6A_1+7B_1=15. \end{aligned}$$

Solving this system yields $A_0=1$, $B_0=-2$. Substituting $A_0=1$, $A_1=7$, $B_0=-2$, and $B_1=1$ into Equation 4.7.22shows that

$$u_p=(1+7x)\cos 3x-(2-x)\sin 3x$$

is a particular solution of Equation 4.7.21. Therefore

$$y_p=e^{-2x}\left[(1+7x)\cos 3x-(2-x)\sin 3x\right]$$

is a particular solution of Equation 4.7.20.

Example 4.7.6

Find a particular solution of

$$y''+2y'+5y=e^{-x}\left[(6-16x)\cos 2x-(8+8x)\sin 2x\right]. \tag{4.7.24}$$

Solution

Let $y=ue^{-x}$. Then

$$\begin{aligned} y''+2y'+5y &= e^{-x}\left[(u''-2u'+u)+2(u'-u)+5u\right] \\ &= e^{-x}(u''+4u) \\ &= e^{-x}\left[(6-16x)\cos 2x-(8+8x)\sin 2x\right] \end{aligned}$$

if

$$u''+4u=(6-16x)\cos 2x-(8+8x)\sin 2x. \tag{4.7.25}$$

Since $\cos 2x$ and $\sin 2x$ are solutions of the complementary equation

$$u''+4u=0,$$

Theorem 4.7.1tells us to look for a particular solution of Equation 4.7.25of the form

$$u_p=(A_0x+A_1x^2)\cos 2x+(B_0x+B_1x^2)\sin 2x.$$

Then

$$\begin{aligned} u_p' &= \left[A_0+(2A_1+2B_0)x+2B_1x^2\right]\cos 2x \\ &\quad +\left[B_0+(2B_1-2A_0)x-2A_1x^2\right]\sin 2x \end{aligned}$$

and

$$\begin{aligned} u_p'' &= \left[2A_1+4B_0-(4A_0-8B_1)x-4A_1x^2\right]\cos 2x \\ &\quad +\left[2B_1-4A_0-(4B_0+8A_1)x-4B_1x^2\right]\sin 2x, \end{aligned}$$

so

$$u_p''+4u_p=(2A_1+4B_0+8B_1x)\cos 2x+(2B_1-4A_0-8A_1x)\sin 2x.$$

Equating the coefficients of $x\cos 2x$, $x\sin 2x$, $\cos 2x$, and $\sin 2x$ here with the corresponding coefficients on the right side of Equation 4.7.25 shows that u_p is a solution of Equation 4.7.25if

$$\begin{aligned} 8B_1 &= -16 \\ -8A_1 &= -\ 8 \\ 4B_0+2A_1 &= 6 \\ -4A_0+2B_1 &= -8. \end{aligned} \tag{4.7.26}$$

The solution of this system is $A_1=1$, $B_1=-2$, $B_0=1$, $A_0=1$. Therefore

$$u_p=x[(1+x)\cos 2x+(1-2x)\sin 2x]$$

is a particular solution of Equation 4.7.25, and

$$y_p=xe^{-x}\left[(1+x)\cos 2x+(1-2x)\sin 2x\right]$$

is a particular solution of Equation 4.7.24

You can also find a particular solution of Equation 4.7.24by substituting

$$y_p=xe^{-x}\left[(A_0+A_1x)\cos 2x+(B_0+B_1x)\sin 2x\right]$$

for y in Equation 4.7.24and equating the coefficients of $xe^{-x}\cos 2x$, $xe^{-x}\sin 2x$, $e^{-x}\cos 2x$, and $e^{-x}\sin 2x$ in the resulting expression for

$$y_p''+2y_p'+5y_p$$

with the corresponding coefficients on the right side of Equation 4.7.24. (See *Exercise 5.5.38*). This leads to the same system Equation 4.7.26 of equations for A_0, A_1, B_0, and B_1 that we obtained in Example 4.7.6. However, if you try this approach you'll see that deriving Equation 4.7.26this way is much more tedious than the way we did it in Example 4.7.6.

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4.7E: The Method of Undetermined Coefficients II (Exercises)

Q4.7.1

In *Exercises 4.7.1-4.7.17* find a particular solution.

1. $y''+3y'+2y=7\cos x-\sin x$
2. $y''+3y'+y=(2-6x)\cos x-9\sin x$
3. $y''+2y'+y=e^x(6\cos x+17\sin x)$
4. $y''+3y'-2y=-e^{2x}(5\cos 2x+9\sin 2x)$
5. $y''-y'+y=e^x(2+x)\sin x$
6. $y''+3y'-2y=e^{-2x}\left[(4+20x)\cos 3x+(26-32x)\sin 3x\right]$
7. $y''+4y=-12\cos 2x-4\sin 2x$
8. $y''+y=(-4+8x)\cos x+(8-4x)\sin x$
9. $4y''+y=-4\cos x/2-8x\sin x/2$
10. $y''+2y'+2y=e^{-x}(8\cos x-6\sin x)$
11. $y''-2y'+5y=e^x\left[(6+8x)\cos 2x+(6-8x)\sin 2x\right]$
12. $y''+2y'+y=8x^2\cos x-4x\sin x$
13. $y''+3y'+2y=(12+20x+10x^2)\cos x+8x\sin x$
14. $y''+3y'+2y=(1-x-4x^2)\cos 2x-(1+7x+2x^2)\sin 2x$
15. $y''-5y'+6y=-e^x\left[(4+6x-x^2)\cos x-(2-4x+3x^2)\sin x\right]$
16. $y''-2y'+y=-e^x\left[(3+4x-x^2)\cos x+(3-4x-x^2)\sin x\right]$
17. $y''-2y'+2y=e^x\left[(2-2x-6x^2)\cos x+(2-10x+6x^2)\sin x\right]$

Q4.7.2

In *Exercises 4.7.18-4.7.22* solve the initial value problem.

18. $y''-7y'+6y=-e^x(17\cos x-7\sin x),\quad y(0)=4,\ y'(0)=2$
19. $y''-2y'+2y=-e^x(6\cos x+4\sin x),\quad y(0)=1,\ y'(0)=4$
20. $y''+6y'+10y=-40e^x\sin x,\quad y(0)=2,\quad y'(0)=-3$
21. $y''-6y'+10y=-e^{3x}(6\cos x+4\sin x),\quad y(0)=2,\quad y'(0)=7$
22. $y''-3y'+2y=e^{3x}\left[21\cos x-(11+10x)\sin x\right],\ y(0)=0,\quad y'(0)=6$

Q4.7.3

In *Exercises 4.7.23-4.7.28* use the principle of superposition to find a particular solution. Where indicated, solve the initial value problem.

23. $y''-2y'-3y=4e^{3x}+e^x(\cos x-2\sin x)$
24. $y''+y=4\cos x-2\sin x+xe^x+e^{-x}$
25. $y''-3y'+2y=xe^x+2e^{2x}+\sin x$
26. $y''-2y'+2y=4xe^x\cos x+xe^{-x}+1+x^2$
27. $y''-4y'+4y=e^{2x}(1+x)+e^{2x}(\cos x-\sin x)+3e^{3x}+1+x$
28. $y''-4y'+4y=6e^{2x}+25\sin x,\quad y(0)=5,\ y'(0)=3$

Q4.7.4

In *Exercises 4.7.29-4.7.31* solve the initial value problem and graph the solution.

29. $y''+4y=-e^{-2x}\left[(4-7x)\cos x+(2-4x)\sin x\right],\ y(0)=3,\quad y'(0)=1$

30. $y''+4y'+4y=2\cos 2x+3\sin 2x+e^{-x},\quad y(0)=-1,\ y'(0)=2$

31. $y''+4y=e^{x}(11+15x)+8\cos 2x-12\sin 2x,\quad y(0)=3,\ y'(0)=5$

Q4.7.5

32. This exercise presents a method for evaluating the integral

$$y=\int e^{\lambda x}\left(P(x)\cos\omega x+Q(x)\sin\omega x\right)\,dx \tag{4.7E.1}$$

where $\omega\neq 0$ and

$$P(x)=p_0+p_1x+\cdots+p_kx^k,\quad Q(x)=q_0+q_1x+\cdots+q_kx^k. \tag{4.7E.2}$$

a. Show that $y=e^{\lambda x}u$, where

$$u'+\lambda u=P(x)\cos\omega x+Q(x)\sin\omega x. \tag{A}$$

b. Show that (A) has a particular solution of the form

$$u_p=A(x)\cos\omega x+B(x)\sin\omega x, \tag{4.7E.3}$$

where

$$A(x)=A_0+A_1x+\cdots+A_kx^k,\quad B(x)=B_0+B_1x+\cdots+B_kx^k, \tag{4.7E.4}$$

and the pairs of coefficients (A_k,B_k), (A_{k-1},B_{k-1}), ...,(A_0,B_0) can be computed successively as the solutions of pairs of equations obtained by equating the coefficients of $x^r\cos\omega x$ and $x^r\sin\omega x$ for $r=k$, $k-1$, ..., 0.

c. Conclude that

$$\int e^{\lambda x}\left(P(x)\cos\omega x+Q(x)\sin\omega x\right)\,dx=e^{\lambda x}\left(A(x)\cos\omega x+B(x)\sin\omega x\right)+c, \tag{4.7E.5}$$

where c is a constant of integration.

33. Use the method of *Exercise 4.7.32* to evaluate the integral.

a. $\int x^2\cos x\,dx$
b. $\int x^2e^x\cos x\,dx$
c. $\int xe^{-x}\sin 2x\,dx$
d. $\int x^2e^{-x}\sin x\,dx$
e. $\int x^3e^x\sin x\,dx$
f. $\int e^x[x\cos x-(1+3x)\sin x]dx$
g. $\int e^{-x}[(1+x^2)\cos x+(1_x^2)\sin x]dx$

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4.8: Undetermined Coefficients for Higher Order Equations

In this section we consider the constant coefficient equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = F(x), \tag{4.8.1}$$

where $n \geq 3$ and F is a linear combination of functions of the form

$$e^{\alpha x}\left(p_0 + p_1 x + \cdots + p_k x^k\right)$$

or

$$e^{\lambda x}\left[\left(p_0 + p_1 x + \cdots + p_k x^k\right)\cos\omega x + \left(q_0 + q_1 x + \cdots + q_k x^k\right)\sin\omega x\right].$$

From Theorem 9.1.5, the general solution of Equation 4.8.1 is $y = y_p + y_c$, where y_p is a particular solution of Equation 4.8.1 and y_c is the general solution of the complementary equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0.$$

In Section 9.2 we learned how to find y_c. Here we will learn how to find y_p when the forcing function has the form stated above. The procedure that we use is a generalization of the method that we used in Sections 5.4 and 5.5, and is again called **method of undetermined coefficients**. Since the underlying ideas are the same as those in these section, we'll give an informal presentation based on examples.

Forcing Functions of the Form $e^{\alpha x}(p_0 + p_1 x + \ldots + p_k x^k)$

We first consider equations of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = e^{\alpha x}\left(p_0 + p_1 x + \cdots + p_k x^k\right).$$

Example 9.3.1

Find a particular solution of

$$y''' + 3y'' + 2y' - y = e^x(21 + 24x + 28x^2 + 5x^3). \tag{4.8.2}$$

Solution

Substituting

$$\begin{aligned} y &= ue^x, \\ y' &= e^x(u' + u), \\ y'' &= e^x(u'' + 2u' + u), \\ y''' &= e^x(u''' + 3u'' + 3u' + u) \end{aligned}$$

into Equation 4.8.2 and canceling e^x yields

$$(u''' + 3u'' + 3u' + u) + 3(u'' + 2u' + u) + 2(u' + u) - u = 21 + 24x + 28x^2 + 5x^3,$$

or

$$u''' + 6u'' + 11u' + 5u = 21 + 24x + 28x^2 + 5x^3. \tag{4.8.3}$$

Since the unknown u appears on the left, we can see that Equation 4.8.3 has a particular solution of the form

$$u_p = A + Bx + Cx^2 + Dx^3.$$

Then

$$\begin{aligned} u_p' &= B + 2Cx + 3Dx^2 \\ u_p'' &= 2C + 6Dx \\ u_p''' &= 6D. \end{aligned}$$

Substituting from the last four equations into the left side of Equation 4.8.3 yields

$$\begin{aligned} u_p''' + 6u_p'' + 11u_p' + 5u_p &= 6D + 6(2C + 6Dx) + 11(B + 2Cx + 3Dx^2) \\ &\quad + 5(A + Bx + Cx^2 + Dx^3) \\ &= (5A + 11B + 12C + 6D) + (5B + 22C + 36D)x \\ &\quad + (5C + 33D)x^2 + 5Dx^3. \end{aligned}$$

Comparing coefficients of like powers of x on the right sides of this equation and Equation 4.8.3 shows that u_p satisfies Equation 4.8.3 if

$$\begin{aligned} 5D &= 5 \\ 5C + 33D &= 28 \\ 5B + 22C + 36D &= 24 \\ 5A + 11B + 12C + 6D &= 21. \end{aligned}$$

Solving these equations successively yields $D = 1$, $C = -1$, $B = 2$, $A = 1$. Therefore

$$u_p = 1 + 2x - x^2 + x^3$$

is a particular solution of Equation 4.8.3, so

$$y_p = e^x u_p = e^x(1 + 2x - x^2 + x^3)$$

is a particular solution of Equation 4.8.2 (Figure 9.3.1).

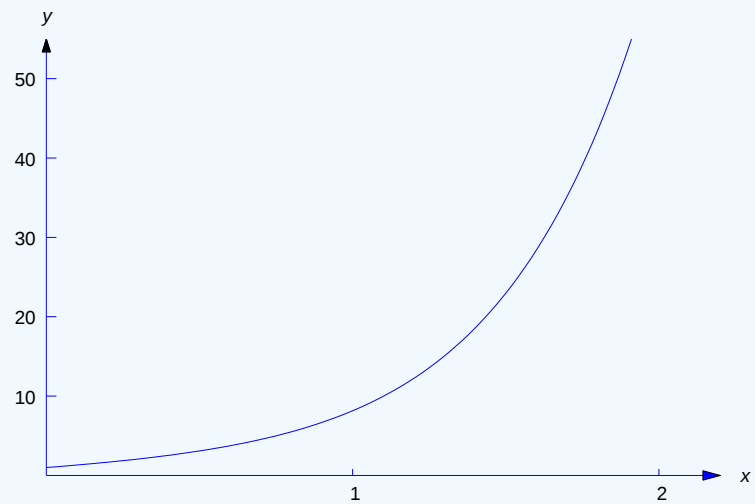

Figure 9.3.1 : $y_p = e^x(1 + 2x - x^2 + x^3)$

Example 9.3.2

Find a particular solution of

$$y^{(4)} - y''' - 6y'' + 4y' + 8y = e^{2x}(4 + 19x + 6x^2). \tag{4.8.4}$$

Solution

Substituting

$$
\begin{aligned}
y &= ue^{2x}, \\
y' &= e^{2x}(u'+2u), \\
y'' &= e^{2x}(u''+4u'+4u), \\
y''' &= e^{2x}(u'''+6u''+12u'+8u), \\
y^{(4)} &= e^{2x}(u^{(4)}+8u'''+24u''+32u'+16u)
\end{aligned}
$$

into Equation 4.8.4 and canceling e^{2x} yields

$$
\begin{aligned}
&(u^{(4)}+8u'''+24u''+32u'+16u)-(u'''+6u''+12u'+8u) \\
&\quad -6(u''+4u'+4u)+4(u'+2u)+8u=4+19x+6x^2,
\end{aligned}
$$

or

$$
u^{(4)}+7u'''+12u''=4+19x+6x^2. \tag{4.8.5}
$$

Since neither u nor u' appear on the left, we can see that Equation 4.8.5 has a particular solution of the form

$$
u_p=Ax^2+Bx^3+Cx^4. \tag{4.8.6}
$$

Then

$$
\begin{aligned}
u_p' &= 2Ax+3Bx^2+4Cx^3 \\
u_p'' &= 2A+6Bx+12Cx^2 \\
u_p''' &= 6B+24Cx \\
u_p^{(4)} &= 24C.
\end{aligned}
$$

Substituting u_p'', u_p''', and $u_p^{(4)}$ into the left side of Equation 4.8.5 yields

$$
\begin{aligned}
u_p^{(4)}+7u_p'''+12u_p'' &= 24C+7(6B+24Cx)+12(2A+6Bx+12Cx^2) \\
&= (24A+42B+24C)+(72B+168C)x+144Cx^2.
\end{aligned}
$$

Comparing coefficients of like powers of x on the right sides of this equation and Equation 4.8.5 shows that u_p satisfies Equation 4.8.5 if

$$
\begin{aligned}
144C &= 6 \\
72B+168C &= 19 \\
24A+42B+\ 24C &= 4.
\end{aligned}
$$

Solving these equations successively yields $C=1/24$, $B=1/6$, $A=-1/6$. Substituting these into Equation 4.8.6 shows that

$$
u_p=\frac{x^2}{24}(-4+4x+x^2)
$$

is a particular solution of Equation 4.8.5, so

$$
y_p=e^{2x}u_p=\frac{x^2e^{2x}}{24}(-4+4x+x^2)
$$

is a particular solution of Equation 4.8.4 (Figure 9.3.2).

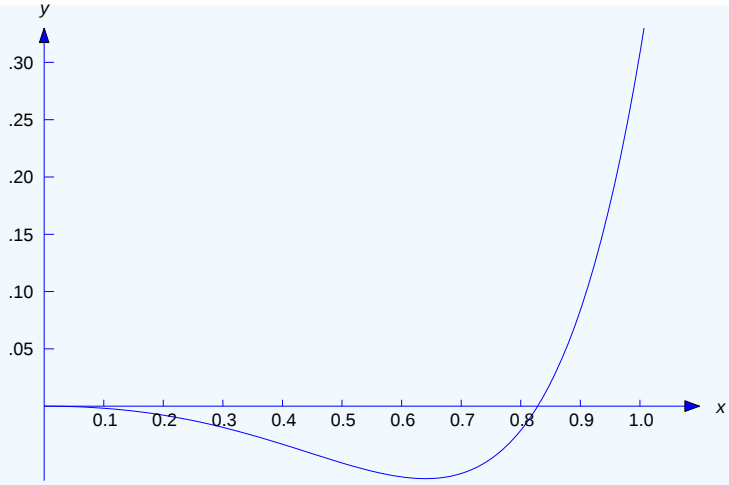

Figure 9.3.2 : $y_p = \dfrac{x^2 e^{2x}}{24}(-4+4x+x^2)$

Forcing Functions of the Form $e^{\alpha x}(P(x)\cos\omega x + Q(x)\sin\omega x)$

We now consider equations of the form

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right),$$

where P and Q are polynomials.

Example 9.3.3

Find a particular solution of

$$y''' + y'' - 4y' - 4y = e^x[(5-5x)\cos x + (2+5x)\sin x]. \tag{4.8.7}$$

Solution

Substituting

$$\begin{aligned} y &= ue^x, \\ y' &= e^x(u'+u), \\ y'' &= e^x(u''+2u'+u), \\ y''' &= e^x(u'''+3u''+3u'+u) \end{aligned}$$

into Equation 4.8.7 and canceling e^x yields

$$(u'''+3u''+3u'+u) + (u''+2u'+u) - 4(u'+u) - 4u = (5-5x)\cos x + (2+5x)\sin x,$$

or

$$u''' + 4u'' + u' - 6u = (5-5x)\cos x + (2+5x)\sin x. \tag{4.8.8}$$

Since $\cos x$ and $\sin x$ are not solutions of the complementary equation

$$u''' + 4u'' + u' - 6u = 0,$$

a theorem analogous to Theorem 5.5.1 implies that Equation 4.8.8 has a particular solution of the form

$$u_p = (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x. \tag{4.8.9}$$

Then

$$\begin{aligned} u_p' &= (A_1 + B_0 + B_1 x)\cos x + (B_1 - A_0 - A_1 x)\sin x, \\ u_p' &= (2B_1 - A_0 - A_1 x)\cos x - (2A_1 + B_0 + B_1 x)\sin x, \\ u_p''' &= -(3A_1 + B_0 + B_1 x)\cos x - (3B_1 - A_0 - A_1 x)\sin x, \end{aligned}$$

so

$$u_p''' + 4u_p'' + u_p' - 6u_p = -\left[10A_0 + 2A_1 - 8B_1 + 10A_1 x\right]\cos x - \left[10B_0 + 2B_1 + 8A_1 + 10B_1 x\right]\sin x.$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in Equation 4.8.8 shows that u_p is a solution of Equation 4.8.8 if

$$\begin{aligned} -10A_1 &= -5 \\ -10B_1 &= 5 \\ -10A_0 - 2A_1 + 8B_1 &= 5 \\ -10B_0 - 2B_1 - 8A_1 &= 2. \end{aligned}$$

Solving the first two equations yields $A_1 = 1/2$, $B_1 = -1/2$. Substituting these into the last two equations yields

$$\begin{aligned} -10A_0 &= 5 + 2A_1 - 8B_1 = 10 \\ -10B_0 &= 2 + 2B_1 + 8A_1 = 5, \end{aligned}$$

so $A_0 = -1$, $B_0 = -1/2$. Substituting $A_0 = -1$, $A_1 = 1/2$, $B_0 = -1/2$, $B_1 = -1/2$ into Equation 4.8.9 shows that

$$u_p = -\frac{1}{2}\left[(2-x)\cos x + (1+x)\sin x\right]$$

is a particular solution of Equation 4.8.8, so

$$y_p = e^x u_p = -\frac{e^x}{2}\left[(2-x)\cos x + (1+x)\sin x\right]$$

is a particular solution of Equation 4.8.7 (Figure 9.3.3).

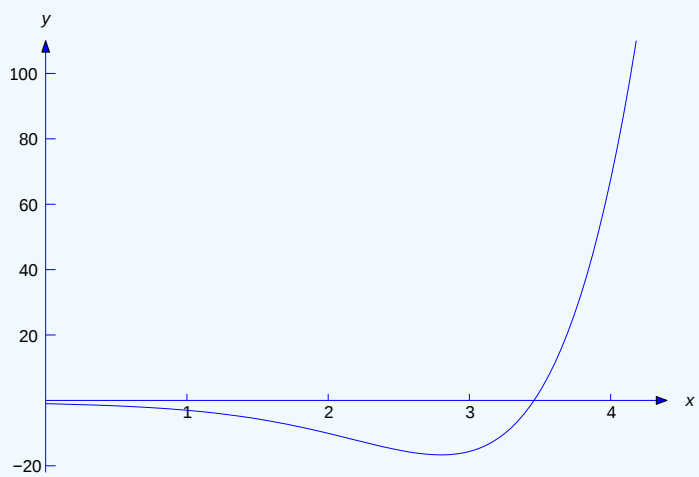

Figure 9.3.3 : $y_p = \{e^x u_p = -\frac{e^x}{2}[(2-x)\cos x + (1+x)\sin x]$

Example 9.3.4

Find a particular solution of

$$y''' + 4y'' + 6y' + 4y = e^{-x}\left[(1-6x)\cos x - (3+2x)\sin x\right]. \tag{4.8.10}$$

Solution

Substituting

$$\begin{aligned} y &= ue^{-x}, \\ y' &= e^{-x}(u'-u), \\ y'' &= e^{-x}(u''-2u'+u), \\ y''' &= e^{-x}(u'''-3u''+3u'-u) \end{aligned}$$

into Equation 4.8.10 and canceling e^{-x} yields

$$(u'''-3u''+3u'-u)+4(u''-2u'+u)+6(u'-u)+4u=(1-6x)\cos x-(3+2x)\sin x,$$

or

$$u'''+u''+u'+u=(1-6x)\cos x-(3+2x)\sin x. \tag{4.8.11}$$

Since $\cos x$ and $\sin x$ are solutions of the complementary equation

$$u'''+u''+u'+u=0,$$

a theorem analogous to Theorem 5.5.1 implies that Equation 4.8.11 has a particular solution of the form

$$u_p=(A_0x+A_1x^2)\cos x+(B_0x+B_1x^2)\sin x. \tag{4.8.12}$$

Then

$$\begin{aligned} u_p' = & \ [A_0+(2A_1+B_0)x+B_1x^2]\cos x+[B_0+(2B_1-A_0)x-A_1x^2]\sin x, \\ u_p'' = & [2A_1+2B_0-(A_0-4B_1)x-A_1x^2]\cos x \\ & +[2B_1-2A_0-(B_0+4A_1)x-B_1x^2]\sin x, \\ u_p''' = & -[3A_0-6B_1+(6A_1+B_0)x+B_1x^2]\cos x \\ & -[3B_0+6A_1+(6B_1-A_0)x-A_1x^2]\sin x, \end{aligned}$$

so

$$\begin{aligned} u_p'''+u_p''+u_p'+u_p = & -[2A_0-2B_0-2A_1-6B_1+(4A_1-4B_1)x]\cos x \\ & -[2B_0+2A_0-2B_1+6A_1+(4B_1+4A_1)x]\sin x. \end{aligned}$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in Equation 4.8.11 shows that u_p is a solution of Equation 4.8.11 if

$$\begin{aligned} -4A_1+4B_1 &= -6 \\ -4A_1-4B_1 &= -2 \\ -2A_0+2B_0+2A_1+6B_1 &= 1 \\ -2A_0-2B_0-6A_1+2B_1 &= -3. \end{aligned}$$

Solving the first two equations yields $A_1=1$, $B_1=-1/2$. Substituting these into the last two equations yields

$$\begin{aligned} -2A_0+2B_0 &= 1-2A_1-6B_1=2 \\ -2A_0-2B_0 &= -3+6A_1-2B_1=4, \end{aligned}$$

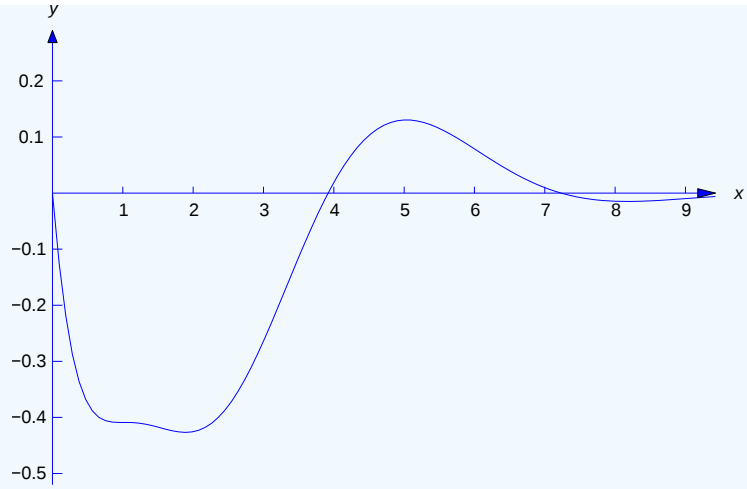

Figure 9.3.4 : $y_p = -\dfrac{xe^{-x}}{2}[(3-2x)\cos x + (1+x)\sin x]$

so $A_0 = -3/2$ and $B_0 = -1/2$. Substituting $A_0 = -3/2$, $A_1 = 1$, $B_0 = -1/2$, $B_1 = -1/2$ into Equation 4.8.12 shows that

$$u_p = -\frac{x}{2}[(3-2x)\cos x + (1+x)\sin x]$$

is a particular solution of Equation 4.8.11, so

$$y_p = e^{-x}u_p = -\frac{xe^{-x}}{2}[(3-2x)\cos x + (1+x)\sin x]$$

Figure 9.3.4 is a particular solution of Equation 4.8.10.

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4.8E: Undetermined Coefficients for Higher Order Equations (Exercises)

Q4.8.1

In *Exercises 4.8.1-4.8.9* find the general solution.

1. $y''' - y'' - y' + y = e^{2x}(10 + 3x)$
2. $y''' + y'' - 2y = -e^{3x}(9 + 67x + 17x^2)$
3. $y''' - 6y'' + 11y' - 6y = e^{2x}(5 - 4x - 3x^2)$
4. $y''' + 2y'' + y' = -2e^{-x}(7 - 18x + 6x^2)$
5. $y''' - 3y'' + 3y' - y = e^{x}(1 + x)$
6. $y^{(4)} - 2y'' + y = -e^{-x}(4 - 9x + 3x^2)$
7. $y''' + 2y'' - y' - 2y = e^{-2x}\left[(23 - 2x)\cos x + (8 - 9x)\sin x\right]$
8. $y^{(4)} - 3y''' + 4y'' - 2y' = e^{x}\left[(28 + 6x)\cos 2x + (11 - 12x)\sin 2x\right]$
9. $y^{(4)} - 4y''' + 14y'' - 20y' + 25y = e^{x}\left[(2 + 6x)\cos 2x + 3\sin 2x\right]$
10. $y''' + 3y'' + 3y' + y = 12e^{-x} + 9\cos 2x - 13\sin 2x$

Q4.8.2

In *Exercises 4.8.10-4.8.15* solve the initial value problem and graph the solution.

11. $y''' - 2y'' - 5y' + 6y = 2e^{x}(1 - 6x), \quad y(0) = 2, \quad y'(0) = 7, \quad y''(0) = 9$
12. $y''' - y'' - y' + y = -e^{-x}(4 - 8x), \quad y(0) = 2, \quad y'(0) = 0, \quad y''(0) = 0$
13. $4y''' - 3y' - y = e^{-x/2}(2 - 3x), \quad y(0) = -1, \quad y'(0) = 15, \quad y''(0) = -17$
14. $y^{(4)} + 2y''' + 2y'' + 2y' + y = e^{-x}(20 - 12x),\ y(0) = 3,\ y'(0) = -4,\ y''(0) = 7,\ y'''(0) = -22$
15. $y''' + 2y'' + y' + 2y = 30\cos x - 10\sin x, \quad y(0) = 3, \quad y'(0) = -4, \quad y''(0) = 16$
16. $y^{(4)} - 3y''' + 5y'' - 2y' = -2e^{x}(\cos x - \sin x),\ y(0) = 2,\ y'(0) = 0,\ y''(0) = -1,\ y'''(0) = -5$

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4.9: Reduction of Order

In this section we give a method for finding the general solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{4.9.1}$$

if we know a nontrivial solution y_1 of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{4.9.2}$$

The method is called **reduction of order** because it reduces the task of solving Equation 4.9.1 to solving a first order equation. Unlike the method of undetermined coefficients, it does not require P_0, P_1, and P_2 to be constants, or F to be of any special form.

By now you shoudn't be surprised that we look for solutions of Equation 4.9.1 in the form

$$y = uy_1 \tag{4.9.3}$$

where u is to be determined so that y satisfies Equation 4.9.1. Substituting Equation 4.9.3 and

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

into Equation 4.9.1 yields

$$P_0(x)(u''y_1 + 2u'y_1' + uy_1'') + P_1(x)(u'y_1 + uy_1') + P_2(x)uy_1 = F(x).$$

Collecting the coefficients of u, u', and u'' yields

$$(P_0y_1)u'' + (2P_0y_1' + P_1y_1)u' + (P_0y_1'' + P_1y_1' + P_2y_1)u = F. \tag{4.9.4}$$

However, the coefficient of u is zero, since y_1 satisfies Equation 4.9.2. Therefore Equation 4.9.4 reduces to

$$Q_0(x)u'' + Q_1(x)u' = F, \tag{4.9.5}$$

with

$$Q_0 = P_0y_1 \quad \text{and} \quad Q_1 = 2P_0y_1' + P_1y_1.$$

(It isn't worthwhile to memorize the formulas for Q_0 and Q_1!) Since Equation 4.9.5 is a linear first order equation in u', we can solve it for u' by variation of parameters as in Section 1.2, integrate the solution to obtain u, and then obtain y from Equation 4.9.3.

Example 5.6.1

a. Find the general solution of

$$xy'' - (2x+1)y' + (x+1)y = x^2, \tag{4.9.6}$$

given that $y_1 = e^x$ is a solution of the complementary equation

$$xy'' - (2x+1)y' + (x+1)y = 0. \tag{4.9.7}$$

b. As a byproduct of (a), find a fundamental set of solutions of Equation 4.9.7.

Solution

a. If $y = ue^x$, then $y' = u'e^x + ue^x$ and $y'' = u''e^x + 2u'e^x + ue^x$, so

$$\begin{aligned} xy'' - (2x+1)y' + (x+1)y &= x(u''e^x + 2u'e^x + ue^x) - (2x+1)(u'e^x + ue^x) + (x+1)ue^x \\ &= (xu'' - u')e^x. \end{aligned}$$

Therefore $y = ue^x$ is a solution of Equation 4.9.6 if and only if

$$(xu'' - u')e^x = x^2,$$

which is a first order equation in u'. We rewrite it as

$$u'' - \frac{u'}{x} = xe^{-x}. \tag{4.9.8}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that Equation 4.9.8 becomes

$$z' - \frac{z}{x} = xe^{-x}. \tag{4.9.9}$$

We leave it to you to show (by separation of variables) that $z_1 = x$ is a solution of the complementary equation

$$z' - \frac{z}{x} = 0$$

for Equation 4.9.9. By applying variation of parameters as in Section 1.2, we can now see that every solution of Equation 4.9.9 is of the form

$$z = vx \quad \text{where} \quad v'x = xe^{-x}, \quad \text{so} \quad v' = e^{-x} \quad \text{and} \quad v = -e^{-x} + C_1.$$

Since $u' = z = vx$, u is a solution of Equation 4.9.8 if and only if

$$u' = vx = -xe^{-x} + C_1 x.$$

Integrating this yields

$$u = (x+1)e^{-x} + \frac{C_1}{2}x^2 + C_2.$$

Therefore the general solution of Equation 4.9.6 is

$$y = ue^x = x + 1 + \frac{C_1}{2}x^2e^x + C_2e^x. \tag{4.9.10}$$

b. By letting $C_1 = C_2 = 0$ in Equation 4.9.10, we see that $y_{p_1} = x + 1$ is a solution of Equation 4.9.6. By letting $C_1 = 2$ and $C_2 = 0$, we see that $y_{p_2} = x + 1 + x^2e^x$ is also a solution of Equation 4.9.6. Since the difference of two solutions of Equation 4.9.6 is a solution of Equation 4.9.7, $y_2 = y_{p_1} - y_{p_2} = x^2e^x$ is a solution of Equation 4.9.7. Since y_2/y_1 is nonconstant and we already know that $y_1 = e^x$ is a solution of Equation 4.9.6, Theorem 5.1.6 implies that $\{e^x, x^2e^x\}$ is a fundamental set of solutions of Equation 4.9.7.

Although Equation 4.9.10 is a correct form for the general solution of Equation 4.9.6, it is silly to leave the arbitrary coefficient of x^2e^x as $C_1/2$ where C_1 is an arbitrary constant. Moreover, it is sensible to make the subscripts of the coefficients of $y_1 = e^x$ and $y_2 = x^2e^x$ consistent with the subscripts of the functions themselves. Therefore we rewrite Equation 4.9.10 as

$$y = x + 1 + c_1e^x + c_2x^2e^x$$

by simply renaming the arbitrary constants. We'll also do this in the next two examples, and in the answers to the exercises.

Example 5.6.2

a. Find the general solution of

$$x^2y'' + xy' - y = x^2 + 1,$$

given that $y_1 = x$ is a solution of the complementary equation

$$x^2y'' + xy' - y = 0. \tag{4.9.11}$$

As a byproduct of this result, find a fundamental set of solutions of Equation 4.9.11.

b. Solve the initial value problem

$$x^2y'' + xy' - y = x^2 + 1, \quad y(1) = 2,\ y'(1) = -3. \tag{4.9.12}$$

Solution

a. If $y = ux$, then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y''+xy'-y &= x^2(u''x+2u')+x(u'x+u)-ux \\ &= x^3u''+3x^2u'. \end{aligned}$$

Therefore $y=ux$ is a solution of Equation 4.9.12 if and only if

$$x^3u''+3x^2u'=x^2+1,$$

which is a first order equation in u'. We rewrite it as

$$u''+\frac{3}{x}u'=\frac{1}{x}+\frac{1}{x^3}. \tag{4.9.13}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z=u'$, so that Equation 4.9.13 becomes

$$z'+\frac{3}{x}z=\frac{1}{x}+\frac{1}{x^3}. \tag{4.9.14}$$

We leave it to you to show by separation of variables that $z_1=1/x^3$ is a solution of the complementary equation

$$z'+\frac{3}{x}z=0$$

for Equation 4.9.14. By variation of parameters, every solution of Equation 4.9.14 is of the form

$$z=\frac{v}{x^3} \quad \text{where} \quad \frac{v'}{x^3}=\frac{1}{x}+\frac{1}{x^3}, \quad \text{so} \quad v'=x^2+1 \quad \text{and} \quad v=\frac{x^3}{3}+x+C_1.$$

Since $u'=z=v/x^3$, u is a solution of Equation 4.9.14 if and only if

$$u'=\frac{v}{x^3}=\frac{1}{3}+\frac{1}{x^2}+\frac{C_1}{x^3}.$$

Integrating this yields

$$u=\frac{x}{3}-\frac{1}{x}-\frac{C_1}{2x^2}+C_2.$$

Therefore the general solution of Equation 4.9.12 is

$$y=ux=\frac{x^2}{3}-1-\frac{C_1}{2x}+C_2x. \tag{4.9.15}$$

Reasoning as in the solution of Example 4.9.1a, we conclude that $y_1=x$ and $y_2=1/x$ form a fundamental set of solutions for Equation 4.9.11.

As we explained above, we rename the constants in Equation 4.9.15 and rewrite it as

$$y=\frac{x^2}{3}-1+c_1x+\frac{c_2}{x}. \tag{4.9.16}$$

b. Differentiating Equation 4.9.16 yields

$$y'=\frac{2x}{3}+c_1-\frac{c_2}{x^2}. \tag{4.9.17}$$

Setting $x=1$ in Equation 4.9.16 and Equation 4.9.17 and imposing the initial conditions $y(1)=2$ and $y'(1)=-3$ yields

$$\begin{aligned} c_1+c_2 &= \frac{8}{3} \\ c_1-c_2 &= -\frac{11}{3}. \end{aligned}$$

Solving these equations yields $c_1=-1/2$, $c_2=19/6$. Therefore the solution of Equation 4.9.12 is

$$y=\frac{x^2}{3}-1-\frac{x}{2}+\frac{19}{6x}.$$

Using reduction of order to find the general solution of a homogeneous linear second order equation leads to a homogeneous linear first order equation in u' that can be solved by separation of variables. The next example illustrates this.

Example 5.6.3

Find the general solution and a fundamental set of solutions of

$$x^2y'' - 3xy' + 3y = 0, \tag{4.9.18}$$

given that $y_1 = x$ is a solution.

Solution

If $y = ux$ then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(u''x + 2u') - 3x(u'x + u) + 3ux \\ &= x^3u'' - x^2u'. \end{aligned}$$

Therefore $y = ux$ is a solution of Equation 4.9.18if and only if

$$x^3u'' - x^2u' = 0.$$

Separating the variables u' and x yields

$$\frac{u''}{u'} = \frac{1}{x},$$

so

$$\ln|u'| = \ln|x| + k, \quad \text{or equivalently} \quad u' = C_1x.$$

Therefore

$$u = \frac{C_1}{2}x^2 + C_2,$$

so the general solution of Equation 4.9.18is

$$y = ux = \frac{C_1}{2}x^3 + C_2x,$$

which we rewrite as

$$y = c_1x + c_2x^3.$$

Therefore $\{x, x^3\}$ is a fundamental set of solutions of Equation 4.9.18.

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4.9E: Reduction of Order (Exercises)

Q4.9.1

In *Exercises 4.9.1-4.9.17* find the general solution, given that y_1 satisfies the complementary equation. As a byproduct, find a fundamental set of solutions of the complementary equation.

1. $(2x+1)y''-2y'-(2x+3)y=(2x+1)^2; \quad y_1=e^{-x}$
2. $x^2y''+xy'-y=\frac{4}{x^2}; \quad y_1=x$
3. $x^2y''-xy'+y=x; \quad y_1=x$
4. $y''-3y'+2y=\frac{1}{1+e^{-x}}; \quad y_1=e^{2x}$
5. $y''-2y'+y=7x^{3/2}e^x; \quad y_1=e^x$
6. $4x^2y''+(4x-8x^2)y'+(4x^2-4x-1)y=4x^{1/2}e^x(1+4x); \quad y_1=x^{1/2}e^x$
7. $y''-2y'+2y=e^x\sec x; \quad y_1=e^x\cos x$
8. $y''+4xy'+(4x^2+2)y=8e^{-x(x+2)}; \quad y_1=e^{-x^2}$
9. $x^2y''+xy'-4y=-6x-4; \quad y_1=x^2$
10. $x^2y''+2x(x-1)y'+(x^2-2x+2)y=x^3e^{2x}; \quad y_1=xe^{-x}$
11. $x^2y''-x(2x-1)y'+(x^2-x-1)y=x^2e^x; \quad y_1=xe^x$
12. $(1-2x)y''+2y'+(2x-3)y=(1-4x+4x^2)e^x; \quad y_1=e^x$
13. $x^2y''-3xy'+4y=4x^4; \quad y_1=x^2$
14. $2xy''+(4x+1)y'+(2x+1)y=3x^{1/2}e^{-x}; \quad y_1=e^{-x}$
15. $xy''-(2x+1)y'+(x+1)y=-e^x; \quad y_1=e^x$
16. $4x^2y''-4x(x+1)y'+(2x+3)y=4x^{5/2}e^{2x}; \quad y_1=x^{1/2}$
17. $x^2y''-5xy'+8y=4x^2; \quad y_1=x^2$

Q4.9.2

In *Exercises 4.9.18-4.9.30* find a fundamental set of solutions, given that y_1 is a solution.

18. $xy''+(2-2x)y'+(x-2)y=0; \quad y_1=e^x$
19. $x^2y''-4xy'+6y=0; \quad y_1=x^2$
20. $x^2(\ln|x|)^2y''-(2x\ln|x|)y'+(2+\ln|x|)y=0; \quad y_1=\ln|x|$
21. $4xy''+2y'+y=0; \quad y_1=\sin\sqrt{x}$
22. $xy''-(2x+2)y'+(x+2)y=0; \quad y_1=e^x$
23. $x^2y''-(2a-1)xy'+a^2y=0; \quad y_1=x^a$
24. $x^2y''-2xy'+(x^2+2)y=0; \quad y_1=x\sin x$
25. $xy''-(4x+1)y'+(4x+2)y=0; \quad y_1=e^{2x}$
26. $4x^2(\sin x)y''-4x(x\cos x+\sin x)y'+(2x\cos x+3\sin x)y=0; \quad y_1=x^{1/2}$
27. $4x^2y''-4xy'+(3-16x^2)y=0; \quad y_1=x^{1/2}e^{2x}$
28. $(2x+1)xy''-2(2x^2-1)y'-4(x+1)y=0; \quad y_1=1/x$
29. $(x^2-2x)y''+(2-x^2)y'+(2x-2)y=0; \quad y_1=e^x$
30. $xy''-(4x+1)y'+(4x+2)y=0; \quad y_1=e^{2x}$

Q4.9.3

In *Exercises 4.9.31-4.9.33* solve the initial value problem, given that y_1 satisfies the complementary equation.

31. $x^2y'' - 3xy' + 4y = 4x^4, \quad y(-1) = 7, \quad y'(-1) = -8; \quad y_1 = x^2$

32. $(3x-1)y'' - (3x+2)y' - (6x-8)y = 0, \quad y(0) = 2,\ y'(0) = 3; \quad y_1 = e^{2x}$

33. $(x+1)^2y'' - 2(x+1)y' - (x^2+2x-1)y = (x+1)^3e^x, \quad y(0) = 1, \quad y'(0) = -1; \quad y_1 = (x+1)e^x$

Q4.9.4

In *Exercises 4.9.34* and *4.9.35* solve the initial value problem and graph the solution, given that y_1 satisfies the complementary equation.

34. $x^2y'' + 2xy' - 2y = x^2, \quad y(1) = \frac{5}{4},\ y'(1) = \frac{3}{2}; \quad y_1 = x$

35. $(x^2-4)y'' + 4xy' + 2y = x+2, \quad y(0) = -\frac{1}{3}, \quad y'(0) = -1; \quad y_1 = \frac{1}{x-2}$

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4.10: Variation of Parameters

In this section we give a method called **variation of parameters** for finding a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{4.10.1}$$

if we know a fundamental set $\{y_1, y_2\}$ of solutions of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{4.10.2}$$

Having found a particular solution y_p by this method, we can write the general solution of Equation 4.10.1 as

$$y = y_p + c_1y_1 + c_2y_2.$$

Since we need only one nontrivial solution of Equation 4.10.2 to find the general solution of Equation 4.10.1 by reduction of order, it is natural to ask why we are interested in variation of parameters, which requires two linearly independent solutions of Equation 4.10.2 to achieve the same goal. Here's the answer:

- If we already know two linearly independent solutions of Equation 4.10.2 then variation of parameters will probably be simpler than reduction of order.
- Variation of parameters generalizes naturally to a method for finding particular solutions of higher order linear equations (Section 9.4) and linear systems of equations (Section 10.7), while reduction of order doesn't.
- Variation of parameters is a powerful theoretical tool used by researchers in differential equations. Although a detailed discussion of this is beyond the scope of this book, you can get an idea of what it means from **Exercises 5.7.37–5.7.39**.

We'll now derive the method. As usual, we consider solutions of Equation 4.10.1 and Equation 4.10.2 on an interval (a, b) where P_0, P_1, P_2, and F are continuous and P_0 has no zeros. Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation Equation 4.10.2. We look for a particular solution of Equation 4.10.1 in the form

$$y_p = u_1y_1 + u_2y_2 \tag{4.10.3}$$

where u_1 and u_2 are functions to be determined so that y_p satisfies Equation 4.10.1. You may not think this is a good idea, since there are now two unknown functions to be determined, rather than one. However, since u_1 and u_2 have to satisfy only one condition (that y_p is a solution of Equation 4.10.1), we can impose a second condition that produces a convenient simplification, as follows.

Differentiating Equation 4.10.3 yields

$$y_p' = u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2. \tag{4.10.4}$$

As our second condition on u_1 and u_2 we require that

$$u_1'y_1 + u_2'y_2 = 0. \tag{4.10.5}$$

Then Equation 4.10.4 becomes

$$y_p' = u_1y_1' + u_2y_2'; \tag{4.10.6}$$

that is, Equation 4.10.5 permits us to differentiate y_p (once!) as if u_1 and u_2 are constants. Differentiating Equation 4.10.4 yields

$$y_p'' = u_1y_1'' + u_2y_2'' + u_1'y_1' + u_2'y_2'. \tag{4.10.7}$$

(There are no terms involving u_1'' and u_2'' here, as there would be if we hadn't required Equation 4.10.5.) Substituting Equation 4.10.3, Equation 4.10.6, and Equation 4.10.7 into Equation 4.10.1 and collecting the coefficients of u_1 and u_2 yields

$$u_1(P_0y_1'' + P_1y_1' + P_2y_1) + u_2(P_0y_2'' + P_1y_2' + P_2y_2) + P_0(u_1'y_1' + u_2'y_2') = F.$$

As in the derivation of the method of reduction of order, the coefficients of u_1 and u_2 here are both zero because y_1 and y_2 satisfy the complementary equation. Hence, we can rewrite the last equation as

$$P_0(u_1'y_1' + u_2'y_2') = F. \tag{4.10.8}$$

Therefore y_p in Equation 4.10.3 satisfies Equation 4.10.1 if

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \tfrac{F}{P_0}, \end{aligned} \tag{4.10.9}$$

where the first equation is the same as Equation 4.10.5 and the second is from Equation 4.10.8.

We'll now show that you can always solve Equation 4.10.9 for u_1' and u_2'. (The method that we use here will always work, but simpler methods usually work when you're dealing with specific equations.) To obtain u_1', multiply the first equation in Equation 4.10.9 by y_2' and the second equation by y_2. This yields

$$\begin{aligned} u_1'y_1y_2' + u_2'y_2y_2' &= 0 \\ u_1'y_1'y_2 + u_2'y_2'y_2 &= \frac{Fy_2}{P_0}. \end{aligned}$$

Subtracting the second equation from the first yields

$$u_1'(y_1y_2' - y_1'y_2) = -\frac{Fy_2}{P_0}. \tag{4.10.10}$$

Since $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 4.10.2 on (a, b), Theorem 5.1.6 implies that the Wronskian $y_1y_2' - y_1'y_2$ has no zeros on (a, b). Therefore we can solve Equation 4.10.10 for u_1', to obtain

$$u_1' = -\frac{Fy_2}{P_0(y_1y_2' - y_1'y_2)}. \tag{4.10.11}$$

We leave it to you to start from Equation 4.10.9 and show by a similar argument that

$$u_2' = \frac{Fy_1}{P_0(y_1y_2' - y_1'y_2)}. \tag{4.10.12}$$

We can now obtain u_1 and u_2 by integrating u_1' and u_2'. The constants of integration can be taken to be zero, since any choice of u_1 and u_2 in Equation 4.10.3 will suffice.

You should not memorize Equation 4.10.11 and Equation 4.10.12 On the other hand, you don't want to rederive the whole procedure for every specific problem. We recommend the a compromise:

a. Write

$$y_p = u_1y_1 + u_2y_2 \tag{4.10.13}$$

to remind yourself of what you're doing.

b. Write the system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \tfrac{F}{P_0} \end{aligned} \tag{4.10.14}$$

for the specific problem you're trying to solve.

c. Solve Equation 4.10.14 for u_1' and u_2' by any convenient method.
d. Obtain u_1 and u_2 by integrating u_1' and u_2', taking the constants of integration to be zero.
e. Substitute u_1 and u_2 into Equation 4.10.13 to obtain y_p.

Example 5.7.1

Find a particular solution y_p of

$$x^2y'' - 2xy' + 2y = x^{9/2}, \tag{4.10.15}$$

given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation

$$x^2y'' - 2xy' + 2y = 0.$$

Then find the general solution of Equation 4.10.15.

Solution

We set

$$y_p = u_1 x + u_2 x^2,$$

where

$$\begin{aligned} u_1' x + \quad u_2' x^2 &= 0 \\ u_1' \quad + 2u_2' x \quad &= \frac{x^{9/2}}{x^2} = x^{5/2}. \end{aligned}$$

From the first equation, $u_1' = -u_2' x$. Substituting this into the second equation yields $u_2' x = x^{5/2}$, so $u_2' = x^{3/2}$ and therefore $u_1' = -u_2' x = -x^{5/2}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -\frac{2}{7} x^{7/2} \quad \text{and} \quad u_2 = \frac{2}{5} x^{5/2}.$$

Therefore

$$y_p = u_1 x + u_2 x^2 = -\frac{2}{7} x^{7/2} x + \frac{2}{5} x^{5/2} x^2 = \frac{4}{35} x^{9/2},$$

and the general solution of Equation 4.10.15is

$$y = \frac{4}{35} x^{9/2} + c_1 x + c_2 x^2.$$

Example 5.7.2

Find a particular solution y_p of

$$(x-1)y'' - xy' + y = (x-1)^2, \tag{4.10.16}$$

given that $y_1 = x$ and $y_2 = e^x$ are solutions of the complementary equation

$$(x-1)y'' - xy' + y = 0.$$

Then find the general solution of Equation 4.10.16.

Solution

We set

$$y_p = u_1 x + u_2 e^x,$$

where

$$\begin{aligned} u_1' x + u_2' e^x &= 0 \\ u_1' \quad + u_2' e^x &= \frac{(x-1)^2}{x-1} = x - 1. \end{aligned}$$

Subtracting the first equation from the second yields $-u_1'(x-1) = x-1$, so $u_1' = -1$. From this and the first equation, $u_2' = -xe^{-x} u_1' = xe^{-x}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -x \quad \text{and} \quad u_2 = -(x+1)e^{-x}.$$

Therefore

$$y_p = u_1 x + u_2 e^x = (-x)x + (-(x+1)e^{-x})e^x = -x^2 - x - 1,$$

so the general solution of Equation 4.10.16is

$$y = y_p + c_1 x + c_2 e^x = -x^2 - x - 1 + c_1 x + c_2 e^x = -x^2 - 1 + (c_1 - 1)x + c_2 e^x. \tag{4.10.17}$$

However, since c_1 is an arbitrary constant, so is c_1-1; therefore, we improve the appearance of this result by renaming the constant and writing the general solution as

$$y=-x^2-1+c_1x+c_2e^x. \tag{4.10.18}$$

There's nothing **wrong** with leaving the general solution of Equation 4.10.16 in the form Equation 4.10.17; however, we think you'll agree that Equation 4.10.18 is preferable. We can also view the transition from Equation 4.10.17 to Equation 4.10.18 differently. In this example the particular solution $y_p=-x^2-x-1$ contained the term $-x$, which satisfies the complementary equation. We can drop this term and redefine $y_p=-x^2-1$, since $-x^2-x-1$ is a solution of Equation 4.10.16 and x is a solution of the complementary equation; hence, $-x^2-1=(-x^2-x-1)+x$ is also a solution of Equation 4.10.16. In general, it is always legitimate to drop linear combinations of $\{y_1,y_2\}$ from particular solutions obtained by variation of parameters. (See **Exercise 5.7.36** for a general discussion of this question.) We'll do this in the following examples and in the answers to exercises that ask for a particular solution. Therefore, don't be concerned if your answer to such an exercise differs from ours only by a solution of the complementary equation.

Example 5.7.3

Find a particular solution of

$$y''+3y'+2y=\frac{1}{1+e^x}. \tag{4.10.19}$$

Then find the general solution.

Solution

The characteristic polynomial of the complementary equation

$$y''+3y'+2y=0 \tag{4.10.20}$$

is $p(r)=r^2+3r+2=(r+1)(r+2)$, so $y_1=e^{-x}$ and $y_2=e^{-2x}$ form a fundamental set of solutions of Equation 4.10.20. We look for a particular solution of Equation 4.10.19 in the form

$$y_p=u_1e^{-x}+u_2e^{-2x},$$

where

$$\begin{aligned} u_1'e^{-x}+\ \ u_2'e^{-2x}&=0\\ -u_1'e^{-x}-2u_2'e^{-2x}&=\frac{1}{1+e^x}. \end{aligned}$$

Adding these two equations yields

$$-u_2'e^{-2x}=\frac{1}{1+e^x},\quad \text{so}\quad u_2'=-\frac{e^{2x}}{1+e^x}.$$

From the first equation,

$$u_1'=-u_2'e^{-x}=\frac{e^x}{1+e^x}.$$

Integrating by means of the substitution $v=e^x$ and taking the constants of integration to be zero yields

$$u_1=\int\frac{e^x}{1+e^x}\,dx=\int\frac{dv}{1+v}=\ln(1+v)=\ln(1+e^x)$$

and

$$\begin{aligned} u_2&=-\int\frac{e^{2x}}{1+e^x}\,dx=-\int\frac{v}{1+v}\,dv=\int\left[\frac{1}{1+v}-1\right]\,dv\\ &=\ln(1+v)-v=\ln(1+e^x)-e^x. \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= u_1 e^{-x} + u_2 e^{-2x} \\ &= [\ln(1+e^x)]e^{-x} + [\ln(1+e^x) - e^x]\, e^{-2x}, \end{aligned}$$

so

$$y_p = \left(e^{-x} + e^{-2x}\right) \ln(1+e^x) - e^{-x}.$$

Since the last term on the right satisfies the complementary equation, we drop it and redefine

$$y_p = \left(e^{-x} + e^{-2x}\right) \ln(1+e^x).$$

The general solution of Equation 4.10.19is

$$y = y_p + c_1 e^{-x} + c_2 e^{-2x} = \left(e^{-x} + e^{-2x}\right) \ln(1+e^x) + c_1 e^{-x} + c_2 e^{-2x}.$$

Example 5.7.4

Solve the initial value problem

$$(x^2-1)y'' + 4xy' + 2y = \frac{2}{x+1}, \quad y(0) = -1, \quad y'(0) = -5, \tag{4.10.21}$$

given that

$$y_1 = \frac{1}{x-1} \quad \text{and} \quad y_2 = \frac{1}{x+1}$$

are solutions of the complementary equation

$$(x^2-1)y'' + 4xy' + 2y = 0.$$

Solution

We first use variation of parameters to find a particular solution of

$$(x^2-1)y'' + 4xy' + 2y = \frac{2}{x+1}$$

on $(-1,1)$ in the form

$$y_p = \frac{u_1}{x-1} + \frac{u_2}{x+1},$$

where

$$\frac{u_1'}{x-1} + \frac{u_2'}{x+1} = 0 \tag{4.10.22}$$

$$-\frac{u_1'}{(x-1)^2} - \frac{u_2'}{(x+1)^2} = \frac{2}{(x+1)(x^2-1)}$$

Multiplying the first equation by $1/(x-1)$ and adding the result to the second equation yields

$$\left[\frac{1}{x^2-1} - \frac{1}{(x+1)^2}\right] u_2' = \frac{2}{(x+1)(x^2-1)}. \tag{4.10.23}$$

Since

$$\left[\frac{1}{x^2-1} - \frac{1}{(x+1)^2}\right] = \frac{(x+1)-(x-1)}{(x+1)(x^2-1)} = \frac{2}{(x+1)(x^2-1)},$$

Equation 4.10.23implies that $u_2' = 1$. From Equation 4.10.22,

$$u_1' = -\frac{x-1}{x+1} u_2' = -\frac{x-1}{x+1}.$$

Integrating and taking the constants of integration to be zero yields

$$\begin{aligned} u_1 &= -\int \frac{x-1}{x+1}\, dx = -\int \frac{x+1-2}{x+1}\, dx \\ &= \int \left[\frac{2}{x+1} - 1\right] dx = 2\ln(x+1) - x \end{aligned}$$

and

$$u_2 = \int dx = x.$$

Therefore

$$\begin{aligned} y_p &= \frac{u_1}{x-1} + \frac{u_2}{x+1} = [2\ln(x+1) - x]\frac{1}{x-1} + x\frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + x\left[\frac{1}{x+1} - \frac{1}{x-1}\right] = \frac{2\ln(x+1)}{x-1} - \frac{2x}{(x+1)(x-1)}. \end{aligned}$$

However, since

$$\frac{2x}{(x+1)(x-1)} = \left[\frac{1}{x+1} + \frac{1}{x-1}\right]$$

is a solution of the complementary equation, we redefine

$$y_p = \frac{2\ln(x+1)}{x-1}.$$

Therefore the general solution of Equation 4.10.24is

$$y = \frac{2\ln(x+1)}{x-1} + \frac{c_1}{x-1} + \frac{c_2}{x+1}. \tag{4.10.24}$$

Differentiating this yields

$$y' = \frac{2}{x^2-1} - \frac{2\ln(x+1)}{(x-1)^2} - \frac{c_1}{(x-1)^2} - \frac{c_2}{(x+1)^2}.$$

Setting $x = 0$ in the last two equations and imposing the initial conditions $y(0) = -1$ and $y'(0) = -5$ yields the system

$$\begin{aligned} -c_1 + c_2 &= -1 \\ -2 - c_1 - c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = 2,\ c_2 = 1$. Substituting these into Equation 4.10.24yields

$$\begin{aligned} y &= \frac{2\ln(x+1)}{x-1} + \frac{2}{x-1} + \frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1} \end{aligned}$$

as the solution of Equation 4.10.21. Figure 5.7.1 is a graph of the solution.

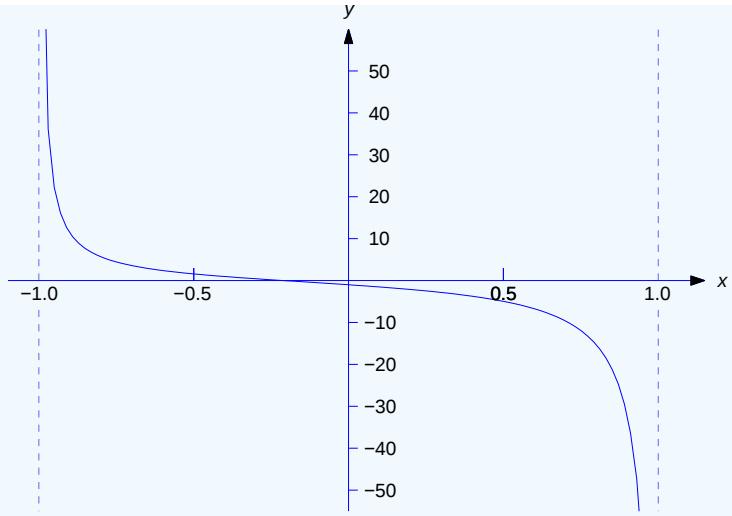

Figure 5.7.1 : $y = \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1}$

We've now considered three methods for solving nonhomogeneous linear equations: undetermined coefficients, reduction of order, and variation of parameters. It's natural to ask which method is best for a given problem. The method of undetermined coefficients should be used for constant coefficient equations with forcing functions that are linear combinations of polynomials multiplied by functions of the form $e^{\alpha x}$, $e^{\lambda x}\cos\omega x$, or $e^{\lambda x}\sin\omega x$. Although the other two methods can be used to solve such problems, they will be more difficult except in the most trivial cases, because of the integrations involved.

If the equation isn't a constant coefficient equation or the forcing function isn't of the form just specified, the method of undetermined coefficients does not apply and the choice is necessarily between the other two methods. The case could be made that reduction of order is better because it requires only one solution of the complementary equation while variation of parameters requires two. However, variation of parameters will probably be easier if you already know a fundamental set of solutions of the complementary equation.

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4.10E: Variation of Parameters (Exercises)

Q4.10.1

In *Exercises 4.10.1–4.10.6* use variation of parameters to find a general solution.

1. $y''+9y=\tan 3x$
2. $y''+4y=\sin 2x\sec^2 2x$
3. $y''-3y'+2y=\frac{4}{1+e^{-x}}$
4. $y''-2y'+2y=3e^x\sec x$
5. $y''-2y'+y=14x^{3/2}e^x$
6. $y''-y=\frac{4e^{-x}}{1-e^{-2x}}$

Q4.10.2

In *Exercises 4.10.7- 4.10.29* use variation of parameters to find a particular solution, given the solutions y_1, y_2 of the complementary equation.

7. $x^2y''+xy'-y=2x^2+2;\quad y_1=x,\quad y_2=\frac{1}{x}$
8. $xy''+(2-2x)y'+(x-2)y=e^{2x};\quad y_1=e^x,\quad y_2=\frac{e^x}{x}$
9. $4x^2y''+(4x-8x^2)y'+(4x^2-4x-1)y=4x^{1/2}e^x,\quad x>0\ ;y_1=x^{1/2}e^x,\ y_2=x^{-1/2}e^x$
10. $y''+4xy'+(4x^2+2)y=4e^{-x(x+2)};\quad y_1=e^{-x^2},\quad y_2=xe^{-x^2}$
11. $x^2y''-4xy'+6y=x^{5/2},\ x>0;\quad y_1=x^2,\ y_2=x^3$
12. $x^2y''-3xy'+3y=2x^4\sin x;\quad y_1=x,\ y_2=x^3$
13. $(2x+1)y''-2y'-(2x+3)y=(2x+1)^2e^{-x};\quad y_1=e^{-x},\quad y_2=xe^x$
14. $4xy''+2y'+y=\sin\sqrt{x};\quad y_1=\cos\sqrt{x},\quad y_2=\sin\sqrt{x}$
15. $xy''-(2x+2)y'+(x+2)y=6x^3e^x;\quad y_1=e^x,\quad y_2=x^3e^x$
16. $x^2y''-(2a-1)xy'+a^2y=x^{a+1};\quad y_1=x^a,\quad y_2=x^a\ln x$
17. $x^2y''-2xy'+(x^2+2)y=x^3\cos x;\quad y_1=x\cos x,\quad y_2=x\sin x$
18. $xy''-y'-4x^3y=8x^5;\quad y_1=e^{x^2},\ y_2=e^{-x^2}$
19. $(\sin x)y''+(2\sin x-\cos x)y'+(\sin x-\cos x)y=e^{-x};\quad y_1=e^{-x},\quad y_2=e^{-x}\cos x$
20. $4x^2y''-4xy'+(3-16x^2)y=8x^{5/2};\quad y_1=\sqrt{x}e^{2x},\ y_2=\sqrt{x}e^{-2x}$
21. $4x^2y''-4xy'+(4x^2+3)y=x^{7/2};\quad y_1=\sqrt{x}\sin x,\ y_2=\sqrt{x}\cos x$
22. $x^2y''-2xy'-(x^2-2)y=3x^4;\quad y_1=xe^x,\ y_2=xe^{-x}$
23. $x^2y''-2x(x+1)y'+(x^2+2x+2)y=x^3e^x;\quad y_1=xe^x,\quad y_2=x^2e^x$
24. $x^2y''-xy'-3y=x^{3/2};\quad y_1=1/x,\quad y_2=x^3$
25. $x^2y''-x(x+4)y'+2(x+3)y=x^4e^x;\quad y_1=x^2,\quad y_2=x^2e^x$
26. $x^2y''-2x(x+2)y'+(x^2+4x+6)y=2xe^x;\quad y_1=x^2e^x,\quad y_2=x^3e^x$
27. $x^2y''-4xy'+(x^2+6)y=x^4;\quad y_1=x^2\cos x,\quad y_2=x^2\sin x$
28. $(x-1)y''-xy'+y=2(x-1)^2e^x;\quad y_1=x,\quad y_2=e^x$
29. $4x^2y''-4x(x+1)y'+(2x+3)y=x^{5/2}e^x;\quad y_1=\sqrt{x},\quad y_2=\sqrt{x}e^x$

Q4.10.3

In *Exercises 4.10.30–4.10.32* use variation of parameters to solve the initial value problem, given y_1, y_2 are solutions of the complementary equation.

30. $(3x-1)y''-(3x+2)y'-(6x-8)y=(3x-1)^2e^{2x}, \quad y(0)=1,\ y'(0)=2 \ ; y_1=e^{2x},\ y_2=xe^{-x}$

31. $(x-1)^2y''-2(x-1)y'+2y=(x-1)^2, \quad y(0)=3, \quad y'(0)=-6 \ ;$

$y_1=x-1$, $y_2=x^2-1$

32. $(x-1)^2y''-(x^2-1)y'+(x+1)y=(x-1)^3e^x, \quad y(0)=4, \quad y'(0)=-6 \ ;$

$y_1=(x-1)e^x, \quad y_2=x-1$

Q4.10.4

In *Exercises 4.10.33- 4.10.35* use variation of parameters to solve the initial value problem and graph the solution, given that y_1, y_2 are solutions of the complementary equation.

33. $(x^2-1)y''+4xy'+2y=2x, \quad y(0)=0,\ y'(0)=-2; \quad y_1=\frac{1}{x-1},\ y_2=\frac{1}{x+1}$

34. $x^2y''+2xy'-2y=-2x^2, \quad y(1)=1,\ y'(1)=-1; \quad y_1=x,\ y_2=\frac{1}{x^2}$

35. $(x+1)(2x+3)y''+2(x+2)y'-2y=(2x+3)^2, \quad y(0)=0, \quad y'(0)=0 \ ; y_1=x+2, \quad y_2=\frac{1}{x+1}$

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4.11: Variation of Parameters for Higher Order Equations

Derivation of the method

We assume throughout this section that the nonhomogeneous linear equation

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=F(x) \tag{4.11.1}$$

is normal on an interval (a,b). We'll abbreviate this equation as $Ly=F$, where

$$Ly=P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y.$$

When we speak of solutions of this equation and its complementary equation $Ly=0$, we mean solutions on (a,b). We'll show how to use the method of variation of parameters to find a particular solution of $Ly=F$, provided that we know a fundamental set of solutions $\{y_1,y_2,\ldots,y_n\}$ of $Ly=0$.

We seek a particular solution of $Ly=F$ in the form

$$y_p=u_1y_1+u_2y_2+\cdots+u_ny_n \tag{4.11.2}$$

where $\{y_1,y_2,\ldots,y_n\}$ is a known fundamental set of solutions of the complementary equation

$$P_0(x)y^{(n)}+P_1(x)y^{(n-1)}+\cdots+P_n(x)y=0$$

and u_1, u_2, …, u_n are functions to be determined. We begin by imposing the following $n-1$ conditions on $u_1,u_2,\ldots,u_n$:

$$\begin{aligned}
u_1'y_1+u_2'y_2+&\quad\cdots\quad+u_n'y_n=0\\
u_1'y_1'+u_2'y_2'+&\quad\cdots\quad+u_n'y_n'=0\\
&\qquad\vdots\\
u_1'y_1^{(n-2)}+u_2'y_2^{(n-2)}+&\quad\cdots\quad+u_n'y_n^{(n-2)}=0.
\end{aligned} \tag{4.11.3}$$

These conditions lead to simple formulas for the first $n-1$ derivatives of y_p:

$$y_p^{(r)}=u_1y_1^{(r)}+u_2y_2^{(r)}\cdots+u_ny_n^{(r)},\ 0\le r\le n-1. \tag{4.11.4}$$

These formulas are easy to remember, since they look as though we obtained them by differentiating Equation 4.11.2 $n-1$ times while treating u_1, u_2, …, u_n as constants. To see that Equation 4.11.3 implies Equation 4.11.4, we first differentiate Equation 4.11.2 to obtain

$$y_p'=u_1y_1'+u_2y_2'+\cdots+u_ny_n'+u_1'y_1+u_2'y_2+\cdots+u_n'y_n,$$

which reduces to

$$y_p'=u_1y_1'+u_2y_2'+\cdots+u_ny_n'$$

because of the first equation in Equation 4.11.3. Differentiating this yields

$$y_p''=u_1y_1''+u_2y_2''+\cdots+u_ny_n''+u_1'y_1'+u_2'y_2'+\cdots+u_n'y_n',$$

which reduces to

$$y_p''=u_1y_1''+u_2y_2''+\cdots+u_ny_n''$$

because of the second equation in Equation 4.11.3. Continuing in this way yields Equation 4.11.4.

The last equation in Equation 4.11.4 is

$$y_p^{(n-1)}=u_1y_1^{(n-1)}+u_2y_2^{(n-1)}+\cdots+u_ny_n^{(n-1)}.$$

Differentiating this yields

$$y_p^{(n)}=u_1y_1^{(n)}+u_2y_2^{(n)}+\cdots+u_ny_n^{(n)}+u_1'y_1^{(n-1)}+u_2'y_2^{(n-1)}+\cdots+u_n'y_n^{(n-1)}.$$

Substituting this and Equation 4.11.4 into Equation 4.11.1 yields

$$u_1Ly_1+u_2Ly_2+\cdots+u_nLy_n+P_0(x)\left(u_1'y_1^{(n-1)}+u_2'y_2^{(n-1)}+\cdots+u_n'y_n^{(n-1)}\right)=F(x).$$

Since $Ly_i=0\ (1\le i\le n)$, this reduces to

$$u_1'y_1^{(n-1)}+u_2'y_2^{(n-1)}+\cdots+u_n'y_n^{(n-1)}=\frac{F(x)}{P_0(x)}.$$

Combining this equation with Equation 4.11.3 shows that

$$y_p=u_1y_1+u_2y_2+\cdots+u_ny_n$$

is a solution of Equation 4.11.1 if

$$\begin{aligned}
u_1'y_1+u_2'y_2+\quad&\cdots\quad+u_n'y_n=0\\
u_1'y_1'+u_2'y_2'+\quad&\cdots\quad+u_n'y_n'=0\\
&\vdots\\
u_1'y_1^{(n-2)}+u_2'y_2^{(n-2)}+\quad&\cdots\quad+u_n'y_n^{(n-2)}=0\\
u_1'y_1^{(n-1)}+u_2'y_2^{(n-1)}+\quad&\cdots\quad+u_n'y_n^{(n-1)}=F/P_0,
\end{aligned}$$

which can be written in matrix form as

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n\\ y_1' & y_2' & \cdots & y_n'\\ \vdots & \vdots & \ddots & \vdots\\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)}\\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}\end{bmatrix}\begin{bmatrix} u_1'\\ u_2'\\ \vdots\\ u_{n-1}'\\ u_n'\end{bmatrix}=\begin{bmatrix} 0\\ 0\\ \vdots\\ 0\\ F/P_0\end{bmatrix}. \tag{4.11.5}$$

The determinant of this system is the Wronskian W of the fundamental set of solutions $\{y_1,y_2,\ldots,y_n\}$, which has no zeros on (a,b), by Theorem 9.1.4. Solving Equation 4.11.5 by Cramer's rule yields

$$u_j'=(-1)^{n-j}\frac{FW_j}{P_0W},\quad 1\le j\le n, \tag{4.11.6}$$

where W_j is the Wronskian of the set of functions obtained by deleting y_j from $\{y_1,y_2,\ldots,y_n\}$ and keeping the remaining functions in the same order. Equivalently, W_j is the determinant obtained by deleting the last row and j-th column of W.

Having obtained u_1', u_2', $\ldots$, u_n', we can integrate to obtain u_1, u_2, $\ldots$, u_n. As in Section 5.7, we take the constants of integration to be zero, and we drop any linear combination of $\{y_1,y_2,\ldots,y_n\}$ that may appear in y_p.

Note

For efficiency, it's best to compute $W_1,W_2,\cdots,W_n$ first, and then compute W by expanding in cofactors of the last row; thus,

$$W=\sum_{j=1}^{n}(-1)^{n-j}y_j^{(n-1)}W_j.$$

Third Order Equations

If $n=3$, then

$$W=\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

Therefore

$$W_1=\begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix}, \quad W_2=\begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix}, \quad W_3=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$

and Equation 4.11.6 becomes

$$u_1'=\frac{FW_1}{P_0W}, \quad u_2'=-\frac{FW_2}{P_0W}, \quad u_3'=\frac{FW_3}{P_0W}. \tag{4.11.7}$$

Example 9.4.1

Find a particular solution of

$$xy'''-y''-xy'+y=8x^2e^x, \tag{4.11.8}$$

given that $y_1=x$, $y_2=e^x$, and $y_3=e^{-x}$ form a fundamental set of solutions of the complementary equation. Then find the general solution of Equation 4.11.8.

Solution

We seek a particular solution of Equation 4.11.8 of the form

$$y_p=u_1x+u_2e^x+u_3e^{-x}.$$

The Wronskian of $\{y_1,y_2,y_3\}$ is

$$W(x)=\begin{vmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix},$$

so

$$\begin{aligned} W_1 &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}=-2, \\ W_2 &= \begin{vmatrix} x & e^{-x} \\ 1 & -e^{-x} \end{vmatrix}=-e^{-x}(x+1), \\ W_3 &= \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}=e^x(x-1). \end{aligned}$$

Expanding W by cofactors of the last row yields

$$W=0W_1-e^xW_2+e^{-x}W_3=0(-2)-e^x\left(-e^{-x}(x+1)\right)+e^{-x}e^x(x-1)=2x.$$

Since $F(x)=8x^2e^x$ and $P_0(x)=x$,

$$\frac{F}{P_0W}=\frac{8x^2e^x}{x\cdot 2x}=4e^x.$$

Therefore, from Equation 4.11.7

$$\begin{aligned} u_1' &= 4e^xW_1= 4e^x(-2)=-8e^x, \\ u_2' &= -4e^xW_2=-4e^x\left(-e^{-x}(x+1)\right)=4(x+1), \\ u_3' &= 4e^xW_3= 4e^x\left(e^x(x-1)\right)=4e^{2x}(x-1). \end{aligned}$$

Integrating and taking the constants of integration to be zero yields

$$u_1 = -8e^x, \quad u_2 = 2(x+1)^2, u_3 = e^{2x}(2x-3).$$

Hence,

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 + u_3 y_3 \\ &= (-8e^x)x + e^x(2(x+1)^2) + e^{-x}\left(e^{2x}(2x-3)\right) \\ &= e^x(2x^2 - 2x - 1). \end{aligned}$$

Since $-e^x$ is a solution of the complementary equation, we redefine

$$y_p = 2xe^x(x-1).$$

Therefore the general solution of Equation 4.11.8 is

$$y = 2xe^x(x-1) + c_1 x + c_2 e^x + c_3 e^{-x}.$$

Fourth Order Equations

If $n = 4$, then

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix},$$

Therefore

$$W_1 = \begin{vmatrix} y_2 & y_3 & y_4 \\ y_2' & y_3' & y_4' \\ y_2'' & y_3'' & y_4'' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & y_3 & y_4 \\ y_1' & y_3' & y_4' \\ y_1'' & y_3'' & y_4'' \end{vmatrix},$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & y_4 \\ y_1' & y_2' & y_4' \\ y_1'' & y_2'' & y_4'' \end{vmatrix}, \quad W_4 = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix},$$

and Equation 4.11.6 becomes

$$u_1' = -\frac{FW_1}{P_0 W}, \quad u_2' = \frac{FW_2}{P_0 W}, \quad u_3' = -\frac{FW_3}{P_0 W}, \quad u_4' = \frac{FW_4}{P_0 W}. \tag{4.11.9}$$

Example 9.4.2

Find a particular solution of

$$x^4 y^{(4)} + 6x^3 y''' + 2x^2 y'' - 4xy' + 4y = 12x^2, \tag{4.11.10}$$

given that $y_1 = x$, $y_2 = x^2$, $y_3 = 1/x$ and $y_4 = 1/x^2$ form a fundamental set of solutions of the complementary equation. Then find the general solution of Equation 4.11.10 on $(-\infty, 0)$ and $(0, \infty)$.

Solution

We seek a particular solution of Equation 4.11.10 of the form

$$y_p = u_1 x + u_2 x^2 + \frac{u_3}{x} + \frac{u_4}{x^2}.$$

The Wronskian of $\{y_1, y_2, y_3, y_4\}$ is

$$W(x)=\begin{vmatrix} x & x^2 & 1/x & -1/x^2 \\ 1 & 2x & -1/x^2 & -2/x^3 \\ 0 & 2 & 2/x^3 & 6/x^4 \\ 0 & 0 & -6/x^4 & -24/x^5 \end{vmatrix},$$

so

$$W_1=\begin{vmatrix} x^2 & 1/x & 1/x^2 \\ 2x & -1/x^2 & -2/x^3 \\ 2 & 2/x^3 & 6/x^4 \end{vmatrix}=-\frac{12}{x^4},$$

$$W_2=\begin{vmatrix} x & 1/x & 1/x^2 \\ 1 & -1/x^2 & -2/x^3 \\ 0 & 2/x^3 & 6/x^4 \end{vmatrix}=-\frac{6}{x^5},$$

$$W_3=\begin{vmatrix} x & x^2 & 1/x^2 \\ 1 & 2x & -2/x^3 \\ 0 & 2 & 6/x^4 \end{vmatrix}=\frac{12}{x^2},$$

$$W_4=\begin{vmatrix} x & x^2 & 1/x \\ 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \end{vmatrix}=\frac{6}{x}.$$

Expanding W by cofactors of the last row yields

$$\begin{aligned} W &= -0W_1+0W_2-\left(-\frac{6}{x^4}\right)W_3+\left(-\frac{24}{x^5}\right)W_4 \\ &= \frac{6}{x^4}\frac{12}{x^2}-\frac{24}{x^5}\frac{6}{x}=-\frac{72}{x^6}. \end{aligned}$$

Since $F(x)=12x^2$ and $P_0(x)=x^4$,

$$\frac{F}{P_0W}=\frac{12x^2}{x^4}\left(-\frac{x^6}{72}\right)=-\frac{x^4}{6}.$$

Therefore, from Equation 4.11.9,

$$\begin{aligned} u_1' &= -\left(-\frac{x^4}{6}\right)W_1=\frac{x^4}{6}\left(-\frac{12}{x^4}\right)=-2, \\ u_2' &= -\frac{x^4}{6}W_2=-\frac{x^4}{6}\left(-\frac{6}{x^5}\right)=\frac{1}{x}, \\ u_3' &= -\left(-\frac{x^4}{6}\right)W_3=\frac{x^4}{6}\frac{12}{x^2}=2x^2, \\ u_4' &= -\frac{x^4}{6}W_4=-\frac{x^4}{6}\frac{6}{x}=-x^3. \end{aligned}$$

Integrating these and taking the constants of integration to be zero yields

$$u_1=-2x,\quad u_2=\ln|x|,\quad u_3=\frac{2x^3}{3}, u_4=-\frac{x^4}{4}.$$

Hence,

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 + u_3 y_3 + u_4 y_4 \\ &= (-2x)x + (\ln|x|)x^2 + \frac{2x^3}{3}\frac{1}{x} + \left(-\frac{x^4}{4}\right)\frac{1}{x^2} \\ &= x^2 \ln|x| - \frac{19x^2}{12}. \end{aligned}$$

Since $-19x^2/12$ is a solution of the complementary equation, we redefine

$$y_p = x^2 \ln|x|.$$

Therefore

$$y = x^2 \ln|x| + c_1 x + c_2 x^2 + \frac{c_3}{x} + \frac{c_4}{x^2}$$

is the general solution of Equation 4.11.10 on $(-\infty, 0)$ and $(0, \infty)$.

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4.11E: Variation of Parameters for Higher Order Equations (Exercises)

Q4.11.1

In *Exercises 4.11.1-4.11.21* find a particular solution, given the fundamental set of solutions of the complementary equation.

1. $x^3y''' - x^2(x+3)y'' + 2x(x+3)y' - 2(x+3)y = -4x^4 \;\; ; \{x,\, x^2,\, xe^x\}$
2. $y''' + 6xy'' + (6+12x^2)y' + (12x+8x^3)y = x^{1/2}e^{-x^2} \;\; ; \{e^{-x^2},\, xe^{-x^2},\, x^2e^{-x^2}\}$
3. $x^3y''' - 3x^2y'' + 6xy' - 6y = 2x \; ; \{x, x^2, x^3\}$
4. $x^2y''' + 2xy'' - (x^2+2)y' = 2x^2 \;; \{1,\, e^x/x,\, e^{-x}/x\}$
5. $x^3y''' - 3x^2(x+1)y'' + 3x(x^2+2x+2)y' - (x^3+3x^2+6x+6)y = x^4e^{-3x} \;\; ; \{xe^x,\, x^2e^x,\, x^3e^x\}$
6. $x(x^2-2)y''' + (x^2-6)y'' + x(2-x^2)y' + (6-x^2)y = 2(x^2-2)^2 \;\; ; \{e^x,\, e^{-x},\, 1/x\}$
7. $xy''' - (x-3)y'' - (x+2)y' + (x-1)y = -4e^{-x} \;\; ; \{e^x,\, e^x/x,\, e^{-x}/x\}$
8. $4x^3y''' + 4x^2y'' - 5xy' + 2y = 30x^2 \; ; \{\sqrt{x},\, 1/\sqrt{x},\, x^2\}$
9. $x(x^2-1)y''' + (5x^2+1)y'' + 2xy' - 2y = 12x^2 \;\; ; \{x,\, 1/(x-1),\, 1/(x+1)\}$
10. $x(1-x)y''' + (x^2-3x+3)y'' + xy' - y = 2(x-1)^2 \;\; ; \{x,\, 1/x, e^x/x\}$
11. $x^3y''' + x^2y'' - 2xy' + 2y = x^2 \;\; ; \{x,\, x^2,\, 1/x\}$
12. $xy''' - y'' - xy' + y = x^2 \;\; ; \{x,\, e^x,\, e^{-x}\}$
13. $xy^{(4)} + 4y''' = 6\ln|x|; \{1,\, x,\, x^2,\, 1/x\}$
14. $16x^4y^{(4)} + 96x^3y''' + 72x^2y'' - 24xy' + 9y = 96x^{5/2} \; ; \{\sqrt{x},\, 1/\sqrt{x},\, x^{3/2},\, x^{-3/2}\}$
15. $x(x^2-6)y^{(4)} + 2(x^2-12)y''' + x(6-x^2)y'' + 2(12-x^2)y' = 2(x^2-6)^2 \;\; ; \{1,\, 1/x,\, e^x,\, e^{-x}\}$
16. $x^4y^{(4)} - 4x^3y''' + 12x^2y'' - 24xy' + 24y = x^4 \; ; \{x,\, x^2,\, x^3,\, x^4\}$
17. $x^4y^{(4)} - 4x^3y''' + 2x^2(6-x^2)y'' + 4x(x^2-6)y' + (x^4-4x^2+24)y = 4x^5e^x \;\; ; \{xe^x,\, x^2e^x,\, xe^{-x},\, x^2e^{-x}\}$
18. $x^4y^{(4)} + 6x^3y''' + 2x^2y'' - 4xy' + 4y = 12x^2 \;\; ; \{x, x^2, 1/x, 1/x^2\}$
19. $xy^{(4)} + 4y''' - 2xy'' - 4y' + xy = 4e^x \;\; ; \{e^x,\, e^{-x},\, e^x/x,\, e^{-x}/x\}$
20. $xy^{(4)} + (4-6x)y''' + (13x-18)y'' + (26-12x)y' + (4x-12)y = 3e^x \;\; ; \{e^x,\, e^{2x},\, e^x/x,\, e^{2x}/x\}$
21. $x^4y^{(4)} - 4x^3y''' + x^2(12-x^2)y'' + 2x(x^2-12)y' + 2(12-x^2)y = 2x^5 \;\; ; \{x,\, x^2,\, xe^x,\, xe^{-x}\}$

Q4.11.2

In *Exercises 4.11.22-4.11.33* solve the initial value problem, given the fundamental set of solutions of the complementary equation. Graph the solution for *Exercises 4.11.22, 4.11.26, 4.11.29,* and *4.11.30.*

22. $x^3y''' - 2x^2y'' + 3xy' - 3y = 4x, \quad y(1) = 4, \quad y'(1) = 4, \quad y''(1) = 2 \; ; \{x,\, x^3,\, x\ln x\}$
23. $x^3y''' - 5x^2y'' + 14xy' - 18y = x^3, \quad y(1) = 0, \quad y'(1) = 1, \quad y''(1) = 7 \; ; \{x^2,\, x^3,\, x^3\ln x\}$
24. $(5-6x)y''' + (12x-4)y'' + (6x-23)y' + (22-12x)y = -(6x-5)^2e^x \quad \{y(0) = -4, \quad y'(0) = -\frac{3}{2},$
    $y''(0) = -19; \{e^x,\, e^{2x},\, xe^{-x}\}$
25. $x^3y''' - 6x^2y'' + 16xy' - 16y = 9x^4, \quad y(1) = 2, \quad y'(1) = 1, \quad y''(1) = 5 \;; \{x,\, x^4,\, x^4\ln|x|\}$
26. $(x^2-2x+2)y''' - x^2y'' + 2xy' - 2y = (x^2-2x+2)^2, \quad y(0) = 0, \quad y'(0) = 5 \quad ,y''(0) = 0; \{x,\, x^2,\, e^x\}$
27. $x^3y''' + x^2y'' - 2xy' + 2y = x(x+1), \quad y(-1) = -6, \quad y'(-1) = \frac{43}{6}, \quad y''(-1) = -\frac{5}{2}; \{x,\, x^2,\, 1/x\}$
28. $(3x-1)y''' - (12x-1)y'' + 9(x+1)y' - 9y = 2e^x(3x-1)^2, \quad y(0) = \frac{3}{4}, \quad y'(0) = \frac{5}{4},$
    $y''(0) = \frac{1}{4}; \{x+1,\, e^x,\, e^{3x}\}$
29. $(x^2-2)y''' - 2xy'' + (2-x^2)y' + 2xy = 2(x^2-2)^2, \quad y(0) = 1, \quad y'(0) = -5 \quad ,y''(0) = 5; \{x^2,\, e^x,\, e^{-x}\}$

30. $x^4y^{(4)}+3x^3y'''-x^2y''+2xy'-2y=9x^2, \quad y(1)=-7, \quad y'(1)=-11, \quad y''(1)=-5, \quad y'''(1)=6;$
$\{x,\ x^2,\ 1/x,\ x\ln x\}$

31. $(2x-1)y^{(4)}-4xy'''+(5-2x)y''+4xy'-4y=6(2x-1)^2, \quad y(0)=\frac{55}{4}, \quad y'(0)=0, \quad y''(0)=13,$
$y'''(0)=1; \{x,\ e^x,\ e^{-x},\ e^{2x}\}$

32. $4x^4y^{(4)}+24x^3y'''+23x^2y''-xy'+y=6x, \quad y(1)=2, \quad y'(1)=0, \quad y''(1)=4,$
$y'''(1)=-\frac{37}{4}; \{x, \sqrt{x}, 1/x, 1/\sqrt{x}\}$

33. $x^4y^4+5x^3y'''-3x^2y''-6xy'+6y=40x^3, \quad y(-1)=-1,\ y'(-1)=-7$,
$y''(-1)=-1, \quad y'''(-1)=-31\,; \{x,\ x^3,\ 1/x,\ 1/x^2\}$

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CHAPTER OVERVIEW

Chapter 5: Applications of Linear Second Order Equations

In this Chapter, we study applications of linear second order equations.

5.1: Spring Problems I

5.1E: Spring Problems I (Exercises)

5.2: Spring Problems II

5.2E: Spring Problems II (Exercises)

5.3: The RLC Circuit

5.3E: The RLC Circuit (Exercises)

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5.1: Spring Problems I

We consider the motion of an object of mass m, suspended from a spring of negligible mass. We say that the spring–mass system is in **equilibrium** when the object is at rest and the forces acting on it sum to zero. The position of the object in this case is the **equilibrium position**. We define y to be the displacement of the object from its equilibrium position (Figure 6.1.1), measured positive upward.

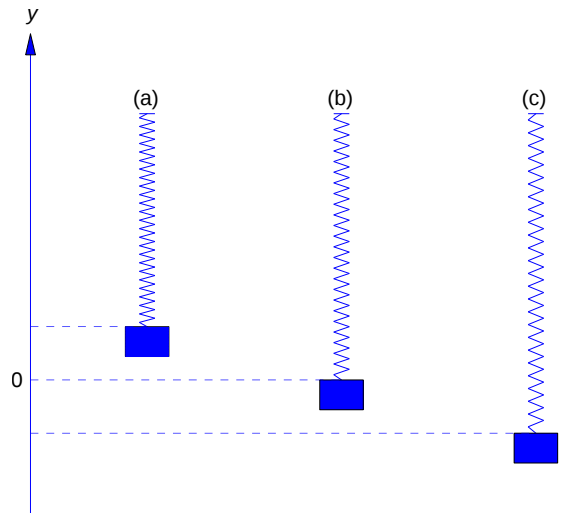

Figure 6.1.1 : (a) $y > 0$ (b) $y = 0$, (c) $y < 0$

Our model accounts for the following kinds of forces acting on the object:

- The force $-mg$, due to gravity.
- A force F_s exerted by the spring resisting change in its length. The **natural length** of the spring is its length with no mass attached. We assume that the spring obeys **Hooke's law**: If the length of the spring is changed by an amount ΔL from its natural length, then the spring exerts a force $F_s = k\Delta L$, where k is a positive number called the **spring constant**. If the spring is stretched then $\Delta L > 0$ and $F_s > 0$, so the spring force is upward, while if the spring is compressed then $\Delta L < 0$ and $F_s < 0$, so the spring force is downward.
- A **damping force** $F_d = -cy'$ that resists the motion with a force proportional to the velocity of the object. It may be due to air resistance or friction in the spring. However, a convenient way to visualize a damping force is to assume that the object is rigidly attached to a piston with negligible mass immersed in a cylinder (called a **dashpot**) filled with a viscous liquid (Figure 6.1.1). As the piston moves, the liquid exerts a damping force. We say that the motion is **undamped** if $c = 0$, or **damped** if $c > 0$.
- An external force F, other than the force due to gravity, that may vary with t, but is independent of displacement and velocity. We say that the motion is **free** if $F \equiv 0$, or **forced** if $F \not\equiv 0$.

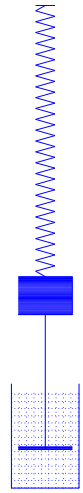

Figure 6.1.2 : A spring – mass system with damping

From Newton's second law of motion,

$$my'' = -mg + F_d + F_s + F = -mg - cy' + F_s + F. \tag{5.1.1}$$

We must now relate F_s to y. In the absence of external forces the object stretches the spring by an amount Δl to assume its equilibrium position (Figure 6.1.3). Since the sum of the forces acting on the object is then zero, Hooke's Law implies that $mg = k\Delta l$. If the object is displaced y units from its equilibrium position, the total change in the length of the spring is $\Delta L = \Delta l - y$, so Hooke's law implies that

$$F_s = k\Delta L = k\Delta l - ky.$$

Substituting this into Equation 5.1.1 yields

$$my'' = -mg - cy' + k\Delta L - ky + F.$$

Since $mg = k\Delta l$ this can be written as

$$my'' + cy' + ky = F. \tag{5.1.2}$$

We call this **the equation of motion**.

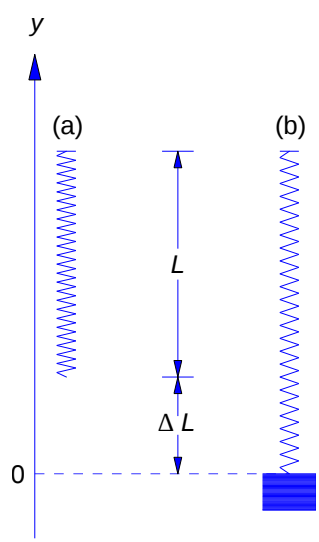

Figure 6.1.3 : (a) Natural length of spring (b) Spring stretched by mass

Simple Harmonic Motion

Throughout the rest of this section we'll consider spring–mass systems without damping; that is, $c = 0$. We'll consider systems with damping in the next section. We first consider the case where the motion is also free; that is, F=0. We begin with an example.

Example 6.1.1

An object stretches a spring 6 inches in equilibrium.

a. Set up the equation of motion and find its general solution.
b. Find the displacement of the object for $t > 0$ if it is initially displaced 18 inches above equilibrium and given a downward velocity of 3 ft/s.

Solution a:

Setting $c = 0$ and $F = 0$ in Equation 5.1.2 yields the equation of motion

$$my'' + ky = 0,$$

which we rewrite as

$$y'' + \frac{k}{m}y = 0. \tag{5.1.3}$$

Although we would need the weight of the object to obtain k from the equation $mg = k\Delta l$ we can obtain k/m from Δl alone; thus, $k/m = g/\Delta l$. Consistent with the units used in the problem statement, we take $g = 32$ ft/s^2. Although Δl is stated in inches, we must convert it to feet to be consistent with this choice of g; that is, $\Delta l = 1/2$ ft. Therefore

$$\frac{k}{m} = \frac{32}{1/2} = 64$$

and Equation 5.1.3 becomes

$$y'' + 64y = 0. \tag{5.1.4}$$

The characteristic equation of Equation 5.1.4 is

$$r^2 + 64 = 0,$$

which has the zeros $r = \pm 8i$. Therefore the general solution of Equation 5.1.4 is

$$y = c_1 \cos 8t + c_2 \sin 8t. \tag{5.1.5}$$

Solution b:

The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus,

$$y(0) = \frac{3}{2} \quad \text{and} \quad y'(0) = -3.$$

Differentiating Equation 5.1.5 yields

$$y' = -8c_1 \sin 8t + 8c_2 \cos 8t. \tag{5.1.6}$$

Setting $t = 0$ in Equation 5.1.5 and Equation 5.1.6 and imposing the initial conditions shows that $c_1 = 3/2$ and $c_2 = -3/8$. Therefore

$$y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t,$$

where y is in feet (Figure 6.1.4).

We'll now consider the equation

$$my'' + ky = 0$$

where m and k are arbitrary positive numbers. Dividing through by m and defining $\omega_0 = \sqrt{k/m}$ yields

$$y'' + \omega_0^2 y = 0.$$

The general solution of this equation is

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t. \tag{5.1.7}$$

We can rewrite this in a more useful form by defining

$$R = \sqrt{c_1^2 + c_2^2}, \tag{5.1.8}$$

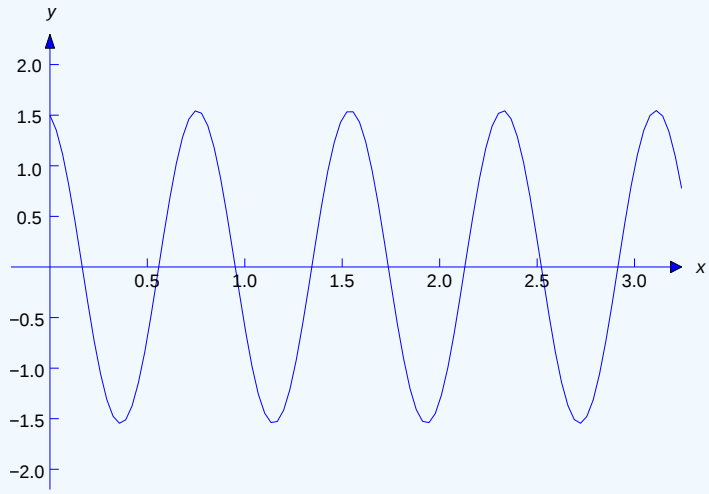

Figure 6.1.4 : $y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t$

and

$$c_1 = R\cos\phi \quad \text{and} \quad c_2 = R\sin\phi. \tag{5.1.9}$$

Substituting from Equation 5.1.9 into Equation 5.1.7 and applying the identity

$$\cos \omega_0 t \cos \phi + \sin \omega_0 t \sin \phi = \cos(\omega_0 t - \phi)$$

yields

$$y = R\cos(\omega_0 t - \phi). \tag{5.1.10}$$

From Equation 5.1.8 and Equation 5.1.9 we see that the R and ϕ can be interpreted as polar coordinates of the point with rectangular coordinates (c_1, c_2) (Figure 6.1.5). Given c_1 and c_2, we can compute R from Equation 5.1.8. From Equation 5.1.8 and Equation 5.1.9, we see that ϕ is related to c_1 and c_2 by

$$\cos\phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \quad \text{and} \quad \sin\phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}.$$

There are infinitely many angles ϕ, differing by integer multiples of 2π, that satisfy these equations. We will always choose ϕ so that $-\pi \leq \phi < \pi$.

The motion described by Equation 5.1.7 or Equation 5.1.10 is **simple harmonic motion**. We see from either of these equations that the motion is periodic, with period

$$T = 2\pi/\omega_0.$$

This is the time required for the object to complete one full cycle of oscillation (for example, to move from its highest position to its lowest position and back to its highest position). Since the highest and lowest positions of the object are $y = R$ and $y = -R$, we say that R is the **amplitude** of the oscillation. The angle ϕ in Equation 5.1.10 is the **phase angle**. It's measured in radians. Equation Equation 5.1.10 is the **amplitude–phase form** of the displacement. If t is in seconds then ω_0 is in radians per second (rad/s); it is the **frequency** of the motion. It is also called the **natural frequency** of the spring–mass system without damping.

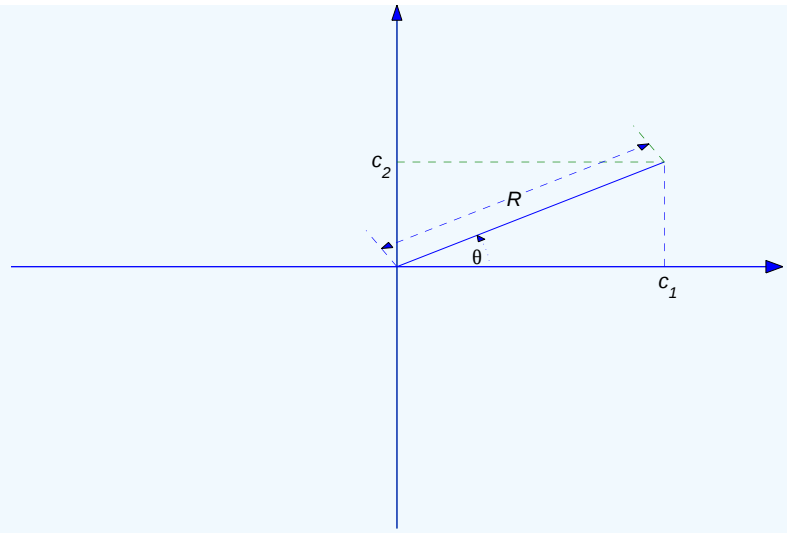

Figure 6.1.5 : $R=\sqrt{c_1^2+c_2^2}$; $c_1=R\cos\phi$; $c_2=R\sin\phi$

Example 6.1.2

We found the displacement of the object in Example 6.1.1 to be

$$y=\frac{3}{2}\cos 8t-\frac{3}{8}\sin 8t.$$

Find the frequency, period, amplitude, and phase angle of the motion.

Solution:

The frequency is $\omega_0=8$ rad/s, and the period is $T=2\pi/\omega_0=\pi/4$ s. Since $c_1=3/2$ and $c_2=-3/8$, the amplitude is

$$R=\sqrt{c_1^2+c_2^2}=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{3}{8}\right)^2}=\frac{3}{8}\sqrt{17}.$$

The phase angle is determined by

$$\cos\phi=\frac{\frac{3}{2}}{\frac{3}{8}\sqrt{17}}=\frac{4}{\sqrt{17}} \tag{5.1.11}$$

and

$$\sin\phi=\frac{-\frac{3}{8}}{\frac{3}{8}\sqrt{17}}=-\frac{1}{\sqrt{17}}. \tag{5.1.12}$$

Using a calculator, we see from Equation 5.1.11that

$$\phi\approx\pm.245 \text{ rad}.$$

Since $\sin\phi<0$ (see Equation 5.1.12), the minus sign applies here; that is,

$$\phi\approx-.245 \text{ rad}.$$

Example 6.1.3

The natural length of a spring is 1 m. An object is attached to it and the length of the spring increases to 102 cm when the object is in equilibrium. Then the object is initially displaced downward 1 cm and given an upward velocity of 14 cm/s. Find the displacement for $t>0$. Also, find the natural frequency, period, amplitude, and phase angle of the resulting motion. Express the answers in cgs units.

Solution:

In cgs units $g = 980$ cm/s^2. Since $\Delta l = 2$ cm, $\omega_0^2 = g/\Delta l = 490$. Therefore

$$y'' + 490y = 0, \quad y(0) = -1, \quad y'(0) = 14.$$

The general solution of the differential equation is

$$y = c_1 \cos 7\sqrt{10}t + c_2 \sin 7\sqrt{10}t,$$

so

$$y' = 7\sqrt{10}\left(-c_1 \sin 7\sqrt{10}t + c_2 \cos 7\sqrt{10}t\right).$$

Substituting the initial conditions into the last two equations yields $c_1 = -1$ and $c_2 = 2/\sqrt{10}$. Hence,

$$y = -\cos 7\sqrt{10}t + \frac{2}{\sqrt{10}} \sin 7\sqrt{10}t.$$

The frequency is $7\sqrt{10}$ rad/s, and the period is $T = 2\pi/(7\sqrt{10})$ s. The amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + \left(\frac{2}{\sqrt{10}}\right)^2} = \sqrt{\frac{7}{5}}\text{cm}$$

The phase angle is determined by

$$\cos\phi = \frac{c_1}{R} = -\sqrt{\frac{5}{7}} \quad \text{and} \quad \sin\phi = \frac{c_2}{R} = \sqrt{\frac{2}{7}}$$

Therefore ϕ is in the second quadrant and

$$\phi = \cos^{-1}\left(-\sqrt{\frac{5}{7}}\right) \approx 2.58\text{rad}$$

Undamped Forced Oscillation

In many mechanical problems a device is subjected to periodic external forces. For example, soldiers marching in cadence on a bridge cause periodic disturbances in the bridge, and the engines of a propeller driven aircraft cause periodic disturbances in its wings. In the absence of sufficient damping forces, such disturbances – even if small in magnitude – can cause structural breakdown if they are at certain critical frequencies. To illustrate, this we'll consider the motion of an object in a spring–mass system without damping, subject to an external force

$$F(t) = F_0 \cos\omega t$$

where F_0 is a constant. In this case the equation of motion Equation 5.1.2 is

$$my'' + ky = F_0 \cos\omega t,$$

which we rewrite as

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos\omega t \tag{5.1.13}$$

with $\omega_0 = \sqrt{k/m}$. We'll see from the next two examples that the solutions of Equation 5.1.13 with $\omega \neq \omega_0$ behave very differently from the solutions with $\omega = \omega_0$.

Example 6.1.4

Solve the initial value problem

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos\omega t, \quad y(0) = 0, \quad y'(0) = 0, \tag{5.1.14}$$

given that $\omega \neq \omega_0$.

Solution:

We first obtain a particular solution of Equation 5.1.13 by the method of undetermined coefficients. Since $\omega \neq \omega_0$, $\cos \omega t$ isn't a solution of the complementary equation

$$y'' + \omega_0^2 y = 0.$$

Therefore Equation 5.1.13 has a particular solution of the form

$$y_p = A \cos \omega t + B \sin \omega t.$$

Since

$$y_p'' = -\omega^2 (A \cos \omega t + B \sin \omega t),$$

$$y_p'' + \omega_0^2 y_p = \frac{F_0}{m} \cos \omega t$$

if and only if

$$(\omega_0^2 - \omega^2)(A \cos \omega t + B \sin \omega t) = \frac{F_0}{m} \cos \omega t.$$

This holds if and only if

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad B = 0,$$

so

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

The general solution of Equation 5.1.13 is

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t, \tag{5.1.15}$$

so

$$y' = \frac{-\omega F_0}{m(\omega_0^2 - \omega^2)} \sin \omega t + \omega_0 (-c_1 \sin \omega_0 t + c_2 \cos \omega_0 t).$$

The initial conditions $y(0) = 0$ and $y'(0) = 0$ in Equation 5.1.14 imply that

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad c_2 = 0.$$

Substituting these into Equation 5.1.15 yields

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t). \tag{5.1.16}$$

It is revealing to write this in a different form. We start with the trigonometric identities

$$\begin{aligned} \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta. \end{aligned}$$

Subtracting the second identity from the first yields

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin \alpha \sin \beta \tag{5.1.17}$$

Now let

$$\alpha - \beta = \omega t \quad \text{and} \quad \alpha + \beta = \omega_0 t, \tag{5.1.18}$$

so that

$$\alpha = \frac{(\omega_0 + \omega)t}{2} \quad \text{and} \quad \beta = \frac{(\omega_0 - \omega)t}{2}. \tag{5.1.19}$$

Substituting Equation 5.1.18and Equation 5.1.19into Equation 5.1.17yields

$$\cos \omega t - \cos \omega_0 t = 2 \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2},$$

and substituting this into Equation 5.1.16yields

$$y = R(t) \sin \frac{(\omega_0 + \omega)t}{2}, \tag{5.1.20}$$

where

$$R(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}. \tag{5.1.21}$$

From Equation 5.1.20we can regard y as a sinusoidal variation with frequency $(\omega_0 + \omega)/2$ and variable amplitude $|R(t)|$. In Figure 6.1.6 the dashed curve above the t axis is $y = |R(t)|$, the dashed curve below the t axis is $y = -|R(t)|$, and the displacement y appears as an oscillation bounded by them. The oscillation of y for t on an interval between successive zeros of $R(t)$ is called a **beat**.

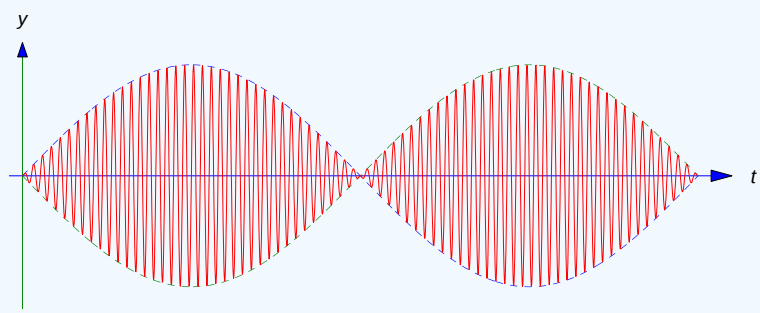

Figure 6.1.6 : Undamped oscillation with beats

You can see from Equation 5.1.20and Equation 5.1.21that

$$|y(t)| \le \frac{2|F_0|}{m|\omega_0^2 - \omega^2|};$$

moreover, if $\omega + \omega_0$ is sufficiently large compared with $\omega - \omega_0$, then $|y|$ assumes values close to (perhaps equal to) this upper bound during each beat. However, the oscillation remains bounded for all t. (This assumes that the spring can withstand deflections of this size and continue to obey Hooke's law.) The next example shows that this isn't so if $\omega = \omega_0$.

Example 6.1.5

Find the general solution of

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t. \tag{5.1.22}$$

Solution:

We first obtain a particular solution y_p of Equation 5.1.22. Since $\cos \omega_0 t$ is a solution of the complementary equation, the form for y_p is

$$y_p = t(A \cos \omega_0 t + B \sin \omega_0 t). \tag{5.1.23}$$

Then

$$y_p' = A \cos \omega_0 t + B \sin \omega_0 t + \omega_0 t(-A \sin \omega_0 t + B \cos \omega_0 t)$$

and

$$y_p'' = 2\omega_0(-A \sin \omega_0 t + B \cos \omega_0 t) - \omega_0^2 t(A \cos \omega_0 t + B \sin \omega_0 t). \tag{5.1.24}$$

From Equation 5.1.23and Equation 5.1.24, we see that y_p satisfies Equation 5.1.22if

$$-2A\omega_0 \sin\omega_0 t + 2B\omega_0 \cos\omega_0 t = \frac{F_0}{m}\cos\omega_0 t;$$

that is, if

$$A = 0 \quad \text{and} \quad B = \frac{F_0}{2m\omega_0}.$$

Therefore

$$y_p = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t$$

is a particular solution of Equation 5.1.22 The general solution of Equation 5.1.22is

$$y = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t + c_1\cos\omega_0 t + c_2\sin\omega_0 t.$$

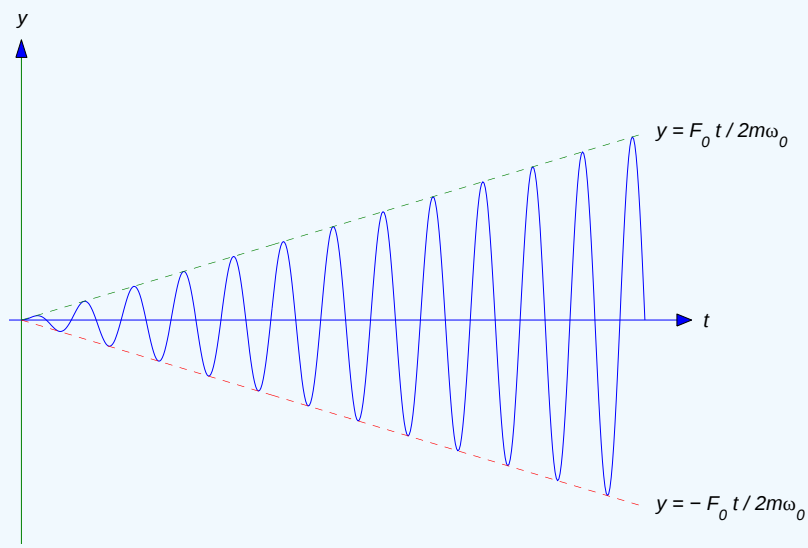

Figure 6.1.7 : Undamped displacement due to resonance.

The graph of y_p is shown in Figure 6.1.7 , where it can be seen that y_p oscillates between the dashed lines

$$y = \frac{F_0 t}{2m\omega_0} \quad \text{and} \quad y = -\frac{F_0 t}{2m\omega_0}$$

with increasing amplitude that approaches ∞ as $t \to \infty$. Of course, this means that the spring must eventually fail to obey Hooke's law or break.

This phenomenon of unbounded displacements of a spring–mass system in response to a periodic forcing function at its natural frequency is called **resonance**. More complicated mechanical structures can also exhibit resonance–like phenomena. For example, rhythmic oscillations of a suspension bridge by wind forces or of an airplane wing by periodic vibrations of reciprocating engines can cause damage or even failure if the frequencies of the disturbances are close to critical frequencies determined by the parameters of the mechanical system in question.

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5.1E: Spring Problems I (Exercises)

In the following exercises assume that there's no damping.

Q5.1.1

1. An object stretches a spring 4 inches in equilibrium. Find and graph its displacement for $t > 0$ if it is initially displaced 36 inches above equilibrium and given a downward velocity of 2 ft/s.

2. An object stretches a string 1.2 inches in equilibrium. Find its displacement for $t > 0$ if it is initially displaced 3 inches below equilibrium and given a downward velocity of 2 ft/s.

3. A spring with natural length $.5$ m has length 50.5 cm with a mass of 2 gm suspended from it. The mass is initially displaced 1.5 cm below equilibrium and released with zero velocity. Find its displacement for $t > 0$.

4. An object stretches a spring 6 inches in equilibrium. Find its displacement for $t > 0$ if it is initially displaced 3 inches above equilibrium and given a downward velocity of 6 inches/s. Find the frequency, period, amplitude and phase angle of the motion.

5. An object stretches a spring 5 cm in equilibrium. It is initially displaced 10 cm above equilibrium and given an upward velocity of $.25$ m/s. Find and graph its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

6. A 10 kg mass stretches a spring 70 cm in equilibrium. Suppose a 2 kg mass is attached to the spring, initially displaced 25 cm below equilibrium, and given an upward velocity of 2 m/s. Find its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

7. A weight stretches a spring 1.5 inches in equilibrium. The weight is initially displaced 8 inches above equilibrium and given a downward velocity of 4 ft/s. Find its displacement for $t > 0$.

8. A weight stretches a spring 6 inches in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/s. Find its displacement for $t > 0$.

9. A spring–mass system has natural frequency $7\sqrt{10}$ rad/s. The natural length of the spring is $.7$ m. What is the length of the spring when the mass is in equilibrium?

10. A 64 lb weight is attached to a spring with constant $k = 8$ lb/ft and subjected to an external force $F(t) = 2\sin t$. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 2 ft/s. Find its displacement for $t > 0$.

11. A unit mass hangs in equilibrium from a spring with constant $k = 1/16$. Starting at $t = 0$, a force $F(t) = 3\sin t$ is applied to the mass. Find its displacement for $t > 0$.

12. A 4 lb weight stretches a spring 1 ft in equilibrium. An external force $F(t) = .25\sin 8t$ lb is applied to the weight, which is initially displaced 4 inches above equilibrium and given a downward velocity of 1 ft/s. Find and graph its displacement for $t > 0$.

13. A 2 lb weight stretches a spring 6 inches in equilibrium. An external force $F(t) = \sin 8t$ lb is applied to the weight, which is released from rest 2 inches below equilibrium. Find its displacement for $t > 0$.

14. A 10 gm mass suspended on a spring moves in simple harmonic motion with period 4 s. Find the period of the simple harmonic motion of a 20 gm mass suspended from the same spring.

15. A 6 lb weight stretches a spring 6 inches in equilibrium. Suppose an external force $F(t) = \frac{3}{16}\sin\omega t + \frac{3}{8}\cos\omega t$ lb is applied to the weight. For what value of ω will the displacement be unbounded? Find the displacement if ω has this value. Assume that the motion starts from equilibrium with zero initial velocity.

16. A 6 lb weight stretches a spring 4 inches in equilibrium. Suppose an external force $F(t) = 4\sin\omega t - 6\cos\omega t$ lb is applied to the weight. For what value of ω will the displacement be unbounded? Find and graph the displacement if ω has this value. Assume that the motion starts from equilibrium with zero initial velocity.

17. A mass of one kg is attached to a spring with constant $k = 4$ N/m. An external force $F(t) = -\cos\omega t - 2\sin\omega t$ n is applied to the mass. Find the displacement y for $t > 0$ if ω equals the natural frequency of the spring–mass system. Assume that the mass is initially displaced 3 m above equilibrium and given an upward velocity of 450 cm/s.

18. An object is in simple harmonic motion with frequency ω_0, with $y(0) = y_0$ and $y'(0) = v_0$. Find its displacement for $t > 0$. Also, find the amplitude of the oscillation and give formulas for the sine and cosine of the initial phase angle.

19. Two objects suspended from identical springs are set into motion. The period of one object is twice the period of the other. How are the weights of the two objects related?

20. Two objects suspended from identical springs are set into motion. The weight of one object is twice the weight of the other. How are the periods of the resulting motions related?

21. Two identical objects suspended from different springs are set into motion. The period of one motion is 3 times the period of the other. How are the two spring constants related?

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5.2: Spring Problems II

Free Vibrations with Damping

In this section we consider the motion of an object in a spring–mass system with damping. We start with unforced motion, so the equation of motion is

$$my'' + cy' + ky = 0. \tag{5.2.1}$$

Now suppose the object is displaced from equilibrium and given an initial velocity. Intuition suggests that if the damping force is sufficiently weak the resulting motion will be oscillatory, as in the undamped case considered in the previous section, while if it is sufficiently strong the object may just move slowly toward the equilibrium position without ever reaching it. We'll now confirm these intuitive ideas mathematically. The characteristic equation of Equation 5.2.1 is

$$mr^2 + cr + k = 0.$$

The roots of this equation are

$$r_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}. \tag{5.2.2}$$

We saw in Section 5.2 that the form of the solution of Equation 5.2.1 depends upon whether $c^2 - 4mk$ is positive, negative, or zero. We'll now consider these three cases.

Underdamped Motion

We say the motion is **underdamped** if $c < \sqrt{4mk}$. In this case r_1 and r_2 in Equation 5.2.2 are complex conjugates, which we write as

$$r_1 = -\frac{c}{2m} - \omega_1 i \quad \text{and} \quad r_2 = -\frac{c}{2m} + \omega_1 i,$$

where

$$\omega_1 = \frac{\sqrt{4mk - c^2}}{2m}.$$

The general solution of Equation 5.2.1 in this case is

$$y = e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

By the method used in Section 6.1 to derive the amplitude–phase form of the displacement of an object in simple harmonic motion, we can rewrite this equation as

$$y = Re^{-ct/2m} \cos(\omega_1 t - \phi), \tag{5.2.3}$$

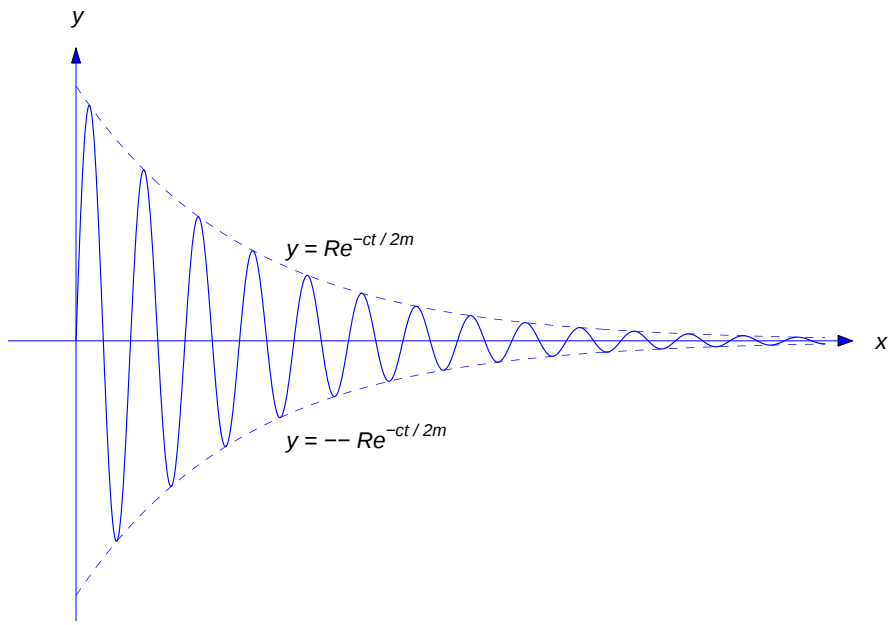

Figure 6.2.1 : Underdamped motion

where

$$R=\sqrt{c_1^2+c_2^2}, \quad R\cos\phi=c_1, \quad \text{and} \quad R\sin\phi=c_2.$$

The factor $Re^{-ct/2m}$ in Equation 5.2.3 is called the **time–varying amplitude** of the motion, the quantity ω_1 is called the **frequency**, and $T=2\pi/\omega_1$ (which is the period of the cosine function in Equation 5.2.3 is called the **quasi–period**. A typical graph of Equation 5.2.3 is shown in Figure 6.2.1 . As illustrated in that figure, the graph of y oscillates between the dashed exponential curves $y=\pm Re^{-ct/2m}$.

Overdamped Motion

We say the motion is **overdamped** if $c>\sqrt{4mk}$. In this case the zeros r_1 and r_2 of the characteristic polynomial are real, with $r_1<r_2<0$ (see 5.2.2), and the general solution of 5.2.1 is

$$y=c_1e^{r_1t}+c_2e^{r_2t}.$$

Again $\lim_{t\to\infty} y(t)=0$ as in the underdamped case, but the motion isn't oscillatory, since y can't equal zero for more than one value of t unless $c_1=c_2=0$. (**Exercise 6.2.23**.)

Critically Damped Motion

We say the motion is **critically damped** if $c=\sqrt{4mk}$. In this case $r_1=r_2=-c/2m$ and the general solution of Equation 5.2.1 is

$$y=e^{-ct/2m}(c_1+c_2t).$$

Again $\lim_{t\to\infty} y(t)=0$ and the motion is nonoscillatory, since y can't equal zero for more than one value of t unless $c_1=c_2=0$. (**Exercise 6.2.22**).

Example 6.2.1

Suppose a 64 lb weight stretches a spring 6 inches in equilibrium and a dashpot provides a damping force of c lb for each ft/sec of velocity.

a. Write the equation of motion of the object and determine the value of c for which the motion is critically damped.
b. Find the displacement y for $t>0$ if the motion is critically damped and the initial conditions are $y(0)=1$ and $y'(0)=20$.
c. Find the displacement y for $t>0$ if the motion is critically damped and the initial conditions are $y(0)=1$ and $y'(0)=-20$.

Solution a

Here $m = 2$ slugs and $k = 64/.5 = 128$ lb/ft. Therefore the equation of motion Equation 5.2.1 is

$$2y'' + cy' + 128y = 0. \tag{5.2.4}$$

The characteristic equation is

$$2r^2 + cr + 128 = 0,$$

which has roots

$$r = \frac{-c \pm \sqrt{c^2 - 8 \cdot 128}}{4}.$$

Therefore the damping is critical if

$$c = \sqrt{8 \cdot 128} = 32 \text{ lb-sec/ft}.$$

Solution b

Setting $c = 32$ in Equation 5.2.4 and cancelling the common factor 2 yields

$$y'' + 16y + 64y = 0.$$

The characteristic equation is

$$r^2 + 16r + 64y = (r+8)^2 = 0.$$

Hence, the general solution is

$$y = e^{-8t}(c_1 + c_2 t). \tag{5.2.5}$$

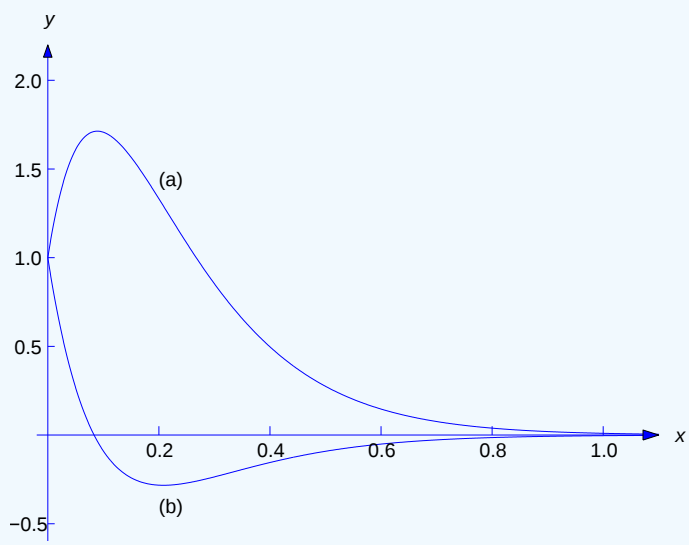

Figure 6.2.2 : (a) $y = e^{-8t}(1 + 28t)$ and (b) $y = e^{-8t}(1 - 12t)$

Differentiating this yields

$$y' = -8y + c_2 e^{-8t}. \tag{5.2.6}$$

6.2.2 Imposing the initial conditions $y(0) = 1$ and $y'(0) = 20$ in the last two equations shows that $1 = c_1$ and $20 = -8 + c_2$. Hence, the solution of the initial value problem is

$$y = e^{-8t}(1 + 28t).$$

Therefore the object approaches equilibrium from above as $t \to \infty$. There's no oscillation.

Solution c

Imposing the initial conditions $y(0)=1$ and $y'(0)=-20$ in Equation 5.2.5 and Equation 5.2.6 yields $1=c_1$ and $-20=-8+c_2$. Hence, the solution of this initial value problem is

$$y=e^{-8t}(1-12t).$$

Therefore the object moves downward through equilibrium just once, and then approaches equilibrium from below as $t\to\infty$. Again, there's no oscillation. The solutions of these two initial value problems are graphed in Figure 6.2.2 .

Example 6.2.2

Find the displacement of the object in Example 6.2.1 if the damping constant is $c=4$ lb–sec/ft and the initial conditions are $y(0)=1.5$ ft and $y'(0)=-3$ ft/sec.

Solution

With $c=4$, the equation of motion Equation 5.2.4 becomes

$$y''+2y'+64y=0 \tag{5.2.7}$$

after cancelling the common factor 2. The characteristic equation

$$r^2+2r+64=0$$

has complex conjugate roots

$$r=\frac{-2\pm\sqrt{4-4\cdot 64}}{2}=-1\pm 3\sqrt{7}i.$$

Therefore the motion is underdamped and the general solution of Equation 5.2.7is

$$y=e^{-t}(c_1\cos 3\sqrt{7}t+c_2\sin 3\sqrt{7}t).$$

Differentiating this yields

$$y'=-y+3\sqrt{7}e^{-t}(-c_1\sin 3\sqrt{7}t+c_2\cos 3\sqrt{7}t).$$

Imposing the initial conditions $y(0)=1.5$ and $y'(0)=-3$ in the last two equations yields $1.5=c_1$ and $-3=-1.5+3\sqrt{7}c_2$. Hence, the solution of the initial value problem is

$$y=e^{-t}\left(\frac{3}{2}\cos 3\sqrt{7}t-\frac{1}{2\sqrt{7}}\sin 3\sqrt{7}t\right). \tag{5.2.8}$$

The amplitude of the function in parentheses is

$$R=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{1}{2\sqrt{7}}\right)^2}=\sqrt{\frac{9}{4}+\frac{1}{4\cdot 7}}=\sqrt{\frac{64}{4\cdot 7}}=\frac{4}{\sqrt{7}}$$

Therefore we can rewrite Equation 5.2.8as

$$y=\frac{4}{\sqrt{7}}e^{-t}\cos(3\sqrt{7}t-\phi),$$

where

$$\cos\phi=\frac{3}{2R}=\frac{3\sqrt{7}}{8}\quad\text{and}\quad\sin\phi=-\frac{1}{2\sqrt{7}R}=-\frac{1}{8}.$$

Therefore $\phi\cong -0.125$ radians.

Example 6.2.3

Let the damping constant in Example 6.2.1 be $c=40$ lb–sec/ft. Find the displacement y for $t>0$ if $y(0)=1$ and $y'(0)=1$.

Solution

With $c=40$, the equation of motion Equation 5.2.4 reduces to

$$y''+20y'+64y=0 \tag{5.2.9}$$

after cancelling the common factor 2. The characteristic equation

$$r^2+20r+64=(r+16)(r+4)=0$$

has the roots $r_1=-4$ and $r_2=-16$. Therefore the general solution of Equation 5.2.9 is

$$y=c_1e^{-4t}+c_2e^{-16t}. \tag{5.2.10}$$

Differentiating this yields

$$y'=-4e^{-4t}-16c_2e^{-16t}.$$

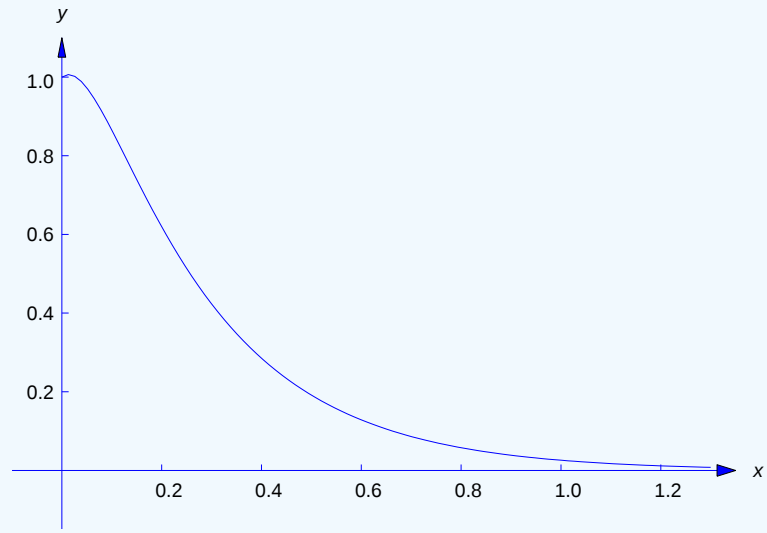

Figure 6.2.3 : $y=\frac{17}{12}e^{-4t}-\frac{5}{12}e^{-16t}$

The last two equations and the initial conditions $y(0)=1$ and $y'(0)=1$ imply that

$$\begin{aligned} c_1 \quad + \quad c_2 \quad &= 1 \\ -4c_1 \quad - \quad 16c_2 \quad &= 1. \end{aligned}$$

The solution of this system is $c_1=17/12$, $c_2=-5/12$. Substituting these into Equation 5.2.10 yields

$$y=\frac{17}{12}e^{-4t}-\frac{5}{12}e^{-16t}$$

as the solution of the given initial value problem (Figure 6.2.3).

Forced Vibrations with Damping

Now we consider the motion of an object in a spring-mass system with damping, under the influence of a periodic forcing function $F(t)=F_0\cos\omega t$, so that the equation of motion is

$$my''+cy'+ky=F_0\cos\omega t. \tag{5.2.11}$$

In Section 6.1 we considered this equation with $c=0$ and found that the resulting displacement y assumed arbitrarily large values in the case of resonance (that is, when $\omega=\omega_0=\sqrt{k/m}$). Here we'll see that in the presence of damping the displacement remains bounded for all t, and the initial conditions have little effect on the motion as $t\to\infty$. In fact, we'll see that for large t the displacement is closely approximated by a function of the form

$$y=R\cos(\omega t-\phi), \tag{5.2.12}$$

where the amplitude R depends upon m, c, k, F_0, and ω. We're interested in the following question:

QUESTION

Assuming that m, c, k, and F_0 are held constant, what value of ω produces the largest amplitude R in Equation 5.2.12, and what is this largest amplitude?

To answer this question, we must solve Equation 5.2.11 and determine R in terms of F_0, ω_0, ω, and c. We can obtain a particular solution of Equation 5.2.11 by the method of undetermined coefficients. Since $\cos \omega t$ does not satisfy the complementary equation

$$my'' + cy' + ky = 0,$$

we can obtain a particular solution of Equation 5.2.11 in the form

$$y_p = A\cos\omega t + B\sin\omega t. \tag{5.2.13}$$

Differentiating this yields

$$y_p' = \quad \omega(-A\sin\omega t + B\cos\omega t)$$

and

$$y_p'' = -\omega^2(A\cos\omega t + B\sin\omega t).$$

From the last three equations,

$$my_p'' + cy_p' + ky_p = (-m\omega^2 A + c\omega B + kA)\cos\omega t + (-m\omega^2 B - c\omega A + kB)\sin\omega t,$$

so y_p satisfies Equation 5.2.11 if

$$\begin{aligned} (k - m\omega^2)A + \quad c\omega B \quad &= F_0 \\ -c\omega A \quad + (k - m\omega^2)B \quad &= 0. \end{aligned}$$

Solving for A and B and substituting the results into Equation 5.2.13 yields

$$y_p = \frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2}\left[(k - m\omega^2)\cos\omega t + c\omega\sin\omega t\right],$$

which can be written in amplitude–phase form as

$$\tag{5.2.14}$$

where

$$\cos\phi = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \text{and} \quad \sin\phi = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}. \tag{5.2.15}$$

To compare this with the undamped forced vibration that we considered in Section 6.1 it is useful to write

$$k - m\omega^2 = m\left(\frac{k}{m} - \omega^2\right) = m(\omega_0^2 - \omega^2), \tag{5.2.16}$$

where $\omega_0 = \sqrt{k/m}$ is the natural angular frequency of the undamped simple harmonic motion of an object with mass m on a spring with constant k. Substituting Equation 5.2.16 into Equation 5.2.14 yields

$$y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}\cos(\omega t - \phi). \tag{5.2.17}$$

The solution of an initial value problem

$$my'' + cy' + ky = F_0\cos\omega t, \quad y(0) = y_0, \quad y'(0) = v_0,$$

is of the form $y = y_c + y_p$, where y_c has one of the three forms

$$\begin{aligned} y_c &= e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t), \\ y_c &= e^{-ct/2m}(c_1 + c_2 t), \\ y_c &= c_1 e^{r_1 t} + c_2 e^{r_2 t}\ (r_1, r_2 < 0). \end{aligned}$$

In all three cases $\lim_{t\to\infty} y_c(t) = 0$ for any choice of c_1 and c_2. For this reason we say that y_c is the **transient component** of the solution y. The behavior of y for large t is determined by y_p, which we call the **steady state component** of y. Thus, for large t the motion is like simple harmonic motion at the frequency of the external force.

The amplitude of y_p in Equation 5.2.17 is

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}, \tag{5.2.18}$$

which is finite for all ω; that is, the presence of damping precludes the phenomenon of resonance that we encountered in studying undamped vibrations under a periodic forcing function. We'll now find the value $\omega_{\max}$ of ω for which R is maximized. This is the value of ω for which the function

$$\rho(\omega) = m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2$$

in the denominator of Equation 5.2.18 attains its minimum value. By rewriting this as

$$\rho(\omega) = m^2(\omega_0^4 + \omega^4) + (c^2 - 2m^2\omega_0^2)\omega^2, \tag{5.2.19}$$

you can see that ρ is a strictly increasing function of ω^2 if

$$c \geq \sqrt{2m^2\omega_0^2} = \sqrt{2mk}.$$

(Recall that $\omega_0^2 = k/m$). Therefore $\omega_{\max} = 0$ if this inequality holds. From Equation 5.2.15, you can see that $\phi = 0$ if $\omega = 0$. In this case, Equation 5.2.14 reduces to

$$y_p = \frac{F_0}{\sqrt{m^2\omega_0^4}} = \frac{F_0}{k},$$

which is consistent with Hooke's law: if the mass is subjected to a constant force F_0, its displacement should approach a constant y_p such that $ky_p = F_0$. Now suppose $c < \sqrt{2mk}$. Then, from Equation 5.2.19,

$$\rho'(\omega) = 2\omega(2m^2\omega^2 + c^2 - 2m^2\omega_0^2),$$

and $\omega_{\max}$ is the value of ω for which the expression in parentheses equals zero; that is,

$$\omega_{\max} = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}} = \sqrt{\frac{k}{m}\left(1 - \frac{c^2}{2km}\right)}.$$

(To see that $\rho(\omega_{\max})$ is the minimum value of $\rho(\omega)$, note that $\rho'(\omega) < 0$ if $\omega < \omega_{\max}$ and $\rho'(\omega) > 0$ if $\omega > \omega_{\max}$.) Substituting $\omega = \omega_{\max}$ in Equation 5.2.18 and simplifying shows that the maximum amplitude $R_{\max}$ is

$$R_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}} \quad \text{if} \quad c < \sqrt{2mk}.$$

We summarize our results as follows.

Theorem 6.2.1

Suppose we consider the amplitude R of the steady state component of the solution of

$$my'' + cy' + ky = F_0 \cos \omega t$$

as a function of ω.

a. If $c \geq \sqrt{2mk}$, the maximum amplitude is $R_{\max} = F_0/k$ and it is attained when $\omega = \omega_{\max} = 0$.

b. If $c < \sqrt{2mk}$, the maximum amplitude is

$$R_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}}, \tag{5.2.20}$$

and it is attained when

$$\omega = \omega_{\max} = \sqrt{\frac{k}{m}\left(1 - \frac{c^2}{2km}\right)}. \tag{5.2.21}$$

Note that $R_{\max}$ and $\omega_{\max}$ are continuous functions of c, for $c \geq 0$, since Equation 5.2.20 and Equation 5.2.21 reduce to $R_{\max} = F_0/k$ and $\omega_{\max} = 0$ if $c = \sqrt{2km}$.

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5.2E: Spring Problems II (Exercises)

Q5.2.1

1. A 64 lb object stretches a spring 4 ft in equilibrium. It is attached to a dashpot with damping constant $c = 8$ lb-sec/ft. The object is initially displaced 18 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement and time–varying amplitude for $t > 0$.

2. A 16 lb weight is attached to a spring with natural length 5 ft. With the weight attached, the spring measures 8.2 ft. The weight is initially displaced 3 ft below equilibrium and given an upward velocity of 2 ft/sec. Find and graph its displacement for $t > 0$ if the medium resists the motion with a force of one lb for each ft/sec of velocity. Also, find its time–varying amplitude.

3. An 8 lb weight stretches a spring 1.5 inches. It is attached to a dashpot with damping constant $c = 8$ lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given an upward velocity of 6 ft/sec. Find and graph its displacement for $t > 0$.

4. A 96 lb weight stretches a spring 3.2 ft in equilibrium. It is attached to a dashpot with damping constant c=18 lb-sec/ft. The weight is initially displaced 15 inches below equilibrium and given a downward velocity of 12 ft/sec. Find its displacement for $t > 0$.

5. A 16 lb weight stretches a spring 6 inches in equilibrium. It is attached to a damping mechanism with constant c. Find all values of c such that the free vibration of the weight has infinitely many oscillations.

6. An 8 lb weight stretches a spring $.32$ ft. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 4 ft/sec. Find its displacement for $t > 0$ if the medium exerts a damping force of 1.5 lb for each ft/sec of velocity.

7. A 32 lb weight stretches a spring 2 ft in equilibrium. It is attached to a dashpot with constant $c = 8$ lb-sec/ft. The weight is initially displaced 8 inches below equilibrium and released from rest. Find its displacement for $t > 0$.

8. A mass of 20 gm stretches a spring 5 cm. The spring is attached to a dashpot with damping constant 400 dyne sec/cm. Determine the displacement for $t > 0$ if the mass is initially displaced 9 cm above equilibrium and released from rest.

9. A 64 lb weight is suspended from a spring with constant $k = 25$ lb/ft. It is initially displaced 18 inches above equilibrium and released from rest. Find its displacement for $t > 0$ if the medium resists the motion with 6 lb of force for each ft/sec of velocity.

10. A 32 lb weight stretches a spring 1 ft in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/sec. Find its displacement for $t > 0$ if the medium resists the motion with a force equal to 3 times the speed in ft/sec.

11. An 8 lb weight stretches a spring 2 inches. It is attached to a dashpot with damping constant $c = 4$ lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement for $t > 0$.

12. A 2 lb weight stretches a spring $.32$ ft. The weight is initially displaced 4 inches below equilibrium and given an upward velocity of 5 ft/sec. The medium provides damping with constant $c = 1/8$ lb-sec/ft. Find and graph the displacement for $t > 0$.

13. An 8 lb weight stretches a spring 8 inches in equilibrium. It is attached to a dashpot with damping constant $c = .5$ lb-sec/ft and subjected to an external force $F(t) = 4\cos 2t$ lb. Determine the steady state component of the displacement for $t > 0$.

14. A 32 lb weight stretches a spring 1 ft in equilibrium. It is attached to a dashpot with constant $c = 12$ lb-sec/ft. The weight is initially displaced 8 inches above equilibrium and released from rest. Find its displacement for $t > 0$.

15. A mass of one kg stretches a spring 49 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 4 N for each m/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.

16. A mass of 100 grams stretches a spring 98 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 600 dynes for each cm/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.

17. A 192 lb weight is suspended from a spring with constant $k = 6$ lb/ft and subjected to an external force $F(t) = 8\cos 3t$ lb. Find the steady state component of the displacement for $t > 0$ if the medium resists the motion with a force equal to 8 times the speed in ft/sec.

18. A 2 gm mass is attached to a spring with constant 20 dyne/cm. Find the steady state component of the displacement if the mass is subjected to an external force $F(t)=3\cos 4t-5\sin 4t$ dynes and a dashpot supplies 4 dynes of damping for each cm/sec of velocity.

19. A 96 lb weight is attached to a spring with constant 12 lb/ft. Find and graph the steady state component of the displacement if the mass is subjected to an external force $F(t)=18\cos t-9\sin t$ lb and a dashpot supplies 24 lb of damping for each ft/sec of velocity.

20. A mass of one kg stretches a spring 49 cm in equilibrium. It is attached to a dashpot that supplies a damping force of 4 N for each m/sec of speed. Find the steady state component of its displacement if it is subjected to an external force $F(t)=8\sin 2t-6\cos 2t$ N.

21. A mass m is suspended from a spring with constant k and subjected to an external force $F(t)=\alpha\cos\omega_0 t+\beta\sin\omega_0 t$, where ω_0 is the natural frequency of the spring–mass system without damping. Find the steady state component of the displacement if a dashpot with constant c supplies damping.

22. Show that if c_1 and c_2 are not both zero then

$$y=e^{r_1 t}(c_1+c_2 t) \tag{5.2E.1}$$

can't equal zero for more than one value of t.

23. Show that if c_1 and c_2 are not both zero then

$$y=c_1 e^{r_1 t}+c_2 e^{r_2 t} \tag{5.2E.2}$$

can't equal zero for more than one value of t.

24. Find the solution of the initial value problem

$$my''+cy'+ky=0, \quad y(0)=y_0,\ y'(0)=v_0, \tag{5.2E.3}$$

given that the motion is underdamped, so the general solution of the equation is

$$y=e^{-ct/2m}(c_1\cos\omega_1 t+c_2\sin\omega_1 t). \tag{5.2E.4}$$

25. Find the solution of the initial value problem

$$my''+cy'+ky=0, \quad y(0)=y_0,\ y'(0)=v_0, \tag{5.2E.5}$$

given that the motion is overdamped, so the general solution of the equation is

$$y=c_1 e^{r_1 t}+c_2 e^{r_2 t}\ (r_1, r_2<0). \tag{5.2E.6}$$

26. Find the solution of the initial value problem

$$my''+cy'+ky=0, \quad y(0)=y_0,\ y'(0)=v_0, \tag{5.2E.7}$$

given that the motion is critically damped, so that the general solution of the equation is of the form

$$y=e^{r_1 t}(c_1+c_2 t)\ (r_1<0). \tag{5.2E.8}$$

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5.3: The RLC Circuit

In this section we consider the $\boldsymbol{RLC}$ circuit, shown schematically in Figure 6.3.1 . As we'll see, the RLC circuit is an electrical analog of a spring-mass system with damping.

Nothing happens while the switch is open (dashed line). When the switch is closed (solid line) we say that the **circuit is closed**. Differences in electrical potential in a closed circuit cause current to flow in the circuit. The battery or generator in Figure 6.3.1 creates a difference in electrical potential $E = E(t)$ between its two terminals, which we've marked arbitrarily as positive and negative. (We could just as well interchange the markings.) We'll say that $E(t) > 0$ if the potential at the positive terminal is greater than the potential at the negative terminal, $E(t) < 0$ if the potential at the positive terminal is less than the potential at the negative terminal, and $E(t) = 0$ if the potential is the same at the two terminals. We call E the **impressed voltage**.

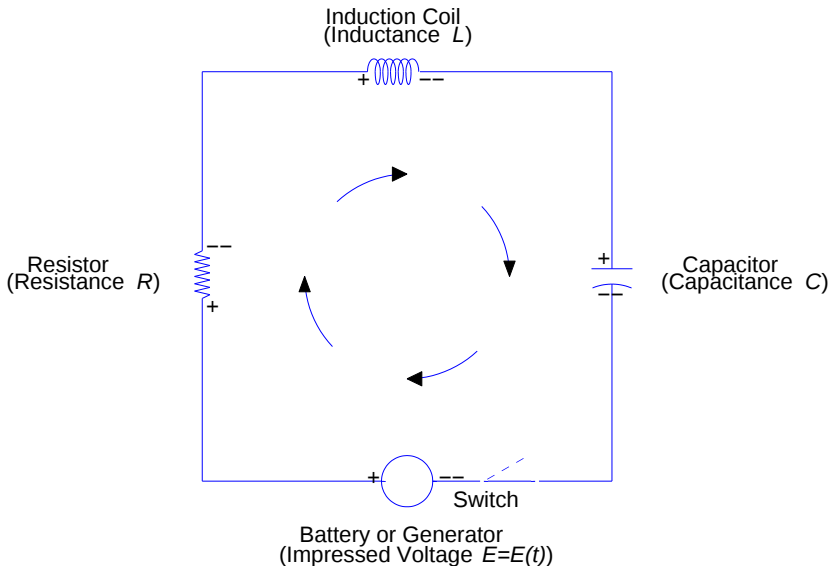

Figure 6.3.1 : An RLC circuit

At any time t, the same current flows in all points of the circuit. We denote current by $I = I(t)$. We say that $I(t) > 0$ if the direction of flow is around the circuit from the positive terminal of the battery or generator back to the negative terminal, as indicated by the arrows in Figure 6.3.1 $I(t) < 0$ if the flow is in the opposite direction, and $I(t) = 0$ if no current flows at time t.

Differences in potential occur at the resistor, induction coil, and capacitor in Figure 6.3.1 . Note that the two sides of each of these components are also identified as positive and negative. The **voltage drop across** each component is defined to be the potential on the positive side of the component minus the potential on the negative side. This terminology is somewhat misleading, since "drop" suggests a decrease even though changes in potential are signed quantities and therefore may be increases. Nevertheless, we'll go along with tradition and call them voltage drops. The voltage drop across the resistor in Figure 6.3.1 is given by

$$V_R = IR, \tag{5.3.1}$$

where I is current and R is a positive constant, the **resistance** of the resistor. The voltage drop across the induction coil is given by

$$V_I = L\frac{dI}{dt} = LI', \tag{5.3.2}$$

where L is a positive constant, the **inductance** of the coil.

A capacitor stores electrical charge $Q = Q(t)$, which is related to the current in the circuit by the equation

$$Q(t) = Q_0 + \int_0^t I(\tau)\, d\tau, \tag{5.3.3}$$

where Q_0 is the charge on the capacitor at $t = 0$. The voltage drop across a capacitor is given by

$$V_C = \frac{Q}{C}, \tag{5.3.4}$$

where C is a positive constant, the **capacitance** of the capacitor.

Table 6.3.1 names the units for the quantities that we've discussed. The units are defined so that

$$\begin{aligned} 1\text{volt} &= 1\text{ampere} \cdot 1\text{ohm} \\ &= 1\text{henry} \cdot 1\text{ ampere/second} \\ &= 1\text{coulomb/farad} \end{aligned}$$

and

$$1\text{ampere} = 1\text{coulomb/second}.$$

Table 6.3.1 : Electrical Units

Symbol	Name	Unit
E	Impressed Voltage	volt
I	Current	ampere
Q	Charge	coulomb
R	Resistance	ohm
L	Inductance	henry
C	Capacitance	farad

According to **Kirchoff's law**, the sum of the voltage drops in a closed RLC circuit equals the impressed voltage. Therefore, from Equation 5.3.1, Equation 5.3.2, and Equation 5.3.4,

$$LI' + RI + \frac{1}{C}Q = E(t). \tag{5.3.5}$$

This equation contains two unknowns, the current I in the circuit and the charge Q on the capacitor. However, Equation 5.3.3 implies that $Q' = I$, so Equation 5.3.5 can be converted into the second order equation

$$LQ'' + RQ' + \frac{1}{C}Q = E(t) \tag{5.3.6}$$

in Q. To find the current flowing in an RLC circuit, we solve Equation 5.3.6 for Q and then differentiate the solution to obtain I.

In Sections 6.1 and 6.2 we encountered the equation

$$my'' + cy' + ky = F(t) \tag{5.3.7}$$

in connection with spring-mass systems. Except for notation this equation is the same as Equation 5.3.6. The correspondence between electrical and mechanical quantities connected with Equation 5.3.6 and Equation 5.3.7 is shown in Table 6.3.2 .

Table 6.3.2 : Electrical and Mechanical Units

Electrical	Mechanical
charge Q	displacement y
current I	velocity y'
impressed voltage $E(t)$	external force $F(t)$
inductance L	mass m
resistance R	damping c
1/capacitance $1/C$	spring constant k

The equivalence between Equation 5.3.6 and Equation 5.3.7 is an example of how mathematics unifies fundamental similarities in diverse physical phenomena. Since we've already studied the properties of solutions of Equation 5.3.7 in Sections 6.1 and 6.2, we can obtain results concerning solutions of Equation 5.3.6 by simply changing notation, according to Table 6.3.1 .

Free Oscillations

We say that an RLC circuit is in **free oscillation** if $E(t)=0$ for $t>0$, so that Equation 5.3.6 becomes

$$LQ''+RQ'+\frac{1}{C}Q=0. \tag{5.3.8}$$

The characteristic equation of Equation 5.3.8 is

$$Lr^2+Rr+\frac{1}{C}=0,$$

with roots

$$r_1=\frac{-R-\sqrt{R^2-4L/C}}{2L} \quad \text{and} \quad r_2=\frac{-R+\sqrt{R^2-4L/C}}{2L}. \tag{5.3.9}$$

There are three cases to consider, all analogous to the cases considered in Section 6.2 for free vibrations of a damped spring-mass system.

Case 1: Underdamped Motion

The oscillation is **underdamped** if $R<\sqrt{4L/C}$. In this case, r_1 and r_2 in Equation 5.3.9 are complex conjugates, which we write as

$$r_1=-\frac{R}{2L}+i\omega_1 \quad \text{and} \quad r_2=-\frac{R}{2L}-i\omega_1,$$

where

$$\omega_1=\frac{\sqrt{4L/C-R^2}}{2L}.$$

The general solution of Equation 5.3.8 is

$$Q=e^{-Rt/2L}(c_1\cos\omega_1 t+c_2\sin\omega_1 t),$$

which we can write as

$$Q=Ae^{-Rt/2L}\cos(\omega_1 t-\phi), \tag{5.3.10}$$

where

$$A=\sqrt{c_1^2+c_2^2}, \quad A\cos\phi=c_1, \quad \text{and} \quad A\sin\phi=c_2.$$

In the idealized case where $R=0$, the solution Equation 5.3.10 reduces to

$$Q=A\cos\left(\frac{t}{\sqrt{LC}}-\phi\right),$$

which is analogous to the simple harmonic motion of an undamped spring-mass system in free vibration.

> Actual RLC circuits are usually underdamped, so the case we've just considered is the most important. However, for completeness we'll consider the other two possibilities.

Case 2: Overdamped Motion

The oscillation is **overdamped** if $R>\sqrt{4L/C}$. In this case, the zeros r_1 and r_2 of the characteristic polynomial are real, with $r_1<r_2<0$ (see 5.3.9), and the general solution of 5.3.8 is

$$Q = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{5.3.11}$$

Case 3: Critically Damped

The oscillation is **critically damped** if $R = \sqrt{4L/C}$. In this case, $r_1 = r_2 = -R/2L$ and the general solution of Equation 5.3.8 is

$$Q = e^{-Rt/2L}(c_1 + c_2 t). \tag{5.3.12}$$

If $R \neq 0$, the exponentials in Equation 5.3.10, Equation 5.3.11, and Equation 5.3.12 are negative, so the solution of any homogeneous initial value problem

$$LQ'' + RQ' + \frac{1}{C}Q = 0, \quad Q(0) = Q_0, \quad Q'(0) = I_0,$$

approaches zero exponentially as $t \to \infty$. Thus, all such solutions are **transient**, in the sense defined Section 6.2 in the discussion of forced vibrations of a spring-mass system with damping.

Example 6.3.1

At $t = 0$ a current of 2 amperes flows in an RLC circuit with resistance $R = 40$ ohms, inductance $L = .2$ henrys, and capacitance $C = 10^{-5}$ farads. Find the current flowing in the circuit at $t > 0$ if the initial charge on the capacitor is 1 coulomb. Assume that $E(t) = 0$ for $t > 0$.

Solution

The equation for the charge Q is

$$\frac{1}{5}Q'' + 40Q' + 10000Q = 0,$$

or

$$Q'' + 200Q' + 50000Q = 0. \tag{5.3.13}$$

Therefore we must solve the initial value problem

$$Q'' + 200Q' + 50000Q = 0, \quad Q(0) = 1, \quad Q'(0) = 2. \tag{5.3.14}$$

The desired current is the derivative of the solution of this initial value problem.

The characteristic equation of Equation 5.3.13 is

$$r^2 + 200r + 50000 = 0,$$

which has complex zeros $r = -100 \pm 200i$. Therefore the general solution of Equation 5.3.13 is

$$Q = e^{-100t}(c_1 \cos 200t + c_2 \sin 200t). \tag{5.3.15}$$

Differentiating this and collecting like terms yields

$$Q' = -e^{-100t}\left[(100c_1 - 200c_2)\cos 200t + (100c_2 + 200c_1)\sin 200t\right]. \tag{5.3.16}$$

To find the solution of the initial value problem Equation 5.3.14, we set $t = 0$ in Equation 5.3.15 and Equation 5.3.16 to obtain

$$c_1 = Q(0) = 1 \quad \text{and} \quad -100c_1 + 200c_2 = Q'(0) = 2;$$

therefore, $c_1 = 1$ and $c_2 = 51/100$, so

$$Q = e^{-100t}\left(\cos 200t + \frac{51}{100}\sin 200t\right)$$

is the solution of Equation 5.3.14. Differentiating this yields

$$I = e^{-100t}(2\cos 200t - 251\sin 200t).$$

Forced Oscillations With Damping

An initial value problem for Equation 5.3.6 has the form

$$LQ'' + RQ' + \frac{1}{C}Q = E(t), \quad Q(0) = Q_0, \quad Q'(0) = I_0, \tag{5.3.17}$$

where Q_0 is the initial charge on the capacitor and I_0 is the initial current in the circuit. We've already seen that if $E \equiv 0$ then all solutions of Equation 5.3.17 are transient. If $E \not\equiv 0$, we know that the solution of Equation 5.3.17 has the form $Q = Q_c + Q_p$, where Q_c satisfies the complementary equation, and approaches zero exponentially as $t \to \infty$ for any initial conditions, while Q_p depends only on E and is independent of the initial conditions. As in the case of forced oscillations of a spring-mass system with damping, we call Q_p the **steady state** charge on the capacitor of the RLC circuit. Since $I = Q' = Q_c' + Q_p'$ and Q_c' also tends to zero exponentially as $t \to \infty$, we say that $I_c = Q_c'$ is the **transient** current and $I_p = Q_p'$ is the **steady state** current. In most applications we are interested only in the steady state charge and current.

Example 6.3.2

Find the amplitude-phase form of the steady state current in the RLC circuit in Figure 6.3.1 if the impressed voltage, provided by an alternating current generator, is $E(t) = E_0 \cos \omega t$.

Solution

We'll first find the steady state charge on the capacitor as a particular solution of

$$LQ'' + RQ' + \frac{1}{C}Q = E_0 \cos \omega t.$$

To do, this we'll simply reinterpret a result obtained in Section 6.2, where we found that the steady state solution of

$$my'' + cy' + ky = F_0 \cos \omega t$$

is

$$y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \phi),$$

where

$$\cos \phi = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \text{and} \quad \sin \phi = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}.$$

(see Equations 5.3.14 and Equation 5.3.15.) By making the appropriate changes in the symbols (according to Table 6.3.2) yields the steady state charge

$$Q_p = \frac{E_0}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \cos(\omega t - \phi),$$

where

$$\cos \phi = \frac{1/C - L\omega^2}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \quad \text{and} \quad \sin \phi = \frac{R\omega}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}}.$$

Therefore the steady state current in the circuit is

$$I_p = Q_p' = -\frac{\omega E_0}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \sin(\omega t - \phi).$$

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5.3E: The RLC Circuit (Exercises)

Q5.3.1

In *Exercises 6.3.1-6.3.5* find the current in the RLC circuit, assuming that $E(t)=0$ for $t>0$.

1. $R=3$ ohms; $L=.1$ henrys; $C=.01$ farads; $Q_0=0$ coulombs; $I_0=2$ amperes.

2. $R=2$ ohms; $L=.05$ henrys; $C=.01$ farads'; $Q_0=2$ coulombs; $I_0=-2$ amperes.

3. $R=2$ ohms; $L=.1$ henrys; $C=.01$ farads; $Q_0=2$ coulombs; $I_0=0$ amperes.

4. $R=6$ ohms; $L=.1$ henrys; $C=.004$ farads'; $Q_0=3$ coulombs; $I_0=-10$ amperes.

5. $R=4$ ohms; $L=.05$ henrys; $C=.008$ farads; $Q_0=-1$ coulombs; $I_0=2$ amperes.

Q5.3.2

In *Exercises 6.3.6-6.3.10* find the steady state current in the circuit described by the equation.

6. $\frac{1}{10}Q''+3Q'+100Q=5\cos 10t-5\sin 10t$

7. $\frac{1}{20}Q''+2Q'+100Q=10\cos 25t-5\sin 25t$

8. $\frac{1}{10}Q''+2Q'+100Q=3\cos 50t-6\sin 50t$

9. $\frac{1}{10}Q''+6Q'+250Q=10\cos 100t+30\sin 100t$

10. $\frac{1}{20}Q''+4Q'+125Q=15\cos 30t-30\sin 30t$

Q5.3.3

11. Show that if $E(t)=U\cos\omega t+V\sin\omega t$ where U and V are constants then the steady state current in the RLC circuit shown in Figure 6.3.1 is

$$I_p=\frac{\omega^2 RE(t)+(1/C-L\omega^2)E'(t)}{\Delta}, \tag{5.3E.1}$$

where

$$\Delta=(1/C-L\omega^2)^2+R^2\omega^2. \tag{5.3E.2}$$

12. Find the amplitude of the steady state current I_p in the RLC circuit shown in Figure 6.3.1 if $E(t)=U\cos\omega t+V\sin\omega t$, where U and V are constants. Then find the value ω_0 of ω maximizes the amplitude, and find the maximum amplitude.

Q5.3.4

In *Exercises 6.3.13-6.3.17* plot the amplitude of the steady state current against ω. Estimate the value of ω that maximizes the amplitude of the steady state current, and estimate this maximum amplitude. HINT: You can confirm your results by doing *Exercise 6.3.12*.

13. $\frac{1}{10}Q''+3Q'+100Q=U\cos\omega t+V\sin\omega t$

14. $\frac{1}{20}Q''+2Q'+100Q=U\cos\omega t+V\sin\omega t$

15. $\frac{1}{10}Q''+2Q'+100Q=U\cos\omega t+V\sin\omega t$

16. $\frac{1}{10}Q''+6Q'+250Q=U\cos\omega t+V\sin\omega t$

17. $\frac{1}{20}Q''+4Q'+125Q=U\cos\omega t+V\sin\omega t$

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CHAPTER OVERVIEW

Chapter 6: Series Solutions of Linear Second Order Equations

In this Chapter, we study a class of second order differential equations that occur in many applications, but cannot be solved in closed form in terms of elementary functions.

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6.1: Prelude to Series Solutions of Linear Second Order Equations

In this Chapter, we study a class of second order differential equations that occur in many applications, but cannot be solved in closed form in terms of elementary functions. Here are some examples:

1. **Bessel's equation** which occurs in problems displaying cylindrical symmetry, such as diffraction of light through a circular aperture, propagation of electromagnetic radiation through a coaxial cable, and vibrations of a circular drum head.

$$x^2y''+xy'+(x^2-\nu^2)y=0,$$

2. **Airy's equation** which occurs in astronomy and quantum physics.

$$y''-xy=0,$$

3. **Legendre's equation** which occurs in problems displaying spherical symmetry, particularly in electromagnetism.

$$(1-x^2)y''-2xy'+\alpha(\alpha+1)y=0,$$

These equations and others considered in this chapter can be written in the form

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0, \tag{6.1.1}$$

where P_0, P_1, and P_2 are polynomials with no common factor. For most equations that occur in applications, these polynomials are of degree two or less. We'll impose this restriction, although the methods that we'll develop can be extended to the case where the coefficient functions are polynomials of arbitrary degree, or even power series that converge in some circle around the origin in the complex plane.

Since Equation 6.1.1 does not in general have closed form solutions, we seek series representations for solutions. We'll see that if $P_0(0)\neq 0$ then solutions of (A) can be written as power series

$$y=\sum_{n=0}^{\infty}a_nx^n$$

that converge in an open interval centered at $x=0$.

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6.2: Review of Power Series

Many applications give rise to differential equations with solutions that can't be expressed in terms of elementary functions such as polynomials, rational functions, exponential and logarithmic functions, and trigonometric functions. The solutions of some of the most important of these equations can be expressed in terms of power series. We'll study such equations in this chapter. In this section we review relevant properties of power series. We'll omit proofs, which can be found in any standard calculus text.

Definition 6.2.1

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n, \tag{6.2.1}$$

where x_0 and $a_0, a_1, \ldots, a_n, \ldots$are constants, is called a *power series in* $x-x_0$. We say that the power series Equation 6.2.1 *converges* for a given x if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n (x-x_0)^n$$

exists; otherwise, we say that the power series *diverges* for the given x.

A power series in $x-x_0$ must converge if $x=x_0$, since the positive powers of $x-x_0$ are all zero in this case. This may be the only value of x for which the power series converges. However, the next theorem shows that if the power series converges for some $x \neq x_0$ then the set of all values of x for which it converges forms an interval.

Theorem 6.2.2

For any power series

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n,$$

exactly one of the these statements is true:

I. The power series converges only for $x=x_0$.
II. The power series converges for all values of x.
III. There's a positive number R such that the power series converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$.

In case (iii) we say that R is the *radius of convergence* of the power series. For convenience, we include the other two cases in this definition by defining $R=0$ in case (i) and $R=\infty$ in case (ii). We define the *open interval of convergence* of $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ to be

$$(x_0-R, x_0+R) \quad \text{if} \quad 0<R<\infty, \quad \text{or} \quad (-\infty,\infty) \quad \text{if} \quad R=\infty.$$

If R is finite, no general statement can be made concerning convergence at the endpoints $x=x_0 \pm R$ of the open interval of convergence; the series may converge at one or both points, or diverge at both.

Recall from calculus that a series of constants $\sum_{n=0}^{\infty} \alpha_n$ is said to *converge absolutely* if the series of absolute values $\sum_{n=0}^{\infty} |\alpha_n|$ converges. It can be shown that a power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ with a positive radius of convergence R converges absolutely in its open interval of convergence; that is, the series

$$\sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

of absolute values converges if $|x-x_0|<R$. However, if $R<\infty$, the series may fail to converge absolutely at an endpoint $x_0 \pm R$, even if it converges here.

The next theorem provides a useful method for determining the radius of convergence of a power series. It's derived in calculus by applying the ratio test to the corresponding series of absolute values. For related theorems see *Exercises 7.2.2* and *7.2.4*.

Theorem 6.2.3

Suppose there's an integer N such that $a_n \neq 0$ if $n \geq N$ and

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L,$$

where $0 \leq L \leq \infty$. Then the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $R = 1/L$, which should be interpreted to mean that $R = 0$ if $L = \infty$, or $R = \infty$ if $L = 0$.

Example 6.2.1

Find the radius of convergence of the series:

a.
$$\sum_{n=0}^{\infty} n!x^n$$

b.
$$\sum_{n=10}^{\infty} (-1)^n \frac{x^n}{n!}$$

c.
$$\sum_{n=0}^{\infty} 2^n n^2 (x-1)^n.$$

Solution a

Here $a_n = n!$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty.$$

Hence, $R = 0$.

Solution b

Here $a_n = (1)^n/n!$ for $n \geq N = 10$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Hence, $R = \infty$.

Solution c

Here $a_n = 2^n n^2$, so

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}(n+1)^2}{2^n n^2} = 2\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 2.$$

Hence, $R = 1/2$.

Taylor Series

If a function f has derivatives of all orders at a point $x = x_0$, then the *Taylor series of f about x_0* is defined by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

In the special case where $x_0 = 0$, this series is also called the *Maclaurin series of f*.

Taylor series for most of the common elementary functions converge to the functions on their open intervals of convergence. For example, you are probably familiar with the following Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty \tag{6.2.2}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty \tag{6.2.3}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad -\infty < x < \infty \tag{6.2.4}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad -1 < x < 1 \tag{6.2.5}$$

Differentiation of Power Series

A power series with a positive radius of convergence defines a function

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

on its open interval of convergence. We say that the series *represents* f on the open interval of convergence. A function f represented by a power series may be a familiar elementary function as in Equations 6.2.2 - 6.2.5; however, it often happens that f isn't a familiar function, so the series actually *defines* f.

The next theorem shows that a function represented by a power series has derivatives of all orders on the open interval of convergence of the power series, and provides power series representations of the derivatives.

Theorem 6.2.4: A power series

A power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with positive radius of convergence R has derivatives of all orders in its open interval of convergence, and successive derivatives can be obtained by repeatedly differentiating term by term; that is,

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \tag{6.2.6}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}, \tag{6.2.7}$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x - x_0)^{n-k}. \tag{6.2.8}$$

Moreover, all of these series have the same radius of convergence R.

Example 6.2.2

Let $f(x) = \sin x$. From Equation 6.2.3,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

From Equation 6.2.6,

$$f'(x)=\sum_{n=0}^{\infty}(-1)^n\frac{d}{dx}\left[\frac{x^{2n+1}}{(2n+1)!}\right]=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!},$$

which is the series Equation 6.2.4 for $\cos x$.

Uniqueness of Power Series

The next theorem shows that if f is *defined* by a power series in $x-x_0$ with a positive radius of convergence, then the power series is the Taylor series of f about x_0.

Theorem 6.2.5

If the power series

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

has a positive radius of convergence, then

$$a_n=\frac{f^{(n)}(x_0)}{n!}; \tag{6.2.9}$$

that is, $\sum_{n=0}^{\infty}a_n(x-x_0)^n$ is the Taylor series of f about x_0.

This result can be obtained by setting $x=x_0$ in Equation 6.2.8, which yields

$$f^{(k)}(x_0)=k(k-1)\cdots 1\cdot a_k=k!a_k.$$

This implies that

$$a_k=\frac{f^{(k)}(x_0)}{k!}.$$

Except for notation, this is the same as Equation 6.2.9.

The next theorem lists two important properties of power series that follow from Theorem 6.2.4.

Theorem 6.2.6

a. If

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=\sum_{n=0}^{\infty}b_n(x-x_0)^n$$

for all x in an open interval that contains x_0, then $a_n=b_n$ for $n=0,1,2,\ldots$.

b. If

$$\sum_{n=0}^{\infty}a_n(x-x_0)^n=0$$

for all x in an open interval that contains x_0, then $a_n=0$ for $n=0,1,2,\ldots$.

To obtain (a) we observe that the two series represent the same function f on the open interval; hence, Theorem 6.2.4 implies that

$$a_n=b_n=\frac{f^{(n)}(x_0)}{n!},\quad n=0,1,2,\ldots.$$

(b) can be obtained from (a) by taking $b_n=0$ for $n=0,1,2,\ldots$.

Taylor Polynomials

If f has N derivatives at a point x_0, we say that

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is the N*-th Taylor polynomial of* f *about* x_0. This definition and Theorem $6.2.4$ imply that if

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where the power series has a positive radius of convergence, then the Taylor polynomials of f about x_0 are given by

$$T_N(x) = \sum_{n=0}^{N} a_n (x - x_0)^n.$$

In numerical applications, we use the Taylor polynomials to approximate f on subintervals of the open interval of convergence of the power series. For example, Equation 6.2.2 implies that the Taylor polynomial T_N of $f(x) = e^x$ is

$$T_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}.$$

The solid curve in Figure $6.2.1$ is the graph of $y = e^x$ on the interval $[0, 5]$. The dotted curves in Figure $6.2.1$ are the graphs of the Taylor polynomials T_1, …, T_6 of $y = e^x$ about $x_0 = 0$. From this figure, we conclude that the accuracy of the approximation of $y = e^x$ by its Taylor polynomial T_N improves as N increases.

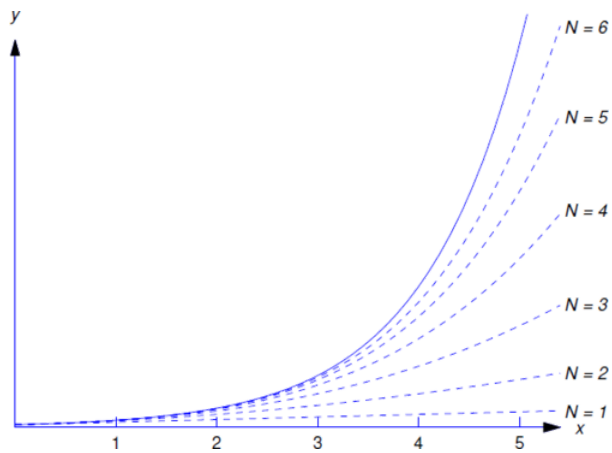

Figure $6.2.1$: Approximation of $y = e^x$ by Taylor polynomials about $x = 0$

Shifting the Summation Index

In Definition $6.2.1$ of a power series in $x - x_0$, the n-th term is a constant multiple of $(x - x_0)^n$. This isn't true in Equation 6.2.6, Equation 6.2.7, and Equation 6.2.8, where the general terms are constant multiples of $(x - x_0)^{n-1}$, $(x - x_0)^{n-2}$, and $(x - x_0)^{n-k}$, respectively. However, these series can all be rewritten so that their n-th terms are constant multiples of $(x - x_0)^n$. For example, letting $n = k + 1$ in the series in Equation 6.2.6 yields

$$f'(x) = \sum_{k=0}^{\infty} (k+1) a_{k+1} (x - x_0)^k, \tag{6.2.10}$$

where we start the new summation index k from zero so that the first term in Equation 6.2.10 (obtained by setting $k = 0$) is the same as the first term in Equation 6.2.6 (obtained by setting $n = 1$). However, the sum of a series is independent of the symbol used to denote the summation index, just as the value of a definite integral is independent of the symbol used to denote the variable of integration. Therefore we can replace k by n in Equation 6.2.10 to obtain

$$f'(x)=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_0)^n, \tag{6.2.11}$$

where the general term is a constant multiple of $(x-x_0)^n$.

It isn't really necessary to introduce the intermediate summation index k. We can obtain Equation 6.2.11 directly from Equation 6.2.6 by replacing n by $n+1$ in the general term of Equation 6.2.6 and subtracting 1 from the lower limit of Equation 6.2.6. More generally, we use the following procedure for shifting indices.

Shifting the Summation Index in a Power Series

For any integer k, the power series

$$\sum_{n=n_0}^{\infty} b_n(x-x_0)^{n-k}$$

can be rewritten as

$$\sum_{n=n_0-k}^{\infty} b_{n+k}(x-x_0)^n$$

that is, replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation leaves the series unchanged.

Example 6.2.3

Rewrite the following power series from Equation 6.2.7 and Equation 6.2.8 so that the general term in each is a constant multiple of $(x-x_0)^n$:

$$(a)\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} \quad (b)\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}.$$

Solution a

Replacing n by $n+2$ in the general term and subtracting 2 from the lower limit of summation yields

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n.$$

Solution b

Replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation yields

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}=\sum_{n=0}^{\infty}(n+k)(n+k-1)\cdots(n+1)a_{n+k}(x-x_0)^n.$$

Example 6.2.4

Given that

$$f(x)=\sum_{n=0}^{\infty} a_n x^n,$$

write the function xf'' as a power series in which the general term is a constant multiple of x^n.

Solution

From Theorem 6.2.4 with $x_0=0$,

$$f''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$xf''(x)=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}.$$

Replacing n by $n+1$ in the general term and subtracting 1 from the lower limit of summation yields

$$xf''(x)=\sum_{n=1}^{\infty}(n+1)na_{n+1}x^n.$$

We can also write this as

$$xf''(x)=\sum_{n=0}^{\infty}(n+1)na_{n+1}x^n,$$

since the first term in this last series is zero. (We'll see later that sometimes it is useful to include zero terms at the beginning of a series.)

Linear Combinations of Power Series

If a power series is multiplied by a constant, then the constant can be placed inside the summation; that is,

$$c\sum_{n=0}^{\infty}a_n(x-x_0)^n=\sum_{n=0}^{\infty}ca_n(x-x_0)^n.$$

Two power series

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n \quad \text{and} \quad g(x)=\sum_{n=0}^{\infty}b_n(x-x_0)^n$$

with positive radii of convergence can be added term by term at points common to their open intervals of convergence; thus, if the first series converges for $|x-x_0|<R_1$ and the second converges for $|x-x_0|<R_2$, then

$$f(x)+g(x)=\sum_{n=0}^{\infty}(a_n+b_n)(x-x_0)^n$$

for $|x-x_0|<R$, where R is the smaller of R_1 and R_2. More generally, linear combinations of power series can be formed term by term; for example,

$$c_1f(x)+c_2f(x)=\sum_{n=0}^{\infty}(c_1a_n+c_2b_n)(x-x_0)^n.$$

Example 6.2.5

Find the Maclaurin series for $\cosh x$ as a linear combination of the Maclaurin series for e^x and e^{-x}.

Solution

By definition,

$$\cosh x=\frac{1}{2}e^x+\frac{1}{2}e^{-x}.$$

Since

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \text{and} \quad e^{-x}=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!},$$

it follows that

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{2}[1+(-1)^n]\frac{x^n}{n!}. \tag{6.2.12}$$

Since

$$\frac{1}{2}[1+(-1)^n] = \begin{cases} 1 & \text{if } n = 2m, \text{ an even integer,} \\ 0 & \text{if } n = 2m+1, \text{ an odd integer,} \end{cases}$$

we can rewrite Equation 6.2.12 more simply as

$$\cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}.$$

This result is valid on $(-\infty, \infty)$, since this is the open interval of convergence of the Maclaurin series for e^x and e^{-x}.

Example 6.2.6

Suppose

$$y = \sum_{n=0}^{\infty} a_n x^n$$

on an open interval I that contains the origin.

a. Express

$$(2-x)y'' + 2y$$

as a power series in x on I.

b. Use the result of (a) to find necessary and sufficient conditions on the coefficients $\{a_n\}$ for y to be a solution of the homogeneous equation

$$(2-x)y'' + 2y = 0 \tag{6.2.13}$$

on I.

Solution a

From Equation 6.2.7 with $x_0 = 0$,

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$\begin{aligned} (2-x)y'' + 2y &= 2y'' - xy' + 2y \\ &= \textstyle\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n. \end{aligned} \tag{6.2.14}$$

To combine the three series we shift indices in the first two to make their general terms constant multiples of x^n; thus,

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n \tag{6.2.15}$$

and

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1)na_{n+1}x^n = \sum_{n=0}^{\infty} (n+1)na_{n+1}x^n, \tag{6.2.16}$$

where we added a zero term in the last series so that when we substitute from Equation 6.2.15 and Equation 6.2.16 into Equation 6.2.14 all three series will start with $n = 0$; thus,

$$(2-x)y''+2y=\sum_{n=0}^{\infty}[2(n+2)(n+1)a_{n+2}-(n+1)na_{n+1}+2a_n]x^n. \tag{6.2.17}$$

Solution b

From Equation 6.2.17 we see that y satisfies Equation 6.2.13 on I if

$$2(n+2)(n+1)a_{n+2}-(n+1)na_{n+1}+2a_n=0, \quad n=0,1,2,\ldots. \tag{6.2.18}$$

Conversely, Theorem 6.2.5b implies that if $y=\sum_{n=0}^{\infty}a_nx^n$ satisfies Equation 6.2.13 on I, then Equation 6.2.18 holds.

Example 6.2.7

Suppose

$$y=\sum_{n=0}^{\infty}a_n(x-1)^n$$

on an open interval I that contains $x_0=1$. Express the function

$$(1+x)y''+2(x-1)^2y'+3y \tag{6.2.19}$$

as a power series in $x-1$ on I.

Solution

Since we want a power series in $x-1$, we rewrite the coefficient of y'' in Equation 6.2.19 as $1+x=2+(x-1)$, so Equation 6.2.19 becomes

$$2y''+(x-1)y''+2(x-1)^2y'+3y.$$

From Equation 6.2.6 and Equation 6.2.7 with $x_0=1$,

$$y'=\sum_{n=1}^{\infty}na_n(x-1)^{n-1} \quad \text{and} \quad y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}.$$

Therefore

$$\begin{aligned}
2y'' &= \sum_{n=2}^{\infty}2n(n-1)a_n(x-1)^{n-2},\\
(x-1)y'' &= \sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-1},\\
2(x-1)^2y' &= \sum_{n=1}^{\infty}2na_n(x-1)^{n+1},\\
3y &= \sum_{n=0}^{\infty}3a_n(x-1)^n.
\end{aligned}$$

Before adding these four series we shift indices in the first three so that their general terms become constant multiples of $(x-1)^n$. This yields

$$2y'' = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n, \tag{6.2.20}$$

$$(x-1)y'' = \sum_{n=0}^{\infty} (n+1)na_{n+1}(x-1)^n, \tag{6.2.21}$$

$$2(x-1)^2y' = \sum_{n=1}^{\infty} 2(n-1)a_{n-1}(x-1)^n, \tag{6.2.22}$$

$$3y = \sum_{n=0}^{\infty} 3a_n(x-1)^n, \tag{6.2.23}$$

where we added initial zero terms to the series in Equation 6.2.21 and Equation 6.2.22. Adding Equation 6.2.20 –Equation 6.2.23 yields

$$\begin{aligned}(1+x)y''+2(x-1)^2y'+3y &= 2y''+(x-1)y''+2(x-1)^2y'+3y \\ &= \sum_{n=0}^{\infty} b_n(x-1)^n,\end{aligned}$$

where

$$b_0 = 4a_2+3a_0, \tag{6.2.24}$$

$$b_n = 2(n+2)(n+1)a_{n+2}+(n+1)na_{n+1}+2(n-1)a_{n-1}+3a_n,\ n \geq 1. \tag{6.2.25}$$

The formula Equation 6.2.24 for b_0 can't be obtained by setting $n=0$ in Equation 6.2.25, since the summation in Equation 6.2.22 begins with $n=1$, while those in Equation 6.2.20, Equation 6.2.21, and Equation 6.2.23 begin with $n=0$.

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6.2.1: Review of Power Series (Exercises)

Q6.2.1

1.For each power series find the center, the interval and the radius of convergence.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n}(x-1)^n$

b. $\sum_{n=0}^{\infty} 2^n n(x-2)^n$

c. $\sum_{n=0}^{\infty} \frac{n!}{9^n} x^n$

d. $\sum_{n=0}^{\infty} \frac{n(n+1)}{16^n}(x-2)^n$

e. $\sum_{n=0}^{\infty} (-1)^n \frac{7^n}{n!} x^n$

f. $\sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}(n+1)^2}(x+7)^n$

Q6.2.2

For *Exercises 6.2.2-6.2.5* verify that the given power series is a solution of the indicated differential equation.

2. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, $y''+y=0$

3. $\sum_{n=0}^{\infty} \frac{1}{2^n n!} x^{2n}$, $y''-xy'-y=0$

4. $\sum_{n=0}^{\infty} (-1)^n x^n$, $(x+1)y''+2y'=0$, $-1<x<1$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$, $(x+1)y''+y'=0$, $-1<x\leq 1$

Q6.2.3

Suppose $y(x)=\sum_{n=0}^{\infty} a_n x^n$ on an open interval that contains $x_0 = 0$. For *Exercises 6.2.6-6.2.10* find a power series in x.

6. $(2+x)y''+xy'+3y$

7. $(1+3x^2)y''+3x^2y'-2y$

8. $(1+2x^2)y''+(2-3x)y'+4y$

9. $(1+x^2)y''+(2-x)y'+3y$

10. $(1+3x^2)y''-2xy'+4y$

Q6.2.4

11. Suppose $y(x)=\sum_{n=0}^{\infty} a_n(x+1)^n$ on an open interval that contains $x_0 = -1$. Find a power series in $x+1$ for

$$xy''+(4+2x)y'+(2+x)y.$$

12. Suppose $y(x)=\sum_{n=0}^{\infty} a_n(x-2)^n$ on an open interval that contains $x_0 = 2$. Find a power series in $x-2$ for

$$x^2y'' + 2xy' - 3xy.$$

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6.3: Series Solutions Near an Ordinary Point I

Many physical applications give rise to second order homogeneous linear differential equations of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \tag{6.3.1}$$

where P_0, P_1, and P_2 are polynomials. Usually the solutions of these equations can't be expressed in terms of familiar elementary functions. Therefore we'll consider the problem of representing solutions of Equation 6.3.1 with series.

We assume throughout that P_0, P_1 and P_2 have no common factors. Then we say that x_0 is an **ordinary point** of Equation 6.3.1 if $P_0(x_0) \neq 0$, or a **singular point** if $P_0(x_0) = 0$. For Legendre's equation,

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \tag{6.3.2}$$

$x_0 = 1$ and $x_0 = -1$ are singular points and all other points are ordinary points. For Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

$x_0 = 0$ is a singular point and all other points are ordinary points. If P_0 is a nonzero constant as in Airy's equation,

$$y'' - xy = 0, \tag{6.3.3}$$

then every point is an ordinary point.

Since polynomials are continuous everywhere, P_1/P_0 and P_2/P_0 are continuous at any point x_0 that isn't a zero of P_0. Therefore, if x_0 is an ordinary point of Equation 6.3.1 and a_0 and a_1 are arbitrary real numbers, then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = a_0, \quad y'(x_0) = a_1 \tag{6.3.4}$$

has a unique solution on the largest open interval that contains x_0 and does not contain any zeros of P_0. To see this, we rewrite the differential equation in Equation 6.3.4 as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem 5.1.1 with $p = P_1/P_0$ and $q = P_2/P_0$. In this section and the next we consider the problem of representing solutions of Equation 6.3.1 by power series that converge for values of x near an ordinary point x_0.

We state the next theorem without proof.

Theorem 7.3.1

Suppose P_0, P_1, and P_2 are polynomials with no common factor and P_0 isn't identically zero. Let x_0 be a point such that $P_0(x_0) \neq 0$, and let ρ be the distance from x_0 to the nearest zero of P_0 in the complex plane. (If P_0 is constant, then $\rho = \infty$.) Then every solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{6.3.5}$$

can be represented by a power series

$$y = \sum_{n=0}^{\infty} a_n(x-x_0)^n \tag{6.3.6}$$

that converges at least on the open interval $(x_0 - \rho, x_0 + \rho)$. (If P_0 is nonconstant, so that ρ is necessarily finite, then the open interval of convergence of (6.3.6) may be larger than $(x_0 - \rho, x_0 + \rho)$. If P_0 is constant then $\rho = \infty$ and $(x_0 - \rho, x_0 + \rho) = (-\infty, \infty)$.

We call Equation 6.3.6 a **power series solution in $x - x_0$** of Equation 6.3.5. We'll now develop a method for finding power series solutions of Equation 6.3.5. For this purpose we write Equation 6.3.5 as $Ly = 0$, where

$$Ly = P_0y'' + P_1y' + P_2y. \tag{6.3.7}$$

Theorem 7.3.1 implies that every solution of $Ly = 0$ on $(x_0 - \rho, x_0 + \rho)$ can be written as

$$y=\sum_{n=0}^{\infty} a_n(x-x_0)^n.$$

Setting $x=x_0$ in this series and in the series

$$y'=\sum_{n=1}^{\infty} n a_n(x-x_0)^{n-1}$$

shows that $y(x_0)=a_0$ and $y'(x_0)=a_1$. Since every initial value problem Equation 6.3.4 has a unique solution, this means that a_0 and a_1 can be chosen arbitrarily, and a_2, a_3, …are uniquely determined by them.

To find a_2, a_3, …, we write P_0, P_1, and P_2 in powers of $x-x_0$, substitute

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n(x-x_0)^n, \\ y' &= \sum_{n=1}^{\infty} n a_n(x-x_0)^{n-1}, \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n(x-x_0)^{n-2} \end{aligned}$$

into Equation 6.3.7, and collect the coefficients of like powers of $x-x_0$. This yields

$$Ly=\sum_{n=0}^{\infty} b_n(x-x_0)^n, \tag{6.3.8}$$

where $\{b_0, b_1, \ldots, b_n, \ldots\}$ are expressed in terms of $\{a_0, a_1, \ldots, a_n, \ldots\}$ and the coefficients of P_0, P_1, and P_2, written in powers of $x-x_0$. Since Equation 6.3.8 and (a) of Theorem 7.1.6 imply that $Ly=0$ if and only if $b_n=0$ for $n \geq 0$, all power series solutions in $x-x_0$ of $Ly=0$ can be obtained by choosing a_0 and a_1 arbitrarily and computing a_2, a_3, …, successively so that $b_n=0$ for $n \geq 0$. For simplicity, we call the power series obtained this way **the power series in $x-x_0$** for the general solution of $Ly=0$, without explicitly identifying the open interval of convergence of the series.

Example 7.3.1

Let x_0 be an arbitrary real number. Find the power series in $x-x_0$ for the general solution of Equations like Equations 6.3.11, 6.3.12, and 6.3.13, which define a given coefficient in the sequence $\{a_n\}$ in terms of one or more coefficients with lesser indices are called **recurrence relations**. When we use a recurrence relation to compute terms of a sequence we are computing **recursively**.

In the remainder of this section we consider the problem of finding power series solutions in $x-x_0$ for equations of the form

$$\left(1+\alpha(x-x_0)^2\right) y''+\beta(x-x_0) y'+\gamma y=0. \tag{6.3.9}$$

Many important equations that arise in applications are of this form with $x_0=0$, including Legendre's equation Equation 6.3.2, Airy's equation Equation 6.3.3, **Chebyshev's equation**,

$$(1-x^2) y''-x y'+\alpha^2 y=0,$$

and **Hermite's equation**,

$$y''-2 x y'+2 \alpha y=0.$$

Since

$$P_0(x)=1+\alpha(x-x_0)^2$$

in Equation 6.3.9, the point x_0 is an ordinary point of Equation 6.3.9, and Theorem 7.3.1 implies that the solutions of Equation 6.3.9 can be written as power series in $x-x_0$ that converge on the interval $(x_0-1/\sqrt{|\alpha|}, x_0+1/\sqrt{|\alpha|})$ if $\alpha \neq 0$, or on $(-\infty, \infty)$ if $\alpha=0$. We'll see that the coefficients in these power series can be obtained by methods similar to the one used in Example 7.3.1 .

To simplify finding the coefficients, we introduce some notation for products:

$$\prod_{j=r}^{s} b_j = b_r b_{r+1} \cdots b_s \quad \text{if} \quad s \geq r.$$

Thus,

$$\prod_{j=2}^{7} b_j = b_2 b_3 b_4 b_5 b_6 b_7,$$

$$\prod_{j=0}^{4} (2j+1) = (1)(3)(5)(7)(9) = 945,$$

and

$$\prod_{j=2}^{2} j^2 = 2^2 = 4.$$

We define

$$\prod_{j=r}^{s} b_j = 1 \quad \text{if} \quad s < r,$$

no matter what the form of b_j.

Example 7.3.2

Find the power series in x for the general solution of

$$y'' + y = 0. \tag{6.3.10}$$

Solution

Here

$$Ly = y'' + y.$$

If

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

so

$$Ly = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} + \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

To collect coefficients of like powers of $x - x_0$, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n + \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

with

$$b_n = (n+2)(n+1) a_{n+2} + a_n.$$

Therefore $Ly=0$ if and only if

$$a_{n+2}=\frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0, \tag{6.3.11}$$

where a_0 and a_1 are arbitrary. Since the indices on the left and right sides of Equation 6.3.11 differ by two, we write Equation 6.3.11 separately for n even $(n=2m)$ and n odd $(n=2m+1)$. This yields

$$a_{2m+2}=\frac{-a_{2m}}{(2m+2)(2m+1)}, \quad m \geq 0, \tag{6.3.12}$$

and

$$a_{2m+3}=\frac{-a_{2m+1}}{(2m+3)(2m+2)}, \quad m \geq 0. \tag{6.3.13}$$

Computing the coefficients of the even powers of $x-x_0$ from Equation 6.3.12 yields

$$\begin{aligned}
a_2 &= -\frac{a_0}{2 \cdot 1} \\
a_4 &= -\frac{a_2}{4 \cdot 3}=-\frac{1}{4 \cdot 3}\left(-\frac{a_0}{2 \cdot 1}\right)=\frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}, \\
a_6 &= -\frac{a_4}{6 \cdot 5}=-\frac{1}{6 \cdot 5}\left(\frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}\right)=-\frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1},
\end{aligned}$$

and, in general,

$$a_{2m}=(-1)^m \frac{a_0}{(2m)!}, \quad m \geq 0. \tag{6.3.14}$$

Computing the coefficients of the odd powers of $x-x_0$ from Equation 6.3.13 yields

$$\begin{aligned}
a_3 &= -\frac{a_1}{3 \cdot 2} \\
a_5 &= -\frac{a_3}{5 \cdot 4}=-\frac{1}{5 \cdot 4}\left(-\frac{a_1}{3 \cdot 2}\right)=\frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, \\
a_7 &= -\frac{a_5}{7 \cdot 6}=-\frac{1}{7 \cdot 6}\left(\frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}\right)=-\frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2},
\end{aligned}$$

and, in general,

$$a_{2m+1}=\frac{(-1)^m a_1}{(2m+1)!} \quad m \geq 0. \tag{6.3.15}$$

Thus, the general solution of Equation 6.3.10 can be written as

$$y=\sum_{m=0}^{\infty} a_{2m}(x-x_0)^{2m}+\sum_{m=0}^{\infty} a_{2m+1}(x-x_0)^{2m+1},$$

or, from Equation 6.3.14 and Equation 6.3.15, as

$$y=a_0 \sum_{m=0}^{\infty}(-1)^m \frac{(x-x_0)^{2m}}{(2m)!}+a_1 \sum_{m=0}^{\infty}(-1)^m \frac{(x-x_0)^{2m+1}}{(2m+1)!}. \tag{6.3.16}$$

If we recall from calculus that

$$\sum_{m=0}^{\infty}(-1)^m \frac{(x-x_0)^{2m}}{(2m)!}=\cos(x-x_0) \quad \text{and} \quad \sum_{m=0}^{\infty}(-1)^m \frac{(x-x_0)^{2m+1}}{(2m+1)!}=\sin(x-x_0),$$

then Equation 6.3.16 becomes

$$y=a_0 \cos(x-x_0)+a_1 \sin(x-x_0),$$

which should look familiar.

$$(1+2x^2)y''+6xy'+2y=0. \tag{6.3.17}$$

Solution

Here

$$Ly=(1+2x^2)y''+6xy'+2y.$$

If

$$y=\sum_{n=0}^{\infty}a_nx^n$$

then

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1} \quad \text{and} \quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2},$$

so

$$\begin{aligned} Ly &= (1+2x^2)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+6x\sum_{n=1}^{\infty}na_nx^{n-1}+2\sum_{n=0}^{\infty}a_nx^n \\ &= \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=0}^{\infty}\left[2n(n-1)+6n+2\right]a_nx^n \\ &= \sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+2\sum_{n=0}^{\infty}(n+1)^2a_nx^n. \end{aligned}$$

To collect coefficients of x^n, we shift the summation index in the first sum. This yields

$$Ly=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+2\sum_{n=0}^{\infty}(n+1)^2a_nx^n=\sum_{n=0}^{\infty}b_nx^n,$$

with

$$b_n=(n+2)(n+1)a_{n+2}+2(n+1)^2a_n, \quad n\geq 0.$$

To obtain solutions of Equation 6.3.17, we set $b_n=0$ for $n\geq 0$. This is equivalent to the recurrence relation

$$a_{n+2}=-2\frac{n+1}{n+2}a_n, \quad n\geq 0. \tag{6.3.18}$$

Since the indices on the left and right differ by two, we write Equation 6.3.18 separately for $n=2m$ and $n=2m+1$, as in Example 7.3.1 . This yields

$$a_{2m+2}=-2\frac{2m+1}{2m+2}a_{2m}=-\frac{2m+1}{m+1}a_{2m}, \quad m\geq 0, \tag{6.3.19}$$

and

$$a_{2m+3}=-2\frac{2m+2}{2m+3}a_{2m+1}=-4\frac{m+1}{2m+3}a_{2m+1}, \quad m\geq 0. \tag{6.3.20}$$

Computing the coefficients of even powers of x from Equation 6.3.19 yields

$$
\begin{aligned}
a_2 &= -\frac{1}{1}a_0,\\
a_4 &= -\frac{3}{2}a_2 = \left(-\frac{3}{2}\right)\left(-\frac{1}{1}\right)a_0 = \frac{1\cdot 3}{1\cdot 2}a_0,\\
a_6 &= -\frac{5}{3}a_4 = -\frac{5}{3}\left(\frac{1\cdot 3}{1\cdot 2}\right)a_0 = -\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}a_0,\\
a_8 &= -\frac{7}{4}a_6 = -\frac{7}{4}\left(-\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\right)a_0 = \frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 3\cdot 4}a_0.
\end{aligned}
$$

In general,

$$
a_{2m} = (-1)^m \frac{\prod_{j=1}^m (2j-1)}{m!}a_0, \quad m \ge 0. \tag{6.3.21}
$$

(Note that Equation 6.3.21 is correct for $m=0$ because we defined $\prod_{j=1}^0 b_j = 1$ for any b_j.)

Computing the coefficients of odd powers of x from Equation 6.3.20 yields

$$
\begin{aligned}
a_3 &= -4\,\frac{1}{3}a_1,\\
a_5 &= -4\,\frac{2}{5}a_3 = -4\,\frac{2}{5}\left(-4\frac{1}{3}\right)a_1 = 4^2\frac{1\cdot 2}{3\cdot 5}a_1,\\
a_7 &= -4\,\frac{3}{7}a_5 = -4\,\frac{3}{7}\left(4^2\frac{1\cdot 2}{3\cdot 5}\right)a_1 = -4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}a_1,\\
a_9 &= -4\,\frac{4}{9}a_7 = -4\,\frac{4}{9}\left(4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)a_1 = 4^4\frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9}a_1.
\end{aligned}
$$

In general,

$$
a_{2m+1} = \frac{(-1)^m 4^m m!}{\prod_{j=1}^m (2j+1)}a_1, \quad m \ge 0. \tag{6.3.22}
$$

From Equation 6.3.21 and Equation 6.3.22,

$$
y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^m (2j-1)}{m!}x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{4^m m!}{\prod_{j=1}^m (2j+1)}x^{2m+1}.
$$

is the power series in x for the general solution of Equation 6.3.17. Since $P_0(x) = 1+2x^2$ has no real zeros, Theorem 5.1.1 implies that every solution of Equation 6.3.17 is defined on $(-\infty,\infty)$. However, since $P_0(\pm i/\sqrt{2}) = 0$, Theorem 7.3.1 implies only that the power series converges in $(-1/\sqrt{2}, 1/\sqrt{2})$ for any choice of a_0 and a_1.

The results in Examples 7.3.1 and 7.3.2 are consequences of the following general theorem.

Theorem 7.3.2

The coefficients $\{a_n\}$ in any solution $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$ of

$$
\left(1+\alpha(x-x_0)^2\right)y'' + \beta(x-x_0)y' + \gamma y = 0 \tag{6.3.23}
$$

satisfy the recurrence relation

$$
a_{n+2} = -\frac{p(n)}{(n+2)(n+1)}a_n, \quad n \ge 0, \tag{6.3.24}
$$

where

$$
p(n) = \alpha n(n-1) + \beta n + \gamma. \tag{6.3.25}
$$

Moreover, the coefficients of the even and odd powers of $x-x_0$ can be computed separately as

$$a_{2m+2}=-\frac{p(2m)}{(2m+2)(2m+1)}a_{2m}, \quad m\geq 0 \tag{6.3.26}$$

$$a_{2m+3}=-\frac{p(2m+1)}{(2m+3)(2m+2)}a_{2m+1}, \quad m\geq 0, \tag{6.3.27}$$

where a_0 and a_1 are arbitrary.

Proof

Here

$$Ly=(1+\alpha(x-x_0)^2)y''+\beta(x-x_0)y'+\gamma y.$$

If

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n,$$

then

$$y'=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1} \quad \text{and} \quad y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}.$$

Hence,

$$\begin{aligned} Ly &= \textstyle\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}+\sum_{n=0}^{\infty}\left[\alpha n(n-1)+\beta n+\gamma\right]a_n(x-x_0)^n \\ &= \textstyle\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}+\sum_{n=0}^{\infty}p(n)a_n(x-x_0)^n, \end{aligned}$$

from Equation 6.3.25 To collect coefficients of powers of $x-x_0$, we shift the summation index in the first sum. This yields

$$Ly=\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+p(n)a_n\right](x-x_0)^n.$$

Thus, $Ly=0$ if and only if

$$(n+2)(n+1)a_{n+2}+p(n)a_n=0, \quad n\geq 0,$$

which is equivalent to Equation 6.3.24 Writing Equation 6.3.24separately for the cases where $n=2m$ and $n=2m+1$ yields Equation 6.3.26and Equation 6.3.27.

Example 7.3.3

Find the power series in $x-1$ for the general solution of

$$(2+4x-2x^2)y''-12(x-1)y'-12y=0. \tag{6.3.28}$$

Solution

We must first write the coefficient $P_0(x)=2+4x-x^2$ in powers of $x-1$. To do this, we write $x=(x-1)+1$ in $P_0(x)$ and then expand the terms, collecting powers of $x-1$; thus,

$$\begin{aligned} 2+4x-2x^2 &= 2+4[(x-1)+1]-2[(x-1)+1]^2 \\ &= 4-2(x-1)^2. \end{aligned}$$

Therefore we can rewrite Equation 6.3.28as

$$\left(4-2(x-1)^2\right)y''-12(x-1)y'-12y=0,$$

or, equivalently,

$$\left(1-\frac{1}{2}(x-1)^2\right)y''-3(x-1)y'-3y=0.$$

This is of the form Equation 6.3.23 with $\alpha=-1/2$, $\beta=-3$, and $\gamma=-3$. Therefore, from Equation 6.3.25

$$p(n)=-\frac{n(n-1)}{2}-3n-3=-\frac{(n+2)(n+3)}{2}.$$

Hence, Theorem7.3.2 implies that

$$\begin{aligned} a_{2m+2} &= -\frac{p(2m)}{(2m+2)(2m+1)}a_{2m} \\ &= \frac{(2m+2)(2m+3)}{2(2m+2)(2m+1)}a_{2m}=\frac{2m+3}{2(2m+1)}a_{2m}, \quad m\ge 0 \end{aligned}$$

and

$$\begin{aligned} a_{2m+3} &= -\frac{p(2m+1)}{(2m+3)(2m+2)}a_{2m+1} \\ &= \frac{(2m+3)(2m+4)}{2(2m+3)(2m+2)}a_{2m+1}=\frac{m+2}{2(m+1)}a_{2m+1}, \quad m\ge 0. \end{aligned}$$

We leave it to you to show that

$$a_{2m}=\frac{2m+1}{2^m}a_0 \quad \text{and} \quad a_{2m+1}=\frac{m+1}{2^m}a_1, \quad m\ge 0,$$

which implies that the power series in $x-1$ for the general solution of Equation 6.3.28 is

$$y=a_0\sum_{m=0}^{\infty}\frac{2m+1}{2^m}(x-1)^{2m}+a_1\sum_{m=0}^{\infty}\frac{m+1}{2^m}(x-1)^{2m+1}.$$

In the examples considered so far we were able to obtain closed formulas for coefficients in the power series solutions. In some cases this is impossible, and we must settle for computing a finite number of terms in the series. The next example illustrates this with an initial value problem.

Example 7.3.4

Compute a_0, a_1, …, a_7 in the series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of the initial value problem

$$(1+2x^2)y''+10xy'+8y=0, \quad y(0)=2, \quad y'(0)=-3. \tag{6.3.29}$$

Solution

Since $\alpha=2$, $\beta=10$, and $\gamma=8$ in Equation 6.3.29,

$$p(n)=2n(n-1)+10n+8=2(n+2)^2.$$

Therefore

$$a_{n+2}=-2\frac{(n+2)^2}{(n+2)(n+1)}a_n=-2\frac{n+2}{n+1}a_n, \quad n\ge 0.$$

Writing this equation separately for $n=2m$ and $n=2m+1$ yields

$$a_{2m+2}=-2\frac{(2m+2)}{2m+1}a_{2m}=-4\frac{m+1}{2m+1}a_{2m}, \quad m\ge 0 \tag{6.3.30}$$

$$a_{2m+3}=-2\frac{2m+3}{2m+2}a_{2m+1}=-\frac{2m+3}{m+1}a_{2m+1}, \quad m\ge 0. \tag{6.3.31}$$

Starting with $a_0 = y(0) = 2$, we compute a_2, a_4, and a_6 from Equation 6.3.30:

$$\begin{aligned} a_2 &= -4\,\frac{1}{1}2 = -8, \\ a_4 &= -4\,\frac{2}{3}(-8) = \frac{64}{3}, \\ a_6 &= -4\,\frac{3}{5}\left(\frac{64}{3}\right) = -\frac{256}{5}. \end{aligned}$$

Starting with $a_1 = y'(0) = -3$, we compute a_3, a_5 and a_7 from Equation 6.3.31:

$$\begin{aligned} a_3 &= -\frac{3}{1}(-3) = 9, \\ a_5 &= -\frac{5}{2}9 = -\frac{45}{2}, \\ a_7 &= -\frac{7}{3}\left(-\frac{45}{2}\right) = \frac{105}{2}. \end{aligned}$$

Therefore the solution of Equation 6.3.29 is

$$y = 2 - 3x - 8x^2 + 9x^3 + \frac{64}{3}x^4 - \frac{45}{2}x^5 - \frac{256}{5}x^6 + \frac{105}{2}x^7 + \cdots\ .$$

Using Technology

Computing coefficients recursively as in Example 7.3.4 is tedious. We recommend that you do this kind of computation by writing a short program to implement the appropriate recurrence relation on a calculator or computer. You may wish to do this in verifying examples and doing exercises (identified by the symbol C) in this chapter that call for numerical computation of the coefficients in series solutions. We obtained the answers to these exercises by using software that can produce answers in the form of rational numbers. However, it is perfectly acceptable - and more practical - to get your answers in decimal form. You can always check them by converting our fractions to decimals.

If you're interested in actually using series to compute numerical approximations to solutions of a differential equation, then whether or not there's a simple closed form for the coefficients is essentially irrelevant. For computational purposes it is usually more efficient to start with the given coefficients $a_0 = y(x_0)$ and $a_1 = y'(x_0)$, compute a_2, …, a_N recursively, and then compute approximate values of the solution from the Taylor polynomial

$$T_N(x) = \sum_{n=0}^{N} a_n (x - x_0)^n.$$

The trick is to decide how to choose N so the approximation $y(x) \approx T_N(x)$ is sufficiently accurate on the subinterval of the interval of convergence that you're interested in. In the computational exercises in this and the next two sections, you will often be asked to obtain the solution of a given problem by numerical integration with software of your choice (see Section 10.1 for a brief discussion of one such method), and to compare the solution obtained in this way with the approximations obtained with T_N for various values of N. This is a typical textbook kind of exercise, designed to give you insight into how the accuracy of the approximation $y(x) \approx T_N(x)$ behaves as a function of N and the interval that you're working on. In real life, you would choose one or the other of the two methods (numerical integration or series solution). If you choose the method of series solution, then a practical procedure for determining a suitable value of N is to continue increasing N until the maximum of $|T_N - T_{N-1}|$ on the interval of interest is within the margin of error that you're willing to accept.

In doing computational problems that call for numerical solution of differential equations you should choose the most accurate numerical integration procedure your software supports, and experiment with the step size until you're confident that the numerical results are sufficiently accurate for the problem at hand.

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6.3.1: Series Solutions Near an Ordinary Point I (Exercises)

Q6.3.1

In *Exercises 6.3.1-6.3.8* find the general solution about the ordinary point x = 0. Also, find a fundamental set of solutions.

1. $(1+x^2)y''+6xy'+6y=0$
2. $(1+x^2)y''+2xy'-2y=0$
3. $(1+x^2)y''-8xy'+20y=0$
4. $(1-x^2)y''-8xy'-12y=0$
5. $(1+2x^2)y''+7xy'+2y=0$
6. $(1+x^2)y''+2xy'+\frac{1}{4}y=0$
7. $(x+2)y''+xy'-y=0$
8. $(1+x^2)y''-10xy'+28y=0$

Q6.3.2

In *Exercises 6.3.11-6.3.13* find $a_0,\ldots,a_N$ for N at least 7 in the power series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of the initial value problem.

9. $(1+x^2)y''+xy'+y=0,\quad y(0)=2,\quad y'(0)=-1$
10. $(1+2x^2)y''-9xy'-6y=0,\quad y(0)=1,\quad y'(0)=-1$
11. $(1+8x^2)y''+2y=0,\quad y(0)=2,\quad y'(0)=-1$

Q6.3.3

In *Exercises 6.3.12-6.3.16* find the power series in $x-x_0$ for the general solution.

12. $y''-y=0;\quad x_0=3$
13. $y''-(x-3)y'-y=0;\quad x_0=3$
14. $(1-4x+2x^2)y''+10(x-1)y'+6y=0;\quad x_0=1$
15. $(11-8x+2x^2)y''-16(x-2)y'+36y=0;\quad x_0=2$
16. $(5+6x+3x^2)y''+9(x+1)y'+3y=0;\quad x_0=-1$

Q6.3.4

In *Exercises 6.3.17-6.3.22* find $a_0,\ldots a_N$ for N at least 7 in the power series $y=\sum_{n=0}^{\infty}a_n(x-x_0)^n$ for the solution of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

17. $(x^2-4)y''-xy'-3y=0,\quad y(0)=-1,\quad y'(0)=2$
18. $y''+(x-3)y'+3y=0,\quad y(3)=-2,\quad y'(3)=3$
19. $(5-6x+3x^2)y''+(x-1)y'+12y=0,\quad y(1)=-1,\quad y'(1)=1$
20. $(4x^2-24x+37)y''+y=0,\quad y(3)=4,\quad y'(3)=-6$
21. $(x^2-8x+14)y''-8(x-4)y'+20y=0,\quad y(4)=3,\quad y'(4)=-4$
22. $(2x^2+4x+5)y''-20(x+1)y'+60y=0,\quad y(-1)=3,\quad y'(-1)=-3$

Q6.3.5

23.

a. Find a power series in x for the general solution of

$$(1+x^2)y''+4xy'+2y=0. \tag{A}$$

b. Use (a) and the formula

$$\frac{1}{1-r}=1+r+r^2+\cdots+r^n+\cdots \quad (-1<r<1)$$

for the sum of a geometric series to find a closed form expression for the general solution of (A) on $(-1,1)$.

c. Show that the expression obtained in (b) is actually the general solution of of (A) on $(-\infty,\infty)$.

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6.4: Regular Singular Points Euler Equations

In the next three sections we'll continue to study equations of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{6.4.1}$$

where P_0, P_1, and P_2 are polynomials, but the emphasis will be different from that of Sections 7.2 and 7.3, where we obtained solutions of Equation 6.4.1 near an ordinary point x_0 in the form of power series in $x - x_0$. If x_0 is a singular point of Equation 6.4.1 (that is, if $P(x_0) = 0$), the solutions can't in general be represented by power series in $x - x_0$. Nevertheless, it is often necessary in physical applications to study the behavior of solutions of Equation 6.4.1 near a singular point. Although this can be difficult in the absence of some sort of assumption on the nature of the singular point, equations that satisfy the requirements of the next definition can be solved by series methods discussed in the next three sections. Fortunately, many equations arising in applications satisfy these requirements.

Theorem 6.4.1: regular singular points

Let P_0, P_1, and P_2 be polynomials with no common factor and suppose $P_0(x_0) = 0$. Then x_0 is a *regular singular point* of the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{6.4.2}$$

if (6.4.2) can be written as

$$(x - x_0)^2 A(x)y'' + (x - x_0)B(x)y' + C(x)y = 0 \tag{6.4.3}$$

where A, B, and C are polynomials and $A(x_0) \neq 0$; otherwise, x_0 is an *irregular* singular point of Equation 6.4.2.

Example 6.4.1: Bessel's equation

Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \tag{6.4.4}$$

has the singular point $x_0 = 0$. Since this equation is of the form Equation 6.4.3 with $x_0 = 0$, $A(x) = 1$, $B(x) = 1$, and $C(x) = x^2 - \nu^2$, it follows that $x_0 = 0$ is a regular singular point of Equation 6.4.4.

Example 6.4.2: Legendre's equation

Legendre's equation,

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \tag{6.4.5}$$

has the singular points $x_0 = \pm 1$. Mutiplying through by $1 - x$ yields

$$(x - 1)^2(x + 1)y'' + 2x(x - 1)y' - \alpha(\alpha + 1)(x - 1)y = 0,$$

which is of the form Equation 6.4.3 with $x_0 = 1$, $A(x) = x + 1$, $B(x) = 2x$, and $C(x) = -\alpha(\alpha + 1)(x - 1)$. Therefore $x_0 = 1$ is a regular singular point of Equation 6.4.5. We leave it to you to show that $x_0 = -1$ is also a regular singular point of Equation 6.4.5.

Example 6.4.3

The equation

$$x^3y'' + xy' + y = 0$$

has an irregular singular point at $x_0 = 0$. (Verify.)

For convenience we restrict our attention to the case where $x_0 = 0$ is a regular singular point of Equation 6.4.2. This isn't really a restriction, since if $x_0 \neq 0$ is a regular singular point of Equation 6.4.2 then introducing the new independent variable $t = x - x_0$

and the new unknown $Y(t)=y(t+x_0)$ leads to a differential equation with polynomial coefficients that has a regular singular point at $t_0=0$. This is illustrated in *Exercise 7.4.22* for Legendre's equation, and in *Exercise 7.4.23* for the general case.

Euler Equations

The simplest kind of equation with a regular singular point at $x_0=0$ is the Euler equation, defined as follows.

Theorem 6.4.2: Euler equation

An *Euler equation* is an equation that can be written in the form

$$ax^2y''+bxy'+cy=0, \tag{6.4.6}$$

where a, b, and c are real constants and $a\neq 0$.

Theorem 5.1.1 implies that Equation 6.4.6 has solutions defined on $(0,\infty)$ and $(-\infty,0)$, since Equation 6.4.6 can be rewritten as

$$ay''+\frac{b}{x}y'+\frac{c}{x^2}y=0.$$

For convenience we'll restrict our attention to the interval $(0,\infty)$. (*Exercise 7.4.19* deals with solutions of Equation 6.4.6 on $(-\infty,0)$.) The key to finding solutions on $(0,\infty)$ is that if $x>0$ then x^r is defined as a real-valued function on $(0,\infty)$ for all values of r, and substituting $y=x^r$ into Equation 6.4.6 produces

$$\begin{aligned} ax^2(x^r)''+bx(x^r)'+cx^r &= ax^2r(r-1)x^{r-2}+bxrx^{r-1}+cx^r \\ &= [ar(r-1)+br+c]x^r. \end{aligned} \tag{6.4.7}$$

The polynomial

$$p(r)=ar(r-1)+br+c$$

is called the *indicial polynomial* of Equation 6.4.6, and $p(r)=0$ is its *indicial equation*. From Equation 6.4.7 we can see that $y=x^r$ is a solution of Equation 6.4.6 on $(0,\infty)$ if and only if $p(r)=0$. Therefore, if the indicial equation has distinct real roots r_1 and r_2 then $y_1=x^{r_1}$ and $y_2=x^{r_2}$ form a fundamental set of solutions of Equation 6.4.6 on $(0,\infty)$, since $y_2/y_1=x^{r_2-r_1}$ is nonconstant. In this case

$$y=c_1x^{r_1}+c_2x^{r_2}$$

is the general solution of Equation 6.4.6 on $(0,\infty)$.

Example 6.4.4

Find the general solution of

$$x^2y''-xy'-8y=0 \tag{6.4.8}$$

on $(0,\infty)$.

Solution

The indicial polynomial of Equation 6.4.8 is

$$p(r)=r(r-1)-r-8=(r-4)(r+2).$$

Therefore $y_1=x^4$ and $y_2=x^{-2}$ are solutions of Equation 6.4.8 on $(0,\infty)$, and its general solution on $(0,\infty)$ is

$$y=c_1x^4+\frac{c_2}{x^2}.$$

Example 6.4.5

Find the general solution of

$$6x^2y''+5xy'-y=0 \tag{6.4.9}$$

on $(0,\infty)$.

Solution

The indicial polynomial of Equation 6.4.9 is

$$p(r)=6r(r-1)+5r-1=(2r-1)(3r+1).$$

Therefore the general solution of Equation 6.4.9 on $(0,\infty)$ is

$$y=c_1x^{1/2}+c_2x^{-1/3}.$$

If the indicial equation has a repeated root r_1, then $y_1=x^{r_1}$ is a solution of

$$ax^2y''+bxy'+cy=0, \tag{6.4.10}$$

on $(0,\infty)$, but Equation 6.4.10 has no other solution of the form $y=x^r$. If the indicial equation has complex conjugate zeros then Equation 6.4.10 has no real–valued solutions of the form $y=x^r$. Fortunately we can use the results of Section 5.2 for constant coefficient equations to solve Equation 6.4.10 in any case.

Theorem 6.4.3

Suppose the roots of the indicial equation

$$ar(r-1)+br+c=0 \tag{6.4.11}$$

are r_1 and r_2. Then the general solution of the Euler equation

$$ax^2y''+bxy'+cy=0 \tag{6.4.12}$$

on $(0,\infty)$ is

$$\begin{aligned} y &= c_1x^{r_1}+c_2x^{r_2} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers ;} \\ y &= x^{r_1}(c_1+c_2\ln x) \text{ if } r_1=r_2 \text{ ;} \\ y &= x^{\lambda}\left[c_1\cos(\omega\ln x)+c_2\sin(\omega\ln x)\right] \text{ if } r_1,r_2=\lambda\pm i\omega \text{ with } \omega>0. \end{aligned}$$

Proof

We first show that $y=y(x)$ satisfies Equation 6.4.12 on $(0,\infty)$ if and only if $Y(t)=y(e^t)$ satisfies the constant coefficient equation

$$a\frac{d^2Y}{dt^2}+(b-a)\frac{dY}{dt}+cY=0 \tag{6.4.13}$$

on $(-\infty,\infty)$. To do this, it is convenient to write $x=e^t$, or, equivalently, $t=\ln x$; thus, $Y(t)=y(x)$, where $x=e^t$. From the chain rule,

$$\frac{dY}{dt}=\frac{dy}{dx}\frac{dx}{dt}$$

and, since

$$\frac{dx}{dt}=e^t=x,$$

it follows that

$$\frac{dY}{dt}=x\frac{dy}{dx}. \tag{6.4.14}$$

Differentiating this with respect to t and using the chain rule again yields

$$\begin{aligned}\frac{d^2Y}{dt^2} &= \frac{d}{dt}\left(\frac{dY}{dt}\right) = \frac{d}{dt}\left(x\frac{dy}{dx}\right)\\ &= \frac{dx}{dt}\frac{dy}{dx} + x\frac{d^2y}{dx^2}\frac{dx}{dt}\\ &= x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2} \quad \left(\text{since } \frac{dx}{dt} = x\right).\end{aligned}$$

From this and Equation 6.4.14,

$$x^2\frac{d^2y}{dx^2} = \frac{d^2Y}{dt^2} - \frac{dY}{dt}.$$

Substituting this and Equation 6.4.14 into Equation 6.4.12 yields Equation 6.4.13. Since Equation 6.4.11 is the characteristic equation of Equation 6.4.13, Theorem 5.2.1 implies that the general solution of Equation 6.4.13 on $(-\infty,\infty)$ is

$$\begin{aligned}Y(t) &= c_1e^{r_1t} + c_2e^{r_2t} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers;}\\ Y(t) &= e^{r_1t}(c_1 + c_2t) \text{ if } r_1 = r_2;\\ Y(t) &= e^{\lambda t}\left(c_1\cos\omega t + c_2\sin\omega t\right) \text{ if } r_1, r_2 = \lambda \pm i\omega \text{ with } \omega \neq 0.\end{aligned}$$

Since $Y(t) = y(e^t)$, substituting $t = \ln x$ in the last three equations shows that the general solution of Equation 6.4.12 on $(0,\infty)$ has the form stated in the theorem.

Example 6.4.6

Find the general solution of

$$x^2y'' - 5xy' + 9y = 0 \tag{6.4.15}$$

on $(0,\infty)$.

Solution

The indicial polynomial of Equation 6.4.15 is

$$p(r) = r(r-1) - 5r + 9 = (r-3)^2.$$

Therefore the general solution of Equation 6.4.15 on $(0,\infty)$ is

$$y = x^3(c_1 + c_2\ln x).$$

Example 6.4.7

Find the general solution of

$$x^2y'' + 3xy' + 2y = 0 \tag{6.4.16}$$

on $(0,\infty)$.

Solution

The indicial polynomial of Equation 6.4.16 is

$$p(r) = r(r-1) + 3r + 2 = (r+1)^2 + 1.$$

The roots of the indicial equation are $r = -1 \pm i$ and the general solution of Equation 6.4.16 on $(0,\infty)$ is

$$y = \frac{1}{x}\left[c_1\cos(\ln x) + c_2\sin(\ln x)\right].$$

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6.4.1: Regular Singular Points Euler Equations (Exercises)

Q6.4.1

In *Exercises 7.4.1-7.4.18* find the general solution of the given Euler equation on $(0, \infty)$.

1. $x^2y'' + 7xy' + 8y = 0$
2. $x^2y'' - 7xy' + 7y = 0$
3. $x^2y'' - xy' + y = 0$
4. $x^2y'' + 5xy' + 4y = 0$
5. $x^2y'' + xy' + y = 0$
6. $x^2y'' - 3xy' + 13y = 0$
7. $x^2y'' + 3xy' - 3y = 0$
8. $12x^2y'' - 5xy'' + 6y = 0$
9. $4x^2y'' + 8xy' + y = 0$
10. $3x^2y'' - xy' + y = 0$
11. $2x^2y'' - 3xy' + 2y = 0$
12. $x^2y'' + 3xy' + 5y = 0$
13. $9x^2y'' + 15xy' + y = 0$
14. $x^2y'' - xy' + 10y = 0$
15. $x^2y'' - 6y = 0$
16. $2x^2y'' + 3xy' - y = 0$
17. $x^2y'' - 3xy' + 4y = 0$
18. $2x^2y'' + 10xy' + 9y = 0$

Q6.4.2

19.

a. Adapt the proof of Theorem 7.4.3 to show that $y = y(x)$ satisfies the Euler equation

$$ax^2y'' + bxy' + cy = 0 \tag{A}$$

on $(-\infty, 0)$ if and only if $Y(t) = y(-e^t)$

$$a\frac{d^2Y}{dt^2} + (b-a)\frac{dY}{dt} + cY = 0.$$

on $(-\infty, \infty)$.

b. Use (a) to show that the general solution of Equation A on $(-\infty, 0)$ is

$$\begin{aligned} y &= c_1|x|^{r_1} + c_2|x|^{r_2} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers;} \\ y &= |x|^{r_1}(c_1 + c_2 \ln|x|) \text{ if } r_1 = r_2; \\ y &= |x|^{\lambda}\left[c_1\cos(\omega\ln|x|) + c_2\sin(\omega\ln|x|)\right] \text{ if } r_1, r_2 = \lambda \pm i\omega \text{ with } \omega > 0. \end{aligned}$$

20. Use reduction of order to show that if

$$ar(r-1) + br + c = 0$$

has a repeated root r_1 then $y = x^{r_1}(c_1 + c_2 \ln x)$ is the general solution of

$$ax^2y'' + bxy' + cy = 0$$

on $(0, \infty)$.

21. A nontrivial solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

is said to be *oscillatory* on an interval (a, b) if it has infinitely many zeros on (a, b). Otherwise y is said to be *nonoscillatory* on (a, b). Show that the equation

$$x^2y'' + ky = 0 \quad (k = \text{ constant})$$

has oscillatory solutions on $(0, \infty)$ if and only if $k > 1/4$.

22. In Example 7.4.2 we saw that $x_0 = 1$ and $x_0 = -1$ are regular singular points of Legendre's equation

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0. \tag{A}$$

a. Introduce the new variables $t = x - 1$ and $Y(t) = y(t + 1)$, and show that y is a solution of (A) if and only if Y is a solution of

$$t(2 + t)\frac{d^2Y}{dt^2} + 2(1 + t)\frac{dY}{dt} - \alpha(\alpha + 1)Y = 0,$$

which has a regular singular point at $t_0 = 0$.

b. Introduce the new variables $t = x + 1$ and $Y(t) = y(t - 1)$, and show that y is a solution of (A) if and only if Y is a solution of

$$t(2 - t)\frac{d^2Y}{dt^2} + 2(1 - t)\frac{dY}{dt} + \alpha(\alpha + 1)Y = 0,$$

which has a regular singular point at $t_0 = 0$.

23. Let P_0, P_1, and P_2 be polynomials with no common factor, and suppose $x_0 \neq 0$ is a singular point of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{A}$$

Let $t = x - x_0$ and $Y(t) = y(t + x_0)$.

a. Show that y is a solution of (A) if and only if Y is a solution of

$$R_0(t)\frac{d^2Y}{dt^2} + R_1(t)\frac{dY}{dt} + R_2(t)Y = 0. \tag{B}$$

where

$$R_i(t) = P_i(t + x_0), \quad i = 0, 1, 2.$$

b. Show that R_0, R_1, and R_2 are polynomials in t with no common factors, and $R_0(0) = 0$; thus, $t_0 = 0$ is a singular point of (B).

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6.5: The Method of Frobenius I

Sections 7.5-7.7 deal with three distinct cases satisfying the assumptions introduced in Section 7.4. In all three cases, (A) has at least one solution of the form

$$y_1 = x^r \sum_{n=0}^{\infty} a_n x^n,$$

where r need not be an integer. The problem is that there are three possibilities - each requiring a different approach - for the form of a second solution y_2 such that $\{y_1, y_2\}$ is a fundamental pair of solutions of (A).

In this section we begin to study series solutions of a homogeneous linear second order differential equation with a regular singular point at $x_0 = 0$, so it can be written as

$$x^2 A(x) y'' + x B(x) y' + C(x) y = 0, \tag{6.5.1}$$

where A, B, C are polynomials and $A(0) \neq 0$.

We'll see that Equation 6.5.1 always has at least one solution of the form

$$y = x^r \sum_{n=0}^{\infty} a_n x^n$$

where $a_0 \neq 0$ and r is a suitably chosen number. The method we will use to find solutions of this form and other forms that we'll encounter in the next two sections is called **the method of Frobenius**, and we'll call them **Frobenius solutions**.

It can be shown that the power series $\sum_{n=0}^{\infty} a_n x^n$ in a Frobenius solution of Equation 6.5.1 converges on some open interval $(-\rho, \rho)$, where $0 < \rho \leq \infty$. However, since x^r may be complex for negative x or undefined if $x = 0$, we'll consider solutions defined for positive values of x. Easy modifications of our results yield solutions defined for negative values of x. (**Exercise 7.5.54**).

We'll restrict our attention to the case where A, B, and C are polynomials of degree not greater than two, so Equation 6.5.1 becomes

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2) y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2) y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2) y = 0, \tag{6.5.2}$$

where α_i, β_i, and γ_i are real constants and $\alpha_0 \neq 0$. Most equations that arise in applications can be written this way. Some examples are

$$\alpha x^2 y'' + \beta x y' + \gamma y = 0 \qquad \text{(Euler's equation)}$$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \qquad \text{(Bessel's equation)}$$

and

$$x y'' + (1 - x) y' + \lambda y = 0$$

where we would multiply the last equation by x to put it in the form Equation 6.5.2. However, the method of Frobenius can be extended to the case where A, B, and C are functions that can be represented by power series in x on some interval that contains zero, and $A_0(0) \neq 0$ (**Exercises 7.5.57** and **7.5.58**).

The next two theorems will enable us to develop systematic methods for finding Frobenius solutions of Equation 6.5.2.

Theorem 7.6.1

Let

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2) y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2) y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2) y,$$

and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1)+\beta_0 r+\gamma_0, \\ p_1(r) &= \alpha_1 r(r-1)+\beta_1 r+\gamma_1, \\ p_2(r) &= \alpha_2 r(r-1)+\beta_2 r+\gamma_2. \end{aligned}$$

Suppose the series

$$y=\sum_{n=0}^{\infty} a_n x^{n+r} \tag{6.5.3}$$

converges on $(0, \rho)$. Then

$$Ly=\sum_{n=0}^{\infty} b_n x^{n+r} \tag{6.5.4}$$

on $(0, \rho)$, where

$$\begin{aligned} b_0 &= p_0(r) a_0 \\ b_1 &= p_0(r+1) a_1+p_1(r) a_0 \\ b_n &= p_0(n+r) a_n+p_1(n+r-1) a_{n-1}+p_2(n+r-2) a_{n-2}, \quad n \geq 2 \end{aligned} \tag{6.5.5}$$

Proof

We begin by showing that if y is given by Equation 6.5.3and α, β, and γ are constants, then

$$\alpha x^2 y''+\beta x y'+\gamma y=\sum_{n=0}^{\infty} p(n+r) a_n x^{n+r}, \tag{6.5.6}$$

where

$$p(r)=\alpha r(r-1)+\beta r+\gamma.$$

Differentiating twice yields

$$y'=\sum_{n=0}^{\infty}(n+r) a_n x^{n+r-1} \tag{6.5.7}$$

and

$$y''=\sum_{n=0}^{\infty}(n+r)(n+r-1) a_n x^{n+r-2}. \tag{6.5.8}$$

Multiplying Equation 6.5.7by x and Equation 6.5.8by x^2 yields

$$x y'=\sum_{n=0}^{\infty}(n+r) a_n x^{n+r}$$

and

$$x^2 y''=\sum_{n=0}^{\infty}(n+r)(n+r-1) a_n x^{n+r}.$$

Therefore

$$\begin{aligned} \alpha x^2 y''+\beta x y'+\gamma y &= \sum_{n=0}^{\infty}[\alpha(n+r)(n+r-1)+\beta(n+r)+\gamma] a_n x^{n+r} \\ &= \sum_{n=0}^{\infty} p(n+r) a_n x^{n+r}, \end{aligned}$$

which proves Equation 6.5.6

Multiplying Equation 6.5.6 by x yields

$$x(\alpha x^2y''+\beta xy'+\gamma y)=\sum_{n=0}^{\infty}p(n+r)a_nx^{n+r+1}=\sum_{n=1}^{\infty}p(n+r-1)a_{n-1}x^{n+r}. \tag{6.5.9}$$

Multiplying Equation 6.5.6 by x^2 yields

$$x^2(\alpha x^2y''+\beta xy'+\gamma y)=\sum_{n=0}^{\infty}p(n+r)a_nx^{n+r+2}=\sum_{n=2}^{\infty}p(n+r-2)a_{n-2}x^{n+r}. \tag{6.5.10}$$

To use these results, we rewrite

$$Ly=x^2(\alpha_0+\alpha_1x+\alpha_2x^2)y''+x(\beta_0+\beta_1x+\beta_2x^2)y'+(\gamma_0+\gamma_1x+\gamma_2x^2)y$$

as

$$\begin{aligned} Ly &= \left(\alpha_0x^2y''+\beta_0xy'+\gamma_0y\right)+x\left(\alpha_1x^2y''+\beta_1xy'+\gamma_1y\right) \\ &\quad + x^2\left(\alpha_2x^2y''+\beta_2xy'+\gamma_2y\right). \end{aligned} \tag{6.5.11}$$

From Equation 6.5.6 with $p=p_0$,

$$\alpha_0x^2y''+\beta_0xy'+\gamma_0y=\sum_{n=0}^{\infty}p_0(n+r)a_nx^{n+r}.$$

From Equation 6.5.9 with $p=p_1$,

$$x\left(\alpha_1x^2y''+\beta_1xy'+\gamma_1y\right)=\sum_{n=1}^{\infty}p_1(n+r-1)a_{n-1}x^{n+r}.$$

From Equation 6.5.10 with $p=p_2$,

$$x^2\left(\alpha_2x^2y''+\beta_2xy'+\gamma_2y\right)=\sum_{n=2}^{\infty}p_2(n+r-2)a_{n-2}x^{n+r}.$$

Therefore we can rewrite Equation 6.5.11 as

$$\begin{aligned} Ly=\sum_{n=0}^{\infty}p_0(n+r)a_nx^{n+r}&+\sum_{n=1}^{\infty}p_1(n+r-1)a_{n-1}x^{n+r} \\ &+\sum_{n=2}^{\infty}p_2(n+r-2)a_{n-2}x^{n+r}, \end{aligned}$$

or

$$\begin{aligned} Ly &= p_0(r)a_0x^r+\left[p_0(r+1)a_1+p_1(r)a_0\right]x^{r+1} \\ &\quad +\sum_{n=2}^{\infty}\left[p_0(n+r)a_n+p_1(n+r-1)a_{n-1}+p_2(n+r-2)a_{n-2}\right]x^{n+r}, \end{aligned}$$

which implies Equation 6.5.4 with $\{b_n\}$ defined as in Equation 6.5.5.

Theorem 7.6.2

Let

$$Ly=x^2(\alpha_0+\alpha_1x+\alpha_2x^2)y''+x(\beta_0+\beta_1x+\beta_2x^2)y'+(\gamma_0+\gamma_1x+\gamma_2x^2)y,$$

where $\alpha_0\neq 0$, and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1)+\beta_0 r+\gamma_0, \\ p_1(r) &= \alpha_1 r(r-1)+\beta_1 r+\gamma_1, \\ p_2(r) &= \alpha_2 r(r-1)+\beta_2 r+\gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n. Define

$$a_0(r)=1, \tag{6.5.12}$$

$$a_1(r)=-\frac{p_1(r)}{p_0(r+1)}, \tag{6.5.13}$$

$$a_n(r)=-\frac{p_1(n+r-1)a_{n-1}(r)+p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)}, \quad n \geq 2. \tag{6.5.14}$$

Then the Frobenius series

$$y(x,r)=x^r \sum_{n=0}^{\infty} a_n(r)x^n \tag{6.5.15}$$

converges and satisfies

$$Ly(x,r)=p_0(r)x^r \tag{6.5.16}$$

on the interval $(0,\rho)$, where ρ is the distance from the origin to the nearest zero of $A(x)=\alpha_0+\alpha_1 x+\alpha_2 x^2$ in the complex plane (if A is constant, then $\rho=\infty$.)

If $\{a_n(r)\}$ is determined by the recurrence relation Equation 6.5.12 then substituting $a_n=a_n(r)$ into Equation 6.5.5 yields $b_0=p_0(r)$ and $b_n=0$ for $n \geq 1$, so Equation 6.5.4 reduces to Equation 6.5.16. We omit the proof that the series Equation 6.5.15 converges on $(0,\rho)$.

If $\alpha_i=\beta_i=\gamma_i=0$ for $i=1,2$, then $Ly=0$ reduces to the Euler equation

$$\alpha_0 x^2 y''+\beta_0 x y'+\gamma_0 y=0.$$

Theorem 7.4.3 shows that the solutions of this equation are determined by the zeros of the indicial polynomial

$$p_0(r)=\alpha_0 r(r-1)+\beta_0 r+\gamma_0.$$

Since Equation 6.5.16 implies that this is also true for the solutions of $Ly=0$, we'll also say that p_0 is the **indicial polynomial** of Equation 6.5.2, and that $p_0(r)=0$ is the **indicial equation** of $Ly=0$. We'll consider only cases where the indicial equation has real roots r_1 and r_2, with $r_1 \geq r_2$.

Theorem 7.6.3

Let L and $\{a_n(r)\}$ be as in Theorem 7.6.2 , and suppose the indicial equation $p_0(r)=0$ of $Ly=0$ has real roots r_1 and r_2, where $r_1 \geq r_2$. Then

$$y_1(x)=y(x,r_1)=x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n$$

is a Frobenius solution of $Ly=0$. Moreover, if r_1-r_2 is not an integer then

$$y_2(x)=y(x,r_2)=x^{r_2} \sum_{n=0}^{\infty} a_n(r_2)x^n$$

is also a Frobenius solution of $Ly=0$, and $\{y_1,y_2\}$ is a fundamental set of solutions.

Proof

Since r_1 and r_2 are roots of $p_0(r)=0$, the indicial polynomial can be factored as

$$p_0(r)=\alpha_0(r-r_1)(r-r_2). \tag{6.5.17}$$

Therefore

$$p_0(n+r_1)=n\alpha_0(n+r_1-r_2),$$

which is nonzero if $n>0$, since $r_1-r_2\geq 0$. Therefore the assumptions of Theorem 7.6.2 hold with $r=r_1$, and Equation 6.5.16implies that $Ly_1=p_0(r_1)x^{r_1}=0$.

Now suppose r_1-r_2 is not an integer. From Equation 6.5.17,

$$p_0(n+r_2)=n\alpha_0(n-r_1+r_2)\neq 0 \quad \text{if} \quad n=1,2,\cdots.$$

Hence, the assumptions of Theorem 7.6.2 hold with $r=r_2$, and Equation 6.5.16implies that $Ly_2=p_0(r_2)x^{r_2}=0$. We leave the proof that $\{y_1,y_2\}$ is a fundamental set of solutions as an **Exercise 7.5.52**.

It is not always possible to obtain explicit formulas for the coefficients in Frobenius solutions. However, we can always set up the recurrence relations and use them to compute as many coefficients as we want. The next example illustrates this.

Example 7.6.1

Find a fundamental set of Frobenius solutions of

$$2x^2(1+x+x^2)y''+x(9+11x+11x^2)y'+(6+10x+7x^2)y=0. \tag{6.5.18}$$

Compute just the first six coefficients a_0,…, a_5 in each solution.

Solution

For the given equation, the polynomials defined in Theorem 7.6.2 are

$$\begin{aligned} p_0(r) &= 2r(r-1)+9r+6 &&= (2r+3)(r+2),\\ p_1(r) &= 2r(r-1)+11r+10 &&= (2r+5)(r+2),\\ p_2(r) &= 2r(r-1)+11r+7 &&= (2r+7)(r+1). \end{aligned}$$

The zeros of the indicial polynomial p_0 are $r_1=-3/2$ and $r_2=-2$, so $r_1-r_2=1/2$. Therefore Theorem 7.6.3 implies that

$$y_1=x^{-3/2}\sum_{n=0}^{\infty}a_n(-3/2)x^n \quad \text{and} \quad y_2=x^{-2}\sum_{n=0}^{\infty}a_n(-2)x^n \tag{6.5.19}$$

form a fundamental set of Frobenius solutions of Equation 6.5.18. To find the coefficients in these series, we use the recurrence relation of Theorem 7.6.2 ; thus,

$$\begin{aligned} a_0(r) &= 1,\\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)}=-\frac{(2r+5)(r+2)}{(2r+5)(r+3)}=-\frac{r+2}{r+3},\\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1}+p_2(n+r-2)a_{n-2}}{p_0(n+r)}\\ &= -\frac{(n+r+1)(2n+2r+3)a_{n-1}(r)+(n+r-1)(2n+2r+3)a_{n-2}(r)}{(n+r+2)(2n+2r+3)}\\ &= -\frac{(n+r+1)a_{n-1}(r)+(n+r-1)a_{n-2}(r)}{n+r+2}, \quad n\geq 2. \end{aligned}$$

Setting $r=-3/2$ in these equations yields

$$\begin{aligned} a_0(-3/2) &= 1,\\ a_1(-3/2) &= -1/3,\\ a_n(-3/2) &= -\tfrac{(2n-1)a_{n-1}(-3/2)+(2n-5)a_{n-2}(-3/2)}{2n+1}, \quad n\geq 2, \end{aligned} \tag{6.5.20}$$

and setting $r=-2$ yields

$$\begin{aligned} a_0(-2) &= 1, \\ a_1(-2) &= 0, \\ a_n(-2) &= -\frac{(n-1)a_{n-1}(-2)+(n-3)a_{n-2}(-2)}{n}, \quad n \geq 2. \end{aligned} \tag{6.5.21}$$

Calculating with Equation 6.5.20 and Equation 6.5.21 and substituting the results into Equation 6.5.19 yields the fundamental set of Frobenius solutions

$$\begin{aligned} y_1 &= x^{-3/2}\left(1-\frac{1}{3}x+\frac{2}{5}x^2-\frac{5}{21}x^3+\frac{7}{135}x^4+\frac{76}{1155}x^5+\cdots\right), \\ y_2 &= x^{-2}\left(1+\frac{1}{2}x^2-\frac{1}{3}x^3+\frac{1}{8}x^4+\frac{1}{30}x^5+\cdots\right). \end{aligned}$$

Special Cases With Two Term Recurrence Relations

For $n \geq 2$, the recurrence relation Equation 6.5.12 of Theorem 7.6.2 involves the three coefficients $a_n(r)$, $a_{n-1}(r)$, and $a_{n-2}(r)$. We'll now consider some special cases where Equation 6.5.12 reduces to a two term recurrence relation; that is, a relation involving only $a_n(r)$ and $a_{n-1}(r)$ or only $a_n(r)$ and $a_{n-2}(r)$. This simplification often makes it possible to obtain explicit formulas for the coefficents of Frobenius solutions.

We first consider equations of the form

$$x^2(\alpha_0+\alpha_1 x)y''+x(\beta_0+\beta_1 x)y'+(\gamma_0+\gamma_1 x)y=0$$

with $\alpha_0 \neq 0$. For this equation, $\alpha_2=\beta_2=\gamma_2=0$, so $p_2 \equiv 0$ and the recurrence relations in Theorem 7.6.2 simplify to

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r), \quad n \geq 1. \end{aligned} \tag{6.5.22}$$

Example 7.6.2

Find a fundamental set of Frobenius solutions of

$$x^2(3+x)y''+5x(1+x)y'-(1-4x)y=0. \tag{6.5.23}$$

Give explicit formulas for the coefficients in the solutions.

Solution

For this equation, the polynomials defined in Theorem 7.6.2 are

$$\begin{aligned} p_0(r) &= 3r(r-1)+5r-1 &&= (3r-1)(r+1), \\ p_1(r) &= r(r-1)+5r+4 &&= (r+2)^2, \\ p_2(r) &= 0. \end{aligned}$$

The zeros of the indicial polynomial p_0 are $r_1=1/3$ and $r_2=-1$, so $r_1-r_2=4/3$. Therefore Theorem 7.6.3 implies that

$$y_1=x^{1/3}\sum_{n=0}^{\infty}a_n(1/3)x^n \quad \text{and} \quad y_2=x^{-1}\sum_{n=0}^{\infty}a_n(-1)x^n$$

form a fundamental set of Frobenius solutions of Equation 6.5.23. To find the coefficients in these series, we use the recurrence relationss Equation 6.5.22; thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\tfrac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ &= -\tfrac{(n+r+1)^2}{(3n+3r-1)(n+r+1)} a_{n-1}(r) \\ &= -\tfrac{n+r+1}{3n+3r-1} a_{n-1}(r), \quad n \geq 1. \end{aligned} \tag{6.5.24}$$

Setting $r = 1/3$ in Equation 6.5.24 yields

$$\begin{aligned} a_0(1/3) &= 1, \\ a_n(1/3) &= -\frac{3n+4}{9n} a_{n-1}(1/3), \quad n \geq 1. \end{aligned}$$

By using the product notation introduced in Section 7.2 and proceeding as we did in the examples in that section yields

$$a_n(1/3) = \frac{(-1)^n \prod_{j=1}^n (3j+4)}{9^n n!}, \quad n \geq 0.$$

Therefore

$$y_1 = x^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^n (3j+4)}{9^n n!} x^n$$

is a Frobenius solution of Equation 6.5.23.

Setting $r = -1$ in Equation 6.5.24 yields

$$\begin{aligned} a_0(-1) &= 1, \\ a_n(-1) &= -\frac{n}{3n-4} a_{n-1}(-1), \quad n \geq 1, \end{aligned}$$

so

$$a_n(-1) = \frac{(-1)^n n!}{\prod_{j=1}^n (3j-4)}.$$

Therefore

$$y_2 = x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\prod_{j=1}^n (3j-4)} x^n$$

is a Frobenius solution of Equation 6.5.23, and $\{y_1, y_2\}$ is a fundamental set of solutions.

We now consider equations of the form

$$x^2(\alpha_0 + \alpha_2 x^2) y'' + x(\beta_0 + \beta_2 x^2) y' + (\gamma_0 + \gamma_2 x^2) y = 0 \tag{6.5.25}$$

with $\alpha_0 \neq 0$. For this equation, $\alpha_1 = \beta_1 = \gamma_1 = 0$, so $p_1 \equiv 0$ and the recurrence relations in Theorem 7.6.2 simplify to

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= 0, \\ a_n(r) &= -\frac{p_2(n+r-2)}{p_0(n+r)} a_{n-2}(r), \quad n \geq 2. \end{aligned}$$

Since $a_1(r) = 0$, the last equation implies that $a_n(r) = 0$ if n is odd, so the Frobenius solutions are of the form

$$y(x, r) = x^r \sum_{m=0}^{\infty} a_{2m}(r) x^{2m},$$

where

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r), \quad m \geq 1. \end{aligned} \tag{6.5.26}$$

Example 7.6.3

Find a fundamental set of Frobenius solutions of

$$x^2(2-x^2)y'' - x(3+4x^2)y' + (2-2x^2)y = 0. \tag{6.5.27}$$

Give explicit formulas for the coefficients in the solutions.

Solution

For this equation, the polynomials defined in Theorem 7.6.2 are

$$\begin{aligned} p_0(r) &= 2r(r-1) - 3r + 2 &&= (r-2)(2r-1), \\ p_1(r) &= 0 \\ p_2(r) &= -[r(r-1) + 4r + 2] &&= -(r+1)(r+2). \end{aligned}$$

The zeros of the indicial polynomial p_0 are $r_1 = 2$ and $r_2 = 1/2$, so $r_1 - r_2 = 3/2$. Therefore Theorem 7.6.3 implies that

$$y_1 = x^2 \sum_{m=0}^{\infty} a_{2m}(1/3) x^{2m} \quad \text{and} \quad y_2 = x^{1/2} \sum_{m=0}^{\infty} a_{2m}(1/2) x^{2m}$$

form a fundamental set of Frobenius solutions of Equation 6.5.27. To find the coefficients in these series, we use the recurrence relation Equation 6.5.26; thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r) \\ &= \frac{(2m+r)(2m+r-1)}{(2m+r-2)(4m+2r-1)} a_{2m-2}(r), \quad m \geq 1. \end{aligned} \tag{6.5.28}$$

Setting $r = 2$ in Equation 6.5.28 yields

$$\begin{aligned} a_0(2) &= 1, \\ a_{2m}(2) &= \frac{(m+1)(2m+1)}{m(4m+3)} a_{2m-2}(2), \quad m \geq 1, \end{aligned}$$

so

$$a_{2m}(2) = (m+1) \prod_{j=1}^{m} \frac{2j+1}{4j+3}.$$

Therefore

$$y_1 = x^2 \sum_{m=0}^{\infty} (m+1) \left(\prod_{j=1}^{m} \frac{2j+1}{4j+3} \right) x^{2m}$$

is a Frobenius solution of Equation 6.5.27.

Setting $r = 1/2$ in Equation 6.5.28 yields

$$\begin{aligned} a_0(1/2) &= 1, \\ a_{2m}(1/2) &= \frac{(4m-1)(4m+1)}{8m(4m-3)} a_{2m-2}(1/2), \quad m \geq 1, \end{aligned}$$

so

$$a_{2m}(1/2)=\frac{1}{8^m m!}\prod_{j=1}^{m}\frac{(4j-1)(4j+1)}{4j-3}.$$

Therefore

$$y_2=x^{1/2}\sum_{m=0}^{\infty}\frac{1}{8^m m!}\left(\prod_{j=1}^{m}\frac{(4j-1)(4j+1)}{4j-3}\right)x^{2m}$$

is a Frobenius solution of Equation 6.5.27 and $\{y_1,y_2\}$ is a fundamental set of solutions.

Note

Thus far, we considered only the case where the indicial equation has real roots that don't differ by an integer, which allows us to apply Theorem 7.6.3 . However, for equations of the form Equation 6.5.25, the sequence $\{a_{2m}(r)\}$ in Equation 6.5.26 is defined for $r=r_2$ if r_1-r_2 isn't an even integer. It can be shown **Exercise 7.5.56** that in this case

$$y_1=x^{r_1}\sum_{m=0}^{\infty}a_{2m}(r_1)x^{2m}\quad\text{and}\quad y_2=x^{r_2}\sum_{m=0}^{\infty}a_{2m}(r_2)x^{2m}$$

form a fundamental set of Frobenius solutions of Equation 6.5.25.

Using Technology

As we said at the end of Section 7.2, if you're interested in actually using series to compute numerical approximations to solutions of a differential equation, then whether or not there's a simple closed form for the coefficents is essentially irrelevant; recursive computation is usually more efficient. Since it is also laborious, we encourage you to write short programs to implement recurrence relations on a calculator or computer, even in exercises where this is not specifically required.

In practical use of the method of Frobenius when $x_0=0$ is a regular singular point, we are interested in how well the functions

$$y_N(x,r_i)=x^{r_i}\sum_{n=0}^{N}a_n(r_i)x^n,\quad i=1,2,$$

approximate solutions to a given equation when r_i is a zero of the indicial polynomial. In dealing with the corresponding problem for the case where $x_0=0$ is an ordinary point, we used numerical integration to solve the differential equation subject to initial conditions $y(0)=a_0,\quad y'(0)=a_1$, and compared the result with values of the Taylor polynomial

$$T_N(x)=\sum_{n=0}^{N}a_nx^n.$$

We can't do that here, since in general we can't prescribe arbitrary initial values for solutions of a differential equation at a singular point. Therefore, motivated by Theorem 7.6.2 (specifically, Equation 6.5.16), we suggest the following procedure.

Verification Procedure

Let L and $Y_n(x;r_i)$ be defined by

$$L_y=x^2(\alpha_0+\alpha_1x+\alpha_2x^2)y''+x(\beta_0+\beta_1x+\beta_2x^2)y'+(\gamma_0+\gamma_1x+\gamma_2x^2)y$$

and

$$y_N(x;r_i)=x^{r_i}\sum_{n=0}^{N}a_n(r_i)x^n$$

where the coefficients $\{a_n(r_i)\}_{n=0}^N$ are computed as in Equation 6.5.12, Theorem 7.6.2 . Compute the error

$$E_N(x;r_i)=x^{-r_i}L_{yN}(x;r_i)/\alpha_0 \tag{6.5.29}$$

for various values of N and various values of x in the interval $(0,\rho)$ with ρ as defined in Theorem 7.6.2

The multiplier x^{-r_i}/α_0 on the right of Equation 6.5.29 eliminates the effects of small or large values of x^{r_i} near $x=0$, and of multiplication by an arbitrary constant. In some exercises you will be asked to estimate the maximum value of $E_N(x;r_i)$ on an interval $(0,\delta]$ by computing $E_N(x_m;r_i)$ at the M points $x_m=m\delta/M$, $m=1$, 2, …, M, and finding the maximum of the absolute values:

$$\sigma_N(\delta)=\max\{|E_N(x_m;r_i)|,\ m=1,2,\ldots,M\}. \tag{6.5.30}$$

(For simplicity, this notation ignores the dependence of the right side of the equation on i and M.)

To implement this procedure, you'll have to write a computer program to calculate $\{a_n(r_i)\}$ from the applicable recurrence relation, and to evaluate $E_N(x;r_i)$.

The next exercise set contains five exercises specifically identified by that ask you to implement the verification procedure. These particular exercises were chosen arbitrarily you can just as well formulate such laboratory problems for any of the equations in any of the **Exercises 7.5.1-7.5.10, 7.5.14-7.4.25**, and **7.5.28-7.5.51 .**

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6.5.1: The Method of Frobenius I (Exercises)

This set contains exercises specifically identified by that ask you to implement the verification procedure. These particular exercises were chosen arbitrarily you can just as well formulate such laboratory problems for any of the equations in *Exercises 7.5.1-7.5.10*, 7.5.**14-7.5.25**, and **7.5.28-7.5.51**.

Q7.5.1

In **Exercises 7.5.1-7.5.10** find a fundamental set of Frobenius solutions. Compute $a_0, a_1, \ldots, a_N$ for N at least 7 in each solution.

1. $2x^2(1+x+x^2)y''+x(3+3x+5x^2)y'-y=0$
2. $3x^2y''+2x(1+x-2x^2)y'+(2x-8x^2)y=0$
3. $x^2(3+3x+x^2)y''+x(5+8x+7x^2)y'-(1-2x-9x^2)y=0$
4. $4x^2y''+x(7+2x+4x^2)y'-(1-4x-7x^2)y=0$
5. $12x^2(1+x)y''+x(11+35x+3x^2)y'-(1-10x-5x^2)y=0$
6. $x^2(5+x+10x^2)y''+x(4+3x+48x^2)y'+(x+36x^2)y=0$
7. $8x^2y''-2x(3-4x-x^2)y'+(3+6x+x^2)y=0$
8. $18x^2(1+x)y''+3x(5+11x+x^2)y'-(1-2x-5x^2)y=0$
9. $x(3+x+x^2)y''+(4+x-x^2)y'+xy=0$
10. $10x^2(1+x+2x^2)y''+x(13+13x+66x^2)y'-(1+4x+10x^2)y=0$

Q7.5.2

11. The Frobenius solutions of

$$2x^2(1+x+x^2)y''+x(9+11x+11x^2)y'+(6+10x+7x^2)y=0$$

obtained in Example 7.5.1 are defined on $(0,\rho)$, where ρ is defined in Theorem 7.5.2. Find ρ. Then do the following experiments for each Frobenius solution, with $M=20$ and $\delta=.5\rho$, $.7\rho$, and $.9\rho$ in the verification procedure described at the end of this section.

a. Compute $\sigma_N(\delta)$ (see Equation 7.5.28) for $N=5$, 10, 15,…, 50.
b. Find N such that $\sigma_N(\delta)<10^{-5}$.
c. Find N such that $\sigma_N(\delta)<10^{-10}$.

12. By Theorem 7.5.2 the Frobenius solutions of the equation in **Exercise 7.5.4** are defined on $(0,\infty)$. Do experiments (a), (b), and (c) of *Exercise 7.5.11* for each Frobenius solution, with $M=20$ and $\delta=1$, 2, and 3 in the verification procedure described at the end of this section.

13. The Frobenius solutions of the equation in *Exercise 7.5.6* are defined on $(0,\rho)$, where ρ is defined in Theorem 7.5.2. Find ρ and do experiments (a), (b), and (c) of *Exericse 7.5.11* for each Frobenius solution, with $M=20$ and $\delta=.3\rho$, $.4\rho$, and $.5\rho$, in the verification procedure described at the end of this section.

Q7.5.3

In **Exercises 7.5.14-7.5.25** find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients in each solution.

14. $2x^2y''+x(3+2x)y'-(1-x)y=0$
15. $x^2(3+x)y''+x(5+4x)y'-(1-2x)y=0$
16. $2x^2y''+x(5+x)y'-(2-3x)y=0$
17. $3x^2y''+x(1+x)y'-y=0$
18. $2x^2y''-xy'+(1-2x)y=0$

19. $9x^2y''+9xy'-(1+3x)y=0$

20. $3x^2y''+x(1+x)y'-(1+3x)y=0$

21. $2x^2(3+x)y''+x(1+5x)y'+(1+x)y=0$

22. $x^2(4+x)y''-x(1-3x)y'+y=0$

23. $2x^2y''+5xy'+(1+x)y=0$

24. $x^2(3+4x)y''+x(5+18x)y'-(1-12x)y=0$

25. $6x^2y''+x(10-x)y'-(2+x)y=0$

Q7.5.4

26. By Theorem 7.5.2 the Frobenius solutions of the equation in *Exercise 7.5.17* are defined on $(0,\infty)$. Do experiments (a), (b), and (c) of *Exercise 7.5.11* for each Frobenius solution, with $M=20$ and $\delta=3$, 6, 9, and 12 in the verification procedure described at the end of this section.

27. The Frobenius solutions of the equation in *Exercise 7.5.22* are defined on $(0,\rho)$, where ρ is defined in Theorem 7.5.2. Find ρ and do experiments (a), (b), and (c) of *Exercise 7.5.11* for each Frobenius solution, with $M=20$ and $\delta=.25\rho$, $.5\rho$, and $.75\rho$ in the verification procedure described at the end of this section.

Q7.5.5

In **Exercises 7.5.28- 7.5.32** find a fundamental set of Frobenius solutions. Compare coefficients $a_0,\ldots,a_N$ for N at least 7 in each solution.

28. $x^2(8+x)y''+x(2+3x)y'+(1+x)y=0$

29. $x^2(3+4x)y''+x(11+4x)y'-(3+4x)y=0$

30. $2x^2(2+3x)y''+x(4+11x)y'-(1-x)y=0$

31. $x^2(2+x)y''+5x(1-x)y'-(2-8x)y$

32. $x^2(6+x)y''+x(11+4x)y'+(1+2x)y=0$

Q7.5.6

In **Exercises 7.5.33-7.5.36** find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients in each solution.

33. $8x^2y''+x(2+x^2)y'+y=0$

34. $8x^2(1-x^2)y''+2x(1-13x^2)y'+(1-9x^2)y=0$

35. $x^2(1+x^2)y''-2x(2-x^2)y'+4y=0$

36. $x(3+x^2)y''+(2-x^2)y'-8xy=0$

37. $4x^2(1-x^2)y''+x(7-19x^2)y'-(1+14x^2)y=0$

38. $3x^2(2-x^2)y''+x(1-11x^2)y'+(1-5x^2)y=0$

39. $2x^2(2+x^2)y''-x(12-7x^2)y'+(7+3x^2)y=0$

40. $2x^2(2+x^2)y''+x(4+7x^2)y'-(1-3x^2)y=0$

41. $2x^2(1+2x^2)y''+5x(1+6x^2)y'-(2-40x^2)y=0$

42. $3x^2(1+x^2)y''+5x(1+x^2)y'-(1+5x^2)y=0$

43. $x(1+x^2)y''+(4+7x^2)y'+8xy=0$

44. $x^2(2+x^2)y''+x(3+x^2)y'-y=0$

45. $2x^2(1+x^2)y''+x(3+8x^2)y'-(3-4x^2)y=0$

46. $9x^2y''+3x(3+x^2)y'-(1-5x^2)y=0$

Q7.5.7

In **Exercises 7.5.47-7.5.51** find a fundamental set of Frobenius solutions. Compare the coefficients $a_0, \ldots, a_{2M}$ for M at least 7 in each solution.

47. $6x^2y'' + x(1+6x^2)y' + (1+9x^2)y = 0$

48. $x^2(8+x^2)y'' + 7x(2+x^2)y' - (2-9x^2)y = 0$

49. $9x^2(1+x^2)y'' + 3x(3+13x^2)y' - (1-25x^2)y = 0$

50. $4x^2(1+x^2)y'' + 4x(1+6x^2)y' - (1-25x^2)y = 0$

51. $8x^2(1+2x^2)y'' + 2x(5+34x^2)y' - (1-30x^2)y = 0$

Q7.5.8

52. Suppose $r_1 > r_2$, $a_0 = b_0 = 1$, and the Frobenius series

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad y_2 = x^{r_2}\sum_{n=0}^{\infty} b_n x^n$$

both converge on an interval $(0, \rho)$.

a. Show that y_1 and y_2 are linearly independent on $(0, \rho)$. HINT: Show that if c_1 and c_2 are constants such that $c_1y_1 + c_2y_2 \equiv 0$ on $(0, \rho)$, then

$$c_1x^{r_1-r_2}\sum_{n=0}^{\infty} a_n x^n + c_2\sum_{n=0}^{\infty} b_n x^n = 0, \quad 0 < x < \rho$$

b. Use the result of (b) to complete the proof of Theorem 7.5.3.

53. The equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{A}$$

is **Bessel's equation of order ν**. (Here ν is a parameter, and this use of "order" should not be confused with its usual use as in "the order of the equation.") The solutions of Equation A are **Bessel functions of order** ν.

a. Assuming that ν isn't an integer, find a fundamental set of Frobenius solutions of Equation A.
b. If $\nu = 1/2$, the solutions of Equation A reduce to familiar elementary functions. Identify these functions.

54.

a. Verify that

$$\frac{d}{dx}\left(|x|^r x^n\right) = (n+r)|x|^r x^{n-1} \quad \text{and} \quad \frac{d^2}{dx^2}\left(|x|^r x^n\right) = (n+r)(n+r-1)|x|^r x^{n-2}$$

if $x \neq 0$.

b. Let

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0.$$

Show that if $x^r\sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ on $(0, \rho)$ then $|x|^r\sum_{n=0}^{\infty} a_n x^n$ is a solution on $(-\rho, 0)$ and $(0, \rho)$.

55.

a. Deduce from Equation 7.5.20 that

$$a_n(r) = (-1)^n\prod_{j=1}^{n}\frac{p_1(j+r-1)}{p_0(j+r)}.$$

b. Conclude that if $p_0(r) = \alpha_0(r-r_1)(r-r_2)$ where $r_1 - r_2$ is not an integer, then

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n \quad \text{and} \quad y_2 = x^{r_2}\sum_{n=0}^{\infty} a_n(r_2)x^n$$

form a fundamental set of Frobenius solutions of

$$x^2(\alpha_0+\alpha_1 x)y''+x(\beta_0+\beta_1 x)y'+(\gamma_0+\gamma_1 x)y=0.$$

c. Show that if p_0 satisfies the hypotheses of (b) then

$$y_1=x^{r_1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\prod_{j=1}^n(j+r_1-r_2)}\left(\frac{\gamma_1}{\alpha_0}\right)^n x^n$$

and

$$y_2=x^{r_2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\prod_{j=1}^n(j+r_2-r_1)}\left(\frac{\gamma_1}{\alpha_0}\right)^n x^n$$

form a fundamental set of Frobenius solutions of

$$\alpha_0 x^2y''+\beta_0 xy'+(\gamma_0+\gamma_1 x)y=0.$$

56. Let

$$Ly=x^2(\alpha_0+\alpha_2x^2)y''+x(\beta_0+\beta_2x^2)y'+(\gamma_0+\gamma_2x^2)y=0$$

and define

$$p_0(r)=\alpha_0r(r-1)+\beta_0r+\gamma_0 \quad \text{and} \quad p_2(r)=\alpha_2r(r-1)+\beta_2r+\gamma_2.$$

a. Use Theorem 7.5.2 to show that if

$$\begin{aligned} a_0(r) &= 1, \\ p_0(2m+r)a_{2m}(r)+p_2(2m+r-2)a_{2m-2}(r) &= 0, \quad m\ge 1, \end{aligned} \tag{A}$$

then the Frobenius series $y(x,r)=x^r\sum_{m=0}^{\infty}a_{2m}x^{2m}$ satisfies $Ly(x,r)=p_0(r)x^r$.

b. Deduce from Equation A that if $p_0(2m+r)$ is nonzero for every positive integer m then

$$a_{2m}(r)=(-1)^m\prod_{j=1}^m\frac{p_2(2j+r-2)}{p_0(2j+r)}.$$

c. Conclude that if $p_0(r)=\alpha_0(r-r_1)(r-r_2)$ where r_1-r_2 is not an even integer, then

$$y_1=x^{r_1}\sum_{m=0}^{\infty}a_{2m}(r_1)x^{2m} \quad \text{and} \quad y_2=x^{r_2}\sum_{m=0}^{\infty}a_{2m}(r_2)x^{2m}$$

form a fundamental set of Frobenius solutions of $Ly=0$.

d. Show that if p_0 satisfies the hypotheses of (c) then

$$y_1=x^{r_1}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!\prod_{j=1}^m(2j+r_1-r_2)}\left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m}$$

and

$$y_2=x^{r_2}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^m m!\prod_{j=1}^m(2j+r_2-r_1)}\left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m}$$

form a fundamental set of Frobenius solutions of

$$\alpha_0x^2y''+\beta_0xy'+(\gamma_0+\gamma_2x^2)y=0.$$

57. Let

$$Ly=x^2q_0(x)y''+xq_1(x)y'+q_2(x)y,$$

where

$$q_0(x)=\sum_{j=0}^{\infty}\alpha_j x^j,\quad q_1(x)=\sum_{j=0}^{\infty}\beta_j x^j,\quad q_2(x)=\sum_{j=0}^{\infty}\gamma_j x^j,$$

and define

$$p_j(r)=\alpha_j r(r-1)+\beta_j r+\gamma_j,\quad j=0,1,\ldots.$$

Let $y=x^r\sum_{n=0}^{\infty}a_n x^n$. Show that

$$Ly=x^r\sum_{n=0}^{\infty}b_n x^n,$$

where

$$b_n=\sum_{j=0}^{n}p_j(n+r-j)a_{n-j}.$$

58.

a. Let L be as in **Exercise 7.5.57**. Show that if

$$y(x,r)=x^r\sum_{n=0}^{\infty}a_n(r)x^n$$

where

$$\begin{aligned} a_0(r)&= 1,\\ a_n(r)&= -\frac{1}{p_0(n+r)}\sum_{j=1}^{n}p_j(n+r-j)a_{n-j}(r),\quad n\ge 1,\end{aligned}$$

then

$$Ly(x,r)=p_0(r)x^r.$$

b. Conclude that if

$$p_0(r)=\alpha_0(r-r_1)(r-r_2)$$

where r_1-r_2 isn't an integer then $y_1=y(x,r_1)$ and $y_2=y(x,r_2)$ are solutions of $Ly=0$.

59. Let

$$Ly=x^2(\alpha_0+\alpha_q x^q)y''+x(\beta_0+\beta_q x^q)y'+(\gamma_0+\gamma_q x^q)y$$

where q is a positive integer, and define

$$p_0(r)=\alpha_0 r(r-1)+\beta_0 r+\gamma_0\quad\text{and}\quad p_q(r)=\alpha_q r(r-1)+\beta_q r+\gamma_q.$$

a. Show that if

$$y(x,r)=x^r\sum_{m=0}^{\infty}a_{qm}(r)x^{qm}$$

where

$$\begin{aligned} a_0(r)&= 1,\\ a_{qm}(r)&= -\tfrac{p_q(q(m-1)+r)}{p_0(qm+r)}a_{q(m-1)}(r),\quad m\ge 1,\end{aligned}\tag{A}$$

then

$$Ly(x,r)=p_0(r)x^r.$$

b. Deduce from Equation A that

$$a_{qm}(r)=(-1)^m\prod_{j=1}^m\frac{p_q\left(q(j-1)+r\right)}{p_0(qj+r)}.$$

c. Conclude that if $p_0(r)=\alpha_0(r-r_1)(r-r_2)$ where r_1-r_2 is not an integer multiple of q, then

$$y_1=x^{r_1}\sum_{m=0}^\infty a_{qm}(r_1)x^{qm}\quad\text{and}\quad y_2=x^{r_2}\sum_{m=0}^\infty a_{qm}(r_2)x^{qm}$$

form a fundamental set of Frobenius solutions of $Ly=0$.

d. Show that if p_0 satisfies the hypotheses of (c) then

$$y_1=x^{r_1}\sum_{m=0}^\infty\frac{(-1)^m}{q^m m!\prod_{j=1}^m(qj+r_1-r_2)}\left(\frac{\gamma_q}{\alpha_0}\right)^m x^{qm}$$

and

$$y_2=x^{r_2}\sum_{m=0}^\infty\frac{(-1)^m}{q^m m!\prod_{j=1}^m(qj+r_2-r_1)}\left(\frac{\gamma_q}{\alpha_0}\right)^m x^{qm}$$

form a fundamental set of Frobenius solutions of

$$\alpha_0x^2y''+\beta_0xy'+(\gamma_0+\gamma_qx^q)y=0.$$

60.

a. Suppose α_0,α_1, and α_2 are real numbers with $\alpha_0\neq0$, and $\{a_n\}_{n=0}^\infty$ is defined by

$$\alpha_0a_1+\alpha_1a_0=0$$

and

$$\alpha_0a_n+\alpha_1a_{n-1}+\alpha_2a_{n-2}=0,\quad n\geq2.$$

Show that

$$(\alpha_0+\alpha_1x+\alpha_2x^2)\sum_{n=0}^\infty a_nx^n=\alpha_0a_0,$$

and infer that

$$\sum_{n=0}^\infty a_nx^n=\frac{\alpha_0a_0}{\alpha_0+\alpha_1x+\alpha_2x^2}.$$

b. With α_0,α_1, and α_2 as in (a), consider the equation

$$x^2(\alpha_0+\alpha_1x+\alpha_2x^2)y''+x(\beta_0+\beta_1x+\beta_2x^2)y'+(\gamma_0+\gamma_1x+\gamma_2x^2)y=0,\tag{A}$$

and define

$$p_j(r)=\alpha_jr(r-1)+\beta_jr+\gamma_j,\quad j=0,1,2.$$

Suppose

$$\frac{p_1(r-1)}{p_0(r)}=\frac{\alpha_1}{\alpha_0},\qquad\frac{p_2(r-2)}{p_0(r)}=\frac{\alpha_2}{\alpha_0},$$

and

$$p_0(r)=\alpha_0(r-r_1)(r-r_2),$$

where $r_1>r_2$. Show that

$$y_1=\frac{x^{r_1}}{\alpha_0+\alpha_1x+\alpha_2x^2}\quad\text{and}\quad y_2=\frac{x^{r_2}}{\alpha_0+\alpha_1x+\alpha_2x^2}$$

form a fundamental set of Frobenius solutions of Equation A on any interval $(0, \rho)$ on which $\alpha_0 + \alpha_1 x + \alpha_2 x^2$ has no zeros.

Q7.5.9

In **Exercises 7.5.61-7.5.68** use the method suggested by **Exercise 7.5.60** to find the general solution on some interval $(0, \rho)$.

61. $2x^2(1+x)y'' - x(1-3x)y' + y = 0$

62. $6x^2(1+2x^2)y'' + x(1+50x^2)y' + (1+30x^2)y = 0$

63. $28x^2(1-3x)y'' - 7x(5+9x)y' + 7(2+9x)y = 0$

64. $9x^2(5+x)y'' + 9x(5+3x)y' - (5-8x)y = 0$

65. $8x^2(2-x^2)y'' + 2x(10-21x^2)y' - (2+35x^2)y = 0$

66. $4x^2(1+3x+x^2)y'' - 4x(1-3x-3x^2)y' + 3(1-x+x^2)y = 0$

67. $3x^2(1+x)^2y'' - x(1-10x-11x^2)y' + (1+5x^2)y = 0$

68. $4x^2(3+2x+x^2)y'' - x(3-14x-15x^2)y' + (3+7x^2)y = 0$

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6.6: The Method of Frobenius II

In this section we discuss a method for finding two linearly independent Frobenius solutions of a homogeneous linear second order equation near a regular singular point in the case where the indicial equation has a repeated real root. As in the preceding section, we consider equations that can be written as

$$x^2(\alpha_0+\alpha_1 x+\alpha_2 x^2)y''+x(\beta_0+\beta_1 x+\beta_2 x^2)y'+(\gamma_0+\gamma_1 x+\gamma_2 x^2)y=0 \tag{6.6.1}$$

where $\alpha_0 \neq 0$. We assume that the indicial equation $p_0(r)=0$ has a repeated real root r_1. In this case Theorem 7.5.3 implies that Equation 6.6.1 has one solution of the form

but does not provide a second solution y_2 such that $\{y_1, y_2\}$ is a fundamental set of solutions. The following extension of Theorem 7.5.2 provides a way to find a second solution.

Theorem 7.7.1

Let

$$Ly=x^2(\alpha_0+\alpha_1 x+\alpha_2 x^2)y''+x(\beta_0+\beta_1 x+\beta_2 x^2)y'+(\gamma_0+\gamma_1 x+\gamma_2 x^2)y, \tag{6.6.2}$$

where $\alpha_0 \neq 0$ and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1)+\beta_0 r+\gamma_0, \\ p_1(r) &= \alpha_1 r(r-1)+\beta_1 r+\gamma_1, \\ p_2(r) &= \alpha_2 r(r-1)+\beta_2 r+\gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n, and define

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1}(r)+p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)}, \quad n \geq 2. \end{aligned}$$

Then the Frobenius series

$$y(x,r)=x^r \sum_{n=0}^{\infty} a_n(r)x^n \tag{6.6.3}$$

satisfies

$$Ly(x,r)=p_0(r)x^r. \tag{6.6.4}$$

Moreover,

$$\frac{\partial y}{\partial r}(x,r)=y(x,r)\ln x+x^r \sum_{n=1}^{\infty} a_n'(r)x^n, \tag{6.6.5}$$

and

$$L\left(\frac{\partial y}{\partial r}(x,r)\right)=p_0'(r)x^r+x^r p_0(r)\ln x. \tag{6.6.6}$$

Proof

Theorem 7.5.2 implies Equation 6.6.4 Differentiating formally with respect to r in Equation 6.6.3yields

$$\begin{aligned}\frac{\partial y}{\partial r}(x,r) &= \frac{\partial}{\partial r}(x^r)\sum_{n=0}^{\infty}a_n(r)x^n + x^r\sum_{n=1}^{\infty}a_n'(r)x^n \\ &= x^r\ln x\sum_{n=0}^{\infty}a_n(r)x^n + x^r\sum_{n=1}^{\infty}a_n'(r)x^n \\ &= y(x,r)\ln x + x^r\sum_{n=1}^{\infty}a_n'(r)x^n,\end{aligned}$$

which proves Equation 6.6.5.

To prove that $\partial y(x,r)/\partial r$ satisfies Equation 6.6.6, we view y in Equation 6.6.2 as a function $y=y(x,r)$ of two variables, where the prime indicates partial differentiation with respect to x; thus,

$$y' = y'(x,r) = \frac{\partial y}{\partial x}(x,r) \quad \text{and} \quad y'' = y''(x,r) = \frac{\partial^2 y}{\partial x^2}(x,r).$$

With this notation we can use Equation 6.6.2 to rewrite Equation 6.6.4 as

$$x^2q_0(x)\frac{\partial^2 y}{\partial x^2}(x,r) + xq_1(x)\frac{\partial y}{\partial x}(x,r) + q_2(x)y(x,r) = p_0(r)x^r, \tag{6.6.7}$$

where

$$\begin{aligned}q_0(x) &= \alpha_0+\alpha_1x+\alpha_2x^2, \\ q_1(x) &= \beta_0+\beta_1x+\beta_2x^2, \\ q_2(x) &= \gamma_0+\gamma_1x+\gamma_2x^2.\end{aligned}$$

Differentiating both sides of Equation 6.6.7 with respect to r yields

$$x^2q_0(x)\frac{\partial^3 y}{\partial r\partial x^2}(x,r) + xq_1(x)\frac{\partial^2 y}{\partial r\partial x}(x,r) + q_2(x)\frac{\partial y}{\partial r}(x,r) = p_0'(r)x^r + p_0(r)x^r\ln x.$$

By changing the order of differentiation in the first two terms on the left we can rewrite this as

$$x^2q_0(x)\frac{\partial^3 y}{\partial x^2\partial r}(x,r) + xq_1(x)\frac{\partial^2 y}{\partial x\partial r}(x,r) + q_2(x)\frac{\partial y}{\partial r}(x,r) = p_0'(r)x^r + p_0(r)x^r\ln x,$$

or

$$x^2q_0(x)\frac{\partial^2}{\partial x^2}\left(\frac{\partial y}{\partial r}(x,r)\right) + xq_1(x)\frac{\partial}{\partial r}\left(\frac{\partial y}{\partial x}(x,r)\right) + q_2(x)\frac{\partial y}{\partial r}(x,r) = p_0'(r)x^r + p_0(r)x^r\ln x,$$

which is equivalent to Equation 6.6.6.

Theorem 7.7.2

Let L be as in Theorem 7.7.1 and suppose the indicial equation $p_0(r)=0$ has a repeated real root r_1. Then

$$y_1(x) = y(x,r_1) = x^{r_1}\sum_{n=0}^{\infty}a_n(r_1)x^n$$

and

$$y_2(x) = \frac{\partial y}{\partial r}(x,r_1) = y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty}a_n'(r_1)x^n \tag{6.6.8}$$

form a fundamental set of solutions of $Ly=0$.

Proof

Since r_1 is a repeated root of $p_0(r)=0$, the indicial polynomial can be factored as

$$p_0(r)=\alpha_0(r-r_1)^2,$$

so

$$p_0(n+r_1)=\alpha_0 n^2,$$

which is nonzero if $n>0$. Therefore the assumptions of Theorem 7.7.1 hold with $r=r_1$, and Equation 6.6.4 implies that $Ly_1=p_0(r_1)x^{r_1}=0$. Since

$$p_0'(r)=2\alpha(r-r_1)$$

it follows that $p_0'(r_1)=0$, so Equation 6.6.6 implies that

$$Ly_2=p_0'(r_1)x^{r_1}+x^{r_1}p_0(r_1)\ln x=0.$$

This proves that y_1 and y_2 are both solutions of $Ly=0$. We leave the proof that $\{y_1,y_2\}$ is a fundamental set as an *Exercise 7.6.53*.

Example 7.7.1

Find a fundamental set of solutions of

$$x^2(1-2x+x^2)y''-x(3+x)y'+(4+x)y=0. \tag{6.6.9}$$

Compute just the terms involving x^{n+r_1}, where $0\le n\le 4$ and r_1 is the root of the indicial equation.

Solution

For the given equation, the polynomials defined in Theorem 7.7.1 are

$$\begin{aligned}
p_0(r) &= r(r-1)-3r+4\\
&= (r-2)^2,\\
p_1(r) &= -2r(r-1)-r+1\\
&= -(r-1)(2r+1),\\
p_2(r) &= r(r-1).
\end{aligned}$$

Since $r_1=2$ is a repeated root of the indicial polynomial p_0, Theorem 7.7.2 implies that

$$y_1=x^2\sum_{n=0}^{\infty}a_n(2)x^n \quad \text{and} \quad y_2=y_1\ln x+x^2\sum_{n=1}^{\infty}a_n'(2)x^n \tag{6.6.10}$$

form a fundamental set of Frobenius solutions of Equation 6.6.9. To find the coefficients in these series, we use the recurrence formulas from Theorem 7.7.1 :

$$\begin{aligned}
a_0(r) &= 1,\\
a_1(r) &= -\frac{p_1(r)}{p_0(r+1)}=-\frac{(r-1)(2r+1)}{(r-1)^2}=\frac{2r+1}{r-1},\\
a_n(r) &= -\frac{p_1(n+r-1)a_{n-1}(r)+p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)}\\
&= \frac{(n+r-2)\left[(2n+2r-1)a_{n-1}(r)-(n+r-3)a_{n-2}(r)\right]}{(n+r-2)^2}\\
&= \frac{(2n+2r-1)}{(n+r-2)}a_{n-1}(r)-\frac{(n+r-3)}{(n+r-2)}a_{n-2}(r), \quad n\ge 2.
\end{aligned}$$

Differentiating yields

$$
\begin{aligned}
a_1'(r) &= -\frac{3}{(r-1)^2}, \\
a_n'(r) &= \frac{2n+2r-1}{n+r-2} a_{n-1}'(r) - \frac{n+r-3}{n+r-2} a_{n-2}'(r) \\
&\quad - \frac{3}{(n+r-2)^2} a_{n-1}(r) - \frac{1}{(n+r-2)^2} a_{n-2}(r), \quad n \geq 2.
\end{aligned}
$$

Setting $r=2$ in Equation ??? and Equation ??? yields

$$
\begin{aligned}
a_0(2) &= 1, \\
a_1(2) &= 5, \\
a_n(2) &= \tfrac{(2n+3)}{n} a_{n-1}(2) - \tfrac{(n-1)}{n} a_{n-2}(2), \quad n \geq 2
\end{aligned}
\tag{6.6.11}
$$

and

$$
\begin{aligned}
a_1'(2) &= -3, \\
a_n'(2) &= \tfrac{2n+3}{n} a_{n-1}'(2) - \tfrac{n-1}{n} a_{n-2}'(2) - \tfrac{3}{n^2} a_{n-1}(2) - \tfrac{1}{n^2} a_{n-2}(2), \quad n \geq 2.
\end{aligned}
\tag{6.6.12}
$$

Computing recursively with Equation 6.6.11 and Equation 6.6.12 yields

$$
a_0(2) = 1,\ a_1(2) = 5,\ a_2(2) = 17,\ a_3(2) = \frac{143}{3},\ a_4(2) = \frac{355}{3},
$$

and

$$
a_1'(2) = -3,\ a_2'(2) = -\frac{29}{2},\ a_3'(2) = -\frac{859}{18},\ a_4'(2) = -\frac{4693}{36}.
$$

Substituting these coefficients into Equation 6.6.10 yields

$$
y_1 = x^2 \left(1 + 5x + 17x^2 + \frac{143}{3} x^3 + \frac{355}{3} x^4 + \cdots\right)
$$

and

$$
y_2 = y_1 \ln x - x^3 \left(3 + \frac{29}{2} x + \frac{859}{18} x^2 + \frac{4693}{36} x^3 + \cdots\right).
$$

Since the recurrence formula Equation ??? involves three terms, it is not possible to obtain a simple explicit formula for the coefficients in the Frobenius solutions of Equation 6.6.9. However, as we saw in the preceding sections, the recurrence formula for $\{a_n(r)\}$ involves only two terms if either $\alpha_1 = \beta_1 = \gamma_1 = 0$ or $\alpha_2 = \beta_2 = \gamma_2 = 0$ in Equation 6.6.1. In this case, it is often possible to find explicit formulas for the coefficients. The next two examples illustrate this.

Example 7.7.2

Find a fundamental set of Frobenius solutions of

$$
2x^2(2+x)y'' + 5x^2 y' + (1+x)y = 0. \tag{6.6.13}
$$

Give explicit formulas for the coefficients in the solutions.

Solution

For the given equation, the polynomials defined in Theorem 7.7.1 are

$$
\begin{aligned}
p_0(r) &= 4r(r-1) + 1 &&= (2r-1)^2, \\
p_1(r) &= 2r(r-1) + 5r + 1 &&= (r+1)(2r+1), \\
p_2(r) &= 0.
\end{aligned}
$$

Since $r_1 = 1/2$ is a repeated zero of the indicial polynomial p_0, Theorem 7.7.2 implies that

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} a_n(1/2) x^n \tag{6.6.14}$$

and

$$y_2 = y_1 \ln x + x^{1/2} \sum_{n=1}^{\infty} a_n'(1/2) x^n \tag{6.6.15}$$

form a fundamental set of Frobenius solutions of Equation 6.6.13. Since $p_2 \equiv 0$, the recurrence formulas in Theorem 7.7.1 reduce to

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r), \\ &= -\frac{(n+r)(2n+2r-1)}{(2n+2r-1)^2} a_{n-1}(r), \\ &= -\frac{n+r}{2n+2r-1} a_{n-1}(r), \quad n \geq 0. \end{aligned}$$

We leave it to you to show that

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{j+r}{2j+2r-1}, \quad n \geq 0. \tag{6.6.16}$$

Setting $r = 1/2$ yields

$$\begin{aligned} a_n(1/2) \quad &= (-1)^n \textstyle\prod_{j=1}^{n} \frac{j+1/2}{2j} = (-1)^n \prod_{j=1}^{n} \frac{2j+1}{4j}, \\ &= \frac{(-1)^n \prod_{j=1}^{n}(2j+1)}{4^n n!}, \quad n \geq 0. \end{aligned} \tag{6.6.17}$$

Substituting this into Equation 6.6.14 yields

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^{n}(2j+1)}{4^n n!} x^n.$$

To obtain y_2 in Equation 6.6.15, we must compute $a_n'(1/2)$ for $n = 1, 2, \ldots$. We'll do this by logarithmic differentiation. From Equation 6.6.16,

$$|a_n(r)| = \prod_{j=1}^{n} \frac{|j+r|}{|2j+2r-1|}, \quad n \geq 1.$$

Therefore

$$\ln|a_n(r)| = \sum_{j=1}^{n} \left(\ln|j+r| - \ln|2j+2r-1|\right).$$

Differentiating with respect to r yields

$$\frac{a_n'(r)}{a_n(r)} = \sum_{j=1}^{n} \left(\frac{1}{j+r} - \frac{2}{2j+2r-1}\right).$$

Therefore

$$a_n'(r) = a_n(r) \sum_{j=1}^{n} \left(\frac{1}{j+r} - \frac{2}{2j+2r-1}\right).$$

Setting $r=1/2$ here and recalling Equation 6.6.17 yields

$$a_n'(1/2)=\frac{(-1)^n\prod_{j=1}^n(2j+1)}{4^n n!}\left(\sum_{j=1}^n\frac{1}{j+1/2}-\sum_{j=1}^n\frac{1}{j}\right). \tag{6.6.18}$$

Since

$$\frac{1}{j+1/2}-\frac{1}{j}=\frac{j-j-1/2}{j(j+1/2)}=-\frac{1}{j(2j+1)},$$

Equation 6.6.18 can be rewritten as

$$a_n'(1/2)=-\frac{(-1)^n\prod_{j=1}^n(2j+1)}{4^n n!}\sum_{j=1}^n\frac{1}{j(2j+1)}.$$

Therefore, from Equation 6.6.15,

$$y_2=y_1\ln x-x^{1/2}\sum_{n=1}^\infty\frac{(-1)^n\prod_{j=1}^n(2j+1)}{4^n n!}\left(\sum_{j=1}^n\frac{1}{j(2j+1)}\right)x^n.$$

Example 7.7.3

Find a fundamental set of Frobenius solutions of

$$x^2(2-x^2)y''-2x(1+2x^2)y'+(2-2x^2)y=0. \tag{6.6.19}$$

Give explicit formulas for the coefficients in the solutions.

Solution

For Equation 6.6.19, the polynomials defined in Theorem 7.7.1 are

$$\begin{aligned}
p_0(r)&=2r(r-1)-2r+2 &&=2(r-1)^2,\\
p_1(r)&=0,\\
p_2(r)&=-r(r-1)-4r-2 &&=-(r+1)(r+2).
\end{aligned}$$

As in Section 7.5, since $p_1\equiv 0$, the recurrence formulas of Theorem 7.7.1 imply that $a_n(r)=0$ if n is odd, and

$$\begin{aligned}
a_0(r)&=1,\\
a_{2m}(r)&=-\frac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r)\\
&=\frac{(2m+r-1)(2m+r)}{2(2m+r-1)^2}a_{2m-2}(r)\\
&=\frac{2m+r}{2(2m+r-1)}a_{2m-2}(r),\quad m\ge 1.
\end{aligned}$$

Since $r_1=1$ is a repeated root of the indicial polynomial p_0, Theorem 7.7.2 implies that

$$y_1=x\sum_{m=0}^\infty a_{2m}(1)x^{2m} \tag{6.6.20}$$

and

$$y_2=y_1\ln x+x\sum_{m=1}^\infty a_{2m}'(1)x^{2m} \tag{6.6.21}$$

form a fundamental set of Frobenius solutions of Equation 6.6.19. We leave it to you to show that

$$a_{2m}(r)=\frac{1}{2^m}\prod_{j=1}^m\frac{2j+r}{2j+r-1}. \tag{6.6.22}$$

Setting $r=1$ yields

$$a_{2m}(1)=\frac{1}{2^m}\prod_{j=1}^m\frac{2j+1}{2j}=\frac{\prod_{j=1}^m(2j+1)}{4^m m!}, \tag{6.6.23}$$

and substituting this into Equation 6.6.20 yields

$$y_1=x\sum_{m=0}^\infty\frac{\prod_{j=1}^m(2j+1)}{4^m m!}x^{2m}.$$

To obtain y_2 in Equation 6.6.21, we must compute $a'_{2m}(1)$ for $m=1,2,\ldots$. Again we use logarithmic differentiation. From Equation 6.6.22,

$$|a_{2m}(r)|=\frac{1}{2^m}\prod_{j=1}^m\frac{|2j+r|}{|2j+r-1|}.$$

Taking logarithms yields

$$\ln|a_{2m}(r)|=-m\ln 2+\sum_{j=1}^m\left(\ln|2j+r|-\ln|2j+r-1|\right).$$

Differentiating with respect to r yields

$$\frac{a'_{2m}(r)}{a_{2m}(r)}=\sum_{j=1}^m\left(\frac{1}{2j+r}-\frac{1}{2j+r-1}\right).$$

Therefore

$$a'_{2m}(r)=a_{2m}(r)\sum_{j=1}^m\left(\frac{1}{2j+r}-\frac{1}{2j+r-1}\right).$$

Setting $r=1$ and recalling Equation 6.6.23 yields

$$a'_{2m}(1)=\frac{\prod_{j=1}^m(2j+1)}{4^m m!}\sum_{j=1}^m\left(\frac{1}{2j+1}-\frac{1}{2j}\right). \tag{6.6.24}$$

Since

$$\frac{1}{2j+1}-\frac{1}{2j}=-\frac{1}{2j(2j+1)},$$

Equation 6.6.24 can be rewritten as

$$a'_{2m}(1)=-\frac{\prod_{j=1}^m(2j+1)}{2\cdot 4^m m!}\sum_{j=1}^m\frac{1}{j(2j+1)}$$

Substituting this into Equation 6.6.21 yields

$$y_2=y_1\ln x-\frac{x}{2}\sum_{m=1}^\infty\frac{\prod_{j=1}^m(2j+1)}{4^m m!}\left(\sum_{j=1}^m\frac{1}{j(2j+1)}\right)x^{2m}.$$

If the solution $y_1=y(x,r_1)$ of $Ly=0$ reduces to a finite sum, then there's a difficulty in using logarithmic differentiation to obtain the coefficients $\{a'_n(r_1)\}$ in the second solution. The next example illustrates this difficulty and shows how to overcome it.

Example 7.7.4

Find a fundamental set of Frobenius solutions of

$$x^2y'' - x(5-x)y' + (9-4x)y = 0. \tag{6.6.25}$$

Give explicit formulas for the coefficients in the solutions.

Solution

For Equation 6.6.25 the polynomials defined in Theorem 7.7.1 are

$$\begin{aligned} p_0(r) &= r(r-1) - 5r + 9 \qquad = (r-3)^2, \\ p_1(r) &= r-4, \\ p_2(r) &= 0. \end{aligned}$$

Since $r_1 = 3$ is a repeated zero of the indicial polynomial p_0, Theorem 7.7.2 implies that

$$y_1 = x^3 \sum_{n=0}^{\infty} a_n(3) x^n \tag{6.6.26}$$

and

$$y_2 = y_1 \ln x + x^3 \sum_{n=1}^{\infty} a_n'(3) x^n \tag{6.6.27}$$

are linearly independent Frobenius solutions of Equation 6.6.25. To find the coefficients in Equation 6.6.26 we use the recurrence formulas

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ &= -\frac{n+r-5}{(n+r-3)^2} a_{n-1}(r), \quad n \ge 1. \end{aligned}$$

We leave it to you to show that

$$a_n(r) = (-1)^n \prod_{j=1}^{n} \frac{j+r-5}{(j+r-3)^2}. \tag{6.6.28}$$

Setting $r = 3$ here yields

$$a_n(3) = (-1)^n \prod_{j=1}^{n} \frac{j-2}{j^2},$$

so $a_1(3) = 1$ and $a_n(3) = 0$ if $n \ge 2$. Substituting these coefficients into Equation 6.6.26 yields

$$y_1 = x^3(1+x).$$

To obtain y_2 in Equation 6.6.27 we must compute $a_n'(3)$ for $n = 1, 2, \ldots$. Let's first try logarithmic differentiation. From Equation 6.6.28,

$$|a_n(r)| = \prod_{j=1}^{n} \frac{|j+r-5|}{|j+r-3|^2}, \quad n \ge 1,$$

so

$$\ln|a_n(r)| = \sum_{j=1}^{n} \left(\ln|j+r-5| - 2\ln|j+r-3|\right).$$

Differentiating with respect to r yields

$$\frac{a_n'(r)}{a_n(r)} = \sum_{j=1}^{n} \left(\frac{1}{j+r-5} - \frac{2}{j+r-3} \right).$$

Therefore

$$a_n'(r) = a_n(r) \sum_{j=1}^{n} \left(\frac{1}{j+r-5} - \frac{2}{j+r-3} \right). \tag{6.6.29}$$

However, we can't simply set $r=3$ here if $n \geq 2$, since the bracketed expression in the sum corresponding to $j=2$ contains the term $1/(r-3)$. In fact, since $a_n(3)=0$ for $n \geq 2$, the formula Equation 6.6.29 for $a_n'(r)$ is actually an indeterminate form at $r=3$.

We overcome this difficulty as follows. From Equation 6.6.28 with $n=1$,

$$a_1(r) = -\frac{r-4}{(r-2)^2}.$$

Therefore

$$a_1'(r) = \frac{r-6}{(r-2)^3},$$

so

$$a_1'(3) = -3. \tag{6.6.30}$$

From Equation 6.6.28 with $n \geq 2$,

$$a_n(r) = (-1)^n (r-4)(r-3) \frac{\prod_{j=3}^{n}(j+r-5)}{\prod_{j=1}^{n}(j+r-3)^2} = (r-3)c_n(r),$$

where

$$c_n(r) = (-1)^n (r-4) \frac{\prod_{j=3}^{n}(j+r-5)}{\prod_{j=1}^{n}(j+r-3)^2}, \quad n \geq 2.$$

Therefore

$$a_n'(r) = c_n(r) + (r-3)c_n'(r), \quad n \geq 2,$$

which implies that $a_n'(3) = c_n(3)$ if $n \geq 3$. We leave it to you to verify that

$$a_n'(3) = c_n(3) = \frac{(-1)^{n+1}}{n(n-1)n!}, \quad n \geq 2.$$

Substituting this and Equation 6.6.30 into Equation 6.6.27 yields

$$y_2 = x^3(1+x)\ln x - 3x^4 - x^3 \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)n!} x^n.$$

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6.6.1: The Method of Frobenius II (Exercises)

Q7.6.1

In **Exercises 7.6.1-7.6.11** find a fundamental set of Frobenius solutions. Compute the terms involving x^{n+r_1}, where $0 \le n \le N$ (N at least 7) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $N > 7$.

1. $x^2y'' - x(1-x)y' + (1-x^2)y = 0$
2. $x^2(1+x+2x^2)y' + x(3+6x+7x^2)y' + (1+6x-3x^2)y = 0$
3. $x^2(1+2x+x^2)y'' + x(1+3x+4x^2)y' - x(1-2x)y = 0$
4. $4x^2(1+x+x^2)y'' + 12x^2(1+x)y' + (1+3x+3x^2)y = 0$
5. $x^2(1+x+x^2)y'' - x(1-4x-2x^2)y' + y = 0$
6. $9x^2y'' + 3x(5+3x-2x^2)y' + (1+12x-14x^2)y = 0$
7. $x^2y'' + x(1+x+x^2)y' + x(2-x)y = 0$
8. $x^2(1+2x)y'' + x(5+14x+3x^2)y' + (4+18x+12x^2)y = 0$
9. $4x^2y'' + 2x(4+x+x^2)y' + (1+5x+3x^2)y = 0$
10. $16x^2y'' + 4x(6+x+2x^2)y' + (1+5x+18x^2)y = 0$
11. $9x^2(1+x)y'' + 3x(5+11x-x^2)y' + (1+16x-7x^2)y = 0$

Q7.6.2

In **Exercises 7.6.12-7.6.22** find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

12. $4x^2y'' + (1+4x)y = 0$
13. $36x^2(1-2x)y'' + 24x(1-9x)y' + (1-70x)y = 0$
14. $x^2(1+x)y'' - x(3-x)y' + 4y = 0$
15. $x^2(1-2x)y'' - x(5-4x)y' + (9-4x)y = 0$
16. $25x^2y'' + x(15+x)y' + (1+x)y = 0$
17. $2x^2(2+x)y'' + x^2y' + (1-x)y = 0$
18. $x^2(9+4x)y'' + 3xy' + (1+x)y = 0$
19. $x^2y'' - x(3-2x)y' + (4+3x)y = 0$
20. $x^2(1-4x)y'' + 3x(1-6x)y' + (1-12x)y = 0$
21. $x^2(1+2x)y'' + x(3+5x)y' + (1-2x)y = 0$
22. $2x^2(1+x)y'' - x(6-x)y' + (8-x)y = 0$

Q7.6.3

In **Exercises 7.6.23-7.6.27** find a fundamental set of Frobenius solutions. Compare the terms involving x^{n+r_1}, where $0 \le n \le N$ (N at least 7) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $N > 7$.

23. $x^2(1+2x)y'' + x(5+9x)y' + (4+3x)y = 0$
24. $x^2(1-2x)y'' - x(5+4x)y' + (9+4x)y = 0$
25. $x^2(1+4x)y'' - x(1-4x)y' + (1+x)y = 0$
26. $x^2(1+x)y'' + x(1+2x)y' + xy = 0$
27. $x^2(1-x)y'' + x(7+x)y' + (9-x)y = 0$

Q7.6.4

In **Exercises 7.6.28-7.6.38** find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

28. $x^2y''-x(1-x^2)y'+(1+x^2)y=0$

29. $x^2(1+x^2)y''-3x(1-x^2)y'+4y=0$

30. $4x^2y''+2x^3y'+(1+3x^2)y=0$

31. $x^2(1+x^2)y''-x(1-2x^2)y'+y=0$

32. $2x^2(2+x^2)y''+7x^3y'+(1+3x^2)y=0$

33. $x^2(1+x^2)y''-x(1-4x^2)y'+(1+2x^2)y=0$

34. $4x^2(4+x^2)y''+3x(8+3x^2)y'+(1-9x^2)y=0$

35. $3x^2(3+x^2)y''+x(3+11x^2)y'+(1+5x^2)y=0$

36. $4x^2(1+4x^2)y''+32x^3y'+y=0$

37. $9x^2y''-3x(7-2x^2)y'+(25+2x^2)y=0$

38. $x^2(1+2x^2)y''+x(3+7x^2)y'+(1-3x^2)y=0$

Q7.6.5

In **Exercises 7.6.39-7.6.43** find a fundamental set of Frobenius solutions. Compute the terms involving x^{2m+r_1}, where $0\le m\le M$ (M at least 3) and r_1 is the root of the indicial equation. Optionally, write a computer program to implement the applicable recurrence formulas and take $M>3$.

39. $x^2(1+x^2)y''+x(3+8x^2)y'+(1+12x^2)y$

40. $x^2y''-x(1-x^2)y'+(1+x^2)y=0$

41. $x^2(1-2x^2)y''+x(5-9x^2)y'+(4-3x^2)y=0$

42. $x^2(2+x^2)y''+x(14-x^2)y'+2(9+x^2)y=0$

43. $x^2(1+x^2)y''+x(3+7x^2)y'+(1+8x^2)y=0$

Q7.6.6

In **Exercises 7.6.44-7.6.52** find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

44. $x^2(1-2x)y''+3xy'+(1+4x)y=0$

45. $x(1+x)y''+(1-x)y'+y=0$

46. $x^2(1-x)y''+x(3-2x)y'+(1+2x)y=0$

47. $4x^2(1+x)y''-4x^2y'+(1-5x)y=0$

48. $x^2(1-x)y''-x(3-5x)y'+(4-5x)y=0$

49. $x^2(1+x^2)y''-x(1+9x^2)y'+(1+25x^2)y=0$

50. $9x^2y''+3x(1-x^2)y'+(1+7x^2)y=0$

51. $x(1+x^2)y''+(1-x^2)y'-8xy=0$

52. $4x^2y''+2x(4-x^2)y'+(1+7x^2)y=0$

Q7.6.7

53. Under the assumptions of Theorem 7.6.2, suppose the power series

$$\sum_{n=0}^{\infty}a_n(r_1)x^n \quad \text{and} \quad \sum_{n=1}^{\infty}a_n'(r_1)x^n$$

converge on $(-\rho,\rho)$.

a. Show that

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n \quad \text{and} \quad y_2 = y_1 \ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n$$

are linearly independent on $(0,\rho)$. HINT: Show that if c_1 and c_2 are constants such that $c_1y_1 + c_2y_2 \equiv 0$ on $(0,\rho)$, then

$$(c_1 + c_2 \ln x)\sum_{n=0}^{\infty} a_n(r_1)x^n + c_2\sum_{n=1}^{\infty} a_n'(r_1)x^n = 0, \quad 0 < x < \rho$$

Then let $x \to 0+$ to conclude that $c_2 = 0$.

b. Use the result of (a) to complete the proof of Theorem 7.6.2.

54. Let

$$Ly = x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_1(r) = \alpha_1 r(r-1) + \beta_1 r + \gamma_1.$$

Theorem 7.6.1 and **Exercise 7.5.55a**

imply that if

$$y(x,r) = x^r\sum_{n=0}^{\infty} a_n(r)x^n$$

where

$$a_n(r) = (-1)^n\prod_{j=1}^{n}\frac{p_1(j+r-1)}{p_0(j+r)},$$

then

$$Ly(x,r) = p_0(r)x^r.$$

Now suppose $p_0(r) = \alpha_0(r-r_1)^2$ and $p_1(k+r_1) \neq 0$ if k is a nonnegative integer.

a. Show that $Ly = 0$ has the solution

$$y_1 = x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n,$$

where

$$a_n(r_1) = \frac{(-1)^n}{\alpha_0^n(n!)^2}\prod_{j=1}^{n} p_1(j+r_1-1).$$

b. Show that $Ly = 0$ has the second solution

$$y_2 = y_1 \ln x + x^{r_1}\sum_{n=1}^{\infty} a_n(r_1)J_n x^n,$$

where

$$J_n = \sum_{j=1}^{n}\frac{p_1'(j+r_1-1)}{p_1(j+r_1-1)} - 2\sum_{j=1}^{n}\frac{1}{j}.$$

c. Conclude from (a) and (b) that if $\gamma_1 \neq 0$ then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\gamma_1}{\alpha_0}\right)^n x^n$$

and

$$y_2 = y_1 \ln x - 2x^{r_1} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{\gamma_1}{\alpha_0}\right)^n \left(\sum_{j=1}^{n} \frac{1}{j}\right) x^n$$

are solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_1 x) y = 0.$$

(The conclusion is also valid if $\gamma_1 = 0$. Why?)

55. Let

$$Ly = x^2(\alpha_0 + \alpha_q x^q) y'' + x(\beta_0 + \beta_q x^q) y' + (\gamma_0 + \gamma_q x^q) y$$

where q is a positive integer, and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0 \quad \text{and} \quad p_q(r) = \alpha_q r(r-1) + \beta_q r + \gamma_q.$$

Suppose

$$p_0(r) = \alpha_0 (r - r_1)^2 \quad \text{and} \quad p_q(r) \not\equiv 0.$$

a. Recall from **Exercise 7.5.59** that $Ly = 0$ has the solution

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{qm}(r_1) x^{qm},$$

where

$$a_{qm}(r_1) = \frac{(-1)^m}{(q^2\alpha_0)^m (m!)^2} \prod_{j=1}^{m} p_q\left(q(j-1) + r_1\right).$$

b. Show that $Ly = 0$ has the second solution

$$y_2 = y_1 \ln x + x^{r_1} \sum_{m=1}^{\infty} a'_{qm}(r_1) J_m x^{qm},$$

where

$$J_m = \sum_{j=1}^{m} \frac{p'_q\left(q(j-1) + r_1\right)}{p_q\left(q(j-1) + r_1\right)} - \frac{2}{q} \sum_{j=1}^{m} \frac{1}{j}.$$

c. Conclude from (a) and (b) that if $\gamma_q \neq 0$ then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\gamma_q}{q^2\alpha_0}\right)^m x^{qm}$$

and

$$y_2 = y_1 \ln x - \frac{2}{q} x^{r_1} \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{\gamma_q}{q^2\alpha_0}\right)^m \left(\sum_{j=1}^{m} \frac{1}{j}\right) x^{qm}$$

are solutions of

$$\alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_q x^q) y = 0.$$

56. The equation

$$xy'' + y' + xy = 0$$

is **Bessel's equation of order zero.** (See **Exercise 7.5.53**.) Find two linearly independent Frobenius solutions of this equation.

57. Suppose the assumptions of **Exercise 7.5.53** hold, except that

$$p_0(r) = \alpha_0(r - r_1)^2.$$

Show that

$$y_1 = \frac{x^{r_1}}{\alpha_0 + \alpha_1 x + \alpha_2 x^2} \quad \text{and} \quad y_2 = \frac{x^{r_1} \ln x}{\alpha_0 + \alpha_1 x + \alpha_2 x^2}$$

are linearly independent Frobenius solutions of

$$x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y = 0$$

on any interval $(0, \rho)$ on which $\alpha_0 + \alpha_1 x + \alpha_2 x^2$ has no zeros.

Q7.6.8

58. $4x^2(1+x)y'' + 8x^2y' + (1+x)y = 0$
59. $9x^2(3+x)y'' + 3x(3+7x)y' + (3+4x)y = 0$
60. $x^2(2-x^2)y'' - x(2+3x^2)y' + (2-x^2)y = 0$
61. $16x^2(1+x^2)y'' + 8x(1+9x^2)y' + (1+49x^2)y = 0$
62. $x^2(4+3x)y'' - x(4-3x)y' + 4y = 0$
63. $4x^2(1+3x+x^2)y'' + 8x^2(3+2x)y' + (1+3x+9x^2)y = 0$
64. $x^2(1-x)^2y'' - x(1+2x-3x^2)y' + (1+x^2)y = 0$
65. $9x^2(1+x+x^2)y'' + 3x(1+7x+13x^2)y' + (1+4x+25x^2)y = 0$

Q7.6.9

66.

a. Let L and $y(x, r)$ be as in **Exercises 7.5.57** and **7.5.58**. Extend Theorem 7.6.1 by showing that

$$L\left(\frac{\partial y}{\partial r}(x, r)\right) = p_0'(r)x^r + x^r p_0(r) \ln x.$$

b. Show that if

$$p_0(r) = \alpha_0(r - r_1)^2$$

then

$$y_1 = y(x, r_1) \quad \text{and} \quad y_2 = \frac{\partial y}{\partial r}(x, r_1)$$

are solutions of $Ly = 0$.

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6.7: The Method of Frobenius III

This page is a draft and is under active development.

In Sections 7.5 and 7.6, we discussed methods for finding Frobenius solutions of a homogeneous linear second order equation near a regular singular point in the case where the indicial equation has a repeated root or distinct real roots that don't differ by an integer. In this section we consider the case where the indicial equation has distinct real roots that differ by an integer. We'll limit our discussion to equations that can be written as

$$x^2(\alpha_0+\alpha_1 x)y''+x(\beta_0+\beta_1 x)y'+(\gamma_0+\gamma_1 x)y=0 \tag{6.7.1}$$

or

$$x^2(\alpha_0+\alpha_2 x^2)y''+x(\beta_0+\beta_2 x^2)y'+(\gamma_0+\gamma_2 x^2)y=0,$$

where the roots of the indicial equation differ by a positive integer.

We begin with a theorem that provides a fundamental set of solutions of equations of the form Equation 6.7.1.

Theorem 7.8.1

Let

$$Ly=x^2(\alpha_0+\alpha_1 x)y''+x(\beta_0+\beta_1 x)y'+(\gamma_0+\gamma_1 x)y,$$

where $\alpha_0 \neq 0,$ and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1)+\beta_0 r+\gamma_0, \\ p_1(r) &= \alpha_1 r(r-1)+\beta_1 r+\gamma_1. \end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n, and define

$$\begin{aligned} a_0(r) &= 1 \\ a_n(r) &= \frac{p_1(n+r-1)}{p_0(n+r)} \quad n \geq 1 \end{aligned} \tag{6.7.2}$$

Let r_1 and r_2 be the roots of the indicial equation $p_0(r)=0$, and suppose $r_1=r_2+k,$ where k is a positive integer. Then

$$y_1=x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n$$

is a Frobenius solution of $Ly=0$. Moreover, if we define

$$\begin{aligned} a_0(r_2) &= 1, \\ a_n(r_2) &= -\frac{p_1(n+r_2-1)}{p_0(n+r_2)} a_{n-1}(r_2), \quad 1 \leq n \leq k-1, \end{aligned} \tag{6.7.3}$$

and

$$C=-\frac{p_1(r_1-1)}{k\alpha_0} a_{k-1}(r_2), \tag{6.7.4}$$

then

$$y_2=x^{r_2}\sum_{n=0}^{k-1} a_n(r_2)x^n+C\left(y_1 \ln x+x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n\right) \tag{6.7.5}$$

is also a solution of $Ly=0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Proof

Theorem 7.5.3 implies that $Ly_1 = 0$. We'll now show that $Ly_2 = 0$. Since L is a linear operator, this is equivalent to showing that

$$L\left(x^{r_2}\sum_{n=0}^{k-1} a_n(r_2)x^n\right) + CL\left(y_1 \ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n\right) = 0. \tag{6.7.6}$$

To verify this, we'll show that

$$L\left(x^{r_2}\sum_{n=0}^{k-1} a_n(r_2)x^n\right) = p_1(r_1-1)a_{k-1}(r_2)x^{r_1} \tag{6.7.7}$$

and

$$L\left(y_1 \ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n\right) = k\alpha_0 x^{r_1}. \tag{6.7.8}$$

This will imply that $Ly_2 = 0$, since substituting Equation 6.7.7and Equation 6.7.8into Equation 6.7.6and using Equation 6.7.4yields

$$\begin{aligned} Ly_2 &= \left[p_1(r_1-1)a_{k-1}(r_2) + Ck\alpha_0\right] x^{r_1} \\ &= \left[p_1(r_1-1)a_{k-1}(r_2) - p_1(r_1-1)a_{k-1}(r_2)\right] x^{r_1} = 0. \end{aligned}$$

We'll prove Equation 6.7.8first. From Theorem 7.6.1,

$$L\left(y(x,r)\ln x + x^r\sum_{n=1}^{\infty} a_n'(r)x^n\right) = p_0'(r)x^r + x^r p_0(r)\ln x.$$

Setting $r = r_1$ and recalling that $p_0(r_1) = 0$ and $y_1 = y(x, r_1)$ yields

$$L\left(y_1 \ln x + x^{r_1}\sum_{n=1}^{\infty} a_n'(r_1)x^n\right) = p_0'(r_1)x^{r_1}. \tag{6.7.9}$$

Since r_1 and r_2 are the roots of the indicial equation, the indicial polynomial can be written as

$$p_0(r) = \alpha_0(r-r_1)(r-r_2) = \alpha_0\left[r^2 - (r_1+r_2)r + r_1r_2\right].$$

Differentiating this yields

$$p_0'(r) = \alpha_0(2r - r_1 - r_2).$$

Therefore $p_0'(r_1) = \alpha_0(r_1 - r_2) = k\alpha_0$, so Equation 6.7.9implies Equation 6.7.8

Before proving Equation 6.7.7, we first note $a_n(r_2)$ is well defined by Equation 6.7.3 for $1 \le n \le k-1$, since $p_0(n+r_2) \ne 0$ for these values of n. However, we can't define $a_n(r_2)$ for $n \ge k$ with Equation 6.7.3, since $p_0(k+r_2) = p_0(r_1) = 0$. For convenience, we define $a_n(r_2) = 0$ for $n \ge k$. Then, from Theorem 7.5.1,

$$L\left(x^{r_2}\sum_{n=0}^{k-1} a_n(r_2)x^n\right) = L\left(x^{r_2}\sum_{n=0}^{\infty} a_n(r_2)x^n\right) = x^{r_2}\sum_{n=0}^{\infty} b_n x^n, \tag{6.7.10}$$

where $b_0 = p_0(r_2) = 0$ and

$$b_n = p_0(n+r_2)a_n(r_2) + p_1(n+r_2-1)a_{n-1}(r_2), \quad n \ge 1.$$

If $1 \le n \le k-1$, then Equation 6.7.3 implies that $b_n = 0$. If $n \ge k+1$, then $b_n = 0$ because $a_{n-1}(r_2) = a_n(r_2) = 0$. Therefore Equation 6.7.10reduces to

$$L\left(x^{r_2}\sum_{n=0}^{k-1} a_n(r_2)x^n\right) = \left[p_0(k+r_2)a_k(r_2) + p_1(k+r_2-1)a_{k-1}(r_2)\right] x^{k+r_2}.$$

Since $a_k(r_2)=0$ and $k+r_2=r_1$, this implies Equation 6.7.7.

We leave the proof that $\{y_1,y_2\}$ is a fundamental set as an **Exercise 7.7.41**.

Example 7.8.1 :

Find a fundamental set of Frobenius solutions of

$$2x^2(2+x)y''-x(4-7x)y'-(5-3x)y=0.$$

Give explicit formulas for the coefficients in the solutions.

Solution

For the given equation, the polynomials defined in Theorem 7.8.1 are

$$\begin{aligned} p_0(r) &= 4r(r-1)-4r-5 &&= (2r+1)(2r-5), \\ p_1(r) &= 2r(r-1)+7r+3 &&= (r+1)(2r+3). \end{aligned}$$

The roots of the indicial equation are $r_1=5/2$ and $r_2=-1/2$, so $k=r_1-r_2=3$. Therefore Theorem 7.8.1 implies that

$$y_1=x^{5/2}\sum_{n=0}^{\infty}a_n(5/2)x^n \tag{6.7.11}$$

and

$$y_2=x^{-1/2}\sum_{n=0}^{2}a_n(-1/2)+C\left(y_1\ln x+x^{5/2}\sum_{n=1}^{\infty}a_n'(5/2)x^n\right) \tag{6.7.12}$$

(with C as in Equation 6.7.4) form a fundamental set of solutions of $Ly=0$. The recurrence formula Equation 6.7.2 is

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\tfrac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r) \\ &= -\tfrac{(n+r)(2n+2r+1)}{(2n+2r+1)(2n+2r-5)}a_{n-1}(r), \\ &= -\tfrac{n+r}{2n+2r-5}a_{n-1}(r),\ n\ge 1, \end{aligned} \tag{6.7.13}$$

which implies that

$$a_n(r)=(-1)^n\prod_{j=1}^{n}\frac{j+r}{2j+2r-5},\ n\ge 0. \tag{6.7.14}$$

Therefore

$$a_n(5/2)=\frac{(-1)^n\prod_{j=1}^{n}(2j+5)}{4^n n!}. \tag{6.7.15}$$

Substituting this into Equation 6.7.11 yields

$$y_1=x^{5/2}\sum_{n=0}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(2j+5)}{4^n n!}x^n.$$

To compute the coefficients $a_0(-1/2), a_1(-1/2)$ and $a_2(-1/2)$ in y_2, we set $r=-1/2$ in Equation 6.7.13 and apply the resulting recurrence formula for $n=1,2$; thus,

$$\begin{aligned} a_0(-1/2) &= 1, \\ a_n(-1/2) &= -\frac{2n-1}{4(n-3)}a_{n-1}(-1/2),\ n=1,2. \end{aligned}$$

The last formula yields

$$a_1(-1/2)=1/8 \quad \text{and} \quad a_2(-1/2)=3/32.$$

Substituting $r_1=5/2, r_2=-1/2, k=3$, and $\alpha_0=4$ into Equation 6.7.4 yields $C=-15/128$. Therefore, from Equation 6.7.12,

$$y_2=x^{-1/2}\left(1+\frac{1}{8}x+\frac{3}{32}x^2\right)-\frac{15}{128}\left(y_1\ln x+x^{5/2}\sum_{n=1}^{\infty}a_n'(5/2)x^n\right). \tag{6.7.16}$$

We use logarithmic differentiation to obtain obtain $a_n'(r)$. From Equation 6.7.14,

$$|a_n(r)|=\prod_{j=1}^{n}\frac{|j+r|}{|2j+2r-5|},\ n\geq 1.$$

Therefore

$$\ln|a_n(r)|=\sum_{j=1}^{n}\left(\ln|j+r|-\ln|2j+2r-5|\right).$$

Differentiating with respect to r yields

$$\frac{a_n'(r)}{a_n(r)}=\sum_{j=1}^{n}\left(\frac{1}{j+r}-\frac{2}{2j+2r-5}\right).$$

Therefore

$$a_n'(r)=a_n(r)\sum_{j=1}^{n}\left(\frac{1}{j+r}-\frac{2}{2j+2r-5}\right).$$

Setting $r=5/2$ here and recalling Equation 6.7.15 yields

$$a_n'(5/2)=\frac{(-1)^n\prod_{j=1}^{n}(2j+5)}{4^n n!}\sum_{j=1}^{n}\left(\frac{1}{j+5/2}-\frac{1}{j}\right). \tag{6.7.17}$$

Since

$$\frac{1}{j+5/2}-\frac{1}{j}=-\frac{5}{j(2j+5)},$$

we can rewrite Equation 6.7.17 as

$$a_n'(5/2)=-5\frac{(-1)^n\prod_{j=1}^{n}(2j+5)}{4^n n!}\left(\sum_{j=1}^{n}\frac{1}{j(2j+5)}\right).$$

Substituting this into Equation 6.7.16 yields

$$\begin{aligned} y_2 &= x^{-1/2}\left(1+\frac{1}{8}x+\frac{3}{32}x^2\right)-\frac{15}{128}y_1\ln x \\ &\quad +\frac{75}{128}x^{5/2}\sum_{n=1}^{\infty}\frac{(-1)^n\prod_{j=1}^{n}(2j+5)}{4^n n!}\left(\sum_{j=1}^{n}\frac{1}{j(2j+5)}\right)x^n. \end{aligned}$$

If $C=0$ in Equation 6.7.4, there's no need to compute

$$y_1\ln x+x^{r_1}\sum_{n=1}^{\infty}a_n'(r_1)x^n$$

in the formula Equation 6.7.5 for y_2. Therefore it is best to compute C before computing $\{a_n'(r_1)\}_{n=1}^{\infty}$. This is illustrated in the next example. (See also **Exercises 7.7.44** and **7.7.45**.)

Example 7.8.2

Find a fundamental set of Frobenius solutions of

$$x^2(1-2x)y''+x(8-9x)y'+(6-3x)y=0.$$

Give explicit formulas for the coefficients in the solutions.

Solution

For the given equation, the polynomials defined in Theorem 7.8.1 are

$$\begin{aligned} p_0(r) &= r(r-1)+8r+6 &&= (r+1)(r+6) \\ p_1(r) &= -2r(r-1)-9r-3 &&= -(r+3)(2r+1). \end{aligned}$$

The roots of the indicial equation are $r_1=-1$ and $r_2=-6$, so $k=r_1-r_2=5$. Therefore Theorem 7.8.1 implies that

$$y_1=x^{-1}\sum_{n=0}^{\infty}a_n(-1)x^n \tag{6.7.18}$$

and

$$y_2=x^{-6}\sum_{n=0}^{4}a_n(-6)+C\left(y_1\ln x+x^{-1}\sum_{n=1}^{\infty}a_n'(-1)x^n\right) \tag{6.7.19}$$

(with C as in Equation 6.7.4) form a fundamental set of solutions of $Ly=0$. The recurrence formula Equation 6.7.2 is

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\tfrac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r) \\ &= \tfrac{(n+r+2)(2n+2r-1)}{(n+r+1)(n+r+6)}a_{n-1}(r),\ n\ge 1, \end{aligned} \tag{6.7.20}$$

which implies that

$$\begin{aligned} a_n(r) &= \textstyle\prod_{j=1}^{n}\tfrac{(j+r+2)(2j+2r-1)}{(j+r+1)(j+r+6)} \\ &= \left(\textstyle\prod_{j=1}^{n}\tfrac{j+r+2}{j+r+1}\right)\left(\textstyle\prod_{j=1}^{n}\tfrac{2j+2r-1}{j+r+6}\right). \end{aligned} \tag{6.7.21}$$

Since

$$\prod_{j=1}^{n}\frac{j+r+2}{j+r+1}=\frac{(r+3)(r+4)\cdots(n+r+2)}{(r+2)(r+3)\cdots(n+r+1)}=\frac{n+r+2}{r+2}$$

because of cancellations, Equation 6.7.21 simplifies to

$$a_n(r)=\frac{n+r+2}{r+2}\prod_{j=1}^{n}\frac{2j+2r-1}{j+r+6}.$$

Therefore

$$a_n(-1)=(n+1)\prod_{j=1}^{n}\frac{2j-3}{j+5}.$$

Substituting this into Equation 6.7.18 yields

$$y_1=x^{-1}\sum_{n=0}^{\infty}(n+1)\left(\prod_{j=1}^{n}\frac{2j-3}{j+5}\right)x^n.$$

To compute the coefficients $a_0(-6), \ldots, a_4(-6)$ in y_2, we set $r=-6$ in Equation 6.7.20 and apply the resulting recurrence formula for $n=1, 2, 3, 4$; thus,

$$\begin{aligned} a_0(-6) &= 1, \\ a_n(-6) &= \frac{(n-4)(2n-13)}{n(n-5)} a_{n-1}(-6), \; n=1,2,3,4. \end{aligned}$$

The last formula yields

$$a_1(-6) = -\frac{33}{4}, \; a_2(-6) = \frac{99}{4}, \; a_3(-6) = -\frac{231}{8}, \; a_4(-6) = 0.$$

Since $a_4(-6)=0$, Equation 6.7.4 implies that the constant C in Equation 6.7.19 is zero. Therefore Equation 6.7.19 reduces to

$$y_2 = x^{-6}\left(1 - \frac{33}{4}x + \frac{99}{4}x^2 - \frac{231}{8}x^3\right).$$

We now consider equations of the form

$$x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y = 0,$$

where the roots of the indicial equation are real and differ by an even integer. The case where the roots are real and differ by an odd integer can be handled by the method discussed in *Exercise 7.5.56*.

The proof of the next theorem is similar to the proof of Theorem 7.8.1 (**Exercise 7.7.43**).

Theorem 7.8.2

Let

$$Ly = x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y,$$

where $\alpha_0 \neq 0$, and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1) + \beta_0 r + \gamma_0, \\ p_2(r) &= \alpha_2 r(r-1) + \beta_2 r + \gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(2m+r)$ is nonzero for all positive integers m, and define

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\tfrac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r), \quad m \geq 1. \end{aligned} \tag{6.7.22}$$

Let r_1 and r_2 be the roots of the indicial equation $p_0(r)=0$, and suppose $r_1 = r_2 + 2k$, where k is a positive integer. Then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{2m}(r_1) x^{2m}$$

is a Frobenius solution of $Ly=0$. Moreover, if we define

$$\begin{aligned} a_0(r_2) &= 1, \\ a_{2m}(r_2) &= -\frac{p_2(2m+r_2-2)}{p_0(2m+r_2)} a_{2m-2}(r_2), \quad 1 \leq m \leq k-1 \end{aligned}$$

and

$$C = -\frac{p_2(r_1-2)}{2k\alpha_0} a_{2k-2}(r_2), \tag{6.7.23}$$

then

$$y_2 = x^{r_2}\sum_{m=0}^{k-1} a_{2m}(r_2)x^{2m} + C\left(y_1 \ln x + x^{r_1}\sum_{m=1}^{\infty} a'_{2m}(r_1)x^{2m}\right) \tag{6.7.24}$$

is also a solution of $Ly = 0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Example 7.8.3

Find a fundamental set of Frobenius solutions of

$$x^2(1+x^2)y'' + x(3+10x^2)y' - (15-14x^2)y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution

For the given equation, the polynomials defined in Theorem 7.8.2 are

$$\begin{aligned} p_0(r) &= r(r-1)+3r-15 &&= (r-3)(r+5) \\ p_2(r) &= r(r-1)+10r+14 &&= (r+2)(r+7). \end{aligned}$$

The roots of the indicial equation are $r_1 = 3$ and $r_2 = -5$, so $k = (r_1 - r_2)/2 = 4$. Therefore Theorem 7.8.2 implies that

$$y_1 = x^3\sum_{m=0}^{\infty} a_{2m}(3)x^{2m} \tag{6.7.25}$$

and

$$y_2 = x^{-5}\sum_{m=0}^{3} a_{2m}(-5)x^{2m} + C\left(y_1 \ln x + x^3\sum_{m=1}^{\infty} a'_{2m}(3)x^{2m}\right)$$

(with C as in Equation 6.7.23) form a fundamental set of solutions of $Ly = 0$. The recurrence formula Equation 6.7.22 is

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\tfrac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r) \\ &= -\tfrac{(2m+r)(2m+r+5)}{(2m+r-3)(2m+r+5)}a_{2m-2}(r) \\ &= -\tfrac{2m+r}{2m+r-3}a_{2m-2}(r),\ m \ge 1, \end{aligned} \tag{6.7.26}$$

which implies that

$$a_{2m}(r) = (-1)^m\prod_{j=1}^{m}\frac{2j+r}{2j+r-3},\ m \ge 0. \tag{6.7.27}$$

Therefore

$$a_{2m}(3) = \frac{(-1)^m\prod_{j=1}^{m}(2j+3)}{2^m m!}. \tag{6.7.28}$$

Substituting this into Equation 6.7.25 yields

$$y_1 = x^3\sum_{m=0}^{\infty}\frac{(-1)^m\prod_{j=1}^{m}(2j+3)}{2^m m!}x^{2m}.$$

To compute the coefficients $a_2(-5)$, $a_4(-5)$, and $a_6(-5)$ in y_2, we set $r = -5$ in Equation 6.7.26 and apply the resulting recurrence formula for $m = 1, 2, 3$; thus,

$$a_{2m}(-5) = -\frac{2m-5}{2(m-4)}a_{2m-2}(-5),\ m = 1, 2, 3.$$

This yields

$$a_2(-5)=-\frac{1}{2},\ a_4(-5)=\frac{1}{8},\ a_6(-5)=\frac{1}{16}.$$

Substituting $r_1=3$, $r_2=-5$, $k=4$, and $\alpha_0=1$ into Equation 6.7.23 yields $C=-3/16$. Therefore, from Equation 6.7.24,

$$y_2=x^{-5}\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{16}x^6\right)-\frac{3}{16}\left(y_1\ln x+x^3\sum_{m=1}^{\infty}a'_{2m}(3)x^{2m}\right). \tag{6.7.29}$$

To obtain $a'_{2m}(r)$ we use logarithmic differentiation. From Equation 6.7.27,

$$|a_{2m}(r)|=\prod_{j=1}^{m}\frac{|2j+r|}{|2j+r-3|},\ m\geq 1.$$

Therefore

$$\ln|a_{2m}(r)|=\sum_{j=1}^{n}\left(\ln|2j+r|-\ln|2j+r-3|\right).$$

Differentiating with respect to r yields

$$\frac{a'_{2m}(r)}{a_{2m}(r)}=\sum_{j=1}^{m}\left(\frac{1}{2j+r}-\frac{1}{2j+r-3}\right).$$

Therefore

$$a'_{2m}(r)=a_{2m}(r)\sum_{j=1}^{n}\left(\frac{1}{2j+r}-\frac{1}{2j+r-3}\right).$$

Setting $r=3$ here and recalling Equation 6.7.28 yields

$$a'_{2m}(3)=\frac{(-1)^m\prod_{j=1}^{m}(2j+3)}{2^m m!}\sum_{j=1}^{m}\left(\frac{1}{2j+3}-\frac{1}{2j}\right). \tag{6.7.30}$$

Since

$$\frac{1}{2j+3}-\frac{1}{2j}=-\frac{3}{2j(2j+3)},$$

we can rewrite Equation 6.7.30 as

$$a'_{2m}(3)=-\frac{3}{2}\frac{(-1)^n\prod_{j=1}^{m}(2j+3)}{2^m m!}\left(\sum_{j=1}^{n}\frac{1}{j(2j+3)}\right).$$

Substituting this into Equation 6.7.29 yields

$$\begin{aligned}y_2&=x^{-5}\left(1-\frac{1}{2}x^2+\frac{1}{8}x^4+\frac{1}{16}x^6\right)-\frac{3}{16}y_1\ln x\\&\quad+\frac{9}{32}x^3\sum_{m=1}^{\infty}\frac{(-1)^m\prod_{j=1}^{m}(2j+3)}{2^m m!}\left(\sum_{j=1}^{m}\frac{1}{j(2j+3)}\right)x^{2m}.\end{aligned}$$

Example 7.8.4

Find a fundamental set of Frobenius solutions of

$$x^2(1-2x^2)y''+x(7-13x^2)y'-14x^2y=0.$$

Give explicit formulas for the coefficients in the solutions.

Solution

For the given equation, the polynomials defined in Theorem 7.8.2 are

$$\begin{aligned} p_0(r) &= r(r-1)+7r &&= r(r+6), \\ p_2(r) &= -2r(r-1)-13r-14 &&= -(r+2)(2r+7). \end{aligned}$$

The roots of the indicial equation are $r_1=0$ and $r_2=-6$, so $k=(r_1-r_2)/2=3$. Therefore Theorem 7.8.2 implies that

$$y_1=\sum_{m=0}^{\infty} a_{2m}(0)x^{2m}, \tag{6.7.31}$$

and

$$y_2=x^{-6}\sum_{m=0}^{2} a_{2m}(-6)x^{2m}+C\left(y_1\ln x+\sum_{m=1}^{\infty} a'_{2m}(0)x^{2m}\right) \tag{6.7.32}$$

(with C as in Equation 6.7.23) form a fundamental set of solutions of $Ly=0$. The recurrence formulas Equation 6.7.22 are

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\tfrac{p_2(2m+r-2)}{p_0(2m+r)}a_{2m-2}(r) \\ &= \tfrac{(2m+r)(4m+2r+3)}{(2m+r)(2m+r+6)}a_{2m-2}(r) \\ &= \tfrac{4m+2r+3}{2m+r+6}a_{2m-2}(r),\ m\ge 1, \end{aligned} \tag{6.7.33}$$

which implies that

$$a_{2m}(r)=\prod_{j=1}^{m}\frac{4j+2r+3}{2j+r+6}.$$

Setting $r=0$ yields

$$a_{2m}(0)=6\frac{\prod_{j=1}^{m}(4j+3)}{2^m(m+3)!}.$$

Substituting this into Equation 6.7.31 yields

$$y_1=6\sum_{m=0}^{\infty}\frac{\prod_{j=1}^{m}(4j+3)}{2^m(m+3)!}x^{2m}.$$

To compute the coefficients $a_0(-6)$, $a_2(-6)$, and $a_4(-6)$ in y_2, we set $r=-6$ in Equation 6.7.33 and apply the resulting recurrence formula for $m=1,2$; thus,

$$\begin{aligned} a_0(-6) &= 1, \\ a_{2m}(-6) &= \frac{4m-9}{2m}a_{2m-2}(-6),\ m=1,2. \end{aligned}$$

The last formula yields

$$a_2(-6)=-\frac{5}{2} \quad \text{and} \quad a_4(-6)=\frac{5}{8}.$$

Since $p_2(-2)=0$, the constant C in Equation 6.7.23 is zero. Therefore Equation 6.7.32 reduces to

$$y_2=x^{-6}\left(1-\frac{5}{2}x^2+\frac{5}{8}x^4\right).$$

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6.7.1: The Method of Frobenius III (Exercises)

Q7.7.1

In **Exercises 7.7.1-7.7.40** find a fundamental set of Frobenius solutions. Give explicit formulas for the coefficients.

1. $x^2y''-3xy'+(3+4x)y=0$
2. $xy''+y=0$
3. $4x^2(1+x)y''+4x(1+2x)y'-(1+3x)y=0$
4. $xy''+xy'+y=0$
5. $2x^2(2+3x)y''+x(4+21x)y'-(1-9x)y=0$
6. $x^2y''+x(2+x)y'-(2-3x)y=0$
7. $4x^2y''+4xy'-(9-x)y=0$
8. $x^2y''+10xy'+(14+x)y=0$
9. $4x^2(1+x)y''+4x(3+8x)y'-(5-49x)y=0$
10. $x^2(1+x)y''-x(3+10x)y'+30xy=0$
11. $x^2y''+x(1+x)y'-3(3+x)y=0$
12. $x^2y''+x(1-2x)y'-(4+x)y=0$
13. $x(1+x)y''-4y'-2y=0$
14. $x^2(1+2x)y''+x(9+13x)y'+(7+5x)y=0$
15. $4x^2y''-2x(4-x)y'-(7+5x)y=0$
16. $3x^2(3+x)y''-x(15+x)y'-20y=0$
17. $x^2(1+x)y''+x(1-10x)y'-(9-10x)y=0$
18. $x^2(1+x)y''+3x^2y'-(6-x)y=0$
19. $x^2(1+2x)y''-2x(3+14x)y'+(6+100x)y=0$
20. $x^2(1+x)y''-x(6+11x)y'+(6+32x)y=0$
21. $4x^2(1+x)y''+4x(1+4x)y'-(49+27x)y=0$
22. $x^2(1+2x)y''-x(9+8x)y'-12xy=0$
23. $x^2(1+x^2)y''-x(7-2x^2)y'+12y=0$
24. $x^2y''-x(7-x^2)y'+12y=0$
25. $xy''-5y'+xy=0$
26. $x^2y''+x(1+2x^2)y'-(1-10x^2)y=0$
27. $x^2y''-xy'-(3-x^2)y=0$
28. $4x^2y''+2x(8+x^2)y'+(5+3x^2)y=0$
29. $x^2y''+x(1+x^2)y'-(1-3x^2)y=0$
30. $x^2y''+x(1-2x^2)y'-4(1+2x^2)y=0$
31. $4x^2y''+8xy'-(35-x^2)y=0$
32. $9x^2y''-3x(11+2x^2)y'+(13+10x^2)y=0$
33. $x^2y''+x(1-2x^2)y'-4(1-x^2)y=0$
34. $x^2y''+x(1-3x^2)y'-4(1-3x^2)y=0$

35. $x^2(1+x^2)y''+x(5+11x^2)y'+24x^2y=0$

36. $4x^2(1+x^2)y''+8xy'-(35-x^2)y=0$

37. $x^2(1+x^2)y''-x(5-x^2)y'-(7+25x^2)y=0$

38. $x^2(1+x^2)y''+x(5+2x^2)y'-21y=0$

39. $x^2(1+2x^2)y''-x(3+x^2)y'-2x^2y=0$

40. $4x^2(1+x^2)y''+4x(2+x^2)y'-(15+x^2)y=0$

Q7.7.2

41.

a. Under the assumptions of Theorem 7.7.1, show that

$$y_1=x^{r_1}\sum_{n=0}^{\infty}a_n(r_1)x^n$$

and

$$y_2=x^{r_2}\sum_{n=0}^{k-1}a_n(r_2)x^n+C\left(y_1\ln x+x^{r_1}\sum_{n=1}^{\infty}a_n'(r_1)x^n\right)$$

are linearly independent.

b. Use the result of (a) to complete the proof of Theorem 7.7.1.

42. Find a fundamental set of Frobenius solutions of Bessel's equation

$$x^2y''+xy'+(x^2-\nu^2)y=0$$

in the case where ν is a positive integer.

43. Prove Theorem 7.7.2.

44. Under the assumptions of Theorem 7.7.1, show that $C=0$ if and only if $p_1(r_2+)=0$ for some integer ℓ in $\{0,1,\ldots,k-1\}$.

45. Under the assumptions of Theorem 7.7.2, show that $C=0$ if and only if $p_2(r_2+2)=0$ for some integer ℓ in $\{0,1,\ldots,k-1\}$.

46. Let

$$Ly=\alpha_0x^2y''+\beta_0xy'+(\gamma_0+\gamma_1x)y$$

and define

$$p_0(r)=\alpha_0r(r-1)+\beta_0r+\gamma_0.$$

Show that if

$$p_0(r)=\alpha_0(r-r_1)(r-r_2)$$

where $r_1-r_2=k$, a positive integer, then $Ly=0$ has the solutions

$$y_1=x^{r_1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\prod_{j=1}^n(j+k)}\left(\frac{\gamma_1}{\alpha_0}\right)^nx^n$$

and

$$\begin{aligned}y_2&=x^{r_2}\sum_{n=0}^{k-1}\frac{(-1)^n}{n!\prod_{j=1}^n(j-k)}\left(\frac{\gamma_1}{\alpha_0}\right)^nx^n\\&\quad-\frac{1}{k!(k-1)!}\left(\frac{\gamma_1}{\alpha_0}\right)^k\left(y_1\ln x-x^{r_1}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!\prod_{j=1}^n(j+k)}\left(\frac{\gamma_1}{\alpha_0}\right)^n\left(\sum_{j=1}^n\frac{2j+k}{j(j+k)}\right)x^n\right).\end{aligned}$$

47. Let

$$Ly = \alpha_0 x^2 y'' + \beta_0 x y' + (\gamma_0 + \gamma_2 x^2) y$$

and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0.$$

Show that if

$$p_0(r) = \alpha_0 (r - r_1)(r - r_2)$$

where $r_1 - r_2 = 2k$, an even positive integer, then $Ly = 0$ has the solutions

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} \frac{(-1)^m}{4^m m! \prod_{j=1}^m (j+k)} \left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m}$$

and

$$\begin{aligned} y_2 = {} & x^{r_2} \sum_{m=0}^{k-1} \frac{(-1)^m}{4^m m! \prod_{j=1}^m (j-k)} \left(\frac{\gamma_2}{\alpha_0}\right)^m x^{2m} \\ & - \frac{2}{4^k k!(k-1)!} \left(\frac{\gamma_2}{\alpha_0}\right)^k \left(y_1 \ln x - \frac{x^{r_1}}{2} \sum_{m=1}^{\infty} \frac{(-1)^m}{4^m m! \prod_{j=1}^m (j+k)} \left(\frac{\gamma_2}{\alpha_0}\right)^m \left(\sum_{j=1}^m \frac{2j+k}{j(j+k)} \right) x^{2m} \right). \end{aligned}$$

48. Let L be as in **Exercises 7.5.57** and **7.5.58**, and suppose the indicial polynomial of $Ly = 0$ is

$$p_0(r) = \alpha_0 (r - r_1)(r - r_2),$$

with $k = r_1 - r_2$, where k is a positive integer. Define $a_0(r) = 1$ for all r. If r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n, define

$$a_n(r) = -\frac{1}{p_0(n+r)} \sum_{j=1}^n p_j(n+r-j) a_{n-j}(r),\ n \geq 1,$$

and let

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n.$$

Define

$$a_n(r_2) = -\frac{1}{p_0(n+r_2)} \sum_{j=1}^n p_j(n+r_2-j) a_{n-j}(r_2) \text{ if} n \geq 1 \text{ and } n \neq k,$$

and let $a_k(r_2)$ be arbitrary.

a. Conclude from *Exercise 7.6.66* that

$$L\left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n \right) = k \alpha_0 x^{r_1}.$$

b. Conclude from *Exercise 7.5.57* that

$$L\left(x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n \right) = A x^{r_1},$$

where

$$A = \sum_{j=1}^k p_j(r_1 - j) a_{k-j}(r_2).$$

c. Show that y_1 and

$$y_2 = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n - \frac{A}{k\alpha_0} \left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1) x^n \right)$$

form a fundamental set of Frobenius solutions of $Ly = 0$.

d. Show that choosing the arbitrary quantity $a_k(r_2)$ to be nonzero merely adds a multiple of y_1 to y_2. Conclude that we may as well take $a_k(r_2) = 0$.

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CHAPTER OVERVIEW

Chapter 7: Laplace Transforms

IN THIS CHAPTER we study the method of Laplace transforms, which illustrates one of the basic problem solving techniques in mathematics: transform a difficult problem into an easier one, solve the latter, and then use its solution to obtain a solution of the original problem. The method discussed here transforms an initial value problem for a constant coefficient equation into an algebraic equation whose solution can then be used to solve the initial value problem. In some cases this method is merely an alternative procedure for solvingproblems that can be solved equally well by methods that we considered previously; however, in other cases the method of Laplace transforms is more efficient than the methods previously discussed. This is especially true in physical problems dealing with discontinuous forcing functions.

- 7.1: Introduction to the Laplace Transform
  - 7.1E: Introduction to the Laplace Transform (Exercises)
- 7.2: The Inverse Laplace Transform
  - 7.2E: The Inverse Laplace Transform (Exercises)
- 7.3: Solution of Initial Value Problems
  - 7.3E: Solution of Initial Value Problems (Exercises)
- 7.4: The Unit Step Function
  - 7.4E: The Unit Step Function (Exercises)
- 7.5: Constant Coefficient Equations with Piecewise Continuous Forcing Functions
  - 7.5E: Constant Coefficient Equations with Piecewise Continuous Forcing Functions (Exercises)
- 7.6: Convolution
  - 7.6E: Convolution (Exercises)
- 7.7: A Brief Table of Laplace Transforms

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7.1: Introduction to the Laplace Transform

Definition of the Laplace Transform

To define the Laplace transform, we first recall the definition of an improper integral. If g is integrable over the interval $[a, T]$ for every $T > a$, then the **improper integral of g over** $[a, \infty)$ is defined as

$$\int_a^\infty g(t)\,dt = \lim_{T\to\infty}\int_a^T g(t)\,dt. \tag{7.1.1}$$

We say that the improper integral **converges** if the limit in Equation 7.1.1 exists; otherwise, we say that the improper integral **diverges** or **does not exist**. Here's the definition of the Laplace transform of a function f.

Defintion 8.1.1 : Laplace Transform

Let f be defined for $t \geq 0$ and let s be a real number. Then the **Laplace transform** of f is the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \tag{7.1.2}$$

for those values of s for which the improper integral converges.

It is important to keep in mind that the variable of integration in Equation 7.1.2 is t, while s is a parameter independent of t. We use t as the independent variable for f because in applications the Laplace transform is usually applied to functions of time.

The Laplace transform can be viewed as an operator $\mathcal{L}$ that transforms the function $f = f(t)$ into the function $F = F(s)$. Thus, Equation 7.1.2 can be expressed as

$$F = \mathcal{L}(f).$$

The functions f and F form a **transform pair**, which we'll sometimes denote by

$$f(t) \leftrightarrow F(s).$$

It can be shown that if $F(s)$ is defined for $s = s_0$ then it is defined for all $s > s_0$.

Computation of Some Simple Laplace Transforms

Example 8.1.1

Find the Laplace transform of $f(t) = 1$.

Solution

From Equation 7.1.2 with $f(t) = 1$,

$$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{T\to\infty}\int_0^T e^{-st}\,dt.$$

If $s \neq 0$ then

$$\int_0^T e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^T = \frac{1-e^{-sT}}{s}. \tag{7.1.3}$$

Therefore

$$\lim_{T\to\infty}\int_0^T e^{-st}dt = \begin{cases} \frac{1}{s}, & s > 0, \\ \infty, & s < 0. \end{cases} \tag{7.1.4}$$

If $s = 0$ the integrand reduces to the constant 1, and

$$\lim_{T\to\infty}\int_0^T 1\,dt = \lim_{T\to\infty}\int_0^T 1\,dt = \lim_{T\to\infty} T = \infty.$$

Therefore $F(0)$ is undefined, and

$$F(s) = \int_0^\infty e^{-st}\,dt = \frac{1}{s}, \quad s > 0.$$

This result can be written in operator notation as

$$\mathcal{L}(1) = \frac{1}{s}, \quad s > 0,$$

or as the transform pair

$$1 \leftrightarrow \frac{1}{s}, \quad s > 0.$$

Note

It is convenient to combine the steps of integrating from 0 to T and letting $T \to \infty$. Therefore, instead of writing Equation 7.1.3 and 7.1.4 as separate steps we write

$$\int_0^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \begin{cases} \frac{1}{s}, & s > 0 \\ \infty, & s < 0 \end{cases}$$

We'll follow this practice throughout this chapter.

Example 8.1.2

Find the Laplace transform of $f(t) = t$.

Solution

From Equation 7.1.2 with $f(t) = t$,

$$F(s) = \int_0^\infty e^{-st}\,t\,dt. \tag{7.1.5}$$

If $s \neq 0$, integrating by parts yields

$$\begin{aligned}\int_0^\infty e^{-st}\,t\,dt &= -\frac{te^{-st}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = -\left[\frac{t}{s} + \frac{1}{s^2}\right]e^{-st}\Big|_0^\infty \\ &= \begin{cases} \frac{1}{s^2}, & s > 0, \\ \infty, & s < 0. \end{cases}\end{aligned}$$

If $s = 0$, the integral in Equation 7.1.5 becomes

$$\int_0^\infty t\,dt = \frac{t^2}{2}\Big|_0^\infty = \infty.$$

Therefore $F(0)$ is undefined and

$$F(s) = \frac{1}{s^2}, \quad s > 0.$$

This result can also be written as

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad s > 0,$$

or as the transform pair

$$t \leftrightarrow \frac{1}{s^2}, \quad s > 0.$$

Example 8.1.3

Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution

From Equation 7.1.2 with $f(t) = e^{at}$,

$$F(s) = \int_0^\infty e^{-st} e^{at}\, dt.$$

Combining the exponentials yields

$$F(s) = \int_0^\infty e^{-(s-a)t}\, dt.$$

However, we know from Example 8.1.1 that

$$\int_0^\infty e^{-st}\, dt = \frac{1}{s}, \quad s > 0.$$

Replacing s by $s - a$ here shows that

$$F(s) = \frac{1}{s-a}, \quad s > a.$$

This can also be written as

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad s > a, \text{ or } e^{at} \leftrightarrow \frac{1}{s-a}, \quad s > a.$$

Example 8.1.4

[Find the Laplace transforms of $f(t) = \sin \omega t$ and $g(t) = \cos \omega t$, where ω is a constant.

Solution

Define

$$F(s) = \int_0^\infty e^{-st} \sin \omega t\, dt \tag{7.1.6}$$

and

$$G(s) = \int_0^\infty e^{-st} \cos \omega t\, dt. \tag{7.1.7}$$

If $s > 0$, integrating Equation 7.1.6 by parts yields

$$F(s) = -\frac{e^{-st}}{s} \sin \omega t \Big|_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t\, dt,$$

so

$$F(s) = \frac{\omega}{s} G(s). \tag{7.1.8}$$

If $s > 0$, integrating Equation 7.1.7 by parts yields

$$G(s) = -\frac{e^{-st} \cos \omega t}{s} \Big|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t\, dt,$$

so

$$G(s)=\frac{1}{s}-\frac{\omega}{s}F(s).$$

Now substitute from Equation 7.1.8 into this to obtain

$$G(s)=\frac{1}{s}-\frac{\omega^2}{s^2}G(s).$$

Solving this for $G(s)$ yields

$$G(s)=\frac{s}{s^2+\omega^2},\quad s>0.$$

This and Equation 7.1.8 imply that

$$F(s)=\frac{\omega}{s^2+\omega^2},\quad s>0.$$

Tables of Laplace Transforms

Extensive tables of Laplace transforms have been compiled and are commonly used in applications. The brief table of Laplace transforms in the Appendix will be adequate for our purposes.

Example 8.1.5

Use the table of Laplace transforms to find $\mathcal{L}(t^3e^{4t})$.

Solution

The table includes the transform pair

$$t^ne^{at}\leftrightarrow\frac{n!}{(s-a)^{n+1}}.$$

Setting $n=3$ and $a=4$ here yields

$$\mathcal{L}(t^3e^{4t})=\frac{3!}{(s-4)^4}=\frac{6}{(s-4)^4}.$$

We'll sometimes write Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms.

Linearity of the Laplace Transform

The next theorem presents an important property of the Laplace transform.

Theorem 8.1.2 Linearity Property

Suppose $\mathcal{L}(f_i)$ is defined for $s>s_i,\ 1\le i\le n)$. Let s_0 be the largest of the numbers s_1, s_2, …,s_n, and let c_1, c_2,…, c_n be constants. Then

$$\mathcal{L}(c_1f_1+c_2f_2+\cdots+c_nf_n)=c_1\mathcal{L}(f_1)+c_2\mathcal{L}(f_2)+\cdots+c_n\mathcal{L}(f_n)\text{ for }s>s_0.$$

Proof

We give the proof for the case where $n=2$. If $s>s_0$ then

$$\begin{aligned}\mathcal{L}(c_1f_1+c_2f_2)&=\int_0^\infty e^{-st}\left(c_1f_1(t)+c_2f_2(t)\right)\,dt\\&=c_1\int_0^\infty e^{-st}f_1(t)\,dt+c_2\int_0^\infty e^{-st}f_2(t)\,dt\\&=c_1\mathcal{L}(f_1)+c_2\mathcal{L}(f_2).\end{aligned}$$

Example 8.1.6

Use Theorem 8.1.2 and the known Laplace transform

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

to find $\mathcal{L}(\cosh bt)\ (b \neq 0)$.

Solution

By definition,

$$\cosh bt = \frac{e^{bt} + e^{-bt}}{2}.$$

Therefore

$$\begin{aligned} \mathcal{L}(\cosh bt) &= \mathcal{L}\left(\tfrac{1}{2}e^{bt} + \tfrac{1}{2}e^{-bt}\right) \\ &= \tfrac{1}{2}\mathcal{L}(e^{bt}) + \tfrac{1}{2}\mathcal{L}(e^{-bt}) \qquad \text{(linearity property)} \\ &= \tfrac{1}{2}\,\tfrac{1}{s-b} + \tfrac{1}{2}\,\tfrac{1}{s+b}, \end{aligned} \tag{7.1.9}$$

where the first transform on the right is defined for $s > b$ and the second for $s > -b$; hence, both are defined for $s > |b|$. Simplifying the last expression in Equation 7.1.9 yields

$$\mathcal{L}(\cosh bt) = \frac{s}{s^2 - b^2}, \quad s > |b|.$$

The next theorem enables us to start with known transform pairs and derive others. (For other results of this kind, see **Exercises 8.1.6** and **8.1.13**.)

Theorem 8.1.3 First Shifting Theorem

If

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \tag{7.1.10}$$

is the Laplace transform of $f(t)$ for $s > s_0$, then $F(s-a)$ is the Laplace transform of $e^{at} f(t)$ for $s > s_0 + a$.

Proof

Replacing s by $s-a$ in Equation 7.1.10 yields

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt \tag{7.1.11}$$

if $s - a > s_0$; that is, if $s > s_0 + a$. However, Equation 7.1.11 can be rewritten as

$$F(s-a) = \int_0^\infty e^{-st}\left(e^{at} f(t)\right)\, dt,$$

which implies the conclusion.

Example 8.1.7

Use Theorem 8.1.3 and the known Laplace transforms of 1, t, $\cos \omega t$, and $\sin \omega t$ to find

$$\mathcal{L}(e^{at}), \quad \mathcal{L}(te^{at}), \quad \mathcal{L}(e^{\lambda t}\sin \omega t), \text{and } \mathcal{L}(e^{\lambda t}\cos \omega t).$$

Solution

In the following table the known transform pairs are listed on the left and the required transform pairs listed on the right are obtained by applying Theorem 8.1.3 .

Table 8.1.1

$f(t) \leftrightarrow F(s)$	$e^{at} f(t) \leftrightarrow F(s-a)$
$1 \leftrightarrow \frac{1}{s}, \quad s>0$	$e^{at} \leftrightarrow \frac{1}{(s-a)}, \quad s>a$
$t \leftrightarrow \frac{1}{s^2}, \quad s>0$	$te^{at} \leftrightarrow \frac{1}{(s-a)^2}, \quad s>a$
$\sin \omega t \leftrightarrow \frac{\omega}{s^2+\omega^2}, \quad s>0$	$e^{\lambda t} \sin \omega t \leftrightarrow \frac{\omega}{(s-\lambda)^2+\omega^2}, \quad s>\lambda$
$\cos \omega t \leftrightarrow \frac{s}{s^2+\omega^2}, \quad s>0$	$e^{\lambda t} \sin \omega t \leftrightarrow \frac{s-\lambda}{(s-\lambda)^2+\omega^2}, \quad s>\lambda$

Existence of Laplace Transforms

Not every function has a Laplace transform. For example, it can be shown (**Exercise 8.1.3**) that

$$\int_0^\infty e^{-st} e^{t^2} dt = \infty$$

for every real number s. Hence, the function $f(t) = e^{t^2}$ does not have a Laplace transform.

Our next objective is to establish conditions that ensure the existence of the Laplace transform of a function. We first review some relevant definitions from calculus.

Recall that a limit

$$\lim_{t \to t_0} f(t)$$

exists if and only if the one-sided limits

$$\lim_{t \to t_0-} f(t) \quad \text{and} \quad \lim_{t \to t_0+} f(t)$$

both exist and are equal; in this case,

$$\lim_{t \to t_0} f(t) = \lim_{t \to t_0-} f(t) = \lim_{t \to t_0+} f(t).$$

Recall also that f is continuous at a point t_0 in an open interval (a, b) if and only if

$$\lim_{t \to t_0} f(t) = f(t_0),$$

which is equivalent to

$$\lim_{t \to t_0+} f(t) = \lim_{t \to t_0-} f(t) = f(t_0). \tag{7.1.12}$$

For simplicity, we define

$$f(t_0+) = \lim_{t \to t_0+} f(t) \quad \text{and} \quad f(t_0-) = \lim_{t \to t_0-} f(t),$$

so Equation 7.1.12 can be expressed as

$$f(t_0+) = f(t_0-) = f(t_0).$$

If $f(t_0+)$ and $f(t_0-)$ have finite but distinct values, we say that f has a **jump discontinuity** at t_0, and

$$f(t_0+) - f(t_0-)$$

is called the **jump** in f at t_0 (Figure 8.1.1).

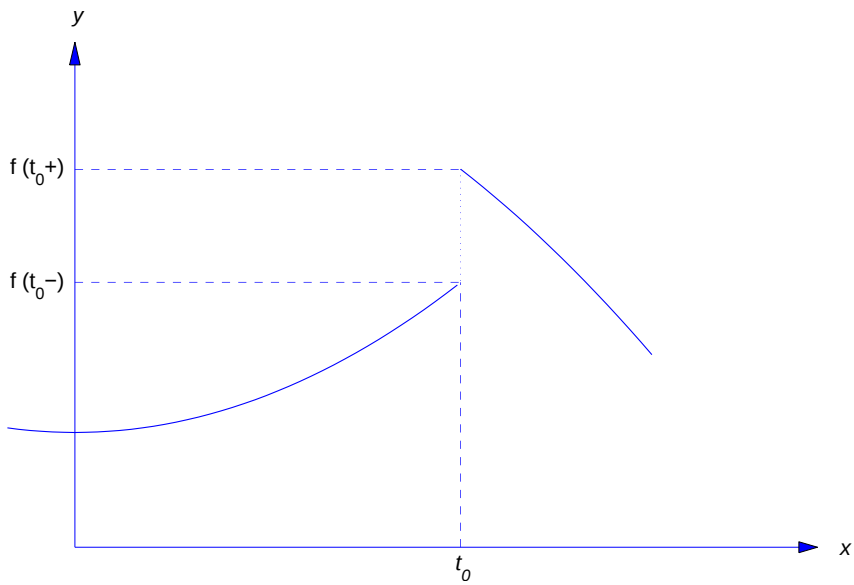

Figure 8.1.1 : A jump discontinuity.

If $f(t_0+)$ and $f(t_0-)$ are finite and equal, but either f isn't defined at t_0 or it is defined but

$$f(t_0) \neq f(t_0+) = f(t_0-),$$

we say that f has a **removable discontinuity** at t_0 (Figure 8.1.2). This terminology is appropriate since a function f with a removable discontinuity at t_0 can be made continuous at t_0 by defining (or redefining)

$$f(t_0) = f(t_0+) = f(t_0-).$$

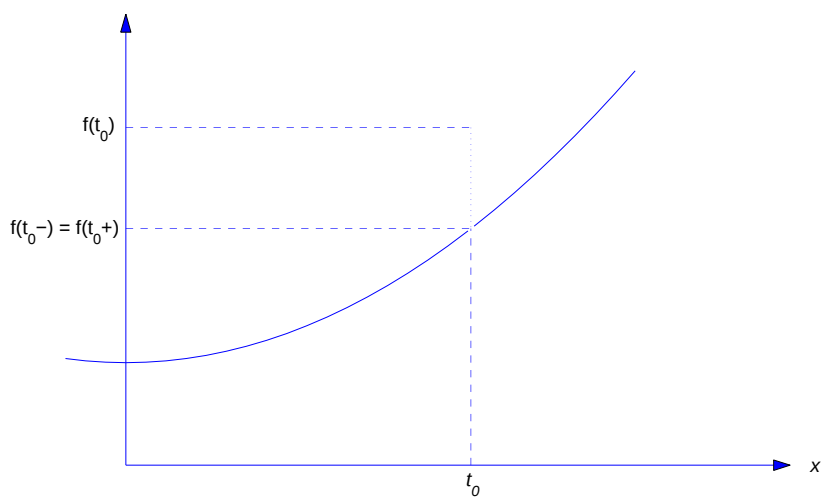

Figure 8.1.2

Note

We know from calculus that a definite integral is not affected by changing the values of its integrand at isolated points. Therefore, redefining a function f to make it continuous at removable discontinuities does not change $\mathcal{L}(f)$.

Definition 8.1.4 : Piecewise Continuous

- A function f is said to be **piecewise continuous** on a finite closed interval $[0, T]$ if $f(0+)$ and $f(T-)$ are finite and f is continuous on the open interval $(0, T)$ except possibly at finitely many points, where f may have jump discontinuities or removable discontinuities.
- A function f is said to be **piecewise continuous** on the infinite interval $[0, \infty)$ if it is piecewise continuous on $[0, T]$ for every $T > 0$.

Figure 8.1.3 shows the graph of a typical piecewise continuous function.

It is shown in calculus that if a function is piecewise continuous on a finite closed interval then it is integrable on that interval. But if f is piecewise continuous on $[0, \infty)$, then so is $e^{-st} f(t)$, and therefore

$$\int_0^T e^{-st} f(t)\, dt$$

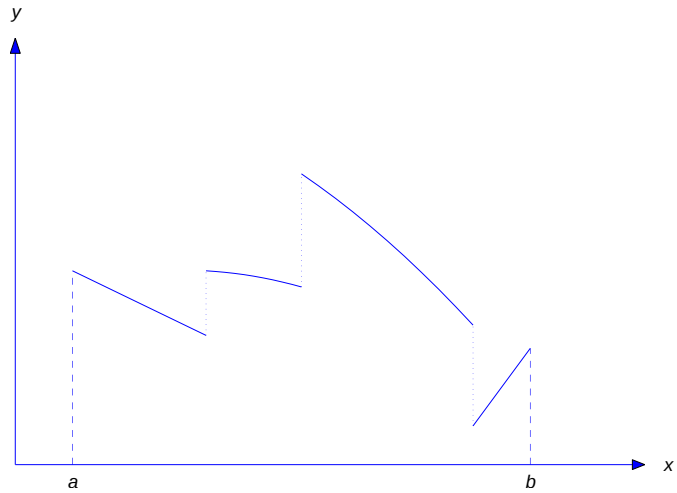

Figure 8.1.3

exists for every $T > 0$. However, piecewise continuity alone does not guarantee that the improper integral

$$\int_0^\infty e^{-st} f(t)\, dt = \lim_{T\to\infty} \int_0^T e^{-st} f(t)\, dt \tag{7.1.13}$$

converges for s in some interval (s_0, ∞). For example, we noted earlier that Equation 7.1.13 diverges for all s if $f(t) = e^{t^2}$. Stated informally, this occurs because e^{t^2} increases too rapidly as $t \to \infty$. The next definition provides a constraint on the growth of a function that guarantees convergence of its Laplace transform for s in some interval (s_0, ∞).

Definition 8.1.5 : of exponential order

A function f is said to be **of exponential order** s_0 if there are constants M and t_0 such that

$$|f(t)| \le M e^{s_0 t}, \quad t \ge t_0. \tag{7.1.14}$$

In situations where the specific value of s_0 is irrelevant we say simply that f is **of exponential order**.

The next theorem gives useful sufficient conditions for a function f to have a Laplace transform. The proof is sketched in **Exercise 8.1.10**.

Theorem 8.1.6

If f is piecewise continuous on $[0, \infty)$ and of exponential order s_0, then $\mathcal{L}(f)$ is defined for $s > s_0$.

Note

We emphasize that the conditions of Theorem 8.1.6 are sufficient, but **not necessary**, for f to have a Laplace transform. For example, **Exercise 8.1.14(c)** shows that f may have a Laplace transform even though f isn't of exponential order

Example 8.1.8

If f is bounded on some interval $[t_0, \infty)$, say

$$|f(t)| \le M, \quad t \ge t_0,$$

then Equation 7.1.14 holds with $s_0 = 0$, so f is of exponential order zero. Thus, for example, $\sin \omega t$ and $\cos \omega t$ are of exponential order zero, and Theorem 8.1.6 implies that $\mathcal{L}(\sin \omega t)$ and $\mathcal{L}(\cos \omega t)$ exist for $s > 0$. This is consistent with the conclusion of Example 8.1.4 .

Example 8.1.9

It can be shown that if $\lim_{t\to\infty} e^{-s_0 t} f(t)$ exists and is finite then f is of exponential order s_0 (**Exercise 8.1.9**). If α is any real number and $s_0 > 0$ then $f(t) = t^\alpha$ is of exponential order s_0, since

$$\lim_{t\to\infty} e^{-s_0 t} t^\alpha = 0,$$

by L'Hôpital's rule. If $\alpha \ge 0$, f is also continuous on $[0, \infty)$. Therefore **Exercise 8.1.9** and Theorem 8.1.6 imply that $\mathcal{L}(t^\alpha)$ exists for $s \ge s_0$. However, since s_0 is an arbitrary positive number, this really implies that $\mathcal{L}(t^\alpha)$ exists for all $s > 0$. This is consistent with the results of Example 8.1.2 and **Exercises 8.1.6** and **8.1.8**.

Example 8.1.10

Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -3e^{-t}, & t \ge 1. \end{cases}$$

Solution

Since f is defined by different formulas on $[0, 1)$ and $[1, \infty)$, we write

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}(1)\, dt + \int_1^\infty e^{-st}(-3e^{-t})\, dt.$$

Since

$$\int_0^1 e^{-st}\, dt = \begin{cases} \frac{1-e^{-s}}{s} & s \ne 0 \\ 1 & s = 0 \end{cases}$$

and

$$\int_1^\infty e^{-st}(-3e^{-t})\, dt = -3\int_1^\infty e^{-(s+1)t}\, dt = -\frac{3e^{-(s+1)}}{s+1}, \quad s > -1,$$

it follows that

$$F(s) = \begin{cases} \frac{1-e^{-s}}{s} - 3\frac{e^{-(s+1)}}{s+1} & s > -1, s \ne 0 \\ 1 - \frac{3}{e} & s = 0 \end{cases}$$

This is consistent with Theorem 8.1.6 , since

$$|f(t)| \le 3e^{-t}, \quad t \ge 1,$$

and therefore f is of exponential order $s_0 = -1$.

Note

In Section 8.4 we'll develop a more efficient method for finding Laplace transforms of piecewise continuous functions.

Example 8.1.11

We stated earlier that

$$\int_0^\infty e^{-st} e^{t^2}\, dt = \infty$$

for all s, so Theorem 8.1.6 implies that $f(t) = e^{t^2}$ is not of exponential order, since

$$\lim_{t\to\infty} \frac{e^{t^2}}{Me^{s_0 t}} = \lim_{t\to\infty} \frac{1}{M} e^{t^2 - s_0 t} = \infty,$$

so

$$e^{t^2} > Me^{s_0 t}$$

for sufficiently large values of t, for any choice of M and s_0 (**Exercise 8.1.3**).

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7.1E: Introduction to the Laplace Transform (Exercises)

Q7.1.1

1. Find the Laplace transforms of the following functions by evaluating the integral $F(s)=\int_0^\infty e^{-st}f(t)\,dt$.

a. t
b. te^{-t}
c. $\sinh bt$
d. $e^{2t}-3e^t$
e. t^2
f. t^2e^{-3t}
g. $e^{2t}(t+1)^2$
h. $e^{-t}\sin 4t$

2. Find the Laplace transforms of the following functions.

a. $\cosh t\sin t$
b. $\sin^2 t$
c. $\cos^2 2t$
d. $\cosh^2 t$
e. $t\sinh 2t$
f. $\sin t\cos t$
g. $\sin\left(t+\frac{\pi}{4}\right)$
h. $\cos 2t-\cos 3t$
i. $\sin 2t+\cos 4t$

3. Show that

$$\int_0^\infty e^{-st}e^{t^2}\,dt=\infty$$

for every real number s.

4. Graph the following piecewise continuous functions and evaluate $f(t+)$, $f(t-)$, and $f(t)$ at each point of discontinuity.

a. $f(t)=\begin{cases}-t, & 0\le t<2,\\ t-4, & 2\le t<3,\\ 1, & t\ge 3.\end{cases}$

b. $f(t)=\begin{cases}t^2+2, & 0\le t<1,\\ 4, & t=1,\\ t, & t>1.\end{cases}$

c. $f(t)=\begin{cases}\sin t, & 0\le t<\pi/2,\\ 2\sin t, & \pi/2\le t<\pi,\\ \cos t, & t\ge\pi.\end{cases}$

d. $f(t)=\begin{cases}t, & 0\le t<1,\\ 2, & t=1,\\ 2-t, & 1<t<2,\\ 3, & t=2,\\ 6, & t>2.\end{cases}$

5. Find the Laplace transform:

a. $f(t)=\begin{cases}e^{-t}, & 0\le t<1,\\ e^{-2t}, & t\ge 1.\end{cases}$

b. $f(t)=\begin{cases}1, & 0\le t<4,\\ t, & t\ge 4.\end{cases}$

c. $f(t)=\begin{cases} t, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$

d. $f(t)=\begin{cases} te^t, & 0 \le t < 1, \\ e^t, & t \ge 1. \end{cases}$

6. Recall the next theorem from calculus.

Theorem $7.1E.1$

Let g be integrable on $[0,T]$ for every $T>0$. Suppose there's a function w defined on some interval $[\tau,\infty)$ (with $\tau \ge 0$) such that $|g(t)| \le w(t)$ for $t \ge \tau$ and $\int_\tau^\infty w(t)\,dt$ converges. Then $\int_0^\infty g(t)\,dt$ converges.

Use Theorem $7.1E.1$ to show that if f is piecewise continuous on $[0,\infty)$ and of exponential order s_0, then f has a Laplace transform $F(s)$ defined for $s > s_0$.

7. Prove: If f is piecewise continuous and of exponential order then $\lim_{s\to\infty} F(s) = 0$.

8. Prove: If f is continuous on $[0,\infty)$ and of exponential order $s_0>0$, then

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}\mathcal{L}(f), \quad s > s_0.$$

HINT: *Use integration by parts to evaluate the transform on the left.*

9. Suppose f is piecewise continuous and of exponential order, and that $\lim_{t\to 0+} f(t)/t$ exists. Show that

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(r)\,dr.$$

HINT: *Use the results of Exercises 8.1.6 and 8.1.11.*

10. Use the result of *Exercise 7.1.9* to find the Laplace transforms of the following functions.

a. $\frac{\sin \omega t}{t} \quad (\omega > 0)$

b. $\frac{\cos \omega t - 1}{t} \quad (\omega > 0)$

c. $\frac{e^{at}-e^{bt}}{t}$

d. $\frac{\cosh t - 1}{t}$

e. $\frac{\sinh^2 t}{t}$

11. The *gamma function* is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx,$$

which can be shown to converge if $\alpha > 0$.

a. Use integration by parts to show that

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \quad \alpha > 0.$$

b. Show that $\Gamma(n+1) = n!$ if $n = 1, 2, 3, \ldots$.

c. From (b) and the table of Laplace transforms,

$$\mathcal{L}(t^\alpha) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}, \quad s > 0,$$

if α is a nonnegative integer. Show that this formula is valid for any $\alpha > -1$. HINT: *Change the variable of integration in the integral for* $\Gamma(\alpha+1)$.

12. Suppose f is continuous on $[0,T]$ and $f(t+T) = f(t)$ for all $t \ge 0$. (We say in this case that f is *periodic with period* T.)

a. Conclude from Theorem 7.1.6 that the Laplace transform of f is defined for $s > 0$.

b. Show that

$$F(s)=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt,\quad s>0.$$

HINT: *Write*

$$F(s)=\sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st}f(t)dt$$

Then show that

$$\int_{nT}^{(n+1)T} e^{-st}f(t)dt=e^{-nsT}\int_0^T e^{-st}f(t)dt$$

and recall the formula for the sum of a geometric series.

13. Use the formula given in *Exercise 7.1.12b* to find the Laplace transforms of the given periodic functions. Graph each function over one period.

a. $f(t)=\begin{cases} t, & 0\le t<1,\\ 2-t, & 1\le t<2,\end{cases}\qquad f(t+2)=f(t),\quad t\ge 0$

b. $f(t)=\begin{cases} 1, & 0\le t<\frac{1}{2},\\ -1, & \frac{1}{2}\le t<1,\end{cases}\qquad f(t+1)=f(t),\quad t\ge 0$

c. $f(t)=|\sin t|$

d. $f(t)=\begin{cases} \sin t, & 0\le t<\pi,\\ 0, & \pi\le t<2\pi,\end{cases}\qquad f(t+2\pi)=f(t)$

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7.2: The Inverse Laplace Transform

Definition of the Inverse Laplace Transform

In Section 8.1 we defined the Laplace transform of f by

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt.$$

We'll also say that f is an **inverse Laplace Transform** of F, and write

$$f = \mathcal{L}^{-1}(F).$$

To solve differential equations with the Laplace transform, we must be able to obtain f from its transform F. There's a formula for doing this, but we can't use it because it requires the theory of functions of a complex variable. Fortunately, we can use the table of Laplace transforms to find inverse transforms that we'll need.

Example 8.2.1

Use the table of Laplace transforms to find

a. $$\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right)$$

b. $$\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right).$$

Solution a

Setting $b = 1$ in the transform pair

$$\sinh bt \leftrightarrow \frac{b}{s^2-b^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) = \sinh t.$$

Solution b

Setting $\omega = 3$ in the transform pair

$$\cos \omega t \leftrightarrow \frac{s}{s^2+\omega^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t.$$

The next theorem enables us to find inverse transforms of linear combinations of transforms in the table. We omit the proof.

Theorem 8.2.1 : Linearity Property

If $F_1, F_2, \ldots, F_n$ are Laplace transforms and $c_1, c_2, \ldots, c_n$ are constants, then

$$\mathcal{L}^{-1}(c_1F_1 + c_2F_2 + \cdots + c_nF_n) = c_1\mathcal{L}^{-1}(F_1) + c_2\mathcal{L}^{-1}(F_2) + \cdots + c_n\mathcal{L}^{-1}F_n.$$

Example 8.2.2

Find

$$\mathcal{L}^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right).$$

Solution

From the table of Laplace transforms in Section 8.8,,

$$e^{at} \leftrightarrow \frac{1}{s-a} \quad \text{and} \quad \sin \omega t \leftrightarrow \frac{\omega}{s^2+\omega^2}.$$

Theorem 8.2.1 with $a=-5$ and $\omega=\sqrt{3}$ yields

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right) &= 8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)+7\mathcal{L}^{-1}\left(\frac{1}{s^2+3}\right)\\ &= 8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)+\frac{7}{\sqrt{3}}\mathcal{L}^{-1}\left(\frac{\sqrt{3}}{s^2+3}\right)\\ &= 8e^{-5t}+\frac{7}{\sqrt{3}}\sin\sqrt{3}t.\end{aligned}$$

Example 8.2.3

Find

$$\mathcal{L}^{-1}\left(\frac{3s+8}{s^2+2s+5}\right).$$

Solution

Completing the square in the denominator yields

$$\frac{3s+8}{s^2+2s+5}=\frac{3s+8}{(s+1)^2+4}.$$

Because of the form of the denominator, we consider the transform pairs

$$e^{-t}\cos 2t \leftrightarrow \frac{s+1}{(s+1)^2+4} \quad \text{and} \quad e^{-t}\sin 2t \leftrightarrow \frac{2}{(s+1)^2+4},$$

and write

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{3s+8}{(s+1)^2+4}\right) &= \mathcal{L}^{-1}\left(\frac{3s+3}{(s+1)^2+4}\right)+\mathcal{L}^{-1}\left(\frac{5}{(s+1)^2+4}\right)\\ &= 3\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right)+\frac{5}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right)\\ &= e^{-t}(3\cos 2t+\frac{5}{2}\sin 2t).\end{aligned}$$

Note

We'll often write inverse Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms in Section 8.8.

Inverse Laplace Transforms of Rational Functions

Using the Laplace transform to solve differential equations often requires finding the inverse transform of a rational function

$$F(s)=\frac{P(s)}{Q(s)},$$

where P and Q are polynomials in s with no common factors. Since it can be shown that $\lim_{s\to\infty} F(s)=0$ if F is a Laplace transform, we need only consider the case where $\text{degree}(P)<\text{degree}(Q)$. To obtain $\mathcal{L}^{-1}(F)$, we find the partial fraction expansion of F, obtain inverse transforms of the individual terms in the expansion from the table of Laplace transforms, and use the linearity property of the inverse transform. The next two examples illustrate this.

Example 8.2.4

Find the inverse Laplace transform of

$$F(s)=\frac{3s+2}{s^2-3s+2}. \tag{7.2.1}$$

Solution

(Method 1)

Factoring the denominator in Equation 7.2.1 yields

$$F(s)=\frac{3s+2}{(s-1)(s-2)}. \tag{7.2.2}$$

The form for the partial fraction expansion is

$$\frac{3s+2}{(s-1)(s-2)}=\frac{A}{s-1}+\frac{B}{s-2}. \tag{7.2.3}$$

Multiplying this by $(s-1)(s-2)$ yields

$$3s+2=(s-2)A+(s-1)B.$$

Setting $s=2$ yields $B=8$ and setting $s=1$ yields $A=-5$. Therefore

$$F(s)=-\frac{5}{s-1}+\frac{8}{s-2}$$

and

$$\mathcal{L}^{-1}(F)=-5\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)+8\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)=-5e^t+8e^{2t}.$$

(Method 2) We don't really have to multiply Equation 7.2.3 by $(s-1)(s-2)$ to compute A and B. We can obtain A by simply ignoring the factor $s-1$ in the denominator of Equation 7.2.2 and setting $s=1$ elsewhere; thus,

$$A=\left.\frac{3s+2}{s-2}\right|_{s=1}=\frac{3\cdot 1+2}{1-2}=-5. \tag{7.2.4}$$

Similarly, we can obtain B by ignoring the factor $s-2$ in the denominator of Equation 7.2.2 and setting $s=2$ elsewhere; thus,

$$B=\left.\frac{3s+2}{s-1}\right|_{s=2}=\frac{3\cdot 2+2}{2-1}=8. \tag{7.2.5}$$

To justify this, we observe that multiplying Equation 7.2.3 by $s-1$ yields

$$\frac{3s+2}{s-2}=A+(s-1)\frac{B}{s-2},$$

and setting $s=1$ leads to Equation 7.2.4. Similarly, multiplying Equation 7.2.3 by $s-2$ yields

$$\frac{3s+2}{s-1}=(s-2)\frac{A}{s-2}+B$$

and setting $s=2$ leads to Equation 7.2.5. (It isn't necesary to write the last two equations. We wrote them only to justify the shortcut procedure indicated in Equation 7.2.4 and Equation 7.2.5.)

The shortcut employed in the second solution of Example 8.2.4 is **Heaviside's method**. The next theorem states this method formally. For a proof and an extension of this theorem, see **Exercise 8.2.10**.

Theorem 8.2.2

Suppose

$$F(s)=\frac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)}, \tag{7.2.6}$$

where $s_1, s_2, \ldots, s_n$ are distinct and P is a polynomial of degree less than n. Then

$$F(s)=\frac{A_1}{s-s_1}+\frac{A_2}{s-s_2}+\cdots+\frac{A_n}{s-s_n},$$

where A_i can be computed from Equation 7.2.6 by ignoring the factor $s-s_i$ and setting $s=s_i$ elsewhere.

Example 8.2.5

Find the inverse Laplace transform of

$$F(s)=\frac{6+(s+1)(s^2-5s+11)}{s(s-1)(s-2)(s+1)}. \tag{7.2.7}$$

Solution

The partial fraction expansion of Equation 7.2.7 is of the form

$$F(s)=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{s-2}+\frac{D}{s+1}. \tag{7.2.8}$$

To find A, we ignore the factor s in the denominator of Equation 7.2.7 and set $s=0$ elsewhere. This yields

$$A=\frac{6+(1)(11)}{(-1)(-2)(1)}=\frac{17}{2}.$$

Similarly, the other coefficients are given by

$$B=\frac{6+(2)(7)}{(1)(-1)(2)}=-10,$$

$$C=\frac{6+3(5)}{2(1)(3)}=\frac{7}{2},$$

and

$$D=\frac{6}{(-1)(-2)(-3)}=-1.$$

Therefore

$$F(s)=\frac{17}{2}\frac{1}{s}-\frac{10}{s-1}+\frac{7}{2}\frac{1}{s-2}-\frac{1}{s+1}$$

and

$$\begin{aligned}\mathcal{L}^{-1}(F) &= \frac{17}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right)-10\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)+\frac{7}{2}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right)-\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)\\ &= \frac{17}{2}-10e^t+\frac{7}{2}e^{2t}-e^{-t}.\end{aligned}$$

Note

We didn't "multiply out" the numerator in Equation 8.2.7 before computing the coefficients in Equation 8.2.8 , since it wouldn't simplify the computations.

Example 8.2.6

Find the inverse Laplace transform of

$$F(s)=\frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2}. \tag{7.2.9}$$

Solution

The form for the partial fraction expansion is

$$F(s)=\frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{(s+2)^2}. \tag{7.2.10}$$

Because of the repeated factor $(s+2)^2$ in Equation 7.2.9, Heaviside's method doesn't work. Instead, we find a common denominator in Equation 7.2.10. This yields

$$F(s)=\frac{A(s+2)^2+B(s+1)(s+2)+C(s+1)}{(s+1)(s+2)^2}. \tag{7.2.11}$$

If Equation 7.2.9 and Equation 7.2.11 are to be equivalent, then

$$A(s+2)^2+B(s+1)(s+2)+C(s+1)=8-(s+2)(4s+10). \tag{7.2.12}$$

The two sides of this equation are polynomials of degree two. From a theorem of algebra, they will be equal for all s if they are equal for any three distinct values of s. We may determine A, B and C by choosing convenient values of s.

The left side of Equation 7.2.12 suggests that we take $s=-2$ to obtain $C=-8$, and $s=-1$ to obtain $A=2$. We can now choose any third value of s to determine B. Taking $s=0$ yields $4A+2B+C=-12$. Since $A=2$ and $C=-8$ this implies that $B=-6$. Therefore

$$F(s)=\frac{2}{s+1}-\frac{6}{s+2}-\frac{8}{(s+2)^2}$$

and

$$\begin{aligned}\mathcal{L}^{-1}(F) &= 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)-6\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)-8\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)\\ &= 2e^{-t}-6e^{-2t}-8te^{-2t}.\end{aligned}$$

Example 8.2.7

Find the inverse Laplace transform of

$$F(s)=\frac{s^2-5s+7}{(s+2)^3}.$$

Solution

The form for the partial fraction expansion is

$$F(s)=\frac{A}{s+2}+\frac{B}{(s+2)^2}+\frac{C}{(s+2)^3}.$$

The easiest way to obtain A, B, and C is to expand the numerator in powers of $s+2$. This yields

$$s^2-5s+7=[(s+2)-2]^2-5[(s+2)-2]+7=(s+2)^2-9(s+2)+21.$$

Therefore

$$F(s) = \frac{(s+2)^2 - 9(s+2) + 21}{(s+2)^3}$$
$$= \frac{1}{s+2} - \frac{9}{(s+2)^2} + \frac{21}{(s+2)^3}$$

and

$$\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 9\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) + \frac{21}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+2)^3}\right)$$
$$= e^{-2t}\left(1 - 9t + \frac{21}{2}t^2\right).$$

Example 8.2.8

Find the inverse Laplace transform of

$$F(s) = \frac{1 - s(5+3s)}{s\left[(s+1)^2+1\right]}. \tag{7.2.13}$$

Solution

One form for the partial fraction expansion of F is

$$F(s) = \frac{A}{s} + \frac{Bs+C}{(s+1)^2+1}. \tag{7.2.14}$$

However, we see from the table of Laplace transforms that the inverse transform of the second fraction on the right of Equation 7.2.14 will be a linear combination of the inverse transforms

$$e^{-t}\cos t \quad \text{and} \quad e^{-t}\sin t$$

of

$$\frac{s+1}{(s+1)^2+1} \quad \text{and} \quad \frac{1}{(s+1)^2+1}$$

respectively. Therefore, instead of Equation 7.2.14 we write

$$F(s) = \frac{A}{s} + \frac{B(s+1)+C}{(s+1)^2+1}. \tag{7.2.15}$$

Finding a common denominator yields

$$F(s) = \frac{A\left[(s+1)^2+1\right] + B(s+1)s + Cs}{s\left[(s+1)^2+1\right]}. \tag{7.2.16}$$

If Equation 7.2.13 and Equation 7.2.16 are to be equivalent, then

$$A\left[(s+1)^2+1\right] + B(s+1)s + Cs = 1 - s(5+3s).$$

This is true for all s if it is true for three distinct values of s. Choosing $s = 0$, -1, and 1 yields the system

$$\begin{aligned} 2A &= 1 \\ A - C &= 3 \\ 5A + 2B + C &= -7. \end{aligned}$$

Solving this system yields

$$A = \frac{1}{2}, \quad B = -\frac{7}{2}, \quad C = -\frac{5}{2}.$$

Hence, from Equation 7.2.15,

$$F(s)=\frac{1}{2s}-\frac{7}{2}\,\frac{s+1}{(s+1)^2+1}-\frac{5}{2}\,\frac{1}{(s+1)^2+1}.$$

Therefore

$$\begin{aligned}\mathcal{L}^{-1}(F)&=\frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right)-\frac{7}{2}\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right)-\frac{5}{2}\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right)\\&=\frac{1}{2}-\frac{7}{2}e^{-t}\cos t-\frac{5}{2}e^{-t}\sin t.\end{aligned}$$

Example 8.2.9

Find the inverse Laplace transform of

$$F(s)=\frac{8+3s}{(s^2+1)(s^2+4)}. \tag{7.2.17}$$

Solution

The form for the partial fraction expansion is

$$F(s)=\frac{A+Bs}{s^2+1}+\frac{C+Ds}{s^2+4}.$$

The coefficients A, B, C and D can be obtained by finding a common denominator and equating the resulting numerator to the numerator in Equation 7.2.17. However, since there's no first power of s in the denominator of Equation 7.2.17, there's an easier way: the expansion of

$$F_1(s)=\frac{1}{(s^2+1)(s^2+4)}$$

can be obtained quickly by using Heaviside's method to expand

$$\frac{1}{(x+1)(x+4)}=\frac{1}{3}\left(\frac{1}{x+1}-\frac{1}{x+4}\right)$$

and then setting $x=s^2$ to obtain

$$\frac{1}{(s^2+1)(s^2+4)}=\frac{1}{3}\left(\frac{1}{s^2+1}-\frac{1}{s^2+4}\right).$$

Multiplying this by $8+3s$ yields

$$F(s)=\frac{8+3s}{(s^2+1)(s^2+4)}=\frac{1}{3}\left(\frac{8+3s}{s^2+1}-\frac{8+3s}{s^2+4}\right).$$

Therefore

$$\mathcal{L}^{-1}(F)=\frac{8}{3}\sin t+\cos t-\frac{4}{3}\sin 2t-\cos 2t.$$

Using Technology

Some software packages that do symbolic algebra can find partial fraction expansions very easily. We recommend that you use such a package if one is available to you, but only after you've done enough partial fraction expansions on your own to master the technique.

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7.2E: The Inverse Laplace Transform (Exercises)

Q7.2.1

1. Find the inverse Laplace transform.

a. $\frac{3}{(s-7)^4}$
b. $\frac{2s-4}{s^2-4s+13}$
c. $\frac{1}{s^2+4s+20}$
d. $\frac{2}{s^2+9}$
e. $\frac{s^2-1}{(s^2+1)^2}$
f. $\frac{1}{(s-2)^2-4}$
g. $\frac{12s-24}{(s^2-4s+85)^2}$
h. $\frac{2}{(s-3)^2-9}$
i. $\frac{s^2-4s+3}{(s^2-4s+5)^2}$

2. Find the inverse Laplace transform.

a. $\frac{2s+3}{(s-7)^4}$
b. $\frac{s^2-1}{(s-2)^6}$
c. $\frac{s+5}{s^2+6s+18}$
d. $\frac{2s+1}{s^2+9}$
e. $\frac{s}{s^2+2s+1}$
f. $\frac{s+1}{s^2-9}$
g. $\frac{s^3+2s^2-s-3}{(s+1)^4}$
h. $\frac{2s+3}{(s-1)^2+4}$
i. $\frac{1}{s}-\frac{s}{s^2+1}$
j. $\frac{3s+4}{s^2-1}$
k. $\frac{3}{s-1}+\frac{4s+1}{s^2+9}$
l. $\frac{3}{(s+2)^2}-\frac{2s+6}{s^2+4}$

3. Find the inverse Laplace transform.

a. $\frac{-s^2+s+5}{s^3+s^2-4s-4}$
b. $\frac{5s-4}{s^3-s^2-2s}$
c. $\frac{3s^2+2s+1}{(s^2+1)(s^2+2s+2)}$
d. $\frac{-s+1}{(4s^2+1)(s^2+1)}$
e. $\frac{34-17s}{(2s-1)(s^2-2s+5)}$
f. $\frac{s-6}{(s^2-1)(s^2+4)}$
g. $\frac{3s+2}{(s^2+1)(s-1)^2}$

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7.3: Solution of Initial Value Problems

Laplace Transforms of Derivatives

In the rest of this chapter we'll use the Laplace transform to solve initial value problems for constant coefficient second order equations. To do this, we must know how the Laplace transform of f' is related to the Laplace transform of f. The next theorem answers this question.

Theorem 8.3.1

Suppose f is continuous on $[0,\infty)$ and of exponential order s_0, and f' is piecewise continuous on $[0,\infty)$. Then f and f' have Laplace transforms for $s>s_0$, and

$$\mathcal{L}(f')=s\mathcal{L}(f)-f(0). \tag{7.3.1}$$

Proof

We know from Theorem 8.1.6 that $\mathcal{L}(f)$ is defined for $s>s_0$. We first consider the case where f' is continuous on $[0,\infty)$. Integration by parts yields

$$\begin{aligned}\int_0^T e^{-st}f'(t)\,dt &= e^{-st}f(t)\Big|_0^T+s\int_0^T e^{-st}f(t)\,dt\\ &= e^{-sT}f(T)-f(0)+s\int_0^T e^{-st}f(t)\,dt\end{aligned} \tag{7.3.2}$$

for any $T>0$. Since f is of exponential order s_0, $\lim\limits_{T\to\infty} e^{-sT}f(T)=0$ and the integral in on the right side of Equation 7.3.2converges as $T\to\infty$ if $s>s_0$. Therefore

$$\begin{aligned}\int_0^\infty e^{-st}f'(t)\,dt &= -f(0)+s\int_0^\infty e^{-st}f(t)\,dt\\ &= -f(0)+s\mathcal{L}(f),\end{aligned}$$

which proves Equation 7.3.1.

Suppose $T>0$ and f' is only piecewise continuous on $[0,T]$, with discontinuities at $t_1<t_2<\cdots<t_{n-1}$. For convenience, let $t_0=0$ and $t_n=T$. Integrating by parts yields

$$\begin{aligned}\int_{t_{i-1}}^{t_i} e^{-st}f'(t)\,dt &= e^{-st}f(t)\Big|_{t_{i-1}}^{t_i}+s\int_{t_{i-1}}^{t_i} e^{-st}f(t)\,dt\\ &= e^{-st_i}f(t_i)-e^{-st_{i-1}}f(t_{i-1})+s\int_{t_{i-1}}^{t_i} e^{-st}f(t)\,dt.\end{aligned}$$

Summing both sides of this equation from $i=1$ to n and noting that

$$\begin{aligned}&\left(e^{-st_1}f(t_1)-e^{-st_0}f(t_0)\right)+\left(e^{-st_2}f(t_2)-e^{-st_1}f(t_1)\right)+\cdots+\left(e^{-st_N}f(t_N)-e^{-st_{N-1}}f(t_{N-1})\right)\\ &\qquad = e^{-st_N}f(t_N)-e^{-st_0}f(t_0)=e^{-sT}f(T)-f(0)\end{aligned}$$

yields Equation 7.3.2, so Equation 7.3.1follows as before.

Example 8.3.1

In Example 8.1.4 we saw that

$$\mathcal{L}(\cos\omega t)=\frac{s}{s^2+\omega^2}.$$

Solution

Applying Equation 7.3.1 with $f(t)=\cos\omega t$ shows that

$$\mathcal{L}(-\omega\sin\omega t)=s\frac{s}{s^2+\omega^2}-1=-\frac{\omega^2}{s^2+\omega^2}.$$

Therefore

$$\mathcal{L}(\sin\omega t)=\frac{\omega}{s^2+\omega^2},$$

which agrees with the corresponding result obtained in 8.1.4.

In Section 2.1 we showed that the solution of the initial value problem

$$y'=ay,\quad y(0)=y_0, \tag{7.3.3}$$

is $y=y_0e^{at}$. We'll now obtain this result by using the Laplace transform.

Let $Y(s)=\mathcal{L}(y)$ be the Laplace transform of the unknown solution of Equation 7.3.3. Taking Laplace transforms of both sides of Equation 7.3.3 yields

$$\mathcal{L}(y')=\mathcal{L}(ay),$$

which, by Theorem 8.3.1 , can be rewritten as

$$s\mathcal{L}(y)-y(0)=a\mathcal{L}(y),$$

or

$$sY(s)-y_0=aY(s).$$

Solving for $Y(s)$ yields

$$Y(s)=\frac{y_0}{s-a},$$

so

$$y=\mathcal{L}^{-1}(Y(s))=\mathcal{L}^{-1}\left(\frac{y_0}{s-a}\right)=y_0\mathcal{L}^{-1}\left(\frac{1}{s-a}\right)=y_0e^{at},$$

which agrees with the known result.

We need the next theorem to solve second order differential equations using the Laplace transform.

Theorem 8.3.2

Suppose f and f' are continuous on $[0,\infty)$ and of exponential order s_0, and that f'' is piecewise continuous on $[0,\infty)$. Then f, f', and f'' have Laplace transforms for $s>s_0$,

$$\mathcal{L}(f')=s\mathcal{L}(f)-f(0), \tag{7.3.4}$$

and

$$\mathcal{L}(f'')=s^2\mathcal{L}(f)-f'(0)-sf(0). \tag{7.3.5}$$

Proof

Theorem 8.3.1 implies that $\mathcal{L}(f')$ exists and satisfies Equation 7.3.4 for $s>s_0$. To prove that $\mathcal{L}(f'')$ exists and satisfies Equation 7.3.5 for $s>s_0$, we first apply Theorem 8.3.1 to $g=f'$. Since g satisfies the hypotheses of Theorem 8.3.1 , we conclude that $\mathcal{L}(g')$ is defined and satisfies

$$\mathcal{L}(g')=s\mathcal{L}(g)-g(0)$$

for $s>s_0$. However, since $g'=f''$, this can be rewritten as

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0).$$

Substituting Equation 7.3.4into this yields Equation 7.3.5

Solving Second Order Equations with the Laplace Transform

We'll now use the Laplace transform to solve initial value problems for second order equations.

Example 8.3.2

Use the Laplace transform to solve the initial value problem

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2, \quad y'(0) = 3. \tag{7.3.6}$$

Solution

Taking Laplace transforms of both sides of the differential equation in Equation 7.3.6 yields

$$\mathcal{L}(y'' - 6y' + 5y) = \mathcal{L}\left(3e^{2t}\right) = \frac{3}{s-2},$$

which we rewrite as

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 5\mathcal{L}(y) = \frac{3}{s-2}. \tag{7.3.7}$$

Now denote $\mathcal{L}(y) = Y(s)$. Theorem 8.3.2 and the initial conditions in Equation 7.3.6 imply that

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 2$$

and

$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0) = s^2Y(s) - 3 - 2s.$$

Substituting from the last two equations into Equation 7.3.7 yields

$$\left(s^2Y(s) - 3 - 2s\right) - 6\left(sY(s) - 2\right) + 5Y(s) = \frac{3}{s-2}.$$

Therefore

$$(s^2 - 6s + 5)Y(s) = \frac{3}{s-2} + (3 + 2s) + 6(-2), \tag{7.3.8}$$

so

$$(s-5)(s-1)Y(s) = \frac{3 + (s-2)(2s-9)}{s-2},$$

and

$$Y(s) = \frac{3 + (s-2)(2s-9)}{(s-2)(s-5)(s-1)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = -\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-5} + \frac{5}{2}\frac{1}{s-1},$$

and taking the inverse transform of this yields

$$y = -e^{2t} + \frac{1}{2}e^{5t} + \frac{5}{2}e^{t}$$

as the solution of Equation 7.3.6.

It isn't necessary to write all the steps that we used to obtain Equation 7.3.8. To see how to avoid this, let's apply the method of Example 8.3.2 to the general initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{7.3.9}$$

Taking Laplace transforms of both sides of the differential equation in Equation 7.3.9 yields

$$a\mathcal{L}(y'') + b\mathcal{L}(y') + c\mathcal{L}(y) = F(s). \tag{7.3.10}$$

Now let $Y(s) = \mathcal{L}(y)$. Theorem 8.3.2 and the initial conditions in Equation 7.3.9 imply that

$$\mathcal{L}(y') = sY(s) - k_0 \quad \text{and} \quad \mathcal{L}(y'') = s^2Y(s) - k_1 - k_0 s.$$

Substituting these into Equation 7.3.10 yields

$$a\left(s^2Y(s) - k_1 - k_0 s\right) + b\left(sY(s) - k_0\right) + cY(s) = F(s). \tag{7.3.11}$$

The coefficient of $Y(s)$ on the left is the characteristic polynomial

$$p(s) = as^2 + bs + c$$

of the complementary equation for Equation 7.3.9. Using this and moving the terms involving k_0 and k_1 to the right side of Equation 7.3.11 yields

$$p(s)Y(s) = F(s) + a(k_1 + k_0 s) + bk_0. \tag{7.3.12}$$

This equation corresponds to Equation 7.3.8 of Example 8.3.2 . Having established the form of this equation in the general case, it is preferable to go directly from the initial value problem to this equation. You may find it easier to remember Equation 7.3.12 rewritten as

$$p(s)Y(s) = F(s) + a\left(y'(0) + sy(0)\right) + by(0). \tag{7.3.13}$$

Example 8.3.3

Use the Laplace transform to solve the initial value problem

$$2y'' + 3y' + y = 8e^{-2t}, \quad y(0) = -4, \; y'(0) = 2. \tag{7.3.14}$$

Solution

The characteristic polynomial is

$$p(s) = 2s^2 + 3s + 1 = (2s+1)(s+1)$$

and

$$F(s) = \mathcal{L}(8e^{-2t}) = \frac{8}{s+2},$$

so Equation 7.3.13 becomes

$$(2s+1)(s+1)Y(s) = \frac{8}{s+2} + 2(2-4s) + 3(-4).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{4\left(1 - (s+2)(s+1)\right)}{(s+1/2)(s+1)(s+2)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = \frac{4}{3}\frac{1}{s+1/2} - \frac{8}{s+1} + \frac{8}{3}\frac{1}{s+2},$$

so the solution of Equation 7.3.14 is

$$y = \mathcal{L}^{-1}(Y(s)) = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$$

(Figure 8.3.1).

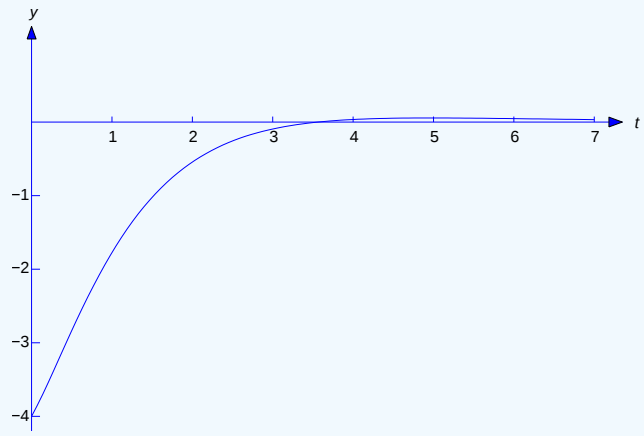

Figure 8.3.1 : $y = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$

Example 8.3.4

Solve the initial value problem

$$y'' + 2y' + 2y = 1, \quad y(0) = -3,\ y'(0) = 1. \tag{7.3.15}$$

Solution

The characteristic polynomial is

$$p(s) = s^2 + 2s + 2 = (s+1)^2 + 1$$

and

$$F(s) = \mathcal{L}(1) = \frac{1}{s},$$

so Equation 7.3.13becomes

$$\left[(s+1)^2 + 1\right] Y(s) = \frac{1}{s} + 1 \cdot (1 - 3s) + 2(-3).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{1 - s(5+3s)}{s\left[(s+1)^2 + 1\right]}.$$

In Example 8.2.8 we found the inverse transform of this function to be

$$y = \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t$$

(Figure 8.3.2), which is therefore the solution of Equation 7.3.15.

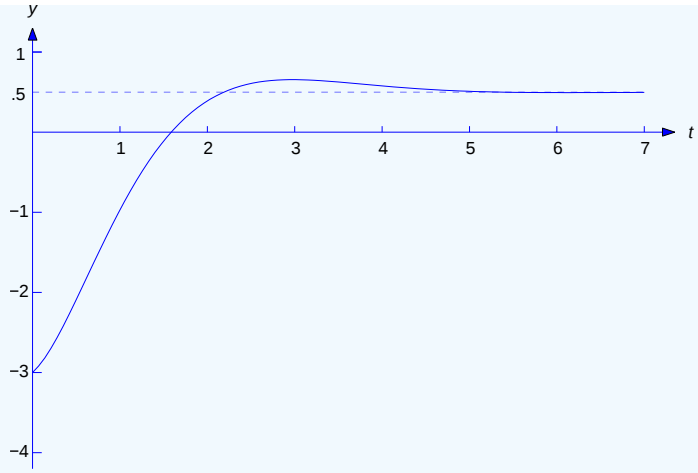

Figure 8.3.2 : $y = \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t$

Note

In our examples we applied Theorems 8.3.1 and 8.3.2 without verifying that the unknown function y satisfies their hypotheses. This is characteristic of the formal manipulative way in which the Laplace transform is used to solve differential equations. Any doubts about the validity of the method for solving a given equation can be resolved by verifying that the resulting function y is the solution of the given problem.

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7.3E: Solution of Initial Value Problems (Exercises)

Q7.3.1

In *Exercises 7.3.1-7.3.31* use the Laplace transform to solve the initial value problem.

1. $y''+3y'+2y=e^t, \quad y(0)=1, \quad y'(0)=-6$
2. $y''-y'-6y=2, \quad y(0)=1, \quad y'(0)=0$
3. $y''+y'-2y=2e^{3t}, \quad y(0)=-1, \quad y'(0)=4$
4. $y''-4y=2e^{3t}, \quad y(0)=1, \quad y'(0)=-1$
5. $y''+y'-2y=e^{3t}, \quad y(0)=1, \quad y'(0)=-1$
6. $y''+3y'+2y=6e^t, \quad y(0)=1, \quad y'(0)=-1$
7. $y''+y=\sin 2t, \quad y(0)=0, \quad y'(0)=1$
8. $y''-3y'+2y=2e^{3t}, \quad y(0)=1, \quad y'(0)=-1$
9. $y''-3y'+2y=e^{4t}, \quad y(0)=1, \quad y'(0)=-2$
10. $y''-3y'+2y=e^{3t}, \quad y(0)=-1, \quad y'(0)=-4$
11. $y''+3y'+2y=2e^t, \quad y(0)=0, \quad y'(0)=-1$
12. $y''+y'-2y=-4, \quad y(0)=2, \quad y'(0)=3$
13. $y''+4y=4, \quad y(0)=0, \quad y'(0)=1$
14. $y''-y'-6y=2, \quad y(0)=1, \quad y'(0)=0$
15. $y''+3y'+2y=e^t, \quad y(0)=0, \quad y'(0)=1$
16. $y''-y=1, \quad y(0)=1, \quad y'(0)=0$
17. $y''+4y=3\sin t, \quad y(0)=1, \quad y'(0)=-1$
18. $y''+y'=2e^{3t}, \quad y(0)=-1, \quad y'(0)=4$
19. $y''+y=1, \quad y(0)=2, \quad y'(0)=0$
20. $y''+y=t, \quad y(0)=0, \quad y'(0)=2$
21. $y''+y=t-3\sin 2t, \quad y(0)=1, \quad y'(0)=-3$
22. $y''+5y'+6y=2e^{-t}, \quad y(0)=1, \quad y'(0)=3$
23. $y''+2y'+y=6\sin t-4\cos t, \quad y(0)=-1,\ y'(0)=1$
24. $y''-2y'-3y=10\cos t, \quad y(0)=2, \quad y'(0)=7$
25. $y''+y=4\sin t+6\cos t, \quad y(0)=-6,\ y'(0)=2$
26. $y''+4y=8\sin 2t+9\cos t, \quad y(0)=1, \quad y'(0)=0$
27. $y''-5y'+6y=10e^t\cos t, \quad y(0)=2, \quad y'(0)=1$
28. $y''+2y'+2y=2t, \quad y(0)=2, \quad y'(0)=-7$
29. $y''-2y'+2y=5\sin t+10\cos t, \quad y(0)=1,\ y'(0)=2$
30. $y''+4y'+13y=10e^{-t}-36e^t, \quad y(0)=0,\ y'(0)=-16$
31. $y''+4y'+5y=e^{-t}(\cos t+3\sin t), \quad y(0)=0, \quad y'(0)=4$

Q7.3.2

32. $2y''-3y'-2y=4e^t, \quad y(0)=1,\ y'(0)=-2$
33. $6y''-y'-y=3e^{2t}, \quad y(0)=0,\ y'(0)=0$

34. $2y'' + 2y' + y = 2t, \quad y(0) = 1,\ y'(0) = -1$

35. $4y'' - 4y' + 5y = 4\sin t - 4\cos t, \quad y(0) = 0,\ y'(0) = 11/17$

36. $4y'' + 4y' + y = 3\sin t + \cos t, \quad y(0) = 2,\ y'(0) = -1$

37. $9y'' + 6y' + y = 3e^{3t}, \quad y(0) = 0,\ y'(0) = -3$

38. Suppose a, b, and c are constants and $a \neq 0$. Let

$$y_1 = \mathcal{L}^{-1}\left(\frac{as+b}{as^2+bs+c}\right) \quad \text{and} \quad y_2 = \mathcal{L}^{-1}\left(\frac{a}{as^2+bs+c}\right).$$

Show that

$$y_1(0) = 1, \quad y_1'(0) = 0 \quad \text{and} \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

HINT: *Use the Laplace transform to solve the initial value problems*

$$\begin{aligned} ay'' + by' + cy = 0, \quad y(0) = 1, \quad y'(0) = 0 \\ ay'' + by' + cy = 0, \quad y(0) = 0, \quad y'(0) = 1 \end{aligned}$$

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7.4: The Unit Step Function

In the next section we'll consider initial value problems

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

where a, b, and c are constants and f is piecewise continuous. In this section we'll develop procedures for using the table of Laplace transforms to find Laplace transforms of piecewise continuous functions, and to find the piecewise continuous inverses of Laplace transforms.

Example 8.4.1

Use the table of Laplace transforms to find the Laplace transform of

$$f(t) = \begin{cases} 2t+1, & 0 \le t < 2, \\ 3t, & t \ge 2 \end{cases} \tag{7.4.1}$$

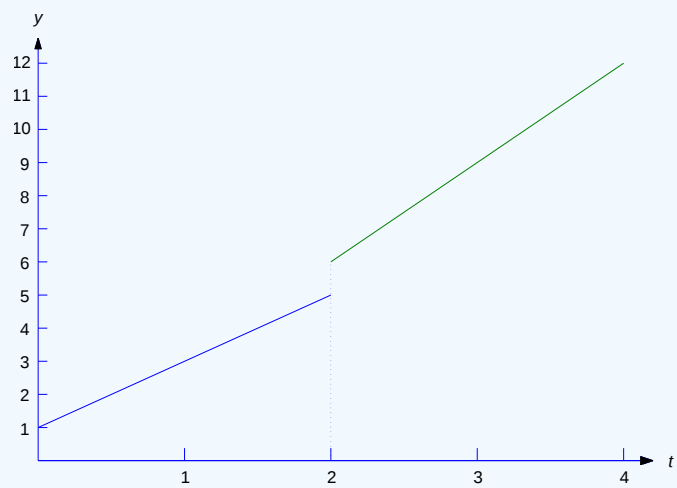

Figure 8.4.1 : The piecewise continuous function Equation 7.4.1.

Solution

Since the formula for f changes at $t = 2$, we write

$$\begin{aligned} \mathcal{L}(f) &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^2 e^{-st}(2t+1)\, dt + \int_2^\infty e^{-st}(3t)\, dt. \end{aligned} \tag{7.4.2}$$

To relate the first term to a Laplace transform, we add and subtract

$$\int_2^\infty e^{-st}(2t+1)\, dt$$

in Equation 7.4.2 to obtain

$$\begin{aligned} \mathcal{L}(f) &= \int_0^\infty e^{-st}(2t+1)\, dt + \int_2^\infty e^{-st}(3t-2t-1)\, dt \\ &= \int_0^\infty e^{-st}(2t+1)\, dt + \int_2^\infty e^{-st}(t-1)\, dt \\ &= \mathcal{L}(2t+1) + \int_2^\infty e^{-st}(t-1)\, dt. \end{aligned} \tag{7.4.3}$$

To relate the last integral to a Laplace transform, we make the change of variable $x = t - 2$ and rewrite the integral as

$$\int_2^\infty e^{-st}(t-1)\,dt = \int_0^\infty e^{-s(x+2)}(x+1)\,dx$$
$$= e^{-2s}\int_0^\infty e^{-sx}(x+1)\,dx.$$

Since the symbol used for the variable of integration has no effect on the value of a definite integral, we can now replace x by the more standard t and write

$$\int_2^\infty e^{-st}(t-1)\,dt = e^{-2s}\int_0^\infty e^{-st}(t+1)\,dt = e^{-2s}\mathcal{L}(t+1).$$

This and Equation 7.4.3 imply that

$$\mathcal{L}(f) = \mathcal{L}(2t+1) + e^{-2s}\mathcal{L}(t+1).$$

Now we can use the table of Laplace transforms to find that

$$\mathcal{L}(f) = \frac{2}{s^2} + \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right).$$

Laplace Transforms of Piecewise Continuous Functions

We'll now develop the method of Example 8.4.1 into a systematic way to find the Laplace transform of a piecewise continuous function. It is convenient to introduce the **unit step function**, defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases} \tag{7.4.4}$$

Thus, $u(t)$ "steps" from the constant value 0 to the constant value 1 at $t = 0$. If we replace t by $t - \tau$ in Equation 7.4.4, then

$$u(t-\tau) = \begin{cases} 0, & t < \tau, \\ 1, & t \geq \tau \end{cases};$$

that is, the step now occurs at $t = \tau$ (Figure 8.4.2).

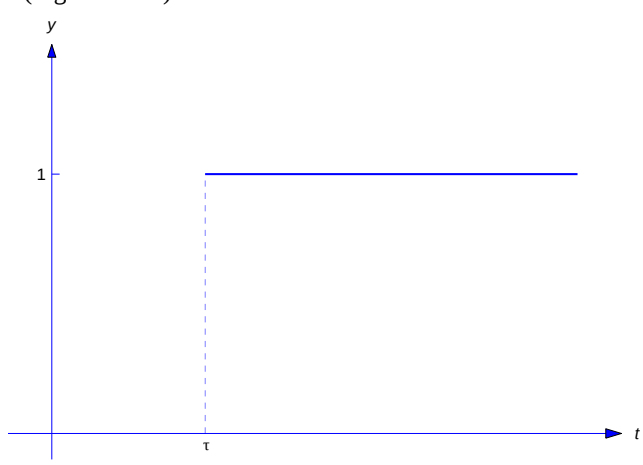

Figure 8.4.2 : $y = u(t-\tau)$

The step function enables us to represent piecewise continuous functions conveniently. For example, consider the function

$$f(t) = \begin{cases} f_0(t), & 0 \leq t < t_1, \\ f_1(t), & t \geq t_1, \end{cases} \tag{7.4.5}$$

where we assume that f_0 and f_1 are defined on $[0, \infty)$, even though they equal f only on the indicated intervals. This assumption enables us to rewrite Equation 7.4.5 as

$$f(t) = f_0(t) + u(t-t_1)\left(f_1(t) - f_0(t)\right). \tag{7.4.6}$$

To verify this, note that if $t < t_1$ then $u(t-t_1) = 0$ and Equation 7.4.6 becomes

$$f(t) = f_0(t) + (0)\left(f_1(t) - f_0(t)\right) = f_0(t).$$

If $t \geq t_1$ then $u(t-t_1) = 1$ and Equation 7.4.6 becomes

$$f(t) = f_0(t) + (1)\left(f_1(t) - f_0(t)\right) = f_1(t).$$

We need the next theorem to show how Equation 7.4.6 can be used to find $\mathcal{L}(f)$.

Theorem 8.4.1

Let g be defined on $[0, \infty)$. Suppose $\tau \geq 0$ and $\mathcal{L}\left(g(t+\tau)\right)$ exists for $s > s_0$. Then $\mathcal{L}\left(u(t-\tau)g(t)\right)$ exists for $s > s_0$, and

$$\mathcal{L}(u(t-\tau)g(t)) = e^{-s\tau}\mathcal{L}\left(g(t+\tau)\right).$$

Proof

By definition,

$$\mathcal{L}\left(u(t-\tau)g(t)\right) = \int_0^\infty e^{-st}u(t-\tau)g(t)\,dt.$$

From this and the definition of $u(t-\tau)$,

$$\mathcal{L}\left(u(t-\tau)g(t)\right) = \int_0^\tau e^{-st}(0)\,dt + \int_\tau^\infty e^{-st}g(t)\,dt.$$

The first integral on the right equals zero. Introducing the new variable of integration $x = t - \tau$ in the second integral yields

$$\mathcal{L}\left(u(t-\tau)g(t)\right) = \int_0^\infty e^{-s(x+\tau)}g(x+\tau)\,dx = e^{-s\tau}\int_0^\infty e^{-sx}g(x+\tau)\,dx.$$

Changing the name of the variable of integration in the last integral from x to t yields

$$\mathcal{L}\left(u(t-\tau)g(t)\right) = e^{-s\tau}\int_0^\infty e^{-st}g(t+\tau)\,dt = e^{-s\tau}\mathcal{L}(g(t+\tau)).$$

Example 8.4.2

Find

$$\mathcal{L}\left(u(t-1)(t^2+1)\right).$$

Solution

Here $\tau = 1$ and $g(t) = t^2 + 1$, so

$$g(t+1) = (t+1)^2 + 1 = t^2 + 2t + 2.$$

Since

$$\mathcal{L}\left(g(t+1)\right) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s},$$

Theorem 8.4.1 implies that

$$\mathcal{L}\left(u(t-1)(t^2+1)\right) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right).$$

Example 8.4.3

Use Theorem 8.4.1 to find the Laplace transform of the function

$$f(t)=\begin{cases} 2t+1, & 0\le t<2,\\ 3t, & t\ge 2,\end{cases}$$

from Example 8.4.1 .

Solution

We first write f in the form Equation 7.4.6 as

$$f(t)=2t+1+u(t-2)(t-1).$$

Therefore

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}(2t+1)+\mathcal{L}\left(u(t-2)(t-1)\right)\\ &= \mathcal{L}(2t+1)+e^{-2s}\mathcal{L}(t+1) \quad \text{(from Theorem 7.4.1)}\\ &= \frac{2}{s^2}+\frac{1}{s}+e^{-2s}\left(\frac{1}{s^2}+\frac{1}{s}\right),\end{aligned}$$

which is the result obtained in Example 8.4.1 .

Formula Equation 7.4.6 can be extended to more general piecewise continuous functions. For example, we can write

$$f(t)=\begin{cases} f_0(t), & 0\le t<t_1,\\ f_1(t), & t_1\le t<t_2,\\ f_2(t), & t\ge t_2,\end{cases}$$

as

$$f(t)=f_0(t)+u(t-t_1)\left(f_1(t)-f_0(t)\right)+u(t-t_2)\left(f_2(t)-f_1(t)\right)$$

if f_0, f_1, and f_2 are all defined on $[0,\infty)$.

Example 8.4.4

Find the Laplace transform of

$$f(t)=\begin{cases} 1, & 0\le t<2,\\ -2t+1, & 2\le t<3,\\ 3t, & 3\le t<5,\\ t-1, & t\ge 5\end{cases} \tag{7.4.7}$$

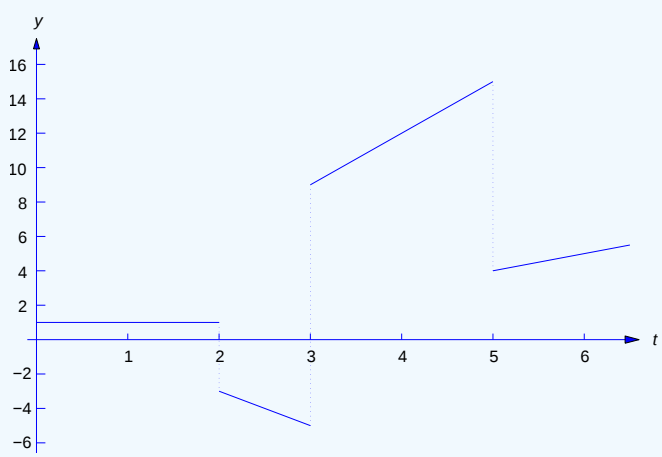

Figure 8.4.3 : The piecewise continuous function Equation 7.4.7

Solution

In terms of step functions,

$$\begin{aligned} f(t) &= 1+u(t-2)(-2t+1-1)+u(t-3)(3t+2t-1) \\ &= +u(t-5)(t-1-3t), \end{aligned}$$

or

$$f(t)=1-2u(t-2)t+u(t-3)(5t-1)-u(t-5)(2t+1).$$

Now Theorem 8.4.1 implies that

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(1)-2e^{-2s}\mathcal{L}(t+2)+e^{-3s}\mathcal{L}\left(5(t+3)-1\right)-e^{-5s}\mathcal{L}\left(2(t+5)+1\right) \\ &= \mathcal{L}(1)-2e^{-2s}\mathcal{L}(t+2)+e^{-3s}\mathcal{L}(5t+14)-e^{-5s}\mathcal{L}(2t+11) \\ &= \frac{1}{s}-2e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right)+e^{-3s}\left(\frac{5}{s^2}+\frac{14}{s}\right)-e^{-5s}\left(\frac{2}{s^2}+\frac{11}{s}\right). \end{aligned}$$

The trigonometric identities

$$\sin(A+B)=\sin A\cos B+\cos A\sin B \tag{7.4.8}$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B \tag{7.4.9}$$

are useful in problems that involve shifting the arguments of trigonometric functions. We'll use these identities in the next example.

Example 8.4.5

Find the Laplace transform of

$$f(t)=\begin{cases} \sin t, & 0\le t<\frac{\pi}{2} \\ \cos t-3\sin t, & \frac{\pi}{2}\le t<\pi \\ 3\cos t, & t\ge\pi \end{cases} \tag{7.4.10}$$

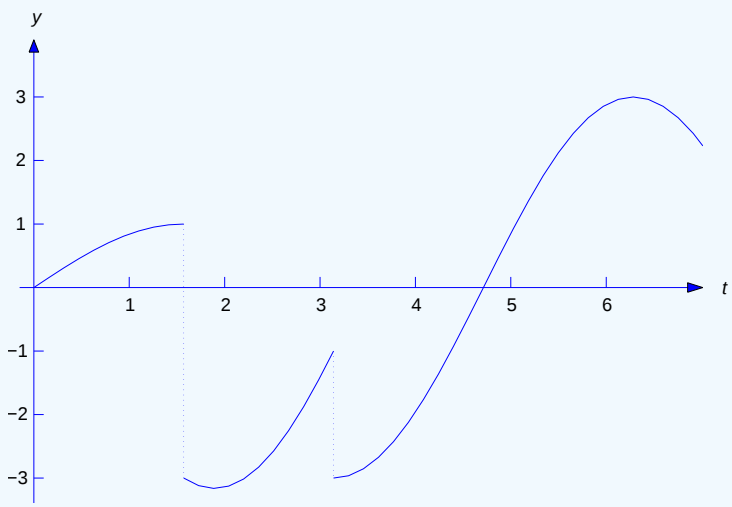

Figure 8.4.4 : The piecewise continuous function Equation 7.4.10.

Solution

In terms of step functions,

$$f(t)=\sin t+u(t-\pi/2)(\cos t-4\sin t)+u(t-\pi)(2\cos t+3\sin t).$$

Now Theorem 8.4.1 implies that

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}(\sin t)+e^{-\frac{\pi}{2}s}\mathcal{L}\left(\cos\left(t+\tfrac{\pi}{2}\right)-4\sin\left(t+\tfrac{\pi}{2}\right)\right)\\ &\quad +e^{-\pi s}\mathcal{L}\left(2\cos(t+\pi)+3\sin(t+\pi)\right).\end{aligned} \tag{7.4.11}$$

Since

$$\cos\left(t+\frac{\pi}{2}\right)-4\sin\left(t+\frac{\pi}{2}\right)=-\sin t-4\cos t$$

and

$$2\cos(t+\pi)+3\sin(t+\pi)=-2\cos t-3\sin t,$$

we see from Equation 7.4.11 that

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}(\sin t)-e^{-\pi s/2}\mathcal{L}(\sin t+4\cos t)-e^{-\pi s}\mathcal{L}(2\cos t+3\sin t)\\ &= \frac{1}{s^2+1}-e^{-\frac{\pi}{2}s}\left(\frac{1+4s}{s^2+1}\right)-e^{-\pi s}\left(\frac{3+2s}{s^2+1}\right).\end{aligned}$$

The Second Shifting Theorem

Replacing $g(t)$ by $g(t-\tau)$ in Theorem 8.4.1 yields the next theorem.

Theorem 8.4.2 : Second Shifting Theorem

If $\tau\geq 0$ and $\mathcal{L}(g)$ exists for $s>s_0$ then $\mathcal{L}\left(u(t-\tau)g(t-\tau)\right)$ exists for $s>s_0$ and

$$\mathcal{L}(u(t-\tau)g(t-\tau))=e^{-s\tau}\mathcal{L}(g(t)),$$

or, equivalently,

$$\text{if } g(t)\leftrightarrow G(s),\ \text{then } u(t-\tau)g(t-\tau)\leftrightarrow e^{-s\tau}G(s). \tag{7.4.12}$$

Note

Recall that the First Shifting Theorem (Theorem 8.1.3) states that multiplying a function by e^{at} corresponds to shifting the argument of its transform by a units. Theorem 8.4.2 states that multiplying a Laplace transform by the exponential $e^{-\tau s}$ corresponds to shifting the argument of the inverse transform by τ units.

Example 8.4.6

Use Equation 7.4.12 to find

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right).$$

Solution

To apply Equation 7.4.12 we let $\tau=2$ and $G(s)=1/s^2$. Then $g(t)=t$ and Equation 7.4.12 implies that

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right)=u(t-2)(t-2).$$

Example 8.4.7

Find the inverse Laplace transform h of

$$H(s)=\frac{1}{s^2}-e^{-s}\left(\frac{1}{s^2}+\frac{2}{s}\right)+e^{-4s}\left(\frac{4}{s^3}+\frac{1}{s}\right),$$

and find distinct formulas for h on appropriate intervals.

Solution

Let

$$G_0(s)=\frac{1}{s^2},\quad G_1(s)=\frac{1}{s^2}+\frac{2}{s},\quad G_2(s)=\frac{4}{s^3}+\frac{1}{s}.$$

Then

$$g_0(t)=t,\ g_1(t)=t+2,\ g_2(t)=2t^2+1.$$

Hence, Equation 7.4.12 and the linearity of $\mathcal{L}^{-1}$ imply that

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left(G_0(s)\right)-\mathcal{L}^{-1}\left(e^{-s}G_1(s)\right)+\mathcal{L}^{-1}\left(e^{-4s}G_2(s)\right) \\ &= t-u(t-1)\left[(t-1)+2\right]+u(t-4)\left[2(t-4)^2+1\right] \\ &= t-u(t-1)(t+1)+u(t-4)(2t^2-16t+33), \end{aligned}$$

which can also be written as

$$h(t)=\begin{cases} t, & 0\le t<1, \\ -1, & 1\le t<4, \\ 2t^2-16t+32, & t\ge 4. \end{cases}$$

Example 8.4.8

Find the inverse transform of

$$H(s)=\frac{2s}{s^2+4}-e^{-\frac{\pi}{2}s}\frac{3s+1}{s^2+9}+e^{-\pi s}\frac{s+1}{s^2+6s+10}.$$

Solution

Let

$$G_0(s)=\frac{2s}{s^2+4},\quad G_1(s)=-\frac{(3s+1)}{s^2+9},$$

and

$$G_2(s)=\frac{s+1}{s^2+6s+10}=\frac{(s+3)-2}{(s+3)^2+1}.$$

Then

$$g_0(t)=2\cos 2t,\quad g_1(t)=-3\cos 3t-\frac{1}{3}\sin 3t,$$

and

$$g_2(t)=e^{-3t}(\cos t-2\sin t).$$

Therefore Equation 7.4.12 and the linearity of $\mathcal{L}^{-1}$ imply that

$$\begin{aligned} h(t) &= 2\cos 2t-u(t-\pi/2)\left[3\cos 3(t-\pi/2)+\frac{1}{3}\sin 3\left(t-\frac{\pi}{2}\right)\right] \\ &= +u(t-\pi)e^{-3(t-\pi)}\left[\cos(t-\pi)-2\sin(t-\pi)\right]. \end{aligned}$$

Using the trigonometric identities Equation 7.4.8 and Equation 7.4.9, we can rewrite this as

$$h(t) = 2\cos 2t+u(t-\pi/2)\left(3\sin 3t-\frac{1}{3}\cos 3t\right) \tag{7.4.13}$$

$$= -u(t-\pi)e^{-3(t-\pi)}(\cos t-2\sin t) \tag{7.4.14}$$

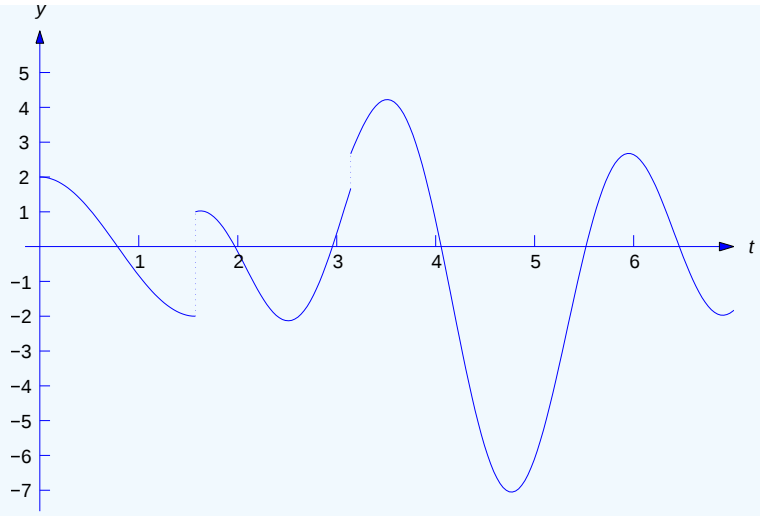

Figure 8.4.5 : The piecewise continuous function Equation 7.4.13.

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7.4E: The Unit Step Function (Exercises)

Q7.4.1

In *Exercises 7.4.1-7.4.6* find the Laplace transform. Then express the given function f in terms of unit step functions and find $\mathcal{L}(f)$. Graph f for *Exercises 7.4.3* and *7.4.4*.

1. $f(t)=\begin{cases}1, & 0 \leq t<4, \\ t, & t \geq 4.\end{cases}$

2. $f(t)=\begin{cases}t, & 0 \leq t<1, \\ 1, & t \geq 1.\end{cases}$

3. $f(t)=\begin{cases}2t-1, & 0 \leq t<2, \\ t, & t \geq 2.\end{cases}$

4. $f(t)=\begin{cases}1, & 0 \leq t<1, \\ t+2, & t \geq 1.\end{cases}$

5. $f(t)=\begin{cases}t-1, & 0 \leq t<2, \\ 4, & t \geq 2.\end{cases}$

6. $f(t)=\begin{cases}t^2, & 0 \leq t<1, \\ 0, & t \geq 1.\end{cases}$

Q7.4.2

In *Exercises 7.4.7-7.4.18* express the given function f in terms of unit step functions and find $\mathcal{L}(f)$. Graph f for *Exercises 7.4.15-7.4.18*.

7. $f(t)=\begin{cases}0, & 0 \leq t<2, \\ t^2+3t, & t \geq 2.\end{cases}$

8. $f(t)=\begin{cases}t^2+2, & 0 \leq t<1, \\ t, & t \geq 1.\end{cases}$

9. $f(t)=\begin{cases}te^t, & 0 \leq t<1, \\ e^t, & t \geq 1.\end{cases}$

10. $f(t)=\begin{cases}e^{-t}, & 0 \leq t<1, \\ e^{-2t}, & t \geq 1.\end{cases}$

11. $f(t)=\begin{cases}-t, & 0 \leq t<2, \\ t-4, & 2 \leq t<3, \\ 1, & t \geq 3.\end{cases}$

12. $f(t)=\begin{cases}0, & 0 \leq t<1, \\ t, & 1 \leq t<2, \\ 0, & t \geq 2.\end{cases}$

13. $f(t)=\begin{cases}t, & 0 \leq t<1, \\ t^2, & 1 \leq t<2, \\ 0, & t \geq 2.\end{cases}$

14. $f(t)=\begin{cases} t, & 0\le t<1, \\ 2-t, & 1\le t<2, \\ 6, & t>2. \end{cases}$

15. $f(t)=\begin{cases} \sin t, & 0\le t<\frac{\pi}{2} \\ 2\sin t, & \frac{\pi}{2}\le t<\pi \\ \cos t, & t\ge\pi \end{cases}$

16. $f(t)=\begin{cases} 2, & 0\le t<1, \\ -2t+2, & 1\le t<3, \\ 3t, & t\ge 3. \end{cases}$

17. $f(t)=\begin{cases} 3, & 0\le t<2, \\ 3t+2, & 2\le t<4, \\ 4t, & t\ge 4. \end{cases}$

18. $f(t)=\begin{cases} (t+1)^2, & 0\le t<1, \\ (t+2)^2, & t\ge 1. \end{cases}$

Q7.4.3

In *Exercises 7.4.19-7.4.28* express the inverse transforms in terms of step functions, and then find distinct formulas the for inverse transforms on the appropriate intervals, as in Example 7.4.7. Graph the inverse transform for *Exercises 7.4.21*, *7.4.22*, and *7.4.25*.

19. $H(s)=\frac{e^{-2s}}{s-2}$

20. $H(s)=\frac{e^{-s}}{s(s+1)}$

21. $H(s)=\frac{e^{-s}}{s^3}+\frac{e^{-2s}}{s^2}$

22. $H(s)=\left(\frac{2}{s}+\frac{1}{s^2}\right)+e^{-s}\left(\frac{3}{s}-\frac{1}{s^2}\right)+e^{-3s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$

23. $H(s)=\left(\frac{5}{s}-\frac{1}{s^2}\right)+e^{-3s}\left(\frac{6}{s}+\frac{7}{s^2}\right)+\frac{3e^{-6s}}{s^3}$

24. $H(s)=\frac{e^{-\pi s}(1-2s)}{s^2+4s+5}$

25. $H(s)=\left(\frac{1}{s}-\frac{s}{s^2+1}\right)+e^{-\frac{\pi}{2}s}\left(\frac{3s-1}{s^2+1}\right)$

26. $H(s)=e^{-2s}\left[\frac{3(s-3)}{(s+1)(s-2)}-\frac{s+1}{(s-1)(s-2)}\right]$

27. $H(s)=\frac{1}{s}+\frac{1}{s^2}+e^{-s}\left(\frac{3}{s}+\frac{2}{s^2}\right)+e^{-3s}\left(\frac{4}{s}+\frac{3}{s^2}\right)$

28. $H(s)=\frac{1}{s}-\frac{2}{s^3}+e^{-2s}\left(\frac{3}{s}-\frac{1}{s^3}\right)+\frac{e^{-4s}}{s^2}$

Q7.4.4

29. Find $\mathcal{L}\left(u(t-\tau)\right)$.

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7.5: Constant Coefficient Equations with Piecewise Continuous Forcing Functions

We'll now consider initial value problems of the form

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1, \tag{7.5.1}$$

where a, b, and c are constants ($a \neq 0$) and f is piecewise continuous on $[0, \infty)$. Problems of this kind occur in situations where the input to a physical system undergoes instantaneous changes, as when a switch is turned on or off or the forces acting on the system change abruptly.

It can be shown (*Exercises 8.5.23* and *8.5.24*) that the differential equation in Equation 7.5.1 has no solutions on an open interval that contains a jump discontinuity of f. Therefore we must define what we mean by a solution of Equation 7.5.1 on $[0, \infty)$ in the case where f has jump discontinuities. The next theorem motivates our definition. We omit the proof.

Theorem 7.5.1

Suppose a, b, and c are constants $(a \neq 0)$, and f is piecewise continuous on $[0, \infty)$. with jump discontinuities at $t_1, \ldots, t_n$, where

$$0 < t_1 < \cdots < t_n.$$

Let k_0 and k_1 be arbitrary real numbers. Then there is a unique function y defined on $[0, \infty)$ with these properties:

a. $y(0) = k_0$ and $y'(0) = k_1$.
b. y and y' are continuous on $[0, \infty)$.
c. y'' is defined on every open subinterval of $[0, \infty)$ that does not contain any of the points $t_1, \ldots, t_n$, and

$$ay'' + by' + cy = f(t)$$

on every such subinterval.
d. y'' has limits from the right and left at $t_1, \ldots, t_n$.

We define the function y of Theorem $7.5.1$ to be the solution of the initial value problem Equation 7.5.1.

We begin by considering initial value problems of the form

$$ay'' + by' + cy = \begin{cases} f_0(t), & 0 \le t < t_1, \\ f_1(t), & t \ge t_1, \end{cases} \qquad y(0) = k_0, \quad y'(0) = k_1, \tag{7.5.2}$$

where the forcing function has a single jump discontinuity at t_1.

Howto: Solve Constant Coefficient Equations with Piecewise Continuous Forcing Functions

We can solve Equation 7.5.2 by the these steps:

- **Step 1.** Find the solution y_0 of the initial value problem

$$ay'' + by' + cy = f_0(t), \quad y(0) = k_0, \quad y'(0) = k_1.$$

- **Step 2.** Compute $c_0 = y_0(t_1)$ and $c_1 = y_0'(t_1)$.
- **Step 3.** Find the solution y_1 of the initial value problem

$$ay'' + by' + cy = f_1(t), \quad y(t_1) = c_0, \quad y'(t_1) = c_1.$$

- **Step 4.** Obtain the solution y of Equation 7.5.2 as

$$y = \begin{cases} y_0(t), & 0 \le t < t_1 \\ y_1(t), & t \ge t_1. \end{cases}$$

It is shown in *Exercise 8.5.23* that y' exists and is continuous at t_1. The next example illustrates this procedure.

Example 7.5.1

Solve the initial value problem

$$y''+y=f(t), \quad y(0)=2,\ y'(0)=-1, \tag{7.5.3}$$

where

$$f(t)=\begin{cases} 1, & 0\le t<\frac{\pi}{2}, \\ -1, & t\ge\frac{\pi}{2}. \end{cases}$$

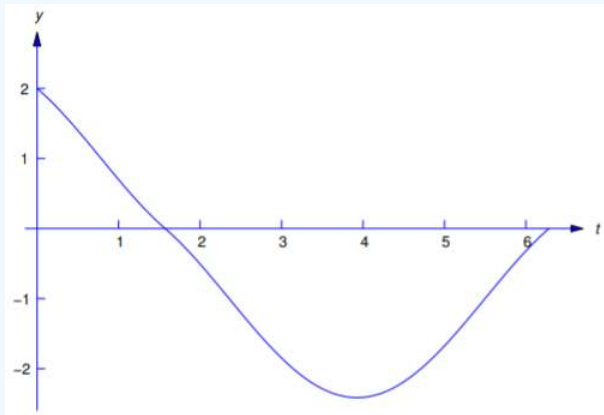

Figure 7.5.1: Graph of Equation 7.5.4

Solution

The initial value problem in Step 1 is

$$y''+y=1, \quad y(0)=2, \quad y'(0)=-1.$$

We leave it to you to verify that its solution is

$$y_0=1+\cos t-\sin t.$$

Doing Step 2 yields $y_0(\pi/2)=0$ and $y_0'(\pi/2)=-1$, so the second initial value problem is

$$y''+y=-1, \quad y\left(\frac{\pi}{2}\right)=0,\ y'\left(\frac{\pi}{2}\right)=-1.$$

We leave it to you to verify that the solution of this problem is

$$y_1=-1+\cos t+\sin t.$$

Hence, the solution of Equation ??? is

$$y=\begin{cases} 1+\cos t-\sin t, & 0\le t<\frac{\pi}{2}, \\ -1+\cos t+\sin t, & t\ge\frac{\pi}{2} \end{cases} \tag{7.5.4}$$

If f_0 and f_1 are defined on $[0,\infty)$, we can rewrite Equation 7.5.2 as

$$ay''+by'+cy=f_0(t)+u(t-t_1)\left(f_1(t)-f_0(t)\right), \quad y(0)=k_0, \quad y'(0)=k_1,$$

and apply the method of Laplace transforms. We'll now solve the problem considered in Example [example:8.5.1} by this method.

Example 7.5.2

Use the Laplace transform to solve the initial value problem

$$y''+y=f(t), \quad y(0)=2,\ y'(0)=-1, \tag{7.5.5}$$

where

$$f(t)=\begin{cases} 1, & 0\le t<\frac{\pi}{2}, \\ -1, & t\ge\frac{\pi}{2}. \end{cases}$$

Solution

Here

$$f(t)=1-2u\left(t-\frac{\pi}{2}\right),$$

so Theorem $7.5.1$ (with $g(t)=1$) implies that

$$\mathcal{L}(f)=\frac{1-2e^{-\pi s/2}}{s}.$$

Therefore, transforming Equation 7.5.5 yields

$$(s^2+1)Y(s)=\frac{1-2e^{-\pi s/2}}{s}-1+2s,$$

so

$$Y(s)=(1-2e^{-\pi s/2})G(s)+\frac{2s-1}{s^2+1}, \tag{7.5.6}$$

with

$$G(s)=\frac{1}{s(s^2+1)}.$$

The form for the partial fraction expansion of G is

$$\frac{1}{s(s^2+1)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}. \tag{7.5.7}$$

Multiplying through by $s(s^2+1)$ yields

$$A(s^2+1)+(Bs+C)s=1,$$

or

$$(A+B)s^2+Cs+A=1.$$

Equating coefficients of like powers of s on the two sides of this equation shows that $A=1$, $B=-A=-1$ and $C=0$. Hence, from Equation 7.5.7,

$$G(s)=\frac{1}{s}-\frac{s}{s^2+1}.$$

Therefore

$$g(t)=1-\cos t.$$

From this, Equation 7.5.6, and Theorem 8.4.2,

$$y=1-\cos t-2u\left(t-\frac{\pi}{2}\right)\left(1-\cos\left(t-\frac{\pi}{2}\right)\right)+2\cos t-\sin t.$$

Simplifying this (recalling that $\cos(t-\pi/2)=\sin t$) yields

$$y=1+\cos t-\sin t-2u\left(t-\frac{\pi}{2}\right)(1-\sin t),$$

or

$$y=\begin{cases} 1+\cos t-\sin t, & 0\le t<\frac{\pi}{2} \\ -1+\cos t+\sin t, & t\ge\frac{\pi}{2} \end{cases}$$

which is the result obtained in Example 7.5.1.

Note

It isn't obvious that using the Laplace transform to solve Equation 7.5.2 as we did in Example 7.5.2 yields a function y with the properties stated in Theorem 7.5.1; that is, such that y and y' are continuous on $[0,\infty)$ and y'' has limits from the right and left at t_1. However, this is true if f_0 and f_1 are continuous and of exponential order on $[0,\infty)$. A proof is sketched in *Exercises 8.6.11–8.6.13*.

Example 7.5.3

Solve the initial value problem

$$y''-y=f(t), \quad y(0)=-1,\ y'(0)=2, \tag{7.5.8}$$

where

$$f(t)=\begin{cases} t, & 0\le t<1, \\ 1, & t\ge 1. \end{cases}$$

Solution

Here

$$f(t)=t-u(t-1)(t-1),$$

so

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(t)-\mathcal{L}\left(u(t-1)(t-1)\right) \\ &= \mathcal{L}(t)-e^{-s}\mathcal{L}(t)\ (from\ Theorem\ 8.4.1) \\ &= \frac{1}{s^2}-\frac{e^{-s}}{s^2}. \end{aligned}$$

Since transforming Equation 7.5.8 yields

$$(s^2-1)Y(s)=\mathcal{L}(f)+2-s,$$

we see that

$$Y(s)=(1-e^{-s})H(s)+\frac{2-s}{s^2-1}, \tag{7.5.9}$$

where

$$H(s)=\frac{1}{s^2(s^2-1)}=\frac{1}{s^2-1}-\frac{1}{s^2};$$

therefore

$$h(t)=\sinh t-t. \tag{7.5.10}$$

Since

$$\mathcal{L}^{-1}\left(\frac{2-s}{s^2-1}\right)=2\sinh t-\cosh t,$$

we conclude from Equation 7.5.9, Equation 7.5.10, and Theorem 7.5.1 that

$$y=\sinh t-t-u(t-1)\left(\sinh(t-1)-t+1\right)+2\sinh t-\cosh t,$$

or

$$y=3\sinh t-\cosh t-t-u(t-1)\left(\sinh(t-1)-t+1\right) \tag{7.5.11}$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1 = 1$.

Example 7.5.4

Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0,\ y'(0) = 0, \tag{7.5.12}$$

where

$$f(t) = \begin{cases} 0, & 0 \le t < \frac{\pi}{4} \\ \cos 2t & \frac{\pi}{4} \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

Solution

Here

$$f(t) = u(t - \pi/4)\cos 2t - u(t - \pi)\cos 2t,$$

so

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}\left(u(t - \pi/4)\cos 2t\right) - \mathcal{L}\left(u(t - \pi)\cos 2t\right) \\ &= e^{-\pi s/4}\mathcal{L}\left(\cos 2(t + \pi/4)\right) - e^{-\pi s}\mathcal{L}\left(\cos 2(t + \pi)\right) \\ &= -e^{-\pi s/4}\mathcal{L}(\sin 2t) - e^{-\pi s}\mathcal{L}(\cos 2t) \\ &= -\frac{2e^{-\pi s/4}}{s^2 + 4} - \frac{se^{-\pi s}}{s^2 + 4}. \end{aligned}$$

Since transforming Equation 7.5.12 yields

$$(s^2 + 1)Y(s) = \mathcal{L}(f),$$

we see that

$$Y(s) = e^{-\pi s/4}H_1(s) + e^{-\pi s}H_2(s), \tag{7.5.13}$$

where

$$H_1(s) = -\frac{2}{(s^2 + 1)(s^2 + 4)} \quad \text{and} \quad H_2(s) = -\frac{s}{(s^2 + 1)(s^2 + 4)}. \tag{7.5.14}$$

To simplify the required partial fraction expansions, we first write

$$\frac{1}{(x + 1)(x + 4)} = \frac{1}{3}\left[\frac{1}{x + 1} - \frac{1}{x + 4}\right].$$

Setting $x = s^2$ and substituting the result in Equation 7.5.14 yields

$$H_1(s) = -\frac{2}{3}\left[\frac{1}{s^2 + 1} - \frac{1}{s^2 + 4}\right] \quad \text{and} \quad H_2(s) = -\frac{1}{3}\left[\frac{s}{s^2 + 1} - \frac{s}{s^2 + 4}\right].$$

The inverse transforms are

$$h_1(t) = -\frac{2}{3}\sin t + \frac{1}{3}\sin 2t \quad \text{and} \quad h_2(t) = -\frac{1}{3}\cos t + \frac{1}{3}\cos 2t.$$

From Equation 7.5.13 and Theorem 8.4.2,

$$y = u\left(t - \frac{\pi}{4}\right)h_1\left(t - \frac{\pi}{4}\right) + u(t - \pi)h_2(t - \pi). \tag{7.5.15}$$

Since

$$\begin{aligned} h_1\left(t-\frac{\pi}{4}\right) &= -\frac{2}{3}\sin\left(t-\frac{\pi}{4}\right)+\frac{1}{3}\sin 2\left(t-\frac{\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{3}(\sin t-\cos t)-\frac{1}{3}\cos 2t \end{aligned}$$

and

$$\begin{aligned} h_2(t-\pi) &= -\frac{1}{3}\cos(t-\pi)+\frac{1}{3}\cos 2(t-\pi) \\ &= \frac{1}{3}\cos t+\frac{1}{3}\cos 2t, \end{aligned}$$

Equation 7.5.15 can be rewritten as

$$y=-\frac{1}{3}u\left(t-\frac{\pi}{4}\right)\left(\sqrt{2}(\sin t-\cos t)+\cos 2t\right)+\frac{1}{3}u(t-\pi)(\cos t+\cos 2t)$$

or

$$\begin{cases} 0, & 0\le t<\frac{\pi}{4} \\ -\frac{\sqrt{2}}{3}(\sin t-\cos t)-\frac{1}{3}\cos 2t, & \frac{\pi}{4}\le t<\pi, \\ -\frac{\sqrt{2}}{3}\sin t+\frac{1+\sqrt{2}}{3}\cos t, & t\ge\pi \end{cases} \tag{7.5.16}$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1=\pi/4$ and $t_2=\pi$ (Figure 7.5.2).

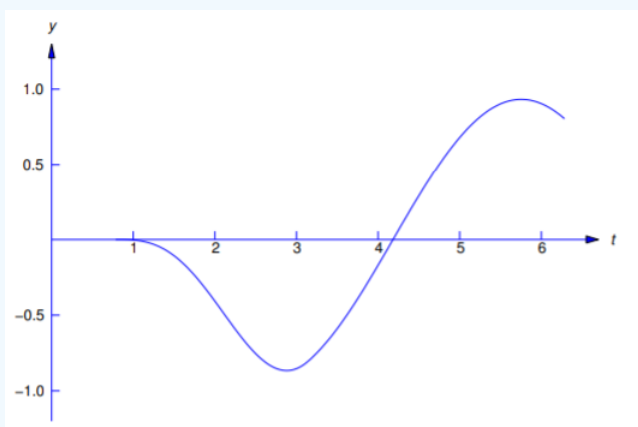

Figure 7.5.2: Graph of Equation 7.5.16

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7.5E: Constant Coefficient Equations with Piecewise Continuous Forcing Functions (Exercises)

Q7.5.1

In *Exercises 7.5.1-7.5.20* use the Laplace transform to solve the initial value problem. Graph the solution for *Exercise 7.5.6, 7.5.9, 7.5.13,* and *7.5.19.*

1. $y''+y=\begin{cases}3, & 0\le t<\pi,\\ 0, & t\ge\pi,\end{cases}\qquad y(0)=0,\quad y'(0)=0$

2. $y''+y=\begin{cases}3, & 0\le t<4,\\ 2t-5, & t>4,\end{cases}\qquad y(0)=1,\quad y'(0)=0$

3. $y''-2y'=\begin{cases}4, & 0\le t<1,\\ 6, & t\ge 1,\end{cases}\qquad y(0)=-6,\quad y'(0)=1$

4. $y''-y=\begin{cases}e^{2t}, & 0\le t<2,\\ 1, & t\ge 2,\end{cases}\qquad y(0)=3,\quad y'(0)=-1$

5. $y''-3y'+2y=\begin{cases}0, & 0\le t<1,\\ 1, & 1\le t<2,\\ -1, & t\ge 2,\end{cases}\qquad y(0)=-3,\quad y'(0)=1$

6. $y''+4y=\begin{cases}|\sin t|, & 0\le t<2\pi,\\ 0, & t\ge 2\pi,\end{cases}\qquad y(0)=-3,\quad y'(0)=1$

7. $y''-5y'+4y=\begin{cases}1, & 0\le t<1\\ -1, & 1\le t<2,\\ 0, & t\ge 2,\end{cases}\qquad y(0)=3,\quad y'(0)=-5$

8. $y''+9y=\begin{cases}\cos t, & 0\le t<\frac{3\pi}{2},\\ \sin t, & t\ge\frac{3\pi}{2},\end{cases}\qquad y(0)=0,\ y'(0)=0$

9. $y''+4y=\begin{cases}t, & 0\le t<\frac{\pi}{2},\\ \pi, & t\ge\frac{\pi}{2},\end{cases}\qquad y(0)=0,\ y'(0)=0$

10. $y''+y=\begin{cases}t, & 0\le t<\pi,\\ -t, & t\ge\pi,\end{cases}\qquad y(0)=0,\ y'(0)=0$

11. $y''-3y'+2y=\begin{cases}0, & 0\le t<2,\\ 2t-4, & t\ge 2,\end{cases},\qquad y(0)=0,\quad y'(0)=0$

12. $y''+y=\begin{cases}t, & 0\le t<2\pi,\\ -2t, & t\ge 2\pi,\end{cases}\qquad y(0)=1,\quad y'(0)=2$

13. $y''+3y'+2y=\begin{cases}1, & 0\le t<2,\\ -1, & t\ge 2,\end{cases}\qquad y(0)=0,\ y'(0)=0$

14. $y''-4y'+3y=\begin{cases}-1, & 0\le t<1,\\ 1, & t\ge 1,\end{cases}\qquad y(0)=0,\ y'(0)=0$

15. $y''+2y'+y=\begin{cases}e^t, & 0\le t<1,\\ e^t-1, & t\ge 1,\end{cases}\qquad y(0)=3,\ y'(0)=-1$

16. $y''+2y'+y=\begin{cases}4e^t, & 0\le t<1,\\ 0, & t\ge 1,\end{cases}\qquad y(0)=0,\ y'(0)=0$

17. $y''+3y'+2y=\begin{cases} e^{-t}, & 0\le t<1, \\ 0, & t\ge 1, \end{cases} \quad y(0)=1,\ y'(0)=-1$

18. $y''-4y'+4y=\begin{cases} e^{2t}, & 0\le t<2, \\ -e^{2t}, & t\ge 2, \end{cases} \quad y(0)=0,\ y'(0)=-1$

19. $y''=\begin{cases} t^2, & 0\le t<1, \\ -t, & 1\le t<2, \\ t+1, & t\ge 2, \end{cases} \quad y(0)=1,\ y'(0)=0$

20. $y''+2y'+2y=\begin{cases} 1, & 0\le t<2\pi, \\ t, & 2\pi\le t<3\pi, \\ -1, & t\ge 3\pi, \end{cases} \quad y(0)=2, \quad y'(0)=-1$

Q7.5.2

21. Solve the initial value problem

$$y''=f(t), \quad y(0)=0, \quad y'(0)=0,$$

where

$$f(t)=m+1, \quad m\le t<m+1, \quad m=0,1,2,\ldots.$$

22. Solve the given initial value problem and find a formula that does not involve step functions and represents y on each interval of continuity of f.

a. $y''+y=f(t), \quad y(0)=0, \quad y'(0)=0$;
$f(t)=m+1, \quad m\pi\le t<(m+1)\pi, \quad m=0,1,2,\ldots.$

b. $y''+y=f(t), \quad y(0)=0, \quad y'(0)=0$;
$f(t)=(m+1)t, \quad 2m\pi\le t<2(m+1)\pi, \quad m=0,1,2,\ldots$ HINT: *You'll need the formula*

$$1+2+\cdots+m=\frac{m(m+1)}{2}.$$

c. $y''+y=f(t), \quad y(0)=0, \quad y'(0)=0$;
$f(t)=(-1)^m, \quad m\pi\le t<(m+1)\pi, \quad m=0,1,2,\ldots.$

d. $y''-y=f(t), \quad y(0)=0, \quad y'(0)=0$;
$f(t)=m+1, \quad m\le t<(m+1), \quad m=0,1,2,\ldots.$
HINT: *You will need the formula*

$$1+r+\ldots+r^m=\frac{1-r^{m+1}}{1-r}(r\ne 1).$$

e. $y''+2y'+2y=f(t), \quad y(0)=0, \quad y'(0)=0$;
$f(t)=(m+1)(\sin t+2\cos t), \quad 2m\pi\le t<2(m+1)\pi, \quad m=0,1,2,\ldots.$
(See the hint in d.)

f. $y''-3y'+2y=f(t), \quad y(0)=0, \quad y'(0)=0$;

g. $f(t)=m+1, \quad m\le t<m+1, \quad m=0,1,2,\ldots.$
(See the hints in b and d.)

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7.6: Convolution

In this section we consider the problem of finding the inverse Laplace transform of a product $H(s)=F(s)G(s)$, where F and G are the Laplace transforms of known functions f and g. To motivate our interest in this problem, consider the initial value problem

$$ay''+by'+cy=f(t), \quad y(0)=0, \quad y'(0)=0.$$

Taking Laplace transforms yields

$$(as^2+bs+c)Y(s)=F(s),$$

so

$$Y(s)=F(s)G(s), \tag{7.6.1}$$

where

$$G(s)=\frac{1}{as^2+bs+c}.$$

Until now wen't been interested in the factorization indicated in Equation 7.6.1, since we dealt only with differential equations with specific forcing functions. Hence, we could simply do the indicated multiplication in Equation 7.6.1 and use the table of Laplace transforms to find $y=\mathcal{L}^{-1}(Y)$. However, this isn't possible if we want a **formula** for y in terms of f, which may be unspecified.

To motivate the formula for $\mathcal{L}^{-1}(FG)$, consider the initial value problem

$$y'-ay=f(t), \quad y(0)=0, \tag{7.6.2}$$

which we first solve without using the Laplace transform. The solution of the differential equation in Equation 7.6.2 is of the form $y=ue^{at}$ where

$$u'=e^{-at}f(t).$$

Integrating this from 0 to t and imposing the initial condition $u(0)=y(0)=0$ yields

$$u=\int_0^t e^{-a\tau}f(\tau)\,d\tau.$$

Therefore

$$y(t)=e^{at}\int_0^t e^{-a\tau}f(\tau)\,d\tau=\int_0^t e^{a(t-\tau)}f(\tau)\,d\tau. \tag{7.6.3}$$

Now we'll use the Laplace transform to solve Equation 7.6.2 and compare the result to Equation 7.6.3. Taking Laplace transforms in Equation 7.6.2 yields

$$(s-a)Y(s)=F(s),$$

so

$$Y(s)=F(s)\frac{1}{s-a},$$

which implies that

$$y(t)=\mathcal{L}^{-1}\left(F(s)\frac{1}{s-a}\right). \tag{7.6.4}$$

If we now let $g(t)=e^{at}$, so that

$$G(s)=\frac{1}{s-a},$$

then Equation 7.6.3 and Equation 7.6.4 can be written as

$$y(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau$$

and

$$y=\mathcal{L}^{-1}(FG),$$

respectively. Therefore

$$\mathcal{L}^{-1}(FG)=\int_0^t f(\tau)g(t-\tau)\,d\tau \tag{7.6.5}$$

in this case.

This motivates the next definition.

Definition 8.6.1 : Convolution

The **convolution** $f*g$ of two functions f and g is defined by

$$(f*g)(t)=\int_0^t f(\tau)g(t-\tau)\,d\tau.$$

It can be shown (**Exercise 8.6.6**) that $f*g=g*f$; that is,

$$\int_0^t f(t-\tau)g(\tau)\,d\tau=\int_0^t f(\tau)g(t-\tau)\,d\tau.$$

Equation 7.6.5 shows that $\mathcal{L}^{-1}(FG)=f*g$ in the special case where $g(t)=e^{at}$. This next theorem states that this is true in general.

Theorem 8.6.2 : The Convolution Theorem

If $\mathcal{L}(f)=F$ and $\mathcal{L}(g)=G,$ then

$$\mathcal{L}(f*g)=FG.$$

A complete proof of the convolution theorem is beyond the scope of this book. However, we'll assume that $f*g$ has a Laplace transform and verify the conclusion of the theorem in a purely computational way. By the definition of the Laplace transform,

$$\mathcal{L}(f*g)=\int_0^\infty e^{-st}(f*g)(t)\,dt=\int_0^\infty e^{-st}\int_0^t f(\tau)g(t-\tau)\,d\tau\,dt.$$

This iterated integral equals a double integral over the region shown in Figure 8.6.1 . Reversing the order of integration yields

$$\mathcal{L}(f*g)=\int_0^\infty f(\tau)\int_\tau^\infty e^{-st}g(t-\tau)\,dt\,d\tau. \tag{7.6.6}$$

However, the substitution $x=t-\tau$ shows that

$$\begin{aligned}\int_\tau^\infty e^{-st}g(t-\tau)\,dt &= \int_0^\infty e^{-s(x+\tau)}g(x)\,dx\\ &= e^{-s\tau}\int_0^\infty e^{-sx}g(x)\,dx=e^{-s\tau}G(s).\end{aligned}$$

Substituting this into Equation 7.6.6 and noting that $G(s)$ is independent of τ yields

$$\begin{aligned}\mathcal{L}(f*g) &= \int_0^\infty e^{-s\tau}f(\tau)G(s)\,d\tau\\ &= G(s)\int_0^\infty e^{-st}f(\tau)\,d\tau=F(s)G(s).\end{aligned}$$

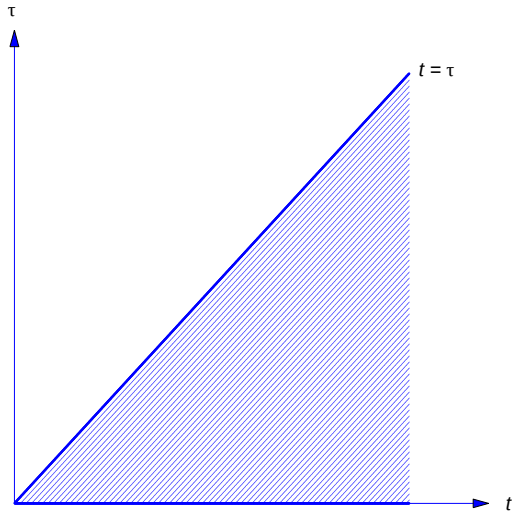

Figure 8.6.1

Example 8.6.1

Let

$$f(t) = e^{at} \quad \text{and} \quad g(t) = e^{bt} \qquad (a \neq b).$$

Verify that $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$, as implied by the convolution theorem.

Solution

We first compute

$$\begin{aligned}(f * g) &= \int_0^t e^{a\tau} e^{b(t-\tau)}\, d\tau \qquad &&= e^{bt} \int_0^t e^{(a-b)\tau} d\tau \\ &= e^{bt} \frac{e^{a-b}\tau}{a-b}\bigg|_0^t &&= \frac{e^{bt}\left[e^{(a-b)t} - 1\right]}{a-b} \\ &= \frac{e^{at} - e^{bt}}{a-b}\end{aligned}$$

Since

$$e^{at} \leftrightarrow \frac{1}{s-a} \quad \text{and} \quad e^{bt} \leftrightarrow \frac{1}{s-b},$$

it follows that

$$\begin{aligned}\mathcal{L}(f * g) &= \frac{1}{a-b}\left[\frac{1}{s-a} - \frac{1}{s-b}\right] \\ &= \frac{1}{(s-a)(s-b)} \\ &= \mathcal{L}(e^{at})\mathcal{L}(e^{bt}) = \mathcal{L}(f)\mathcal{L}(g).\end{aligned}$$

A Formula for the Solution of an Initial Value Problem

The convolution theorem provides a formula for the solution of an initial value problem for a linear constant coefficient second order equation with an unspecified. The next three examples illustrate this.

Example 8.6.2

Find a formula for the solution of the initial value problem

$$y'' - 2y' + y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{7.6.7}$$

Solution

Taking Laplace transforms in Equation 7.6.7 yields

$$(s^2 - 2s + 1)Y(s) = F(s) + (k_1 + k_0 s) - 2k_0.$$

Therefore

$$\begin{aligned} Y(s) &= \frac{1}{(s-1)^2} F(s) + \frac{k_1 + k_0 s - 2k_0}{(s-1)^2} \\ &= \frac{1}{(s-1)^2} F(s) + \frac{k_0}{s-1} + \frac{k_1 - k_0}{(s-1)^2}. \end{aligned}$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1}\left(\frac{k_0}{s-1} + \frac{k_1 - k_0}{(s-1)^2}\right) = e^t\left(k_0 + (k_1 - k_0)t\right).$$

Since

$$\frac{1}{(s-1)^2} \leftrightarrow te^t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2} F(s)\right) = \int_0^t \tau e^\tau f(t-\tau)\, d\tau.$$

Therefore the solution of Equation 7.6.7 is

$$y(t) = e^t\left(k_0 + (k_1 - k_0)t\right) + \int_0^t \tau e^\tau f(t-\tau)\, d\tau.$$

Example 8.6.3

Find a formula for the solution of the initial value problem

$$y'' + 4y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{7.6.8}$$

Solution

Taking Laplace transforms in Equation 7.6.8 yields

$$(s^2 + 4)Y(s) = F(s) + k_1 + k_0 s.$$

Therefore

$$Y(s) = \frac{1}{(s^2+4)} F(s) + \frac{k_1 + k_0 s}{s^2 + 4}.$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1}\left(\frac{k_1 + k_0 s}{s^2 + 4}\right) = k_0 \cos 2t + \frac{k_1}{2} \sin 2t.$$

Since

$$\frac{1}{(s^2+4)} \leftrightarrow \frac{1}{2} \sin 2t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+4)}F(s)\right)=\frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau\,d\tau.$$

Therefore the solution of Equation 7.6.8 is

$$y(t)=k_0\cos 2t+\frac{k_1}{2}\sin 2t+\frac{1}{2}\int_0^t f(t-\tau)\sin 2\tau\,d\tau.$$

Example 8.6.4

Find a formula for the solution of the initial value problem

$$y''+2y'+2y=f(t),\quad y(0)=k_0,\quad y'(0)=k_1. \tag{7.6.9}$$

Solution

Taking Laplace transforms in Equation 7.6.9 yields

$$(s^2+2s+2)Y(s)=F(s)+k_1+k_0s+2k_0.$$

Therefore

$$\begin{aligned}Y(s)&=\frac{1}{(s+1)^2+1}F(s)+\frac{k_1+k_0s+2k_0}{(s+1)^2+1}\\&=\frac{1}{(s+1)^2+1}F(s)+\frac{(k_1+k_0)+k_0(s+1)}{(s+1)^2+1}\end{aligned}$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1}\left(\frac{(k_1+k_0)+k_0(s+1)}{(s+1)^2+1}\right)=e^{-t}\left((k_1+k_0)\sin t+k_0\cos t\right).$$

Since

$$\frac{1}{(s+1)^2+1}\leftrightarrow e^{-t}\sin t\quad\text{and}\quad F(s)\leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}F(s)\right)=\int_0^t f(t-\tau)e^{-\tau}\sin\tau\,d\tau.$$

Therefore the solution of Equation 7.6.9 is

$$y(t)=e^{-t}\left((k_1+k_0)\sin t+k_0\cos t\right)+\int_0^t f(t-\tau)e^{-\tau}\sin\tau\,d\tau. \tag{7.6.10}$$

Evaluating Convolution Integrals

We'll say that an integral of the form $\int_0^t u(\tau)v(t-\tau)\,d\tau$ is a **convolution integral**. The convolution theorem provides a convenient way to evaluate convolution integrals.

Example 8.6.5

Evaluate the convolution integral

$$h(t)=\int_0^t (t-\tau)^5\tau^7\,d\tau.$$

Solution

We could evaluate this integral by expanding $(t-\tau)^5$ in powers of τ and then integrating. However, the convolution theorem provides an easier way. The integral is the convolution of $f(t)=t^5$ and $g(t)=t^7$. Since

$$t^5 \leftrightarrow \frac{5!}{s^6} \quad \text{and} \quad t^7 \leftrightarrow \frac{7!}{s^8},$$

the convolution theorem implies that

$$h(t) \leftrightarrow \frac{5!7!}{s^{14}} = \frac{5!7!}{13!}\frac{13!}{s^{14}},$$

where we have written the second equality because

$$\frac{13!}{s^{14}} \leftrightarrow t^{13}.$$

Hence,

$$h(t) = \frac{5!7!}{13!}t^{13}.$$

Example 8.6.6

Use the convolution theorem and a partial fraction expansion to evaluate the convolution integral

$$h(t) = \int_0^t \sin a(t-\tau)\cos b\tau\, d\tau \quad (|a| \neq |b|).$$

Solution

Since

$$\sin at \leftrightarrow \frac{a}{s^2+a^2} \quad \text{and} \quad \cos bt \leftrightarrow \frac{s}{s^2+b^2},$$

the convolution theorem implies that

$$H(s) = \frac{a}{s^2+a^2}\frac{s}{s^2+b^2}.$$

Expanding this in a partial fraction expansion yields

$$H(s) = \frac{a}{b^2-a^2}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right].$$

Therefore

$$h(t) = \frac{a}{b^2-a^2}(\cos at - \cos bt).$$

Volterra Integral Equations

An equation of the form

$$y(t) = f(t) + \int_0^t k(t-\tau)y(\tau)\, d\tau \tag{7.6.11}$$

is a **Volterra integral equation**. Here f and k are given functions and y is unknown. Since the integral on the right is a convolution integral, the convolution theorem provides a convenient formula for solving Equation 7.6.11. Taking Laplace transforms in Equation 7.6.11 yields

$$Y(s) = F(s) + K(s)Y(s),$$

and solving this for $Y(s)$ yields

$$Y(s)=\frac{F(s)}{1-K(s)}.$$

We then obtain the solution of Equation 7.6.11 as $y=\mathcal{L}^{-1}(Y)$.

Example 8.6.7

Solve the integral equation

$$y(t)=1+2\int_0^t e^{-2(t-\tau)}y(\tau)\,d\tau. \tag{7.6.12}$$

Solution

Taking Laplace transforms in Equation 7.6.12 yields

$$Y(s)=\frac{1}{s}+\frac{2}{s+2}Y(s),$$

and solving this for $Y(s)$ yields

$$Y(s)=\frac{1}{s}+\frac{2}{s^2}.$$

Hence,

$$y(t)=1+2t.$$

Transfer Functions

The next theorem presents a formula for the solution of the general initial value problem

$$ay''+by'+cy=f(t),\quad y(0)=k_0,\quad y'(0)=k_1,$$

where we assume for simplicity that f is continuous on $[0,\infty)$ and that $\mathcal{L}(f)$ exists. In **Exercises 8.6.11-8.6.14** it is shown that the formula is valid under much weaker conditions on f.

Theorem 8.6.3

Suppose f is continuous on $[0,\infty)$ and has a Laplace transform. Then the solution of the initial value problem

$$ay''+by'+cy=f(t),\quad y(0)=k_0,\quad y'(0)=k_1, \tag{7.6.13}$$

is

$$y(t)=k_0y_1(t)+k_1y_2(t)+\int_0^t w(\tau)f(t-\tau)\,d\tau, \tag{7.6.14}$$

where y_1 and y_2 satisfy

$$ay_1''+by_1'+cy_1=0,\quad y_1(0)=1,\quad y_1'(0)=0, \tag{7.6.15}$$

and

$$ay_2''+by_2'+cy_2=0,\quad y_2(0)=0,\quad y_2'(0)=1, \tag{7.6.16}$$

and

$$w(t)=\frac{1}{a}y_2(t). \tag{7.6.17}$$

Proof

Taking Laplace transforms in Equation 7.6.13 yields

$$p(s)Y(s)=F(s)+a(k_1+k_0s)+bk_0,$$

where

$$p(s) = as^2 + bs + c.$$

Hence,

$$Y(s) = W(s)F(s) + V(s) \tag{7.6.18}$$

with

$$W(s) = \frac{1}{p(s)} \tag{7.6.19}$$

and

$$V(s) = \frac{a(k_1 + k_0 s) + bk_0}{p(s)}. \tag{7.6.20}$$

Taking Laplace transforms in Equation 7.6.15and Equation 7.6.16shows that

$$p(s)Y_1(s) = as + b \quad \text{and} \quad p(s)Y_2(s) = a.$$

Therefore

$$Y_1(s) = \frac{as + b}{p(s)}$$

and

$$Y_2(s) = \frac{a}{p(s)}. \tag{7.6.21}$$

Hence, Equation 7.6.20can be rewritten as

$$V(s) = k_0 Y_1(s) + k_1 Y_2(s).$$

Substituting this into Equation 7.6.18yields

$$Y(s) = k_0 Y_1(s) + k_1 Y_2(s) + \frac{1}{a} Y_2(s) F(s).$$

Taking inverse transforms and invoking the convolution theorem yields Equation 7.6.14 Finally, Equation 7.6.19 and Equation 7.6.21imply Equation 7.6.17.

It is useful to note from Equation 7.6.14that y is of the form

$$y = v + h,$$

where

$$v(t) = k_0 y_1(t) + k_1 y_2(t)$$

depends on the initial conditions and is independent of the forcing function, while

$$h(t) = \int_0^t w(\tau) f(t - \tau)\, d\tau$$

depends on the forcing function and is independent of the initial conditions. If the zeros of the characteristic polynomial

$$p(s) = as^2 + bs + c$$

of the complementary equation have negative real parts, then y_1 and y_2 both approach zero as $t \to \infty$, so $\lim_{t\to\infty} v(t) = 0$ for any choice of initial conditions. Moreover, the value of $h(t)$ is essentially independent of the values of $f(t - \tau)$ for large τ, since $\lim_{\tau\to\infty} w(\tau) = 0$. In this case we say that v and h are **transient** and **steady state components**, respectively, of the solution y of Equation 7.6.13. These definitions apply to the initial value problem of Example 8.6.4 , where the zeros of

$$p(s)=s^2+2s+2=(s+1)^2+1$$

are $-1 \pm i$. From Equation 7.6.10, we see that the solution of the general initial value problem of Example 8.6.4 is $y=v+h$, where

$$v(t)=e^{-t}\left((k_1+k_0)\sin t+k_0\cos t\right)$$

is the transient component of the solution and

$$h(t)=\int_0^t f(t-\tau)e^{-\tau}\sin\tau\,d\tau$$

is the steady state component. The definitions don't apply to the initial value problems considered in Examples 8.6.2 and 8.6.3 , since the zeros of the characteristic polynomials in these two examples don't have negative real parts.

In physical applications where the input f and the output y of a device are related by Equation 7.6.13, the zeros of the characteristic polynomial usually do have negative real parts. Then $W=\mathcal{L}(w)$ is called the **transfer function** of the device. Since

$$H(s)=W(s)F(s),$$

we see that

$$W(s)=\frac{H(s)}{F(s)}$$

is the ratio of the transform of the steady state output to the transform of the input.

Because of the form of

$$h(t)=\int_0^t w(\tau)f(t-\tau)\,d\tau,$$

w is sometimes called the **weighting function** of the device, since it assigns weights to past values of the input f. It is also called the **impulse response** of the device, for reasons discussed in the next section.

Formula Equation 7.6.14 is given in more detail in **Exercises 8.6.8-8.6.10** for the three possible cases where the zeros of $p(s)$ are real and distinct, real and repeated, or complex conjugates, respectively.

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7.6E: Convolution (Exercises)

Q7.6.1

1. Express the inverse transform as an integral.

a. $\frac{1}{s^2(s^2+4)}$
b. $\frac{s}{(s+2)(s^2+9)}$
c. $\frac{s}{(s^2+4)(s^2+9)}$
d. $\frac{s}{(s^2+1)^2}$
e. $\frac{1}{s(s-a)}$
f. $\frac{1}{(s+1)(s^2+2s+2)}$
g. $\frac{1}{(s+1)^2(s^2+4s+5)}$
h. $\frac{1}{(s-1)^3(s+2)^2}$
i. $\frac{s-1}{s^2(s^2-2s+2)}$
j. $\frac{s(s+3)}{(s^2+4)(s^2+6s+10)}$
k. $\frac{1}{(s-3)^5s^6}$
l. $\frac{1}{(s-1)^3(s^2+4)}$
m. $\frac{1}{s^2(s-2)^3}$
n. $\frac{1}{s^7(s-2)^6}$

2. Find the Laplace transform.

a. $\int_0^t \sin a\tau \cos b(t-\tau)\,d\tau$
b. $\int_0^t e^\tau \sin a(t-\tau)\,d\tau$
c. $\int_0^t \sinh a\tau \cosh a(t-\tau)\,d\tau$
d. $\int_0^t \tau(t-\tau)\sin\omega\tau\cos\omega(t-\tau)\,d\tau$
e. $e^t\int_0^t \sin\omega\tau\cos\omega(t-\tau)\,d\tau$
f. $e^t\int_0^t \tau^2(t-\tau)e^\tau\,d\tau$
g. $e^{-t}\int_0^t e^{-\tau}\tau\cos\omega(t-\tau)\,d\tau$
h. $e^t\int_0^t e^{2\tau}\sinh(t-\tau)\,d\tau$
i. $\int_0^t \tau e^{2\tau}\sin 2(t-\tau)\,d\tau$
j. $\int_0^t (t-\tau)^3e^\tau\,d\tau$
k. $\int_0^t \tau^6e^{-(t-\tau)}\sin 3(t-\tau)\,d\tau$
l. $\int_0^t \tau^2(t-\tau)^3\,d\tau$
m. $\int_0^t (t-\tau)^7e^{-\tau}\sin 2\tau\,d\tau$
n. $\int_0^t (t-\tau)^4\sin 2\tau\,d\tau$

3. Find a formula for the solution of the initial value problem.

a. $y''+3y'+y=f(t),\quad y(0)=0,\quad y'(0)=0$
b. $y''+4y=f(t),\quad y(0)=0,\quad y'(0)=0$
c. $y''+2y'+y=f(t),\quad y(0)=0,\quad y'(0)=0$
d. $y''+k^2y=f(t),\quad y(0)=1,\quad y'(0)=-1$
e. $y''+6y'+9y=f(t),\quad y(0)=0,\quad y'(0)=-2$
f. $y''-4y=f(t),\quad y(0)=0,\quad y'(0)=3$
g. $y''-5y'+6y=f(t),\quad y(0)=1,\quad y'(0)=3$
h. $y''+\omega^2y=f(t),\quad y(0)=k_0,\quad y'(0)=k_1$

4. Solve the integral equation.

a. $y(t)=t-\int_0^t(t-\tau)y(\tau)\,d\tau$
b. $y(t)=\sin t-2\int_0^t\cos(t-\tau)y(\tau)\,d\tau$
c. $y(t)=1+2\int_0^t y(\tau)\cos(t-\tau)\,d\tau$
d. $y(t)=t+\int_0^t y(\tau)e^{-(t-\tau)}\,d\tau$
e. $y'(t)=t+\int_0^t y(\tau)\cos(t-\tau)\,d\tau,\ y(0)=4$
f. $y(t)=\cos t-\sin t+\int_0^t y(\tau)\sin(t-\tau)\,d\tau$

5. Use the convolution theorem to evaluate the integral.

a. $\int_0^t(t-\tau)^7\tau^8\,d\tau$
b. $\int_0^t(t-\tau)^{13}\tau^7\,d\tau$
c. $\int_0^t(t-\tau)^6\tau^7\,d\tau$
d. $\int_0^t e^{-\tau}\sin(t-\tau)\,d\tau$
e. $\int_0^t\sin\tau\cos 2(t-\tau)\,d\tau$

6. Show that

$$\int_0^t f(t-\tau)g(\tau)\,d\tau=\int_0^t f(\tau)g(t-\tau)\,d\tau$$

by introducing the new variable of integration $x=t-\tau$ in the first integral.

7. Use the convolution theorem to show that if $f(t)\leftrightarrow F(s)$ then

$$\int_0^t f(\tau)\,d\tau\leftrightarrow\frac{F(s)}{s}.$$

8. Show that if $p(s)=as^2+bs+c$ has distinct real zeros r_1 and r_2 then the solution of

$$ay''+by'+cy=f(t),\quad y(0)=k_0,\quad y'(0)=k_1$$

is

$$\begin{aligned}y(t)\ =\ & k_0\frac{r_2e^{r_1t}-r_1e^{r_2t}}{r_2-r_1}+k_1\frac{e^{r_2t}-e^{r_1t}}{r_2-r_1}\\ &+\frac{1}{a(r_2-r_1)}\int_0^t(e^{r_2\tau}-e^{r_1\tau})f(t-\tau)\,d\tau.\end{aligned}$$

9. Show that if $p(s)=as^2+bs+c$ has a repeated real zero r_1 then the solution of

$$ay''+by'+cy=f(t),\quad y(0)=k_0,\quad y'(0)=k_1$$

is

$$y(t)=\ k_0(1-r_1t)e^{r_1t}+k_1te^{r_1t}+\frac{1}{a}\int_0^t\tau e^{r_1\tau}f(t-\tau)\,d\tau.$$

10. Show that if $p(s)=as^2+bs+c$ has complex conjugate zeros $\lambda\pm i\omega$ then the solution of

$$ay''+by'+cy=f(t),\quad y(0)=k_0,\quad y'(0)=k_1$$

is

$$\begin{aligned}y(t)\ =\ & e^{\lambda t}\left[k_0(\cos\omega t-\frac{\lambda}{\omega}\sin\omega t)+\frac{k_1}{\omega}\sin\omega t\right]\\ &+\frac{1}{a\omega}\int_0^t e^{\lambda t}f(t-\tau)\sin\omega\tau\,d\tau.\end{aligned}$$

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7.7: A Brief Table of Laplace Transforms

This page is a draft and is under active development.

Table 7.7.1

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$(s>0)$
t^n	$\frac{n!}{s^{n+1}}$	$(s>0)$
$(n=\text{ integer }>0)$		
$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{(p+1)}}$	$(s>0)$
e^{at}	$\frac{1}{s-a}$	$(s>a)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$(s>0)$
$(n=\text{ integer }>0)$		
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$(s>0)$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$(s>0)$
$e^{\lambda t} \cos \omega t$	$\frac{s-\lambda}{(s-\lambda)^2+\omega^2}$	$(s>\lambda)$
$e^{\lambda t} \sin \omega t$	$\frac{\omega}{(s-\lambda)^2+\omega^2}$	$(s>\lambda)$
$\cosh bt$	$\frac{s}{s^2-b^2}$	$(s>\lvert b\rvert)$
$\sinh bt$	$\frac{b}{s^2-b^2}$	$(s>\lvert b\rvert)$
$t \cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$(s>0)$
$t \sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$(s>0)$
$\sin \omega t-\omega t \cos \omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$	$(s>0)$
$\omega t-\sin \omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$	$(s>0)$
$\frac{1}{t} \sin \omega t$	$\arctan\left(\frac{\omega}{s}\right)$	$(s>0)$
$e^{at} f(t)$	$F(s-a)$	
$t^k f(t)$	$(-1)^k F^{(k)}(s)$	
$f(\omega t)$	$\frac{1}{\omega} F\left(\frac{s}{\omega}\right), \quad \omega>0$	

$f(t)$	$F(s)$	
$u(t-\tau)$	$\frac{e^{-\tau s}}{s}$	$(s>0)$
$u(t-\tau)f(t-\tau)\ (\tau>0)$	$e^{-\tau s}F(s)$	
$\int_o^t f(\tau)g(t-\tau)\,d\tau$	$F(s)\cdot G(s)$	
$\delta(t-a)$	e^{-as}	$(s>0)$

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CHAPTER OVERVIEW

Chapter 8: Boundary Value Problems and Fourier Expansions

IN THIS CHAPTER we develop series representations of functions that will be used to solve partial differential equations in Chapter 9.

8.1: Eigenvalue Problems for $y'' + \lambda y = 0$

8.1E: Eigenvalue Problems for $y'' + \lambda y = 0$ (Exercises)

8.2: Fourier Series I

8.2E: Fourier Series I (Exercises)

8.3: Fourier Series II

8.3E: Fourier Series II (Exercises)

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8.1: Eigenvalue Problems for y'' + λy = 0

In Chapter 9 we'll study partial differential equations that arise in problems of heat conduction, wave propagation, and potential theory. The purpose of this chapter is to develop tools required to solve these equations. In this section we consider the following problems, where λ is a real number and $L>0$:

- Problem 1: $y''+\lambda y=0, \quad y(0)=0, \quad y(L)=0$
- Problem 2: $y''+\lambda y=0, \quad y'(0)=0, \quad y'(L)=0$
- Problem 3: $y''+\lambda y=0, \quad y(0)=0, \quad y'(L)=0$
- Problem 4: $y''+\lambda y=0, \quad y'(0)=0, \quad y(L)=0$
- Problem 5: $y''+\lambda y=0, \quad y(-L)=y(L), \quad y'(-L)=y'(L)$

In each problem the conditions following the differential equation are called *boundary conditions*. Note that the boundary conditions in Problem 5, unlike those in Problems 1-4, don't require that y or y' be zero at the boundary points, but only that y have the same value at $x=\pm L$, and that y' have the same value at $x=\pm L$. We say that the boundary conditions in Problem 5 are *periodic*.

Obviously, $y \equiv 0$ (the trivial solution) is a solution of Problems 1-5 for any value of λ. For most values of λ, there are no other solutions. The interesting question is this:

For what values of λ does the problem have nontrivial solutions, and what are they?

A value of λ for which the problem has a nontrivial solution is an *eigenvalue* of the problem, and the nontrivial solutions are λ-*eigenfunctions*, or *eigenfunctions associated with* λ. Note that a nonzero constant multiple of a λ-eigenfunction is again a λ-eigenfunction.

Problems 1-5 are called *eigenvalue problems*. *Solving* an eigenvalue problem means finding all its eigenvalues and associated eigenfunctions. We'll take it as given here that all the eigenvalues of Problems 1-5 are real numbers. This is proved in a more general setting in Section 13.2.

Theorem 8.1.1

Problems 1–5 have no negative eigenvalues. Moreover, $\lambda=0$ is an eigenvalue of Problems 2 and 5, with associated eigenfunction $y_0=1$, but $\lambda=0$ isn't an eigenvalue of Problems $1, 3$, or 4.

Proof

We consider Problems 1-4, and leave Problem 5 to you (*Exercise 11.1.1*). If $y''+\lambda y=0$, then $y(y''+\lambda y)=0$, so

$$\int_0^L y(x)(y''(x)+\lambda y(x))\,dx=0;$$

therefore,

$$\lambda \int_0^L y^2(x)\,dx=-\int_0^L y(x)y''(x)\,dx. \tag{8.1.1}$$

Integration by parts yields

$$\begin{aligned} \textstyle\int_0^L y(x)y''(x)\,dx &= y(x)y'(x)|_0^L-\textstyle\int_0^L(y'(x))^2\,dx \\ &= y(L)y'(L)-y(0)y'(0)-\textstyle\int_0^L(y'(x))^2\,dx. \end{aligned} \tag{8.1.2}$$

However, if y satisfies any of the boundary conditions of Problems 1-4, then

$$y(L)y'(L)-y(0)y'(0)=0;$$

hence, Equation 8.1.1and Equation 8.1.2imply that

$$\lambda \int_0^L y^2(x)\,dx=\int_0^L(y'(x))^2\,dx.$$

If $y \not\equiv 0$, then $\int_0^L y^2(x)\,dx > 0$. Therefore $\lambda \geq 0$ and, if $\lambda = 0$, then $y'(x) = 0$ for all x in $(0, L)$ (why?), and y is constant on $(0, L)$. Any constant function satisfies the boundary conditions of Problem 2, so $\lambda = 0$ is an eigenvalue of Problem 2 and any nonzero constant function is an associated eigenfunction. However, the only constant function that satisfies the boundary conditions of Problems 1, 3, or 4 is $y \equiv 0$. Therefore $\lambda = 0$ isn't an eigenvalue of any of these problems.

Example 8.1.1

Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0. \tag{8.1.3}$$

Solution

From Theorem 8.1.1, any eigenvalues of Equation 8.1.3 must be positive. If y satisfies Equation 8.1.3 with $\lambda > 0$, then

$$y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x,$$

where c_1 and c_2 are constants. The boundary condition $y(0) = 0$ implies that $c_1 = 0$. Therefore $y = c_2 \sin\sqrt{\lambda}\,x$. Now the boundary condition $y(L) = 0$ implies that $c_2 \sin\sqrt{\lambda}\,L = 0$. To make $c_2 \sin\sqrt{\lambda}\,L = 0$ with $c_2 \neq 0$, we must choose $\sqrt{\lambda} = n\pi/L$, where n is a positive integer. Therefore $\lambda_n = n^2\pi^2/L^2$ is an eigenvalue and

$$y_n = \sin\frac{n\pi x}{L}$$

is an associated eigenfunction.

For future reference, we state the result of Example 8.1.1 as a theorem.

Theorem 8.1.2

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_n = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

There are no other eigenvalues.

We leave it to you to prove the next theorem about Problem 2 by an argument like that of *Example* 8.1.1 (*Exercise 11.1.17*).

Theorem 8.1.3

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

has the eigenvalue $\lambda_0 = 0$, with associated eigenfunction $y_0 = 1$, and infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_n = \cos\frac{n\pi x}{L}, n = 1, 2, 3 \ldots.$$

There are no other eigenvalues.

Example 8.1.2

Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0. \tag{8.1.4}$$

Solution:

From Theorem 8.1.1, any eigenvalues of Equation 8.1.4 must be positive. If y satisfies Equation 8.1.4 with $\lambda > 0$, then

$$y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x,$$

where c_1 and c_2 are constants. The boundary condition $y(0) = 0$ implies that $c_1 = 0$. Therefore $y = c_2 \sin\sqrt{\lambda}\, x$. Hence, $y' = c_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$ and the boundary condition $y'(L) = 0$ implies that $c_2 \cos\sqrt{\lambda}\, L = 0$. To make $c_2 \cos\sqrt{\lambda}\, L = 0$ with $c_2 \neq 0$ we must choose

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2L},$$

where n is a positive integer. Then $\lambda_n = (2n-1)^2\pi^2/4L^2$ is an eigenvalue and

$$y_n = \sin\frac{(2n-1)\pi x}{2L}$$

is an associated eigenfunction.

For future reference, we state the result of Example 8.1.2 as a theorem.

Theorem 8.1.4

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = (2n-1)^2\pi^2/4L^2$, with associated eigenfunctions

$$y_n = \sin\frac{(2n-1)\pi x}{2L}, \quad n = 1, 2, 3, \ldots.$$

There are no other eigenvalues.

We leave it to you to prove the next theorem about Problem 4 by an argument like that of Example 8.1.2 (*Exercise 11.1.18*).

Theorem 8.1.5

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = (2n-1)^2\pi^2/4L^2$, with associated eigenfunctions

$$y_n = \cos\frac{(2n-1)\pi x}{2L}, \quad n = 1, 2, 3, \ldots.$$

There are no other eigenvalues.

Example 8.1.3

Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L). \tag{8.1.5}$$

Solution

From Theorem 8.1.1, $\lambda = 0$ is an eigenvalue of Equation 8.1.5 with associated eigenfunction $y_0 = 1$, and any other eigenvalues must be positive. If y satisfies Equation 8.1.5 with $\lambda > 0$, then

$$y = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x, \tag{8.1.6}$$

where c_1 and c_2 are constants. The boundary condition $y(-L) = y(L)$ implies that

$$c_1 \cos(-\sqrt{\lambda}\, L) + c_2 \sin(-\sqrt{\lambda}\, L) = c_1 \cos\sqrt{\lambda}\, L + c_2 \sin\sqrt{\lambda}\, L. \tag{8.1.7}$$

Since

$$\cos(-\sqrt{\lambda}\, L) = \cos\sqrt{\lambda}\, L \quad \text{and} \quad \sin(-\sqrt{\lambda}\, L) = -\sin\sqrt{\lambda}\, L, \tag{8.1.8}$$

Equation 8.1.7 implies that

$$c_2 \sin\sqrt{\lambda}\, L = 0. \tag{8.1.9}$$

Differentiating Equation 8.1.6 yields

$$y' = \sqrt{\lambda}\left(-c_1 \sin\sqrt{\lambda}x + c_2 \cos\sqrt{\lambda}x\right).$$

The boundary condition $y'(-L) = y'(L)$ implies that

$$-c_1 \sin(-\sqrt{\lambda}\, L) + c_2 \cos(-\sqrt{\lambda}\, L) = -c_1 \sin\sqrt{\lambda}\, L + c_2 \cos\sqrt{\lambda}\, L,$$

and Equation 8.1.8 implies that

$$c_1 \sin\sqrt{\lambda}\, L = 0. \tag{8.1.10}$$

Eqns. Equation 8.1.9 and Equation 8.1.10 imply that $c_1 = c_2 = 0$ unless $\sqrt{\lambda} = n\pi/L$, where n is a positive integer. In this case Equation 8.1.9 and Equation 8.1.10 both hold for arbitrary c_1 and c_2. The eigenvalue determined in this way is $\lambda_n = n^2\pi^2/L^2$, and each such eigenvalue has the linearly independent associated eigenfunctions

$$\cos\frac{n\pi x}{L} \quad \text{and} \quad \sin\frac{n\pi x}{L}.$$

For future reference we state the result of Example 8.1.3 as a theorem.

Theorem 8.1.6

The eigenvalue problem

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L),$$

has the eigenvalue $\lambda_0 = 0$, with associated eigenfunction $y_0 = 1$ and infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_{1n} = \cos\frac{n\pi x}{L} \quad \text{and} \quad y_{2n} = \sin\frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

There are no other eigenvalues.

Orthogonality

We say that two integrable functions f and g are *orthogonal* on an interval $[a, b]$ if

$$\int_a^b f(x)g(x)\, dx = 0.$$

More generally, we say that the functions ϕ_1, ϕ_2, …, ϕ_n, …(finitely or infinitely many) are orthogonal on $[a, b]$ if

$$\int_a^b \phi_i(x)\phi_j(x)\, dx = 0 \quad \text{whenever} \quad i \neq j.$$

The importance of orthogonality will become clear when we study Fourier series in the next two sections.

Example 8.1.4

Show that the eigenfunctions

$$1,\ \cos\frac{\pi x}{L},\ \sin\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \sin\frac{2\pi x}{L},\ldots,\cos\frac{n\pi x}{L},\ \sin\frac{n\pi x}{L},\ldots \tag{8.1.11}$$

of Problem 5 are orthogonal on $[-L, L]$.

Solution

We must show that

$$\int_{-L}^{L} f(x)g(x)\,dx = 0 \tag{8.1.12}$$

whenever f and g are distinct functions from Equation 8.1.11. If r is any nonzero integer, then

$$\int_{-L}^{L} \cos\frac{r\pi x}{L}\,dx = \frac{L}{r\pi}\sin\frac{r\pi x}{L}\bigg|_{-L}^{L} = 0. \tag{8.1.13}$$

and

$$\int_{-L}^{L} \sin\frac{r\pi x}{L}\,dx = -\frac{L}{r\pi}\cos\frac{r\pi x}{L}\bigg|_{-L}^{L} = 0.$$

Therefore Equation 8.1.12 holds if $f \equiv 1$ and g is any other function in Equation 8.1.11.

If $f(x) = \cos m\pi x/L$ and $g(x) = \cos n\pi x/L$ where m and n are distinct positive integers, then

$$\int_{-L}^{L} f(x)g(x)\,dx = \int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx. \tag{8.1.14}$$

To evaluate this integral, we use the identity

$$\cos A\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

with $A = m\pi x/L$ and $B = n\pi x/L$. Then Equation 8.1.14 becomes

$$\int_{-L}^{L} f(x)g(x)\,dx = \frac{1}{2}\left[\int_{-L}^{L}\cos\frac{(m-n)\pi x}{L}\,dx + \int_{-L}^{L}\cos\frac{(m+n)\pi x}{L}\,dx\right].$$

Since $m-n$ and $m+n$ are both nonzero integers, Equation 8.1.13 implies that the integrals on the right are both zero. Therefore Equation 8.1.12 is true in this case.

If $f(x) = \sin m\pi x/L$ and $g(x) = \sin n\pi x/L$ where m and n are distinct positive integers, then

$$\int_{-L}^{L} f(x)g(x)\,dx = \int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx. \tag{8.1.15}$$

To evaluate this integral, we use the identity

$$\sin A\sin B = \frac{1}{2}[\cos(A-B)-\cos(A+B)]$$

with $A = m\pi x/L$ and $B = n\pi x/L$. Then Equation 8.1.15 becomes

$$\int_{-L}^{L} f(x)g(x)\,dx = \frac{1}{2}\left[\int_{-L}^{L}\cos\frac{(m-n)\pi x}{L}\,dx - \int_{-L}^{L}\cos\frac{(m+n)\pi x}{L}\,dx\right] = 0.$$

If $f(x) = \sin m\pi x/L$ and $g(x) = \cos n\pi x/L$ where m and n are positive integers (not necessarily distinct), then

$$\int_{-L}^{L} f(x)g(x)\,dx = \int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$$

> because the integrand is an odd function and the limits are symmetric about $x = 0$.

Exercises 11.1.19-11.1.22 ask you to verify that the eigenfunctions of Problems 1-4 are orthogonal on $[0, L]$. However, this also follows from a general theorem that we'll prove in Chapter 13.

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8.1E: Eigenvalue Problems for y" + λy = 0 (Exercises)

Q8.1.1

1. Prove that $\lambda = 0$ is an eigenvalue of Problem 5 with associated eigenfunction $y_0 = 1$, and that any other eigenvalues must be positive.

Q8.1.2

In *Exercises 8.1.2-8.1.16* solve the eigenvalue problem.

2. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0$
3. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(\pi) = 0$
4. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$
5. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi) = 0$
6. $y'' + \lambda y = 0, \quad y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi)$
7. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(1) = 0$
8. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(1) = 0$
9. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0$
10. $y'' + \lambda y = 0, \quad y(-1) = y(1), \quad y'(-1) = y'(1)$
11. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1) = 0$
12. $y'' + \lambda y = 0, \quad y(-2) = y(2), \quad y'(-2) = y'(2)$
13. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y(2) = 0$
14. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(3) = 0$
15. $y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(1/2) = 0$
16. $y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(5) = 0$

Q8.1.3

17. Prove Theorem 8.1.3.

18. Prove Theorem 8.1.5.

19. Verify that the eigenfunctions

$$\sin \frac{\pi x}{L}, \sin \frac{2\pi x}{L}, \ldots, \sin \frac{n\pi x}{L}, \ldots$$

of Problem 1 are orthogonal on $[0, L]$.

20. Verify that the eigenfunctions

$$1, \cos \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \ldots, \cos \frac{n\pi x}{L}, \ldots$$

of Problem 2 are orthogonal on $[0, L]$.

21. Verify that the eigenfunctions

$$\sin \frac{\pi x}{2L}, \sin \frac{3\pi x}{2L}, \ldots, \sin \frac{(2n-1)\pi x}{2L}, \ldots$$

of Problem 3 are orthogonal on $[0, L]$.

22. Verify that the eigenfunctions

$$\cos\frac{\pi x}{2L},\ \cos\frac{3\pi x}{2L},\ldots,\ \cos\frac{(2n-1)\pi x}{2L},\ldots$$

of Problem 4 are orthogonal on $[0, L]$.

Q8.1.4

In *Exercises 8.1.23-8.1.26* solve the eigenvalue problem.

23. $y''+\lambda y=0, \quad y(0)=0, \quad \int_0^L y(x)\,dx=0$

24. $y''+\lambda y=0, \quad y'(0)=0, \quad \int_0^L y(x)\,dx=0$

25. $y''+\lambda y=0, \quad y(L)=0, \quad \int_0^L y(x)\,dx=0$

26. $y''+\lambda y=0, \quad y'(L)=0, \quad \int_0^L y(x)\,dx=0$

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8.2: Fourier Series I

In Example 8.1.4 and *Exercises 8.1.4-8.1.22* we saw that the eigenfunctions of Problem 5 are orthogonal on $[-L, L]$ and the eigenfunctions of Problems 1–4 are orthogonal on $[0, L]$. In this section and the next we introduce some series expansions in terms of these eigenfunctions. We'll use these expansions to solve partial differential equations in Chapter 9.

Theorem 8.2.1

Suppose the functions $\phi_1, \phi_2, \phi_3, \ldots$, are orthogonal on $[a, b]$ and

$$\int_a^b \phi_n^2(x)\,dx \neq 0, \quad n = 1, 2, 3, \ldots. \tag{8.2.1}$$

Let $c_1, c_2, c_3, \ldots$be constants such that the partial sums $f_N(x) = \sum_{m=1}^N c_m \phi_m(x)$ satisfy the inequalities

$$|f_N(x)| \leq M, \quad a \leq x \leq b, \quad N = 1, 2, 3, \ldots$$

for some constant $M < \infty$. Suppose also that the series

$$f(x) = \sum_{m=1}^{\infty} c_m \phi_m(x) \tag{8.2.2}$$

converges and is integrable on $[a, b]$. Then

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}, \quad n = 1, 2, 3, \ldots. \tag{8.2.3}$$

Proof

Multiplying Equation 8.2.2by ϕ_n and integrating yields

$$\int_a^b f(x)\phi_n(x)\,dx = \int_a^b \phi_n(x)\left(\sum_{m=1}^{\infty} c_m \phi_m(x)\right)dx. \tag{8.2.4}$$

It can be shown that the boundedness of the partial sums $\{f_N\}_{N=1}^{\infty}$ and the integrability of f allow us to interchange the operations of integration and summation on the right of Equation 8.2.4, and rewrite Equation 8.2.4as

$$\int_a^b f(x)\phi_n(x)\,dx = \sum_{m=1}^{\infty} c_m \int_a^b \phi_n(x)\phi_m(x)\,dx. \tag{8.2.5}$$

(This isn't easy to prove.) Since

$$\int_a^b \phi_n(x)\phi_m(x)\,dx = 0 \quad \text{if} \quad m \neq n,$$

Equation 8.2.5reduces to

$$\int_a^b f(x)\phi_n(x)\,dx = c_n \int_a^b \phi_n^2(x)\,dx.$$

Now Equation 8.2.1implies Equation 8.2.3

Theorem 8.2.1 motivates the next definition.

Theorem 8.2.2

Suppose $\phi_1, \phi_2, \ldots, \phi_n, \ldots$are orthogonal on $[a, b]$ and $\int_a^b \phi_n^2(x)\,dx \neq 0$, $n = 1, 2, 3, \ldots$. Let f be integrable on $[a, b]$, and define

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}, \quad n = 1, 2, 3, \ldots. \tag{8.2.6}$$

Then the infinite series $\sum_{n=1}^{\infty} c_n\phi_n(x)$ is called the *Fourier expansion of* f in terms of the orthogonal set $\{\phi_n\}_{n=1}^{\infty}$, and c_1, c_2, …, c_n, … are called the *Fourier coefficients of* f with respect to $\{\phi_n\}_{n=1}^{\infty}$. We indicate the relationship between f and its Fourier expansion by

$$f(x) \sim \sum_{n=1}^{\infty} c_n\phi_n(x), \quad a \le x \le b. \tag{8.2.7}$$

You may wonder why we don't write

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \quad a \le x \le b,$$

rather than Equation 8.2.7. Unfortunately, this isn't always true. The series on the right may diverge for some or all values of x in $[a, b]$, or it may converge to $f(x)$ for some values of x and not for others. So, for now, we'll just think of the series as being associated with f because of the definition of the coefficients $\{c_n\}$, and we'll indicate this association informally as in Equation 8.2.7.

Fourier Series

We'll now study Fourier expansions in terms of the eigenfunctions

$$1,\ \cos\frac{\pi x}{L},\ \sin\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ \sin\frac{2\pi x}{L}, \ldots, \cos\frac{n\pi x}{L},\ \sin\frac{n\pi x}{L}, \ldots.$$

of Problem 5. If f is integrable on $[-L, L]$, its Fourier expansion in terms of these functions is called the *Fourier series of* f on $[-L, L]$. Since

$$\int_{-L}^{L} 1^2\,dx = 2L,$$

$$\int_{-L}^{L} \cos^2\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(1 + \cos\frac{2n\pi x}{L}\right)dx = \frac{1}{2}\left(x + \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right)\Bigg|_{-L}^{L} = L,$$

and

$$\int_{-L}^{L} \sin^2\frac{n\pi x}{L}\,dx = \frac{1}{2}\int_{-L}^{L}\left(1 - \cos\frac{2n\pi x}{L}\right)dx = \frac{1}{2}\left(x - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right),\Bigg|_{-L}^{L} = L,$$

we see from Equation 8.2.6 that the Fourier series of f on $[-L, L]$ is

$$a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right),$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx,\ n \ge 1.$$

Note that a_0 is the average value of f on $[-L, L]$, while a_n and b_n (for $n \ge 1$) are twice the average values of

$$f(x)\cos\frac{n\pi x}{L} \quad \text{and} \quad f(x)\sin\frac{n\pi x}{L}$$

on $[-L, L]$, respectively.

Convergence of Fourier Series

The question of convergence of Fourier series for arbitrary integrable functions is beyond the scope of this book. However, we can state a theorem that settles this question for most functions that arise in applications.

Theorem 8.2.3

A function f is said to be *piecewise smooth* on $[a,b]$ if:

a. f has at most finitely many points of discontinuity in (a,b);
b. f' exists and is continuous except possibly at finitely many points in (a,b);
c. $f(x_0+)=\lim_{x\to x_0+} f(x)$ and $f'(x_0+)=\lim_{x\to x_0+} f'(x)$ exist if $a\le x_0<b$;
d. $f(x_0-)=\lim_{x\to x_0-} f(x)$ and $f'(x_0-)=\lim_{x\to x_0-} f'(x)$ exist if $a<x_0\le b$.

Since f and f' are required to be continuous at all but finitely many points in $[a,b]$, $f(x_0+)=f(x_0-)$ and $f'(x_0+)=f'(x_0-)$ for all but finitely many values of x_0 in (a,b). Recall from Section 8.1 that f is said to have a *jump discontinuity* at x_0 if $f(x_0+)\ne f(x_0-)$.

The next theorem gives sufficient conditions for convergence of a Fourier series. The proof is beyond the scope of this book.

Theorem 8.2.4

If f is a piecewise smooth on $[-L,L]$, then the Fourier series

$$F(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right) \tag{8.2.8}$$

of f on $[-L,L]$ converges for all x in $[-L,L]$; moreover,

$$F(x)=\begin{cases} f(x) & if\ -L<x<L\ and\ f\ is\ continuous\ at\ x \\ \frac{f(x-)+f(x+)}{2} & if\ -L<x<L\ and\ f\ is\ discontinuous\ at\ x \\ \frac{f(-L+)+f(L-)}{2} & if\ x=L\ or\ x=-L \end{cases}$$

Since $f(x+)=f(x-)$ if f is continuous at x, we can also say that

$$F(x)=\begin{cases} \frac{f(x+)+f(x-)}{2} & if\ -L<x<L \\ \frac{f(L-)+f(-L+)}{2} & if\ x=\pm L \end{cases}$$

Note that F is itself piecewise smooth on $[-L,L]$, and $F(x)=f(x)$ at all points in the open interval $(-L,L)$ where f is continuous. Since the series in Equation 8.2.8 converges to $F(x)$ for all x in $[-L,L]$, you may be tempted to infer that the error

$$E_N(x)=\left|F(x)-a_0-\sum_{n=1}^{N}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)\right|$$

can be made as small as we please for all x in $[-L,L]$ by choosing N sufficiently large. However, this isn't true if f has a discontinuity somewhere in $(-L,L)$, or if $f(-L+)\ne f(L-)$. Here's the situation in this case.

If f has a jump discontinuity at a point α in $(-L,L)$, there will be sequences of points $\{u_N\}$ and $\{v_N\}$ in $(-L,\alpha)$ and (α,L), respectively, such that

$$\lim_{N\to\infty} u_N=\lim_{N\to\infty} v_N=\alpha$$

and

$$E_N(u_N)\approx .09|f(\alpha-)-f(\alpha+)| \quad \text{and} \quad E_N(v_N)\approx .09|f(\alpha-)-f(\alpha+)|.$$

Thus, the maximum value of the error $E_N(x)$ near α does not approach zero as $N\to\infty$, but just occurs closer and closer to (and on both sides of) α, and is essentially independent of N.

If $f(-L+) \neq f(L-)$, then there will be sequences of points $\{u_N\}$ and $\{v_N\}$ in $(-L, L)$ such that

$$\lim_{N\to\infty} u_N = -L, \quad \lim_{N\to\infty} v_N = L,$$

$$E_N(u_N) \approx .09|f(-L+) - f(L-)| \quad \text{and} \quad E_N(v_N) \approx .09|f(-L+) - f(L-)|.$$

This is the *Gibbs phenomenon*. Having been alerted to it, you may see it in Figures 8.2.2 - 8.2.4, below; however, we'll give a specific example at the end of this section.

Example 8.2.1

Find the Fourier series of the piecewise smooth function

$$f(x) = \begin{cases} -x, & -2 < x < 0, \\ \frac{1}{2}, & 0 < x < 2 \end{cases}$$

on $[-2, 2]$ (Figure 8.2.1). Determine the sum of the Fourier series for $-2 \leq x \leq 2$.

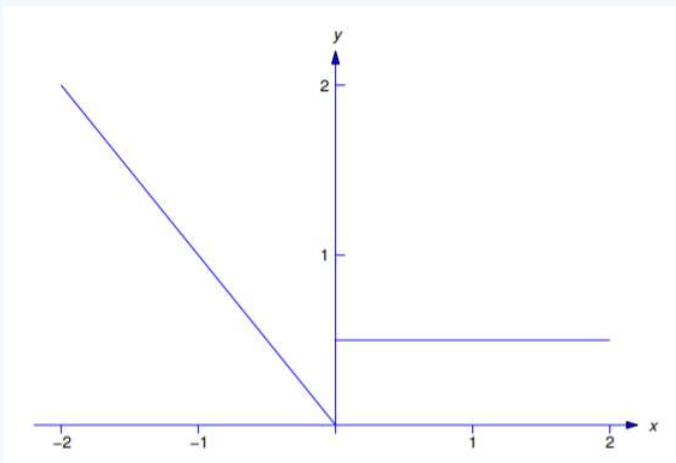

Figure 8.2.1

Solution

Note that wen't bothered to define $f(-2)$, $f(0)$, and $f(2)$. No matter how they may be defined, f is piecewise smooth on $[-2, 2]$, and the coefficients in the Fourier series

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2}\right)$$

are not affected by them. In any case, Theorem 8.2.4 implies that $F(x) = f(x)$ in $(-2, 0)$ and $(0, 2)$, where f is continuous, while

$$F(-2) = F(2) = \frac{f(-2+) + f(2-)}{2} = \frac{1}{2}\left(2 + \frac{1}{2}\right) = \frac{5}{4}$$

and

$$F(0) = \frac{f(0-) + f(0+)}{2} = \frac{1}{2}\left(0 + \frac{1}{2}\right) = \frac{1}{4}.$$

To summarize,

$$F(x)=\begin{cases} \frac{5}{4}, & x=-2 \\ -x, & -2<x<0, \\ \frac{1}{4}, & x=0, \\ \frac{1}{2}, & 0<x<2, \\ \frac{5}{4}, & x=2. \end{cases}$$

We compute the Fourier coefficients as follows:

$$a_0=\frac{1}{4}\int_{-2}^{2} f(x)\,dx=\frac{1}{4}\left[\int_{-2}^{0}(-x)\,dx+\int_{0}^{2}\frac{1}{2}\,dx\right]=\frac{3}{4}.$$

If $n\geq 1$, then

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx=\frac{1}{2}\left[\int_{-2}^{0}(-x)\cos\frac{n\pi x}{2}\,dx+\int_{0}^{2}\frac{1}{2}\cos\frac{n\pi x}{2}\,dx\right] \\ &= \frac{2}{n^2\pi^2}(\cos n\pi-1), \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx=\frac{1}{2}\left[\int_{-2}^{0}(-x)\sin\frac{n\pi x}{2}\,dx+\int_{0}^{2}\frac{1}{2}\sin\frac{n\pi x}{2}\,dx\right] \\ &= \frac{1}{2n\pi}(1+3\cos n\pi). \end{aligned}$$

Therefore

$$F(x)=\frac{3}{4}+\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi-1}{n^2}\cos\frac{n\pi x}{2}+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1+3\cos n\pi}{n}\sin\frac{n\pi x}{2}.$$

Figure $8.2.2$ shows how the partial sum

$$F_m(x)=\frac{3}{4}+\frac{2}{\pi^2}\sum_{n=1}^{m}\frac{\cos n\pi-1}{n^2}\cos\frac{n\pi x}{2}+\frac{1}{2\pi}\sum_{n=1}^{m}\frac{1+3\cos n\pi}{n}\sin\frac{n\pi x}{2}$$

approximates $f(x)$ for $m=5$ (dotted curve), $m=10$ (dashed curve), and $m=15$ (solid curve).

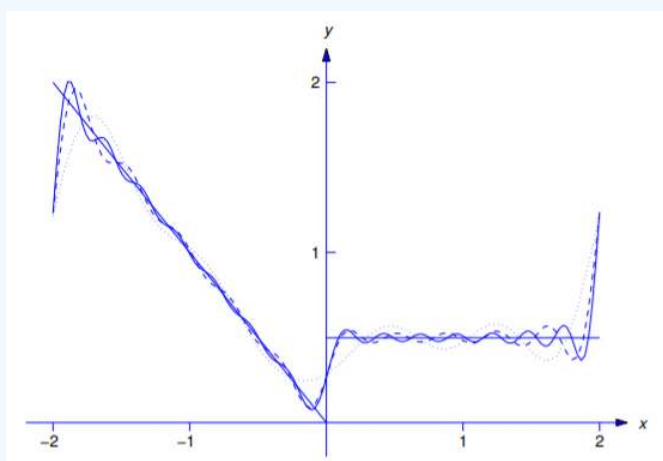

Figure 8.2.2

Even and Odd Functions

Computing the Fourier coefficients of a function f can be tedious; however, the computation can often be simplified by exploiting symmetries in f or some of its terms. To focus on this, we recall some concepts that you studied in calculus. Let u and v be defined on $[-L,L]$ and suppose that

$$u(-x)=u(x) \quad \text{and} \quad v(-x)=-v(x), \quad -L \leq x \leq L.$$

Then we say that u is an *even* function and v is an *odd function*. Note that:

- The product of two even functions is even.
- The product of two odd functions is even.
- The product of an even function and an odd function is odd.

Example 8.2.2

The functions $u(x)=\cos \omega x$ and $u(x)=x^2$ are even, while $v(x)=\sin \omega x$ and $v(x)=x^3$ are odd. The function $w(x)=e^x$ is neither even nor odd.

You learned parts (a) and (b) of the next theorem in calculus, and the other parts follow from them (*Exercise 11.2.1*).

Theorem 8.2.5

Suppose u is even and v is odd on $[-L, L]$. Then:

a. $\int_{-L}^{L} u(x)\, dx=2 \int_0^L u(x)\, dx$,
b. $\int_{-L}^{L} v(x)\, dx=0$,
c. $\int_{-L}^{L} u(x) \cos \frac{n \pi x}{L}\, dx=2 \int_0^L u(x) \cos \frac{n \pi x}{L}\, dx$,
d. $\int_{-L}^{L} v(x) \sin \frac{n \pi x}{L}\, dx=2 \int_0^L v(x) \sin \frac{n \pi x}{L}\, dx$,
e. $\int_{-L}^{L} u(x) \sin \frac{n \pi x}{L}\, dx=0$
f. $\int_{-L}^{L} v(x) \cos \frac{n \pi x}{L}\, dx=0$.

Example 8.2.3

Find the Fourier series of $f(x)=x^2-x$ on $[-2,2]$, and determine its sum for $-2 \leq x \leq 2$.

Solution

Since $L=2$,

$$F(x)=a_0+\sum_{n=1}^{\infty}\left(a_n \cos \frac{n \pi x}{2}+b_n \sin \frac{n \pi x}{2}\right)$$

where

$$a_0=\frac{1}{4} \int_{-2}^{2}\left(x^2-x\right) dx, \tag{8.2.9}$$

$$a_n=\frac{1}{2} \int_{-2}^{2}\left(x^2-x\right) \cos \frac{n \pi x}{2}\, dx, \quad n=1,2,3, \ldots, \tag{8.2.10}$$

and

$$b_n=\frac{1}{2} \int_{-2}^{2}\left(x^2-x\right) \sin \frac{n \pi x}{2}\, dx, \quad n=1,2,3, \ldots. \tag{8.2.11}$$

We simplify the evaluation of these integrals by using Theorem [thmtype:11.2.5} with $u(x)=x^2$ and $v(x)=x$; thus, from Equation 8.2.9,

$$a_0=\frac{1}{2} \int_0^2 x^2\, dx=\left.\frac{x^3}{6}\right|_0^2=\frac{4}{3}.$$

From Equation 8.2.10,

$$\begin{aligned} a_n &= \int_0^2 x^2 \cos \frac{n\pi x}{2}\, dx = \frac{2}{n\pi}\left[x^2 \sin \frac{n\pi x}{2}\bigg|_0^2 - 2\int_0^2 x \sin \frac{n\pi x}{2}\, dx\right] \\ &= \frac{8}{n^2\pi^2}\left[x \cos \frac{n\pi x}{2}\bigg|_0^2 - \int_0^2 \cos \frac{n\pi x}{2}\, dx\right] \\ &= \frac{8}{n^2\pi^2}\left[2\cos n\pi - \frac{2}{n\pi}\sin \frac{n\pi x}{2}\bigg|_0^2\right] = (-1)^n \frac{16}{n^2\pi^2}. \end{aligned}$$

From Equation 8.2.11,

$$\begin{aligned} b_n &= -\int_0^2 x \sin \frac{n\pi x}{2}\, dx = \frac{2}{n\pi}\left[x \cos \frac{n\pi x}{2}\bigg|_0^2 - \int_0^2 \cos \frac{n\pi x}{2}\, dx\right] \\ &= \frac{2}{n\pi}\left[2\cos n\pi - \frac{2}{n\pi}\sin \frac{n\pi x}{2}\bigg|_0^2\right] = (-1)^n \frac{4}{n\pi}. \end{aligned}$$

Therefore

$$F(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{2}.$$

Theorem 8.2.4 implies that

$$F(x) = \begin{cases} 4, & x = -2, \\ x^2 - x, & -2 < x < 2, \\ 4, & x = 2. \end{cases}$$

Figure 8.2.3 shows how the partial sum

$$F_m(x) = \frac{4}{3} + \frac{16}{\pi^2}\sum_{n=1}^{m} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2} + \frac{4}{\pi}\sum_{n=1}^{m} \frac{(-1)^n}{n} \sin \frac{n\pi x}{2}$$

approximates $f(x)$ for $m = 5$ (dotted curve), $m = 10$ (dashed curve), and $m = 15$ (solid curve).

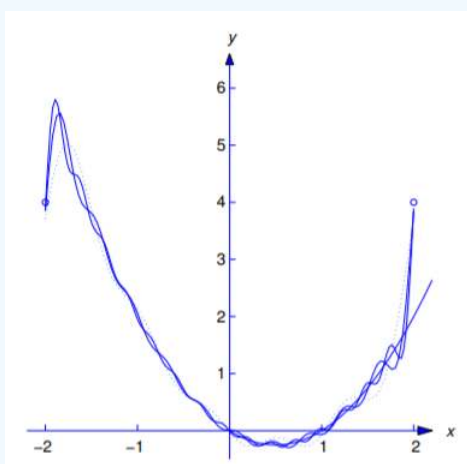

Figure 8.2.3: Approximation of $f(x) = x^2 - x$ *by partial sums of its Fourier series on* $[-2, 2]$

Theorem 8.2.5 ilmplies the next theorem follows.

Theorem 8.2.6

Suppose f is integrable on $[-L, L]$.

a. If f is even, the Fourier series of f on $[-L, L]$ is

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n \ge 1.$$

b. If f is odd, the Fourier series of f on $[-L, L]$ is

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Example 8.2.4

Find the Fourier series of $f(x) = x$ on $[-\pi, \pi]$, and determine its sum for $-\pi \le x \le \pi$.

Solution

Since f is odd and $L = \pi$,

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi x\sin nx\,dx = -\frac{2}{n\pi}\left[x\cos nx\big|_0^\pi - \int_0^\pi \cos nx\,dx\right] \\ &= -\frac{2}{n}\cos n\pi + \frac{2}{n^2\pi}\sin nx\Big|_0^\pi = (-1)^{n+1}\frac{2}{n}. \end{aligned}$$

Therefore

$$F(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

Theorem 8.2.4 implies that

$$F(x) = \begin{cases} 0, & x = -\pi, \\ x, & -\pi < x < \pi, \\ 0, & x = \pi. \end{cases}$$

Figure 8.2.4 shows how the partial sum

$$F_m(x) = -2\sum_{n=1}^{m}\frac{(-1)^n}{n}\sin nx$$

approximates $f(x)$ for $m = 5$ (dotted curve), $m = 10$ (dashed curve), and $m = 15$ (solid curve).

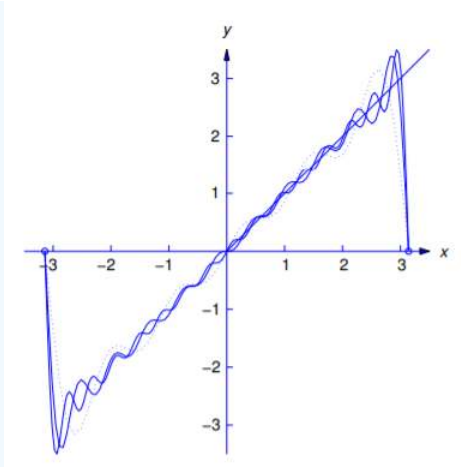

Figure 8.2.4: Approximation of $f(x) = x$ *by partial sums of its Fourier series on* $[-\pi, \pi]$

Example 8.2.5

Find the Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$ and determine its sum for $-\pi \le x \le \pi$.

Solution:

Since f is even and $L = \pi$,

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

Since $f(x) = x$ if $x \ge 0$,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{x^2}{2\pi}\bigg|_0^{\pi} = \frac{\pi}{2}$$

and, if $n \ge 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{n\pi}\left[x\sin nx|_0^{\pi} - \int_0^{\pi}\sin nx\,dx\right] \\ &= \frac{2}{n^2\pi}\cos nx\bigg|_0^{\pi} = \frac{2}{n^2\pi}(\cos n\pi - 1) = \frac{2}{n^2\pi}[(-1)^n - 1]. \end{aligned}$$

Therefore

$$F(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=0} \frac{(-1)^n - 1}{n^2}\cos nx. \tag{8.2.12}$$

However, since

$$(-1)^n - 1 = \begin{cases} 0 & \text{if } n = 2m, \\ -2 & \text{if } n = 2m+1, \end{cases}$$

the terms in Equation 8.2.12 for which $n = 2m$ are all zeros. Therefore we only to include the terms for which $n = 2m+1$; that is, we can rewrite Equation 8.2.12 as

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}\cos(2m+1)x.$$

However, since the name of the index of summation doesn't matter, we prefer to replace m by n, and write

$$F(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x.$$

Since $|x|$ is continuous for all x and $|-\pi|=|\pi|$, Theorem 8.2.4 implies that $F(x)=|x|$ for all x in $[-\pi,\pi]$.

Example 8.2.6

Find the Fourier series of $f(x)=x(x^2-L^2)$ on $[-L,L]$, and determine its sum for $-L\leq x\leq L$.

Solution:

Since f is odd,

$$F(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L},$$

where

$$\begin{aligned}
b_n &= \frac{2}{L}\int_0^L x(x^2-L^2)\sin\frac{n\pi x}{L}\,dx\\
&= -\frac{2}{n\pi}\left[x(x^2-L^2)\cos\frac{n\pi x}{L}\bigg|_0^L-\int_0^L(3x^2-L^2)\cos\frac{n\pi x}{L}\,dx\right]\\
&= \frac{2L}{n^2\pi^2}\left[(3x^2-L^2)\sin\frac{n\pi x}{L}\bigg|_0^L-6\int_0^L x\sin\frac{n\pi x}{L}\,dx\right]\\
&= \frac{12L^2}{n^3\pi^3}\left[x\cos\frac{n\pi x}{L}\bigg|_0^L-\int_0^L\cos\frac{n\pi x}{L}\,dx\right]=(-1)^n\frac{12L^3}{n^3\pi^3}.
\end{aligned}$$

Therefore

$$F(x)=\frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\frac{n\pi x}{L}.$$

Theorem 8.2.4 implies that $F(x)=x(x^2-L^2)$ for all x in $[-L,L]$.

Example 8.2.7 Gibbs Phenomenon

The Fourier series of

$$f(x)=\begin{cases}0, & -1<x<-\frac{1}{2},\\ 1, & -\frac{1}{2}<x<\frac{1}{2},\\ 0, & \frac{1}{2}<x<1\end{cases}$$

on $[-1,1]$ is

$$F(x)=\frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\cos(2n-1)\pi x.$$

(Verify.) According to Theorem 8.2.4,

$$F(x)=\begin{cases}0, & -1\leq x<-\frac{1}{2},\\ \frac{1}{2}, & x=-\frac{1}{2},\\ 1, & -\frac{1}{2}<x<\frac{1}{2},\\ \frac{1}{2}, & x=\frac{1}{2},\\ 0, & \frac{1}{2}<x\leq 1;\end{cases}$$

thus, F (as well as f) has unit jump discontinuities at $x=\pm\frac{1}{2}$. Figures 8.2.5-8.2.7 show the graphs of $y=f(x)$ and

$$y = F_{2N-1}(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{N}\frac{(-1)^{n-1}}{2n-1}\cos(2n-1)\pi x$$

for $N = 10$, 20, and 30. You can see that although F_{2N-1} approximates F (and therefore f) well on larger intervals as N increases, the maximum absolute values of the errors remain approximately equal to .09, but occur closer to the discontinuities $x = \pm\frac{1}{2}$ as N increases.

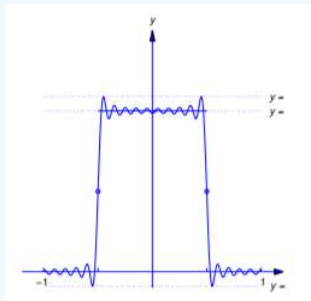

Figure 8.2.5: The Gibbs Phenomenon Example 8.2.7, $N = 10$

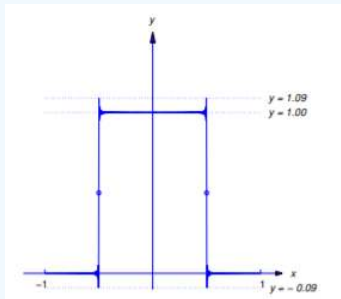

Figure 8.2.6: The Gibbs Phenomenon: Example 8.2.7, $N = 20$

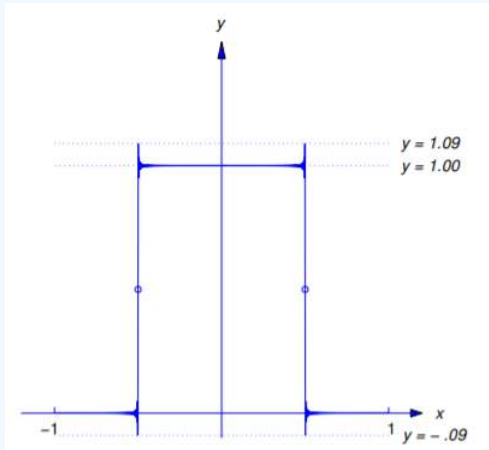

Figure 8.2.7: The Gibbs Phenomenon: Example 8.2.7, $N = 30$

Using Technology

The computation of Fourier coefficients will be tedious in many of the exercises in this chapter and the next. To learn the technique, we recommend that you do some exercises in each section "by hand," perhaps using the table of integrals at the front of the book. However, we encourage you to use your favorite symbolic computation software in the more difficult problems.

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8.2E: Fourier Series I (Exercises)

Q8.2.1

1. Prove Theorem 8.1.5.

Q8.2.2

In *Exercises 8.2.2-8.2.16* find the Fourier series of f on $[-L, L]$ and determine its sum for $-L \leq x \leq L$. Graph f for *Exercises 8.2.2, 8.2.6, 8.2.8, 8.2.15*, and *8.2.16* and

$$F_m(x) = a_0 + \sum_{n=1}^{m} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

on the same axes for variable values of m.

2. $L = 1;\ f(x) = 2 - x$

3. $L = \pi;\ f(x) = 2x - 3x^2$

4. $L = 1;\ f(x) = 1 - 3x^2$

5. $L = \pi;\ f(x) = |\sin x|$

6. $L = \pi;\ f(x) = x \cos x$

7. $L = \pi;\ f(x) = |x| \cos x$

8. $L = \pi;\ f(x) = x \sin x$

9. $L = \pi;\ f(x) = |x| \sin x$

10. $L = 1;\ f(x) = \begin{cases} 0, & -1 < x < -\frac{1}{2}, \\ \cos \pi x, & -\frac{1}{2} < x < \frac{1}{2}, \\ 0, & \frac{1}{2} < x < 1 \end{cases}$

11. $L = 1;\ f(x) = \begin{cases} 0, & -1 < x < -\frac{1}{2}, \\ x \cos \pi x, & -\frac{1}{2} < x < \frac{1}{2}, \\ 0, & \frac{1}{2} < x < 1 \end{cases}$

12. $L = 1;\ f(x) = \begin{cases} 0, & -1 < x < -\frac{1}{2}, \\ \sin \pi x, & -\frac{1}{2} < x < \frac{1}{2}, \\ 0, & \frac{1}{2} < x < 1 \end{cases}$

13. $L = 1;\ f(x) = \begin{cases} 0, & -1 < x < -\frac{1}{2}, \\ |\sin \pi x|, & -\frac{1}{2} < x < \frac{1}{2}, \\ 0, & \frac{1}{2} < x < 1 \end{cases}$

14. $L = 1;\ f(x) = \begin{cases} 0, & -1 < x < -\frac{1}{2}, \\ x \sin \pi x, & -\frac{1}{2} < x < \frac{1}{2}, \\ 0, & \frac{1}{2} < x < 1 \end{cases}$

15. $L = 4;\ f(x) = \begin{cases} 0, & -4 < x < 0, \\ x, & 0 < x < 4 \end{cases}$

16. $L = 1;\ f(x) = \begin{cases} x^2, & -1 < x < 0, \\ 1 - x^2, & 0 < x < 1 \end{cases}$

Q8.2.3

17. Verify the Gibbs phenomenon for $f(x)=\begin{cases} 2, & -2<x<-1, \\ 1, & -1<x<1, \\ -1, & 1<x<2. \end{cases}$

18. Verify the Gibbs phenomenon for $f(x)=\begin{cases} 2, & -3<x<-2, \\ 3, & -2<x<2, \\ 1, & 2<x<3. \end{cases}$

19. Deduce from Example 11.2.5 that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}=\frac{\pi^2}{8}.$$

20.

a. Find the Fourier series of $f(x)=e^x$ on $[-\pi, \pi]$.
b. Deduce from (a) that

$$\sum_{n=0}^{\infty} \frac{1}{n^2+1}=\frac{\pi \coth \pi-1}{2}.$$

21. Find the Fourier series of $f(x)=(x-\pi) \cos x$ on $[-\pi, \pi]$.

22. Find the Fourier series of $f(x)=(x-\pi) \sin x$ on $[-\pi, \pi]$.

23. Find the Fourier series of $f(x)=\sin kx$ ($k \neq$ integer) on $[-\pi, \pi]$.

24. Find the Fourier series of $f(x)=\cos kx$ ($k \neq$ integer) on $[-\pi, \pi]$.

25.

a. Suppose $f(-L)=f(L)$, $f'(-L)=f'(L)$, f' is continuous, and f'' is piecewise continuous on $[-L, L]$. Use Theorem 8.2.4 and integration by parts to show that

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n \cos \frac{n \pi x}{L}+b_n \sin \frac{n \pi x}{L}\right), \quad -L \leq x \leq L,$$

with

$$a_0=\frac{1}{2 L} \int_{-L}^{L} f(x)\, dx,$$

$$a_n=-\frac{L}{n^2 \pi^2} \int_{-L}^{L} f''(x) \cos \frac{n \pi x}{L}\, dx, \quad \text{and} \quad b_n=-\frac{L}{n^2 \pi^2} \int_{-L}^{L} f''(x) \sin \frac{n \pi x}{L}\, dx,\ n \geq 1.$$

b. Show that if, in addition to the assumptions in (a), f'' is continuous and f''' is piecewise continuous on $[-L, L]$, then

$$a_n=\frac{L^2}{n^3 \pi^3} \int_{-L}^{L} f'''(x) \sin \frac{n \pi x}{L}\, dx.$$

26. Show that if f is integrable on $[-L, L]$ and

$$f(x+L)=f(x), \quad -L<x<0$$

(Figure 8.2.8), then the Fourier series of f on $[-L, L]$ has the form

$$A_0+\sum_{n=1}^{\infty}\left(A_n \cos \frac{2 n \pi}{L}+B_n \sin \frac{2 n \pi}{L}\right)$$

where

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx,$$

and

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx, \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots.$$

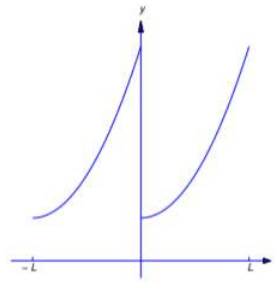

Figure 8.2.8: $y = f(x)$, *where* $f(x+L) = f(x), -L < x < 0$

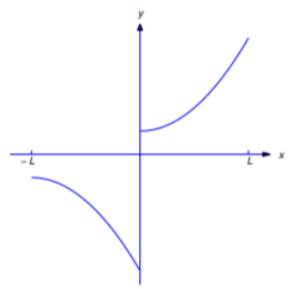

Figure 8.2.9: $y = f(x)$, *where* $f(x+L) = -f(x), -L < x < 0$

27. Show that if f is integrable on $[-L, L]$ and

$$f(x+L) = -f(x), \quad -L < x < 0$$

(Figure 8.2.9), then the Fourier series of f on $[-L, L]$ has the form

$$\sum_{n=1}^{\infty}\left(A_n\cos\frac{(2n-1)\pi x}{L} + B_n\sin\frac{(2n-1)\pi x}{L}\right),$$

where

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{L}\,dx \quad \text{and} \quad B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots.$$

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8.3: Fourier Series II

In this section we discuss Fourier expansions in terms of the eigenfunctions of Problems 1-4 for Section 8.1.

Fourier Cosine Series

From *Exercise 8.1.20*, the eigenfunctions

$$1,\ \cos\frac{\pi x}{L},\ \cos\frac{2\pi x}{L},\ldots,\ \cos\frac{n\pi x}{L},\ldots$$

of the boundary value problem

$$y''+\lambda y=0,\quad y'(0)=0,\quad y'(L)=0 \tag{8.3.1}$$

(Problem 2) are orthogonal on $[0,L]$. If f is integrable on $[0,L]$ then the Fourier expansion of f in terms of these functions is called the *Fourier cosine series of* f on $[0,L]$. This series is

$$a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L},$$

where

$$a_0=\frac{\int_0^L f(x)\,dx}{\int_0^L dx}=\frac{1}{L}\int_0^L f(x)\,dx$$

and

$$a_n=\frac{\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx}{\int_0^L\cos^2\frac{n\pi x}{L}\,dx}=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,\quad n=1,2,3,\ldots.$$

Comparing this definition with Theorem 8.2.6a shows that the Fourier cosine series of f on $[0,L]$ is the Fourier series of the function

$$f_1(x)=\begin{cases} f(-x), & -L<x<0,\\ f(x), & 0\le x\le L\end{cases}$$

obtained by extending f over $[-L,L]$ as an even function (Figure 8.3.1).

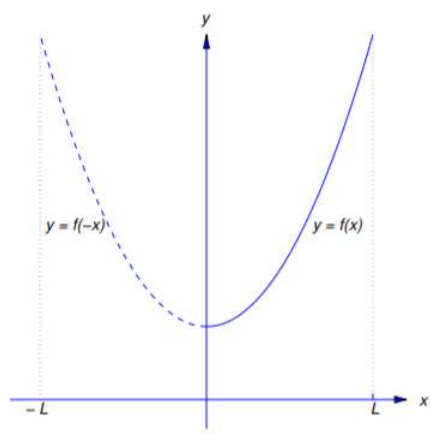

Figure 8.3.1

Applying Theorem 8.2.4 to f_1 yields the next theorem.

Theorem 8.3.1

If f is piecewise smooth on $[0, L]$, then the Fourier cosine series

$$C(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

of f on $[0, L]$, with

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots,$$

converges for all x in $[0, L]$; moreover,

$$C(x) = \begin{cases} f(0+) & if\ x = 0 \\ f(x) & if\ 0 < x < L\ and\ f\ is\ continuous\ at\ x \\ \frac{f(x-)+f(x+)}{2} & if\ 0 < x < L\ and\ f\ is\ discontinuous\ at\ x \\ f(L-) & if\ x = L \end{cases}$$

Example 8.3.1

Find the Fourier cosine series of $f(x) = x$ on $[0, L]$.

The coefficients are

$$a_0 = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\frac{x^2}{2}\bigg|_0^L = \frac{L}{2}$$

and, if $n \geq 1$

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx = \frac{2}{n\pi}\left[x\sin\frac{n\pi x}{L}\bigg|_0^L - \int_0^L \sin\frac{n\pi x}{L}\,dx\right] \\ &= -\frac{2}{n\pi}\int_0^L \sin\frac{n\pi x}{L}\,dx = \frac{2L}{n^2\pi^2}\cos\frac{n\pi x}{L}\bigg|_0^L = \frac{2L}{n^2\pi^2}[(-1)^n - 1] \\ &= \begin{cases} -\frac{4L}{(2m-1)^2\pi^2} & \text{if } n = 2m-1, \\ 0 & \text{if } n = 2m. \end{cases} \end{aligned}$$

Therefore

$$C(x) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{L}$$

Theorem 8.3.1 implies that

$$C(x) = x, \quad 0 \leq x \leq L.$$

Fourier Sine Series

From *Exercise 8.1.19*, the eigenfunctions

$$\sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots, \sin\frac{n\pi x}{L}, \ldots$$

of the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

(Problem 1) are orthogonal on $[0, L]$. If f is integrable on $[0, L]$ then the Fourier expansion of f in terms of these functions is called the *Fourier sine series of* f on $[0, L]$. This series is

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx}{\int_0^L \sin^2 \frac{n\pi x}{L}\, dx} = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, 3, \ldots.$$

Comparing this definition with Theorem 11.2.6b shows that the Fourier sine series of f on $[0, L]$ is the Fourier series of the function

$$f_2(x) = \begin{cases} -f(-x), & -L < x < 0 \\ f(x), & 0 \le x \le L, \end{cases}$$

obtained by extending f over $[-L, L]$ as an odd function (Figure $8.3.2$).

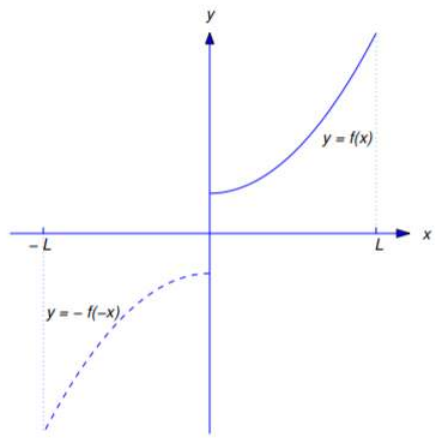

Figure 8.3.2

Applying Theorem 8.2.4 to f_2 yields the next theorem.

Theorem $8.3.2$

If f is piecewise smooth on $[0, L]$, then the Fourier sine series

$$S(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

of f on $[0, L]$, with

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx,$$

converges for all x in $[0, L]$; moreover,

$$S(x) = \begin{cases} 0 & if\ x = 0 \\ f(x) & if\ 0 < x < L\ and\ f\ is\ continuous\ at\ x \\ \frac{f(x-)+f(x+)}{2} & if\ 0 < x < L\ and\ f\ is\ discontinuous\ at\ x \\ 0 & if\ x = L \end{cases}$$

Example $8.3.2$

Find the Fourier sine series of $f(x) = x$ on $[0, L]$.

Solution

The coefficients are

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx = -\frac{2}{n\pi}\left[x\cos\frac{n\pi x}{L}\bigg|_0^L - \int_0^L \cos\frac{n\pi x}{L}\,dx\right] \\ &= (-1)^{n+1}\frac{2L}{n\pi} + \frac{2L}{n^2\pi^2}\sin\frac{n\pi x}{L}\bigg|_0^L = (-1)^{n+1}\frac{2L}{n\pi}. \end{aligned}$$

Therefore

$$S(x) = -\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{L}.$$

Theorem $8.3.2$ implies that

$$S(x) = \begin{cases} x, & 0 \le x < L, \\ 0, & x = L. \end{cases}$$

Mixed Fourier Cosine Series

From *Exercise 8.1.22*, the eigenfunctions

$$\cos\frac{\pi x}{2L},\ \cos\frac{3\pi x}{2L},\ldots,\ \cos\frac{(2n-1)\pi x}{2L},\ldots$$

of the boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0 \tag{8.3.2}$$

(Problem 4) are orthogonal on $[0, L]$. If f is integrable on $[0, L]$ then the Fourier expansion of f in terms of these functions is

$$\sum_{n=1}^{\infty} c_n \cos\frac{(2n-1)\pi x}{2L},$$

where

$$c_n = \frac{\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx}{\int_0^L \cos^2\frac{(2n-1)\pi x}{L}\,dx} = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx.$$

We'll call this expansion the *mixed Fourier cosine series* of f on $[0, L]$, because the boundary conditions of (Equation $8.3.2$) are "mixed" in that they require y to be zero at one boundary point and y' to be zero at the other. By contrast, the "ordinary" Fourier cosine series is associated with (Equation ???), where the boundary conditions require that y' be zero at both endpoints.

It can be shown (*Exercise 8.3.57*) that the mixed Fourier cosine series of f on $[0, L]$ is simply the restriction to $[0, L]$ of the Fourier cosine series of

$$f_3(x) = \begin{cases} f(x), & 0 \le x \le L, \\ -f(2L - x), & L < x \le 2L \end{cases}$$

on $[0, 2L]$ (Figure $8.3.3$).

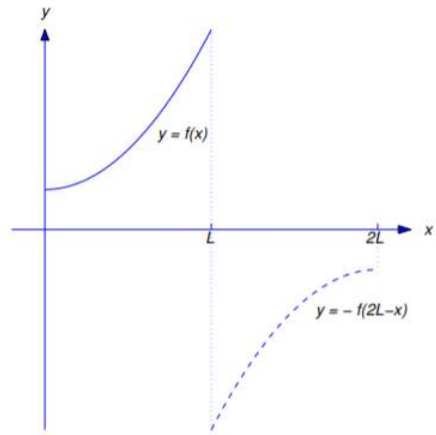

Figure 8.3.3

Applying Theorem $8.3.1$ with f replaced by f_3 and L replaced by $2L$ yields the next theorem.

Theorem $8.3.3$

If f is piecewise smooth on $[0, L]$, then the mixed Fourier cosine series

$$C_M(x) = \sum_{n=1}^{\infty} c_n \cos \frac{(2n-1)\pi x}{2L}$$

of f on $[0, L]$, with

$$c_n = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L}\, dx,$$

converges for all x in $[0, L]$; moreover,

$$C_M(x) = \begin{cases} f(0+) & if\ x = 0 \\ f(x) & if\ 0 < x < L\ and\ f\ is\ continuous\ at\ x \\ \frac{f(x-)+f(x+)}{2} & if\ 0 < x < L\ and\ f\ is\ discontinuous\ at\ x \\ 0 & if\ x = L \end{cases}$$

Example $8.3.3$

Find the mixed Fourier cosine series of $f(x) = x - L$ on $[0, L]$.

Solution

The coefficients are

$$\begin{aligned} c_n &= \frac{2}{L} \int_0^L (x-L) \cos \frac{(2n-1)\pi x}{2L}\, dx \\ &= \frac{4}{(2n-1)\pi} \left[(x-L) \sin \frac{(2n-1)\pi x}{2L} \Big|_0^L - \int_0^L \sin \frac{(2n-1)\pi x}{2L}\, dx \right] \\ &= \frac{8L}{(2n-1)^2\pi^2} \cos \frac{(2n-1)\pi x}{2L} \Big|_0^L = -\frac{8L}{(2n-1)^2\pi^2}. \end{aligned}$$

Therefore

$$C_M(x) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{2L}.$$

Theorem $8.3.3$ implies that

$$C_M(x) = x - L, \quad 0 \le x \le L.$$

Mixed Fourier Sine Series

From *Exercise 8.1.21*, the eigenfunctions

$$\sin\frac{\pi x}{2L}, \sin\frac{3\pi x}{2L}, \ldots, \sin\frac{(2n-1)\pi x}{2L}, \ldots$$

of the boundary value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$$

(Problem 3) are orthogonal on $[0, L]$. If f is integrable on $[0, L]$, then the Fourier expansion of f in terms of these functions is

$$\sum_{n=1}^{\infty} d_n \sin\frac{(2n-1)\pi x}{2L},$$

where

$$d_n = \frac{\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx}{\int_0^L \sin^2\frac{(2n-1)\pi x}{2L}\,dx} = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx.$$

We'll call this expansion the *mixed Fourier sine series* of f on $[0, L]$.

It can be shown (*Exercise 8.3.58*) that the mixed Fourier sine series of f on $[0, L]$ is simply the restriction to $[0, L]$ of the Fourier sine series of

$$f_4(x) = \begin{cases} f(x), & 0 \le x \le L, \\ f(2L - x), & L < x \le 2L, \end{cases}$$

on $[0, 2L]$ (Figure $8.3.4$).

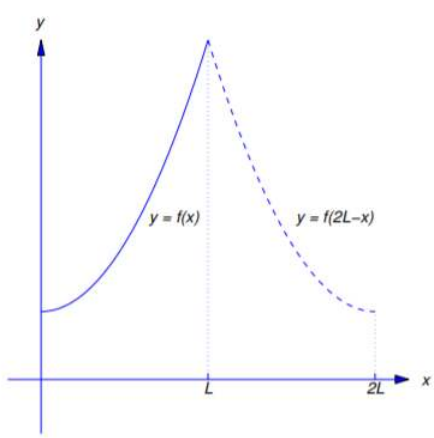

Figure 8.3.4

Applying Theorem $8.3.2$ with f replaced by f_4 and L replaced by $2L$ yields the next theorem.

Theorem 8.3.4

If f is piecewise smooth on $[0, L]$, then the mixed Fourier sine series

$$S_M(x) = \sum_{n=1}^{\infty} d_n \sin\frac{(2n-1)\pi x}{2L}$$

of f on $[0, L]$, with

$$d_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx,$$

converges for all x in $[0, L]$; moreover,

$$SM(x) = \begin{cases} 0 & if\ x = 0 \\ f(x) & if\ 0 < x < L\ and\ f\ is\ continuous\ at\ x \\ \frac{f(x-)+f(x+)}{2} & if\ 0 < x < L\ and\ f\ is\ discontinuous\ at\ x \\ f(L-) & if\ x = L \end{cases}$$

Example 8.3.4

Find the mixed Fourier sine series of $f(x) = x$ on $[0, L]$.

Solution

The coefficients are

$$\begin{aligned} d_n &= \frac{2}{L}\int_0^L x\sin\frac{(2n-1)\pi x}{2L}\,dx \\ &= -\frac{4}{(2n-1)\pi}\left[x\cos\frac{(2n-1)\pi x}{2L}\bigg|_0^L - \int_0^L \cos\frac{(2n-1)\pi x}{2L}\,dx\right] \\ &= \frac{4}{(2n-1)\pi}\int_0^L \cos\frac{(2n-1)\pi x}{2L}\,dx \\ &= \frac{8L}{(2n-1)^2\pi^2}\sin\frac{(2n-1)\pi x}{2L}\bigg|_0^L = (-1)^{n+1}\frac{8L}{(2n-1)^2\pi^2}. \end{aligned}$$

Therefore

$$S_M(x) = -\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{2L}.$$

Theorem 8.3.4 implies that

$$S_M(x) = x, \quad 0 \le x \le L.$$

A Useful Observation

In applications involving expansions in terms of the eigenfunctions of Problems 1-4, the functions being expanded are often polynomials that satisfy the boundary conditions of the problem under consideration. In this case the next theorem presents an efficient way to obtain the coefficients in the expansion.

Theorem 8.3.5

a. If $f'(0) = f'(L) = 0$, f'' is continuous, and f''' is piecewise continuous on $[0, L]$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad 0 \le x \le L, \tag{8.3.3}$$

with

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n = \frac{2L^2}{n^3\pi^3}\int_0^L f'''(x)\sin\frac{n\pi x}{L}\,dx, \quad n \ge 1. \tag{8.3.4}$$

Now suppose f' is continuous and f'' is piecewise continuous on $[0, L]$.

b. If $f(0) = f(L) = 0$, then

$$f(x)=\sum_{n=1}^{\infty} b_n \sin \frac{n \pi x}{L}, \quad 0 \leq x \leq L,$$

with

$$b_n=-\frac{2 L}{n^2 \pi^2} \int_0^L f''(x) \sin \frac{n \pi x}{L} \, dx. \tag{8.3.5}$$

c. If $f'(0)=f(L)=0$, then

$$f(x)=\sum_{n=1}^{\infty} c_n \cos \frac{(2 n-1) \pi x}{2 L}, \quad 0 \leq x \leq L,$$

with

$$c_n=-\frac{8 L}{(2 n-1)^2 \pi^2} \int_0^L f''(x) \cos \frac{(2 n-1) \pi x}{2 L} \, dx. \tag{8.3.6}$$

d. If $f(0)=f'(L)=0$, then

$$f(x)=\sum_{n=1}^{\infty} d_n \sin \frac{(2 n-1) \pi x}{2 L}, \quad 0 \leq x \leq L,$$

with

$$d_n=-\frac{8 L}{(2 n-1)^2 \pi^2} \int_0^L f''(x) \sin \frac{(2 n-1) \pi x}{2 L} \, dx. \tag{8.3.7}$$

Proof

We'll prove (a) and leave the rest to you (*Exercises 8.3.35, 8.3.42*, and 8.3.50). Since f is continuous on $[0, L]$, Theorem 8.3.1implies (Equation 8.3.3) with a_0, a_1, a_2,... as defined in Theorem 8.3.1. We already know that a_0 is as in (Equation 8.3.4). If $n \geq 1$, integrating twice by parts yields

$$\begin{aligned}
a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n \pi x}{L} \, dx \\
&= \frac{2}{n \pi}\left[f(x) \sin \frac{n \pi x}{L}\Big|_0^L-\int_0^L f'(x) \sin \frac{n \pi x}{L} \, dx\right] \\
&= -\frac{2}{n \pi} \int_0^L f'(x) \sin \frac{n \pi x}{L} \, dx \text{ (since } \sin 0=\sin n \pi=0) \\
&= \frac{2 L}{n^2 \pi^2}\left[f'(x) \cos \frac{n \pi x}{L}\Big|_0^L-\int_0^L f''(x) \cos \frac{n \pi x}{L}\right] \, dx \\
&= -\frac{2 L}{n^2 \pi^2} \int_0^L f''(x) \cos \frac{n \pi x}{L} \, dx \text{ (since } f'(0)=f'(L)=0) \\
&= -\frac{2 L^2}{n^3 \pi^3}\left[f''(x) \sin \frac{n \pi x}{L}\Big|_0^L-\int_0^L f'''(x) \sin \frac{n \pi x}{L} \, dx\right] \\
&= \frac{2 L^2}{n^3 \pi^3} \int_0^L f'''(x) \sin \frac{n \pi x}{L} \, dx \text{ (since } \sin 0=\sin n \pi=0).
\end{aligned}$$

(By an argument similar to one used in the proof of Theorem 8.3.1, the last integration by parts is legitimate in the case where f''' is undefined at finitely many points in $[0, L]$, so long as it is piecewise continuous on $[0, L]$.) This completes the proof.

Example 8.3.5

Find the Fourier cosine expansion of $f(x)=x^2(3L-2x)$ on $[0,L]$.

Solution

Here

$$a_0=\frac{1}{L}\int_0^L(3Lx^2-2x^3)\,dx=\frac{1}{L}\left(Lx^3-\frac{x^4}{2}\right)\Bigg|_0^L=\frac{L^3}{2}$$

and

$$a_n=\frac{2}{L}\int_0^L(3Lx^2-2x^3)\cos\frac{n\pi x}{L}\,dx,\quad n\geq 1.$$

Evaluating this integral directly is laborious. However, since $f'(x)=6Lx-6x^2$, we see that $f'(0)=f'(L)=0$. Since $f'''(x)=-12$, we see from (Equation 8.3.4) that if $n\geq 1$ then

$$\begin{aligned}a_n&=-\frac{24L^2}{n^3\pi^3}\int_0^L\sin\frac{n\pi x}{L}\,dx=\frac{24L^3}{n^4\pi^4}\cos\frac{n\pi x}{L}\Bigg|_0^L=\frac{24L^3}{n^4\pi^4}[(-1)^n-1]\\&=\begin{cases}-\frac{48L^3}{(2m-1)^4\pi^4} & \text{if } n=2m-1,\\ 0 & \text{if } n=2m.\end{cases}\end{aligned}$$

Therefore

$$C(x)=\frac{L^3}{2}-\frac{48L^3}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\cos\frac{(2n-1)\pi x}{L}.$$

Example 8.3.6

Find the Fourier sine expansion of $f(x)=x(x^2-3Lx+2L^2)$ on $[0,L]$.

Solution

Since $f(0)=f(L)=0$ and $f''(x)=6(x-L)$, we see from (Equation 8.3.5) that

$$\begin{aligned}b_n&=-\frac{12L}{n^2\pi^2}\int_0^L(x-L)\sin\frac{n\pi x}{L}\,dx\\&=\frac{12L^2}{n^3\pi^3}\left[(x-L)\cos\frac{n\pi x}{L}\Bigg|_0^L-\int_0^L\cos\frac{n\pi x}{L}\,dx\right]\\&=\frac{12L^2}{n^3\pi^3}\left[L-\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Bigg|_0^L\right]=\frac{12L^3}{n^3\pi^3}.\end{aligned}$$

Therefore

$$S(x)=\frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{L}$$

Example 8.3.7

Find the mixed Fourier cosine expansion of $f(x)=3x^3-4Lx^2+L^3$ on $[0,L]$.

Solution

Since $f'(0)=f(L)=0$ and $f''(x)=2(9x-4L)$, we see from (Equation 8.3.6) that

$$
\begin{aligned}
c_n &= -\frac{16L}{(2n-1)^2\pi^2}\int_0^L (9x-4L)\cos\frac{(2n-1)\pi x}{2L}\,dx \\
&= -\frac{32L^2}{(2n-1)^3\pi^3}\left[(9x-4L)\sin\frac{(2n-1)\pi x}{2L}\bigg|_0^L - 9\int_0^L \sin\frac{(2n-1)\pi x}{2L}\right]dx \\
&= -\frac{32L^2}{(2n-1)^3\pi^3}\left[(-1)^{n+1}5L + \frac{18L}{(2n-1)\pi}\cos\frac{(2n-1)\pi x}{2L}\bigg|_0^L\right] \\
&= \frac{32L^3}{(2n-1)^3\pi^3}\left[(-1)^n 5 + \frac{18}{(2n-1)\pi}\right].
\end{aligned}
$$

Therefore

$$
C_M(x) = \frac{32L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[(-1)^n 5 + \frac{18}{(2n-1)\pi}\right]\cos\frac{(2n-1)\pi x}{2L}
$$

Example 8.3.8

Find the mixed Fourier sine expansion of

$$
f(x) = x(2x^2 - 9Lx + 12L^2)
$$

on $[0, L]$.

Solution

Since $f(0) = f'(L) = 0$, and $f''(x) = 6(2x-3L)$, we see from (Equation 8.3.7) that

$$
\begin{aligned}
d_n &= -\frac{48L}{(2n-1)^2\pi^2}\int_0^L (2x-3L)\sin\frac{(2n-1)\pi x}{2L}dx \\
&= \frac{96L^2}{(2n-1)^3\pi^3}\left[(2x-3L)\cos\frac{(2n-1)\pi x}{2L}\bigg|_0^L - 2\int_0^L \cos\frac{(2n-1)\pi x}{2L}dx\right] \\
&= \frac{96L^2}{(2n-1)^3\pi^3}\left[3L - \frac{4L}{(2n-1)\pi}\sin\frac{(2n-1)\pi x}{2L}\bigg|_0^L\right] \\
&= \frac{96L^3}{(2n-1)^3\pi^3}\left[3 + (-1)^n\frac{4}{(2n-1)\pi}\right]
\end{aligned}
$$

Therefore

$$
S_M(x) = \frac{96L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\left[3 + (-1)^n\frac{4}{(2n-1)\pi}\right]\sin\frac{(2n-1)\pi x}{2L}
$$

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8.3E: Fourier Series II (Exercises)

In *Exercise 8.3.2* graph f and some partial sums of the required series.

Q8.3.1

In *Exercises 8.3.1-8.3.10* find the Fourier cosine series.

1. $f(x)=x^2$; $[0,L]$
2. $f(x)=1-x$; $[0,1]$
3. $f(x)=x^2-2Lx$; $[0,L]$
4. $f(x)=\sin kx$ $(k\neq \text{integer})$;$[0,\pi]$
5. $f(x)=\begin{cases}1, & 0\leq x\leq \frac{L}{2}\\ 0, & \frac{L}{2}<x<L;\end{cases}$ $[0,L]$
6. $f(x)=x^2-L^2$; $[0,L]$
7. $f(x)=(x-1)^2$; $[0,1]$
8. $f(x)=e^x$; $[0,\pi]$
9. $f(x)=x(L-x)$; $[0,L]$
10. $f(x)=x(x-2L)$; $[0,L]$

Q8.3.2

In *Exercises 8.3.11-8.3.17* find the Fourier sine series

11. $f(x)=1$; $[0,L]$
12. $f(x)=1-x$; $[0,1]$
13. $f(x)=\cos kx$ $(k\neq \text{integer})$; $[0,\pi]$
14. $f(x)=\begin{cases}1, & 0\leq x\leq \frac{L}{2}\\ 0, & \frac{L}{2}<x<L;\end{cases}$ $[0,L]$
15. $f(x)=\begin{cases}x, & 0\leq x\leq \frac{L}{2},\\ L-x, & \frac{L}{2}\leq x\leq L;\end{cases}$ $[0,L]$.
16. $f(x)=x\sin x$; $[0,\pi]$
17. $f(x)=e^x$; $[0,\pi]$

Q8.3.3

In *Exercises 8.3.18-8.3.24* find the mixed Fourier cosine series.

18. $f(x)=1$; $[0,L]$
19. $f(x)=x^2$; $[0,L]$
20. $f(x)=x$; $[0,1]$
21. $f(x)=\begin{cases}1, & 0\leq x\leq \frac{L}{2}\\ 0, & \frac{L}{2}<x<L;\end{cases}$ $[0,L]$
22. $f(x)=\cos x$; $[0,\pi]$
23. $f(x)=\sin x$; $[0,\pi]$
24. $f(x)=x(L-x)$; $[0,L]$

Q8.3.4

In *Exercises 8.3.25-8.3.30* find the mixed Fourier sine series.

25. $f(x)=1;\ [0,L]$

26. $f(x)=x^2;\ [0,L]$

27. $f(x)=\begin{cases}1, & 0\le x\le \frac{L}{2}\\ 0, & \frac{L}{2}<x<L;\end{cases}\quad [0,L]$

28. $f(x)=\cos x;\ [0,\pi]$

29. $f(x)=\sin x;\ [0,\pi]$

30. $f(x)=x(L-x);\ [0,L]$.

Q8.3.5

In *Exercises 8.3.31-8.3.34* use Theorem 8.3.5a to find the Fourier cosine series of f on $[0,L]$.

31. $f(x)=3x^2(x^2-2L^2)$

32. $f(x)=x^3(3x-4L)$

33. $f(x)=x^2(3x^2-8Lx+6L^2)$

34. $f(x)=x^2(x-L)^2$

Q8.3.6

35.

a. Prove Theorem 8.3.5b.
b. In addition to the assumptions of Theorem 8.3.5b, suppose $f''(0)=f''(L)=0$, f''' is continuous, and $f^{(4)}$ is piecewise continuous on $[0,L]$. Show that

$$b_n=\frac{2L^3}{n^4\pi^4}\int_0^L f^{(4)}(x)\sin\frac{n\pi x}{L}\,dx,\quad n\ge 1.$$

Q8.3.7

In *Exercises 8.3.36-8.3.41* use Theorem 8.3.5b or, where applicable, *Exercises 8.1.35b* to find the Fourier sine series of f on $[0,L]$.

36. $f(x)=x(L-x)$

37. $f(x)=x^2(L-x)$

38. $f(x)=x(L^2-x^2)$

39. $f(x)=x(x^3-2Lx^2+L^3)$

40. $f(x)=x(3x^4-10L^2x^2+7L^4)$

41. $f(x)=x(3x^4-5Lx^3+2L^4)$

Q8.3.8

42.

a. Prove Theorem 8.3.5c.
b. In addition to the assumptions of Theorem 8.3.5c, suppose $f''(L)=0$, f'' is continuous, and f''' is piecewise continuous on $[0,L]$. Show that

$$c_n=\frac{16L^2}{(2n-1)^3\pi^3}\int_0^L f'''(x)\sin\frac{(2n-1)\pi x}{2L}\,dx,\quad n\ge 1.$$

Q8.3.9

In *Exercises 8.3.43-8.3.49* use Theorem 8.3.5c, or where applicable, *Exercise 8.1.42b*, to find the mixed Fourier cosine series of f on $[0, L]$.

43. $f(x) = x^2(L - x)$

44. $f(x) = L^2 - x^2$

45. $f(x) = L^3 - x^3$

46. $f(x) = 2x^3 + 3Lx^2 - 5L^3$

47. $f(x) = 4x^3 + 3Lx^2 - 7L^3$

48. $f(x) = x^4 - 2Lx^3 + L^4$

49. $f(x) = x^4 - 4Lx^3 + 6L^2x^2 - 3L^4$

Q8.3.10

50.

a. Prove Theorem 8.3.5d.
b. In addition to the assumptions of Theorem 8.3.5d, suppose $f''(0) = 0$, f'' is continuous, and f''' is piecewise continuous on $[0, L]$. Show that

$$d_n = -\frac{16L^2}{(2n-1)^3\pi^3} \int_0^L f'''(x) \cos \frac{(2n-1)\pi x}{2L}\, dx, \quad n \geq 1.$$

Q8.3.11

In *Exercises 8.3.51-8.3.56* use Theorem 8.3.5d or, where applicable, *Exercise 8.3.50b*, to find the mixed Fourier sine series of the f on $[0, L]$.

51. $f(x) = x(2L - x)$

52. $f(x) = x^2(3L - 2x)$

53. $f(x) = (x - L)^3 + L^3$

54. $f(x) = x(x^2 - 3L^2)$

55. $f(x) = x^3(3x - 4L)$

56. $f(x) = x(x^3 - 2Lx^2 + 2L^3)$

Q8.3.12

57. Show that the mixed Fourier cosine series of f on $[0, L]$ is the restriction to $[0, L]$ of the Fourier cosine series of

$$f_3(x) = \begin{cases} f(x), & 0 \leq x \leq L, \\ -f(2L - x), & L < x \leq 2L \end{cases}$$

on $[0, 2L]$. Use this to prove Theorem 8.3.3.

58. Show that the mixed Fourier sine series of f on $[0, L]$ is the restriction to $[0, L]$ of the Fourier sine series of

$$f_4(x) = \begin{cases} f(x), & 0 \leq x \leq L, \\ f(2L - x), & L < x \leq 2L \end{cases}$$

on $[0, 2L]$. Use this to prove Theorem 8.3.4.

59. Show that the Fourier sine series of f on $[0, L]$ is the restriction to $[0, L]$ of the Fourier sine series of

$$f_3(x) = \begin{cases} f(x), & 0 \leq x \leq L, \\ -f(2L - x), & L < x \leq 2L \end{cases}$$

on $[0, 2L]$.

60. Show that the Fourier cosine series of f on $[0, L]$ is the restriction to $[0, L]$ of the Fourier cosine series of

$$f_4(x) = \begin{cases} f(x), & 0 \le x \le L, \\ f(2L - x), & L < x \le 2L \end{cases}$$

on $[0, 2L]$.

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CHAPTER OVERVIEW

Chapter 9: Fourier Solutions of Partial Differential Equations

In this chapter, we use the series discussed in Chapter 11 to solve partial differential equations that arise in problems of mathematical physics.

9.1: The Heat Equation

9.1E: The Heat Equation (Exercises)

9.2: The Wave Equation

9.2E: The Wave Equation (Exercises)

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9.1: The Heat Equation

We begin the study of partial differential equations with the problem of heat flow in a uniform bar of length L, situated on the x axis with one end at the origin and the other at $x = L$ (Figure 12.1.1).

We assume that the bar is perfectly insulated except possibly at its endpoints, and that the temperature is constant on each cross section and therefore depends only on x and t. We also assume that the thermal properties of the bar are independent of x and t. In this case, it can be shown that the temperature $u = u(x,t)$ at time t at a point x units from the origin satisfies the partial differential equation

$$u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0,$$

where a is a positive constant determined by the thermal properties. This is the **heat equation**.

Figure 12.1.1 : A uniform bar of length L

To determine u, we must specify the temperature at every point in the bar when $t = 0$, say

$$u(x,0) = f(x), \quad 0 \le x \le L.$$

We call this the **initial** condition. We must also specify **boundary conditions** that u must satisfy at the ends of the bar for all $t > 0$. We'll call this problem an **initial-boundary value problem.**

We begin with the boundary conditions $u(0,t) = u(L,t) = 0$, and write the initial-boundary value problem as

$$\begin{aligned} u_t &= a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L. \end{aligned} \tag{9.1.1}$$

Our method of solving this problem is called **separation of variables** (not to be confused with method of separation of variables used in Section 2.2 for solving ordinary differential equations). We begin by looking for functions of the form

$$v(x,t) = X(x)T(t)$$

that are not identically zero and satisfy

$$v_t = a^2 v_{xx}, \quad v(0,t) = 0, \quad v(L,t) = 0$$

for all (x,t). Since

$$v_t = XT' \quad \text{and} \quad v_{xx} = X''T,$$

$v_t = a^2 v_{xx}$ if and only if

$$XT' = a^2 X''T,$$

which we rewrite as

$$\frac{T'}{a^2 T} = \frac{X''}{X}.$$

Since the expression on the left is independent of x while the one on the right is independent of t, this equation can hold for all (x,t) only if the two sides equal the same constant, which we call a **separation constant**, and write it as $-\lambda$; thus,

$$\frac{X''}{X} = \frac{T'}{a^2 T} = -\lambda.$$

This is equivalent to

$$X'' + \lambda X = 0$$

and

$$T' = -a^2 \lambda T. \tag{9.1.2}$$

Since $v(0,t) = X(0)T(t) = 0$ and $v(L,t) = X(L)T(t) = 0$ and we don't want T to be identically zero, $X(0) = 0$ and $X(L) = 0$. Therefore λ must be an eigenvalue of the boundary value problem

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(L) = 0, \tag{9.1.3}$$

and X must be a λ-eigenfunction. From Theorem 11.1.2, the eigenvalues of Equation 9.1.3 are $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$X_n = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots.$$

Substituting $\lambda = n^2\pi^2/L^2$ into Equation 9.1.2 yields

$$T' = -(n^2\pi^2 a^2/L^2)T,$$

which has the solution

$$T_n = e^{-n^2\pi^2 a^2 t/L^2}.$$

Now let

$$v_n(x,t) = X_n(x)T_n(t) = e^{-n^2\pi^2 a^2 t/L^2} \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \ldots$$

Since

$$v_n(x,0) = \sin \frac{n\pi x}{L},$$

v_n satisfies Equation 9.1.1 with $f(x) = \sin n\pi x/L$. More generally, if $\alpha_1, \ldots, \alpha_m$ are constants and

$$u_m(x,t) = \sum_{n=1}^{m} \alpha_n e^{-n^2\pi^2 a^2 t/L^2} \sin \frac{n\pi x}{L},$$

then u_m satisfies Equation 9.1.1 with

$$f(x) = \sum_{n=1}^{m} \alpha_n \sin \frac{n\pi x}{L}.$$

This motivates the next definition.

Definition 12.1.1

The **formal solution** of the initial-boundary value problem

$$\begin{gathered} u_t = a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0,t) = 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) = f(x), \quad 0 \le x \le L \end{gathered} \tag{9.1.4}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-n^2\pi^2 a^2 t/L^2} \sin \frac{n\pi x}{L}, \tag{9.1.5}$$

where

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0, L]$; that is,

$$\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

We use the term "formal solution" in this definition because it is not in general true that the infinite series in Equation 9.1.5 actually satisfies all the requirements of the initial-boundary value problem Equation 9.1.4 when it does, we say that it is an **actual solution** of Equation 9.1.4.

Because of the negative exponentials in Equation 9.1.5, u converges for all (x,t) with $t>0$. Since each term in Equation 9.1.5 satisfies the heat equation and the boundary conditions in Equation 9.1.4, u also has these properties if u_t and u_{xx} can be obtained by differentiating the series in Equation 9.1.5 term by term once with respect to t and twice with respect to x, for $t>0$. However, it is not always legitimate to differentiate an infinite series term by term. The next theorem gives a useful sufficient condition for legitimacy of term by term differentiation of an infinite series. We omit the proof.

Theorem 12.1.2

A convergent infinite series

$$W(z) = \sum_{n=1}^{\infty} w_n(z)$$

can be differentiated term by term on a closed interval $[z_1, z_2]$ to obtain

$$W'(z) = \sum_{n=1}^{\infty} w_n'(z)$$

(where the derivatives at $z=z_1$ and $z=z_2$ are one-sided) provided that w_n' is continuous on $[z_1,z_2]$ and

$$|w_n'(z)| \le M_n, \quad z_1 \le z \le z_2, \quad n=1,2,3,\ldots,$$

where M_1, M_2, …, M_n, …, are constants such that the series $\sum_{n=1}^{\infty} M_n$ converges.

Theorem 12.1.2 , applied twice with $z=x$ and once with $z=t$, shows that u_{xx} and u_t can be obtained by differentiating u term by term if $t>0$ (**Exercise 12.1.54**). Therefore u satisfies the heat equation and the boundary conditions in Equation 9.1.4 for $t>0$. Therefore, since $u(x,0)=S(x)$ for $0\le x\le L$, u is an actual solution of Equation 9.1.4 if and only if $S(x)=f(x)$ for $0\le x\le L$. From Theorem 11.3.2, this is true if f is continuous and piecewise smooth on $[0,L]$, and $f(0)=f(L)=0$.

In this chapter we'll define formal solutions of several kinds of problems. When we ask you to solve such problems, we always mean that you should find a formal solution.

Example 12.1.1

Solve Equation 9.1.4 with $f(x)=x(x^2-3Lx+2L^2)$.

Solution

From Example [example:11.3.6}, the Fourier sine series of f on $[0,L]$ is

$$S(x) = \frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi x}{L}$$

Therefore

$$u(x,t) = \frac{12L^3}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}e^{-n^2\pi^2a^2t/L^2}\sin\frac{n\pi x}{L}$$

If both ends of bar are insulated so that no heat can pass through them, then the boundary conditions are

$$u_x(0,t)=0, \quad u_x(L,t)=0, \quad t>0.$$

We leave it to you (**Exercise 12.1.1**) to use the method of separation of variables and Theorem 11.1.3 to motivate the next definition.

Definition 12.1.3 : Formal Solution

The **formal solution** of the initial-boundary value problem

$$\begin{aligned} u_t &= a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u_x(0,t) &= 0, \quad u_x(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L \end{aligned} \tag{9.1.6}$$

is

$$u(x,t) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n e^{-n^2\pi^2 a^2 t/L^2} \cos \frac{n\pi x}{L},$$

where

$$C(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi x}{L}$$

is the Fourier cosine series of f on $[0, L]$; that is,

$$\alpha_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad \alpha_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad n = 1, 2, 3, \ldots.$$

Example 12.1.2

Solve Equation 9.1.6 with $f(x) = x$.

Solution

From Example 11.3.1, the Fourier cosine series of f on $[0, L]$ is

$$C(x) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}$$

Therefore

$$u(x,t) = \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-(2n-1)^2\pi^2 a^2 t/L^2} \cos \frac{(2n-1)\pi x}{L}.$$

We leave it to you (**Exercise 12.1.2**) to use the method of separation of variables and Theorem 11.1.4 to motivate the next definition.

Definition 12.1.4

The formal solution of the initial-boundary value problem

$$\begin{aligned} u_t &= a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u(0,t) &= 0, \quad u_x(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L \end{aligned} \tag{9.1.7}$$

is

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-(2n-1)^2\pi^2 a^2 t/4L^2} \sin \frac{(2n-1)\pi x}{2L},$$

where

$$S_M(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{(2n-1)\pi x}{2L}$$

is the mixed Fourier sine series of f on $[0, L]$; that is,

$$\alpha_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L}\, dx.$$

Example 12.1.3

Solve Equation 9.1.7 with $f(x) = x$.

Solution

From Example 11.3.4, the mixed Fourier sine series of f on $[0, L]$ is

$$S_M(x) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2L}.$$

Therefore

$$u(x,t) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} e^{-(2n-1)^2 \pi^2 a^2 t/4L^2} \sin \frac{(2n-1)\pi x}{2L}.$$

Figure 12.1.2 shows a graph of $u = u(x,t)$ plotted with respect to x for various values of t. The line $y = x$ corresponds to $t = 0$. The other curves correspond to positive values of t. As t increases, the graphs approach the line $u = 0$.

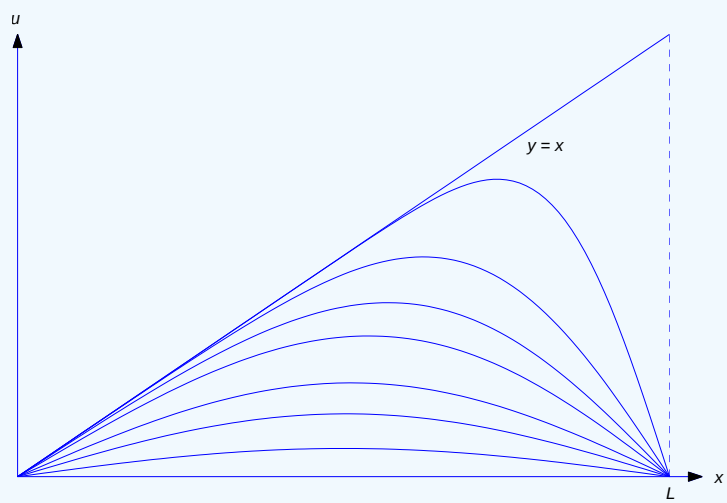

Figure 12.1.2

We leave it to you (**Exercise 12.1.3**) to use the method of separation of variables and Theorem 11.1.5 to motivate the next definition.

Definition 12.1.5

The formal solution of the initial-boundary value problem

$$\begin{aligned} u_t &= a^2 u_{xx}, \quad 0 < x < L, \quad t > 0, \\ u_x(0,t) &= 0, \quad u(L,t) = 0, \quad t > 0, \\ u(x,0) &= f(x), \quad 0 \le x \le L \end{aligned} \tag{9.1.8}$$

is

$$u(x,t)=\sum_{n=1}^{\infty}\alpha_n e^{-(2n-1)^2\pi^2a^2t/4L^2}\cos\frac{(2n-1)\pi x}{2L},$$

where

$$C_M(x)=\sum_{n=1}^{\infty}\alpha_n\cos\frac{(2n-1)\pi x}{2L}$$

is the mixed Fourier cosine series of f on $[0,L]$; that is,

$$\alpha_n=\frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx.$$

Example 12.1.4

Solve Equation 9.1.8 with $f(x)=x-L$.

Solution

From Example 11.3.3, the mixed Fourier cosine series of f on $[0,L]$ is

$$C_M(x)=-\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{2L}.$$

Therefore

$$u(x,t)=-\frac{8L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2a^2t/4L^2}\cos\frac{(2n-1)\pi x}{2L}.$$

Nonhomogeneous Problems

A problem of the form

$$\begin{aligned}&u_t=a^2u_{xx}+h(x),\quad 0<x<L,\quad t>0,\\&u(0,t)=u_0,\quad u(L,t)=u_L,\quad t>0,\\&u(x,0)=f(x),\quad 0\le x\le L\end{aligned}\tag{9.1.9}$$

can be transformed to a problem that can be solved by separation of variables. We write

$$u(x,t)=v(x,t)+q(x),\tag{9.1.10}$$

where q is to be determined. Then

$$u_t=v_t\quad\text{and}\quad u_{xx}=v_{xx}+q''$$

so u satisfies Equation 9.1.9 if v satisfies

$$\begin{aligned}&v_t=a^2v_{xx}+a^2q''(x)+h(x),\quad 0<x<L,\quad t>0,\\&v(0,t)=u_0-q(0),\quad v(L,t)=u_L-q(L),\quad t>0,\\&v(x,0)=f(x)-q(x),\quad 0\le x\le L.\end{aligned}$$

This reduces to

$$\begin{aligned}&v_t=a^2v_{xx},\quad 0<x<L,\quad t>0,\\&v(0,t)=0,\quad v(L,t)=0,\quad t>0,\\&v(x,0)=f(x)-q(x),\quad 0\le x\le L\end{aligned}\tag{9.1.11}$$

if

$$a^2q'' + h(x) = 0, \quad q(0) = u_0, \quad q(L) = u_L.$$

We can obtain q by integrating $q'' = -h/a^2$ twice and choosing the constants of integration so that $q(0) = u_0$ and $q(L) = u_L$. Then we can solve Equation 9.1.11 for v by separation of variables, and Equation 9.1.10 is the solution of Equation 9.1.9.

Example 12.1.5

Solve

$$u_t = u_{xx} - 2, \quad 0 < x < 1, \quad t > 0,$$
$$u(0,t) = -1, \quad u(1,t) = 1, \quad t > 0,$$
$$u(x,0) = x^3 - 2x^2 + 3x - 1, \quad 0 \le x \le 1.$$

Solution

We leave it to you to show that

$$q(x) = x^2 + x - 1$$

satisfies

$$q'' - 2 = 0, \quad q(0) = -1, \quad q(1) = 1.$$

Therefore

$$u(x,t) = v(x,t) + x^2 + x - 1,$$

where

$$v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0,$$
$$v(0,t) = 0, \quad v(1,t) = 0, \quad t > 0,$$

and

$$v(x,0) = x^3 - 2x^2 + 3x - 1 - x^2 - x + 1 = x(x^2 - 3x + 2).$$

From Example 12.1.1 with $a = 1$ and $L = 1$,

$$v(x,t) = \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2\pi^2 t} \sin n\pi x.$$

Therefore

$$u(x,t) = x^2 + x - 1 + \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2\pi^2 t} \sin n\pi x.$$

A similar procedure works if the boundary conditions in Equation 9.1.11 are replaced by mixed boundary conditions

$$u_x(0,t) = u_0, \quad u(L,t) = u_L, \quad t > 0$$

or

$$u(0,t) = u_0, \quad u_x(L,t) = u_L, \quad t > 0;$$

however, this isn't true in general for the boundary conditions

$$u_x(0,t) = u_0, \quad u_x(L,t) = u_L, \quad t > 0.$$

Using Technology

Numerical experiments can enhance your understanding of the solutions of initial-boundary value problems. To be specific, consider the formal solution

$$u(x,t)=\sum_{n=1}^{\infty}\alpha_n e^{-n^2\pi^2a^2t/L^2}\sin\frac{n\pi x}{L},$$

of Equation 9.1.4, where

$$S(x)=\sum_{n=1}^{\infty}\alpha_n\sin\frac{n\pi x}{L}$$

is the Fourier sine series of f on $[0,L]$. Consider the m-th partial sum

$$u_m(x,t)=\sum_{n=1}^{m}\alpha_n e^{-n^2\pi^2a^2t/L^2}\sin\frac{n\pi x}{L}. \tag{9.1.12}$$

For several fixed values of t (including $t=0$), graph $u_m(x,t)$ versus t. In some cases it may be useful to graph the curves corresponding to the various values of t on the same axes in other cases you may want to graph the various curves successively (for increasing values of t), and create a primitive motion picture on your monitor. Repeat this experiment for several values of m, to compare how the results depend upon m for small and large values of t. However, keep in mind that the meanings of "small" and "large" in this case depend upon the constants a^2 and L^2. A good way to handle this is to rewrite Equation 9.1.12 as

$$u_m(x,t)=\sum_{n=1}^{m}\alpha_n e^{-n^2\tau}\sin\frac{n\pi x}{L},$$

where

$$\tau=\frac{\pi^2a^2t}{L^2}, \tag{9.1.13}$$

and graph u_m versus x for selected values of τ.

These comments also apply to the situations considered in Definitions 12.1.3 -12.1.5 , except that Equation 9.1.13 should be replaced by

$$\tau=\frac{\pi^2a^2t}{4L^2},$$

in Definitions 12.1.4 and 12.1.5 .

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9.1E: The Heat Equation (Exercises)

In some of the exercises we say "perform numerical experiments." This means that you should perform the computations just described with the formal solution obtained in the exercise.

Q9.1.1

1. Explain Theorem 9.1.3.

2. Explain Theorem 9.1.4.

3. Explain Theorem 9.1.5.

4. Perform numerical experiments with the formal solution obtained in Example 9.1.1.

5. Perform numerical experiments with the formal solution obtained in Example 9.1.2.

6. Perform numerical experiments with the formal solution obtained in Example 9.1.3.

7. Perform numerical experiments with the formal solution obtained in Example 9.1.4.

Q9.1.2

In *Exercises 9.2.8-9.2.19* solve the initial-boundary value problem.

8. $u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(1-x), \quad 0 \le x \le 1$

9. $u_t = 9u_{xx}, \quad 0 < x < 4, \quad t > 0.$
$u(0,t) = 0, \quad u(4,t) = 0, \quad t > 0,$
$u(x,0) = 1, \quad 0 \le x \le 4$

10. $u_t = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0.$
$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x \sin x, \quad 0 \le x \le \pi$

11. $u_t = 9u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u(0,t) = 0, \quad u(2,t) = 0, \quad t > 0,$
$u(x,0) = x^2(2-x), \quad 0 \le x \le 2$

12. $u_t = 4u_{xx}, \quad 0 < x < 3, \quad t > 0,$
$u(0,t) = 0, \quad u(3,t) = 0, \quad t > 0,$
$u(x,0) = x(9-x^2), \quad 0 \le x \le 3$

13. $u_t = 4u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u(0,t) = 0, \quad u(2,t) = 0, \quad t > 0,$
$u(x,0) = \begin{cases} x, & 0 \le x \le 1, \\ 2-x, & 1 \le x \le 2. \end{cases}$

14. $u_t = 4u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(2,t) = 0, \quad t > 0,$
$u(x,0) = x(x-4), \quad 0 \le x \le 2$

15. $u_t = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0,$
$u(x,0) = x(1-x), \quad 0 \le x \le 1$

26. $u_t = 3u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x(\pi - x), \quad 0 \le x \le \pi$

17. $u_t = 5u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u(0,t) = 0, \quad u_x(2,t) = 0, \quad t > 0,$

$u(x,0)=x(4-x), \quad 0 \leq x \leq 2$

18. $u_t=16u_{xx}, \quad 0<x<2\pi, \quad t>0,$
$u_x(0,t)=0, \quad u(2\pi,t)=0, \quad t>0,$
$u(x,0)=4, \quad 0 \leq x \leq 2\pi$

19. $u_t=3u_{xx}, \quad 0<x<1, \quad t>0,$
$u_x(0,t)=0, \quad u(1,t)=0, \quad t>0,$
$u(x,0)=1-x, \quad 0 \leq x \leq 1$

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9.2: The Wave Equation

In this section we consider initial-boundary value problems of the form

$$\begin{gathered} u_{tt}=a^2 u_{xx}, \quad 0<x<L, \quad t>0, \\ u(0,t)=0, \quad u(L,t)=0, \quad t>0, \\ u(x,0)=f(x), \quad u_t(x,0)=g(x), \quad 0 \leq x \leq L, \end{gathered} \tag{9.2.1}$$

where a is a constant and f and g are given functions of x. The partial differential equation $u_{tt}=a^2 u_{xx}$ is called the **wave equation**. It is necessary to specify both f and g because the wave equation is a second order equation in t for each fixed x. This equation and its generalizations

$$u_{tt}=a^2\left(u_{xx}+u_{yy}\right) \quad \text{and} \quad u_{tt}=a^2\left(u_{xx}+u_{yy}+u_{zz}\right)$$

to two and three space dimensions have important applications to the the propagation of electromagnetic, sonic, and water waves.

The Vibrating String

We motivate the study of the wave equation by considering its application to the vibrations of a string – such as a violin string – tightly stretched in equilibrium along the x-axis in the xu-plane and tied to the points $(0,0)$ and $(L,0)$ (Figure 12.2.1).

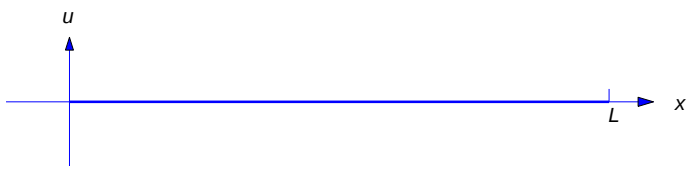

Figure 12.2.1 : A stretched string

If the string is plucked in the vertical direction and released at time $t=0$, it will oscillate in the xu-plane. Let $u(x,t)$ denote the displacement of the point on the string above (or below) the abscissa x at time t.

We'll show that it is reasonable to assume that u satisfies the wave equation under the following assumptions:

1. The mass density (mass per unit length) ρ of the string is constant throughout the string.
2. The tension T induced by tightly stretching the string along the x-axis is so great that all other forces, such as gravity and air resistance, can be neglected.
3. The tension at any point on the string acts along the tangent to the string at that point, and the magnitude of its horizontal component is always equal to T, the tension in the string in equilibrium.
4. The slope of the string at every point remains sufficiently small so that we can make the approximation

$$\sqrt{1+u_x^2} \approx 1. \tag{9.2.2}$$

Figure 12.2.2 shows a segment of the displaced string at a time $t>0$. (Don't think that the figure is necessarily inconsistent with Assumption 4; we exaggerated the slope for clarity.)

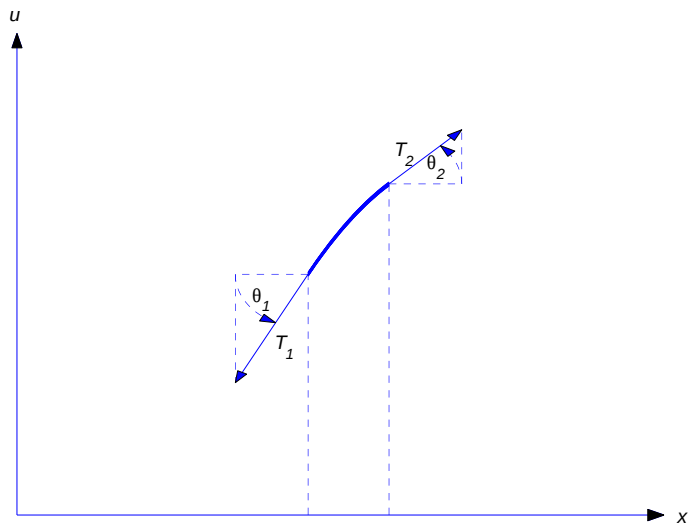

Figure 12.2.2 : A segment of the displaced string

The vectors $\mathbf{T}_1$ and $\mathbf{T}_2$ are the forces due to tension, acting along the tangents to the segment at its endpoints. From Newton's second law of motion, $\mathbf{T}_1 - \mathbf{T}_2$ is equal to the mass times the acceleration of the center of mass of the segment. The horizontal and vertical components of $\mathbf{T}_1 - \mathbf{T}_2$ are

$$|\mathbf{T}_2| \cos\theta_2 - |\mathbf{T}_1| \cos\theta_1 \quad \text{and} \quad |\mathbf{T}_2| \sin\theta_2 - |\mathbf{T}_1| \sin\theta_1,$$

respectively. Since

$$|\mathbf{T}_2| \cos\theta_2 = |\mathbf{T}_1| \cos\theta_1 = T \tag{9.2.3}$$

by assumption, the net horizontal force is zero, so there's no horizontal acceleration. Since the initial horizontal velocity is zero, there's no horizontal motion.

Applying Newton's second law of motion in the vertical direction yields

$$|\mathbf{T}_2| \sin\theta_2 - |\mathbf{T}_1| \sin\theta_1 = \rho \Delta s \, u_{tt}(\overline{x}, t), \tag{9.2.4}$$

where Δs is the length of the segment and $\overline{x}$ is the abscissa of the center of mass; hence,

$$x < \overline{x} < x + \Delta x.$$

From calculus, we know that

$$\Delta s = \int_x^{x+\Delta x} \sqrt{1 + u_x^2(\sigma, t)} \, d\sigma;$$

however, because of Equation 9.2.2, we make the approximation

$$\Delta s \approx \int_x^{x+\Delta x} 1 \, d\sigma = \Delta x,$$

so Equation 9.2.4 becomes

$$|\mathbf{T}_2| \sin\theta_2 - |\mathbf{T}_1| \sin\theta_1 = \rho \Delta x \, u_{tt}(\overline{x}, t).$$

Therefore

$$\frac{|\mathbf{T}_2| \sin\theta_2 - |\mathbf{T}_1| \sin\theta_1}{\Delta x} = \rho u_{tt}(\overline{x}, t).$$

Recalling Equation 9.2.3, we divide by T to obtain

$$\frac{\tan\theta_2 - \tan\theta_1}{\Delta x} = \frac{\rho}{T} u_{tt}(\overline{x}, t). \tag{9.2.5}$$

Since $\tan\theta_1 = u_x(x, t)$ and $\tan\theta_2 = u_x(x + \Delta x, t)$, Equation 9.2.5 is equivalent to

$$\frac{u_x(x+\Delta x)-u_x(x,t)}{\Delta x}=\frac{\rho}{T}u_{tt}(\overline{x},t).$$

Letting $\Delta x \to 0$ yields

$$u_{xx}(x,t)=\frac{\rho}{T}u_{tt}(x,t),$$

which we rewrite as $u_{tt}=a^2u_{xx}$, with $a^2=T/\rho$.

The Formal Solution

As in Section 12.1, we use separation of variables to obtain a suitable definition for the formal solution of Equation 9.2.1. We begin by looking for functions of the form $v(x,t)=X(x)T(t)$ that are not identically zero and satisfy

$$v_{tt}=a^2v_{xx}, \quad v(0,t)=0, \quad v(L,t)=0$$

for all (x,t). Since

$$v_{tt}=XT'' \quad \text{and} \quad v_{xx}=X''T,$$

$v_{tt}=a^2v_{xx}$ if and only if

$$XT''=a^2X''T,$$

which we rewrite as

$$\frac{T''}{a^2T}=\frac{X''}{X}.$$

For this to hold for all (x,t), the two sides must equal the same constant; thus,

$$\frac{X''}{X}=\frac{T''}{a^2T}=-\lambda,$$

which is equivalent to

$$X''+\lambda X=0$$

and

$$T''+a^2\lambda T=0. \tag{9.2.6}$$

Since $v(0,t)=X(0)T(t)=0$ and $v(L,t)=X(L)T(t)=0$ and we don't want T to be identically zero, $X(0)=0$ and $X(L)=0$. Therefore λ must be an eigenvalue of

$$X''+\lambda X=0, \quad X(0)=0, \quad X(L)=0, \tag{9.2.7}$$

and X must be a λ-eigenfunction. From Theorem 11.1.2, the eigenvalues of Equation 9.2.7 are $\lambda_n=n^2\pi^2/L^2$, with associated eigenfunctions

$$X_n=\sin\frac{n\pi x}{L}, \quad n=1,2,3,\ldots.$$

Substituting $\lambda=n^2\pi^2/L^2$ into Equation 9.2.6 yields

$$T''+(n^2\pi^2a^2/L^2)T=0,$$

which has the general solution

$$T_n=\alpha_n\cos\frac{n\pi at}{L}+\frac{\beta_nL}{n\pi a}\sin\frac{n\pi at}{L},$$

where α_n and β_n are constants. Now let

$$v_n(x,t)=X_n(x)T_n(t)=\left(\alpha_n\cos\frac{n\pi at}{L}+\frac{\beta_nL}{n\pi a}\sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L}.$$

Then

$$\frac{\partial v_n}{\partial t}(x,t)=\left(-\frac{n\pi a}{L}\alpha_n\sin\frac{n\pi at}{L}+\beta_n\cos\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L},$$

so

$$v_n(x,0)=\alpha_n\sin\frac{n\pi x}{L}\quad\text{and}\quad\frac{\partial v_n}{\partial t}(x,0)=\beta_n\sin\frac{n\pi x}{L}.$$

Therefore v_n satisfies Equation 9.2.1 with $f(x)=\alpha_n\sin n\pi x/L$ and $g(x)=\beta_n\cos n\pi x/L$. More generally, if α_1, α_2, …, α_m and β_1, β_2,…, β_m are constants and

$$u_m(x,t)=\sum_{n=1}^m\left(\alpha_n\cos\frac{n\pi at}{L}+\frac{\beta_nL}{n\pi a}\sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L},$$

then u_m satisfies Equation 9.2.1 with

$$f(x)=\sum_{n=1}^m\alpha_n\sin\frac{n\pi x}{L}\quad\text{and}\quad g(x)=\sum_{n=1}^m\beta_n\sin\frac{n\pi x}{L}.$$

This motivates the next definition.

Definition 12.2.1

If f and g are piecewise smooth of $[0,L]$, then the formal solution of Equation 9.2.1 is

$$u(x,t)=\sum_{n=1}^\infty\left(\alpha_n\cos\frac{n\pi at}{L}+\frac{\beta_nL}{n\pi a}\sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L},\tag{9.2.8}$$

where

$$S_f(x)=\sum_{n=1}^\infty\alpha_n\sin\frac{n\pi x}{L}\quad\text{and}\quad S_g(x)=\sum_{n=1}^\infty\beta_n\sin\frac{n\pi x}{L}$$

are the Fourier sine series of f and g on $[0,L]$; that is,

$$\alpha_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\quad\text{and}\quad\beta_n=\frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx.$$

Since there are no convergence-producing factors in Equation 9.2.8 like the negative exponentials in t that appear in formal solutions of initial-boundary value problems for the heat equation, it is not obvious that Equation 9.2.8 even converges for any values of x and t, let alone that it can be differentiated term by term to show that $u_{tt}=a^2u_{xx}$. However, the next theorem guarantees that the series converges not only for $0\le x\le L$ and $t\ge 0$, but for $-\infty<x<\infty$ and $-\infty<t<\infty$.

Theorem 12.2.2

If f and g are piecewise smooth on $[0,L]$, then u in Equation 9.2.1 converges for all (x,t), and can be written as

$$u(x,t)=\frac{1}{2}[S_f(x+at)+S_f(x-at)]+\frac{1}{2a}\int_{x-at}^{x+at}S_g(\tau)\,d\tau.\tag{9.2.9}$$

Proof

Setting $A=n\pi x/L$ and $B=n\pi at/L$ in the identities

$$\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

and

$$\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$$

yields

$$\cos\frac{n\pi at}{L}\sin\frac{n\pi x}{L} = \frac{1}{2}\left[\sin\frac{n\pi(x+at)}{L} + \sin\frac{n\pi(x-at)}{L}\right] \tag{9.2.10}$$

and

$$\begin{aligned} \sin\tfrac{n\pi at}{L}\sin\tfrac{n\pi x}{L} &= -\tfrac{1}{2}\left[\cos\tfrac{n\pi(x+at)}{L} - \cos\tfrac{n\pi(x-at)}{L}\right] \\ &= \tfrac{n\pi}{2L}\int_{x-at}^{x+at}\sin\tfrac{n\pi\tau}{L}d\tau \end{aligned} \tag{9.2.11}$$

From Equation 9.2.10,

$$\begin{aligned} \textstyle\sum_{n=1}^{\infty}\alpha_n\cos\tfrac{n\pi at}{L}\sin\tfrac{n\pi x}{L} &= \tfrac{1}{2}\textstyle\sum_{n=1}^{\infty}\alpha_n\left(\sin\tfrac{n\pi(x+at)}{L} + \sin\tfrac{n\pi(x-at)}{L}\right) \\ &= \tfrac{1}{2}[S_f(x+at) + S_f(x-at)] \end{aligned} \tag{9.2.12}$$

Since it can be shown that a Fourier sine series can be integrated term by term between any two limits, Equation 9.2.11 implies that

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{\beta_n L}{n\pi a}\sin\frac{n\pi at}{L}\sin\frac{n\pi x}{L} &= \frac{1}{2a}\sum_{n=1}^{\infty}\beta_n\int_{x-at}^{x+at}\sin\frac{n\pi\tau}{L}d\tau \\ &= \frac{1}{2a}\int_{x-at}^{x+at}\left(\sum_{n=1}^{\infty}\beta_n\sin\frac{n\pi\tau}{L}\right)d\tau \\ &= \frac{1}{2a}\int_{x-at}^{x+at}S_g(\tau)d\tau \end{aligned}$$

This and Equation 9.2.12imply Equation 9.2.9, which completes the proof.

As we'll see below, if S_g is differentiable and S_f is twice differentiable on $(-\infty,\infty)$, then Equation 9.2.9 satisfies $u_{tt} = a^2u_{xx}$ for all (x,t). We need the next theorem to formulate conditions on f and g such that S_f and S_g to have these properties.

Theorem 12.2.3

Suppose h is differentiable on $[0,L]$; that is, $h'(x)$ exists for $0<x<L$, and the one-sided derivatives

$$h'_+(0) = \lim_{x\to 0+}\frac{h(x)-h(0)}{x} \quad \text{and} \quad h'_-(L) = \lim_{x\to L-}\frac{h(x)-h(L)}{x-L}$$

both exist.

a. (a) Let p be the odd periodic extension of h to $(-\infty,\infty)$; that is,

$$p(x) = \begin{cases} h(x), & 0\le x\le L, \\ -h(-x), & -L<x<0, \end{cases} \quad \textit{and } p(x+2L) = p(x), \quad -\infty<x<\infty.$$

Then p is differentiable on $(-\infty,\infty)$ if and only if

$$h(0) = h(L) = 0 \tag{9.2.13}$$

b. Let q be the even periodic extension of h to $(-\infty,\infty)$; that is,

$$p(x) = \begin{cases} h(x), & 0\le x\le L, \\ h(-x), & -L<x<0, \end{cases} \quad \textit{and } q(x+2L) = q(x), \quad -\infty<x<\infty.$$

Then pq is differentiable on $(-\infty,\infty)$ if and only if

$$h'_+(0) = h'_-(L) = 0 \tag{9.2.14}$$

Proof

Throughout this proof, k denotes an integer. Since f is differentiable on the open interval $(0, L)$, both p and q are differentiable on every open interval $((k-1)L, kL)$. Thus, we need only to determine whether p and q are differentiable at $x = kL$ for every k.

(a) From Figure 12.2.3 , p is discontinuous at $x = 2kL$ if $h(0) \neq 0$ and discontinuous at $x = (2k-1)L$ if $h(L) \neq 0$. Therefore p is not differentiable on $(-\infty, \infty)$ unless $h(0) = h(L) = 0$. From Figure 12.2.4 , if $h(0) = h(L) = 0$, then

$$p'(2kL) = h'_+(0) \quad \text{and} \quad p'((2k-1)L) = h'_-(L)$$

for every k; therefore, p is differentiable on $(-\infty, \infty)$.

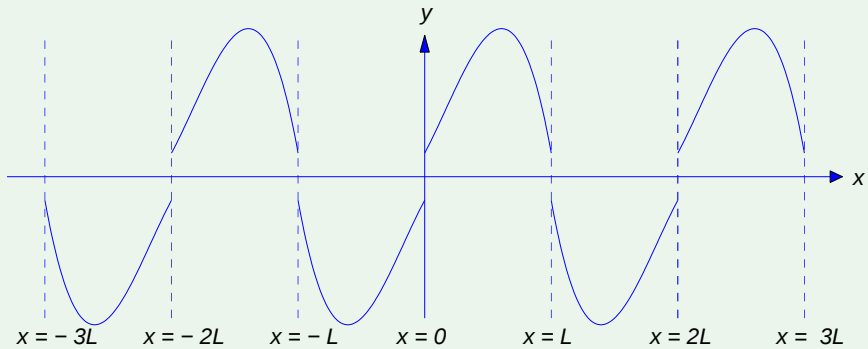

Figure 12.2.3 : The odd extension of a function that does not satisfy 9.2.13

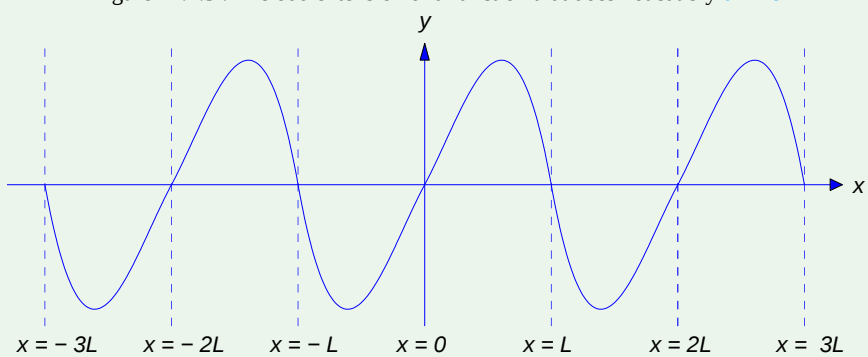

Figure 12.2.4 : The odd extension of a function that satisfies 9.2.13

(b) From Figure 12.2.5 ,

$$q'_-(2kL) = -h'_+(0) \quad \text{and} \quad q'_+(2kL) = h'_+(0),$$

so q is differentiable at $x = 2kL$ if and only if $h'_+(0) = 0$. Also,

$$q'_-((2k-1)L) = h'_-(L) \quad \text{and} \quad q'_+((2k-1)L) = -h'_-(L),$$

so q is differentiable at $x = (2k-1)L$ if and only if $h'_-(L) = 0$. Therefore q is differentiable on $(-\infty, \infty)$ if and only if $h'_+(0) = h\prime(L) = 0$, as in Figure 12.2.6 . This completes the proof.

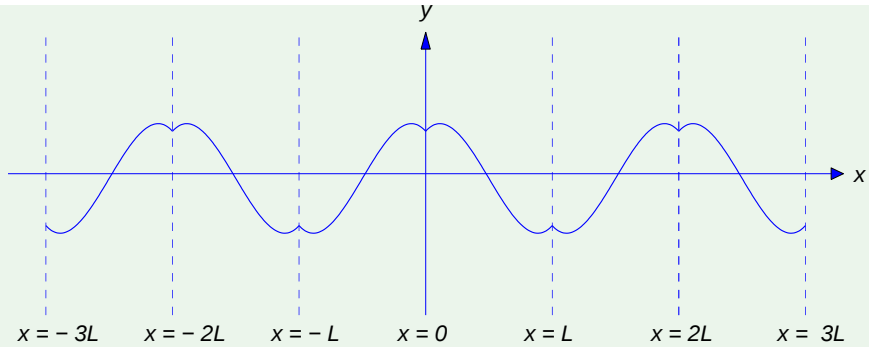

Figure 12.2.5 : The even extension of a function that doesn't satisfy 9.2.14

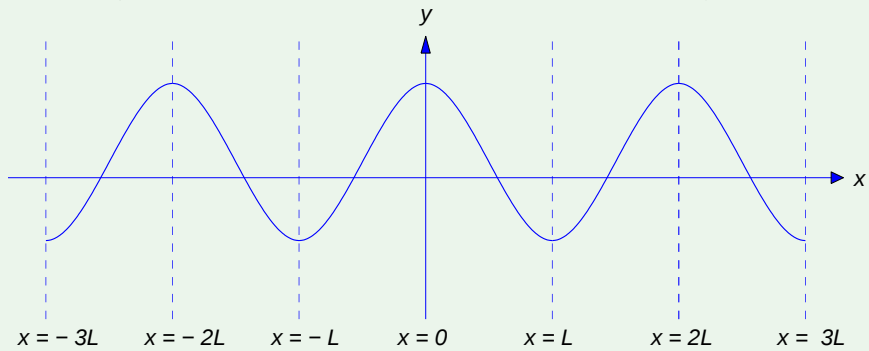

Figure 12.2.6 : The even extension of a function that satisfies 9.2.14

Theorem 12.2.4

The formal solution of (9.2.1) is an actual solution if g is differentiable on $[0, L]$ and

$$g(0) = g(L) = 0, \tag{9.2.15}$$

while f is twice differentiable on $[0, L]$ and

$$f(0) = f(L) = 0 \tag{9.2.16}$$

and

$$f''_+(0) = f''_-(L) = 0. \tag{9.2.17}$$

Proof

We first show that S_g is differentiable and S_f is twice differentiable on $(-\infty, \infty)$. We'll then differentiate Equation 9.2.9 twice with respect to x and t and verify that Equation 9.2.9is an actual solution of Equation 9.2.1.

Since f and g are continuous on $(0, L)$, Theorem 11.3.2 implies that $S_f(x) = f(x)$ and $S_g(x) = g(x)$ on $[0, L]$. Therefore S_f and S_g are the odd periodic extensions of f and g. Since f and g are differentiable on $[0, L]$, Equation 9.2.15, Equation 9.2.16, and Theorem 12.2.3 a imply that S_f and S_g are differentiable on $(-\infty, \infty)$.

Since $S'_f(x) = f'(x)$ on $[0, L]$ (one-sided derivatives at the endpoints), and S'_f is even (the derivative of an odd function is even), S'_f is the even periodic extension of f'. By assumption, f' is differentiable on $[0, L]$. Because of Equation 9.2.17, Theorem 12.2.3 b with $h = f'$ and $q = S'_f$ implies that S''_f exists on $(-\infty, \infty)$.

Now we can differentiate Equation 9.2.9twice with respect to x and t:

$$u_x(x,t) = \frac{1}{2}[S'_f(x+at) + S'_f(x-at)] + \frac{1}{2a}[S_g(x+at) - S_g(x-at)],$$

$$u_{xx}(x,t) = \frac{1}{2}[S''_f(x+at) + S''_f(x-at)] + \frac{1}{2a}[S'_g(x+at) - S'_g(x-at)], \tag{9.2.18}$$

$$u_t(x,t)=\frac{a}{2}[S_f'(x+at)-S_f'(x-at)]+\frac{1}{2}[S_g(x+at)+S_g(x-at)], \tag{9.2.19}$$

and

$$u_{tt}(x,t)=\frac{a^2}{2}[S_f''(x+at)-S_f''(x-at)]+\frac{a}{2}[S_g'(x+at)-S_g'(x-at)]. \tag{9.2.20}$$

Comparing Equation 9.2.18and Equation 9.2.20shows that $u_{tt}(x,t)=a^2u_{xx}(x,t)$ for all (x,t).

From Equation 9.2.8, $u(0,t)=u(L,t)=0$ for all t. From Equation 9.2.9, $u(x,0)=S_f(x)$ for all x, and therefore, in particular,

$$u(x,0)=f(x), \quad 0\le x<L.$$

From Equation 9.2.19, $u_t(x,0)=S_g(x)$ for all x, and therefore, in particular,

$$u_t(x,0)=g(x), \quad 0\le x<L.$$

Therefore u is an actual solution of Equation 9.2.1. This completes the proof.

Equation 9.2.9 is called **d'Alembert's solution** of Equation 9.2.1. Although d'Alembert's solution was useful for proving Theorem 12.2.4

and is very useful in a slightly different context (**Exercises 12.2.63 - 12.2.68**), Equation 9.2.8 is preferable for computational purposes.

Example 12.2.1

Solve Equation 9.2.1 with

$$f(x)=x(x^3-2Lx^2+L^2) \quad \text{and} \quad g(x)=x(L-x).$$

Solution

We leave it to you to verify that f and g satisfy the assumptions of Theorem 12.2.4 .

From **Exercise 11.3.39**,

$$S_f(x)=\frac{96L^4}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\sin\frac{(2n-1)\pi x}{L}.$$

From **Exercise 11.3.36**,

$$S_g(x)=\frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{L}.$$

From Equation 9.2.8,

$$\begin{aligned}u(x,t)=&\frac{96L^4}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\cos\frac{(2n-1)\pi at}{L}\sin\frac{(2n-1)\pi x}{L}\\&+\frac{8L^3}{a\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi at}{L}\sin\frac{(2n-1)\pi x}{L}\end{aligned}$$

Theorem 12.1.2 implies that u_{xx} and u_{tt} can be obtained by term by term differentiation, for all (x,t), so $u_{tt}=a^2u_{xx}$ for all (x,t) (**Exercise 12.2.62**). Moreover, Theorem 11.3.2 implies that $S_f(x)=f(x)$ and $S_g(x)=g(x)$ if $0\le x\le L$. Therefore $u(x,0)=f(x)$ and $u_t(x,0)=g(x)$ if $0\le x\le L$. Hence, u is an actual solution of the initial-boundary value problem.

Note

In solving a specific initial-boundary value problem (Equation 9.2.1), it is convenient to solve the problem with $g \equiv 0$, then with $f \equiv 0$, and add the solutions to obtain the solution of the given problem. Because of this, either $f \equiv 0$ or $g \equiv 0$ in all the specific initial-boundary value problems in the exercises.

The Plucked String

If f and g don't satisfy the assumptions of Theorem 12.2.4 , then Equation 9.2.8 is not an actual solution of Equation 9.2.1 in fact, it can be shown that Equation 9.2.1 doesn't have an actual solution in this case. Nevertheless, u is defined for all (x,t), and we can see from Equation 9.2.18 and Equation 9.2.20 that $u_{tt}(x,t) = a^2 u_{xx}(x,t)$ for all (x,t) such that $S_f''(x \pm at)$ and $S_g'(x \pm at)$ exist. Moreover, u may still provide a useful approximation to the vibration of the string; a laboratory experiment can confirm or deny this.

We'll now consider the initial-boundary value problem Equation 9.2.1 with

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{L}{2}, \\ L - x, & \frac{L}{2} \le x \le L \end{cases} \tag{9.2.21}$$

and $g \equiv 0$. Since f is not differentiable at $x = L/2$, it does'nt satisfy the assumptions of Theorem 12.2.4 , so the formal solution of Equation 9.2.1 can't be an actual solution. Nevertheless, it is instructive to investigate the properties of the formal solution.

The graph of f is shown in Figure 12.2.7 . Intuitively, we are plucking the string by half its length at the middle. You're right if you think this is an extraordinarily large displacement; however, we could remove this objection by multiplying the function in Figure 12.2.7 by a small constant. Since this would just multiply the formal solution by the same constant, we'll leave f as we've defined it. Similar comments apply to the exercises.

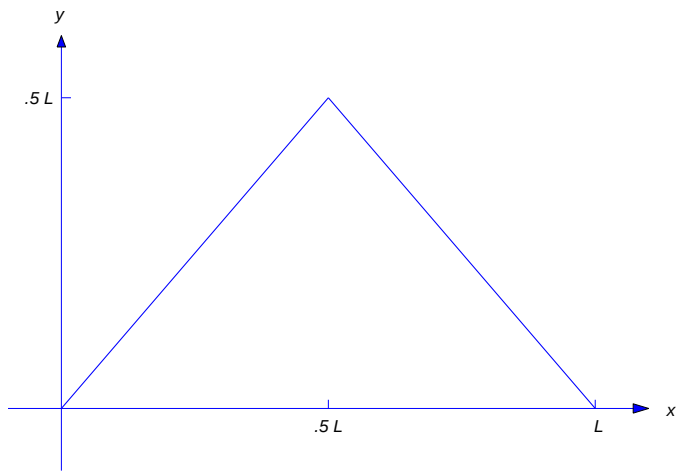

Figure 12.2.7 : Graph of Equation 9.2.21

From **Exercise 11.3.15**, the Fourier sine series of f on $[0, L]$ is

$$S_f(x) = \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{L}$$

which converges to f for all x in $[0, L]$, by Theorem 11.3.2. Therefore

$$u(x,t) = \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \cos \frac{(2n-1)\pi a t}{L} \sin \frac{(2n-1)\pi x}{L} \tag{9.2.22}$$

This series converges absolutely for all (x,t) by the comparison test, since the series

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

converges. Moreover, Equation 9.2.22 satisfies the boundary conditions

$$u(0,t)=u(L,t)=0, \quad t>0,$$

and the initial condition

$$u(x,0)=f(x), \quad 0 \le x \le L.$$

However, we can't justify differentiating Equation 9.2.22 term by term even once, and formally differentiating it twice term by term produces a series that diverges for all (x,t). (Verify.). Therefore we use d'Alembert's form

$$u(x,t)=\frac{1}{2}[S_f(x+at)+S_f(x-at)] \tag{9.2.23}$$

for u to study its derivatives. Figure 12.2.8 shows the graph of S_f, which is the odd periodic extension of f. You can see from the graph that S_f is differentiable at x (and $S_f'(x)=\pm 1$) if and only if x is not an odd multiple of $L/2$.

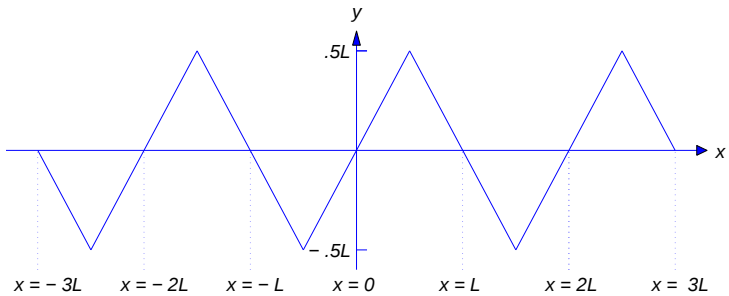

Figure 12.2.8 : The odd periodic extension of 9.2.21

In Figure 12.2.9 the dashed and solid curves are the graphs of $y=S_f(x-at)$ and $y=S_f(x+at)$ respectively, for a fixed value of t. As t increases the dashed curve moves to the right and the solid curve moves to the left. For this reason, we say that the functions $u_1(x,t)=S_f(x+at)$ and $u_2(x,t)=S_f(x-at)$ are **traveling waves**. Note that u_1 satisfies the wave equation at (x,t) if $x+at$ is not an odd multiple of $L/2$ and u_2 satisfies the wave equation at (x,t) if $x-at$ is not an odd multiple of $L/2$. Therefore Equation 9.2.23 (or, equivalently, Equation 9.2.22) satisfies $u_{tt}(x,t)=a^2u_{xx}(x,t)=0$ for all (x,t) such that neither $x-at$ nor $x+at$ is an odd multiple of $L/2$.

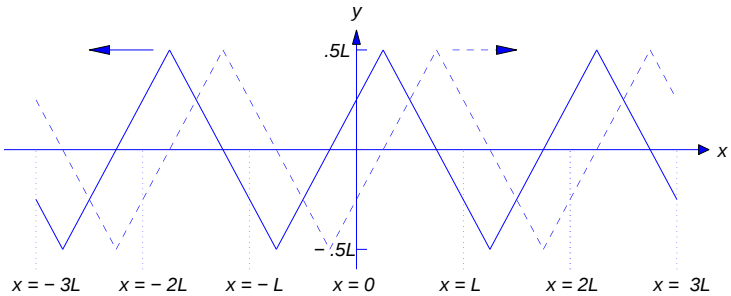

Figure 12.2.9 : Graphs of $y=S_f(x-at)$ (dashed) and $y=S_f(x-at)$ (solid), with f as in 9.2.21

We conclude by finding an explicit formula for $u(x,t)$ under the assumption that

$$0 \le x \le L \quad \text{and} \quad 0 \le t \le L/2a. \tag{9.2.24}$$

To see how this formula can be used to compute $u(x,t)$ for $0 \le x \le L$ and arbitrary t, we refer you to **Exercise 12.2.16**.

From Figure 12.2.10 ,

$$S_f(x-at)=\begin{cases} x-at, & 0 \le x \le \frac{L}{2}+at, \\ L-x+at, & \frac{L}{2}+at \le x \le L \end{cases}$$

and

$$S_f(x+at)=\begin{cases} x+at, & 0 \le x \le \frac{L}{2}-at, \\ L-x-at, & \frac{L}{2}-at \le x \le L \end{cases}$$

if (x,t) satisfies Equation 9.2.24.

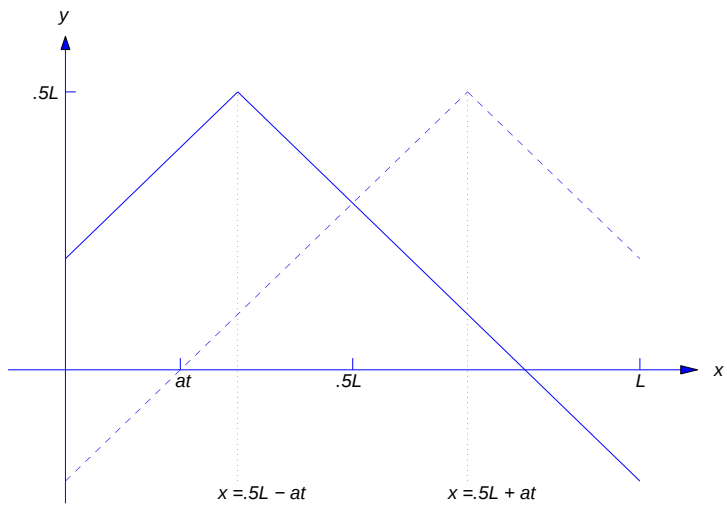

Figure 12.2.9 on $[0, L]$

Therefore, from Equation 9.2.23,

$$u(x,t)=\begin{cases} x, & 0\le x\le \frac{L}{2}-at, \\ \frac{L}{2}-at, & \frac{L}{2}-at\le x\le \frac{L}{2}+at, \\ L-x, & \frac{L}{2}-at\le x\le L \end{cases}$$

if (x,t) satisfies Equation 9.2.24. Figure 12.2.11 is the graph of this function on $[0, L]$ for a fixed t in $(0, L/2a)$.

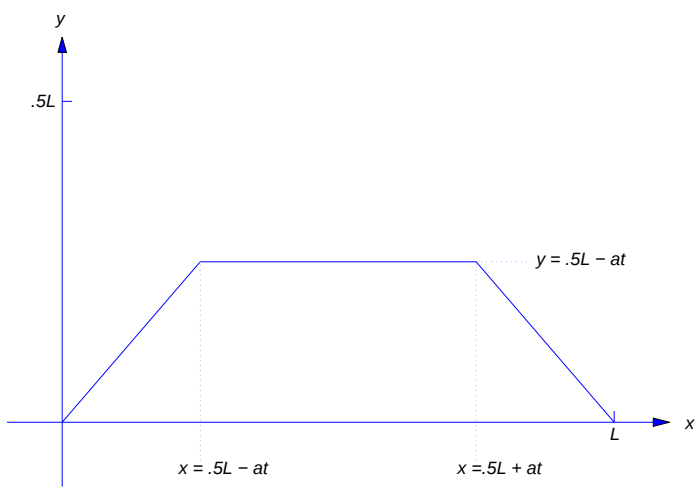

Figure 12.2.11 : The graph of 9.2.23 on $[0, L]$ for a fixed t in $(0, L/2a)$

Using Technology

Although the formal solution

$$u(x,t)=\sum_{n=1}^{\infty}\left(\alpha_n\cos\frac{n\pi at}{L}+\frac{\beta_n L}{n\pi a}\sin\frac{n\pi at}{L}\right)\sin\frac{n\pi x}{L}$$

of Equation 9.2.1 is defined for all (x,t), we are mainly interested in its behavior for $0\le x\le L$ and $t\ge 0$. In fact, it is sufficient to consider only values of t in the interval $0\le t<2L/a$, since

$$u(x,t+2kL/a)=u(x,t)$$

for all (x,t) if k is an integer. (Verify.)

You can create an animation of the motion of the string by performing the following numerical experiment.

Let m and k be positive integers. Let

$$t_j = \frac{2Lj}{ka}, \quad j = 0, 1, \ldots k;$$

thus, t_0, t_1, $\ldots t_k$ are equally spaced points in $[0, 2L/a]$. For each $j = 0$, 1,2, $\ldots k$, graph the partial sum

$$u_m(x, t_j) = \sum_{n=1}^{m} \left(\alpha_n \cos \frac{n\pi a t_j}{L} + \frac{\beta_n L}{n\pi a} \sin \frac{n\pi a t_j}{L} \right) \sin \frac{n\pi x}{L}$$

on $[0, L]$ as a function of x. Write your program so that each graph remains displayed on the monitor for a short time, and is then deleted and replaced by the next. Repeat this procedure for various values of m and k.

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9.2E: The Wave Equation (Exercises)

Q9.2.1

In *Exercises 9.2.1-9.2.8* solve the initial-boundary value problem.

1. $u_{tt} = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 1-x, & \frac{1}{2} \le x \le 1 \end{cases}, \quad 0 \le x \le 1$

2. $u_{tt} = 9u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x(1-x), \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

3. $u_{tt} = 9u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0,t) = 0, \quad u(2,t) = 0, \quad t > 0,$
$u(x,0) = 4 - x^2, \quad u_t(x,0) = 0, \quad 0 \le x \le 2$

4. $u_{tt} = 4u_{xx}, \quad 0 < x < 1, \quad t > 0,$
$u_x(0,t) = 0, \quad u(1,t) = 0, \quad t > 0,$
$u(x,0) = x^2(1-x), \quad u_t(x,0) = 0, \quad 0 \le x \le 1$

5. $u_{tt} = 64u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = x(2\pi - x), \quad u_t(x,0) = 0, \quad 0 \le x \le \pi$

6. $u_{tt} = 64u_{xx}, \quad 0 < x < \pi, \quad t > 0,$
$u(0,t) = 0, \quad u_x(\pi,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = x(2\pi - x), \quad 0 \le x \le \pi$

7. $u_{tt} = 5u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(2,t) = 0, \quad t > 0,$
$u(x,0) = 2x^2(3-x), \quad u_t(x,0) = 0, \quad 0 \le x \le 2$

8. $u_{tt} = 5u_{xx}, \quad 0 < x < 2, \quad t > 0,$
$u_x(0,t) = 0, \quad u_x(2,t) = 0, \quad t > 0,$
$u(x,0) = 0, \quad u_t(x,0) = 2x^2(3-x), \quad 0 \le x \le 2$

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CHAPTER OVERVIEW

Chapter 10: Linear Systems of Differential Equations

IN THIS CHAPTER we consider systems of differential equations involving more than one unknown function. Such systems arise in many physical applications.SECTION 10.1 presents examples of physical situations that lead to systems of differential equations. SECTION 10.2 discusses linear systems of differential equations. SECTION 10.3 deals with the basic theory of homogeneous linear systems. SECTIONS 10.4, 10.5, AND 10.6 present the theory of constant coefficient homogeneous systems. SECTION 10.7 presents the method of variation of parameters for nonhomogeneous linear systems.

10.1: Introduction to Systems of Differential Equations

10.1E: Introduction to Systems of Differential Equations (Exercises)

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10.1: Introduction to Systems of Differential Equations

Many physical situations are modelled by systems of n differential equations in n unknown functions, where $n \geq 2$. The next three examples illustrate physical problems that lead to systems of differential equations. In these examples and throughout this chapter we'll denote the independent variable by t.

Example 10.1.1 : Flow

Tanks T_1 and T_2 contain 100 gallons and 300 gallons of salt solutions, respectively. Salt solutions are simultaneously added to both tanks from external sources, pumped from each tank to the other, and drained from both tanks (Figure 10.1.1). A solution with 1 pound of salt per gallon is pumped into T_1 from an external source at 5 gal/min, and a solution with 2 pounds of salt per gallon is pumped into T_2 from an external source at 4 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 3 gal/min. T_1 is drained at 6 gal/min and T_2 is drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2. Assume that both mixtures are well stirred.

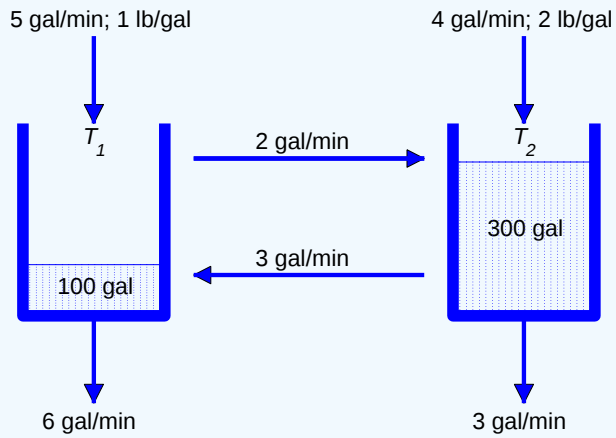

Figure 10.1.1

As in Section 4.2, let **rate in** and **rate out** denote the rates (lb/min) at which salt enters and leaves a tank; thus,

$$\begin{aligned} Q_1' &= (\text{rate in})_1 - (\text{rate out})_1, \\ Q_2' &= (\text{rate in})_2 - (\text{rate out})_2. \end{aligned}$$

Note that the volumes of the solutions in T_1 and T_2 remain constant at 100 gallons and 300 gallons, respectively.

T_1 receives salt from the external source at the rate of

$$(1 \text{ lb/gal}) \times (5\text{-gal/min}) = 5 \text{ lb/min},$$

and from T_2 at the rate of

$$\begin{aligned} (\text{lb/gal in } T_2) \times (3\text{-gal/min}) &= \frac{1}{300} Q_2 \times 3 \\ &= \frac{1}{100} Q_2 \text{ lb/min}. \end{aligned}$$

Therefore

$$(\text{rate in})_1 = 5 + \frac{1}{100} Q_2. \tag{10.1.1}$$

Solution leaves T_1 at the rate of 8 gal/min, since 6 gal/min are drained and 2 gal/min are pumped to T_2; hence,

$$(\text{rate out})_1 = (\text{ lb/gal in } \mathrm{T}_1) \times (8\text{-gal/min}) = \frac{1}{100} Q_1 \times 8 = \frac{2}{25} Q_1. \tag{10.1.2}$$

Equations 10.1.1 and Equation 10.1.2 imply that

$$Q_1' = 5 + \frac{1}{100}Q_2 - \frac{2}{25}Q_1. \tag{10.1.3}$$

T_2 receives salt from the external source at the rate of

$$(2 \text{ lb/gal}) \times (4\text{-gal/min}) = 8 \text{ lb/min},$$

and from T_1 at the rate of

$$\begin{aligned}(\text{lb/gal in } T_1) \times (2\text{-gal/min}) &= \frac{1}{100}Q_1 \times 2 \\ &= \frac{1}{50}Q_1 \text{ lb/min}.\end{aligned}$$

Therefore

$$(\text{rate in})_2 = 8 + \frac{1}{50}Q_1. \tag{10.1.4}$$

Solution leaves T_2 at the rate of 6 gal/min, since 3 gal/min are drained and 3 gal/min are pumped to T_1; hence,

$$(\text{rate out})_2 = (\text{ lb/gal in } \mathrm{T}_2) \times (6\text{-gal/min}) = \frac{1}{300}Q_2 \times 6 = \frac{1}{50}Q_2. \tag{10.1.5}$$

Equations 10.1.4 and 10.1.5 imply that

$$Q_2' = 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2. \tag{10.1.6}$$

We say that Equations 10.1.3 and 10.1.6 form a **system of two first order equations in two unknowns**, and write them together as

$$\begin{aligned}Q_1' &= 5 - \frac{2}{25}Q_1 + \frac{1}{100}Q_2 \\ Q_2' &= 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2.\end{aligned}$$

Example 10.1.2 : Coupled Springs

A mass m_1 is suspended from a rigid support on a spring S_1 and a second mass m_2 is suspended from the first on a spring S_2 (Figure 10.1.2). The springs obey Hooke's law, with spring constants k_1 and k_2. Internal friction causes the springs to exert damping forces proportional to the rates of change of their lengths, with damping constants c_1 and c_2. Let $y_1 = y_1(t)$ and $y_2 = y_2(t)$ be the displacements of the two masses from their equilibrium positions at time t, measured positive upward. Derive a system of differential equations for y_1 and y_2, assuming that the masses of the springs are negligible and that vertical external forces F_1 and F_2 also act on the objects.

Solution

In equilibrium, S_1 supports both m_1 and m_2 and S_2 supports only m_2. Therefore, if $\Delta\ell_1$ and $\Delta\ell_2$ are the elongations of the springs in equilibrium then

$$(m_1 + m_2)g = k_1\Delta\ell_1 \quad \text{and} \quad m_2 g = k_2\Delta\ell_2. \tag{10.1.7}$$

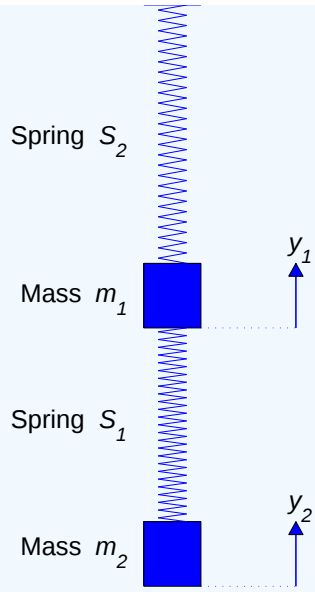

Figure 10.1.2

Let H_1 be the Hooke's law force acting on m_1, and let D_1 be the damping force on m_1. Similarly, let H_2 and D_2 be the Hooke's law and damping forces acting on m_2. According to Newton's second law of motion,

$$\begin{aligned} m_1 y_1'' &= -m_1 g + H_1 + D_1 + F_1, \\ m_2 y_2'' &= -m_2 g + H_2 + D_2 + F_2. \end{aligned} \tag{10.1.8}$$

When the displacements are y_1 and y_2, the change in length of S_1 is $-y_1 + \Delta\ell_1$ and the change in length of S_2 is $-y_2 + y_1 + \Delta\ell_2$. Both springs exert Hooke's law forces on m_1, while only S_2 exerts a Hooke's law force on m_2. These forces are in directions that tend to restore the springs to their natural lengths. Therefore

$$H_1 = k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \quad \text{and} \quad H_2 = k_2(-y_2 + y_1 + \Delta\ell_2). \tag{10.1.9}$$

When the velocities are y_1' and y_2', S_1 and S_2 are changing length at the rates $-y_1'$ and $-y_2' + y_1'$, respectively. Both springs exert damping forces on m_1, while only S_2 exerts a damping force on m_2. Since the force due to damping exerted by a spring is proportional to the rate of change of length of the spring and in a direction that opposes the change, it follows that

$$D_1 = -c_1 y_1' + c_2(y_2' - y_1') \quad \text{and} \quad D_2 = -c_2(y_2' - y_1'). \tag{10.1.10}$$

From Equations 10.1.8, 10.1.9, and 10.1.10,

$$\begin{aligned} &= -m_1 g + k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) - c_1 y_1' + c_2(y_2' - y_1') + F_1 \\ &= -(m_1 g - k_1\Delta\ell_1 + k_2\Delta\ell_2) - k_1 y_1 + k_2(y_2 - y_1) - c_1 y_1' + c_2(y_2' - y_1') + F_1 \end{aligned} \tag{10.1.11}$$

and

$$\begin{aligned} m_2 y_2'' \quad &= -m_2 g + k_2(-y_2 + y_1 + \Delta\ell_2) - c_2(y_2' - y_1') + F_2 \\ &= -(m_2 g - k_2\Delta\ell_2) - k_2(y_2 - y_1) - c_2(y_2' - y_1') + F_2. \end{aligned} \tag{10.1.12}$$

From Equation 10.1.7,

$$m_1 g - k_1\Delta\ell_1 + k_2\Delta\ell_2 = -m_2 g + k_2\Delta\ell_2 = 0.$$

Therefore we can rewrite Equation 10.1.11and Equation 10.1.12as

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2) y_1' + c_2 y_2' - (k_1 + k_2) y_1 + k_2 y_2 + F_1 \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + F_2. \end{aligned}$$

Example 10.1.3 : Gravity

Let $\mathbf{X} = \mathbf{X}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ be the position vector at time t of an object with mass m, relative to a rectangular coordinate system with origin at Earth's center (Figure 10.1.3).

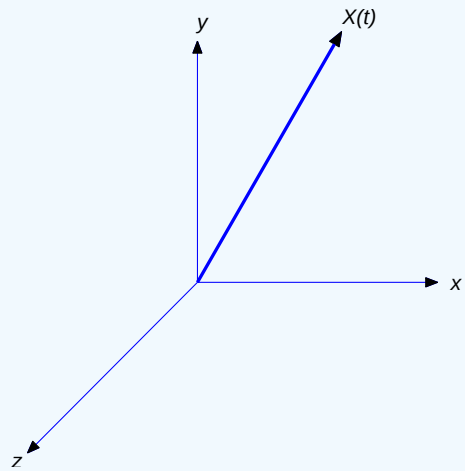

Figure 10.1.3

According to Newton's law of gravitation, Earth's gravitational force $\mathbf{F} = \mathbf{F}(x, y, z)$ on the object is inversely proportional to the square of the distance of the object from Earth's center, and directed toward the center; thus,

$$\mathbf{F} = \frac{\mathbf{K}}{||\mathbf{X}||^2}\left(-\frac{\mathbf{X}}{||\mathbf{X}||}\right) = -\mathbf{K}\frac{\mathbf{x}\,\mathbf{i} + \mathbf{y}\,\mathbf{j} + \mathbf{z}\,\mathbf{k}}{\left(\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2\right)^{3/2}}, \tag{10.1.13}$$

where K is a constant. To determine K, we observe that the magnitude of $\mathbf{F}$ is

$$\|\mathbf{F}\| = K\frac{\|\mathbf{X}\|}{\|\mathbf{X}\|^3} = \frac{K}{\|\mathbf{X}\|^2} = \frac{K}{(x^2 + y^2 + z^2)}.$$

Let R be Earth's radius. Since $\|\mathbf{F}\| = mg$ when the object is at Earth's surface,

$$mg = \frac{K}{R^2}, \quad \text{so} \quad K = mgR^2.$$

Therefore we can rewrite Equation 10.1.13as

$$\mathbf{F} = -mgR^2\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\left(x^2 + y^2 + z^2\right)^{3/2}}.$$

Now suppose $\mathbf{F}$ is the only force acting on the object. According to Newton's second law of motion, $\mathbf{F} = m\mathbf{X}''$; that is,

$$m(x''\,\mathbf{i} + y''\,\mathbf{j} + z''\,\mathbf{k}) = -mgR^2\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\left(x^2 + y^2 + z^2\right)^{3/2}}.$$

Cancelling the common factor m and equating components on the two sides of this equation yields the system

$$\begin{aligned} x'' &= -\frac{gR^2x}{(x^2 + y^2 + z^2)^{3/2}} \\ y'' &= -\frac{gR^2y}{(x^2 + y^2 + z^2)^{3/2}} \\ z'' &= -\frac{gR^2z}{(x^2 + y^2 + z^2)^{3/2}}. \end{aligned} \tag{10.1.14}$$

Rewriting Higher Order Systems as First Order Systems

A system of the form

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2, \ldots, y_n) \\ y_2' &= g_2(t, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= g_n(t, y_1, y_2, \ldots, y_n) \end{aligned} \tag{10.1.15}$$

is called a **first order system**, since the only derivatives occurring in it are first derivatives. The derivative of each of the unknowns may depend upon the independent variable and all the unknowns, but not on the derivatives of other unknowns. When we wish to emphasize the number of unknown functions in Equation 10.1.15we will say that Equation 10.1.15is an $n \times n$ system.

Systems involving higher order derivatives can often be reformulated as first order systems by introducing additional unknowns. The next two examples illustrate this.

Example 10.1.4

Rewrite the system

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2) y_1' + c_2 y_2' - (k_1 + k_2) y_1 + k_2 y_2 + F_1 \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + F_2 . \end{aligned} \tag{10.1.16}$$

derived in Example 10.1.2 as a system of first order equations.

Solution

If we define $v_1 = y_1'$ and $v_2 = y_2'$, then $v_1' = y_1''$ and $v_2' = y_2''$, so Equation 10.1.16becomes

$$\begin{aligned} m_1 v_1' &= -(c_1 + c_2) v_1 + c_2 v_2 - (k_1 + k_2) y_1 + k_2 y_2 + F_1 \\ m_2 v_2' &= c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + F_2 . \end{aligned}$$

Therefore $\{y_1, y_2, v_1, v_2\}$ satisfies the 4×4 first order system

$$\begin{aligned} y_1' &= v_1 \\ y_2' &= v_2 \\ v_1' &= \frac{1}{m_1}[-(c_1 + c_2) v_1 + c_2 v_2 - (k_1 + k_2) y_1 + k_2 y_2 + F_1] \\ v_2' &= \frac{1}{m_2}[c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + F_2] . \end{aligned} \tag{10.1.17}$$

Note

The difference in form between Equation 10.1.15and Equation 10.1.17, due to the way in which the unknowns are denoted in the two systems, isn't important; Equation 10.1.17is a first order system, in that each equation in Equation 10.1.17expresses the first derivative of one of the unknown functions in a way that does not involve derivatives of any of the other unknowns.

Example 10.1.5

Rewrite the system

$$\begin{aligned} x'' &= f(t, x, x', y, y', y'') \\ y''' &= g(t, x, x', y, y' y'') \end{aligned}$$

as a first order system.

Solution

We regard x, x', y, y', and y'' as unknown functions, and rename them

$$x = x_1,\ x' = x_2, \quad y = y_1, \quad y' = y_2, \quad y'' = y_3.$$

These unknowns satisfy the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= f(t, x_1, x_2, y_1, y_2, y_3) \\ y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= g(t, x_1, x_2, y_1, y_2, y_3). \end{aligned}$$

Rewriting Scalar Differential Equations as Systems

In this chapter we'll refer to differential equations involving only one unknown function as **scalar** differential equations. Scalar differential equations can be rewritten as systems of first order equations by the method illustrated in the next two examples.

Example 10.1.6

Rewrite the equation

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0 \tag{10.1.18}$$

as a 4×4 first order system.

Solution

We regard y, y', y'', and y''' as unknowns and rename them

$$y = y_1, \quad y' = y_2, \quad y'' = y_3, \quad \text{and} \quad y''' = y_4.$$

Then $y^{(4)} = y_4'$, so Equation 10.1.18can be written as

$$y_4' + 4y_4 + 6y_3 + 4y_2 + y_1 = 0.$$

Therefore $\{y_1, y_2, y_3, y_4\}$ satisfies the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= -4y_4 - 6y_3 - 4y_2 - y_1. \end{aligned}$$

Example 10.1.7

Rewrite

$$x''' = f(t, x, x', x'')$$

as a system of first order equations.

Solution

We regard x, x', and x'' as unknowns and rename them

$$x = y_1, \quad x' = y_2, \quad \text{and} \quad x'' = y_3.$$

Then

$$y_1' = x' = y_2, \quad y_2' = x'' = y_3, \quad \text{and} \quad y_3' = x'''.$$

Therefore $\{y_1, y_2, y_3\}$ satisfies the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= f(t, y_1, y_2, y_3). \end{aligned}$$

Since systems of differential equations involving higher derivatives can be rewritten as first order systems by the method used in Examples 10.1.5 -10.1.7 , we'll consider only first order systems.

Numerical Solution of Systems

The numerical methods that we studied in Chapter 3 can be extended to systems, and most differential equation software packages include programs to solve systems of equations. We will not go into detail on numerical methods for systems; however, for illustrative purposes we'll describe the Runge-Kutta method for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20} \end{aligned}$$

at equally spaced points t_0, t_1, …, $t_n = b$ in an interval $[t_0, b]$. Thus,

$$t_i = t_0 + ih, \quad i = 0, 1, \ldots, n,$$

where

$$h = \frac{b - t_0}{n}.$$

We'll denote the approximate values of y_1 and y_2 at these points by $y_{10}, y_{11}, \ldots, y_{1n}$ and $y_{20}, y_{21}, \ldots, y_{2n}$. The Runge-Kutta method computes these approximate values as follows: given y_{1i} and y_{2i}, compute

$$\begin{aligned} I_{1i} &= g_1(t_i, y_{1i}, y_{2i}), \\ J_{1i} &= g_2(t_i, y_{1i}, y_{2i}), \\ I_{2i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{1i}, y_{2i} + \frac{h}{2} J_{1i}\right), \\ J_{2i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{1i}, y_{2i} + \frac{h}{2} J_{1i}\right), \\ I_{3i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{2i}, y_{2i} + \frac{h}{2} J_{2i}\right), \\ J_{3i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{2i}, y_{2i} + \frac{h}{2} J_{2i}\right), \\ I_{4i} &= g_1(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}), \\ J_{4i} &= g_2(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}), \end{aligned}$$

and

$$\begin{aligned} y_{1,i+1} &= y_{1i} + \frac{h}{6}(I_{1i} + 2I_{2i} + 2I_{3i} + I_{4i}), \\ y_{2,i+1} &= y_{2i} + \frac{h}{6}(J_{1i} + 2J_{2i} + 2J_{3i} + J_{4i}) \end{aligned}$$

for $i = 0$, …, $n - 1$. Under appropriate conditions on g_1 and g_2, it can be shown that the global truncation error for the Runge-Kutta method is $O(h^4)$, as in the scalar case considered in Section 3.3.

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10.1E: Introduction to Systems of Differential Equations (Exercises)

Q10.1.1

1. Tanks T_1 and T_2 contain 50 gallons and 100 gallons of salt solutions, respectively. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 1 gal/min, and a solution with 3 pounds of salt per gallon is pumped into T_2 from an external source at 2 gal/min. The solution from T_1 is pumped into T_2 at 3 gal/min, and the solution from T_2 is pumped into T_1 at 4 gal/min. T_1 is drained at 2 gal/min and T_2 is drained at 1 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t>0$. Derive a system of differential equations for Q_1 and Q_2. Assume that both mixtures are well stirred.

2. Two 500 gallon tanks T_1 and T_2 initially contain 100 gallons each of salt solution. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 6 gal/min, and a solution with 1 pound of salt per gallon is pumped into T_2 from an external source at 5 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 1 gal/min. Both tanks are drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t>0$. Derive a system of differential equations for Q_1 and Q_2 that's valid until a tank is about to overflow. Assume that both mixtures are well stirred.

3. A mass m_1 is suspended from a rigid support on a spring S_1 with spring constant k_1 and damping constant c_1. A second mass m_2 is suspended from the first on a spring S_2 with spring constant k_2 and damping constant c_2, and a third mass m_3 is suspended from the second on a spring S_3 with spring constant k_3 and damping constant c_3. Let $y_1=y_1(t)$, $y_2=y_2(t)$, and $y_3=y_3(t)$ be the displacements of the three masses from their equilibrium positions at time t, measured positive upward. Derive a system of differential equations for y_1, y_2 and y_3, assuming that the masses of the springs are negligible and that vertical external forces F_1, F_2, and F_3 also act on the masses.

4. Let $\mathbf{X}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$ be the position vector of an object with mass m, expressed in terms of a rectangular coordinate system with origin at Earth's center (Figure 10.1.3). Derive a system of differential equations for x, y, and z, assuming that the object moves under Earth's gravitational force (given by Newton's law of gravitation, as in Example 10.1.3) and a resistive force proportional to the speed of the object. Let α be the constant of proportionality.

5. Rewrite the given system as a first order system.

a. $\begin{array}{l} x'''=f(t,x,y,y') \\ y''=g(t,y,y') \end{array}$

b. $\begin{array}{l} u'=f(t,u,v,v',w') \\ v''=g(t,u,v,v',w) \\ w''=h(t,u,v,v',w,w') \end{array}$

c. $y'''=f(t,y,y',y'')$

d. $y^{(4)}=f(t,y)$

e. $\begin{array}{l} x''=f(t,x,y) \\ y''=g(t,x,y) \end{array}$

6. Rewrite the system Equation 10.1.14 of differential equations derived in Example 10.1.3 as a first order system.

7. Formulate a version of Euler's method (Section 3.1) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t,y_1,y_2), \quad y_1(t_0)=y_{10}, \\ y_2' &= g_2(t,y_1,y_2), \quad y_2(t_0)=y_{20}, \end{aligned}$$

on an interval $[t_0,b]$.

8. Formulate a version of the improved Euler method (Section 3.2) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t,y_1,y_2), \quad y_1(t_0)=y_{10}, \\ y_2' &= g_2(t,y_1,y_2), \quad y_2(t_0)=y_{20}, \end{aligned}$$

on an interval $[t_0,b]$.

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