MATH 011: MATH CONCEPTS FOR ELEMENTARY SCHOOL TEACHERS – NUMBER SYSTEMS



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Melissa Flora

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Detailed Licensing

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

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1.1: Common Core

What is Common Core?

"Common Core" refers to the Common Core State Standards (CCSS) that are used in many states in the U.S. There are Common Core standards for Mathematics and English. Here we focus on the Mathematics CCSS. For information about the standards and history of California's adoption of them, see the California Department of Education Common Core State Standards website.

How to Read and Identify Common Core Standards

The standards have changed and will continue to change, but what is important for now is simply being able to read and identify them. There is no way to know for sure what standards schools will be using in the future, so what we care about right now is being able to use them. Do not memorize any of them! So that we are all looking at the same set of standards, we will use the CCSS attached here.

While it is recommended that you read the full introduction to the standards, it is not required. What you *definitely* should look at is guide on page 5 showing all the components of the standards. Note that you do not have to memorize any of this (you can and should look up this information often), but instead focus on understanding and being able to tell the difference between a **standard**, a **cluster**, and a **domain**; how the standards are labeled, and how these components relate to the organization of standards.

Example 1.1.1

Find standard 1.MD.3. Identify the standard, cluster, and domain.

Do this, then check your answer below.

Solution

This standard can be found on page 16 of the document. How did I find it? The 1 in 1.MD.3 means that it's a first grade standard. So, I went to the section of the document that covers first grade standards. The MD in 1.MD.3 is an abbreviation of the name of the domain that this standard is in. So, I looked for a domain name that would reasonably be abbreviated as MD. Sure enough, one of the domains is "Measurement and Data". The 3 in 1.MD.3 means that it's the third standard in the domain. The standard is reproduced below.

> 1.MD.3 Tell and write time in hours and half-hours using analog and digital clocks.

Note that the standard in the example is its own cluster, while 1.MD.1 and 1.MD.2 are in another cluster and 1.MD.4 is a third cluster (it is also by itself). Look at all four of these standards and see if you agree that they all belong in the domain Measurement and Data. See if you also agree that 1.MD.1 and 1.MD.2 belong in the same cluster while 1.MD.3 and 1.MD.4 belong in their own singular clusters.

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1.2: Introduction to Problem Solving

The Common Core State Standards for Mathematics identify eight “Mathematical Practices” — the kinds of expertise that all teachers should try to foster in their students.

The very first Mathematical Practice is:

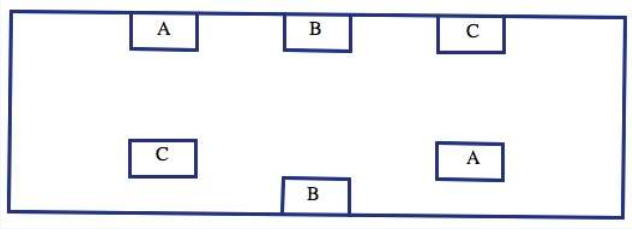
Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary.

This chapter will help you develop these very important mathematical skills, so that you will be better prepared to help your future students develop them. Let’s start with solving a problem!

✔ (ABC)

Draw curves connecting the A box in the top row to the A box in the bottom row, the B box in the top row to the B box in the bottom row, and the C box in the top row to the C box in the bottom row. Your curves cannot cross or even touch each other, they cannot cross through any of the boxes around the letters, and they cannot go outside the large box or even touch it’s sides.



? Think / Pair / Share

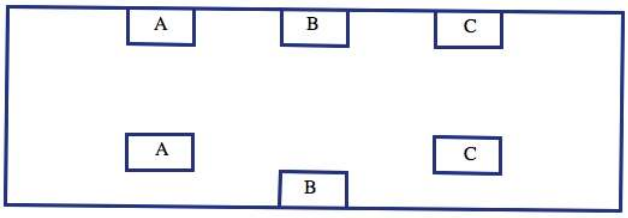
After you have worked on the problem on your own for a while, talk through your ideas with a partner if possible (even if you have not solved it).

- What did you try?
- What makes this problem difficult?
- Can you change the problem slightly so that it would be easier to solve?

A Problem Solving Strategy: Wishful Thinking

Do you wish something in the problem was different? Would it then be easier to solve the problem?

For example, what if ABC problem had a picture like this:



Can you solve this case and use it to help you solve the original case? Think about moving the boxes around once the lines are already drawn.

Here is one possible solution.



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1.3: Problem or Exercise?

The main activity of mathematics is solving problems. However, most people in most American mathematics classrooms actually do practice exercises. An exercise is different from a problem.

In a **problem**, you probably don't know at first how to approach solving it. You don't know what mathematical ideas might be used in the solution. Part of solving a problem is understanding what is being asked, and knowing what a solution should look like. Problems often involve false starts, making mistakes, and lots of scratch paper!

In an **exercise**, you are often practicing a skill. You may have seen a teacher demonstrate a technique, or you may have read a worked example in the book. You then practice on very similar assignments, with the goal of mastering that skill.

> **Note**
>
> What is a problem for some people may be an exercise for other people who have more background knowledge! For a young student just learning addition, this might be a problem:
>
> $$\textit{Fill in the blank to make a true statement } \_\_\_ + 4 = 7.$$
>
> But for you, that is an exercise!

Both problems and exercises are important in mathematics learning. But we should never forget that the ultimate goal is to develop more and better skills (often through exercises) so that we can solve harder and more interesting problems.

Learning math is a bit like learning to play a sport. You can practice a lot of skills:

- hitting hundreds of forehands in tennis so that you can place them in a particular spot in the court,
- breaking down strokes into the component pieces in swimming so that each part of the stroke is more efficient,
- keeping control of the ball while making quick turns in soccer,
- shooting free throws in basketball,
- catching high fly balls in baseball,

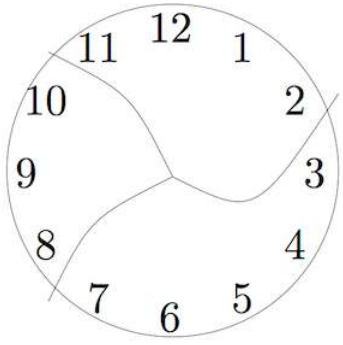
and so on.

But the point of the sport is to play the game. You practice the skills so that you are better at playing the game. In mathematics, solving problems is playing the game!

On Your Own

For each question below, decide if it is a *problem* or an *exercise*. (You do not need to solve the problems! Just decide which category it fits for you.) After you have labeled each one, compare your answers with a study partner or ask a tutor or your teacher for feedback.

1. This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. (Note: **Consecutive numbers** are integers that appear one after the other, such as 1, 2, 3, 4 or 13, 14, 15.)



Can you break another clock into a different number of pieces so that the sums are consecutive numbers? Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2).

2. A soccer coach began the year with a $500 budget. By the end of December, the coach spent $450. How much money in the budget was not spent?
3. What is the product of 4,500 and 27?
4. Arrange the digits 1, 2, 3, 4, 5, and 6 into a “difference triangle” where each number in the row below is the difference of the two numbers above it.
5. Simplify the following expression: $\frac{2+2(5^3-4^2)^5-2^2}{2(5^3-4^2)}$.
6. What is the sum of $\frac{5}{2}$ and $\frac{3}{13}$?
7. You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?



8. How many squares, of any possible size, are on a standard 8 × 8 chess board? Recall that a **square** is a rectangle with four sides of equal length.
9. What number is 3 more than half of 20?
10. Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1’s are separated by one digit, the 2’s are separated by two digits, the 3’s by three digits, and the 4’s by four digits.

There are many subtypes of problems and exercises. Of particular importance are **story problems**. These are problems that involve some sort of story.

Example 1.3.1

Write two problems for a young student just learning addition, one of which is a story problem and one of which is not a story problem, that are solved by adding 5 and 11.

Solution

Problem 1 (not a story problem)

> Fill in the blank: 5+11 = __

Problem 2 (a story problem)

> Mo has 5 colored pencils. Her mother gives her 11 more colored pencils. How many colored pencils does she have after receiving her mother's gift?

While all kinds of exercises and problems are useful at times, story problems are particularly valuable as a teaching tool because they attach meaning and narrative to the concepts involved. They often can also prepare learners for what are often referred to as "real-world" or application problems.

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1.4: Quantitative Analysis

Quantitative Analysis: An Illustrative Example

Quantitative Analysis is the practice of analyzing the quantities in a situation. It is an important part of solving application problems because one cannot truly understand a problem that involves quantities (something one should do before trying to solve a problem) without understanding what each quantity represents and how it relates to other quantities.

Let's look at Exercise 58 from Section 3.1 of the OpenStax textbook *Elementary Algebra, Second Edition*.

> Travis bought a pair of boots on sale for $25 off the original price. He paid $60 for the boots. What was the original price of the boots?

To analyze the quantities, we must be able to identify them.

Example 1.4.1

Identify the quantities in this exercise. Determine if each quantity is relevant to the problem we are trying to solve. If the value of the quantity is known, state the value.

> Travis bought a pair of boots on sale for $25 off the original price. He paid $60 for the boots. What was the original price of the boots?

Solution

1. Travis bought a pair of boots. So, there are two boots. This is not relevant to the problem we are trying to solve, however, because everything is about the pair, not each boot individually.
2. The boots had an original price. This is relevant to the problem since that is what we are asked to find. We do not know this value.
3. The boots had a sale price. This is relevant to the problem because it is related to the original price. The sale price is $60.
4. The boots were discounted. This is relevant to the problem since the discount tells us how the original price and the sale price are related. The discount is $25.

Sometimes people like to draw pictures to show relationships between quantities. This is not necessary to every use of Quantitative Analysis, but it is an option. Someone might draw something somewhat literal, in this case maybe stacks of money or even individual bills. Someone might instead draw something more abstract. Let's see an example of that.

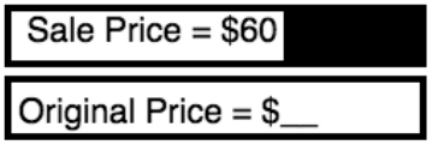


Figure 1.4.1: The sale price compared to the original price. (CC BY 4.0; Melissa Flora)

In the image above, we see a representation of the sale price, the original price, and the fact that the sale price is less than the original price. The amount by which the sale price is less than the original price is represented by the black rectangle.

With or without the picture, we want to ask ourselves, how the quantities are related. In our identification of the quantities, we noted that the discount tells us how the other two important quantities (the original price and the sale price) are related. Since the boot are "on sale for $25 off the original price", this means that the sale price is the result of taking the discount away from the original price. Let's describe that mathematically.

Example 1.4.1

Write a mathematical equation representing the fact that the sale price (which is $60) is the result of taking the discount (which is $25) away from the original price (which is unknown).

Solution

$60 = Original Price – $25

Now that we have represented the relationship between the quantities and we know which quantity we are trying to find, we can find that quantity using one of many strategies. In this course, you will learn about many strategies from the perspective of a teacher. But you have already learned this, so use whichever strategy you like to find the original price. Then check your answer below.

Example 1.4.1

Solve the exercise.

> Travis bought a pair of boots on sale for $25 off the original price. He paid $60 for the boots. What was the original price of the boots?

Solution

Since $60 = Original Price – $25, we know $60 +$25 = Original Price. 60+25 = 85, so the original price of the boots is $85.

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1.5: Word Problems

Try Different Numbers

Word problems can be more difficult than non-word problems because they require reading comprehension in addition to mathematical comprehension. Sometimes simply having to do both at the same time can make things more difficult than expected. One useful strategy for solving word problems is to try a similar problem with "nicer" numbers so that the mathematical comprehension is easier. Then once the easier version is solved, the learner can analyze what they did and use that as a model for the original version. Consider the problem below.

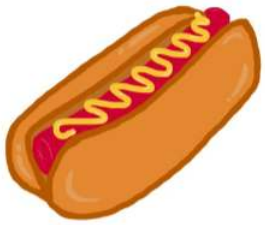


Figure 2.3.1: A hot dog in a bun with mustard

Rosie ate $2\frac{3}{4}$ hot dogs. Edward ate $1\frac{1}{2}$ hot dogs. What part of all the hot dogs together did Rosie eat by herself?

Example 1.5.1

Come up with an easier version of the hot dog problem. Then solve that problem.

Solution

Easier Version: Rosie ate five hot dogs. Edward ate three hot dogs. What part of all the hot dogs together did Rosie eat by herself.

Solution to Easier Version: Together, they ate 5+3 = 8 hot dogs. Rosie ate 5 out of 8 hot dogs, so she ate 5/8 of the hot dogs.

Now, mimic the same process for the easier version to solve the original hot dog problem that includes fractions.

Try Different words

Another useful strategy for solving word problems is to try rewriting the problem with less complicated sentences and simpler names to make the reading comprehension is easier. You may also want to get rid of some abbreviations (such as changing "mph" to "miles/hour", which is done below). Consider the problem below.



Figure 2.3.3: Train

Cara is riding in the South Train leaving Los Angeles and heading towards San Diego. Madeline is riding in the North Train leaving San Diego and heading towards Los Angeles. The South Train is traveling 84 mph while the North Train is traveling 92 mph. The distance between the two cities is 132 miles. Assuming the trains leave at the exact same time, how long will it take them to meet (crossing harmlessly on different rails)?

Example 1.5.1

Rewrite the train problem to make it more straightforward.

Solution

Problem Rewrite: The South Train leaves point A towards point B. The North Train leaves point B towards point A. The South Train's speed is 84 miles/hour. The North Train's speed is 92 miles/hour. The two points are 132 miles apart. Assuming the trains leave at the exact same time, how long will it take them to meet?

Now we can take the information from that and draw a picture more easily. On a sheet of paper, draw two points, labeled A and B, and a line segment between them. Before drawing a point on that line segment to indicate where the trains meet, let's see if we can think ahead to make our picture more accurate. Will they meet closer to point A or point B? Another way of asking the question is, "Will the South Train have traveled more or less than the North Train?" Think about your answer to this question. Then read on.

Since the South Train's speed is less than the North Train's speed, the North Train will have traveled a longer distance in the amount of time it takes them to meet. So, this means the point where they meet should show a distance between that point and point B as greater than the distance between that point and point A. In other words, the meeting point should be *closer to point A*. Draw that meeting point.

Label the length of the line segment (from A to B) as representing 132 miles. We do not know how long it will take them to meet. That's the question we are trying to answer! So, let's give that time a name. How about t hours? How do speed, distance, and time relate to each other? You have learned that Distance = Rate x Time and that speed is a type of rate. But even if you don't remember that, you might notice that in this problem, the units of distance are miles, the units of speed are miles/hour, and the units of time are hours. So, using what we know about fraction multiplication, we can see that miles/hour x hours = miles/hour x hours/1 = miles. In other words, Speed x Time = Distance!

Since the South Train's distance traveled is the part of the line between point A and the meeting point, we can use its speed, 84 miles/hour, and the unknown elapsed time, t hours, to label that part of the figure. Do that. Now use the same idea to label the part of the line between the meeting point and point B. Your drawing with labels should look something like the diagram below (yours might have a different orientation, possibly representing north and south).

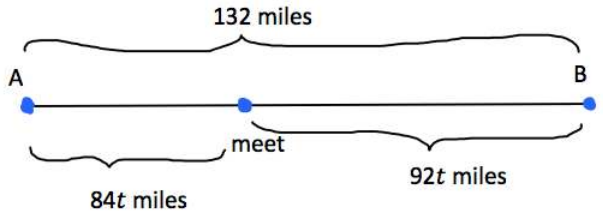


Figure 1.5.1: One possible diagram for the train problem (CC BY 4.0; Melissa Flora)

Now, use the diagram as your guide to set up an equation and solve for t to find how long after the trains leave they meet.

Try Diagrams Even Without a Story Context

People (especially children) tend to only want to draw pictures when solving problems if the problem has a visual component or some part of a narrative that is naturally visualized. But we need not limit ourselves this way! In the problem below, there is nothing that is naturally visual about it. But we can give the abstract concept of two numbers visual representations anyway.

One number is four times as large as another number, and their sum is 5285. What are the two numbers?

Example 1.5.1

Draw a picture representing the number problem that two elements: one representing the larger number and one representing the smaller number. Make the drawing so that the relationship between the two numbers is reflected.

Solution

For this problem, we can use strips, squares, or anything else where we can represent the smaller number as one "thing" and the larger number as a "thing" or "things" 4 times larger. One example is below.

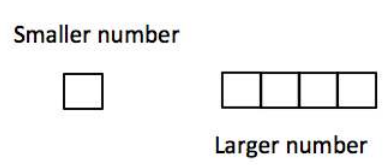


Figure 1.5.1: The two numbers (CC BY 4.0; Melissa Flora)

Since the sum of the two numbers is 5285, this means 5 squares (using the diagram in the example) represents 5285. Use this to determine how much one square represents. Then use that to determine the values of the smaller and larger numbers.

Practice Problems

Try using one or more of the strategies described above to solve these practice problems. If none of them seem to help, what strategy or strategies do work?

1. *Grade 1:* Some bicycles and tricycles are on the playground. There are seven seats and nineteen wheels. How many bicycles are there? How many tricycles are there? *(Solve without using Algebra.)*
2. *Grade 2:* Pam has three quarters. She wants to buy candy, which costs 54¢. How much change will she have after she buys the candy?
3. *Grade 3:* It takes a rocket 8 seconds to travel one mile. How long will it take the rocket to travel 7 miles?
4. *Grade 4:* Irwin decided to create designs using his cereal. In total, he created nine designs and used 63 pieces of cereal. How many cereal pieces were used in each design? Was there an equal number of pieces in each design?
5. *Grade 5:* Leo has 10 animals at his ranch, which are turtles, bunnies, and chickens. If there were 5 turtles, and 3 bunnies, what fraction of the animals are chickens?
6. *Grade 6:* Ellen wants to build a rectangular fenced area in her yard that is 10 meters by 8 meters. The fence will be made of glass panes between posts. The hardware store owner suggests installing a post every 2 meters. The posts cost $3.42 each. Ellen will also need to buy a glass pane to place between each pair of posts. The glass panes cost $50.84 per meter long. She also needs to buy one gate for $31.99. How much do all of the materials cost before tax?

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1.6: Problem Solving Strategies

Think back to the first problem in this chapter, the ABC Problem. What did you do to solve it? Even if you did not figure it out completely by yourself, you probably worked towards a solution and figured out some things that *did not* work.

Unlike exercises, there is never a simple recipe for solving a problem. You can get better and better at solving problems, both by building up your background knowledge and by simply practicing. As you solve more problems (and learn how other people solve them), you learn strategies and techniques that can be useful. But no single strategy works every time.

How to Solve It

George Pólya was a great champion in the field of *teaching* effective problem solving skills. He was born in Hungary in 1887, received his Ph.D. at the University of Budapest, and was a professor at Stanford University (among other universities). He wrote many mathematical papers along with three books, most famously, *How to Solve it*. Pólya died at the age 98 in 1985.[1]



George Pólya, circa 1973

1. Image of Pólya by Thane Plambeck from Palo Alto, California (Flickr) [CC BY 2.0 (http://creativecommons.org/licenses/by/2.0)], via Wikimedia Commons ↵

In 1945, Pólya published the short book *How to Solve It*, which gave a four-step method for solving mathematical problems:

1. First, you have to understand the problem.
2. After understanding, then make a plan.
3. Carry out the plan.
4. Look back on your work. How could it be better?

This is all well and good, but how do you actually do these steps?!?! Steps 1. and 2. are particularly mysterious! How do you "make a plan?" That is where you need some tools in your toolbox, and some experience to draw upon.

Much has been written since 1945 to explain these steps in more detail, but the truth is that they are more art than science. This is where math becomes a creative endeavor (and where it becomes so much fun). We will articulate some useful problem solving strategies, but no such list will ever be complete. This is really just a start to help you on your way. The best way to become a skilled problem solver is to learn the background material well, and then to solve a lot of problems!

We have already seen one problem solving strategy, which we call "Wishful Thinking." Do not be afraid to change the problem! Ask yourself "what if" questions:

- What if the picture was different?
- What if the numbers were simpler?
- What if I just made up some numbers?

You need to be sure to go back to the original problem at the end, but wishful thinking can be a powerful strategy for getting started.

This brings us to the most important problem solving strategy of all:

A Problem Solving Strategy: Try Something!

If you are really trying to solve a problem, the whole point is that you do not know what to do right out of the starting gate. You need to just try something! Put pencil to paper (or stylus to screen or chalk to board or whatever!) and try something. This is often

an important step in understanding the problem; just mess around with it a bit to understand the situation and figure out what is going on.

Note that being "good at mathematics" is not about doing things right the first time. It is about figuring things out. Practice being okay with having done something incorrectly. Try to avoid using an eraser and just lightly cross out incorrect work (do not black out the entire thing). This way if it turns out that you did something useful, you still have that work to reference! If what you tried first does not work, try something else! Play around with the problem until you have a feel for what is going on.

✔ (Payback)

Last week, Alex borrowed money from several of his friends. He finally got paid at work, so he brought cash to school to pay back his debts. First he saw Brianna, and he gave her 1/4 of the money he had brought to school. Then Alex saw Chris and gave him 1/3 of what was left after paying Brianna. Finally, Alex saw David and gave him 1/2 of the remaining money. Who got the most money from Alex?

? Think/Pair/Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner if possible (even if you have not solved it). What did you try? What did you figure out about the problem? This problem lends itself to two particular strategies. Did you try either of these as you worked on the problem? If not, read about the strategy and then try it out before watching the solution.

A Problem Solving Strategy: Draw a Picture

Some problems are obviously about a geometric situation, and it is clear you want to draw a picture and mark down all of the given information before you try to solve it. But even for a problem that is not geometric, like this one, thinking visually can help! Can you represent something in the situation by a picture?

Draw a square to represent all of Alex's money. Then shade 1/4 of the square — that's what he gave away to Brianna. How can the picture help you finish the problem?

After you have worked on the problem yourself using this strategy (or if you are completely stuck), you can watch someone else's solution.



A Problem Solving Strategy: Make Up Numbers

Part of what makes this problem difficult is that it is about money, but there are no numbers given. That means the numbers must not be important. So just make them up!

Try this: Assume (that is, pretend) Alex had some specific amount of money when he showed up at school, say $100. Then figure out how much he gives to each person.

Or try working backward: suppose Alex has some specific amount left at the end, say $10. Since he gave David half of what he had before seeing David, that means he had $20 before running into David. Now, work backwards and figure out how much each

person got.

Watch the solution only after you tried this strategy for yourself.



If you use the “Make Up Numbers” strategy, it is really important to remember what the original problem was asking! You do not want to answer something like “Everyone got $10.” That is not true in the original problem; that is an artifact of the numbers you made up. So after you work everything out, be sure to re-read the problem and answer what was asked!

✔ (Squares on a Chess Board)

How many squares, of any possible size, are on a 8 × 8 chess board? (The answer is not 64... It’s a lot bigger!)

Remember Pólya’s first step is to understand the problem. If you are not sure what is being asked, or why the answer is not just 64, be sure to ask someone!

? Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner if possible (even if you have not solved it). What did you try? What did you figure out about the problem, even if you have not solved it completely?

Most people want to draw a picture for this problem, but even with the picture it can be hard to know if you have found the correct answer. The numbers get big, and it can be hard to keep track of your work. Your goal at the end is to be absolutely positive that you found the right answer. Instead of asking the teacher, “Is this right?”, you should be ready to justify it and say, “Here’s my answer, and here is how I got it.”

A Problem Solving Strategy: Try a Simpler Problem

Pólya suggested this strategy: “If you can’t solve a problem, then there is an easier problem you can solve: find it.” He also said, “If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem?” In this case, an 8 × 8 chess board is pretty big. Can you solve the problem for smaller boards? Like 1 × 1? 2 × 2? 3 × 3?

The ultimate goal is to solve the original problem. But working with smaller boards might give you some insight and help you devise your plan (that is Pólya’s step (2)).

A Problem Solving Strategy: Work Systematically

If you are working on simpler problems, it is useful to keep track of what you have figured out and what changes as the problem gets more complicated.

For example, in this problem you might keep track of how many 1 × 1 squares are on each board, how many 2 × 2 squares on are each board, how many 3 × 3 squares are on each board, and so on. You could keep track of the information in a table:

size of board	# of 1 × 1 squares	# of 2 × 2 squares	# of 3 × 3 squares	# of 4 × 4 squares	…

1 by 1	1	0	0	0
2 by 2	4	1	0	0
3 by 3	9	4	1	0
…				

A Problem Solving Strategy: Use Manipulatives to Help You Investigate

Sometimes even drawing a picture may not be enough to help you investigate a problem. Having actual materials that you move around can sometimes help a lot!

For example, in this problem it can be difficult to keep track of which squares you have already counted. You might want to cut out 1 × 1 squares, 2 × 2 squares, 3 × 3 squares, and so on. You can actually move the smaller squares across the chess board in a systematic way, making sure that you count everything once and do not count anything twice.



A Problem Solving Strategy: Look for and Explain Patterns

Sometimes the numbers in a problem are so big, there is no way you will actually count everything up by hand. For example, if the problem in this section were about a 100 × 100 chess board, you would not want to go through counting all the squares by hand! It would be much more appealing to find a pattern in the smaller boards and then extend that pattern to solve the problem for a 100 × 100 chess board just with a calculation.

? Think / Pair / Share

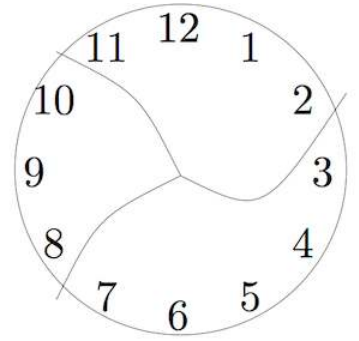
If you have not done so already, extend the table above all the way to an 8 × 8 chess board, filling in all the rows and columns. Use your table to find the total number of squares in an 8 × 8 chess board. Then:

- Describe all of the patterns you see in the table. If possible, actually describe these to a friend.
- Explain and justify any of the patterns you see (if possible, actually do this with a friend). If you don't have a partner to work with, imagine they asked you, "How can you be sure the patterns will continue?"
- Expand this to find what calculation(s) you would perform to find the total number of squares on a 100 × 100 chess board.

(We will come back to this question soon. So if you are not sure right now how to explain and justify the patterns you found, that is OK.)

✓ (Broken Clock)

This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. (**Consecutive numbers** are whole numbers that appear one after the other, such as 1, 2, 3, 4 or 13, 14, 15.)



Can you break another clock into a different number of pieces so that the sums are consecutive numbers? Assume that each piece has at least two numbers and that no number is damaged (e.g. 12 isn't split into two digits 1 and 2).

Remember that your first step is to understand the problem. Work out what is going on here. What are the sums of the numbers on each piece? Are they consecutive?

? Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner if possible (even if you have not solved it). What did you try? What progress have you made?

A Problem Solving Strategy: Find the Math, Remove the Context

Sometimes the problem has a lot of details in it that are unimportant, or at least unimportant for getting started. The goal is to find the underlying math problem, then come back to the original question and see if you can solve it using the math.

In this case, worrying about the clock and exactly how the pieces break is less important than worrying about finding consecutive numbers that sum to the correct total. Ask yourself:

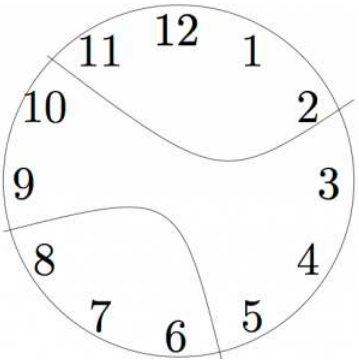
- What is the sum of all the numbers on the clock's face?
- Can I find two consecutive numbers that give the correct sum? Or four consecutive numbers? Or some other amount?
- How do I know when I am done? When should I stop looking?

Of course, solving the question about consecutive numbers is not the same as solving the original problem. You have to go back and see if the clock can actually break apart so that each piece gives you one of those consecutive numbers. Maybe you can solve the math problem, but it does not translate into solving the clock problem.

A Problem Solving Strategy: Check Your Assumptions

When solving problems, it is easy to limit your thinking by adding extra assumptions that are not in the problem. Be sure you ask yourself: Am I constraining my thinking too much?

In the clock problem, because the first solution has the clock broken radially (all three pieces meet at the center, so it looks like slicing a pie), many people assume that is how the clock must break. But the problem does not require the clock to break radially. It might break into pieces like this:



Were you assuming the clock would break in a specific way? Try to solve the problem now, if you have not already.

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CHAPTER OVERVIEW

2: Solving Problems

2.1: George Polya's Four Step Problem Solving Process
2.2: Beware of Patterns!
2.3: Practice
2.4: Asking Questions

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2.1: George Polya's Four Step Problem Solving Process

Step 1: Understand the Problem

- Do you understand all the words?
- Can you restate the problem in your own words?
- Do you know what is given?
- Do you know what the goal is?
- Is there enough information?
- Is there extraneous information?
- Is this problem similar to another problem you have solved?

Step 2: Devise a Plan: Below are some strategies one might use to solve a problem. Can one (or more) of the following strategies be used? (A strategy is defined as an artful means to an end.)

1. Guess and test.	11. Solve an equivalent problem.
2. Use a variable.	12. Work backwards
3. Draw a picture.	13. Use cases.
4. Look for a pattern.	14. Solve an equation.
5. Make a list.	15. Look for a formula.
6. Solve a simpler problem.	16. Do a simulation.
7. Draw a diagram.	17. Use a model
8. Use direct reasoning.	

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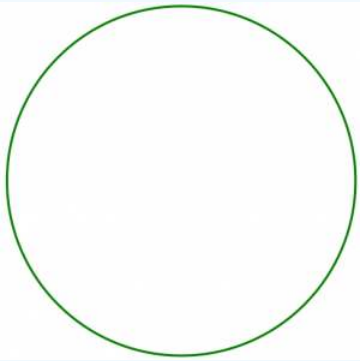
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2.2: Beware of Patterns!

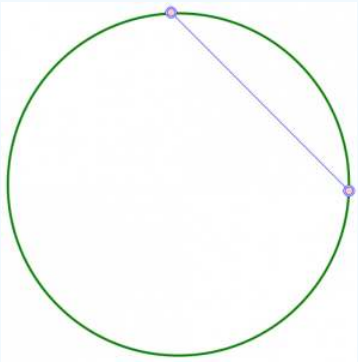
The “Look for Patterns” strategy can be particularly appealing, but you have to be careful! Do not forget the “and Explain” part of the strategy. Not all patterns are obvious, and not all of them will continue.

(Dots on a Circle)

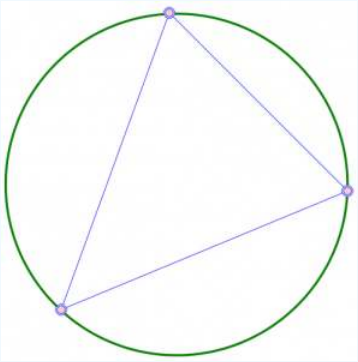
Start with a circle.



If I put two dots on the circle and connect them, the line divides the circle into two pieces.



If I put three dots on the circle and connect each pair of dots, the lines divides the circle into four pieces.



Suppose you put one hundred dots on a circle and connect each pair of dots, meaning every dot is connected to 99 other dots. How many pieces will you get? Lines may cross each other, but assume the points are chosen so that three or more lines never meet at a single point.

? Think / Pair / Share

After you have worked on the problem on your own for a while, talk through your ideas with a partner (even if you have not solved it). What strategies did you try? What did you figure out? What questions do you still have?

The natural way to work on this problem is to use smaller numbers of dots and look for a pattern, right? If you have not already, try it. How many pieces when you have four dots? Five dots? How would you describe the pattern?

Now try six dots. You will want to draw a big circle and space out the six dots to make your counting easier. Then carefully count up how many pieces you get. It is probably a good idea to work with a partner so you can check each other's work. Make sure you count every piece once and do not count any piece twice. How can you be sure that you do that?

Were you surprised? For the first several steps, it seems to be the case that when you add a dot you double the number of pieces. But that would mean that for six dots, you should get 32 pieces, and you only get 30 or 31, depending on how the dots are arranged. No matter what you do, you cannot get 32 pieces. The pattern simply does not hold up.

Mathematicians love looking for patterns and finding them. We get excited by patterns. But we are also very skeptical of patterns! If we cannot explain why a pattern would occur, then we are not willing to just believe it.

For example, if my number pattern starts out: 2, 4, 8, ... I can find lots of ways to continue the pattern, each of which makes sense in some contexts. Here are some possibilities:

- 2, 4, 8, 2, 4, 8, 2, 4, 8, 2, 4, 8, ...

This is a a repeating pattern, cycling through the numbers 2, 4, 8 and then starting over with 2.

- 2, 4, 8, 32, 256, 8192, ...

To get the next number, multiply the previous two numbers together.

- 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
- 2, 4, 8, 14, 22, 32, 44, 58, 74 ...

? Think / Pair / Share

- For the last two patterns above, describe in words how the number sequence is being created.
- Find at least two other ways to continue the sequence 2, 4, 8, . . . that looks different from all the ones you have seen so far. Write your rule in words, and write the next five terms of the number sequence.

So how can you be sure your pattern fits the problem? You have to tie them together! Remember the "Squares on a Chess Board" problem? You might have noticed a pattern like this one:

If the chess board has 5 squares on a side, then there are

- 5 × 5 = 25 squares of size 1 × 1.
- 4 × 4 = 16 squares of size 2 × 2.
- 3 × 3 = 9 squares of side 3 × 3.
- 2 × 2 = 4 squares of size 4 × 4.
- 1 × 1 = 1 squares of size 5 × 5.

So there are a total of

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$$

squares on a 5 × 5 chess board. You can probably guess how to continue the pattern to any size board, but how can you be absolutely sure the pattern continues in this way? What if this is like "Dots on a Circle," and the obvious pattern breaks down after a few steps? You have to tie the pattern to the problem, so that it is clear why the pattern must continue in that way.

The first step in explaining a pattern is writing it down clearly. This brings us to another problem solving strategy.

Problem Solving Strategy 11 (Use a Variable!).

One of the most powerful tools we have is the use of a variable. If you find yourself doing calculations on things like "the number of squares," or "the number of dots," give those quantities a name! They become much easier to work with.

? Think/Pair/Share

For now, just work on describing the pattern with variables.

- Stick with a 5 × 5 chess board for now, and consider a small square of size $k \times k$. Describe the pattern: How many squares of size $k \times k$ fit on a chess board of size 5 × 5?
- What if the chess board is bigger? Based on the pattern above, how many squares of size $k \times k$ should fit on a chess board of size 10 × 10?
- What if you do not know how big the chess board is? Based on the pattern above, how many squares of size $k \times k$ should fit on a chess board of size $n \times n$?

Now comes the tough part: explaining the pattern. Let us focus on an 8 × 8 board. Since it measures 8 squares on each side, we can see that we get 8× 8 = 64 squares of size 1× 1. And since there is just a single board, we get just one square of size 8 × 8. But what about all the sizes in-between?

? Think/Pair/Share

Using the Chess Board video in the previous chapter as a model, work with a partner to carefully explain why the number of 3 × 3 squares will be $6 \cdot 6 = 36$, and why the number of 4 × 4 squares will be $5 \cdot 5 = 25$.

There are many different explanations other than what is found in the video. Try to find your own explanation.

Here is what a final justification might look like (*watch the Chess Board video as a concrete example of this solution*):

Solution (Chess Board Pattern).

Let n be the side of the chess board and let k be the side of the square. If the square is going to fit on the chess board at all, it must be true that $k \leq n$. Otherwise, the square is too big.

If I put the $k \times k$ square in the upper left corner of the chess board, it takes up k spaces across and there are $(n - k)$ spaces to the right of it. So I can slide the $k \times k$ square to the right $(n - k)$ times, until it hits the top right corner of the chess board. The square is in $(n - k + 1)$ different positions, counting the starting position.

If I move the $k \times k$ square back to the upper left corner, I can shift it down one row and repeat the whole process again. Since there are $(n - k)$ rows below the square, I can shift it down $(n - k)$ times until it hits the bottom row. This makes $(n - k + 1)$ total rows that the square moves across, counting the top row.

So, there are $(n - k + 1)$ rows with $(n - k + 1)$ squares in each row. That makes $(n - k + 1)^2$ total squares.

Thus, the solution is the sum of $(n - k + 1)^2$ for all $k \leq n$. In symbols:

$$\text{number of squares on an n} \times \text{n board} = \sum_{k=1}^{n} (n - k + 1)^2.$$

Once we are sure the pattern continues, we can use it to solve the problem. So go ahead!

- How many squares on a 10 × 10 chess board?
- What calculation would you do to solve that problem for a 100 × 100 chess board?

There is a number pattern that describes the number of pieces you get from the "Dots on a Circle" problem. If you want to solve the problem, go for it! Think about all of your problem solving strategies. But be sure that when you find a pattern, you can explain *why it is the right pattern for this problem*, and not just another pattern that seems to work but might not continue.

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2.3: Practice

You have several problem solving strategies to work with. Here are the ones we have described so far (and you probably came up with even more of your own strategies as you worked on problems).

1. Wishful Thinking.
2. Try Something!
3. Draw a Picture.
4. Make Up Numbers.
5. Try a Simpler Problem.
6. Work Systematically.
7. Use Manipulatives to Help you Investigate.
8. Look for and Explain Patterns.
9. Find the Math, Remove the Context.
10. Check Your Assumptions.
11. Use a Variable.

Try your hand at some of these problems, keeping these strategies in mind. If you are stuck on a problem, come back to this list and ask yourself which of the strategies might help you make some progress.

Problem 6

You have eight coins and a balance scale. The coins look alike, but one of them is a counterfeit. The counterfeit coin is lighter than the others. You may only use the balance scale two times. How can you find the counterfeit coin?



Problem 7

You have five coins, no two of which weigh the same. In seven weighings on a balance scale, can you put the coins in order from lightest to heaviest? That is, can you determine which coin is the lightest, next lightest, . . . , heaviest.

Problem 8

You have ten bags of coins. Nine of the bags contain good coins weighing one ounce each. One bag contains counterfeit coins weighing 1.1 ounces each. You have a regular (digital) scale, not a balance scale. The scale is correct to one-tenth of an ounce. In one weighing, can you determine which bag contains the bad coins?

Problem 9

Suppose you have a balance scale. You have three different weights, and you are able to weigh every whole number from 1 gram to 13 grams using just those three weights. What are the three weights?

Problem 10

There are a bunch of coins on a table in front of you. Your friend tells you how many of the coins are heads-up. You are blindfolded and cannot see a thing, but you can move the coins around, and you can flip them over. However, you cannot tell just by feeling them if the coins are showing heads or tails. Your job: separate the coins into two piles so that the same number of heads are showing in each pile.

Problem 11

The digital root of a number is the number obtained by repeatedly adding the digits of the number. If the answer is not a one-digit number, add those digits. Continue until a one-digit sum is reached. This one digit is the digital root of the number.

For example, the digital root of 98 is 8, since 9 + 8 = 17 and 1 + 7 = 8.

Record the digital roots of the first 30 integers and find as many patterns as you can. Can you explain any of the patterns?

Problem 12

If this lattice were continued, what number would be directly to the right of 98? How can you be sure you're right?

	3		6		9		12		…
1	2	4	5	7	8	10	11	13	…

Problem 13

Arrange the digits 0 through 9 so that the first digit is divisible by 1, the first two digits are divisible by 2, the first three digits are divisible by 3, and continuing until you have the first 9 digits divisible by 9 and the whole 10-digit number divisible by 10.

Problem 14

There are 25 students and one teacher in class. After an exam, everyone high-fives everyone else to celebrate how well they did. How many high- fives were there?

Problem 15

In cleaning out your old desk, you find a whole bunch of 3¢ and 7¢ stamps. Can you make exactly 11¢ of postage? Can you make exactly 19¢ of postage? What is the largest amount of postage you cannot make?

Problem 16

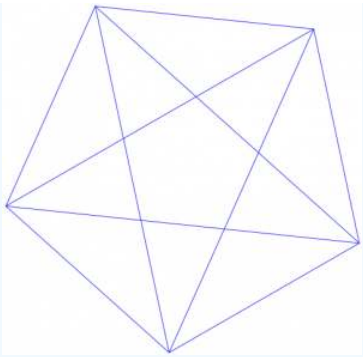
Find the largest eight-digit number made up of the digits 1, 1, 2, 2, 3, 3, 4, and 4 such that the 1's are separated by one digit, the 2's are separated by two digits, the 3's by three digits, and the 4's by four digits.

Problem 17

Kami has ten pockets and 44 dollar bills. She wants to have a different amount of money in each pocket. Can she do it?

Problem 18

How many triangles of all possible sizes and shapes are in this picture?



Problem 19

Arrange the digits 1–6 into a “difference triangle” where each number in the row below is the difference of the two numbers above it.

Example: Below is a difference triangle, but it does not work because it uses 1 twice and does not have a 6:

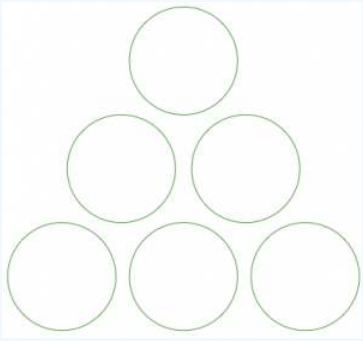
4		5		3
	1		2	
		1		

Problem 20

Certain pipes are sold in lengths of 6 inches, 8 inches, and 10 inches. How many different lengths can you form by attaching three sections of pipe together?

Problem 21

Place the digits 1, 2, 3, 4, 5, 6 in the circles so that the sum on each side of the triangle is 12. Each circle gets one digit, and each digit is used exactly once.



Problem 22

Find a way to cut a circular pizza into 11 pieces using just four straight cuts

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2.4: Asking Questions

A lot of people — from Polya to the writers of the Common Core State Standards and a lot of people in between — talk about problem solving in mathematics. One fact is rarely acknowledged, except by many professional mathematicians: Asking good questions is as valuable (and as difficult) as solving mathematical problems.

After solving a mathematical problem and explaining your solution to someone else, it is a very good mathematical habit to ask yourself: What other questions can I ask?

Example 2.4.1: Squares on a Chess Board

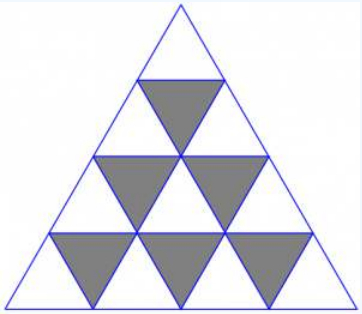
Recall Problem 3, "Squares on a Chess Board":

> *How many squares of any possible size are on a standard 8 × 8 chess board? (The answer is not 64! It's a lot bigger!)*

We have already talked about some obvious follow-up questions like "What about a 10 × 10 chess board? Or 100 × 100? Or $n \times n$?"

But there are a lot of interesting (and less obvious . . . and harder) questions you might ask:

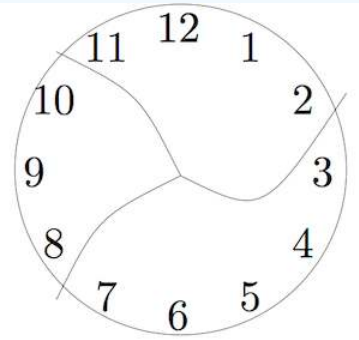
- How many rectangles of any size and shape can you find on a standard 8 × 8 chess board? (This is a lot harder, because the rectangles come in all different sizes, like 1 × 2 and 5 × 3. How could you possibly count them all?)
- How many triangles of any size and shape can you find in this picture?



Example 2.4.2: Broken Clock

Recall Problem 4, "Broken Clock":

> *This clock has been broken into three pieces. If you add the numbers in each piece, the sums are consecutive numbers. Can you break another clock into a different number of pieces so that the sums are consecutive numbers?*



The original problem only asks if you can find one other way. The obvious follow-up question: "Find every possibly way to break the clock into some number of pieces so that the sums of the numbers on each piece are consecutive numbers. Justify

that you have found every possibility.”

? Think / Pair / Share

Choose a problem from the Problem Bank (preferably a problem you have worked on, but that is not strictly necessary). What follow-up or similar questions could you ask?

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CHAPTER OVERVIEW

3: Representing Quantities Without Place Values

3.1: Number Systems

3.2: Roman Numerals

3.3: Numeration Systems

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3.1: Number Systems

Our number system is a western adaptation of the Hindu-Arabic numeral system developed somewhere between the first and fourth centuries CE (Common Era, previously referred to as "AD"). However, numbers have been recorded with tally marks throughout history. The Ishango Bone[1] from Africa is about 25,000 years old. It's the lower leg bone from a baboon, and contains tally marks. We know the marks were used for counting because they appear in distinct groups.



This reindeer antler[2] from France is about 15,000 years old, and also shows clearly grouped tally marks.



Of course, we still use tally marks today![3]



Base ten numbers (the ones you have probably been using your whole life), and base b numbers (where b is not 10) are both positional number systems.

Definition

A **positional number system** (also called place-value notation) is one way of writing numbers. It has unique symbols for 1 through $b - 1$, where b is the base of the system. Modern positional number systems also include a symbol for 0.

The **positional value** of each symbol depends on its position in the number:

- The positional value of a symbol in the first position (usually the furthest to the right when dealing with natural numbers) is just its face value.
- The positional value of a symbol in the second position is b times its value.
- The positional value of a symbol in the third position is b^2 times its value.
- And so on.

The value of a number is the sum of the positional values of its digits.

Definition

In an **additive number system,** the value of a written number is the sum of the face values of the symbols that make up the number. The only symbol necessary for an additive number system is a symbol for 1, however many additive number systems contain other symbols.

History: Roman numerals

The ancient Romans used a version of an additive number systems. The Romans represented numbers this way:

Number	Roman Numeral
1	I
5	V
10	X
50	L
100	C

Number	Roman Numeral
500	D
1,000	M

So the number 2013 would be represented as MMXIII because it represents 2,000 (two M's), one ten (one X), and three ones (three I's).

For any additive number system very large numbers become impractical to write without inventing more and more symbols. To represent the number one million in Roman numerals it would take one thousand M's or defining a new symbol (which some people have done)!

Classical Roman numerals were completely additive and written with the largest values on the left and smallest values on the right. However, the Roman numerals later developed one efficiency advantage by making the order of the symbols matter. If a symbol to the left was smaller than the symbol to the right, it would be subtracted instead of added. So for example nine is represented as IX rather than the Classical Roman notation VIIII.

? Think / Pair / Share

If you don't already know how to use Roman numerals, research it a little bit. Then answer these questions.

- Write the numbers 1–20 in Roman numerals.
- What is the maximum number of symbols needed to write any number between 1 and 1,000 in Roman Numerals? Justify your answer.

The earliest positional number systems are attributed to the Babylonians (base 60) and the Mayans (base 20). These positional systems were both developed before they had a symbol or a clear concept for zero. Instead of using 0, a blank space was used to indicate skipping a particular place value. This could lead to ambiguity.

Suppose we didn't have a symbol for 0, and someone wrote the number

$$2 \quad 3.$$

It would be impossible to tell if they mean 23, 203, 2003, or maybe two separate numbers (two and three).

Leonardo Pisano Bigollo, more commonly known as **Fibonacci**[4], played a pivotal role in guiding Europe out of a long period in which the importance and development of math was in marked decline. He was born in Italy around 1170 CE to Guglielmo Bonacci, a successful merchant. Guglielmo brought his son with him to what is now Algeria, and Leonardo was educated in mathematics there.



Fibonacci

At the time, Roman Numerals dominated Europe, and the official means of calculations was the abacus. Muḥammad ibn Mūsā al-Khwārizmī[5] described the use of Hindu-Arabic system in his book *On the Calculation with Hindu Numerals* in 825 CE,

but it was not well-known in Europe.



Statue of al-Khwarizmi at Amirkabir University of Technology

Fibonacci's book *Liber Abaci* described the Hindu-Arabic system and its business applications for a European readership. His book was well-received throughout Europe, and it marked the beginning of a reawakening of European mathematics.

History: Numbers in Hawai'i

The Hindu-Arabic number system is now used nearly exclusively throughout the globe. But many cultures had their own number systems before contact and trade with other countries spread the work of al-Khwārizmī throughout the world.

There is evidence that people in Hawai'i prior to contact with non-Polynesians actually used two different number systems. Depending on what they were counting, they might have used base 4 (or a mixed base-10 and base-4 system). One theory is that certain objects (fish, taro, etc.) were often put in bundles of 4, so were more natural to count by 4's than by 10's. The number four also had spiritual significance in Hawaiian culture.

Humans have 5 fingers on each hand, making base ten a natural choice for counting. But there are 4 gaps between the fingers, meaning that a hand can carry four fish or taro plants or similar objects, making base four a natural choice for some cultures.

In the mixed base system, instead of powers of 10, numbers are broken down into sums of numbers that look like 4 times a power of 10 (40, 400, 4000, etc.).

1	ʻekahi
2	ʻelua
3	ʻekolu
4	ʻehā (or kauna)
5	ʻelima
6	ʻeono
7	ʻehiku
8	ʻewalu
9	ʻeiwa

10	‘umi
11–19	‘umi kumamā {kahi, lua, kolu, hā, etc.}
20	iwakālua
21–29	Iwakālua kumamā {kahi, lua, kolu, hā, etc.}
30	kanakolu
31–39	kanakolu kumamā {kahi, lua, etc.}
40	kanahā
400	lau
4,000	mano
40,000	kini
400,000	lehu

Here are a few examples (refer to the table above for the Hawaiian names of the numbers):

Example:

‘ekolu kini, ‘ewalu lau me ‘ekahi

translates to three 40,000’s, eight 400’s, and one;

$3 \cdot 40000 + 8 \cdot 400 + 1 = 123201$

Example:

$5207 = 1 \cdot 4000 + 3 \cdot 400 + 7$

would be ‘ekahi mano, ‘ekolu lau me ‘ehiku

On Your Own

Work on the following exercises on your own or with a partner.

1. Translate this Hawaiian number to English and then write it in base ten. ‘ekahi kanahā me kanakolu kumamāiwa
2. Translate this base-ten number to Hawaiian: $1,573$

Note: there is no need to try to memorize the information about Roman and Hawaiian numbers. The important thing is to comprehend how they work. You will always be able to look up the details.

Think / Pair / Share

How is learning about different number systems (including representing numbers in different bases) valuable to you as a future teacher?

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5. Image of al-Khwarizmi statue by M. Tomczak [CC BY-SA 3.0], via Wikimedia Commons. ↵

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3.2: Roman Numerals

Why study Roman Numerals?

Even though Roman Numerals are rare in today's society, they are still used and expected to be understood. They are taught in grades three through five, depending on the district. They can be seen in clocks, the Super Bowl, Film Credits for the copyright date like MCMLXII, preface of textbooks and others like Star Wars Episode VI and WWII.

2.4.1: Basic Table of Roman Numerals

Value	Symbol
1	I
5	V
10	X
50	L
100	C
500	D
1000	M

2.4.2: The Complete Table of Roman Numerals

	1	2	3	4	5	6	7	8	9
ONES	I	II	III	IV	V	VI	VII	VIII	IX
TENS	X	XX	XXX	XL	L	LX	LXX	LXXX	XC
HUNDREDS	C	CC	CCC	CD	D	DC	DCC	DCCC	CM
THOUSANDS	M	MM	MMM	$\overline{\text{IV}}$	$\overline{\text{V}}$	$\overline{\text{VI}}$	$\overline{\text{VII}}$	$\overline{\text{VIII}}$	$\overline{\text{IX}}$
TEN THOUSANDS	$\bar{X}$	$\overline{XX}$	$\overline{XXX}$	$\overline{XL}$	$\bar{L}$	$\overline{LX}$	$\overline{LXX}$	$\overline{LXXX}$	$\overline{XC}$
HUNDRED THOUSANDS	$\bar{C}$	$\overline{CC}$	$\overline{CCC}$	$\overline{CD}$	$\bar{D}$	$\overline{DC}$	$\overline{DCC}$	$\overline{DCCC}$	$\overline{CM}$

Example 3.2.1

For all numbers except 4 and 9, we **ADD** the Roman Numerals together, in order from left to right, greatest value to lowest value.

$$\begin{aligned} 11 &= 10+1 = XI \\ 8 &= 5+1+1+1 = VIII \\ 123 &= 100+10+10+1+1+1 = CXXIII \\ 3816 &= 1000+1000+1000+500+100+100+100+10+5+1 = MMMDCCCXVI \\ 7002 &= 5000+1000+1000+1+1 = \overline{VIII}I \end{aligned}$$

Example 3.2.2

For any number that includes a 4 or a 9, we subtract. When we are looking at a Roman Number expression and we see a Roman character **OUT OF ORDER**, which is the clue to **SUBTRACT**!

$$9 = 10 - 1 = IX$$
$$4 = 5 - 1 = IV$$
$$1400 = 1000 + (500 - 100) = MCD$$
$$29452 = 10000 + 10000 + 10000 - 1000 + (500 - 100) + 50 + 1 + 1 = \overline{IXXX}CDLII$$

Partner Activity 1

Convert Between Roman Numerals and Our Current Decimal System:

1. MMXLV = _____
2. MDCCLXXXIX = _____
3. 1993 = _____
4. 5495 = _____

Practice Problems

Convert Between Roman Numerals and Our Current Decimal System:

1. 84759
2. MMXX
3. MCMLXXXII
4. 1764
5. 50000
6. $\overline{XX}LXXXIX$

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3.3: Numeration Systems

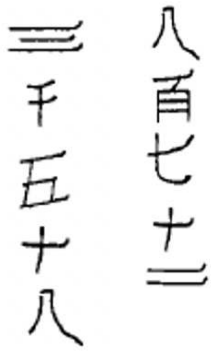
In set of practice problems, you will learn about two ancient numeration systems: those of the Chinese and Maya.

Chinese Numerals

Chinese numerals are still used today. Symbols for some Chinese numerals are shown below.

1	2	3	4	5	6	7	8	9	10	100	1,000
一	二	三	四	五	六	七	八	九	十	百	千

Chinese numerals are formed by writing the symbols vertically and using the multiplicative principle, which simplifies the recording of numerals by eliminating the repetition of symbols. For instance, the Chinese write the numeral for 3,058 by thinking $3 \times 1000 + 5 \times 10 + 8$ and write down the symbols 3, 1000, 5, 10 and 8 in that order to represent that number. Even though you can think of the 8 as 8×1, the 1 is not written down. This Chinese numeral (3,058) is shown to the left. The Chinese numeral for 872 is shown to the right.



Below are some more Chinese numerals. Make sure you understand how to read all of them before trying the exercises.

6,400	87	9,531	2,605	4,011	7,000
六千四百	八十七	九千五百三十一	二千六百五	四千一十一	七千

Practice Problem 1

Write each Hindu-Arabic numeral as a Chinese numeral.

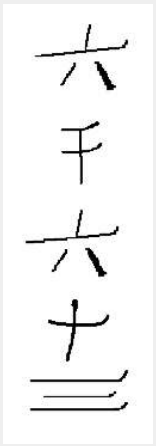
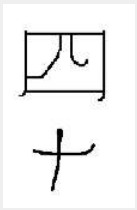
a. 5,093
b. 610
c. 427
d. 8,008

So you won't have to keep going back to look up these symbols, here again are the Chinese numerals you probably need to look at to do the following exercise.

1	2	3	4	5	6	7	8	9	10	100	1000
											

Practice Problem 2

Rewrite each Chinese numeral in Hindu-Arabic numeral .

a. ____	b. ____	c. ____	d. ____	e. ____
				

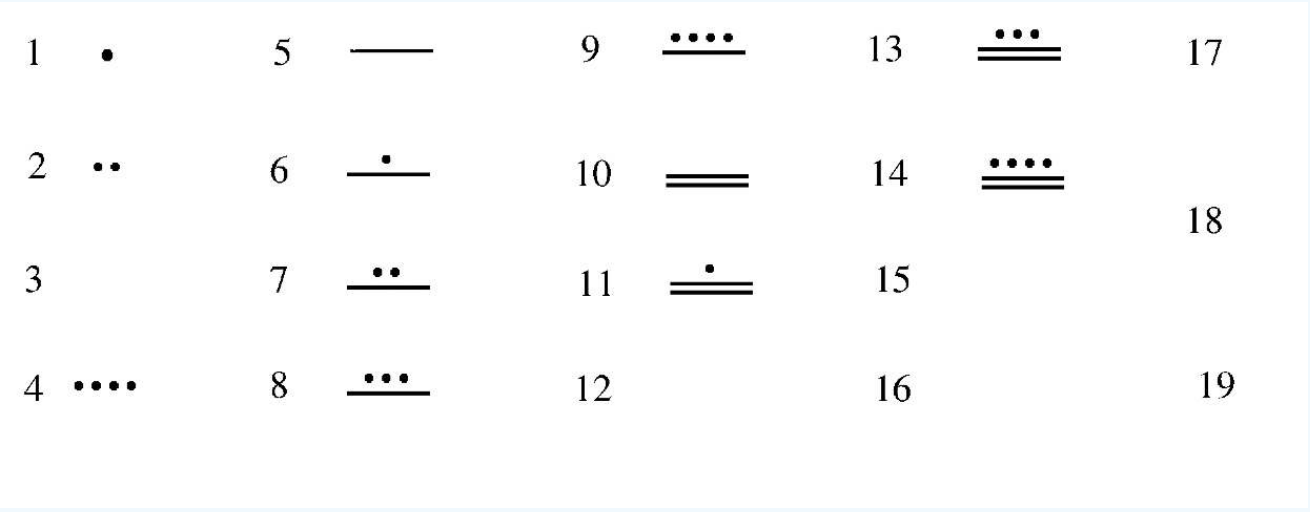
Notice in the Chinese system that numbers over nine have symbols written in pairs. To write 800, you must write the symbol for 8 above the symbol for 100. This is true even if there is only a "1" in the place value, see 2d above. Although the multiplicative principle allows you to write down fewer symbols than a simple additive system for most numerals, a further simplification would allow us to skip writing the second numeral of each pair. This would work if we used the *position* of the symbol to indicate the size of that group (10, 100, 1000, etc.). In other words, we could use a **positional numeration system.** In order to keep track of a position where no digit is used, a symbol for zero is necessary. Although the Chinese system doesn't need a symbol for zero, a circle was introduced to represent zero in the 1200s.

Maya Numerals

The Maya numeration system uses a positional system and is similar to the Chinese system in that the symbols for the numerals are written from top to bottom. Maya numerals were developed by the Maya priests of southern Mexico and Central America around 300 BCE. It is believed to be the earliest positional numeration system incorporating a zero and using it for a placeholder.

? Pracitce Problem 3

Some Maya numerals are shown below. Try to figure out the pattern and then fill in the missing numerals.



? Practice Problem 4

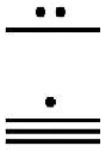
Explain what symbols are in this system, what they stand for and how the system works for at least the numerals one to nineteen.

From what you have seen of this system so far, it might look like a simple additive system. One might guess that the numeral for 20 would be four line segments and that the numeral for 103 would be twenty line segments and three dots. At first glance, it is very similar to the tally system. However, the Maya used a vertical *positional* system. The bottom level represented how many units (or ones), the second level up represented how many 20's, the third level up represented how many 360's (20x18), the fourth level up represented how many 7200's (20x18x20), the fifth level up represented how many 144,000's (20x18x20x20), etc. Except between the second and third level, each place value increased by a multiple of 20. It is *almost* a Base Twenty system except for that strange third level. Why the third level is 18 times the second level is explained later. This chart shows the first four place values.

7200s
360s
20s
1s

To try and make sense of all of this, we will look at some Maya numerals that have more than one position now. The numerals from one to nineteen only utilize the bottom level so it isn't apparent that the Maya system is positional until you count past nineteen.

Here is a two-level Maya numeral. There is a 16 in the bottom level, representing 16 ones, or 16 (16×1), plus there is a 7 in the second level up, representing 7 groups of twenty, or 140 (7×20). We add the values of each level, 16 + 140, so the numeral you see represents 156.



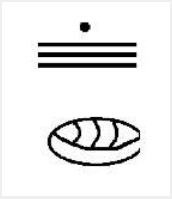
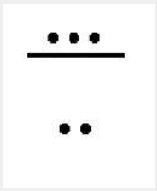
You have seen two basic symbols in this system so far: a dot, which represents the number one and a line segment, which represents the number five. As previously mentioned, there needs to be a symbol for zero to incorporate the use of place value. For that, the Maya used a symbol representing a shell.



This is a two-level Maya numeral with a zero in the bottom level (representing zero ones, or 0) plus a 13 in the second level up (representing 13 groups of twenty, or 260 since $13 \times 20 = 260$). Adding the values together, we have $0 + 260 = 260$.

? Practice Problem 5

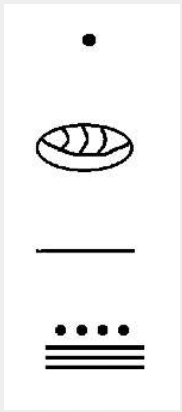
Write the Hindu-Arabic equivalent of each Maya numeral. Show how you obtained your answers. Note that proper space should be left between each place value. Otherwise, someone might incorrectly conclude the number shown for 5a represents 14, which would be the answer if there was no space.

a. ____	b. ____	c. ____
		

Now, we'll go on to some three- and four-level Maya numerals. Remember that the place value for the third level up is 360 and the place value for the fourth level up is 7200. See if you can figure these out on your own first.

? Practice Problem 6

Write the Hindu-Arabic equivalent of each Maya numeral. Show how you obtained your answers.

a. ____	b. ____	c. ____	d. ____	e. ____
				

How did you do? Make sure that if you still have any trouble understanding any of these that you go back and work through them again or ask for help.

It's a little trickier to start with a Hindu-Arabic numeral and convert it to Maya, but with a little patience and practice, you'll be doing it quickly and accurately! Before showing a method for doing this, try the following exercise. Hint: It should be easy and no calculator is required. Think about the place values of the various levels in the Maya system.

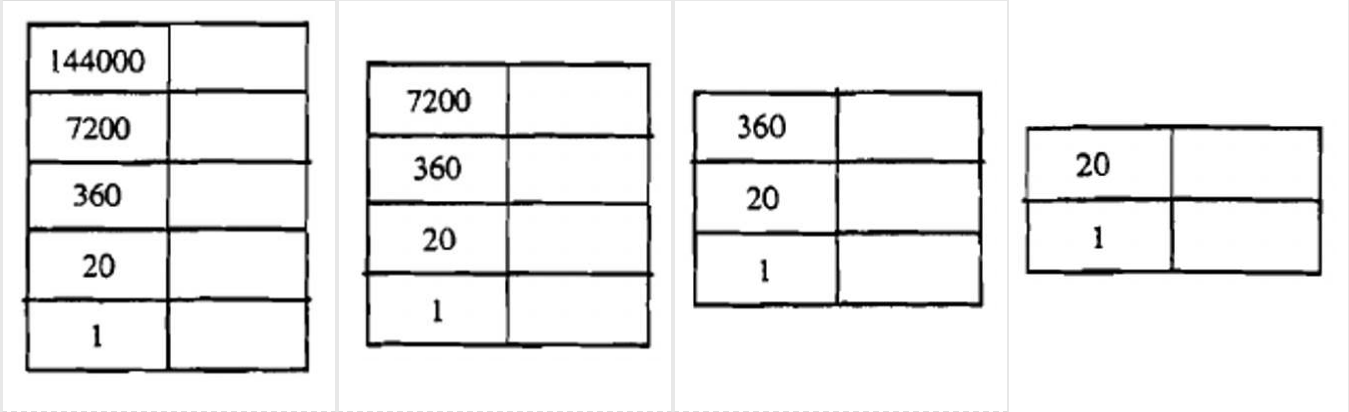
? Practice Problem 7

Write the Maya numeral equivalents for each of the following numbers:

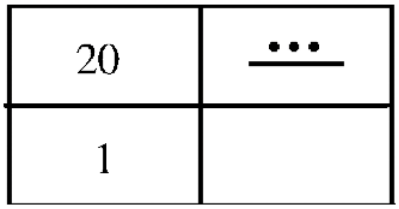
a. 1	b. 20	c. 360	d. 7200	e. 144000

To convert a number to a Maya numeral, the first thing you'll have to determine is how many levels the numeral will have. Remember the levels: 1, 20, 360, 7200, 144000, 2880000, etc. So any numeral less than 20 has one level, a numeral between 20 and 359 has two levels, a numeral between 360 and 7199 has three levels, a numeral between 7200 and 143999 has four levels, and so on.

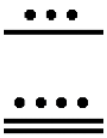
It might help if you set up a table with the correct number of levels set up already with a space to fill in what symbol you'll be using at each level. For instance, look below at the four most common charts you'll be using.



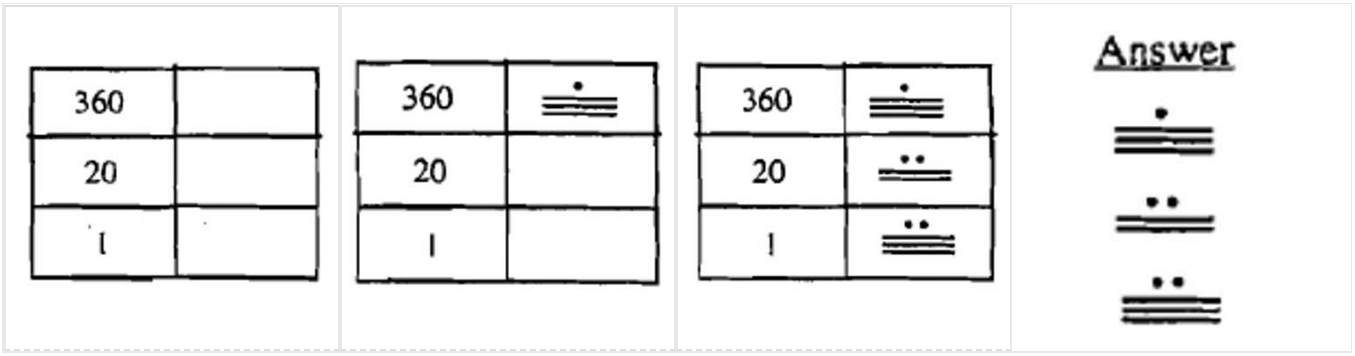
Let's start with the number 174. Convince yourself that this will be a two-level numeral. We'll start at the top, which is the 20s place value. We have to ask ourselves how many 20s are in 174? This is a division question: $174 \div 20 = 8$, remainder 14. We can begin to construct the Maya numeral by starting with the two-level chart and filling an 8 in second level up as shown below.



So far, we have eight groups of twenty, or 160 filled in, which leaves 14 more (the remainder) to accommodate. When you get down to the units place, the remainder is filled in there. So the next step is to fill 14 in the units place. Do that in the vacant place in the chart shown. Before feeling satisfied that everything is correct, check your answer by computing the Maya numeral you have just constructed and see if it is indeed 174. Then, get rid of the chart and write the answer as a Maya numeral as shown to the right.



Let's try another number. We'll convert 6017 to a Maya numeral. This will be a three-level numeral (make sure you understand why!), so we'll start off with a three-level chart and figure out how many 360s are in 6017. We do this division exercise: $6017 \div 360 = 16$, remainder 257. This tells us to put the symbol for 16 in the third level up (360s place). Now we have to take the remainder, 257, and find out how many 20s there are in it to find out what to put in the second level. We do this division exercise: $257 \div 20 = 12$, remainder 17. Since only the units place value is left (the bottom level), the remainder of 17 goes in that level. The sequence of filling in the charts is shown below. Make sure you go back and check your work by converting the numeral back to Hindu-Arabic ($1 \times 17 + 12 \times 20 + 16 \times 360$) and seeing if it really equals 6017.

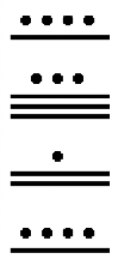


Let's convert 71509 to a Maya number. Fill in the Maya symbols on the chart as we work through the problem.

7200	
360	
20	
1	

This is a four-level Maya numeral (make sure you understand why!). Find what number will go in the 7200s place by performing a division calculation. Did you get 9, remainder 6709?

So, 9 is in the fourth level up. Write the Maya numeral for 9 in the fourth level. Now, do the next calculation. If you did this correctly, you should have $6709 \div 360 = 18$, remainder 229. So, 18 is in the third level up. Write the Maya numeral for 18 there. Do the next computation. You should have $229 \div 20 = 11$, remainder 9. So, 11 is the second level up and that leaves 9 in the units place. Write the Maya numerals for 11 and 9 in the appropriate locations. The final answer is shown.



Let's try a five-level number. How about 430040?

144000	
7200	
360	
20	
1	

Fill in the Maya symbols on the chart to the as we work through the problem. We first make sure you understand why 430040 requires five levels to write with Maya numerals. Now, do the first division calculation.

If you have done this correctly and correctly interpreted the result, you should have a 2 at the fifth level up (the 144000s place) with a remainder of 142040 to represent with the first through fourth levels. This is a big remainder but is less than 144000. Now, do the next calculation and interpret the results.

You should have divided 142040 by 7200 to get 19 at the fourth level up and a remainder of 5240. Do the next calculation and interpret the results.

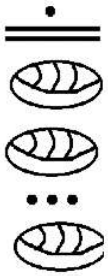
You should have divided 5240 by 360 to get 14 at the third level up and a remainder of 200. What happens next? Think about this before reading on.

There are exactly ten 20s in 200 and no remainder for the units, so we will need a zero at the bottom level. Finally, the numeral we seek is shown. **Make sure you check it** by computing $2 \times 144000 + 19 \times 7200 + 14 \times 360 + 10 \times 20 + 0$. Does it really equal 430040? If so, it's right.



Let's convert 1584060 to Maya. Try this on your own. Then read on.

This is another five-level Maya numeral, so our first division of $1584060 \div 144000$ gives us 11 at the fifth level up with a remainder of 60. There are no 7200s or 360s in 60, so there must be zeroes at those two levels. There are 3 20s in 60 with no remainder, so the units level must also have a zero. The numeral is shown. Please check it. Is it really 1584060?



Practice Problem 8

Think about the highest numeral value that can be on any given level. For instance, the bottom level could go up to 19, or . But what about the second level up? What number would be represented by a two level Maya numeral if there was a 19 at the second level up and a zero at the bottom level? Write the Maya numeral for 380.

Practice Problem 9

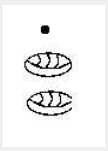
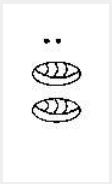
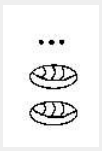
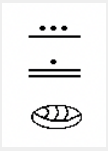
What is the highest Maya numeral value that should be on the second level up? What is the highest Maya symbol that could be on any level except the second level up?

Practice Problem 10

Convert each of the following to Maya numerals. Make sure you check your answers!

a. 1549
b. 4750
c. 53981
d. 145804
e. 1000000

You may be wondering why the Maya chose 360 for the third level up instead of 400, which seems more natural. Their counting system was based on their calendar, which consisted of 18 months of 20 days each, hence 360. The extra five days were treated differently. Their system made it easy to count time since those extra five days were not counted like the rest. For instance, consider these representations of the number of days in each time period:

1 year	2 years	3 years	5 months, 9 days	8 years, 11 months
				

Practice Problem 11

Write the Maya numeral for the number of days (not including the extra 5 in a year) in:

a. 7 Maya months, 15 days
b. 13 Maya years
c. 20 Maya years

Let's compare the Hindu-Arabic, Chinese and Maya numerals.

Practice Problem 12

State how many different symbols a person has to memorize to use each system:

a. Hindu-Arabic: _____
b. Chinese: _____
c. Maya: _____

Practice Problem 13

Write each number as a numeral in Chinese and Maya. Note how many symbols it takes to write the given number in Hindu-Arabic, Chinese and Maya. Show work.

a. 15
b. 100
c. 100
d. 9999

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CHAPTER OVERVIEW

4: Representing Quantities With Place values

4.1: Trading and Place Value
4.2: Other Base Systems
4.3: Different Bases and Their Number Lines
4.4: Place Values with Different Bases
4.5: Supplementary Topics

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4.1: Trading and Place Value

Let's exploring place value in this problem set. We'll start off by working with cups, pints, quarts, half-gallons, and gallons. We are using these because they can represent the relationship between place values in **binary**, the place-value number system with only two digits: 1 and 0. It will help if you create your own manipulatives for this. Cut out many small strips of paper of the same length, and write "C" on them. Those are your cups. Cut out two strips of paper that are each twice the size of the previous strips. Write "P" on them to represent pints. Continue this process until you have two strips representing gallons. If you don't have paper long enough, you can double the width instead of the length. See the examples below. You may also download this manipulatives file and print it. Then just cut out the shapes! If you do not have the ability to do this, you can use drawings. Size is important, so if you use drawings, make sure you keep the sizes consistent.

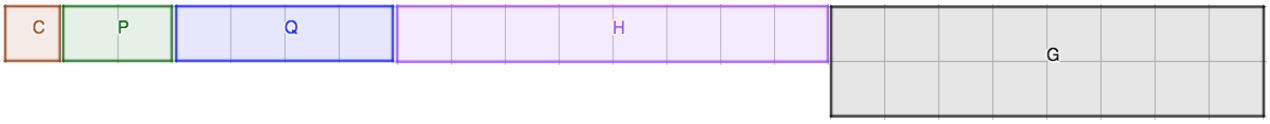


Figure 4.1.1: Strips (or shapes) representing cups, pints, quarts, half-gallons, and gallons

? Problem 1

Fill in the blanks with the correct numeral to represent the relationship for each pair.

a. There are _____ cups in a pint.
b. There are _____ pints in a quart.
c. There are _____ quarts in a half-gallon.
d. There are _____ half-gallons in a gallon.

? Problem 2

Suppose you were sick for nine days and a friend came over three times a day, each time leaving you a cup of soup. Suppose you never ate any of the soup, but instead kept putting each cup in the refrigerator. After those nine days were up, your refrigerator was stocked with all these individual cups of soup. You decide to consolidate these into as few containers as possible. You have five different kinds of containers: those holding cups, pints, quarts, half-gallons and gallons. Fill in each blank with the correct numeral.

a. Using your manipulatives or drawings, represent aside three cups a day for nine days. There are _____ cups represented.
b. Trade in the cups for as many pints as you can. There are now _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup.
c. Trade in the pints for as many quarts as you can. There are now _____ quart(s), _____ pint(s), and _____ cup(s) of soup. This is a total of _____ containers of soup.
d. Trade in the quarts for as many half-gallons as you can. There are now _____ half-gallon(s), _____ quart(s), _____ pint(s), and _____ cup(s) of soup. This is a total of _____ containers of soup.
e. Trade in the half-gallons for as many gallons as you can. There are now _____ gallon(s), _____ half-gallon(s), _____ quart(s), _____ pint(s), and _____ cup(s) of soup. This is a total of _____ containers of soup
f. Write down the information from part e in a consolidated fashion by filling in each space in the chart with the correct numeral.

G	H	Q	P	C

g. Originally, from part (a), there were _____ cups of soup. By consolidating (see part e and f), the number of containers was reduced to only _____ containers of soup!

What is interesting about this problem is you have just done a conversion from a number written in base ten to a number written in base two! The 27 cups, is 27 ones, which is 2 tens and 7 ones. That's why in base ten, the leftmost digit is 2 and the rightmost digit is 7. Within 27, there is 1 sixteen, 1 eight, 0 fours, 1 two, and 1 one. In binary, the first five places (right to left) are ones, twos, fours, eights, and sixteens. So, 27 in base ten is 11011 in binary (or base two). Notice that the number of containers of soup is the sum of digits in the binary number and the binary number is precisely the numbers in the chart you filled in above!

? Problem 3

Repeat Problem 2, but change the circumstances so that the friend only brings over two cups of soup a day for nine days.

a. Using your manipulatives or drawings, represent two cups a day for nine days. There are _____ cups.

b. Trade in the cups for as many pints as you can. There are now _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup.

c. Trade in the pints for as many quarts as you can. There are now _____ quart(s), _____ pint(s), and _____ cup(s) of soup. This is a total of _____ containers of soup.

d. Trade in the quarts for as many half-gallons as you can. There are now _____ half-gallon(s), _____ quart(s), _____ pint(s), and _____ cup(s) of soup. This is a total of _____ containers of soup.

e. Trade in the half-gallons for as many gallons as you can. There are now _____ gallon(s), _____ half-gallon(s), _____ quart(s), _____ pint(s), and _____ cup(s) of soup. This is a total of _____ containers of soup.

f. Write down the information from part e by filling in each space in the chart with the correct number.

G	H	Q	P	C

See if you can notice a pattern with what happens while doing these trades (or conversions). Which base ten number did you just convert into a binary number?

Now try the next problem.

Problem 4

Repeat Problem 2 one more time, but change the circumstances so that the friend brings over five cups of soup a day for nine days. Also, let's assume you also own a 2 gallon container, which we'll refer to as a double-gallon. See if you can do this without manipulatives or drawings. It is okay if you cannot.

a. There are _____ cups.

b. Trade in the cups for as many pints as you can. There are now _____ pint(s) and _____ cup(s) of soup. This is a total of _____ containers of soup.

c. Trade in the pints for as many quarts as you can. There are now _____ quart(s), _____ pint(s), and _____ cup(s) of soup. This is a total of _____ containers of soup.

d. Trade in the quarts for as many half-gallons as you can. There are now _____ half-gallon(s), _____ quart(s), _____ pint(s), and _____ cup(s) of soup. This is a total of _____ containers of soup.

e. Trade in the half-gallons for as many gallons as you can. There are now _____ gallon(s), _____ half-gallon(s), _____ quart(s), _____ pint(s), and _____ cup(s) of soup. This is a total of _____ containers of soup.

f. Trade in the gallons for as many double-gallons as you can. There are now _____ double-gallon(s), _____ gallons, _____ half-gallon(s), _____ quart(s), _____ pint(s), and _____ cup(s) of soup. This is a total of _____ containers of soup.

g. Write down the information from part f by filling in each space in the chart with the correct number.

G	H	Q	P	C

In the previous problems, exchanges or trades were being made any time you had two of one container. You were basically employing the grouping technique we see when counting objects and writing them with base ten numerals. When employing the grouping technique, we are working in a particular base, depending on how many it takes to form a group. In the exercises you just completed, you were working in base two.

To write a numeral in a given base, the base is written out in words to the right and a little below the numeral, as in 11011_{two}. The only time you do not have to write the base is when it is a base ten numeral because we are so used to that. However, it is okay and often good to write the "ten" anyway when doing conversions.

- In Problem 3, the base ten numeral 18_{ten} (two cups for nine days) is written as 10010_{two} in base two (or binary).
- In exercise 4, the base ten numeral 45_{ten} (five cups for nine days) is written as 101101_{two} in base two.

It is imperative that you read and say these numerals in different bases correctly. It's much easier to make mistakes if you say the numeral wrong and much easier to avoid mistakes if you say the numeral correctly. The following paragraph explains the proper way to read or say the numerals.

In base ten, we have abbreviated ways of saying numerals out loud. For instance, "13" is read "thirteen". But, "13_{five}" is read "One, three, base five" and "246201_{eight}" is read "two, four, six, two, zero, one, base eight." *It's extremely important that you learn to read and say* "13_{five}" as "one, three, base five" *and not as* "thirteen, base five"! There is no "thirteen" in base five. Why? When you *say* thirteen, it refers to the base ten numeral meaning one group of ten and three ones ("thirteen" is essentially an abbreviation of "three and ten"). Even though when we read it, we say the word "base", do not write the word "base".

Problem 5

Write out in words how to say each of the following numerals.

a. 302_{six} is said out loud as _____
b. 1011_{two} is said out loud as _____
c. 435_{seven} is said out loud as _____

Problem 6

Write the given numerals

a. "Five, zero, one, six, base eight" is written as _____
b. "one, zero, one, zero, zero, one, base two" is written as _____

When you see a base two (or binary) numeral like 11011_{two}, you need to think about what each place value stands for. *In all bases*, the rightmost place value is the units or ones place. Next, since the "two" to the right of the numeral tells you this is a base two numeral, each place value as you move to the left increases by a multiple of 2. To the right is a chart showing the first eight place values in base two. They are written in base ten.

128	64	32	16	8	4	2	1

To check that the base two numeral 11011_{two} really is 27_{ten}, put it into a chart with just five place values showing (since it's a five-digit numeral) and check the total value of the numeral. Look at the numeral written to the right. The place value for each digit is written below each digit of the numeral. The numeral is written in bold so it does not get confused with the place values.

$$\begin{aligned}(1 \text{ group of } 16) + (1 \text{ group of } 8) + (0 \text{ groups of } 4) + (1 \text{ group of } 2) + (1 \text{ group of } 1) \quad &= (\mathbf{1} \times 16) + (\mathbf{1} \times 8) + (\mathbf{0} \times 4) + (\mathbf{1} \times 2) + (\mathbf{1} \times 1) \\ &= 16 + 8 + 2 + 1 \\ &= \mathbf{27}.\end{aligned}$$

All right! It really works!

Let's convert $\mathbf{10\ 011\ 010_{two}}$ to base ten. Try it on your own, then check your answer below.

This numeral converts to 128 + 16 + 8 + 2 = 154.

Problem 7

Fill in the chart to show what values are missing in this base two place value chart.

				128	64	32	16	8	4	2	1

Problem 8

Convert each base two numeral to base ten.

a. $10\ 011_{\text{two}}$	b. $1\ 000\ 001_{\text{two}}$	c. $111\ 111_{\text{two}}$

Let's convert some numerals in other bases. Consider the base five numeral, 24_{five}. First, we need to establish the place values in base five just like we did for base two.

Problem 9

Fill in the chart to show what values are missing in this base five place value chart.

			25		1

In base five, $\mathbf{24_{five}}$ is a two digit numeral. Note there are **2** groups of 5 and **4** units or $2 \times 5 + 4 \times 1 = 10 + 4 \quad = \mathbf{14}$.

? Problem 10

Convert $\mathbf{13_{five}}$ to a base ten numeral. *Remember not to read this as "thirteen"!!!*

Let's look at a tougher problem (only tougher in that it is bigger and has more room for error). Here is how to convert $\mathbf{31204_{five}}$ to base ten. Try it on your own then continue reading. It is okay if you get stuck; it is important that you try first because it prepares your brain before reading the correct answer.

$$\begin{aligned} \mathbf{3} \times 625 + \mathbf{1} \times 125 + \mathbf{2} \times 25 + \mathbf{0} \times 5 + \mathbf{4} \times 1 \quad &= 1875 + 125 + 50 + 4 \\ &= \mathbf{2054}. \end{aligned}$$

Let's take the base three numerals $\mathbf{1002021_{three}}$ and $\mathbf{22122_{three}}$ and convert to base ten. First, we should make a place value chart for base three.

? Problem 11

Fill in the missing place values for base three.

		243			9		

Try to convert $\mathbf{1002021_{three}}$ on your own, then continue reading. Note that in the solution below, the 0 terms are skipped. If this makes it more confusing, rewrite this solution but with the 0 terms included.

$$\mathbf{1} \times 729 + \mathbf{2} \times 27 + \mathbf{2} \times 3 + \mathbf{1} = 729 + 54 + 6 + 1 = \mathbf{790}.$$

Try too convert $\mathbf{22122_{three}}$ on your own, then continue reading.

$$\mathbf{2} \times 81 + \mathbf{2} \times 27 + \mathbf{1} \times 9 + \mathbf{2} \times 3 + \mathbf{2} = 162 + 54 + 9 + 6 + 2 = \mathbf{233}.$$

? Problem 12

Convert the following base three numerals to base ten.

a. $200\ 112_{\text{three}}$ = ____
b. $12\ 002\ 110_{\text{three}}$ = ____
c. $1\ 111\ 111_{\text{three}}$ = ____

Knowing what the value of *any* place other than the ones place in a numeral can tell you which base you're working in. See if you can determine the base from the given values of the places in the following problem.

? Problem 13

Figure out the base in each place value chart and fill in the missing place values for each

a. Base ____

			36		

b. Base ____

				7	

c. Base ____

			81		

d. Base ____

			64		

e. Base ____

		64			

f. Base ____

	10000				

g. Base ____

				12	

h. Base ____

				11	

Another way to write out the values in a place value chart for a given base is by using exponents. Each place value is a power of the base. When writing it out this way, the actual base ten values are not explicitly written out. The following is another way to write a place value chart for base seven:

7^{11}	7^{10}	7^9	7^8	7^7	7^6	7^5	7^4	7^3	7^2	7^1	7^0

Did you remember that $7^0 = 1$? Any nonzero number raised to the zero power equals 1. For example, $3^0 = 1, 15^0 = 1, 1^0 = 1, 10^0 = 1, 546^0 = 1$ and so on.

Problem 14

Consider base n, where n is any integer greater than 1. Look at the place value chart below and fill in the missing place values for base n.

$\boldsymbol{n^{11}}$				n^7				n^3			n^0

If you are converting very large numbers, it might be useful to write out a place value chart using the base written to a given exponent to save time and space.

For instance, let's say you were asked to convert the numeral $100\ 200\ 020\ 000\ 100_{\text{three}}$ to base ten (some spaces have been added to make it easier to read this number). There are a lot of zeros for most of the place values. In fact, this fifteen digit base three numeral only has four nonzero digits. We know each place value is a power of three, since this is a base three numeral. In base ten, this numeral is:

$$\begin{aligned} 1 \times 3^{14} + 2 \times 3^{11} + 2 \times 3^7 + 1 \times 3^2 &= 1 \times 4782969 + 2 \times 177147 + 2 \times 2187 + 1 \times 9 \\ &= 4782969 + 354294 + 4374 + 9 \\ &= 5,141,646 \end{aligned}$$

Problem 15

$100\ 200\ 020\ 000\ 100_{\text{three}}$ can be written as $1 \times 3^{14} + 2 \times 3^{11} + 2 \times 3^7 + 1 \times 3^2$. This is called expanded notation. Pay close attention to the fact that the 1 in the third place from the right is 3^2 (not 3^3). Write the following numerals in expanded notation:

a. $3\ 000\ 600\ 020\ 000\ 000_{\text{eight}}$ = ____
b. $3\ 000\ 040\ 020\ 000\ 000_{\text{five}}$ = ____
c. $400\ 030\ 000\ 000\ 002_{\text{eleven}}$ = ____
d. $100\ 100\ 000\ 010\ 000_{\text{two}}$ = ____

Problem 16

Refer to the numeral, $3\ 040\ 001\ 000\ 002_{\text{nine}}$. For each nonzero digit in the numeral, write the place value of that digit as a power of nine. *Do not write out the actual base ten numeral.* The first one is done for you.

a. 2 is in the 9^0 place.
b. 1 is in the ____ place.

c. 4 is in the ____ place.
d. 3 is in the ____ place.

If a number is written in expanded notation, you can reverse the process and write it as a number in the base used. For instance, $4 \times 7^{13} + 5 \times 7^{10} + 2 \times 7^{8} + 3 \times 7^{3} + 6 \times 7^{0}$ can be written as the base seven numeral $40\ 050\ 200\ 003\ 006_{\text{seven}}$. Notice that a number in the *ninth* place from the right is really the *eighth* power of the base, as shown by the placement of 2 in the numeral $40\ 050\ 200\ 003\ 006_{\text{seven}}$. This is because the first place represents the base to the zero power. In other words, the counting does not start at one, but at zero.

Problem 17

Rewrite the following numbers that are in expanded notation to a numeral in the base indicated:

a. $2 \times 5^{9} + 4 \times 5^{8} + 1 \times 5^{6} + 3 \times 5^{2}$ is written in base five as ____.
b. $3 \times 8^{10} + 4 \times 8^{7} + 1 \times 8^{3} + 6 \times 8^{2}$ is written in base eight as ____.
c. $7 \times 12^{9} + 8 \times 12^{5} + 4 \times 12^{3}$ is written in base twelve as ____.
d. $2 \times 3^{8} + 1 \times 3^{6} + 1 \times 3^{4} + 2 \times 3^{2}$ is written in base three as ____.

Problem 18

a. If a base six numeral has eighteen digits, what is the place value of the first (leftmost) digit written as a power of six? ____

b. If a base nine numeral has twenty digits, what is the place value of the first (leftmost) digit written as a power of nine? ____

We now consider the number of distinct digits that exist in each base system.

Take another the following numerals:

$1\ 100\ 101_{\text{two}}$	$212\ 201_{\text{three}}$	42101_{seven}	80034_{nine}

Did you notice that there are only 0's and 1's in base two numerals? That's because if there was a 2 or higher for one of the placeholders, it could be traded in for the next place value to the left. Using physical models at the beginning of this problem set, two cups could be exchanged for a pint, two pints could be exchanged for a quart, etc. Although we only went up to the double-gallon which contained 32 cups, there is really no limit to the number of place values a numeral has in a place value system. Also, one doesn't have to come up with a new name for each place value (unlike how new symbols need to be created to represent bigger and bigger numbers in, say, Roman numerals, without writing a lot of symbols.

Think about any base ten numeral. There are ten possible digits (or symbols) for each placeholder in the numeral 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. There is no separate single digit for the number ten (10); it is made up of a 1 and 0, which means one group of ten and zero units! In base two, there are only two possible digits for each place value in the numeral: 0 and 1. There is no need or use for the symbol (or digit) "2" when writing a numeral in base two because it is represented as 10_{two}.

Problem 19

How many possible digits can each place value in base three have? _____

What are those possible digits? ____

Problem 20

How many possible digits can each place value in base four have? _____

What are those possible digits? ____

Problem 21

How many possible digits can each place value in base five have? _____

What are those possible digits? ____

Problem 22

How many possible digits can each place value in base six have? _____

What are those possible digits? ____

Problem 23

How many possible digits can each place value in base eight have? _____

What are those possible digits? ____

Problem 24

Can the digit (symbol) "7" be used in a base seven numeral? _____

Explain your answer.

Problem 25

a. What is the lowest base in which the digit "6" may appear? _____

b. What is the highest base in which the digit "6" may appear? _____

Do you see a pattern developing? Just wait until you see what happens in base eleven, twelve, and thirteen!

Problem 26

How many possible digits for each place value do you need for base eleven? _____

Problem 27

How many possible digits for each place value do you need for base twelve? _____

Problem 28

How many possible digits for each place value do you need for base thirteen? _____

In base eleven, you need eleven different symbols (or digits) for possible placeholders. The problem we encounter is that we only have these ten recognizable digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Remember that 10 is made up of *two separate digits; it isn't a single symbol!* We need to introduce a new symbol to represent ten in base eleven and higher. The convention is to use the letter "T" or "A". Similarly, we use the letter "E" or "B" to represent the number eleven in base twelve and higher and the letter "W" or "C" to represent the number twelve in base thirteen and higher. Since the second place value from the right represents the base you are working in, you never need a separate symbol to represent the value of the base or any number higher than the base. That's why there are only 0s and 1s and *no* 2s in base two and why there are only 0s, 1s, 2s, 3s, and 4s and *no* 5s in base five, etc.

The next few problems are worked exactly like those you have done so far, except the bases are higher than ten. We will use the new "digits" T, E, and W for ten, eleven, and twelve.

Below is a value chart for base twelve.

144	12	1

Let's convert the numeral $TE5_{\text{twelve}}$ to base ten.

$TE5_{\text{twelve}} = 10 \times 144 + 11 \times 12 + 5 \times 1 = 1440 + 132 + 5 = \mathbf{1577}$.

Let's try a pair of examples: $2T9_{\text{eleven}}$ and 2109_{eleven}.

To convert $2T9_{\text{eleven}}$ to base ten, first consider the base eleven place value chart shown.

1331	121	11	1

$2T9_{\text{eleven}} = 2 \times 121 + 10 \times 11 + 9 \times 1 = 242 + 110 + 9 = \underline{361}$.

Now, let's convert 2109_{eleven} to base ten. Looking at the base eleven place value chart, 2109_{eleven} represents $2 \times 1331 + 1 \times 121 + 011 + 9 \times 1 = 2662 + 121 + 0 + 9 = 2792$.

Let's try another one. We will convert TEW_{thirteen} to base ten. First, we should fill in a place value chart for base thirteen.

2197	169	13	1

$TEW_{\text{thirteen}} = 10 \times 169 + 11 \times 13 + 12 \times 1 = 1690 + 143 + 12 = \underline{1845}$

? Problem 29

Convert each of the following to a base ten numeral.

a. $47E_{\text{twelve}}$ =	b. $T74_{\text{eleven}}$ =	c. TTT_{thirteen} =
d. 2034_{twelve} =	e. 1025_{eleven} =	f. 1028_{thirteen} =

? Problem 30

Convert each of the following to a base ten numeral.

a. 110_{two} =	e. 110_{six} =	i. 110_{eleven} =
b. 110_{three} =	f. 110_{seven} =	j. 110_{twelve} =
c. 110_{four} =	g. 110_{eight} =	k. 110_{thirteen} =
d. 110_{five} =	h. 110_{nine} =	l. 110_{twenty} =

When using base systems for numerals, there is an easy way to compare number sizes that did not work for Roman numerals. Two positive integers written in the same base can be compared by first looking at the length of the numbers and, if they are the same length, looking at the digits starting from left to right. But what if they are written in different bases. Sometimes we have to convert them to the same base to compare them. But not always. Let's finish up this problem set by thinking about this.

? Problem 31

Without doing any computations, circle which numeral has the larger value. Explain your reasoning. Try to be mathematically clear in your explanation. Notice the sequence of digits is the same for each numeral. Only the base is different.

$$13\ 201\ 154\ 320\ 050\ 146_{\text{eight}} \textbf{ OR } 13\ 201\ 154\ 320\ 050\ 146_{\text{eleven}}$$

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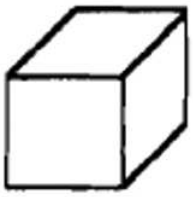
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4.2: Other Base Systems

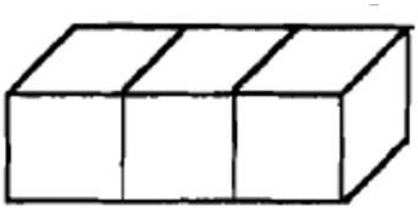
Previously, we used cups, pints, etc. to model the grouping that forms the place values in a base two system. It is helpful to have a more abstract way to model this that can be applied to any base. For this, we have multibase blocks. Most people have seen the blocks that are used for a base ten system because they are used in many classrooms. When the base is known, we can use that when referring to the different types. Below, we will look at base three blocks, which are the multibase blocks used for working in base three.

The ones place is also called the units place. This is why the fundamental type of block in multibase blocks is the unit block. We simply refer to it as a unit. Just like how for the number three we must jump to the threes place in base three, when we group three units together, we create a new kind of block: a long. When we group three longs together, we get a new shape to represent the nines place: a flat. Different people use different terminology for the next type of multibase block. When three flats are grouped together, we get a new shape to represent the twenty-sevens place. Some people call this a "cube", others a "big cube", and still others a "block". For now, we will use "cube". After that, instead of creating new names, we will just put multiple names together. Three cubes will form a "cube-long" (or "long cube" or "long block"). Three cube-longs will form a "cube-flat" (or "flat cube" or "flat block"). At some point (possibly one we have already passed), this becomes tedious. But that's a good time to transition from using manipulatives to using the symbols only. Below is an amateur drawing of what a unit, long, flat and cube or block really look like in base three.

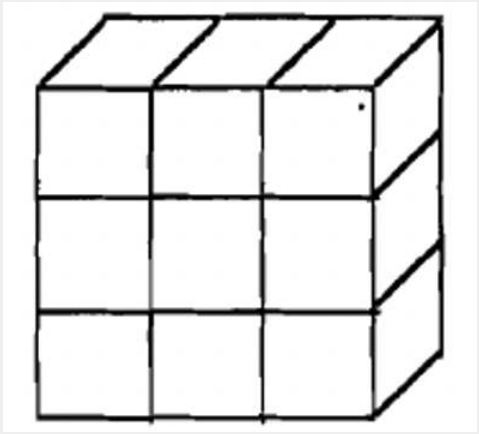
base three unit



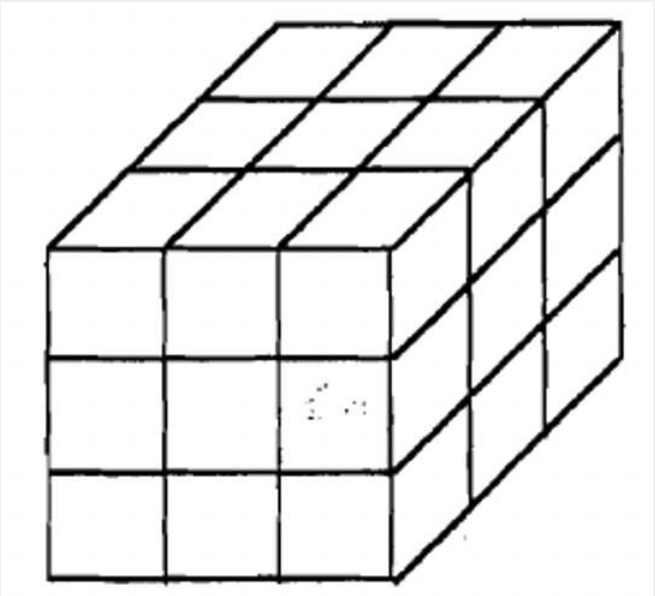
base three long



base three flat



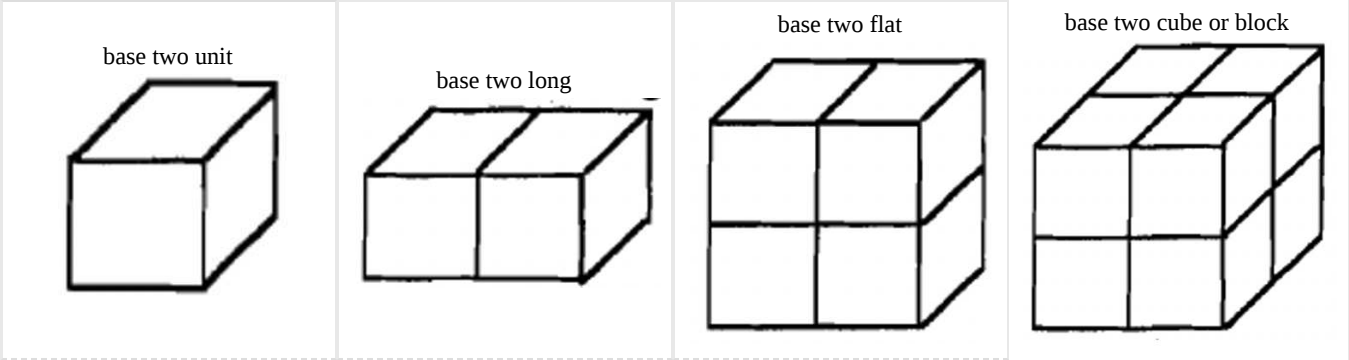
base three cube or block



Notice that in base three, a unit is made up of 1 small cube, a long is made up of 3 units (small cubes), a flat is made up of 9 units (small cubes), a cube or block is made up of 27 units (small cubes), a cube-long (or long cube or long block) is made up of 81 units (small cubes), a cube-flat (or flat cube or flat block) is made up of 243 units (small cubes), a "cube-cube" (or "block block") is made up of 729 units (small cubes), and so on. These are the place values in base three numerals.

Think about how you would form the base two blocks. What would a long look like? A flat? The rest? After thinking about this and possibly drawing pictures, continue reading.

Below is an amateur drawing of a unit, long, flat and block in base two. Note that the pictures are not quite accurately scaled.



In base two, a long is made up of 2 units, a flat is made up of 4 units, a cube or block is made up of 8 units, a cube-long (or long cube or long block) is made up of 16 units, a cube-flat (or flat cube or flat block) is made up of 32 units, a cube-cube (or block block) is made up of 64 units, and so on.

On your own, try to draw a picture of a base four unit, long, flat and cube (or block).

Example 4.2.1

Fill in the blanks.

In base four, it takes ____ units to make a long, it takes ____ longs to make a flat, it takes ____ flats to make a cube, it takes ____ cube to make a cube-long, and it takes ____ cube-longs to make a cube-flat.

In base four, a long is made up of _____ units, a flat is made up of _____ units, a cube is made up of _____ units, and a cube-long is made up of _____ units.

In base five, it takes _____ flats to make a cube.

In base six, a flat is made up of _____ units.

Solution

The blanks should have the following numbers in them: 4, 4, 4, 4, 4, 4, 16, 64, 256, 5, and 36.

Previously you practiced converting from other bases to base ten; now we'll focus on converting from base ten to other bases. There is more than one way to accomplish this task, but we will focus on only one method with an aim to understand what is actually happening.

Example 4.2.1

Represent 42 in base two.

Solution

To model 42 with base two blocks, we have some options. We could use 42 units. But this would be unnecessarily cluttered. Let's imagine exchanging them to have fewer individual pieces.

Imagine exchanging as many units as possible to form longs. Since each long is made of 2 units, we could convert all 42 units into 21 longs. Check to make sure you agree with this. So, we now have 21 longs and 0 units.

Now imagine exchanging as many longs as possible to form flats. Since each flat is made of 2 longs, we could convert 20 longs into 10 flats and have one long left over. Do you agree with this? Now we have 10 flats, 1 long, and 0 units.

Now imagine exchanging as many flats as possible to form cubes. Since each cube is made of 2 flats, we could convert all 10 flats into 5 cubes. Do you agree with this? Now we have 5 cubes, 0 flats, 1 long, and 0 units.

Before we keep going, let's see why we know we can keep going. Since there are 5 cubes and we only need *2* to move up to the next shape size (because we are in base *two*), we keep going!

Now imagine exchanging as many cubes as possible to form cube-longs. Since each cube-long is made of 2 cubes, we could convert 4 cubes into 2 cube-longs with one cube left over. Do you agree with this? Now we have 2 cube-longs, 1 cubes, 0 flats, 1 long, and 0 units.

Now imagine exchanging as many cube-longs as possible to form cube-flats. Since each cube-flat is made of 2 cube-longs, we could convert both cube-longs into 1 cube-flat. Do you agree with this? Now we have 1 cube-flat, 0 cube-longs, 1 cube, 0 flats, 1 long, and 0 units.

We are finished. Make sure you understand why we know this!

This means $42_{\text{ten}} = 101010_{\text{two}}$.

Try to redo the example above but with base three. After you do this, read the next example.

Example 4.2.1

Represent 42 in base three.

Solution

Let's start with 42 units again. Now we imagine we are using (or we actually use) base three blocks.

- Exchange as many units as possible to get longs. Since each long is 3 units, all 42 units can be exchanged for 14 longs. We have 14 longs and 0 units.
- Exchange as many longs as possible to get flats. Since each flat is 3 longs, 12 longs can be exchanged for 4 flats with 2 longs left over. We have 12 flats, 2 longs, and 0 units.
- Exchange as many flats as possible to get cubes. Since each cube is 3 flats, 3 flats can be exchanged for 1 cube with 1 flat left over. We have 1 cube, 1 flat, 2 longs, and 0 units.

We stop now (make sure you understand why!) and have $42_{\text{ten}} = 1120_{\text{three}}$.

Try again but with base four. If you did it correctly, you should have $42_{\text{ten}} = 222_{\text{four}}$.

Now, let's look at that problem (converting to base four) but discuss it with groups of values rather than blocks.

Example 4.2.1

Represent 42 in base four.

Solution

- We start with 42 ones.
- From those 42 ones, create as many groups of 4 ones as possible. Since 42 ones = 40 ones + 2 ones and 40 ones is 10 groups of 4 ones , we can think of 42 as 10 fours and 2 ones.
- From the 10 fours, create as many groups of 4 fours as possible. Since 10 fours = 8 fours + 2 fours and 8 fours is 2 groups of 4 fours, we can think of 10 fours as 2 groups of 4 fours and 2 fours left over. Since 4 fours is sixteen, we have changes the 10 fours to 2 sixteens and 2 fours left over. Put together with the previous work, we have 2 sixteens, 2 fours, and 2 ones.

We stop now because we only have 2 sixteens (not enough to form a group of 4). Thus, $42_{\text{ten}} = 222_{\text{four}}$.

If we only needed to convert relatively small numbers and we always had multibase blocks available in any base we needed to convert to, we could continue to convert to various bases by using the blocks and doing exchanges. But this is not always the case. Consider if you were asked to convert 999 to base four. This would be pretty time-consuming and tedious. So, it is important to see how it works without the manipulatives.

The previous example probably got a little confusing because of all the numbers. We can make this a little easier by keeping things organized in a chart. Let's look at an example.

Example 4.2.1

Represent 82 in base six.

Solution

Let's make a chart that corresponds to place values in base six. Since the place values (from left to right) in base six are ones, sixes, thirty-sixes, two hundred sixteens, etc. we only need three places (since 216 > 82). We will label the columns of a table with the base ten values of the places. We start with 82 in the ones place.

36	6	1
		82

Now, we find how many groups of 6 are in 82. Since $82 \div 6 = 13\frac{4}{6}$, we have 13 groups of 6 and 4 left over. In other words, we have 13 sixes and 4 ones. This means we write 13 in the 6 column, cross out the 82 and replace it with a 4 in the 1 column.

36	6	1
	13	~~82~~ 4

Now, we find how many groups of 6 are in 13. Since $13 \div 6 = 2\frac{1}{6}$, we have 2 groups of 6 and 1 left over. In other words, we have 2 thirty-sixes (since 6 groups of 6 equals 36) and 1 six. This means we write 2 in the 36 column, cross out the 13 and replace it with a 1 in the 6 column.

36	6	1
2	~~13~~ 1	~~82~~ 4

We confirm that we don't need another digit by noticing that 2 < 6. So, we are finished, and $82_{\text{ten}} = 214_{\text{six}}$.

Try to represent 477 in base five. After doing this, read the next example.

Example 4.2.1

Represent 477 in base five.

Solution

Since 477 is greater than $5^3 = 125$ but less than $5^4 = 625$, we will need four places. The chart we create looks like this.

125	25	5	1
3	~~19~~ 4	~~95~~ 0	~~477~~ 2

So, $477_{\text{ten}} = 3402_{\text{five}}$.

Try redoing the previous example but converting to base twelve instead of five. Notice you need one fewer place. Did you get $477_{\text{ten}} = 339\text{twelve}$? If not, see if you can find your error.

Try again but converting to base three. Now you'll need *more* places, 6 of them! Did you get $477_{\text{ten}} = 122200\text{three}$?

Now, let's end with a little extension our exploration a bit. Ask yourself the following questions. Share your responses with a classmate, a tutor, or your teacher and discuss them.

1. In base ten, what is the largest eight digit numeral?
2. In base ten, what is the smallest ten digit numeral?
3. In base three, what is the largest seven digit numeral?
4. In base three, what is the smallest seven digit numeral?
5. In base three, what number comes after 1202010_{three}?

6. In base three, what number comes after 2100212_{three}?
7. In base three, what number comes before 1202221_{three}?
8. In base three, what number comes before 2110100_{three}?
9. What is wrong with the numeral 1022301_{three}?
10. Can you count to 113_{four} in base four without ever leaving base four?
11. Can you count to 10100_{two} in base two (binary) withouth ever leavning base two?

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4.3: Different Bases and Their Number Lines

The way we count, the way we visualize numbers, all starts from a number line. As early as Kindergarten, students learn how to read and use a number line. Our current number system is called the Decimal Number System (or Hindu-Arabic). Notice that Decimal begins with DEC like DECahedron (a 10 sided polygon) and we have 10 characters in our numbering system: zero through nine. So, even though we live in a Base 10 system (because we have characters for zero through nine), the number ten is represented by two characters (digits), one and zero.

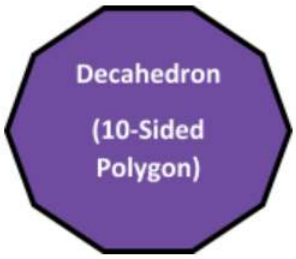


Figure 2.5.1: Decahedron

Below is the number line for **Base 10**. Notice how it is broken up into rows. We will be using the number lines as rows for the sake of our lesson. However, when you teach the number line to the elementary school students, the number line will be one continuous row (to set up for negative numbers in second grade) OR set up like below only going from 1 to 10 or 0 through 10.

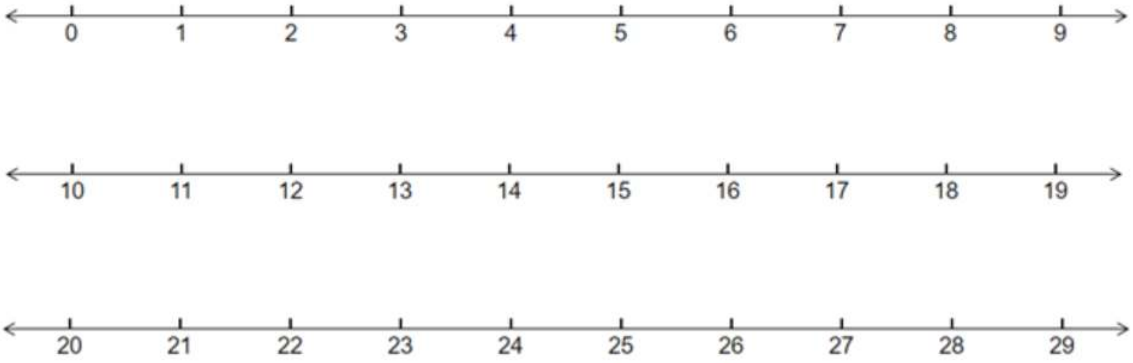


Figure 2.5.2: Number line for Base 10

There are an infinite amount of different bases and an infinite amount of corresponding number lines. Below are three different examples, written like we have Base 10 on the previous page.

Base 2 Number Line (reads from left to right, then top to bottom). The fifth number in the number line is 101_{two}.

Table 2.5.1: Base 2 Number Line

0	1
10	11
100	101
110	111
1000	1001
1010	1011
1100	1101
1110	1111
10000	10001

Base 8 Number Line (reads from left to right, then top to bottom). The 10th number is 12_{eight}.

Table 2.5.2: Base 8 Number Line

0	1	2	3	4	5	6	7
10	11	12	13	14	15	16	17
20	21	22	23	24	25	26	27

Base 12 Number Line (reads from left to right, then top to bottom). The 15th number is 13_{twelve}.

Table 2.5.3: Base 12 Number Line

0	1	2	3	4	5	6	7	8	9	A	B
10	11	12	13	14	15	16	17	18	19	1A	1B
20	21	22	23	24	25	26	27	28	29	2A	2B

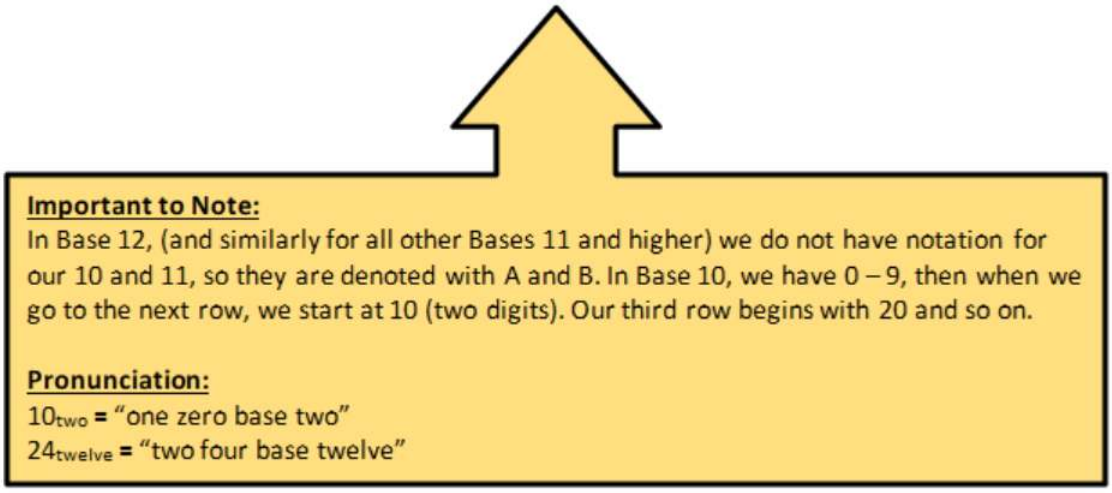


Figure 2.5.3

Where are different bases used?

Base 2: Computers use Base 2, just zeros and ones for all of their programming. Your phone operates in only zeros (OFF) and ones (ON).

Base 5: This was one of the very first systems of counting, since we have five fingers.

Base 8: Think of Base 8 as the mathematics for the Cartoon Universe. Many Cartoons have only eight fingers. We have 10 fingers and live in a Base 10 universe. Cartoon have eight fingers and live in a Base 8 universe. The Cartoon character does not have a character for eight or nine items, as we do in our universe.

Base 12: Base 12 is rare to find within history. However, we currently have 12 hours on the clock and 12 months in the year. There were a few tribes in Africa and India which used the duodecimal (base 12) system.

Base 20: The Mayans used a Base 20 system, and invented the concept of zero.

Base 60: An extreme example is base 60, which was used by the Babylonians (about 4000 years ago) which is now current day Iraq. They had characters for 1 – 59 items. (The concept of "zero" was not discovered yet.) However, this is where our current concept of 60 minutes and 60 seconds come from.

Practice Problems

1. What is wrong about 413_{four}?
2. How do you pronounce 542_{six}?
3. Write four rows worth of the number line of Base 3.

4. If you were not familiar with the Base 10 system, which base system do you think would work best for our society and culture?

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4.4: Place Values with Different Bases

Table 2.7.1: Place Values with different bases

Base 10		Base 2		Base 8		Base 12	
10^0	ones	2^0	ones	8^0	ones	12^0	ones
10^1	tens	2^1	twos	8^1	eights	12^1	twelves
10^2	hundreds	2^2	fours	8^2	sixty-fours	12^2	one-hundred forty-fours
10^3	thousands	2^3	eights	8^3	five-hundred twelves	12^3	one-thousand seven-hundred twenty-eights

Notice that all the words are plural!

Note: $8^3 = 512$ and $12^3 = 1728$

Practice Problems

In our Base 10 system, the place value for 10,000 is called the ten thousands. Write the equivalent name for the same place value for:

1. Base 3
2. Base 5
3. Base 9
4. Base 14

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4.5: Supplementary Topics

*You will need: A **Calculator***

So far, we've studied how to change a whole number in a different base to a whole number in Base Ten. It's also possible to work with a numeral that has a decimal point in another base. Look again at the chart for how the place value system worked for a number in Base n. The pattern continues to the left forever.

n^{11}	n^{10}	n^9	n^8	n^7	n^6	n^5	n^4	n^3	n^2	n^1	n^0

The model above is sufficient for a whole number. If you are looking at a number that does not have a decimal point, it is implied that the decimal point is to the right of the rightmost digit. We can stretch this model to include a number with decimal points. This is analogous to working with fractions. Following the pattern established above, the place values to the right of the decimal point, in order, are $n^{-1}, n^{-2}, n^{-3}, n^{-4}$,...

So, a way to view the various place values in base b is:

$$\underline{b^4}\ \underline{b^3}\ \underline{b^2}\ \underline{b^1}\ \underline{b^0} \bullet \underline{b^{-1}}\ \underline{b^{-2}}\ \underline{b^{-3}}\ \underline{b^{-4}}$$

That big dot in the middle is the decimal point. The place values still go infinitely to the left and also to the right. The above model only shows five places to the left of the decimal point and four places to the right of the decimal point.

Now, you need to recall from algebra what a negative exponent means. In case you've forgotten, here is the definition of a negative exponent and what it means to have an exponent of zero.

If $b \neq 0$ and for any n, $b^{-n} = \frac{1}{b^n}$ Example: $6^{-2} = \frac{1}{6^2} = \frac{1}{36}$	If $b \neq 0$, then $b^0 = 1$ Example: $8^0 = 1$

? Exercise 1

Simplify by first rewriting each problem so that there are no negative exponents. Simplify each fraction.

a. 4^{-1}	b. 4^{-2}	c. 4^{-3}
d. 3^{-1}	e. 5^{-2}	f. 2^{-3}

Let's figure out how to rewrite each of the following numerals in Base Ten.

✔ Example 1

142.3_{five}

Solution

$$\begin{aligned} 142.3_{\text{five}} &= (1 \times 5^2) + (4 \times 5^1) + (2 \times 5^0) + (3 \times 5^{-1}) \\ &= (1 \times 25) + (4 \times 5) + (2 \times 1) + (3 \times \frac{1}{5^1}) \\ &= 25 + 20 + 2 + \frac{3}{5} \\ &= 47\frac{3}{5} \end{aligned}$$

? Exercise 2

Convert each numeral to Base Ten. **Show *each* of the following steps**:

Step 1: Write the numeral in expanded notation.

Step 2: Simplify each number that has a positive exponent; rewrite each number that has a negative exponent as a number with a positive exponent.

Step 3: Simplify each term (*terms are separated by addition signs*).

Step 4: Add the terms together. Simplify any fractions if necessary

a. 43.2_{eight} =
b. 143.7_{eleven} =
c. 1111.1_{two}
d. 123.2_{four} =

Here is an example of converting to base ten when the numeral has more than one place after the decimal point.

Example 2

24.31_{five}

Solution

$$24.31_{\text{five}} = (2 \times 5^1) + (4 \times 5^0) + (3 \times 5^{-1}) + (1 \times 5^{-2})$$
$$= (2 \times 5) + (4 \times 1) + (3 \times \frac{1}{5^{-1}}) + (1 \times \frac{1}{5^2})$$
$$= 10 + 4 + \frac{3}{5} + \frac{1}{25}$$

(To add fractions, you need a common denominator, which is fairly easy in bases since the denominator will be a power of the original base.)

$$= 14 + \frac{15}{25} + \frac{1}{25}$$
$$= 14\frac{16}{25}$$

Exercise 3

Convert each numeral to Base Ten. **Show each of the following steps**:

Step 1: Write the numeral in expanded notation

Step 2: Simplify each number that has a positive exponent; rewrite each number that has a negative exponent as a number with a positive exponent

Step 3: Simplify each term (*terms are separated by addition signs*)

Step 4: Add like terms together; to add fractions, get a common demominator (it will be a power of the original base.)

Step 5: Simplify further if necessary. Reduce any fractions if necessary

a. 43.21_{six} =
b. 231.123_{four} =
c. 156.12_{seven}
d. 111.101_{two} =
e. 222.222_{three}
f. $T2E.0T_{\text{twelve}}$ =

In order to really understand computer technology, you must have the ability to express numbers in base two and sixteen. At first, Base Two (called the Binary System) was used to express computer code. Base Two was the natural choice, since there are only two symbols, 0 and 1. At the most primitive level, electronic computers only know two things, off (0) and on (1). To get a basic feel for how some information gets read into a computer, imagine there are eight switches –switch 1, switch 2, switch 3,..., switch 8. How the computer responds depends on which of the eight switches are turned "on." There are actually 256 (2^8) different configurations possible.

Imagine that switch 7, switch 2 and switch 1 was turned on. We will use an 8-digit code to show this. It looks like an 8-digit numeral, where the rightmost digit represents switch 1 and the leftmost digit represents switch 8. For our example, there should be a 1 in the spots for switches 1, 2 and 7 and a 0 in the spots for switches 3, 4, 5, 6 and 8. The code looks like this: 01000011. This is read: "off-on-off-off-off-off-on-on." The code 01000011 is really in the format of a Base Two numeral and can be converted to Base Ten. When writing binary *code* for the computer, base two is implied, but the numeral is not written as a base two number. Code is written 01000011, whereas the base two numeral is written 1000011_{two}.

? Exercise 4

Using this system of eight switches being either turned on or off, represent the binary code if the switches indicated are turned on. Then, write the base ten numeral (also called decimal numeral) represented by each binary number.

a. switches that are on: 1, 3, 5, 6:	Binary Code: ____ Base ten numeral it represents: ____
b. switches that are on: 2, 4, 5, 8:	Binary Code: ____ Base ten numeral it represents: ____
c. switches that are on: 3, 5, 7:	Binary Code: ____ Base ten numeral it represents: ____
d. No switches are on:	Binary Code: ____ Base ten numeral it represents: ____
e. All eight switches are one:	Binary Code: ____ Base ten numeral it represents: ____

The binary system is fairly simplistic in that all it comes down to is a sequence of on and off switches. Actually, at the present time, the two electrical states usually used in computers are low voltage and high voltage. Whether we think of positive and negative charges, on and off switches, low and high voltages, etc., the main idea is the same –there are only *two states*. Although binary is simple on one hand, the disadvantage of the binary system is that even small numbers are made up of lots of digits. For instance, the base ten numeral 50 is written as 110010_{two} which has 6 digits. The base ten (also called decimal) numeral 1025 is written as 10000000001_{two} which has eleven digits!

An intermediate system of numeration, using a larger base, was developed so that the code for the numerals could be written more compactly, with fewer digits. For convenience, a base that was a power of two (making conversion easy) and one that had a reasonable number (not too many) of primary symbols was used.

In the first example where switches 1, 2 and 7 were on, the binary code was 01000011. Anything that can be typed on a keyboard actually has a special 8-digit code like this one.

A standard code called ASCII (an acronym for "American Standard Code for Information Interchange") was established in 1968 by the American National Standards Institute. Each separate keyboard symbol (and 32 special control functions, such as the "return" key) was assigned a number from 0 to 127, giving 128 different possibilities. When you type the uppercase letter "C" on the keyboard, the computer codes this little piece of information as 01000011 *–more ASCII code is on page 54.*

If I typed my name "JULIE HARLAND" on the keyboard, the computer would code this with 13 eight-digit numerals –one 8-digit numeral for each letter and one for the space in the middle. If I wanted to print out and read the code, I would be looking at 104 digits! To make this task less cumbersome, we can have the computer convert each original 8-digit numeral (which is really just a base two numeral) to a special hexadecimal (base sixteen) numeral, which consists of only two, instead of eight, digits. Note that if I wanted to read the 8-digit numeral as a single-digit numeral, I'd need a base 256 system, which would require 256 different

symbols –that would be way too complicated! Here is how the 8-digit binary code 01000011 is converted to the hexadecimal system, making it easier for me to read the code.

Each 8-digit binary code numeral is broken into two separate 4-digit numerals: 0100 0011. Each 4 digit numeral is converted from base 2 to base 16 (there are 24 , or 16 possibilities for a system with only 4 switches.) But in Base Sixteen, 16 different primary symbols are needed. WE **DO NOT USE** T for ten, E for eleven AND W for twelve IN BASE SIXTEEN! Instead, these are the 16 symbols used for counting in base 16: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F. In other words, A stands for ten, B for eleven, C for twelve, D for thirteen, E for fourteen and F for fifteen. The first group of four digits converts from 0100 in base two to 4 in base ten. 4 in base ten is written as 4 in base sixteen. The second group of four digits converts from 0011 in base 2 to 3 in base ten. 3 in base ten is written as 3 in base sixteen. Therefore, the 8-digit numeral 01000011 is simply written as 43 in hexadecimal.

Base Eight (the octal system) is another common intermediate system used. One only needs to memorize eight symbols for this system (0, 1, 2, 3, 4, 5, 6, and 7). Base Sixteen is more compact and only requires one to memorize six more than the usual ten in base ten. Using A, B, C, D, E and F for 10,11,12,13,14 and 15 is fairly natural for most people to learn. On the other hand, going to a base 32 (the next power of 2) system would be fairly cumbersome because you'd have to incorporate 22 new symbols. We'll limit our discussion to hexadecimal, and not work in the octal system in this exercise set.

One more example:

✔ Example 3

If switches 8, 7, 5, 4 and 2 are switched on, write the code for this configuration in binary and in hexadecimal. What decimal numeral does this represent?

Solution

In binary, the code is ***11011010***.

The first 4-digit sequence, 1101, is thirteen in base ten–D in hex; the second 4-digit sequence, 1010, is ten in base ten–A in hex. Therefore, the code in hex is ***DA***. Convert the binary or the hex to base ten to get 218.

Remember: If writing a "code" then the base is not written it is implied and the binary code is simply written as 11011010 or in hex as DA. If I'm writing a base two numeral, write it as 11011010two or if writing as a hexadecimal numeral, write DAsixteen. ***Pay attention to the directions!***

? Exercise 5

Write the binary and hex code corresponding to these switches being turned on.

Note: You already wrote the binary code for these in exercise 4. To check that you write the correct hexadecimal code, convert the base sixteen numeral back to base ten and see if it matches what you wrote for base ten in exercise 4.

a. switches that are on: 1, 3, 5, 6	Binary Code: ____ Hex Code: ____
b. switches that are on: 2, 4, 5, 8	Binary Code: ____ Hex Code: ____
c. switches that are on: 3, 5, 7	Binary Code: ____ Hex Code: ____
d. No switches are on	Binary Code: ____ Hex Code: ____
e. All eight switches are on	Binary Code: ____ Hex Code: ____

Exercise 6

Convert each hexadecimal code to binary code, and to a base ten numeral.

a. Convert 5E to binary code: ____	and to base ten: ____
b. Convert E5 to binary code: ____	and to base ten: ____
c. Convert 39 to binary code: ____	and to base ten: ____
d. Convert 1F to binary code: ____	and to base ten: ____
e. Convert 98 to binary code: ____	and to base ten: ____
f. Convert 2A to binary code: ____	and to base ten: ____
g. Convert 07 to binary code: ____	and to base ten: ____
h. Convert 40 to binary code: ____	and to base ten: ____

Part of the ASCII code chart (for only 28 of the 128 possible keyboard strokes) is shown below. The keyboard stroke, the decimal (base ten) numeral, the hexadecimal code and the binary code are given for these 28 keyboard strokes.

Key	Base Ten	Hex	Binary
<space>	32	20	00100000
!	33	21	00100001
,	44	2C	00101100
-	45	2D	00101101
.	46	2E	00101110
A	65	41	01000001
B	66	42	01000010
C	67	43	01000011
D	68	44	01000100
E	69	45	01000101
F	70	46	01000110
G	71	47	01000111
H	72	48	01001000
I	73	49	01001001
J	74	4A	01001010
K	75	4B	01001011
L	76	4C	01001100
M	77	4D	01001101
N	78	4E	01001110
O	79	4F	01001111
P	80	50	01010000

Key	Base Ten	Hex	Binary
Q	81	51	01010001
R	82	52	01010010
S	83	53	01010011
T	84	54	01010100
U	85	55	01010101
V	86	56	01010110
W	87	57	01010111

Remember that first numeral we wrote where switches 1, 2 and 7 were on? We wrote this as 01000011, and I mentioned that this is the ASCII code for the letter "C." Look up the code for "C" in the chart –is this starting to make a little more sense?

In any case, let's get back to how the code for "JULIE HARLAND" would look. We'll do it in both hex and binary, separating the code for each space or letter by a comma. Hex is easier, so let's do that first: 4A, 55, 4C, 49, 45, 20, 48, 41, 52, 4C, 41, 4E, 44. That's all there is to it. It's a code. The computer stores it in binary as: 01001010, 01010101, 01001100, 01001001, 01000101, 00100000, 01001000, 01000001, 01010010, 01001100, 01000001, 01001110, 01000100.

Note that I've only given you the ASCII code for some of the upper case letters of the alphabet. Code for a lower case letter is different from code for an upper case letter.

Exercise 7

Decode the following messages. Part a is in binary and part b is in hex.

a. 01001001, 00100000, 01001100, 01001111, 01010110, 01000101, 00100000, 01001101, 01000001, 01010100, 01001000, 00100001

translates to: _____

translates to: ____

Exercise 8

Write each of the following in hexadecimal code and in binary code. Show work.

a. **HELP!**

Hex: ___
Binary: ____

b. **BE HAPPY.** *Note: I bet you can figure out the code for the "Y" if you try!*

Hex: ___
Binary: ____

c. Write your first name in Hex and Binary. Be sure to include your work.

Exercise 9

Convert the following Base Ten numerals to binary and to hexadecimal. In this case, you aren't being asked to write a code, so remember to write the base to the right and a little below each numeral as with converting to any other base. Show all work

a. 73

b. 122

c. 50
d. 250
e. 1000

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CHAPTER OVERVIEW

5: Addition and Subtraction

5.1: Addition and Subtraction
5.2: Combining
5.3: Subtraction
5.4: Number Line Model
5.5: Addition and Subtraction
5.6: Operations in Different Bases

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5.1: Addition and Subtraction

Additive Problems

Any problem that involves addition or subtraction can be called an "additive problem". Both addition and subtraction represented with an equation involving a sum, and just because a problem might be written in a way that refers to addition (or subtraction) doesn't mean everyone will solve the problem with addition (or subtraction). For example, in the following problem, the situation is presented as an addition problem. But most people solve this problem with subtraction.

Example 5.1.1

Viktor has a box of marbles. His friend gives him 9 marbles. Afterward, Viktor counts the total of all the marbles and finds he now has 42 marbles. How may marbles did Viktor have before he got more marbles from his friend?

Solution

One way to answer this is to first note that the problem can be summarized as follows.

Viktor's original marbles + 9 marbles from his friend = 42 marbles total

Since we already know the relationship between addition and subtraction, we can rewrite this as follows.

Viktor's original marbles = 42 marbles total – 9 marbles from his friend

Since 42 – 9 = 33, Viktor had 33 marbles before he got more marbles from his friend.

It is tempting to assume that all problems like the one in the previous example "should" be solved this way. But consider if we simply changed the sizes of the values.

Example 5.1.1

Wally has a box of dice. His friend gives him 52 dice. Afterward, Wally counts the total of all the dice and finds he now has 59 dice. How many dice did Wally have before he got more dice from his friend?

Solution

One way to answer this is to imagine acting it out. Pretend you are Wally and have a box of dice. Next to the box of dice, you have the 52 dice from the friend. When you count them out afterward, suppose you started with the 52 new dice and counted on from there to count the box of dice. Since you know that the final result is 59, there can't be that many dice in the box to begin with. In fact, even if you only knew how to count on your fingers, you could quickly count from 52 to 59 and get that there had to be 7 dice in the box.

So, Wally had 7 dice before he got more dice from his friend.

What these two examples highlight is that if someone is comfortable with multiple mathematical strategies for solving additive problems, they can pick the strategy that works best for each given problem. If a child learns only one strategy for solving additive problems (or even specific types of additive problems), then they are likely to encounter problems for which that one strategy is quite inefficient and prone to producing errors.

Before continuing, it will help if we have a few definitions.

Definitions: Addend, Sum, Subtrahend, Minuend, and Difference

Two values added together are each called **addends**. The total of those values is called the **sum**. For example, in $5+9=14$, 5 and 9 are addends and 14 is the sum.

When one value is subtracted from another, the value being subtracted is called the **subtrahend**, and the value the subtrahend is subtracted from is the **minuend**. The result of the subtraction is called the **difference**. For example, in $7-3=4$, 7 is the minuend, 3 is the subtrahend, and 4 is the difference.

Types of Addition & Subtraction Exercises

For now, let's focus on exercises rather than word problems or story problems. Even by keeping things simple, we have a variety of ways of thinking of addition and subtraction. It is important to note that we will use "?" to represent what is unknown (i.e., what we want to find). This is the foundation of algebra! So, even if you work with very young children, you are helping them lay the foundations of what they will be learning many years later in algebra courses.

Three types of addition exercises:

1. Missing Sum
   i. $7+3=\ ?$
   ii. Story problem example: "Nestor ate 7 pieces of candy and Oscar ate 3 pieces of candy. How many pieces of candy did they eat in total?
2. Missing Start (first addend)
   i. $?\ +3=10$
   ii. Story problem example: Kaitlyn walked some dogs on Saturday and then three more dogs on Sunday. In totall, she walked 10 dogs on Saturday and Sunday. How many dogs had she walked on Saturday?
3. Missing Change (second addend)
   i. $7+\ ?=10$
   ii. Story problem example: Deon rode his bike 7 miles to the park. Afterward, he rode further until his GPS showed he had ridded a total of 10 miles. How much further did he ride after he rode to the park?

There are many sub-types of addition problems, but notice that these three main types really depend on where the unknown value is in an equation with a single addition operation, which has three values: two addends and a sum. Similarly, there are three possibilities for the unknown value in an equation with a single subtraction operation: a minuend, a subtrahend, and a difference.

Three types of subtraction exercises:

1. Missing Minuend
   i. $?\ -3=7$
   ii. Story problem example: “Mary has seven bananas, but she started with more than that and then ate three. How many bananas did she start with?”
2. Missing Subtrahend
   i. $10-\ ?=7$
   ii. Example: “Jennifer has ten oranges. She sold some of them and has 7 left. How many did she sell?”
3. Missing Difference
   i. $10-3=\ ?$
   ii. Example: “Cory has ten oranges and three bananas. How many more oranges does Cory have than bananas?”

Perform the operation $354-89$. Do not use a calculator. Then continue reading.

What kind of exercise did you just do? It's likely that when you think of "subtraction", you think of this type of exercise. We now know there are many kinds of situations that involve subtraction, or involve addition that are solved by performing a subtraction operation.

Additionally, we must consider that there are many ways of performing addition and subtraction. How did you subtract 89 from 354? After you have thought about this, find someone else and ask them to do it. Or maybe try to do it a different way. Maybe pretend that you didn't know whatever method you used to perform the subtraction. Then compare the methods.

Methods of Subtraction

Consider the work of nine 2nd graders, all subtracting 354-89, just like you did. Grade each student, as if you were their teacher, using a scale from 1 – 5, 5 being the best. Having the correct answer is only one point out of five. The other four points come from

the students' procedure and thoughts. Remember, even if you do not understand HOW they arrived at their correct answer, that does not make their procedure incorrect.

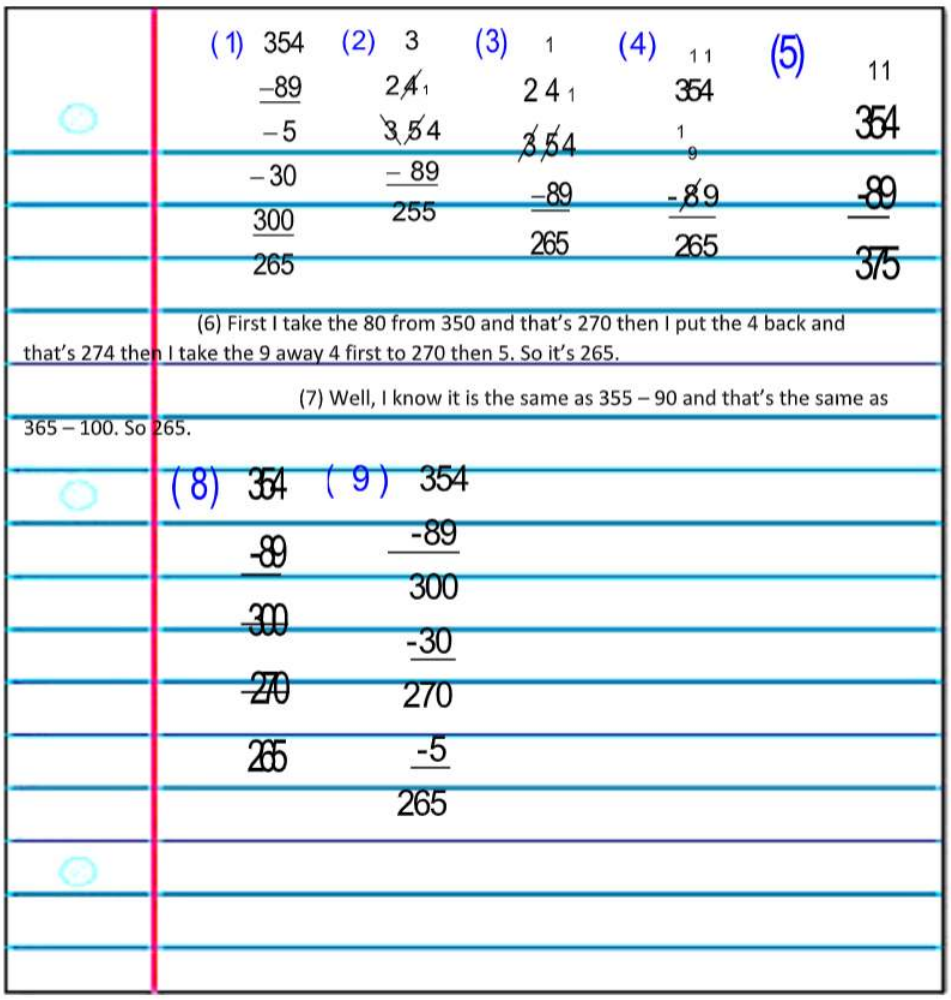


Figure 3.2.2

Now, ask yourself these questions. Were you surprised by any of the children's methods in the work above? Did you figure out how each of them did their work? Did the students who got the wrong answer use a poor method or just make a mistake while using a find method?

Interestingly, all of the children performing $354 - 89$ used methods that involved actually subtracting. They could have used an "adding up" strategy. Consider the following example.

Example 5.1.2

Lance buys some supplies totaling $7.32. He hands the cashier a ten-dollar bill. His change is $2.68. If the cashier counts back his change, how might the cashier do this?

Solution

The cashier will count up in the following say if they use the fewest coins as possible without using a 50¢ coin:

"$7.32 + $1 + $1 + 25¢ + 25¢ + 10¢ + 5¢ + 1¢ + 1¢ + 1¢ = $10.00"

Even if we are only dealing with a subtraction exercise without a context like money, we can still use an adding up strategy.

Example 5.1.3

Subtract 342 – 186 using a number line and adding up.

Solution

One way to do this is to start at 186, add 4 to get to 190, add 10 to get to 200, add 100 to get to 300, and add 42 to get to 342. Since the total added up is 156, we know $186 + 156 = 342$. See the number line below.

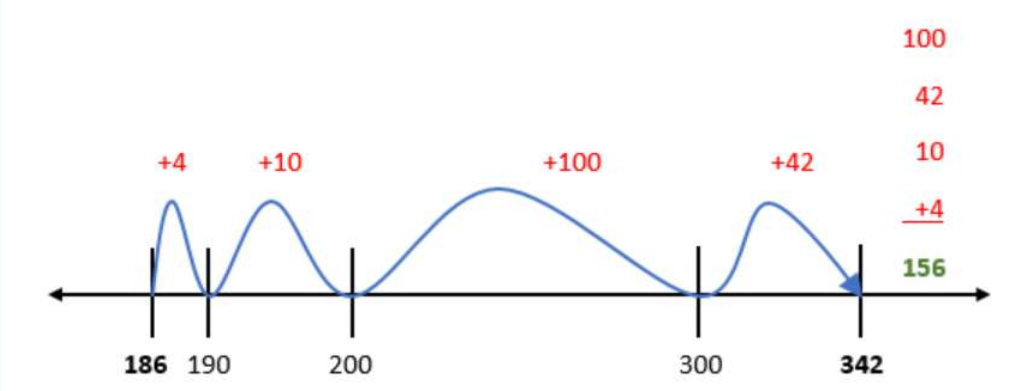


Figure 3.2.3

This means $342 - 186 = 156$.

Notice that one advantage to using an adding up strategy is that you do not have to worry about whether a digit of the subtrahend is smaller than a digit of the minuend. In terms of the standard way children are taught an algorithm for subtracting in the United States, this means there is no need to "borrow".

Practice Problems

Perform the following subtractions by adding up.

1. 753 – 345 = _____
2. 421 – 175 = _____

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5.2: Combining

Combining with Exchanges

Let's pretend we have 20 of each of the following: pennies, dimes, one dollar bills, ten dollar bills, hundred dollar bills, and thousand dollar bills. Instead of drawing pictures of the money, these abbreviations for each kind of coin or bill will be used: penny (A), dime (B), one dollar bill (C), ten dollar bill (D), hundred dollar bill (E), and thousand dollar bill (F).

Before we combine amounts, let's look at how we can do exchanges.

Example 5.2.1

Suppose you had the following amount of currency: AABCCCCCDDDDDDDDDDDDDE. Make any possible exchanges to have fewer pieces of currency.

Solution

Since 10 ten dollar bills forms $100, DDDDDDDDDD = E. So, AABCCCCCDDDDDDDDDDDDDE = AABCCCCCDDDEE.

Now try the following exercise.

Exercise 1

Show what exchanges could be made for each of the following:

a. AAAAAAAAAA = ____	b. CCCCCCCCCC = ____	c. FFFFFFFFFF = ____

At this point, we are better equipped to start combining two or more amounts of money.

Example 5.2.1

In one pile, you have BBCCCCCCCCCCCCCDDDEEEEE. In another pile, you have AAAABCDDDDDDDDDF. What do you have in total? Perform any exchanges possible to have fewer pieces of currency.

Solution

We will proceed by literally combining them (in terms of the letters, this is called concatenation), rearranging the letters to make it more clear about how much we have of each type of currency, and then making the exchanges.

BBCCCCCCCCCCCCCDDDEEEEE + AAAABCDDDDDDDDDF = BBCCCCCCCCCCCCCDDDEEEEEAAAABCDDDDDDDDDF

= AAAABBBCCCCCCCCCCCCCDDDDDDDDDDDDEEEEEF = AAAABBBCCCCDDDDDDDDDDDDEEEEEF

= AAAABBBCCCCDDDEEEEEEF

Now try the following exercise.

Exercise 2

Form each sum by combining the two addends into one lump. Make exchanges as necessary so that the sum is represented by the least number of coins or bills. Show all steps, indicating any exchanges made.

a. CCAADDDCB + BBBBBBBBBAAACC
b. AAAACCCCCCCCCEEE + CCCCCCBBA
c. EEEEEEEEAAAAAAAA + EEEAA
d. DDDDCCCCCBBBBB + DDDDDCCCCBBBBB

To add in the Maya system, the first step is to combine the symbols at each level together. *The symbols at each level must be combined separately.* Try to explain to yourself or another person why this is. Then, at each level, look to see if any exchanges can be made.

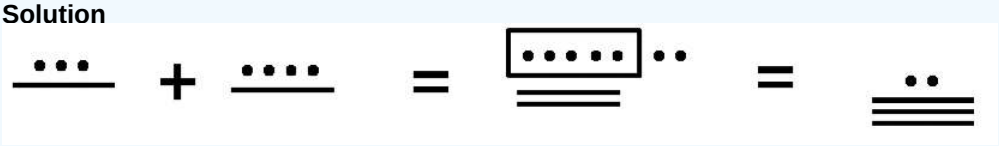
- A group of five dots can always be exchanged for a bar (also referred to as a line segment).
- Exchanges from one level to the next are a little trickier. A group of 20 at one level can be traded for one dot at the next level up *except* from the second to the third level *where a group of **18** at the second level can be traded for one dot at the third level*.

Study the following examples of one level Maya numerals being added together. Any time an exchange is being made, a circle or box will be put around what will be replaced at the next step. An arrow will be used to indicate if an exchange is being made from one level to the next level up. The arrow indicates a dot will be on the next level up in the next step in place of the amount boxed at the original level.

Example 5.2.1

Use Maya numerals to add 8 + 9.

Solution

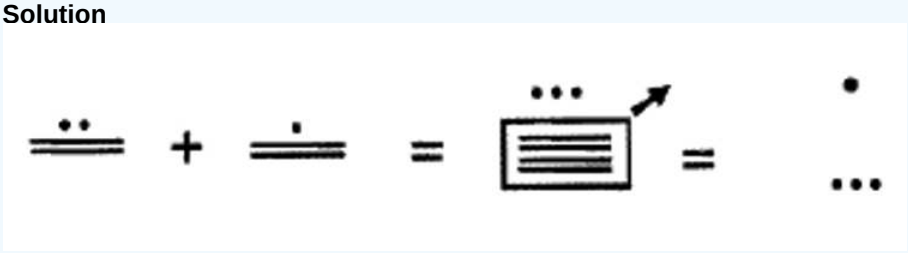


In the previous example, we only needed to exchange 5 dots for 1 bar. Let's see an example in which we will have to "carry" from one level to the next.

Example 5.2.1

Use Maya numerals to add 12 + 11.

Solution



Let's now see an example in which both kinds of exchanges are made.

Example 5.2.1

Use Maya numerals to add 14 + 9.

Solution



The previous example shows two exchanges happen separately, but we could have exchanged 20 in the form of 4 bars and 5 dots all at once. You'll see this in the next example.

Example 5.2.1

Use Maya numerals to add 17 + 19.

Solution



Now, try some on your own.

Exercise 3

Add the following single-level Maya numerals together. Make any necessary exchanges and show all steps.

a.



b.



c.

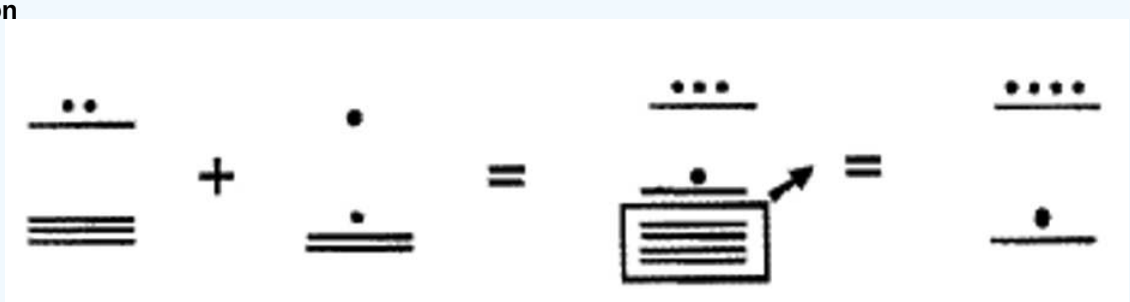


All of the previous examples and exercises involving Maya numerals only used one level each for the addends. Things get tougher when we add more levels both because there are more levels to deal with and we have to look for a new kind of exchange: 18 in the second level can be exchanged for 1 in the third level.

Example 5.2.1

Write 155 and 31 with Maya numerals. Then add 155+31 using Maya numerals. Make any necessary exchanges so that the final value is a properly-written Maya numeral.

Solution

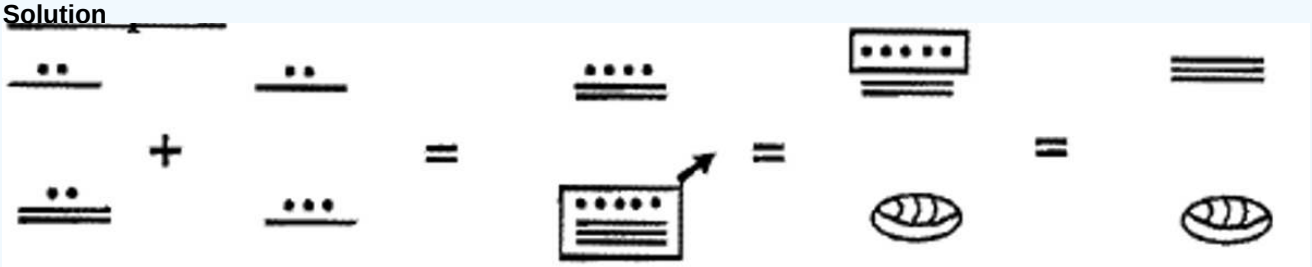


So far, we have not need to use the Maya symbol for zero. Let's see an example of that.

Example 5.2.1

Write 152 and 148 with Maya numerals. Then add 152+148 using Maya numerals. Make any necessary exchanges so that the final value is a properly-written Maya numeral.

Solution



Let's now see what exchanging looks like from the second level to the third level.

Example 5.2.1

Write 210 and 202 with Maya numerals. Then add 210+202 using Maya numerals. Make any necessary exchanges so that the final value is a properly-written Maya numeral.

Solution

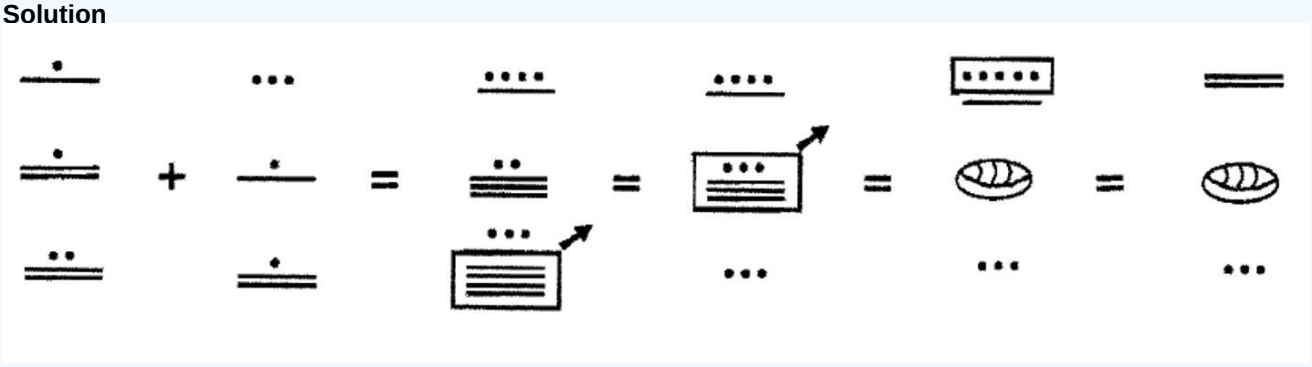


Remember, a group of 18 at the second level can be exchanged for a dot at the third level! Let's now see an example in which we need to use the zero symbol *and* we need to make exchanges at multiple levels.

Example 5.2.1

Write 2392 and 1211 with Maya numerals. Then add 155+31 using Maya numerals. Make any necessary exchanges so that the final value is a properly-written Maya numeral.

Solution



In the next example, notice that something that may look a little strange happens. Instead of exchanging 5 dots for 1 bar, we exchange 1 bar for 5 dots. This is because at that level (the second level), the amount we need to exchange to move up to the next level is 18, not 20. So, we need to be able to see 18. Since 18 is 3 bars and 3 dots, we need to exchange a bar for dots in order to get a group of 18.

Example 5.2.1

Write 76896 and 46754 with Maya numerals. Then add 76896+46754 using Maya numerals. Make any necessary exchanges so that the final value is a properly-written Maya numeral.

Solution

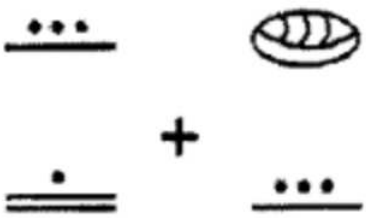


After doing each of the above examples solely in Maya, I checked each problem by adding in the Hindu-Arabic (base ten) and converting the sum into Maya. You can also leave that sum in Hindu-Arabic and convert your Maya sum into Hindu-Arabic to compare those two numbers. When I checked the previous example (76,896 + 46,754 = 123,650), a fairly complicated problem, the answers didn't match. So I reworked the problem until I got it right. It's a good idea to check your work carefully as you go along which requires strong doses of concentration, effort and patience. Otherwise, it's quite easy to make one or more mistakes during computation. Keep in mind that if all you do is convert to Hindu-Arabic, add and convert back to Maya, you won't get the benefit of these examples and exercises. The purpose is to practice the combining, exchanging, etc. that you are required to show when you do the exercises. Try each of the following exercises.

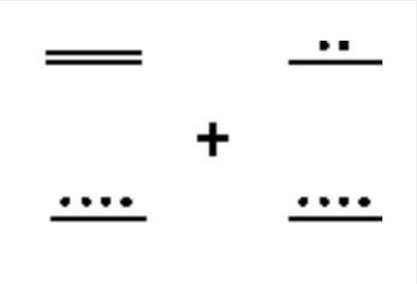
Exercise 4

Add the following Maya numerals together. Make any necessary exchanges and show all steps. It's a very good idea to check your work and that doesn't mean looking up the answer in the solutions! After adding in Maya, convert the answer to Hindu-Arabic. Then, convert the original two numbers to Hindu-Arabic and add together and see if it matches the answer.

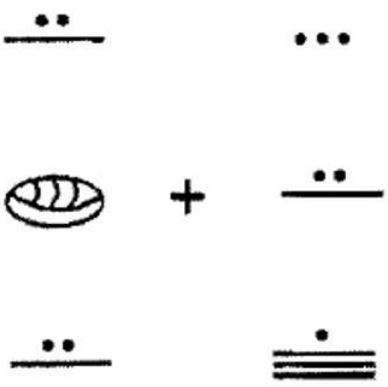
a.



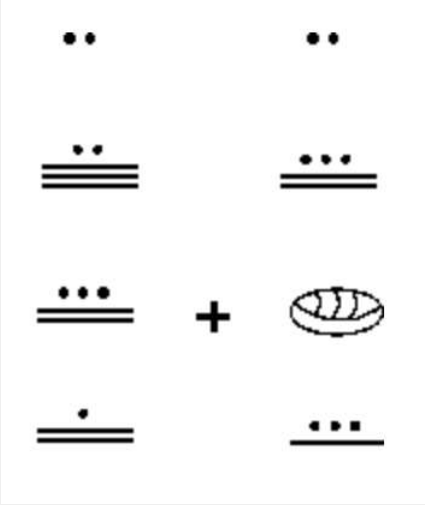
b.



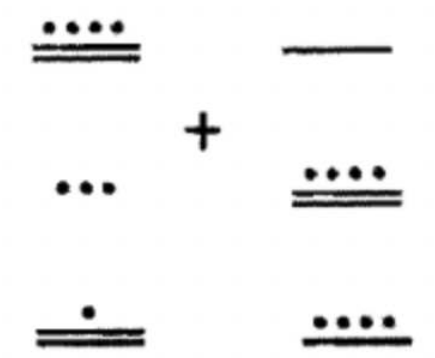
c.



d.



e.



Combining in a Classroom

In a classroom, you won't be using Maya numerals. Instead, you'll be adding Hindu-Arabic numerals, often represented with base ten blocks. As in many cases when learning about elementary school teaching, we will make things harder than just adding base ten numerals (which should be easy for you at this time). In the following example, you will consider a situation in which you have used base four multibase blocks.

Exercise 5

Count out 15 units and make exchanges with base four longs and/or flats. Put them in a pile (or draw them in a group), called Pile A. Set this aside. Since you've made exchanges in base four, write 15 as a base four numeral in the space provided for Pile A below. Now, count out 13 more units and make exchanges with base four longs and/or flats. Put them in a pile (or draw them in a group), called Pile B. Write 13 as a base four numeral in the space provided for Pile B below. To add, simply combine the blocks from Pile A and Pile B to form one big pile (or drawing) of blocks. You are forming the union of Pile A and Pile B! Make all possible exchanges with base four units and longs for longs and flats. Then write the sum as a base four numeral in the space provided for the combined pile.

Combining the two piles is the same as forming the sum of the two numerals. You have just added two numbers together in base four.

Below is the addition problem you have just performed in base four.

_____ [Pile A] + _____ [Pile B] = _____ [Combined Pile]

Let's redo the exercise but in base seven.

Exercise 6

Count out 15 unit blocks and make exchanges with base seven longs and/or flats. Put them in a pile (or draw them in a group), called Pile A. Set this aside. Since you've made exchanges in base seven, write 15 as a base seven numeral in the space provided for Pile A below. Now, count out 13 more units and make exchanges with base seven longs and/or flats. Put them in a pile (or draw them in a group), called Pile B. Write 13 as a base seven numeral on the space provided for Pile B below. To add, combine the blocks from Pile A and Pile B to form one big pile (or drawing) of blocks. Make all possible exchanges with base seven units and longs for longs and flats. Then write the sum as a base seven numeral in the space provided for the combined pile.

Combining the two piles is the same as forming the sum of the two numerals. You have just added two numbers together in base seven.

Below is the addition problem you have just performed in base seven.

_____ [Pile A] + _____ [Pile B] = _____ [Combined Pile]

Let's go to a smaller base. Note that this time, we need more place values (and more versions of multibase blocks).

? Exercise 7

Count out 15 unit blocks and make exchanges with base two longs, flats, etc. Put them in a pile (or draw them in a group), called Pile A. Set this aside. Since you've made exchanges in base two, write 15 as a base two numeral in the space provided for Pile A below. Now, count out 13 more units and make exchanges with base two longs, flats, etc. Put them in a pile (or draw them in a group), called Pile B. Write 13 as a base two numeral on the space provided for Pile B below. To add, combine the blocks from Pile A and Pile B to form one big pile (or drawing) of blocks. Make all possible exchanges with base two units, longs, etc. for longs, flats, etc. Then write the sum as a base two numeral in the space provided for the combined pile.

Combining the two piles is the same as forming the sum of the two numerals. You have just added two numbers together in base two.

Below is the addition problem you have just performed in base two.

_____ [Pile A] + _____ [Pile B] = _____ [Combined Pile]

Write the three addition problems from exercises 5, 6, and 7 in a vertical format like the one in base six shown below. Study this base six problem. Try to figure out a way to do the addition problems using paper and pencil instead of by using the multibase blocks. In other words, try to come up with your own algorithm (method) for doing addition in other bases. Explain your method and show a few examples.

$$\begin{array}{r} 435_{\text{six}} \\ +\ 314_{\text{six}} \\ \hline 1153_{\text{six}} \end{array}$$

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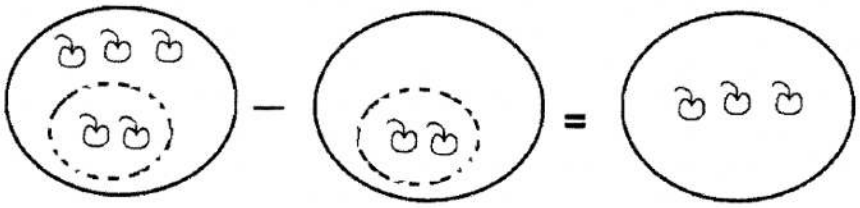
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5.3: Subtraction

Taking Away

How would you explain subtraction? How would you use manipulatives to explain how to *do* subtraction to a young child?

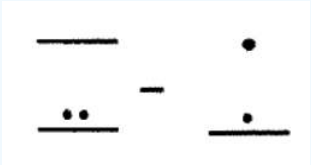
There are three major categories of situations that call for subtraction: taking away (or taking from), taking apart, and comparing. Most of us are default to thinking of subtraction as *taking away*. This is because it is associated with an easily acted out or imagined activity. A typical way someone might introduce the idea of subtraction is by saying "If I put five apples in the bowl and then *take away* two of the apples, how many are left in the bowl?" The illustration below shows this as a subtraction problem where after two apples are removed from the bowl, there are three apples remaining.



Let's use the action of taking away to subtract Maya numerals. It's similar to addition in that you remove a subset *at each level* – exchanges or trades might have to be done before being able to take away at any given level. Pay close attention to which level you are on when making exchanges. A dot at one level can be traded in for a group of 20 at the next level down except from level three to level two – *one dot at the third level is replaced by a group of 18 at the second level*. Study the following examples. See if you can figure out which exchanges are being made. I will indicate exchanges which are being made from one level to the other with a downward arrow.

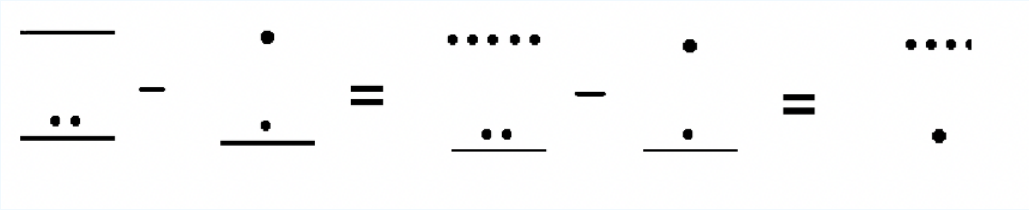
Example 5.3.1

Perform the subtraction below.



Solution

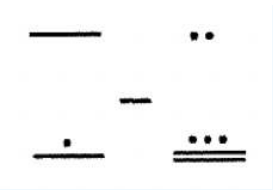
For this difference, in order to take one dot in the second level of subtrahend from the bar in the second level of the minuend, we must exchange the bar for five dots.



Let's try another example, but this time we will need to do the equivalent of what is often called "borrowing" in the standard American subtraction algorithm.

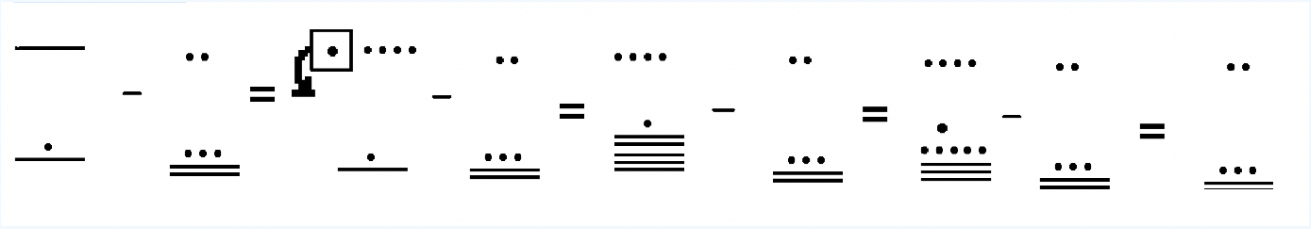
Example 5.3.1

Perform the subtraction below.



Solution

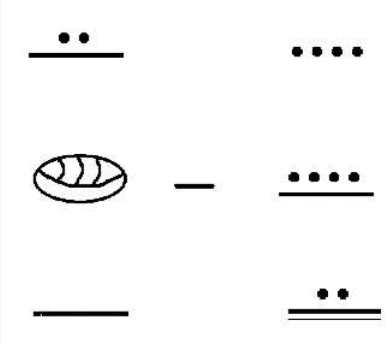
The bar in the second level of the minuend must be exchanged for five dots for two reasons. First, we need dots in order to take the two dots in the second level of the subtrahend away from it. Second, we need to exchange one of those second level dots for a 20 units (in the work below, this is written with four bars) in the first level.



In the previous two examples, note that as exchanges are being made in the minuend (so that you can use the take-away approach), the subtrahend just keeps getting repeated. Alternately, you can first show the exchanges being made for the minuend, make a break (like a dotted line as shown below) and then write the subtraction problem out and subtract.

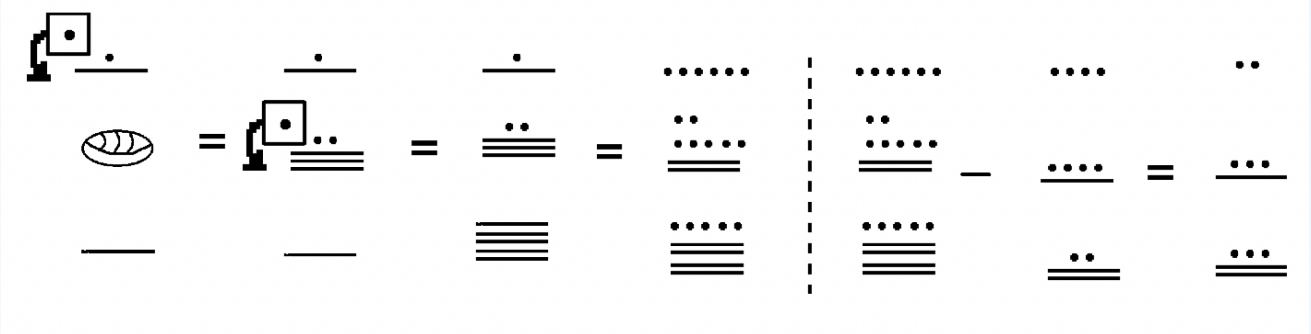
Example 5.3.1

Perform the subtraction below. Note that the values here have three levels, so we must be careful when exchanging between the second and third levels.



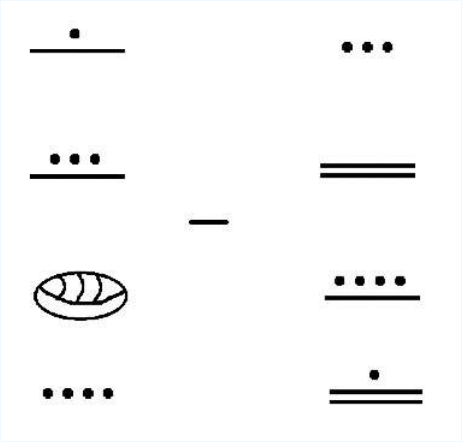
Solution

In this example, we organize our work in the alternative way described above. We work just with the minuend first. Once we have enough dots and bars in each level of the minuend to take the dots in bars of each level of the subtrahend away, we proceed with the subtraction.



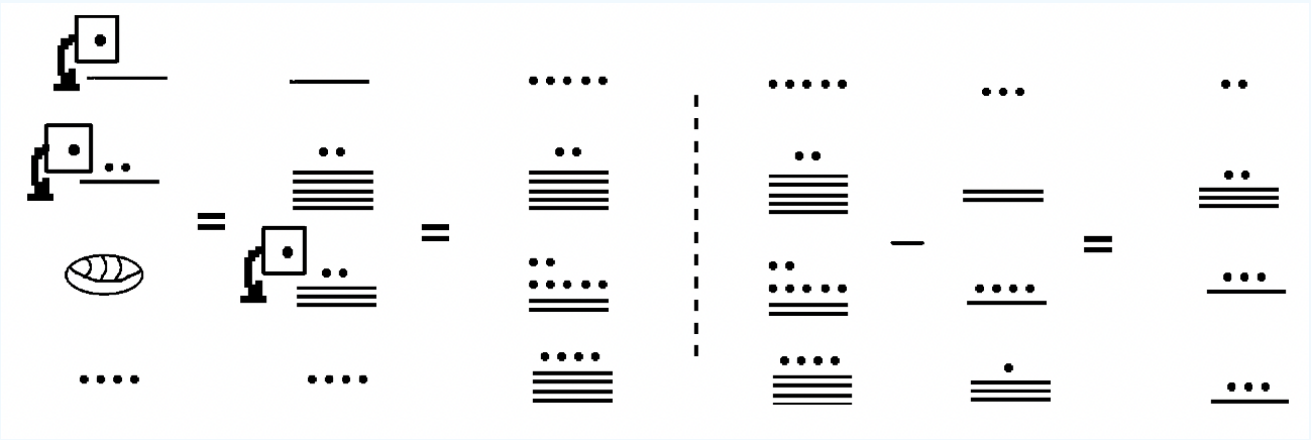
Let's try a difference of two values with four levels.

Example 5.3.1



Solution

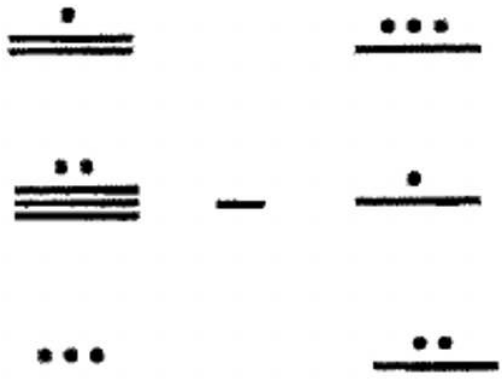
Once again, we must be careful to exchange one unit in the third level for 18 units in the second level.



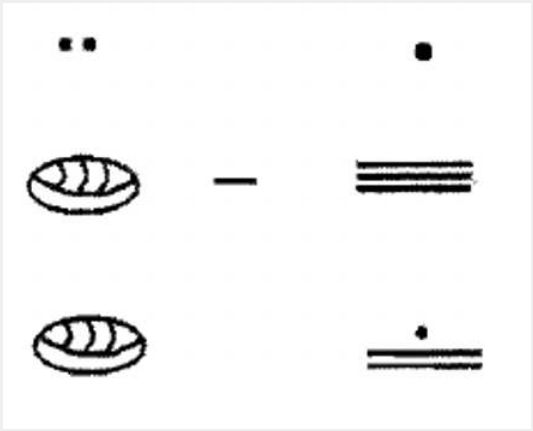
It takes a lot of concentration and effort to work these correctly. Perform each step carefully because it is easy to make mistakes. Each of the previous four examples can be checked by adding the answer to the subtrahend and seeing if the sum is the minuend. The other way to check is convert the minuend and subtrahend to Hindu-Arabic, subtract in Hindu-Arabic, then convert the difference to a Maya numeral. Try doing both forms of checking your answers. If one of them doesn't check out, see if you can find your error(s).

? Example 5.3.1

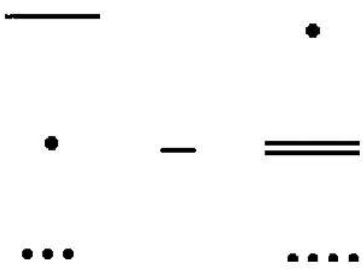
Perform the following subtraction problems using only Maya numerals. Show all steps and check!



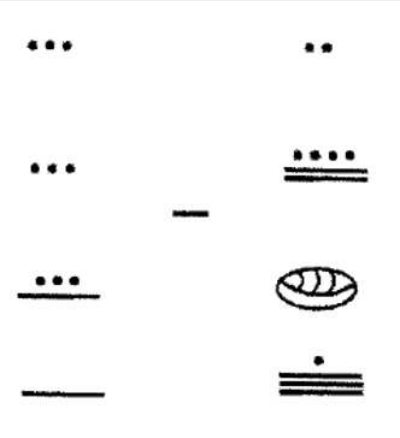
a.



b.



c.



d.

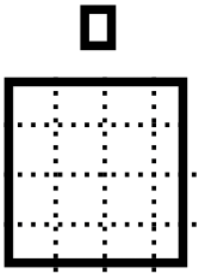
Children living in the U.S. these days will not be writing numbers in Maya, so we use manipulatives that reflect the Hindu-Arabic numeration system we use. To help you understand the difficulty a child just learning subtraction might have, we will work in base four. This means our manipulatives will be base four blocks.

Example 5.3.1

Subtract $101_{\text{four}} - 13_{\text{four}}$ using base four blocks.

Solution

We will make a pile, called Pile A, using the base four blocks to represent 101_{four} and another pile, called Pile B, to represent 13_{four}. Your piles should look something like what you see here.



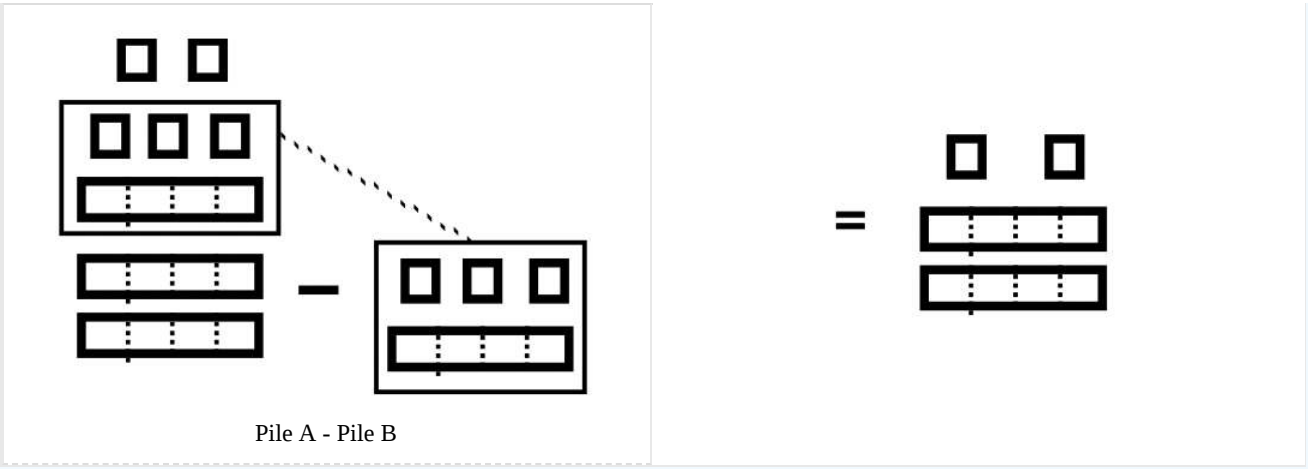
Pile A



Pile B

Below is an illustration of the subtraction problem being performed by taking away the blocks in the subtrahend from the box in the minuend. But to do this, a copy of Pile B must be a contained Pile A.

First, the flat in Pile A must be exchanged for four longs, and one of those longs must be exchanged for four units.There is a box around the subset (Pile B) to be removed from Pile A note how it matches Pile B. After taking those blocks away, 2 longs and 2 units remain. Therefore, the completed subtraction problem is $101_{\text{four}} - 13_{\text{four}} = 22_{\text{four}}$.



Pile A - Pile B

Let's check our answer using the numeration system we are more familiar with. We will write base ten numerals without the "ten" subscript to keep things less cluttered. First, we convert 101_{four} to a Hindu-Arabic numeral.

$$101_{\text{four}} = 1 \cdot 4^2 + 1 \cdot 4^0 = 16 + 1 = 17$$

We do the same for 13_{four}.

$$13_{\text{four}} = 1 \cdot 4^1 + 3 \cdot 4^0 = 4 + 3 = 7$$

So, $101_{\text{four}} - 13_{\text{four}} = 17 - 7 = 10$. In the example, we got 22_{four}. So, we convert that too.

$$22_{\text{four}} = 2 \cdot 4^1 + 2 \cdot 4^0 = 8 + 2 = 10$$

It checks out!

Now it is your turn to do some "taking away" in another base.

? Exercise 1

Follow the directions below to subtract $17 - 9$ in base five. In order for you to get practice converting to other bases, you must first convert 17 and 9 to base five.

Count out 17 unit blocks and make exchanges with base five blocks. Put them in a pile, called Pile A, and save this pile. Since you've made exchanges in base five, write 17 as a base five numeral on the space provided for Pile A below. Now, do the same for 9 to form Pile B and write 9 as a base five numeral on the space provided for Pile B below.

If necessary, make exchanges within Pile A so that an exact copy of Pile B is seen in Pile A. Take away those blocks in Pile A which represent the blocks in Pile B. The blocks left in Pile A are those that are left after subtracting the blocks that were in Pile B. If necessary, make any exchanges with base five blocks and then write this number as a base five numeral on the third blank provided below.

Below is the subtraction problem you have just performed in base five :

_____[Pile A] + _____[Pile B] = _____[Difference of Pile A and B]

Convert the difference (answer) to base ten. It should be 8 since 17 – 9 = 8. Is it?

Let's try another one in a different base.

? Exercise 2

Follow the directions below to subtract $21 - 6$ in base eight.

Count out 21 unit blocks and make exchanges with base eight blocks. Put them in a pile, called Pile A, and save this pile. Since you've made exchanges in base eight, write 21 as a base eight numeral on the space provided for Pile A below. Now, do the same for 6 to form Pile B and write 6 as a base eight numeral on the space provided for Pile B below.

If necessary, make exchanges within Pile A so that an exact copy of Pile B is seen in Pile A. Take away those blocks in Pile A which represent the blocks in Pile B. The blocks left in Pile A are those that are left after subtracting the blocks that were in Pile B. If necessary, make any exchanges with base eight blocks and then write this number as a base eight numeral on the third blank provided below.

Below is the subtraction problem you have just performed in base eight :

_____[Pile A] + _____[Pile B] = _____[Difference of Pile A and B]

Convert the difference (answer) to base ten. It should be 15 since 21 – 6 = 15. Is it?

Let's do one more in a smaller base.

? Exercise 3

Follow the directions below to subtract $21 - 7$ in base three.

Count out 21 unit blocks and make exchanges with base three blocks. Put them in a pile, called Pile A, and save this pile. Since you've made exchanges in base eight, write 21 as a base three numeral on the space provided for Pile A below. Now, do the same for 7 to form Pile B and write 7 as a base three numeral on the space provided for Pile B below.

If necessary, make exchanges within Pile A so that an exact copy of Pile B is seen in Pile A. Take away those blocks in Pile A which represent the blocks in Pile B. The blocks left in Pile A are those that are left after subtracting the blocks that were in Pile B. If necessary, make any exchanges with base three blocks and then write this number as a base eight numeral on the third blank provided below.

Below is the subtraction problem you have just performed in base three :

_____[Pile A] + _____[Pile B] = _____[Difference of Pile A and B]

Convert the difference (answer) to base ten. It should be 14 since 21 – 7 = 14. Is it?

? Exercise 4

Write the three subtraction problems from the previous exercises in a vertical format like the one in base four shown. Study this base four problem as well as the three problems you write down. Try to figure out a way to do the subtraction problems using paper and pencil instead of by using the multibase blocks. In other words, try to come up with your own algorithm (method) for doing subtraction in other bases. Explain your method and show a few examples. Later, you will be learning algorithms for subtraction.

$$\begin{array}{r} 101_{\text{four}} \\ -13_{\text{four}} \\ \hline 22_{\text{four}} \end{array}$$

Think Deeper

Closure

It is important that you are able to think deeply about elementary school level mathematics. This is both because it will help you better understand it and because children will likely ask you some interesting questions in the future. Consider the next question.

Is there a natural number (positive integer) answer for every natural number subtraction problem? In other words, if you subtract a natural number from another natural number, will the difference also be a natural number? Try to answer this and explain your answer. If the answer is no, try to find a counterexample.

An equivalent way to ask that question is, "Is subtraction of natural numbers closed?" The word "closed" in this case means that you stay within that set of numbers. If you answered correctly, your answer was "no". To find a counterexample, we only need to choose a minuend and subtrahend such that the minuend is smaller than the subtrahend. For example, 5 – 7 = -2. Both 5 and 7 are natural numbers, but -2 is not. So, subtraction of natural numbers is not closed. What if we changed "natural numbers" to

"integers". Think about this and once you have an answer, see if you can check your answer by talking with a classmate or searching the internet.

Missing Addends

If you think of subtraction in terms of missing addends, then we say that the statements $a-b=c$ and $a=b+c$ are equivalent to each other. Consider the statement, 8 – 2 = 6. It is equivalent to the statement 8 = 2 + 6. We also know that 2 + 6 = 6 + 2 because of what is called the commutative property of addition. Therefore, 8 = 6 + 2, which in turn is equivalent to the statement 8 – 6 = 2. This gives us four facts about how to relate the numbers 2, 6 and 8 using addition and subtraction

8 – 2 = 6
8 = 2 + 6
8 – 6 = 2
8 = 6 + 2

Some teachers relate subtraction and addition by using the idea of "fact families" such as the four facts above, which is one fact family.

? Exercise 5

Write down two subtraction statements that are equivalent to each addition statement.

a. 9 + 5 = 14	b. 2 + 21 = 23

We can use the equivalency of subtraction and addition equations to form a new approach for solving subtraction problems.

✔ Example 5.3.1

Subtract $18-14$ using a missing addend approach.

Solution

Here, we will use a ? to indicate the value we are trying to find.

$18-14=?$ is the same as $14+?=18$. Most adults quickly see that $14+4=18$. But even if you were a child just learning addition, you could count on your fingers to find that fact. So, $18-14=4$

Try the next subtraction problems.

? Exercise 6

Perform the following subtraction problems by employing the missing addend approach. Fill in the blanks

a. 9 – 7 = ____ because _____
b. 13 – 6 = ____ because _____
c. 88 – 8 = ____ because _____
d. 70 – 14 = ____ because _____

As before, let's put ourselves in the position of children by trying to do the same thing in different bases.

✔ Example 5.3.1

Calculate $143_{\text{six}}-51_{\text{six}}$ using a missing addend approach.

Solution

First, $143_{\text{six}} - 51_{\text{six}} = ?$ is equivalent to $51_{\text{six}} + ? = 143_{\text{six}}$. Let's do this a little bit at a time.

$$51_{\text{six}} + 2_{\text{six}} = 53_{\text{six}}$$

We now have the digit we want to have in the ones place. Now we need to get a 4 in the sixes place. That's a little difficult to do, so let's first just try for getting 0. To do this, we note that in base six, since 5 + 1 = 6, $5_{\text{six}} + 1_{\text{six}} = 10_{\text{six}}$. So, we can now get that 0.

$$53_{\text{six}} + 10_{\text{six}} = 103_{\text{six}}$$

Notice that we got the 1 in the thirty-sixes place already! So, we truly only need to get the 4 in the sixes place now. In any base, adding something to 0 is easy.

$$103_{\text{six}} + 40_{\text{six}} = 143_{\text{six}}$$

So, combining our addends, we have $2_{\text{six}} + 10_{\text{six}} + 40_{\text{six}} = 52_{\text{six}}$. This means $51_{\text{six}} + 52_{\text{six}} = 143_{\text{six}}$, which means we now have our answer.

$$143_{\text{six}} - 51_{\text{six}} = 52_{\text{six}}$$

If you didn't follow that example, try reading it again but using base six blocks to model each step. Keep the numbers being added separate from the 51_{six} so it's clear at the end what you added to the subtrahend to get the 143_{six}.

? Exercise 7

Using multibase blocks and a missing addend approach to perform the following operations.

- $231_{\text{five}} - 140_{\text{five}}$
- $100_{\text{two}} - 11_{\text{two}}$
- $135_{\text{twelve}} - T6_{\text{twelve}}$

The missing addend approach is sometimes called the *Additive (or Austrian) Algorithm.* This approach is very useful for doing subtraction on a number line.

Symmetry

Note that 8 + 4 = 12 is equivalent to 12 = 8 + 4. This is an example of the *Symmetric Property of Equality*.

Definition: Symmetric Property of Equality

The **Symmetric Property of Equality** states that for any equation, if $a = b$, then it is also true that $b = a$.

Therefore, it is perfectly okay to write the four facts in the Missing Addends portion above as the following.

2 + 6 = 8
6 + 2 = 8
8 – 2 = 6
8 – 6 = 2

? Exercise 8

Use the symmetric property of equality to rewrite each equation.

a. 7 – 3 = 4	b. 9 – 1 = 8

We like symmetry. But we cannot assume everything that we like is true. Consider the following.

Exercise 7

Is there a symmetric property of inequality? Explain.

Hint: An example of an inequality is $4 < 9$.

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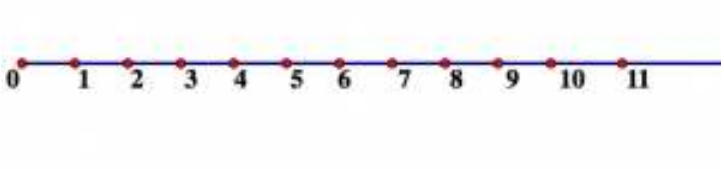
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5.4: Number Line Model

So far, we have focused on quantities that can be counted. For example, the population of humans on Earth at any given time may be known, but we know it's *possible* to count it. But not everything can be counted. Another way we often think about numbers is as abstract quantities that can be measured. For example, the length of a road, the area of a field, and the volume of a lake can all be measured, but not counted.

In a measurement model, you have to pick a *basic unit*. The basic unit is a quantity — such as length, area, or volume — that you assign to the number one. You can then assign numbers to other quantities based on how many of your basic unit fit inside.

For now, we'll focus on length, and we'll work with a number line where the basic unit is already marked off.



A number line continues "forever" in both directions, but we can only draw part of it. In the number line represented above, we see where 0, 1, 2, ..., and 11 are, but all the rest of the real numbers are on the line too. We just either cannot see them (such as -1 or 100) or they are not marked and explicitly labeled (like 1.5 or 12). Number lines are useful for showing relationships between sizes of numbers. For example, in the number line below, we can see a lot of things. We see that $0 < \frac{1}{5} < \frac{2}{5} < \frac{3}{5} < \frac{4}{5} < 1$ because numbers that are shown further to the right on a number line are larger than numbers shown further to the left. Note that this is only a convention. We the line need not be horizontal, and even if it is, we do not *have* to have the larger numbers on the right and smaller numbers on the left, for example. But once there are at least two numbers represented on the number line, we can see how it is oriented. Assuming we are not working in a different scale (for example, a logarithmic scale), we can then find all sorts of other numbers. For example, if I wanted to see where $1\frac{1}{5}$ is, I could just continue the line to the right of 1 using the line segment from 0 to $\frac{1}{5}$ as a rule for how far to go.

We can also see that 1 is five times the size of $\frac{1}{5}$ by noting that five copies of the line segment between 0 and $\frac{1}{5}$ fit between 0 and 1.



Figure 5.4.1: A number line showing fifths between 0 and 1 (CC BY 4.0; Melissa Flora)

But we can do a lot more than just compare values. Number lines can also be used as tools to add and subtract.

Addition and Subtraction on the Number Line

Imagine a person — we'll call him Zed — standing on a number line. We'll say that the distance Zed walks when he takes a step is exactly one unit.



When Zed wants to add or subtract with whole numbers on the number line, he always starts at 0 and faces the positive direction (towards 1). On the number line above, this means he faces to the right. This is usually the positive direction for horizontal number lines. Then what he does depends on the calculation. Before continuing, it is a good idea to make a note of the fact that we are only dealing with nonnegative integers at this time. So, when we write "add two number", we mean "add two nonnegative integers".

If Zed wants to *add* two numbers, he walks forward (to the right of 0 on the number line) however many steps are indicated by the first number (the first addend). Then he walks forward (to your right on the number line) the number of steps indicated by the

second number (the second addend). Where he lands is the sum of the two numbers.

Example 5.4.1

How would Zed add 3+4 on the number line?

Solution:

If Zed wants to add 3 + 4, he starts at 0 and faces towards the positive numbers. He walks forward 3 steps, then he walks forward 4 more steps.

Zed ends at the number 7, so the sum of 3 and 4 is 7. 3 + 4 = 7. (But you knew that of course! The point right now is to make sense of the *number line model*.)



When Zed wants to *subtract* two numbers, he he walks forward (to the right on the number line) however many steps are indicated by the first number (the minuend). Then he walks *backwards* (to the left on the number line) the number of steps indicated by the second number (the subtrahend). Where he lands is the difference of the two numbers.

Example 5.4.1

How would Zed subtract 11 – 3 on the number line?

Solution:

If Zed wants to subtract 11 – 3, he starts at 0 and faces the positive numbers (the right side of the number line). He walks forward 11 steps on the number line, then he walks backwards 3 steps.

Zed ends at the number 8, so the difference of 11 and 3 is 8. 11 – 3 = 8.



It's time for you to try some yourself. Actually draw a number line. Use a ruler to draw one that includes at least the numbers 0, 1, 2, ..., 15.

Think / Pair / Share

- Work out each of these exercises on a number line.
  - 4 + 5
  - 6 + 9
  - 10 – 7
  - 8 – 1
- Why does it make sense to walk forward for addition and walk backwards for subtraction? In what way is this the same as "combining" for addition and "take away" for subtraction"? Once you think you can answer this, try telling a friend and see if they like your answer.
- What happens if you do these subtraction problems on a number line? Explain your answers.
  - 6 – 9
  - 1 – 7
  - 4 – 11
  - 0 – 1

Number Lines for Other Bases

We can use number lines with other bases to do addition and subtraction as well. If we can count in a base, then we can add or subtract. Consider the example below.

Example 5.4.1

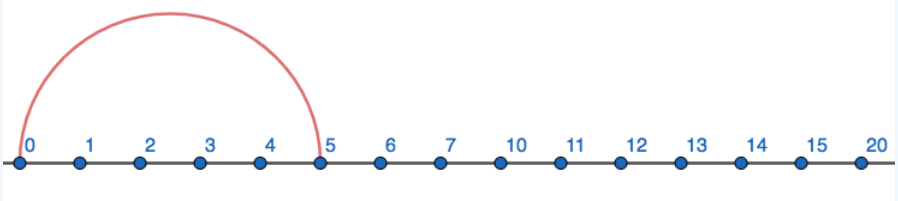
Use a number line to add $5_{\text{eight}} + 6_{\text{eight}}$.

Solution

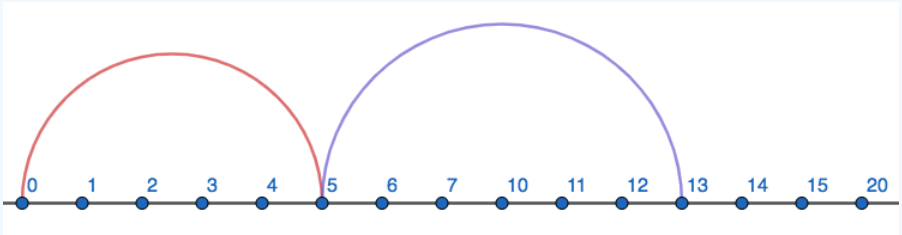
The number line below shows numbers written in base eight. So, 10 on the number line is 10_{eight}, which is equal to 8 in the decimal system.



We begin by pretending to be Zed standing at 0 and moving to the right 5 steps.



Then, we pretend to be Zed adding \(6_{\text{eight\}}\) by taking 6 more steps to the right.



As we can see in the number line above, this means $5_{\text{eight}} + 6_{\text{eight}} = 13_{\text{eight}}$.

Let's try a subtraction problem.

Example 5.4.1

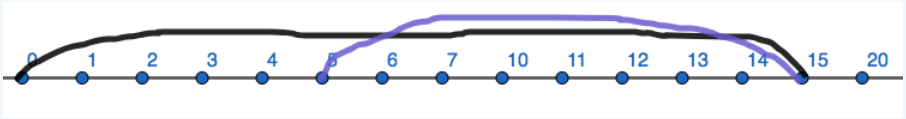
Use a number line to subtract $15_{\text{eight}} - 10_{\text{eight}}$.

Solution

Using the same base eight number line, we start by moving from 0 to 15_{eight}.



Now, since we need to *subtract* 10_{eight} (which is equal to 8 in base ten!), we move to the left 10_{eight} steps.



As we can see, this means $15_{\text{eight}} - 10_{\text{eight}} = 5_{\text{eight}}$.

Open Number Lines

An open number line (also called an empty number line) is a number line that starts with nothing on it, allowing the user to define the unit and pick a starting value as they see fit. They are especially useful when skip-counting.

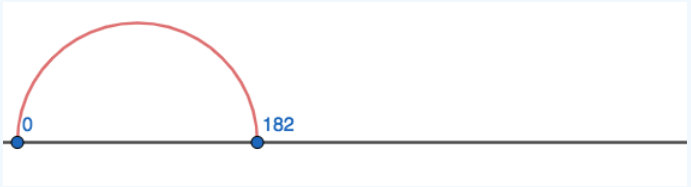
Example 5.4.1

Use an open number line to add $182 + 20$.

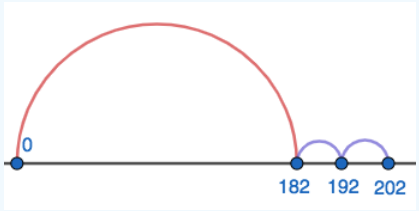
Solution

We begin with just a line.

Since we know the first thing our friend Zed would have to do is take 182 steps from 0 to 182, we do not want to start by labeling 0, 1, 2, 3, etc. In fact, we won't bother labeling anything other than the two numbers we care about right now: 0 and 182. Since we know we will want to move further to the right (because we are adding a positive number to 182), we make sure to leave room to the right of 182.



After we note where Zed would have ended up if he actually took all 182 of his steps, we know that he would take 20 more steps. It would be perfectly find to count out those 20 steps. But, in the effort to take less time counting by ones, we will imagine him taking just 2 big 10-step-long jumps.



We can now see that $182 + 20 = 202$.

The more efficient practice of making large jumps only works if the person using the number line has a good enough number sense. One way to develop such number sense is to use regular number lines and other tools.

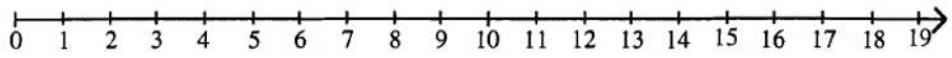
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5.5: Addition and Subtraction

***You will need**: Positive and Negative Counters (Material Cards 18A and 18B)*

Remember that whole numbers are made up of zero and the counting numbers: 0, 1, 2, 3, 4, 5, . . . When school children first see a number line, they usually focus on the whole numbers and the number line looks something like this:

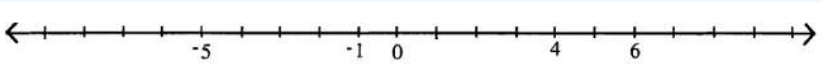


In this model, the number line starts at zero, and the arrow to the right indicates that it goes on indefinitely to the right. Actually, there are infinitely many numbers (or points) between any two numbers on the number line. For example, a few of the infinite number of points between 1 and 2 are 3/2, 4/3, 5/4, 6/5, 7/5, 8/5, etc. On the number line, there are infinitely many irrational numbers as well. Irrational numbers cannot be written as the ratio of two whole numbers, like the way we represent reduced fractions. Examples of some irrational numbers between 3 and 4 are $\sqrt{11}$, $\sqrt{15}$, 3.10110111..., and π. Contrary to popular belief, π *does not* equal 3.14 and π. *does not* equal 22/7. Both 3.14 and 22/7 are rational numbers (unlike π which is an irrational number, NOT a rational number) and 3.14 and 22/7 are only ***approximations*** commonly used when calculating with π. Irrational numbers will be explored and discussed in more detail in a different module.

Okay, let's get back to our discussion of the number line. The number line shown at the beginning of this exercise set is really only a half-line, so to call it a number *line* is really a misnomer. A horizontal line goes on indefinitely in *both* the left and right direction, not just to the right. Numbers to the right of zero are called positive numbers. Numbers to the left of zero are called *negative numbers*. Zero is the only number that has no sign – it is neither positive, nor negative. For each positive number, there is a corresponding negative number, which on a given number line, is the same distance from zero as the positive number, but it is to the left side of zero. Numbers the same distance from but on opposite sides of zero are called "opposites." Negative numbers are represented like this: -5 is read "negative five", -20 is read "negative twenty", etc. So, 5 and -5 are opposites, 20 and -20 are opposites, and zero is the opposite of itself.

? Exercise 1

For each marking on the number line, fill in the missing numbers



Definitions: The set of **positive integers** (which are also commonly called the **natural numbers** or **counting numbers**) is written as: {1, 2, 3, 4, ... }. Positive numbers may also be written with a positive sign before it. For instance, 4 may be written a +4.

The set of **whole numbers** contains both zero and the natural numbers and is written as: {0, 1, 2, 3, ... }.

Definition: The set of negative integers contains exactly all the opposites of all the positive integers.

1. Write the set of **negative integers**:

Definition: The set of **integers** is made up of zero, the positive integers and the negative integers.

2. Write the set of **integers**:

Definition: The **absolute value** of a number on the number line is defined to be the distance that point is from zero. Since absolute value is defined in terms of distance, it can never be negative. To evaluate the absolute value of a number, state the distance it is from zero.

✔ Examples

The absolute value of 5 is 5, since the distance from 5 to 0 is 5 units. The absolute value of -5 is also 5, since the distance from -5 to 0 is also 5 units.

Exercise 3

Evaluate the absolute value of each of the following numbers:

a. 4: ____	b. -8: ____	c. 2: ____	d. -1: ____	e . 0: ____

To indicate the absolute value of a number, we enclose it between two vertical lines. "The absolute value of -15 is 15" is written like this: | -15 | = 15. In order to simplify a problem with an absolute value sign, the part inside the absolute value sign must be simplified first. So, if the part inside the absolute value sign is not simply a numeral, think of the part inside as being in parentheses, and simplify it first. Then, take the absolute value of the numeral. You'll have to do this for part c and f of exercise 4.

Exercise 4

Simplify each of the following:

a. | 7 | = _____		c. | 7 – 3 | = _____
d. $\left\|\frac{3}{7}\right\|$ = _____	e. $\left\|-\frac{3}{7}\right\|$ = _____	f. | 2·5 – 4 | = _____

Exercise 5

Write two numerals that have an absolute value of 6 _____ and _____

Exercise 6

Write two numerals that have an absolute value of 19 _____ and _____

Exercise 7

Write all numerals that have an absolute value of zero _____

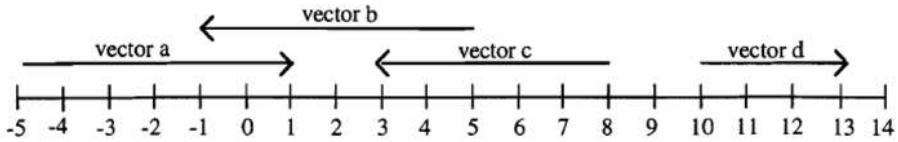
Exercise 8

Write all numerals that have an absolute value of -10 (negative 10) _____

Each positive and negative number can be represented with a **directed line segment** called a **vector**.

Basically, a vector looks like an arrow. It has two properties: it has a certain length (called the "magnitude") and it points in a certain direction. Since we'll be using horizontal number lines, we'll be drawing horizontal vectors. An arrow pointing to the right will denote a positive number and an arrow pointing to the left will denote a negative number.

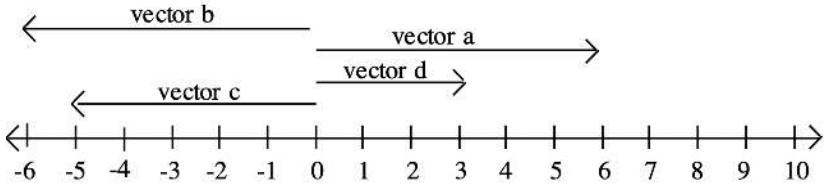
Below is a number line, with some vectors shown above. Vector a is 6 units long and the arrow points to the right. Therefore, it represents the number +6. Vector b is also 6 units long, but it points to the left. Therefore, it represents the number -6. Both vectors have a magnitude of 6.



Exercise 9

a. What number does vector c represent? _____
b. What number does vector d represent? _____

Each vector has an initial point (the starting point of the vector) and a terminal point (the ending point, where the arrow is shown). If the initial point of the vector is placed at zero, the number line can be used as a convenient marker – the terminal point of the vector is simply placed on the number being represented. But it *isn't necessary to* represent it that way, as you could see from all the vectors shown in the above examples. Vectors a, b, c and d could have been drawn with the initial point at zero as shown below. If you draw a vector with its initial point at zero, the number that vector represents is simply the same number on the number line where its terminal point lands.



Another way of thinking about numbers is to think of them in terms of "**actions**". You could think of the number "5" as the action of moving 5 spaces to the right. When you draw the vector representing 5, that is a way of showing that particular action. You could think of the number "-3" as the action of moving 3 spaces to the left. When you draw the vector representing -3, that is a way of showing that particular action.

The number line, along with this idea of thinking about numbers in terms of actions, can be used to add numbers together.

Definition: **To add two numbers, m and n, you DO the first action, m. The addition sign means NOW DO the second action, n. The answer to the problem (which is the sum: m + n) is the single action that could have been done to achieve the same result as doing the separate actions in succession.**

Consider this practical way of thinking about it. Think of walking east as representing a positive number and walking west as representing a negative number. If you walked 5 blocks east, then 7 blocks west, then 3 blocks east, you would be one block east of your original starting point. This can be a way to think about the problem: 5 + (-7) + 3 = 1.

Procedure for using vectors as actions to add two numbers, m and n

1. Pick a starting point and draw the vector m (which is a way of representing the action of m) with its initial point at the designated starting point.
2. The addition sign means starting at the terminal point of the previous action DO the next action. So, draw the vector n, placing the initial point of n at the terminal point of vector m.
3. The sum, m + n, is the same number that would be represented by a vector with its initial point placed on the original starting point and its terminal point on the final terminal point after the last action took place.

For CONVENIENCE only, it is easiest to always use a number line and pick zero as the initial starting point. By doing that, the answer to the problem will be wherever the terminal point of the last action **landed** on the number line.

Before going on, it is important to note that it is not necessary to mark every point on a number line. In fact, this is impossible since there are infinitely many points on a number line. However, it is usually a good idea to mark zero and at least one point on either side of zero. Mark the number line in a way that makes it convenient to do the problem.

We'll use the above procedure to add the following numbers. For these examples and for the exercises, start at zero for convenience. Draw the first vector. Then move up a little to draw the second vector so it is obvious which vector was drawn first and which one was drawn last. There should be one vector shown for each number in the sum. Since we are placing the initial point of the first vector on zero, we can state the answer by noting where the terminal point of the last vector landed on the number line.

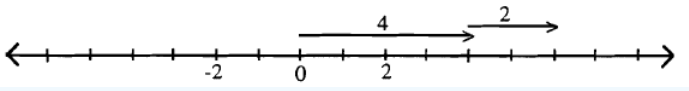
Example: $4 + 2$

4 + 2.

Solution

Starting at zero, DO the action 4 (which is shown by drawing the vector representing 4), then starting at the point where the first action ended, DO the action 2 (shown by drawing the vector representing 2).

The answer is **6** since the terminal point of the last vector landed on 6.



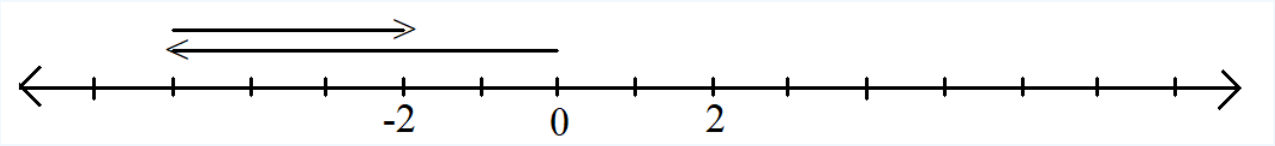
Example: $-5+3$

-5 + 3.

Solution

Starting at zero, DO the action -5 (which is shown by drawing the vector representing -5), then DO the action 3 (shown by drawing the vector representing 3).

The answer is **-2** since the terminal point of the last vector landed on -2.



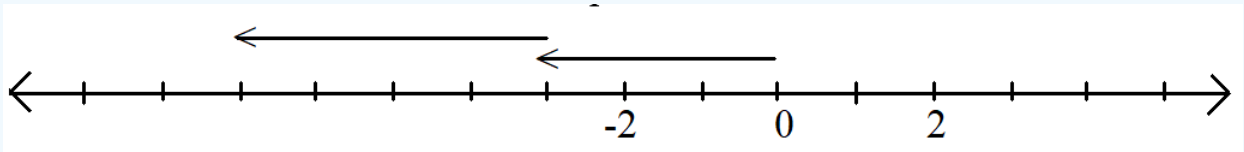
Example: $-3+-4$

-3 + -4.

Solution

Starting at zero, DO the action -3 (shown by drawing the vector representing -3), then DO the action -4 (shown by drawing the vector representing -4).

The answer is **-7** since the terminal point of the last vector landed on -7.



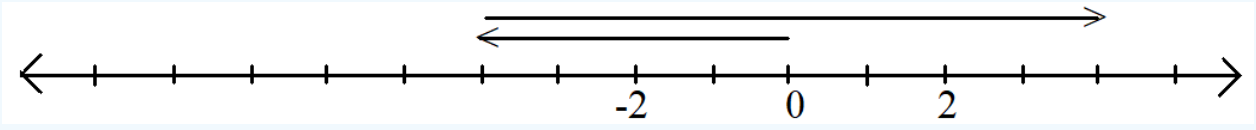
Example: $-4+8$

-4 + 8.

Solution

Starting at zero, DO the action -4 (which is shown by drawing the vector representing -4), then DO the action 8 (shown by drawing the vector representing 8).

The answer is **4** since the terminal point of the last vector landed on 4.

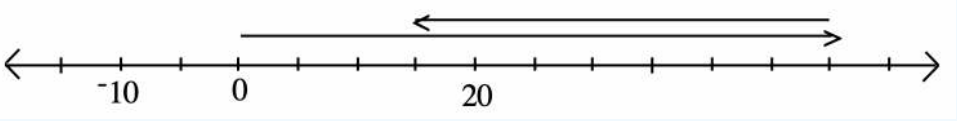


Example: $50+-35$

50 + -35.

Starting at zero, DO the action 50 (shown by drawing the vector representing 50), then DO the action -35 (shown by drawing the vector representing -35).

The answer is **15** since the terminal point of the last vector landed on 15.



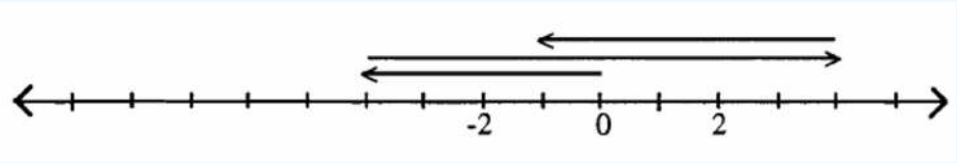
It is IMPERATIVE that you LABEL your number line. Otherwise, it is not clear what each vector represents. Do not assume that each mark stands for 1 unit. Without the label of at least one number on each side of zero on the last example, one might assume the last example shown represented 10 + -7 and that the answer was 3.

? Exercise 10

Use vectors on the number line to add the following numbers together. Mark and LABEL your number line with at least zero and a point on either side.

Here is an example of how to do the following problems:

-4 + 8 + (-5) = -1 since *the terminal point of the last vector landed on -1*.



a. 5 + (-8) = _____ since _____



b. -7 + (-2) = _____ since _____



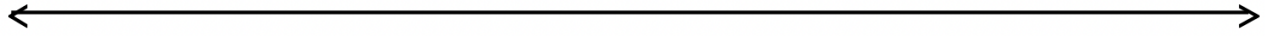
c. -5 + 9 = _____ since _____



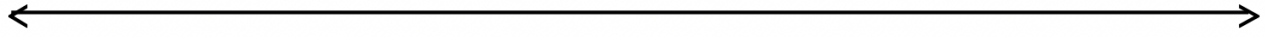
d. 7 + (-6) = _____ since _____



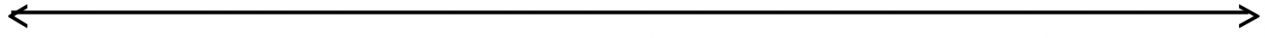
e. -7 + 5 = _____ since _____



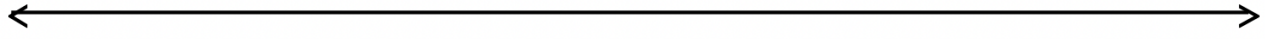
f. -100 + 40 = _____ since _____



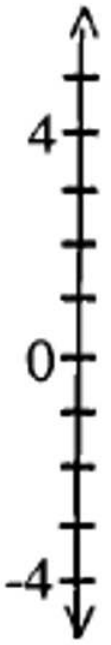
g. -3 + (-2) + 4 = _____ since _____



h. 4 + (-12) + 5 = _____ since _____



Sometimes, you'll see vertical number lines instead of horizontal number lines. In this case, the positive numbers go up, and are represented by vertical vectors with the arrow pointing up. Likewise, negative numbers go down and are represented by vertical vectors pointing down. An example of a vertical number line is shown on the right.



? Exercise 11

There is a common use of using a vertical number line. Can you think of one? See if you can think of any and write any down here.

If you need a hint for Exercise 11, think about the weather.

We'll continue to think about actions as we use vectors to do subtraction problems.

Definition: **To subtract a number, m, you UNDO the action, m which means you DO the OPPOSITE of the action m.**

Whether or not you are doing a subtraction problem, an addition problem, or both in the same problem, the answer to the problem is the single action that could have been done to achieve the same result as doing the separate actions in succession.

You can think of subtraction as undoing some mistake.

Consider a variation on the original example given for addition. To refresh your memory, here was that problem: Think of walking east as representing a positive number and walking west as representing a negative number. If you walked 5 blocks east, then 7 blocks west, then 3 blocks east, you would be one block east of your original starting point. This can be a way to think about the problem: 5 + (-7) + 3 = 1.

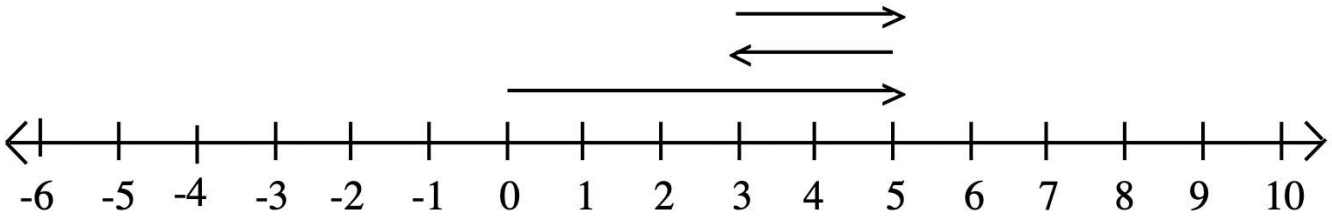
? Exercise 12

Imagine you walked 5 blocks east, then 2 blocks west, and then realized you had made a mistake. You weren't supposed to walk 2 blocks west. You would need to undo that mistake. What would you do to undo that mistake and get back to where you were before you walked 2 blocks west on accident?

In exercise 12, I hope you realized you needed to walk 2 blocks east to undo your mistake. To translate the steps taken in #12, we could write 5 + (-2) – (-2) = 5. Here is how we would explain and show the actions using vectors on the number line.

5 + (-2) – (-2): Starting at zero, DO the action 5 (shown by drawing the vector representing 5), then DO the action -2 (shown by drawing the vector representing -2), then UNDO the action of -2 (which means to DO the OPPOSITE of the action -2, which is to DO the action +2, shown by drawing the vector representing +2).

The answer is **5** since the terminal point of the last vector landed on 5.



Procedure for using vectors to add or subtract two numbers:

1. If the first number is m, pick a starting point and draw vector m (which is a way of representing the action m) with its initial point at the designated starting point.
2. If the next sign is addition, DO the next action by drawing the vector representing the next number, placing the initial point of this vector at the terminal point of the previous vector. If, on the other hand, the next sign is subtraction, UNDO the next action. So, draw the vector representing the opposite of the next number, placing the initial point of this vector at the terminal point of the previous vector.
3. If there is another addition or subtraction sign, go back to step 2. Otherwise, at this point, we can find the answer. The answer is the same number that would be represented by a vector with its initial point placed on the original starting point and its terminal point on the final terminal point after the last action took place.

Again, using zero as the starting point is CONVENIENT because the answer to the problem will be wherever the terminal point of the last action **landed** on the number line.

We'll use the above procedure to compute the following problems. For these examples and for the exercises, start at zero for convenience. Draw the first vector. Then move up a little to draw the second vector so it is obvious which vector was drawn first and which one was drawn last. There should be one vector shown for each number in the problem. Since we are placing the initial point of the first vector on zero, we can state the answer by noting where the terminal point of the last vector landed on the number line.

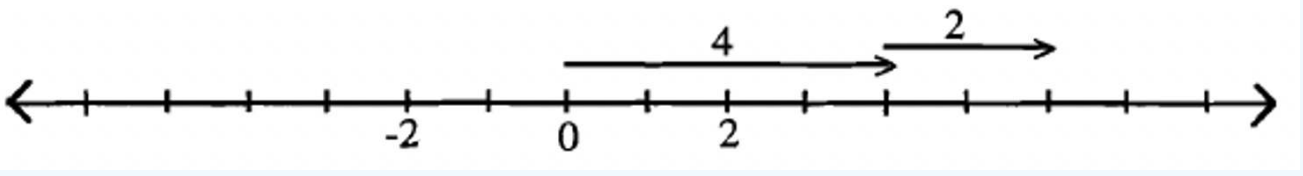
✔ Example

4 – (-2).

Solution

Starting at zero, DO the action 4 (which is shown by drawing the vector representing 4), then UNDO the action -2 (which means to DO the OPPOSITE of the action -2, which is to DO the action +2, shown by drawing the vector representing +2).

The answer is **6** since the terminal point of the last vector landed on 6.



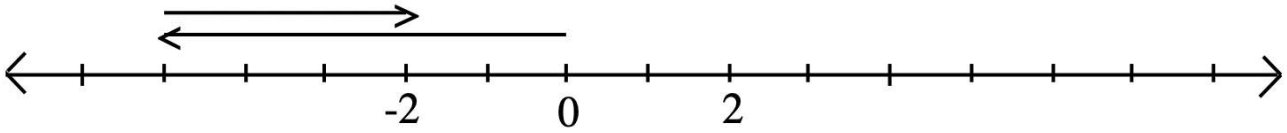
? Example

-5 – (-3).

Solution

Starting at zero, DO the action -5 (shown by drawing the vector representing -5), then UNDO the action -3 (which means to DO the OPPOSITE of the action -3, which is to DO the action +3, shown by drawing the vector representing +3).

The answer is **-2** since the terminal point of the last vector landed on -2.



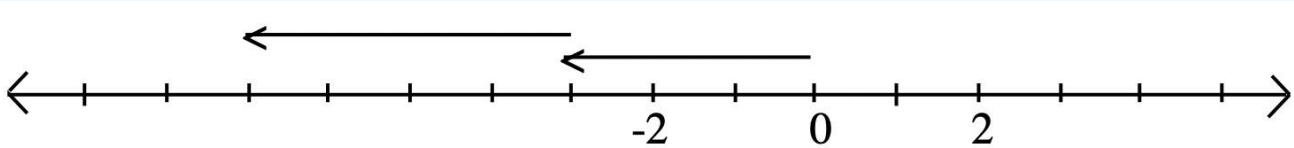
? Example

-3 – (+4).

Solution

Starting at zero, DO the action -3 (shown by drawing the vector representing -3), then UNDO the action +4 (which means to DO the OPPOSITE of the action +4, which is to DO the action -4, shown by drawing the vector representing -4).

The answer is **-7** since the terminal point of the last vector landed on -7.



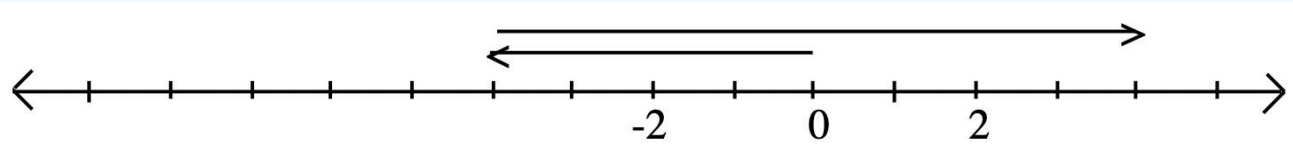
? Example

-4 – (-8).

Solution

Starting at zero, DO the action -4 (shown by drawing the vector representing -4), then UNDO the action -8 (which means to DO the OPPOSITE of the action -8, which is to DO the action +8, shown by drawing the vector representing +8).

The answer is **4** since the terminal point of the last vector landed on 4.



? Example

50 + (-35).

Solution

Starting at zero, DO the action 50 (shown by drawing the vector representing 50), then DO the action -35 (shown by drawing the vector representing -35).

The answer is **15** since the terminal point of the last vector landed on 15.

It is IMPERATIVE that you LABEL your number line. Otherwise, it is not clear what each vector represents. Do not assume that each mark stands for 1 unit. Without the label of at least one number on each side of zero on the last example, one might assume the last example shown represented 10 + (-7) and that the answer was 3.

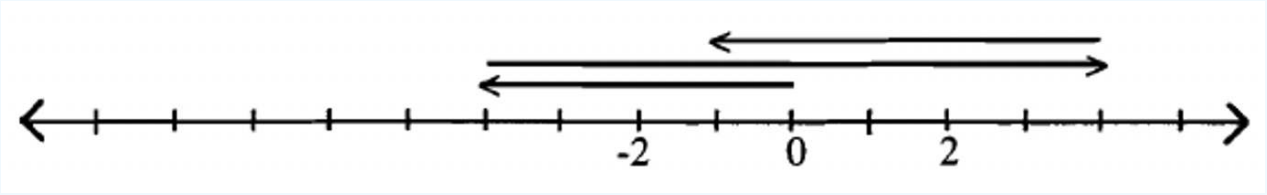
? Exercise 13

Use vectors on the number line to add the following numbers together. Mark and LABEL your number line with at least zero and a point on either side.

Here is an example of how to do the following problems:

? Example

-4 + 8 + (-5) = -1 since *the terminal point of the last vector landed on -1.*



a. 5 – (+8) = _____ since _____ .



b. -7 – (+2) = _____ since _____ .



c. -5 – (-9) = _____ since _____ .



d. 7 – (+6) = _____ since _____ .



e. -7 – (-5) = _____ since _____ .



f. -100 – (-40) = _____ since _____ .



g. -3 – (+2) – (-4) = _____ since _____ .



h. 4 – (+12) + 5 = _____ since _____ .



At this point, it would be beneficial if you looked back at the examples on page 4 and 5 and compared them with the recent examples shown on page 8 and 9. The examples on page 8 and 9 include subtraction, whereas the examples on page 4 and 5 are addition problems. Using the idea of adding and subtracting using vectors and the action approach (either DO or UNDO an action), the problems when done with vectors on the number line look exactly the same. The same is true if you look at the problems in

Exercise 10 and 13. Somehow, then, there seems to a connection between subtraction and addition. We'll put it all together after we explore a very different way of adding and subtracting, this time using manipulatives.

Another way to represent positive and negative numbers is by using a collection of positive and negative counters. You need to use counters of one color to represent positive numbers (each counter = +1) and counters of another color to represent negative numbers. If you use the material cards, each green square will represent +1 and each red square will represent -1. We'll discover that there are many ways to represent the same number using these counters. Get out your counters to do the following exercises.

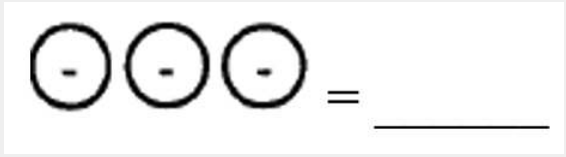
The simplest way to represent a positive number, x, is to have a collection of x green counters. The simplest way to represent a negative number, -y, is to have a collection of y red counters. For example, 4 red counters will represent -4 and 3 green counters will represent +3. In these exercises, I'll sometimes represent a positive counter by either a square or circle with a positive sign (+) or a "G" inside, or by just a "G". Similarly, a negative counter will be represented by either a square or circle with a negative sign (-) or an "R" inside, or by just an "R".

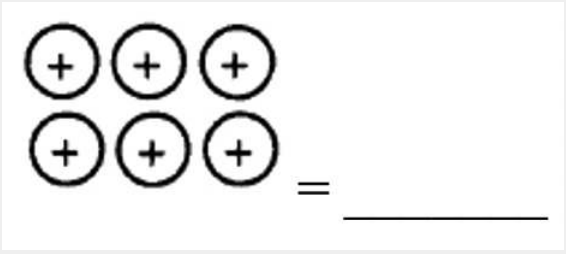
Exercise 14

What number is represented by each of the following collections?

a. 

b. RRRRRRRRR = _____

c. 

d. 

Exercise 15

Under each number, represent the number by showing a collection of counters.

	b. 2	

What number do you think the collection below represents? _____ Explain why.



In exercise 14, there were 5 positives and 5 negatives. Did you come up with an answer of zero? If so, you're on the right track. Basically, when a positive counter is paired with a negative counter, they nullify each other. Any time there are the same number of positive counters as negative counters, you have a representation of the number, zero.

Exercise 17

Using positive and negative counters, show three different representations of zero.

a.	b.	c.

Let's see if you can figure out what number each collection represents.

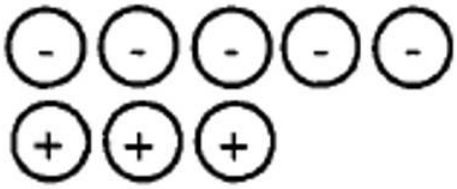
Exercise 18

State the number represented by each collection

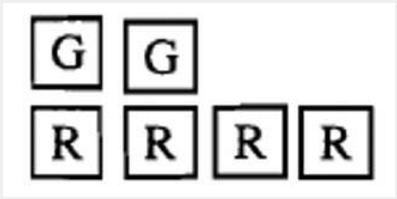
a. The collection below represents _____



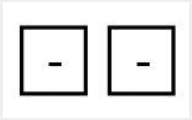
b. The collection below represents _____



c. The collection below represents _____



d. The collection below represents _____



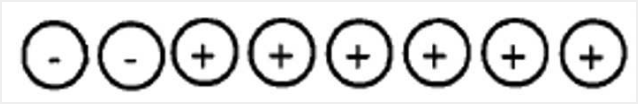
e. The collection below represents _____
RRRRRRGGG

f. The collection below represents _____
GRRGRGRRGGGGG

g. The collection below represents _____



h. The collection below represents _____



How do you think you did? Hopefully, you came up with an answer of -2 for a-d. If so, you should be convinced that different collections can represent the same number. Each pairing of a negative with a positive basically "cancels each other out". Therefore, to figure out what number a particular collection represents, simply pair as many positives with negatives as possible (red-green pairs), and remove or ignore those counters. Then, count the counters of the one "color" or "sign" that remains. You may have noticed it was easier to compute a-d because the pairings were more easily seen.

Exercise 19

Using positive and negative counters, find 3 different representations of 4. Do these with your actual positive and negative manipulatives first. Then, show it on paper.

a.	b.	c.

Exercise 20

Using positive (green) and negative (red) counters, show 3 different representations of -4. Then, show your three different representations below.

a.	b.	c.

Let's move on to how we can use the counters to add integers. To add integers, simply make a collection of counters representing each number to be added and combine the counters. Then, find the number that the resulting collection represents. Do this by first pairing off any positives and negatives and removing them from the pile.

Example

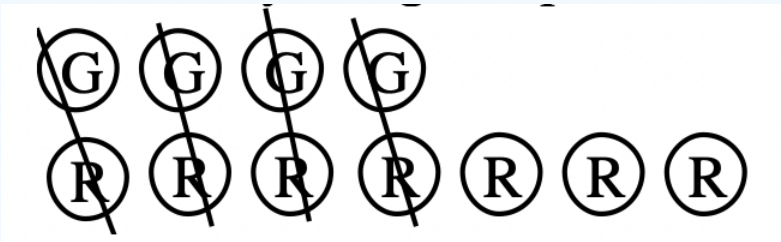
Show two different ways to use the counters to add the 4 + (-7).

First Way:

First, combine a collection representing 4 (4G) and a collection representing -7 (7R):



Second, remove any red-green pairs:



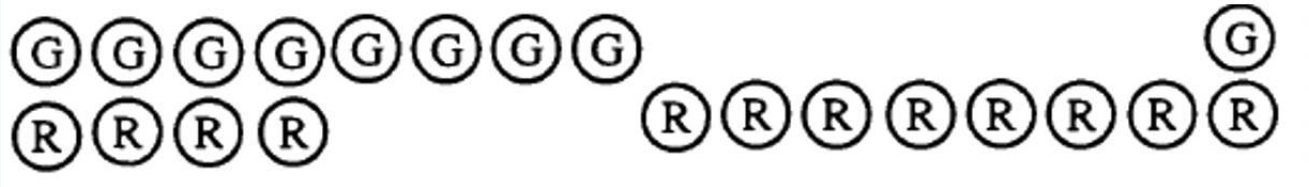
Third, count the counters remaining:



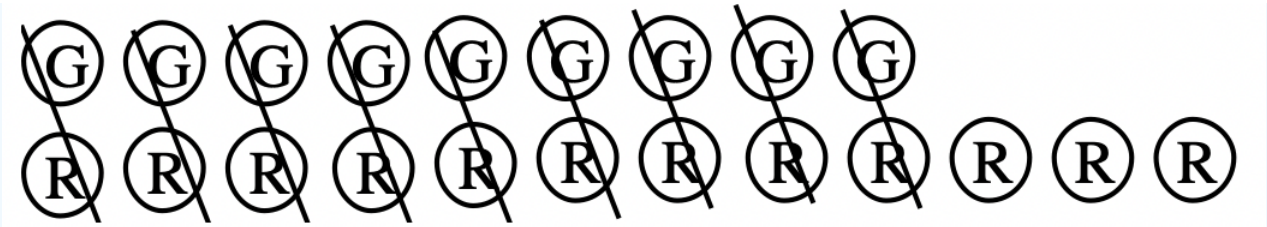
Therefore, 4 + (-7) = -3.

Second Way:

First, combine a collection representing 4 (8G and 4R) and a collection representing 7 (8R and 1G):



Second, remove any red-green pairs:



Third, count what counters are remaining:



Therefore, 4 + (-7) = -3.

Of course, the simpler and more direct method is the first way shown, but the result is the same even when complicated representations of the numbers are used in the first step.

Exercise 21

Use red and green counters to add the following integers. **Do these problems using the actual manipulatives**. Then, ***explain*** and ***show*** the steps below

Although subtraction is a little trickier using counters, they provide a really interesting way of showing how to subtract negative numbers.

Using the counters, m – n means remove a collection of counters representing n from a collection of counters representing m. The number that the resulting collection represents is the difference (answer).

This means we have to start with a collection of counters representing m. In order to remove a collection representing n, it must be possible to remove a collection representing n from the collection used for m. Sometimes, the original collection chosen for m will not suffice, and the collection must be altered by adding some red-green pairs to the collection representing m. Usually, we start by putting in a simple representation of m counters and see if it is possible to remove or taking away a representation of n counters. If so, we do than and find the number that the resulting collection represents. Let's try this on some examples using the physical models (manipulatives) and then practice showing it on paper.

Example: $4 - 2$

4 – 2

This means to remove a collection representing 2 from a collection representing 4.

Step 1: Make a collection representing 4 (4 green counters): $\not{G}\not{G}$GG

Step 2: Take out a collection representating 2 (2 green counters):

Step 3: The collection remaining (GG) represents 2. So, 4 – 2 = 2.

Example: $-5-(-3)$

-5 – (-3)

This means to remove a collection representing -3 from a collection representing -5.

Step 1: Make a collection representing -5 (5 red counters): RRRRR

Step 2: Take out a collection representing -3 (3 red counters):

Step 3: The collection remaining (RR) represents -2. So, -5 – (-3) = -2.

Example: $3-5$

3 – 5

This means to remove a collection representing 5 from a collection representing 3.

Step 1: Make a collection representing 3 (3 green counters): GGG

OH NO! *The next step would be to take out a representation of 5 (5 green counters), but notice this is impossible because there are only 3 green counters to begin with.* ***An extra step must be inserted here.*** *We need to alter the representation of 3 so that 5 greens can be taken out. It should make sense that we can always insert some green-red pairs to the representation of 3 (each green-red pair is the effect of adding zero) without changing the representation of 3. In this example, it makes sense to add 2 more red-green pairs, so that there will be 5 greens and 2 reds in the collection. It will still be a collection representing 3. This extra step is often necessary when doing subtraction problems with integers.*

Step 2: Add 2 red-green pairs to the collection representing 3: GGGRGRG

Step 3: Take out a collection representing 5 (5 green counters): $\boxed{\text{GGGGG}}$ RR

Step 4: The collection remaining (RR) represents -2. So, 3 – 5 = -2

Exercise 22

For each problem, state what the problem means (For instance, 4 – (- 9) means remove a collection of counters representing -9 from a collection of counters representing 4), and then explain and show each step you need to take to find the answer. Each problem requires 3 or 4 steps. You should do these problems using the manipulatives, and as you do it, explain and show the steps on paper.

a. 6 – 3
This means
Show all of the steps below.

b. 4 – 6
This means
Show all of the steps below.

c. -7 – (- 6)
This means
Show all of the steps below.

d. -3 – (- 7)
This means
Show all of the steps below.

e. 4 – (- 3)
This means
Show all of the steps below.

f. -2 – 5
This means
Show all of the steps below.

g. 5 – 5
This means
Show all of the steps below.

h. 5 – (- 5)
This means
Show all of the steps below.

i. -4 – (- 4)
This means
Show all of the steps below.

j. -6 – 6
This means
Show all of the steps below.

Exercise 23

State what the problem means in terms of using positive and negative counters and show all steps to (a) subtract 5 - 8, and then (b) add 5 + (-8).

a. 5 – 8

c. What did you notice about your answers to a and b?

Hopefully, you noticed that removing a collection representing 8 from a collection representing 5 produced the same result as combining a collection representing 5 and -8.

In other words, subtracting 8 produced the same result as adding -8.

Exercise 24

a. Use the actual manipulatives. Start with a collection of at least 4 red and some green counters and write what you have below:
b. Remove 4 red counters from your collection. Show this step by crossing out or circling 4 red counters in your collection above. After removing those red counters, what number does the collection in part a now represent? _____
c. Start over. Begin by combining the original collection you had in part a with 4 green counters (add 4 green counters to your original collection). Below, show what the collection now looks like.
d. What number does your final collection represent? _____

Exercises 21 and 22 illustrate a very important fact about subtraction. Subtracting a number gives the same result as adding the opposite number. Exercises 10 and 13, and the examples preceding each of those exercises also illustrated this fact.

Note that red and green counters, or positive and negative counters are opposites.

Example

Explain what 7 – 4 means, and then state another way to get the same result.

Solution

7 – 4 means remove a collection representing 4 from a collection representing 7. This gives the same result as combining a collection representing 7 with a collection representing the opposite of 4, which is to combine a collection representing 7 with

a collection representing -4 (which means combine 7G with 4R.) So, 7 – 4 = 7 + (- 4).

Example

Explain what 2 – (- 6) means, and then state another way to get the same result.

Solution

2 – (- 6) means remove a collection representing -6 from a collection

representing 2. This gives the same result as combining a collection representing 2 with a collection representing the opposite of -6, which is to combine a collection representing 2 with a collection representing 6 (which means combine 2G with 6G.) So, 2 – (- 6) = 2 + 6.

Exercise 25

Explain what each problem means and then state another way to get the same result

	b. 4 – 8		

Exercise 26

Rewrite each subtraction as an equivalent addition problem.

a. 4 – 5 = _____		

Exercise 27

Rewrite each problem so it is only in terms of addition. In other words, only change each subtraction part to an equivalent addition part. Do not compute the answer.

Exercise 28

Fill in the blanks:

a. Subtracting 6 is the same as adding _____ .
b. Subtracting -8 is the same as adding _____ .
c. Subtracting a positive number is the same as adding a _____ number, where that _____ number is the opposite of the original positive number.
d. Subtracting a negative number is the same as adding a _____ number, where that _____ number is the opposite of the original negative number.
e. Subtracting a number is the same as adding the _____ of that number.

The one thing you must pay careful attention to is the difference between the subtraction sign and the number being subtracted. When you rewrite a subtraction problem as an equivalent addition problem, you have to remember to change the number that was being

subtracted to its opposite. It is certainly not true that 6 – 3 is the same as 6 + 3. But it is true that 6 minus (positive 3) is equal to 6 plus (negative 3); or 6 – 3 = 6 + (- 3).

It isn't necessary to change subtraction problems to equivalent addition problems. It is, however, sometimes very useful because the addition problem may be easier to work with. And if all subtraction can be written as addition, then one only needs to learn rules

for addition and apply them to all addition or subtraction problems. Let me say that although it's a good idea to know the rules for addition and subtraction, it's even more important to be able to go back to the basic concepts and be able to use manipulatives to understand what addition and subtraction really mean.

If you filled in the correct answer to 28e, then you've got it. The relationship between addition and subtraction lies in this fact.

Every subtraction problem can be rewritten as an equivalent addition problem. Instead of subtracting a number, you can add the opposite of that number.

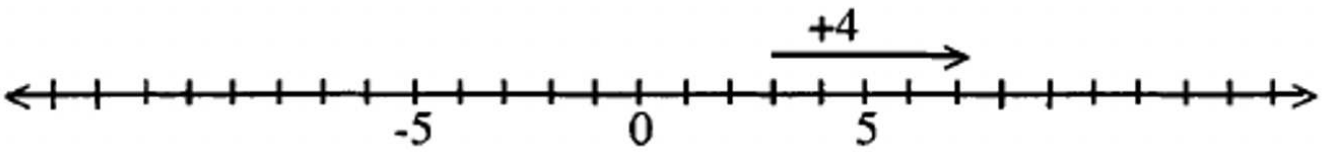
Definition: **Let m and n be any integer. Then, m – n = m + (- n). This is called Adding the Opposite.** Remember: -n is not necessarily negative! If n is positive, -n is negative but if n happens to be negative (n is just a variable), then -n would be positive.

Let's return to the number line and using vectors in conjunction with the *missing addend approach* to compute a subtraction problem. Note that this is an entirely different approach from the way we did it at the beginning of the exercise set, when we did it with doing and undoing actions. First, let's review the definition of subtraction using the missing addend approach.

Definition: Subtraction *(Missing Addend Approach)* The difference of two numbers, m and n, written **m – n** is the number **c** such that **n + c = m**. In other words, if **c** is added to the subtrahend, **n**, then the sum is the minuend, **m**. The answer, **c,** is called the *missing addend*.

Consider the problem 7 – 3. The missing addend approach states the answer to this subtraction problem will be the number that when added to the number shown after the subtraction sign equals the number that precedes the subtraction sign. Since 3 is the number after the subtraction sign, we have to find a number to add to 3 that will give an answer of 7. Using a number line, this means if you place the initial point of a vector at 3, what vector must be drawn (the missing addend can still be thought of as which action must take place) so that the terminal point will land on 7? Draw a vector with its initial point at 3 and its terminal point at 7. Make sure the arrow on the end of the vector *points* to 7.

The number that the vector represents is the answer to the problem. We are trying to find that missing addend, which is the vector we draw. The illustration for 7 – 3 is shown below. The vector has length 4 and points to the right, so the answer is 4, which I've written above the vector (labeled and marked the missing addend) to indicate the answer to the problem.



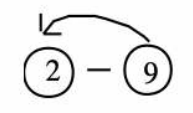
Again, note that this is a **completely different approach to subtraction**. It is not at all like the way we did it at the beginning of the exercise set, when we did it by doing one action, and then undoing another and then seeing what one action could have achieved the same result.

Using the missing addend approach, only one vector is shown, and *the number that vector represents is the answer* to the problem. In other words, the vector *represents* the missing addend you are looking for, which is the difference of the two numbers. You can find the answer to the subtraction problem by drawing a vector, placing initial point of the vector on the subtrahend (the number shown right after the subtraction sign), and placing the terminal point on the minuend (the number preceding the subtraction sign). Note that this definition only make sense when subtracting two numbers, as opposed to adding and/or subtracting several numbers.

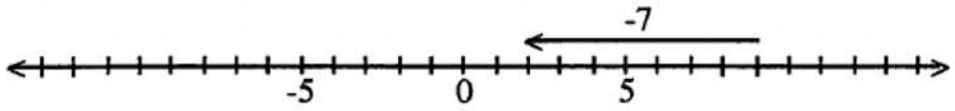
Procedure for subtracting two integers using vectors and the missing addend approach: To find the difference of m and n, draw a vector with its initial point on n and its terminal point on m. The vector drawn is the missing addend, and is the answer to the problem: m – n.

It's important to distinguish between the subtraction sign and the subtrahend (the number shown after the subtraction sign). Also it's important to remember to put the initial point of the vector that will represent the missing addend on the subtrahend (the number AFTER the subtraction sign, and the terminal point on the minuend (the number shown PRECEDING the subtraction sign). One method I sometimes use is to circle the minuend and subtrahend and then draw an arrow from the subtrahend to the minuend so I remember to draw the vector with its initial point on the subtrahend and its terminal point on the minuend.

Let's try the problem 2 – 9. Then, 9 + ____ = 2. Circle the minuend and subtrahend and draw an arrow from the subtrahend to the minuend before you draw the vector.



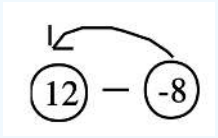
On the number line, draw a vector with its initial point at 9 and its terminal point at 2. The directed vector drawn represents the answer: -7.



Example

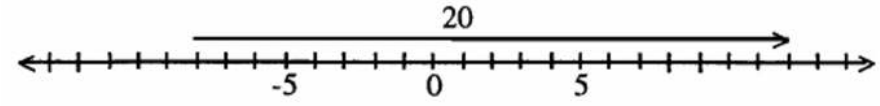
12 – (- 8). Then, -8 + ____ = 12.

Circle the minuend and subtrahend and draw an arrow from the subtrahend to the minuend before you draw the vector.



Note when you draw an arrow from the subtrahend to the minuend, the arrow points the the left in the actual subtraction problem. But that has NOTHING to do with whether or not the arrow on the vector you draw on the number line will point left or right. It points to the minuend. For this example, note that on the missing addend vector drawn below, it points to the right, and that the answer to this problem is a positive number.

Place the initial point of the vector on -8 and the terminal point on 12. The vector drawn below shows the answer: 20.



Exercise 29

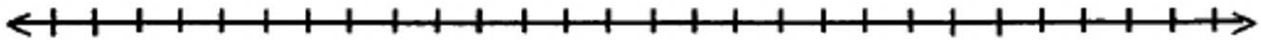
Before going on, think about the possibility of using the missing addend approach and not even getting a vector.

a. In other words, what if the initial and terminal point of the vector was on the same number? It would look like a point, not a vector. When would this happen?
b. What would be the answer to a problem where this happened?

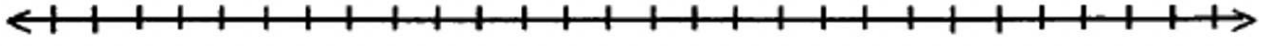
Exercise 30

Find the answer to the following subtraction problems by *using vectors on the number line in conjunction with the missing addend approach*. Draw an arrow from the subtrahend to the minuend as shown in the last two examples. Before drawing the missing addend representing the answer to the problem, label and mark your number line with at least 0 and one point on each side of zero. Then, draw the vector that represents the answer to each problem and **label** the vector with that number.

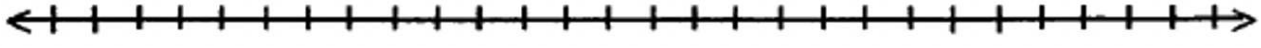
a. 10 – 4 = _____



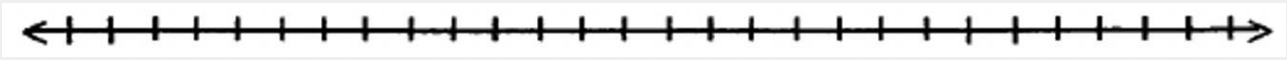
b. -7 – 5 = _____



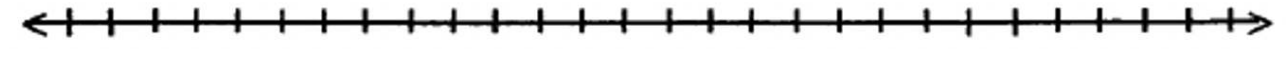
c. 8 – (- 3) = _____



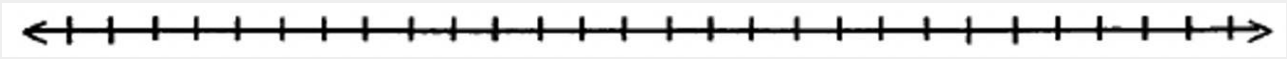
d. -9 – (- 1) = _____



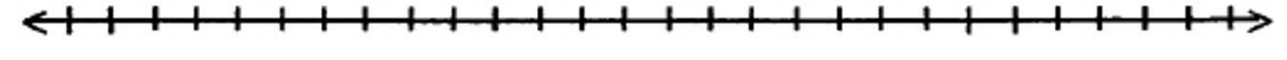
e. 3 – 10 = _____



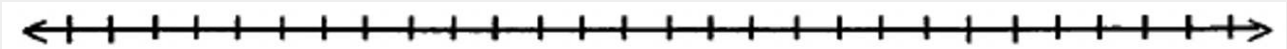
f. -5 – (- 8) = _____



g. 6 – (- 6) = _____



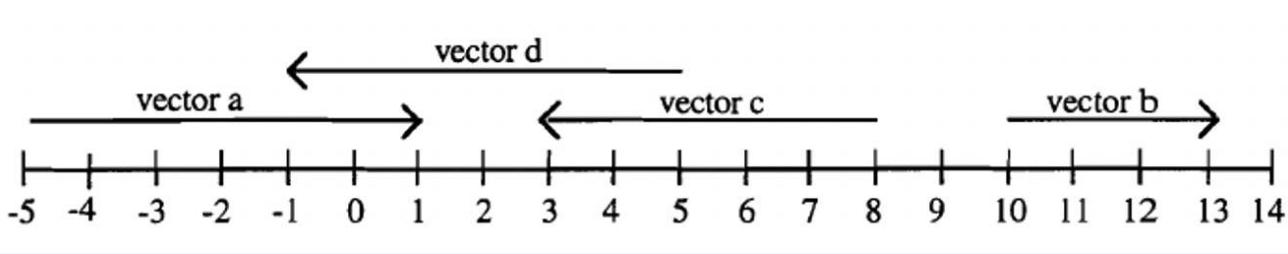
h. 9 – 9 = _____



Exercise 31

By looking at the vectors a - c below, figure out what subtraction problem was being performed, and then state the answer. For instance, vector d came from doing the problem -1 – 5 since the initial point of that vector is on 5 and the terminal point is on -1. Since vector d is 6 units long and goes left, it represents the number -6. Therefore, vector d would be: -1 – 5 = -6.

a. Vector a:	b. Vector b:	c. Vector c:



One thing I love about using the missing addend model to show subtraction is that it works equally well when subtracting with larger integers. Just approximate where the integers are on the number line in relation to themselves and zero. The answer makes sense using the model.

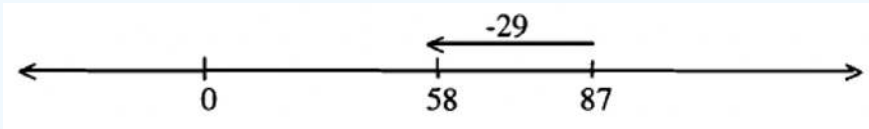
The following examples show how to use vectors on the number line in conjunction with the missing addend approach to find these differences: 58 – 87, -130 – 55 and -130 – -55.

Example 1: 58 – 87

58 – 87.

Solution

Draw a vector from 87 to 58. Since the vector is pointing to the left, the answer is negative and the length of the vector is 29 units long, 87 – 58. So, 58 – 87 = -29.

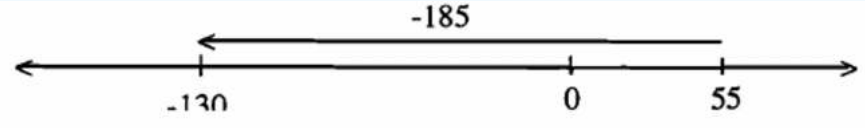


Example 2: $-130 - 55$

-130 – 55.

Solution

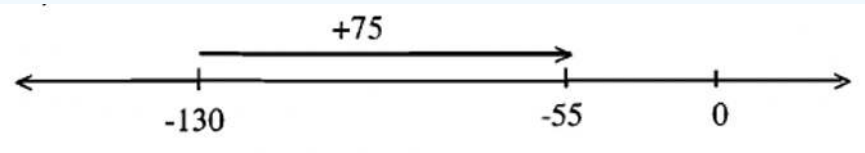
Draw a vector from 55 to -130. Since the vector is pointing to the left, the answer is negative and the length of the vector is 185 units long, since it is 130 units from -130 to 0 plus another 55 units from 0 to 55. So, -130 – 55 = -185.



Example 3: $-55 - (-130)$

-55 – (-130).

Draw a vector from -130 to -55. Since the vector is pointing to the right, the answer is positive and the length of the vector is 130 – 55 , or 75 units long. So, -55 – (-130) = 75.



Exercise 32

Mark and label a number line with at least zero, the minuend (the number in front of the subtraction sign) and the subtrahend (the number being subtracted). Then, use a vector to find the answer to the problem. Label the vector that is the answer

a. -92 – (-71) = _____



b. 92 – (- 81) = _____



c. -110 – (- 200) = _____



d. -122 – 76 = _____



e. 208 – 389 = _____



Finally, here are a set of rules that can always be used to add and subtract integers without the use of manipulatives. While these rules will always work, it is better to first understand where these rules came from and why they work. From doing the exercises in this exercise set, you should now have a better understanding of what it means to add and subtract with positive and negative numbers.

- **Any subtraction problem can be rewritten as an equivalent addition problem using the definition of adding the opposite. Once all subtraction problems are rewritten as equivalent addition problems, there is only addition to be done, and so therefore, we only need rules for addition, which follow.**
- **To add numbers that have like signs (either all positive or all negative), add the absolute values of the numbers. The sign of the answer will be the sign of the original numbers.**
- **To add two numbers with unlike signs (one positive and one negative), take the difference of the absolute values of the two numbers. The sign of the answer will be the same as the sign of the number with the larger absolute value. If the absolute values of the numbers are the same, the answer is zero.**

I never really liked the way these definitions were worded, but this is the mathematical way of expressing exactly how to get the answer. This is how **I** think about the rules.

First, I do usually rewrite each subtraction as an equivalent addition problem, unless it is two whole numbers being subtracted. In that case, if the first whole number is larger than the second whole number, I do regular whole number subtraction. If the larger whole number is being subtracted from the smaller whole number, I know the answer is negative and simply put a negative number in front of the answer.

*This is how **I** think about adding numbers that have like signs:*

- Let's say you were adding 8 + 9. These are both positive. So you just add the two whole numbers together as usual, and the answer is still positive. It's like having 8 dollars in the bank, and depositing 9 more dollars. You now have 17 dollars in the bank. So, I hope it's no surprise that 8 + 9 = 17.
- Let's say you were adding -8 + (- 9). These are both negative. This is like being in debt, and going deeper into dept. So you just add the two whole numbers together as usual, but the answer is negative. It's like owing someone 8 dollars, and then borrowing 9 more dollars. You now owe that person 17 dollars. So, -8 + (- 9) = - 17.

*This is how **I** think about adding numbers that have unlike signs.*

- Let's say you were adding 24 + (-10). Now this is an example of adding two numbers with unlike signs. Since the signs are DIFFERENT, I ignore their signs and take their DIFFERENCE – *that's the same thing as just subtracting their absolute values*. When I ignore their signs, I subtract the smaller number from the larger number. Then, to determine if the sign of the answer is positive or negative, I see which one outweighs the other (are there more negatives or positives?) – *that's the same thing as taking the sign of the one with the absolute value*. 24 + (-10) could be thought of as having 24 dollars in the bank and writing a check for 10 dollars. You still have $14 left in the bank. So, 24 + (-10) = 14.
- If you were adding -24 + 10, the problem would be done the same as the previous one except that in this case, the negatives outweigh the positives, so the final answer is negative. It's like owing someone 24 dollars and paying them 10 dollars. Now, you only owe them 14 dollars. So, -24 + 10 = -14.
- Let's say you were adding 14 + -14 or adding -83 + 83. In both of these examples, I'm adding opposites. In a way, one undoes the other. So both equal zero. This also could be done by thinking about which sign outweighs the other. But neither outweighs the other, and that is another reason the answer is neither negative nor positive, but is zero.

Using the rules at the beginning of the previous page, or the way I think about it, or maybe some other way you think about it that you are sure works and makes sense, you should be able to add and subtract integers with confidence. You should be able to explain the meaning of the operations, and be able to use manipulatives and number lines to help others learn how to add and subtract with integers and understand

? Exercise 33

Perform the following arithmetic problems by first changing subtraction to adding the opposite. Then add two numbers at a time, working from left to right. Show your steps.

Exercise 34

Perform the following arithmetic problems by first changing subtraction to adding the opposite. Then add numbers with like signs together so that you have the sum of a positive and a negative number. Then add those two numbers together. Show your steps.

Remember that for a set to be closed under addition, the sum of any two elements in the set will always be in the set. For a set to be closed under subtraction, the difference of any two elements in the set will always be in the set. To prove a set is not closed under addition or subtraction, you need to provide a counterexample.

Exercise 35

For each of the following sets, determine if the set is closed under the operation given. Provide a counterexample if it is not closed

a. Integers under addition?	b. Integers under subtraction?
c. Positive integers under addition?	d. Positive integers under subtraction?
e. Negative integers under addition?	f. Negative integers under subtraction?
g. {-1, 0, 1} under addition?	h. {-1, 0, 1} under subtraction?

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5.6: Operations in Different Bases

First, let us set up the number lines. We plan to add in base six and then (separate exercise) base three. We are not adding mixed bases in this book, but yes, it is a thing. Google it.

Second, add it vertically. Rewrite the problem. Carry your numbers that are greater than or equal to 10, just like you did in elementary school. But count using the number (actually count it with your finger, swallow your pride and do it).

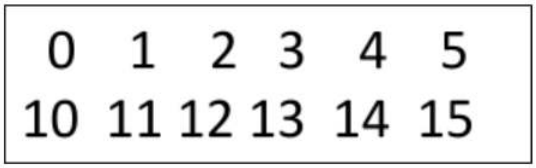


Figure 2.8.1: Base Six Number Line

Example 5.6.1

$$5415_{\text{SIX}} + 3042_{\text{SIX}}$$

$$\begin{array}{r} 5415_{\text{six}} \\ +3042_{\text{six}} \\ \hline 12501_{\text{six}} \end{array}$$

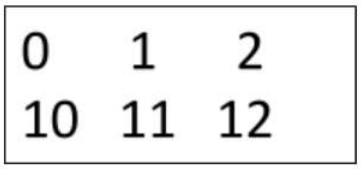


Figure 2.8.2: Base Three Number Line

Example 5.6.2

$$1202_{\text{THREE}} + 1022_{\text{THREE}}$$

$$\begin{array}{r} 1202_{\text{three}} \\ +1022_{\text{three}} \\ \hline 10001_{\text{three}} \end{array}$$

Now, we will subtract in different bases. For this example, we will use base seven and eight respectively. Below are the corresponding number lines. Again, use your fingers to count on the number line (everyone else is doing it). Borrow, when appropriate, like you did in elementary school.

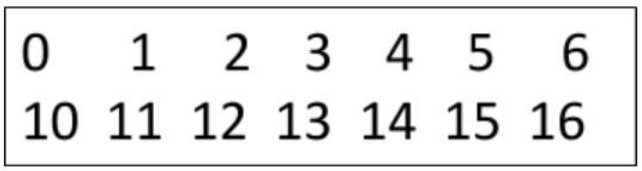


Figure 2.8.3: Base Seven Number Line

Example 5.6.3

$$5631234_{\text{SEVEN}} - 2564310_{\text{SEVEN}}$$

$$\begin{array}{r} 5631234_{\text{seven}} \\ -2564310_{\text{seven}} \\ \hline 3033624_{\text{seven}} \end{array}$$

0	1	2	3	4	5	6	7
10	11	12	13	14	15	16	17

Figure 2.8.4: Base Eight Number Line

Example 5.6.4

$$3741_{\text{EIGHT}} - 1465_{\text{EIGHT}}$$

$$\begin{array}{r} 3741_{\text{eight}} \\ -1465_{\text{eight}} \\ \hline 2254_{\text{eight}} \end{array}$$

Partner Activity 1

Add or subtract the following numbers:

1. $65_{\text{nine}} + 41_{\text{nine}} =$
2. $123_{\text{four}} + 301_{\text{four}} =$
3. $540_{\text{six}} - 23_{\text{six}} =$
4. $220_{\text{three}} - 122_{\text{three}} =$
5. $1001001_{\text{two}} - 1011_{\text{two}} =$

Practice Problems

Add or subtract the following expressions:

1. $435_{\text{six}} - 31_{\text{six}}$
2. $10010_{\text{two}} - 111_{\text{two}}$
3. $2100_{\text{three}} + 21_{\text{three}}$
4. $57643_{\text{eight}} + 24677_{\text{eight}}$

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CHAPTER OVERVIEW

6: Multiplication

6.1: Multiplication
6.2: Area Model for Multiplication
6.3: Multiplication Algorithms
6.4: Additional Multiplication Topics
6.5: Practice

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6.1: Multiplication

Just like how we could think of addition and subtraction from various perspectives (e.g., add on, put together, compare, etc.), there are many ways we can view multiplication. Before continuing, recall that the result of multiplication is called the **product** of the two values being multiplied together. It is most likely that you first learned multiplication in terms of addition.

Repeated addition

Definition: Multiplication as Repeated Addition

The operation of **multiplication** on two known values is represented as $\times$ or $\cdot$ or with parentheses. If n is any positive integer and m is any real number, then $n \times m$ is equal to the sum of n addends equal to m. This is the **repeated addition perspective of multiplication**.

Let's see an example of using the repeated addition perspective to answer a multiplication problem.

Example 6.1.1

Watermelons cost \$3 per pound. How much does a watermelon that weighs five pounds cost?

Solution

Based on the context, many children would not even need to view this as multiplication. Since they know there are five pounds, they might draw a diagram with 5 regions, each representing a pound. It might be a watermelon shape divided into 5 regions, or it might be abstract and simply be 5 separate circles or other shapes. Regardless, there are 5 portions, and each portion represents a pound that costs \$3. So, when they look for the total cost, a child is likely to simply think of it as $\$3 + \$3 + \$3 + \$3 + \$3 = \$15$.

Notice that in the example above, we added 5 copies of 3 instead of 3 copies of 5. This is because of the context. Think about how you might write a problem in which a child would naturally solve it by adding $5 + 5 + 5 = 15$ instead.

Example 6.1.1

Are we able to use repeated addition when multiplying decimals such as $0.3 \times 6 = 1.8$?

Solution

Yes, in some cases.

For example, if we were asked to find the total weight of six boxes that weigh 0.3 pounds each, it would make sense to add $0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3 = 1.8$.

What is 30% of 6 pounds?

$6 \times 0.3 = 0.3 + 0.3 + 0.3 + 0.3 + 0.3 + 0.3 = 1.8$ (6 groups of 0.3)

Note that in the example above, the answer was "Yes, in *some* cases." If we change the context, repeated addition does not make quite as much sense.

Example 6.1.1

Peaches cost \$2 per pound. You buy 3.7 pounds of peaches. How much do you have to pay for the peaches?

Solution

We cannot add 3.7 copies of something (see more on this in the next perspective), so we cannot add 3.7 of \$2. While it is possible to add 3.7 + 3.7, that does not make sense contextually. So, in order to answer this problem with repeated addition, we have remove the context. This is difficult for young learners because abstraction itself is difficult.

Once the context is removed, we can change the question to one more easily answered.

New question: What is the product of 2 and 3.7?

$2 \times 3.7 = 3.7 + 3.7 = 7.4$

Now we can return to the context and answer the original question. 3.7 pounds of peaches cost $7.4.

Another situation in which repeated addition does not work is when we simply have two values that are both not positive integers.

Example 6.1.1

Apples cost $2.99 per pound. How much does 3.2 pounds of apples cost?

Solution

Even with abstraction, there is no way to solve this problem with repeated addition. You cannot add 2.99 copies of 3.2, nor can you add 3.2 copies of 2.99. We must switch to a different perspective or think of the multiplication in a completely abstract way and use an algorithm. For example, most adults would not use any particular perspective and just think of the contextless question, "What is the product of 2.99 and 3.2?" They would answer this by using whatever multiplication algorithm they are most comfortable with. If they use that multiplication algorithm correctly, they would get that $2.99 \times 3.2 = 9.568$. Returning to the context, we would have that they cost $9.568 or $9.57 when rounded to the nearest cent.

As noted above, there are other perspectives. Let's look at another one.

Portion of a Quantity

Definition: Multiplication as a Portion of a Quantity

If n and m are any positive real numbers, then $n \times m$ is equal to the value that represents an n portion of m. This is the **portion of a quantity perspective of multiplication**.

The portion referred to above may be none of, a fractional part of, exactly equal to, or more than the quantity we are considering. Let's look at each situation, starting with none.

Example 6.1.1

Spot's favorite dog treats each have 5 calories. Spot isn't given any treats today. How many calories does Spot get from treats today?

Solution

In this case, we are looking at the product 0×5 as no portion of 5 calories. Having no portion of the 5 calories at all means Spot gets 0 calories from treats today.

When the portion is greater than 0 but not a positive integer, then we are dealing with a fractional part of a quantity.

Example 6.1.1

Two friends go into business together. Based on the amount of work they each do, Frances gets 30% of all profits and Garfield gets 70% of all profits. The average profit from each sale is $6. How much does Frances get from a profit of $6?

Solution

Here, we are looking for the fractional part (30 or $\frac{30}{100}$ or 0.3) of a quantity ($6). We want to find the product $0.3 \cdot 6$ or, equivalently $\frac{30}{100} \cdot 6$. One way to do this is to literally use fractions. We think of splitting 6 into 100 portions and take 30 of those portions. Or we can use the reduced fraction $\frac{3}{10}$ instead to get that our answer is equal to the same value as 3 copies of one-tenth of 6. Let's use that one.

$$\frac{1}{10} \text{ of } 6 = \frac{6}{10}$$

$$3 \text{ copies of } \frac{6}{10} = \frac{6}{10} + \frac{6}{10} + \frac{6}{10}$$

$$= \frac{18}{10} = \frac{9}{5} = 1\frac{4}{5}$$

Since $\frac{4}{5} = \frac{80}{100}$ that portion of a dollar is 80¢. So, Frances' share of the profit is $1.80.

You might have noticed that the repeated addition perspective popped up in the problem solution above. That does sometimes happen! It is okay to mix perspectives.

When the portion is exactly 1, then we have a very simple situation.

Example 6.1.1

Toni has 16 ounces of water in his water bottle. He drinks all of it. How much water did he drink?

Solution

This question is essentially asking for the product of 1 and 16. This is because 1 represents the entire portion of the quantity at hand. Just like how $1 \times 16 = 16$, the answer to the question is that Toni drank 16 ounces of water.

Finally, we can have a portion greater than one.

Example 6.1.1

Last year, Lupe had to pay $24 for a parking permit to park in the "A" parking lots on her school's campus. This year's parking permit prices are 150% of last year's prices. How much will the same parking permit to park in the "A" parking lots cost this year?

Solution

In this case, we are looking for 150% of the price $24. There are a few ways this can be done, but one is to imagine 24 one-dollar bills. 150% of those would be all 24 plus an additional amount equal to 50% of 24. Thus, the new price is $24+$12 = $36.

Note that in none of these cases did we use values that are negative. We will deal more with negative values in the future. But keep in mind that the portion of a quantity perspective of multiplication does not apply very well to products involving negative numbers! Also note that although these kinds of problems can get quite difficult, it is possible to create problems for young learners that encourage viewing multiplication this way. For example, children learn "half" much earlier than they learn the general concept of fractions. So, surprisingly young children can understand what the following problem is asking.

Example 6.1.1

Gabrielle took 20 minutes to do her hair yesterday morning. It only took her half as long to do it this morning. How long did it take her to do her hair this morning?

Solution

One way a child might solve this is to use a clock model. They might shade in 20 minutes of a clock (such as one made from an image such as this blank clock face with numbers provided by the Florida Center for Instructional Technology) and use that to find what half of 20 minutes is. Then they can count out the 10 minutes that they get in order to find the answer: Gabrielle took 10 minutes to do her hair this morning.

Arrays and Area

While the previous two perspectives can be adapted to fit some situations involving non-integers, we still haven't discussed a perspective of multiplication that can be apply to multiplying *any* two positive real numbers. We also have not discussed a perspective that is inherently visual or physical. This is where the array and area perspective of multiplication comes in.

Definition: Multiplication as a Measure of an Array or Area

If n and m are any positive integers, then $n \times m$ is equal to the number of objects that form an array of n columns and m rows. This is the **array perspective of multiplication**.

If n and m are any positive real numbers, then $n \times m$ square units is equal to the value that represents the area of a rectangle of length n units and width m units. This is the **area perspective of multiplication**.

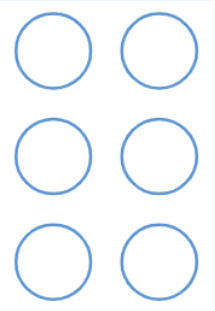
The array perspective pairs well with the repeated addition perspective because we can think of the number of objects in an array as a collection of equal-sized groups.

Example 6.1.1

A crafter puts his beads into small piles. There are 3 rows of piles, and each row has 2 piles in it. How many piles did he make?

Solution

If a child draws or directly models this situation, they are may solve this problem the same way they solve a repeated addition problem. In other words, they might look at a drawing like the one below and simply add 2+2+2 = 6.



The child could count the 6 circles individually instead of thinking of it as addition, however. In any case, the answer the child will find is that there are 6 piles of beads.

As noted in the example above, arrays do not *have* to be thought of as models of repeated addition. But they are more likely to be used that way than area.

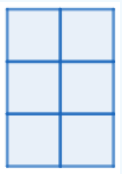
Example 6.1.2

Model and find the areas of rectangles that are...

1. 3 units by 2 units.
2. 2 units by 3 units.

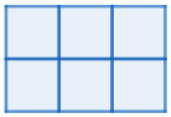
Solution

A rectangle that is 3 units by 2 units is sketched below.



We can see that since there are 6 small squares that form this rectangle, the area of the rectangle is 6 square units.

A rectangle that is 3 units by 2 units is sketched below.



We can see that since there are 6 small squares that form this rectangle, the area of the rectangle is 6 square units.

The previous example illustrates an additional advantage of the array or area perspective: simply changing the orientation of a picture models the commutativity of multiplication. That is, rotating an $n \times m$ rectangle or array to form an $m \times n$ rectangle or array shows that $n \times m = m \times n$.

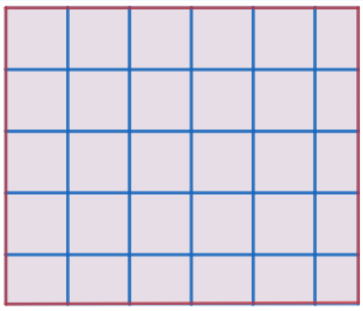
Let's finish this perspective up with an example illustrating the usefulness of the area perspective with non-integers.

Example 6.1.1

A rectangular room is 5.7 meters long and 4.8 meters wide. The carpet you like costs $20 per square meter. How much will it cost to buy the carpet for this room?

Solution

We can model this by drawing a rectangle in which we can see 5 whole units plus 0.7 of a unit along one dimension and 4 whole units plus 0.8 of a unit along the other dimension. The rectangle below shows the whole units along with the partial units.



A benefit to using such models is that even learners who have not learned how to multiply with decimals can use a model like this to estimate the answer. They can see the 20 complete small squares and note that putting together the partial squares can form a few more complete squares. It is even possible to use small grid lines for them to not just estimate the answer but find it exactly!

For now, we will simply multiply since the purpose of this example is to emphasize the fact that the product 5.7×4.8 can be modeled with a rectangle, unlike an array or repeated addition.

$5.7 \times 4.8 = 27.36$

So, the cost of the carpet is $20(27.36) = $547.20.

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6.2: Area Model for Multiplication

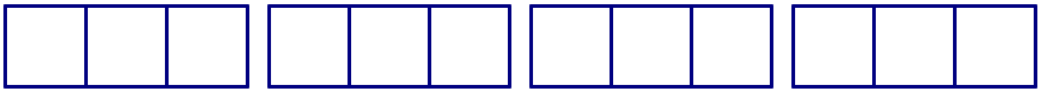
When people refer to "measurement", often they think of using a ruler, yardstick, or tape measure. That is because one of the first forms of measurement we learn are based on a linear measurement model. In mathematics, we see this in the fact that often the first way we model relationships between numbers abstractly is with the number line.

We know that multiplication and area are related. But it is important to explicitly note what area actually is: a measurement describing two-dimensional space. It is also important that we understand why the area perspective of multiplication makes sense.

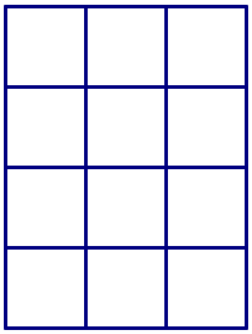
Suppose our basic unit is one square, as shown below.



So, you *could* describe 3 units using three squares. Using the repeated addition perspective of multiplication, we can picture 4 × 3 as 4 groups of 3 squares each, as shown below.



But if we stack up the groups, we would have 4 rows, with 3 squares in each row.



So we can think about 4 × 3 as a rectangle that has length 3 and width 4. The product, 12, is the total number of squares in that rectangle. Since each unit was actually a square, we refer to the area using "square units". That is, the area of the 4 x 3 rectangle is 12 square units.

The Distributive Property

In addition to many other benefits to thinking of multiplication in terms of area, one thing area makes very clear is that the Distributive Property makes sense.

Definition: The Distributive Property

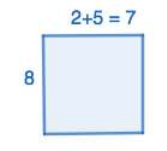
For any three real numbers, the **distributive property** states that the product of one and the sum of the other two is equal to the sum of the products of the one with each of the other two. Symbolically, we write that if a, b, and c are any real numbers, then $a \cdot (b+c) = a \cdot b + a \cdot c$. Sometimes we summarize this by stating that **multiplication distributes over addition**.

Example 6.2.1

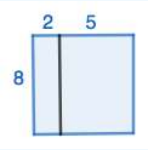
Use a rectangle to show that $8 \cdot (2+5) = 8 \cdot 2 + 8 \cdot 5$. Then calculate the value of the expression.

Solution

First, we will draw an 8 x 7 rectangle. The 7 is because $2+5=7$.



Now, we will split the side that is 7 units long into two pieces: one that is 2 units long and one that is 5 units long.



Since it's the same rectangle, the area is the same. The area of the first version is $8 \cdot (2+5)$. The area of the second version is the same value, but we can think of it as the sum of the areas of the two smaller rectangles that form it: $8 \cdot 2 + 8 \cdot 5$.

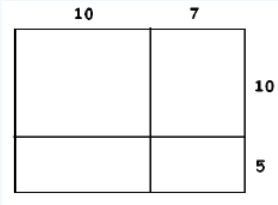
Thus, $8 \cdot (2+5) = 8 \cdot 2 + 8 \cdot 5 = 16 + 40 = 56$.

If you already know that $8 \times 7 = 56$, then it is easy to see that the answer above for $8 \cdot (2+5) = 56$ is correct. What if both dimensions of a rectangle are split into pieces?

Try the following problem on your own and share it with a classmate, friend, or tutor.

? Think / Pair / Share

Vera drew this picture as a model for 15 × 17. Use her picture to help you compute 15 × 17. Explain your work.



The problem above is an example of the following fact: distribution can be applied multiple times in one expression!

✔ Example 6.2.1

Use the distributive property twice to write $(a+b) \cdot (c+d)$ as a sum of products.

Solution

First, we use the distributive property to distribute the $(a+b)$.

$$(a+b) \cdot (c+d) = (a+b) \cdot c + (a+b) \cdot d$$

Now, although we have not explcitily stated so before, we can use the commutativity of multiplication to rewirte the producst with the $(a+b)$ factors second.

$$(a+b) \cdot (c+d) = (a+b) \cdot c + (a+b) \cdot d = c \cdot (a+b) + d \cdot (a+b)$$

This means the distributive property can be used again, but this time we distribute c and d in their respective terms.

$$\begin{aligned}(a+b) \cdot (c+d) &= (a+b) \cdot c + (a+b) \cdot d = c \cdot (a+b) + d \cdot (a+b) \\ &= c \cdot a + c \cdot b + d \cdot a + d \cdot b\end{aligned}$$

We use the pictures and the distrubitive property to make many calculations easier. Try the following exercises, keeping in mind that it is easy to multiply number that end in 0. For the third product, it may be useful to think about the fact that you can split a rectangle into as many small rectangles are you want!

Exercise 1

Draw pictures like Vera's for each of these multiplication exercises. Use your pictures to find the products without using a calculator or the standard American multiplication algorithm.

- 23×37
- 8×43
- 371×42

The Standard Algorithm for Multiplication

How were you taught to compute 83 × 27 in school? Were you taught to write something like the following?

$$\begin{array}{r} 83 \\ \times \quad 27 \\ \hline 21 \\ 56 \\ 6 \\ 16 \\ \hline 2241 \end{array}$$

Or maybe you were taught to leave in the zeros rather than leaving them out?

$$\begin{array}{r} 83 \\ \times \quad 27 \\ \hline 21 \\ 560 \\ 60 \\ 1600 \\ \hline 2241 \end{array}$$

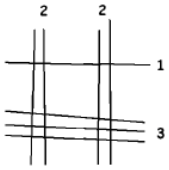
This is really no different than drawing the rectangle and using Vera's picture for calculating! Try the following and discuss your answers with a classmate, friend, or tutor.

Think / Pair / Share

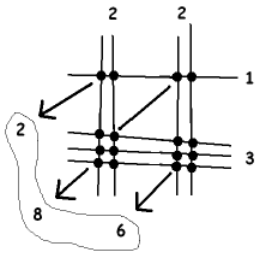
- Use 83×27 to explain why Vera's rectangle method and the standard algorithm are really the same.
- Calculate the products below using both methods. Explain where you're computing the same pieces in each algorithm.
  - 23×14
  - 106×21
  - 213×31

Other Algorithms: Lines and Intersections

Here's an unusual way to perform multiplication. To compute 22 × 13, for example, draw two sets of vertical lines, the left set containing two lines and the right set two lines (for the digits in 22) and two sets of horizontal lines, the upper set containing one line and the lower set three (for the digits in 13).



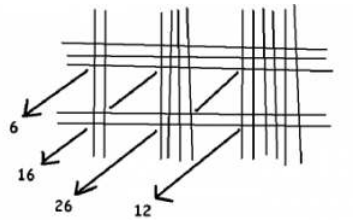
There are four sets of intersection points. Count the number of intersections in each and add the results diagonally as shown:



The answer 286 appears!

Use an area model like Vera's to calculate the same product. Can you see where the values 2, 2, 6, and 6 are in the area model?

There is one possible glitch in this lines & intersections algorithm, as illustrated by the computation 246 × 32 below:



Although the answer 6 thousands, 16 hundreds, 26 tens, and 12 ones is absolutely correct, one needs to carry digits and translate this as 7,872. Once again, use an area model to calculate the same product and see if you can find where the numbers of intersections appear in the area model. Try the following now.

Exercise 2

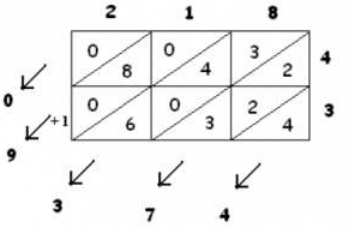
1. Compute 131 × 122 using the lines and intersections algorithm. Check your answer using another method.
2. Compute 15 × 1332 using the lines and intersections algorithm. Check your answer using another method.
3. Can you adapt the method to compute 102 × 3054?
4. Why does the method work in general?

Other Algorithms: Lattice Multiplication

Throughout history, many people have used what is now commonly known as the *lattice method*.

To multiply 43 and 218, for example, draw a 2 × 3 grid. Write the digits of one number along the right side of the grid and the digits of the other number along the top.

Divide each cell of the grid diagonally and write in the product of the column digit and row digit of that cell, separating the tens from the units across the diagonal of that cell. If the product is a one digit answer, place a 0 in the tens place.



To get the answer, add the entries in each diagonal, carrying tens digits over to the next diagonal if necessary. In our example, we have

$$218 \times 43 = 9374$$

Now try the following.

Exercise 3

1. Compute 5763 × 345 using the lattice method.
2. Explain why the lattice method is really the standard American multiplication algorithm in disguise.
3. What is the specific function of the diagonal lines in the grid?

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6.3: Multiplication Algorithms

You will need: ***Base Blocks*** *(Material Cards 4-15)*

There are many algorithms for multiplication. Personally, I think the way most of us learned it in school is one of the hardest algorithms. It's an uphill battle getting new methods into the school system. But, it's worth pursuing. It's a good thing to know what multiplication *means*, and how to multiply any two numbers together. However, there is no reason that everyone should have to use the same method to obtain the answer. In this exercise set, you'll learn some new algorithms for multiplication. You'll also learn how to multiply in different bases.

Many people use repeated addition all the time to multiply. Consider the multiplication problem 26 times 17. Basically, this means add 26 seventeens added together. This could be done by adding them together one at a time. That would take a long time.

Another approach would be to add 10 seventeens together first to get 170, then add 10 more seventeens together to get 170 again, then add 5 more seventeens together which is 85, and then add one more seventeen. Adding 10 seventeens (170), 10 seventeens (170), 5 seventeens (85) and 1 seventeen (17) gives 170 + 170 + 85 + 17 = 442, which is exactly the answer to 26 times 17. That should be no surprise since we added 26 (10+10+5+1) seventeens together.

Someone else might add 10 seventeens (170), 10 seventeens (170), 3 seventeens (51) and 3 more seventeens (51) to get 170 + 170 + 51 + 51 = 340 + 102 =442.

There are several other ways people might break up the number 26 and still get the correct answer. This approach may be one you frequently or maybe this is the first time you've thought of it this way. In any case, it's different from just doing it the same old way.

Because of the commutative property, you could have done 17 times 26 instead. Maybe you'd add 10 twenty-sixes (260), then 5 more twenty-sixes (130) and 2 more twenty-sixes (52) to get 260 + 130 + 52 = 442.

One reason it's advantageous to do multiplication using repeated addition is that many problems can be done in your head this way. But, of course, every new method takes practice and getting used to, and it's common for people to revert back to using their old, familiar methods even if it takes longer and requires paper. That's okay, too. It is, however, very important that a teacher knows several ways so that he or she can provide more opportunities for students. What seems obvious to one student is foreign to another, so you give alternatives and present different ways of looking at things.

? Exercise 1

Use some form of repeated addition to do the following multiplications. Explain how you do each one.

a. 23×13

b. 14×14

One new algorithm (for most of you reading this) is called **Duplation**. This is actually a very old algorithm used in ancient times by the Egyptians. It also uses the principle that multiplication is simply repeated addition. It is called duplation because it uses the idea of doubling numbers to get the answer. Let's look again at the first multiplication we did in this section. It was 26×17. Since 26 seventeens must be added together, the number 26 is broken down into powers of two. It's easy to get powers of two: simply start with 1, then double it to get 2, then double it to get 4, then double it to get 8, then double it to get 16, etc. To figure out how far you have to go in the doubling process, stop if doubling gives a number bigger than the number you are trying to break down (26, in this case). Now, all this sounds more complicated than it really is. Before explaining too much more, notice how you'll write it. Write 26 at the top and underline it. Underneath, start with 1, then, double continuously, stopping at 16 since if you doubled another time, you'd get a number bigger than 26. To the right of 26, write the number 17 and underline it.

Underneath it, first write 17, then double 17 to get 34, then double 34 to get 68, double 68 to get 136, etc., until you have the same amount of numbers under the 17 as you do under the 26.

26	×	17
1	17	
2	34	
4	68	
8	136	
16	272	
.....	.....	

Do you see the corresponding numbers? 1 corresponds with 17, because 17 is 1×17, 2 corresponds with 34, because 34 is 2×17, 4 corresponds with 68, because 68 is 4×17, 8 corresponds with 136, because 136 is 8×17, and 16 corresponds with 272, because 272 is 8×17. Now, you may need to think about this for a few minutes. The left side keeps track of how many of some number you are adding together. So, the 1×17 should be clear. So, 2×17 is simply 17 doubled. Now if 2×17 is 34, then 4×17 is twice as many as 2×17, so double 34 to get 68. This continues on and on. So, if 8×17 is 136, then if we double 136, we should have 16×17. Isn't it neat how we know that $16 \times 17 = 272$ and we just double a few numbers to get there?

Okay, now we only need to add 26 seventeens together. Simply start at the bottom of the first column, and check off numbers that add up to 26 (this is like doing it in base two). After you check off the numbers in the left column, circle or point to their corresponding numbers in the right column. It's those numbers in the right column that you add together to get the answer. Watch how the rest of the problem is done:

26	×	17
1		17
✓2	----->	34
4		68
✓8	----->	136
✓16	----->	272

272 + 136 + 34 = **442** (yep, it's the same answer again!)

Let's do the same problem over again, but use the commutative property of multiplication. In other words, use the duplation method to compute: 17×26. This time, in the left column, you check off numbers that add up to 17 it's coincidental that I had to double to 16 again and stop. In the right hand column, start with 26 and double away.

17	×	26
✓1	----->	26
2		52
4		104
8		208
✓16	----->	416

416 + 26 = **442** (yep, it's the same answer again!)

Here are two more examples for you to study. I'll do 15×35 and 35×15.

15	×	35
✓1	----->35	
✓2	----->70	
✓4	----->140	
✓8	----->280	
		280+140+70+35=**525**

35	×	15
✓1	----->15	
✓2	----->30	
4		60
8		120
16		240
✓32	----->480	
		480+30+15=**525**

Of course, because of the commutative property of multiplication, the answer is 525, no matter which way you do it. Are you ready to try a couple of duplation problems of your own? Well, ready or not, here they come!

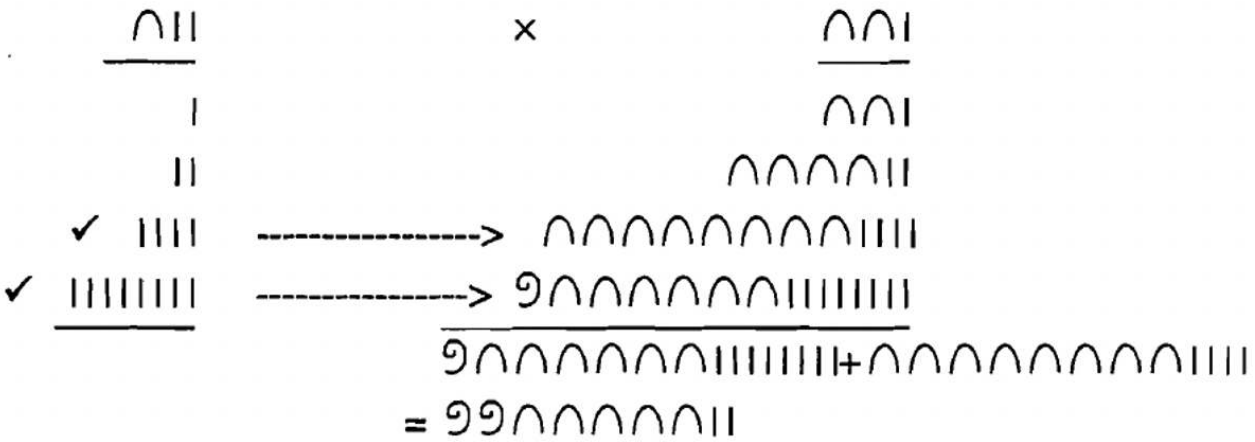
? Exercise 2

Multiply 13×29 using duplation. Then, multiply 29×13 using duplation. Show all of your work. Make it clear what is added together for each problem.

? Exercise 3

Use duplation to multiply 27 by 14, and then 14 by 27. Show your work.

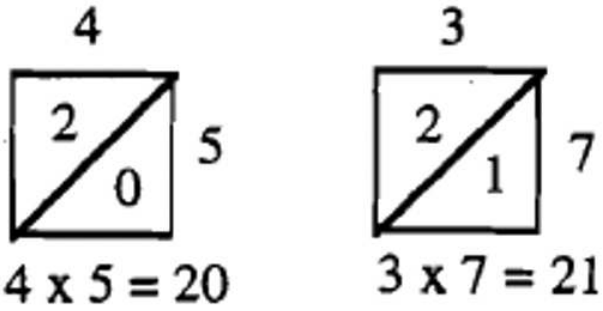
Here is an example of how the Egyptians multiplied using duplation.



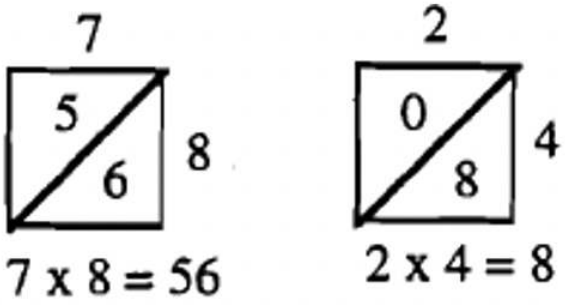
? Exercise 4

Use duplation to multiply  .

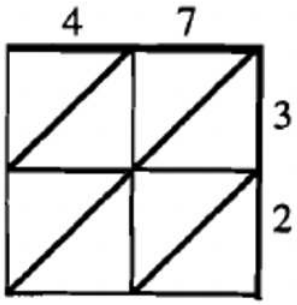
Do the problem entirely in Egyptian.



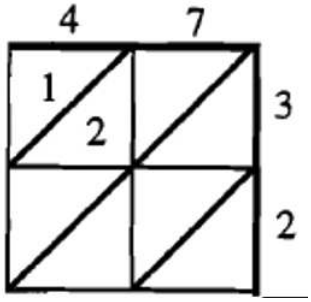
You are going to learn a cool and easy way to multiply numbers together. But first, I'll show you how to multiply 2 single digits together. Make a square with a diagonal in it. Put one number on top and the other on the right of the square. Multiply the two digits together and put the answer in the box with the diagonal. Look at the examples to the right.



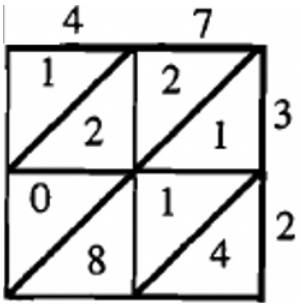
The method you are now going to learn is called the **LATTICE METHOD** and it could be used for multiplying 2 digit or 3 digit or even bigger numbers together. The example well do is 47×32.



STEP 1: To begin, put one of the numbers at the top (47) of a rectangle (that has a space for each digit in the number) and the other number along the side (32) of the same rectangle (that has a space for each digit). In this case, we need a rectangle that has two spaces across the top and 2 spaces along the side. Then, you make diagonals. Look to the right.

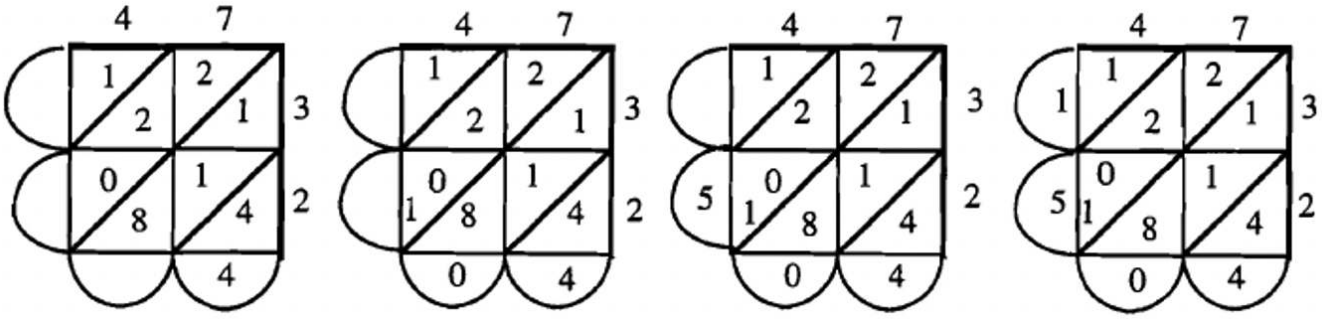


STEP 2: Multiply each of the digits on the top by each of the digits along the side, and put the answer where they would meet. For instance, look where the 12 goes when you multiply 4 times 3.



STEP 3: There are three other multiplications to do: 7 times 3, 4 times 2 and 7 times 2. Do you see how all the answers are filled in? It doesn't matter which ones you multiply and put in first.

STEP 4: There are four diagonals (made up of little triangles) slanting to the right inside the rectangle. Starting with the one in the right bottom corner, add the digits inside each diagonal and fill put the sum in the "cup" under its diagonal. In the first diagonal is 4, so 4 is put in the "cup" under its diagonal (bottom right of rectangle). In the next diagonal are three digits a 1, 1 and 8 which adds up to 10, so put a 0 in the cup under that diagonal (at the bottom to the left) and a 1 (for the carry) in one of the triangles to the left. The next diagonal has a 2, 2, 0 and 1, which adds up to 5, and goes in the "cup" under its diagonal (left, bottom). In the last diagonal, there is just a 1, so a 1 is put in the "cup"under its diagonal (left, top). Look at how I filled in the numbers one at a time.



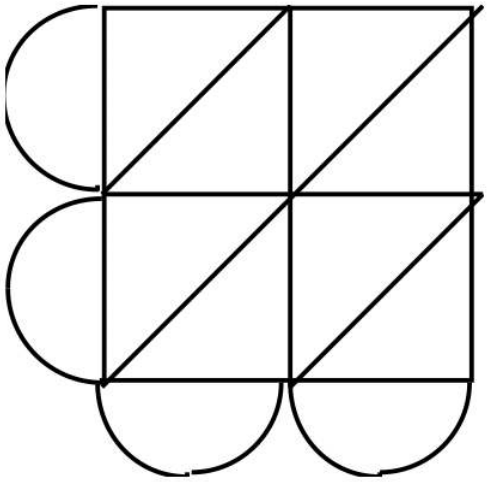
STEP 5: The numbers you filled in under each diagonal in the "cups" (outside the rectangle) form the answer. It is read by starting at the left top and reading down and then across the bottom. In this case, the answer is 1504. Isn't that cool? That's all there is to it!

Most of my students love this very old Lattice method of multiplying. It works as easily for small numbers as it does for large numbers. First, you simply do a series of simple single digit multiplications, and then you add, with carrying. Actually you could use the left to right (with underlining for carrying) method of addition for the last step. The main disadvantage is that you have to draw a lattice, which is a small price to pay to make multiplication easier and less complicated. Most likely, it takes less time than using the standard algorithm with carrying, because there's less chance of error and confusion with all that multiplying and carrying, etc. I propose all students learn multiplication this way first, and let them stay with it if they like it. Who needs to shorten it up and make it more complicated than it really is? Ah, well, I take it you get my point.

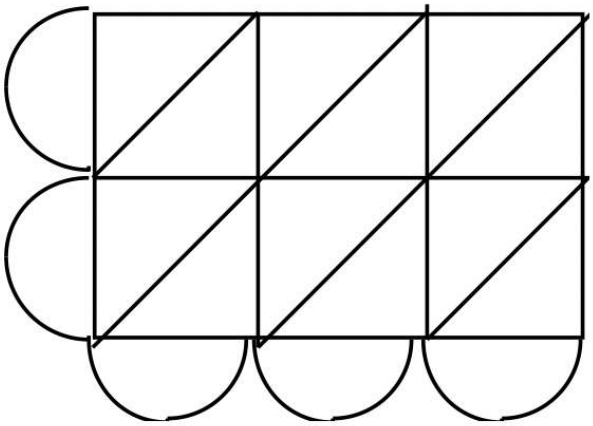
? Exercise 5

Use the lattices provided in a - d to multiply the given numbers. Then write the answer in the space provided. For e, draw your own lattice to compute the multiplication. By the way, you don't have to draw the little cups. That was my own little invention so it would be easy to identify the answer. There are two more problems (f and g) on the next page. Check your answers with a calculator

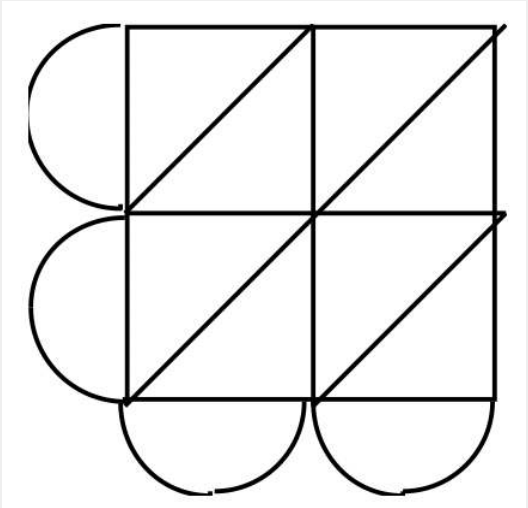
a. $63 \times 57 =$ ____



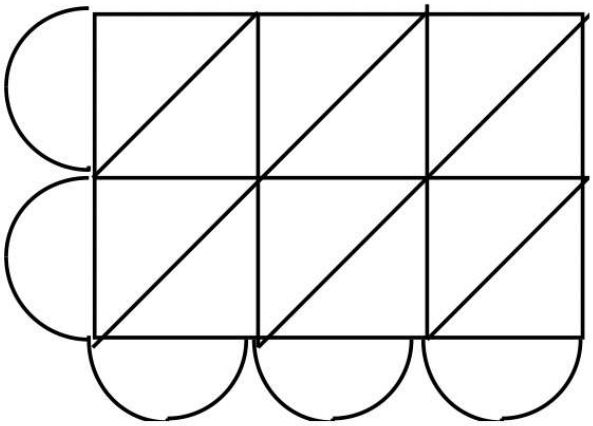
b. $342 \times 85 =$ ____



c. Make up a problem:



d. Make up a problem:



e. $58 \times 67 =$ ____ Draw your own lattice.

Exercise 6

a. Get out your Base Four Blocks. Take 41 units, and trade them in using your Base Four blocks. You should have some flats, longs and units which you can write as a numeral in Base Four.

Write the Base Four numeral here:

b. If you wanted to multiply 2 times that number, you would need to make two piles representing that number, join the two equal piles together (since multiplication is repeated addition) and then take the new pile and make trade-ins. Make two equal piles with the blocks, where each pile is the same as the one you formed from trade-ins in part a.Draw a picture of what the two piles look like.

c. Join the two piles together and make trade-ins to get a new base four numeral. Write this Base Four numeral here:

d. Fill in the first blank with what you got for part a and the second blank with what you got for part c. You just did a multiplication problem in Base Four using repeated addition!

$2_{\text{four}} \times$ _____ $_{\text{four}} =$ _____ $_{\text{four}}$

Exercise 7

a. Write the Base Four numeral from 6a here:

b. If you wanted to multiply 3 times that number, you would need to make three piles representing that number, join the three equal piles together (since multiplication is repeated addition) and then take the new pile and make trade ins. Make three equal piles with the blocks, where each pile is the same as the one you formed from trade-ins in part a. Draw a picture of what the three piles look like.

c. Join the three piles together and make trade-ins to get a new base four numeral. What Base Four numeral did you get?

d. Fill in the first blank with what you got for part a and the second blank with what you got for part c. You just did another multiplication problem in Base Four using repeated addition!

$3_{\text{four}} \times$ _____ four $=$ _____ four

Exercise 8

Using the same procedure as in 6 and 7, take out your base three blocks and do the following multiplication: $2_{\text{three}} \times 221_{\text{three}}$. Explain your work, and show pictures.

Exercise 9

If you wanted to do $10_{\text{three}} \times 221_{\text{three}}$, you would need to trade the long (10_{three}) in for three units, make three piles using Base three blocks representing 221_{three}, and do the same thing you did in exercise 7. Explain how you do this and show pictures.

This method of repeated addition with the blocks would be too cumbersome if we were working with higher numbers. For instance, if you were trying to use this method to multiply $321_{\text{four}} \times 312_{\text{four}}$, the first number when broken down into single units would consist of more than 50 units, and that would be way too many piles to make! Instead, we'll think about and figure out what it means to multiply using larger blocks than units. For instance, we'll figure out what a flat times a long is, or what a long times a flat is in any base! We'll be using these abbreviations: U for unit, L for long, F for flat, B for block, LB for long block, FB for flat block, and BB for block block.

Exercise 10

a. Get out your base three blocks. You're going to find the product of a long and a long similar to how you did products with the C-strips in Exercise Set 1. $L \times L$ is obtained by putting one long in the horizontal position and the second long in the vertical position, underneath and perpendicular to the first long. Then form a rectangle of longs underneath. Take the rectangle (don't include the top horizontal long) and trade up in base three, if possible. What is the product of long and a long in base three?

b. Get out your base four blocks. Find the product of long and a long in base four using the method explained in part a. What is the product of long and a long in base four?

c. Get out your base five blocks. Find the product of long and a long in base five using the method explained in part a. What is the product of long and a long in base five?

d. In any base, what is the product of a long and a long? $L \times L =$

Exercise 11

a. Get out your base three blocks. You're going to find the product of a flat and a long now. First, we need to convert the flat to longs so that we can make a horizontal train across the top of the T. Do this. Then, put a long in the vertical position, underneath and perpendicular to the long train of longs making the top of the T. Then form a rectangle of longs underneath. Take the rectangle (don't include any longs from the top horizontal part of the T) and trade up in base three, if possible. What is the product of flat and a long in base three?

b. Get out your base four blocks. Find the product of flat and a long in base four using the method explained in part a. What is the product of flat and a long in base four?

c. Get out your base six blocks. Find the product of flat and a long in base six using the method explained in part a. What is the product of flat and a long in base six?

d. In any base, what is the product of a flat and a long? $F \times L =$

? Exercise 12

a. Get out your base three blocks. You're going to find the product of a block and a long now. First, we need to convert the block to flats, and then the flat to longs so that we can make a horizontal train across the top of the T. Do this. Then, put a long in the vertical position, underneath and perpendicular to the long train of longs making the top of the T. Then form a rectangle of longs underneath. Take the rectangle (don't include any longs from the top horizontal part of the T) and trade up in base three, if possible. What is the product of block and a long in base three?

b. Get out your base four blocks. Find the product of block and a long in base four using the method explained in part a. What is the product of block and a long in base four?

c. Get out your base two blocks. Find the product of block and a long in base two using the method explained in part a. What is the product of block and a long in base two?

d. In any base, what is the product of a block and a long? $B \times L =$

? Exercise 13

Summarize the results from part d of exercises 10, 11 and 12.

a. $L \times L =$ _____

b. $L \times F =$ _____

c. $L \times B =$ _____

Use the commutative property of multiplication to compute the following:

d. $F \times L =$ _____

e. $B \times L =$ _____

? Exercise 14

In your own words, what happens when something is multiplied by a long?

Did you notice that when something is multiplied by a long, it bumps up to the next place value. This is analogous to multiplying by 10 in base 10, since 10 is a long in base ten. For instance, in base 10, 347 represents 3 hundreds, 4 tens and 7 ones; when you multiply 347×10, you get 3470, 3 thousands, 4 hundreds and 7 tens, the place value for each digit moved up one place value.

? Exercise 15

Consider multiplying $312_{\text{four}} \times 10_{\text{four}}$. This means you are multiplying a number in base four by a long in base four. Get out your base four blocks and represent the number 312_{four}.

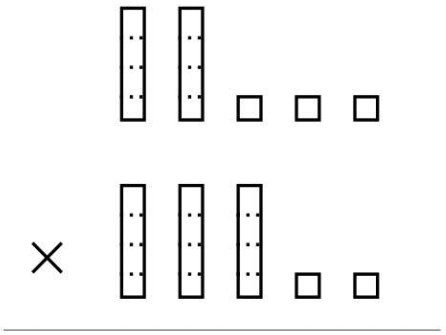
a. You have ____ flats, _____ longs and _____ units.

b. If you multiply each of these by the long, now how many of each type of place value block do you have in base four?

To multiply a two-digit numeral, X, in a base by some number, Y, in that base, the idea of repeated addition can be used. For instance, if we represent the number with base blocks, we'll want to create an "answer pile" where we put the blocks representing the product as we multiply by the two-digit numeral. Every unit in X multiplied by Y gives us one set of base blocks contained in

Y in the answer pile. Every long in X multiplied by Y gives us the next place value up for each of the base blocks contained in Y. This is much more complicated to explain than to do with the blocks. So, let's do an example.

Get out your Base Four Blocks. We'll multiply $23_{\text{four}} \times 32_{\text{four}}$. With your blocks, represent this as a multiplication problem. It should look something like this:



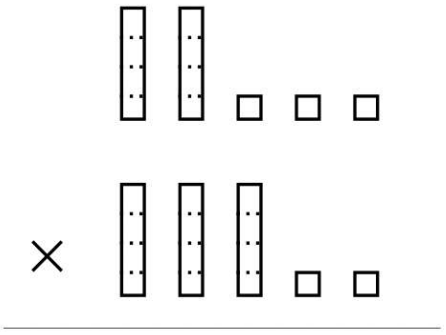
or you might abbreviate it on paper by writing it like this:

L L U U U

x L L L U U

Recall that each long in the bottom multiplied by the top makes each top piece move up to the next higher place value block. So, when the first bottom long is multiplied by the top number, 2 flats and 3 longs go into the answer pile. When the second bottom long is multiplied by the top number, another 2 flats and 3 longs go into the answer pile. When the third bottom long is multiplied by the top number, yet another 2 flats and 3 longs go into the answer pile. When I do this with the blocks, I usually move each part of the multiplier (bottom number) I've done so far to the right or out of the way. In any case, in your answer pile, you should now have 6 flats and 9 longs in the answer pile. We still have to multiply by each of the two units. Each unit multiplied by the top number gives a set of the top number; it's like multiplying by 1. So, when the first bottom unit is multiplied by the top number, 2 longs and 3 units go into the answer pile. When the second bottom unit is multiplied by the top number, another 2 longs and 3 units go into the answer pile. Create a place for the answer pile and do this with your blocks. Try it yourself, then check on the next page.

I'll show the answer pile below: each row shows what I got as I multiplied each bottom piece (3 longs and 2 units) by the top. By the way, you could have multiplied by the units first and then by the longs I'll do it that way for the next example. It all just eventually ends up in the answer pile and all exchanges have to made in the end, so you can write it in Base Four. This is $23_{\text{four}} \times 32_{\text{four}}$:

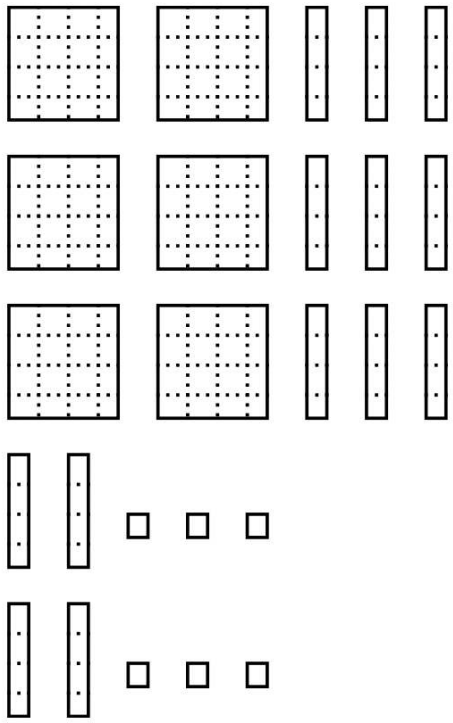


or you might abbreviate it on paper by writing it like this:

L L U U U

x L L L U U

THE Answer PILE IS SHOWN BELOW:

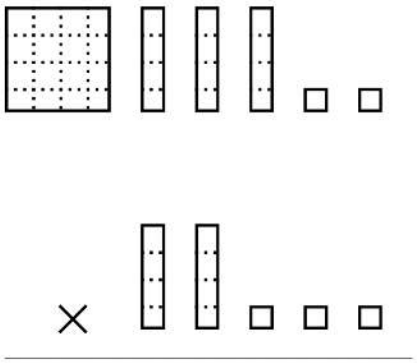


or you might abbreviate it on paper by writing it like this:

F F L L L
F F L L L
F F L L L
L L U U U
L L U U U

Now, make all your trade-ins with these pieces. You should end up with 2 blocks, 1 flat, 2 longs and 2 units, which is the number 2122_{four}, written in base four.

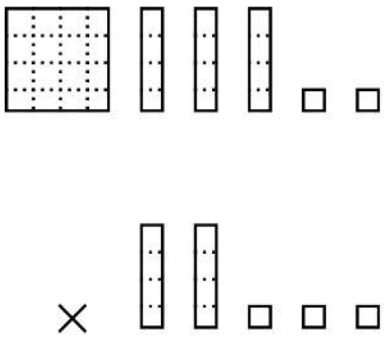
Next, we'll multiply $132_{\text{four}} \times 23_{\text{four}}$. With your blocks, represent this as a multiplication problem. It should look something like this:



or you might abbreviate it on paper by writing it like this:

F L L L U U
x L L U U U

Each unit multiplied by the top number gives a set of the top number it's like multiplying by 1. So, when the first bottom unit is multiplied by the top number, a flat, 3 longs and 2 units go into the answer pile. When the second bottom unit is multiplied by the top number, another flat, 3 longs and 2 units go into the answer pile. When the third bottom unit is multiplied by the top number, yet another flat, 3 longs and 2 units go into the answer pile. Create a place for the answer pile and do this with your blocks. When I do this with the blocks, I usually move each part of the multiplier I've done so far to the right or out of the way. In any case, in your answer pile, you should now have 3 flats, 9 longs and 6 units in the answer pile. We still have to multiply by each of the two longs. Recall that each long multiplied by the top makes each top piece move up to the next higher place value block. So, when the first bottom long is multiplied by the top number, a block, 3 flats and 2 longs go into the answer pile. When the second bottom long is multiplied by the top number, another block, 3 flats and 2 longs go into the answer pile. I'll show the answer pile below: each row shows what I got as I multiplied each bottom piece (3 units and 2 longs) by the top. By the way, you could have multiplied by the longs first and then by the units. It all just eventually ends up in the answer pile and all exchanges have to made in the end, so you can write it in Base Four. This is $132_{\text{four}} \times 23_{\text{four}}$:

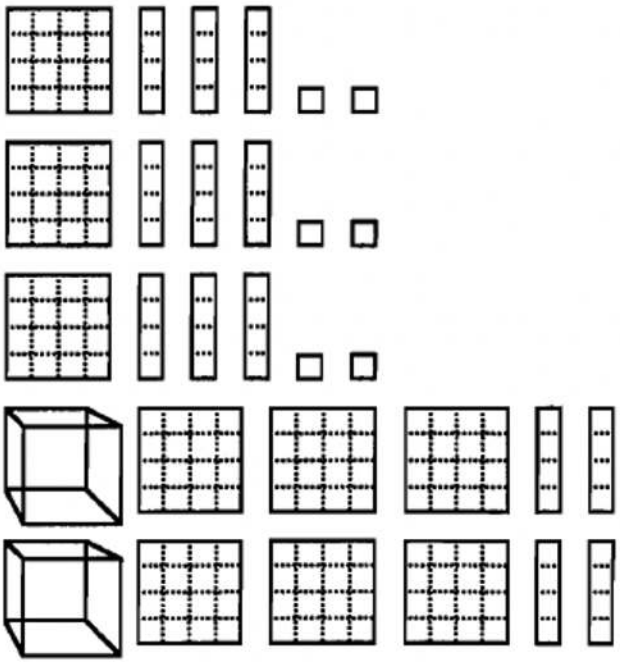


or you might abbreviate it on paper by writing it like this:

F L L L U U

x L L U U U

THE Answer PILE IS SHOWN BELOW:



or you might abbreviate it on paper by writing it like this:

F L L L U U

F L L L U U

F L L L U U

B F F F L L

B F F F L L

Now, make all your trade-ins with these pieces. You should end up with 1 long block, 1 block, 2 longs and 2 units, which is the number 11022_{four}, written in base four. Make sure you do and understand these last two problems before going on.

Exercise 16

Multiply each of the following using base blocks. Then explain and draw the steps in the space below as done in the previous two examples.

a. Use Base Four blocks to multiply $12_{\text{four}} \times 31_{\text{four}}$

b. Use Base Three blocks to multiply $112_{\text{three}} \times 21_{\text{three}}$

Exercise 17

a. Get out your base three blocks. You're going to find the product of a flat and a flat now. We need to convert the first flat to longs, make a train, and put this horizontal train across the top of the T. Do this. Then, convert the second flat to longs, make a train, put this train in the vertical position, underneath and perpendicular to the long train of longs making the top of the T. Then form a rectangle of a bunch of longs underneath. Take the rectangle (don't include any longs from the top horizontal part of the T) and trade up as much as possible in base three, if possible. What is the product of flat and a flat in base three?

b. Here's another way to find the product of a flat and a flat in base three. For each unit in the first flat, put a flat in an answer pile. Take all those flats in the answer pile and trade them in to see what you get. What is the product of flat and a flat in base

three?

c. Get out your base two blocks. Find the product of flat and a flat in base two using the method explained in part a or b. What is the product of flat and a flat in base two?

d. Complete the following: In any base, $F \times F =$ _____

Another way to think about what happens when you multiply by a flat is to think of a flat as a long times a long. Let's say you are multiplying something by a flat. First, multiply it by a long, which means it moves up one place value, and then multiply that answer by a long once more, which means it moves up one more place value. So, anything multiplied by a long moves up one place value, and anything multiplied by a flat moves up two place values. Multiplying by a flat is analogous to multiplying by 100 in base ten. 100b in any base b represents a flat. Think about what happens when a block is multiplied by anything.

Exercise 18

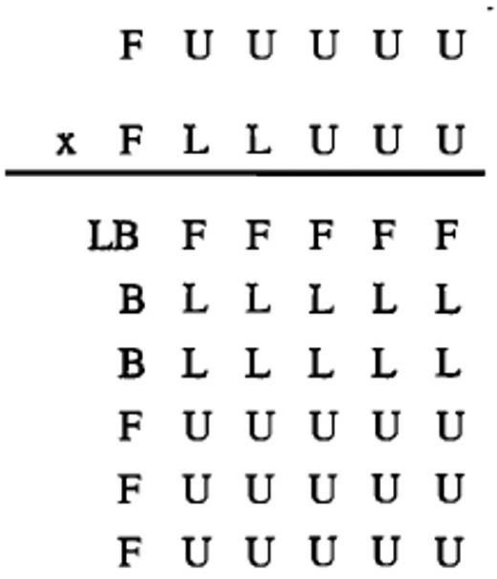
Complete the following:

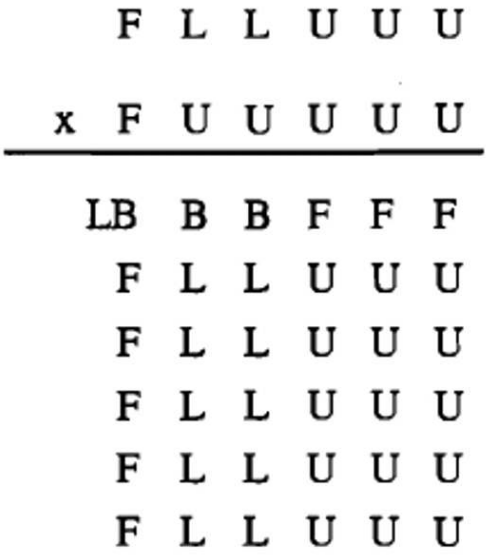
a. $U \times U =$ ____	f. $F \times U =$ _____
b. $U \times L =$ _____	g. $B \times U =$ _____
c. $U \times F =$ _____	h. $L \times L =$ _____
d. $U \times B =$ _____	i. $L \times F =$ _____
e. $L \times U =$ _____	j. $L \times B =$ _____

Exercise 19

Use base six blocks to multiply the following. $105_{\text{six}} \times 123_{\text{six}}$. If you have trouble, go on to the next page, where it is explained and then try it again. Show the steps and diagram.

If you had trouble with #19, below is an abbreviated picture of of the steps. The multiplication shown on the left is $105_{\text{six}} \times 123_{\text{six}}$ and the multiplication shown on the right is $123_{\text{six}} \times 105_{\text{six}}$ In both cases, I started multiplying with the biggest block at the bottom and moved left to right. In both cases, the final result is the same, and after all exchanges have been made, there is 1 long block, 3 blocks, 4 flats and 3 units, which is 13403_{six}, written in base six. It takes a lot of practice to really do this with the blocks. But it's fun once you get it!





Exercise 20

Use base three blocks to multiply $102_{\text{three}} \times 120_{\text{three}}$. Show steps like done in the example shown above.

We can shorten up this process of multiplying with the blocks even further. In exercise 18, you showed the products obtained when multiplying two base blocks together. The interesting thing is that a flat times a long is a block, no matter what base you are in. You can actually make up a multiplication chart for using blocks.

Exercise 21

Complete the rest of the multiplication chart below.

$\times$	**U**	**L**	**F**	**B**
U				
L	L		B	
F				FB
B		LB		

In our first example with the blocks, we computed $23_{\text{four}} \times 32_{\text{four}}$. Since we are multiplying 2 longs and 3 units by 3 longs and 2 units, the multiplication can be written as $(2L+3U) \times (3L+2U)$. Applying the distributive property (or FOIL), we get

$(2L+3U) \times (3L+2U)$

= $2L \times 3L + 2L \times 2U + 3U \times 3L + 3U \times 2U$

= $\mathbf{(2 \times 3)(L \times L) + (2 \times 2)(L \times U) + (3 \times 3)(L \times U) + (3 \times 2)(U \times U)}$

= 6F + 4L + 9L + 6U

= 6F + 13L + 6U **

At this point exchanges have to be made. In Base Four,

6F = 1B + 2F, 13L = 3F + 1L and 6U = 1L + 2U

= 1B + 2F + 3F + 1L + 1L + 2U

= 1B + 5F + 2L + 2U

The 5 flats are now exchanged for a block and a flat

= 1B + 1B + 1F + 2L + 2U

= 2B + 1F + 2L + 2U

which is written as 2122_{four} in base four. This is the same answer we got before.

**If I were doing 23×32 in a different base, all of steps up to this point are exactly the same. For instance, on the next page, the same basic problem is done in base six. All of the steps are the same, but notice that once exchanges are made, they are done with base six blocks in mind.

Also, it isn't necessary to write out the step that is in bold type. That's an application of the commutative and associative properties, and I wanted to show that step for clarification.

Here is the same basic problem done in Base Six.

$23_{\text{six}} \times 32_{\text{six}} = (2L + 3U) \times (3L + 2U)$
$= 2L \times 3L + 2L \times 2U + 3U \times 3L + 3U \times 2U$
$= (2 \times 3)(L \times L) + (2 \times 2)(L \times U) + (3 \times 3)(L \times U) + (3 \times 2)(U \times U)$
$= 6F + 4L + 9L + 6U$
$= 6F + 13L + 6U$
$= 1B + 2F + 1L + 1L$
$= 1B + 2F + 2L$
$= 1220_{\text{six}}$

Here is the same basic problem done in Base Nine.

$23_{\text{nine}} \times 32_{\text{nine}} = (2L + 3U) \times (3L + 2U)$
$= 2L \times 3L + 2L \times U3U \times 3L + 3U \times 2U$
$= 6F + 4L + 9L + 6U$
$= 6F + 13L + 6U$
$= 6F + 1F + 4L + 6U$
$= 7F + 4L + 6U$
$= 746_{\text{nine}}$

? Exercise 22

Show the steps using the blocks as shown in the previous examples to multiply the following.

a. $23_{\text{five}} \times 32_{\text{five}}$

b. $42_{\text{eight}} \times 53_{\text{eight}}$

Here is an example of using this method for numerals with more than two digits each. You still use the distributive property. Each term in the first parentheses is multiplied by each term in the second parentheses.

$212_{\text{five}} \times 102_{\text{five}} = (2F + 1L + 2U) \times (1F + 2U)$
$= 2F \times 1F + 2F \times 2U + 1L \times 1F + 1L \times 2U + 2U \times 1F + 2U \times 2U$
$= 2LB + 4F + 1B + 2L + 2F + 4U$
$= 2LB + 1B + 6F + 2L + 4U$
$= 2LB + 1B + 1B + 1F + 2L + 4U$
$= 2LB + 2B + 1F + 2L + 4U$
$= 22124_{\text{five}}$

$212_{\text{eight}} \times 102_{\text{eight}} = (2F + 1L + 2U) \times (1F + 2U)$
$= 2F \times 1F + 2F \times 2U + 1L \times 1F + 1L \times 2U + 2U \times 1F + 2U \times 2U$
$= 2LB + 4F + 1B + 2L + 2F + 4U$
$= 2LB + 1B + 6F + 2L + 4U$
$= 21624_{\text{eight}}$

In this last example, we got away without having to make any exchanges. We aren't so lucky in this next example, which is in base three. Base Three is harder to work with since it only takes three of something before you have to make an exchange. Sometimes it seems that all you are doing is exchanging!

$212_{\text{three}} \times 102_{\text{three}} = (2F + 1L + 2U) \times (1F + 2U)$
$= 2F \times 1F + 2F \times 2U + 1L \times 1F + 1L \times 2U + 2U \times 1F + 2U \times 2U$
$= 2LB + 4F + 1B + 2L + 2F + 4U$
$= 2LB + 1B + 6F + 2L + 4U$
$= 2LB + 1B + 2B + 2L + 1L + 1U$
$= 2LB + 3B + 3L + 1U$
$= 2LB + 1LB + 1F + 1U$
$= 3LB + 1F + 1U$
$= 1FB + 1F + 1U$
$= 100101_{\text{three}}$

? Exercise 23

Show the steps using the blocks as shown in the previous examples to multiply:

$212_{\text{four}} \times 102_{\text{four}}$

? Exercise 24

Show the steps using the blocks as shown in the previous examples to multiply:

a. $361_{\text{nine}} \times 15_{\text{nine}}$

b. $111_{\text{two}} \times 11_{\text{two}}$

Another way to multiply with the blocks is by using a chart. This is similar to how we did it for addition. It's actually very similar to doing it using the regular multiplication algorithm, except we don't have to write the carry as we go. You simply multiply each digit of one numeral by each digit in the other numeral, and put it in the proper column. Then, you add the like terms, and still do exchanges. Here is an example writing it out with the distributive property. Below, contrast with the method using a chart.

$23_{\text{six}} \times 32_{\text{six}} = (2L + 3U) \times (3L + 2U)$
$= 2L \times 3L + 2L \times 2U + 3U \times 3L + 3U \times 2U$
$= 6F + 4L + 9L + 6U$
$= 6F + 13L + 6U$
$= 1B + 2F + 1L + 1L$
$= 1B + 2F + 2L$
$= 1220_{\text{six}}$

In step one, the individual multiplications are done. If you get a 2-digit numeral, just write it in. No need to carry. In step 2, the like terms are added. Then, exchanges are made. I did the exchanges by exchanging the units first. In step 3, When 6 units were exchanged, I had 0 units and 1 more long, so I crossed off the 13 and put 14 in the next column to the left. In step 4, I exchanged 14 longs for 2 flats and 2 longs, so I crossed off 6 in the flat column, and added 2 more to get 8, and crossed off 14 in the long column since there were only 2 left. In the last step, 8 flats are exchanged for 1 block and 2 flats. I've shown the individual steps, all the work is shown in Step 5. That's all the space it takes.

Step 1

	B	F	L	U
			2	3
x			3	2
			4	6
		6	9	

Step 2

	B	F	L	U
			2	3
x			3	2
			4	6
		6	9	
		6	13	6

Step 3

	B	F	L	U
			2	3
x			3	2
			4	6
		6	9	
		6	~~13~~ 14	~~6~~ 0

Step 4

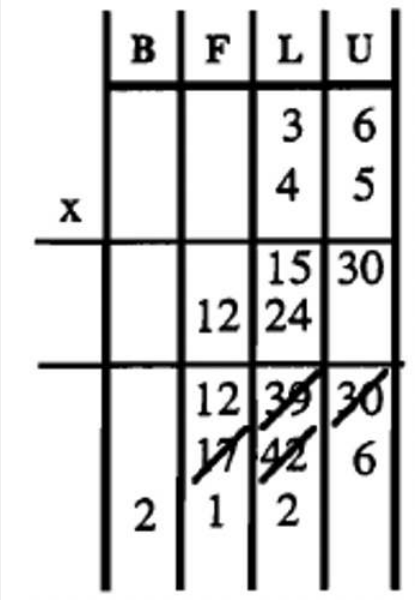
	B	F	L	U
			2	3
x			3	2
			4	6
		6	9	
		~~6~~ 8	~~13~~ ~~14~~ 2	~~6~~ 0

Step 5

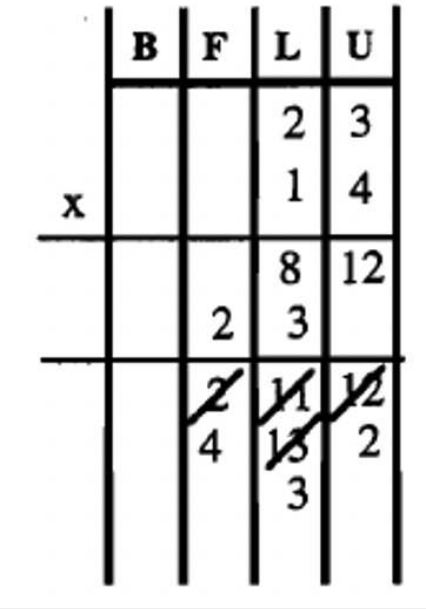
	B	F	L	U
			2	3
x			3	2
			4	6
		6	9	
	1	~~6~~ ~~8~~ 2	~~13~~ ~~14~~ 2	~~6~~ 0

Here are three more examples of using the charts to multiply with the blocks.

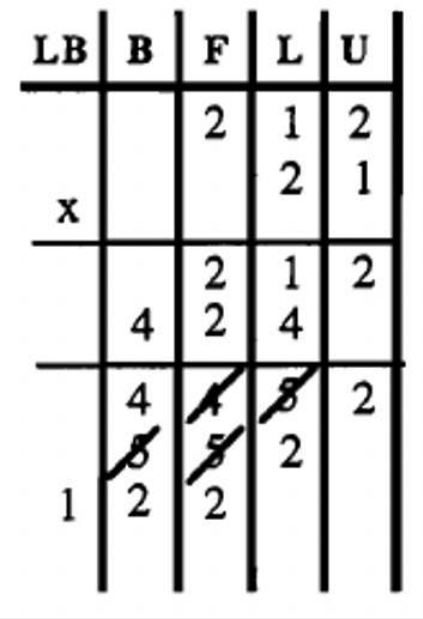
$36_{\text{eight}} \times 45_{\text{eight}}$	$23_{\text{five}} \times 14_{\text{five}}$	$212_{\text{three}} \times 22_{\text{three}}$



$= 2126_{\text{eight}}$



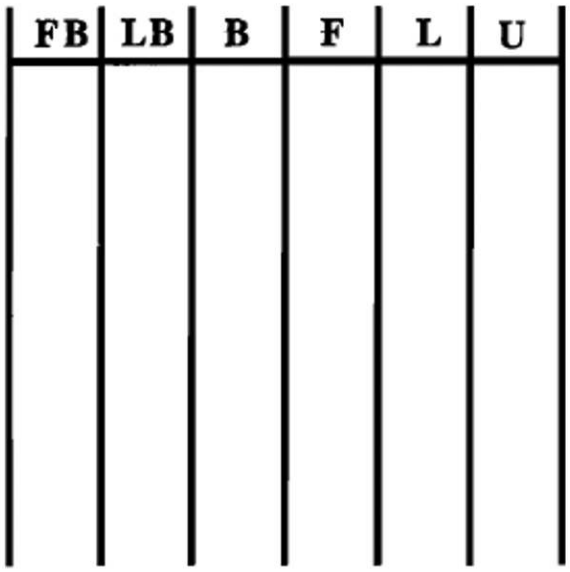
$= 432_{\text{five}}$



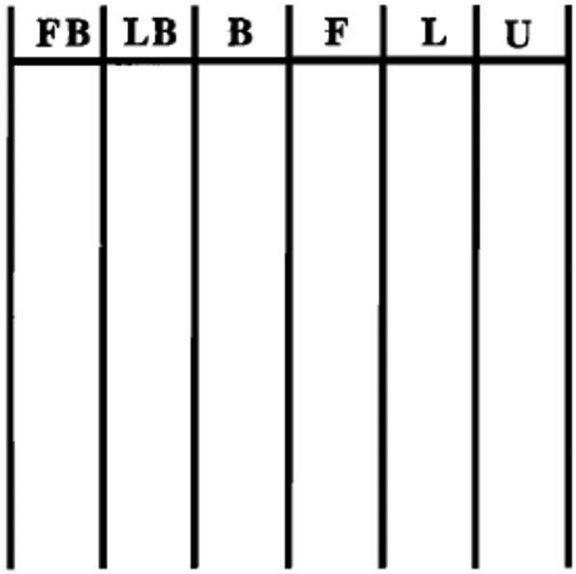
$= 12222_{\text{three}}$

Exercise 25

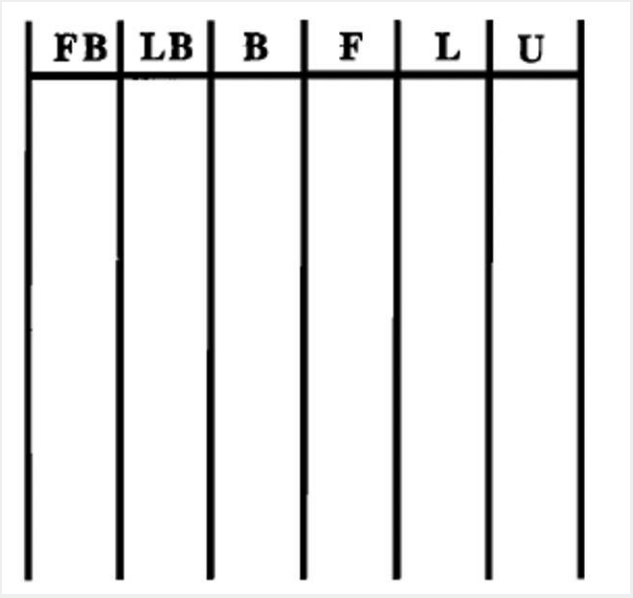
Use the base block charts, writing the numbers in terms of actual base blocks at first, as shown in the above examples, to compute the following. Show all your work and steps below.



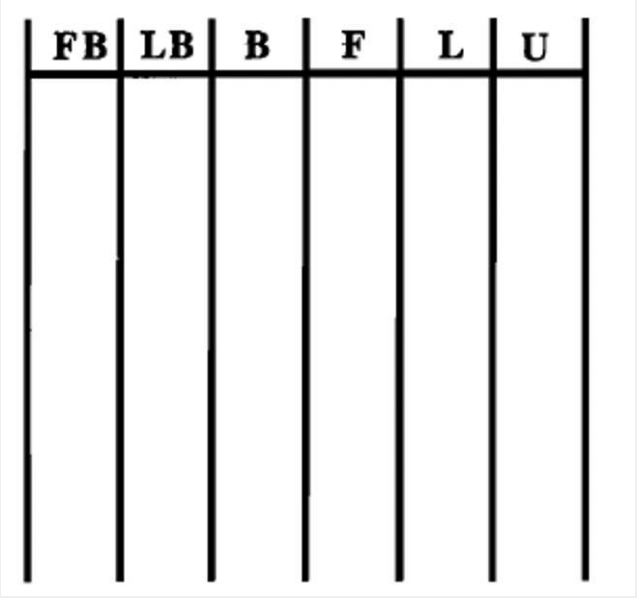
a. $34_{\text{six}} \times 25_{\text{six}} =$



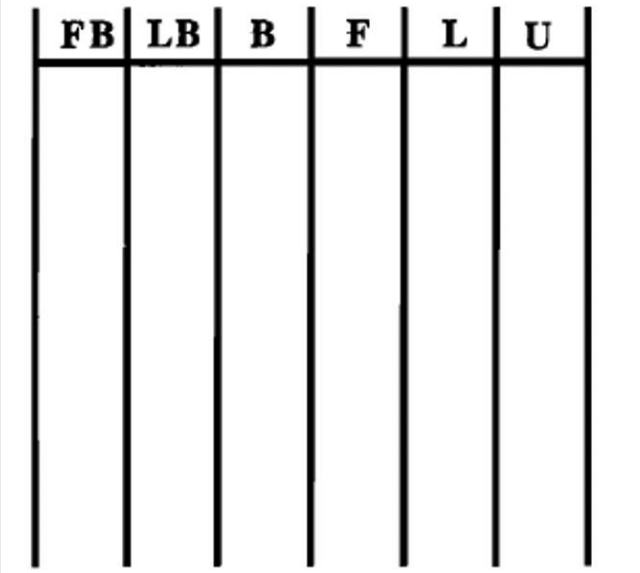
b. $32_{\text{eleven}} \times 4T_{\text{eleven}} =$



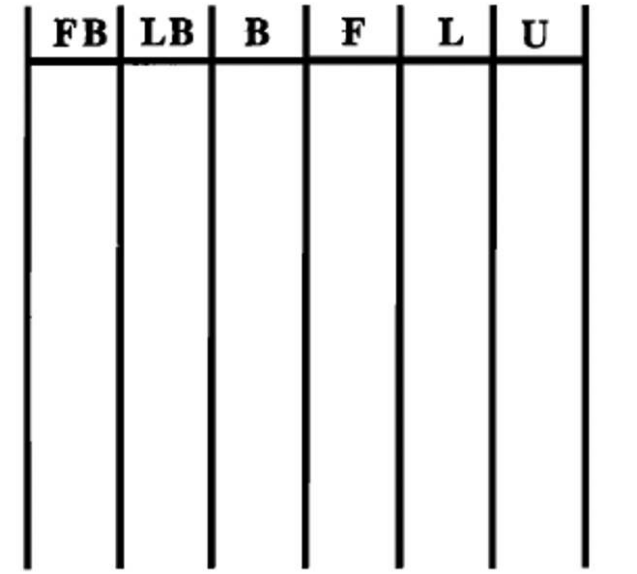
c. $111_{\text{two}} \times 101_{\text{two}} =$



d. $1E_{\text{twelve}} \times 53_{\text{twelve}} =$



e. $212_{\text{three}} \times 22_{\text{three}} =$



f. $23_{\text{four}} \times 32_{\text{four}} =$

Eventually, it would be nice to be able to do the multiplications without having to think in terms of the actual blocks. The traditional algorithm could be used. To do this, you have to make sure you immediately convert to the given base before writing the answer. For instance, $4_{\text{five}} \times 4_{\text{five}} = 31_{\text{five}}$ because 4 units times 4 units is 16 units, which in base five converts to 3 longs and 1 unit, which is written as 31_{five} in base five. A good start is to make up some multiplication tables in other bases. A base seven multiplication table is shown on the next page. Keep in mind that there are seven digits in base seven, so you'll have a table to fill in that is 7 rows by 7 columns. Because of the commutative property, there is symmetry in the table, which means many answers are duplicated. Furthermore, multiplying by 0 or 1 is trivial. At the top of the table, we'll indicate it is in base seven, and then leave off writing the base to the right and below each numeral.

Table 6.3.1: Base Seven Multiplication Table

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0

1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	31	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

Make sure you don't read these as base ten numerals. 26 means "two six, base seven", which is 2 longs and 6 units, or 20 base ten units! An easy way to get started is to do the two rows and two columns where you multiply by 0 and 1, then, go down the diagonal, from upper left to bottom right. Then, fill in the numbers above the diagonal. Using the commutative property, fill in the bottom of the diagonal.

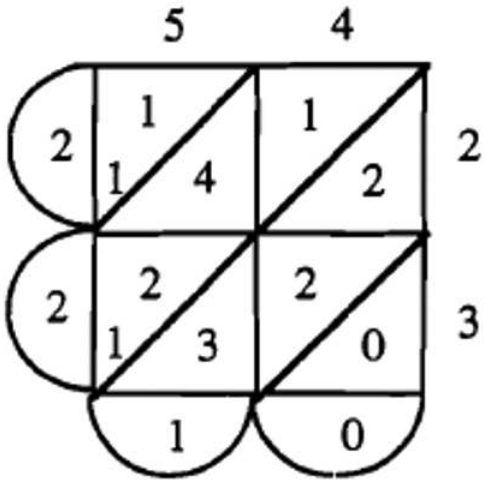
? Exercise 26

Make up a multiplication table for each base indicated.

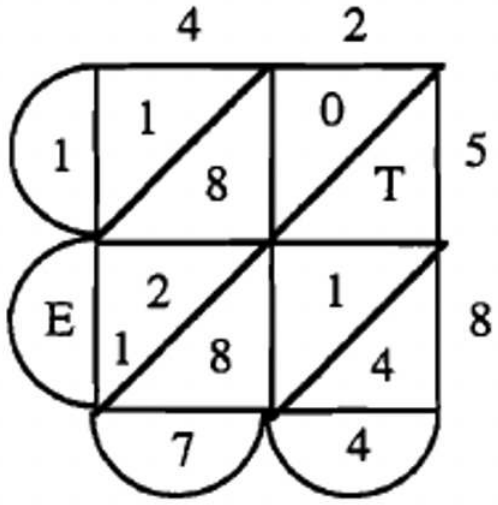
a. Base Two Multiplication Table	b. Base Three Multiplication Table
c. Base Six Multiplication Table	d. Base Eight Multiplication Table
e. Base Twelve Multiplication Table	

If you can quickly figure out the answer to multiplying two single digits in any given base, you can now use a lattice or the standard algorithm to do multiplication in different bases. Here are some examples of multiplying numbers in different bases using the lattice method.

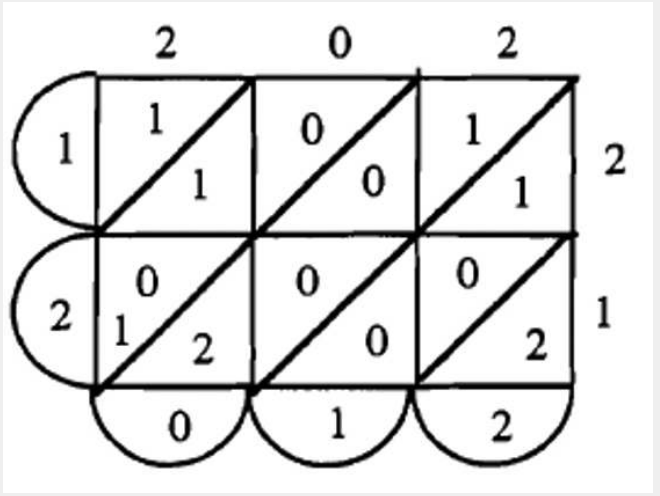
$54_{\text{six}} \times 23_{\text{six}} = 2210_{\text{six}}$



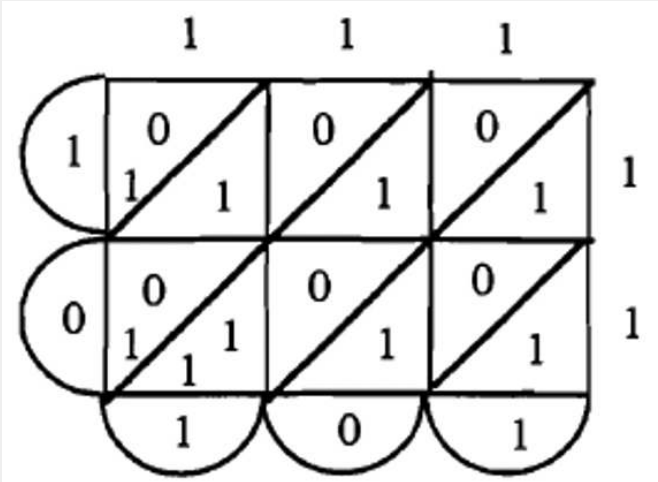
$42_{\text{twelve}} \times 58_{\text{twelve}} = 1E74_{\text{twelve}}$



$202_{\text{three}} \times 21_{\text{three}} = 12012_{\text{three}}$

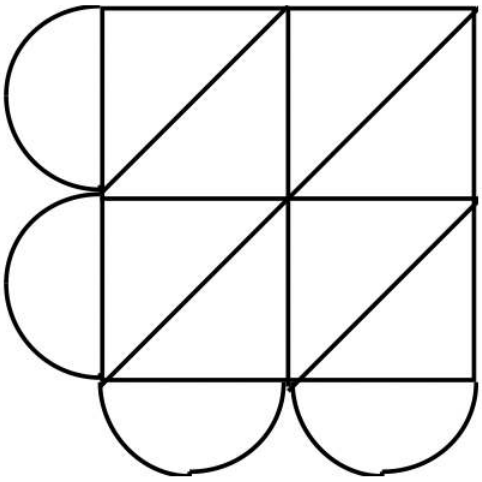


$111_{\text{two}} \times 11_{\text{two}} = 10101_{\text{two}}$

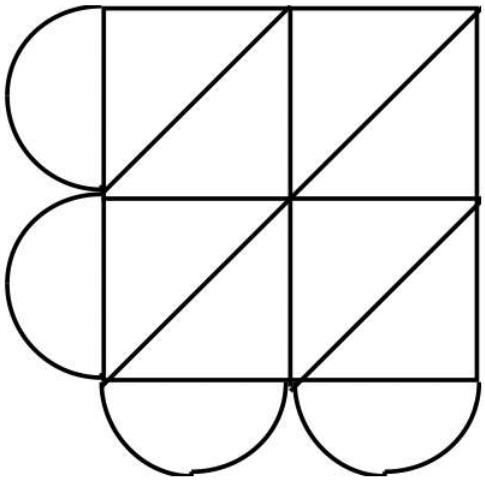


Exercise 27

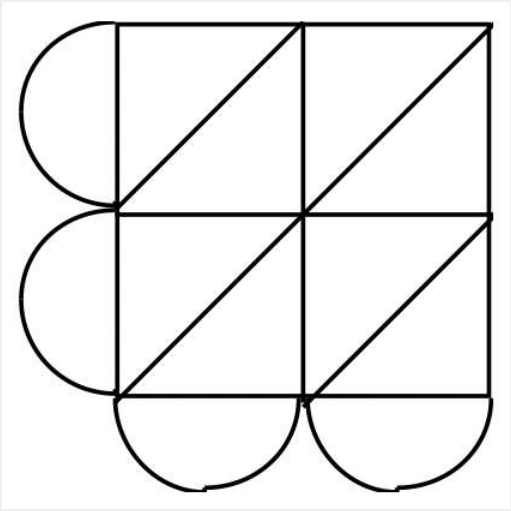
Use a lattice to multiply each of the numbers in the base given.



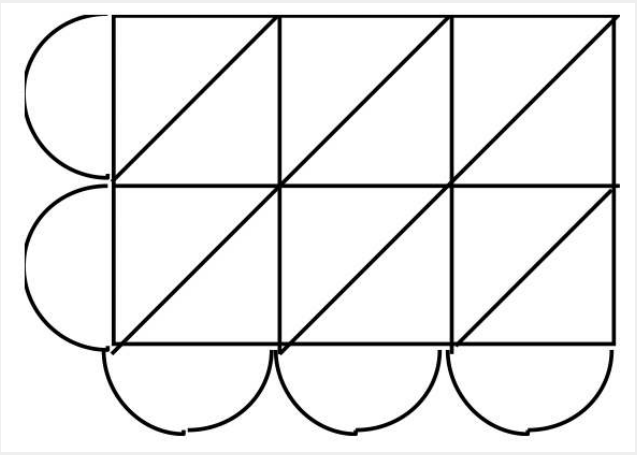
a. $32_{\text{five}} \times 14_{\text{five}} =$



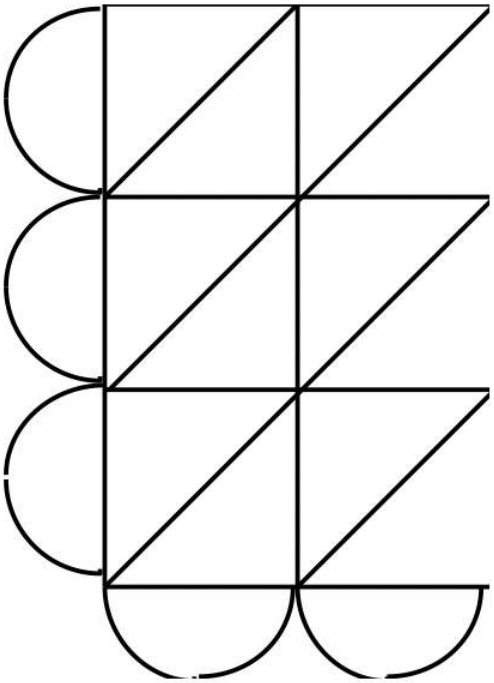
b. $93_{\text{thirteen}} \times 27_{\text{thirteen}} =$



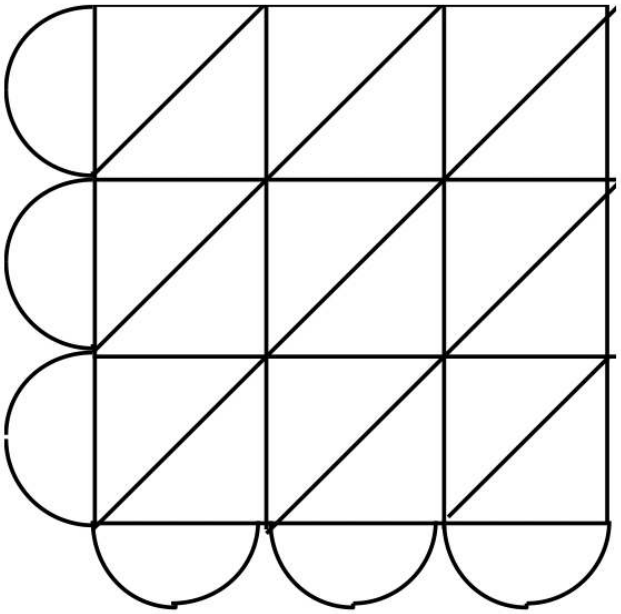
c. $12_{\text{three}} \times 21_{\text{three}}$ =



d. $524_{\text{seven}} \times 23_{\text{seven}}$ =



e. $23_{\text{seven}} \times 524_{\text{seven}}$ =



f. $212_{\text{four}} \times 132_{\text{four}}$ =

Finally, you could use the standard multiplication algorithm with carrying. This is the shortest algorithm in terms of space on the paper, but you have to be fairly adept at mental calculations, which doesn't come naturally to most of us when working in other bases.

Before going on to the standard multiplication algorithm, we need to notice a few more things. First of all, remember when we were working with the blocks to figure out a $L \times L$, etc.? When something was multiplied by a long, everything moved up one place value. In base ten, when you multiply a whole number by 10, the effect is that the answer is the original number with a zero tacked on to the end. That is, every digit moved up a place value. This is true in all bases. For example, $524_{\text{seven}} \times 10_{\text{seven}} = 5240_{\text{seven}}$ and $1T6_{\text{twelve}} \times 10_{\text{twelve}} = 1T60_{\text{twelve}}$. If you are multiplying by 100 in any base, two zeroes are tacked on to the end, and so on. This principle can be used to multiply 2000×400 by taking advantage of the commutative and associative properties of multiplication as shown below:

$2000 \times 400 = 2 \times 1000 \times 4 \times 100 = 2 \times 4 \times 1000 \times 100 = 8 \times 100000 = 800000$.

Of course, there is no need to write all the steps out. Simply multiply 2 times 4 and tack on five zeroes. Here's one in base eight: $3000_{\text{eight}} \times 500_{\text{eight}} = 1700000_{\text{eight}}$.

The **Partial Products Algorithm** uses the above fact. Doing the partial products algorithm is similar to how we used the distributive property to multiply with the blocks (back in exercise 22). For instance, to multiply, rewrite the problem like this:

$$\begin{aligned} 346 \times 72 &= (300+40+6) \times (70+2) \\ &= (300+40+6) \times 70 + (300+40+6) \times 2 \\ &= 300 \times 70 + 40 \times 70 + 6 \times 70 + 300 \times 2 + 40 \times 2 + 6 \times 2 \\ &= 21000 + 2800 + 420 + 600 + 80 + 12 \\ &= 24912 \end{aligned}$$

The partial products are shown in the third line in the example above. They are:

$300 \times 70, 40 \times 70, 6 \times 70, 300 \times 2, 40 \times 2$ and 6×2

It's easier to write this in a vertical format. I'll show this problem two ways. It doesn't matter which partial products you multiply first. The second way it is shown is more similar to our standard algorithm, where we start with the unit's value of the bottom number and eventually work up to the larger place values.

346	
$\times \underline{72}$	
21000	(300×70)
2800	(40×70)
420	(6×70)
600	(300×2)
80	(40×2)
$\underline{12}$	(6×2)
24912	

346	
$\times \underline{72}$	
12	(2×6)
80	(2×40)
600	(2×300)
420	(70×6)
2800	(70×40)
$\underline{21000}$	(70×300)
24912	

On the left the first three partial products are 346×70 and the second three partial products are 346×2. On the right, the first three partial products are 2×346 and the second three partial products are 70×346.

Our standard algorithm is simply a shortening up of the partial products algorithm. We don't write all the zeroes and we doing the carrying involved with adding more than one partial product at a time in our head. Sure, it's shorter, but it is easier to make mistakes.

346	
$\times \underline{72}$	
692	(2×346)
$\underline{24220}$	(70×346)
24912	

Usually we just move over one place to the left to account for the zero in the one's place and don't write the zero in 24220. We "think" 7 times 346.

? Exercise 28

For each problem, write out the partial products in a vertical format to multiply the numbers together. Show your work, identifying the multiplication for each partial product as shown in the example on the bottom of the previous page.

a. 538 $\times \underline{34}$	b. 257 $\times \underline{941}$

Finally, I'll provide some examples of what the multiplication looks like when the standard algorithm is used to multiply numbers in different bases. You have to remember to write down the number and all carries in the base you are working in. We tend to always put the numeral with the most digits on the top, but this is not necessary. Here are three ways to multiply $563_{\text{eight}} \times 24_{\text{eight}}$.

IMPORTANT NOTE:

Each numeral below is in base eight, even though the "eight" isn't written in each intermediate step. E.g., 4×3 in the first multiplication below, is done in base 8; therefore $4 \times 3 = 14$.

$$\begin{array}{r} 563_{\text{eight}} \\ \times \underline{24_{\text{eight}}} \\ 2714 \\ \underline{13460} \\ 16374_{\text{eight}} \end{array}$$

$$\begin{array}{r} 24_{\text{eight}} \\ \times \underline{563_{\text{eight}}} \\ 74 \\ 1700 \\ \underline{14400} \\ 16374_{\text{eight}} \end{array}$$

$$\begin{array}{rr} 563_{\text{eight}} & \\ \times \underline{24_{\text{eight}}} & \\ 14 & (4 \times 2) \\ 300 & (4 \times 60) \\ 2400 & (4 \times 500) \\ 60 & (20 \times 3) \\ 1400 & (20 \times 60) \\ \underline{12000} & (20 \times 500) \\ 16374_{\text{eight}} & \end{array}$$

Multiply the following numbers together using **two different algorithms** (or switching the order of the numbers around as shown above). Show your work. Do not do it by converting each number to base ten, multiplying them together, and then convert back to the given base. Use each algorithm at least one time; lattice, partial products, standard algorithm, using the blocks with charts, using the distributive property with or without the blocks. You have lots of choices, but always keep in mind what base you are working in. You need to know all the algorithms for the exam so be sure to try them all. Okay, have fun!

Exercise 29

$23_{\text{five}} \times 43_{\text{five}}$

First way:	Second way:

Exercise 30

$612_{\text{seven}} \times 35_{\text{seven}}$

First way:	Second way:

Exercise 31

$1011_{\text{two}} \times 1101_{\text{two}}$

First way:	Second way:

Exercise 32

$2121_{\text{three}} \times 202_{\text{three}}$

First way:	Second way:

Exercise 33

$4T1_{\text{tweleve}} \times 27_{\text{tweleve}}$

First way:	Second way:

Exercise 34

$423_{textsix} \times 145_{\text{six}}$

First way:

Second way:

Exercise 35

$2212_{\text{three}} \times 210_{\text{three}}$

First way:

Second way:

Exercise 36

$5421_{\text{six}} \times 44_{\text{six}}$

First way:

Second way:

Exercise 37

$10110_{\text{two}} \times 1111_{\text{two}}$

First way:

Second way:

Exercise 38

$E4T_{\text{twelve}} \times E2_{\text{twelve}}$

First way:

Second way:

Exercise 39

$6661_{\text{eight}} \times 72_{\text{eight}}$

First way:

Second way:

Exercise 40

$1212_{\text{four}} \times 302_{\text{four}}$

First way:

Second way:

Exercise 41

$4234_{\text{five}} \times 244_{\text{five}}$

First way:

Second way:

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6.4: Additional Multiplication Topics

***You will need**: Positive and Negative Counters (Material Cards 18A and 18B)*

Now, we'll explore how to multiply positive and negative numbers using the counters. Let's look again at a definition of multiplication for whole numbers.

Definition: Multiplication of Whole Numbers using the Repeated-Addition Approach

If m and n are whole numbers, then

$$m \times n = n + n + n + n + \ldots + n,$$

where there are m add ends of n in this sum.

This definition makes sense as long as m and n are positive numbers. In fact, we can even make sense of and modify this definition using counters as long as m is positive by rewording the definition slightly, as stated below:

Definition: Multiplication of a Whole Number times an Integer using the Repeated-Addition Approach, using positive and negative counters

If m is a whole number and n is any integer, $m \times n$ is obtained by combining m subsets of a collection of counters representing n. The number that the resulting collection represents is the answer to the problem $m \times n$.

Exercise 1

This exercise shows you one way to use the above definition to multiply 3×-4. In this problem, the definition can be used because 3 is a whole number and -4 is an integer. We need to combine 3 subsets of a collection of counters that represent -4.

a. Use your positive and negative counters to represent a collection of counters that represent -4. For this exercise, choose a collection of 5 negatives and 1 positive. Show what your collection looks like below:
b. Now, form two more collections (for a total of 3 subsets) of counters that you had for part a. Combine the counters together and show what the large collection looks like below:
c. After removing any red-green pairs (zero) from your collection in part b, show the collection that remains below. What number does this represent? _____

Exercise 2

Do 3×-4 again, using a different collection of counters to represent -4.

a. Use your positive and negative counters to represent a collection of counters that represent -4. This time, choose a collection of 6 negatives and 2 positives. Show what your collection looks like below:
b. Now, form two more collections (for a total of 3 subsets) of counters that you had for part a. Combine the counters together and show what the large collection looks like below:
c. After removing any red-green pairs (zero) from your collection in part b, show the collection that remains below. What number does this represent? _____

Exercise 3

Okay, let's do 3×-4 one more time, choosing the easiest way to represent -4.

a. Form a collection of counters to represent -4. Do it the easy, natural way, using the least number of counters possible. Show what your collection looks like below:
b. Now, form two more collections (for a total of 3 subsets) of counters that you had for part a. Combine the counters together and show what the large collection looks like below:
c. What number does the collection in part b represent? _____

Well, I hope you got the answer of -12 for exercises 1, 2 and 3, since $3 \times -4 = -12$! This illustrates that it doesn't matter exactly which collection of counters you use to represent -4, as long as the collection really is -4. To compute 3×-4, you could combine 3 subsets of a collection of 8 reds and 4 greens, or you could combine 3 subsets of a collection of 7 reds and 3 greens, etc. You'll always end up with a collection that represents -12. For exercise 3, did you choose 4 negatives as your representation? If so, did you notice you didn't have to remove any red-green pairs to answer part c? From now on, let's do it the easy way, using the simplest collection possible.

? Exercise 4

Use your counters to do each of the following multiplication problems using the definition of multiplying a whole number by an integer. Then, explain what the multiplication problem given *means* in terms of the counters, and explain and show the individual steps. Use the following example as a model.

? Example

Multiply $\mathbf{2 \times -6}$.

Solution

Multiplying 2×-6 means to combine 2 subsets of 6 red counters. The number that the resulting collection represents is the product (answer). 2×-6 = RRRRRR + RRRRRR = RRRRRRRRRRRR = -12

a. Multiply 4×-2

This means to

b. Multiply 3×5

This means to

c. Multiply 5×-3

This means to

d. Multiply 7×2

This means to

e. Multiply 0×-3

This means to

Okay, now that you've mastered how to multiply a whole number by an integer, let's work on how we can use the counters to multiply a negative integer by an integer. Let's look once more at the definition for multiplying $m \times n$, when m is a whole number.

Definition: Multiplication of a Whole Number times an Integer using the Repeated-Addition Approach, using positive and negative counters

If m is a whole number and n is any integer, $m \times n$ is obtained by combining m subsets of a collection of counters representing n. The number that the resulting collection represents is the answer to the problem $m \times n$.

If m is negative, this definition doesn't make sense since you certainly can't combine a negative number of subsets! The way we'll revise this definition to include the possibility that m may be negative is to agree that if m is negative, we REMOVE m subsets of a collection of counters representing n. The trick to doing this is to remove the subsets from a collection of counters representing zero. So, here is the comprehensive definition for multiplying any two *integers*, using positive and negative counters.

Multiplication of two Integers, using positive and negative counters

- ***Case 1:*** If m is a whole number and n is any integer, $m \times n$ is obtained by combining m subsets of a collection of counters representing n. The product of m and n, $m \times n$, is the number that the resulting collection represents.

- ***Case 2:*** If m is negative and n is any integer, $m \times n$ is obtained by removing |m| subsets of a collection of counters representing n from a collection of counters representing zero. The product of m and n, $m \times n$, is the number that the resulting collection represents.

Exercise 5

This exercise shows you how to use the above definition to multiply -4×3. For this problem, we need to remove 4 subsets of a collection of counters that represents 3 from a collection of counters representing zero. The simplest collection to represent 3 is 3 positives, or 3 green counters. (We *could* use a more complicated collection and still arrive at the same answer. We did this kind of an exercise in exercises 1 - 3. Convince yourself it wouldn't matter here, either.)

a. We first need to form a collection of counters that represents zero so that it will be possible to remove 4 subsets of 3 green counters. For this example, make a collection of 14 red and 14 green counters. Write down what your collection looks like here:
b. From your collection, remove a subset of 3 green counters. Then, remove 3 more subsets of 3 green counters. You have just removed 4 subsets of 3 green counters from zero. To show it on paper, circle a subset of 3 green counters in the collection above in part a. Then, circle 3 more subsets of 3 green counters so that four separate subsets are circled. After you remove the counters (which are shown by what you circled in part a), show what is left in your collection below.
c. Remove any red-green pairs (zero) from your remaining collection. Show this on paper by crossing off or circling any red-green pairs (zero) from your collection shown in part b. Show the collection that remains below. What number does this collection of counters represent?

Exercise 6

Use the definition again to multiply -4×3. Remember, you need to remove 4 subsets of a collection of counters that represents 3.

a. We first need to form a collection of counters that represents zero so that it will be possible to remove 4 subsets of 3 green counters. This time, put in the smallest collection of counters possible so that you'll be able to remove 4 subsets of 3 green counters. Write down what your collection looks like here:
b. From your collection, remove four subsets of 3 green counters (take out one subset of 3 green counters at a time). You have just removed 4 subsets of 3 green counters from zero. To show it on paper, circle 4 different subsets of 3 green counters in the collection shown in part a. After you remove the counters (which are shown by what you circled in part a), show what is left in your collection below.
c. What number does the above collection represent?

Did you notice that if you start out with a minimal collection to represent zero, you don't have to remove any red-green pairs when you get to part c? That is the easiest way to do it because you know how to calculate what to take out before you start the problem. However, if you were teaching this to someone who couldn't figure that out ahead of time, you might always start out with zero being 20 (or some other agreed upon number) of each counter.

On the next page, the steps are shown for the example below– how to use your counters to do a multiplication problem when the first number is a negative integer. It's necessary to explain what the multiplication problem given means in terms of the counters, and then explain and show the individual steps. Use the example as a model for the exercises that follow.

Example

Multiply -4×-3 by using the definition of multiplying integers.

Solution

Multiplying -4×-3 means to remove 4 subsets of 3 red counters from a representation of zero. The number that the resulting collection represents is the answer.

Since there are 12 greens remaining, the answer is +12. Didn't you always wonder why a negative number times a negative number equaled a positive number? Using the definition of multiplying integers with counters, you can really see why it's true. After you do

a few more problems, feel free to go out and show all of your friends who still don't understand it or just memorized the rules because someone told them that's the way it is and to just accept it. Okay, enough of that unbridled enthusiasm from me for now.

It's time for you to work a few problems. Some have negative numbers before the multiplication sign (start with a representation of zero and remove subsets as in the last example) and some have whole numbers before the multiplication sign (so just combine subsets together like you did in the earlier exercises of this exercise set). It's the number *before* the multiplication sign that will indicate which case of the definition you will use.

Exercise 7

Use your counters to do each of the following multiplication problems using the definition of multiplying two integers with positive and negative counters. Then, *explain what the multiplication problem given* ***means*** in terms of the counters, and explain and show each of the individual steps. Use the example above as a model when the first number is negative.

a. -5×3 ____ This means to _____

Show work and all steps below. Then, state the answer to the problem.

b. -3×2 ____ This means to _____

Show work and all steps below. Then, state the answer to the problem.

c. 2×-3 ____ This means to _____

Show work and all steps below. Then, state the answer to the problem.

NOTE: Although the answer to part b is the same as part c due to the commutative property of multiplication, the problems **mean** different things, the steps are not alike and the problems are done differently.

d. -2×3 ____ This means to _____

Show work and all steps below. Then, state the answer to the problem.

e. 3×2 ____ This means to _____

Show work and all steps below. Then, state the answer to the problem.

f. 0×-4 ____ This means to _____

Show work and all steps below. Then, state the answer to the problem.

g. -4×0 (this means something different than 0×-4) ____ This means to _____

Show work and all steps below. Then, state the answer to the problem.

Let's do some more problems when the first integer is negative, using a chart to keep track of the counters. The first column will be the multiplication problem, the second will show what representation of zero is being used, the third will explain the meaning of the multiplication problem (what has to be done in all of these cases, subsets are being removed from the representation of zero), the fourth will show how many of each of the counters are left after the removal of the subsets, and the last column will be the answer, obtained from the representation shown in the fourth column.

Exercise 8

Fill in all of the blanks. FILL IN THE ENTIRE TABLE ALL THE WAY TO THE BOTTOM!! You should be using your real manipulatives (red and green counters) as you do (most of) these problems. I've done the first one for you to use as a model.

problem	Counters for Zero	**The meaning of the problem**	counters remaining	answer
-3×6	20G 20R	remove 3 sets of 6 G from zero	2G 20R	-18
-3×6	23G 23R			
-3×6	18G 18R			

problem	Counters for Zero	**The meaning of the problem**	counters remaining	answer
-4×4	18G 18R			
-4×-4	18G 18R			
-5×2	14G 14R			
-2×5	13G 13R			
-5×-2	11G 11R			
-2×-5				
-3×3	10G 10R			
-3×-3	12G 12R			
-7×2				
-2×8	18G 18R			
-2×-8	18G 18R			

After doing all of the exercises in this Exercise Set, the rule for multiplication of integers should make sense.

Rule for Multiplying Two Integers Together:

- **To multiply two integers, first multiply the absolute values of the integers. To determine the sign of the product: it is positive if both integers have the same sign (both positive or both negative); otherwise it is negative (if one of the integers is positive and the other integer is negative). If one of the integers is zero, the answer is zero.**

If you are multiplying more than two integers together, the sign of the product can be determinined by how many negative numbers are being multiplied. For every two negative numbers being multiplied together, the answer is positive. Therefore, if there are an even number of negative numbers being multiplied, the sign of the product is positive. If there are an odd number of negative numbers being multiplied, the sign of the product is negative. If one of the integers is zero, the answer is zero.

Although we haven't covered division in this exercise set, the rule for determining the sign of a quotient is the same as the rule for determining the sign of a product.

? Exercise 9

Determine if each product is negative (–), positive (+) or zero (0). Do not compute

Remember that for a set to be closed under multiplication, the product of any two elements in the set must be in the set. To prove a set is not closed under multiplication, you need to provide a counterexample.

? Exercise 10

For each of the following sets, determine if the set is closed under multiplication. Provide a counterexample if it is not closed.

a. Integers
b. Positive Integers
c. Negative Integers
d. {-1, 0}
e. {1, -1}
f. {-1, 0, 1}

Less than (<) and Greater than (>) signs are used to order numbers.](a < b\) if a is to the left of b on the number line. $a < b$ (read "a is less than b") can also be written as $b > a$ (read "b is greater than a.)

Exercise 11

Decide which of the following are true if a, b and c are any integers, p is a positive integer and n is a negative integer. Provide a counterexample if it is false.

a. If $a < b$ and $b < c$, then $a < c$
b. If $a < b$, then $a + c < b + c$
c. If $a < b$, then $ap < bp$
d. If $a < b$, then $ap > bp$
e. If $a < b$, then $an > bn$
f. If $a < b$, then $an < bn$

Exercise 11a, 11b, 11c, and 11e are the four properties of Ordering for Integers. 11a is called the Transitive Property for Less Than. 11b is called the Property of Less Than and Addition. 11c is the Property of Less Than and Multiplication by a Positive 11e. is the Property of Less Than and Multiplication by a Negative. If the less than symbols in 11a, 11b and 11c were replaced with greater than symbols, you would have the corresponding properties of greater than. For part 11e, if both signs were switched, you would have the Property of Greater Than and Multiplication by a Negative.

Exercise 12

Fill in the following properties, if a, b and c are any integers, p is a positive integer and n is a negative integer.

a. Transitive Property for Greater Than:
b. Property of Greater Than and Addition:
c. Property of Greater Than and Multiplication by a Positive:
d. Property of Greater Than and Multiplication by a Negative:

11.a,b and c are similar to properties of equality. The difference between equalities and inequalities (whether there is a less than or greater than symbol) if that when both sides of an inequality are multiplied by a negative number, the inequality sign changes directions.

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6.5: Practice

- Submit homework separately from this workbook and staple all pages together. (One staple for the entire submission of all the unit homework)
- Start a new module on the front side of a new page and write the module number on the top center of the page.
- Answers without supporting work will receive no credit.
- Some solutions are given in the solutions manual.
- You may work with classmates but do your own work.

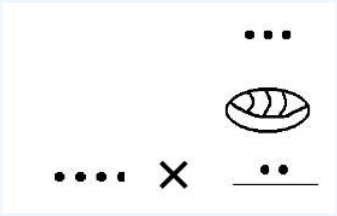
HW #1

Use the repeated addition definition of multiplication to compute the following. First, write out the meaning of the multiplication, and then compute the answer.

a. 8×4	b. 4×11

HW #2

Use the repeated addition definition of multiplication to compute the following. Make sure you write out the meaning of the multiplication **in the system given** showing all of the work and exchanges. Do not do the problem in base ten.



HW #3

Use the definition of multiplication for trains to compute the following. Then translate to make an equation using numbers.

a. $P \times W =$ _____ translates to _____
b. $K \times W =$ _____ translates to _____

HW #4

Write the Cartesian product

a. {3, x} $\times$ {0, 1, 6}	b. {a, b, 0} $\times$ {1, 2}

HW #5

Use the set theory definition of multiplication to verify $3 \times 2 = 6$

HW #6

Complete the following using your base blocks. Show all work.

a. $F \times L$	b. $B \times L$	c. $F \times F$	d. $F \times B$	e. $B \times F$

? HW #7

Write the base four multiplication table

? HW #8

Compute the following $2506_{\text{seven}} \times 451_{\text{seven}}$ using the lattice method.

a. 2506×451	b. $2506_{\text{seven}} \times 451_{\text{seven}}$

? HW #9

Compute 19×24 and 24×19 using the Duplation method. Show all steps.

? HW #10

For each set of three numbers given, illustrate an example of the associative property of multiplication, and then illustrate an example of the distributive property of multiplication over addition. Follow procedures outlined in this module

a. $2_{\text{five}}, 3_{\text{five}}, 4_{\text{five}}$	b. $3_{\text{six}}, 4_{\text{six}}, 5_{\text{six}}$

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CHAPTER OVERVIEW

7: Division

7.1: Division
7.2: Division Algorithms
7.3: Division in other Bases
7.4: Practice

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7.1: Division

Division can be represented with may symbols. In this section, instead of $p \div q$ or p/q, we will use $\frac{p}{q}$.

Definitions: Quotient, Dividend, Divisor, and Remainder

If p is any real number and q is any *nonzero* real number, then $\frac{p}{q}$ is the **quotient** of p and q. In this division equation, p is called the **dividend**, and q is called the **divisor**. If p and q are positive integers (still with $p \neq 0$, then there exists nonnegative integers d and r such that $0 \leq r < q$ and $\frac{p}{q} = d + \frac{r}{q}$, or equivalently, $p = dq + r$. The nonnegative integer r is called the **remainder**.

As there were multiple perspectives of multiplication, there are multiple perspectives of division. Many children first learn learn division in terms of sharing, but the method most children in the U.S. use for performing division is based on subtraction. Let's consider that perspective first.

Repeated Subtraction

Definition: Division Repeated Subtraction

If p, q, and d are positive integers and r is a nonnegative integer less than q such that $\frac{p}{q} = d + \frac{r}{q}$, then d is the number of times q can be subtracted from p without going below 0. This is the **repeated subtraction perspective of division**.

At some point, you have seen long division. Long division is based on the repeated subtraction perspective. Consider the following example.

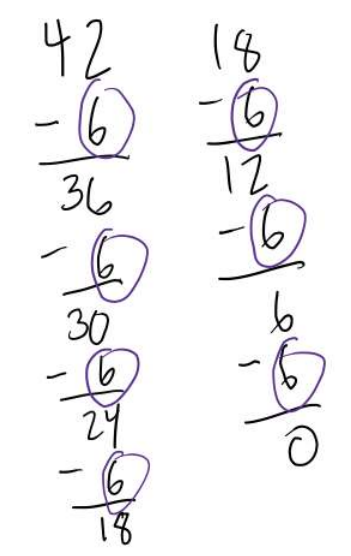
Example 7.1.1

Dee has 42 dice. She puts them into piles of 6 dice each. How many piles does she end up with?

Solution

We will solve this two ways. First, let's imagine we are directly modeling Dee's actions. Consider actually counting out 42 dice (or blocks or marbles or any other object) and then forming the piles. The first pile uses up 6 dice, leaving 36 left to make piles from. The next pile uses up 6 more dice, leaving 30 dice left. The process continues until 6 piles of 6 dice each have been created, leaving exactly 6 left, which forms the last pile. There would be 7 piles, which is the answer to the question.

The imagined procedure above involves actually subtracting 6 from the larger values repeatedly. As a second method, let's do this abstractly by simply subtracting 6 repeatedly. Below is how a young learner might do this.



The subtrahends (all sixes) are circled as the learner counts the number of times they could subtract 6. Each of those circled sixes represents a pile of 6 dice. There are 7 circled sixes, so there are 7 piles!

In the previous example, the one performing the subtraction must subtract 6 seven times. This is tedious, but not impossible. However, if we already have a good grasp of multiplication, we can do this faster. Simply knowing that $6 \times 7 = 42$ means that we could subtract all 7 sixes at once. That is precisely what long division does!

Thinking of the division $\frac{p}{q} = d + \frac{r}{q}$ using the same letters as at the beginning of this section as repeated subtraction is equivalent to thinking that d is the number of measures of q that are in p. That is why this is also called the **measurement perspective of division** or just **measurement division**. Some people also refer to this as **quotitive division**. Many children will abstractly model this method on a number line, which emphasizes the "measurement" concept. Try the example again by starting at 42 on a number line and making leaps of 6 units to the left until you reach 0. How many leaps did you make? It should be 7 if you did it correctly!

Before moving on, think about this: could this perspective of division be applied to non-integers? What about negative numbers? These are important questions to ask.

Sharing

Example 7.1.2

Charlie has 15 balloons that he will give to three children to be shared equally. How many balloons does each child receive?

Solution

A young learner modeling this directly would probably use 15 blocks (or any other object) and "deal" them out into 3 piles until he runs out. When finished, they would count how much is in each pile (they may count each pile to make sure they are equal). The result would be that there are 5 blocks in each pile. Thus, each child receives five balloons.

Notice that we already started knowing how many piles there would be in the example above. This is the opposite of the previous example. In the example with the dice, we knew how many dice would be in each pile but did not know how many piles there would be. This leads us to another perspective of division.

Definition: Division as Sharing Equally

Suppose p, q, and d are positive integers and r is a nonnegative integer less than q such that $\frac{p}{q} = d + \frac{r}{q}$. This means that if p is split evenly into q groups (with r left over), there will be d in each group. This is the **sharing equally perspective of division**.

This is also sometimes called **partitive division**. Also, sometimes the word "distribute" is used instead of "share". This is commonly how children first get exposed to division because it pairs nicely with lessons on sharing. In this way, children see a mathematical operation linked to a social activity! One might say that division is a friendly operation. Unfortunately, sometimes the fact that this division is mathematically equivalent to the quotitive division discussed previously is not always made clear. These are two very different actions when modeled directly. Thus, to most children, they feel like completely different operations. Children must be exposed to both viewpoints early on in order to make the connections between them.

Factors

After children have learned about quotitive and partitive division, but before they learn long division, it is important that they connect division to multiplication.

Definition: Division as Finding a Factor

Suppose p, q, and d are positive integers and r is a nonnegative integer less than q such that $\frac{p}{q} = d + \frac{r}{q}$. This is equivalently written as $p = dq + r$. Because d is the unknown factor in the product dq, this is the **unknown or missing factor perspective of division**.

In addition to this perspective being foundational to using long division instead of the long-hand repeated subtraction shown in the first example, thinking of division in terms of multiplication makes it much clearer as to how division can work with non-integer and negative numbers. It also allows us to succinctly model what happens if we try to let q equal 0. The following example is more advanced than what children first learning division would learn, but it is important to see where this perspective leads to later in their education.

Example 7.1.1

A. Use multiplication to divide 10 by $\frac{1}{2}$.

B. Use multiplication to show that it is impossible to divide 10 by 0.

Solution

A. We know that the quotient of 10 and $\frac{1}{2}$ is the value of d such that $d \cdot \frac{1}{2} = 10$. If we were to use a guess-and-check strategy, we might start with $2 \cdot \frac{1}{2} = 1$. We either know or would quickly find out that we need to only bother trying even numbers, so the next try might be 4: $4 \cdot \frac{1}{2} = 2$. Seeing that we need something bigger, we might try 8: $8 \cdot \frac{1}{2} = 4$. This is still too small and our product is not increasing fast enough, so we might make a bigger leap, like 30: $30 \cdot \frac{1}{2} = 15$. This is too big, so we try something smaller, such as 20: $20 \cdot \frac{1}{2} = 10$. This is exactly what we want! So, since $20 \cdot \frac{1}{2} = 10$, we know that $\frac{10}{1/2} = 20$.

Of course, many people do not need to guess-and-check. They might notice that $d \cdot \frac{1}{2} = 10$ is the same as $\frac{d}{2} = 10$, which means $d = 20$. In this way, we can see that sometimes thinking about division as multiplication leads to an easier division problem!

B. We will show that it is impossible to divide 10 by 0 by pretending it is and then showing why this does not work. Let's suppose there is some real number, say x that is the quotient of 10 and 0: $\frac{10}{0} = x$. If we think of this as multiplication, we would have that $x \cdot 0 = 10$. But we know that $x \cdot 0 = 0$. So, this only way this could be true is if $0 = 10$. Since that is not true, our original supposition was incorrect. There is no quotient of 10 and 0.

Practice Problems

1. Sophie needs to pass out papers to her coworkers. She has 28 papers and each coworker needs four papers. How many coworkers does she have? What type of division is this?
2. Later that day, Sophie speaks to ten other coworkers who need different papers. She runs off 30 copies. How many papers did each coworker receive? What type of division is this?

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7.2: Division Algorithms

As with other operations, there are many ways of performing division (not just many perspectives). You are likely aware of the fact that we can directly model (if the numbers are fairly small and usually integers), repeatedly subtract the divisor, or use long division. But there are other ways of performing division. Let's start with a division algorithm that is "between" repeated subtraction of the divisor and long division.

Scaffold Division

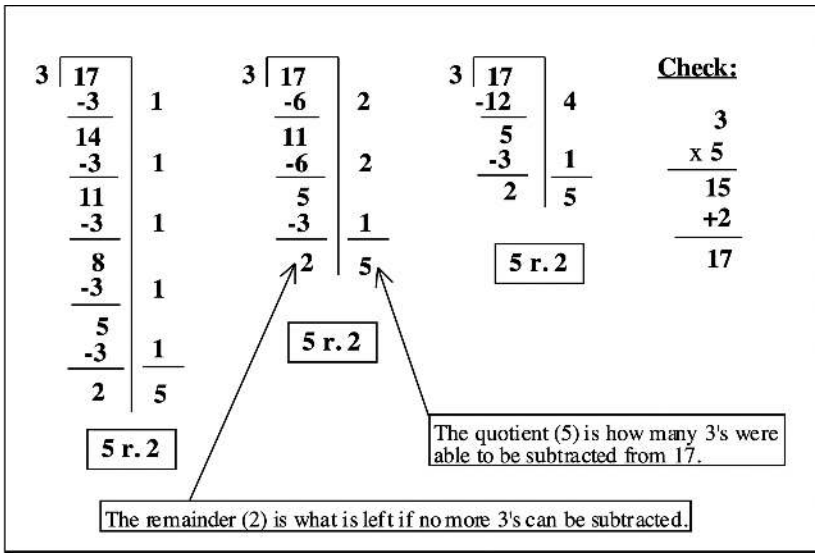
Let's consider $\frac{17}{3}$. We will do this three different ways using what's called a **scaffold**. To make a scaffold, write the dividend, then draw a line starting to the left of the dividend and going up and around to the right of the dividend, extending as far downward as needed (it will not be clear how far this is until we are finished, so start short and simply extend downward as you continue through the problem). Write the divisor to the left of the part of the scaffold that goes around the dividend. See below.

In the first example below, we subtract 3 repeatedly (no short cuts!). We keep track of how many threes we subtract at each step to the right of the scaffold. Since 3 is subtracted one at a time, and a total of five 3's are subtracted before we get to 2 (any more and we would get a negative value as a result, so we stop), we have 5 ones to the right of the scaffold.

In the second one, first 2 threes are subtracted at once, then 2 more, and then 1 more. Notice that the numbers 2, 2, and 1 are written to the right of the scaffold. We stop because the last difference is 2, which means we cannot take away another 3. That means, again, we subtracted a total of 5 threes.

In the third one, first 4 threes are subtracted at once (and 4 is written to the right of the scaffold), and then only 1 more (and 1 is written to the right of the scaffold). Again, we stop because the difference is 2. And again this means we subtracted a total of 5 threes.

In each case, we got that $17 \div 3 = 5$ with a remainder of 2.



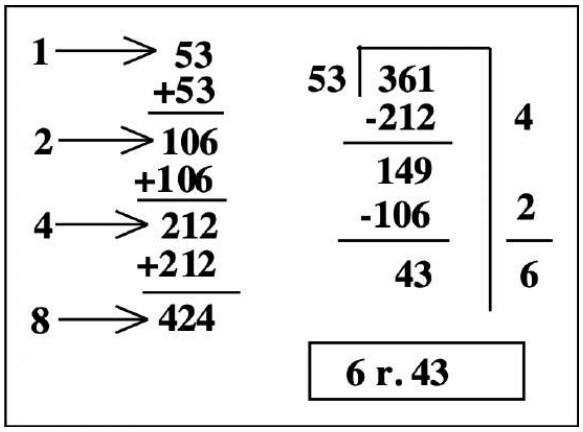
Long division with large numbers is often difficult because you have to multiply in your head or use a lot of space for scratch work to find the largest possible factor for each multiplication step. It's easy to make mistakes! But, by using scaffold division, you can keep the scratch work organized, and you do not have to find the *largest* number times one number "goes into" another number. Try the following problem on your own using scaffold division: $361 \div 53$.

Now that you have tried it, let's do it together. The way we will do this might look different than what you did, but that does not mean your work is incorrect! Scaffold division lets the user choose what works best for them.

Before we do division, let's make a quick partial multiplication table up for 53, by using repeated addition! This table is your prep work for division. A simple way to make up a partial multiplication table for 53 without ever multiplying entails doubling numbers. We know that $1 \times 53 = 53$. To figure out 2×53, just add two copies of 53 together. Two copies of 53 added to two more copies of 53 would equal four copies of 53, or 4×53. Four copies of 53 added to four more copies of 53 would equal eight copies of 53, or 8×53. Okay, so how do you show this? Start with one 53 and double it to find 2×53. Double that number to find 4×53. Double

that number to find 8×53. Look in the left part of the box shown below. There is a 1 pointing to 53 because that is the result of 1×53. If you double the 53, (53 + 53), then you now know $2 \times 53 = 106$. So a 2 points to 106. If you double that (106 + 106), you now know $4 \times 53 = 212$. So a 4 points to 212. And if you double this number (212 + 212), then you now know $8 \times 53 = 424$. So an 8 points to 424. You now know $1 \times 53, 2 \times 53, 4 \times 53,$, and 8×53.

Instead of guessing how many times 53 "goes into" 361, we note how many copies of 53 can be subtracted from 361 where underestimating is okay. Since $8 \times 53 = 424$ and $424 > 361$, we don't bother guessing 8. The biggest number in our list products we already found that can be subtracted from 361 is 212. Subtract 212 from 361 and write 4 on the right because you just subtracted 4 copies of 53. The resulting difference is 149. The biggest number you can subtract from 149 is 106. If you do that, you've subtracted 2 more copies of 53. So, write a 2 on the right. After subtracting, 43 is left. Since $43 < 53$, no more copies of 53 can be subtracted. Therefore, 43 is the remainder. Add the numbers on the right to see how many copies of 53 you subtracted. Since 6 copies of 53 were subtracted, we have $361 \div 53 = 6 + \frac{43}{53}$ or $361 \div 53 = 6$ remainder 3.



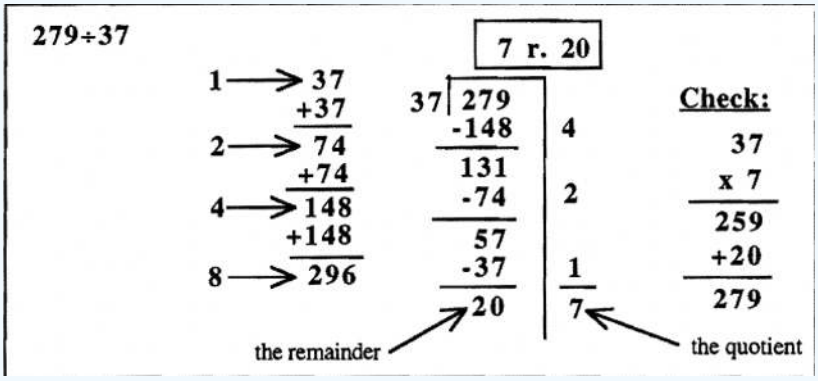
To check, multiply 53×6 and add 43. You should have gotten 361.

Here are some more examples for you to study. In each example, a partial multiplication table is made up first using the doubling technique. This is not necessary, and you will see more examples in a bit that do not do this. This prep work which involves a few additions makes the division much easier. Then, the problem is done using a scaffold. After adding up how many times the divisor was subtracted from the dividend (the numbers to the right of the scaffold), the answer (quotient and remainder) is written in a box up above the problem. Finally, the check is shown. Also, check to make sure the remainder is less than the divisor.

Example 7.2.1

Divide $\frac{279}{37}$ using a scaffold.

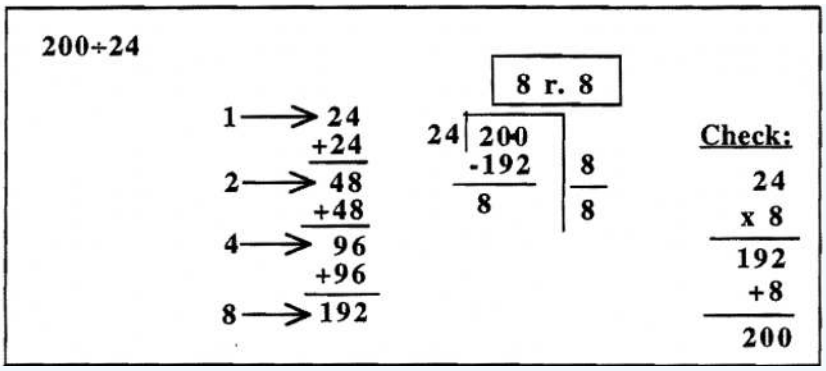
Solution



Example 7.2.1

Divide $\frac{200}{24}$ using a scaffold.

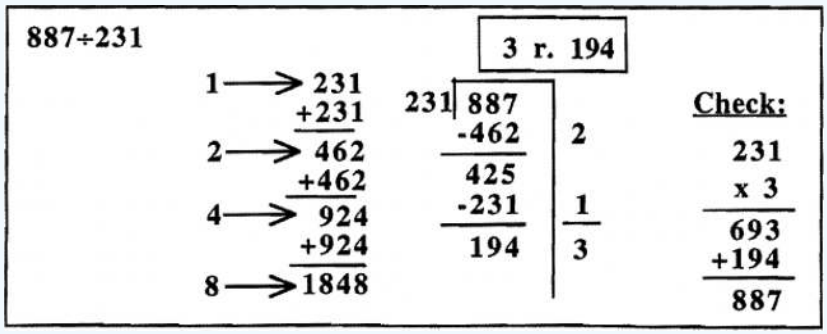
Solution



Example 7.2.1

Divide $\frac{887}{231}$ using a scaffold.

Solution



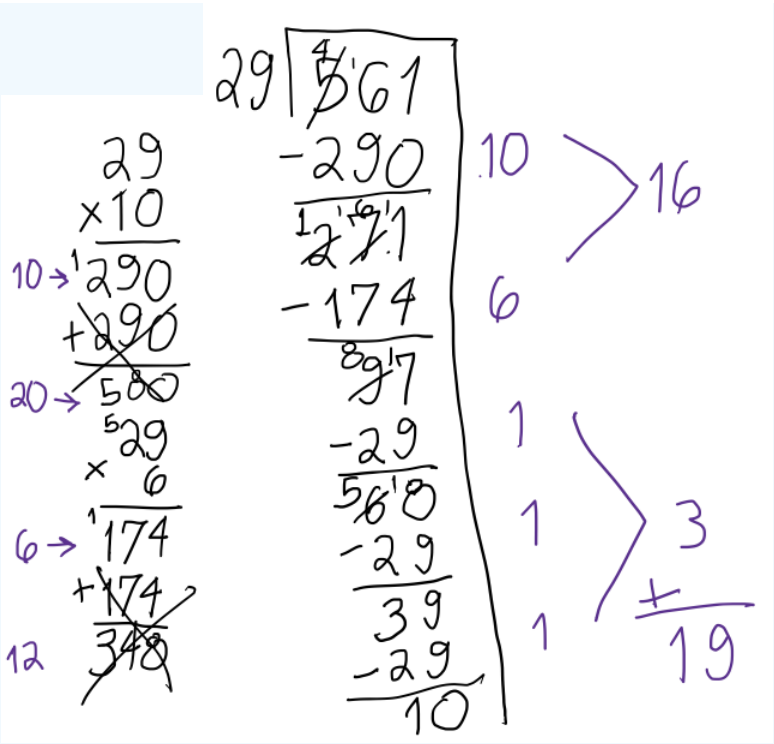
In the examples so far, all of the quotients were less than 10. It's possible to continue doubling the divisor in the partial multiplication table, so that you can find the divisor times any power of 2 — 1, 2, 4, 8, 16, 32, 64, etc. You would stop doubling if you notice that doubling once more gives a number bigger than the quotient. We do not need to use this doubling method, however. Let's see how two children perform the division in the example below.

Example 7.2.1

Divide $\frac{561}{29}$ using a scaffold.

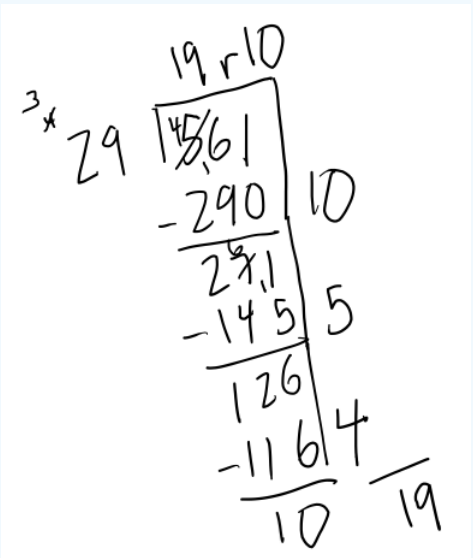
Solution

Child A:



This child used a multiplication table but did not use only doubling. Notice that they crossed out the two times when the result was bigger than what they wanted. This tells us that they tried doubling 174 after they had already subtracted $561 - 290 = 271$ because $348 > 271$. The child never wrote their answer, but if they interpreted their work correctly, they would have that $\frac{561}{29} = 19 + \frac{10}{29}$ or $\frac{561}{29} = 19$ remainder 10.

Child B:



This child did not use a multiplication table at all. Their work is a little more cluttered because they did the multiplication inside the scaffold (note the crossed out 4 and the 3 above and to the left of the divisor). They also wrote their answer, 19 remainder 10, above the scaffold.

In each case, the child should have checked their work. If they did, they would get that $19 \times 29 + 10 = 551 + 10 = 561$.

It's time for you to try using the scaffold method.

Exercises

Solve each problem twice. First by doing prep work with the doubling method to make a partial multiplication table first before drawing the scaffold. Then, redo it either with a different multiplication table or without using a multiplication table at all. Check your answers.

A. $415 \div 72$ = _____

B. $1235 \div 214$ = _____

C. $3128 \div 321$ = _____

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7.3: Division in other Bases

You will need: ***Base Blocks*** *(Material Cards 3 - 15)*

We will now work on dividing in other bases. First, we will divide by using repeated subtraction with the actual base blocks. Later, we'll employ the same algorithm we did in the previous exercise set to do the problems using repeated subtraction with the partial multiplication table, and the scaffold.

Get out your Base Three base blocks. We are going to use them with the repeated subtraction method to divide $221_{\text{three}} \div 21_{\text{three}}$. The dividend is 221_{three}, which consists of 2 flats, 2 longs and 1 unit. We need to find out how many times we can repeatedly subtract the divisor, 21_{three}, from the dividend. To do this, we can convert the dividend into only longs and units by making some exchanges, and then form as many subsets containing 1 long and 2 units as possible. The number of subsets formed is how many times the divisor could be subtracted from the dividend. Get out your blocks to do this problem with me.

Step 1: Form the dividend in base three. You should actually get out your base three blocks and do the problems with the blocks. I will be using the initials for the base blocks (U, L, F, B, etc.) to show what is going on. The dividend can be shown like this:

F F L L U

Step 2: Make exchanges so it is possible to form as many subsets as possible, each containing the divisor, which is 2 longs and 1 unit. Each flat can be exchanged for 3 longs to get:

L L L L L L L L U

Then, one of the longs can be exchanged for 3 units to get L L L L L L L U U U U. Let's see if we can subtract out some equal subsets of 2 longs and 1 unit from the dividend. As you take out each 2 longs and a unit, form them in a subset.

I can take out and form 3 equal subsets of 2 longs and 1 unit each from the dividend. In the dividend, one long and 1 unit is left. The subsets formed look like this:

LLU	LLU	LLU

The part left in the dividend, which is the remainder looks like this:

LU

Step 3: Each equal subset of 2 longs and 1 unit that was subtracted out of the dividend counts as 1 for the quotient. Give yourself a unit for each equal subset you formed this is the quotient. The long and unit left over are the remainder because it is less than the divisor.

Quotient: U U U	Remainder: L U

Step 4: In the quotient and/or remainder, make any exchanges necessary so the quotient and remainder can be written in base three.

Quotient: L	Remainder: L U

Step 5: Write the quotient and remainder in base three: $10_{\text{three}} r.\ 11_{\text{three}}$.

Therefore, $221_{\text{three}} \div 21_{\text{three}} = 10_{\text{three}} r.\ 11_{\text{three}}$

Step 6: Check the answer. This is shown to the right.

? Exercise 1

Following the example on the previous page, do the following division step by step. Use base three blocks to do this exercise. You can either use initials, or draw pictures of the blocks to show how you do the division.

Do the following division: $222_{\text{three}} \div 12_{\text{three}}$

a. Form the dividend in base three. Draw a picture of what the dividend looks like:

b. Make exchanges so it is possible to form as many subsets as possible, each containing the divisor, which is 1 long and 2 units. Subtract out some equal subsets of 1 long and 2 units from the dividend. As you take out each long and 2 units, form them in a subset.Draw a picture of the all the equal subsets you were able to form by subtracting from the dividend, then draw a separate picture of what was left in the dividend, which will be the remainder (this must be less than 1 long and 2 units).

i) Show all of the equal subsets formed:

ii) Remainder:

c. For each of the equal subsets of 1 long and 2 units that was subtracted out of the dividend and formed (shown in b.i.), give yourself a unit — this is the quotient. Draw a picture of the quotient in terms of units:

d. In the quotient (part c) and/or remainder (part b.ii), make any exchanges necessary so the quotient and remainder can be written in base three. Draw a picture of what the quotient and remainder looks like after making any and all exchanges possible.

Quotient:	Remainder:

Check:
$$\begin{array}{r} 21_{\text{three}} \\ \times 10_{\text{three}} \\ \hline 210_{\text{three}} \\ +11_{\text{three}} \\ \hline 221_{\text{three}} \end{array}$$

e. Write the quotient and remainder in base three: _____

f. Check the answer.

Exercise 2

Use base blocks in the bases indicated to find the quotient and remainder for each problem. Then check your answer in the base given. Show your work. You can multiply using any multiplication algorithm you prefer — you might like to try the lattice.

a. $1204_{\text{five}} \div 42_{\text{five}}$ = Check:	c. $1345_{\text{six}} \div 25_{\text{six}}$ = Check:
b. $323_{\text{four}} \div 23_{\text{four}}$ = Check:	d. $11111_{\text{two}} \div 101_{\text{two}}$ = Check:

You may have noticed in parts c and d that you had to make a lot of piles, and that the repeated subtraction was somewhat tedious when you could only subtract one subset of divisor from the dividend at a time. This is where it is useful to use the idea that a long times the divisor is similar to multiplying by 10 in base ten. Remember that 10b is a long for some base, b. For instance, when you multiply 25 times 10, you get 250. In base six, 25_{six} means 2 longs and 5 units. When you multiply this by a long (10_{six}), you get 250_{six} which is 2 flats and 5 longs — every piece shifts up a place value. We can use this in exercise 2c to make the subtraction go faster. We'll do exercise 2c using this idea on the next page.

Exercise 2c, $1345_{\text{six}} \div 25_{\text{six}}$, can be done using a more efficient method. Here are the steps:

Draw a picture of the dividend, 1345_{six}, using Base Six blocks.

B F F F L L L L U U U U U

You want to subtract as many subsets of the divisor, 25_{six}, from the dividend as you can. This would take a lot of steps. You'd have to break down the block and three flats into lots of longs and units. Now, 25_{six}, means 2 longs and 5 units. If you multiplied this by

a long, you would have 250_{six}, which is 2 flats and 5 longs. If you subtracted 2 flats and 5 longs from the dividend at once, it would be the same as subtracting 2 longs and 5 units from the dividend six times! And it is possible to subtract 2 flats and 5 longs at once.

Let's trade some of the pieces in the dividend for smaller pieces, so it is possible to take subtract and make subsets of 2 flats and 5 longs. I started by trading the block in for 6 flats and two of the flats in for 12 longs:

F F F F F F F L L L L L L L L L L L L L L U U U U U

Now, I'll form as many subsets of 2 flats and 5 longs as possible.

F F L L L L L	F F L L L L L	F F L L L L L

I still have F L U U U U U left in the dividend, and can't take any more subsets of 2 flats and 5 longs out. So now, I try to subtract subsets of the divisor from what's left in the dividend. Again, I need to make some trades. I'll trade the flat for 5 longs and 6 units, so I have:

L L L L L L L U U U U U U U U U U U U

Two subsets can be formed:

L L U U U U U U	L L U U U U U U

All that is left in the dividend is a unit and 2 longs, which is less than the divisor, and is the remainder. So the remainder, LLU, is 21_{six}.

Now, we have to figure out the quotient. The first 3 subsets I formed were each a long times the divisor, so for the quotient, I put in a long for each of the subsets formed. I put in a unit for each of the next two subsets I formed. So I have 3 longs and 2 units for the quotient, and no exchanges need to be made.

The quotient is L L L U U, or 32_{six}, and the remainder is 21_{six}.

Therefore, $1345_{six} \div 25_{six} = 32_{six}$ r. 21_{six}

You should have gotten the same answer when you did exercise 2c, but you would have subtracted and formed 13 subsets of the divisor, and then traded 13 units in for the answer in base six.

(By the way, if the divisor, 25_{six}, was multiplied by a flat, you would have 2 blocks and 5 flats, which is bigger than the dividend, so we can't take that many of the divisor away at one time. So we start by subtracting subsets of a long times the divisor, one at a time.)

? Exercise 3

Divide the following using base blocks. You can use the method described on the previous page, or subtract one subset of the divisor at a time. **Explain or draw pictures with the blocks to show how you did the problem.** Check your answers using any multiplication algorithm.

a. $1046_{seven} \div 31_{seven}$ = _____
Check:

b. $10021_{three} \div 22_{three}$ = _____
Check:

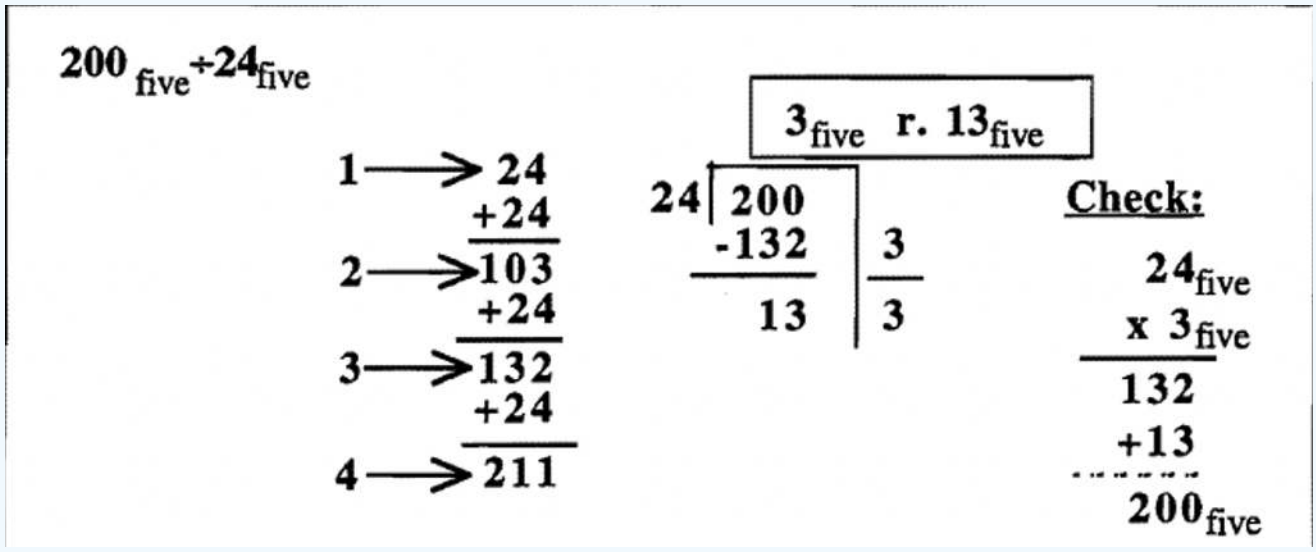
We're now going to work on doing division in different bases using the same repeated subtraction algorithm we did in Exercise Set 2. Since most if not all of us aren't familiar enough with different bases to estimate in our head, making up a partial multiplication table becomes extremely useful for these problems.

Because each base has a different number of digits, we may opt to do the doubling method or some partial table for larger bases like base 11, 12 and 13. For smaller bases, you might consider making a complete table. In base five, you would only have to figure

out 2, 3 and 4 times the divisor (since 4 is the highest digit in base five). Study the following examples of using the scaffold method in other bases.

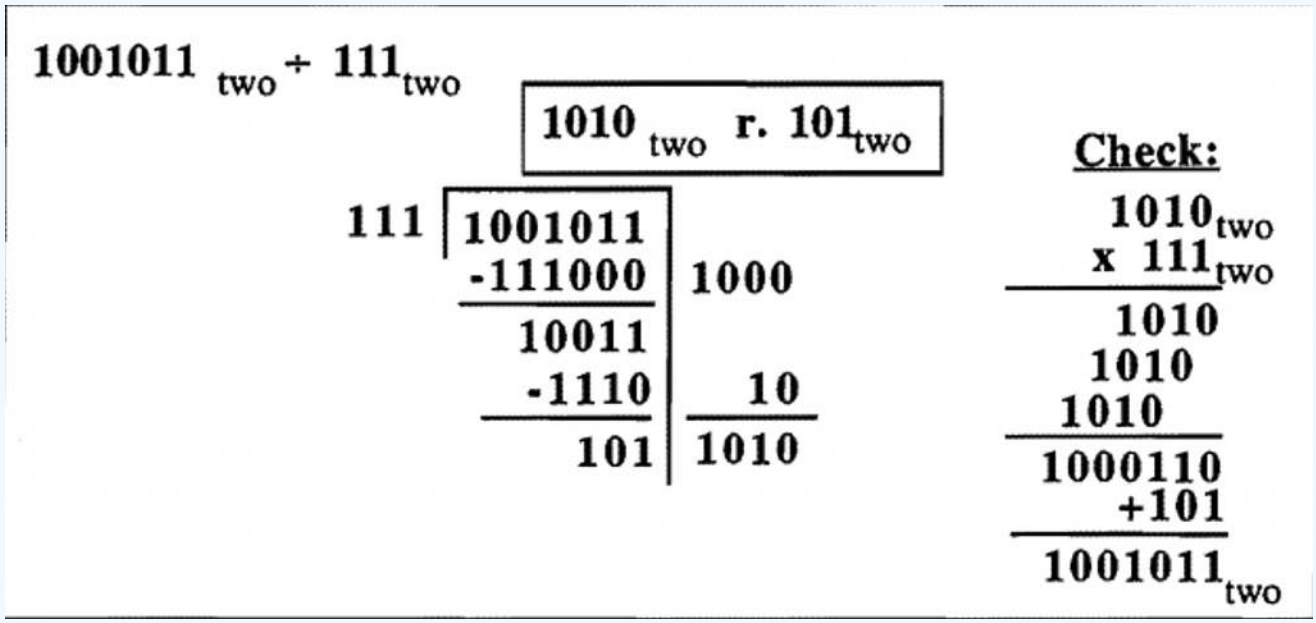
Example 1

Division problem in base five: $200_{\text{five}} \div 24_{\text{five}}$. I leave off writing "five" except for the answer (boxed) and the check. You have to be very careful to remember you are in base five when you are adding (the prep work at the beginning, and the addition at the end of the check when you add the remainder), subtracting (doing the repeated subtraction), and multiplying (in the check). For the check, you could use the lattice method, or any other method you prefer.



Example 2

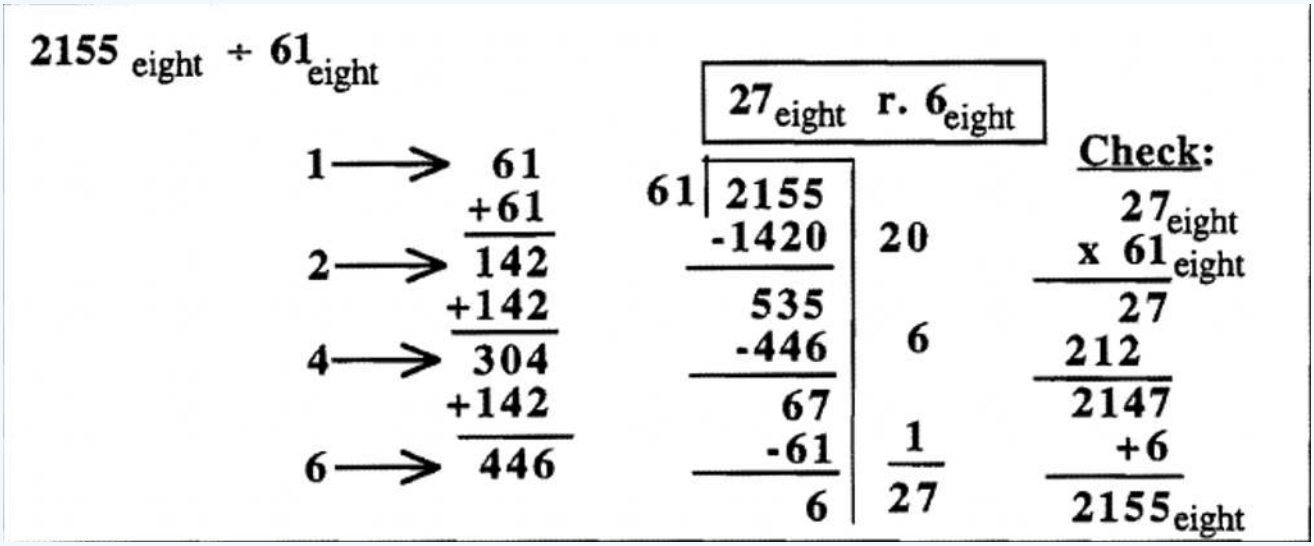
Division problem in base two: $1001011_{\text{two}} \div 111_{\text{two}}$. There is really no prep work here, because the only digits in base two are 0 and 1.



For examples 3 and 4, I chose to do a different kind of partial multiplication table for each.

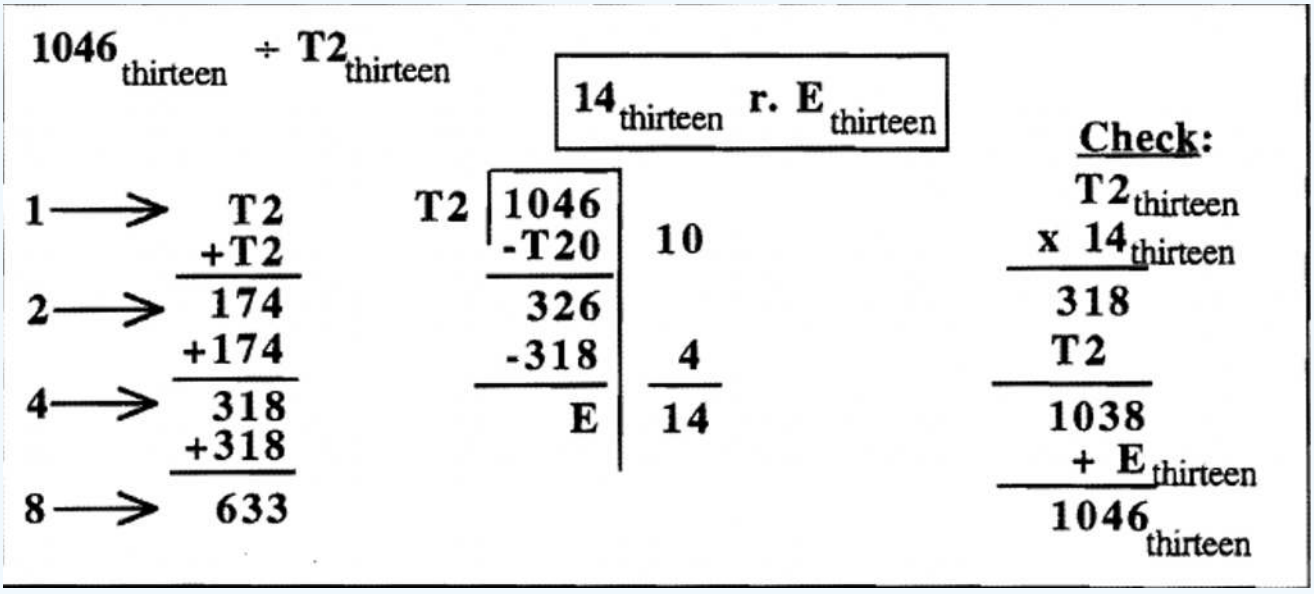
Example 3

Division problem in base eight: $2155_{\text{eight}} \div 61_{\text{eight}}$. Since this is a problem in base eight, I decided to make a partial table to figure out the divisor times 1, 2, 4, and 6. Another person might choose to do a table for only 1, 2, and 4. Remember that 7 is the highest digit in base eight. Yet another student might do a full table for all of the digits in base eight: 1, 2, 3, 4, 5, 6, and 7.



Example 4

Division problem in base thirteen: $1046_{\text{thirteen}} \div T2_{\text{thirteen}}$. Since this is a problem in base thirteen, I decided to make a partial table to figure out the divisor times 1, 2, 4, and 8. Remember that W is the highest digit in base eight. Another person might choose to do a table for 1, 2, 4, 6, 8, T, and W. Remember that W is the highest digit in base thirteen. Another student might make a full table for all of the digits in base thirteen, but that's a lot of prep work that usually doesn't pay off.



Make sure you practice the previous four examples before doing the exercises that follow on the next pages.

For each of the division problems in the remainder of this section, show your work, using a model similar to the last four examples. First make up a partial or full multiplication table (prep for division using repeated subtraction). Then, divide using the scaffold method with the repeated subtraction method. Write your answer in a box above the scaffold. Then, check your answer by multiplying the divisor by the quotient and adding the remainder. Make sure you pay close attention to what base you are working in as you add, subtract and/or multiply in that base. Remember to write the base when you write the quotient and remainder.

Exercise 4

$222_{\text{three}} \div 12_{\text{three}}$

Exercise 5

$1204_{\text{five}} \div 42_{\text{five}}$

Exercise 6

$323_{\text{four}} \div 23_{\text{four}}$

Exercise 7

$1345_{\text{six}} \div 25_{\text{six}}$

Exercise 8

$11111_{\text{two}} \div 101_{\text{two}}$

Exercise 9

$1046_{\text{seven}} \div 31_{\text{seven}}$

Exercise 10

$10021_{\text{three}} \div 22_{\text{three}}$

Exercise 11

$200_{\text{five}} \div 3_{\text{five}}$

Exercise 12

$1001011_{\text{two}} \div 1010_{\text{two}}$

Exercise 13

$2155_{\text{eight}} \div 27_{\text{eight}}$

Exercise 14

$1046_{\text{thirteen}} \div 14_{\text{thirteen}}$

Exercise 15

$101\text{E}_{\text{twelve}} \div 54_{\text{twelve}}$

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7.4: Practice

- **Submit homework separately from this workbook and staple all pages together. (One staple for the entire submission of all the unit homework)**
- **Start a new module on the front side of a new page and write the module number on the top center of the page.**
- **Answers without supporting work will receive no credit.**
- **Some solutions are given in the solutions manual.**
- **You may work with classmates but do your own work.**

? HW #1

Mae had 72 lollipops and wanted to sell packages that had 4 lollipops in each package. How many packages could she make?

a. What division problem do you need to do to find the answer?

b. What interpretation of division does this word problem represent?

c. Explain how to use the correct interpretation to draw a picture to find the answer.

d. Draw a picture using the correct interpretation to help find the answer.

e. Write the multiplication problem shown by the picture:

f. Explain how you use the picture to find the answer to the problem. Be specific.

g. What is the answer to this problem?

? HW #2

Tina had 72 marbles and wanted to divide them up evenly amongst the 9 children in her class. How many marbles should each child receive?

a. What division problem do you need to do to find the answer? :

b. What interpretation of division does this word problem represent?

c. Explain how to use the correct interpretation to draw a picture to find the answer.

d. Draw a picture using the correct interpretation to help find the answer.

e. Write the multiplication problem shown by the picture:

f. Explain how you use the picture to find the answer to the problem. Be specific.

g. What is the answer to this problem?

? HW #3

Joan had 12 cups of sugar. A cookie recipe required 2/3 of a cup of sugar. How many recipes could she make?

a. What division problem do you need to do to find the answer? :

b. What interpretation of division does this word problem represent?

c. Explain how to use the correct interpretation to draw a picture to find the answer.

d. Draw a picture using the correct interpretation to help find the answer.

e. Write the multiplication problem shown by the picture:

f. Explain how you use the picture to find the answer to the problem. Be specific.

g. What is the answer to this problem?

HW #4

a. Make up a word problem that would require using the partitioning of subsets interpretation of division and the division problem: $20 \div 5$.

b. Draw a picture using this interpretation to find the answer.

c. Explain how you use the picture to find the answer to the problem. Be specific.

d. Write the multiplication problem shown by the picture.

e. What is the answer to this problem?

HW #5

a. Make up a word problem that would require using the repeated subtraction interpretation of division and the division problem: $20 \div 5$.

b. Draw a picture using this interpretation to find the answer.

c. Explain how you use the picture to find the answer to the problem. Be specific.

d. Write the multiplication problem shown by the picture.

e. What is the answer to this problem?

HW #6

a. Make up a word problem that would require using the repeated subtraction interpretation of division and the division problem: $6 \div 2/3$.

b. Draw a picture using this interpretation to find the answer.

c. Explain how you use the picture to find the answer to the problem. Be specific.

d. Write the multiplication problem shown by the picture.

e. What is the answer to this problem? _____

HW #7

Draw a picture to do the division $30 \div 2$ using

a. partitioning into subsets

b. repeated subtraction.

c. Which interpretation is easiest to draw and why?

HW #8

Draw a picture to do the division $32 \div 16$ using

a. partitioning into subsets

b. repeated subtraction.

c. Which interpretation is easiest to draw and why?

HW #9

Fill in the blanks: $637 \div 14$ = 45 r.7 means _____ $\cdots$ _____ + _____ = _____

HW #10

Someone was solving this problem: “Mark had 42 pencils and wanted to divide them up equally amongst his 3 children. How many pencils should he give each child?”

Which picture best represents a picture he might draw to find the solution?



HW #11

To use the division interpretation of “partitioning into subsets”

a. would the number you are dividing by be the number of subsets or the amount in each subset?

b. After doing the division, would the answer be represented by the number of subsets or the amount in each subset?

HW #12

To use the division interpretation of “repeated subtraction”,

a. would the number you are dividing by be the number of subsets or the amount in each subset?

b. After doing the division, would the answer be represented by the number of subsets or the amount in each subset?

For problems 13 – 17

a. Divide in the base given using repeated subtraction. All of your work must be shown. **Show ALL steps**.

b. Check each answer by multiplying the divisor by the quotient and then adding the remainder to see if this equals the dividend. **All of your work must be shown. Do not skip any steps.**

HW #13

$4320_{\text{seven}} \div 52_{\text{seven}}$

HW #14

$2316_{\text{nine}} \div 7_{\text{nine}}$

HW #15

$60382_{\text{eleven}} \div 21\text{T}_{\text{eleven}}$

HW #16

$43401_{\text{five}} \div 444_{\text{five}}$ = _____

HW #17

$2020100_{\text{three}} \div 220_{\text{three}}$ = _____

HW #18

Decide which of the following sets are closed under division. If it is closed under division, write "closed" and provide at least one example and compelling reason why you are sure it is closed. If it is not closed, write "not closed" and provide a counterexample.

a. {-1}	b. {0}	c. {0, 1}
d. {-1, 1}	e. {1, 2, 3, 4, ...}	

HW #19

Is division commutative?

Support your answer with either an example or a counterexample.

HW #20

Is division associative?

Support your answer with either an example or a counterexample

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CHAPTER OVERVIEW

8: Fractions

8.1: What is a Fraction?
8.2: A Key Property of Fractions
8.3: Rational Numbers
8.4: Adding and Subtracting Fractions
8.5: What is a Fraction? Revisited
8.6: Multiplying Fractions
8.7: Introduction to Dividing Fractions
8.8: Dividing Fractions Methods
8.9: Fractions Involving Zero
8.10: Practice
8.11: Egyptian Fractions
8.12: Algebra Connections
8.13: What is a Fraction? Looking Back

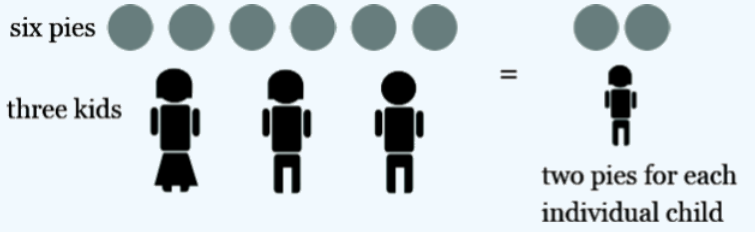
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8.1: What is a Fraction?

One of the things that makes fractions such a difficult concept to teach and to learn is that you have to think about them in a lot of different ways, depending on the problem at hand. For now, we are going to think of a fraction as the answer to a division problem.

Example: Pies Per child

Suppose 6 pies are to be shared equally among 3 children. This yields 2 pies per kid. We can abbreviate this as $\frac{6\text{pies}}{3\text{children}} = 2\text{pies per child}$. Without the units, this is $\frac{6}{3} = 2$.



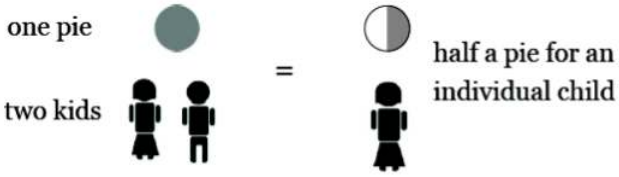
The fraction $\frac{6}{3}$ is equivalent to the quotient $6 \div 3$, which equals 2. It represents the number of pies one whole child receives when three kids share six pies equally.

In the same way …

- Sharing 10 pies among 2 kids yields $\frac{10}{2} = 5$ pies per kid.
- Sharing 8 pies among 2 children yields $\frac{8}{2} = 4$ pies per child.
- Sharing 5 pies among 5 kids yields $\frac{5}{5} = 1$ pie per kid.
- Sharing 1 pie among 2 children yields $\frac{1}{2}$, which we call “one-half.”

This final example is actually saying something! It also represents how fractions are usually first taught to students:

If one pie is shared *equally* between two kids, then each child receives a portion of a pie which we choose to call “half.”



Thus students are taught to associate the number “$\frac{1}{2}$” to the picture  .

In the same way, the picture  is said to represent “one-third,” that is, $\frac{1}{3}$. This is indeed the amount of pie an individual child would receive if one pie is shared among three. The picture  is called “one-fifth” and is indeed $\frac{1}{5}$, the amount of pie an individual receives if three pies are shared among five children. And the picture  is called “three-fifths”, which represents $\frac{3}{5}$, the amount of pie an individual receives if three pies are shared among five children.

Think / Pair / Share

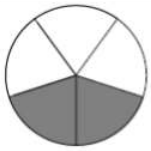
Carefully explain to another person why this is true: If five kids share three pies equally, each child receives an amount that looks like this:  .

Your explanation will probably require both words and pictures.

On Your Own

Work on the following exercises on your own or with a partner.

1. Draw a picture associated with the fraction $\frac{1}{6}$.
2. Draw a picture associated with the fraction $\frac{3}{7}$. Is your picture really the amount of pie an individual would receive if three pies are shared among seven kids? Be very clear on this!
3. Let's work backwards! Here's the answer to a division problem:



   This represents the amount of pie an individual kid receives if some number of pies is shared among some number of children. How many pies? How many children? How can you justify your answers?

4. Here's another answer to a division problem:



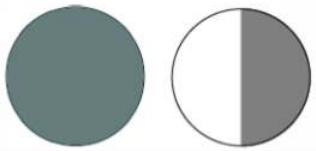
   How many pies? How many children? How can you justify your answers?

5. Here is another answer to a division problem:



   How many pies? How many children? How can you justify your answers?

6. Leigh says that "$\frac{3}{5}$ is three times as big as $\frac{1}{5}$." Is this right? Explain your answer.
7. Draw a picture for the answer to the division problem $\frac{4}{8}$. Describe what you notice about the answer.
8. Draw a picture for the answer to the division problem $\frac{2}{10}$. Describe what you notice about the answer.
9. What does the division problem $\frac{1}{1}$ represent? How much pie does an individual child receive?
10. What does the division problem $\frac{5}{1}$ represent? How much pie does an individual child receive?
11. What does the division problem $\frac{5}{5}$ represent? How much pie does an individual child receive?
12. Here is the answer to another division problem. This is the amount of pie an individual child receives:



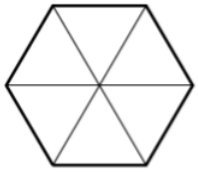
   How many pies were in the division problem? How many kids were in the division problem? Justify your answers.

13. Here is the answer to another division problem. This is the amount of pie an individual child receives:



How many pies were in the division problem? How many kids were in the division problem? Justify your answers

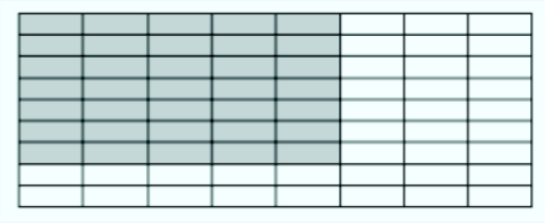
14. Many teachers have young students divide differently shaped pies into fractions. For example, a hexagonal pie is good for illustrating the fractions: $\frac{1}{6}$, $\frac{2}{6}$, $\frac{3}{6}$, $\frac{4}{6}$, $\frac{5}{6}$, and $\frac{6}{6}$.



- Why is this shape used? What does $\frac{1}{6}$ of a pie look like?
- What does $\frac{6}{6}$ of a pie look like?
- What shape pie would be good for illustrating the fractions $\frac{1}{8}$ up to $\frac{8}{8}$?

Exercise 1

Some rectangular pies are distributed to some number of kids. This picture represents the amount of pie an individual child receives. The large rectangle represents one whole pie.



How many pies? How many kids? Carefully justify your answers!

Pies Per Child Model

In our model, a fraction $\frac{a}{b}$ represents the amount of pie an individual child receives when a pies are shared equally by b kids. Of course, this only makes sense when a and b are both positive integers.

$$\begin{matrix}\#\text{pies} \longrightarrow \\ \#\text{kids} \longrightarrow\end{matrix} \frac{a}{b} = \text{amount per individual child}$$

Try to answer the following questions and explain your answers to a friend or classmate.

Think / Pair / Share

- What is $\frac{2}{2}$? What is $\frac{7}{7}$? What is $\frac{100}{100}$? How can you use the "Pies Per Child Model" to make sense of $\frac{a}{a}$ for any positive integer a?
- What is $\frac{2}{1}$? What is $\frac{7}{1}$? What is $\frac{1876}{1}$? How can you use the "Pies Per Child Model" to make sense of $\frac{b}{1}$ for any positive integer b?

- Write the answer to this division problem: “I have no pies to share among thirteen kids.” How can you generalize this division problem to make a general statement about fractions?

Before we continue, recall the following definitions so that we can use precise terminology.

Definition

For a fraction $\frac{a}{b}$, the top number a is called the **numerator** of the fraction, and the bottom number a, is called the **denominator** of the fraction.

Many people insist that the numerator and denominator each be positive, but they do not have to be. Try to answer these questions below and explain your answers to a friend or classmate.

Think / Pair / Share

To understand why the numerator and denominator need not be integers, let's avoid discussing division of children and think about a plate of bacteria culture (which can be divided up however we please). Since the plate is circular, we will use a circle. Let's suppose the culture is red. So, the following will represent one whole plate of the culture.



- What would half a plate of the culture look like?
- What would one-third of a plate of the culture look like?
- What would three-fifths of a plate of the culture look like?

So, what would $\frac{1}{\left(\frac{1}{2}\right)}$ represent?

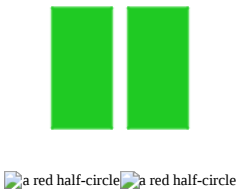
To answer that, we will consider what happens when we divide food among the bacteria culture. Let's say the bacteria get pellets of plant material to consume, which we will represent with a green rectangle.



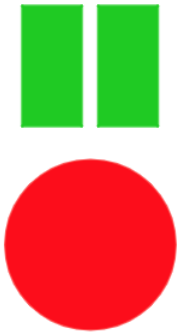
This means $\frac{1}{\left(\frac{1}{2}\right)}$ would be the result of dividing one pellet of plant material among a “group” of half a plate of the culture.



So how much would a whole plate of the culture receive? Well, since each half-place gets a full pellet, we can represent this as below.



But if we rotate the right half-circle around, we clearly get one whole circle. In other words, we have a whole plate of the culture.



The whole plate of the bacteria culture gets two pellets. So, we have $\frac{1}{\left(\frac{1}{2}\right)} = \frac{2}{1} = 2$. Now try to answer the following using the pellets per plate of bacteria culture model. Try to explain each, possibly with pictures, to a friend or classmate.

? Think / Pair / Share

Draw pictures for these problems if it helps!

1. What does $\frac{1}{\left(\frac{1}{3}\right)}$ represent?
2. What is $\frac{1}{\left(\frac{1}{6}\right)}$?
3. Explain why the fraction $\frac{5}{\left(\frac{1}{2}\right)}$ represents the number 10.
4. What is $\frac{4}{\left(\frac{1}{3}\right)}$?
5. **Challenge:** Two-and-a-half pellets are to be shared equally among four-and-a-half plates of bacteria culture. How much of a pellet does a whole plate of culture receive?

Improper Fractions

A fraction with a numerator smaller than its denominator is called a *proper fraction*. For example, $\frac{45}{58}$ is "proper."

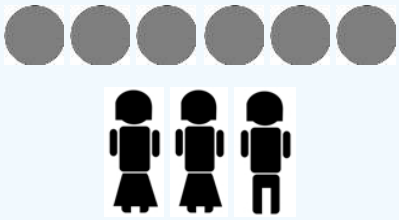
A fraction with numerator larger than its denominator is called an *improper fraction*. For example, $\frac{7}{3}$ is "improper."

For some reason, improper fractions are considered, well, *improper* by some teachers. So students are often asked to write improper fractions as a combination of a whole number and a proper fraction (often called "mixed numbers"). Despite their name, improper fractions are useful nonetheless!

With a mixed number, you have a good sense of the overall size of the number: "a little more than five," or "a bit less than 17."

Example: $\frac{7}{3}$

If seven pies are shared among three kids, then each kid will certainly receive two whole pies, leaving one pie to share among the three children.



Thus, $\frac{7}{3}$ equals 2 plus $\frac{1}{3}$. People write: $\frac{7}{3} = 2\frac{1}{3}$ and call the result a *mixed number*. One can also write: $2 + \frac{1}{3}$, which is what $2\frac{1}{3}$ really means. But most people choose to omit the plus sign.

But it is often easier to do calculations with improper fractions. Why do you think that is? This will be easier to answer once we learn more about calculating with fractions. But a hint for now is that it relates to the fact that there really is a hidden "+" between the integer and the fraction in a mixed number. So, think about what happens when you want to do calculations with a "number" that actually is itself a sum.

Students are often asked to memorize the names “proper fractions,” “improper fractions,” and “mixed number” so that they can follow directions. But, to a mathematician, these names are not at all important! There is no “correct” way to express an answer (assuming, that the answer is mathematically the right value). We often wish to express our answer in a "nicer" form, but sometimes the context will tell you what form is “nicer” and what form is more complicated.

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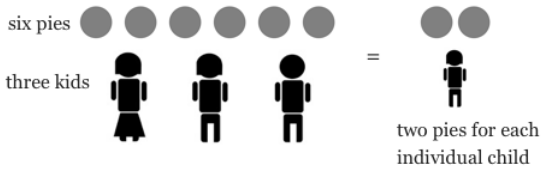
8.2: A Key Property of Fractions

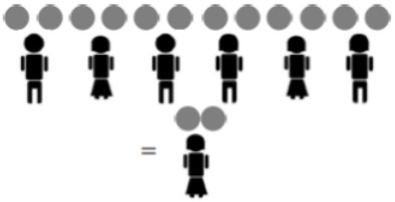
We know that $\frac{a}{b}$ is the answer to a division problem; for example, if a and b are both positive integers, then $\frac{a}{b}$ represents the amount of pie an individual child receives when a pies are shared equally by b children.

What happens if we double the number of pie and double the number of kids? Nothing! The amount of pie per child is still the same. In other words, we have the following fact.

$$\frac{2a}{2b} = \frac{a}{b}$$

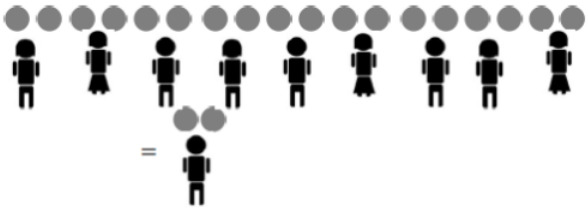
For example, as the picture shows, $\frac{6}{3}$ and $\frac{12}{6}$ both give two pies for each child.





And tripling the number of pies and the number of children also does not change the final amount of pies per child, nor does quadrupling each number, or one trillion-billion-tupling the numbers!

$$\frac{6}{3} = \frac{12}{6} = \frac{18}{9} = \ldots = \text{ two pies per child}\}$$



This leads us to want to believe what turns out to be a key property of fractions.

The Fundamental Property of Fractions

For any real numbers a, b, and x such that $b \neq 0$ and $x \neq 0$, $\frac{xa}{xb} = \frac{a}{b}$.

Note that we have only thought about this for the special case where a, b, and x are positive integers; but it is actually true for many more possibilities.

We say that the fractions $\frac{xa}{xb}$ and $\frac{a}{b}$ are **equivalent**.

Think / Pair / Share

Write down at least 3 fractions equivalent to each of the following. Share your answers with another person and see if they agree that your answers are appropriate.

1. $\frac{1}{2}$
2. $\frac{10}{3}$

3. 1

Note that in each case above, we started with smaller values in the numerator and denominator and then considered cases in which the values are larger. What if we went the other way.

Example: Going Backwards

$\frac{20}{32}$ is the value equal to the amount an individual child gets when 32 kids share 20 pies.

This is the same result for when 8 kids share 5 pies.

$$\frac{20}{32} = \frac{5 \cdot 4}{8 \cdot 4} = \frac{5}{8}$$

Many people say we have *cancelled* the 4. Unfortunately, the word "cancelled" gets used for many situations in mathematics and can hide what is actually going on. It can also make it seem like we just get to take the 4 out because it's a weird trick or magic. Instead, the following phrases are better.

- $\frac{20}{32} = \frac{5 \cdot 4}{8 \cdot 4} = \frac{5}{8}$ shows what happens when we *divide* out the common factor 4 from the numerator and denominator.
- $\frac{20}{32} = \frac{5 \cdot 4}{8 \cdot 4} = \frac{5}{8}$ shows the process of *reducing* the fraction to lowest terms.
- $\frac{20}{32} = \frac{5 \cdot 4}{8 \cdot 4} = \frac{5}{8}$ illustrates ;*simplifying* the fraction.

Each of these phrasings have advantages. It is a good idea to be comfortable with all of them and to try to use them instead of "cancelling".

Example: How Low Can You Go?

As another example, $\frac{280}{350}$ can certainly be simplified by noticing that there is a common factor of 10 in both the numerator and the denominator.

$$\frac{280}{350} = \frac{28 \cdot 10}{35 \cdot 10} = \frac{28}{35}$$

We can go further as 28 and 35 are both multiples of 7.

$$\frac{28}{35} = \frac{4 \cdot 7}{5 \cdot 7} = \frac{4}{5}$$

Thus, sharing 280 pies among 350 children gives the same result as sharing 4 pies among 5 children!

$$\frac{280}{350} = \frac{4}{5}$$

Since 4 and 5 share no common factors, this is as far as we can go with this example (while keeping the numerator and denominator equal to positive integers).

On Your Own

Exercise 8.2.1

Mix and Match: Which simplified fraction goes with each non-simplified fraction fraction? (Notice that there are fewer simplified fractions than non-simplified frctions!)

1. $\frac{10}{20}$	a. $\frac{2}{3}$
2. $\frac{50}{75}$	b. $\frac{9}{16}$
3. $\frac{24000}{36000}$	c. $\frac{12}{7}$
4. $\frac{24}{14}$	d. $\frac{1}{40}$
5. $\frac{18}{32}$	e. $\frac{1}{2}$
6. $\frac{1}{40}$	

See if you can come up with a good explanation for why all of this works. Then share it with another person and see if they believe your explanation.

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8.3: Rational Numbers

"Rational numbers" is simply the formal name for what people tend to call simply "fractions". The problem is that "fractions" is much more general. Below is a formal definition of a rational number.

Definition: Rational Number

A **rational number** is a real number that can be written as the quotient of an integer, m, and a nonzero integer, n, and is written as $\frac{m}{n}$.

Note that another name for "quotient" is "ratio", which makes the use of the word "rational" a little more reasonable in this context.

To get a better understanding of this definition, let's look at a few numbers and determine if they are rational numbers.

Example 8.3.1

Determine if each number is a rational number or not.

- $-\frac{9}{2}$
- 100
- $\frac{\pi}{2}$
- $\frac{\sqrt{-1}}{2}$

Solution

The first two numbers are rational numbers, and the second two are not. Let's see why.

- $-\frac{9}{2}$ can be written as a quotient of an integer and a nonzero integer. In this case, $-\frac{9}{2} = \frac{-9}{2}$ works. Note that we could use a different quotient, such as $\frac{9}{-2}$ or $\frac{-27}{6}$.
- 100 can be written as a quotient of an integer and a nonzero integer. $100 = \frac{100}{1}$ is the easiest way to see this. But, again, we could choose a different (though equivalent) quotient instead.
- $\frac{\pi}{2}$ *cannot* be written as a quotient of an integer and a nonzero integer. The reason for this is that the number π is an **irrational** number. Johann Heinrich Lambert proved the irrationality of π in 1761 in "Mémoire sur quelques propriétés remarquables des quantités transcendentes circulaires et logarithmiques".
- $\frac{\sqrt{-1}}{2}$ is not even a real number, so it cannot be a rational number.

The point of the four numbers in the previous example is to make it clear that "rational number" and "fraction" are not interchangeable. 100 is not a fraction, but it can be written as one! $\frac{\pi}{2}$ is a fraction, but it's not a fraction of an integer over a nonzero integer!

Parts of Wholes

So far, we have focused on fractions as the result of division (i.e., fractions are quotients). When the numerator and denominator must be integers means that we can use a part-whole definition of a fraction using discrete objects. We will stick with positive values and leave negative values to be defined as their "opposites". The following is one way to represent a rational number, $\frac{m}{n}$.

Definition: Unit Fractions as a Part of a Whole

Suppose m and n are positive integers. Given a defined whole (i.e., some object or set of objects that represents 1), which we call a **unit**, if the unit is divided into n equal parts, then each part represents the **unit fraction** $\frac{1}{n}$. A collection of m copies of the parts represents the rational number $\frac{m}{n}$.

Let's try this.

Example 8.3.1

For each rational number, $\frac{m}{n}$, do the following.

a. Define a unit in terms of dots. State the value of each dot.
   i. Show a representation of $\frac{1}{n}$.
   ii. Show a representation of $\frac{m}{n}$.

b. Define the unit with a different number of dots, and do parts i and ii again.
   i. $\frac{3}{8}$
   ii. $\frac{2}{5}$

Solution

$\frac{3}{8}$

a. Define one unit to be the 8 dots below. Each dot represents 1/8.

• • • •
• • • •

   i. Below is a representation of 1/8.

   •

   ii. Below is a representation of 3/8.

   • • •

b. Define one unit to be the 16 dots below. Each dot represents 1/16.

• • • •
• • • •
• • • •
• • • •

   i. In order to find 1/8, let's divide the whole into 8 equal portions. We will do this using brackets [ ] around dots. We need 8 sets of brackets.
   [• •] [• •]
   [• •] [• •]
   [• •] [• •]
   [• •] [• •]
   Since there are 8 of them, one portion is 1/8. Below is a representation of 1/8.

   • •

   ii. Below is a representation of 3/8.

   • • • •
   • •

$\frac{2}{5}$

a. Define one unit to be the 5 dots below. Each dot represents 1/5.

• • • • •

   i. Below is a representation of 1/5.

   •

   ii. Below is a representation of 2/5.

   • •

b. Define one unit to be the 10 dots below. Each dot represents 1/10.

• • • • •
• • • • •

i. On your own, divide the whole into 5 equal portions. If you have done it correctly, each portion should have 2 dots. Below is a representation of 1/5.

• •

ii. Below is a representation of 2/5.

• • • •

It is crucial to define your unit before beginning any exercise involving the representation of fractions! Try this on your own using dots to represent $\frac{3}{4}$.

After you do this, see if you can answer these questions.

- Using this dots method of defining a unit, how many dots would you use in order to represent $\frac{29}{100}$?
- If you used a regular octagon as the unit instead, what would 1/8 be?
- What if you tried to use the same octagon as a unit but wanted to represent 1/5?
- What other ways can you represent units?

Comparing Fractions

We are now going to work again with dots (because they are easy to represent in a text document). This time we will use them to compare two fractions.

Example 8.3.1

Use dot models to compare $\frac{2}{5}$ and $\frac{3}{7}$.

Solution

Note that we need a dot model that can work for both fifths and sevenths. We saw in the earlier example that fifths can be modeled with a unit consisting of 5 dots. But that would be difficult to use as the unit for dealing with sevenths. We also saw earlier that fifths can be modeled with a unit consisting of 10 dots. This still won't do it. You might have noticed that the reason 10 dots was chosen is because $10 = 5 \cdot 2$. You also probably figured out that the smallest number of dots used to model a fraction is based on the denominator, and we would probably start with 7 dots as the unit for representing sevenths. But that won't work for fifths! Using the fact that the multiple of 5 worked for fifths, we will try a multiple of both 5 and 7. Let's use $5 \cdot 7 = 35$ for our unit.

Let 1 unit be the 35 dots below.

• • • • • • •
• • • • • • •
• • • • • • •
• • • • • • •
• • • • • • •

That means each individual dot is 1/35 of a unit.

To find 1/5, we divide the collection into 5 equal portions. Notice that we can do this by simply using the rows.

[• • • • • • •]
[• • • • • • •]
[• • • • • • •]
[• • • • • • •]
[• • • • • • •]

Each row of 7 dots is 1/5 of the unit! In other words, 1/5 is represented below.

• • • • • • •

This means 2/5 is two copies of this. 2/5 is represented below.

• • • • • • •

• • • • • • •

To find 1/7, we divide the collection into 7 equal portions. Notice that we can do this by simply using the columns. Each column of 5 dots is 1/7 of the unit. In other words, 1/7 is represented below.

•

•

•

•

•

This means that 3/7 is 3 copies of this. 3/7 is represented below.

• • •

• • •

• • •

• • •

• • •

Now, we can compare them. We compare the 14 dots that form 2/5 with the 15 dots that form 3/7.

• • • • • • • • • •

• • • • • • • • • •

• • •

• • •

• • •

Since 14 is less than 15, we can see that it is reasonable to say that $\frac{2}{5} < \frac{3}{7}$.

Try to use the process to do the following.

? Exercise 8.3.1

Compare 3/4 and 4/5 using dot models

Before moving on, notice that the method of finding the appropriate number of dots to use as the unit is similar to finding a common denominator. This is not just a coincidence!

Adding and Subtracting Fractions

We will continue to use our dots model method to add and subtract fractions, using add-to and take-from perspectives of addition and subtraction.

Example 8.3.1

Use dot models to *find the sum of* $\frac{2}{5}$ and $\frac{3}{7}$.

Solution

As in the previous example, let's use 35 dots for the unit.



As before, 2/5 is represented below.



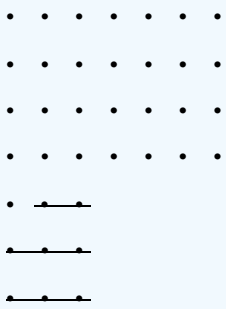
Similarly, 3/7 is represented below.



We will represent $\frac{2}{5}+\frac{3}{7}$ by starting with the 14 dots that form 2/5 and then adding the 15 dots that form 3/7 to that. Let's start by just putting them together without thinking about how we represented the unit.



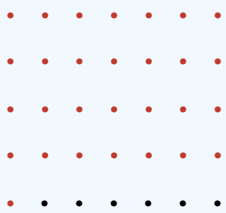
Now, we know that moving dots around does not change the number of dots and, thus, does not change the size of the portion we represent. So, let's move the lower dots up to form complete rows of 7 dots.



In short, $\frac{2}{5}+\frac{3}{7}$ is represented below.



If we compare this to the unit, we see that this is 29 out of the 35 dots that form the unit.



Therefore, $\frac{2}{5}+\frac{3}{7}=\frac{29}{35}$.

Try to use the process to do the following.

? Exercise 8.3.1

Use dot models to find the sum of $\frac{1}{4}$ and $\frac{2}{5}$.

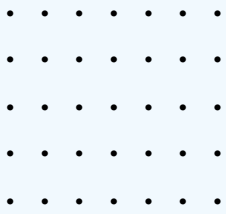
Okay, let's go on to an example of subtraction.

✔ Example 8.3.1

Use dot models to *find the difference of* $\frac{3}{7}$ and $\frac{2}{5}$.

Solution

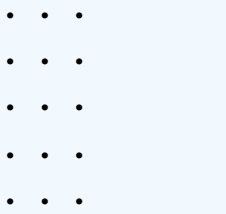
As in the previous examples, let's use 35 dots for the unit.



As before, 2/5 is represented below.



Similarly, 3/7 is represented below.



We will represent $\frac{2}{5}-\frac{3}{7}$ by starting with the 15 dots that form 3/7 and then taking 14 dots that form 2/5 away from that. To do this, let's remind ourselves that we can rearrange dots. So, we will take the 14 dots that form 2/5 in columns of 5. So, now 2/5 is represented below.

• • •

• • •

• • •

• • •

• •

This makes it easier to see that when we take that away from 3/7, we have only one left!

• • •

• • •

• • •

• • •

• • •

In short, $\frac{3}{7} - \frac{2}{5}$ is represented below.

•

If we compare this to the unit, we see that this is 1 out of the 35 dots that form the unit.

• • • • • • •

• • • • • • •

• • • • • • •

• • • • • • •

• • • • • • •

Therefore, $\frac{3}{7} - \frac{2}{5} = \frac{1}{35}$.

Try to use the process to do the following.

? Exercise 8.3.1

Use dot models to subtract $\frac{7}{8} - \frac{2}{3}$.

Multiplying Fractions

Multiplying fractions is a little bit trickier. This is partly because the repeated addition perspective (what we often think of as the simplest perspective of multiplication) doesn't make sense when you are multiplying two fractions. Instead, we will use the portion of a quantity perspective. When you see two fractions multiplied together, like $\frac{2}{5} \cdot \frac{3}{7}$, think of it the first fraction *of* the second fraction. E.g., $\frac{2}{5} \cdot \frac{3}{7}$ is 2/5 *of* 3/7. What does this mean? Part of the problem is we also need to think about what the fraction $\frac{3}{7}$ means. That fraction means 3/7 of a unit. So, $\frac{2}{5} \cdot \frac{3}{7}$ is actually 2/5 *of* 3/7 *of a unit*! That means to multiply $\frac{2}{5} \cdot \frac{3}{7}$, you must take 3/7 of a unit and then take 2/5 of that.

Note that because of the way the English preposition "of" works, it makes sense that we have to first figure out what 3/7 is before dealing with the 2/5. Just like if you want to find "the brother of the best friend of Tom", you must find out who Tom's best friend is before you can even consider finding out who that person's brother is.

✔ Example 8.3.1

Use dot models to find the *product* of $\frac{2}{5}$ and $\frac{3}{7}$.

Solution

As in the previous examples, let's use 35 dots for the unit.



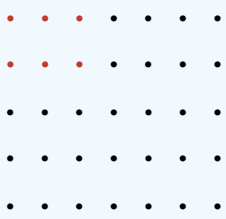
So, 3/7 of a unit is represented below.



We now need to find 2/5 of this! To do this, let's divide this portion into fifths. As before, since there are 5 rows of 3 in this representation of 3/7, each row of 3 is 1/5 of 3/7.

This means that 2/5 of 3/7 is 2 copies of that.

We now compare this to our unit.



Therefore, $\frac{2}{5} \cdot frac37 = \frac{6}{35}$.

Okay, let's see what it looks like if we switch the order of the fractions.

Example 8.3.1

Use dot models to find the product of $\frac{3}{7}$ and $\frac{2}{5}$.

Solution

We will start off the same, using 35 dots for the unit.



But now we need to know what 2/5 of the unit is. 2/5 of a unit is represented below.

• • • • • • •

We now need to find 3/7 of this. To do this, we divide this portion into sevenths. Since there are 7 rows of 2 in this representation of 2/5, each column of 2 is 1/7 of 2/5.

•

•

This means that 3/7 of 2/5 is 3 copies of that.

• • •

• • •

This is the same as what we got for $\frac{2}{5} \cdot frac37$.

Therefore, $\frac{3}{7} \cdot frac25 = \frac{6}{35}$.

Although the answer is the same for the previous two examples, the sequence of steps is not. This is a reminder that *the commutativity of multiplication is not obvious*! Thinking of multiplication as the result of area helps, but we also must pair this with acceptance of the fact that all the perspectives of multiplication do, indeed, describe the same operation. This is also not obvious and is one of many reasons why people learning arithmetic for the first time need time and perseverance to build mathematical foundations. Speaking of perseverance, try to do the following on your own.

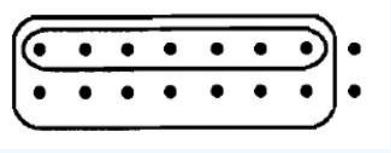
? Exercise 8.3.1

Use dot models to find the product of $\frac{4}{5}$ and $\frac{2}{3}$ and the product of $\frac{3}{4}$ and $\frac{5}{6}$.

One way to challenge yourself is to try working backwards.

✔ Example 8.3.1

Which multiplication is modeled below: $\frac{1}{2} \cdot \frac{7}{8}$ or $\frac{7}{8} \cdot \frac{1}{2}$?



Solution

Notice that we see that the unit is represented with 16 dots.

• • • • • • • •

• • • • • • • •

14 of the dots are selected from the unit. These 14 dots consist of 7 columns of 2 dots each. Each column is 1/8 of the unit.

• • • • • • • •

• • • • • • • •

So, the 14 dots that are encircled represent 7 copies of 1/8. That is, they are 7/8 of the unit!

Then half of that selection is encircled. That's half of the 7/8. So, the inner-most encircled portion is 1/2 of 7/8 of the unit.

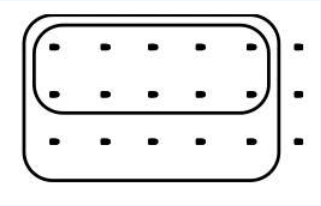
• • • • • • •

In other words, this models $\frac{1}{2} \cdot \frac{7}{8} = \frac{7}{16}$.

Now try one on your own.

Exercise 8.3.1

Which multiplication is modeled below: $\frac{5}{6} \cdot \frac{2}{3}$ or $\frac{2}{3} \cdot \frac{5}{6}$?



Dividing Fractions

To think about division with rational numbers, we can use the repeated subtraction (also called the measurement or quotitive) perspective of division. This perspective tells us that the quotient $a \div b$ can be obtained by answering this question: "How many sets of b are contained in a?"

Example 8.3.1

Use dot models to find the *quotient* of 3 and $\frac{1}{2}$.

Solution

Since we will be caring about wholes and $\frac{1}{2}$ of a whole, let's use two dots for a unit.

• •

So, 3, which is 3 copies of a unit, is represented below.

• •

• •

• •

On the other hand, $\frac{1}{2}$ of a unit is represented below.

•

We want to know how many $\frac{1}{2}$s are in 3. Since $\frac{1}{2}$ is •, that is the same as asking how many • are in the representation of 3 below.

• •

• •

• •

Clearly, there are 6. So, $3 \div \frac{1}{2} = 6$.

Let's do another, but this time involving fractions as both the divisor and the dividend. We will still keep it a little simple by only using unit fractions.

Example 8.3.1

Use dot models to find the *quotient* of $\frac{1}{2}$ and $\frac{1}{4}$.

Solution

Since we will be caring about both 1/2 and 1/4, let's use four dots for a unit.

• • • •

So, 1/2 of a unit is represented below.

• •

On the other hand, 1/4 of a unit is represented below.

•

We want to know how many 1/4s are in 1/2. Since 1/4 is •, that is the same as asking how many • are in the representation of 1/2 below.

• •

Clearly, there are 2, so $\frac{1}{2} \div \frac{1}{4} = 2$.

Try to do a similar problem on your own.

? Exercise 8.3.1

Use dot models to find $\frac{1}{5} \div \frac{1}{10}$.

You might have noticed that this process seemed surprisingly easy compared to, say, multiplication. This is because we have still make this easier than in general. In particular, note that in the previous example and exercise, the denominator of the dividend is a factor of the denominator of the divisor. It turns out that was more important than the fact that the dividend was a unit fraction. It should be pretty straightforward to expand this method to perform division when we change the dividend to a non-unit fraction. Try the following.

? Exercise 8.3.1

Use dot models to find $\frac{2}{3} \div \frac{1}{6}$

To perform division in general may take a little more effort. The reason is that when we ask "How many b are in a?", the answer might not be an integer. There might be a nonzero remainder. Let's leave that for a later time though since we have already covered quite a bit!

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8.4: Adding and Subtracting Fractions

Fractions with the Same Denominators

Here are two very similar fractions: $\frac{2}{7}$ and $\frac{3}{7}$. What might it mean to add them? It might seem reasonable to say:

$$\frac{2}{7} \text{ represents 2 pies shared by 7 kids.}$$

$$\frac{3}{7} \text{ represents 3 pies shared by 7 kids.}$$

So maybe $\frac{2}{7}+\frac{3}{7}$ represents 5 pies among 14 kids, giving the answer $\frac{5}{14}$. It is very tempting to say that "adding fractions" means "adding pies and adding kids."

The trouble is that a fraction is not a pie, and a fraction is not a child. So adding pies and adding children is not actually adding fractions. A fraction is something different. It is related to pies and kids, but something more subtle. A fraction is an *amount of pie per child*.

One cannot add pies, one cannot add children. One must add instead the amounts individual kids receive.

Example: 2/7 + 3/7

Let us take it slowly. Consider the fraction $\frac{2}{7}$. Here is a picture of the amount an individual child receives when two pies are given to seven kids:



Consider the fraction $\frac{3}{7}$. Here is the picture of the amount an individual child receives when three pies are given to seven children:

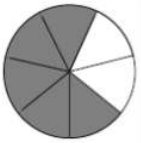


The sum $\frac{2}{7}+\frac{3}{7}$ corresponds to the sum:



The answer, from the picture, is $\frac{5}{7}$.

Remember that $\frac{5}{7}$ means "the amount of pie that one child gets when five pies are shared by seven children." Carefully explain why that is the same as the picture given by the sum.



Many people read this as "two sevenths plus three sevenths gives five sevenths" and think that the problem is just as easy as saying "two apples plus three apples gives five apples." And, in the end, they are right! Sevenths (or any other size of number) can be

thought of as objects just like apples are objects.



This is how the addition of fractions is first taught to students: Adding fractions with the same denominator seems just as easy as adding apples.

4 tenths + 3 tenths + 8 tenths = 15 tenths:

$$\frac{4}{10}+\frac{3}{10}+\frac{8}{10}=\frac{15}{10}.$$

(And, if you like, $\frac{15}{10}=\frac{5\cdot 3}{5\cdot 2}=\frac{3}{2}$.)

82 sixty-fifths + 91 sixty-fifths = 173 sixty-fifths:

$$\frac{82}{65}+\frac{91}{65}=\frac{173}{65}.$$

We are really adding **amounts per child** not amounts, but the answers match the same way.

We can use the “pies per child" model to explain *why* adding fractions with like denominators works in this way.

Example: 2/7 + 3/7

Think about the addition problem $\frac{2}{7}+\frac{3}{7}$:

$$\begin{array}{r}\text{amount of pie each kids gets when 7 kids share 2 pies}\\ +\ \text{amount of pie each kids gets when 7 kids share 3 pies}\\ \hline ?????\end{array}$$

Since in both cases we have 7 kids sharing the pies, we can imagine that it is the same 7 kids in both cases. First, they share 2 pies. Then they share 3 more pies. The total each child gets by the time all the pie-sharing is done is the same as if the 7 kids had just shared 5 pies to begin with. That is:

$$\begin{array}{r}\text{amount of pie each kids gets when 7 kids share 2 pies}\\ +\ \text{amount of pie each kids gets when 7 kids share 3 pies}\\ \hline \text{amount of pie each kids gets when 7 kids share 5 pies.}\end{array}$$

$$\frac{2}{7}+\frac{3}{7}=\frac{5}{7}.$$

Now let us think about the general case. Our claim is the following.

$$\frac{a}{d}+\frac{b}{d}=\frac{a+b}{d}.$$

Translating into our model, we have d kids. First, they share a pies between them, and $\frac{a}{d}$ represents the amount each child gets. Then they share b more pies, so the additional amount of pie each child gets is $\frac{b}{d}$. The total each kid gets is $\frac{a}{d}+\frac{b}{d}$.

But it does not really matter that the kids first share a pies and then share b pies. The amount each child gets is the same as if they had started with all of the pies — all $a+b$ of them — and shared them equally. That amount of pie is represented by $\frac{a+b}{d}$.

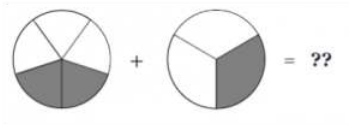
Try to answer the following questions and share them with a classmate, friend, or any other person. Do they agree with your answers?

? Think / Pair / Share

- How can you *subtract* fractions with the same denominator? For example, what is $\frac{400}{903} - \frac{170}{903}$?
- *Why* does $\frac{a}{d} - \frac{b}{d} = \frac{a-b}{d}$? Use the "pies per phild" model.
- *Why* is the fact that the denominators are the same *essential* to this addition and subtraction method? Where is that fact used in the explanations?

Fractions with Different Denominators

This approach to adding fractions suddenly becomes tricky if the denominators involved are not the same common value. For example, what is $\frac{2}{5} + \frac{1}{3}$?



Let us phrase this question in terms of pies and kids.

> *Suppose Poindexter is part of a team of five kids that shares two pies. Then later he is part of a team of three kids that shares one pie. How much pie does Poindexter receive in total?*

Try to answer the following questions and share your answer with a classmate or friend. It is actually very difficult! Especially think about the situation where the person you are talking to does not already know about adding fractions.

? Think / Pair / Share

1. Do you see that the Poindexter problem is the same problem as computing $\frac{2}{5} + \frac{1}{3}$?
2. What might be the best approach to answering the problem?

One way to think about answering this addition question is to write $\frac{2}{5}$ in a series of alternative forms using our key fraction rule (that is, multiply the numerator and denominator each by 2, and then each by 3, and then each by 4, and so on) and to do the same for $\frac{1}{3}$.

$$\begin{array}{cc} \frac{2}{5} & + \frac{1}{3} \\ \frac{4}{10} & \frac{2}{6} \\ \frac{6}{15} & \frac{3}{9} \\ \frac{8}{20} & \frac{4}{12} \\ \frac{10}{25} & \frac{5}{15} \\ \vdots & \vdots \end{array}$$

We see that the problem $\frac{2}{5} + \frac{1}{3}$ is actually the same as $\frac{6}{15} + \frac{5}{15}$. So we can find the answer using the same-denominator method.

$$\frac{2}{5} + \frac{1}{3} = \frac{6}{15} + \frac{5}{15} = \frac{11}{15}.$$

✔ Example: 3/8 + 3/10

Here is another example of adding fractions with unlike denominators: $\frac{3}{8} + \frac{3}{10}$. In this case, Valerie is part of a group of 8 kids who share 3 pies. Later she is part of a group of 10 kids who share 3 different pies. How much total pie did Valerie get?

$$\begin{array}{cc} \frac{3}{8} + \frac{3}{10} \\ \frac{6}{16} \quad \frac{6}{20} \\ \frac{9}{24} \quad \frac{9}{30} \\ \frac{12}{32} \quad \frac{12}{40} \\ \frac{15}{40} \quad \frac{15}{50} \\ \vdots \quad \vdots \end{array}$$

$$\frac{3}{8} + \frac{3}{10} = \frac{15}{40} + \frac{12}{40} = \frac{27}{40}.$$

Of course, you do not need to list all of the equivalent forms of each fraction in order to find a common denominator. If you can see a denominator right away (or think of a faster method that always works), go for it!

? Think / Pair / Share

Cassie suggests the following method for the example above:

> *When the denominators are the same, we just add the numerators. So when the numerators are the same, shouldn't we just add the denominators? Like this:*
>
> $$\frac{3}{8} + \frac{3}{10} = \frac{3}{18}.$$

What do you think of Cassie's suggestion? Does it make sense? What would you say if you were Cassie's teacher?

On Your Own

Try these exercises on your own. For each addition exercise, also write down a "pies per child" interpretation of the problem. You might also want to draw a picture.

1. What is $\frac{1}{2} + \frac{1}{3}$?
2. What is $\frac{2}{5} + \frac{37}{10}$?
3. What is $\frac{1}{2} + \frac{3}{10}$?
4. What is $\frac{2}{3} + \frac{5}{7}$?
5. What is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$?
6. What is $\frac{3}{10} + \frac{4}{25} + \frac{7}{20} + \frac{3}{5} + \frac{49}{50}$?

Now try these subtraction exercises.

1. What is $\frac{7}{10} - \frac{3}{10}$?
2. What is $\frac{7}{10} - \frac{3}{20}$?
3. What is $\frac{1}{3} - \frac{1}{5}$?
4. What is $\frac{2}{35} - \frac{2}{7} + \frac{2}{5}$?
5. What is $\frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16}$?

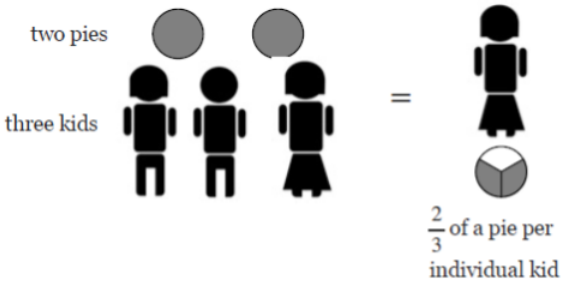
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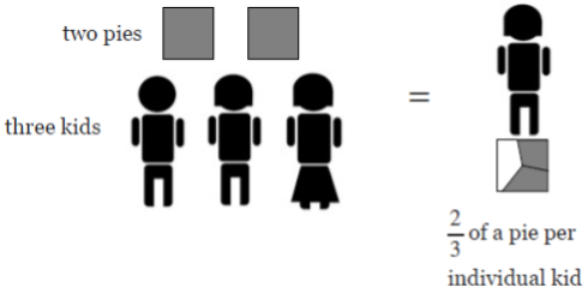
8.5: What is a Fraction? Revisited

Models

We have been thinking about a fraction as the answer to a division problem. For example, $\frac{2}{3}$ is the result of dividing 2 into 3 portions. Like sharing two pies among three children.



Of course, pies do not have to be round. We can have square pies, or triangular pies or squiggly pies or any shape you please. An issue arises, however, when we recall that the division should be fair. For example, in the image below, are we sure that the two shaded pieces shown for that child really are 2/3 of a square?



So, we have stuck with circular pies to make this a little easier. That said, depending on the numbers of children and pies, even a circular pie is not easy to divide. See if you can draw the amount of pie each child gets if 4 pies are shared among 11 children.

Despite this, when the numbers are friendly, the “Pies Per Child Model” has served us perfectly well in thinking about the meaning of fractions, equivalent fractions, and even adding and subtracting fractions. However, there is no way to use this model to make sense of multiplying fractions! What would this mean?

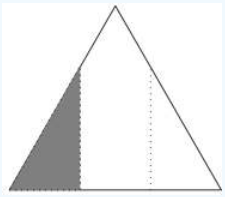


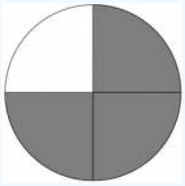
This is a good time to remind ourselves that there is no reason to force ourselves to stick with one model. We can switch models and think about fractions in a new way. Of course, there is actually a reason we tend to try sticking with one model. The more familiar a model is, the more comfortable we are using it! Think about students learning this for the first time. We keep switching concepts and models, and speak of fractions in each case as though all is naturally linked and obvious. None of this is obvious, it is all absolutely confusing. This is just one of the reasons that fractions can be such a difficult concept to teach and to learn in elementary school! Let's get in the mindset of someone just learning about fractions for the first time. Try to answer the two questions below and explain your ansers to another person. Does that person "buy" your answers?

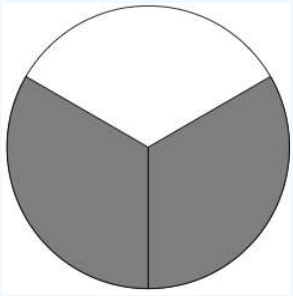
? Think / Pair / Share (What's wrong here?)

For each of the following visual representations of fractions, there is a corresponding **incorrect** symbolic expression.

- Why is the symbolic representation incorrect?
- What might elementary students find confusing in these visual representations?







$\frac{1}{3}$ $\frac{2}{3} > \frac{3}{4}$





$\frac{1}{3} \neq 13$

Units and unitizing

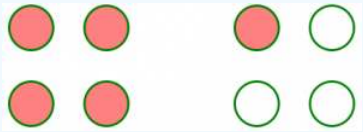
In thinking about fractions, it is important to remember that there are always *units* attached to a fraction, even if the units are hidden. If you see the number $\frac{1}{2}$ in a problem, you should ask yourself "half of what?" The answer to that question is your unit, the amount that equals 1. In other words, you can think of "$\frac{1}{2}$" as actually shorthand for "half *of one*".

For our pie model, our units have been consistent: the "whole" (or unit) was one whole pie (which we treated as single circle), and fractions were represented by pies cut into equal-sized pieces. But this is just a model, and we can take anything, cut it into equal-sized pieces, and talk about fractions *of that whole*.

One thing that can make fraction problems so difficult is that the fractions in the problem may be given in different *units* (they may be "parts" of different "wholes").

Example: Everyone is right!

Mr. Li shows this picture to his class and asks what number is shown by the shaded region.



- Kendra says the shaded region represents the number 5.
- Dylan says it represents $2\frac{1}{2}$.
- Kiana says it represents $\frac{5}{8}$.
- Nate says it is $1\frac{1}{4}$.

Mr. Li exclaims, "Everyone is right!"

Do you believe Mr. Li in the example above? His statement is accurate. See if you can justify each answer by explaining what each student thought was the *unit* in Mr. Li's picture. Try the following and share your answers with another person. Did you and the other person get the same answers?

Think / Pair / Share

Now look at this picture:



- If the shaded region represents $3\frac{2}{3}$, what is the unit?
- Find three other numbers that could be represented by the shaded region, and explain what the unit is for each answer.

What other models could we use? How about line segments? Suppose the picture below represents $\frac{2}{3}$. Let's make sense of that claim. There are three important characterisitcs of the picture. There is the black horizontal line segment, there are two vertical line segments, and there is a thick grey horizontal line segment. The black horizontal line segment can be used as the whole (the *unit*) because it is split into three equal-length pieces by the vertical line segments and two of those three equal pieces correspond to the length of the thick grey horizontal line segment.



Think / Pair / Share

For each picture below, say what fraction it represents and how you know you are right. Share with a friend and see if they accept your answers.





Ordering Fractions

If we think about fractions as “portions of a line segment,” then we can talk about their locations on a number line. We can start to treat fractions like numbers on their own. In the back of our minds, we should remember that fractions are always relative to some unit. But on a number line, the unit is clear: it is the distance between 0 and 1.



This number line model makes it much easier to tackle questions about the relative size of fractions: the sizes are based on where they appear on the number line. We can mark off different fractions as parts of the unit segment. Just as with whole numbers, fractions that appear farther to the right are greater.



3/5 and 5/8 are very close, but 5/8 is just a bit bigger. Try to do the following on your own and then share with another person. Did your answers make sense to them?

? Think / Pair / Share (Ordering fractions)

1. What quick method can you use to determine if a fraction is greater than 1?
2. What quick method can you use to determine if a fraction is greater than $\frac{1}{2}$?
3. Organize these fractions from smallest to largest using *benchmarks*: 0 to $\frac{1}{2}$, $\frac{1}{2}$ to 1, and greater than 1. Justify your choices.

$$\frac{25}{23}, \quad \frac{4}{7}, \quad \frac{17}{35}, \quad \frac{2}{9}, \quad \frac{14}{15}.$$

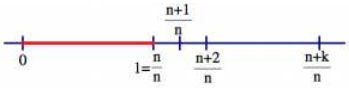
4. Arrange each group of fractions in *ascending order*. Keep track of your thinking and your methods.

- $\frac{7}{17}, \quad \frac{4}{17}, \quad \frac{12}{17}.$
- $\frac{3}{7}, \quad \frac{3}{4}, \quad \frac{3}{8}.$
- $\frac{5}{6}, \quad \frac{7}{8}, \quad \frac{3}{4}.$
- $\frac{8}{13}, \quad \frac{12}{17}, \quad \frac{1}{6}.$
- $\frac{5}{6}, \quad \frac{10}{11}, \quad \frac{2}{3}.$

Fraction Intuition

Below there are a lot of different "rules" you may have discovered over the years by just noticing patterns. But it is also likely that you never really wrote them down. For each one, think about if you ever discovered this on your own and, if so, would you have described it the same way.

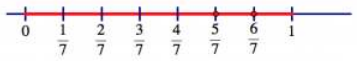
Greater than 1: A fraction is greater than 1 if its numerator is greater than its denominator. How can we see this? Well, the denominator represents how many pieces in one whole (one unit). The numerator represents how many pieces in your portion. So if the numerator is bigger, that means you have more than the number of pieces needed to make one whole.



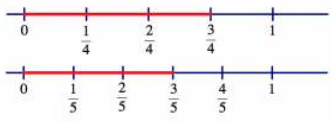
Greater than $\frac{1}{2}$: A fraction is greater than $\frac{1}{2}$ if the numerator is more than half the denominator. Another way to check (which might be an easier calculation): a fraction is greater than $\frac{1}{2}$ if twice the numerator is greater than the denominator.

Why? See if you can justify this to yourself.

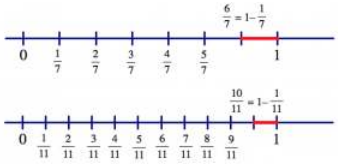
Same denominators: If two fractions have the same denominator, just compare the numerators. The fractions will be in the *same order as the numerators*. For example, $\frac{5}{7} < \frac{6}{7}$. Why? Well, the pieces are the same size since the denominators are the same. If you have more pieces of the same size, you have a bigger number.



Same numerators: If the numerators of two fractions are the same, just compare the denominators. The fractions should be in the *reverse order of the denominators*. For example, $\frac{3}{4} > \frac{3}{5}$. The justification for this one is a little trickier: The denominator tells you how many pieces make up one whole. If there are more pieces in a whole (if the denominator is bigger), then the individual pieces must be smaller. And if you take the same number of pieces (same numerator), then the collection with the bigger pieces is greater.



Numerator = denominator-1: You can easily compare two fractions whose numerators are both one less than their denominators. The fractions will be in the same order as the denominators. Think of each fraction as a pie with one piece missing. The greater the denominator, the smaller the missing piece, so the greater the amount remaining. For example, $\frac{6}{7} < \frac{10}{11}$, since $\frac{6}{7} = 1 - \frac{1}{7}$ and $\frac{10}{11} = 1 - \frac{1}{11}$.



Numerator = denominator − constant: You can extend the test above to fractions whose numerators are both the same amount less than their denominators. The fractions will again be in the same order as the denominators, for exactly the same reason. For example, $\frac{3}{7} < \frac{7}{11}$, because both are four “pieces” less than one whole, and the $\frac{1}{11}$ pieces are smaller than the $\frac{1}{7}$ pieces.

Equivalent fractions: Find equivalent fractions that lets you compare numerators or denominators, and then use one of the above "rules".

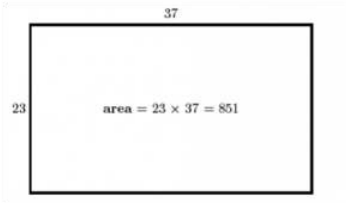
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8.6: Multiplying Fractions

Area Model

One of our models for multiplying whole numbers was an area model. For example, the product 23×37 is the area (number of 1×1 squares) of a 23-by-37 rectangle:

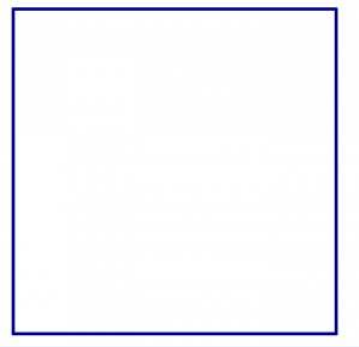


So the product of two fractions, say, $\frac{4}{7} \times \frac{2}{3}$ should also correspond to an area problem.

Example: (4/7 × 2/3)

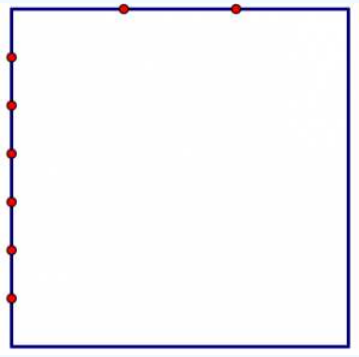
Let us start with a segment of some length that we call 1 unit:

Now, build a square that has one unit on each side:

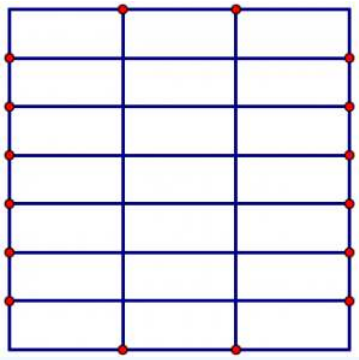


The area of the square, of course, is $1 \times 1 = 1$ square unit.

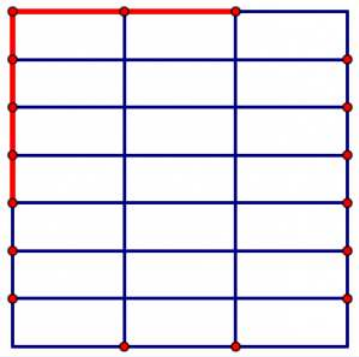
Now, let us divide the segment on top into three equal-sized pieces. (So each piece is $\frac{1}{3}$.) And we will divide the segment on the side into seven equal-sized pieces. (So each piece is $\frac{1}{7}$.)



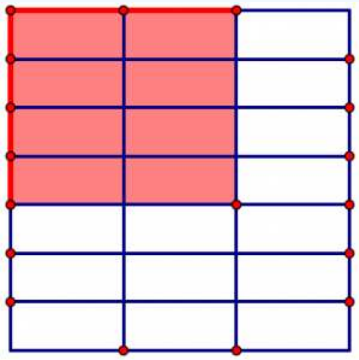
We can use those marks to divide the whole square into small, equal-sized rectangles. (Each rectangle has one side that measures $\frac{1}{3}$ and another side that measures $\frac{1}{7}$.)



We can now mark off four sevenths on one side and two thirds on the other side.



The result of the multiplication should be the area of the rectangle with $\frac{4}{7}$ on one side and $\frac{2}{3}$ on the other. What is that area?



Remember, the whole square was one unit. That one-unit square is divided into 21 equal-sized pieces, and our rectangle (the one with sides $\frac{4}{7}$ and $\frac{2}{3}$) contains eight of those rectangles. Since the shaded area is the answer to our multiplication problem we conclude that

$$\frac{4}{7} \times \frac{2}{3} = \frac{8}{21}.$$

Think / Pair / Share

1. Use are model to compute each of the following products. Draw the picture to see the answer clearly.
   $\frac{3}{4} \times \frac{5}{6}$, $\frac{3}{8} \times \frac{4}{5}$, $\frac{5}{8} \times \frac{3}{7}$.
2. The area problem $\frac{4}{7} \times \frac{2}{3}$ yielded a diagram with a *total* of 21 small rectangles. Explain why 21 appears as the total number of equal-sized rectangles.
3. The area problem $\frac{4}{7} \times \frac{2}{3}$ yielded a diagram with 8 small *shaded* rectangles. Explain why 8 appears as the number of shaded rectangles.

Problem 5

How can you extend the area model for fractions greater than 1? Try to draw a picture for each of these:

$$\frac{3}{4}\cdot\frac{3}{2}, \qquad \frac{2}{5}\cdot\frac{4}{3}, \qquad \frac{3}{10}\cdot\frac{5}{4}, \qquad \frac{5}{2}\cdot\frac{7}{4}.$$

On Your Own

Work on the following exercises on your own or with a partner.

1. Compute the following products, simplifying each of the answers as much as possible. You do not need to draw pictures, but you may certainly choose to do so if it helps! $\frac{5}{11}\times\frac{7}{12}$, $\frac{4}{7}\times\frac{4}{8}$, $\frac{1}{2}\times\frac{1}{3}$, $\frac{2}{1}\times\frac{3}{1}$, $\frac{1}{5}\times\frac{5}{1}$.
2. Compute the following products. (Do not work too hard!) $\frac{3}{4}\times\frac{1}{3}\times\frac{2}{5}$, $\frac{5}{5}\times\frac{7}{8}$, $\frac{88}{88}\times\frac{541}{788}$, $\frac{77876}{311}\times\frac{311}{77876}$.
3. Try this one. Can you make use of the fraction rule $\frac{xa}{xb}=\frac{a}{b}$ to help you calculate? How?
   $\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{5}{6}\times\frac{6}{7}\times\frac{7}{8}\times\frac{8}{9}\times\frac{9}{10}$.

Think / Pair / Share

How are these two problems different? Draw a picture of each.

1. Pam had $\frac{2}{3}$ of a cake in her refrigerator, and she ate $\frac{1}{2}$ of it. How much total cake did she eat?
2. On Monday, Pam ate $\frac{2}{3}$ of a cake. On Tuesday, Pam ate $\frac{1}{2}$ of a cake. Both cakes were the same size. How much total cake did she eat?

When a problem includes a phrase like "$\frac{2}{3}$ of …," students are taught to treat "of" as multiplication, and to use that to solve the problem. As the above problems show, in some cases this makes sense, and in some cases it does not. It is important to read carefully and understand what a problem is asking, not memorize rules about "translating" word problems.

Explaining the Rule

You probably simplified your work in the exercises above by using a multiplication rule like the following.

Multiplying Fractions

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}.$$

Of course, you may then choose to simplify the final answer, but the answer is always *equivalent* to this one. Why? The area model can help us explain what is going on.

First, let us clearly write out how the area model says to multiply $\frac{a}{b}\cdot\frac{c}{d}$. We want to build a rectangle where one side has length $\frac{a}{b}$ and the other side has length $\frac{c}{d}$. We start with a square, one unit on each side.

- Divide the top segment into b equal-sized pieces. Shade a of those pieces. (This will be the side of the rectangle with length $\frac{a}{b}$.)
- Divide the left segment into d equal-sized pieces. Shade c of those pieces. (This will be the side of the rectangle with length $\frac{c}{d}$.)
- Divide the whole rectangle according to the tick marks on the sides, making equal-sized rectangles.
- Shade the rectangle bounded by the shaded segments.



If the answer is $\frac{a \cdot c}{b \cdot d}$, that means there are $b \cdot d$ total equal-sized pieces in the square, and $a \cdot c$ of them are shaded. We can see from the model why this is the case:

- The top segment was divided into b equal-sized pieces. So there are b columns in the rectangle.
- The side segment was divided into d equal-sized pieces. So there are d rows in the rectangle.
- A rectangle with b columns and d rows has $b \cdot d$ pieces. (The area model for whole-number multiplication!)

? Think / Pair / Share

Stick with the general multiplication rule

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

Write a clear explanation for why $a \cdot c$ of the small rectangles will be shaded.

Multiplying Fractions by Whole Numbers

Often, elementary students are taught to multiply fractions by whole numbers using the fraction rule.

✓ Example: Multiply Fractions

For example, to multiply $2 \cdot \frac{3}{7}$, we think of "2" as $\frac{2}{1}$, and compute this way

$$2 \cdot \frac{3}{7} = \frac{2}{1} \cdot \frac{3}{7} = \frac{2 \cdot 3}{1 \cdot 7} = \frac{6}{7}.$$

We can also think in terms of our original "Pies Per Child" model to answer questions like this.

✓ Example: Pies per Child

We know that $\frac{3}{7}$ means the amount of pie each child gets when 7 children evenly share 3 pies**.**

If we compute $2 \cdot \frac{3}{7}$ that means we double the amount of pie each kid gets. We can do this by doubling the number of pies. So the answer is the same as $\frac{6}{7}$: the amount of pie each child gets when 7 children evenly share 6 pies.

Finally, we can think in terms of units and unitizing.

✓ Example: Units

The fraction $\frac{3}{7}$ means that I have 7 equal pieces (of *something*), and I take 3 of them.

So $2 \cdot \frac{3}{7}$ means do that twice. If I take 3 pieces and then 3 pieces again, I get a total of 6 pieces. There are still 7 equal pieces in the whole, so the answer is $\frac{6}{7}$.

? Think / Pair / Share

1. Use all three methods to explain how to find each product: $3 \cdot \frac{2}{5}$, $4 \cdot \frac{3}{8}$, $6 \cdot \frac{1}{5}$.
2. Compare these different ways of thinking about fraction multiplication. Are any of them more natural to you? Does one make more sense than the others? Do the particular numbers in the problem affect your answer? Does your partner agree?

Explaining the Key Fraction Rule

Roy says that the fraction rule

$$\frac{xa}{xb} = \frac{a}{b}$$

is "obvious" if you think in terms of multiplying fractions. He reasons as follows:

We know multiplying anything by 1 does not change a number:

$$\begin{aligned} 1 \cdot 4 &= 4 \\ 1 \cdot 2014 &= 2014 \\ 1 \cdot \frac{5}{7} &= \frac{5}{7} \end{aligned}$$

So, in general,

$$1 \cdot \frac{a}{b} = \frac{a}{b}.$$

Now, $\frac{2}{2} = 1$*, so that means that*

$$\frac{2}{2} \cdot \frac{a}{b} = 1 \cdot \frac{a}{b} = \frac{a}{b},$$

which means

$$\frac{2a}{2b} = \frac{a}{b}.$$

By the same reasoning, $\frac{3}{3} = 1$*, so that means that*

$$\frac{3}{3} \cdot \frac{a}{b} = 1 \cdot \frac{a}{b} = \frac{a}{b},$$

which means

$$\frac{3a}{3b} = \frac{a}{b}.$$

? Think / Pair / Share

What do you think about Roy's reasoning? Does it make sense? How would Roy explain the general rule for positive whole numbers x:

$$\frac{xa}{xb} = \frac{a}{b}?$$

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8.7: Introduction to Dividing Fractions

Dividing fractions is one of the hardest ideas in elementary school mathematics. By now, you are used to the rule: to divide by a fraction, multiply by its reciprocal. ("invert and multiply"). But ask yourself: Why does this rule work? Does it really make sense to you? Can you explain why it makes sense to a third grader?

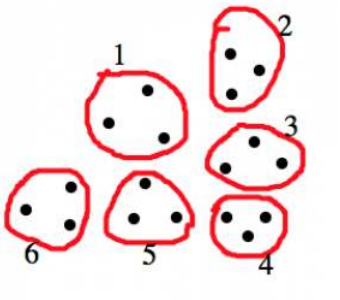
We are going to build up to the "invert and multiply" rule, but along the way, we'll find some more meaningful ways to understand division of fractions. So please play along: **pretend that you don't already know the "invert and multiply" rule,** and solve the problems in this chapter with other methods.

Groups of Equal Size

Remember the quotative model for division: $18 \div 3$ means:

> *How many groups of 3 can I find in 18?*

We start with 18 dots (or candy bars or molecules), and we make groups of 3 dots (or 3 whatevers). We ask: how many groups can we make?



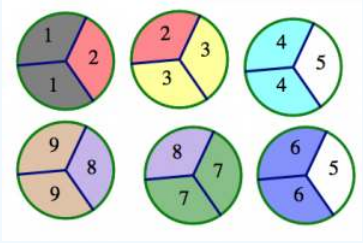
18 dots, split into groups of 3 dots. Since there are 6 groups, we have 18 : 3 = 6.

This same idea applies when we divide fractions. For example, $6 \div 23$ means:

> *How many groups of $\frac{2}{3}$ can I find in 6?*

Example: 6 ÷ 2/3

Let's draw a picture of 6 pies, and see how many groups of $\frac{2}{3}$ we can find:



We found nine equal groups of size $\frac{2}{3}$, so we conclude that

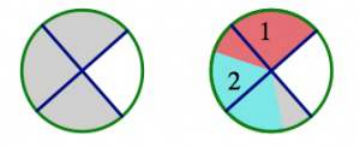
$$6 \div \frac{2}{3} = 9.$$

Unfortunately, it's not always quite so straightforward to find the equal groups. For example, $\frac{3}{4} \div \frac{1}{3}$ asks the question:

> *How many groups of $\frac{1}{3}$ can I find in $\frac{3}{4}$?*

Example: 3/4 ÷ 1/3

Let's draw a picture of $\frac{3}{4}$ of a pie, and see how many groups of $\frac{1}{3}$ we can find:

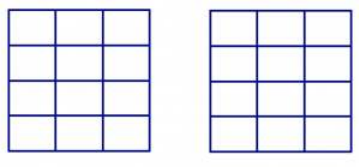


The first pictures shows $\frac{3}{4}$ of a pie. The second picture shows two equal groups of $\frac{1}{3}$ inside of $\frac{3}{4}$, but there's a little bit left over. We conclude $\frac{3}{4} \div \frac{1}{3} = 2 + \text{a tiny bit more}$. But how much more? Can we figure it out exactly?

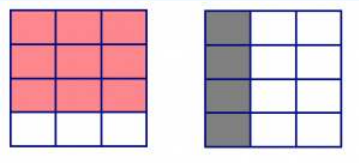
Here's a method that will let you do the computation exactly. We'll use rectangular pies, and divide them up into rows and columns based on the denominators of the numbers we're dividing.

Example: 3/4 ÷ 1/3

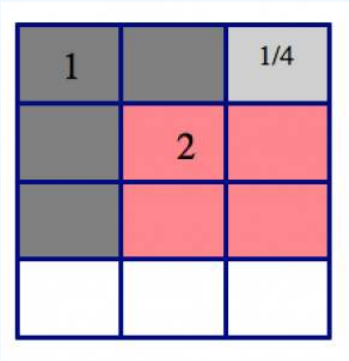
Start by drawing two identical rectangles, each with 4 rows (from the denominator of $\frac{3}{4}$ and 3 columns (from the denominator of $\frac{1}{3}$).



Shade $\frac{3}{4}$ of the first rectangle (this is exactly three rows), and shade $\frac{1}{3}$ of the second rectangle (so that's one column).



Now ask: how many copies of $\frac{1}{3}$ can I find in $\frac{3}{4}$? Well, $\frac{1}{3}$ is equal to four of the smaller squares. So we find groups equal to that:



In the picture of $\frac{3}{4}$, we can find:

- two groups of four squares (two groups of $\frac{1}{3}$), and
- one square left over, which is $\frac{1}{4}$ of the group we're looking for.

We conclude:

$$\frac{3}{4} \div \frac{1}{3} = 2\frac{1}{4}.$$

Think / Pair / Share

Use either method above to find the following quotients. Remember, pretend that you don't know any method to divide fractions except finding equal-sized groups.

$$\frac{3}{4} \div \frac{1}{2} \qquad \frac{1}{3} \div \frac{1}{2} \qquad \frac{4}{9} \div \frac{1}{3} \qquad \frac{4}{5} \div \frac{1}{3} \qquad \frac{3}{5} \div \frac{3}{4} \qquad \frac{3}{2} \div \frac{1}{2} \qquad \frac{2}{3} \div \frac{1}{2}$$

Common Denominator Method

Think / Pair / Share

Solve each of the following fraction division problems using the "groups of equal size" method: $\frac{6}{4} \div \frac{3}{4}, \frac{6}{10} \div \frac{3}{10}, \frac{8}{9} \div \frac{4}{9}, \frac{6}{33} \div \frac{2}{33}, \frac{5}{4} \div \frac{2}{4}, \frac{5}{2} \div \frac{2}{2}, \frac{5}{10} \div \frac{2}{10}$ What do you notice?

This leads to our first fraction division method:

Common denominator method

If two fractions have the same denominator, then when you divide them, you can just divide the numerators. In symbols,

$$\frac{a}{d} \div \frac{b}{d} = \frac{a}{b}.$$

Think / Pair / Share

- Use the common denominator method to find these quotients: $\frac{1}{3} \div \frac{2}{3}, \frac{5}{8} \div \frac{3}{8}, \frac{3}{8} \div \frac{5}{8}, \frac{15}{33} \div \frac{1}{33}$
- What if the fractions do not have a common denominator? Is the method useless, or can you find a way to make it work? Can you solve these problems? $\frac{3}{5} \div \frac{3}{4}, \frac{3}{4} \div \frac{8}{7}, \frac{2}{3} \div \frac{1}{2}, \frac{5}{8} \div \frac{1}{4}$.

Missing Factor Method

We know that we can always turn a division problem into a "missing factor" multiplication problem. Can that help us compute fraction division? Sometimes!

Think / Pair / Share

For each division problem, rewrite it as a missing factor multiplication question. Then find the quotient using what you know about multiplying fractions.

$$\frac{9}{10} \div \frac{3}{5}, \qquad \frac{7}{8} \div \frac{1}{4}, \qquad \frac{6}{7} \div \frac{3}{7}, \qquad \frac{10}{9} \div \frac{2}{3}, \qquad \frac{25}{12} \div \frac{5}{6}.$$

Unfortunately, the missing factor method doesn't always work out so nicely. For example,

$$\frac{3}{4} \div \frac{1}{3} = \_\_$$

can be rewritten as

$$\frac{1}{3} \cdot \_\_ = \frac{3}{4}.$$

There isn't a nice ratio of whole numbers that obviously fills in the blank, but we'll come back to this idea and resolve it soon.

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8.8: Dividing Fractions Methods

The missing factor method is a particularly nice way to understand fraction division. It builds on what we know about multiplication and division, reinforcing that these operations have the same relationship whether the numbers are whole number, fractions, or anything else. It makes sense. But we've seen that it doesn't always work out nicely. For example,

$$\frac{3}{4} \div \frac{1}{3} = \_\_$$

can be rewritten as

$$\frac{1}{3} \cdot \_\_ = \frac{3}{4}.$$

You want to ask:

- For the numerator: $1 \cdot \_\_ = 3$. We can fill in the blank with a 3.
- For the denominator: $3 \cdot \_\_ = 4$. We can fill in the blank with $\frac{4}{3}$. (Why does that work?)

So we have:

$$\frac{1}{3} \cdot \frac{3}{\frac{4}{3}} = \frac{3}{4}.$$

You learned about fractions like

$$\frac{3}{\frac{4}{3}}$$

back in the "What is a Fraction?" chapter. This means that each $\frac{4}{3}$ of a kid gets 3 pies. So how much does an individual kid (one whole kid) get? You could draw a picture to help you figure it out. But we can also use the key fraction rule to help us out.

$$\frac{3}{\frac{4}{3}} = \frac{3 \cdot 3}{3 \cdot \frac{4}{3}} = \frac{9}{4}.$$

This process is going to be key to understanding why the "invert and multiply" rule for fraction division actually makes sense.

Simplify an "Ugly" Fraction

Example 8.8.1:

$7\frac{2}{3}$ pies are shared equally by $5\frac{1}{4}$ children. How much pie does each child get?

Technically, we could just write down the answer as $\frac{7\frac{2}{3}}{5\frac{1}{4}}$ and be done! The answer is equivalent to this fraction, so why not?

Is there a way to make this look friendlier? Well, if we change those mixed numbers to "improper" fractions, it helps a little:

$$\frac{7\frac{2}{3}}{5\frac{1}{4}} = \frac{\frac{23}{3}}{\frac{21}{4}}$$

That's a bit better, but it's still not clear how much pie each kid gets. Let's use the key fraction rule to make the fraction even friendlier. Let's multiply the numerator and denominator each by 3. (Why three?) Remember, this means we're multiplying the fraction by $\frac{3}{3}$, which is just a special form of 1, so we don't change its value.

$$\frac{3 \cdot \frac{23}{3}}{3 \cdot \frac{21}{4}} = \frac{23}{\frac{63}{4}}.$$

Now multiply numerator and denominator each by 4. (Why four?)

$$\frac{4 \cdot 23}{4 \cdot \frac{63}{4}} = \frac{92}{63}.$$

We now see that the answer is $\frac{92}{63}$. That means that sharing $7\frac{2}{3}$ pies among $5\frac{1}{4}$ children is the same as sharing 92 pies among 63 children. (In both situations, the individual child get exactly the same amount of pie.)

Example 8.8.2:

Let's forget the context now and just focus on the calculations so that we can see what is going on more clearly. Try this one:

$$\frac{\frac{3}{5}}{\frac{2}{3}}.$$

Multiplying the numerator and denominator each by 5 (why did we choose 5?) gives

$$\frac{\frac{3}{5}}{\frac{2}{3}} = \frac{5 \cdot \frac{3}{5}}{5 \cdot \frac{2}{3}} = \frac{3}{\frac{10}{3}}.$$

Now multiply the numerator and denominator each by 3 (why did we choose 3?):

$$\frac{3 \cdot 3}{3 \cdot \frac{10}{3}} = \frac{9}{10}.$$

On Your Own

1. Each of the following is a perfectly nice fraction, but it could be written in a simpler form. So do that! Write each of them in a simpler form following the examples above. $\frac{\frac{2}{3}}{\frac{1}{3}}$, $\frac{2\frac{1}{5}}{2\frac{1}{4}}$, $\frac{\frac{5}{7}}{\frac{3}{5}}$, $\frac{\frac{3}{7}}{\frac{4}{5}}$.

Think / Pair / Share

- Jessica calculated the second exercise above this way: $\frac{2\frac{1}{5}}{2\frac{1}{4}} = \frac{\frac{1}{5}}{\frac{1}{4}} = \frac{\frac{1}{5} \cdot 4}{\frac{1}{4} \cdot 4} = \frac{\frac{4}{5}}{1} = \frac{4}{5}$. Is her solution correct, or is she misunderstanding something? Carefully explain what is going on with her solution, and what you would do as Jessica's teacher.
- Isaac calculated the last exercise above this way: $\frac{\frac{3}{7}}{\frac{4}{5}} = \frac{\frac{3}{7} \cdot 7}{\frac{4}{5} \cdot 5} = \frac{3}{4}$. Is his solution correct, or is he misunderstanding something? Carefully explain what is going on with his solution, and what you would do as Isaac's teacher.

Perhaps without realizing it, you have just found another method to divide fractions.

Example: 3/5 ÷ 4/7

Consider $\frac{3}{5} \div \frac{4}{7}$. We know that a fraction is the answer to a division problem, meaning

$$\frac{3}{5} \div \frac{4}{7} = \frac{\frac{3}{5}}{\frac{4}{7}}.$$

And now we know how to simplify ugly fractions like this one! Multiply the numerator and denominator each by 5:

$$\frac{(\frac{3}{5}) \cdot 5}{(\frac{4}{7}) \cdot 5} = \frac{3}{\frac{20}{7}}.$$

Now multiply them each by 7:

$$\frac{(3) \cdot 7}{(\frac{20}{7}) \cdot 7} = \frac{21}{20}.$$

Done! So

$$\frac{3}{5} \div \frac{4}{7} = \frac{21}{20}.$$

Example: 5/9 ÷ 8/11

Let's do another! Consider $\frac{5}{9} \div \frac{8}{11}$:

$$\frac{5}{9} \div \frac{8}{11} = \frac{\frac{5}{9}}{\frac{8}{11}}.$$

Let's multiply numerator and denominator each by 9 and by 11 at the same time. (Why not?)

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{(\frac{5}{9}) \cdot 9 \cdot 11}{(\frac{8}{11}) \cdot 9 \cdot 11} = \frac{5 \cdot 11}{8 \cdot 9}.$$

(Do you see what happened here?)

So we have

$$\frac{\frac{5}{9}}{\frac{8}{11}} = \frac{5 \cdot 11}{8 \cdot 9} = \frac{55}{72}.$$

On Your Own

Compute each of the following, using the simplification technique in the examples above.

$$\frac{1}{2} \div \frac{1}{3}, \qquad \frac{4}{5} \div \frac{3}{7}, \qquad \frac{2}{3} \div \frac{1}{5}, \qquad \frac{45}{59} \div \frac{902}{902}, \qquad \frac{10}{13} \div \frac{2}{13}.$$

Invert and multiply

Consider the problem $\frac{5}{12} \div \frac{7}{11}$. Janine wrote:

$$\frac{\frac{5}{12}}{\frac{7}{11}} = \frac{\frac{5}{12} \cdot 12 \cdot 11}{\frac{7}{11} \cdot 12 \cdot 11} = \frac{5 \cdot 11}{7 \cdot 12} = \frac{5}{12} \cdot \frac{11}{7}.$$

She stopped before completing her final step and exclaimed: "Dividing one fraction by another is the same as multiplying the first fraction with the second fraction upside down!"

? Think / Pair / Share

First check each step of Janine's work here and make sure that she is correct in what she did up to this point. Then answer these questions:

- Do you understand what Janine is saying? Explain it very clearly.
- Work out $\frac{\frac{3}{7}}{\frac{4}{13}}$ using the simplification method. Is the answer the same as $\frac{3}{7} \cdot \frac{13}{4}$?
- Work out $\frac{\frac{2}{5}}{\frac{3}{10}}$ using the simplification method. Is the answer the same as $\frac{2}{5} \cdot \frac{10}{3}$?
- Work out $\frac{\frac{a}{b}}{\frac{c}{d}}$ using the simplification method. Is the answer the same as $\frac{a}{b} \cdot \frac{d}{c}$?
- Is Janine right? Is dividing two fractions always the same as multiplying the two fractions with the second one turned upside down? What do you think? (Do not just think about examples. This is a question if something is *always true*.)

Summary

We now have several methods for solving problems that require dividing fractions:

Dividing fractions:

- Draw a picture using the rectangle method, and use that to solve the division problem.
- Find a common denominator and divide the numerators.
- Rewrite the division as a missing factor multiplication problem, and solve that problem.
- Simplify an ugly fraction.
- Invert the second fraction (the dividend) and then multiply.

Think / Pair / Share

Discuss your opinions about our four methods for solving fraction division problems with a partner:

- Which method for division of fractions is the easiest to *understand why it works* ?
- Which method for division of fractions is the easiest to *use in computations*?
- What are the benefits and drawbacks of each method? (Think both as a future teacher and as someone solving math problems here.)

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8.9: Fractions Involving Zero

Zero in the Numerator

? Think / Pair Share

Does the fraction $\frac{0}{11}$ make sense?

- Write a "pies per child" story for the fraction $\frac{0}{11}$. Does it make sense? How much pie does each individual child receive in your story?
- Think of $\frac{0}{11}$ as the answer to a division problem. What is that division problem? Can you solve it?

It seems pretty clear that zero pies among eleven kids gives zero pies per child:

$$\frac{0}{11} = 0.$$

The same reasoning would lead us to say:

$$\frac{0}{b} = 0 \text{ for any positive number } b.$$

The "Pies Per Child Model" offers one explanation: If there are no pies for us to share, no one gets any pie. It does not matter how many children there are. No pie is no pie is no pie.

We can also justify this claim by thinking about a missing factor multiplication problem:

$$\frac{0}{b} \text{ is asking us to fill in the blank : } \_\_ \cdot\ b = 0.$$

The only way to fill that in and make a true statement is with 0, so $\frac{0}{b} = 0$.

Zero in the Denominator

What happens if things are flipped the other way round?

? Think / Pair / Share

Does the fraction $\frac{11}{0}$ make sense?

- Write a "pies per child" story for the fraction $\frac{11}{0}$. Does it make sense? How much pie does each individual child receive in your story?
- Think of $\frac{11}{0}$ as the answer to a division problem. What is that division problem? Can you solve it?

Students often learn in school that "dividing by 0 is undefined." But they learn this as a rule, rather than thinking about why it makes sense or how it connects to other ideas in mathematics. In this case, the most natural connection is to a multiplication fact, the zero property for multiplication:

$$\text{any number} \cdot 0 = 0.$$

That says we can never find solutions to problems like

$$\_\_ \cdot 0 = 5, \qquad \_\_ \cdot 0 = 17, \qquad \_\_ \cdot 0 = 1.$$

Using the connection between fractions and division, and the connection between division and multiplication, that means there is no number $\frac{5}{0}$. There is no number $\frac{17}{0}$. And there is no number $\frac{1}{0}$. They are all "undefined" because they are not equal to any number at all.

Think / Pair / Share

Can we give meaning to $\frac{0}{0}$ at least? After all, a zero would appear on both sides of that equation!

- Cyril says that $\frac{0}{0} = 2$ since $0 \cdot 2 = 0$.
- Ethel says that $\frac{0}{0} = 17$ since $0 \cdot 17 = 0$.
- Wonhi says that $\frac{0}{0} = 887231243$ since $0 \cdot 887231243 = 0$.

Who is right? Can they all be correct? What do you think?

Cyril says that $\frac{0}{0} = 2$, and he believes he is correct because it passes the check: $0 \cdot 2 = 0$.

But 17 also passes the check, and so does 887231243. In fact, I can choose any number for x, and $0 \cdot x = 0$ will pass the check!

The trouble with the expression $\frac{a}{0}$ (with a not zero) is that there is *no meaningful value* to assign to it. The trouble with $\frac{0}{0}$ is different: There are *too many possible values* to give it!

Dividing by zero is simply too problematic to be done! It is best to avoid doing so and never will we allow zero as the denominator of a fraction. (But all is fine with 0 as a numerator.)

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8.10: Practice

Problem 6

Harriet is part of a group of five children who share four pies. Jeff is part of a group of seven children who share four pies. Jean is part of a group of seven children who share six pies.

1. Who gets more pie, Harriet or Jeff? Justify your answer!
2. Who gets more pie, Jeff or Jean? Justify your answer!
3. Who gets more pie, Harriet or Jean? Justify your answer!

Problem 7

Yesterday was Zoe's birthday, and she had a big rectangular cake. Today, $\frac{2}{5}$ of the cake is left. The **leftover cake** is shown here.



Draw a picture of the original (whole) cake and explain your work.

Problem 8

Use benchmarks and intuitive methods to arrange the fractions below in ascending order. Explain how you decided. (The point of this problem is to think more and compute less!):

$$\frac{2}{5}, \quad \frac{1}{3}, \quad \frac{5}{8}, \quad \frac{1}{4}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{7}.$$

Problem 9

Which of these fractions has the larger value? Justify your choice.

$$\frac{10001}{10002} \quad \textit{or} \quad \frac{10000001}{10000002}.$$

Problem 10

Solve each division problem. Look for a shortcut, and explain your work.

$$\frac{251+251+251+251}{4}$$

$$\frac{377+377+377+377+377}{5}$$

$$\frac{123123+123123+123123+123123+123123+123123}{3}$$

Problem 11

Yoko says

$$\frac{16}{64} = \frac{1}{4}$$

because she cancels the sixes:

$$\frac{16}{64} = \frac{1}{4}.$$

But note:

$$\frac{16}{64} = \frac{1 \cdot 16}{4 \cdot 16} = \frac{1}{4}$$

So is Yoko right? Does her cancelation rule always work? If it does not always work, can you find any other example where it works? Can you find every example where it works?

Problem 12

Jimmy says that a fraction does not change in value if you add the same amount to the numerator and the denominator. Is he right? If you were Jimmy's teacher, how would you respond?

Problem 13

1. Shelly says that if $ab < cd$ then $\frac{a}{b} < \frac{c}{d}$. Is Shelly's claim always true, sometimes true, or never true? If you were Shelly's teacher, what would you say to her?
2. Rob says that if $ad < bc$ then $\frac{a}{b} < \frac{c}{d}$. Is Rob's claim always true, sometimes true, or never true? If you were Rob's teacher, what would you say to him?

Problem 14

Jill, her brother, and another partner own a pizza restaurant. If Jill owns $\frac{1}{3}$ of the restaurant and her brother owns $\frac{1}{4}$ of the restaurant, what fraction does the third partner own?

Problem 15

John spent a quarter of his life as a boy growing up, one-sixth of his life in college, and one-half of his life as a teacher. He spent his last six years in retirement. How old was he when he died?

Problem 16

Nana was planning to make a red, white, and blue quilt. One-third was to be red and two-fifths was to be white. If the area of the quilt was to be 30 square feet, how many square feet would be blue?[1]



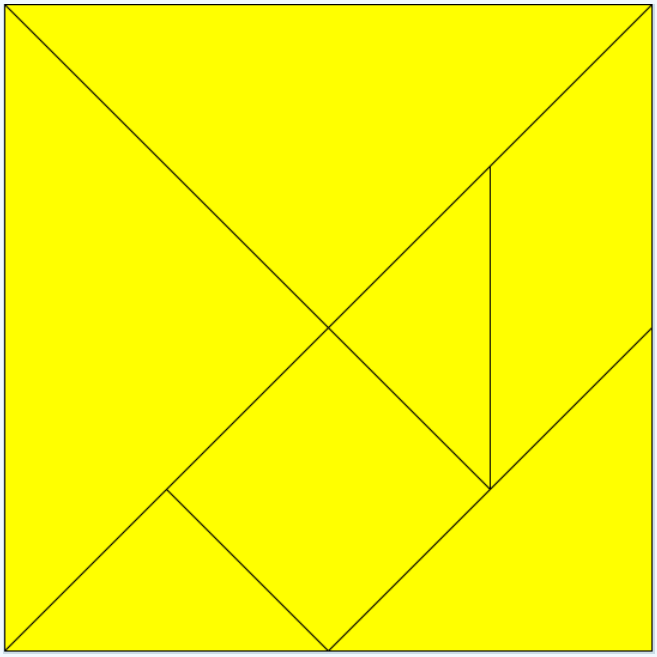
Ku'u Hae Aloha (My Beloved Flag), Hawaiian cotton quilt from Waimea, before 1918, Honolulu Academy of Arts.

Problem 17

Rafael ate one-fourth of a pizza and Rocco ate one-third of it. What fraction of the pizza did they eat?

Problem 18

Tangrams[2] are a seven-piece puzzle, and the seven pieces can be assembled into a big square.



1. If the large square shown above is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.
2. The tangram puzzle contains a small square. If the small square (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.
3. The tangram set contains two large triangles. If a large triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.
4. The tangram set contains one medium triangle. If the medium triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers.
5. The tangram set contains two small triangles. If a small triangle (the single tangram piece) is one whole, assign a fraction value to each of the seven tangram pieces. Justify your answers

Problem 19

Mikiko said her family made two square pizzas at home. One of the pizzas was 8 inches on each side, and the other was 12 inches on each side. Mikiko ate $\frac{1}{4}$ of the small pizza and $\frac{1}{12}$ of the large pizza. So she said that she ate

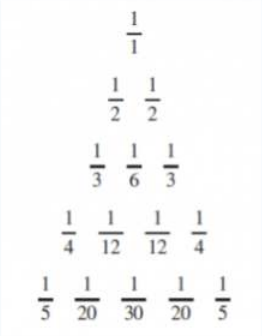
$$\frac{1}{4}+\frac{1}{12}=\frac{3}{12}+\frac{1}{12}=\frac{4}{12}=\frac{1}{3}$$

of a pizza. Do you agree with Mikiko's calculation? Did she eat $\frac{1}{3}$ of a whole pizza? Carefully justify your answer. (This question is tricky. It's probably a good idea to draw a picture!)

Problem 20

Look at the triangle of numbers. There are lots of patterns here! Find as many as you can. In particular, try to answer these questions:

1. What pattern describes the first number in each row?
2. How is each fraction related to the two fractions below it?
3. Can you write down the next two rows of the triangle?



Problem 21

Marie made a sheet cake at home, but she saved some to bring to work and share with her co-workers the next day. Answer these questions about Marie's cake. (Draw a picture!)

1. Suppose Marie saved $\frac{1}{2}$ of the cake for her co-workers and the co-workers ate $\frac{3}{4}$ of this. What fraction of the entire cake did they eat?
2. What if Marie saved $\frac{1}{6}$ instead, and they ate $\frac{2}{3}$ of this?
3. What if she saved $\frac{5}{7}$ of the cake and they ate $\frac{1}{2}$ of this?

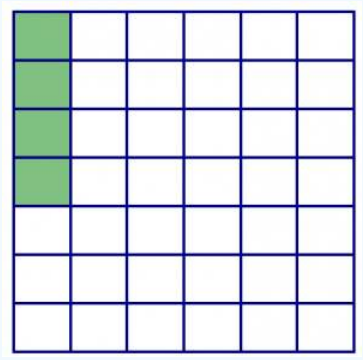
Problem 22

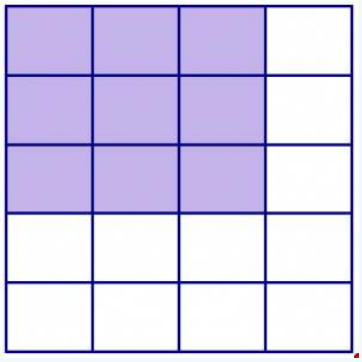
An elementary school held a "Family Math Night" event, and 405 students showed up. Two-thirds of the students who showed up won a door prize. How many students won prizes?

Problem 23

For each picture shown:

- What multiplication problem is represented?
- What is the product?





Problem 24

For each problem, use only the digits 0, 1, 2,. . . , 9 at most once each in place of the variables. Find the value closest to 1. Note that $a$can be a different value in each of the three problems. Justify your answer: How do you know it is the closest to 1?

1. $\frac{a}{b} \ldotp$
2. $\frac{a}{b} \cdot \frac{c}{d} \ldotp$
3. $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} \ldotp$

Problem 25

A town plans to build a community garden that will cover $\frac{2}{3}$ of a square mile on an old farm. One side of the garden area will be along an existing fence that is $\frac{3}{4}$ of a mile long. If the garden is a rectangle, how long is the other side?

Problem 26

Nate used $90\frac{1}{2}$ pounds of seed to plant $\frac{1}{4}$ acres of wheat. How many pounds of seed did he use per acre?

Problem 27

The family-sized box of laundry detergent contains 35 cups of detergent. Your family's machine requires $1\frac{1}{4}$ cup per load. How many loads of laundry can your family do with one box of detergent?

Problem 28

Jessica bikes to campus every day. When she is one-third of the way between her home and campus, she passes a grocery store. When she is halfway to school, she passes a Subway sandwich shop. This morning, Jessica passed the grocery store at 8:30am, and she passed Subway at 8:35am. What time did she get to campus?

Problem 29

If you place a full container of flour on a balance scale and place on the other side a $\frac{1}{3}$ pound weight plus a container of flour (the same size) that is $\frac{3}{4}$ full, then the scale balances. How much does the full container of flour weigh?



Problem 30

Geoff spent $\frac{3}{4}$ of his allowance on a movie. He spent $\frac{11}{18}$ of what was left on snacks at school. He also spent $3 on a magazine, and that left him with $\frac{1}{24}$ of his total allowance, which he put into his savings account. How much money did Geoff save that week?

Problem 31

Lily was flying to San Francisco from Honolulu. Halfway there, she fell asleep. When she woke up, the distance remaining was half the distance traveled while she slept. For what fraction of the trip was Lily asleep?

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2. Tangram image from Wikimedia Commons, public domain. ↵

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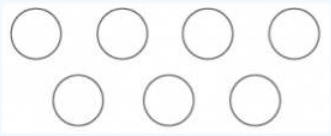
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8.11: Egyptian Fractions

Example: Egyptian fraction for 7/12

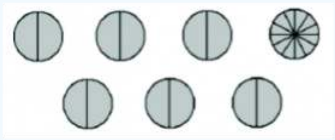
Consider the problem: Share 7 pies equally among 12 kids. Of course, given our model for fractions, each child is to receive the quantity "$\frac{7}{12}$" But this answer has little intuitive feel.

Suppose we took this task as a very practical problem. Here are the seven pies:



Is it possible to give each of the kids a whole pie? No.

How about the next best thing — can each child get half a pie? Yes! There are certainly 12 half pies to dole out. There is also one pie left over yet to be shared among the 12 kids. Divide this into twelfths and hand each kid an extra piece.



So each child gets $\frac{1}{2}+\frac{1}{12}$ of a pie, and it is indeed true that

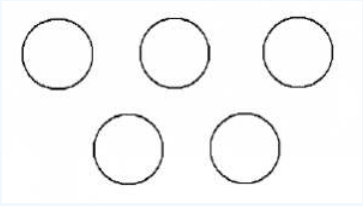
$$\frac{7}{12}=\frac{1}{2}+\frac{1}{12}.$$

(Check that calculation. . . don't just believe it!)

This seems quite reasonable. Instead of seven pieces each of size $\frac{1}{12}$, each kid gets a piece that is $\frac{1}{2}$ and a piece that is $\frac{1}{12}$. It's a lot less cutting, and a lot less messy!

Problem 32

1. Suppose you want to share five pies among six children, but you want each child to get a small number of (relatively) large pieces rather than five pieces of size $\frac{1}{6}$. Following the example above, how could you do it?



2. Using similar ideas, how could you share 4 pies among 7 kids?

History: Rhind Papyrus

The Egyptians (probably) were not particularly concerned with splitting up pies. But in fact, they did have a very strange (to us) way of expressing fractions. We know this by examining the Rhind Papyrus. This ancient document indicates that fractions were in use as many as four thousand years ago in Egypt, but the Egyptians seem to have worked primarily with **unit fractions**. They insisted on writing all of their fractions as sums of fractions with numerators equal to 1, and they insisted that the denominators of the fractions were all different.

> *Accurate reckoning for inquiring into things, and the knowledge of all things, mysteries… all secrets.*

The Rhind Papyrus is an ancient account of Egyptian mathematics named after Alexander Henry Rhind. Rhind was a Scotsman who acquired the ancient papyrus in 1858 in Luxor, Egypt.

The papyrus dates back to around 1650 B.C. It was copied by a scribe named Ahmes (the earliest known contributor to the field of mathematics!) from a lost text written during the reign of king Amenehat III. The opening quote is taken from Ahmes introduction to the Rhind Papyrus[1]. The papyrus covers topics relating to fractions, volume, area, pyramids, and more.



Rhind Papyrus

Egyptian Fractions

To write a fraction as an Egyptian fraction, you must rewrite the fraction as:

- a **sum of unit fractions** (that means the numerator is 1), and
- the denominators must all be different.

Example: Egyptian fractions for 3/10 and 5/7

The Egyptians would not write $\frac{3}{10}$, and they would not even write $\frac{1}{10}+\frac{1}{10}+\frac{1}{10}$. Instead, they wrote

$$\frac{1}{4}+\frac{1}{20}.$$

The Egyptians would not write $\frac{5}{7}$, and they would not even write $\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}$. Instead, they wrote

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{70}.$$

(You should check that the sums above give the correct resulting fractions!)

Problem 33

Write the following as a sum of two *different* unit fractions. Be sure to check your answers.

$$\frac{2}{3}, \qquad \frac{2}{5}, \qquad \frac{2}{7}, \qquad \frac{2}{9}.$$

Can you find a general rule for how to write $\frac{2}{n}$ as an Egyptian fraction? (Assume n is an odd number.)

Problem 34

Write the following as a sum of distinct unit fractions. (“Distinct” means the fractions must have different denominators.) Note that you may need to use more than two unit fractions in some of the sums. Be sure to check your answers.

$$\frac{3}{4}, \quad \frac{5}{6}, \quad \frac{3}{5}, \quad \frac{5}{9}.$$

Can you find a general process for fractions bigger than $\frac{1}{2}$?

Example 8.11.1:

Write the following fractions as Egyptian fractions.

$$\frac{17}{20}, \quad \frac{3}{7}.$$

Can you find a general algorithm that will turn *any fraction at all*into an Egyptian fraction?

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1. Image of Rhind Papyrus from Wikimedia Commons, public domain. ↵

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8.12: Algebra Connections

In an advanced algebra course students are often asked to work with complicated expressions like:

$$\frac{\frac{1}{x}+1}{\frac{3}{x}}$$

We can make it look friendlier by using the key fraction rule, exactly the same technique we used in the chapter on "Dividing Fractions: Invert and Multiply." In this example, let us multiply the numerator and denominator each by x. (Do you see why this is a good choice?) We obtain:

$$\frac{(\frac{1}{x}+1)\cdot x}{(\frac{1}{x})\cdot x}=\frac{1+x}{3},$$

and $\frac{1+x}{3}$ is much less scary.

Notice that expressions like

$$\frac{1}{x}$$

cannot be rewritten as a decimal. Expressions like this arise in numerous applications, so it is important for math and science students to be able to work with fractions in fraction form, without always resorting to converting to decimals.

Example 8.12.1:

As another example, given:

$$\frac{\frac{1}{a}-\frac{1}{b}}{ab},$$

one might find it helpful to multiply the numerator and the denominator each by a and then each by b:

$$\frac{(\frac{1}{a}-\frac{1}{b})\cdot a\cdot b}{(ab)\cdot a\cdot b}=\frac{b-a}{a^2b^2}.$$

Example 8.12.2:

For

$$\frac{\frac{1}{(w+1)^2}-2}{\frac{1}{w+1)^2}+5},$$

it might be good to multiply numerator and denominator each by $(w+1)^2$. (Why?)

$$\frac{(\frac{1}{(w+1)^2}-2)\cdot(w+1)^2}{(\frac{1}{w+1)^2}+5)\cdot(w+1)^2}=\frac{1-2(w+1)^2}{1+5(w+1)^2}.$$

On Your Own

Can you make each of these expressions look less scary?

$$\frac{2-\frac{1}{x}}{1+\frac{1}{x}},\qquad \frac{\frac{1}{x+h}+3}{\frac{1}{x+h}},\qquad \frac{1}{\frac{1}{a}+\frac{1}{b}},\qquad \frac{\frac{1}{x+a}-\frac{1}{x}}{a}.$$

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8.13: What is a Fraction? Looking Back

So far, we have no single model that makes sense of fractions in all contexts. Sometimes a fraction is an action ("Cut this in half.") Sometimes it is a quantity ("We each get 2/3 of a pie!") And sometimes we want to treat fractions like numbers, like ticks on the number line in-between whole numbers.

We could say that a fraction is just a pair of numbers *a* and *b*, where we require that $b \neq 0$. We just happen to write the pair as $\frac{a}{b}$.

But again this is not quite right, since a whole infinite collection of pairs of numbers represent the same fraction! For example:

$$\frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \ldots$$

So a single fraction is actually a whole infinite class of pairs of numbers that we consider "equivalent."

How do mathematicians think about fractions? Well, in exactly this way. They think of pairs of numbers written as $\frac{a}{b}$, where we remember two important facts:

- $b \neq 0$, and
- $\frac{a}{b}$ is really shorthand for a whole infinite class of pairs that look like $\frac{xa}{xb}$ for all $x \neq 0$.

This is a hefty shift of thinking: The notion of a "number" has changed from being a specific combination of symbols to a whole class of combinations of symbols that are deemed equivalent.

Mathematicians then *define* the addition of fractions to be given by the daunting rule:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

This is obviously motivated by something like the "Pies Per Child Model." But if we just define things this way, we must worry about *proving* that choosing different representations for $\frac{a}{b}$ and $\frac{c}{d}$ lead to the same final answer.

For example, it is not immediately obvious that

$$\frac{2}{3} + \frac{4}{5} \quad and \quad \frac{4}{6} + \frac{40}{50}$$

give answers that are equivalent. (Check that they do!)

They also *define* the product of fractions as:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Again, if we start from here, we have to *prove* that you get equivalent answers for different choices of fractions equivalent to $\frac{a}{b}$ and $\frac{c}{d}$.

Then mathematicians establish that the axioms of an arithmetic system hold with these definitions and carry on from there! (That is, they check that addition and multiplication are both commutative and associative, that the distributive law holds, that all representations of 0 act like an additive identity, and so on.)

This is abstract, dry, and not at all the best first encounter to offer students on the topic of fractions. And, moreover, this approach completely avoids the question as to what a fraction really means in the "real world." But it is the best one can do if one is to be completely honest.

? Think / Pair / Share

So… what is a fraction, really? How do you think about them? And what is the best way to talk about them with elementary school students?

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CHAPTER OVERVIEW

9: Decimals

9.1: Decimals
9.2: "Deci"-mals in Other Bases
9.3: Terminating or Repeating?
9.4: Orders of Magnitude
9.5: Practice

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9.1: Decimals

The meaning of decimals is best understood once one has a real understanding of place value and fractions. The decimal notation we commonly use is an extension of place value in base ten. The decimal point indicates that succeeding digits represent tenths, hundredths, thousandths, etc. So a person must understand what these fractions mean in order to understand and make the connection to decimals.

The key to understanding the relationship between decimals and fractions (or mixed numbers) begins with READING the decimal correctly. Most people read 5.3 as "five point three" which doesn't help one understand its meaning. It should be read "five and three tenths." In doing so, the connection between the decimal 5.3 and the mixed number $5\frac{3}{10}$ is clear. Similarly, 18.035 should be read "Eighteen and thirty-five thousandths" and corresponds to the mixed number $18\frac{35}{1000}$. Here are a few more:

a. 0.309 and $\frac{309}{1000}$ are both correctly read "three hundred nine thousandths"
b. 10.04 and $10\frac{4}{100}$ are both correctly read "ten and four hundredths"

Note that the decimal point is read as "and". The decimal point (and the word "and") separates the whole part from the fractional part of a mixed number. This is the only correct use of the word "and" when reading numbers. 760 is read "seven hundred sixty." A common mistake is to read 760 as "seven hundred and sixty." If there is no decimal point, don't say the word "and".

People just learning about decimals (like elementary school children) should NOT read the number 5.4 as "five point four". This is a shortcut way of reading the number that is only appropriate to use once one really understands the connection between decimals and fractions. Remember that a number written in decimal form is really just a different way to write a mixed number where the denominator of the fractional part is a power of ten! The name of the fractional part (tenths, hundredths, thousandths, etc.) is the place value of the last digit of the number after the decimal point, which also happens to be the denominator of the number written in fractional form.

Look again at 0.309 and $\frac{309}{1000}$. In the decimal, there are three digits (or place values) after the decimal point. In the fraction, there are three zeroes after the 1 (which is the number 1000 in the denominator). The same applies to 18.035 and $18\frac{35}{1000}$. For 5.3 and $5\frac{3}{10}$, the decimal has one digit after the decimal point, and the fraction has one zero after the 1. It's best if students are allowed to discover this fact for themselves. It's always more meaningful to discover relationships (that often become rules) on your own. Students who are regularly asked to read decimals, fractions and mixed numbers the correct way are more likely to make this particular discovery by themselves.

Mixed numbers like 5.3 and 18.035 can also be written immediately as improper fractions. The denominator of the fraction will still be the same as if it were written as a mixed number. The numerator is the number without a decimal point at all. For instance, 5.3 can be written as $5\frac{3}{10}$ or $\frac{53}{10}$; and 18.035 can be written as $18\frac{35}{1000}$ or $\frac{18035}{1000}$.

Note that when you first write a decimal as a fraction, it isn't necessarily in simplest (or reduced) form.

? Exercise 1

On the line, write in words how to read each of the following decimals. Do not use the word "point". Then, underneath, write each decimal as a fraction. If the number is greater than or equal to 1, first write the fraction as a mixed number and then write it as an improper fraction. *Simplify any fraction that is not in simplest form.* **Show all steps** (including original fractions before simplifying).

a. 0.4 _____

0.4 =

b. 0.26 _____

0.26 =

c. 3.08 _____

3.08 =

d. 9.85 _____

9.85 =

e. 17.305 _____

17.305 =

Exercise 2

Write each proper fraction as a decimal. Write each improper fraction first as a mixed number (don't reduce), and then also as a decimal. Do not use your calculator.

a. $\frac{14}{100}$ = _____

b. $\frac{8}{1000}$ = _____

c. $\frac{435}{100}$ = _____ = _____

d. $\frac{5638}{10}$ = _____ = _____

e. $\frac{305}{100}$ = _____ = _____

The rule you may remember for multiplying fractions is to multiply the numbers together as if there were no decimal point, and then move the decimal point in from the right the total number of places it is in for both numbers. For instance, (8)(0.4) is done by multiplying 8 times 4 and then moving the decimal point in one place to get 3.2. Similarly, (0.06)(0.7) is done by multiplying 6 times 7 and moving the decimal point in three places (two for 0.06 and one more for 0.7 for a total of three) to get 0.042. Many people make this harder than it really is and don't realize they can easily multiply 0.3 and 0.4 in their head. It's as simple as $3 \cdot 4$ and moving the decimal in two places to get 0.12. Now, how about 1.1 times 1.2? It's simply 11 times 12 with the decimal moved in two places: 1.32.

Let's observe why this rule for multiplying decimals works by rewriting the numbers as fractions first. The key here is to write numbers greater than or equal to one as improper fractions. Then multiply (without canceling or reducing) the fractions. Lastly, rewrite the fraction as a decimal.

Exercise Example 1

(8)(0.4) = $\frac{8}{1} \cdot \frac{4}{10} = \frac{32}{10}$ = 3.2 (notice the decimal point is one place in for .4)

Exercise Example 2

(1.2)(1.01) = $\frac{12}{10} \cdot \frac{101}{100} = \frac{1212}{1000}$ = 1.212

(notice the decimal point is three places in, one for 1.2 plus two more for 1.01)

Exercise 3

Multiply the following decimals mentally, and write the answer on the blank. Then do it again by showing the same steps as shown in the previous two examples where each decimal is **first** written as a fraction, then multiply the numerators and denominators, and then convert that answer (don't simplify) to a decimal. Do not use your calculator.

a. (0.4)(.07) = _____

(0.4)(.07) =

b. (1.6)(0.2) = _____

(1.6)(0.2) =

c. (0.25)(0.3) = _____

(0.25)(0.3) =

d. (2.2)(0.3) = _____

(2.2)(0.3) =

Recognizing Equivalent Decimals and Comparing Decimals

Zeroes, which are behind a decimal's last non-zero digit, can be added or removed without changing the value of the decimal. If you look at some equivalent fraction, you'll see why this should be true. For example,

$$\frac{14}{10} = \frac{140}{1000} = \frac{1400}{10000} = \frac{14000}{100000}.$$

These are all equivalent to $\frac{14}{100}$ because the numerator and denominator was multiplied by some power of 10 (10, 100 or 1000) to get one of the other equivalent fractions. If we replace each of the four fractions above with their decimal equivalents, we get 0.14 = 0.140 = 0.1400 = 0.14000. Let's call any zeroes at the end of a decimal's last non-zero digit "**trailing zeroes**". Then, we could conclude that any number starting with .14 that has trailing zeroes is also equivalent, like 0.1400000.

Two decimals are equal only if one can be made to look identical to the other by adding or removing trailing zeroes. You can also determine if they are equivalent by removing any excess trailing zeroes from each to see if they are identical.

? Exercise 4

For the first decimal given, circle any of the next four decimals that are equal to it.

a.	1.900;	1.0900	1.9	1.90000	0.190
b.	4.034;	4.0340	4.0334	4.0034	4.3040

To compare two or more decimals that are not equal, but have the same number of digits after the decimal point, you can write each as a fraction with the same denominator and then compare the numerator. For instance 0.14 is less than 0.21 since fourteen hundredths is less than twenty-one hundredths. Basically, in this case, it's just like comparing whole numbers. You'll be able to determine which is larger by comparing each number as if there was no decimal point. But, keep in mind this only makes sense if the numbers you are looking at have the same number of digits after the decimal point.

? Exercise 5

Compare each of the following decimals using <, = or >.

a.	3.5	0.9
b.	35.06	35.0600
c.	0.089	0.098

To compare two or more decimals that are not equal that do not all have the same number of digits after the decimal point, first write each decimal with the same number of digits after the decimal point (by adding trailing zeroes to one or more if necessary). By doing that, you are comparing tenths with tenths, or hundredths with hundredth, etc., as you did in exercise 5.

? Exercise 6

Compare each of the following decimals using <, = or >.

a.	3.51	3.488
b.	35.061	35.35
c.	0.08933	0.0894

If a fraction is written with a power of ten in the denominator, it's basic to write the same number in decimal form. You did that in Exercise 2. Any fraction that is written with a power of 10 in the denominator can be written as a terminal decimal. This means that

it's possible to write the number with trailing zeroes. But what if the fraction doesn't have a power of 10 (like 10, 100, 1000, etc.) in the denominator? Sometimes those can be tricky!

If a fraction WITHOUT a power of ten in the denominator CAN be written as an equivalent fraction WITH a power of ten in the denominator, then it can be written as a terminating decimal.

For instance, $\frac{1}{2}$ can be written as $\frac{5}{10}$ (by multiplying the numerator and denominator by 5)

Therefore, $\frac{1}{2} = \frac{5}{10} = 0.5$ (remember to read this as five-tenths)

Well, that one wasn't too hard, but what about $\frac{7}{80}$? The question is whether you can multiply the denominator, 80, by something to get 10, 100, 1000, 10000, etc. There is no whole number you can multiply 80 by to get 10, 100 or 1000. But, if you multiply 80 times 125, it equals 100000. So, by multiplying both 7 and 80 by 125, we get the equivalent fraction $\frac{875}{10000}$ for $\frac{7}{80}$ which can now be written as the decimal 0.0875.

How about writing $\frac{5}{6}$ as a terminating decimal? Well, there is nothing you can multiply 6 by to get 10 or 100 or 1000 or 10000. Is there maybe some number we could multiply 6 by to get some higher power of ten? Well, that's a good question! Actually, there isn't, but how could you be sure? You certainly can't try every power of ten since there are an infinite number of them to try.

It would be nice if there was an easy way to determine if any given fraction could be written as a terminating decimal. The key is to consider the factors of the denominator of a fraction that can be written as a terminating decimal. If a fraction can be written as a terminating decimal, then there is some equivalent fraction where the denominator must be a power of ten: 10 or 100 or 1000, etc.

? Exercise 7

Write the prime factorization of each of the following:

a. 10 = _____

b. 100 = _____

c. 1000 = _____

d. 10000 = _____

e. 100000 = _____

? Exercise 8

What are the only prime factors of powers of 10? _____

? Exercise 9

If a power of ten has three factors of 5, how many factors of 2 does it have? _____

? Exercise 10

If a power of ten has two factors of 2, how many factors of 5 does it have? _____

Powers of ten only have 2s and 5s as its prime factors and nothing else.

Let's go back to our three numbers , $\frac{1}{2}$, $\frac{7}{80}$, and $\frac{5}{6}$, that we were trying to write as terminating decimals and analyze the situation.

$\frac{1}{2}$ is simplified, and in its prime factored form, there is exactly one 2 in the denominator. To write as an equivalent fraction with a denominator that is a power of ten, it must have only 2s and 5s as prime factors in the denominator, and the same number of each! Therefore, multiplying by one more 5 in the numerator and denominator did the trick!

Prime factor the numerator and denominator of this reduced fraction: $\frac{7}{80} = \frac{7}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 5}$

We must determine if it is possible to multiply the denominator by something so that the resulting denominator will be made up of only 2s and 5s and the same number of each. Well, there are four factors of 2 and one factor of 5. Since we need the same number

of each factor, making an equivalent fraction by multiplying the numerator and denominator by three more factors of 5 will do the trick.

$$\frac{7}{80} = \frac{7}{2\cdot2\cdot2\cdot2\cdot5} \cdot \frac{5\cdot5\cdot5}{5\cdot5\cdot5} = \frac{875}{10000} = 0.0875$$

In both examples, note we either multiplied by extra factors of 2 or 5, but not both!

What about $\frac{5}{6}$? Well, it's reduced, and the prime factorization of the denominator is $2 \cdot 3$. No matter what the denominator is multiplied by, we'll be stuck with a factor of 3 in the denominator. Since the only prime factors of powers of 10 are 2 and 5, there can't be a prime factor of 3 in the denominator if we want to end up with only a power of 10 in the denominator. Therefore, since it is impossible to write $\frac{5}{6}$ as an equivalent fraction with a power of ten in the denominator, it cannot be written as a terminating decimal.

How about $\frac{63}{72}$? If we prime factor the denominator, we get $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$.

? Exercise 11

Well, what do you think? Can $\frac{63}{72}$ be written as a terminating decimal? _____

Explain why or why not.

One way to check that you are obtaining the correct results is to use a calculator. For $\frac{1}{2}$, we got 0.5, which you can check by doing the division $1 \div 2$ on your calculator.

? Exercise 12

Use your calculator to find the decimal equivalent for each of the following:

a. $\frac{7}{80}$ = _____

b. $\frac{63}{72}$ = _____

? Exercise 13

Hmmm, did you get 0.0875 for part a? _____

If you did part b on your calculator, did you get a terminating decimal of 0.875?

Is that what you expected? Why or why not?

The reason $\frac{63}{72}$ can be written as a terminating decimal is because in its ***simplified*** form, it only has 2's and/or 5's as its prime factors. Here is how to finish this problem by simplifying first, and then multiplying by any needed factors of 2 or 5 to get the same number of each:

$$\frac{63}{72} = \frac{3\cdot3\cdot7}{2\cdot2\cdot2\cdot3\cdot3} = \frac{7}{2\cdot2\cdot2} \cdot \frac{5\cdot5\cdot5}{5\cdot5\cdot5} = \frac{875}{10000} = 0.875$$

? Exercise 14

For each fraction, determine if it can be written as an equivalent fraction with a power of ten in the denominator. If a fraction cannot be written as a terminal decimal, ***explain*** why not. Otherwise, show **ALL** of the steps (as shown in the previous examples) to write it as a terminal decimal. The steps are listed below.

a. Simplify if possible
b. Prime factor the denominator
c. Multiply the numerator and denominator by an appropriate number of factors of 2 or 5 so that the denominator will be a power of 10
d. Simplify the numerator and denominator
e. Write as a terminating decimal

Then, **check your answer with a calculator** by taking the original fraction and dividing the numerator by the denominator. You should get the same decimal obtained by doing the five steps outlined above.

a. $\frac{3}{4}$

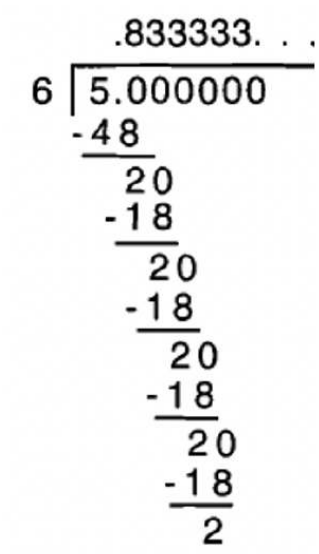
b. $\frac{9}{20}$

c. $\frac{9}{15}$

d. $\frac{18}{25}$

e. $\frac{5}{14}$

Okay, now we have to deal with those fractions that cannot be written as terminating decimals. Any simplified fraction that has at least one prime factor that isn't a 2 or 5 is in this category. Let's look at $\frac{5}{6}$ again. One way to write this as a decimal is to divide 5 by 6.



As you could see, I keep dividing 6 into 20, write 3, multiply to get 18, subtract from 20, get 2, bring down the 0, and start all over again. This can go on forever and ever. So, the 3's at the end will trail on forever. This is NOT a terminating decimal since the number cannot be written with trailing zeroes. In this case, there are trailing 3s. The **three dots at the end (called ellipses)** of the number are to show that the 3s repeat forever. The answer can be written as 0.8333333... or $0.8\bar{3}$. *Remember to put the ellipses (three dots) at the end of the number!* The bar over the 3 indicates that the 3 repeats forever and ever. Here are other ways to represent the same thing: $0.8\overline{33}$ or $0.83333\bar{3}$. In the first case, it says 33 repeats forever. In the second case, it says after the first three 3's, the 3 repeats forever. In both cases, when you write it out in long form, it looks like 0.8333333... so it's the same number. Usually, we'll write 0.8333333... or $0.8\bar{3}$.

? Exercise 15

Write 0.8333333... or $0.8\bar{3}$ in three more ways, different than $0.8\overline{33}$ *or* $0.83333\bar{3}$

_____ , _____, and _____

One thing to notice about what happened when we divided 5 by 6 is that I kept getting 2 after each subtraction in the division, this is like the remainder. Also remember that when you divide, each remainder must be less than or equal than what you are dividing by.

Exercise 16

Write the possible remainders for each number.

a. 6: _____

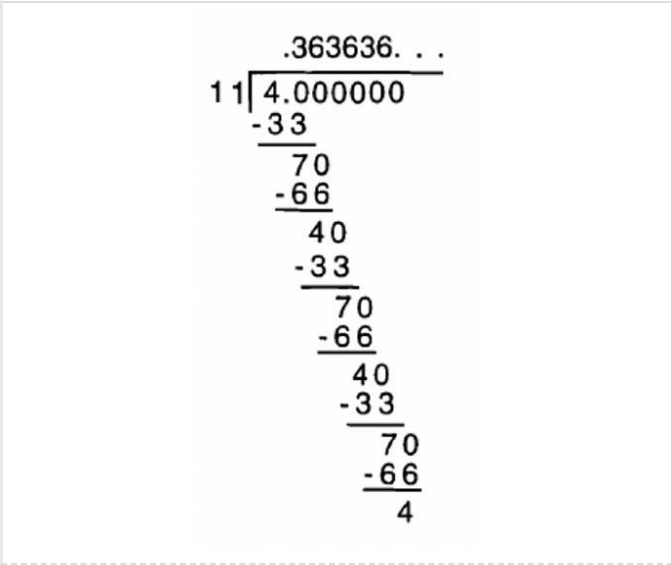
b. 7: _____

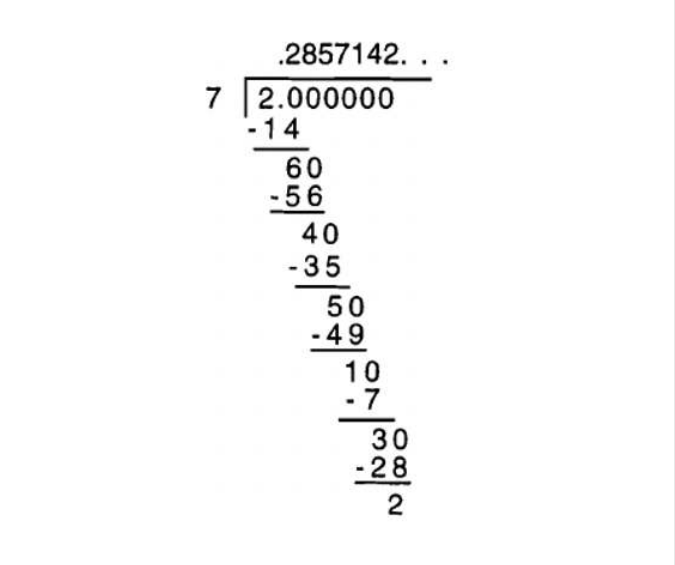
c. 9: _____

d. 11: _____

e. 3: _____

Let's now look at what happens when you do long division compute $\frac{4}{11}$ and $\frac{2}{7}$.





In both of these cases, eventually, you get a remainder that you got previously, so the computation repeats itself. In the case of dividing 4 by 11, two remainders come up before there is some repetition 4 and 7. From 16d, you should have realized that the only possible remainders for 11 are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. Now if you got a remainder of 0 when dividing, you'd have a terminating decimal. So, when dividing by 11, the most remainders you might get in a row before one repeats is 10 in a row. But in the case of $\frac{4}{11}$, the remainders repeat after the 4 and 7. This means you keep dividing into 40 or 70 (since you bring the zero down) and that's why in the quotient the digits start repeating. ***Remember to put the ellipses (three dots) at the end of the number if there is a pattern of repeating digits.***

Now in the case of dividing 2 by 7, the remainders you get as you go along are 6, then 4, then 5, then 1, then 3, and then 2. When you bring down the 0, you are dividing 7 into 20 again, and hence the remainders repeat. Notice that the sequence of remainders (6, 4, 5, 1, 3, and 2) is different than the sequence of digits that repeat in the quotient. The digits that repeat in the quotient are 285714, so $\frac{2}{7}$ = 0.285714285714... or $0.\overline{285714}$

Since there are only six possible remainders other than zero when dividing by 7, only a sequence of 6 digits could possibly repeat. In this case, all six possible remainders of 7 appeared in the long division.

There are many ways to express the infinite or repeating decimal 0.285714285714... or $0.\overline{285714}$ First of all, to establish what is repeating, you would want to see the sequence of digits repeating at least two times through. Therefore, 0.285714285714...is the shortest possible way of showing it when the ellipses (three dots) are used. If you simply wrote 0.285714..., it wouldn't be clear whether or not the 4 repeated or the 14 repeated, etc.

Here are a few other ways to write 0.285714285714... besides $0.\overline{285714}$

0.285714285714285714285714...

0.285714285714285714...

(in this case, you see it as the 857142 repeating from this point on)

0.2857142857142857...

(in this case, you see it as the 142857 repeating from this point on)

$0.2\overline{8571428}$

(in this case, you see it as the 571428 repeating from this point on)

? Exercise 17

Write 0.285714285714... two more ways using ellipses (the three dots) and two more ways using a bar over a repeating sequence of digits.

? Exercise 18

Determine which of the following is equivalent to 0.383432432432...

One way to do this is write the number out in long form by carefully continuing the pattern and then check the digits one place value at a time.

a. 0.3834324324...
b. 0.383432432...
c. 0.3834323432...
d. $0.3834\overline{3243}$
e. $0.383\overline{4324}$

Keep in mind that if you are finding a decimal equivalent for a fraction with 17 in the denominator that there may be up to sixteen digits in a row before you see any repetition. Now, obviously, there are only 10 digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) in base ten. Note that the sequence of digits repeating in the quotient is totally different than the possible remainders you might get when you divide by 17 since a remainder may be more than a single-digit number.

? Exercise 19

a. How many possible remainders could there be if you divide a number by 33?

b. Do long division to write $\frac{31}{33}$ as a repeating decimal using ellipses and then using the bar over the repeating digits on this line: _____

Show work here:

c. In the long division, how many digits repeat? _____

d. In the long division, what were the remainders you would keep getting if you continued the division forever and ever? _____

? Exercise 20

Use your calculator to write each as a repeating decimal. Write the answer two ways, first with a bar over the repeating digits, and then in long form, **using ellipses**. Note that your calculator may or may not round the last digit shown. It can't show digits repeating forever, so you have to be savvy enough to know whether the decimal showing on the display is terminal or if it is an approximation.

a. $\frac{5}{9}$ = _____ = _____
b. $\frac{5}{7}$ = _____ = _____
c. $\frac{1}{6}$ = _____ = _____
d. $\frac{2}{3}$ = _____ = _____

e. $\frac{7}{11}$ = _____ = _____

f. $\frac{5}{12}$ = _____ = _____

g. $\frac{8}{15}$ = _____ = _____

h. $\frac{16}{45}$ = _____ = _____

i. $\frac{5}{66}$ = _____ = _____

Exercise 21

If you are changing a simplified fraction to a decimal where the denominator is x, how many digits at most can repeat in the quotient? _____

Well, you've learned how to write terminating decimals as fractions, and how to write fractions as decimals. You should be able to determine whether a fraction can be written as a terminating or repeating decimal before doing the actual division.

Rational Numbers are defined to be numbers which can be expressed as the ratio of two integers. Fractions as we usually refer to them (without decimal points or square roots, etc. in the numerator or denominator) are rational numbers. Since all fractions can be written as terminating or nonterminating (infinite, repeating) decimals, then all terminating and repeating decimals are also rational numbers. In Exercise 1, you wrote some terminating decimals as fractions. Since all nonterminating decimals came from a fraction, now we want to be able to go the other way around and write a repeating decimal as a fraction. We've got a nifty trick for doing just that!

Let's say we wanted to write $0.\overline{72}$ or 0.727272727272... as a fraction.

First, notice this is a very different number from 0.72, since 0.72 is seventy-two hundredths, or $\frac{72}{100}$. This simplifies to $\frac{18}{25}$. So,0 .72 and $\frac{18}{25}$ are the same number. You can check to make sure $\frac{18}{25}$ is the correct fraction by using a calculator and dividing 18 by 25. The calculator should read 0.72, which is where we started.

Okay, so how do we write $0.\overline{72}$ as a fraction? Since this is a repeating decimal that has infinitely many digits, there is no single power of ten we can put in the denominator. The trick is to use algebra to write a number in a way that eliminates the infinitely repeating part of the decimal. First, let's write out $0.\overline{72}$ the long way:

$0.\overline{72}$ = .727272727272... and call it the number x. So x = 0.727272727272...

Remember that if you multiply a decimal by 10, the decimal point moves one place to the right, and when you multiply a decimal by 100, the decimal point moves two places to the right, etc.

Exercise 22

If x = 0.727272727272..., then write out what 10x, 100x and 1000x equals. Write it out **without** using the bar over the repeating digits. Use the ellipses (three dots).

10x = _____

100x = _____

1000x = _____

Exercise 23

Okay, let's work with x = 0.727272727272... and 100x = 72.727272727272...

a. In algebra, what is 100x - x? _____ (Hint: Subtract coefficients.)

b. Compute 100x - x another way: Line up the decimal points and subtract 0.727272727272... from 72.727272727272... Notice that if the decimals are lined up, the "tails" at the end of both repeating decimals are exactly the same so it should be easy to subtract. Show the work below.

c. Write an equation so that the answer to part a equals the answer to part b. Then use algebra to solve for x. Show work.

d. Simplify the fraction you got for x in part c. Show work.

e. Use a calculator to rewrite the fraction you got in part c as a decimal:

Is it equal to $0.\overline{72}$? _____

If so, you must have written $0.\overline{72}$ as the correct fraction!

Here's the trick for changing repeating decimals to simplified fractions:

Call the number you are trying to write as a decimal a variable, like n or x.

If there is one repeating digit, compute 10x; if there are two repeating digits, compute 100x; if there are three repeating digits, compute 1000x, etc. This aligns the repeating decimals up with each other so that the tail of x and the other number (10x, 100x, 1000x, etc.) is the same.

Then subtract x from the other number (10x, 100x, 1000x, etc.) The tails of both numbers will come off and you should have an algebraic problem to solve at this point. Make sure you write your answer as a reduced fraction with only integers in the numerator and denominator.

Here are two examples. *If you write the number in decimal form without the bar over the number, remember to put the ellipses (three dots) at the end!*

Write $0.\bar{2}$ as a simplified fraction.

Let x = $0.\bar{2}$ = .222222... Since only one digit is repeating, multiply x by 10.

Then 10x = 2.222222...

$$\begin{aligned} 10x &= 2.22222\ldots \\ -x &= .2222\ldots \\ \hline 9x &= 2 \end{aligned}$$

$$x = \frac{2}{9}$$

Use a calculator to check that $\frac{2}{9} = 0.\bar{2}$

Write $0.\overline{54}$ as a simplified fraction.

Let x = $0.\overline{54}$ = .545454... Since two digits are repeating, multiply x by 100.

Then 100x = 54.545454...

$$\begin{aligned} 100x &= 54.545454\ldots \\ -x &= .545454\ldots \\ \hline 99x &= 54 \end{aligned}$$

$$x = \frac{54}{99} = \frac{6}{11}$$

Use a calculator to check that $\frac{6}{11} = 0.\overline{54}$

? Exercise 24

Rewrite each of the following decimals as simplified fractions. For repeating decimals, use the technique shown in the previous examples. Then, check your answer using a calculator by dividing the numerator by the denominator to see if the result matches the original problem.

a. 0.4

b. $0.\bar{4}$ Begin by letting x = $0.\bar{4}$ or x = 0.44444...

c. 0.06 =

d. $0.\overline{06}$ Begin by letting x = $0.\overline{06}$ or x = 0.060606...

e. 0.9

f. $0.\bar{9}$ Begin by letting x = $0.\bar{9}$ or x = 0.9999... (this answer might surprise you)

g. 0.45

h. $0.\overline{45}$ Begin by letting x = $0.\overline{45}$ or x = 0.454545...

i. 0.084

j. $0.\overline{084}$ Begin by letting x = $0.\overline{084}$ or 0.084084084...

Sometimes, the arithmetic gets a little more challenging. Consider writing 0.14444... as a fraction. The 4 repeats starting two places after the decimal point. We plunge forward as before, but there's a little glitch at the end because one side of the equation will have a decimal point in it. If we were to divide by 9, the fraction will have a decimal point in the numerator so it isn't a reduced rational number; both the numerator and denominator have to be integers. Here are basic steps up until that point.

$10n = 1.444444\ldots$and $n = 0.144444$

Subtracting, we get $9n = 1.3$

One way to remedy this situation is to multiply both sides of the equation by 10 (or 100 or 1000 as needed) to eliminate the decimal. This is like clearing fractions by multiplying both sides of an equation by the least common denominator.

So multiply both sides of $9n = 1.3$ by 10 to get $90n = 13$. Then divide by 90 to get $\frac{13}{90}$.

Another way to remedy this situation is to write the right side of the equation as a fraction. Remember that 1.3 is $\frac{13}{10}$. So the equation is $9n = \frac{13}{10}$.

You can cross multiply to get $90n = 13$ and then divide by 90 to get $n = \frac{13}{90}$

In any case, remember that a reduced fraction is the ratio of two integers that are relatively prime.

? Exercise 25

Rewrite each repeating decimal, use the technique shown in the previous example, as a simplified fraction. Then, using your calculator, divide the numerator by the denominator and see if the result matches the original problem.

a. 0.02828...

b. 0.2888...

c. 0.00666...

d. 0.1011011...

e. 0.3999...

So, what did you think about the answer to 24f and 25e? Both of those are little mind-boggling. It's kind of hard to accept, but 0.999999... is really the same number as 1. It's not less than 1 it's exactly 1! For 24f, if you let $10n = 9.999\ldots$ and $n = 0.999\ldots$, then subtracting n from $10n$ yields $9n = 9$, so $n = 1$. That's hard to swallow, but it's the truth! Another way to see this is to realize that 1/3 + 2/3 = 1. But 1/3 = 0.333333.... and 2/3 = 0.666666... Then, 1/3 + 2/3 = 0.33333... + 0.66666... = 0.99999.... We know 1/4 + 2/3 = 1, so 0.99999... must also equal 1, as well. Oh, this stuff is just too cool!

For 0.39999..., the result of repeating 9s after a decimal makes 0.39999...= 0.4 = $\frac{4}{10} = \frac{2}{5}$.

So, basically, any number with a bunch of trailing 9s ends up being a terminating decimal.

All of the numbers we've been dealing with so far fractions, terminating decimals, and repeating decimals make up the rational numbers. Every rational number can be written as the ratio of two relatively prime integers, and can also be written as a terminating or repeating decimal. Conversely, every terminating and repeating decimal is a rational number.

Oh, but there's more...much more!

The rationals make up a very small part of the real numbers. To complete the real number system, we have to talk about the irrational numbers. Every real number is either rational or irrational. Those numbers which cannot be written as the ratio of two relatively prime integers is irrational. Those decimal numbers which neither terminate nor repeat are irrational. So, what do they look like?

One of the most commonly known irrational numbers is π. It is the number that is the ratio of the circumference of a circle to its diameter. Sounds like it's a ratio of two integers, but it's not! π can only be approximated. The most common approximation is 3.14 or 22/7. Neither of these is equal to π because both of these are rational numbers, and π is not!

Other irrational numbers are square roots of numbers that are not perfect squares, or cube roots of numbers that are not perfect cubes, etc.

For instance, these numbers are irrational: $\sqrt{5}, \sqrt{3}, \sqrt{12}, \sqrt{20}, \sqrt[3]{35}, \sqrt[4]{72}$

Exercise 26

Write five irrational numbers not already listed above _____

Exercise 27

Is $\sqrt{9}$ irrational? _____ Why or why not? _____

Another way to express an irrational number in decimal form is to make up a decimal that perhaps has some pattern to it, but never terminates or repeats. Two examples of this type are 2.12112111211112... and 5.010203040506070809010011012...

Exercise 28

Write four irrational numbers in decimal form that shows a clear pattern.

Exercise 29

Write both a rational number and an irrational number in decimal form that is between 0.53 and 0.54.

Exercise 30

Write both a rational number and an irrational number in decimal form that is between 0.53333... and 0.54444...

Rational _____

Irrational _____

Exercise 31

Classify each of the following numbers as rational or irrational.

a. 0.428222... _____

b. 0.283848... _____

c. $\frac{5}{13}$ _____

d. $\sqrt{80}$ _____

e. $\sqrt{100}$ _____

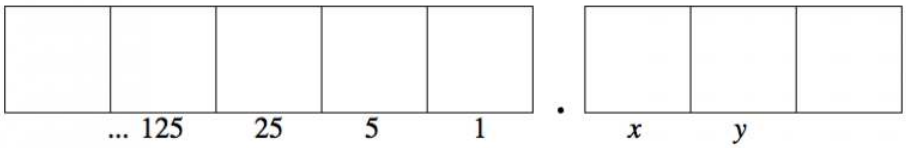
f. π _____

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9.2: "Deci"-mals in Other Bases

Just like in base 10, we can add boxes to the right of the decimal point in other bases, like base 5.

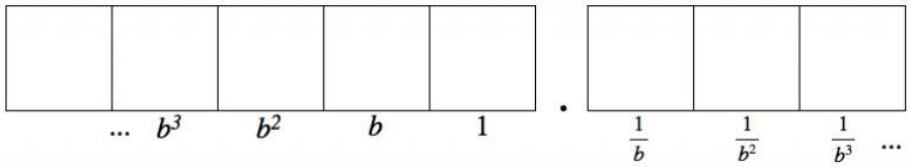


However, the prefix “dec” in “decimal point” means ten. So we really shouldn’t call it a decimal point anymore. Maybe a “pentimal point”? (In fact, the general term is **radix point**.)

? Think / Pair / Share

- Use reasoning like you saw on page 6 for the base ten system to think about other number systems:
- Figure out the values of *x* and *y* in the picture of the base-5 system above. Be sure you can explain your reasoning.
- Draw a base-4 “Dots & Boxes” model, including a radix point and some boxes to the right. Label at least three boxes to the left of the ones place and three boxes to the right of the ones place.
- Draw a base-6 “Dots & Boxes” model, including a radix point and some boxes to the right. Label at least three boxes to the left of the ones place and three boxes to the right of the ones place

In general, in a base-*b* system, the boxes to the left of the ones place represent positive powers of the base *b*. Boxes to the right of the ones place represent reciprocals of those powers.



On Your Own

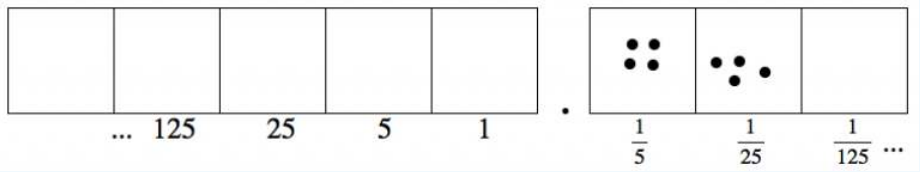
Work on the following exercises on your own or with a partner.

1. Draw a “Dots & Boxes” picture of each number. $(a)\; 0.03_{five} \quad (b)\; 0.22_{six} \quad (c)\; 0.103_{four} \quad (d)\; 0.002_{three} \ldots$
2. Find a familiar (base-10) fraction value for each number. $(a)\; 0.04_{five} \quad (b)\; 0.3_{six} \quad (c)\; 0.02_{four} \quad (d)\; 0.03_{nine} \ldots$
3. Find a familiar (base-10) fraction value for each number. (You might want to re-read the example of 0.31 in base ten from the previous section.) $(a)\; 0.13_{five} \quad (b)\; 0.25_{six} \quad (c)\; 0.101_{two} \quad (d)\; 0.24_{seven} \quad (e)\; 0.55_{eight} \ldots$

? Think / Pair / Share

Tami and Courtney were working on converting 0.44_{five} to a familiar base-10 fraction. Courtney said this:

> *The places in base five to the right of the point are like* $\frac{1}{5}$ *and then* $\frac{1}{25}$*. Since this has two places, the answer should be* $\frac{44}{25}$*.*



Tami thought about what Courtney said and replied:

> *I don't know what the right answer is, but I know that can't be right. The number* 0.44_{five} *is less than one, since there are no numbers in the ones place and no explosions that we can do. But the fraction* $\frac{44}{25}$ *is more than one. It's almost two. So they can't be the same number.*

- Who makes the most sense, Courtney or Tami? Why do you think so?
- Find the right answer to the problem Courtney and Tami were working on.

Problem 1

Find the "decimal" representation of $\frac{1}{4}$ in each of the following bases. Be sure that you can justify your answer. (You might want to review the example of $12\frac{3}{4}$ in the previous section.)

$base\ 2$	$base\ 4$	$base\ 6$
$base\ 8$	$base\ 10$	$base\ 12$

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9.3: Terminating or Repeating?

You've seen that when you write a fraction as a decimal, sometimes the decimal *terminates*, like:

$$\frac{1}{2} = 0.5 \quad \text{and} \quad \frac{33}{100} = 0.033.$$

However, some fractions have decimal representations that go on forever in a repeating pattern, like:

$$\frac{1}{3} = 0.33333\ldots \quad \text{and} \quad \frac{6}{7} = 0.857142857142857142857142\ldots$$

It's not totally obvious, but it is true: Those are the only two things that can happen when you write a fraction as a decimal.

Of course, you can *imagine* (but never write down) a decimal that goes on forever but doesn't repeat itself, for example:

$$0.1010010001000010000001\ldots \quad \text{and} \quad \pi = 3.14159265358979\ldots$$

But these numbers can never be written as a nice fraction $\frac{a}{b}$ where a and $b$are whole numbers. They are called *irrational numbers*. The reason for this name: Fractions like $\frac{a}{b}$ are also called *ratios*. Irrational numbers cannot be expressed as a *ratio* of two whole numbers.

For now, we'll think about the question: Which fractions have decimal representations that terminate, and which fractions have decimal representations that repeat forever? We'll focus just on *unit fractions*.

Definition

A **unit fraction** is a fraction that has 1 in the numerator. It looks like $\frac{1}{n}$ for some whole number n.

Think / Pair / Share

- Which of the following fractions have infinitely long decimal representations and which do not? $\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10} \ldots$
- Try some more examples on your own. Do you have a conjecture?

A fraction $\frac{1}{b}$ has an infinitely long decimal expansion if:

_____.

Problem 7

Complete the table below which shows the decimal expansion of unit fractions where the denominator is a power of 2. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Try even more examples until you can make a conjecture: What is the decimal representation of the unit fraction $\frac{1}{2^n}$?

Fraction	Denominator	Decimal
$\frac{1}{2}$	2	0.5
$\frac{1}{4}$	2^2	0.25
$\frac{1}{8}$	2^3	0.125
$\frac{1}{16}$		
$\frac{1}{32}$		
$\frac{1}{64}$		
$\frac{1}{128}$		

$\frac{1}{256}$		

Problem 8

Complete the table below which shows the decimal expansion of unit fractions where the denominator is a power of 5. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Try even more examples until you can make a conjecture: What is the decimal representation of the unit fraction $\frac{1}{5^n}$?

Fraction	Denominator	Decimal
$\frac{1}{5}$	5	0.2
$\frac{1}{25}$	5^2	0.04
$\frac{1}{125}$	5^3	
$\frac{1}{625}$		
$\frac{1}{3125}$		
$\frac{1}{15625}$		

Marcus noticed a pattern in the table from Problem 7, but was having trouble explaining exactly what he noticed. Here's what he said to his group:

> *I remembered that when we wrote fractions as decimals before, we tried to make the denominator into a power of ten. So we can do this: $*
>
> $$\frac{1}{2} = \frac{1}{2} \cdot \frac{5}{5} = \frac{5}{10} = 0.5.$$
> $$\frac{1}{4} = \frac{1}{4} \cdot \frac{25}{25} = \frac{25}{100} = 0.25.$$
> $$\frac{1}{8} = \frac{1}{8} \cdot \frac{125}{125} = \frac{125}{1000} = 0.125.$$
>
> *$When we only have 2's, we can always turn them into 10's by adding enough 5's.*

Think / Pair / Share

- Write out several more examples of what Marcus discovered.
- If Marcus had the unit fraction $\frac{1}{2^n}$, what would be his first step to turn it into a decimal? What would the decimal expansion look like and why?
- Now think about unit fractions with powers of 5 in the denominator. If Marcus had the unit fraction $\frac{1}{5^n}$, what would be his first step to turn it into a decimal? What would the decimal expansion look like and why?

Marcus had a really good insight, but he didn't explain it very well. He doesn't really mean that we "turn 2's into 10's." And he's not doing any addition, so talking about "adding enough 5's" is pretty confusing.

Problem 9

1. Complete the statement below by filling in the numerator of the fraction.

> *The unit fraction $\frac{1}{2^n}$ has a decimal representation that terminates. The representation will have n decimal digits, and will be equivalent to the fraction $\frac{?}{10^n}$.*

2. Write a better version of Marcus's explanation to justify why this fact is true.

Problem 10

Write a statement about the decimal representations of unit fractions $\frac{?}{5^n}$ and justify that your statement is correct. (Use the statement in Problem 9 as a model.)

Problem 11

Each of the fractions listed below has a terminating decimal representation. Explain how you could know this for sure, without actually calculating the decimal representation.

$$\frac{1}{10} \quad \frac{1}{20} \quad \frac{1}{50} \quad \frac{1}{200} \quad \frac{1}{500} \quad \frac{1}{4000}.$$

The Period of a Repeating Decimal

If the denominator of a fraction can be factored into just 2's and 5's, you can always form an equivalent fraction where the denominator is a power of ten.

For example, if we start with the fraction

$$\frac{1}{2^a 5^b},$$

we can form an equivalent fraction

$$\frac{1}{2^a 5^b} = \frac{1}{2^a 5^b} \cdot \frac{2^b 5^a}{2^b 5^a} = \frac{2^b 5^a}{2^{a+b} 5^{a+b}} = \frac{2^b 5^a}{10^{a+b}}.$$

The denominator of this fraction is a power of ten, so the decimal expansion is finite with (at most) $a+b$ places.

What about fractions where the denominator has other prime factors besides 2's and 5's? Certainly we *can't* turn the denominator into a power of 10, because powers of 10 have just 2's and 5's as their prime factors. So in this case the decimal expansion will go on forever. But why will it have a *repeating pattern*? And is there anything else interesting we can say in this case?

Definition

The **period** of a repeating decimal is the smallest number of digits that repeat.

For example, we saw that

$$\frac{1}{3} = 0.33333\cdots = 0.\bar{3}.$$

The repeating part is just the single digit 3, so the period of this repeating decimal is one.

Similarly, we know that

$$\frac{6}{7} = 0.857142857142857142857142\ldots = 0.\overline{857142}.$$

The smallest repeating part is the digits 857142, so the period of this repeating decimal is 6.

You can think of it this way: the *period* is the length of the string of digits under the vinculum (the horizontal bar that indicates the repeating digits).

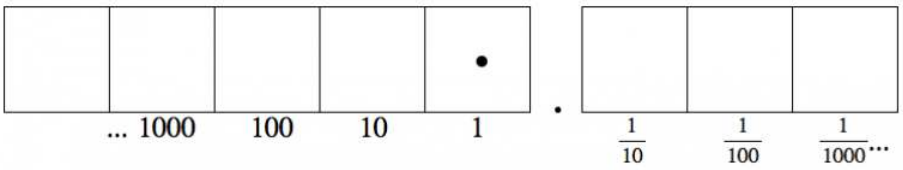
Problem 12

Complete the table below which shows the decimal expansion of unit fractions where the denominator has prime factors besides 2 and 5. (You may want to use a calculator to compute the decimal representations. The point is to look for and then explain a pattern, rather than to compute by hand.)

Try even more examples until you can make a conjecture: What can you say about the period of the fraction $\frac{1}{n}$ when n has prime factors besides 2 and 5?

Fraction	Denominator	Decimal
$\frac{1}{3}$	$0.1\overline{6}$	1
$\frac{1}{6}$	$0.\overline{142857}$	6
$\frac{1}{7}$		
$\frac{1}{9}$		
$\frac{1}{11}$		
$\frac{1}{12}$		
$\frac{1}{13}$		
$\frac{1}{14}$		

Imagine you are doing the "Dots & Boxes" division to compute the decimal representation of a unit fraction like $\frac{1}{6}$. You start with a single dot in the ones box:



To find the decimal expansion, you "unexplode" dots, form groups of six, see how many dots are left, and repeat.

Draw your own pictures to follow along this explanation:

Picture 1: When you unexplode the first dot, you get 10 dots in the $\frac{1}{10}$ box, which gives one group of six with remainder of 4.

Picture 2: When you unexplode those four dots, you get 40 dots in the $\frac{1}{100}$ box, which gives six group of six with remainder of 4.

Picture 3: Unexplode those 4 dots to get 40 in the next box to the right.

Picture 4: Make six groups of 6 dots with remainder 4.

Since the remainder repeated (we got a remainder of 4 again), we can see that the process will now repeat forever:

- unexplode 4 dots to get 40 in the next box to the right,
- make six groups of 6 dots with remainder 4,
- unexplode 4 dots to get 40 in the next box to the right,
- make six groups of 6 dots with remainder 4,
- and so on forever…

On Your Own

Work on the following exercises on your own or with a partner.

1. Use "Dots & Boxes" division to compute the decimal representation of $\frac{1}{11}$. Explain how you know for sure the process will repeat forever.

2. Use “Dots & Boxes” division to compute the decimal representation of $\frac{1}{12}$. Explain how you know for sure the process will repeat forever.
3. What are the possible *remainders* you can get when you use division to compute the fraction $\frac{1}{7}$? How can you be sure the process will eventually repeat?
4. What are the possible *remainders* you can get when you use division to compute the fraction $\frac{1}{9}$? How can you be sure the process will eventually repeat?

Problem 13

Suppose that n is a whole number, and it has some prime factors besides 2’s and 5’s. Write a convincing argument that:

1. The decimal representation of $\frac{1}{n}$ will go on forever (it will not terminate).
2. The decimal representation of $\frac{1}{n}$ will be an infinite *repeating* decimal.
3. The period of the decimal representation of $\frac{1}{n}$ will be less than *n*.

Problem 14

1. Find the “decimal” expansion for $\frac{1}{2}$ in the following bases. Be sure to show your work: $two, \quad three, \quad four, \quad five, \quad six, \quad seven \ldotp$
2. Make a conjecture: If I write the decimal expansion of $\frac{1}{2}$ in base *b*, when will that expansion be finite and when will it be an infinite repeating decimal expansion?
3. Can you prove your conjecture is true?

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9.4: Orders of Magnitude

Problem 17

How old were you when you were one million seconds old? (That's 1,000,000.)

- Before you figure it out, write down a guess. What's your gut instinct? About a day? A week? A month? A year? Have you already reached that age? Or maybe you won't live that long?
- Now figure it out! When was / will be your million-second birthday?

Problem 18

How old were you when you were one *billion* seconds old? (That's 1,000,000,000.)

- Again, before you figure it out, write down a guess.
- Now figure it out! When was / will be your billion-second birthday?

Were you surprised by the answers? People (most people, anyway) tend to have a very good sense for small, everyday numbers, but have very bad instincts about big numbers. One problem is that we tend to think *additively*, as if one billion is about a million plus a million more (give or take). But we need to think *mulitplicatively* in situations like this. One billion is 1,000 × a million.

So you could have just taken your answer to Problem 17 and multiplied it by 1,000 to get your answer to Problem 18. Of course, you would probably still need to do some calculations to make sense of the answer.

Think / Pair / Share

When is your one trillion second birthday? What will you do to celebrate?

Think / Pair / Share

The US debt is total amount the government has borrowed. (This borrowing covers the *deficit* — the difference between what the government spends and what it collects in taxes.) In summer of 2013, the US debt was *on the order of* 10 trillion dollars. (That means more than 10 trillion but less than 100 trillion. If you were to write out the dots-and-boxes picture, the dots would be as far left as the 10,000,000,000,000 place.)

- If the US pays back one penny every second, will the national debt be paid off in your lifetime? Explain your answer.
- A headline from April 2013 said, "US to Pay Down $35 billion in Quarter 2." Suppose the US pays down $35 billion dollars *every*quarter (so four times per year). About how many years would it take to pay of the total national debt?

Here are some big-number problems to think about. Can you solve them?

Problem 19

1. Suppose you have a million jelly beans, and you tile the floor with them. How big of an area will they cover? The classroom? A football field? Something bigger? What if it was a billion jelly beans?
2. Suppose you have a million jelly beans and you stack them up. How tall would it be? As tall as you? As a tree? As a skyscraper? What if it was a billion jelly beans? About how many jelly beans (what o*rder of magnitude*) would you need to stack up to reach the moon? Explain your answers.

Fermi Problems

James Boswell wrote,

> *Knowledge is of two kinds. We know a subject ourselves, or we know where we can find information upon it.*

But math proves this wrong. There is actually a third kind of knowledge: Knowledge that you *figure out for yourself.* In fact, this is what scientists and mathematicians do for a living: they create new knowledge! Starting with what is already known, they ask "what if…" questions. And eventually, they figure out something new, something no one ever knew before!

Even for knowledge that you *could* look up (or ask someone), you can often figure out the answer (or a close approximation to the answer) on your own. You need to use a little knowledge, and a little ingenuity.

Fermi problems, named for the physicist Enrico Fermi, involve using your knowledge, making educated guesses, and doing reasonable calculations to come up with an answer that might at first seem unanswerable.

Example 9.4.1:

Here's a classic Fermi problem: How many elementary school teachers are there in the state of Hawaii?

You might think: How could I possibly answer that? Why not just google it? (But some Fermi problems we meet will have — gasp! — non-googleable answers.)

First let's define our terms. We'll say that we care about classroom teachers (not administrators, supervisors, or other school personnel) who have a permanent position (not a sub, an aide, a resource room teacher, or a student teacher) in a grade K–5 classroom.

But let's stop and think. Do you know the population of Hawaii? It's about 1,000,000 people. (That's not exact, of course. But this is an exercise is estimation. We're trying to get at the *order of magnitude* of the answer.)

How many of those people are elementary school students? Well, what do you know about the population of Hawaii? Or what do you *suspect* is true? A reasonable guess would be that the population is evenly distributed across all age groups. That would give a population that looks something like this:

age range	# of people
0 – 9	125,000
10 – 19	125,000
20 – 29	125,000
30 – 39	125,000
40 – 49	125,000
50 – 59	125,000
60 – 69	125,000
70 – 79	125,000

We'll assume people don't live past 80. Of course some people do! But we're all about making simplifying assumptions right now. That gives us eight age categories, with about 125,000 people in each category.

An even better guess (since we have a large university that draws lots of students) is that there's a "bump" around college age. And some people live past 80, but there are probably fewer people in the older age brackets. Maybe the breakdown is something like this? (If you have better guesses, use them!)

age range	# of people
0 – 9	125,000
10 – 19	130,000
20 – 29	140,000
30 – 39	125,000
40 – 49	125,000

50 – 59	125,000
60 – 69	120,000
> 70	105,000

So, how many K–5 students are in Hawaii? That covers about six years of the 0–9 range. If we are still going with about the same number of people at each age, there should be about 12,500 in each grade for a total of 12,500 × 6 = 75,000 K–5 students.

OK, but we really wanted to know about K–5 *teachers*. One nice thing about elementary school: there tends to be just one teacher per class. So we need an estimate of how many classes, and that will tell us how many teachers.

So, how many students in each class? It probably varies a bit, with smaller kindergarten classes (since they are more rambunctious and need more attention), and larger fifth grade classes. There are also smaller classes in private schools and charter schools, but larger classes in public schools. A reasonable average might be 25 students per class across all grades K–5 and all schools.

So that makes 75,000 ÷ 25 = 3,000 K–5 classrooms in Hawaii. And that should be the same as the number of K–5 teachers.

How good is this estimate? Can you think of a way to check and find out for sure?

So now you see the process for tackling a Fermi problem:

- Define your terms.
- Write down what you know.
- Make some reasonable guesses / estimates.
- Do some simple calculations.

Try your hand at some of these:

Problem 20

How much money does your university earn in parking revenue each year?

Problem 21

How many tourists visit Waikiki in a year?

Problem 22

How much gas would be saved in Hawaii if one out of every ten people switched to a carpool?

Problem 23

How high can a climber go up a mountain on the energy in one chocolate bar?

Problem 24

How much pizza is consumed each month by students at your university?

Problem 25

How much would it cost to provide free day care to every four-year-old in the US?

Problem 26

How many books are in your university's main library?

Problem 27

Make up your own Fermi problem… what would you be interested in calculating? Then try to solve it!

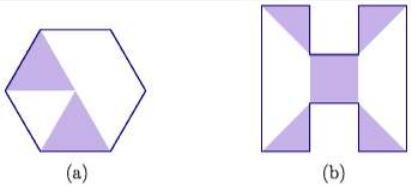
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9.5: Practice

Problem 28

Express the shaded portion of each figure as both a fraction and as a decimal. Justify your answers.



Problem 29

Which number is bigger: 0.135 or 0.14? Justify your answer.

Problem 30

Arrange the digits 1, 2, 3, and 4 in the boxes to create the smallest possible sum. Use each digit exactly once. Justify that your answer is as small as possible.

$$\frac{\square}{\square} + \frac{\square}{\square}$$

Problem 31

Arrange the digits 1, 2, 3, and 4 in the boxes to create the smallest possible (positive) difference. Use each digit exactly once. Justify that your answer is as small as possible.

$$\frac{\square}{\square} - \frac{\square}{\square}$$

Problem 32

Use the "Dots & Boxes" model to show that $\frac{1}{9} = 0.\bar{1}$. Then use this fact to answer these questions and justify your answers.

1. What fraction is given by $0.\bar{2}$?
2. What fraction is given by $0.\bar{5}$?
3. What fraction is given by $0.\bar{6}$?
4. What fraction is given by $0.\bar{8}$?
5. What fraction is given by $0.\bar{9}$?

Problem 33

In this problem, you will focus on the calculation

$$170 \times \square.$$

Your goal is to get a product that is close to 200.

1. Will you multiply 170 by a number greater or less than 1? Greater or less than 2? Justify your answers.
2. Suppose you can use only one decimal place. Fill in the box with a number that gets as close to 200 as possible.
3. Suppose you can use only two decimal places. Fill in the box with a number that gets as close to 200 as possible.
4. Suppose you can use only three decimal places. Fill in the box with a number that gets as close to 200 as possible.

Problem 34

Do each computation below without using a calculator. Explain your thinking.

1. $(23 \times 0.1) + (0.001 \times 55) \ldotp$
2. $18.45 \div (0.63 \div 0.7) \ldotp$
3. $22.65 - (0.03 \cdot 10) \ldotp$

Problem 35

Without actually calculating anything (just use your number sense!), order x, y, and z from smallest to largest. Explain your ordering.

$$x = 0.07 + 0.000001$$
$$y = 0.07 \times 0.000001$$
$$z = 0.07 \div 0.000001$$

Problem 36

For each question below, choose the correct calculation and explain your choice. Then estimate the answer (don't calculate it exactly) and explain why your estimate is a good one.

1. A large pizza has eight slices and costs $15.95. How much does each slice of pizza cost? Should you calculate 15.95 × 8 or 15.95 ÷ 8?
2. There are 2.54 centimeters in an inch. A standard sheet of notebook paper is $8\frac{1}{2}$ inches wide and 11 inches long. How many centimeters wide is the page? Should you calculate 8.5 × 2.54 or 11 × 2.54 or 8.5 ÷ 2.54 or 11 ÷ 2.54?
3. In a model train set, 1.38 inches represents one foot in real life. The height of One World Trade Center in New York City is 1776 feet. How tall would a scale model of the building be? Should you calculate 1776 × 1.38 or 1776 ÷ 1.38?
4. Eight-tenths of a jumprope is 1.75 meters long. How long is the whole rope? Should you calculate 0.8 × 1.75 or 0.8 ÷ 1.75 or 1.75 ÷ 0.8?

Problem 37

Kaimi had no money at all when he cashed his paycheck. As he left the bank, he bought a piece of candy for a nickel from a machine. Later, he realized that the money in his pocket was equal to twice his paycheck. After a quick calculation, he figured out what happened: the teller accidentally switched the dollars and cents. How much was Kaimi supposed to be paid, and what did the teller give him? Justify your answer.

Problem 38

Here are the rules to a card game. Read the rules carefully and then answer the questions below.

- Each player starts with 10 points. The goal is to score as close to 100 points as possible without going over.
- On your turn: draw two cards, which will each have a decimal number on them. Using estimation (no computation), you can choose to multiply or divide your current score by one of the decimal numbers.
- After you decide, compute your new score exactly using a calculator. If your new score is over 100, you lose. If not, the other player takes a turn.
- At the end of your turn, you can decide to end the game. If you do, the other player gets one more turn. Then, the player with the score that is closest to 100 without going over wins the game.

Here are the questions:

1. On your turn, your score is 50. You draw the cards 0.2 and 1.75. Remember that your choices are: $

$$\begin{array}{ll} \text{divide by } 0.2 & \text{multiply by } 0.2 \\ \text{divide by } 1.75 & \text{multiply by } 1.75. \end{array}$$

$What is your best move and why?

2. On your turn, your score is 88. You draw 1.3 and 0.6. What is your best move and why?
3. Your partner has a score of 57, and your score is 89. On her turn, your partner draws 0.8 and 1.8. She says she wants to end the game. On your final turn, you draw 0.7 and 1.2. If you both make the best possible move, who will win the game? Justify your answer.

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CHAPTER OVERVIEW

10: Estimation and Rounding

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10.1: Definitions

What is the difference between estimation and rounding? First, let us look at the definitions from Google:

Definition: Estimation

A rough calculation of the value, number, quantity, or extent of something.

Definition: Rounding

Alter (a number) to one less exact but more convenient for calculations.

Now let us look at some examples and determine which one is which:

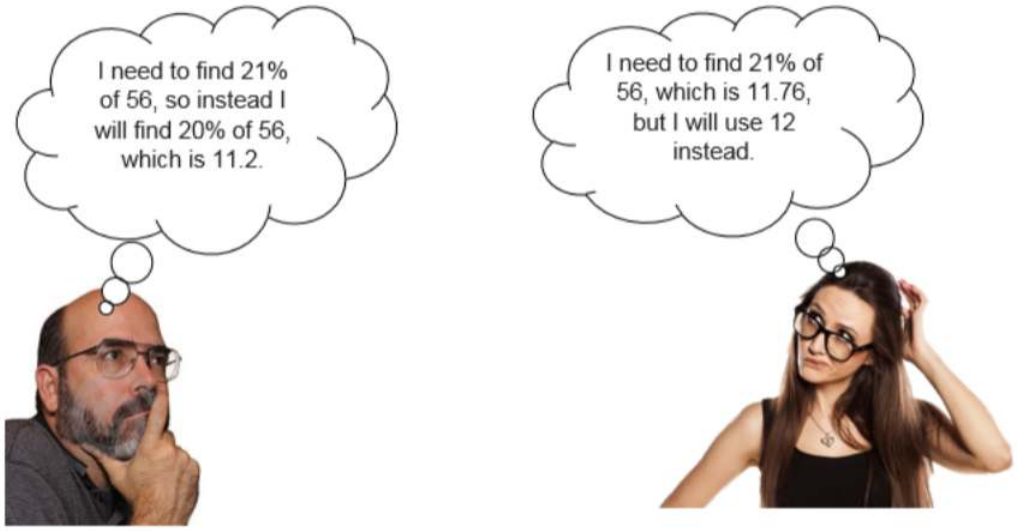


Figure 3.5.1

Irwin's example is **Estimation**. He is estimating an answer by using easier numbers.

Mary's example is **Rounding**. She is rounding up 0.76 to 1, to make the numbers easier for her.

One of the main components of Common Core was to make sure that students could perform mental math exercises with ease. Therefore, there is a strong emphasis on percentages, estimation, rounding and multiplication.

Mental Math Percentages

(use these to help you with Example 10.1.1*)*

10%: Move the decimal once to the left

5%: Find 10% and divide by 2

20%: Find 10% and multiply by 2

1%: Move the decimal twice to the left

Example 10.1.1

Restaurant Split

For dinner, on Friday night, Chris and Maddy dine at Seven Seas Seafood. They decide to leave their phones in the car, so that there are no distractions and actually have some real conversation! The dinner went very well and the bill comes. Maddy decided to be nice and pay the entire bill; after all, she did order the crab buffet. Now, her phone is in the car and she needs to calculate tip and total. Chris says, "I think you should tip 17%. The service was good, but not great. And do it quick, I want to go home."



Figure 3.5.2

Solution

Luckily, Maddy remembered how to use her amazing estimation skills and tipped $20.

$$17\% = 10\% + 5\% + 1\% + 1\%$$
$$10\% = \$11.658 \approx \$12$$
$$5\% \approx \$12 \div 2 = \$6$$
$$1\% = \$1.1658 \approx \$1$$
$$\$12 + \$6 + \$1 + \$1 = \$20$$

Partner Activity 1

Calculate tip for the following restaurant bills using the method presented in Example 10.1.1. Make sure you round to the nearest penny.

1. $85.12
2. $21.47
3. $96.01

Mental Math Multiplication

Example 10.1.2

Growing up, most of us learned the multiplication tables from zero to ten. Maybe, even some of you reached 11 and 12. But what about the higher numbers? Here is Common Core's answer.

We need to multiply **18 × 4**.

Solution

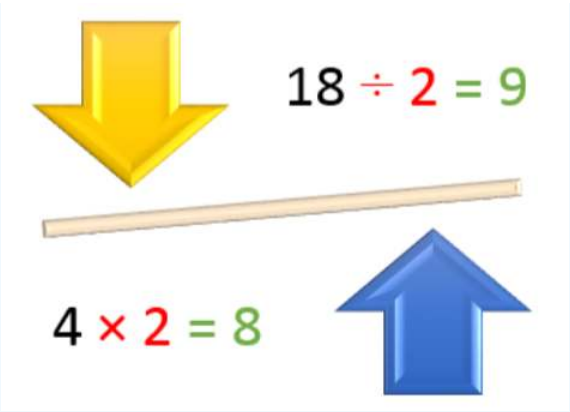


Figure 3.5.3

We can change the expression as long as we keep it balanced. Dividing 18 by 2 AND multiplying 4 by 2 will keep the expression balanced.

Therefore, 18 × 4 is the same as 9 X 8 = 72.

We do not have to use just two. Any number will work! As long as you keep it balanced!

Partner Activity 2

Here are some advanced multiplication expressions. By yourself, write an easier way to evaluate the expression, so you can use mental math. Then solve. Afterwards, compare your methods to your partner's method.

1. 24×3
2. 5×36
3. 16×4

Mental Math Subtraction Trick

Example 10.1.3

One of the hardest things for students to learn is subtracting when the first number (called the minuend) has lots of zeros, like 10,000. There can be confusion with borrowing: which zeros end up being tens and which zeros end up being nines? Common Core has an answer for that, too!

Solution

Let's try subtracting 1000 - 4983 the traditional way, with borrowing:

$$\begin{array}{cccccc} & & 9 & 9 & 9 \\ 0 & \not{1}0 & \not{1}0 & \not{1}0 & 10 \\ \not{1} & \not{0} & \not{0} & \not{0} & \not{0} \\ - & 4 & 9 & 8 & 3 \\ \hline & 5 & 0 & 1 & 7 \end{array}$$

Some students might see a problem like this and be intimidated by the amount of borrowing. An easier way to subtract: First subtract 1 from both numbers.

Going the Distance

1000 → 9999 and 4983 → 4982

$$\begin{array}{r} 9999 \\ -4982 \\ \hline 5017 \end{array}$$

No. More. Borrowing. EVER.

> **Note**
>
> Remember, subtraction is just measuring distance. We are finding the DISTANCE between 10000 and 4983, which is the same DISTANCE as 9999 and 4982

Partner Activity 3

Solve these subtraction problems WITHOUT borrowing! Think about what you should **add or subtract to both numbers.**

1. 1000 – 679
2. 513 – 284
3. 16452 – 999

Mental Math Addition and Subtraction in Schools Today

Common Core changed the way students add and subtract. Some schools still using carrying (addition) and borrowing (subtraction), like traditional mathematics. However, some schools encourage to teach the Common Core methods explained above, which will lead to mental math abilities, after enough practice.

Practice Problems

1. Estimation
   a. What is 11% of 150?
   b. What is 19% of 400?
2. Rounding
   a. Round 16.456 to the nearest hundredth.
   b. Round $49.347 to the nearest penny.
   c. Round 82.42 to the nearest whole number.
3. Tip
   a. The restaurant bill is $43.18. What is a 15% tip?
   b. The restaurant bill is $56.79. What is a 20% tip?
   c. The restaurant bill is $96.42. What is a 17% tip?
4. Multiply big numbers
   a. Use mental math to multiply 4×18.
   b. Use mental math to multiply 24×3.
   c. What did you notice about A and B?
5. Subtract from 10000
   a. Use the trick in Example $10.1.3$ to subtract 10000 – 9134.
   b. Use the trick in Example $10.1.3$ to subtract 100000 – 34678.

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10.2: Rounding Integers

Rounding as an Approximation

A primary use of whole numbers is to keep count of how many objects there are in a collection. Sometimes we're only interested in the approximate number of objects in the collection rather than the precise number. For example, there are *approximately* 20 symbols in the collection below.



The *precise* number of symbols in the above collection is 18.

We often approximate the number of objects in a collection by mentally seeing the collection as occurring in groups of tens, hundreds, thousands, etc. This process of approximation is one method of **rounding**. Rounding is very useful in estimation. When we think of a collection as occurring in groups of tens, we say we're *rounding to the nearest ten*. When we think of a collection as occurring in groups of hundreds, we say we're *rounding to the nearest hundred*. This idea of rounding continues through thousands, ten thousands, hundred thousands, millions, etc.

One Rounding Rule

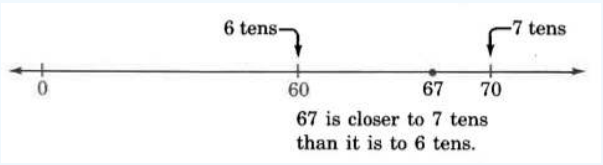
Most people reading this text will be used to one rounding rule. We will think of this as the *standard* rounding rule. The process of rounding whole numbers using this rule is illustrated in the following examples.

Example 10.2.1

Round 67 to the nearest ten.

Solution

On the number line, 67 is more than halfway from 60 to 70. The digit immediately to the right of the tens digit, the round-off digit, is the indicator for this.



Thus, 67, rounded to the nearest ten, is 70.

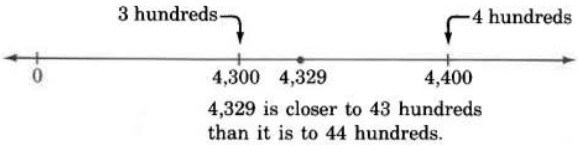
Note that we can also refer to the place value when rounding. The previous example involved *rounding 67 to the tens place*.

Example 10.2.2

Round 4,329 to the nearest hundred.

Solution

On the number line, 4,329 is less than halfway from 4,300 to 4,400. The digit to the immediate right of the hundreds digit, the round-off digit, is the indicator.



Thus, 4,329, rounded to the nearest hundred is 4,300.

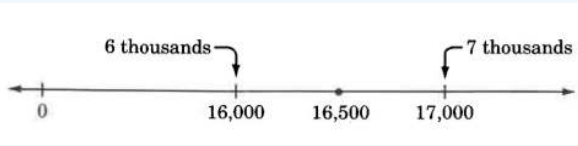
Unfortunately, the word "nearest" can make rounding confusing.

Example 10.2.3

Round 16,500 to the nearest thousand.

Solution

On the number line, 16,500 is exactly halfway from 16,000 to 17,000.



By convention, when the number to be rounded is *exactly halfway* between two numbers, it is rounded to the *higher* number.

Thus, 16,500, rounded to the nearest thousand, is 17,000.

Note the phrase "by convention" in the previous example. What that means is that the community we are discussing has made a decision (consciously or not) to do things one way. For example, by convention, we say that the letter "U" comes before the letter "V" in the English alphabet. But it does not have to be that way. It would be just as reasonable if "V" came before "U".

Most of the time, when we say "round to the nearest ____" we leave out the caveat "unless the number is exactly halfway between two values, in which case choose the larger one". This fact is merely something people decided. In fact, sometimes, we want to choose the lower value. In such cases, we must specifiy that's what we want since that's not the convention. One way a person might state this is by saying, "round down" instead of "round".

Example 10.2.4

A person whose salary is $41,450 per year might tell a friend that she makes $41,000 per year. She has rounded 41,450 to the nearest thousand. The number 41,450 is closer to 41,000 than it is to 42,000.

The Method of Rounding Whole Numbers using the Standard Rule

From the observations made in the preceding examples, we can use the following method to **round a whole number** to a particular position.

1. Mark the position of the round-off digit.
2. Note the digit to the immediate right of the round-off digit.
   a. If it is less than 5, replace it and all the digits to its right with zeros. Leave the round-off digit unchanged.
   b. If it is 5 or larger, replace it and all the digits to its right with zeros. Increase the round-off digit by 1.

In the following example, you will see some cases that can confuse many people.

Example 10.2.1

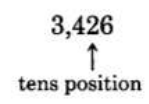
Use the method described above to do the following.

1. Round 3,426 to the nearest ten.
2. Round 9,614,018,007 to the nearest ten million.
3. Round 148,422 to the nearest million.

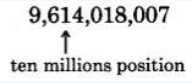
4. Round 397,000 to the nearest ten thousand.

Solution

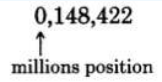
1. We are rounding to the tens position. Mark the digit in the tens position.



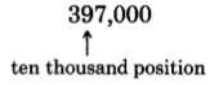
   Observe the digit immediately to the right of the tens position. It is 6. Since 6 is greater than 5, we *round up* by replacing 6 with 0 and adding 1 to the digit in the tens position (the round-off position): 2+1=32+1=3.
   Thus, 3,426 rounded to the nearest ten is 3,430.
2. We are rounding to the nearest ten million.



   Observe the digit immediately to the right of the ten millions position. It is 4. Since 4 is less than 5, we *round down* by replacing 4 and all the digits to its right with zeros.
   Thus, 9,614,018,007 rounded to the nearest ten million is 9,610,000,000.
3. Since we are rounding to the nearest million, we'll have to *imagine* a digit in the millions position. We'll write 148,422 as 0,148,422.



   The digit immediately to the right is 1. Since 1 is less than 5, we'll *round down* by replacing it and all the digits to its right with zeros.
   0,000,000
   Thus, 148,422 rounded to the nearest million is 0.
4. We are rounding to the nearest ten thousand.



   The digit immediately to the right of the ten thousand position is 7. Since 7 is greater than 5, we round up by replacing 7 and all the digits to its right with zeros and adding 1 to the digit in the ten thousands position. But 9+1=109+1=10 and we must carry the 1 to the next (the hundred thousands) position.
   Thus, 397,000 rounded to the nearest ten thousand is 400,000.

Now try to do some on your own.

? Exercise 10.2.1

Use the method described above to do the following.

1. Round 3387 to the nearest hundred.
2. Round 26,515 to the nearest thousand.
3. Round 30,852,900 to the nearest million.
4. Round 39 to the nearest hundred.
5. Round 59,600 to the nearest thousand.

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10.3: Rounding Decimals

Rounding Decimal Numbers

We previously considered the concept of rounding numbers where our concern with rounding was related to positive integers only. With a few minor changes, we can apply the same rules of rounding to decimals (non-integers) and negative integers. To round a value to any integer (positive or negative), we simply apply the same method as before while keeping in mind what it means for negative numbers to be less than or greater than other numbers. We have not discussed negative numbers yet, however. So, we will not focus on this at this time. So, we will focus on non-integers. The method outlined below applies to positive and negative values, but all the examples and exercises will involve positive values to keep things simple for now.

To round a decimal to a particular position:

1. Mark the position of the round-off digit.
2. Note whether the digit to the immediate right of the marked digit is
   a. *less than 5*. If so, leave the round-off digit unchanged.
   b. *5 or greater*. If so, add 1 to the round-off digit.
3. If the round-off digit is
   a. to the right of the decimal point, eliminate all the digits to its right.
   b. to the left of the decimal point, replace all the digits between it and the decimal point with zeros and eliminate the decimal point and all the decimal digits.

Example 10.3.1

Round each decimal to the specified position. (The numbers in parentheses indicate which step is being used.)

Round 32.116 to the nearest hundredth.

Solution

1. 32.116 (hundredths position)
2. The digit immediately to the right is 6, and $6 > 5$, so we add 1 to the round-off digit: $1 + 1 = 2$.
3. The round-off digit is to the right of the decimal point, so we eliminate all digits to its right.

The number 32.116 rounded to the nearest hundredth is 32.12.

Note that the only real change to the method we saw in the previous section is that we need to consider what happens to the right of the round-off digit based on where it is in relation to the decimal point.

Example 10.3.1

Round 633.14216 to the nearest hundred.

Solution

1. 633.14216 (hundreds position)
2. The digit immediately to the right is 3, and $3 < 5$ so we leave the round-off digit unchanged.
3. The round-off digit is to the left of 0, so we replace all the digits between it and the decimal point with zeros and eliminate the decimal point and all the decimal digits.

The number 633.14216 rounded to the nearest hundred is 600.

We can actually combine 3a and 3b to the following: 3* Replace all digits to the right of the round-off digit with 0. This is because we could write literally any number as a number followed by infinitely many 0s. For example, 4.5 = 4.5000... and 670 = 670.000... The reason we usually do not do this is because by using the method with two cases for Step 3, the reader can tell exactly where our round-off digit is. Let's see what we mean by this.

Example 10.3.1

Round 84.996 to the nearest hundredth using both methods (one with Steps 3a and 3b and one with Step 3*).

Solution

1. The round-off digit is the 9 in the hundredths place.
2. The digit to the right of that is 6, so we add 1 to 9. That gives us 10, so we replace the 9 with 0 and add the 1 to the digit in the tenths place. Since the digit in the tenths place is also 9, we get 10 again. So, we replace that 9 with 0 and add the 1 to the digit in the ones place. The digit in the ones place is 4, so we get 5.
3. In the first method, we eliminate the digits to the right of the round-off digit.
   Thus, 84.996 rounded to the nearest hundredth is 85.00.
   In the second method, we replace all digits to the right of the round-off digit with 0.
   Thus, 84.996 rounded to the nearest hundredth is 85.000...

Notice that in the first method, we got 85.00, and the last written 0 is in the hundredths place, which was the round-off digit! In the second method, we get 85.000..., which is equivalent in value to 85.00. But by writing it this way, the person reading our answer does not know that the hundredths place was where we rounded to unless we explicitly state that in words.

The purpose of that previous example is to give a reason for why we use the usual rounding method in which Step 3 is split into 3a and 3b. It is common for people to think, "This is unnecessarily complicated" about rules in mathematics. Sometimes it's because it has to be (in these cases, the "rules" are actually "properties"). Sometimes it's because the rule is based on a reason that may not be evident at first (like what we just saw). But there definitely are times when the "rule" is just a convention with no particular reason behind it. In those cases, there is often disagreement between mathematicians about that rule. Yes, mathematicians argue about math. They do it a lot!

Example 10.3.1

Round each of the following numbers.

- 1,729.63 to the nearest ten
- 1.0144 to the nearest tenth
- 60.98 to the nearest one

Solution

- 1,729.63 rounded to the nearest ten is 1,730.
- 1.0144 rounded to the nearest tenth is 1.0.
- 60.98 rounded to the nearest one is 61.

Sometimes we hear a phrase such as "round to the fourth decimal place" or "round to three decimal places." This phrase means that the round-off digit is the fourth decimal digit (7the digit in the ten-thousandths position) or third decimal digit (the digit in the thousandths position), respectively.

Example 10.3.1

Round each of the following numbers.

- 67.129 to the second decimal place
- 67.129558 to 3 decimal places

Solution

- 67.129 rounded to the second decimal place is 67.13.
- 67.129558 rounded to 3 decimal places is 67.130.

Now, try these on your own.

Exercise 10.3.1

Round each decimal to the specified position.

1. 4.816 to the nearest hundredth.
2. 0.35928 to the nearest ten thousandths.
3. 82.1 to the nearest one.
4. 753.98 to the hundreds place.
5. Round 43.99446 to three decimal places.
6. Round 105.019997 to four decimal places.
7. Round 99.9999 to the second decimal place.

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10.4: Estimation by Rounding

When beginning a computation, it is valuable to have an idea of what value to expect for the result. When a computation is completed, it is valuable to know if the result is reasonable.

In the rounding process, it is important to note two facts:

1. The rounding that is done in estimation does not always follow the rules of rounding discussed previously. Since estimation is concerned with the expected value of a computation, rounding is done using *convenience* as the guide rather than using hard-and-fast rounding rules. For example, if we wish to estimate the result of the division $80 \div 26$ *conveniently* divided by 20 than by 30.
2. Since rounding may occur out of convenience, and different people have different ideas of what may be convenient, results of an estimation done by rounding may vary. For a particular computation, different people may get different estimated results. *Results may vary*.

Definition: Estimation

Estimation is the process of determining an expected value of a computation. Common words used in estimation are *about*, *near*, and *between*.

Estimation by Rounding

The rounding technique estimates the result of a computation by rounding the numbers involved in the computation to one or two nonzero digits.

Example 10.4.1

Estimate the sum: $2,357 + 6,106$.

Solution

Notice that 2,357 is near $\underbrace{2,400}_{\text{two nonzero digits}}$ and that 6,106 is near $\underbrace{6,100}_{\text{two nonzero digits}}$

The sum can be estimated by $2,400 + 6,100 = 8,500$. (It is quick and easy to add 24 and 61.)

Thus, $2,357 + 6,106$ is about 8,400. In *fact*, $2,357 + 6,106 = 8,463$.

Notice that the rounding happened *before* the operation was performed. Typically, a major reason we estimate is to make the computations easier. If you were to add $2,357 + 6,106$ and *then* round the sum, you actually did more work than if you had just added $2,357 + 6,106$ and written the answer. This means that sometimes the estimation is less accurate that we would get if we rounded after the computation. This is okay because we are only looking for estimates. This is one subtle difference between rounding and estimation. Rounding should reflect the reality of the value's size. Estimation might result in something quite different (though reasonable in the sense that it arose from reasonable methods).

Example 10.4.1

Estimate each sum by rounding the addends.

- $4,216 + 3,942$
- $812 + 514$

You may use the $\approx$ symbol to indicate that rounding has taken place.

Solution

- $4,216 + 3,942 \approx 4,200 + 3,900$. So, $4,216 + 3,942$ is about equal to 8,100.
- $812 + 514 \approx 800 + 500$. So, $812 + 514$ is about equal to 1,300.

For each case, find what the actual sum is and compare it to the estimate. How accurate are the estimates?

Try the following on your own. Compare the estimate you get with the actual sum.

Exercise 10.4.1

Estimate the sum: $43,892 + 92,106$

Let's now estimate some differences.

Example 10.4.1

Estimate the difference: $5,203 - 3,015$.

Solution

Notice that 5,203 is near $\underbrace{5,200}_{\text{two nonzero digits}}$ and that 3,015 is near $\underbrace{3,000}_{\text{one nonzero digit}}$

The difference can be estimated by $5,200 - 3,000 = 2,200$.

Thus, $5,203 - 3,015$ is *about* 2,200. In *fact*, $5,203 - 3,015 = 2,188$.

We could make a less accurate estimation by observing that 5,203 is near 5,000. The number 5,000 has only one nonzero digit rather than two (as does 5,200). This fact makes the estimation quicker (but a little less accurate). We then estimate the difference by $5,000 - 3,000 = 2,000$ and conclude that $5,203 - 3,015$ is about 2,000. This is why we say "answers may vary."

After reading the following example, redo it by with rounding to the tens place.

Example 10.4.1

Estimate the difference: $628 - 413$.

Solution

628 − 413 : 600 − 400. About 200. In fact, 215.

Try the following on your own. If more than one method of rounding comes to mind, try them all. Compare your results to the actual differences.

Exercise 10.4.1

Estimate the differences.

- $7,842 - 5,209$
- $73,812 - 28,492$

Let's try deal with multiplication now.

Example 10.4.1

Estimate the product: $73 \cdot 46$.

Solution

Notice that 73 rounded to the nearest ten is 70 and 46 rounded to the nearest ten is 50.

The product can be estimated by $70 \cdot 50 = 3,500$.

Thus, $73 \cdot 46$ is about 3,500. In fact, $73 \cdot 46 = 3,358$.

Remember that we are using rounding to make these estimates, but that is not the only way to make estimates. If you thought of another method, that would be fine. For example, you might note that 73 is near 75 and 46 is near 40, despite 40 not being the

nearest ten. Here's how one person used that method: 73 is near 75 and 46 is near 40, which is four tens. $75 \cdot 10 \cdot 4 = 750 \cdot 4 = 750 \cdot 2 \cdot 2 = 1500 \cdot 2 = 3000$. This requires a lot more comfort in "playing with numbers". So, if you're not feeling ready for this kind of mental math, that is okay. It takes time to develop number sense. Let's return to the rounding method of estimation.

Example 10.4.1

Estimate the product: $87 \cdot 4,316$.

Solution

We will again round each factor so that we only have one nonzero digit. Notice that 87 rounded to the nearest ten is 90 and 4,316 rounded to the nearest thousand is 4,000.

The product can be estimated by $90 \cdot 4,000 = 360,000$

Thus, $87 \cdot 4,316$ is about 360,000. In fact, $87 \cdot 4,316 = 375,492$

Now try the following on your own.

Exercise 10.4.1

Estimate the products.

1. $31 \cdot 87$
2. $18 \cdot 421$
3. $160 \cdot 943$

Let's try division.

Example 10.4.1

Estimate the quotient: $153 \div 17$.

Solution

Notice that 153 is close to $\underbrace{150}_{\text{two nonzero digits}}$ and that 17 is close to $\underbrace{15}_{\text{two nonzero digits}}$.

The quotient can be estimated by $150 \div 15 = 10$.

Thus, $153 \div 17$ is about 10. In fact, $153 \div 17 = 9$.

The quotient in the previous example was made easy by the fact that 150 is a multiple of 10. Let's keep that in mind for the following example.

Example 10.4.1

Estimate the quotient: $742,000 \div 2,400$

Solution

Let's look at two possibilities.

1. Notice that 742,000 is close to $\underbrace{700,000}_{\text{one nonzero digit}}$ and that 2,400 is close to $\underbrace{2,000}_{\text{one nonzero digit}}$.
   The quotient can be estimated by $700,000 \div 2,000 = 350$ Thus, $742,000 \div 2,400$is about 350.
2. Notice that 742,000 is close to $\underbrace{740,000}_{\text{two nonzero digits}}$ and that 2,400 is close to $\underbrace{2,000}_{\text{one nonzero digit}}$.
   The quotient can be estimated by $740,000 \div 2,000 = 370$ Thus, $742,000 \div 2,400$is about 370.

In fact, $742,000 \div 2,400 = 309.1\overline{6}$

Both were reasonable since 74 and 70 are multiples of 2. What if we changed the values just slightly, however?

Example 10.4.1

Estimate the quotient: $742,000 \div 2,900$

Solution

Let's look at two possibilities.

1. Notice that 722,000 is close to $\underbrace{700,000}_{\text{one nonzero digit}}$ and that 2,900 is close to $\underbrace{3,000}_{\text{one nonzero digit}}$.
   The quotient can be estimated by $700,000 \div 3,000$ This is not an easy calculation! So, let's change our tactic.
2. Notice that 722,000 is close to $\underbrace{720,000}_{\text{two nonzero digits}}$ and that 2,900 is close to $\underbrace{3,000}_{\text{one nonzero digit}}$.
   The quotient can be estimated by $720,000 \div 3,000 = 240$ Thus, $722,000 \div 2,900$is about 240.

In addition to this showing how our choice of round-off digit is important, we can see how useful estimation is since actually $722,000 \div 2,900 = 248.\overline{96551724137931034482758620}68$

Try the following on your own. If your first try does not produce an easy calculation, try again.

Exercise 10.4.1

Estimate the quotients.

1. $221 \div 18$
2. $4,079 \div 381$
3. $609,000 \div 16,000$

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10.5: Estimation by Clustering

Learning Objectives

- understand the concept of clustering
- be able to estimate the result of adding more than two numbers when clustering occurs using the clustering technique

Cluster

When more than two numbers are to be added, the sum may be estimated using the clustering technique. The rounding technique could also be used, but if several of the numbers are seen to **cluster** (are seen to be close to) one particular number, the clustering technique provides a quicker estimate. Consider a sum such as

$32+68+29+73$

Notice two things:

There are more than two numbers to be added.

Clustering occurs.

a. Both 68 and 73 cluster around 70, so $68+73$ is close to $80+70=2(70)=140$.

$32+\overbrace{68+29+71}$

b. Both 32 and 29 cluster around 30, so $32+29$ is close to $30+30=2(30)=60$.

The sum may be estimated by

$$\begin{aligned}(2\cdot 30)+(2\cdot 70) &= 60+140\\ &= 200\end{aligned}$$

In fact, $32+68+29+73=202$.

Sample Set A

Estimate each sum. Results may vary.

$27+48+31+52$.

Solution

27 and 31 cluster near 30. Their sum is about $2\cdot 30=60$.

48 and 52 cluster near 50. Their sum is about $2\cdot 50=100$.

Thus, $27+48+31+52$ is about $\begin{aligned}(2\cdot 30)+(2\cdot 50) &= 60+100\\ &= 160\end{aligned}$

In fact, $27+48+31+52=158$. .

Sample Set A

$88+21+19+91$.

Solution

88 and 91 cluster near 90. Their sum is about $2\cdot 90=180$.

21 and 19 cluster near 20. Their sum is about $2\cdot 20=40$.

Thus, $88+21+19+91$ is about $\begin{aligned}(2\cdot 90)+(2\cdot 20) &= 180+40\\ &= 220\end{aligned}$

In fact, $88+21+19+91=219$.

Sample Set A

$17+21+48+18$.

Solution

17, 21, and 18 cluster near 20. Their sum is about $3 \cdot 20 = 60$.

48 is about 50.

Thus, $17+21+48+18$ is about $\begin{aligned}(3 \cdot 20)+50 &= 60+50 \\ &= 110\end{aligned}$

In fact, $17+21+48+18 = 104.$.

Sample Set A

$61+48+39+57+52$.

Solution

61 and 57 cluster near 60. Their sum is about $2 \cdot 60 = 120$.

48, 49, and 52 cluster near 50. Their sum is about $3 \cdot 50 = 150$.

Thus, $61+48+39+57+52$ is about $\begin{aligned}(2 \cdot 60)+(3 \cdot 50) &= 120+150 \\ &= 270\end{aligned}$

In fact, $61+48+39+57+52 = 267.$.

Sample Set A

$706+321+293+684.$

Solution

706 and 684 cluster near 700. Their sum is about $2 \cdot 700 = 1,400.$

321 and 293 cluster near 300. Their sum is about $2 \cdot 300 = 600.$

Thus, $706+321+293+684$ is about $\begin{aligned}(2 \cdot 700)+(2 \cdot 300) &= 1,400+600 \\ &= 2,000\end{aligned}$

In fact, $706+321+293+684 = 2,004.$.

Practice Set A

Use the clustering method to estimate each sum.

$28+51+31+47$

Answer

$(2 \cdot 30)+(2 \cdot 50) = 60+100 = 160$

Practice Set A

$42+39+68+41$

Answer

$(3 \cdot 40)+70 = 120+70 = 190$

Practice Set A

$37+39+83+42+79$

Answer

$(3\cdot 40)+(2\cdot 80)=120+160=280$

Practice Set A

$612+585+830+794$

Answer

$(2\cdot 600)+(2\cdot 800)=1,200+1,600=2,800$

Exercises

Use the clustering method to estimate each sum. Results may vary.

Exercise 10.5.1

$28+51+31+47$

Answer

$2(30)+2(50)=160(157)$

Exercise 10.5.2

$42+19+39+23$

Exercise 10.5.3

$88+62+59+90$

Answer

$2(90)+2(60)=300(299)$

Exercise 10.5.4

$76+29+33+82$

Exercise 10.5.5

$19+23+87+21$

Answer

$3(20)+90=150(150)$

Exercise 10.5.6

$41+28+42+37$

Exercise 10.5.7

$89+32+89+93$

Answer

$3(90)+30=300(303)$

Exercise 10.5.8

$73+72+27+71$

Exercise 10.5.9

$43+62+61+55$

Answer

$40+3(60)=220(221)$

Exercise 10.5.10

$31+77+31+27$

Exercise 10.5.11

$57+34+28+61+62$

Answer

$3(60)+2(30)=240(242)$

Exercise 10.5.12

$94+18+23+91+19$

Exercise 10.5.13

$103+72+66+97+99$

Answer

$3(100)+2(70)=440(437)$

Exercise 10.5.14

$42+121+119+124+41$

Exercise 10.5.15

$19+24+87+23+91+93$

Answer

$3(20)+3(90)=330(337)$

Exercise 10.5.16

$108+61+63+96+57+99$

Exercise 10.5.17

$518+721+493+689$

Answer

$2(500)+2(700)=2,400(2,421)$

Exercise 10.5.18

$981+1208+1214+1006$

Exercise 10.5.19

$23+81+77+79+19+81$

Answer

$2(20)+4(80)=360(360)$

Exercise 10.5.20

$94+68+66+101+106+71+110$

Exercises for Review

Exercise 10.5.21

Specify all the digits greater than 6.

Answer

7, 8, 9

Exercise 10.5.22

Find the product: $\frac{2}{3}\cdot\frac{9}{14}\cdot\frac{7}{12}$.

Exercise 10.5.23

Convert 0.06 to a fraction.

Answer

$\frac{3}{50}$

Exercise 10.5.24

Write the proportion in fractional form: "5 is to 8 as 25 is to 40."

Exercise 10.5.25

Estimate the sum using the method of rounding: $4,882+2,704$.

Answer

$4,900+2,700=7,600(7,586)$

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10.6: Mental Arithmetic and Using the Distributive Property

Learning Objectives

- understand the distributive property
- be able to obtain the exact result of a multiplication using the distributive property

The Distributive Property

Distributive Property

The **distributive property** is a characteristic of numbers that involves both addition and multiplication. It is used often in algebra, and we can use it now to obtain exact results for a multiplication.

Suppose we wish to compute $3(2+5)$. We can proceed in either of two ways, one way which is known to us already (the order of operations), and a new way (the distributive property).

Compute $3(2+5)$ using the order of operations.

$3(2+5)$

Operate inside the parentheses first: $2+5=7$.

$3(2+5)=3\cdot 7$

Now multiply 3 and 7.

$3(2+5)=3\cdot 7=21$

Thus, $3(2+5)=21$.

Compute $3(2+5)$ using the distribution property.

We know that multiplication describes repeated addition. Thus,

$$\begin{aligned} 3(2+5) &= \underbrace{2+5+2+5+2+5}_{2+5 \text{ appears 3 times}} \\ &= 2+2+2+5+5+5 \qquad \text{(by the commutative property of addition)} \\ &= 3\cdot 2+3\cdot 5 \qquad \text{(since multiplication describes repeated addition)} \\ &= 6+15 \\ &= 21 \end{aligned}$$

Thus, $3(2+5)=21$.

Let's look again at this use of the distributive property.

$$3(2+5)=\underbrace{2+5+2+5+2+5}_{2+5 appears 3 times}$$

$$3(2+5)=\underbrace{2+2+2}_{2 appears 3 times}+\underbrace{5+5+5}_{5 appears 3 times}$$

$$3(2+5)=3\cdot 2+3\cdot 5$$

3 times 2 3 times 5

The 3 has been *distributed* to the 2 and 5.

This is the distributive property. We distribute the *factor* to each *addend* in the parentheses. The distributive property works for both sums and differences.

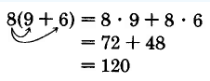
Sample Set A

$$\begin{aligned} 4(6+2) &= 4\cdot 6+4\cdot 2 \\ &= 24+8 \\ &= 32 \end{aligned}$$

Using the order of operations, we get

$$\begin{aligned} 4(6+2) &= 4\cdot 8 \\ &= 32 \end{aligned}$$

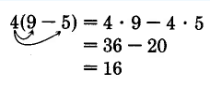
Sample Set A



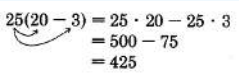
Using the order of operations, we get

$$\begin{aligned} 8(9+6) &= 8 \cdot 15 \\ &= 120 \end{aligned}$$

Sample Set A



Sample Set A



Practice Set A

Use the distributive property to compute each value.

$6(8+4)$

Answer

$6 \cdot 8 + 6 \cdot 4 = 48 + 24 = 72$

Practice Set A

$4(4+7)$

Answer

$4 \cdot 4 + 4 \cdot 7 = 16 + 28 = 44$

Practice Set A

$8(2+9)$

Answer

$8 \cdot 2 + 8 \cdot 9 = 16 + 72 = 88$

Practice Set A

$12(10+3)$

Answer

$12 \cdot 10 + 12 \cdot 3 = 120 + 36 = 156$

Practice Set A

$6(11-3)$

Answer

$6 \cdot 11 - 6 \cdot 3 = 66 - 18 = 48$

Practice Set A

$8(9-7)$

Answer

$8 \cdot 9 - 8 \cdot 7 = 72 - 56 = 16$

Practice Set A

$15(30-8)$

Answer

$15 \cdot 30 - 15 \cdot 8 = 450 - 120 = 330$

Estimation Using the Distributive Property

We can use the distributive property to obtain exact results for products such as $25 \cdot 23$. The distributive property works best for products when one of the factors ends in 0 or 5. We shall restrict our attention to only such products.

Sample Set B

Use the distributive property to compute each value.

$25 \cdot 23$

Solution

Notice that $23 = 20 + 3$. We now write

$$\begin{aligned} 25 \cdot 23 &= 25(20+3) \\ &= 25 \cdot 20 + 25 \cdot 3 \\ &= 500 + 75 \\ &= 575 \end{aligned}$$

Thus, $25 \cdot 23 = 575$

We could have proceeded by writing 23 as $30 - 7$.

$$\begin{aligned} 25 \cdot 23 &= 25(30-7) \\ &= 25 \cdot 30 - 25 \cdot 7 \\ &= 750 - 175 \\ &= 575 \end{aligned}$$

Sample Set B

$15 \cdot 37$

Solution

Notice that $37 = 30 + 7$. We now write

$$\begin{aligned} 15 \cdot 37 &= 15(30+7) \\ &= 15 \cdot 30 + 15 \cdot 7 \\ &= 450 + 105 \\ &= 555 \end{aligned}$$

Thus, $15 \cdot 37 = 555$

We could have proceeded by writing 37 as $40 - 3$.

$$\begin{aligned} 15 \cdot 37 &= 15(40-3) \\ &= 15 \cdot 40 - 15 \cdot 3 \\ &= 600 - 45 \\ &= 555 \end{aligned}$$

Sample Set B

$15 \cdot 86$

Solution

Notice that $86 = 80 + 6$. We now write

$$\begin{aligned} 15 \cdot 86 &= 15(80+6) \\ &= 15 \cdot 80 + 15 \cdot 6 \\ &= 1{,}200 + 90 \\ &= 1{,}290 \end{aligned}$$

We could have proceeded by writing 86 as $90 - 4$.

$$\begin{aligned} 15 \cdot 86 &= 15(90-4) \\ &= 15 \cdot 90 - 15 \cdot 4 \\ &= 1{,}350 - 60 \\ &= 1{,}290 \end{aligned}$$

Practice Set B

Use the distributive property to compute each value.

$25 \cdot 12$

Answer

$25(10+2) = 25 \cdot 10 + 25 \cdot 2 = 250 + 50 = 300$

Practice Set B

$35 \cdot 14$

Answer

$35(10+4) = 35 \cdot 10 + 35 \cdot 4 = 350 + 140 = 490$

Practice Set B

$80 \cdot 58$

Answer

$80(50+8) = 80 \cdot 50 + 80 \cdot 8 = 4{,}000 + 640 = 4{,}640$

Practice Set B

$65 \cdot 62$

Answer

$65(60+2) = 65 \cdot 60 + 65 \cdot 2 = 3,900 + 130 = 4,030$

Exercises

Use the distributive property to compute each product.

Exercise 10.6.1

$15 \cdot 13$

Answer

$15(10+3) = 150 + 45 = 195$

Exercise 10.6.2

$15 \cdot 14$

Exercise 10.6.3

$25 \cdot 11$

Answer

$25(10+1) = 250 + 25 = 275$

Exercise 10.6.4

$25 \cdot 16$

Exercise 10.6.5

$15 \cdot 16$

Answer

$15(20-4) = 300 - 60 = 240$

Exercise 10.6.6

$35 \cdot 12$

Exercise 10.6.7

$45 \cdot 83$

Answer

$45(80+3) = 3600 + 135 = 3735$

Exercise 10.6.8

$45 \cdot 38$

Exercise 10.6.9

$25 \cdot 38$

Answer

$25(40-2) = 1,000 - 50 = 950$

Exercise 10.6.10

$25 \cdot 96$

Exercise 10.6.11

$75 \cdot 14$

Answer

$75(10+4) = 750 + 300 = 1,050$

Exercise 10.6.12

$85 \cdot 34$

Exercise 10.6.13

$65 \cdot 26$

Answer

$65(20+6) = 1,300 + 390 = 1,690$ or $65(30-4) = 1,950 - 260 = 1,690$

Exercise 10.6.14

$55 \cdot 51$

Exercise 10.6.15

$15 \cdot 107$

Answer

$15(100+7) = 1,500 + 105 = 1,605$

Exercise 10.6.16

$25 \cdot 208$

Exercise 10.6.17

$35 \cdot 402$

Answer

$35(400+2) = 14,000+70 = 14,070$

Exercise 10.6.18

$85 \cdot 110$

Exercise 10.6.19

$95 \cdot 12$

Answer

$95(10+2) = 950+190 = 1,140$

Exercise 10.6.20

$65 \cdot 40$

Exercise 10.6.21

$80 \cdot 32$

Answer

$80(30+2) = 2,400+160 = 2,560$

Exercise 10.6.22

$30 \cdot 47$

Exercise 10.6.23

$50 \cdot 63$

Answer

$50(60+3) = 3,000+150 = 3,150$

Exercise 10.6.24

$90 \cdot 78$

Exercise 10.6.25

$40 \cdot 89$

Answer

$40(90-1) = 3,600-40 = 3,560$

Exercises for Review

Exercise 10.6.26

Find the greatest common factor of 360 and 3,780.

Exercise 10.6.27

Reduce $\frac{594}{5,148}$ to lowest terms.

Answer

$\frac{3}{26}$

Exercise 10.6.28

$1\frac{5}{9}$ of $2\frac{4}{7}$ is what number?

Exercise 10.6.29

Solve the proportion: $\frac{7}{15} = \frac{x}{90}$

Answer

$x = 42$

Exercise 10.6.30

Use the clustering method to estimate the sum: $88 + 106 + 91 + 114$.

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10.7: Estimation by Rounding Fractions

Learning Objectives

- be able to estimate the sum of two or more fractions using the technique of rounding fractions

Estimation by rounding fractions is a useful technique for estimating the result of a computation involving fractions. Fractions are commonly rounded to $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 0, and 1. Remember that rounding may cause estimates to vary.

Sample Set A

Make each estimate remembering that results may vary.

Estimate $\frac{3}{5}+\frac{5}{12}$.

Solution

Notice that $\frac{3}{5}$ is about $\frac{1}{2}$, and that $\frac{5}{12}$ is about $\frac{1}{2}$.

Thus, $\frac{3}{5}+\frac{5}{12}$ is about $\frac{1}{2}+\frac{1}{2}=1$. In fact, $\frac{3}{5}+\frac{5}{12}=\frac{61}{60}$, a little more than 1.

Sample Set A

Estimate $5\frac{3}{8}+4\frac{9}{10}+11\frac{1}{5}$.

Solution

Adding the whole number parts, we get 20. Notice that $\frac{3}{8}$ is close to $\frac{1}{4}$, $\frac{9}{10}$ is close to 1, and $\frac{1}{5}$ is close to $\frac{1}{4}$. Then $\frac{3}{8}+\frac{9}{10}+\frac{1}{5}$ is close to $\frac{1}{4}+1+\frac{1}{4}=1\frac{1}{2}$.

Thus, $5\frac{3}{8}+4\frac{9}{10}+11\frac{1}{5}$ is close to $20+1\frac{1}{2}=21\frac{1}{2}$.

In fact, $5\frac{3}{8}+4\frac{9}{10}+11\frac{1}{5}=21\frac{19}{40}$, a little less than $21\frac{1}{2}$.

Practice Set A

Use the method of rounding fractions to estimate the result of each computation. Results may vary.

$\frac{5}{8}+\frac{5}{12}$

Answer

Results may vary. $\frac{1}{2}+\frac{1}{2}=1$. In fact, $\frac{5}{8}+\frac{5}{12}=\frac{25}{24}=1\frac{1}{24}$

Practice Set A

$\frac{7}{9}+\frac{3}{5}$

Answer

Results may vary. $1+\frac{1}{2}=1\frac{1}{2}$. In fact, $\frac{7}{9}+\frac{3}{5}=1\frac{17}{45}$

Practice Set A

$8\frac{4}{15}+3\frac{7}{10}$

Answer

Results may vary. $8\frac{1}{4}+3\frac{3}{4}=11+1=12$. In fact, $8\frac{4}{15}+3\frac{7}{10}=11\frac{29}{30}$

Practice Set A

$16\frac{1}{20}+4\frac{7}{8}$

Answer

Results may vary. $(16+0)+(4+1)=16+5=21$. In fact, $16\frac{1}{20}+4\frac{7}{8}=20\frac{37}{40}$

Exercises

Estimate each sum or difference using the method of rounding. After you have made an estimate, find the exact value of the sum or difference and compare this result to the estimated value. Result may vary.

Exercise 10.7.1

$\frac{5}{6}+\frac{7}{8}$

Answer

$1+1=2(1\frac{17}{24})$

Exercise 10.7.2

$\frac{3}{8}+\frac{11}{12}$

Exercise 10.7.3

$\frac{9}{10}+\frac{3}{5}$

Answer

$1+\frac{1}{2}=1\frac{1}{2}(1\frac{1}{2})$

Exercise 10.7.4

$\frac{13}{15}+\frac{1}{20}$

Exercise 10.7.5

$\frac{3}{20}+\frac{6}{25}$

Answer

$$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}(\frac{39}{100})$$

Exercise 10.7.6

$$\frac{1}{12}+\frac{4}{5}$$

Exercise 10.7.7

$$\frac{15}{16}+\frac{1}{12}$$

Answer

$$1+0=1(1\frac{1}{48})$$

Exercise 10.7.8

$$\frac{29}{30}+\frac{11}{20}$$

Exercise 10.7.9

$$\frac{5}{12}+6\frac{4}{11}$$

Answer

$$\frac{1}{2}+6\frac{1}{2}=7(6\frac{103}{132})$$

Exercise 10.7.10

$$\frac{3}{7}+8\frac{4}{15}$$

Exercise 10.7.11

$$\frac{9}{10}+2\frac{3}{8}$$

Answer

$$1+2\frac{1}{2}=3\frac{1}{2}(3\frac{11}{40})$$

Exercise 10.7.12

$$\frac{19}{20}+15\frac{5}{9}$$

Exercise 10.7.13

$$8\frac{3}{5}+4\frac{1}{20}$$

Answer

$$8\frac{1}{2}+4=12\frac{1}{2}(12\frac{13}{20})$$

Exercise 10.7.14

$5\frac{3}{20}+2\frac{8}{15}$

Exercise 10.7.15

$9\frac{1}{15}+6\frac{4}{5}$

Answer

$$9+7=16(15\frac{13}{15})$$

Exercise 10.7.16

$7\frac{5}{12}+10\frac{1}{16}$

Exercise 10.7.17

$3\frac{11}{20}+2\frac{13}{25}+1\frac{7}{8}$

Answer

$$3\frac{1}{2}+2\frac{1}{2}+2=8\ \ (7\ \frac{189}{200})$$

Exercise 10.7.18

$6\frac{1}{12}+1\frac{1}{10}+5\frac{5}{6}$

Exercise 10.7.19

$\frac{15}{16}-\frac{7}{8}$

Answer

$$1-1=0(\frac{1}{16})$$

Exercise 10.7.20

$\frac{12}{25}-\frac{9}{20}$

Exercises for Review

Exercise 10.7.21

The fact that

$$(\text{a first number} \cdot \text{a second number}) \cdot \text{a third number} = \text{a first number} \cdot (\text{second number} \cdot \text{a third number})$$

is an example of which property of multiplication?

Answer

associative

Exercise 10.7.22

Find the quotient: $\frac{14}{15} \div \frac{4}{45}$.

Exercise 10.7.23

Find the difference: $3\frac{5}{9} - 2\frac{2}{3}$

Answer

$\frac{8}{9}$

Exercise 10.7.24

Find the quotient: $4.6 \div 0.11$.

Exercise 10.7.25

Use the distributive property to compute the product: $25 \cdot 37$.

Answer

$25(40-3) = 1000 - 75 = 925$

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CHAPTER OVERVIEW

11: Logic and Sets

11.1: Careful Use of Language in Mathematics

11.2: The Basics of Sets

11.3: Venn Diagrams

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11.1: Careful Use of Language in Mathematics

This section might seem like a bit of a sidetrack from the idea of problem solving, but in fact it is not. Mathematics is a social endeavor. We do not just solve problems and then put them aside. Problem solving has (at least) three components:

1. Solving the problem. This involves a lot of scratch paper and careful thinking.
2. Convincing yourself that your solution is complete and correct. This involves a lot of self-check and asking yourself questions.
3. Convincing someone else that your solution is complete and correct. This usually involves writing the problem up carefully or explaining your work in a presentation.

If you are not able to do that last step, then you have not really solved the problem. We will talk more about how to write up a solution soon. Before we do that, we have to think about how mathematicians use language (which is, it turns out, a bit different from how language is used in the rest of life).

Mathematical Statements

Definition

A mathematical statement is a complete sentence that is either true or false, but not both at once.

So a "statement" in mathematics cannot be a question, a command, or a matter of opinion. It is a complete, grammatically correct sentence (with a subject, verb, and usually an object). It is important that the statement is either true or false, though you may not know which! (Part of the work of a mathematician is figuring out which sentences are true and which are false.)

Think / Pair / Share

For each English sentence below, decide if it is a mathematical statement or not. If it is, is the statement true or false (or are you unsure)? If it is not a mathematical statement, in what way does it fail?

1. Blue is the prettiest color.
2. 60 is an even number.
3. Is your dog friendly?
4. Honolulu is the capital of Hawaii.
5. This sentence is false.
6. All roses are red.
7. UH Manoa is the best college in the world.
8. 1/2 = 2/4.
9. Go to bed.
10. There are a total of 204 squares on an 8 × 8 chess board.

Now write three mathematical statements and three English sentences that fail to be mathematical statements.

Notice that "1/2 = 2/4" is a perfectly good mathematical statement. It does not look like an English sentence, but read it out loud. The subject is "1/2." The verb is "equals." And the object is "2/4." This is a very good test when you write mathematics: try to read it out loud. Even the equations should read naturally, like English sentences.

Statement (5) is different from the others. It is called a **paradox**: a statement that is self-contradictory. If it is true, then we conclude that it is false. (Why?) If it is false, then we conclude that it is true. (Why?) Paradoxes are no good as mathematical statements, because it cannot be true and it cannot be false.

And / or

Consider this sentence:

After work, I will go to the beach, or I will do my grocery shopping.

In everyday English, that probably means that if I go to the beach, I will not go shopping. I will do one or the other, but not both activities. This is called an "exclusive or."

We can usually tell from context whether a speaker means “either one or the other or both,” or whether he means “either one or the other but not both.” (Some people use the awkward phrase “and/or” to describe the first option.)

Remember that in mathematical communication, though, we have to be very precise. We cannot rely on context or assumptions about what is implied or understood.

Definition

In mathematics, the word “or” always means “one or the other or both.”

The word “and” always means “both are true.”

Think / Pair / Share

For each sentence below:

- Decide if the choice $x = 3$ makes the statement true or false.
- Choose a different value of that makes the statement true (or say why that is not possible).
- Choose a different value of that makes the statement false (or say why that is not possible).

1. x is odd or x is even.
2. x is odd and x is even.
3. x is prime or x is odd.
4. $x > 5$ or $x < 5$.
5. $x > 5$ and $x < 5$.
6. $x + 1 = 7$ or $x - 1 = 7$.
7. $x \cdot 1 = x$ or $x \cdot 0 = x$.
8. $x \cdot 1 = x$ and $x \cdot 0 = x$.
9. $x \cdot 1 = x$ or $x \cdot 0 = 0$.

Quantifiers

Problem 23 (All About the Benjamins)

You are handed an envelope filled with money, and you are told “Every bill in this envelope is a $100 bill.”

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

Suppose you were given a different sentence: “There is a $100 bill in this envelope.”

- What would convince you beyond any doubt that the sentence is true? How could you convince someone else that the sentence is true?
- What would convince you beyond any doubt that the sentence is false? How could you convince someone else that the sentence is false?

Think / Pair / Share

What is the difference between the two sentences? How does that difference affect your method to decide if the statement is true or false?

Some mathematical statements have this form:

- “Every time...”
- “For all numbers. . . ”
- “For every choice. . . ”

- "It's always true that. . . "

These are *universal* statements. Such statements claim that something is always true, no matter what.

- To prove a universal statement is false, you must find an example where it fails. This is called a **counterexample** to the statement.
- To prove a universal statement is true, you must either check every single case, or you must find a logical reason why it would be true. (Sometimes the first option is impossible, because there might be infinitely many cases to check. You would never finish!)

Some mathematical statements have this form:

- "Sometimes..."
- "There is some number. . . "
- "For some choice. . . "
- "At least once..."

These are *existential* statements. Such statements claim there is some example where the statement is true, but it may not always be true.

- To prove an existential statement is true, you may just find the example where it works.
- To prove an existential statement is false, you must either show it fails in every single case, or you must find a logical reason why it cannot be true. (Sometimes the first option is impossible!)

? Think / Pair / Share

For each statement below, do the following:

- Decide if it is a universal statement or an existential statement. (This can be tricky because in some statements the quantifier is "hidden" in the meaning of the words.)
- Decide if the statement is true or false, and do your best to justify your decision.

1. Every odd number is prime.
2. Every prime number is odd.
3. For all positive numbers x, $x^3 > x$.
4. There is some number x such that $x^3 = x$.
5. The points (1, 1), (2, 1), and (3, 0) all lie on the same line.
6. Addition (of real numbers) is commutative.
7. Division (of real numbers) is commutative.

Look back over your work. you will probably find that some of your arguments are sound and convincing while others are less so. In some cases you may "know" the answer but be unable to justify it. That is okay for now! Divide your answers into four categories:

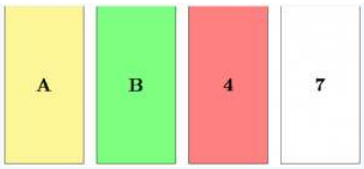
1. I am confident that the justification I gave is good.
2. I am not confident in the justification I gave.
3. I am confident that the justification I gave is not good, or I could not give a justification.
4. I could not decide if the statement was true or false.

Conditional Statements

✓ Problem 24 (Card Logic)

You have a deck of cards where each card has a letter on one side and a number on the other side. Your friend claims: "If a card has a vowel on one side, then it has an even number on the other side."

These cards are on a table.



Which cards must you flip over to be certain that your friend is telling the truth?

? Think / Pair / Share

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Try to come to agreement on an answer you both believe.

Here is another very similar problem, yet people seem to have an easier time solving this one:

✔ Problem 25 (IDs at a Party)

You are in charge of a party where there are young people. Some are drinking alcohol, others soft drinks. Some are old enough to drink alcohol legally, others are under age. You are responsible for ensuring that the drinking laws are not broken, so you have asked each person to put his or her photo ID on the table. At one table, there are four young people:

- One person has a can of beer, another has a bottle of Coke, but their IDs happen to be face down so you cannot see their ages.
- You can, however, see the IDs of the other two people. One is under the drinking age, the other is above it. They both have fizzy clear drinks in glasses, and you are not sure if they are drinking soda water or gin and tonic.

Which IDs and/or drinks do you need to check to make sure that no one is breaking the law?

? Think / Pair / Share

After you have thought about the problem on your own for a while, discuss your ideas with a partner. Do you agree on which cards you must check? Compare these two problems. Which question is easier and why?

Definition

A **conditional statement** can be written in the form

If some statement then some statement.

Where the first statement is the **hypothesis** and the second statement is the **conclusion**.

? Think / Pair / Share

These are each conditional statements, though they are not all stated in “if/then” form. Identify the hypothesis of each statement. (You may want to rewrite the sentence as an equivalent “if/then” statement.)

1. If the tomatoes are red, then they are ready to eat.
   The tomatoes are red. / The tomatoes are ready to eat.
2. An integer n is even if it is a multiple of 2.
   n is even. / n is a multiple of 2.
3. If n is odd, then n is prime.
   n is odd. / n is prime.
4. The team wins when JJ plays.
   The team wins. / JJ plays.

Remember that a mathematical statement must have a definite truth value. It is either true or false, with no gray area (even though we may not be sure which is the case). How can you tell if a conditional statement is true or false? Surely, it depends on whether

the hypothesis and the conclusion are true or false. But how, exactly, can you decide?

The key is to think of a conditional statement like a promise, and ask yourself: under what condition(s) will I have broken my promise?

Example 11.1.1:

Here is a conditional statement:

> *If I win the lottery, then I'll give each of my students $1,000.*

There are four things that can happen:

- **True hypothesis, true conclusion:** I do win the lottery, and I do give everyone in class $1,000. I kept my promise, so the conditional statement is TRUE.
- **True hypothesis, false conclusion:** I do win the lottery, but I decide not to give everyone in class $1,000. I broke my promise, so the conditional statement is FALSE.
- **False hypothesis, true conclusion:** I do not win the lottery, but I am exceedingly generous, so I go ahead and give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.
- **False hypothesis, false conclusion:** I do not win the lottery, so I do not give everyone in class $1,000. I did not break my promise! (Do you see why?) So the conditional statement is TRUE.

What can we conclude from this? **A conditional statement is false only when the hypothesis is true and the conclusion is false.** In every other instance, the promise (as it were) has not been broken. If a mathematical statement is not false, it must be true.

Example 11.1.2:

Here is another conditional statement:

> *If you live in Honolulu, then you live in Hawaii.*

Is this statement true or false? It seems like it should depend on who the pronoun "you" refers to, and whether that person lives in Honolulu or not. Let us think it through:

- Sookim lives in Honolulu, so the hypothesis is true. Since Honolulu is in Hawaii, she does live in Hawaii. The statement is true about Sookim, since both the hypothesis and conclusion are true.
- DeeDee lives in Los Angeles. The statement is true about DeeDee since the hypothesis is false.

So in fact it does not matter! The statement is true either way. The right way to understand such a statement is as a universal statement: "Everyone who lives in Honolulu lives in Hawaii."

This statement is true, and here is how you might justify it: "Pick a random person who lives in Honolulu. That person lives in Hawaii (since Honolulu is in Hawaii), so the statement is true for that person. I do not need to consider people who do not live in Honolulu. The statement is automatically true for those people, because the hypothesis is false!"

Example 11.1.3:

How do we show a (universal) conditional statement is false?

You need to give a specific instance where the hypothesis is true and the conclusion is false. For example:

> *If you are a good swimmer, then you are a good surfer.*

Do you know someone for whom the hypothesis is true (that person is a good swimmer) but the conclusion is false (the person is not a good surfer)? Then the statement is false!

Think / Pair / Share

For each conditional statement, decide if it is true or false. Justify your answer.

1. If $2 \times 2 = 4$ then $1 + 1 = 3$.
2. If $2 \times 2 = 5$ then $1 + 1 = 3$.
3. If $\pi > 3$ then all odd numbers are prime.
4. If $\pi < 3$ then all odd numbers are prime.
5. If a number has a 4 in the one's place, then the number is even.
6. If a number is even, then the number has a 4 in the one's place.
7. If the product of two numbers is 0, then one of the numbers is 0.
8. If the sum of two numbers is 0, then one of the numbers is 0.

Think / Pair / Share (Two truths and a lie)

On your own, come up with two conditional statements that are true and one that is false. Share your three statements with a partner, but do not say which are true and which is false. See if your partner can figure it out!

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11.2: The Basics of Sets

You will need: ***Coins*** *(Material Card 1),* ***A–Blocks*** *(Material Cards 2A–2E)*

? Exercise 1

Have you ever collected anything? If so, what?

? Exercise 2

Name some things people might collect, including anything people in your group collect.

? Exercise 3

Suppose someone collected 25 coins from the years 1964 –1969.

a. What different types of coins might be in their collection?

b. Name one way you might sort the coins into groups.

c. Can you think of a different way to sort them into groups? Explain.

d. How do you think a child would sort them?

Let's assume these are the 25 coins that were collected:

1966 penny, 1967 nickel, 1966 quarter, 1967 penny, 1965 penny, 1966 half dollar, 1967 quarter, 1965 dime, 1967 dime, 1968 quarter, 1964 dime, 1966 nickel, 1965 nickel, 1967 half dollar, 1966 dime, 1964 nickel, 1969 quarter, 1969 half dollar, 1965 half dollar, 1968 penny, 1968 dime, 1964 quarter, 1965 quarter, 1969 dime, 1968 nickel

To simplify writing each coin out, let's abbreviate 1966 penny by $6P$, and 1967 nickel by $7N$, etc. So in our collection, we have the following:

$6P, 7N, 6Q, 7P, 5P, 6H, 7Q, 5D, 7D, 8Q, 4D, 6N, 5N, 7H, 6D, 4N, 9Q, 9H, 5H, 8P, 8D, 4Q, 5Q, 9D$ and $8N$

A physical model for these coins is found on Material Card 1. If you haven't already done so, cut out a set of coins from this Material Card and use them to do several of the following exercises.

? Exercise 4

Sort the coins into five groups; pennies, nickels, dimes, quarters and half dollars. Using the abbreviations on each coin, list the coins that go into each group (such as $6P, 7N, 6Q, 7P, 5P$, etc. each coin must be listed with **a number *and* a letter**. You can't just write a letter or a number, like 6 or 5 or P. Each coin is made up of one of each! It's like writing the initials of a first and last name instead of the full name.)

a. pennies: ___ (did you list 4 pennies?)

b. dimes: ___ (did you list 6 dimes?)

c. half dollars: ___ (did you list 4 of them?)

d. Now sort the coins into groups by year. How many groups are there? List two different groups and which coins go into each group. Remember to list each coin individually each of the 25 coins has a digit followed by a letter.

The collection of coins we have been working with is called a ***set***. A ***set*** is simply a collection of objects. That is the definition of a set (without the word "simply").

Exercise 5

Define ***set***: A ***set*** is

In exercises e and f, you grouped the ***elements*** or ***members*** of the set into *different* ***subsets***. The ***elements*** of a set are the objects in the set. In this case, the ***elements*** of the set are the 25 different coins.

Exercise 6

Define ***elements***: The ***elements*** of a set are

A ***subset*** is also a set. When we talk about ***subsets***, it is in reference to another set. For example, the set of pennies in the collection above is a ***subset*** of the collection of coins.

The following is a formal definition of ***subset***:

A set ***B*** is a ***subset*** of a set ***C*** if every element of set ***B*** is an element of set ***C***.

To show that one set is ***not*** *a subset of another set,* ***there would have to be an element in the first set that is*** **not** ***in the second set.***

There is one thing about sets that confuses a lot of people. By definition, we say that a set is a collection of objects. But in reality, a set can have no elements in it. A set containing no elements is called the ***empty set*** or ***null set***. Look at the definition of a subset very carefully. Is the null set is a subset of every set?

I hope this was your conclusion: ***The null set is a subset of every set.***

Exercise 7

Define ***null set*** : ___

Note that in my definition of a subset on the previous page, I used the letters B and C for my sets. I did not have to use these letters. They are just dummy variables much like when we use x, y and n in algebra!! Here is another perfectly fine way to define subset: A set ***R*** is a ***subset*** of a set ***Q*** if every element of set ***R*** is an element of set ***Q***.

Exercise 8

Use different letters than just used to define ***subset***:

We usually use a lot of abbreviations in math so that it isn't so cumbersome writing everything out. Many times, we pick a capital letter to represent a set or subset and *define* the set by that letter. I will define some sets now: The set of 25 coins we've been working with will be defined as **C**. The subsets of pennies, nickels, dimes, quarters and half dollars will be denoted by **P**, **N**, **D**, **Q** and **H**, respectively. Notice I gave them meaningful letters so it is easy to remember which letter stands for which subset.

One way to describe a set is to list its objects, if possible. (Later, we will discuss why this is not always practical or possible.) When we list the objects, we do so *inside of braces using commas to separate the elements*. The order in which the elements are listed is irrelevant. Using the listing method, one way to write our coin collection, set **C**, is

$$\mathbf{C} = \{6P, 7N, 6Q, 7P, 5P, 6H, 7Q, 5D, 7D, 8Q, 4D, 6N, 5N, 7H, 6D, 4N, 9Q, 9H, 5H, 8P, 8D, 4Q, 5Q, 9D, 8N\}$$

Exercise 9

Let **S** be the subset of coins from 1964, **V** from 1965, **W** from 1966, **X** from 1967, **Y** from 1968, **Z** from 1969 and **T** from 1970. Are **P**, **N**, **D**, **Q**, **H**, **S**, **V**, **W**, **X**, **Y**, **Z** and **T** all subsets of **C**? ___ Using the correct notation, write out these 12 sets using the listing method by listing the elements in each set. *The only elements in* ***C*** *are those listed above this exercise. Note that 6P is one element in* ***P*** *because 6P is a particular penny in* ***C*** *but 6 is not an element and P is not an element.* The difference between how you write the answers here and how you did it in exercise 4 is that you are using proper notation with the braces, and commas in between the elements.

N = ___	W = ___
Q = ___	Y = ___
S = ___	T = ___

The way to express that an element, like 4D, is in set **C** is with the symbol which means “is an element of”. We write: 4D (read "4D is an element of **C**").

Let **A** represent the one dollar coins in our set **C**. Notice that set **A** is empty. We can show that **A** contains no elements by writing **A = {∅}**. The null or empty set is also written ∅. So, we can also write **A** = ∅. In fact, **{ }** = ∅ always. It is correct to write the null or empty set either way. But note that it is ***not correct*** to write the null set like this: {∅}. That would be a set that ***does*** contain at least one element the null set! See how this can be a bit confusing?

Now to some questions about subsets...

Exercise 10

a. Is there *any* element in **N** that is not in **D**? ___ If so, name *one*: ___

b. From your work in part a, answer this question: Is **N** a subset of **D**? ___

c. Is there *any* element in **A** that is not in **C**? ___ If so, name *one*: ___

d. From your work in part a, answer this question: Is **A** a subset of **C** ? ___

The symbol ⊆ means "is a subset of" and so the way to express that **P** is a **subset** of **C** is by writing . Since **N** is also a subset of **C**, we can write **N** ⊆ **C**. If the first set has **fewer** elements than the second set, then the first set is a **proper subset** of the second set. In this case, the symbol ⊆ may be used in place of ⊆. So, in the case of the coins, it is also correct to write P ⊂ C and N ⊂ C. A first set is **not** a subset of a second set if there is *at least one element in the first set* that is *not in the second set*. Think again about why the null set has to be a subset of every set. For it not to be a subset, there would have to be *at least one element in it* that wasn't in the second set. But, the null set is empty!

Exercise 11

P is not a subset of **N**. To show that **P** is not a subset of **N**, we must name at least one element in **P** that is not in **N**. Name an element in **P** that is not in **N**: ___

Think about and discuss the subtle differences between the following statements: $6P \in C$, $\{6P\} \subset C$, $D \subseteq D$ and $P \subset C$ are correct whereas $6P \subset C$, $\{6P\} \in C$, $D \subset D$ and $P \in C$ are **not** correct!

Exercise 12

Decide which of the following statements are true and which ones are false:

a. _____ $\varnothing \subset A$	b. _____ $\varnothing \subseteq \varnothing$	c. _____ $S \subseteq C$
d. _____ $C \subseteq C$	e. _____ $B \subset B.$	f. _____ $7P \in P$
g. _____ $T \subseteq \varnothing$	h. _____ $9D \in Z.$	i. _____ $5H \in N$

Exercise 13

Make up three true statements and three false statements using the newly defined subset and element symbols. Use the null set in at least one true and one false statement.

True Statement	False Statement

Something else to consider when working with sets is how many elements are in a given set. If we count the number of elements in a set, we call that number the **cardinality** or **cardinal number** of that set. To denote the cardinality of set **C**, we write $n(\mathbf{C}) = 25$. Think of the *n* as the ***number** of objects in the set*. Do you remember any functional notation from algebra like $f(x) = 2x + 3$ implies $f(5) = 2(5) + 3 = 10 + 3 = 13$ – that is, when you plug 5 into the function f, out pops 13? Well, when working with the cardinality of a set, it is the same kind of thing and we use similar notation. In this case, when you plug **C** into the function n, out pops the number 25!!! When you are asked about the cardinality of a set, the answer is a number. We say that two sets are **equivalent** if they each have the *same **number** of elements* which is to say they have the *same cardinality*. We use the symbol, ~, to denote equivalence. For instance, since **P** and **H** each contain four elements, we can write **P** ~ **H**. Two sets are **equal** if they have the ***exact same elements***.

? Exercise 14

Compute the following, using the information from exercise 9.

a. n(**P**) = ____ b. n(**N**) = ____ c. n(**Q**) = ____ d. n(**D**) = ____

e. n(**Z**) = ____ f. n(**H**) = ____ g. n(**Y**) = ____ h. n(**T**) = ____

i. n(**S**) = ____ j. n(**W**) = ____ k. n(**X**) = ____ l. n(**V**) = ____

m. n(**C**) = ____ n. n(**A**) = ____ (**A** represents the one dollar coins in our set **C**)

o. Consider the subsets of **C** we have defined so far. Using the equivalence symbol, express which pairs of subsets are equivalent to each other.

What if you were asked which coins were pennies and ***also*** from the year 1965? In set **C**, there is only one coin that satisfies both criteria and that is $5P$. When trying to find the elements that are common to two sets, in this case, **P** (pennies) and **V** (coins from 1965), we use the word **intersection**. A formal definition of **intersection** follows:

The **intersection** of two sets, **A** and **B**, written $A \cap B$, is the set of elements which are found in both **A** and **B**.

Using the correct notation, we could now write $P \cap V = \{5P\}$

? Exercise 15

Complete the following, using correct notation:

a. = ____ X ∩ Y	b. = ____ H ∩ Z	c. = ___ N ∩ S
d. = ____ Q ∩ W	e. = ____ D ∩ P	

? Exercise 16

Two sets are ***disjoint*** if their intersection is empty. For instance, **D** and **Q** are disjoint because $D \cap Q = \varnothing$. Name three other pairs of sets that are disjoint.

? Exercise 17

Define ***disjoint*** sets: ___

? Exercise 18

What if you were asked which coins were either pennies **or** from the year 1965? List which coins would satisfy one or both of these conditions:

Did you get eight different coins this time? ___ Any coin having a 5 or P or both would do it. You just found the ***union*** of the two sets **P** and **V**. A formal definition of union follows:

The ***union*** of two sets, **A** and **B**, written $A \cup B$, is the set of elements that are *either* in **A** *or* **B** *or* **both**.

Using correct notation, we write P $\cup = 6P, 7P, 5P, 5D, 5N, 5H, 8P, 5Q$Remember that the order in which the elements are written is irrelevant! ***IMPORTANT:*** *If an element is in both sets, it is only written down* ***once*** *in the union.*

Exercise 19

Complete the following, using correct notation:

a. N ∪ S = ____
b. Q ∪ W = ____
c. D ∪ P = ____
d. X ∪ Y = _____

Exercise 20

Write the definition for ***union*** and then the definition for ***intersection***. Use different letters than A and B.

Exercise 21

Make up one new problem involving the coins using the intersection symbol and one new problem using the union symbol. Write the solutions to the problems.

Exercise 22

Make up one true and one false statement involving the coins using the intersection and union symbols. State which statement is true and which is false.

Put aside the coin set for now. Let's work with another set. When working with sets, we need to be clear about what we are talking about. Here is another definition for you.

Mathematicians often refer to the set under consideration as the ***universe*** or ***universal set,*** and usually use the letter **U** to represent the universe.

In our coin problem, the ***universe*** wasn't specified. It could have been the *25* coins in the collection, all of the coins made between 1960 and 1970 or maybe all the coins in the world. Without knowing the universe, there are some things we couldn't answer in set theory. Now that we have that out of the way, here are two more important definitions.

The ***complement*** of a set **A**, written $\mathbf{A}^c$ or $\bar{A}$ or **A',** is the set of elements in the universal set that are not in **A**. We read $\mathbf{A}^c$ as "**A complement"** or "**not *A".*** *The universe must be specified in order to compute the complement.* For example, if **U**={1, 2, 3, 4, 5} and **H** = {3, 4}, then H' = {1, 2, 5).

The ***difference*** of two sets **A** and **B**, written **A – B (or A – B) ,** is the set of elements of **A** that are ***not also*** in **B**. (Some people list the elements of **A** and then cross off any that are in **B** to get the answer for **A – B** . So if **F**= {1, 2, 3 ,4} and **G** = {1, 3, 5, 7}, then to find **F – G**, list the elements of **F** and cross off any that are in G: {~~1~~,2,~~3~~,4}. The answer is: {2, 4)

Exercise 23

Try the following problems in which the universe is the first 9 counting numbers. If U represents the universe, we have U = {1, 2, 3, 4, 5, 6, 7, 8, 9} Three subsets of U are defined as follows: **A** = {1, 2, 3, 4, 5} **B** = {2, 4, 6, 8} **C** = {3, 5, 7}

a. **A** ∩ **B** = _____	b. **A** ∩ **C** = ____
c. **B** ∩ **C** = _____	d. **A** ∪ **B** = ____
e. **A** ∪ **C** = _____	f. **B** ∪ **C** = ____
g. $\mathbf{A}^c$ = _____	h. $\mathbf{B}^c$ = ____

i. $\mathbf{C}^c$ = _____	j. **A** – **B** = _____
k. **B** – **A** = ____	1. **A** – **C** = _____
m. **C** – **A** = ____	n. **B** – **C** = _____
o. **C** – **B** = _____	

*Did you notice that the order matters **for** difference, but not for union or intersection?*

Continuation of exercise 23 where **U** = {1, 2, 3, 4, 5, 6, 7, 8, 9}. The subsets are:
A= {1, 2, 3, 4, 5} **B** = {2, 4, 6, 8} **C** = {3, 5, 7}

Try these "more involved" problems. These will take more than one step to do. You might need scratch paper.

p. (**A** ∪ **C**) – **B** = ____	q. (**A** – **C**)^c = ____
r. (**A** ∩ **B**) ∪ **C** = ____	s. (**A** ∪ **C**) ∩ **B** = ____
t. (**A** ∪ **C**)^c = ____	u. $\mathbf{A}^c \cap \mathbf{B}^c$ = ____
v. $A \cap \bar{B}$ = ____	w. $A \cup \bar{B}$ = ____
x. $\mathbf{A}^c \cup (\mathbf{B}^c \cap \mathbf{C}^c)$ = _____	

Answer true or false for the following:

y. ∅ ⊂ **B** ___	z. (**A** – **C**) ~ (**C** – **B**) ___
aa. 5 ∈ (**C**– (**A** ∪ **B**)) ___	bb. n(**A** ∪ **B**) = 9 ___
cc. n(**A**) + n(**B**) = 9 ___	dd. (**A** ∪ **B**)^c = $\mathbf{A}^c \cup \mathbf{B}^c$ ___
ee. (**A** ∪ **B**)^c = $\mathbf{A}^c \cap \mathbf{B}^c$ ___	ff. **A** – **B** = **A** – (**A** ∪ **B**) ___
gg. **A** and **B** are disjoint ___	hh. **B** and **C** are disjoint ___
ii. U – **A** = $\mathbf{A}^c$ ___	jj. {3, 4, 5, 6, 8} ⊂ {4, 5, 6, 7, 8, 9} __

Make sure you can work through all of the problems above perfectly on your own. If you have trouble understanding any of them, check with your classmates or ask your teacher if something still needs clarifying!!! Then, make up some more problems of your own as instructed below before going on.

Make up four of your own "more involved problems" and show the solutions. Use the same sets (**U**, **A**, **B** and **C**) as in exercise 23.

1. ___
2. ___
3. ___
4. ___

? Exercise 24

Now it's time for you to make up some original problems on your own and to provide the solutions. First, decide on a universal set, U, and define three subsets, D, E and F. Then, make up two original and nontrivial (thoughtful and involving two or more steps) problems using combinations of union, complement, intersection and difference

Remember when I mentioned that the listing method wasn't always practical or possible? Consider a universal set that contained all of the whole numbers. These could not be listed. Another way of expressing this set would be by description. You

could say **U** = (the whole numbers) This is a very formal way of writing this same set **U** = {x|x is a whole number} which is read "**U** is the set of all x such that x is a whole number". You may have seen this notation in algebra. We won't be so formal here. Another way to get around the listing problem is to use three consecutive dots which means "and so on". We would write **U** = {0,1,2,3,...}. Here is a way to express all the even numbers between 13 and 509: {14,16,18,...,508}. You could list them out (if you didn't have a life!), but you can see the advantage of using three dots!!!! Using the three dots doesn't always work either. Consider the set of all real numbers. They cannot be listed. You would have to write something like this: {all real numbers) or {x|x is a real number}.

? Exercise 25

Use correct set notation to express the following sets in one or more ways:

a. the letters of the alphabet

b. the even integers between and including 124 and 698

c. the whole numbers strictly in between 100 and 1000

d. the names of the past presidents of the United States up until 1995

e. the names of the presidents of the U.S. from 1981 - 1995

It's now time get out your **A-blocks**. They can be found on Material Cards 2A - 2E. The white cards are value label cards (which you won't use until Exercise Set 3), and the colored objects are the actual A-blocks. There are 24 objects in each set of A-blocks.

We will use the following abbreviations when referring to each of the 24 elements.

S = small	**L** = large	**Y** = yellow	**R** = red	**B** = blue	**G** = green	**Q** = square	**T** = triangle	**C** = circle

We'll use three letters to specify each object in this order - size, color, shape. The small blue square would be denoted by SBQ and the large red circle would be denoted by LRC. These abbreviations are on the objects that you cut out.

Let's define our universe of A-blocks by **A**.

? Exercise 26

Play with the A-blocks a little bit. How do you think children might play with them?

? Exercise 27

Arrange the A-blocks into some piles of subsets. What kinds of piles did you make and how many were in each pile?

? Exercise 28

Divide them up differently. This time, what kinds of piles did you make and how many were in each pile?

? Exercise 29

a. Select a shape and take all the blocks of that shape and put them on a blank piece of paper. Close your eyes and have a friend remove one of objects. Look at the remaining pieces and determine which piece was removed. Think about how you figured it out. How do you think a child would figure it out?

b. Repeat part (a), but this time select a color and put all the blocks of that color on a blank sheet of paper.

c. Repeat part (a), but this time select a size and put all the blocks of that size on a blank sheet of paper.

d. Which was the fastest for you to figure out — shape, color or size? Why do you think that was so?

e. **Extra Credit:** Work this activity with 2 young children. One should be very young (you decide) and the other several years older. Document your findings and share on the Forum along with a picture or video. Compare how they figured out the missing piece to how you did. Be sure to do the activity yourself first.

Exercise 30

In this exercise, parts a through f are not related to each other. Start with the whole set of A-Blocks for each part and then proceed to follow the directions each time.

a. Make subsets of the blocks, dividing them by shape (disregard size)

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets of shapes. State which shape you picked: _____

Using correct set notation, and abbreviations, list its elements

b. Make subsets of the blocks, dividing them by color.

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets you made. State which color you picked: _____

Using correct set notation, and abbreviations, list its elements

c. Make subsets of the blocks, dividing them by size.

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets you made. State which size you picked: _____

Using correct set notation, and abbreviations, list its elements

d. Make subsets of the blocks, dividing them by color shape.

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets you made. State which color and shape you picked: _____

Using correct set notation, and abbreviations, list its elements

e. Make subsets of the blocks, dividing them by color and size.

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets you made. State which color and size you picked: _____

Using correct set notation, and abbreviations, list its elements

f. Make subsets of the blocks, dividing them by size and shape.

How many subsets are there? _____ How many blocks are in each subset? _____

Pick one of the subsets you made. State which size and shape you picked: _____

Using correct set notation, and abbreviations, list its elements

A particular **shape**, **color** or **size** is called a **value**. Let each of the values of A-blocks be defined in terms of subsets **S**, **L**,**R**, **B**, **G**, **Y**, **Q**, **T**, **C** will define small, large, red, blue, green, yellow, square, triangle and circle, respectively.

Exercise 31

a. Have a friend think of two values (like blue and square, for instance). Then have your friend put all the blocks having either of these values on a blank paper, but intentionally leaving one out. Try to figure out which block is missing. What do you have to figure out first?

b. Which values were chosen? ____ Which piece was missing?

Do this exercise two more times, with other values. Try it with all the blocks hidden one time and with all of them in clear sight another time.

c. Is it easier if all the blocks, including the missing block, are in plain view somewhere or if the missing block as well as all the rest are hidden?

d. Try this game by having your friend choose three values. Is this easier or harder? Try it more than once. List the values chosen and the piece missing for one of the games.

e. When making a subset that has either of two values as in part *a*, you are forming the ____ of those two sets. Here's a hint, *it isn't* complement. Using correct set notation, write the set formed by the union of the two values chosen in part a. _____ ∪ _____ = _____

Exercise 32

Choose a value (a particular shape, color or size). Put all the blocks with that value in one pile. Pick a different value and put all the blocks with that value in a separate pile.

a. Which values were chosen? ____

b. Were there any pieces that needed to go in both piles? ____ If so, which pieces?

c. Suppose the values chosen were red and blue. Were there any blocks that needed to go in both piles? ____ If so, which ones?

d. Suppose yellow and large were chosen. Were there any blocks that needed to go in both piles? ____ If so, which ones? ____

e. Elements belonging in both piles must be both yellow AND large. When making a subset where the objects must have both values, you are forming the ____ of those two sets, Using set notation, we write $\mathbf{Y} \cap \mathbf{L}$ =____

Put your A-blocks away for now. Don't lose them!! Before we continue working with the A-blocks, we'll need to learn about and work with Venn diagrams. We will then come back and do more work with our A-blocks.

Exercise 33

Let's consider a situation where a mother, Mary, has three children –Alicia, Brent and Carlos. Mary has red hair. The question we will attempt to answer is this: What possibilities exist regarding which, if any of her children, have red hair? List the possibilities you came up with:

Let's work through the above problem. First of all, let's break the problem down a little bit. It's possible none of her children have red hair, only one has red hair, two out of three have red hair or all three of them ended up as redheads. Within each of these four situations, let's write down each of the possibilities. We will list each possibility as a set, using the childrens' names as the possible elements.

Situation 1: None of the children are redheads: { }

Situation 2: One child is a redhead. This gives us the three possibilities listed below:
{Alicia} {Brent} {Carlos}

Situation 3: Two out of three have red hair. You can also think of this as only one child not having red hair. In any case, that gives us the three possibilities below: {Brent, Carlos} {Alicia, Carlos} {Alicia, Brent}
Situation 4: All three children have red hair: {Alicia, Brent, Carlos}

We came up with eight different possibilities.

The above problem really amounts to a set theory question. Given a set M, where M = {A, B ,C}, list all possible subsets of M. First of all, remember that the null set is a subset of every set and the entire set is a subset of itself (it's just not a proper subset). You did remember those two points, didn't you?

Let's begin by listing the possibilities according to how many elements might be in the subsets either no elements, one element, two elements or three elements. For each of these situations, list all possibilities:

No elements: { }	Two elements: {A, B}, {A, C}, {B, C}
One element: {A}, {B}, {C}	Three elements: {A, B, C}

Here is a list of all possible subsets: { }, {A}, {B}, {C}, {A, B}, {A, C}, {B, C}, {A, B, C}

WOW! Did you see how we came up with the same eight possibilities? This set theory stuff is just amazing, isn't it?

? Exercise 34

List all possible subsets for each of the sets given.

a. ∅: ___
b. {P}: ___
c. {G, F}: ___
d. {X, Y, Z}: ___
e. {1, 2, 3, 4}: ___

? Exercise 35

Using the information you gained from exercise 34, answer the following question.
How many subsets are there for a set containing the given number of elements?

a. no elements: ___
b. one element: ___
c. two elements: ___
d. three elements: ___
e. four elements: ___
f. five elements: ___
g. n elements (use a formula): ___

There is one more operation we will define on sets. So far, you've worked with union, intersection, difference and complement. Now we will learn what it means to take the Cartesian product of two sets, A and B.

Cartesian product

The **Cartesian product** of set **A** with set **B**, which is written **A** $\times$ **B** and is read as '**A** cross **B**" is the set of all possible ordered pairs (a,b), where a $\in$ **A** and b $\in$ **B**.

If you are finding the Cartesian product, the answer is a set that contains ordered pairs. The order matters! The first element must come from the set written to the left of the $\times$ and the second element must come from the set written to the right of the $\times$.

Example 1

If A = {x, y, z} and B = {a, b}, find A $\times$ B and B $\times$ A.

Solution

A $\times$ B = {(x, a), (x, b), (y, a), (y, b), (z, a), (z, b)}
B $\times$ A = {(a, x), (b, x), (a, y), (b, y), (a, z), (b, z)}
Notice there were 3 elements in one set and two in the other. The Cartesian product has six elements in it, six ordered pairs.

Example 2

If E = {1, 2} and F = {2,3}, find E $\times$ F, F $\times$ E, E $\times$ E, and F $\times$ F

Solution

E $\times$ F= {(1, 2), (1.3), (2, 2), (2, 3)}	E $\times$ E = {(1, 1), (1, 2), (2, 1), (2, 2)}
F $\times$ E = {(2, 1), (3, 1), (2, 2), (3, 2)}	F $\times$ F = {(2, 2), (2, 3), (3, 2), (3, 3)}

Notice there were two elements in one set and two in the other. Each Cartesian product has four elements in it – four ordered pairs.

Example 3

Find {4, 5} $\times$ {7, 2, x, #}.

Solution

{4, 5} $\times$ {7, 2, x, #} = {(4, 7), (4, 2), (4, x), (4, #), (5, 7), (5, 2), (5, x), (5, #)}

Notice there were two elements in one set and four in the other. The Cartesian product has eight elements in it – eight ordered pairs.

In Example I, one set contained 3 elements and the other contained 2 elements, The Cartesian product contained 6 elements – 6 ordered pairs. In Example 2, one set contained 2 elements and the other contained 2 elements, The Cartesian product contained 4 elements, 4 ordered pairs. In Example 3, one set contained 2 elements and the other contained 4 elements. The Cartesian product contained 8 elements , 8 ordered pairs.

Exercise 36

If a set **B** contains 5 elements (in other words, n(**B**) = 5) and a set **C** contains 7 elements (or n(**C**) = 7), then how many elements (where each element is an ordered pair) are in the Cartesian product **B** $\times$ **C**?

Exercise 37

If a set **B** contains 5 elements (in other words, n(**B**) = 5) and a set **C** contains 7 elements (or n(**C**) = 7), then how many elements (where each element is an ordered pair) are in the Cartesian product **C** $\times$ **B**?

Exercise 38

In general, for any two sets, F and G, is $n(F \times G) = n(G \times F)$?

Exercise 39

Assume a set B contains b elements (in other words, $n(B) = b$) and a set C contains c elements (or $n(C) = c$). Use this information to compute the following, where you are asked how many elements (where each element is an ordered pair) are in a given Cartesian product.

a. $n(B \times C) =$ ____	b. $n(B \times B) =$ ____	c. $n(C \times C) =$ ____

Exercise 40

In general, if you take any two sets, A and B, is $A \times B = B \times A$? ____ If you answered yes, provide an example of two different sets, A and B, and show that $A \times B = B \times A$. If you answered no, provide an example of two different sets, A and B, and show that $A \times B \neq B \times A$.

Example 4

This one is trickier: Find $\{(4, 3), 5) \times \{(3, 3),\{4, 7, 5, 2\}, 1\}$

Okay, take a breath...you can do this. Look carefully at the first set; it has two elements, the first element happens to be the ordered pair (4, 3) and the second element is the number
5. Look carefully at the second set; it has three elements, the first shown happens to be the ordered pair (3,3), the second happens to be set containing four elements, and the third is the element 1, So the first set contains 2 elements and the second set contains 3 elements. First, think about how many elements are in the Cartesian product. Did you get the idea to multiply 2 by 3? I hope so! There are six elements in the Cartesian product. Each of those elements is an ordered pair, and some of these ordered pairs will look kind of weird. See if you can understand the solution written below.

Solution

$\{(4, 3), 5\} \times \{(3, 3), \{4, 7, 5, 2\}, 1\} = \{((4, 3), (3, 3)), ((4, 3), \{4, 7, 5, 2\}), ((4, 3), 1), (5, (3 , 3)), (5, \{4, 7, 5, 2\}), (5, 1)\}$

Look at this solution carefully. There are six ordered pairs in the solution. The first three ordered pairs have (4, 3) as its first coordinate and the second three ordered pairs have 5 as its first coordinate. Then (3, 3) is the second coordinate for the first and fourth ordered pair, {4, 7, 5, 2} is the second coordinate for the second and fifth ordered pair and 1 is the second coordinate for the third and sixth coordinate. If you can follow this one, you should do great on the next exercise.

Exercise 41

Write the Cartesian product. Each answer is a set containing ordered pairs. Before doing each problem, think about how many ordered pairs will be in the answer. Make sure you write your answer using correct notation, put the ordered pairs in a set. There is a comma between each coordinate in each ordered pair and there is a comma between each element in the set. Use braces { } around the set. One of the answers is the null set!

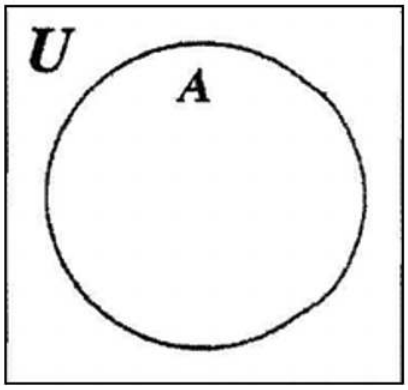
a. $\{3, 4\} \times \{2, 6\} =$ ____
b. $\{6, 7, 8, 9\} \times \{5\} =$ ____
c. $\{r, s, t\} \times \{\ \} =$ ____
d. $\{a\} \times \{a\} =$ ____
e. $\{x, y\} \times \{x, y\} =$ ____
f. $\{1, 3, 5\} \times \{1, 3, 5\} =$ ____
g. $\{(9, 4), C)\} \times \{D, \{a, b, c\}\} =$ ___

h. $\{\{5, 6, 7, 8, 9\}\} \times \{g, \{4, 3\}\} =$ ___

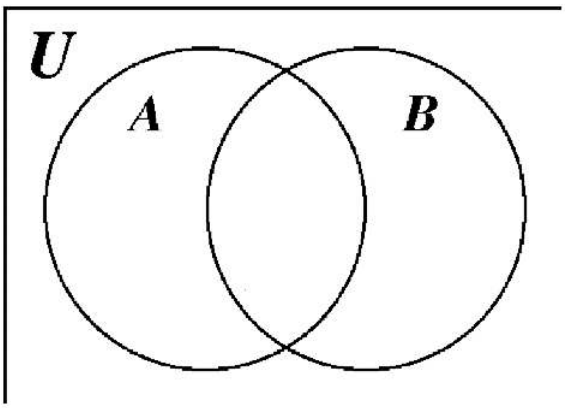
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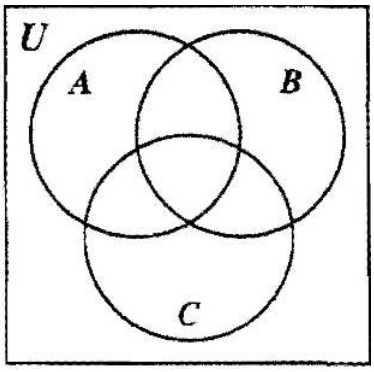
11.3: Venn Diagrams



This is a Venn diagram using only one set,
A



This is a Venn diagram Below using two sets,
A
and
B
.

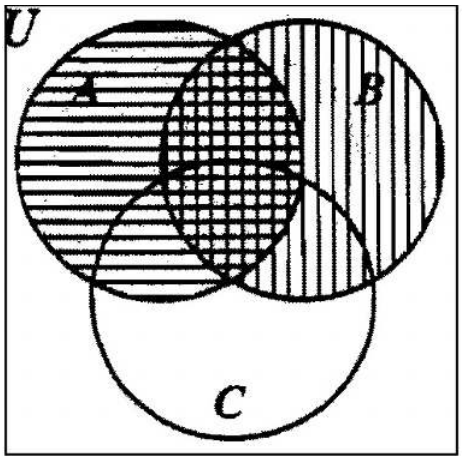


This is a Venn diagram using sets **A**, **B** and **C**.

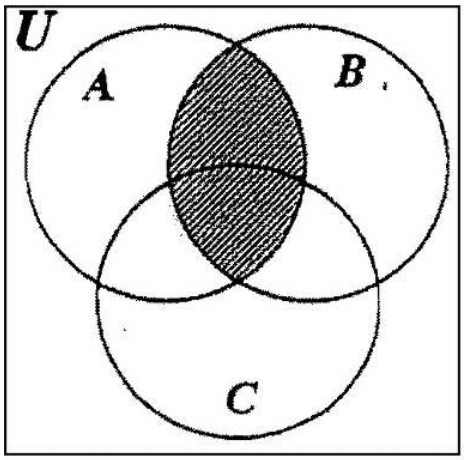
Study the Venn diagrams on this and the following pages. It takes a whole lot of practice to shade or identify regions of Venn diagrams. Be advised that it may be necessary to shade several practice diagrams along the way before you get to the final result.

We shade Venn diagrams to represent sets. We will be doing some very easy, basic Venn diagrams as well as several involved and complicated Venn diagrams.

To find the intersection of two sets, you might try shading one region in a given direction, and another region in a different direction. Then you would look where those shadings overlap. That overlap would be the intersection.



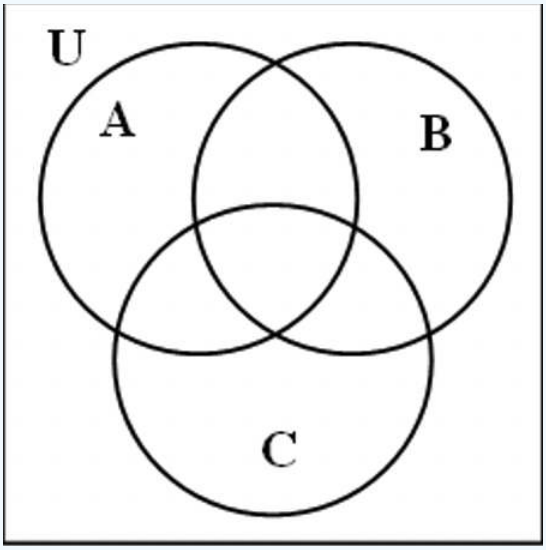
For example, to visualize $A \cap B$, shade A with horizontal lines and B with vertical lines. Then the overlap is $A \cap B$. The diagram on the left would be a first step in getting the answer. The shaded part on the diagram to the right shows the final answer.



Here are two problems for you to try. Only shade in the final answer for each exercise.

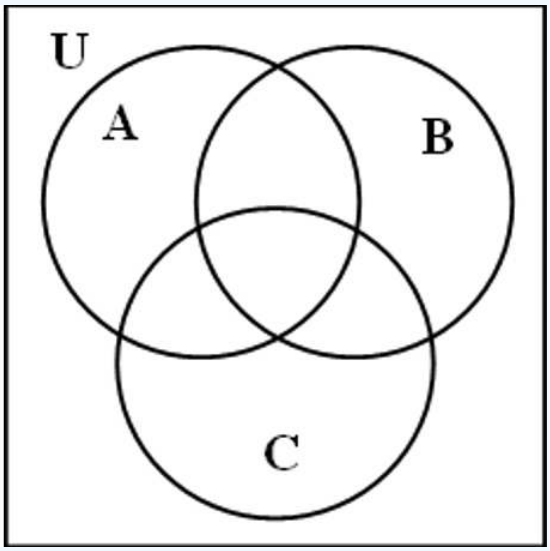
? Exercise 1

Shade the region that represents $A \cap C$

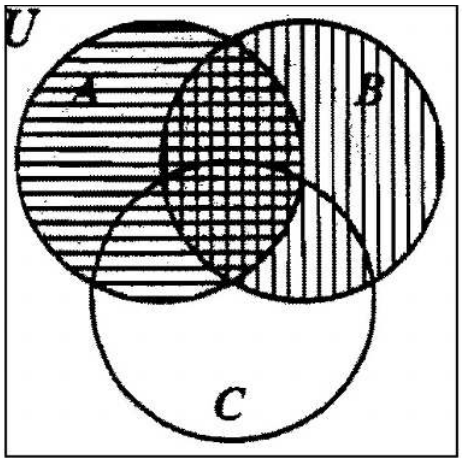


Exercise 2

Shade the region that represents $B \cap C$



To shade the union of two sets, shade each region completely or shade both regions in the same direction. Thus, to find the union of **A** and **B**, shade all of **A** and all of **B**.

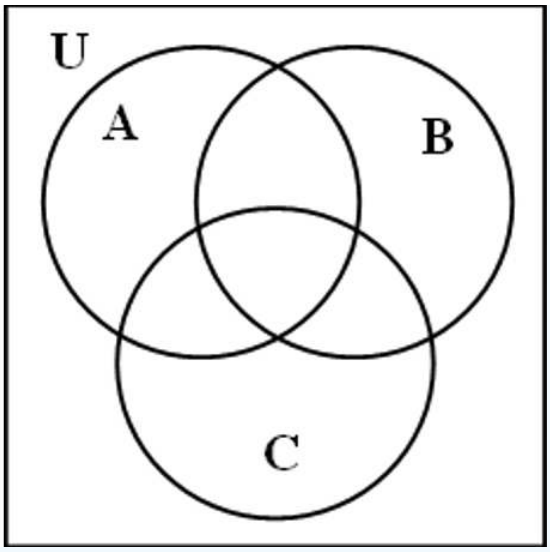


The final answer is represented by the shaded area in the diagram to the right.

Here are two problems for you to try. Only shade in the final answer for each exercise.

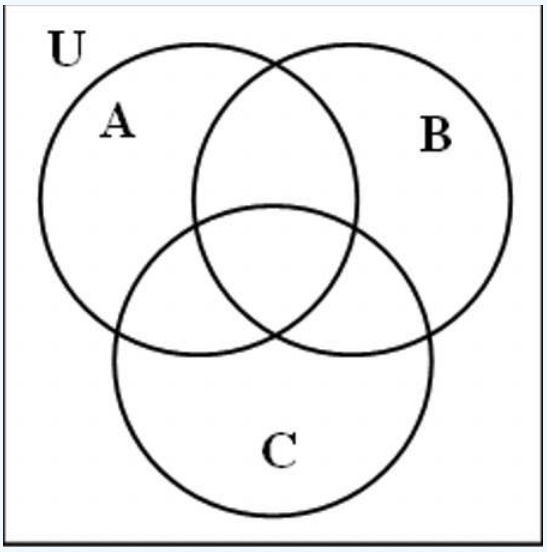
Exercise 3

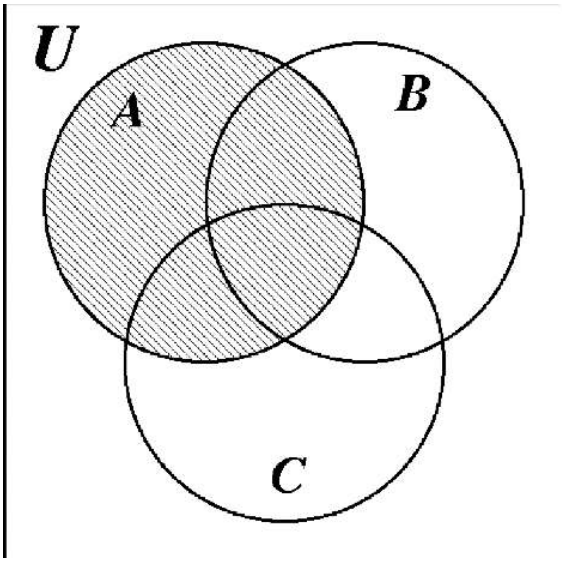
Shade the region that represents $A \cup C$



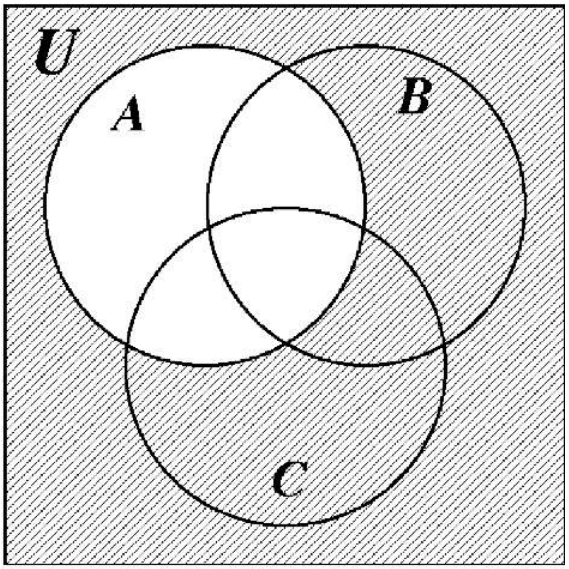
Exercise 4

Shade the region that represents $B \cup C$

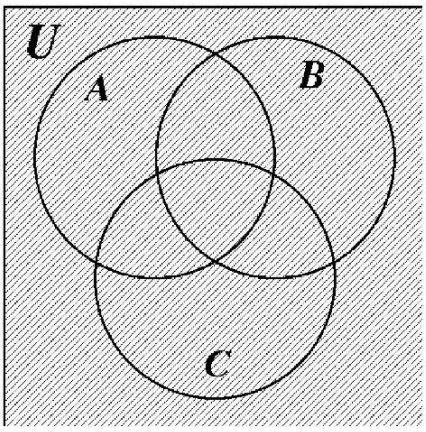




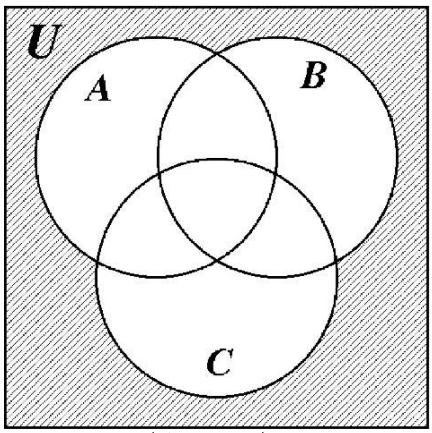
For the complement of a region, shade everything outside the given region. You can think of it as shading everything except that region. On the Venn diagram to the left, the shaded area represents **A**. On the Venn diagram to the right, the shaded area represents .



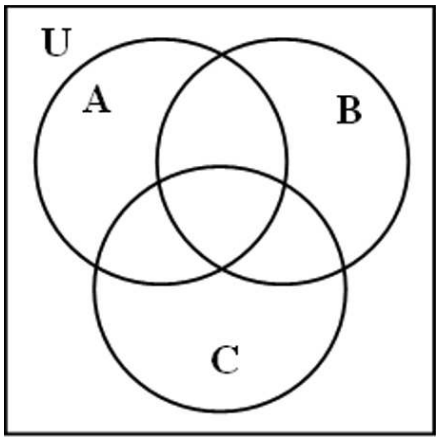
Many people are confused about what part of the Venn diagram represents the universe, **U**. The universe is the ***entire*** Venn diagram, including the sets **A**, **B** and **C**. The three Venn diagrams on the next page illustrate the differences between **U**, U^c and $(A \cup B \cup C)^c$. Carefully note these differences.



U



$(A \cup B \cup C)^c$

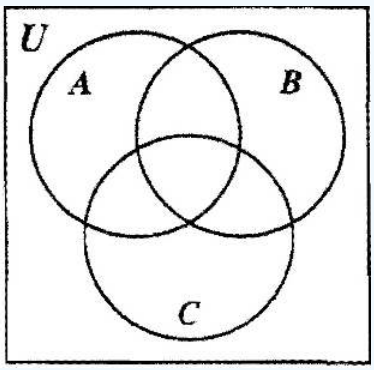


U^c

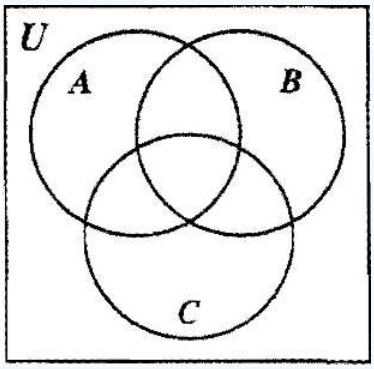
Usually, parentheses are necessary to indicate which operation needs to be done first. If there is only union or intersection involved, this isn't necessary as in $(\mathbf{A} \cup \mathbf{B} \cup \mathbf{C})^c$ above. Convince yourself that $((\mathbf{A} \cup \mathbf{B}) \cup \mathbf{C}) = (\mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}))$. Similarly, convince yourself of the analogous fact for intersection by performing the following steps. On the first Venn diagram below, shade $\mathbf{A} \cap \mathbf{B}$ with horizontal lines and shade **C** with vertical lines. Then, the overlap is $((\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C})$. On the second Venn diagram, shade **A** with lines slanting to the right and $\mathbf{B} \cup \mathbf{C}$ with lines slanting to the left. Then the overlap is $(\mathbf{A} \cap (\mathbf{B} \cap \mathbf{C}))$. Check to see that the final answer, the overlap in this case, is the same for both. Shade the final answer in the third Venn diagram.

Exercise 5

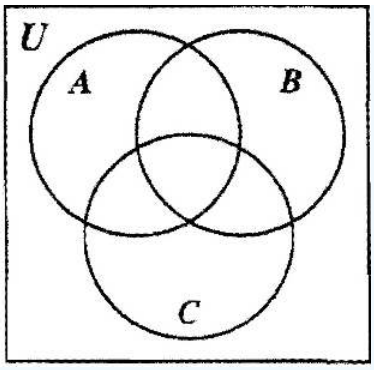
a. $(\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}$



b. $(\mathbf{A} \cup (\mathbf{B} \cup \mathbf{C}))$



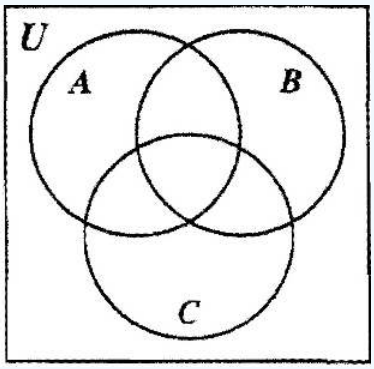
c. Shade final answer here.



Now, it's time for you to try a few more diagrams on your own. It may take more than one step to figure out the answer. You might need to do preliminary drawings on scratch paper first. The shadings you show here should be the final answer only, but you should be able to explain and support how you arrived at your answer. Compare your answers with other people in your class and make sure a consensus is reached on the correct answer. Do this for all the Venn diagrams throughout this exercise set. **Shade in the region that represents what is written above each of the six Venn diagrams on the following page**. Note that in cases involving more than one operation, it is necessary to use parentheses and follow order of operations. Exercises 10 and 11 illustrate why this is necessary.

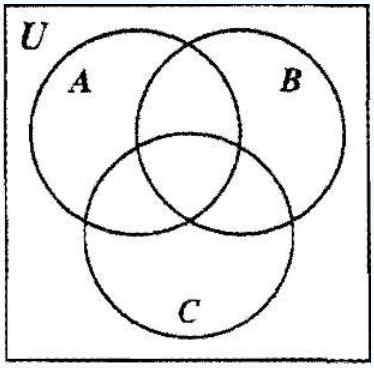
Exercise 6

$\mathbf{B}^c$



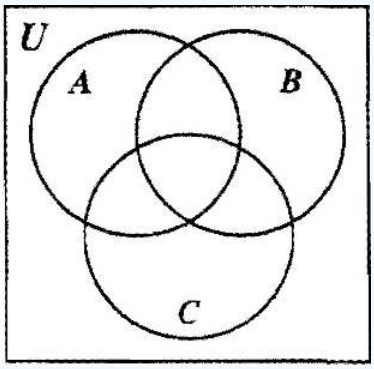
Exercise 7

$(\mathbf{C} \cap \mathbf{A})^c$



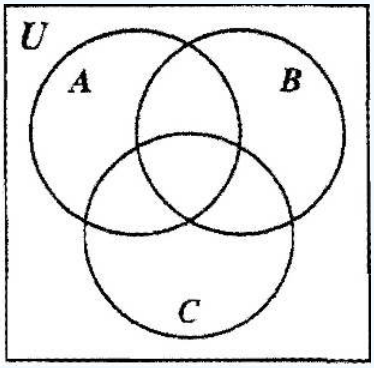
Exercise 8

$(\mathbf{B} \cup \mathbf{C})^c$



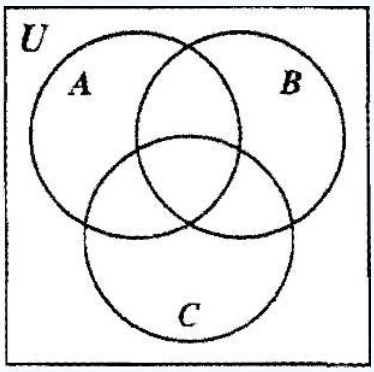
Exercise 9

$(\mathbf{A} \cap \mathbf{B} \cap \mathbf{C})^c$



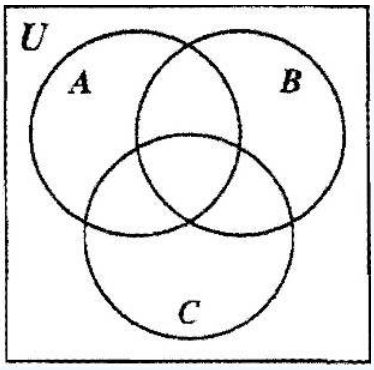
Exercise 10

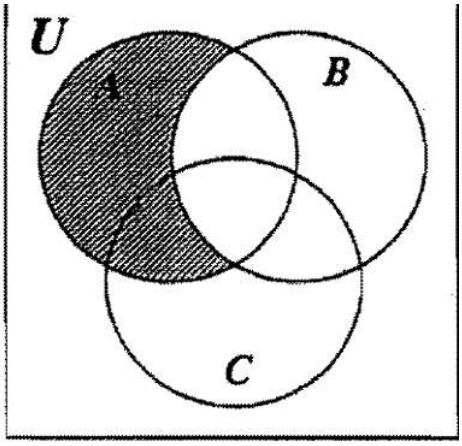
$(\mathbf{A} \cap \mathbf{B}) \cap \mathbf{C}$



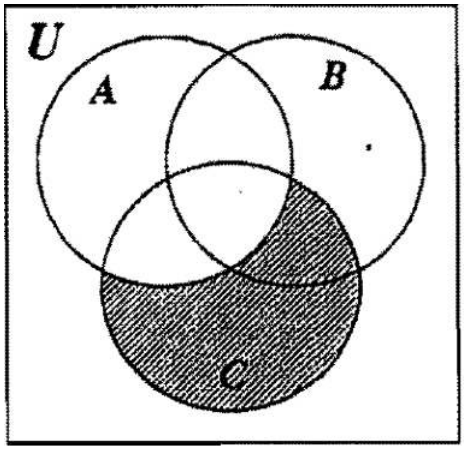
Exercise 11

$(\mathbf{A} \cap (\mathbf{B} \cap \mathbf{C})$





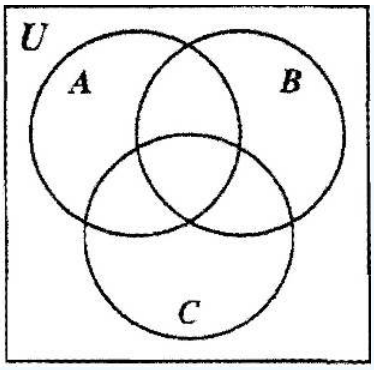
For difference, shade the region coming before the difference sign (–) *but don't include or shade* ***any*** *part of the region that follows the difference sign.* The Venn on the left represents **A–B** and the one on the right represents **C**–**A.**



Here are two problems for you to try. Only shade in the final answer for each exercise.

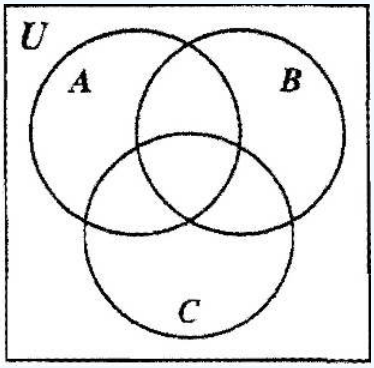
? Exercise 12

Shade the region that represents **A** – **C**

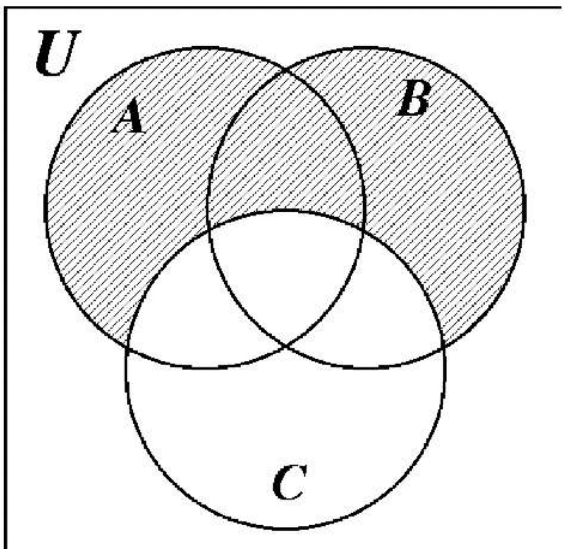


? Exercise 13

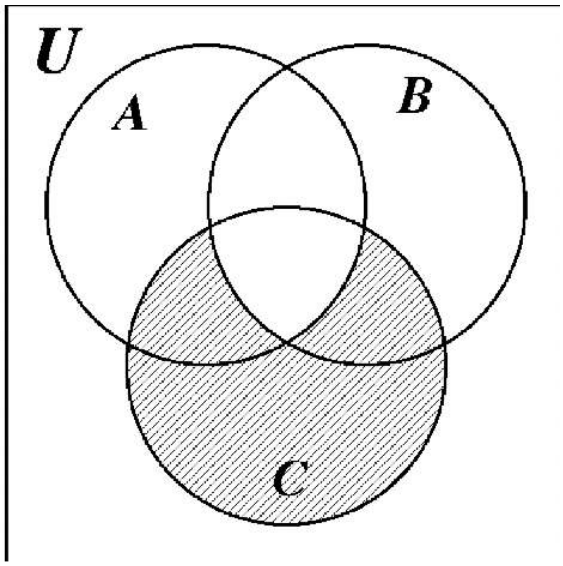
Shade the region that represents **B – C**



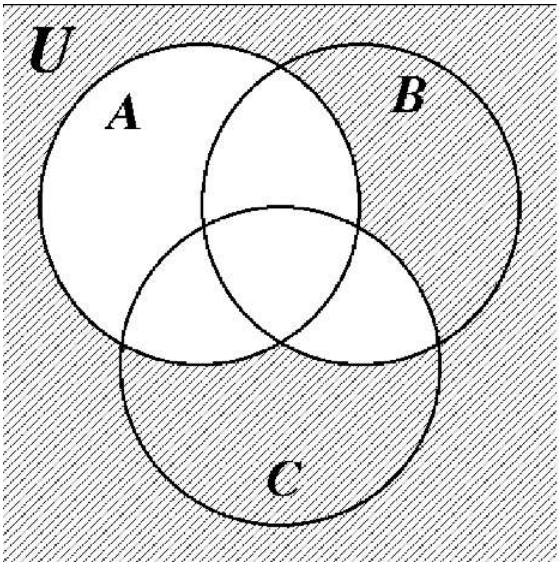
Study the following Venn diagrams. Make sure you understand how to get the answers.



(A ∪ B) – C



C – (**A** ∩ **B**)

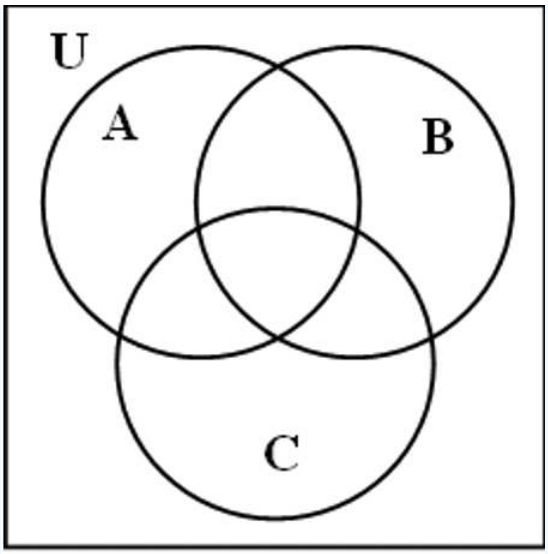


A^c – (**B** ∩ **C**)

It's your turn to shade in the region that represents what is written above each diagram.

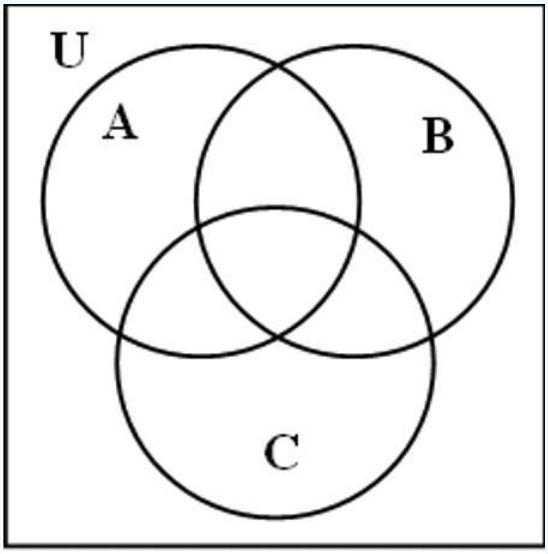
? Exercise 14

(**A** ∩ **C**) – **B**



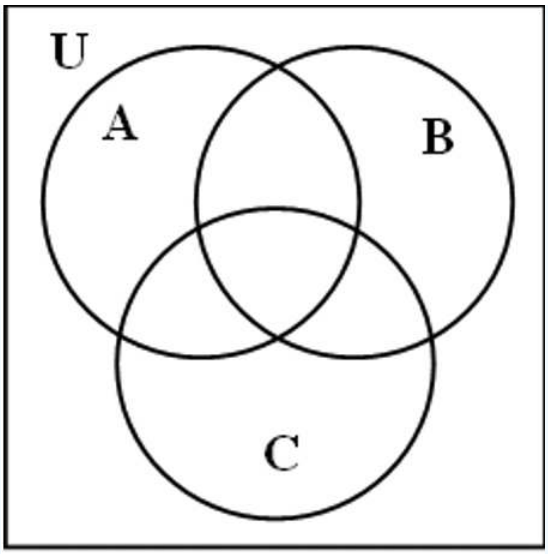
Exercise 15

B – (A ∩ C)



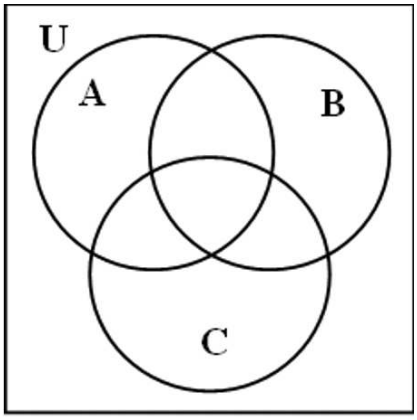
Exercise 16

(A – C) ∪ (B – A)

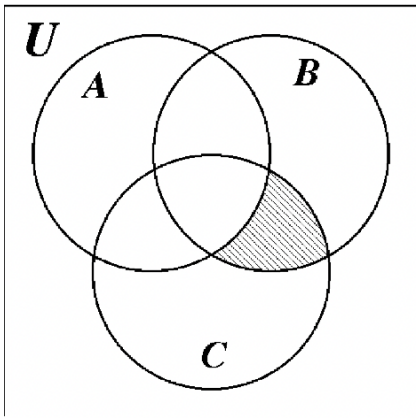


Suppose you wanted to find **((C – A) ∩ B)**^{c}. This would probably take a few steps to get the answer. One approach to finding the correct shading is to notice that the final answer is the complement of **(C – A) ∩ B**. That means we would have to first figure out what (C – **A) ∩ B** looked like. In order to do that, we notice that this is the intersection of two things **C – A** and **B.** On the blank Venn diagram to the left below, shade C – **A** with horizontal lines and B with vertical lines. The overlap would be the intersection. The overlap on your drawing should match the shading shown on the Venn diagram in the middle. Does it? The last step would then be to take the complement of the shading shown on the middle diagram. This is shown on the Venn diagram on the far right. So, it took drawing three Venns to come up with the final answer for this problem. Someone else might be able to do it in fewer steps while someone else might take more steps.

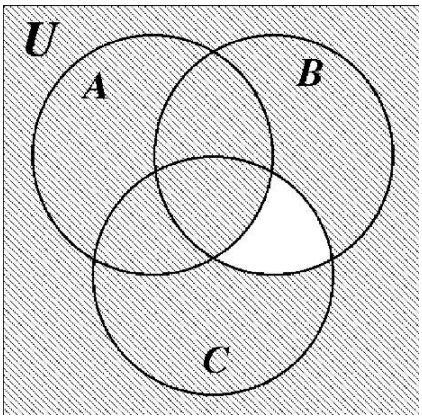
? Exercise 17



(C – A) ∩ B



((C – A) ∩ B)^c



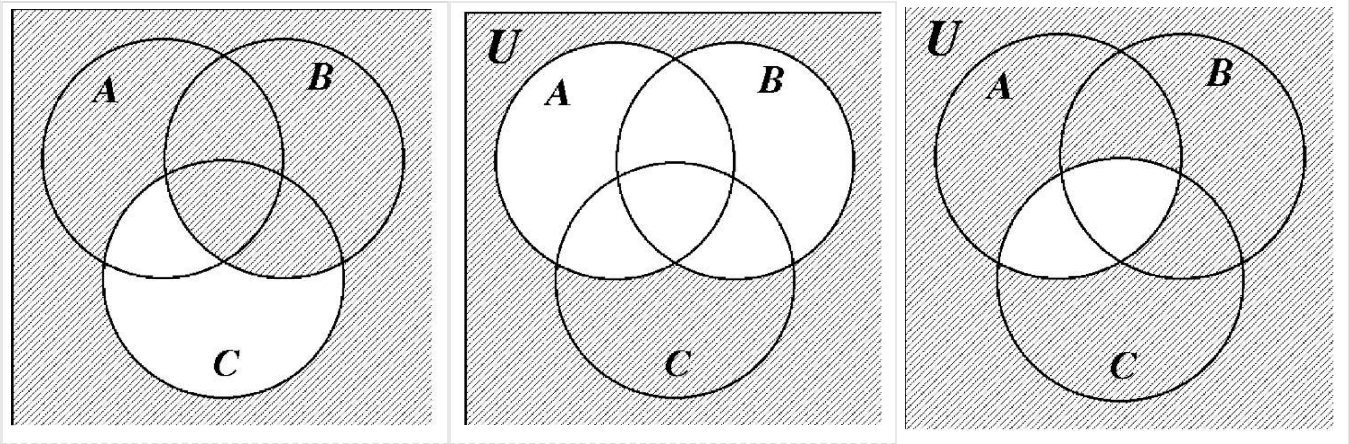
As mentioned previously, it takes a lot of practice to get good at shading Venn diagrams. It's even trickier to look at a Venn diagram and describe it, In fact, there is usually more than one way to describe a Venn diagram. For example, the shading for **((C – A) ∩ B)**^{c} shown on the previous page is the same as it is for **((C ∩ B) – A)**^{c}. What does this mean? We're so used to only having one correct answer. Well, consider if someone asked you to write an arithmetic problem for which the answer was 2. There would be infinitely many possibilities. For example, 5 - 3 or 1 + 1 or 10/5 would all be acceptable answers. Granted, this kind of question on a test would be harder for a teacher to grade because each student's response would have to be checked to see if it would work. There isn't one pat answer. The same goes if a teacher asks you to look at a shading of a Venn diagram and describe it. On the other hand, if a description is given and you are asked to shade the Venn diagram, there is only one correct shading. It is much like being asked to compute an arithmetic problem. The answer to 10 - 8 is 2 and that is the only acceptable answer!

The point of all this is that to master shadings of Venn diagrams and descriptions of Venn diagrams by looking at the shadings takes lots and lots and lots of practice. Give yourself plenty of time to study and work on them and you will accomplish this feat!!!

On the next few pages, you are asked to shade several one, two and three set Venn diagrams. The correct shadings follow. Make sure you try these problems in earnest. Make sure you can explain the steps involved to arrive at the correct shading. After mastering the shadings, see if you can look at a shaded Venn diagram and come up with an accurate description. Again, remember there is more than one way to describe a given Venn diagram.

These Venn diagrams will be helpful when studying for a test. Go back and practice drawing the same Venn diagrams later. Use the answers to see if you can describe them by looking at the picture. Of course, remember that your description might not match exactly since there as more than one way to describe any given Venn diagram. If your description is different, make sure you go through the steps of shading a Venn with your description and see if your shading really matches the Venn diagram you were trying to describe.

Here are a few shaded Venn diagrams. See if you can look at the shadings and come up with a description. I've put some possible answers at the bottom of this page.



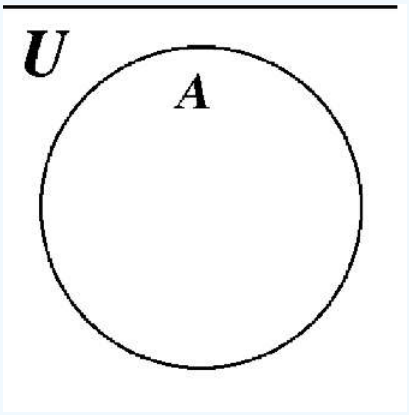
Here are some possible descriptions for the above Venn diagrams:

$(\mathbf{C} - \mathbf{B})^c$	$(\mathbf{A} \cup \mathbf{C})^c$	$(\mathbf{A} \cap \mathbf{C})^c$

Shade the region that represents what is written above each of the one and two set Venn diagrams below. You may need to draw preliminary drawings first for some of them.

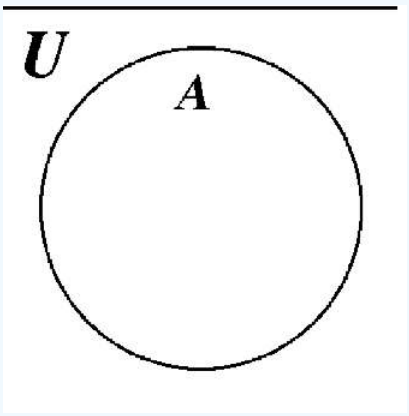
? Exercise 18

A



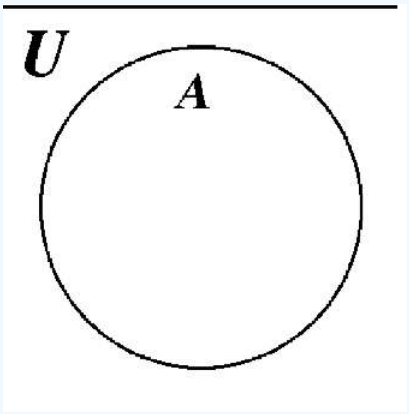
Exercise 19

A^c



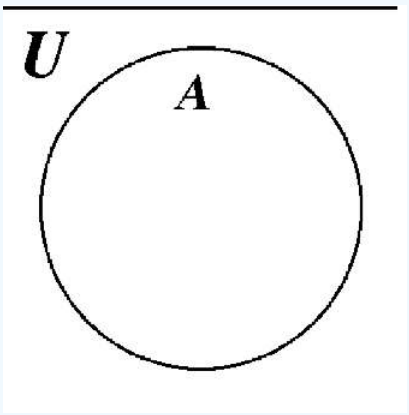
Exercise 20

U



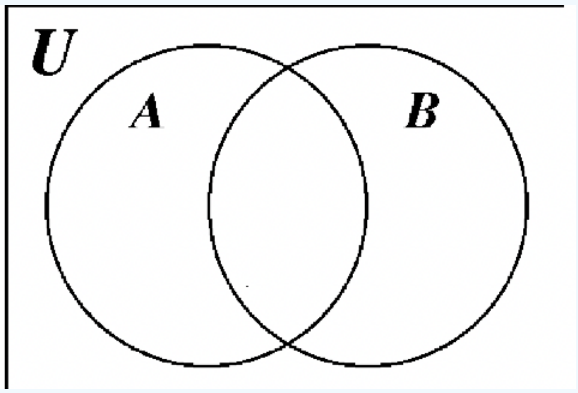
Exercise 21

U^c



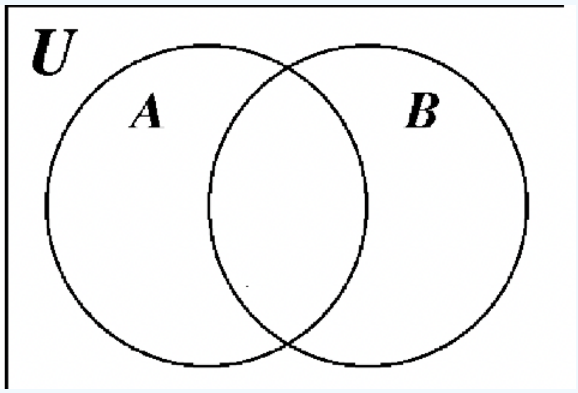
Exercise 22

$\mathbf{A} \cap \mathbf{B}$



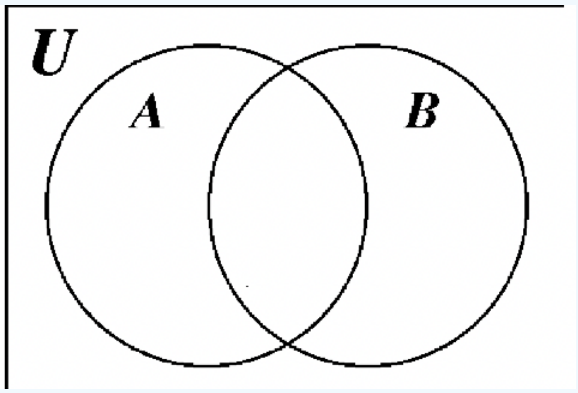
Exercise 23

$\mathbf{A} \cup \mathbf{B}$



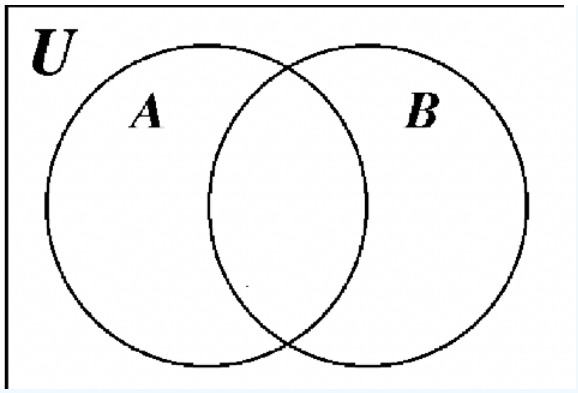
Exercise 24

$A \cup B^c$



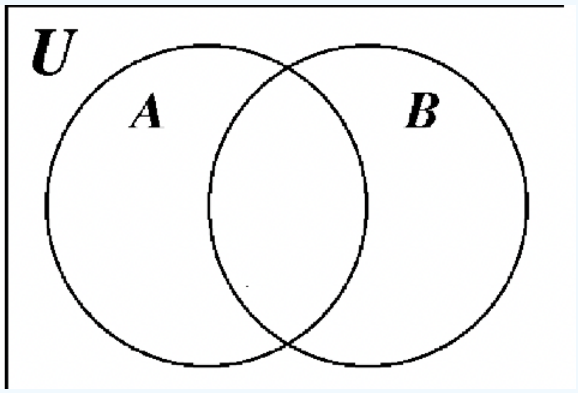
Exercise 25

$(A \cap B)^c$



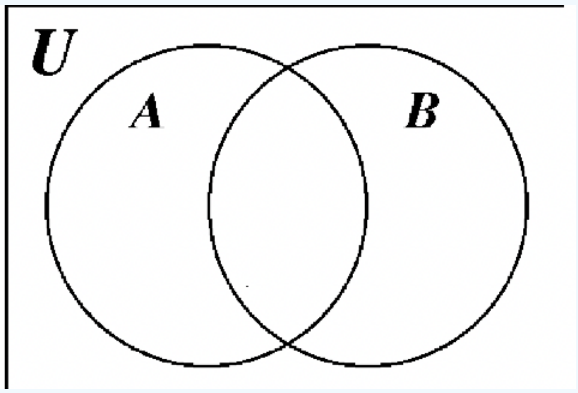
Exercise 26

$(A \cup B)^c$



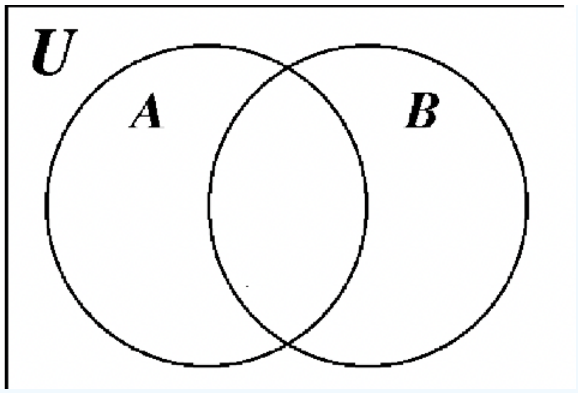
Exercise 27

(A \B) ∪ (B \A)



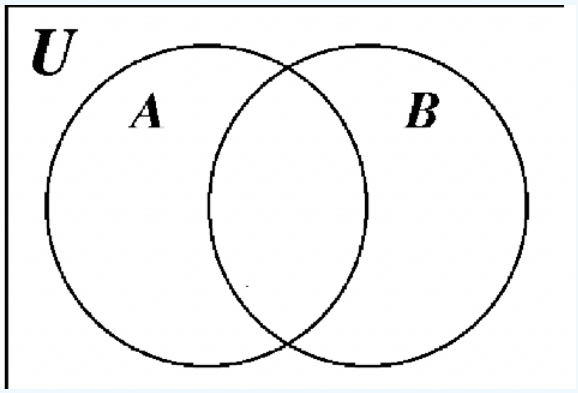
Exercise 28

$A^c \cup B^c$



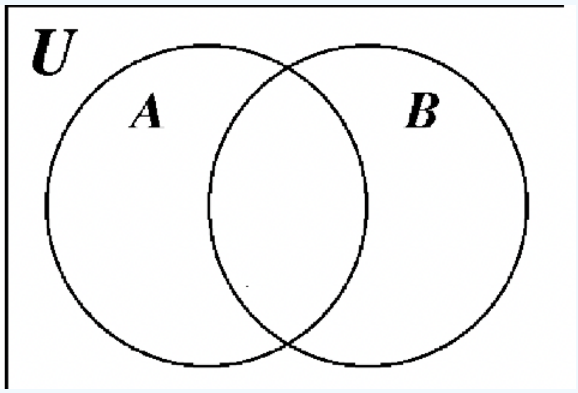
? Exercise 29

$A^c \cap B^c$



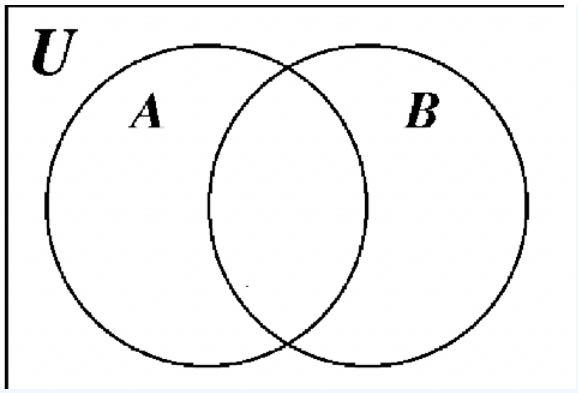
? Exercise 30

$(A \cup B)^c \cup (A \cap B)$



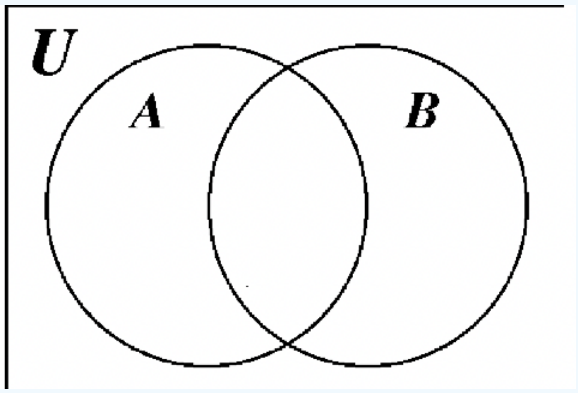
? Exercise 31

B



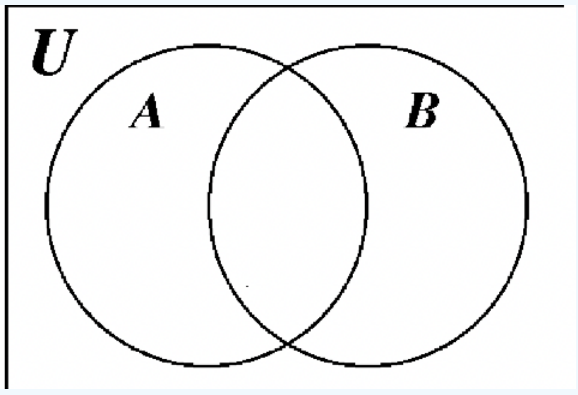
Exercise 32

B - A



Exercise 33

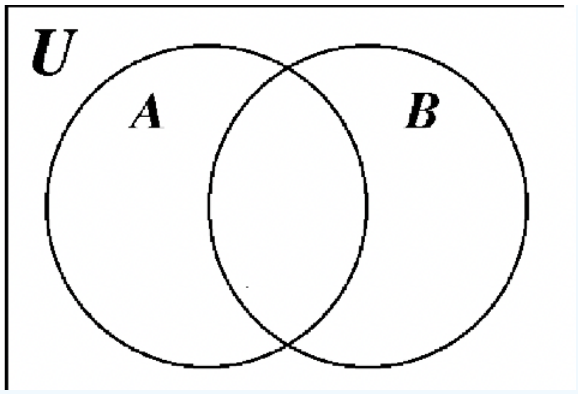
B^c



Shade the region that represents what is written above each of the one and two set Venn diagrams below. You may need to draw preliminary drawings first for some of them.

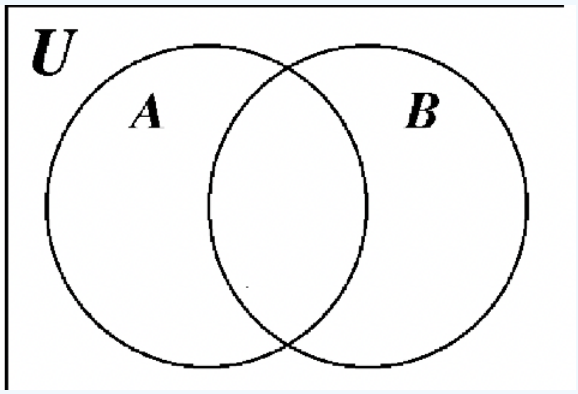
Exercise 34

(A ∩ B) – C



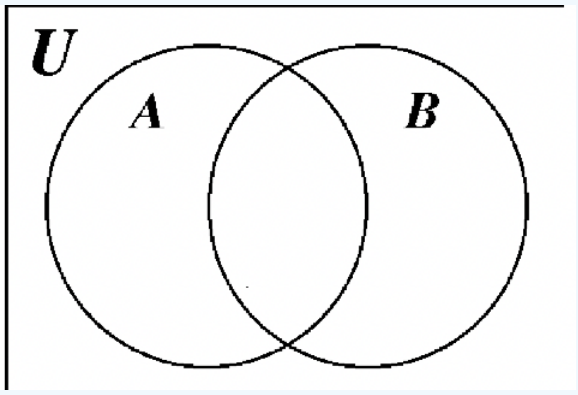
Exercise 35

$(C \cup B) - A$



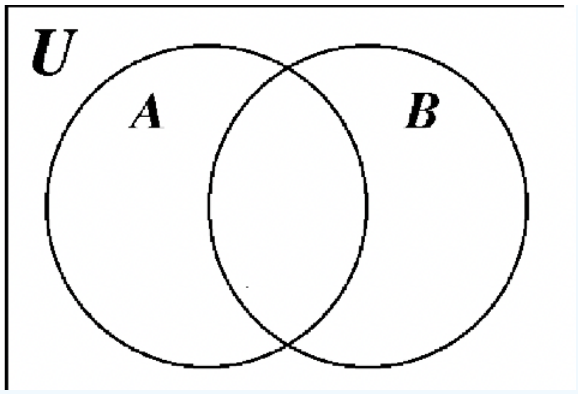
Exercise 36

$(A \cap B) \cup C$



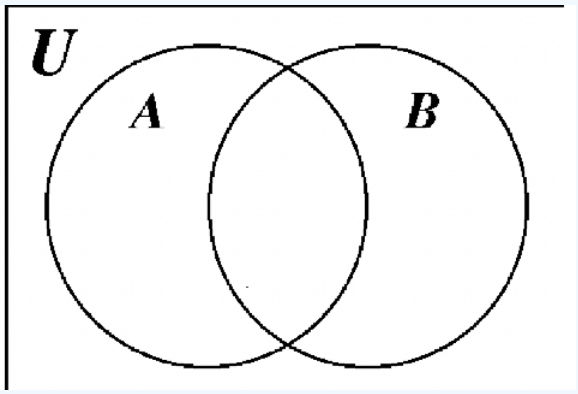
Exercise 37

$(A \cup B) \cap C$



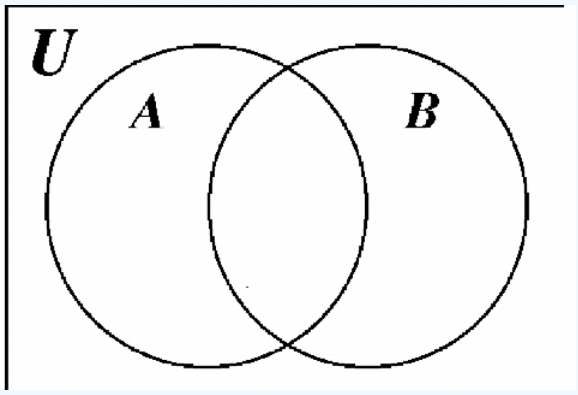
Exercise 38

$A^c - B$



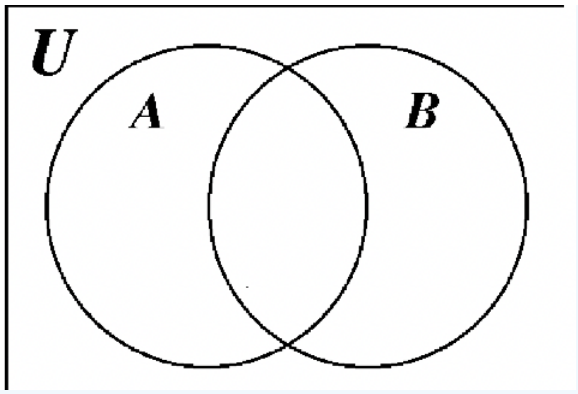
Exercise 39

$A \cap B \cap C) - B$



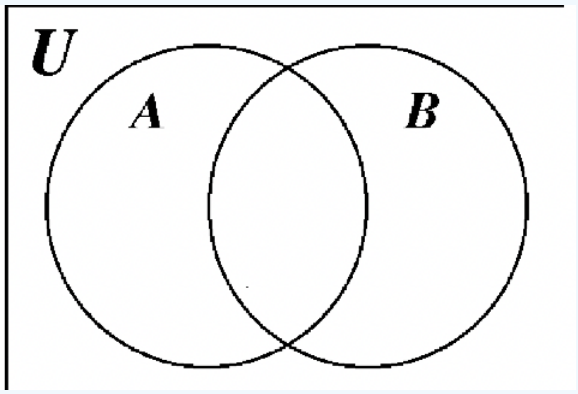
Exercise 40

$B - (A \cup C)$



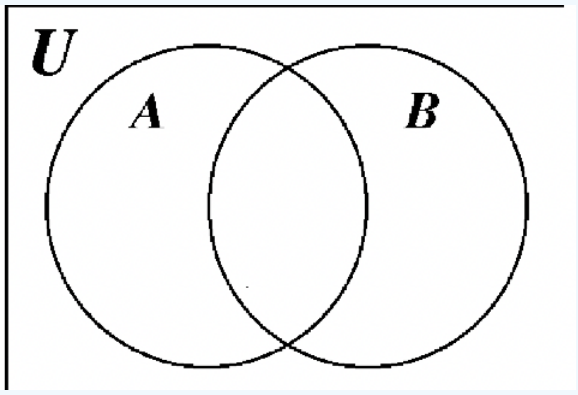
Exercise 41

$\mathbf{C - (A \cap B)}$



Exercise 42

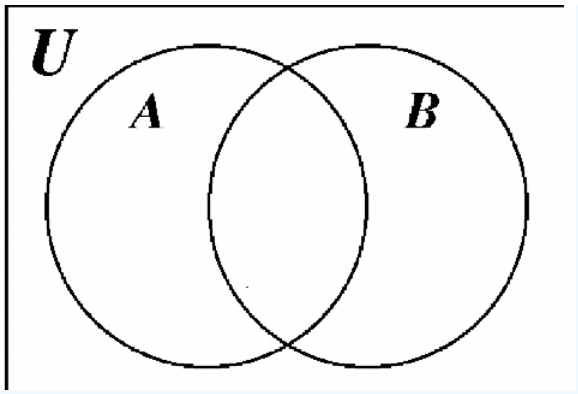
$\mathbf{(B - A) \cap (B - C)}$



Exercise 43

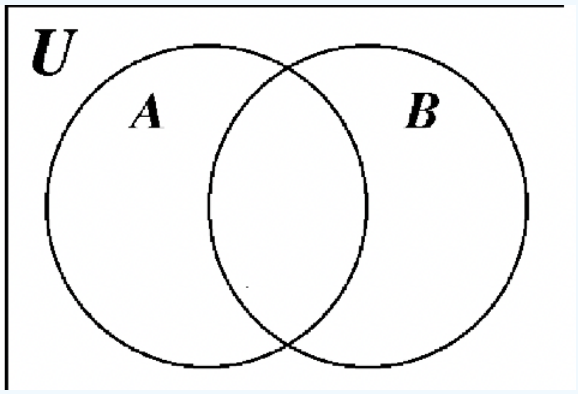
$\mathbf{(B - A) \cup (B - C)}$

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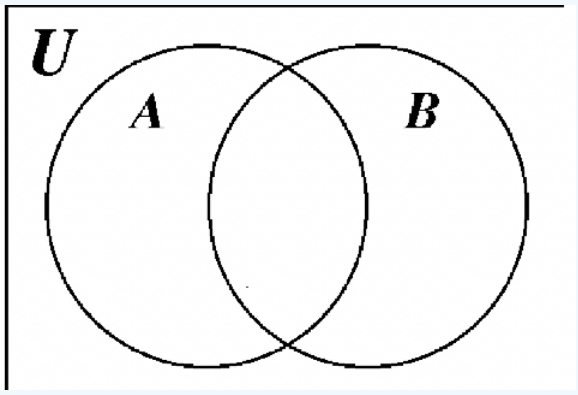
Exercise 44

$(\mathbf{A} \cup \mathbf{B})^c$



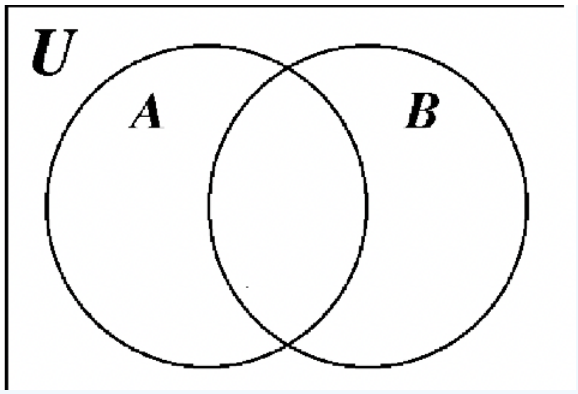
Exercise 45

$\mathbf{A}^c \cap \mathbf{B}^c$



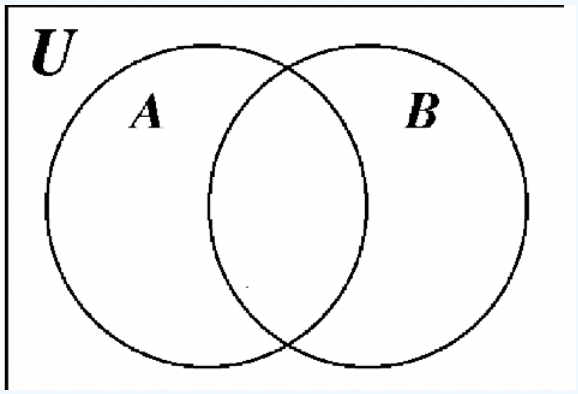
Exercise 46

$\mathbf{A}^c - \mathbf{B}^c$



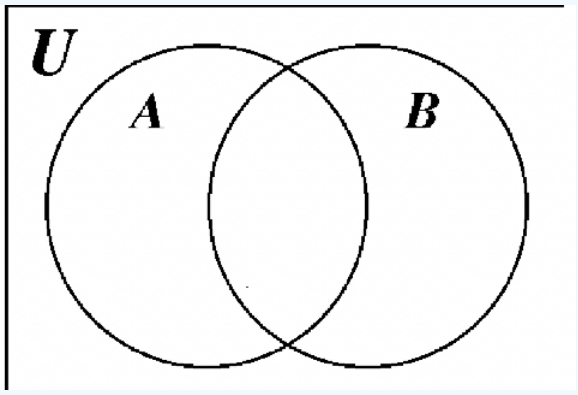
? Exercise 47

$(\mathbf{C} - \mathbf{B})^c$



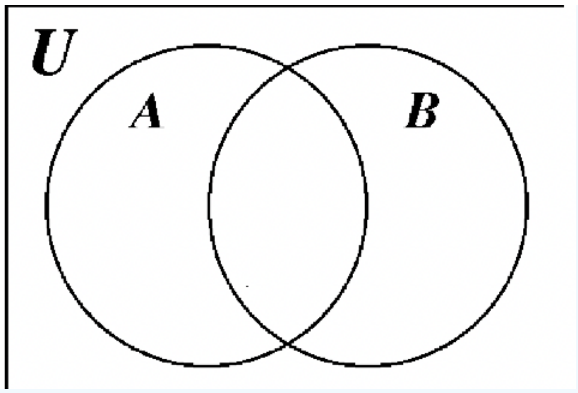
? Exercise 48

$(\mathbf{B}^c \cap \mathbf{C}) - \mathbf{A}$



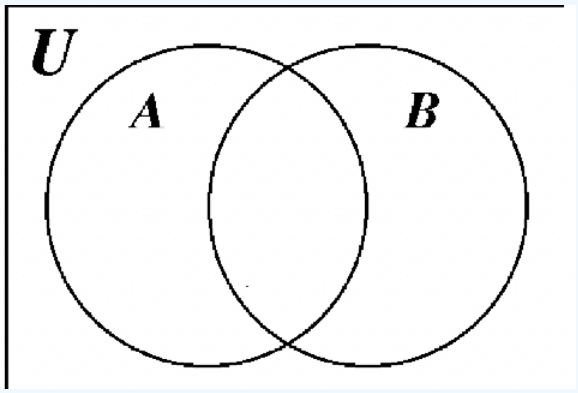
? Exercise 49

$(\mathbf{A} - (\mathbf{B} \cup \mathbf{C})) \cup (\mathbf{B} - (\mathbf{A} \cup \mathbf{C})) \cup (\mathbf{C} - (\mathbf{A} \cup \mathbf{B}))$



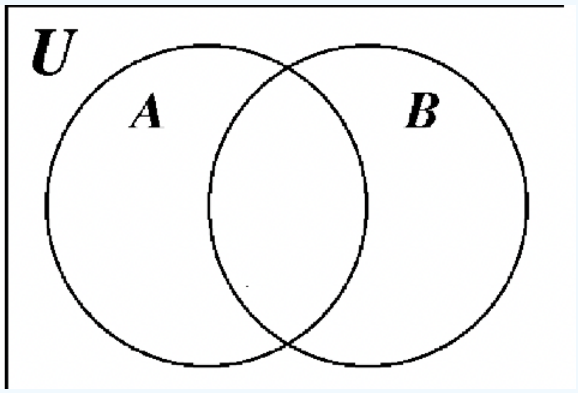
Exercise 50

$(\mathbf{A} \cap \mathbf{C})^c$



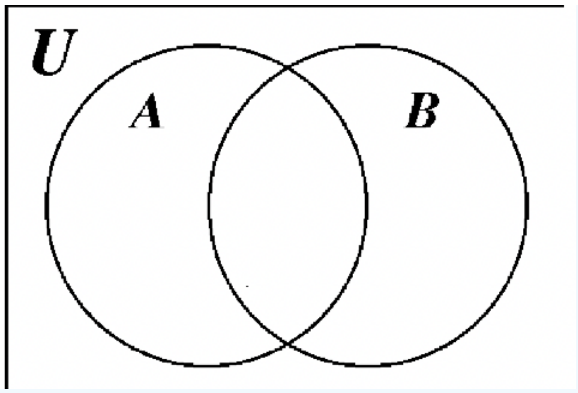
Exercise 51

$((\mathbf{A} \cap \mathbf{B}) - \mathbf{C}) \cap (\mathbf{C} - \mathbf{A})$



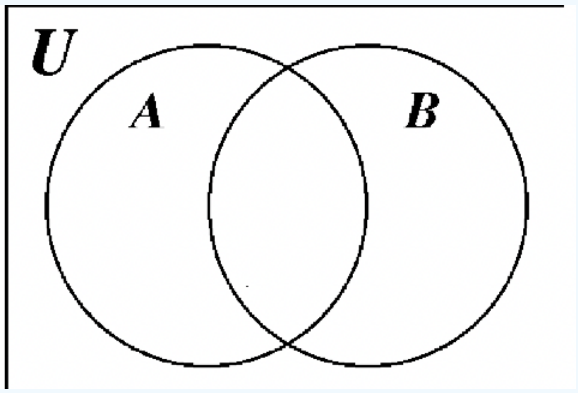
Exercise 52

$\mathbf{A}^c \cup \mathbf{C}^c$

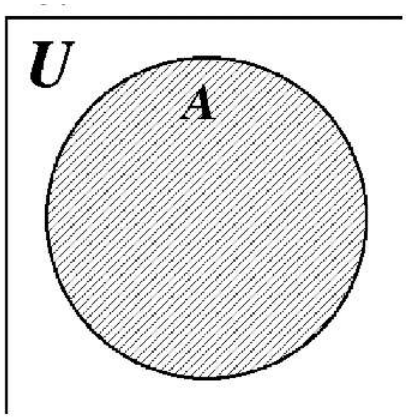


Exercise 53

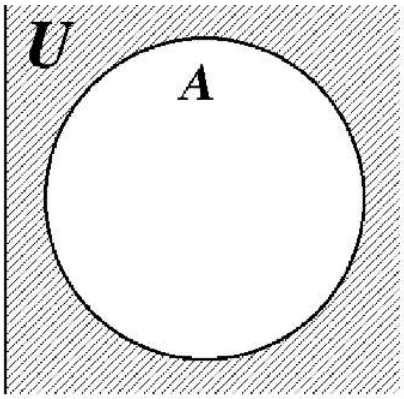
$\mathbf{B} \cap (\mathbf{C} \cup \mathbf{A}^c)$



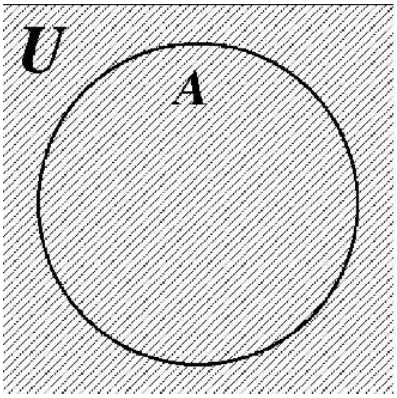
Here are the correct shadings to the exercises on the previous pages. After mastering these shadings, reverse the process by looking at the shadings on this page and try to describe them. It takes practice and patience and remember that there may be more than one way to describe some of these. In fact, many times you'll see there is a simpler way to describe them than was on the original exercise!!



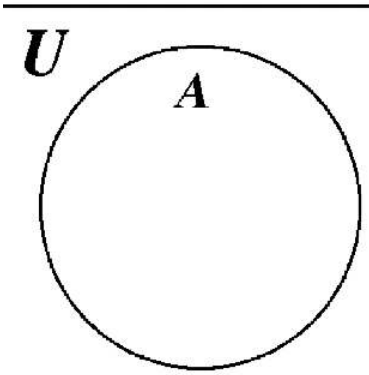
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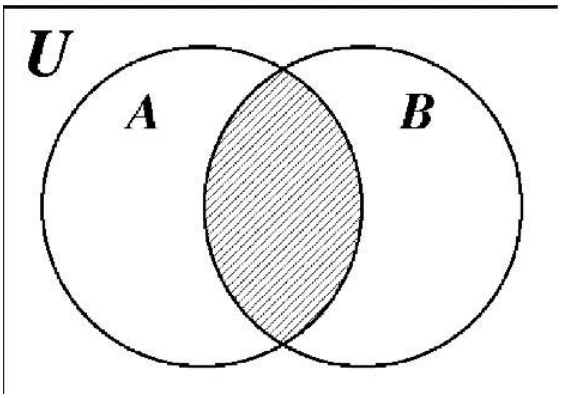
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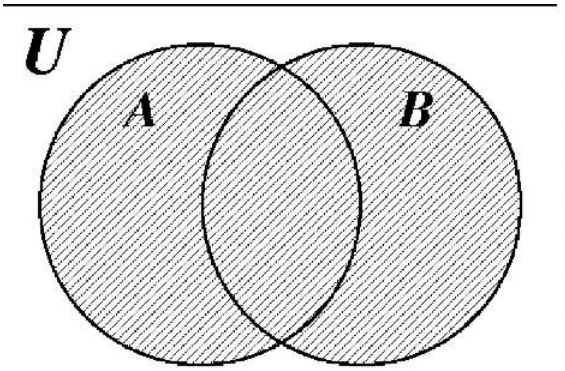
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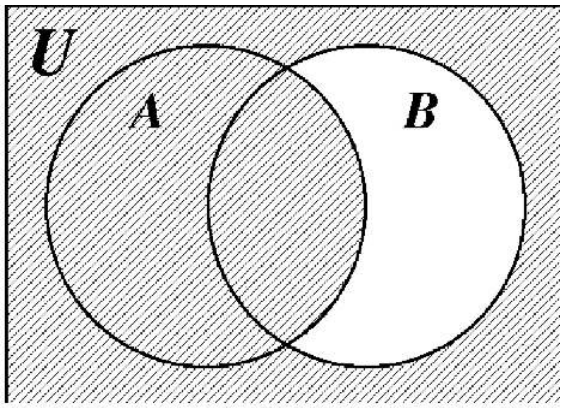
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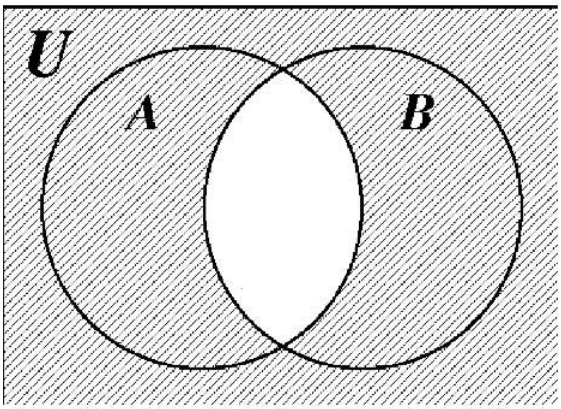
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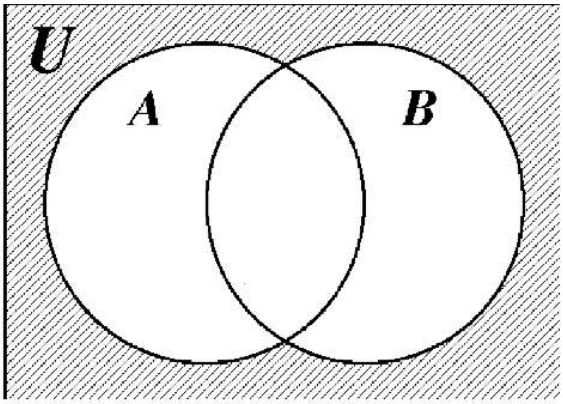
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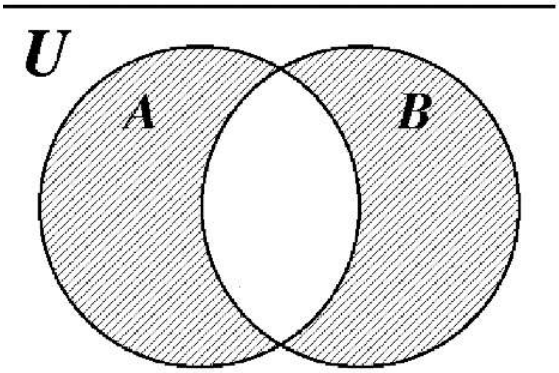
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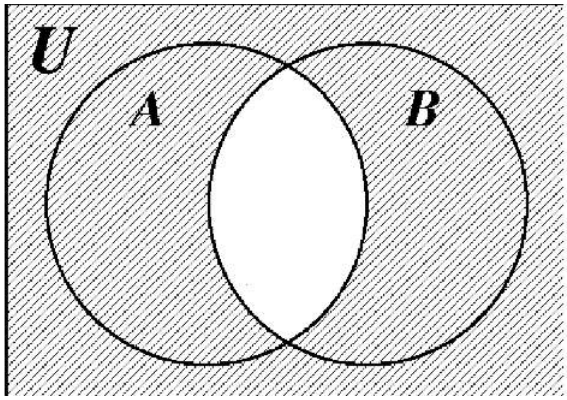
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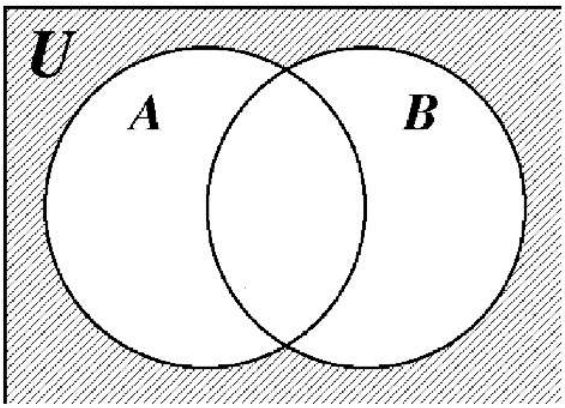
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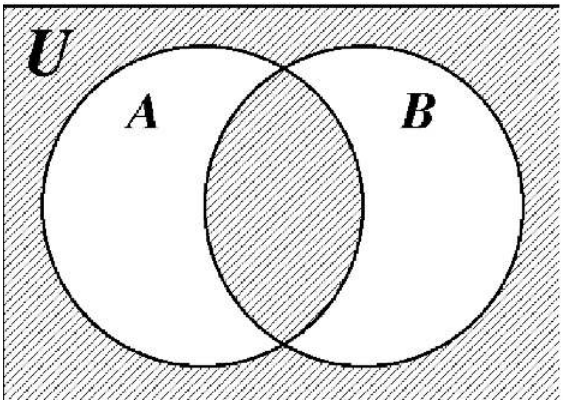
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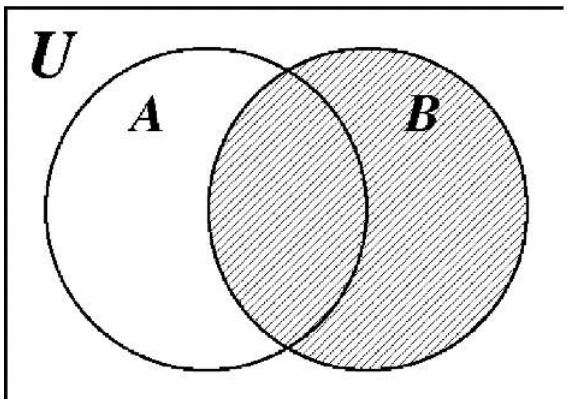
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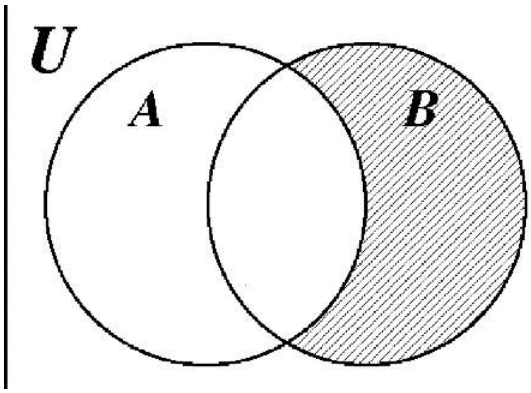
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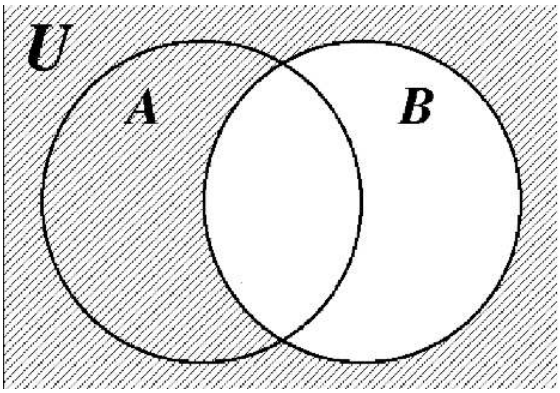
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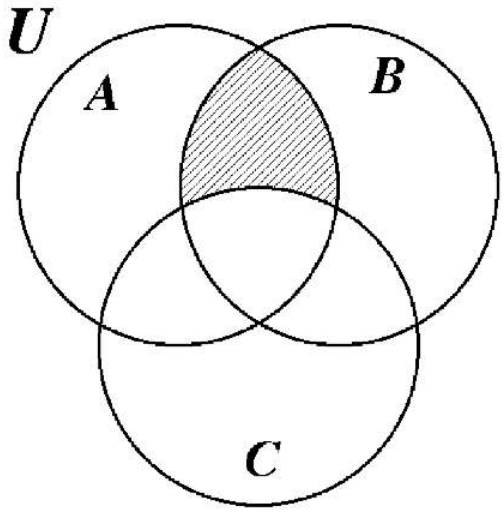
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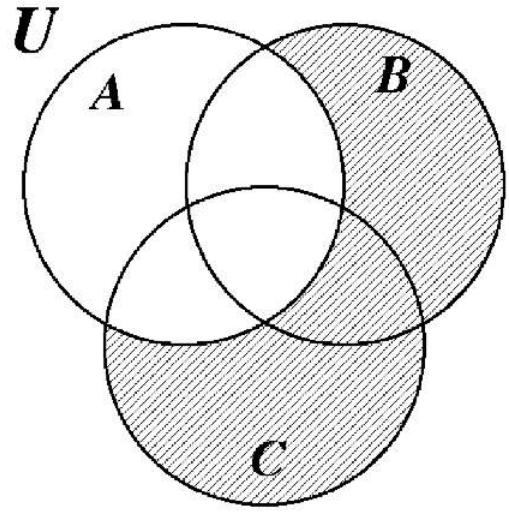
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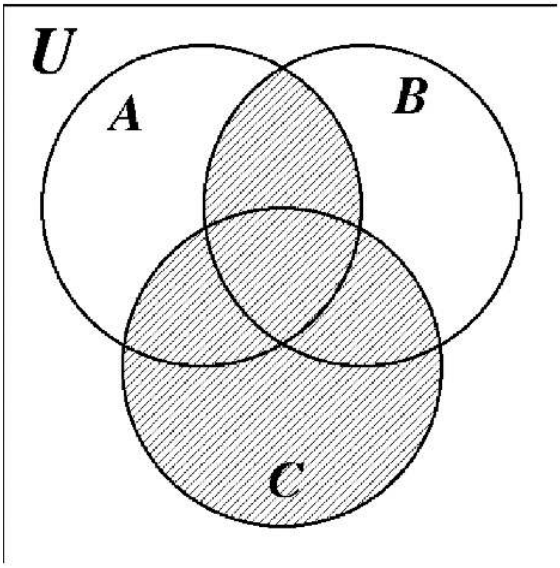
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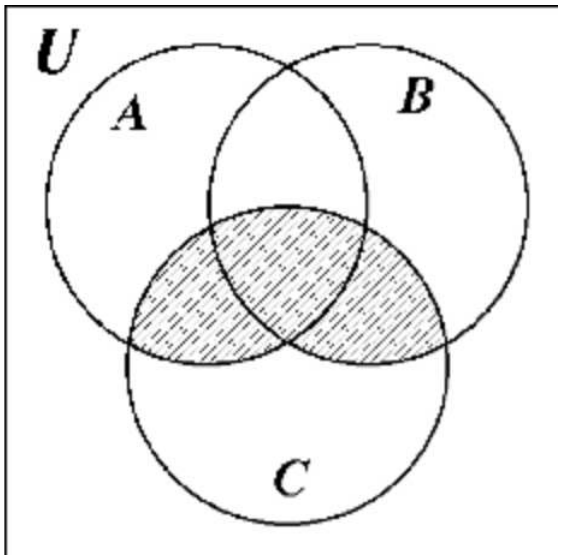
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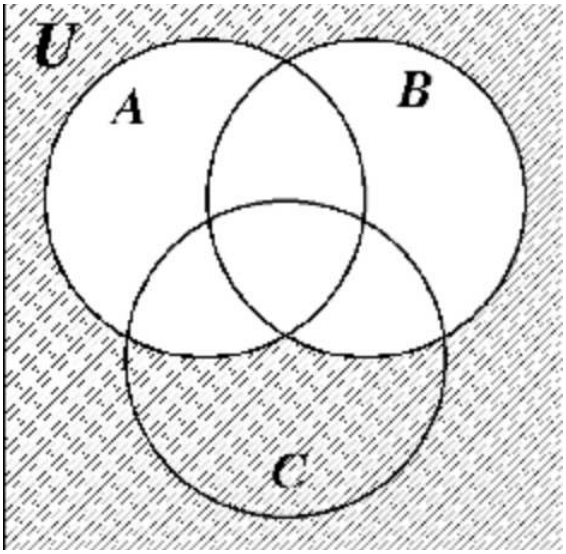
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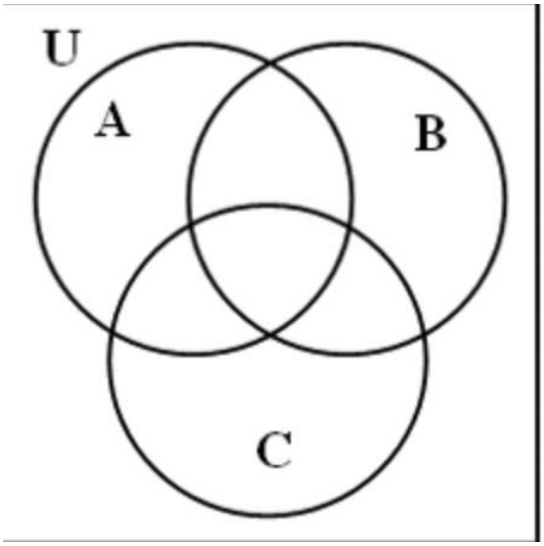
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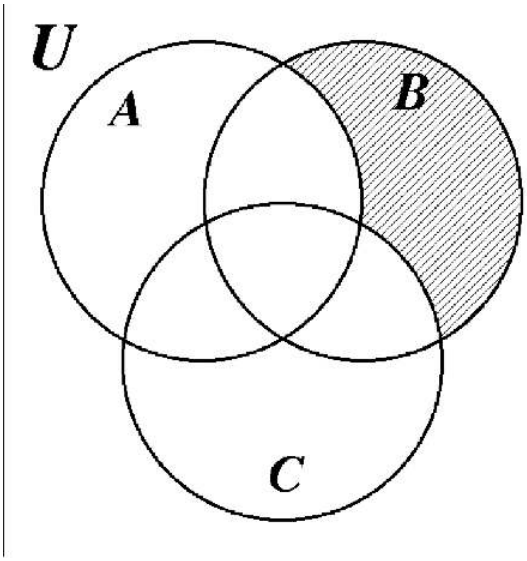


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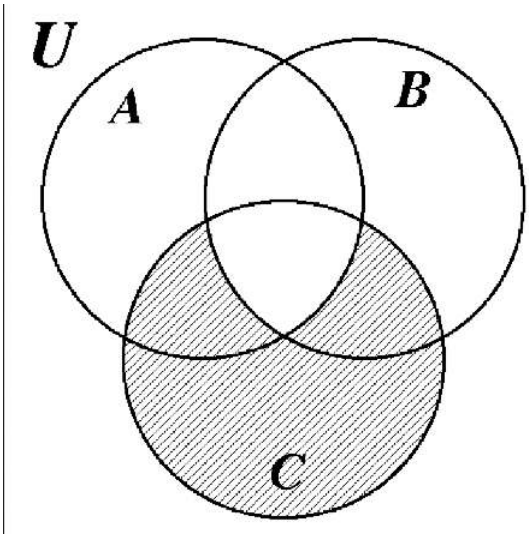


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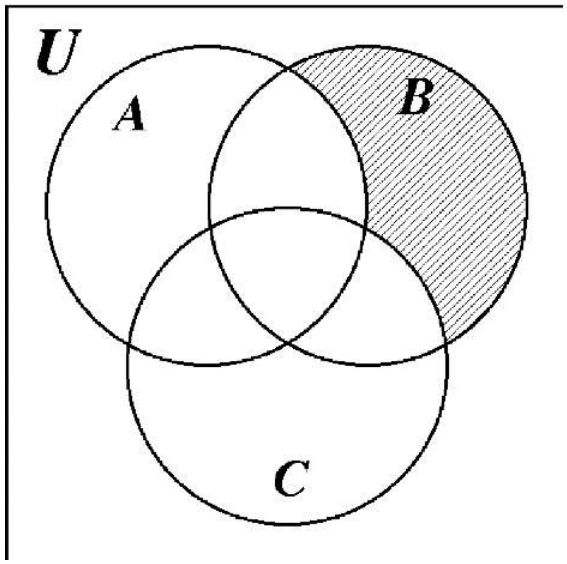
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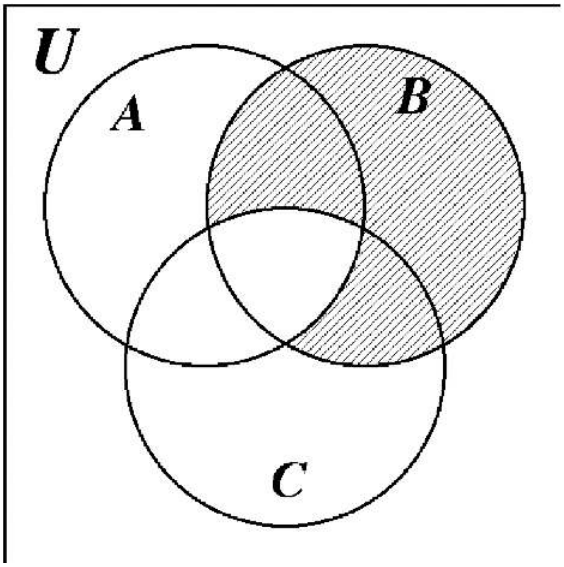
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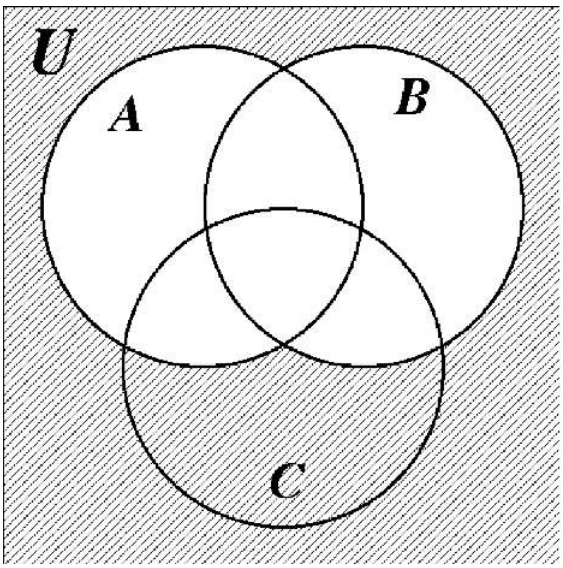
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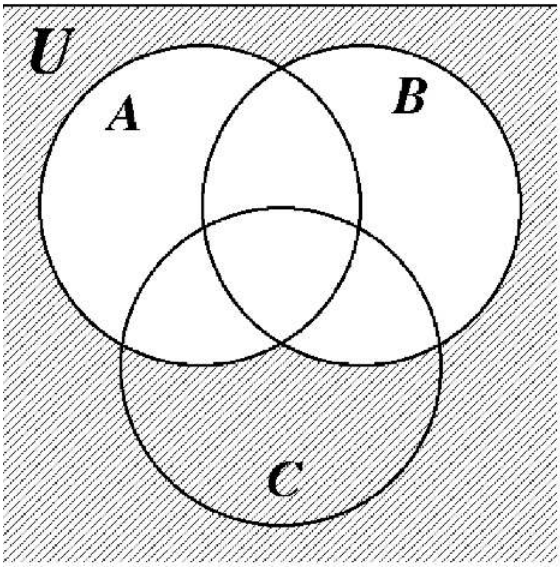
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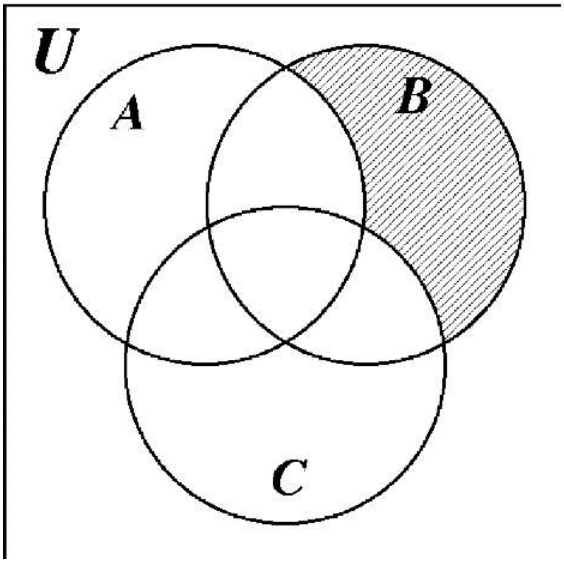
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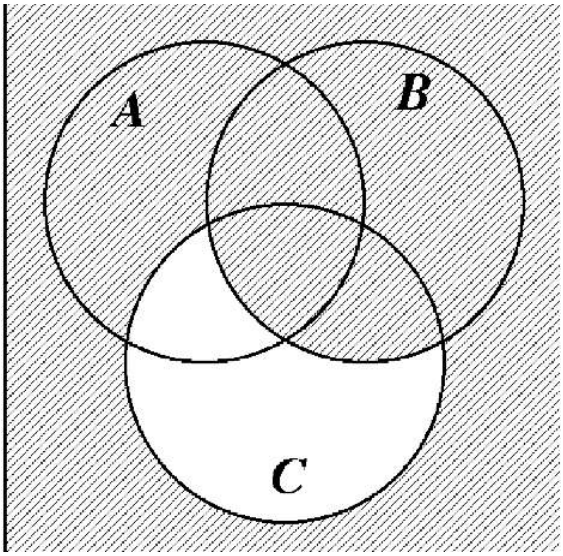
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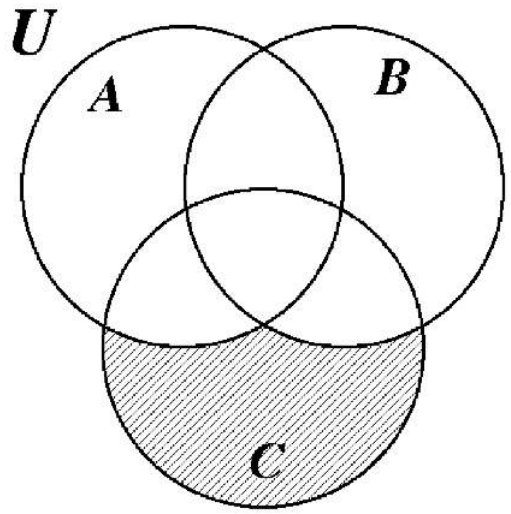
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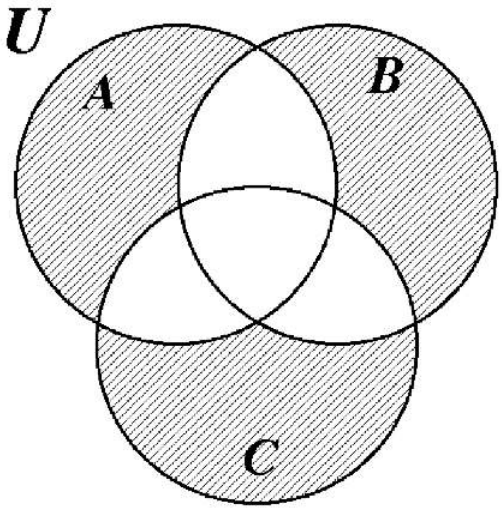
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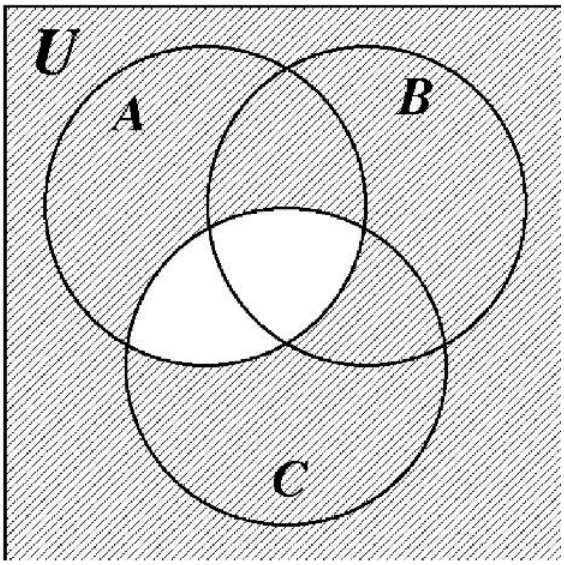
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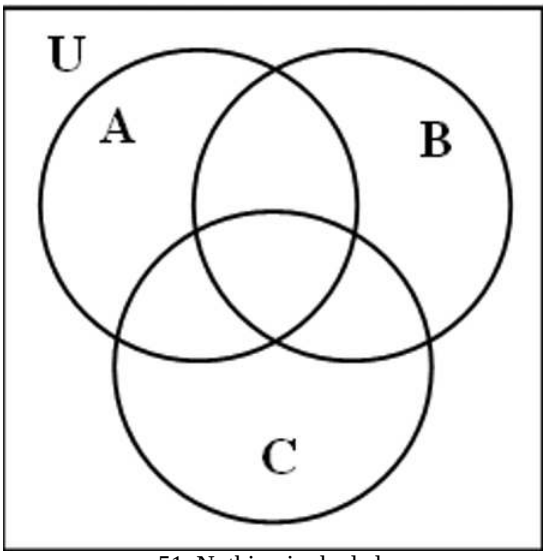
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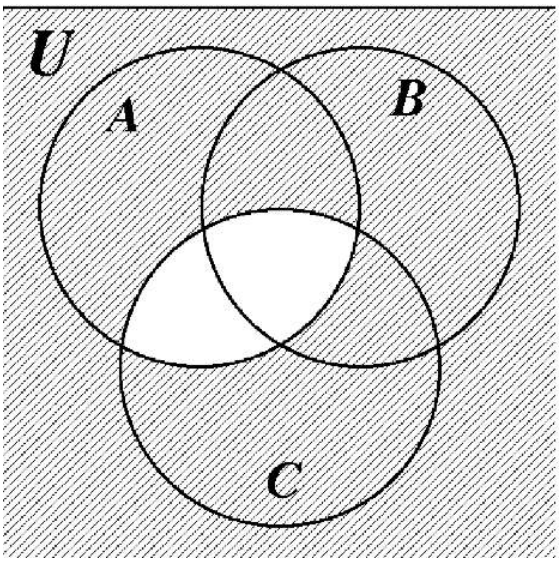
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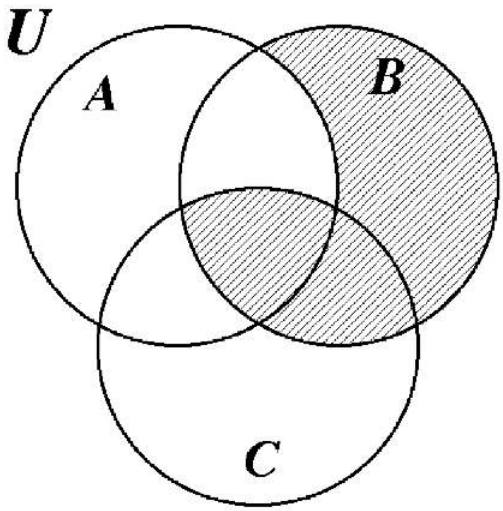
50.



51. Nothing is shaded.



52.



53.

In the Material Card section there are blank Venn diagram templates you can use for practice.

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CHAPTER OVERVIEW

12: Number Theory

12.1: Number Theory
12.2: Divisibility
12.3: Divisibility Tests

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12.1: Number Theory

Factors and Multiples

Number theory is a vast topic in mathematics. For the purposes of this course, we will concern ourselves with only a few topics. The first such topic is the relationship between multiplication and division among the nonnegative integers.

Definition: Factors and Multiples

Let m, n, and p be nonnegative integers such that $mn = p$. Then m and n are **factors** of p, and p is a **multiple** of m and n

Let's use these words with specific values.

Example 12.1.1

- Find all the positive integer factors of 12.
- Find at least 5 positive integer multiples of 12.

Solution

- Factors of 12: 1, 2, 3, 4, 6, & 12
- Multiples of 12: 12, 24, 36, 48, 60, …

Now, try one on your own.

Exercise 12.1.1

- List all the positive integer factors for 30. Hint: there are eight of them!
- List at least 5 positive integers multiples of 30.

Primes and Composites

Definition: Prime number

Any natural number (positive integer) that has exactly two natural number factors is called a **prime number**. Those two factors are 1 and itself.

Some examples of prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, … Notice that 0 is not a positive number, and even if we allowed 0 to count, it has infinitely many factors because 0 times any integer equals 0. So, 0 is definitely *not* prime. Similarly, 1 has only one positive integer factor, which is itself. So 1 is also *not* prime.

Definition: Composite Number

Any natural number (positive integer), which has a positive integer factor other than 1 and itself is called a **composite number**.

Some examples of composite numbers: 4, 6, 8, 9, 10, 12, 14, 15, … Notice that this time we do not include 0 simply by definition since it is not positive.

Relatively Prime Pairs of Numbers

Definition: Relatively Prime

Two or more positive integers whose only common positive integer factor is 1 are **relatively prime**.

Example 12.1.1

Show 7 and 8 are relatively prime.

Solution

The positive integer factors of 7 are 1 and 7 (tit is prime).

The positive integer factors of 8 are 1, 2, 4, and 8.

The only common number between them is 1. So, 7 and 8 are relatively prime.

Now try one on your own.

Exercise 12.1.1

Show 4 and 15 are relatively prime

Can you find other pairs of relatively prime positive integers?

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12.2: Divisibility

It's helpful to understand what is meant by the **DIGITAL ROOT** of a number because they are used in divisibility tests, and are also used for checking arithmetic problem. A **DIGITAL ROOT** of a number is *one* of these digits: **0, 1, 2, 3, 4, 5, 6, 7 or 8**.

Definition: The DIGITAL ROOT of a number is the remainder obtained when a number is divided by 9.

? Exercise 1

Divide each of the following numbers by 9. Then, write the remainder.

a. 25	h. 8
b. 48	i. 54
c. 53	j. 74
d. 829	k. 481
e. 5402	l. 936
f. 3455	m. 8314
g. 47522	n. 647

Below is another easier way to find the **digital root** of a number.

Step 1: Add the individual digits of the number.

Step 2: If there is more than one digit after adding all the digits, repeat this process until you get a single digit. If the final sum is 9, write 0, because 9 and 0 are equivalent in digital roots (since the remainder is a number smaller than 9). That is why the digital root of a number is only one of these digits: 0, 1, 2, 3, 4, 5, 6, 7 or 8. Those are the only possible remainders that can be obtained when a number is divided by 9! It can't be 9. *The single digit you finally end up with is the **DIGITAL ROOT** of the number*.

Examples: Find the digital roots of the following numbers.

✓ Example 1: 34

34: Add the digits: 3 + 4 = 7

*The digital root of 34 is **7***

✓ Example 2: 321

321: Add the digits: 3 + 2 + 1 = 6

*The digital root of 321 is **6***

✓ Example 3: 58

58: Add the digits: 5 + 8 = 13 Add the digits again: 1 + 3 = 4

*The digital root of 58 is **4***

✓ Example 4: 97

97: Add the digits: 9 + 7 = 16 Add the digits again: 1 + 6 = 7

*The digital root of 97 is **7***

Example 5: 72

72: Add the digits: 7 + 2 = 9 In digital roots, 9 is the same as 0

*The digital root of 72 is **0***

Example 6: 346,721

346,721: Add the digits, we get 3 + 4 + 6 + 7 + 2 + 1= 23
Add the digits again: 2 + 3 = 5

*The digital root of 346,725 is **5***

Note that you **should not write 97 = 16 = 7.** 97 **IS NOT EQUAL** to 7!! The *digital root* of 97 is equal to 7. I use dashes, colons or arrows to record the digital root, so it looks something like this: 346,721 $\rightarrow$ 23 $\rightarrow$ 5. **Don't use equal signs!!!**

Another method for finding digital roots uses the fact that 9 and 0 are equivalent in digital roots. The process of finding digital roots is also called **Casting out Nines**. When you add the digits, you don't have to add the digit 9 or any combination of numbers that add up to 9 (like 2 and 7, or 5 and 4, or 2 and 3 and 4, etc.) – you can "cast out" all 9's. Cross them off; then add the remaining digits together. The following example illustrates how "casting out nines" simplifies the process of finding the digital root of a large number.

Example

Find the digital root of 5,624,398.

Without casting out nines: 5 + 6 + 2 + 4 + 3 + 9 + 8 = 37. Add again: 3 + 7 = 10. Add again: 1 + 0 = **1**.

Casting out Nines: Since 5 and 4 add up to 9, cross them off: $\not{5}, 6, \not{2}\not{4}, \not{3}\not{9}\not{8}$. Since 6 and 3 add up to 9, also cross them off: $\not{5}, \not{6}, \not{2}\not{4}, \not{3}\not{9}\not{8}$. Also cross off the 9: $\not{5}, \not{6}, \not{2}\not{4}, \not{3}\not{9}\not{8}$. The only digits to add are the 2 and 8, which is 10. The digital root of 10 is 1. So **1** is the digital root of 5,624,398 which is the same answer obtained without first casting out nines.

Below, the digital roots for examples 4, 5 and 6 on the previous page is computed again using casting out nines. Note that the digital root remains the same.

Example 4: 97

97: Cross off the 9. Only the 7 remains. The digital root is **7**.

Example 5: 72

72: 7 + 2 = 9, so cross them off. Therefore, the digital root is **0**.

Example 6: 346,721

346,721: Cross off the 3 and 6, and also the 7 and 2. The only digits to add are the 4 and 1. Therefore, the digital root is **5**.

Exercise 2

Find the digital roots of the following numbers, using either method. Remember that 9 is not a digital root. Write zero (0) if the sum is 9 (cast out the 9).

a. 25	h. 8
b. 48	i. 54
c. 53	j. 74

d. 829	k. 481
e. 5402	l. 936
f. 3455	m. 8314
g. 47522	n. 647

? Exercise 3

Did you obtain the same answers for exercises 1 and 2?

Note: This method of finding the digital root only has to do with the number 9. You can't find the remainder of a number when dividing by 7 by adding the digits.

Using Digital Roots is to Check Addition and Multiplication Problems.

To check an addition problem, add the digital roots of the addends. Then, check to see if the digital root of that sum is the same as the digital root of the actual sum of the addends. This works whether there are only two addends, or several addends.

Below are some examples of how to check addition. The actual addition is shown to the left. Arrows are used to show the digital roots of the addends and sum. To check, the digital roots of the addends are added together, and then the digital root of that sum is computed. Compare it to the digital root of the actual sum. If they are equal, put a check to indicate the answer is ***probably*** correct. *NOTE: There is a **slight** chance that the digital roots match, but the answer is still not correct due to some other mistake, like transposing digits. For instance, in example 1 below, it's possible someone might write down 1153 for the answer. The digital roots in the check would match. The possibility of this happening is slight, so we generally assume the problem was done correctly if the digital roots check.* On the other hand, if the digital roots do not check, you know it is wrong.

✓ Example 1

$$\begin{array}{r} 723 \rightarrow 3 \\ +412 \rightarrow 7 \\ \hline 1135 \rightarrow 1 \end{array}$$

Ck: 3 + 7 = 10 $\rightarrow$ 1 $\surd$ Correct!

Explanation: To check, add the digital root of the addends (3 + 7 = 10); then find the digital root of 10 (1). Verify that this equals the digital root of the sum, 1135 (1). Since it does, the addition problem was probably done correctly.

✓ Example 2

$$\begin{array}{r} 463 \rightarrow 4 \\ +529 \rightarrow 7 \\ \hline 1630 \rightarrow 1 \end{array}$$

Ck: 4 + 7 = 11 $\rightarrow$ 2 Wrong!

Explanation: To check, add the digital root of the addends (4 + 7 = 11); then find the digital root of 11 (2). This should equal the digital root of the sum, 1630 (1). Since it doesn't , there is a mistake and the addition problem was done incorrectly.

? Exercise 4

Someone did the following addition problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Note that the procedure is the same if there are more than two addends. Add the digital roots of all the addends. Show Work!

a. $\begin{array}{r} 4983 \\ +6829 \\ \hline 10802 \end{array}$	b. $\begin{array}{r} 5567 \\ +4987 \\ \hline 10554 \end{array}$	c. $\begin{array}{r} 3467 \\ +2541 \\ \hline 5908 \end{array}$
d. $\begin{array}{r} 8972 \\ 8876 \\ +5873 \\ \hline 23721 \end{array}$	e. $\begin{array}{r} 5873 \\ 3674 \\ +4763 \\ \hline 13260 \end{array}$	f. $\begin{array}{r} 4789 \\ 9835 \\ +5301 \\ \hline 19925 \end{array}$

For a multiplication problem, the check is similar, except to check, you **multiply** the digital roots of the numbers you are multiplying. To check addition, you ADD the digital roots of the addends and check against the digital root of the sum. To check multiplication, you MULTIPLY the digital root of the numbers being multiplied and check against the digital root of the product. Here are some examples that provide an answer someone might have written down after doing the multiplication problem on another piece of paper.

Example 1

$$\begin{array}{r} 85 \to 13 \to 4 \\ \times 63 \to 0 \\ \hline 5355 \to 18 \to 0 \end{array}$$

Ck: $4 \times 0 = 0$ Correct!

Explanation: To check, multiply the digital roots of the numbers you are multiplying ($4 \times 0 = 0$); then find the digital root of 0 (0). Verify that it equals the digital root of the product, 5355 (0). Since it does, the multiplication problem was probably done correctly.

Example 2

$$\begin{array}{r} 43 \to 7 \\ \times 52 \to 7 \\ \hline 2136 \to 3 \end{array}$$

Ck: $7 \times 7 = 49 \to 4$ Incorrect!

Explanation: To check, multiply the digital roots of the numbers you are multiplying ($7 \times 7 = 49$); then find the digital root of 49 (4). This should equal the digital root of the product, 2136 (3). Since it doesn't , there is a mistake. Therefore, the multiplication problem was done incorrectly.

Exercise 5

Someone did the following multiplication problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show Work!

a. $\begin{array}{r} 74 \\ \times 53 \\ \hline 3922 \end{array}$	b. $\begin{array}{r} 29 \\ \times 36 \\ \hline 944 \end{array}$	c. $\begin{array}{r} 68 \\ \times 28 \\ \hline 1904 \end{array}$
d. $\begin{array}{r} 65 \\ \times 79 \\ \hline 5125 \end{array}$	e. $\begin{array}{r} 81 \\ \times 55 \\ \hline 4455 \end{array}$	f. $\begin{array}{r} 48 \\ \times 52 \\ \hline 2596 \end{array}$
g. $\begin{array}{r} 569 \\ \times 61 \\ \hline 34709 \end{array}$	h. $\begin{array}{r} 653 \\ \times 524 \\ \hline 242172 \end{array}$	i. $\begin{array}{r} 2333 \\ \times 1203 \\ \hline 2806599 \end{array}$

Exercise 6

Add or multiply the following as indicated, then use digital roots to check the answer to each problem. Show Work!

a. $\begin{array}{r} 7362 \\ +5732 \\ \hline \end{array}$	b. $\begin{array}{r} 8308 \\ +956 \\ \hline \end{array}$
d. $\begin{array}{r} 6784 \\ +6835 \\ \hline \end{array}$	e. $\begin{array}{r} 9994 \\ +8721 \\ \hline \end{array}$
f. $\begin{array}{r} 57 \\ \times 8 \\ \hline \end{array}$	g. $\begin{array}{r} 34 \\ \times 7 \\ \hline \end{array}$
g. $\begin{array}{r} 87 \\ \times 52 \\ \hline \end{array}$	h. $\begin{array}{r} 825 \\ \times 13 \\ \hline \end{array}$

Using Digital Roots is to Check Subtraction Problems.

To check subtraction, you usually add the difference (the answer) to the subtrahend (the number that was subtracted) and see if you get the minuend (the number you subtracted from). In other words, to check that 215 – 134 = 81 was done correctly, simply add 81 and 134, which is 215. To check subtraction using digital roots, simply add the *digital root* of the difference to the *digital root* of the subtrahend and see if you get the *digital root* of the minuend. To check that 215 – 134 = 81 was done correctly, simply add the digital roots 81 and 134: 0 + 8 = 8. Verify that 8 is the digital root of the minuend, 215. It is! Here is another example. Let's say someone did the following subtraction problem: 5462 – 2873 = 2589. To check, add the digital roots of 2589 and 2873, or 6 + 2 to get 8. Since 8 is also the digital root of 5462, the answer is probably correct.

Here are some more examples of how to check subtraction using digital roots.

Example 1

$$\begin{array}{r} 7362 \to 0 \\ -5732 \to 8 \\ \hline 1630 \to 1 \end{array}$$

Ck: 1 + 8 = 9 → 0 √ Correct!

Explanation: To check, add the digital root of the difference and subtrahend (1 + 8 = 9). The digital root of 9 is 0. Verify that this equals the digital root of the minuend, 7632 (0). Since it does, the subtraction was probably done correctly.

Example 2

$$\begin{array}{r} 5073 \to 6 \\ -878 \to 5 \\ \hline 4205 \to 2 \end{array}$$

Ck: 2 + 5 = 7 WRONG!

Explanation: To check, add the digital root of the difference and subtrahend (2 + 5 = 7). This should equal the digital root of the minuend, 5073 (6). Since it doesn't, the subtraction was not done correctly.

Exercise 7

Someone did the following subtractions problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show Work!

a. $\begin{array}{r} 572 \\ -356 \\ \hline 216 \end{array}$	b. $\begin{array}{r} 296 \\ -189 \\ \hline 107 \end{array}$	c. $\begin{array}{r} 501 \\ -327 \\ \hline 284 \end{array}$

d. $\begin{array}{r} 3217 \\ -2391 \\ \hline 926 \end{array}$	e. $\begin{array}{r} 3546 \\ -1138 \\ \hline 2408 \end{array}$	f. $\begin{array}{r} 7502 \\ -4729 \\ \hline 2773 \end{array}$

Exercise 8

Subtract, add or multiply the following as indicated, then use digital roots to check the answer to each problem. Show Work!

a. $\begin{array}{r} 7362 \\ -5732 \end{array}$	b. $\begin{array}{r} 8308 \\ -956 \end{array}$
d. $\begin{array}{r} 6784 \\ -6335 \end{array}$	e. $\begin{array}{r} 9994 \\ -8721 \end{array}$
f. $\begin{array}{r} 557 \\ +348 \end{array}$	g. $\begin{array}{r} 834 \\ +767 \end{array}$
h. $\begin{array}{r} 48 \\ \times 6 \end{array}$	i. $\begin{array}{r} 13 \\ \times 29 \end{array}$

Using Digital Roots is to Check Division Problems.

To check division, you multiply the divisor (the number you are dividing by) by the quotient (the answer), and then add the remainder. It is correct if the answer obtained is equal to the dividend (the number you divided into). In other words, here is how to check this division, $431 \div 34 = 12$r. 23: Multiply 34 by 12 and add 23. Here are the steps: $34 \times 12 + 23 = 408 + 23 = 431$. Since 431 is the dividend, this problem was done correctly. To check division using digital roots, simply multiply the *digital root* of the divisor by the *digital root* of the quotient, and add the *digital root* of the remainder. If the digital root of that number equals the *digital root* of the dividend, it is probably correct.

Note: After the digital roots of the divisor and quotient are multiplied, you can first find the digital root of that product before adding the digital root of the remainder.

Here is how to check the division problem we did above by using digital roots. Multiply the digital root of the divisor, 34, by the digital root of the quotient, 12: $7 \times 3 = 21$. Then add the digital root of the remainder, 23: 21 + 5 = 26. Determine the digital root of that number, 26, which is 8. Check to see if 8 matches the digital root of the dividend, 431. It does. As stated in the note, you could determine the digital root of 21 before adding the digital root of the remainder. In this case, it would have looked like this: $7 \times 3 = 21 \rightarrow 3$. Then add the digital root of the remainder, 23: 3 + 5 = 8. Verify that this equals the digital root of the dividend, 431. It does. Therefore, the division was probably done correctly. More examples follow. The check without digital roots is shown to the right. The digital root check is shown under the division problem.

Example 1

Example 1:

$$\begin{array}{r} 0 \leftarrow \quad 45 \text{ r. } 15 \quad \rightarrow 6 \\ 0 \leftarrow 27 \,\big|\, 1230 \rightarrow 6 \end{array}$$

Digital Root Check:

0 x 0 = 0; 0 + 6 = 6 Correct!

Explanation: To check, multiply the digital root of the divisor, 27, by the digital root of the quotient, 45: 0 x 0 = 0. Add the digital root of the remainder, 15: 0 + 6 = 6. Since this equals the digital root of the dividend, 1230, the division was probably done correctly.

Check:

$$\begin{array}{r} 27 \\ \times\ 45 \\ \hline 135 \\ 108 \\ \hline 1215 \\ +15 \\ \hline 1230 \end{array}$$

Example 2

Example 2:

$$\begin{array}{r} 1 \leftarrow \quad 19 \text{ r. } 32 \quad \rightarrow 5 \\ 7 \leftarrow 34 \,\big|\, 678 \rightarrow 21 \rightarrow 3 \end{array}$$

Digital Root Check:

7 x 1 = 7; 7 + 5 = 12 --> 3 Correct!

Explanation: To check, multiply the digital root of the divisor, 34, by the digital root of the quotient, 19: 7 x 1 = 7. Add the digital root of the remainder, 32: 7 + 5 = 12 --> 3. Since this equals the digital root of the dividend, 678, the division was probably done correctly.

Check:

$$\begin{array}{r} 34 \\ \times\ 19 \\ \hline 646 \\ +32 \\ \hline 678 \end{array}$$

Below is one more example for you to study.

Example 3

Example 3:

$$\begin{array}{r} 6 \leftarrow \quad 213 \text{ r. } 271 \quad \rightarrow 1 \\ 8 \leftarrow 458 \,\big|\, 97831 \rightarrow 10 \rightarrow 1 \end{array}$$

Digital Root Check:

8 x 6 = 48 --> 12 --> 3; 3 + 1 = 4 WRONG!

Explanation: To check, multiply the digital root of the divisor, 458, by the digital root of the quotient, 213: 8 x 6 = 48 --> 12 --> 3. Add the digital root of the remainder, 271: 3 + 1 = 4. Since 4 does not equal the digital root of the dividend, 97831, the division was done incorrectly.

Check:

$$\begin{array}{r} 458 \\ \times\ 213 \\ \hline 97554 \\ +271 \\ \hline 97825 \end{array}$$

Incorrect!

Exercise 9

Someone did the following division problems, but only wrote down the answers. Check the answer to each problem by using digital roots. Show work. You don't need to do the actual check as shown to the right of the last three examples. Show work!

a.

$$73\overline{)5609} \quad 76 \text{ r. } 61$$

b.

$$123\overline{)6723} \quad 54 \text{ r. } 81$$

c.

$$26\overline{)5134} \quad 197 \text{ r. } 10$$

d.

$$78\overline{)7988} \quad 102 \text{ r. } 32$$

One more note of caution about using digital roots:

In problem 9d, a common mistake some people make is to forget to put the zero in the quotient, 102. Instead, the answer might have been written as 12 r. 32. Using digital roots, the answer would check. So, you should also check the reasonableness of the answer by approximating. For instance, 78×12 is about 80 times 10 or 800, which is not even close to equaling the dividend, 7988. Another time digital roots might fail is when someone transposes the digits of a number. If the answer to a problem was 465, and it was written as 456, using digital roots wouldn't detect the mistake.

Exercise 10

Do the following division problem, and check the answer using digital roots. Show work.

$23972 \div 156 =$ _____

Later in this module, we will explore prime numbers, composite numbers, the greatest common factor of two or more numbers, and the least common multiple of two or more numbers. Factoring is the method used to find the prime factorization of a composite number, and it can also be used to find the greatest common factor and least common multiple of a set of numbers. One of the problems in factoring large numbers is that sometimes it isn't clear if it is prime or composite. In other words, it isn't clear whether it has any factors other than 1 or itself. Most of us know that if the last digit of a numeral is even, then 2 will divide into it; or if it ends in 0, 10 will divide into it; or if it ends in 0 or 5, that 5 will divide into it. Sometimes, people even have trouble determining if relatively small numbers are prime. For instance, many people think 91 is prime, but in fact it is not. Knowing some divisibility tests makes the task easier, so we'll soon take some time to discuss divisibility tests for several numbers. First, we need to go over some notation concerning divisibility.

When you see 12/3, this means 12 "**divided by**" 3. The slash that slants to the right is another way to write the division sign, ÷. 12/3 (or $12 \div 3$) is a division problem, and the answer is 4.

Here is something altogether different. If I say "3 **divides** 12", I am making a *statement*. "3 divides 12" is not a division problem that needs to be done. It is a statement that happens to be true. The symbol used to represent the word "divides" is a vertical line. So, "3 **divides** 12" can be written "3|12". Again, this is a statement, a fact, not a division problem. The way to express "does not divide" is to put a slash through the symbol: $\nmid$

So, how do I know "3 ***divides*** 12" is a true statement? The definition of "divides" follows:

Definition: **"a divides b" if there exists a whole number, n, such that an = b.** This is simply saying that **"a divides b" means a is a factor of b!** In shorthand notation, this is written **"a|b if there exists a whole number, n, such that an = b, or a|b means that a is a factor of b."**

Okay, so what exactly does "there exists a whole number, n, such that an = b" mean? Well, it means that the first number times some whole number equals the second number. Or you can think "a divides b" is true if $b \div a$ is a whole number.

Let's get back to why "3 **divides** 12" is a true statement. Hmmm...you can be formal and ask yourself: Is there a whole number n such that 3n = 12? Or, you can simply ask yourself: Is 3 a factor (or divisor) of 12? In either case, the answer is yes, so the statement is true.

Let's work a little more on the difference between a statement using "divides", and an actual division problem.

Examples: Determine if each of the following is a statement or if it is a division problem. If it is a statement, state if it is true or false and back up your answer. If it is a division problem, state the answer to the division problem.

✔ Example 1: 4|20

4|20

Solution

This is a true statement, because 4 is a factor of 20 (since $4 \cdot 5 = 20$)

✔ Example 2: 20/6

20/6

Solution

This is a division problem. The answer is 3 r. 2

✔ Example 3: 15|3

15|3

Solution

This is a false statement because 15 is not a factor of 3

✔ Example 4: 6 divides 20

6 divides 20

Solution

This is a false statement because 6 is not a factor of 20

✔ Example 5: 3 divides 21

3 divides 21

Solution

This is a true statement because 3 is a factor of 21 (since $3 \cdot 7 = 21$)

? Exercise 11

Determine if each of the following is a statement or if it is a division problem. If it is a statement, then decide if it is true or false and back up your answer. If it is a division problem, state the answer to the division problem. Study the examples on the previous page if you need help getting started.

a. 35/7
b. 35|7
c. 7|35:
d. 40/7
e. 56|8
f. 7|40:
g. 12 divides 60
h. 80 divided by 30
i. 70 divided by 5
j. 42 divides 3
k. 6 divides 42
l. 80 divided by 10
m. 100/2
n. 4|100
o. 4|90:
p. 25|5

Let's go back to the formal definition of "divides":

Definition: **a|b if there exists a whole number, n, such that an = b.**

We need to use this formal notation in order to do some proofs. Consider the following:

Example 1

Is the following statement true? If a|b and a|c, then a|(b+c).

a|b and a|c means that a is a factor of b and a is also a factor of c. We must determine if that necessarily implies that a is also a factor of the sum, b + c. The first line of strategy is to test it out on a few numbers, and see if you can find a counterexample. If you find a counterexample, the answer is no and you are done. If you can't find a counterexample, maybe it is true. If you think it is true, you must PROVE it by being general and formal.

I would start by picking any number for a, like a = 3, and then choose numbers for b and c for which 3 is a factor, like 15 and 18. Plug them in: "If 3|15 and 3|18, does 3|(15+18)?" Since 3|33, the answer is yes. It looks like this statement might be true.

Exercise 12

Choose different numbers for a, b and c in "If a|b and a|c, then a|(b+c)" to see if the statement seems to be true.

The statement, "If a|b and a|c, then a|(b+c)", happens to be true. You must write a proof to prove that it is always true. Here is one way to write a formal proof:

Example 1

Is the following statement true?: If a|b and a|c, then a|(b+c).

Solution

If a|b, then an = b for some whole number, n. If a|c, then am = c for some whole number, m. Using these substitutions for b and c, we get that a|(b + c) is true if a|(an + am) which is true if a is a factor of an + am. Factor: an + am = a(n + m). This clearly shows that a is indeed a factor of an + am. Therefore, if a|b and a|c, then a|(b+c).

? Exercise 13

Write a formal proof to show that the following is true: If x|y and x|z, then x|(y + z).

✓ Example 2

Is the following statement true? If a|(b+c), then a|b and a|c.

a|(b + c) means that a is a factor of the sum, b + c. The question asks if that necessarily implies that a is a factor of b and also a factor of c. The first line of strategy is to test it out on a few numbers, and see if you can find a counterexample, or if it looks like it is true. If you find a counterexample, the answer is no and you are done. If you think it is true, you must PROVE it by being general and formal.

Start by picking any number for a, like a = 3 and picking a number for the sum b + c, for which 3 is a factor, like 15 or 18, etc. Let b + c = 15. Note that there are many combinations of numbers that add up to 15: 1 + 14, 2 + 13, 3 + 12, 7 + 8, etc. You are being asked if 3 divides into the sum of any two whole numbers, will it necessarily divide into each individual addend as well? For instance, if you broke 15 up as the sum of 9 and 6, this would be the statement: "If 3|(9 + 6), then 3|9 and 3|6." Using these numbers, it is true and you haven't found a counterexample. Try splitting up 15 another way, perhaps as the sum of 10 and 5. Then, the statement becomes: "If 3|(10 + 5), then 3|10 and 3|5." The answer is no, and therefore these numbers may be used as a counterexample, which proves the statement is false. Remember: In order for a statement to be true, it must be for all values of a and b. On the other hand, only one counterexample showing it is not true is sufficient to prove a statement is false.

? Exercise 14

Consider the statement: "If a|(b+c), then a|b and a|c." Provide an example using numbers other than a = 3, b = 9 and c = 6 that makes it look like this statement might be true.

? Exercise 15

Consider the statement: "If a|(b+c), then a|b and a|c." Provide a counterexample using numbers other than a = 3, b = 10 and c = 5 to show this statement is false.

? Exercise 16

Consider the statement: "If a|c and b|c, then (a + b)|c." Provide a counterexample to show this statement is false.

? Exercise 17

Consider the statement: "If (a + b)|c, then a|c and b|c." Provide a counterexample to show this statement is false.

? Exercise 18

Prove that the following statement is true: "If a|b and a|c, then a|(bc)"

✓ Example

Prove that the following statement is true: "If a|b and a|(b + c), then a|c"

Solution

If a|b, then am = b for some whole number, m. If a|(b + c), then an = b + c for some whole number, n. Keep in mind that since b and c are positive, then (b + c) > b, which means n > m. We are trying to prove that a is a factor of c. Since am = b, we can substitute am for b into the equation an = b + c, which means an = am + c. Solving for c, this is equivalent to an – am = c. So if a is a factor of an – am, then a is a factor of c. Factor: an – am = a(n – m). This clearly shows a is a factor of an – am, which

means a is a factor of c. Therefore, the statement "If a|b and a|(b + c), then a|c" is true. (Note: n – m must be a whole number since n and m are whole numbers and n > m.)

? Exercise 19

Prove that the following statement is true: "If c|a and c|(a + b), then c|b"

Okay, finally on to the divisibility tests. We will use our new notation for divides (|).

All counting numbers can be considered factors of zero. In other words, for all counting numbers, m, m|0 is always true since there is always some number times m that equals zero, namely zero itself.

Assume n is a positive whole number. The following are divisibility tests to determine what numbers divide into n, or what numbers are factors of n.

Divisibility Test for 2: 2|n if the last digit of n is even (0, 2, 4, 6, or 8)

✓ Examples

Use the divisibility test for 2 to determine if the following is true or false

1. 2|97

This is false because the last digit (7) is not even

2. 2|356

This is true because the last digit (6) is even

Divisibility Test for 4: (think of 4 as 2^2)

4|n if 4|(the number represented by the last two digits of n)

✓ Examples

Use the divisibility test for 4 to determine if the following is true or false.

1. 4|527

This is false because 4 does not divide 27, (since 4 is not a factor of 27).

2. 4|25,356

This is true because 4|56.

3. 4|624

This is true because 4|24.

Divisibility Test for 8: (think of 8 as 2^3)

8|n if 8|(the number represented by the last three digits of n)

✓ Examples

Use the divisibility test for 8 to determine if the following is true or false.

1. 8|42,527

This is false because 8 does not divide 527

2. 8|25,336

This is true because 8|336

3. 8|7,624

This is true because 8 does divide 624, (since 8 is a factor of 624)

It should be clear to you that if a number is not divisible by 2, it is not divisible by 4 or 8; and if it is not divisible by 4, it is not divisible by 8. Conversely, if a number is divisible by 8, then it is divisible by both 2 and 4; and if it is divisible by 4, it is divisible by 2.

? Exercise 20

Use the divisibility tests for 2, 4 and 8 to determine if the following is true or false. Support your answer with a reason using the appropriate divisibility test.

a. 2|9,712 _____

b. 2|5,643 _____

c. 4|5,690 _____

d. 4|63,868 _____

e. 4|854,100 _____

f. 8|12,345,248 _____

g. 8| 54,094,422 _____

Divisibility Test for 5: 5|n if the last digit of n is 0 or 5

✓ Examples

Use the divisibility test for 5 to determine if the following is true or false.

1. 5|527

This is false because 7 is the last digit of 527

2. 5|25,335

This is true because 5 is the last digit of 25,335

3. 5|7,620

This is true because 0 is the last digit of 7,620

Divisibility Test for 10: 10|n if the last digit of n is 0

✓ Examples

Use the divisibility test for 10 to determine if the following is true or false.

1. 10|527

This is false because 7 is the last digit of 527

2. 10|25,335

This is false because 5 is the last digit of 25,335

3. 10|7,620

This is true because 0 is the last digit of 7,620

Exercise 21

Use the divisibility tests for 5 and 10 to determine if the following is true or false. Support your answer with a reason using the appropriate divisibility test.

a. 5|9,750 _____

b. 5|5,645 _____

c. 5|5,696 _____

d. 10|63,860 _____

e. 10|854,105 _____

Divisibility Test for 3: 3|n if 3|(the digital root of n)

Note: This is equivalent to saying 3|n if the digital root of n is 0, 3 or 6

Examples

Use the divisibility test for 3 to determine if the following is true or false.

1. 3|97

This is false because 3 does not divide 7, which is the digital root of 97

2. 3|356

This is false because 3 does not divide 5, which is the digital root of 356

3. 3|738

This is true because 3|0, where 0 is the digital root of 738

Exercise 22

Use the divisibility test for 3 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 3. Show work.

a. 3|9,750 _____

b. 3|5,645 _____

c. 3|5,696 _____

d. 3|63,860 _____

e. 3|854,115 _____

Divisibility Test for 9: 9|n if 9|(the digital root of n)

Note: **This is equivalent to saying that the digital root of n must equal zero.** Take a moment to think about this divisibility test for 9. In exercise 16, we discovered that the digital root of a number is the same as the remainder you obtain when you divide a number by 9. In order for a number to be divisible by 9, the remainder would be zero, which is exactly what the digital root would be.

Examples

Use the divisibility test for 9 to determine if the following is true or false.

1. 9|627

This is false because 9 does not divide 6, which is the digital root of 627.

2. 9|25,334

This is false since 9 does not divide 8, which is the digital root of 25,334.

3. 9|7,533

This is true because the digital root of 9 is 0.

Exercise 23

Use the divisibility test for 9 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 9. Show work.

a. 9|9,753 _____

b. 9|5,646 _____

c. 9|5,697 _____

d. 9|63,576 _____

e. 9|854,103 _____

Divisibility Test for 6: 6|n if 2|n AND 3|n. (Think of 6 as $2 \cdot 3$)

Important Note: A number is divisible by 6 only if it passes the divisibility test for **both** 2 and 3. If it doesn't pass one of the tests, then it is not divisible by 6. To show it is divisible by 6, you must show it passes both of the tests.

Examples

Use the divisibility test for 6 to determine if the following is true or false.

1. 6|627

This is false because 2 does not divide 7 (the last digit, which is not even).

2. 6|25,334

This is false because 3 does not divide 25,334, since 3 does not divide the digital root, which is 8.

3. 6|7,620

This is true because 2|7,620 (since the last digit, 0, is even) AND 3|7,620 (since 3|6, where 6 is the digital root of 7,620).

Exercise 24

Use the divisibility test for 6 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 6.

a. 6|9,753

b. 6|5,645

c. 6|5,696

d. 6|63,876

e. 6|854,103

Divisibility Test for 15: 15|n if 5|n AND 3|n. (Think of 15 as $5 \cdot 3$)

Important Note: A number is divisible by 15 only if it passes the divisibility test for both 5 and 3. If it doesn't pass one of the tests, then it is not divisible by 15. To show it is divisible by 15, you must show it passes both of the tests.

Examples

Use the divisibility test for 15 to determine if the following is true or false.

1. 15|623: This is false because 5 does not divide 623 (since the last digit is not 0 or 5).

2. 15|24,335: This is false because 3 does not divide 24,335, since 3 does not divide the digital root, which is 8.

3. 15|7,620: This is true because 5|7,620 (since the last digit is 0) AND 3|7,620 (since 3|6, where 6 is the digital root of 7,620).

Exercise 25

Use the divisibility test for 15 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 15.

a. 15|9,753
b. 15|6,645
c. 15|5,690
d. 15|63,872
e. 15|654,105

Divisibility Test for 7: This test isn't easy to describe. Below are the steps.

Step 1: Cross off the one's digit of the number to get a number with one less place value.

Step 2: Double the one's digit you crossed off and subtract from the new number obtained with the one's digit missing.

Step 3: If 7 divides the number you get after subtracting, it divides the original number. Otherwise, it doesn't. If you aren't sure, repeat the procedure on the new number by going back to step 1.

Examples

Use the divisibility test for 7 to determine if the following is true or false. The way you would show the steps using the numbers is shown to the right of the explanation.

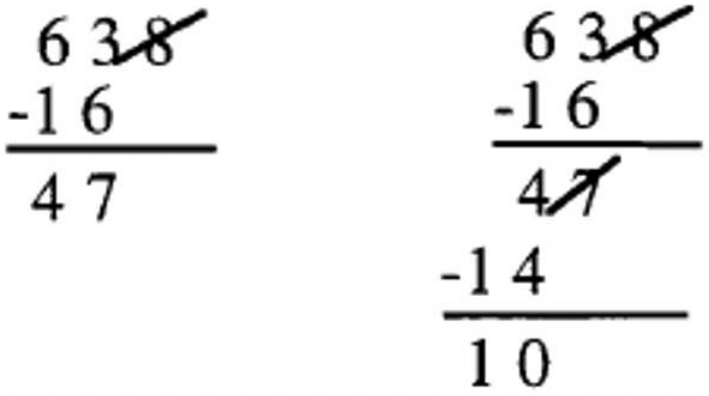
1. 7|91: Cross off the 1, double it (2), and subtract from what is left (9). The answer is 7 (9 – 2 = 7). Since 7|7, 7|91 is true.

$$\begin{array}{r} 9\not{1} \\ -2 \\ \hline 7 \end{array}$$

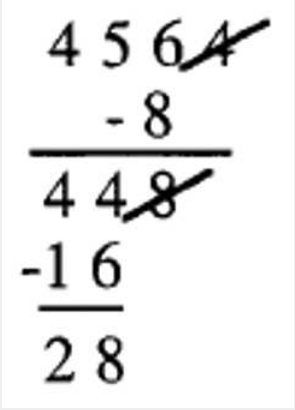
2. 7|96: Cross off the 6, double it (12), and subtract from what is left (9). When you subtract, ignore the sign (just do 12 – 9 = 3). Since 7 does not divide 3, 7|96 is false.

$$\begin{array}{r} 9\not{6} \\ -12 \\ \hline 3 \end{array}$$

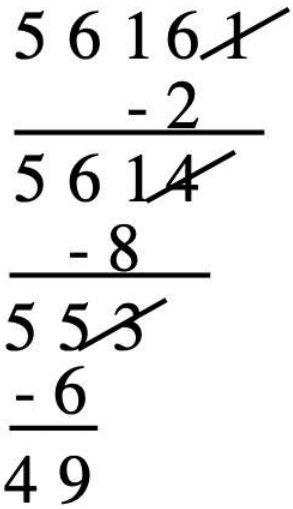
3. 7|638: Cross off the 8, double it (16) and subtract from what is left (63). The answer is 47 (63 – 16 = 47). Since 7 does not divide 47, then 7|638 is false. (If you weren't sure whether or not 7 divided 47, you can take it a step further; this is shown to the right.)



4. 7|4564: Cross off the 4, double it (8) and subtract from what is left (456). The answer is 448 (456 – 8 = 456). Cross off the 8, double it (16) and subtract from what is left (44). The answer is 28 (44 – 16 = 28). Since 7|28, then 7|4564 is true.



5. 7|56161: Cross off the 1, double it (2) and subtract from what is left (5616 – 2 = 5614). Cross off the 4, double it (8) and subtract from what is left (561 – 8 = 553). Cross off the 3, double it (6) and subtract from what is left (55 – 6 = 49). Since 7|49, then 7|56161 is true.



Exercise 26

Use the divisibility test for 7 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 7. Show work.

a. 7|833
b. 7|5,645
c. 7|4,795
d. 7|14,763

Divisibility Test for 11: 11|n if the difference between the sum of the digits in the places that are even powers of 10 and the sum of the digits in the places that are odd powers of 10 is divisible by 11.

This test is more confusing to describe than to do. What you do is add up every other digit. Then, add up the ones you skipped. Then, subtract these two numbers and see if 11 divides this number.

Examples

Use the divisibility test for 11 to determine if the following is true or false.

a. 11|4,365: Add 4 + 6 = 10 Add 3 + 5 = 8 Subtract 10 – 8 = 2 Since 11 doesn't divide 2, then 11 doesn't divide 4365. Therefore, 11|4365 is false.

b. 11|540,879,216: Add 5 + 0 + 7 + 2 + 6 = 20. Add 4 + 8 + 9 + 1 = 22. Subtract 22 – 20 = 2 Since 11 doesn't divide 2, 11|540,879,216 is false.

c. 11|542,879,216: Add 5 + 2 + 7 + 2 + 6 = 22 Add 4 + 8 + 9 + 1 = 22 Subtract 22 – 22 = 0 Since 11|0, then 11|542,879,216 is true.

d. 11|4,052,631: Add 4 + 5 + 6 + 1 = 16 Add 0 + 2 + 3 = 5 Subtract 16 – 5 = 11 Since 11|11, then 11|4,052,631 is true.

Exercise 27

Use the divisibility test for 11 to determine if the following is true or false. Support your answer with a reason using the divisibility test for 11. Show work.

a. 11|9,053
b. 11|63,920,876
c. 11|568,696
d. 11|513,645
e. 11|803,003,808

Listed below are the divisibility tests that you should know. It's important to realize that each of the tests only work for the number specified. In other words, you can't use the divisibility test for 3 to determine is 7 divides a number. The divisibility test for 7 has nothing to do with digital roots!

Divisibility Test for 2: 2|n if the last digit of n is even (0, 2, 4, 6, or 8)

Divisibility Test for 3: 3|n if 3|(the digital root of n); OR 3|n if the digital root of n is 0, 3 or 6

Divisibility Test for 4: 4|n if 4|(the number represented by the last two digits of n)

Divisibility Test for 5: 5|n if the last digit of n is 0 or 5

Divisibility Test for 6: 6|n if 2|n AND 3|n. (Think of 6 as $2 \cdot 3$)

Divisibility Test for 7: The steps of this test are described below.

- Step 1: Cross off the one's digit of the number to get a number with one less place value.
- Step 2: Double the one's digit you crossed off and subtract from the new number without the one's digit.
- Step 3: If 7 divides the number you get after subtracting, it divides the original number. Otherwise, it doesn't. If you aren't sure, repeat the procedure on the new number by going back to step 1.

Divisibility Test for 8: 8|n if 8|(the number represented by the last three digits of n)

Divisibility Test for 9: 9|n if 9|(the digital root of n); OR 9|n if the digital root of n is zero

Divisibility Test for 10: 10|n if the last digit of n is 0

Divisibility Test for 11: 11|n if the difference between the sum of the digits in the places that are even powers of 10 and the sum of the digits in the places that are odd powers of 10 is divisible by 11

Divisibility Test for 15: 15|n if 5|n AND 3|n

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12.3: Divisibility Tests

Divisibility Rules

Table 5.3.1: Divisibility Rules

Divisible by ___?	The Trick!
2	Last digit is even
3	Add up the digits and if the sum is divisible by 3 then so is the original number
4	Divide last 2 digits by 4
5	Ends in 0 or 5
6	Rules for 2 and 3 work
8	Divide last 3 digits by 8
9	Add up the digits and if the sum is divisible by 9 then so is the original number
10	Ends in 0

Example 12.3.1

What divides evenly into 3495?

Solution

Table 5.3.2

Number	Check it!	Yes or No?
2	Is 3495 even?	no
3	Add the digits: 3 + 4 + 9 + 5 = 21 and 21 divides 3 evenly	yes
4	Can 95 divide 4 evenly?	no
5	Ends in 0 or 5?	yes
6	Divisible by both 2 AND 3?	no
8	Can 495 divide 8 evenly?	no
9	Add the digits: 3 + 4 + 9 + 5 = 21 and 21 does not divide 9 evenly	no
10	Ends in 0?	no



Figure 5.3.1

Practice Problems

Test if the numbers in the left column are divisible by the numbers in the top row. Put a “X” in the box where divisibility holds true. Show any work below the table.

Table 5.3.3

	2	3	4	5	6	8	9	10
67820								
512								
49								
3463								

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CHAPTER OVERVIEW

13: Algebraic Thinking

13.1: Introduction
13.2: Examples
13.3: Equality
13.4: Growing Patterns
13.5: Structural and Procedural Algebra
13.6: Practice

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13.1: Introduction

Algebra skills are essential for your future students. Why? Here are just a few reasons:

- Mathematics, and especially algebra, is the language of science and modern technology. Thinking algebraically helps you to make sense of the world, to understand and interact with technology more productively, and to succeed in other fields.
- Algebra is a tool for solving problems. This may not be your experience so far, but it is true. If you are able to "algebratize" a problem, that often helps lead you to a solution.
- Algebra helps you to think abstractly. It is a tool for thinking about operations like addition, subtraction, multiplication, and division separate from doing calculations on particular numbers. Algebra helps you to *understand and explain why* the operations work the way they do, to *describe* their properties clearly, and to *manipulate* expressions to see the bigger picture.

You might wonder why future elementary teachers should master algebra, a topic usually studied (by that name, anyway) in 8th grade and beyond. But the Common Core Standards for School Mathematics has standards in "Operations and Algebraic Thinking" beginning in kindergarten!

Everyone who shows up to school has already learned a lot about abstraction and generalization — the fundamental ideas in algebra. They are all capable of learning to formalize these ideas. Your job as an elementary school teacher will be to provide your students with even more experiences in abstraction and generalization in a mathematical context, so that these ideas will seem quite natural when they get to a class with the name "Algebra."

Let's start with a problem:

✔ Problem 1

I can use four 4's to make 0:

$$44 - 44 = 0.$$

I can also use four 4's to make the number 10:

$$(4 \times 4) - 4 - \sqrt{4} = 10.$$

Your challenge: Use four 4's to make all of the numbers between 0 and 20. (Try to find different solutions for 0 and 10 than the ones provided.) You can use any mathematical operations, but you can't use any digits other than the four 4's.

? Think / Pair / Share

- What does "algebra" mean to you?
- What does Problem 1 have to do with "algebra"?
- What do you imagine when you think about using algebra to solve problems in school?
- Have you ever used algebra to solve problems outside of school?
- What is meant by "algebraic thinking," and what kinds of algebraic thinking can be done by elementary school students?

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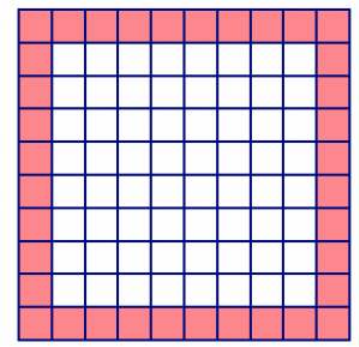
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13.2: Examples

Here's another problem:

Problem 2

Here is a large square made up of 100 smaller unit squares. The unit squares along the border of the large square are colored red. Without counting one-by-one, can you figure out how many red squares there are in the picture?

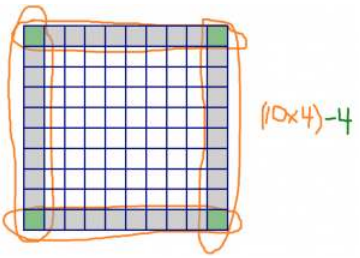


Clearly describe how you figured out the number of red squares, and how you know your answer is correct.

Justin calculated the number of squares as $(10 \times 4) - 4$. He justified his answer this way:

> *Since the dimensions of the big square are* 10×10*, there are 10 squares along each of the four sides. So that gives me 40 red squares. But then each corner is part of two different sides. I've counted each of the corners twice. So I need to make up for that by subtracting 4 at the end.*

Justin showed this picture to justify his work:



Think / Pair / Share

- What do you think about Justin's solution? Are you convinced? Could he have explained it more clearly?
- Was Justin's solution different from your solution or the same?
- Notice the color coding in Justin's picture. What do the colors represent? Why did he use the colors the way he did?

Problem 3

There are lots of different ways to calculate the number of colored squares along the border of a 10×10 square. Below are the calculations several other students did. For each calculation, write a justification and draw a picture to show why it calculates the number of squares correctly. Think about using color in your picture to make your work more clear.

1. Valerie calculated $10 + 10 + 8 + 8$.
2. Kayla calculated 4×9.
3. Linda calculated $(10 \times 10) - (8 \times 8)$.
4. Mark calculated $(4 \times 8) + 4$.

5. Allan calculated $10+9+9+8$.

Problem 4

Now suppose that you have a large 6×6 square with the unit squares along the border colored red. Adapt two of the techniques above to calculate the number of red unit squares.

For each technique you used, write an explanation and include a picture. Think about how to use colors or other methods to make your picture and explanation more clear.

Problem 5

Now suppose that you have a large 25×25 square with the unit squares along the border colored red. Adapt two of the techniques above to calculate the number of red unit squares.

For each technique you used, write an explanation and include a picture. Think about how to use colors or other methods to make your picture and explanation more clear.

Problem 6

1. Suppose that you have 64 red squares. Can you use all of those squares to make the border of a larger square in a picture like the one above? If yes, what are the dimensions of the larger square? If no, why not?
2. What if you have 30 red squares? Same questions.
3. What if you have 256 red squares? Same questions.

Think / Pair / Share

Describe some general rules:

- If you have a large $n \times n$ square with the border squares colored red, how many red squares will there be? Justify your answer with words and a picture.
- If you have k red squares, is there a quick test you can do to decide if you can use all of those squares to make the border of a large square? Can you tell how big the square will be?

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13.3: Equality

The notion of equality is fundamental in mathematics, and especially in algebra and algebraic thinking. The symbol "="' expresses a *relationship*. It is *not* an operation in the way that + and are × operations. It should not be read left-to-right, and it definitely does not mean "… and the answer is …".

For your work to be clear and easily understood by others, it is essential that you use the symbol = appropriately. And for your future students to understand the meaning of the = symbol and use it correctly, it is essential that you are clear and precise in your use of it.

Let's start by working on some problems.

Problem 7

Akira went to visit his grandmother, and she gave him $1.50 to buy a treat.

He went to the store and bought a book for $3.20. After that, he had $2.30 left.

How much money did Akira have before he visited his grandmother?

Problem 8

Examine the following equations. Decide: Is the statement always true, sometimes true, or never true? Justify your answers.

a. $5+3=8$.
b. $\frac{2}{3}+\frac{1}{2}=\frac{3}{5}$.
c. $5+3=y$.
d. $\frac{a}{5}=\frac{5}{a}$.
e. $n+3=m$.
f. $3x=2x+x$.
g. $5k=5k+1$.

Problem 9

Consider the equation

$$18-7=\_\_\_.$$

1. Fill in the blank with something that makes the equation *always true*.
2. Fill in the blank with something that makes the equation *always false*.
3. Fill in the blank with something that makes the equation *sometimes true and sometimes false*.

Problem 10

If someone asked you to *solve* the equations in Problem 8, what would you do in each case and why?

Think / Pair / Share

Kim solved Problem 7 this way this way:

> *Let's see:*
>
> $$2.30+3.20=5.50-1.50=4,$$
>
> *so the answer is 4.*

What do you think about Kim's solution? Did she get the correct answer? Is her solution clear? How could it be better?

Although Kim found the correct numerical answer, her calculation really doesn't make any sense. It is true that

$$2.30 + 3.20 = 5.50.$$

But it is definitely *not* true that

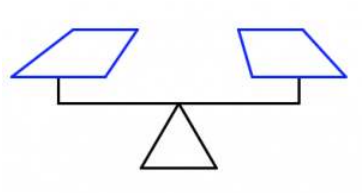
$$2.30 + 3.20 = 5.50 - 1.40.$$

She is incorrectly using the symbol "=", and that makes her calculation hard to understand.

? Think / Pair / Share

- Can you write a good *definition* of the symbol "="? What does it mean and what does it represent?
- Give some examples: When should the symbol "=" be used, and when should it *not* be used?
- Do these two equations express the same relationships or different relationships? Explain your answer.
  $x^2 - 1 = (x+1)(x-1)$. $(x+1)(x-1) = x^2 - 1$.

This picture shows a (very simplistic) two-pan balance scale. Such a scale allows you to *compare* the weight of two objects. Place one object in each pan. If one side is lower than the other, then that side holds heavier objects. If the two sides are balanced, then the objects on each side weigh the same.

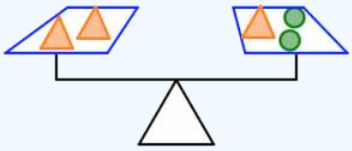


? Think / Pair / Share

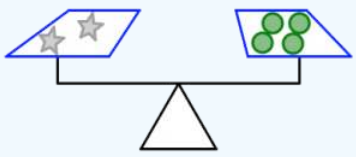
In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The silver stars all weigh the same.
- The scale is balanced.

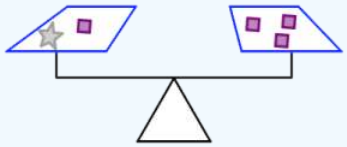
1. In the picture below, what do you know about the weights of the triangles and the circles? How do you know it?



2. In the picture below, what do you know about the weights of the circles and the stars? How do you know it?



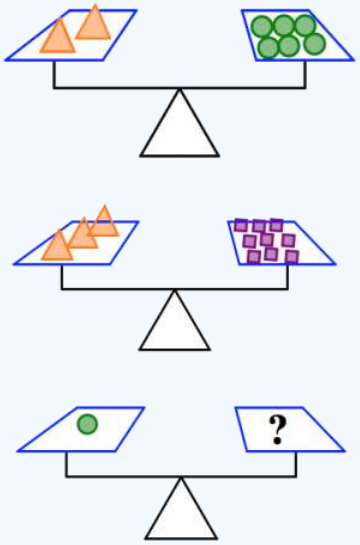
3. In the picture below, what do you know about the weights of the stars and the squares? How do you know it?



Problem 11

In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The scale is balanced.

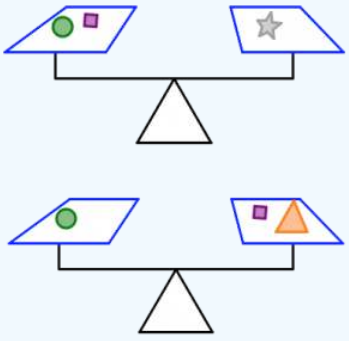


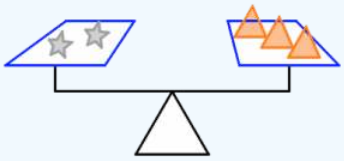
How many purple squares will balance with one circle? Justify your answer.

Problem 12

In the pictures below:

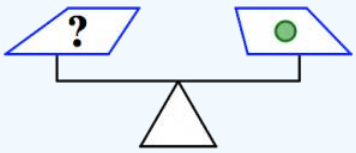
- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The silver stars all weigh the same.
- The scale is balanced.



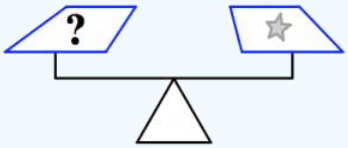


How many purple squares will balance the scale in each case? Justify your answers.

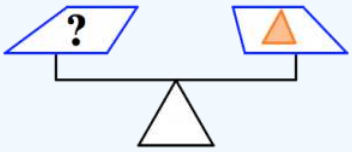
a.



b.



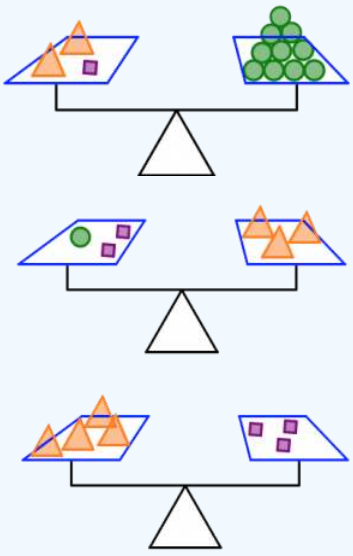
c.

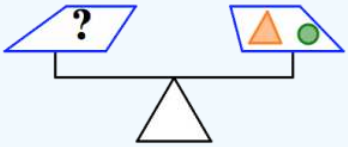


Problem 13

In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The scale is balanced.



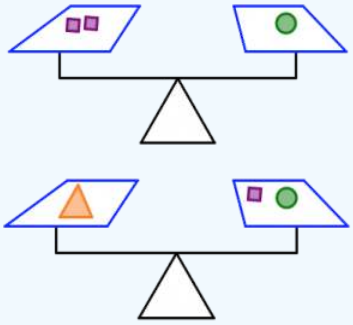


What will balance the last scale? Can you find more than one answer?

Problem 14

In the pictures below:

- The orange triangles all weigh the same.
- The green circles all weigh the same.
- The purple squares all weigh the same.
- The scale is balanced.



1. Which shape weighs the most: the square, the triangle, or the circle? Which shape weighs the least? Justify your answers.
2. Which of the two scales is holding the most total weight? How do you know you're right?

Think / Pair / Share

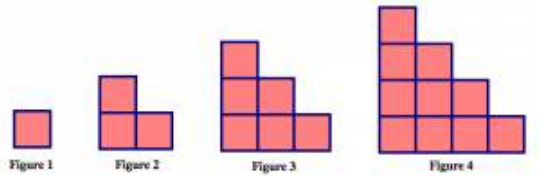
What do Problems 11–14 above have to do with the "=" symbol?

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13.4: Growing Patterns

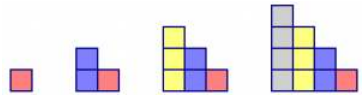
Here is a pattern made from square tiles.



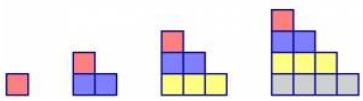
? Think / Pair / Share

- Describe how you see this pattern growing. Be as specific as you can. Draw pictures and write an explanation to make your answer clear.
- Say as much as you can about this growing pattern. Can you draw pictures to extend the pattern?
- What mathematical questions can you ask about this pattern? Can you answer any of them?

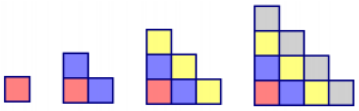
Here are some pictures that students drew to describe how the pattern was growing.



Ali's picture



Michael's picture



Kelli's picture

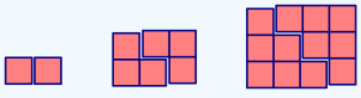
? Think / Pair / Share

Describe in words how each student saw the pattern growing. Use the students' pictures above (or your own method of seeing the growing pattern) to answer the following questions:

- How many tiles would you need to build the 5th figure in the pattern?
- How many tiles would you need to build the 10th figure in the pattern?
- How can you compute the number of tiles in any figure in the pattern?

✓ Problem 15

Hy saw the pattern in a different way from everyone else in class. Here's what he drew:



Hy's picture.

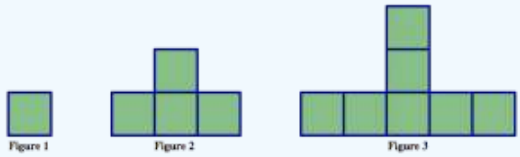
1. Describe in words how Hy saw the pattern grow.
2. How would Hy calculate the number of tiles needed to build the 10th figure in the pattern?
3. How would Hy calculate the number of tiles needed to build the 100th figure in the pattern?

4. How would Hy calculate the number of tiles needed to build any figure in the pattern?

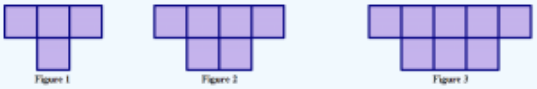
The next few problems present several growing patterns made with tiles. For each problem you work on, do the following:

1. Describe in words and pictures how you see the pattern growing.
2. Calculate the number of tiles you would need to build the 10th figure in the pattern. Justify your answer based on how the pattern grows.
3. Calculate the number of tiles you would need to build the 100th figure in the pattern.
4. Describe how you can figure out the number of tiles in any figure in the pattern. Be sure to justify your answer based on how the pattern grows.
5. Could you make one of the figures in the pattern using exactly 25 tiles? If yes, which figure? If no, why not? Justify your answer.
6. Could you make one of the figures in the pattern using exactly 100 tiles? If yes, which figure? If no, why not? Justify your answer.

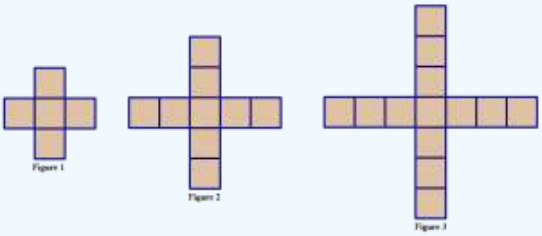
Problem 16



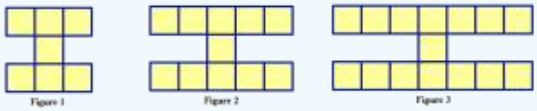
Problem 17



Problem 18



Problem 19



Problem 20

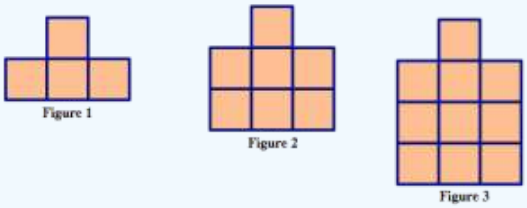


Figure 1

Figure 2

Figure 3

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13.5: Structural and Procedural Algebra

When most people think about algebra from school, they think about "solving for *x*." They imagine lots of equations with varying levels of complexity, but the goal is always the same: find the unknown quantity. This is a *procedural* view of algebra.

Even elementary students can be exposed to ideas in procedural algebra. This happens any time they think about unknown quantities and try to solve for them. For example, when first grade students learn to add and subtract numbers "within 10,'" they should frequently tackle problems like these:

- $3 + \_\_\_ = 7$.
- Find several pairs of numbers that add up to 10.

Although procedural algebra is important, it's not the most important skill, and it's certainly not the whole story.

You also need to foster thinking about *structural algebra* in your students: using symbols to express meaning in a situation. If there is an *x* on your page, you should be able to answer, "what does the *x* mean? What does it represent?"

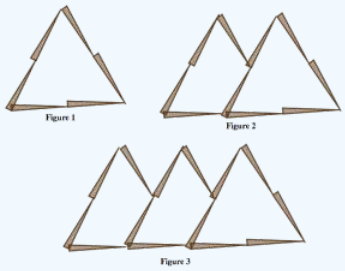
Most of what you've done so far in this chapter is *structural algebra*. You've used letters and symbols not to represent a single unknown quantity, but a *varying* quantity. For example, in Section 4 you used letters to represent the "figure number'" or "case number" in a growing pattern. The letters could take on different values, and the expressions gave you information: how many tiles or toothpicks or stars you needed to build that particular figure in that particular pattern.

? Think / Pair / Share

- Consider the expression $a + 3$. Give a real world situation that could be represented by this expression. Share your answer with your partner. Together, can you come up with even more ideas?
- Suppose the expression $3c + 2$ represents the number of tiles used at any stage of a growing pattern.
- - Evaluate the expression at $c = 1, 2,$ and 3. What do the values tell you about the pattern?
  - Can you describe in words how the pattern is growing?
  - Can you design a pattern with tiles that grows according to this rule?
  - Where do you see the "3" in your pattern? Where do you see the "2"'? Where do you see the "c"?

✔ Problem 21

Krystal was looking at this pattern, which may be familiar to you from the Problem Bank:



She wrote down the equation

$$y = 4x + 2.$$

In Krystal's equation, what does x represent? What does y represent? How do you know?

✔ Problem 22

Candice was thinking about this problem:

> *Today is Jennifer's birthday, and she's twice as old as her brother. When will she be twice as old as him again?*

She wrote down the equation $2n = m$. In Candice's equation, what does n represent? What does m represent? How do you know?

Problem 23

Sarah and David collect old coins. Suppose the variable k stands for the number of coins Sarah has in her collection, and ℓ stands for the number of coins David has in his collection. What would each of these equations say about their coin collections?

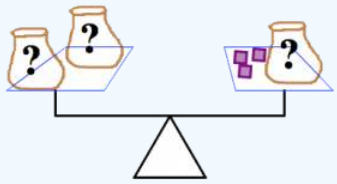
$$(a)\ k = \ell + 1 \qquad (b)\ k = \ell \qquad (c)\ 3k = 2\ell \qquad (d)\ k = \ell - 11$$

Problem 24

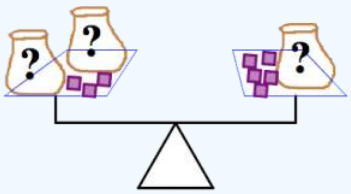
The pictures below show balance scales containing bags and blocks. The bags are marked with a "?"' because they contain some unknown number of blocks. In each picture:

- Each bag contains the same number of blocks.
- The scale is balanced.

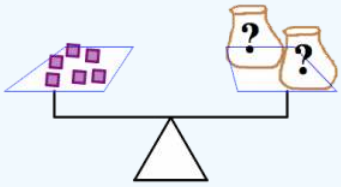
For each picture, determine how many blocks are in each bag. Justify your answers.



(a)



(b)



(c)

Problem 25

When he was working on Problem 24, Kyle wrote down these three equations.

(i)

$$2m = 6.$$

(ii)

$$2x = x + 3.$$

(iii)

$$z+5=2z+3.$$

Match each equation to a picture, and justify your choices. Then solve the equations, and say (in a sentence) what the solution represents.

Problem 26

Draw a balance puzzle that represents the equation

$$2h+3=h+8.$$

Now solve the balance puzzle. Where is the "h" in your puzzle? What does it represent?

Problem 27

Draw a balance puzzle that represents the equation

$$3b+7=3b+2.$$

Now solve the equation. Explain what happens.

Problem 28

Which equation below is most like the one in Problem 27 above? Justify your choice.

a. $5+3=8 \ldotp$
b. $\frac{2}{3} + \frac{1}{2} = \frac{3}{5} \ldotp$
c. $5+3 = y \ldotp$
d. $\frac{a}{5} = \frac{5}{a} \ldotp$
e. $n + 3 = m \ldotp$
f. $3x = 2x+ x \ldotp$
g. $5k = 5k + 1 \ldotp$

Problem 29

Draw a balance puzzle that represents the equation

$$4\ell+7=4\ell+7.$$

Now solve the equation. Explain what happens.

Problem 30

Which equation below is most like the one in Problem 29 above? Justify your choice.

a. $5+3 = 8 \ldotp$
b. $\frac{2}{3} + \frac{1}{2} = \frac{3}{5} \ldotp$
c. $5+3 = y \ldotp$
d. $\frac{a}{5} = \frac{5}{a} \ldotp$
e. $n + 3 = m \ldotp$
f. $3x = 2x + x \ldotp$
g. $5k = 5k + 1 \ldotp$

Problem 31

Create a balance puzzle where the solution is not a whole number of blocks. Can you solve it? Explain your answer.

Problem 32

There are three piles of rocks: pile A, pile B, and pile C. Pile B has two more rocks than pile A. Pile C has four times as many rocks as pile A. The total number of rocks in all three piles is 14.

1. Use x to represent the number of rocks in pile A, and write equations that describe the rules above. Then find the number of rocks in each pile.
2. Use x to represent the number of rocks in pile B, and write equations that describe the rules above. Then find the number of rocks in each pile.
3. Use x to represent the number of rocks in pile C, and write equations that describe the rules above. Then find the number of rocks in each pile.

Think / Pair / Share

Look back at Problems 21–32. Which of them felt like *structural algebraic thinking*? Which felt like *procedural algebraic thinking*? Did any of the problems feel like they involved both kinds of thinking?

Variables and Equations

You have seen that in algebra, letters and symbols can have different meanings depending on the context.

- A symbol could stand for some *unknown quantity*.
- A symbol could stand for some quantity that *varies*. (Hence the term "variable" to describe these symbols.)

In much the same way, *equations* can represent different things.

- They can represent a problem to be solved. This is the traditional procedural algebra type of question.
- They can represent a relationship between two or more quantities. For example, $A = s^2$ represents the relationship between the area of a square and its side length.
- They can represent *identities*: mathematical truths. For example, $x^{2} - 1 = (x + 1)(x - 1)$ is always true, for every value of x. There is nothing to solve for, and no relationship between varying quantities. (If you do try to "solve for x," you will get the equation $0 = 0$, much like you saw in Problem 29. Not very satisfying!)

Think / Pair / Share

Give an example of each type of equation. Be sure to say what the symbols in the equations represent.

Problem 33

Answer the following questions about the equation

$$x^2 - 1 = (x+1)(x-1).$$

1. Evaluate both sides of the above equation for $x = 1, 2, 3, 4,$ and 5. What happens?
2. Use the *distributive property* of multiplication over addition to expand the right side of the equation and simplify it.
3. Use the equation to compute 99^2 quickly, without using a calculator. Explain how you did it.

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13.6: Practice

Problems 34-36 ask you to solve problems about a strange veterinarian who created three mystifying machines.

Cat Machine: Place a cat in the input bin of this machine, press the button, and out jump two dogs and a mouse.

Dog Machine: This machine converts a dog into a cat and a mouse.

Mouse Machine: This machine can convert a mouse into a cat and three dogs.

Each machine can also operate in reverse. For example, if you have two dogs and a mouse, you can use the first machine to convert them into a cat.

Problem 34

The veterinarian hands you two cats, and asks you to convert them into exactly three dogs (no extra dogs and no other animals). Can you do it? If yes, say what process you would use. If no, say why not.

Problem 35

The veterinarian hands you one dog. He says he only wants cats, but he doesn't care how many. Can you help him? How?

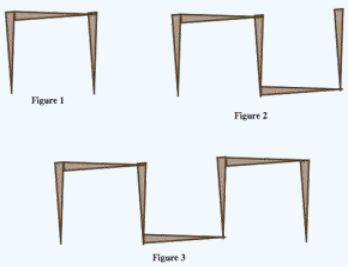
Problem 36

The veterinarian hands you one cat. He says he only wants dogs, but he doesn't care how many. Can you help him? How?

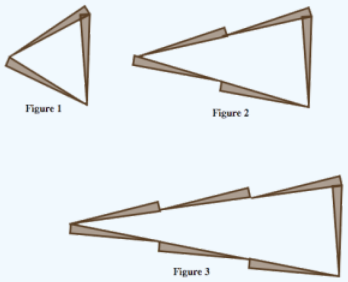
Problems 37-40 present several growing patterns made with toothpicks. For each problem you work on, do the following:

1. Describe in words and pictures how you see the pattern growing.
2. Calculate the number of toothpicks you would need to build the 10th figure in the pattern. Justify your answer based on how the pattern grows.
3. Calculate the number of toothpicks you would need to build the 100th figure in the pattern.
4. Describe how you can figure out the number of toothpicks in any figure in the pattern. Be sure to justify your answer based on how the pattern grows.
5. Could you make one of the figures in the pattern using exactly 25 toothpicks? If yes, which figure? If no, why not? Justify your answer.
6. Could you make one of the figures in the pattern using exactly 100 toothpicks? If yes, which figure? If no, why not? Justify your answer.

Problem 37



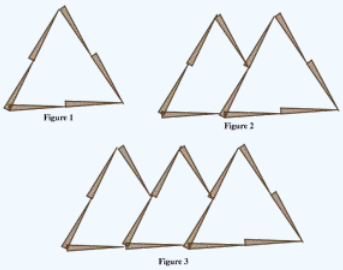
Problem 38



Problem 39



Problem 40



In a *mobile*, the arms must be perfectly balanced for it to hang properly. The artist Alexander Calder was famous for his artistic mobiles.You can view some of his amazing work here. Click “Explore Works.”

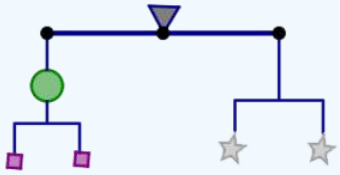
Problems 41-42 present you with mobile puzzles. In these puzzles:

- Objects that are the same shape have the same weight. (So all circles weigh the same, all squares weigh the same, etc.)
- Assume the strings and rods that hold the objects together don’t factor into the total weight.
- Each arm of the mobile must have exactly the same weight.

Problem 41

In this puzzle:

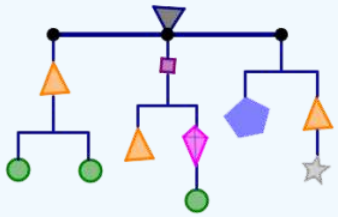
- The total weight is 36 grams.
- All shapes weigh less than 10 grams.
- All of the weights are whole numbers.
- One circle weighs more than one square.



Find the weight of each piece. Is there more than one answer? How do you know you are right?

Problem 42

In this puzzle, the total weight is 54 grams.



Find the weight of each piece. Is there more than one answer? How do you know you are right?

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