PREPARING FOR CALCULUS BOOTCAMP



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Preparing for Calculus Bootcamp

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Licensing

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CHAPTER OVERVIEW

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Page 1.1: Introduction

Welcome to The Preparing for Calculus Boot Camp! The aim of this 8 day boot camp is to review the mathematical foundation needed to do well in your calculus courses (and beyond!).

Before we start learning math, there are a few things to keep in mind regarding your future math courses. Some students may already know this and so I apologize if this information is redundant. However, other students may find this information valuable and so allow us to discuss some general information regarding calculus and the different options which are available to you.

With regards to STEM majors, the book that our department uses for calculus is called Essential Calculus by James Stewart. A full coverage of single-variable calculus means that you have covered Chapters 1 through 8 in the above text. Here at Queens College, we have two different tracks that covers Chapters 1 through 8.

CALCULUS TRACKS

1) We first offer Math 141, 142, and 143. In Math 141, Chapters 1 - 3 are covered. In Math 142, Chapters 4, 5, and 7 are covered. In Math 143, Chapters 6 and 8 are covered. Each class is three credits and the meetings are twice a week for an hour and fifteen minutes.

2) We also offer Math 151 and 152. In Math 151, Chapters 1 - 4 are covered. In Math 152, Chapters 5 - 8 are covered. Each class is four credits and the meetings are twice a week for an hour and fifty minutes.

Thus, regardless of which track you select, you still cover all of single-variable calculus. The question now becomes, what are the differences among these tracks? For those who wish to have a more gentle introduction to calculus, I would personally advise them to take the Math 141, 142, and 143 option. Those who feel they have a strong command of mathematics may wish to take Math 151 and 152.

Please also note that the Math Department offers Math 131 and 132 which is calculus for non-science students. More information may be found here: https://qc-undergraduate.catalog.cuny.edu/courses/0112541.

For a full list of all undergraduate math courses that the Math Department offers, please see here: https://www.qc.cuny.edu/academics/math/undergraduate-courses/.

With that said, allow us to begin our mathematical journey!

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Page 1.1.1: Advice For Success

We first start off our journey by noting that it is important to be aware of how to succeed in mathematics. Although the following is not a comprehensive list, I hope that it serves as a guideline on how to excel in your math classes. We will try and practice many of these skills here in our boot camp! Whenever we do so, we will highlight it and make a note.

STRATEGIES

1) In mathematics, we should try to see many examples and non-examples. In doing so, we obtain a clearer idea on the topic that is being discussed.

2) Whenever we have a definition or a theorem, try to translate the definition or theorem in words. Often, students tend to memorize a formula by repeating the symbols, but instead we should be able to explain the formula in words! For instance, instead of saying "The Pythagorean Theorem states that $a^2+b^2=c^2$ ", a more detailed answer would be "The Pythagorean Theorem states that for any right triangle, the square length of one leg plus the square length of the other leg equals to the square length of the hypotenuse". At first, the second suggestion may seem like it is a lot of work for a little value. However, in your journey in calculus, you will be required to demonstrate a strong grasp on interpreting what a formula may mean.

3) Try to reflect on every rule/theorem. Can you understand why the rule/theorem makes intuitive sense?

4) Form a study group since a good habit to get into is to try discussing the material with someone. Mathematics is a language and if we only write math, then we have never practiced speaking it! There is a rich vocabulary associated with mathematics and the act of forcing yourself to find the right words and explain an idea will give you a better command of the material.

5) Math is not a spectator sport! To learn math, you must practice, practice and practice. It is easy to attend lectures and watch your professor work out all the examples. But just because you watched someone work out, it does not mean that you, yourself, can also work out! Remember, your instructors have an excellent command of the material. Just because they may make a problem look easy, it does not mean you can replicate their process on the day of the exam without practicing first.

6) Although this may sound excessive, perhaps for your math courses we have four separate notebooks. In one notebook, we should copy down the lecture notes. When we go home, we should actively copy these lecture notes into our second notebook. Doing so forces us to go over the lecture and grants us an opportunity to solve the problems done in class. This is important because many students do not realize that it is one thing to read someone's solution and understand it. However, it is a completely different thing to be able to come up with that solution, on your own, from scratch. Students (myself included) often read their notes and think they can imitate the same type of thinking on the test because our instructor makes it look so easy. However, this is often not the case. We really don't know how much we *actually* know until we are forced to come up with an answer on our own. For some students, the first time they are actually trying a problem on their own is during the exam. If the first time we are actually trying a problem from scratch is on the exam, then it is not going to go well! The third notebook should contain all the homework problems worked out in as much details as possible. And finally, the forth notebook should contain summaries/outlines of the textbook. The textbook can offer a different viewpoint from the instructor and serves as another place where you can actively study.

7) Make use of your professor's office hours, the Learning Commons (https://www.qc.cuny.edu/academics/qclc/) and the Math Lab (https://www.qc.cuny.edu/academics/math/getting-help/).

8) Here is a more comprehensive list of tips: https://mathstat.slu.edu/resources/success-in-mathematics.

Again, all of the above is merely advice. It is important to find which combination of strategies works best for you. The one thing to keep in mind is that simply attending the lecture might not guarantee good grades! There is a lot of work to be done *outside* of the classroom and hopefully the above list can point you in the right direction.

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Page 1.2: Real Numbers

Although some of the terminology might be new, the ideas presented in this section should be somewhat familiar to us.

For us, our entire study of calculus will deal with *real numbers*. In order to rigorously define what we mean by a real number, a lot of machinery that is beyond the scope of our calculus courses are required and so we will rely on our intuition to understand real numbers. Allow us to review some types of numbers.

Definition: Natural Numbers

Definition: The natural numbers are defined to be the set containing the numbers $1, 2, 3, 4, \ldots$ We denote the set of natural numbers by $\mathbb{N}$. So $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.

Definition: Integers

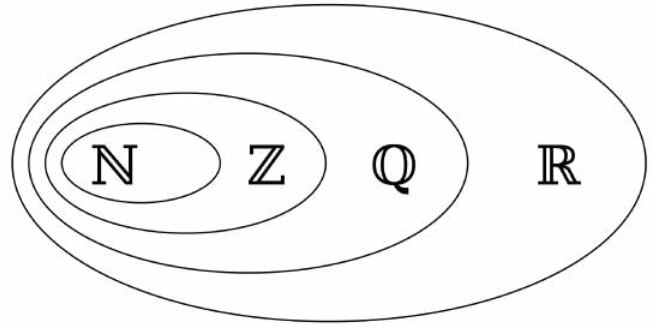
Definition: The integers consist of the natural numbers together with the negatives and zero. We denote the set of integers by $\mathbb{Z}$. So $\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$.

Definition: Rational Numbers

Definition: The rational numbers consist of all numbers of the form $\frac{m}{n}$ where m and n are integers and $n \neq 0$. We denote the set of integers by $\mathbb{Q}$.

The above definition is saying that a rational number is just a ratio of two integers. Here are some examples of rational numbers: $\frac{1}{2}, \frac{9}{10}, 0.61 = \frac{61}{100}, -0.23 = -\frac{23}{100}$.

However, not all real numbers can be expressed as a ratio of two integers. For instance, although we do not prove this here, note that $\sqrt{2}$ cannot be expressed as a ratio of two integers. Additionally, π cannot be expressed as a ratio of two integers. These numbers are called *irrational numbers*. Putting the rational numbers and irrational numbers together yields what we call the set of real numbers which we denote by $\mathbb{R}$.



The real numbers follow certain rules which we are familiar with. These rules are the presented below.

Properties of Real Numbers $Page 1.2.1$

Suppose a, b and c are real numbers. Then the following are true:

1) $a + b = b + a$ (this is called the commutative property of addition)

2) $a \times b = b \times a$ (this is called the commutative property of multiplication)

3) $(a + b) + c = a + (b + c)$ (this is called the associative property of addition)

4) $(a \times b) \times c = a \times (b \times c)$ (this is called the associative property of multiplication)

5) $a(b + c) = ab + ac$ (this is called the distributive property)

In words, the above list is saying when we add two numbers, the order does not matter. The same is true for multiplication. Additionally, when we add three numbers, it does not matter which two we add first. The same can be said for multiplication. And finally, property 5 is saying that when we multiply a number by a sum of two numbers, we can multiply the outside number to each inside number and add the resulting products.

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Page 1.3: Rules of Exponents and Radicals

A big part of learning calculus involves being able to understand the notation that is used. More generally, anytime a new symbol is introduced, we should question it and see several examples and non-examples. Often, the purpose of a symbol is to provide a means of simplifying the notation and this is also the case when it comes to exponents and radicals. For instance, it would be rather cumbersome to write $5 \times 5 \times 5 \times 5$ and even more cumbersome if we wished to multiply 5 by itself twenty times. Thus, we introduce the notion of an *exponent*. Instead of writing $5 \times 5 \times 5 \times 5$, we can simply denote this as 5^4 which is read as "5 raised to the fourth power" and in the case of multiplying 5 by itself twenty times, rather than writing $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$, we may simply denote this by 5^{20}. We call 5 the *base* and we call 20 the *exponent*.

More generally, we make the following definition.

Definition: Exponential Notation

Definition: If a is any real number and n is any positive integer, then the nth power of a is denoted by a^n where

$$a^n = \underbrace{a \times a \times \ldots \times a}_{n \text{ times}} \qquad \text{(Page 1.3.1)}$$

The number a is called the **base** and n is called the **exponent**.

Key Idea

Whenever a definition is provided, we should reflect on the assumptions and requirements of the definition. Doing so will lead to us a deeper understanding of the material.

Here, we notice that the definition requires a to be a real number and pretty much all the numbers we will see in calculus will be real numbers (the main exception to this will be noted later on). Since most of the numbers we see will be real numbers, we will not place too much attention on a. Rather, our focus is on n. We see our definition requires n to be a *positive integer*. This immediately raises two questions:

1. Is it possible for n to be negative? For instance, how do we define 5^{-20}?
2. Is it possible for n to not be an integer? For instance, what does $5^{20.23}$ mean?

We will consider the answer to these two questions in just a moment. For now, allow us to consider some examples.

Example $Page 1.3.1$

Compute the following:

1) $\left(\frac{1}{2}\right)^5$

2) 2^4

3) $(-2)^4$

4) -2^4

5) $(-2)^3$

6) -2^3

Answer

1) $\left(\frac{1}{2}\right)^5 = \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) = \frac{1}{32}$

2) $2^4 = (2)(2)(2)(2) = 16$

3) $(-2)^4 = (-2)(-2)(-2)(-2) = 16$

4) $-2^4 = -(2)(2)(2)(2) = -16$

5) $(-2)^3 = (-2)(-2)(-2) = -8$

6) $-2^3 = -(2)(2)(2) = -8$

We now turn our attention to the first question posed earlier: is it possible for n to be negative? The answer is yes:

Definition: Negative and Zero Exponents

Definition: If $a \neq 0$ is a real number and n is a positive integer, then

$$a^{-n} = \frac{1}{a^n} \text{ and } a^0 = 1$$

As mentioned above, ***whenever a definition is provided, we should reflect on the assumptions and requirements of the definition. Doing so will lead to us a deeper understanding of the material.*** Here, we see that a is not allowed to be equal to zero and we may wonder why. That is, what does 0^0 equal to? The answer is that for us, 0^0 is not defined! We will touch a little bit more upon this in Day 8 but for those who are interested, a good exercise is to graph the functions $f(x) = x^0$ and $g(x) = 0^x$. What happens to the values of f and g as x gets closer and closer to zero from th right hand side? It is okay if we cannot answer this now since the solution will be apparent by Day 8. In the meantime, allow us to consider the following example.

Example $Page 1.3.2$

Compute the following:

1) $\left(\frac{1}{2}\right)^{-5}$

2) 2^{-4}

3) $(-2)^{-4}$

4) -2^{-4}

5) $(-2)^{-3}$

6) -2^{-3}

7) x^{-1} you may assume that $x \neq 0$.

8) x^{-2} you may assume that $x \neq 0$.

9) x^{-3} you may assume that $x \neq 0$.

10) x^{-2023} you may assume that $x \neq 0$.

11) 2023^0

Answer

1) $\left(\frac{1}{2}\right)^{-5} = \frac{1}{(\frac{1}{2})^5} = \frac{1}{\frac{1}{32}} = 32$

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2) $2^{-4} = \dfrac{1}{2^4} = \dfrac{1}{16}$

3) $(-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

4) $-2^{-4} = -\dfrac{1}{2^4} = -\dfrac{1}{16}$

5) $(-2)^{-3} = \dfrac{1}{(-2)^3} = \dfrac{1}{-8} = -\dfrac{1}{8}$

6) $-2^{-3} = -(2^{-3}) = -\dfrac{1}{(2)^3} = -\dfrac{1}{8}$

7) $x^{-1} = \dfrac{1}{x}$

8) $x^{-2} = \dfrac{1}{x^2}$

9) $x^{-3} = \dfrac{1}{x^3}$

10) $x^{-2023} = \dfrac{1}{x^{2023}}$

11) $2023^0 = 1$

It is important that we understand the rules when it comes to working with exponents. These rules are summarized below.

Theorem: The Laws of Exponents $Page 1.3.1$

Theorem: Let a and b be real numbers and m and n be integers. Then we have the following:

1) $a^m a^n = a^{m+n}$

2) $\frac{a^m}{a^n} = a^{m-n}$

3) $\left(a^m\right)^n = a^{mn}$

4) $(ab)^n = a^n b^n$

5) $\left(\dfrac{a}{b}\right)^n = a^n b^n$

Key Idea

Whenever a theorem is presented to you, a good idea is to take a few moments and translate the symbols into words. The above theorem is saying the following:

1) In order to multiply two powers with the same base, we add the exponents.

2) In order to divide two powers with the same base, we subtract the exponents.

3) Whenever we raise a power to a new power, we multiply the exponents.

4) To raise a product to a power, raise each factor to the power.

5) To raise a quotient to a power, raise both numerator and denominator to the power.

With these ideas in mind, allow us to try a few examples.

Example $Page1.3.3$

Perform the given operation using the multiplication properties of exponents and write your answer in simplest form:

a. $b^2 \cdot b^3 = b^{2+3} = b^5$

b. $x^8 x^7 = x^{8+7} = x^{15}$

c. $a^8 a^9 a^{14} = a^{8+9+14} = a^{31}$

d. $4x^4 \cdot 7x^6 = (4 \cdot 7)\left(x^4 \cdot x^6\right) = 28x^{10}$

e. $5xy^3 \cdot 6y = (5 \cdot 6)(x)\left(y^3 \cdot y\right) = 30xy^4$

f. $\left(5x^4y^2\right)\left(2x^7y^3\right) = 10x^{4+7}y^{2+3} = 10x^{11}y^5$

g. $(2x)^3 = 2^3x^3 = 8x^3$

h. $\left(-4a^2b^5\right)^3 = (-4)^3a^{2\cdot3}b^{5\cdot3} = -64a^6b^{15}$

i. $$\begin{aligned}\left(-5r^3s\right)^2 \cdot \left(2r^4s^3\right)^3 \cdot \left(-r^2s^2\right) &= (-5)^2r^{3\cdot2}s^2 \cdot 2^3r^{4\cdot3}s^{3\cdot3} \cdot (-1)r^2s^2 \\ &= 25r^6s^2 \cdot 8r^{12}s^9 \cdot (-1)r^2s^2 \\ &= -200r^{20}s^{13}\end{aligned}$$

j. $\dfrac{x^5}{x^3} = \dfrac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x \cdot x} = \dfrac{x \cdot x}{1} = \dfrac{x^2}{1} = x^2$

k. $\dfrac{x^3}{x^5} = \dfrac{x \cdot x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \dfrac{1}{x \cdot x} = \dfrac{1}{x^2}$

l. $\left(\dfrac{x}{y}\right)^5 = \dfrac{x}{y} \cdot \dfrac{x}{y} \cdot \dfrac{x}{y} \cdot \dfrac{x}{y} \cdot \dfrac{x}{y} = \dfrac{x \cdot x \cdot x \cdot x \cdot x}{y \cdot y \cdot y \cdot y \cdot y} = \dfrac{x^5}{y^5}$

Example $Page1.3.4$

Perform the given operation using the division properties of exponents and state your answer in simplest form:

a. $\dfrac{b^8}{b^7} = b^{8-7} = b^1 = b$

b. $\dfrac{x^{12}y^2}{x^8y} = x^{12-8}y^{2-1} = x^4y$

c. $\dfrac{8x^6y^2}{6x^5y^7} = \dfrac{4x^{6-5}}{3y^{7-2}} = \dfrac{4x^1}{3y^5} = \dfrac{4x}{3y^5}$

d. $\dfrac{\left(m^2\right)^3\left(n^4\right)^5}{\left(m^3\right)^3} = \dfrac{m^6n^{20}}{m^9} = \dfrac{n^{20}}{m^3}$

e. $\left(\dfrac{3a^2b^4}{9c^3}\right)^2 = \left(\dfrac{a^2b^4}{3c^3}\right)^2 = \dfrac{a^{2\cdot2}b^{4\cdot2}}{3^2c^{3\cdot2}} = \dfrac{a^4b^8}{9c^6}$ Another way to simplify this correctly is this:

$\left(\dfrac{3a^2b^4}{9c^3}\right)^2 = \dfrac{3^2a^{2\cdot2}b^{4\cdot2}}{9^2c^{3\cdot2}} = \dfrac{9a^4b^8}{81c^6} = \dfrac{a^4b^8}{9c^6}$

Example $Page1.3.5$

Perform the given operation and write your answer using positive exponents only

a. $\dfrac{x^4y^2}{x^3y^{-3}} = x^{4-3}y^{2-(-3)} = xy^5$

b. $\left(x^2y\right) \cdot \left(xy^{-4}\right) = x^{2+1}y^{1+(-4)} = \dfrac{x^3}{y^3}$

We will now turn our attention to the second question asked earlier. Namely, is it possible for n to not be an integer? For instance, what does $5^{20.23}$ mean? In order to answer this, we must introduce the notion of a radical or root.

Definition: Square Root

Definition: Let a denote a non-negative real number (meaning a is either positive or zero). The square root of a, denoted by $\sqrt{a}$ or $\sqrt[2]{a}$ is said to equal b provided that $b \geq 0$ and $b^2 = a$. That is,

$$\sqrt[2]{a} = b \text{ means } b^2 = a \text{ and } b \geq 0$$

Again, allow us to reflect on this definition. We may wonder, why does a have to be positive or zero? Notice that $a = b^2$ and for all real numbers b, $b^2 \geq 0$. Hence a must also be greater than or equal to 0 and so a must either be positive or zero. Or quite simply, if $b^2 = a$ and since and real number raised to the power of 2 is positive or zero, then a must be positive or zero. This will be important later on!

Additionally, we may wonder why is it that we require $b \geq 0$. The answer to this will be discussed on Day 2 since the explanation requires us to understand what is a *function*. Allow us to now consider an example.

Example $Page 1.3.1$

$$\sqrt{16} = 4 \text{ because } 4^2 = 16 \text{ and } 4 \geq 0$$

Notice that

$$\sqrt{16} \neq -4$$

Although it is true that $(-4)^2 = 16$, it is not true that $-4 \geq 0$ and so $\sqrt{16} \neq -4$. In summary, the square root symbol, $\sqrt{\ }$ means "the positive square root of".

With regards to square roots, we will now pose an interesting question.

Exercise $Page 1.3.1$

Suppose a is some real number. Is it always true that $\sqrt{a^2} = a$? Discuss.

Answer

To answer questions like these, allow us to substitute in a few values for a in order to investigate the statement. If a is any real number, then a can be positive, negative or zero. Let us first choose a positive value for a, say 3. Is it true that $\sqrt{3^2} = 3$? Yes!

Let us now choose a negative value for a, say -3. Is it true that $\sqrt{(-3)^2} = -3$? No!

And so the equation $\sqrt{a^2} = a$ is not always true. In fact, the equation is only true when $a \geq 0$.

More generally, we should note the following:

$$\text{For all real numbers } a, \sqrt{a^2} = |a|$$

However, there is nothing special about the *square* root. That is, we can extend this idea. If

$$\sqrt[2]{a} = b \text{ means } b^2 = a \text{ and } b \geq 0$$

then it makes sense to say

$$\sqrt[4]{a} = b \text{ means } b^4 = a$$

and

$$\sqrt[6]{a} = b \text{ means } b^6 = a \text{ and } b \geq 0$$

and so on. More generally, we present the following definition:

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Definition: n^{th} root where n is even

Definition: Let a denote a non-negative real number (meaning a is either positive or zero) and let n denote any positive even integer. Then the nth root of a is defined as follows:

$$\sqrt[n]{a} = b \text{ means } b^n = a \text{ and } b \geq 0$$

The above definition is saying if n is even, then to find $\sqrt[n]{a}$, we require both a and b to be positive.

Example $Page$1.3.1

Evaluate the following:

1) $\sqrt{25}$

2) $\sqrt[4]{81}$

3) $\sqrt[6]{64}$

4) $\sqrt[6]{-64}$

Answer

1) To find $\sqrt{25}$ we must ask ourselves can we find a real number b such that $b^2 = 25$ and $b \geq 0$? The answer is yes - if we take $b = 5$ then $b^2 = 25$ and $b \geq 0$.

2) To find $\sqrt[4]{81}$ we must ask ourselves can we find a real number b such that $b^4 = 81$ and $b \geq 0$? The answer is yes - if we take $b = 3$ then $b^4 = 81$ and $b \geq 0$.

3) To find $\sqrt[6]{64}$ we must ask ourselves can we find a real number b such that $b^6 = 64$ and $b \geq 0$? The answer is yes - if we take $b = 2$ then $b^6 = 64$ and $b \geq 0$.

4) $\sqrt[6]{-64}$ we must ask ourselves can we find a real number b such that $b^6 = -64$ and $b \geq 0$? The answer here is **no.** Notice for any real number b, b^6 will be positive or zero and so b^6 can never be equal to -64. Hence the answer is that $\sqrt[6]{-64}$ is not defined. Similarly, $\sqrt[2]{-64}$ and $\sqrt[4]{-64}$ are not defined.

Notice that the last part of the above example suggests that for even roots, the number inside the root cannot be negative. Allow us to now discuss odd roots.

Definition: n^{th} root where n is odd

Definition: Let a denote a real number (meaning a is either positive, negative or zero) and let n denote any positive odd integer. Then the nth root of a is defined as follows:

$$\sqrt[n]{a} = b \text{ means } b^n = a$$

Notice in the above definition for odd roots, we now allow the number inside the root to be negative and we dropped the condition that $b \geq 0$.

Example $Page$1.3.1

Evaluate the following:

1) $\sqrt[3]{8}$

2) $\sqrt[3]{-8}$

3) $\sqrt[3]{10^3}$

4) $\sqrt[3]{(-10)^3}$

Answer

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1) To find $\sqrt[3]{8}$ we must ask ourselves can we find a real number b such that $b^3 = 8$? The answer is yes - if we take $b = 2$ then $b^3 = 8$.

2) To find $\sqrt[3]{-8}$ we must ask ourselves can we find a real number b such that $b^3 = -8$? The answer is yes - if we take $b = -2$ then $b^3 = -8$.

3) To find $\sqrt[3]{10^3}$ we must ask ourselves can we find a real number b such that $b^3 = 10^3$? The answer is yes - if we take $b = 10$ then $b^3 = 10^3$.

4) To find $\sqrt[3]{(-10)^3}$ we must ask ourselves can we find a real number b such that $b^3 = (-10)^3$? The answer is yes - if we take $b = -10$ then $b^3 = (-10)^3$.

The above examples can be briefly summarized as the following:

Note

1) $\sqrt[n]{a^n} = a$ if n is a positive odd integer.

2) $\sqrt[n]{a^n} = |a|$ if n is a positive even integer.

2) We cannot take the nth root of a negative number when n is a positive even integer. However, we can take the nth root of a negative number when n is a positive odd integer.

When it comes to n^{th} roots, we should be familiar with their properties:

Theorem $Page1.3.1$

Theorem:

1) $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

2) $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

3) $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

Key Idea

Take a moment and try to translate the above theorem into words.

Answer

1) The nth root of a product is the product of the individual n^{th} roots.

2) The nth root of a quotient is the quotient of the individual n^{th} roots.

3) The mth root of the nth root is the same as the mn^{th} root.

Exercise $Page1.3.1$

Simplifythe following expressions as much as possible:

1) $\sqrt[3]{x^4}$

2) $\sqrt[4]{81x^8y^4}$

Answer

1)

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$$\begin{aligned}\sqrt[3]{x^4} &= \sqrt[3]{x^3 x} \\ &= \sqrt[3]{x^3}\sqrt[3]{x} \\ &= x\sqrt[3]{x}\end{aligned}$$

2)

$$\begin{aligned}\sqrt[4]{81x^8y^4} &= \sqrt[4]{81}\sqrt[4]{x^8}\sqrt[4]{y^4} \\ &= 3\sqrt[4]{(x^2)^4}|y| \\ &= 3|x^2||y| \\ &= 3x^2|y|\end{aligned}$$

We now make one final extension via the following definition.

Definition: Rational Exponents

Definition: For any rational exponent $\frac{m}{n}$ in lowest terms and $n > 0$, we define

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

or equivalently

$$a^{\frac{m}{n}} = (\sqrt[n]{a^m})$$

If n is even, then we require $a \geq 0$.

Exercise $Page 1.3.1$

Express the following roots in terms of a rational exponent.

1) $\sqrt{x}$

2) $\sqrt[3]{x}$

3) $\sqrt[3]{x^4}$

4) $8^{\frac{2}{3}}$

5) $125^{-\frac{1}{3}}$

Answer

1) $\sqrt{x} = x^{\frac{1}{2}}$

2) $\sqrt[3]{x} = x^{\frac{1}{3}}$

3) $\sqrt[3]{x^4} = x^{\frac{4}{3}}$

4) $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ or $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$

5) $125^{-\frac{1}{3}} = \frac{1}{125^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{125}} = \frac{1}{5}$

Note that The Law of Exponents also applies to when the powers are rational numbers and not just integers.

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Theorem: The Laws of Exponents (Generalized) $Page 1.3.1$

Theorem: Let a and b be real numbers and m and n be rational numbers. Then we have the following:

1) $a^m a^n = a^{m+n}$

2) $\frac{a^m}{a^n} = a^{m-n}$

3) $\left(a^m\right)^n = a^{mn}$

4) $(ab)^n = a^n b^n$

5) $\left(\frac{a}{b}\right)^n = a^n b^n$

Allow us to now work through a few examples.

Exercise $Page 1.3.1$

Simplify the following:

1) $x^{\frac{1}{3}} x^{\frac{7}{3}}$

2) $\frac{10x^{\frac{5}{6}}}{2x^{\frac{7}{6}}}$

3) $(4a^5b^7)^{\frac{1}{2}}$

Answer

1)

$$x^{\frac{1}{3}} x^{\frac{7}{3}} = x^{\frac{1}{3}+\frac{7}{3}} = x^{\frac{8}{3}}$$

2)

$$\frac{10x^{\frac{5}{6}}}{2x^{\frac{7}{6}}} = \frac{10}{2}\frac{x^{\frac{5}{6}}}{x^{\frac{7}{6}}} = 5x^{\frac{5}{6}-\frac{7}{6}} = 5x^{-\frac{2}{6}} = 5x^{-\frac{1}{3}}$$

3)

$$(4a^5b^7)^{\frac{1}{2}} = 4^{\frac{1}{2}}(a^5)^{\frac{1}{2}}(b^7)^{\frac{1}{2}} = 2a^{\frac{5}{2}}b^{\frac{7}{2}}$$

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Page 1.4: Algebraic Expressions and Operations

We will often deal with algebraic expressions in calculus. In order to understand algebraic expressions, we must first discuss variables.

Definition: Term

Definition: A **variable** is a letter which denotes a real number.

In calculus, we will often use x and y as our variables but there will be instances when we may use t or s. Note that a real number in front of a variable is called the coefficient.

Definition: Algebraic Expression

Definition: If we start with variables and some real numbers and combine them via addition, subtraction, multiplication, division, powers, or roots, then we obtain an **algebraic expression**.

Example $Page1.4.1$

The following are examples of algebraic expressions:

$$141x^2+142x+143,\ \sqrt{5x}+12,\ \frac{12x^2}{\sqrt[4]{x+5}+9}, 12xyz+15s$$

One common type of algebraic expression we will encounter is a polynomial.

Definition: Polynomial

Definition: A polynomial in the variable x is an expression of the form $a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ where $a_n, a_{n-1}, \ldots, a_1, a_0$ are real numbers.

Here are some examples of polynomials in the variable x: $3x^2$, $7x+5$, $\frac{3}{2}x^3+3x^2-\frac{1}{2}x+1$, $6x^2-4x+7$.

The **degree of a polynomial** is defined to be the highest power of the variable that appears in the polynomial.

Term	Degree
$3x^2$	2
$7x+5$	1
$\frac{3}{2}x^3+3x^2-\frac{1}{2}x+1$	3
$6x^2-4x+7$	2
x	1

Typically, we arrange terms of polynomials in descending order based on the degree of each term. Such an arrangement is called **standard form**. The **leading coefficient** is the coefficient of the variable with the highest power, in this case, a_n.

Example $Page1.4.1$:

Write in standard form: $3x-4x^2+5x^3+7-2x^4$.

Answer:

$-2x^4+5x^3-4x^2+3x+7$

We classify polynomials by the number of terms and the degree:

Expression	Classification	Degree
$5x^7$	**Monomial** (one term)	7
$8x^6 - 1$	**Binomial** (two terms)	6
$-3x^2 + x - 1$	**Trinomial** (three terms)	2
$5x^3 - 2x^2 + 3x - 6$	**Polynomial** (many terms)	3

We can further classify polynomials with one variable by their degree:

Polynomial	Name
5	**Constant** (degree 0)
$2x + 1$	**Linear** (degree 1)
$3x^2 + 5x - 3$	**Quadratic** (degree 2)
$x^3 + x^2 + x + 1$	**Cubic** (degree 3)
$7x^4 + 3x^3 - 7x + 8$	Fourth-degree polynomial

In this text, we call any polynomial of degree $n \geq 4$ an nth-degree polynomial. In other words, if the degree is 4, we call the polynomial a fourth-degree polynomial. If the degree is 5, we call it a fifth-degree polynomial, and so on.

Example $Page1.4.2$

State whether the following polynomial is linear or quadratic and give the leading coefficient: $25 + 4x - x^2$.

Solution

The highest power is 2; therefore, it is a quadratic polynomial. Rewriting in standard form we have

$$-x^2 + 4x + 25$$

Here $-x^2 = -1x^2$ and thus the leading coefficient is -1.

Answer:

Quadratic; leading coefficient: -1

Adding and Subtracting Algebraic Expressions

We begin by simplifying algebraic expressions that look like $+(a+b)$ or $-(a+b)$. Here, the coefficients are actually implied to be $+1$ and -1 respectively and therefore the distributive property applies. Multiply each term within the parentheses by these factors as follows:

$$+(a+b) = +1(a+b) = (+1)a + (+1)b = a + b$$
$$-(a+b) = -1(a+b) = (-1)a + (-1)b = -a - b$$

Use this idea as a means to eliminate parentheses when adding and subtracting polynomials.

Example $Page1.4.3$:

Add: $9x^2 + \left(x^2 - 5\right)$.

Solution

The property $+(a+b) = a+b$ allows us to eliminate the parentheses, after which we can then combine like terms.

$$\begin{aligned} 9x^2 + \left(x^2 - 5\right) &= 9x^2 + x^2 - 5 \\ &= 10x^2 - 5 \end{aligned}$$

Answer:

$10x^2-5$

Example $Page1.4.4$:

Add: $\left(3x^2y^2-4xy+9\right)+\left(2x^2y^2-6xy-7\right)$.

Solution

Remember that the variable parts have to be exactly the same before we can add the coefficients.

$$\begin{aligned}&\left(3x^2y^2-4xy+9\right)+\left(2x^2y^2-6xy-7\right)\\&=3x^2y^2-4xy+9+2x^2y^2-6xy-7\\&=5x^2y^2-10xy+2\end{aligned}$$

Answer:

$5x^2y^2-10xy+2$

When subtracting algebraic expressions, the parentheses become very important.

Example $Page1.4.5$:

Subtract: $4x^2-\left(3x^2+5x\right)$.

Solution

The property $-(a+b)=-a-b$ allows us to remove the parentheses after subtracting each term.

$$\begin{aligned}4x^2-\left(3x^2+5x\right)&=4x^2-3x^2-5x\\&=x^2-5x\end{aligned}$$

Answer:

x^2-5x

Subtracting a quantity is equivalent to multiplying it by -1.

Example $Page1.4.6$:

Subtract: $\left(3x^2-2xy+y^2\right)-\left(2x^2-xy+3y^2\right)$.

Solution

Distribute the -1, remove the parentheses, and then combine like terms. Multiplying the terms of a polynomial by -1 changes all the signs.

$$\left(3x^2-2xy+y^2\right)-\left(2x^2-xy+3y^2\right)$$

$$\begin{aligned}&=3x^2-2xy+y^2-2x^2+xy-3y^2\\&=x^2-xy-2y^2\end{aligned}$$

Answer:

$x^2-xy-2y^2$

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? Exercise $Page1.4.1$

Subtract: $\left(7a^2-2ab+b^2\right)-\left(a^2-2ab+5b^2\right)$.

Answer

$6a^2-4b^2$

www.youtube.com/v/IDtREB_PQ3A

Multiplying Algebraic Expressions

Use the product rule for exponents, $x^m \cdot x^n = x^{m+n}$, to multiply a monomial times a polynomial. In other words, when multiplying two expressions with the same base, add the exponents. To find the product of monomials, multiply the coefficients and add the exponents of variable factors with the same base. For example,

$$\begin{aligned} 7x^4 \cdot 8x^3 &= 7 \cdot 8 \cdot x^4 \cdot x^3 \quad \textit{Commutative property} \\ &= 56x^{4+3} \quad \textit{Product rule for exponents} \\ &= 56x^7 \end{aligned}$$

To multiply a polynomial by a monomial, apply the distributive property, and then simplify each term.

✓ Example $Page1.4.7$:

Multiply: $5xy^2\left(2x^2y^2-xy+1\right)$.

Solution

Apply the distributive property and then simplify.

$$5xy^2\left(2x^2y^2-xy+1\right)$$

$$\begin{aligned} &= 5xy^2 \cdot 2x^2y^2 - 5xy^2 \cdot xy + 5xy^2 \cdot 1 \\ &= 10x^3y^4 - 5x^2y^3 + 5xy^2 \end{aligned}$$

Answer:

$10x^3y^4-5x^2y^3+5xy^2$

To summarize, multiplying a polynomial by a monomial involves the distributive property and the product rule for exponents. Multiply all of the terms of the polynomial by the monomial. For each term, multiply the coefficients and add exponents of variables where the bases are the same.

In the same manner that we used the distributive property to distribute a monomial, we use it to distribute a binomial.

$$\begin{aligned} (a+b)(c+d) &= (a+b) \cdot c + (a+b) \cdot d \\ &= ac+bc+ad+bd \\ &= ac+ad+bc+bd \end{aligned}$$

Here we apply the distributive property multiple times to produce the final result. This same result is obtained in one step if we apply the distributive property to a and b separately as follows:

$$(a+b)(c+d) = ac+ad+bc+bd$$

This is often called the FOIL method. Multiply the first, outer, inner, and then last terms.

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Example $Page1.4.8$:

Multiply: $(6x-1)(3x-5)$.

Solution

Distribute $6x$ and -1 and then combine like terms.

$$\begin{aligned}(6x-1)(3x-5) &= 6x\cdot 3x - 6x\cdot 5 + (-1)\cdot 3x - (-1)\cdot 5\\ &= 18x^2 - 30x - 3x + 5\\ &= 18x^2 - 33x + 5\end{aligned}$$

Answer:

$18x^2 - 33x + 5$

Consider the following two calculations:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2 + ab + ba + b^2\\ &= a^2 + ab + ab + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2 - ab - ba + b^2\\ &= a^2 - ab - ab + b^2\\ &= a^2 - 2ab + b^2\end{aligned}$$

This leads us to two formulas that describe **perfect square trinomials**:

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2\\ (a-b)^2 &= a^2 - 2ab + b^2\end{aligned}$$

We can use these formulas to quickly square a binomial.

Example $Page1.4.9$:

Multiply: $(3x+5)^2$

Solution

Here $a = 3x$ and $b = 5$. Apply the formula:

$$\begin{aligned}(a+b)^2 &= a^2 + 2\,a\cdot b + b^2\\ (3x+5)^2 &= (3x)^2 + 2\cdot(3x)(5) + (5)^2\\ &= 9x^2 + 30x + 25\end{aligned}$$

Answer:

$9x^2 + 30x + 25$

This process should become routine enough to be performed mentally. Our third special product follows:

$$\begin{aligned}(a+b)(a-b) &= a^2 - ab + ba - b^2\\ &= a^2 - ab + ab - b^2\\ &= a^2 - b^2\end{aligned}$$

This product is called **difference of squares**:

$$(a+b)(a-b) = a^2 - b^2$$

The binomials $(a+b)$ and $(a-b)$ are called **conjugate binomials**. When multiplying conjugate binomials the middle terms are opposites and their sum is zero; the product is itself a binomial.

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Example $Page1.4.10$:

Multiply: $(3xy+1)(3xy-1)$.

Solution

$$\begin{aligned}(3xy+1)(3xy-1) &= (3xy)^2 - 3xy + 3xy - 1^2 \\ &= 9x^2y^2 - 1\end{aligned}$$

Answer:

$9x^2y^2 - 1$

Exercise $Page1.4.2$

Multiply: $\left(x^2+5y^2\right)\left(x^2-5y^2\right)$.

Answer

$\left(x^4-25y^4\right)$

www.youtube.com/v/p7R3FdPp6_s

Example $Page1.4.11$:

Multiply: $(5x-2)^3$.

Solution

Here we perform one product at a time.

$$\begin{aligned}\left(5x-2\right)^3 &= \left(5x-2\right)\underbrace{\left(5x-2\right)\left(5x-2\right)}_{\textit{Multiply first.}} \\ &= \left(5x-2\right)\left(25x^2-10x-10x+4\right) \\ &= \left(5x-2\right)\left(25x^2-20x+4\right) \\ &= 125x^3-100x^2+20x-50x^2+40x-8 \\ &= 125x^3-150x^2+60x-8\end{aligned}$$

Answer:

$125x^2-150x^2+60x-8$

Dividing Polynomials

Use the quotient rule for exponents, $\frac{x^m}{x^n} = x^{m-n}$, to divide a polynomial by a monomial. In other words, when dividing two expressions with the same base, subtract the exponents. In this section, we will assume that all variables in the denominator are nonzero.

Example $Page1.4.12$:

Divide: $\frac{24x^7y^5}{8x^3y^2}$.

Solution

Divide the coefficients and apply the quotient rule by subtracting the exponents of the like bases.

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$$\frac{24x^7y^5}{8x^3y^2} = \frac{24}{8}x^{7-3}y^{5-2}$$
$$= 3x^4y^3$$

Answer:

$3x^4y^3$

When dividing a polynomial by a monomial, we may treat the monomial as a common denominator and break up the fraction using the following property:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

Applying this property will result in terms that can be treated as quotients of monomials.

✔ Example $Page1.4.13$:

Divide: $\frac{-5x^4+25x^3-15x^2}{5x^2}$.

Solution

Break up the fraction by dividing each term in the numerator by the monomial in the denominator, and then simplify each term.

$$\frac{-5x^4+25x^3-15x^2}{5x^2} = -\frac{5x^4}{5x^2} + \frac{25x^3}{5x^2} - \frac{15x^2}{5x^2}$$
$$= -\frac{5}{5}x^{4-2} + \frac{25}{5}x^{3-2} - \frac{15}{5}x^{2-2}$$
$$= -1x^2 + 5x^1 - 3x^0$$
$$= -x^2 + 5x - 3 \cdot 1$$

Answer:

$-x^2+5x-3$

We can check our division by multiplying our answer, the quotient, by the monomial in the denominator, the divisor, to see if we obtain the original numerator, the dividend.

Table $Page1.4.7$

$\frac{Dividend}{Divisor} = Quotient$	$\frac{-5x^4+25x^3-15x^2}{5x^2} = -x^2+5x-3$
or	or
$Dividend = Divisor \cdot Quotient$	$-5x^4+25x^3-15x^2 = 5x^2(-x^2+5x-3)$

The same technique outlined for dividing by a monomial *does not* work for polynomials with two or more terms in the denominator. In this section, we will outline a process called **polynomial long division**, which is based on the division algorithm for real numbers. For the sake of clarity, we will assume that all expressions in the denominator are nonzero.

✔ Example $Page1.4.14$:

Divide $\frac{x^3+3x^2-8x-4}{x-2}$:

Solution

Here $x-2$ is the divisor and x^3+3x^2-8x-4 is the dividend. To determine the first term of the quotient, divide the leading term of the dividend by the leading term of the divisor.

$$x-2\overline{)x^3+3x^2-8x-4}$$

We begin by dividing the leading terms.

$$x^3 \div x = x^2$$

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Multiply the first term of the quotient by the divisor, remembering to distribute, and line up like terms with the dividend.

$$\begin{array}{r} x^2 \\ x-2\overline{)\,x^3+3x^2-8x-4} \\ \underline{x^3-2x^2} \end{array} \qquad \textit{Distribute and line up like terms.} \qquad x^2\cdot(x-2)=x^3-2x^2$$

Subtract the resulting quantity from the dividend. Take care to subtract both terms.

$$\begin{array}{r} x^2 \\ x-2\overline{)\,x^3+3x^2-8x-4} \\ \underline{-\left(x^3-2x^2\right)} \\ 5x^2 \end{array} \qquad \textit{Subtract.} \qquad -\left(x^3-2x^2\right)=-x^3+2x^2$$

Bring down the remaining terms and repeat the process.

Notice that the leading term is eliminated and that the result has a degree that is one less. The complete process is illustrated below:

$$\begin{array}{r} x^2+\ 5x+2 \\ x-2\overline{)\,x^3+3x^2-8x-4} \\ \underline{-\left(x^3-2x^2\right)} \\ 5x^2\ -8x\ -4 \\ \underline{-\left(5x^2-10x\right)} \\ 2x\ -\ 4 \\ \underline{-\left(2x-4\right)} \\ 0 \end{array}$$

Polynomial long division ends when the degree of the remainder is less than the degree of the divisor. Here, the remainder is 0. Therefore, the binomial divides the polynomial evenly and the answer is the quotient shown above the division bar.

$$\frac{x^3+3x^2-8x-4}{x-2}=x^2+5x+2$$

To check the answer, multiply the divisor by the quotient to see if you obtain the dividend as illustrated below:

$$x^3+3x^2-8x-4=(x-2)\left(x^2+5x+2\right)$$

This is left to the reader as an exercise.

Answer:

x^2+5x+2

Next, we demonstrate the case where there is a nonzero remainder.

$$\begin{array}{r} \textit{Quotient} \\ \textit{Divisor}\overline{)\,\textit{Dividend}} \\ \vdots \\ \hline \textit{Remainder} \end{array}$$

Just as with real numbers, the final answer adds to the quotient the fraction where the remainder is the numerator and the divisor is the denominator. In general, when dividing we have:

$$\frac{Dividend}{Divisor} = Quotient + \frac{Remainder}{Divisor}$$

If we multiply both sides by the divisor we obtain,

$$Dividend = Quotient \times Divisor + Remainder$$

Example $Page 1.4.15$:

Divide: $\frac{6x^2-5x+3}{2x-1}$.

Solution

Since the denominator is a binomial, begin by setting up polynomial long division.

$$2x-1\overline{)\,6x^2-5x+3}\qquad ?$$

To start, determine what monomial times $2x-1$ results in a leading term $6x^2$.This is the quotient of the given leading terms: $(6x^2)\div(2x)=3x$. Multiply $3x$ times the divisor $2x-1$, and line up the result with like terms of the dividend.

$$\begin{array}{r} 3x \\ 2x-1\overline{)\,6x^2-5x+3} \\ 6x^2-3x \end{array}$$

Subtract the result from the dividend and bring down the constant term $+3$.

$$\begin{array}{r} 3x\quad ? \\ 2x-1\overline{)\,6x^2-5x+3} \\ \underline{-\left(6x^2-3x\right)} \\ -2x+3 \end{array}$$

Subtracting eliminates the leading term. Multiply $2x-1$ by -1 and line up the result.

$$\begin{array}{r} 3x-1 \\ 2x-1\overline{)\,6x^2-5x+3} \\ \underline{-\left(6x^2-3x\right)} \\ -2x+3 \\ -2x+1 \end{array}$$

Subtract again and notice that we are left with a remainder.

$$\begin{array}{r} 3x-1 \\ 2x-1\overline{)\,6x^2-5x+3} \\ \underline{-\left(6x^2-3x\right)} \\ -2x+3 \\ \underline{-\left(-2x+1\right)} \\ 2 \end{array}$$

The constant term 2 has degree 0 and thus the division ends. Therefore,

$$\frac{6x^2-5x+3}{2x-1} = 3x-1+\frac{2}{2x-1}$$

To check that this result is correct, we multiply as follows:

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$$\begin{aligned} quotient \times divisor + remainder &= (3x-1)(2x-1)+2 \\ &= 6x^2 - 3x - 2x + 1 + 2 \\ &= 6x^2 - 5x + 2 = dividend \ \checkmark \end{aligned}$$

Answer:

$3x - 1 + \frac{2}{2x-1}$

Occasionally, some of the powers of the variables appear to be missing within a polynomial. This can lead to errors when lining up like terms. Therefore, when first learning how to divide polynomials using long division, fill in the missing terms with zero coefficients, called **placeholders**.

Example *Page*1.4.16:

Divide: $\frac{27x^3+64}{3x+4}$.

Solution

Notice that the binomial in the numerator does not have terms with degree 2 or 1. The division is simplified if we rewrite the expression with placeholders:

$$27x^3 + 64 = 27x^3 + 0x^2 + 0x + 64$$

Set up polynomial long division:

$$3x+4\overline{)27x^3+0x^2+0x+64}$$

We begin with 27x3÷3x=9x2 and work the rest of the division algorithm.

$$\begin{array}{r} 9x^2 - 12x + 16 \\ 3x+4\overline{)\ 27x^3+0x^2+0x+64} \\ \underline{-\left(27x^3+36x^2\right)} \\ -36x^2 + 0x + 64 \\ \underline{-\left(-36x^2 - 48x\right)} \\ 48x + 64 \\ \underline{-\left(48x + 64\right)} \\ 0 \end{array}$$

Answer:

$9x^2 - 12x + 16$

Example *Page*1.4.17:

Divide: $\frac{3x^4-2x^3+6x^2+23x-7}{x^2-2x+5}$.

Solution

$$x^2-2x+5\overline{)3x^4-2x^3+6x^2+23x-7}$$

Begin the process by dividing the leading terms to determine the leading term of the quotient $3x^4 \div x^2 = 3x^2$. Take care to distribute and line up the like terms. Continue the process until the remainder has a degree less than 2.

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$$
\begin{array}{r}
3x^2+4x-1 \\
x^2-2x+5\overline{)\,3x^4-2x^3+6x^2+23x-7} \\
\underline{-\left(3x^4-6x^3+15x^2\right)} \\
4x^3-9x^2+23x-7 \\
\underline{-\left(4x^3-8x^2+20x\right)} \\
-x^2+3x-7 \\
\underline{-\left(-x^2+2x-5\right)} \\
x-2
\end{array}
$$

The remainder is $x-2$. Write the answer with the remainder:

$\frac{3x^4-2x^3+6x^2+23x-7}{x^2-2x+5}=3x^2+4x-1+\frac{x-2}{x^2-2x+5}$

Answer:

$3x^2+4x-1+\frac{x-2}{x^2-2x+5}$

Polynomial long division takes time and practice to master. Work lots of problems and remember that you may check your answers by multiplying the quotient by the divisor (and adding the remainder if present) to obtain the dividend.

? Exercise *Page*1.4.3

Divide: $\frac{6x^4-13x^3+9x^2-14x+6}{3x-2}$.

Answer

$2x^3-3x^2+x-4-\frac{2}{3x-2}$

www.youtube.com/v/K9NRVMreKAQ

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Page 1.5: Rationalizing Expressions

Introduction

An algebraic expression may sometimes have a radical in the denominator (or numerator). In calculus, a helpful technique would be to get rid of the radical in the denominator (or numerator). The technique used to rid the expression of the radical is called ***rationalizing***. We will first discuss the idea of rationalizing the denominator, followed by rationalizing the numerator.

What is Rationalizing a Denominator?

The idea of rationalizing a denominator makes a bit more sense if you consider the definition of "rationalize." Recall that the numbers 5, $\frac{1}{2}$, and 0.75 are all known as rational numbers because they can each be expressed as a ratio of two integers ($\frac{5}{1}$, $\frac{1}{2}$, and $\frac{3}{4}$ respectively). Some radicals are irrational numbers because they cannot be represented as a ratio of two integers. As a result, the point of rationalizing a denominator is to change the expression so that the denominator becomes a rational number.

Here are some examples of irrational and rational denominators.

Irrational		**Rational**
$\frac{1}{\sqrt{2}}$	=	$\frac{\sqrt{2}}{2}$
$\frac{2+\sqrt{3}}{\sqrt{3}}$	=	$\frac{2\sqrt{3}+3}{3}$

Now let's examine how to get from irrational to rational denominators.

Rationalizing Denominators with One Term

Let's start with the fraction $\frac{1}{\sqrt{2}}$. Its denominator is $\sqrt{2}$, an irrational number.

Notice that multiplying the fraction by 1 does not change the value and so equivalently, multiplying the fraction by $\frac{\sqrt{2}}{\sqrt{2}}$ also does not change it's value. Allow us to now see what happens.

$$\frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2 \cdot 2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$$

The denominator of the new fraction is no longer a radical (notice, however, that the numerator is).

So why choose to multiply $\frac{1}{\sqrt{2}}$ by $\frac{\sqrt{2}}{\sqrt{2}}$? You knew that the square root of a number times itself will be a whole number. In algebraic terms, this idea is represented by $\sqrt{x} \cdot \sqrt{x} = x$. Look back to the denominators in the multiplication of $\frac{1}{\sqrt{2}} \cdot 1$. Do you see where $\sqrt{2} \cdot \sqrt{2} = \sqrt{4} = 2$?

Here are some more examples. Notice how the value of the fraction is not changed at all; it is simply being multiplied by another name for 1.

Example

Rationalize the denominator. $\frac{2+\sqrt{3}}{\sqrt{3}}$

Solution

$\frac{2+\sqrt{3}}{\sqrt{3}}$	The denominator of this fraction is $\sqrt{3}$. To make it into a rational number, multiply it by $\sqrt{3}$, since $\sqrt{3} \cdot \sqrt{3} = 3$.
$\frac{2+\sqrt{3}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$	Multiply the entire fraction by another name for 1, $\frac{\sqrt{3}}{\sqrt{3}}$.
$\frac{\sqrt{3}(2+\sqrt{3})}{\sqrt{3} \cdot \sqrt{3}}$	
$\frac{2\sqrt{3}+\sqrt{3} \cdot \sqrt{3}}{\sqrt{9}}$	Use the Distributive Property to multiply $\sqrt{3}(2+\sqrt{3})$.
$\frac{2\sqrt{3}+\sqrt{9}}{\sqrt{9}}$	Simplify the radicals, where possible. $\sqrt{9} = 3$.

$\frac{2+\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}+3}{3}$

You can use the same method to rationalize denominators to simplify fractions with radicals that contain a variable. As long as you multiply the original expression by another name for 1, you can eliminate a radical in the denominator without changing the value of the expression itself.

Example

Rationalize the denominator. $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}}$**, where** $x \neq 0$

Solution

$\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$	The denominator is $\sqrt{x}$, so the entire expression can be multiplied by $\frac{\sqrt{x}}{\sqrt{x}}$ to get rid of the radical in the denominator.
$\frac{\sqrt{x}(\sqrt{x}+\sqrt{y})}{\sqrt{x}\cdot\sqrt{x}}$	
$\frac{\sqrt{x}\cdot\sqrt{x}+\sqrt{x}\cdot\sqrt{y}}{\sqrt{x}\cdot\sqrt{x}}$	Use the Distributive Property. Simplify the radicals, where possible. Remember that $\sqrt{x}\cdot\sqrt{x} = x$.

$\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}} = \frac{x+\sqrt{xy}}{x}$

Example

Rationalize the denominator and simplify. $\sqrt{\frac{100x}{11y}}$**, where** $y \neq 0$

Solution

$\frac{\sqrt{100x}}{\sqrt{11y}}$	Rewrite $\sqrt{\frac{a}{b}}$ as $\frac{\sqrt{a}}{\sqrt{b}}$.
$\frac{\sqrt{100x}}{\sqrt{11y}} \cdot \frac{\sqrt{11y}}{\sqrt{11y}}$	The denominator is $\sqrt{11y}$, so multiplying the entire expression by $\frac{\sqrt{11y}}{\sqrt{11y}}$ will rationalize the denominator.
$\frac{\sqrt{100\cdot 11xy}}{\sqrt{11y}\cdot\sqrt{11y}}$	Multiply and simplify the radicals, where possible.
$\frac{\sqrt{100}\cdot\sqrt{11xy}}{\sqrt{11y}\cdot\sqrt{11y}}$	100 is a perfect square. Remember that $\sqrt{100} = 10$ and $\sqrt{x}\cdot\sqrt{x} = x$.

$\sqrt{\frac{100x}{11y}} = \frac{10\sqrt{11xy}}{11y}$

Exercise

Rationalize the denominator and simplify. $\frac{3}{\sqrt{7}}$

A. $\frac{3\sqrt{7}}{\sqrt{7}}$
B. 3
C. $\frac{7\sqrt{3}}{3}$
D. $\frac{3\sqrt{7}}{7}$

Answer

A. Incorrect. After you rationalize a denominator, the denominator should not be in radical form. Try multiplying the fraction by $\frac{\sqrt{7}}{\sqrt{7}}$. The correct answer is $\frac{3\sqrt{7}}{7}$.

B. Incorrect. The process of rationalizing does not remove the denominator, it just changes the fraction so that the denominator is no longer a radical. Try multiplying the numerator and denominator by $\frac{\sqrt{7}}{\sqrt{7}}$. The correct answer is $\frac{3\sqrt{7}}{7}$.

C. Incorrect. You may have rationalized the fraction $\frac{7}{\sqrt{3}}$ instead of $\frac{3}{\sqrt{7}}$. Try multiplying the numerator and denominator by $\frac{\sqrt{7}}{\sqrt{7}}$. The correct answer is $\frac{3\sqrt{7}}{7}$.

D. Correct. To rationalize the denominator, multiply $\frac{3}{\sqrt{7}}$ by $\frac{\sqrt{7}}{\sqrt{7}}$. This will create an integer in the denominator, so the fraction will have been rationalized.

Rationalizing Denominators with Two Terms

Denominators do not always contain just one term, as shown in the previous examples. Sometimes, you will see expressions like $\frac{3}{\sqrt{2}+3}$ where the denominator is composed of two terms, $\sqrt{2}$ and $+3$.

Unfortunately, you cannot rationalize these denominators the same way you rationalize single-term denominators. If you multiply $\sqrt{2}+3$ by $\sqrt{2}$, you get $2+3\sqrt{2}$. The original $\sqrt{2}$ is gone, but now the quantity $3\sqrt{2}$ has appeared, but this is no better!

In order to rationalize this denominator, you want to square the radical term and somehow prevent the integer term from being multiplied by a radical. Is this possible?

It is possible, and you have already seen how to do it!

Recall that when binomials of the form $(a+b)(a-b)$ are multiplied, the product is a^2-b^2. So, for example, $(x+3)(x-3)=x^2-3x+3x-9=x^2-9$; notice that the terms $-3x$ and $+3x$ combine to 0. Now for the connection to rationalizing denominators: what if you replaced x with $\sqrt{2}$?

Look at the examples below. Just as $-3x+3x$ combines to 0 in the top example, $-3\sqrt{2}+3\sqrt{2}$ combines to 0 in the bottom example.

$$
\begin{aligned}
&(x+3)(x-3)\\
&=x^2-3x+3x-9\\
&=x^2-9\\
&(\sqrt{2}+3)(\sqrt{2}-3)\\
&=(\sqrt{2})^2-3\sqrt{2}+3\sqrt{2}-9\\
&=(\sqrt{2})^2-9\\
&=2-9\\
&=-7
\end{aligned}
$$

There you have it! Multiplying $\sqrt{2}+3$ by $\sqrt{2}-3$ removed one radical without adding another.

In this example, $\sqrt{2}-3$ is known as a **conjugate**, and $\sqrt{2}+3$ and $\sqrt{2}-3$ are known as a **conjugate pair**. To find the conjugate of a binomial that includes radicals, change the sign of the second term to its opposite as shown in the table below.

Term	Conjugate	Product
$\sqrt{2}+3$	$\sqrt{2}-3$	$(\sqrt{2}+3)(\sqrt{2}-3)=(\sqrt{2})^2-(3)^2=2-9=-$
$\sqrt{x}-5$	$\sqrt{x}+5$	$(\sqrt{x}-5)(\sqrt{x}+5)=(\sqrt{x})^2-(5)^2=x-25$
$8-2\sqrt{x}$	$8+2\sqrt{x}$	$(8-2\sqrt{x})(8+2\sqrt{x})=(8)^2-(2\sqrt{x})^2=64-$
$1+\sqrt{xy}$	$1-\sqrt{xy}$	$(1+\sqrt{xy})(1-\sqrt{xy})=(1)^2-(\sqrt{xy})^2=1-x$

More generally, if a fraction has a denominator of the form $a+b\sqrt{c}$, then the conjugate is $a-b\sqrt{c}$ and vice-versa.

Example

Rationalize the denominator and simplify. $\frac{5-\sqrt{7}}{3+\sqrt{5}}$

Solution

$\frac{5-\sqrt{7}}{3+\sqrt{5}}\cdot\frac{3-\sqrt{5}}{3-\sqrt{5}}$	Find the conjugate of $3+\sqrt{5}$. Then multiply the entire expression by $\frac{3-\sqrt{5}}{3-\sqrt{5}}$.

$\frac{(5-\sqrt{7})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})}$	
$\frac{5\cdot3-5\sqrt{5}-3\sqrt{7}+\sqrt{7}\cdot\sqrt{5}}{3\cdot3-3\sqrt{5}+3\sqrt{5}-\sqrt{5}\cdot\sqrt{5}}$	Use the Distributive Property to multiply the binomials in the numerator and denominator.
$\frac{15-5\sqrt{5}-3\sqrt{7}+\sqrt{35}}{9-3\sqrt{5}+3\sqrt{5}-\sqrt{25}}$	Since you multiplied by the conjugate of the denominator, the radical terms in the denominator will combine to 0.
$\frac{15-5\sqrt{5}-3\sqrt{7}+\sqrt{35}}{9-\sqrt{25}}$ $\frac{15-5\sqrt{5}-3\sqrt{7}+\sqrt{35}}{9-5}$	Simplify radicals where possible.

$\frac{5-\sqrt{7}}{3+\sqrt{5}}=\frac{15-5\sqrt{5}-3\sqrt{7}+\sqrt{35}}{4}$

Example

Rationalize the denominator and simplify. $\frac{\sqrt{x}}{\sqrt{x}+2}$

Solution

$\frac{\sqrt{x}}{\sqrt{x}+2}\cdot\frac{\sqrt{x}-2}{\sqrt{x}-2}$	Find the conjugate of $\sqrt{x}+2$. Then multiply the numerator and denominator by $\frac{\sqrt{x}-2}{\sqrt{x}-2}$.
$\frac{\sqrt{x}(\sqrt{x}-2)}{(\sqrt{x}+2)(\sqrt{x}-2)}$	
$\frac{\sqrt{x}\cdot\sqrt{x}-2\sqrt{x}}{\sqrt{x}\cdot\sqrt{x}-2\sqrt{x}+2\sqrt{x}-2\cdot2}$	Use the Distributive Property to multiply the binomials in the numerator and denominator.
$\frac{\sqrt{x}\cdot\sqrt{x}-2\sqrt{x}}{\sqrt{x}\cdot\sqrt{x}-2\sqrt{x}+2\sqrt{x}-4}$	Simplify. Remember that $\sqrt{x}\cdot\sqrt{x}=x$. Since you multiplied by the conjugate of the denominator, the radical terms in the denominator will combine to 0.

$\frac{\sqrt{x}}{\sqrt{x}+2}=\frac{x-2\sqrt{x}}{x-4}$

One word of caution: this method will work for binomials that include a square root, but not for binomials with roots greater than 2. This is because squaring a root that has an index greater than 2 does not remove the root, as shown below.

$(\sqrt[3]{10}+5)(\sqrt[3]{10}-5)$
$=(\sqrt[3]{10})^2-5\sqrt[3]{10}+5\sqrt[3]{10}-25$
$=(\sqrt[3]{10})^2-25$
$=\sqrt[3]{100}-25$

$\sqrt[3]{100}$ cannot be simplified any further, since its prime factors are $2\cdot2\cdot5\cdot5$. There are no cubed numbers to pull out! Multiplying $\sqrt[3]{10}+5$ by its conjugate does not result in a radical-free expression.

Exercise

Identify the conjugate of the denominator. $\frac{x+2}{\sqrt{2x}-7}$

A. $\sqrt{2x}+7$
B. $\sqrt{2x}-7$
C. $2x+\sqrt{7}$
D. $(2x)^2+7$

Answer

A. Correct. The conjugate will be the binomial that, when multiplied by the denominator, eliminates the radical. The conjugate is $\sqrt{2x}+7$.

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B. Incorrect. The denominator is not the conjugate. Look for the binomial that, when multiplied by the denominator, eliminates the radical. The correct answer is $\sqrt{2x}+7$.

C. Incorrect. Multiplying the denominator by $2x+\sqrt{7}$ will not remove the radical. Look for the binomial that follows the pattern $(a+b)(a-b)=a^2-b^2$. The correct answer is $\sqrt{2x}+7$.

D. Incorrect. Multiplying the denominator by $(2x)^2+7$ will not remove the radical. Look for the binomial that follows the pattern $(a+b)(a-b)=a^2-b^2$. The correct answer is $\sqrt{2x}+7$.

Rationalizing The Numerator

The process of rationalizing the numerator is extremely similar to that of rationalizing the denominator.

? Exercise $Page1.5.1$

Rationalize the numerator:

$$\frac{\sqrt{4+h}-2}{h}$$

You may assume that $h \neq 0$.

Answer

$$\frac{\sqrt{4+h}-2}{h} = \frac{\sqrt{4+h}-2}{h} \times \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4+h}+2}$$

? Exercise $Page1.5.1$

Rationalize the numerator:

$$\frac{\sqrt{t^2+9}-3}{t}$$

You may assume that $t \neq 0$.

Answer

$$\frac{\sqrt{t^2+9}-3}{t} = \frac{\sqrt{t^2+9}-3}{t} \times \frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} = \frac{t}{\sqrt{t^2+9}+3}$$

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Page 1.6: Factoring Polynomials

Suppose we were asked to simplify $7x^2yz(2y^2+3xz^2)$. How would we do this? Quite simply, we would distribute to obtain $14x^2y^3z+21x^3yz^3$.

The question we will now consider is essentially the reverse. That is, starting with $14x^2y^3z+21x^3yz^3$ how can we go back to $7x^2yz(2y^2+3xz^2)$?

The answer is that we simply check the greatest common factor/divisor of each corresponding part. Notice that:

1) $gcd(14,21)=7$

2) $gcd(x^2,x^3)=x^2$

3) $gcd(y^2,y)=y$

4) $gcd(z,z^3)=z$

So each term in $14x^2y^3z+21x^3yz^3$ shares a factor of $7x^2yz$. Hence

$$\frac{14x^2y^3z+21x^3yz^3}{7x^2yz}=\frac{14x^2y^3z}{7x^2yz}+\frac{21x^3yz^3}{7x^2yz}=2y^2+3xz^2$$

That is,

$$\frac{14x^2y^3z+21x^3yz^3}{7x^2yz}=2y^2+3xz^2$$

Cross multiplying yields $14x^2y^3z+21x^3yz^3=7x^2yz(2y^2+3xz^2)$. With enough practice, we will be able to do this in one line.

? Exercise $Page1.6.1$

Factor $8x^2y^3+16xy^4-32x^5y$.

Answer

$8xy(xy^2+2y^3-4x^4)$

? Exercise $Page1.6.1$

Factor $48ab^2-64a^3b^5$.

Answer

$16ab^2(3-4a^2b^3)$

When factoring, we should keep an eye out for special cases. For example, one such case we should look out for is when we are asked to factor the difference of two squares.

More concretely, consider the expression $100x^2-25$ which is a difference of two squares. Observe that $100x^2-25=(100x+5)(100x-5)$.

More generally, $a^2-b^2)=(a-b)(a+b)$ where a and b may represent real numbers or algebraic expressions. A list of some special cases are provided below.

Theorem $Page1.6.1$

Theorem: Let a and b denote real numbers or algebraic expressions. Then

1) $a^2-b^2)=(a+b)(a-b)$

2) $a^2+2ab+b^2=(a+b)^2$

3) $a^2-2ab+b^2=(a-b)^2$

Exercise $Page1.6.1$

Factor the following:

1) $49x^2-81$

2) $4x^2-36$

Answer

1) $49x^2-81=(7x+9)(7x-9)$

2) $4x^2-36=(2x+6)(2x-6)=2(x+3)2(x-3)=4(x+3)(x-3)$

We will often have to factor a polynomial, especially quadratics. Recall from the last section that a **quadratic** is a polynomial of the form $a_2x^2+a_1x+a_0$. However, we will commonly denote a quadratic as ax^2+bx+c . Allow us to now discuss some techniques on how to factor a quadratic.

The AC Method (General Technique)

To factor ax^2+bx+c , we perform the following steps:

1) First multiply a by c.

2) Search for two factors of ac whose sum is b.

3) Rewrite b as a sum in the quadratic and then factor by grouping.

Exercise $Page1.6.1$

Factor x^2-5x+6

Answer

1) We first multiply the a term with the c term. Doing so yields 6.

2) We now must find factors of 6 which sums to -5. We see that -2 and -3 does the trick.

3)

$$\begin{aligned} x^2-5x+6 &= x^2-2x-3x+6 \\ &= (x^2-2x)+(-3x+6) \\ &= x(x-2)-3(x-2) \\ &= (x-2)(x-3) \end{aligned}$$

In your previous studies, you may have seen a more "simple" way to factor quadratics when $a=1$.

The AC Method when $a=1$.

To factor x^2+bx+c the AC Method can be simplified to the following:

1) Search for two factors of c whose sum is b. Let us call these factors r_1 and r_2.

2) Then $x^2+bx+c=(x+r_1)(x+r_2)$.

Exercise $Page1.6.1$

Factor x^2-5x+6

Answer

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1) Since $a = 1$ our first step should be finding factors of 6 which sums to -5. We see that -2 and -3 does the trick. So $r_1 = -2$ and $r_2 = -3$.

2) Then $x^2 - 5x + 6 = (x-2)(x-3)$

Again, this latter technique only works when $a = 1$.

? Exercise $Page 1.6.1$

Factor $2x^2 - 5x - 3$

Answer

1) We first multiply the a term with the c term. Doing so yields -6.

2) We now must find factors of -6 which sums to -5. We see that -6 and 1 does the trick.

3)

$$\begin{aligned} 2x^2 - 5x - 3 &= 2x^2 - 6x + x - 3 \\ &= (2x^2 - 6x) + (x - 3) \\ &= 2x(x-3) + 1(x-3) \\ &= (x-3)(2x+1) \end{aligned}$$

? Exercise $Page 1.6.1$

Factor:

1) $3x^2 + x - 4$

2) $4x^2 + 5x - 6$

3) $2x^2 - 7x + 6$

4) $5x^2 - 9x - 2$

5) $x^2 + 2x - 24$

6) $x^2 + 8x + 15$

Answer

1) $3x^2 + x - 4 = (3x+4)(x-1)$

2) $4x^2 + 5x - 6 = (x+2)(4x-3)$

3)$2x^2 - 7x + 6 = (2x-3)(x-2)$

4) $5x^2 - 9x - 2 = (5x+1)(x-2)$

5) $x^2 + 2x - 24 = (x+6)(x-4)$

6) $x^2 + 8x + 15(x+3)(x+5)$

Solving Quadratic Equations

We will now turn our attention to solving quadratic equations. We first note the following theorem.

Theorem: The Zero Product Property $Page 1.6.1$

Theorem: Suppose a and b are real numbers or algebraic expressions. If $ab = 0$ then either $a = 0$ or $b = 0$.

With The Zero Product Property, we can begin to solve equations such as quadratics. When we say *solve an equation* we mean we can find all values for the variable which will make the equation true.

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Technique 1 for Solving Quadratics

To solve $ax^2+bx+c=0$ we first *attempt* to factor the quadratic. Then we set each factor equal to zero and solve.

Exercise $Page 1.6.1$

Solve the equation $2x^2-5x-3=0$

Answer

1) We first multiply the a term with the c term. Doing so yields -6.

2) We now must find factors of -6 which sums to -5. We see that -6 and 1 does the trick.

3)

$$\begin{aligned} 2x^2-5x-3 &= 2x^2-6x+x-3 \\ &= (2x^2-6x)+(x-3) \\ &= 2x(x-3)+1(x-3) \\ &= (x-3)(2x+1) \end{aligned}$$

4)

$$\begin{aligned} 2x^2-5x-3 &= 0 \\ (x-3)(2x+1) &= 0 \\ x-3=0 \text{ or } 2x+1 &= 0 \\ x=3 \text{ or} x &= -\frac{1}{2} \end{aligned}$$

Exercise $Page 1.6.1$

Solve the equation $x^2-7x=0$

Answer

1) We first factor this as $x(x-7)=0$.

2) We now set each factor to 0 and solve. Hence either $x=0$ or $x=7$.

Exercise $Page 1.6.1$

Solve the equation $x^2+2x-24=0$

Answer

1) We first factor $x^2+2x-24$ as $(x+6)(x-4)$.

2) We now set each factor to 0 and solve. Hence either $x=-6$ or $x=4$.

Let us now modify the above problem ever so slightly. Suppose we were asked to solve the equation $x^2+2x-25=0$. Take a few moments and try our AC technique!

The above request shows that not every quadratic equation can be solved via the AC Method. Allow us to play around with the above equation and let us see what we can do. For now, do not worry as to why we are performing the following steps. Instead, just follow along:

$$
\begin{aligned}
x^2+2x-25 &= 0 && \text{adding 25 to both sides yields}\\
x^2+2x &= 25 && \text{adding 1 to both sides yields}\\
x^2+2x+1 &= 26 && \text{factoring the left hand side yields}\\
(x+1)^2 &= 26 && \text{taking the square roots of both sides yields}\\
\sqrt{(x+1)^2} &= \sqrt{26} && \text{recall that} \sqrt{a^2}=|a|\\
|x+1| &= \sqrt{26} && \text{so either } x+1=\sqrt{26} \text{ or } x+1=-\sqrt{26}
\end{aligned}
$$

If $x+1=\sqrt{26}$ then $x=1-\sqrt{26}$ and if $x+1=-\sqrt{26}$ then $x=-1-\sqrt{26}$.

Allow us to now reflect on the above process. We initially had an equation of the form $ax^2+bx+c=0$ which we could not factor. We then subtracted c from both sides to obtain $ax^2+bx=-c$. Then, we added in a "magical" number, m to both sides and obtained $ax^2+bx+m=-c+m$. The reason this number was "magical" is because the left hand side then turned into a perfect square!

The process outlined above is called **completing the square**. Completing the square is a method which can be used to solve all quadratics - factorable and non-factorable!

One question you may have is how do we determine this magical number m? Allow us to state the process of completing the square which answers this question.

Completing the Square

1) Given an equation of the form $ax^2+bx+c=0$ subtract c from both sides.

2) Divide both sides by a.

3) Add m to both sides where $m=\left(dfrac\text{coefficient on x}2\right)^2$.

4) The left hand side is now a perfect square and so we may solve the equation.

Exercise $Page1.6.1$

Solve $x^2-4x+2=0$.

Answer

1) Subtracting 2 from both sides yields $x^2-4x=-2$.

2) Dividing both sides by 1 does not affect the equation.

3) $m=\left(\frac{-4}{2}\right)^2=4$ and so we add 4 to both sides of the equation.

4) Adding 4 to both sides of the equation yields $x^2-4x+4=2$. The left hand side is precisely $(x-2)^2$ and so we have to solve $(x-2)^2=2$. Taking the square root of both sides yields $|x-2|=\sqrt{2}$ and so either $x-2=sqrt2$ or $x-2=-\sqrt{2}$. Hence $x=2\pm\sqrt{2}$.

Exercise $Page1.6.1$

Solve $5x^2-9x-2=0$.

Answer

1) Adding 2 to both sides yields $5x^2-9x=2$.

2) Dividing both sides by 5 yields $x^2-\frac{9}{5}x=\frac{2}{5}$.

3) $m=\left(\frac{\frac{9}{5}}{2}\right)^2=\frac{81}{100}$ and so we add $\frac{81}{100}$ to both sides of the equation.

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4) Adding $\frac{81}{100}$ to both sides of the equation yields $x^2-\frac{9}{5}x+\frac{81}{100}=\frac{2}{5}+\frac{81}{100}$. The left hand side is precisely $(x-\frac{9}{10})^2$ and so we have to solve $(x-\frac{9}{10})^2=\frac{121}{100}$. Taking the square root of both sides yields $|x-\frac{9}{10}|=\sqrt{\frac{121}{100}}$ and so either $x-\frac{9}{10}=\sqrt{\frac{121}{100}}$ or $x-\frac{9}{10}=-\sqrt{\frac{121}{100}}$. Hence $x=\frac{9}{10}\pm\frac{121}{100}$ which yields $x=2$ or $x=-\frac{1}{5}$.

? Exercise $Page 1.6.1$

Solve $2x^2-10x-3=0$.

Answer

$x=\frac{5}{2}\pm\sqrt{\frac{31}{4}}$.

? Exercise $Page 1.6.1$

Solve $ax^2+bx+c=0$. You may assume that $a\neq 0$.

Answer

1) Subtracting c from both sides yields $ax^2+bx=-c$.

2) Dividing both sides by a yields $x^2+\frac{b}{a}x=-\frac{c}{a}$.

3) $m=\left(\frac{\frac{b}{a}}{2}\right)^2=\frac{b^2}{4a^2}$ and so we add $\frac{b^2}{4a^2}$ to both sides of the equation.

4) Adding $\frac{b^2}{4a^2}$ to both sides of the equation yields $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$. Simplifying the right hand side via a common denominator yields $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$ or equivalently $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$.

Now we will simplify the left hand side and solve. Doing so yields $(x+\frac{b}{2a})^2=\frac{b^2-4ac}{4a^2}$.

Taking the square root of both sides yields

$$\sqrt{(x+\frac{b}{2a})^2}=\sqrt{\frac{b^2-4ac}{4a^2}}$$

. Recalling that $\sqrt{a^2}=|a|$ the above line becomes

$$|x+\frac{b}{2a}|=\sqrt{\frac{b^2-4ac}{4a^2}}$$

Hence

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

and so

$$x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

Simplifying the right hand side yields

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

and adding the two fractions yields

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$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{(Page 1.6.1)}$$

which is precisely the quadratic formula.

The Quadratic Formula

Theorem $Page 1.6.1$

Theorem: The solution(s) to $ax^2 + bx + c = 0$ where $a \neq 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise $Page 1.6.1$

Using the quadratic formula, solve $5x^2 - 9x - 2 = 0$.

Answer

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{9 \pm \sqrt{(-9)^2 - 4(5)(-2)}}{2(5)} = \frac{9 \pm \sqrt{121}}{10} = frac9 \pm 1110$$

Hence $x = 2$ or $x = -\frac{1}{5}$.

Exercise $Page 1.6.1$

Using the quadratic formula, solve $3x^2 - 4x - 2 = 0$.

Answer

$$x = \frac{2 \pm \sqrt{10}}{3}$$

.

Note that in the quadratic formula, we refer to $b^2 - 4ac$ as the **discriminant**. Whenever the discriminant is negative, the quadratic will have no real solutions. For instance, one may check that the solutions to $3x^2 - 4x + 2 = 0$ do not exist among the real numbers.

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CHAPTER OVERVIEW

Chapter 2: Day 2

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Page 2.1: Rational Expressions

Learning Objectives

In this section students will:

- Simplify rational expressions.
- Multiply rational expressions.
- Divide rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

A pastry shop has fixed costs of $\$280$ per week and variable costs of $\$9$ per box of pastries. The shop's costs per week in terms of x, the number of boxes made, is $280+9x$. We can divide the costs per week by the number of boxes made to determine the cost per box of pastries.

$$\frac{280+9x}{x}$$

Notice that the result is a polynomial expression divided by a second polynomial expression. In this section, we will explore quotients of polynomial expressions.

Simplifying Rational Expressions

The quotient of two polynomial expressions is called a rational expression. We can apply the properties of fractions to rational expressions, such as simplifying the expressions by canceling common factors from the numerator and the denominator. To do this, we first need to factor both the numerator and denominator. Let's start with the rational expression shown.

$$\frac{x^2+8x+16}{x^2+11x+28}$$

We can factor the numerator and denominator to rewrite the expression.

$$\frac{(x+4)^2}{(x+4)(x+7)}$$

Then we can simplify that expression by canceling the common factor $(x+4)$.

$$\frac{x+4}{x+7}$$

Howto: Given a rational expression, simplify it

1. Factor the numerator and denominator.
2. Cancel any common factors.

Example $Page 2.1.1$: Simplifying Rational Expressions

Simplify $\dfrac{x^2-9}{x^2+4x+3}$

Solution

$$\frac{(x+3)(x-3)}{(x+3)(x+1)} \qquad \text{Factor the numerator and the denominator}$$

$$\frac{x-3}{x+1} \qquad \text{Cancel common factor } (x+3)$$

Analysis

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We can cancel the common factor because any expression divided by itself is equal to 1.

Q&A

Can the x^2 term be cancelled in the last example?

No. A factor is an expression that is multiplied by another expression. The x^2 term is not a factor of the numerator or the denominator.

Exercise $Page 2.1.1$

Simplify $\dfrac{x-6}{x^2-36}$

Answer

$$\frac{1}{x+6}$$

Multiplying Rational Expressions

Multiplication of rational expressions works the same way as multiplication of any other fractions. We multiply the numerators to find the numerator of the product, and then multiply the denominators to find the denominator of the product. Before multiplying, it is helpful to factor the numerators and denominators just as we did when simplifying rational expressions. We are often able to simplify the product of rational expressions.

Howto: Given two rational expressions, multiply them

1. Factor the numerator and denominator.
2. Multiply the numerators.
3. Multiply the denominators.
4. Simplify.

Example $Page 2.1.2$: Multiplying Rational Expressions

Multiply the rational expressions and show the product in simplest form:

$$\frac{(x+5)(x-1)}{3(x+6)} \times \frac{(2x-1)}{(x+5)}$$

Solution

$$\frac{(x+5)(x-1)}{3(x+6)} \times \frac{(2x-1)}{(x+5)}$$ Factor the numerator and denominator.

$$\frac{(x+5)(x-1)(2x-1)}{3(x+6)(x+5)}$$ Multiply numerators and denominators

$$\frac{(x-1)(2x-1)}{3(x+6)}$$ Cancel common factors to simplify

Exercise $Page 2.1.2$

Multiply the rational expressions and show the product in simplest form:

$$\frac{x^2+11x+30}{x^2+5x+6} \times \frac{x^2+7x+12}{x^2+8x+16}$$

Answer

$$\frac{(x+5)(x+6)}{(x+2)(x+4)}$$

Dividing Rational Expressions

Division of rational expressions works the same way as division of other fractions. To divide a rational expression by another rational expression, multiply the first expression by the reciprocal of the second. Using this approach, we would rewrite $\frac{1}{x} \div \frac{x^2}{3}$ as the product $\frac{1}{x} \cdot \frac{3}{x^2}$. Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.

$$\frac{1}{x} \cdot \frac{3}{x^2} = \frac{3}{x^3}$$

Howto: Given two rational expressions, divide them

1. Rewrite as the first rational expression multiplied by the reciprocal of the second.
2. Factor the numerators and denominators.
3. Multiply the numerators.
4. Multiply the denominators.
5. Simplify.

Example $Page 2.1.3$: Dividing Rational Expressions

Divide the rational expressions and express the quotient in simplest form:

$$\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}$$

Solution

$$\frac{2x^2+x-6}{x^2-1} \div \frac{x^2-4}{x^2+2x+1}$$

$$\frac{2x^2+x-6}{x^2-1} \times \frac{x^2+2x+1}{x^2-4} \qquad \text{Rewrite as a multiplication problem}$$

$$\frac{(2x-3)(x+2)}{(x-1)(x+1)} \times \frac{(x+1)(x+1)}{(x-2)(x+2)} \qquad \text{Factor the numerator and denominator.}$$

$$\frac{(2x-3)(x+2)(x+1)(x+1)}{(x-1)(x+1)(x-2)(x+2)} \qquad \text{Multiply numerators and denominators}$$

$$\frac{(2x-3)(x+1)}{(x-1)(x-2)} \qquad \text{Cancel common factors to simplify}$$

Exercise $Page 2.1.3$

Divide the rational expressions and express the quotient in simplest form:

$$\frac{9x^2-16}{3x^2+17x-28} \div \frac{3x^2-2x-8}{x^2+5x-14}$$

Answer

0

Adding and Subtracting Rational Expressions

Adding and subtracting rational expressions works just like adding and subtracting numerical fractions. To add fractions, we need to find a common denominator. Let's look at an example of fraction addition.

$$\begin{aligned}\frac{5}{24}+\frac{1}{40} &= \frac{25}{120}+\frac{3}{120}\\ &= \frac{28}{120}\\ &= \frac{7}{30}\end{aligned}$$

We have to rewrite the fractions so they share a common denominator before we are able to add. We must do the same thing when adding or subtracting rational expressions.

The easiest common denominator to use will be the **least common denominator**, or LCD. The LCD is the smallest multiple that the denominators have in common. To find the LCD of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance, if the factored denominators were $(x+3)(x+4)$ and $(x+4)(x+5)$, then the LCD would be $(x+3)(x+4)(x+5)$.

Once we find the LCD, we need to multiply each expression by the form of 1 that will change the denominator to the LCD. We would need to multiply the expression with a denominator of $(x+3)(x+4)$ by $\frac{x+5}{x+5}$ and the expression with a denominator of $(x+4)(x+5)$ by $\frac{x+3}{x+3}$.

Howto: Given two rational expressions, add or subtract them

1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD.
4. Add or subtract the numerators.
5. Simplify.

Example $Page2.1.4$: Adding Rational Expressions

Add the rational expressions:

$$\frac{5}{x}+\frac{6}{y}$$

Solution

First, we have to find the LCD. In this case, the LCD will be xy. We then multiply each expression by the appropriate form of 1 to obtain xy as the denominator for each fraction.

$$\begin{aligned}&\frac{5}{x}\times\frac{y}{y}+\frac{6}{y}\times\frac{x}{x}\\ &\frac{5y}{xy}+\frac{6x}{xy}\end{aligned}$$

Now that the expressions have the same denominator, we simply add the numerators to find the sum.

$$\frac{6x+5y}{xy}$$

Analysis

Multiplying by $\frac{y}{y}$ or $\frac{x}{x}$ does not change the value of the original expression because any number divided by itself is 1, and multiplying an expression by 1 gives the original expression.

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Example $Page 2.1.5$: Subtracting Rational Expressions

Subtract the rational expressions:

$$\frac{6}{x^2+4x+4}-\frac{2}{x^2-4} \qquad \text{(Page 2.1.1)}$$

Solution

$\frac{6}{(x+2)^2}-\frac{2}{(x+2)(x-2)}$	Factor
$\frac{6}{(x+2)^2}\times\frac{x-2}{x-2}-\frac{2}{(x+2)(x-2)}\times\frac{x+2}{x+2}$	Multiply each fraction to get LCD as denominator
$\frac{6(x-2)}{(x+2)^2(x-2)}-\frac{2(x+2)}{(x+2)^2(x-2)}$	Multiply
$\frac{6x-12-(2x+4)}{(x+2)^2(x-2)}$	Apply distributive property
$\frac{4x-16}{(x+2)^2(x-2)}$	Subtract
$\frac{4(x-4)}{(x+2)^2(x-2)}$	Simplify

Q&A

Do we have to use the LCD to add or subtract rational expressions?

No. Any common denominator will work, but it is easiest to use the LCD.

Exercise $Page 2.1.4$

Subtract the rational expressions: $\frac{3}{x+5}-\frac{1}{x-3}$

Answer

$$\frac{2(x-7)}{(x+5)(x-3)}$$

Simplifying Complex Rational Expressions

A complex rational expression is a rational expression that contains additional rational expressions in the numerator, the denominator, or both. We can simplify complex rational expressions by rewriting the numerator and denominator as single rational expressions and dividing. The complex rational expression $\frac{a}{\frac{1}{b}+c}$ can be simplified by rewriting the numerator as the fraction $\frac{a}{1}$ and combining the expressions in the denominator as $\frac{1+bc}{b}$. We can then rewrite the expression as a multiplication problem using the reciprocal of the denominator. We get $\frac{a}{1}\cdot\frac{b}{1+bc}$, which is equal to $\frac{ab}{1+bc}$.

Howto: Given a complex rational expression, simplify it

1. Combine the expressions in the numerator into a single rational expression by adding or subtracting.
2. Combine the expressions in the denominator into a single rational expression by adding or subtracting.
3. Rewrite as the numerator divided by the denominator.
4. Rewrite as multiplication.

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5. Multiply.
6. Simplify.

Example $Page 2.1.6$: Simplifying Complex Rational Expressions

Simplify: $\dfrac{y+\dfrac{1}{x}}{\dfrac{x}{y}}$

Solution

Begin by combining the expressions in the numerator into one expression.

$$y \times \frac{x}{x} + \frac{1}{x} \qquad \text{Multiply by } \frac{x}{x} \text{ to get LCD as denominator}$$

$$\frac{xy}{x} + \frac{1}{x}$$

$$\frac{xy+1}{x} \qquad \text{Add numerators}$$

Now the numerator is a single rational expression and the denominator is a single rational expression.

$$\frac{\dfrac{xy+1}{x}}{\dfrac{x}{y}}$$

We can rewrite this as division, and then multiplication.

$$\frac{xy+1}{x} \div \frac{x}{y}$$

$$\frac{xy+1}{x} \times \frac{y}{x} \qquad \text{Rewrite as multiplication}$$

$$\frac{y(xy+1)}{x^2} \qquad \text{Multiply}$$

Exercise $Page 2.1.5$

Simplify: $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{y}$

Answer

$$\frac{x^2-y^2}{xy^2}$$

Q&A

Can a complex rational expression always be simplified?

Yes. We can always rewrite a complex rational expression as a simplified rational expression.

Media

Access these online resources for additional instruction and practice with rational expressions.

1. Simplify Rational Expressions
2. Multiply and Divide Rational Expressions

3. Add and Subtract Rational Expressions
4. Simplify a Complex Fraction

Key Concepts

- Rational expressions can be simplified by cancelling common factors in the numerator and denominator. See Example.
- We can multiply rational expressions by multiplying the numerators and multiplying the denominators. See Example.
- To divide rational expressions, multiply by the reciprocal of the second expression. See Example.
- Adding or subtracting rational expressions requires finding a common denominator. See Example and Example.
- Complex rational expressions have fractions in the numerator or the denominator. These expressions can be simplified. See Example.

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Page 2.2: Avoiding Common Pitfalls

For each statement below, determine if they are true or false.

1) $(a+b)^2 = a^2 + b^2$

2) $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$

3) $\sqrt{a^2+b^2} = a+b$

4) $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$

5) $\frac{a+b}{a} = b$

6) $a^{-1} + b^{-1} = (a+b)^{-1}$

Answer

Each statement above is false! Be careful from this point onwards. Whenever we write down a formula or we perform some sort of simplification, we should always take a moment and ask ourselves if the rule we are using is true! Many statements in math might look true but are not, as seen in the above example.

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Page 2.3: Solving Inequalities

The goal of this section is to understand how to solve inequalities. An inequality looks similar to that of an equation but instead of an equality sign, we have either a $\leq, \geq, <$ or $>$ sign.

When we say we are going to solve an inequality, this means we are going to find all the real numbers which makes the inequality true. In order to solve inequalities, we first present some rules:

Theorem $Page 2.3.1$

Theorem: Suppose a, b, c and d denote real numbers or algebraic expressions.

1) $a \leq b$ if and only if $a+c \leq b+c$.

2) $a \leq b$ if and only if $a-c \leq b-c$.

3) If $c > 0$ then $a \leq b$ if and only if $ac \leq bc$.

4) If $c < 0$ then $a \leq b$ if and only if $ac \geq bc$.

5) If a and b are positive, then $a \leq b$ if and only if $\frac{1}{a} \geq \frac{1}{b}$.

6) If $a \leq b$ and $c \leq d$ then $a+c \leq b+d$.

7) If $a \leq b$ and $b \leq c$ then $a \leq c$.

As usual, it is important to be able to understand these statements in words.

Statements 1 and 2 are saying that we can add the same quantity to each side of an inequality and the direction of the inequality is preserved.

Statement 3 is saying that if we multiply an inequality by a positive number, then the direction of the inequality is also preserved. However, if we multiply the inequality by a negative number, then the direction of the inequality is flipped (Statement 4).

Statement 5 tells us the direction of the inequality is also flipped whenever we are taking reciprocals of each side of an inequality involving positive quantities.

Finally, Statements 6 and 7 are saying that inequalities can be added and that inequalities are transitive.

Using these rules, allow us to work out an example.

Exercise $Page 2.3.1$

Find all values of x such that $3x+5 > 0$.

Answer

For now, do not worry about why we are doing each step. Instead, allow us to focus on the validity of both steps.

$$\begin{aligned} 3x+5 &> 0 && \text{we first subtract 5 from both sides} \\ 3x &> -5 && \text{we now divide both sides by 3} \\ x &> -\frac{5}{3} \end{aligned}$$

And so $x > -\frac{5}{3}$ which means that the solution of the inequality is the interval $(-\frac{5}{3}, \infty)$.

Notice the above inequality is linear meaning that each term is a constant or a multiple of the variable x. In general, to solve linear inequalities, we simply isolate the variable on one side of the inequality sign.

? Exercise $Page 2.3.1$

Find all values of x such that $-3x+5>0$.

Answer

$$\begin{aligned} -3x+5 &> 0 && \text{we first subtract 5 from both sides} \\ -3x &> -5 && \text{we now divide both sides by -3} \\ x &< \frac{5}{3} \end{aligned}$$

And so $x<\frac{5}{3}$ which means that the solution of the inequality is the interval $(-infty, \frac{5}{3})$.

Nonlinear inequalities contain expressions that are nonlinear. Based on things you have had experience with in algebra courses, variables raised to powers and variables in denominators within an inequality would indicate that an inequality is nonlinear.

Some examples of nonlinear inequalities:

$$m^2-8m>9\,, \qquad \frac{x-4}{(3x-2)(x+5)} \leq 0$$

A solution to an inequality is any real number that will produce a true statement when substituted in for the variable.

✔ Example $Page 2.3.1$

Are -3, -2, and -1 solutions to $x^2-x-6\leq 0$?

Solution

Substitute the given value in for x and simplify.

$x^2-x-6\leq 0$	$x^2-x-6\leq 0$	$x^2-x-6\leq 0$
$(-3)^2-(-3)-6\leq 0$	$(-2)^2-(-2)-6\leq 0$	$(-1)^2-(-1)-6\leq 0$
$9+3-6\leq 0$	$4+2-6\leq 0$	$1+1-6\leq 0$
$6\leq 0$ ✗	$0\leq 0$ ✓	$-4\leq 0$ ✓

Answer:

-2 and -1 are solutions and -3 is not.

Solving Nonlinear Inequalities Using Graphs

Graphing the function defined by $y=x^2-x-6$ found in the previous example we have,

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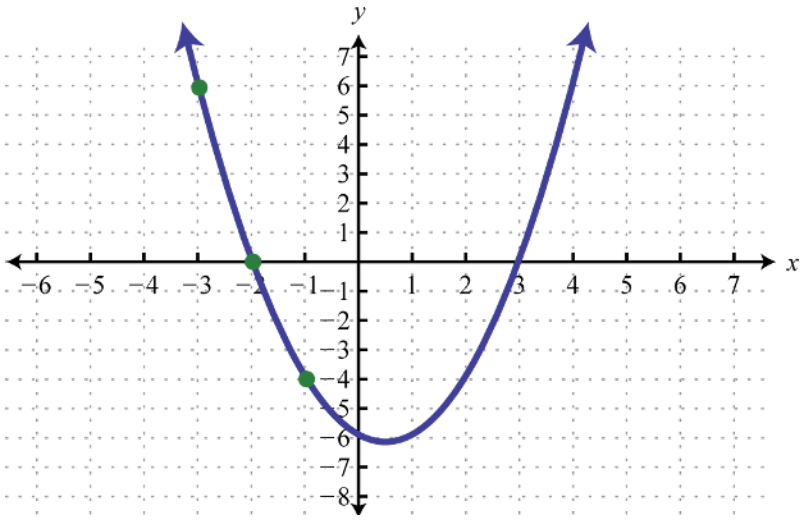


Figure $Page 2.3.3$: Graph of y=x^2-x+6

Notice that the x-intercepts are at $(-2,0)$ and $(3,0)$. Remember that to find the x-intercepts of a function, set $y=0$. Finding them algebraically,

$$\begin{aligned} 0 &= x^2 - x - 6 \\ 0 &= (x+2)(x-3) \\ x+2 &= 0 \quad \text{or} \quad x-3=0 \\ x &= -2 \quad \text{or} \quad x=3 \end{aligned}$$

The values -2 and 3 are called **critical numbers**. These values bound the regions where the function is positive (above the x-axis) or negative (below the x-axis).

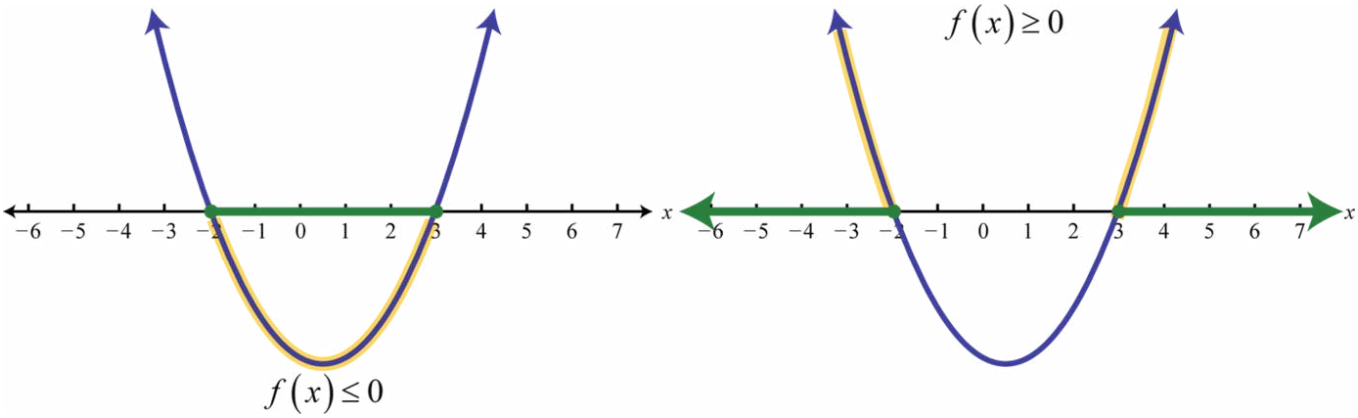


Figure $Page 2.3.4$: Graphs showing intervals where y=x^2-x+6 is positive and negative

We can see that the function is below the x-axis between -2 and 3. Therefore $x^2 - x - 6 \leq 0$ has solutions where $-2 \leq x \leq 3$, using interval notation $[-2,3]$. Furthermore, $x^2 - x - 6 \geq 0$ has solutions where $x \leq -2$ or $x \geq 3$, using interval notation $(-\infty,-2] \cup [-3,\infty)$.

Example $Page 2.3.2$

Given the graph of $f(x)$ determine the solutions to $f(x) > 0$:

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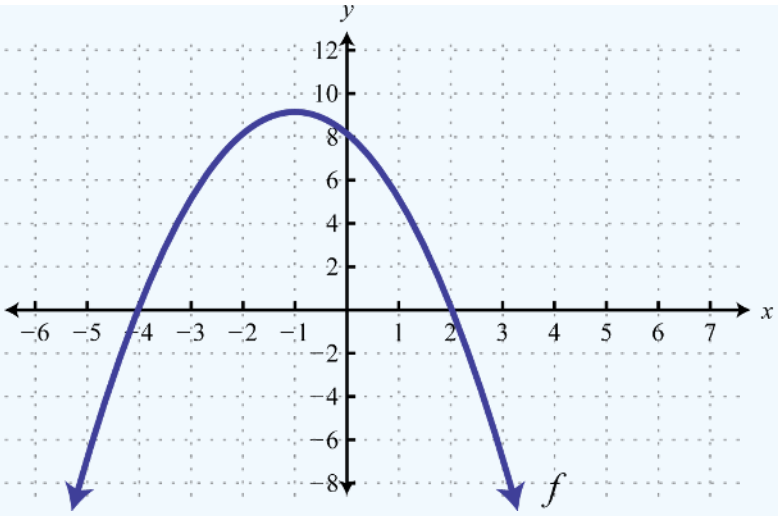


Figure $Page 2.3.5$: Graph of function f(x)

Because of the strict inequality, the solution set is shaded with an open dot on each of the boundaries. This indicates that these critical numbers are not actually included in the solution set. This solution set can be expressed two ways,

$$\{x|-4<x<2\} \qquad \textit{Set Notation}$$
$$(-4,2) \qquad \textit{Interval Notation}$$

In this textbook, we will continue to present answers in interval notation.

Answer:

$(-4, 2)$

? You Try $Page 2.3.1$

Use the graph of f to determine the solutions to $f(x) \leq 0$. State answer in interval notation.

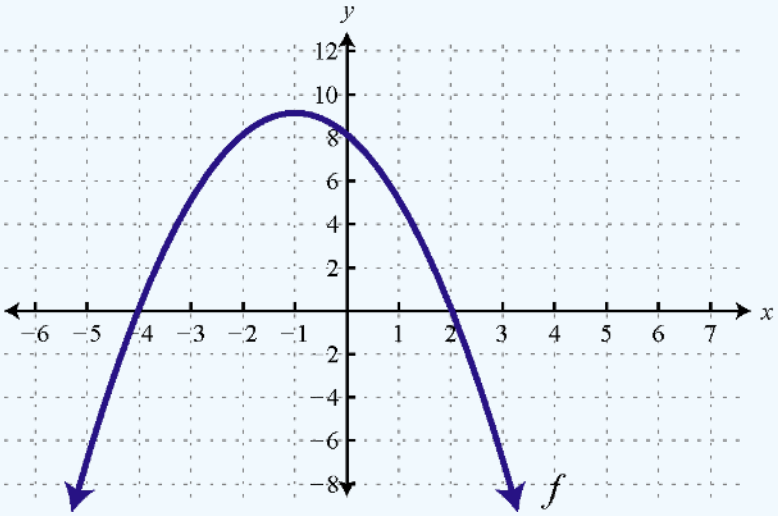


Figure $Page 2.3.6$: Graph of function f(x)

Answer

$(-\infty, -4] \cup [2, \infty)$

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Solving Polynomial Inequalities Using Sign Charts

A **polynomial inequality** is a mathematical statement that relates a polynomial expression as either less than or greater than another.

Some examples of polynomial inequalities:

$$(3x+1)(x-2)<0, \qquad t^3+30 \geq 7t^2+7t$$

We have seen how to solve an inequality graphically. To solve inequalities without graphing, we make use of a **sign chart** which models a function using a number line that represents the x-axis and signs $(+$ or $-)$ to indicate where the function is positive or negative. For example,

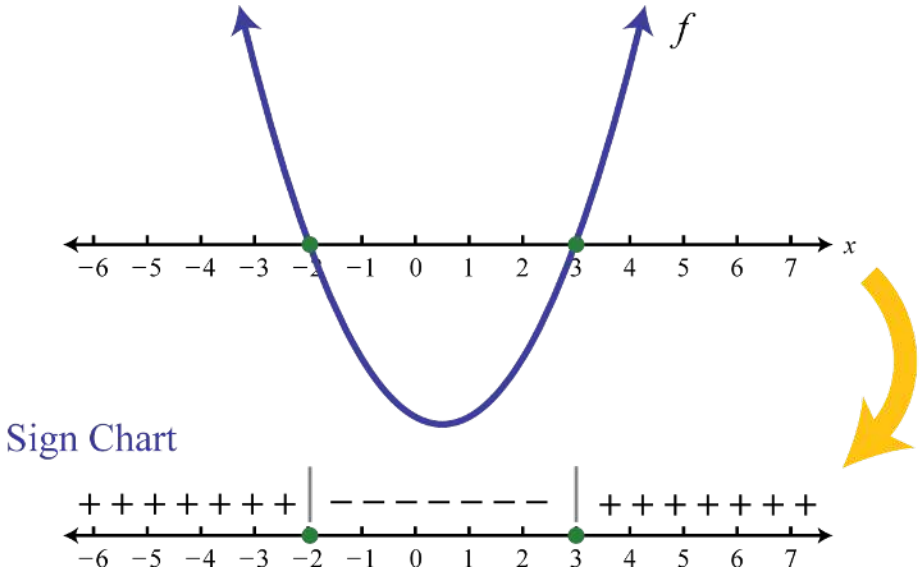


Figure $Page 2.3.7$

The plus signs indicate that the function is positive on the region. The negative signs indicate that the function is negative on the region. The boundaries are the critical numbers, -2 and 3 in this case. Sign charts are useful when a detailed picture of the graph is not needed and are used extensively in higher level mathematics. The steps for solving a polynomial inequality with one variable are outlined in the following example.

Example $Page 2.3.3$

Solve: $-x^2+6x+7 \geq 0$.

Solution

It is important to note that this quadratic polynomial inequality is in standard form, with zero on one side of the inequality. Zero must be on one side of the inequality to be able to find the critical numbers in the next step.

Step 1: Determine the critical numbers. To find them, set the function equal to zero and solve.

$$\begin{aligned} -x^2+6x+7&=0 \\ -\left(x^2-6x-7\right)&=0 \\ -(x+1)(x-7)&=0 \end{aligned}$$

$$\begin{aligned} x+1&=0 \quad \text{or} \quad x-7=0 \\ x&=-1 \qquad\quad x=7 \end{aligned}$$

The critical numbers are -1 and 7.

Step 2: Create a sign chart by putting the critical numbers on a number line. Since the critical numbers bound the regions where the function is positive or negative, we need to test a single value in each region. In this case, the critical numbers partition the number line into three regions. We choose test values $x=-3, x=0$, and $x=10$.

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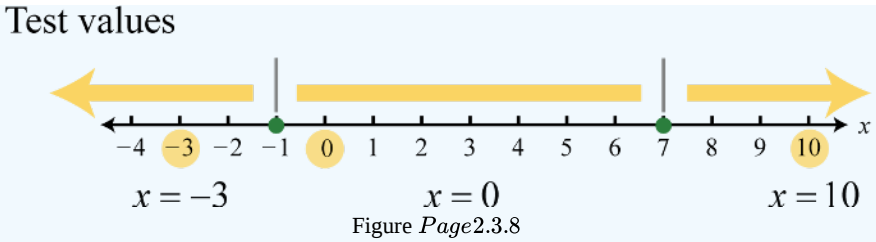


Figure $Page 2.3.8$

Test values may vary. In fact, we need only determine the sign $(+$ or $-)$ of the result when evaluating $f(x) = -x^2 + 6x + 7 = -(x+1)(x-7)$. Here we evaluate using the factored form.

$$\begin{aligned} f(-3) &= -(-3+1)(-3-7) = -(-2)(-10) &&= -Negative \\ f(0) &= -(0+1)(0-7) = -(1)(-7) &&= +Positive \\ f(10) &= -(10+1)(10-7) = -(11)(3) &&= -Negative \end{aligned}$$

Since the result of evaluating for -3 was negative, we place negative signs above the first region. The result of evaluating for 0 was positive, so we place positive signs above the middle region. Finally, the result of evaluating for 10 was negative, so we place negative signs above the last region, and the sign chart is complete.

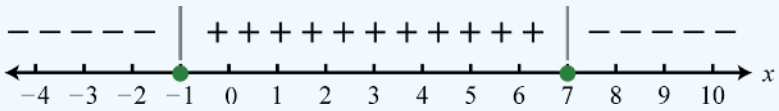


Figure $Page 2.3.9$

Step 3: Use the sign chart to answer the question. In this case, we are asked to determine where $f(x) \geq 0$, or where the function is positive or zero. From the sign chart we see this occurs when x-values are inclusively between -1 and 7.

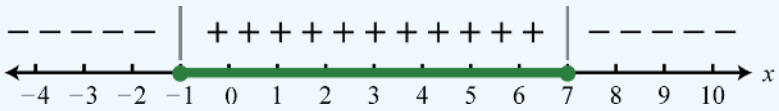


Figure $Page 2.3.9$

Using interval notation, the answer is expressed as $[-1, 7]$.

The graph is not required; however, for the sake of completeness it is provided below.

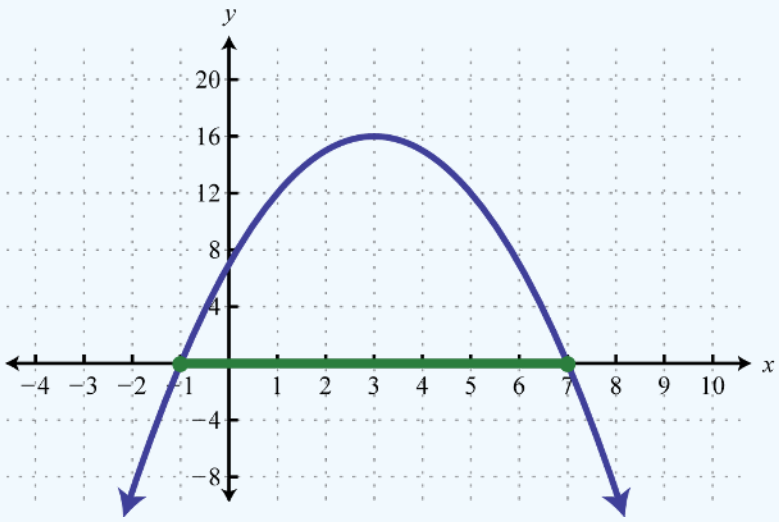


Figure $Page 2.3.10$

Indeed the function is greater than or equal to zero, above or on the x-axis, for x-values in the specified interval.

Answer:

$[-1, 7]$

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Before looking at another example it is important to note that there are multiple ways to set up sign charts. Some like to set up sign charts and evaluate each factor individually. Below is an example of a sign chart set up this way:

	(- ∞, -1)	(-1,7)	(7, ∞)
-1	-	-	-
(x+1)	-	+	+
(x-7)	-	-	+
-(x+1)(x-7)	-	+	-

Figure $Page2.3.11$: Sign chart for -(x+1)(x-7)

Since -1 is a constant factor that is negative, we fill in negative signs everywhere for this factor. Notice that the factor x+1 changes sign, from negative to positive, at the critical number -1. This factor gave this critical value. Similarly, this is also true for the factor x-7 and critical value of 7.

Example $Page2.3.4$

Solve: $2x^2-7x+3>0$.

Solution

Begin by finding the critical numbers, in this case, the roots of $f(x)=2x^2-7x+3$.

$$\begin{aligned} 2x^2-7x+3 &= 0 \\ (2x-1)(x-3) &= 0 \end{aligned}$$

$$\begin{aligned} 2x-1=0 \quad &\text{or} \quad x-3=0 \\ 2x=1 \quad &\qquad\quad x=3 \\ x=\frac{1}{2} \quad & \end{aligned}$$

The critical numbers are $\frac{1}{2}$ and 3. Because of the strict inequality > we will use open dots.

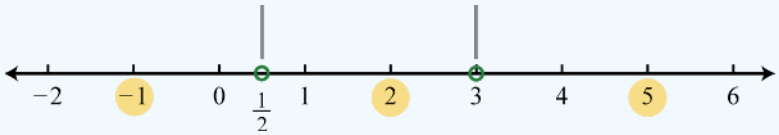


Figure $Page2.3.12$

Next choose a test value in each region and determine the sign after evaluating $f(x)=2x^2-7x+3=(2x-1)(x-3)$. Here we choose test values $-1, 2$, and 5.

$$\begin{aligned} f(-1)&=(2(-1)-1)(-1-3) &&=(-)(-)=+ \\ f(2)&=(2(2)-1)(2-3) &&=(+)(-)=- \\ f(5)&=(2(5)-1)(5-3) &&=(+)(+)=+ \end{aligned}$$

And we can complete the sign chart.

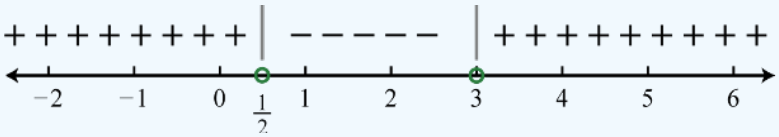


Figure $Page2.3.13$

The question asks us to find the x-values that produce positive results (greater than zero). Therefore, shade in the regions with a $+$ over them. This is the solution set.

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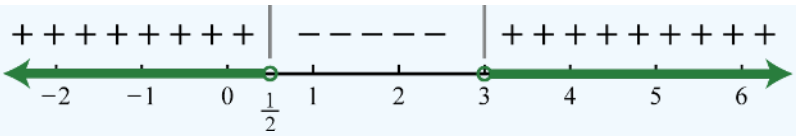


Figure $Page 2.3.14$

Answer:

$\left(-\infty, \frac{1}{2}\right) \cup (3, \infty)$

? Exercise $Page 2.3.1$

Find where the function $f(x) = 12x^3 - 12x^2 - 24x$ is positive.

Answer

We must find when $12x^3 - 12x^2 - 24x > 0$. We already see that all the nonzero terms are on one side and so allow us to set this side equal to zero.

$$12x^3 - 12x^2 - 24x = 0$$

We now must factor this and solve.

$$\begin{aligned} 12x^3 - 12x^2 - 24x &= 0 \\ 12x(x^2 - x - 2) &= 0 \\ 12x(x-2)(x+1) &= 0 \end{aligned}$$

By the Zero Product Property, either $12x = 0$ or $x - 2 = 0$ or $x + 1 = 0$. Hence, $x = 0$ or $x = 2$ or $x = -1$.

Placing these values onto a number line, we see that the number line is divided into four parts: $(-\infty, -1), (-1, 0), (0, 2), (2, \infty)$.

Selecting test values along each interval we see the following:

Interval	$12x$	$x - 2$	$x + 1$	$f(x)$
$(-\infty, -1)$	-	-	-	-
$(-1, 0)$	-	-	+	+
$(0, 2)$	+	-	+	-
$(2, \infty)$	+	+	+	+

Hence f is positive on the intervals $(-1, 0)$ and $(2, \infty)$.

Sometimes the quadratic function does not factor. In this case we can make use of the quadratic formula.

✓ Example $Page 2.3.5$

Solve: $x^2 - 2x - 11 \leq 0$

Solution

Find the critical numbers.

$$x^2 - 2x - 11 = 0$$

Identify a, b, and c for use in the quadratic formula. Here $a = 1, b = -2$, and $c = -11$. Substitute the appropriate values into the quadratic formula and then simplify.

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$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-11)}}{2(1)} \\
&= \frac{2 \pm \sqrt{48}}{2} \\
&= \frac{2 \pm 4\sqrt{3}}{2} \\
&= 1 \pm 2\sqrt{3}
\end{aligned}
$$

Therefore the critical numbers are $1 - 2\sqrt{3} \approx -2.5$ and $1 + 2\sqrt{3} \approx 4.5$. Use a closed dot on the number to indicate that these values will be included in the solution set.

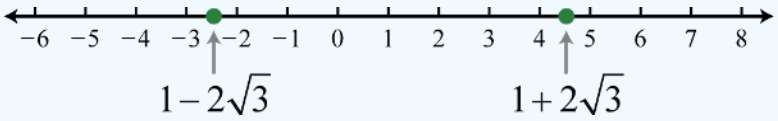


Figure $Page 2.3.15$

Here we will use test values $-5, 0$, and 7.

$$
\begin{aligned}
f(-5) = (-5)^2 - 2(-5) - 11 = 25 + 10 - 11 &= + \\
f(0) = (0)^2 - 2(0) - 11 = 0 + 0 - 11 &= - \\
f(7) = (7)^2 - 2(7) - 11 = 49 - 14 - 11 &= +
\end{aligned}
$$

After completing the sign chart shade in the values where the function is negative as indicated by the question $(f(x) \leq 0)$.

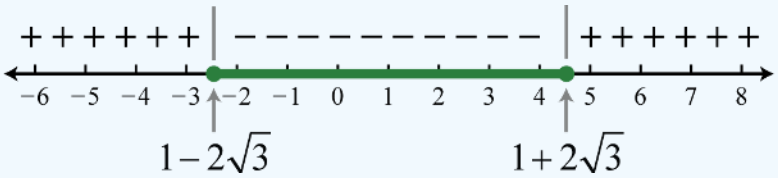


Figure $Page 2.3.16$

Answer:

$[1 - 2\sqrt{3}, 1 + 2\sqrt{3}]$

? You Try $Page 2.3.2$

Solve: $9 - x^2 > 0$.

Answer

$(-3, 3)$

It may be the case that there are no critical numbers.

✔ Example $Page 2.3.6$

Solve: $x^2 - 2x + 3 > 0$.

Solution

To find the critical numbers solve,

Substitute $a = 1, b = -2$, and $c = 3$ into the quadratic formula and then simplify.

Because the solutions are not real, we conclude there are no real roots; hence there are no critical numbers. When this is the case, the graph has no x-intercepts and is completely above or below the x-axis. We can test any value to create a sign chart. Here we choose $x = 0$.

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$$f(0) = (0)^2 - 2(0) + 3 = +$$

Because the test value produced a positive result the sign chart looks as follows:

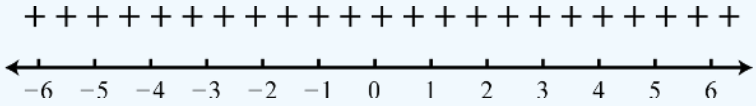


Figure $Page 2.3.17$

We are looking for the values where $f(x) > 0$; the sign chart implies that any real number for x will satisfy this condition.

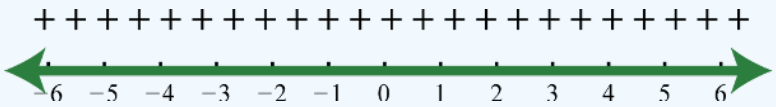


Figure $Page 2.3.18$

Answer:

$(-\infty, \infty)$

The function in the previous example is graphed below.

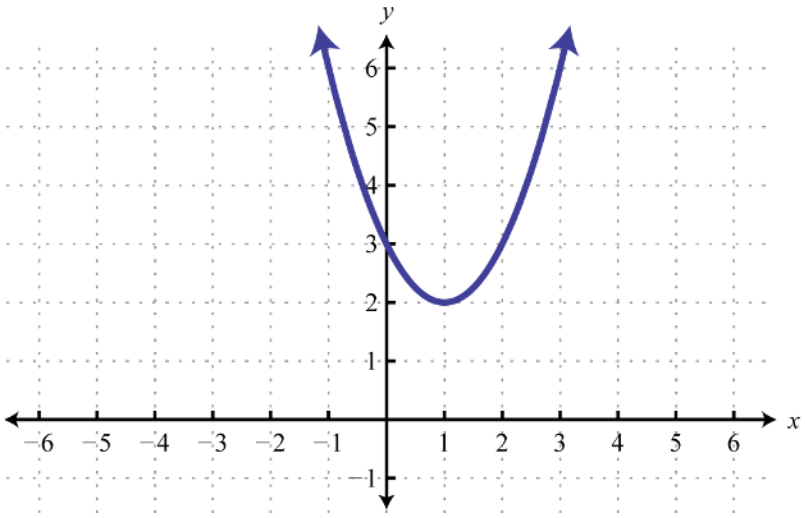


Figure $Page 2.3.19$

We can see that it has no x-intercepts and is always above the x-axis (positive). If the question was to solve $x^2 - 2x + 3 < 0$, then the answer would have been no solution. The function is never negative.

? You Try $Page 2.3.3$

Solve: $9x^2 - 12x + 4 \leq 0$

Answer

One solution, $\frac{2}{3}$.

✔ Example $Page 2.3.7$

Solve $x(x+3)^2(x-4) < 0$.

Solution

Begin by finding the critical numbers. Because $f(x) = x(x+3)^2(x-4)$ is given in its factored form and zero is on one side of the inequality, the critcal numbers are apparent. Here the critical values are: $0, -3$, and 4. Because of the strict inequality, plot them using open dots on a number line.

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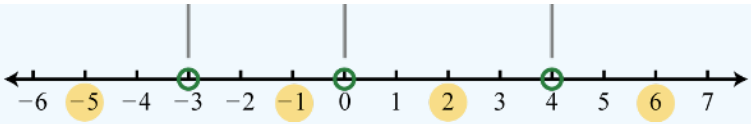


Figure $Page 2.3.20$

In this case, the critical numbers partition the number line into four regions. Test values in each region to determine if f is positive or negative. Here we choose test values $-5, -1, 2$, and 6. Remember that we are only concerned with the sign $(+$ or $-)$ of the result.

$$\begin{aligned} f(-5) &= (-5)(-5+3)^2(-5-4) = (-)(-)^2(-) && = +Positive \\ f(-1) &= (-1)(-1+3)^2(-1-4) = (-)(+)^2(-) && = +Positive \\ f(2) &= (2)(2+3)^2(2-4) = (+)(+)^2(-) && = -Negative \\ f(6) &= (6)(6+3)^2(6-4) = (+)(+)^2(+) && = +Positive \end{aligned}$$

After testing values we can complete a sign chart.

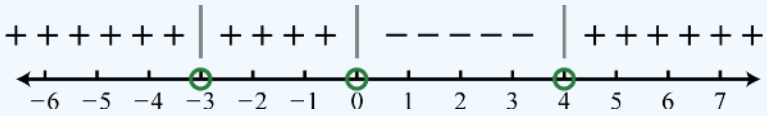


Figure $Page 2.3.21$

The question asks us to find the values where $f(x) < 0$, or where the function is negative. From the sign chart we can see that the function is negative for x-values in between 0 and 4.

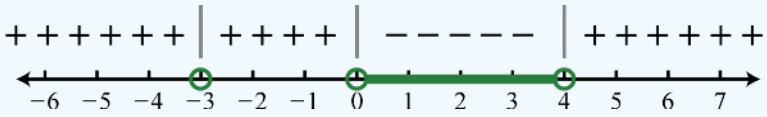


Figure $Page 2.3.22$

Answer:

$(0, 4)$

Although we have not discussed the graphs of polynomial functions yet, the graph of this function is provided below. Compare the graph to its corresponding sign chart.

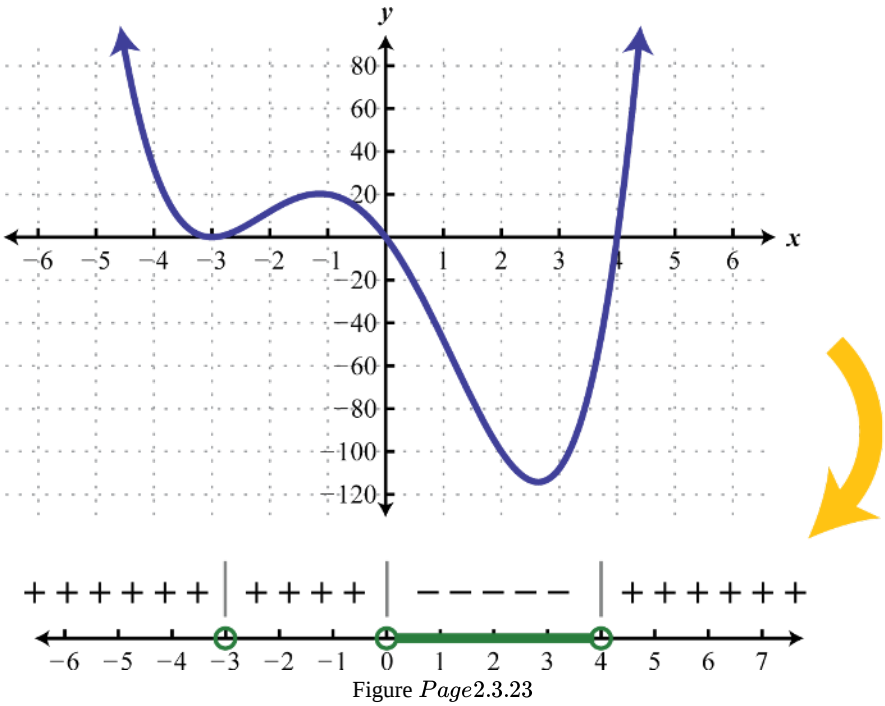


Figure $Page 2.3.23$

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Certainly it may not be the case that the polynomial is factored nor that it has zero on one side of the inequality. To model a function using a sign chart, all of the terms should be on one side and zero on the other. The general steps for solving a polynomial inequality are listed in the following example.

Example $Page2.3.8$

Solve: $2x^4 > 3x^3 + 9x^2$.

Solution

Step 1: Obtain zero on one side of the inequality. In this case, subtract to obtain a polynomial on the left side in standard from.

$$\begin{aligned} 2x^4 &> 3x^3 + 9x^2 \\ 2x^4 - 3x^3 - 9x^2 &> 0 \end{aligned}$$

Step 2: Find the critical numbers. Here we can find the zeros by factoring.

$$\begin{aligned} 2x^4 - 3x^3 - 9x^2 &= 0 \\ x^2\left(2x^2 - 3x - 9\right) &= 0 \\ x^2(2x+3)(x-3) &= 0 \end{aligned}$$

There are three solutions, hence, three critical numbers $-\frac{3}{2}, 0$, and 3. The strict inequality indicates that we should use open dots.

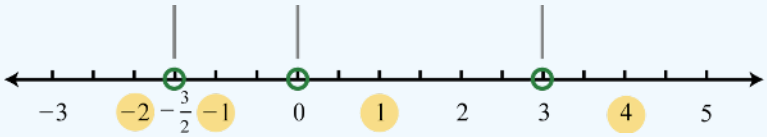


Figure $Page2.3.24$

Step 3: Create a sign chart. In this case use $f(x) = x^2(2x+3)(x-3)$ and test values $-2, -1, 1$, and 4 to determine the sign of the function in each interval.

$$\begin{aligned} f(-2) &= (-2)^2(2(-2)+3)(-2-3) &&= (-)^2(-)(-) = + \\ f(-1) &= (-1)^2(2(-1)+3)(-1-3) &&= (-)^2(+)(-) = - \\ f(1) &= (1)^2(2(1)+3)(1-3) &&= (+)^2(+)(-) = - \\ f(4) &= (4)^2(2(4)+3)(4-3) &&= (+)^2(+)(+) = + \end{aligned}$$

With this information we can complete the sign chart.

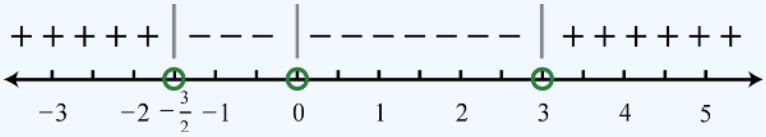


Figure $Page2.3.25$

Step 4: Use the sign chart to answer the question. Here the solution consists of all values for which $f(x) > 0$. Shade in the values that produce positive results and then express this set in interval notation.



Figure $Page2.3.26$

Answer:

$\left(-\infty, -\frac{3}{2}\right) \cup (3, \infty)$

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Example $Page 2.3.9$

Solve: $x^3+x^2 \le 4(x+1)$.

Solution

Begin by rewriting the inequality in standard form, with zero on one side.

$$\begin{aligned} x^3+x^2 &\le 4(x+1) \\ x^3+x^2 &\le 4x+4 \\ x^3+x^2-4x-4 &\le 0 \end{aligned}$$

Next find the critical numbers of $f(x)=x^3+x^2-4x-4$:

$$\begin{aligned} x^3+x^2-4x-4 &= 0 \quad \textit{Factor by grouping.} \\ x^2(x+1)-4(x+1) &= 0 \\ (x+1)\left(x^2-4\right) &= 0 \\ (x+1)(x+2)(x-2) &= 0 \end{aligned}$$

The critical numbers are -2, -1, and 2. Because of the inclusive inequality $(\le)$ we will plot them using closed dots.

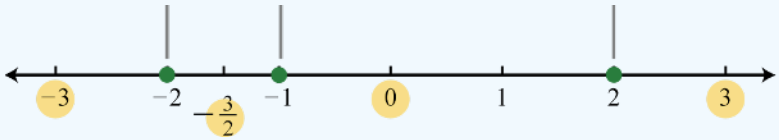


Figure $Page 2.3.27$

Use test values, -3, $-\frac{3}{2}$, 0, and 3 to create a sign chart.

$$\begin{aligned} f(-3) &= (-3+1)(-3+2)(-3-2) &&= (-)(-)(-) = - \\ f\left(-\tfrac{3}{2}\right) &= \left(-\tfrac{3}{2}+1\right)\left(-\tfrac{3}{2}+2\right)\left(-\tfrac{3}{2}-2\right) &&= (-)(+)(-) = + \\ f(0) &= (0+1)(0+2)(0-2) &&= (+)(+)(-) = - \\ f(3) &= (3+1)(3+2)(3-2) &&= (+)(+)(+) = + \end{aligned}$$

And we have

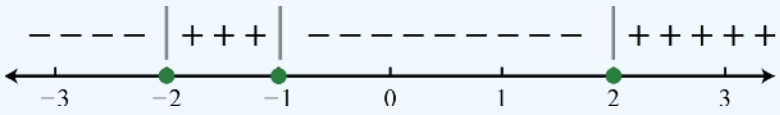


Figure $Page 2.3.28$

Use the sign chart to shade in the values that have negative results $(f(x) \le 0)$.

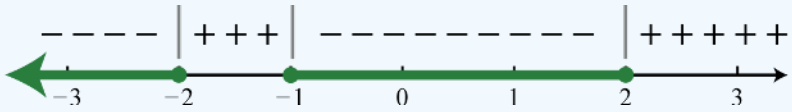


Figure $Page 2.3.29$

Answer

$(-\infty,-2]\cup[-1,2]$

You Try $Page 2.3.4$

Solve $-3x^4+12x^3-9x^2>0$.

Answer

$(1,3)$

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Solving Rational Inequalities Using Sign Charts

A **rational inequality** is a mathematical statement that relates a rational expression as either less than or greater than another. Because rational functions have restrictions to the domain, which we will discuss more in the next section, we must take care when solving rational inequalities. In addition to the zeros, we will include the restrictions to the domain of the function which are values that make the denominator zero in the set of critical numbers.

Example $Page 2.3.10$

Solve: $\dfrac{(x-4)(x+2)}{(x-1)} \geq 0$

Solution

The zeros of a rational function occur when the numerator is zero, provided they are unique to the numerator, and the values that produce zero in the denominator are the restrictions. In this case,

Roots (Numerator)	Restriction(Denominator)
$x-4=0$ or $x+2=0$	$x-1=0$
$x=4 \quad x=-2$	$x=1$

Therefore the critical numbers are $-2, 1$, and 4. Because of the inclusive inequality $(\geq)$ use a closed dot for the roots $-2, 4$ and always use an open dot for restrictions 1. Restrictions are never included in the solution set.

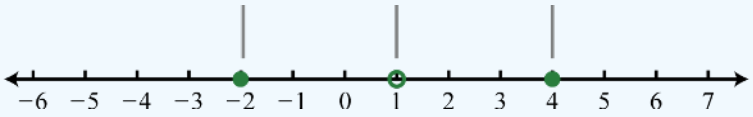


Figure $Page 2.3.30$

Use test values $x=-4, 0, 2, 6$.

$$f(-4)=\frac{(-4-4)(-4+2)}{(-4-1)} \quad =\frac{(-)(-)}{(-)}=-$$

$$f(0)=\frac{(0-4)(0+2)}{(0-1)} \quad =\frac{(-)(+)}{(-)}=+$$

$$f(2)=\frac{(2-4)(2+2)}{(2-1)} \quad =\frac{(-)(+)}{(+)}=-$$

$$f(6)=\frac{(6-4)(6+2)}{(6-1)} \quad =\frac{(+)(+)}{(+)}=+$$

And then complete the sign chart.

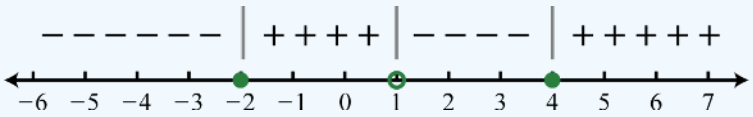


Figure $Page 2.3.31$

The question asks us to find the values for which $f(x) \geq 0$, in other words, positive or zero. Shade in the appropriate regions and present the solution set in interval notation.

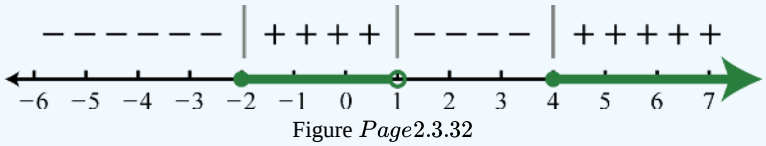


Figure $Page 2.3.32$

Answer

$[-2, 1) \cup [4, \infty)$

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Although we have not learned about the graphs of rational functions yet, the graph of this function is provided below. Compare the graph to its corresponding sign chart above.

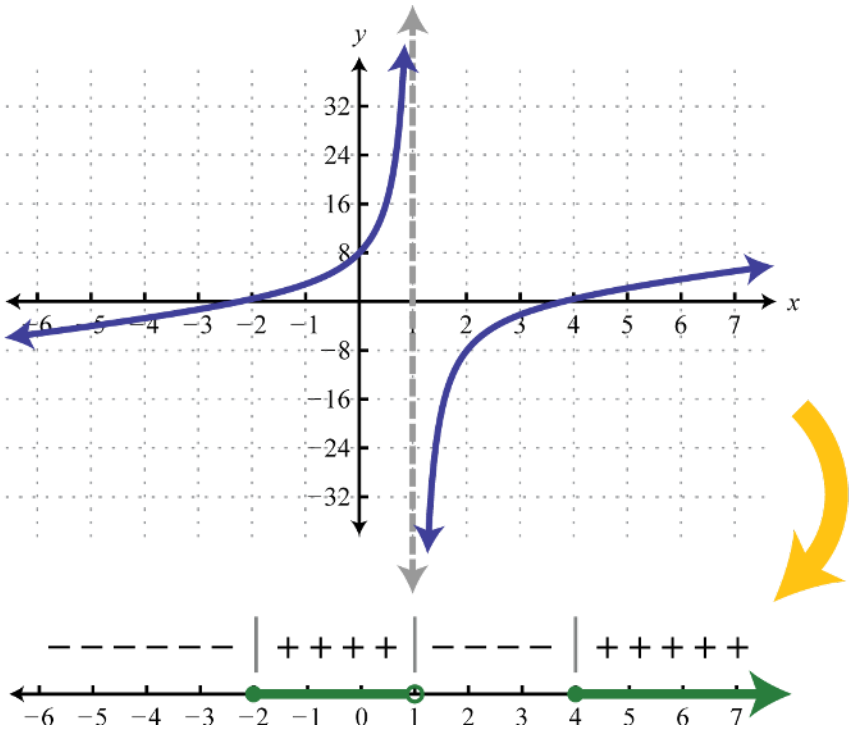


Figure $Page 2.3.33$

Notice that the restriction $x = 1$ corresponds to a place where the function values get very large and also very small. This is called a vertical asymptote which bounds regions where the function changes from positive to negative. While not included in the solution set because it makes the denominator zero, this value is a critical number. Before creating a sign chart we must ensure the inequality has a zero on one side. The general steps for solving a rational inequality are outlined in the following example.

✔ Example $Page 2.3.11$

Solve $\dfrac{7}{x+3} < 2$.

Solution

Step 1: Begin by obtaining zero on the right side.

$$\frac{7}{x+3} < 2$$

$$\frac{7}{x+3} - 2 < 0$$

Step 2: Determine the critical numbers. The critical numbers are the zeros and restrictions. Begin by simplifying to a single algebraic fraction.

$$\frac{7}{x+3} - \frac{2}{1} < 0$$

$$\frac{7-2(x+3)}{x+3} < 0$$

$$\frac{7-2x-6}{x+3} < 0$$

$$\frac{-2x+1}{x+3} < 0$$

https://math.libretexts.org/@go/page/132683

Next find the critical numbers. Set the numerator and denominator equal to zero and solve.

Root	Restriction
$-2x+1=0$	
$-2x=-1$	$x+3=0$
$x=\frac{1}{2}$	$x=-3$

In this case, the strict inequality indicates that we should use an open dot for the root.

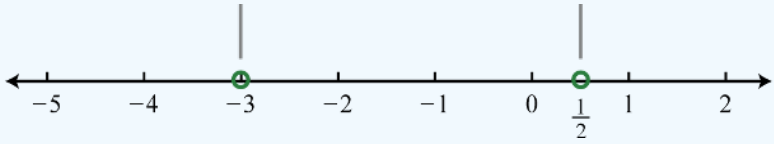


Figure $Page 2.3.34$

Step 3: Create a sign chart. Choose test values -4, 0, and 1.

$$f(-4)=\frac{-2(-4)+1}{-4+3} \quad =\frac{+}{-}=-$$

$$f(0)=\frac{-2(0)+1}{0+3} \quad =\frac{+}{+}=+$$

$$f(1)=\frac{-2(1)+1}{1+3} \quad =\frac{-}{+}=-$$

And we have

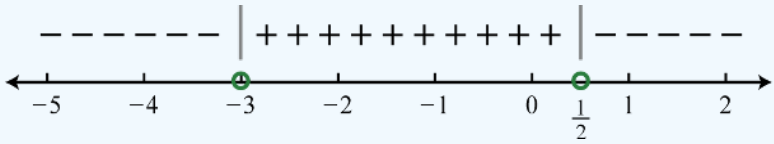


Figure $Page 2.3.35$

Step 4: Use the sign chart to answer the question. In this example we are looking for the values for which the function is negative, $f(x)<0$. Shade the appropriate values and then present your answer using interval notation.

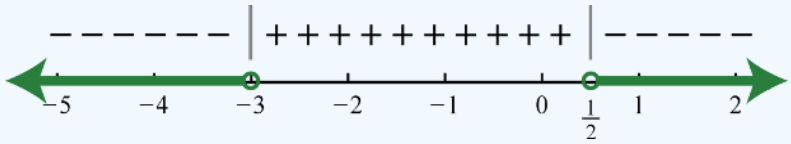


Figure $Page 2.3.36$

Answer:

$(-\infty,-3)\cup\left(\frac{1}{2},\infty\right)$

✔ Example $Page 2.3.12$

Solve: $\frac{1}{x^2-4}\leq\frac{1}{2-x}$.

Solution

Begin by obtaining zero on the right side.

$$\frac{1}{x^2-4}\leq\frac{1}{2-x}$$

$$\frac{1}{x^2-4}-\frac{1}{2-x}\leq 0$$

Next simplify the left side to a single algebraic fraction.

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$$\frac{1}{x^2-4}-\frac{1}{2-x}\le 0$$

$$\frac{1}{(x+2)(x-2)}-\frac{1}{-(x-2)}\le 0$$

$$\frac{1}{(x+2)(x-2)}+\frac{1(x+2)}{(x-2)(x+2)}\le 0$$

$$\frac{1+x+2}{(x+2)(x-2)}\le 0$$

$$\frac{x+3}{(x+2)(x-2)}\le 0$$

The critical numbers are -3, -2, and 2. Note that ± 2 are restrictions and thus we will use open dots when plotting them on a number line. Because of the inclusive inequality we will use a closed dot at the root -3.

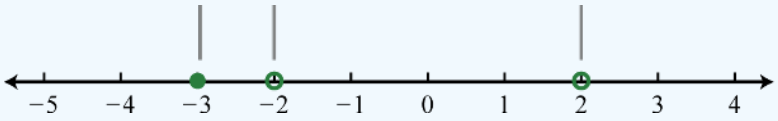


Figure $Page 2.3.37$

Choose test values -4, $-2\frac{1}{2}=-\frac{5}{2}$, 0, and 3.

$$f(-4)=\frac{-4+3}{(-4+2)(-4-2)} \qquad =\frac{(-)}{(-)(-)}=-$$

$$f(-\tfrac{5}{2})=\frac{-\frac{5}{2}+3}{(-\frac{5}{2}+2)(-\frac{5}{2}-2)} \qquad =\frac{(+)}{(-)(-)}=+$$

$$f(0)=\frac{0+3}{(0+2)(0-2)} \qquad =\frac{(+)}{(+)(-)}=-$$

$$f(3)=\frac{3+3}{(3+2)(3-2)} \qquad =\frac{(+)}{(+)(+)}=+$$

Construct a sign chart.

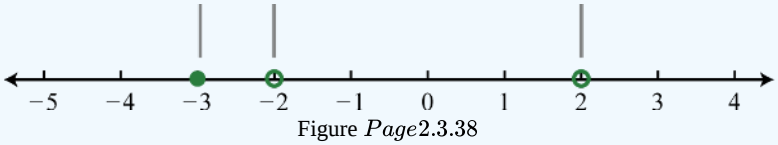


Figure $Page 2.3.38$

Answer the question; in this case, find x where $f(x)\le 0$.

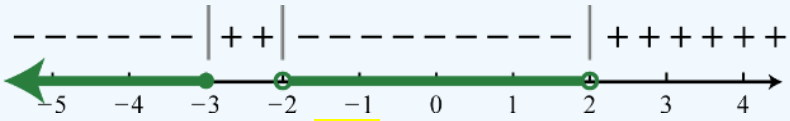


Figure $Page 2.3.39$

Answer

$$(-\infty,-3]\cup(-2,2)$$

? You Try $Page 2.3.5$

Solve: $\dfrac{2x^2}{2x^2+7x-4}\ge\dfrac{x}{x+4}$.

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Answer

$$(-4, 0] \cup \left(\frac{1}{2}, \infty\right)$$

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Page 2.4: Introduction to Functions

In calculus, we will frequently deal with functions and so the goal of this section is to explain, what do we mean by a *function*? In order to answer this question, we first must discuss the notion of a set.

Definition: Set

Definition: A set, typically denoted by a capital letter such as A, refers to an unordered collection of well-defined objects. We call the objects in the set the elements of the set. If x is an element of A, we write $x \in A$. If x is not an element of the set A, then we write $x \notin A$.

There are a few different ways to mathematically denote a set. One way to denote or describe a set is the roster method. The roster method describes a set by explicitly listing its elements. For instance, the set A of all positive integers less than 5 can be written as $A = \{1, 2, 3, 4\}$. Or the set A of all positive integers less than 100 can be written as $A = \{1, 2, 3, \ldots, 99\}$.

However, there is another common way to describe a set. Specifically, we can use what is called set-builder notation. To use set builder notation to describe a set, we describe the property an element must have to be a member of the set. Intuitively, we can think of the set builder notation as being the blueprints for the set. That is, the set builder notation tells us how to build the set. For instance, the set A of all positive integers less than 5 can be written as $A = \{x : x \in \mathbb{Z}^+ \text{ and } x \leq 5\}$ or $A = \{x \mid x \in \mathbb{Z}^+ \text{ and } x \leq 5\}$.

The general form when using set builder notation is to write $\{x | x \text{ has property } P\}$.

Another type of set that we should be familiar with is intervals. In calculus, we will often look at intervals of real numbers. These intervals are also sets and can be described via set builder notation.

1) $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

2) $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$

3) $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

4) $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$

5) $[a, \infty) = \{x \in \mathbb{R} \mid a \leq x\}$

6) $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$

7) $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$

8) $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$

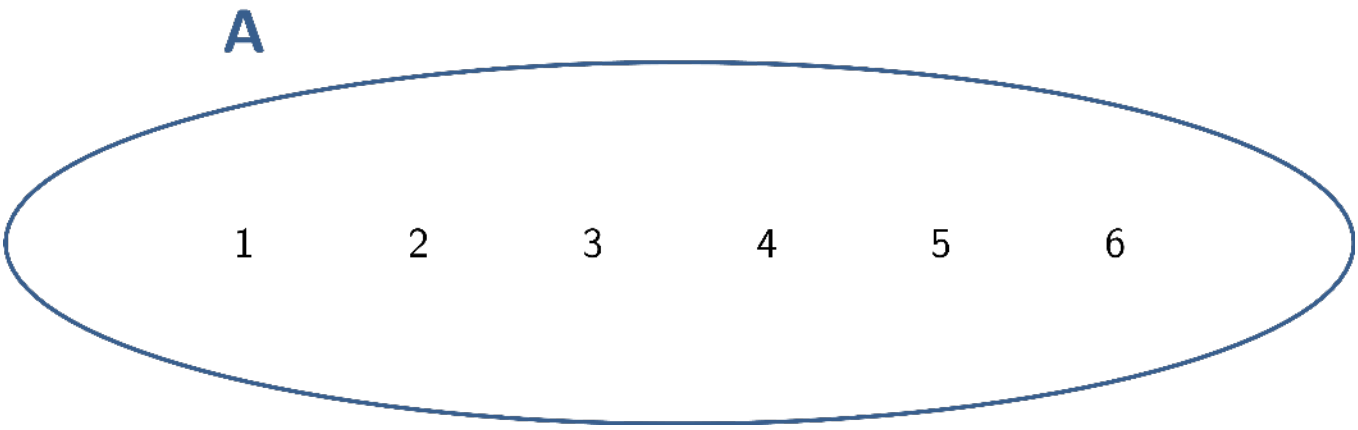
9) $(-\infty, \infty) = \mathbb{R}$

A visual representation of these can be found here: https://math.libretexts.org/Courses/Prince_Georges_Community_College/MAT_2160%3A_Applied_Calculus_I/06%3A_Algebra_Review/6.04%3A_Set-Builder_and_Interval_Notations

Notice that we said a set is just an unordered collection of well-defined objects. We never said these objects have to be numbers and so the following are also examples of sets.

1) $A = \{1, 2, 3, 4, 5, 6\}$.
2) $E = \{x \mid x \text{ is a student enrolled in our Math 241 course}\}$.
3) $F = \{x \in \mathbb{Z} \mid x \leq 10\}$.
4) $G = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$

If we wish to denote a function visually, then we would often construct an image that looks like the following:



We are now in a position to explain what is meant by the word, *function*.

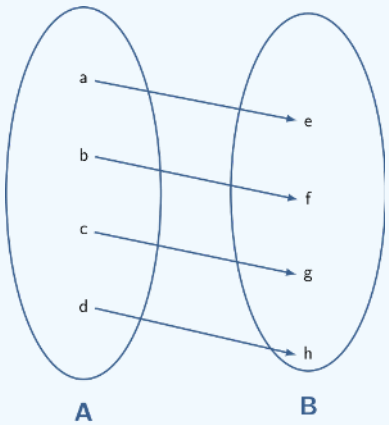
Definition: Function

Definition: Let A and B be nonempty sets. A function, f, from A to B is a rule that assigns to each element in A one and only one element, in B. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \longrightarrow B$ and we call A the domain of f and we call B the codomain of f.

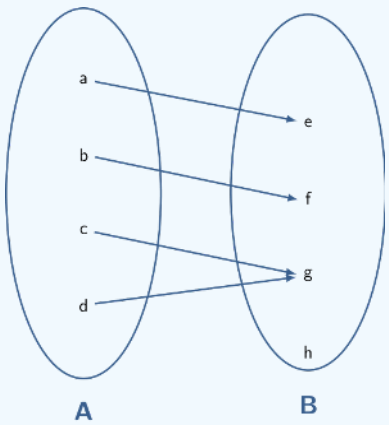
Exercise $Page 2.4.1$

Determine which diagrams below represents a function.

1.

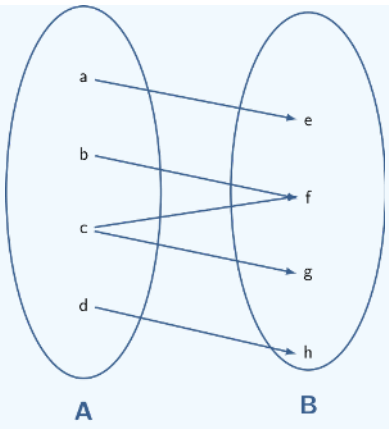


2.

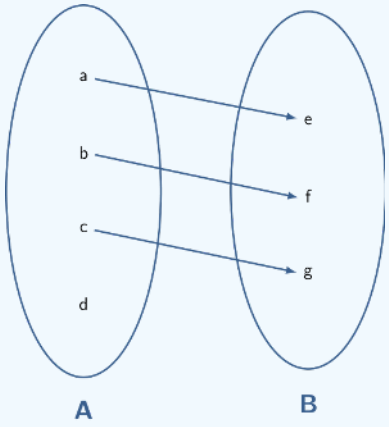


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3.



4.



Answer

1) This satisfies the definition of a function.

2) This still satisfies the definition of a function.

3) This does NOT satisfy the definition of a function.

4) This does NOT satisfy the definition of a function.

We will often let x denote an element in domain and y denote an element in the codomain.

The above definition is saying that for a rule to be regarded as a function, two conditions must be met:

1) Every element in the alleged domain has the capability of being "plugged" into the function (meaning that every element in the domain gets "mapped" to somewhere in B.

2) When that mapping happens, each element in A gets sent to one and ONLY ONE element in B. That is, no x value should ever get sent to two different y values.

In addition to the picture representation of a function, we may also represent a function via an algebraic equation, though a table of values, or by a set of coordinates.

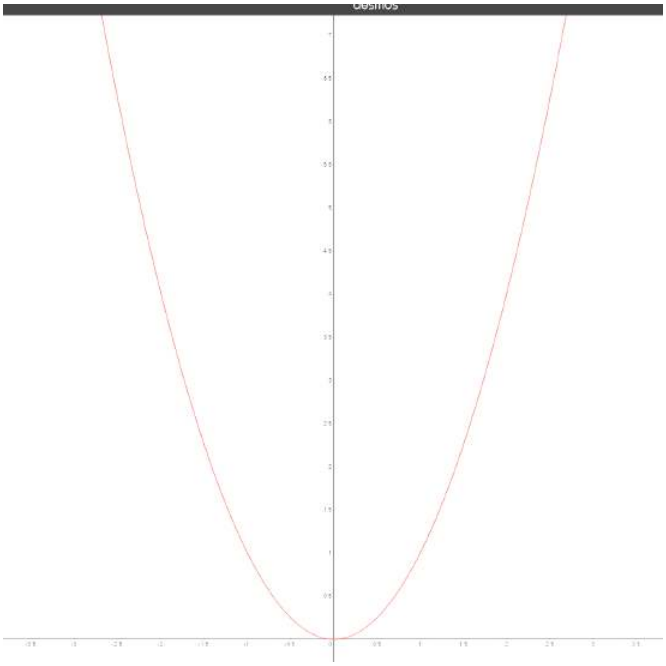
For instance, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. This is the algebraic way to denote the function. Alternatively, we could construct a table of values as follows:

x	y
...	...
-2	4
-1	1

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x	y
0	0
1	1
2	4
...	...

Alternatively, we may graph the function:



or list a set of coordinates: $\{\ldots,(-2,4),(-1,1),(0,0),(1,1),(2,4),\ldots\}=\{(x,x^2)\mid x\in\mathbb{R}\}$

The final way we may represent a function is by drawing it's graph. A quick way to check if a graph is indeed the image of a function is to apply the Vertical Line Test.

Theorem: The Vertical Line Test $Page 2.4.1$

Theorem: A curve in the coordinate plane is the graph of a function provided that every vertical line intersects the curve at most.

Notice if there is some vertical line that intersects the curve more than once, then this is telling us for a single x value, there are multiple y values this particular x value gets mapped to and hence the curve fails to be a function.

Exercise $Page 2.4.1$

Using Desmos, graph the equation $x^2+y^2=25$. Does this equation represent a function?

Answer

No, we see this function fails the Vertical Line Test.

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Exercise $Page2.4.1$

Using Desmos, graph the equation $y = x^2$. Does this equation represent a function?

Answer

Yes, we see this function passes the Vertical Line Test.

Exercise $Page2.4.1$

Determine if each rule stated below is a function.

1) $f : \mathbb{R}^+ \to \mathbb{R}, f(x) = \sqrt{x}$

2) $f : \mathbb{R} \to \mathbb{R}, f(x) = \sqrt{x}$

3) $f : \mathbb{R}^+ \to \mathbb{Z}, f(x) = \sqrt{x}$

Answer

1) This is a function because every element in the alleged domain has the capability of being "plugged" into the function and upon doing so, the function always yields one and only one output.

2) This is NOT a function because we cannot plug in every element in the domain into our function. For instance, $f(-1) = \sqrt{(-1)} \notin \mathbb{R}$.

3) This is NOT a function because there are elements in the domain which do not get mapped to any element in the codomain. For instance, $f(2) = \sqrt{2} \notin \mathbb{Z}$.

Notice in $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, not every possible value in the codomain gets mapped to. For instance, we will never find an element x in the domain such that $f(x) = -1$.

However, consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. Notice for this function, every element in the codomain gets mapped to.

The set of values for which a function maps to is called the *range* of the function.

Definition: Range

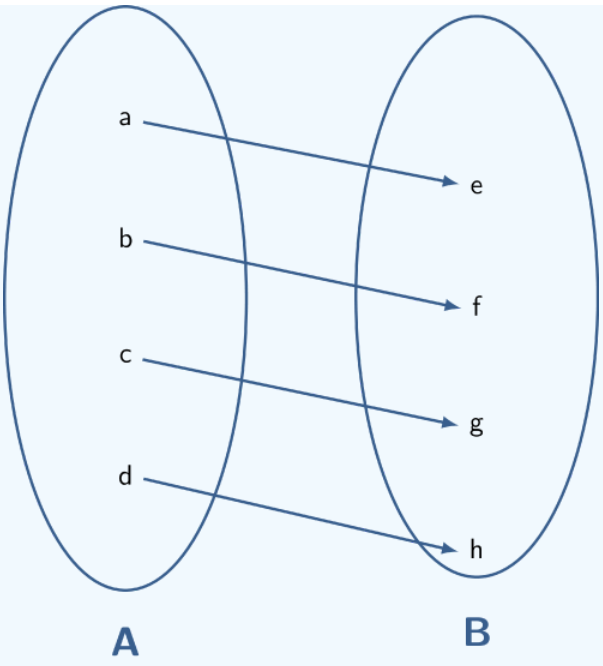
Definition: If $f : A \to B$ then the set $\{f(x) \mid x \in A\}$ is called the **range** of f.

Intuitively, the range of f is the set of all y-values or the set of all outputs for the function.

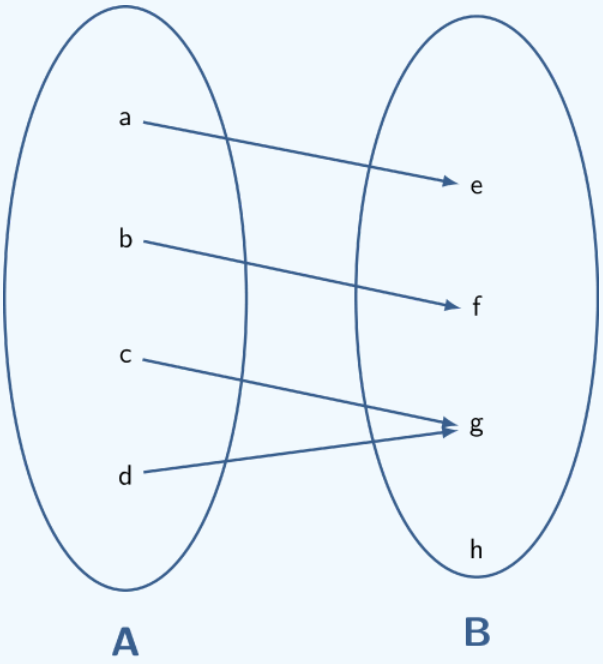
Exercise $Page2.4.1$

For each function below, determine the domain and range.

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1)



2)

3) $f : \mathbb{R} \to \mathbb{R}, f(x) = 3$.

4) $f : \mathbb{R} \to \mathbb{R}, f(x) = x$.

5) $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$.

Answer

1) The domain is $\{a, b, c, d\}$ and the range is $\{e, f, g, h\} = B$.

2) The domain is $\{a, b, c, d\}$ and the range is $\{e, f, g\} \neq B$.

3) The domain is $\mathbb{R}$ and the range is $\{3\}$.

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4) The domain is $\mathbb{R}$ and the range is $\mathbb{R}$.

5) The domain is $\mathbb{R}$ and the range is $[0, \infty)$.

Sometimes the domain of a function is not specified. In such a case, we take the domain to be the set of all real numbers which makes the function valid. Note that it is sometimes easier to ask yourself "when is the function *not* defined?" as opposed to asking "when *is* the function defined?"

? Exercise $Page 2.4.1$

Determine the domain of the following functions.

1) $f(x) = \frac{1}{x}$

2) $f(x) = \sqrt{x-4}$

3) $f(x) = 3x^2 - 6$

4) $f(x) = \frac{1}{x^2-5x+6}$

Answer

1) The domain is $(-\infty, 0) \cup (0, \infty)$.

2) The domain is $4, \infty)$.

3) The domain is $\mathbb{R}$.

4) The domain is $(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

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Page 2.5: Graphing Types of Functions

Learning Objectives

- Graph linear functions.
- Graph quadratic functions.
- Graph radical functions.

Introduction

When both the input (independent variable) and the output (dependent variable) are real numbers, a function can be represented by a coordinate graph. The input is plotted on the x-axis and the output is plotted on the y-axis.

Linear Functions

Definition: Term

Definition: A **linear function**, f, refers to a function $f:\mathbb{R}\to\mathbb{R}$ of the form $f(x)=mx+b$. We call m the slope of the function and b the y-intercept of the function.

Allow us to graph a few linear functions. A helpful first step in graphing a function is to make a table of values. This is particularly useful when you don't know the general shape the function will have. You probably already know that a linear function will be a straight line, but let's make a table first to see how it can be helpful.

Example

Make a table of values for $f(x)=3x+2$.

Solution

x	$f(x)$	
		Make a two-column table. Label the columns x and $f(x)$.
-2		Choose several values for x and put them as separate rows in the x column. **Tip:** It's always good to include 0, positive values, and negative values, if you can.
-1		
0		
1		
3		
-2	-4	Evaluate the function for each value of x, and write the result in the $f(x)$ column next to the x value you used. When $x=0, f(0)=3(0)+2=2$, $f(1)=3(1)+2=5$, $f(-1)=3(-1)+2=-3+2=-1$ and so on.
-1	-1	
0	2	
1	5	
3	11	
-2	-4	(Note that your table of values may be different from someone else's. You may each choose different numbers for x.)
-1	-1	
0	2	
1	5	
3	11	

Now that you have a table of values, you can use them to help you draw both the shape and location of the function. **Important:** The graph of the function will show all possible values of x and the corresponding values of y. This is why the graph is a line and not just the dots that make up the points in our table.

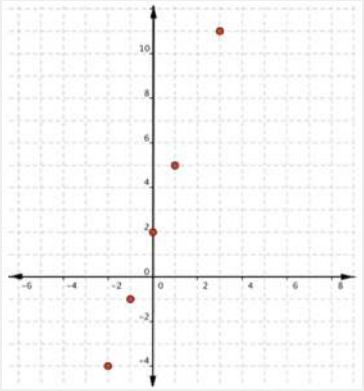
Example

Graph $f(x) = 3x + 2$.

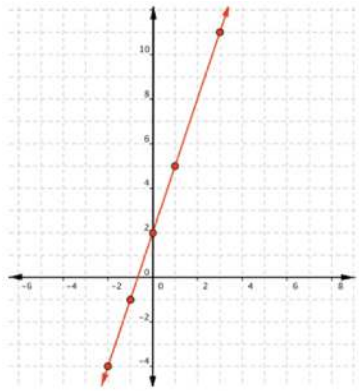
Solution

x	$f(x)$
-2	-4
-1	-1
0	2
1	5
3	11

Start with a table of values, like the one in the previous example. If you think of $f(x)$ as y, each row forms an ordered pair that you can plot on a coordinate grid.



Plot the points.



Since the points lie on a line, use a straight edge to draw the line. Try to go through each point without moving the straight edge.

Let's try another one.

Example

Graph $f(x) = -x + 1$.

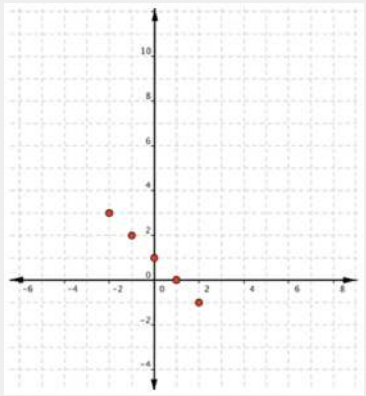
Solution

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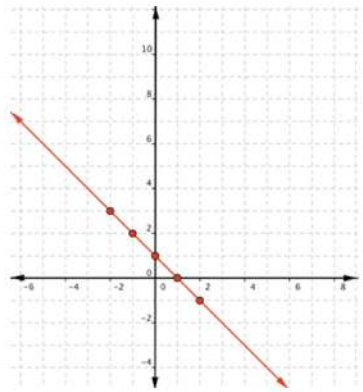
$f(-2) = -(-2) + 1 = 2 + 1 = 3$
$f(-1) = -(-1) + 1 = 1 + 1 = 2$
$f(0) = -(0) + 1 = 0 + 1 = 1$
$f(1) = -(1) + 1 = -1 + 1 = 0$
$f(2) = -(2) + 1 = -2 + 1 = -1$

x	$f(x)$
-2	3
-1	2
0	1
1	0
2	-1

Start with a table of values. You can choose different values for x, but once again, it's helpful to include 0, some positive values, and some negative values.f you think of $f(x)$ as y, each row forms an ordered pair that you can plot on a coordinate grid.



Plot the points.



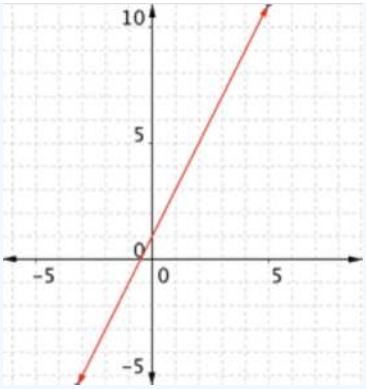
Since the points lie on a line, use a straight edge to draw the line. Try to go through each point without moving the straight edge.

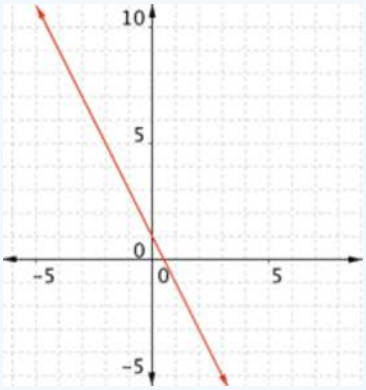
These graphs are representations of a linear function. Remember that a function is a correspondence between two variables, such as x and y.

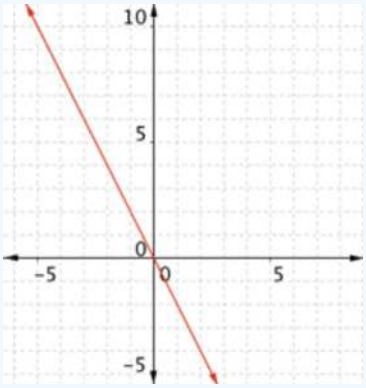
? Exercise

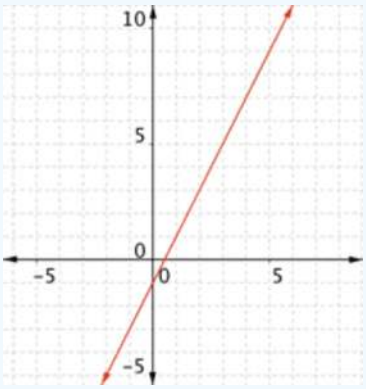
Graph $f(x) = 2x - 1$. Which of the following graphs is correct?

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A. 

B. 

C. 

D. 

Answer

A. Incorrect.

Two good points to check are when $x = 0$ and when $y = 0$. In this case, that would be $(0, -1)$ and $\left(\frac{1}{2}, 0\right)$. The correct answer is Graph D.

B. Incorrect.

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Two good points to check are when $x = 0$ and when $y = 0$. In this case, that would be $(0, -1)$ and $\left(\frac{1}{2}, 0\right)$. The correct answer is Graph D.

C. Incorrect.

Two good points to check are when $x = 0$ and when $y = 0$. In this case, that would be $(0, -1)$ and $\left(\frac{1}{2}, 0\right)$. The correct answer is Graph D.

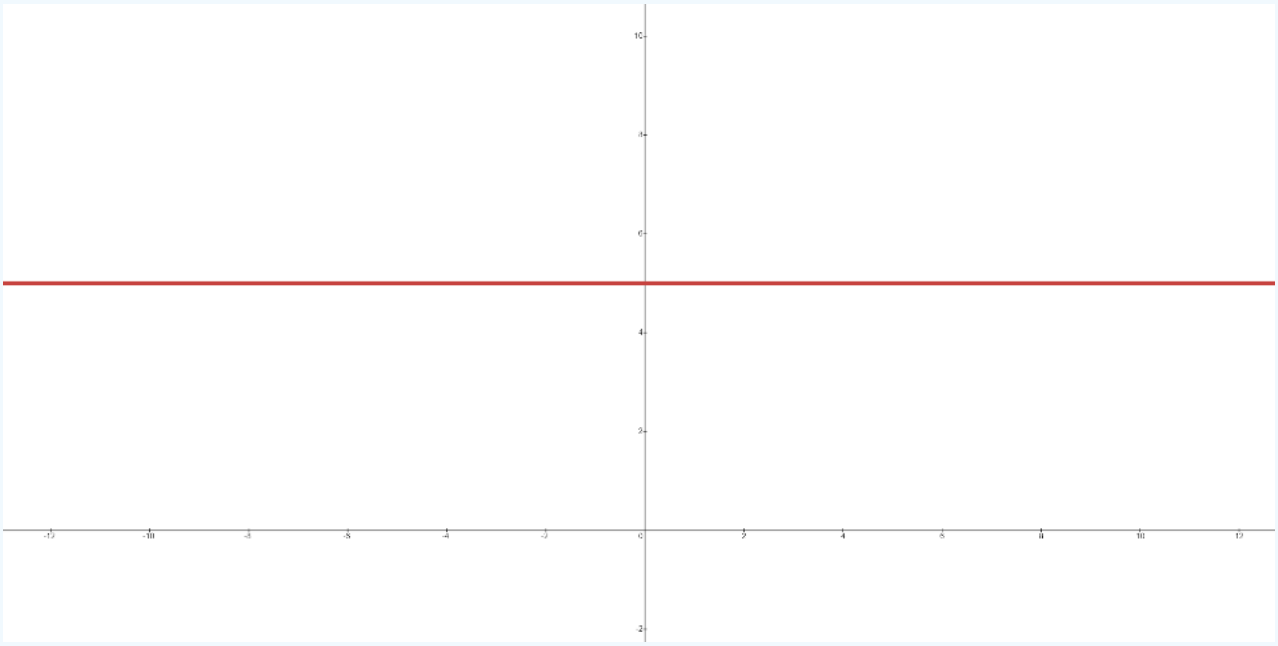
D. Correct.

Two good points to check are when $x = 0$ and when $y = 0$. In this case, that would be $(0, -1)$ and $\left(\frac{1}{2}, 0\right)$. So, Graph D is the correct answer.

? Exercise $Page 2.5.1$

Graph the function $f(x) = 5$.

Answer



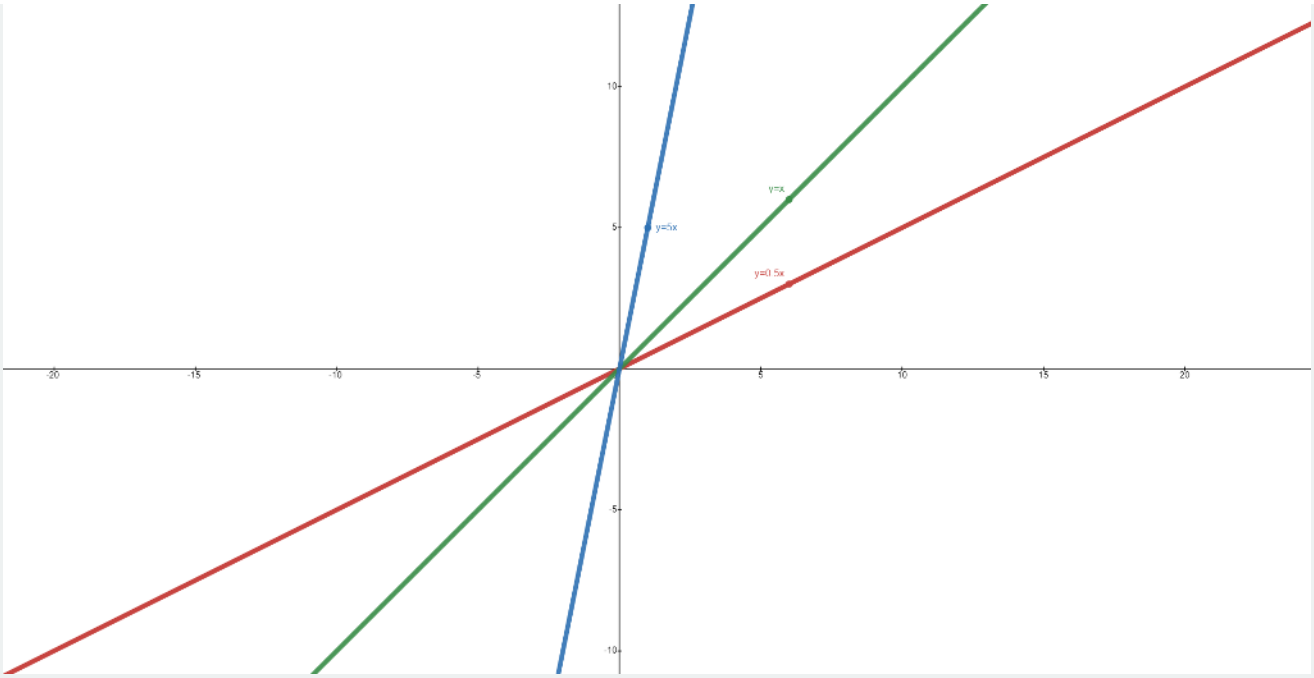
Note that in general, that if c denotes some constant, then the line $y = c$ is a horizontal line passing through c on the y-axis.

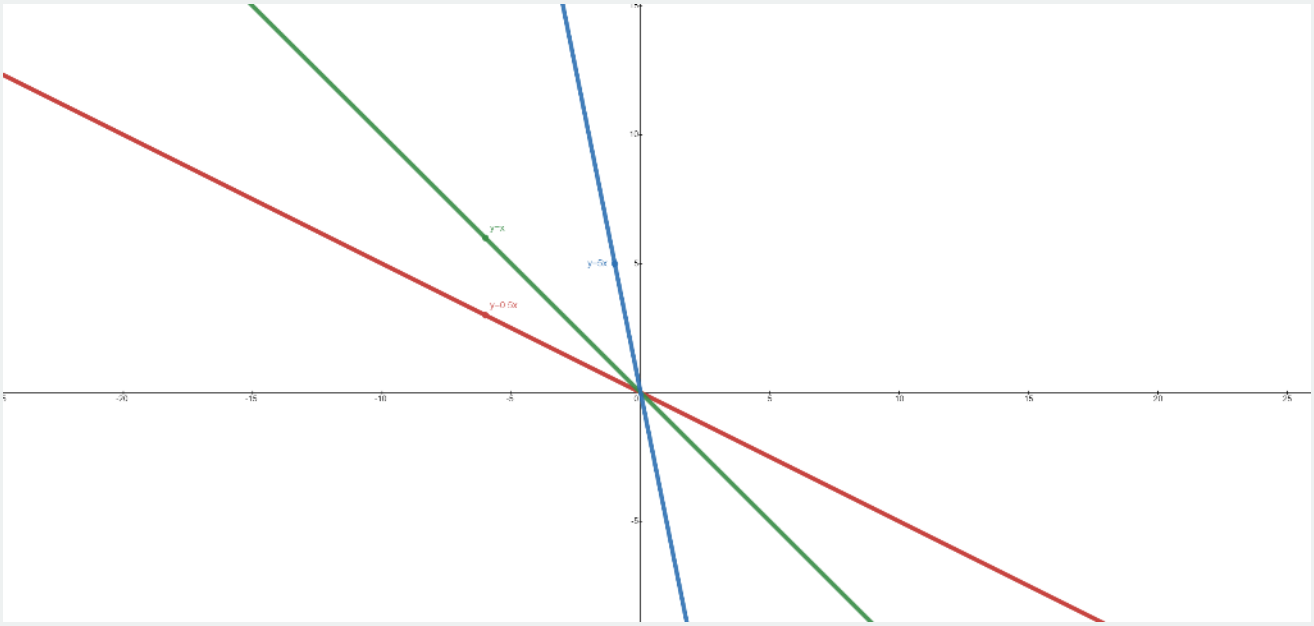
Here is a collection of some helpful facts to remember when it comes to linear lines:

Note

For the graph of $y = mx + b$,

1) the slope, m controls the direction and steepness of the line. If m is positive, then the graph rises upwards from left to right. As m increases, the rise becomes more steep (vertical). On the other hand, if m is negative, then the graph decreases or points down from left to right. As m becomes larger in the negative direction, the graph also becomes more steep (vertical). These properties are illustrated in the following two diagrams:





2) the y-intercept controls where the graph crosses the y axis.

Quadratic Functions

Nonlinear functions can also be graphed. It's helpful to have an idea what the shape should be, so you can be sure that you've chosen enough points to plot as a guide. Let's start with the most basic quadratic function, $f(x) = x^2$.

Example

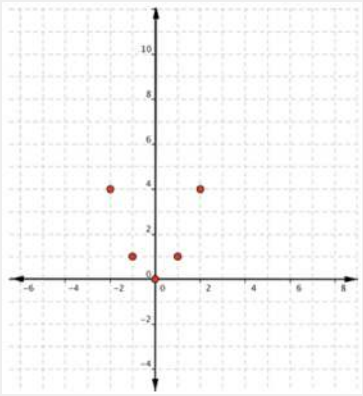
Graph $f(x) = x^2$.

Solution

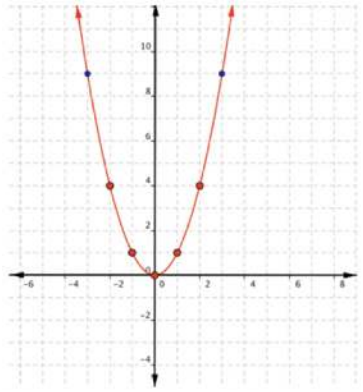
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x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

Start with a table of values. Then think of the table as ordered pairs.



Plot the points.

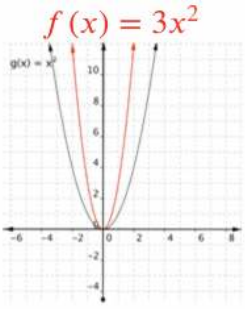


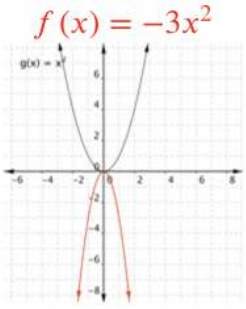
Since the points are *not* on a line, you can't use a straight edge. Connect the points as best you can, using a *smooth curve* (not a series of straight lines). You may want to find and plot additional points (such as the ones in blue here).

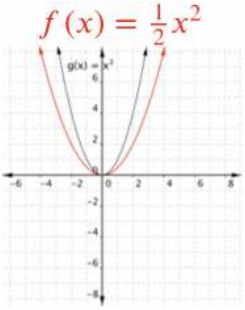
Notice that the shape is like the letter U. This is called a **parabola**. One-half of the parabola is a mirror image of the other half. The line that goes down the middle is called the **line of reflection**. In this case that line is the y-axis.

The equations for quadratic functions have the form $f(x)=ax^2+bx+c$ where $a\neq 0$. In the basic graph above, $a=1$, $b=0$, and $c=0$. Here are some graphs with different values for a, b, and c. Notice how each value changes the shape and location of the parabola.

Changing a changes the width of the parabola and whether it opens up $(a>0)$ or down $(a<0)$.

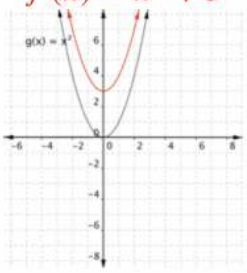




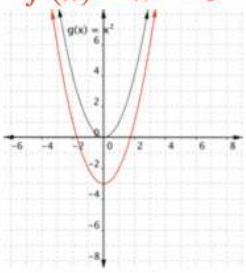


If there is no b term, changing c moves the parabola up or down so that the y intercept is $(0,c)$.

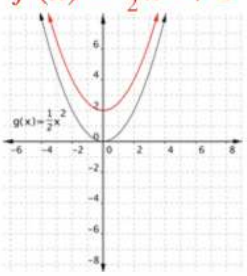
$f(x) = x^2 + 3$



$f(x) = x^2 - 3$

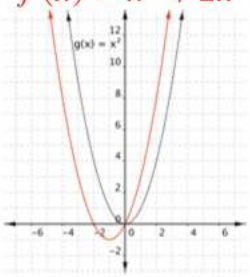


$f(x) = \frac{1}{2}x^2 + 2$

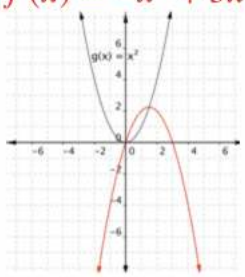


Changing b moves the line of reflection. The way it moves depends also on a.

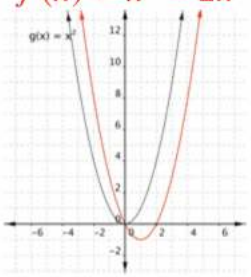
$f(x) = x^2 + 2x$



$f(x) = -x^2 + 3x$



$f(x) = x^2 - 2x$



Properties of a Parabola

For $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers.

- The parabola opens upward if $a > 0$ and downward if $a < 0$.
- a changes the width of the parabola. The parabola gets narrower if $|a| > 1$ and wider if $|a| < 1$.
- The **vertex** depends on the values of a, b, and c. The vertex is $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$.

Example

Graph $f(x) = -2x^2 + 3x - 3$.

Solution

$a = -2$, so the graph will open down and be thinner than $f(x) = x^2$. $c = -3$, so it will move to intercept the y-axis at $(0, -3)$.	Before making a table of values, look at the values of a and c to get a general idea of what the graph should look like.
$\begin{array}{c\|c} x & f(x) \\ \hline -2 & -17 \\ -1 & -8 \\ 0 & -3 \\ 1 & -2 \\ 2 & -5 \end{array}$	Create a table of values. Notice that in this table, the x values increase. The y values increase and then start to decrease again. That indicates a parabola.

Vertex formula $= \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

x-coordinate of vertex: $\frac{-b}{2a} = \frac{-(3)}{2(-2)} = \frac{-3}{-4} = \frac{3}{4}$

y-coordinate of vertex:

$$f\left(\frac{-b}{2a}\right) = f\left(\frac{3}{4}\right)$$

$$f\left(\frac{3}{4}\right) = -2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) - 3$$

$$= -2\left(\frac{9}{16}\right) + \frac{9}{4} - 3$$

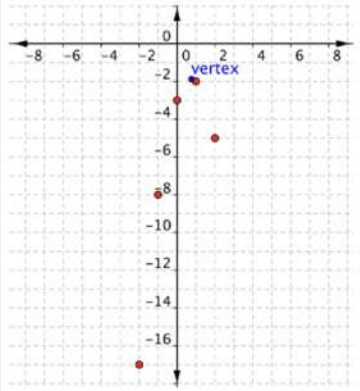
$$= \frac{-18}{16} + \frac{9}{4} - 3$$

$$= \frac{-9}{8} + \frac{18}{8} - \frac{24}{8}$$

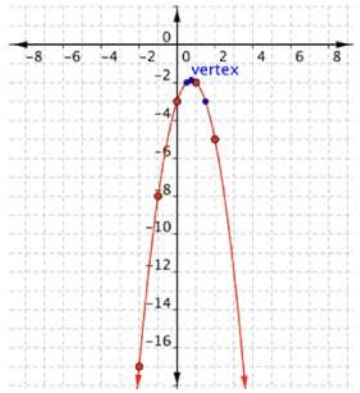
$$= -\frac{15}{8}$$

Vertex: $\left(\frac{3}{4}, -\frac{15}{8}\right)$

To find the vertex of the parabola, use the formula $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$. Finding the vertex may make graphing the parabola easier.



Use the table pairs to plot points. Note the vertex, $\left(\frac{3}{4}, -\frac{15}{8}\right)$, in blue.



Connect the points as best you can, using a *smooth curve*. Remember that the parabola is two mirror images, so if your points don't have pairs with the same value, you may want to include additional points (such as the ones in blue here). Plot points on either side of the vertex.

$x = \frac{1}{2}$ and $x = \frac{3}{2}$ are good values to include.

Exercise

Would the graph for $f(x) = -3x^2 + 2x - 1$ open up or down, and why?

A. Up, because a (the coefficient of x^2) is negative.
B. Down, because a (the coefficient of x^2) is negative.
C. Up, because b (the coefficient of x) is positive.
D. Down, because b (the coefficient of x) is positive.

Answer

A. Up, because a (the coefficient of x^2) is negative.

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Incorrect. The coefficient of x^2 is the correct place to look, but a negative coefficient means the graph opens down. The correct answer is down, because a (the coefficient of x^2) is negative.

B. Down, because a (the coefficient of x^2) is negative.

Correct. A negative coefficient of x^2 means the graph opens down.

C. Up, because b (the coefficient of x) is positive.

Incorrect. Look at the coefficient for x^2, not the coefficient for x. The correct answer is down, because a (the coefficient of x^2) is negative.

D. Down, because b (the coefficient of x) is positive.

Incorrect. Look at the coefficient for x^2, not the coefficient for x. The correct answer is down, because a (the coefficient of x^2) is negative.

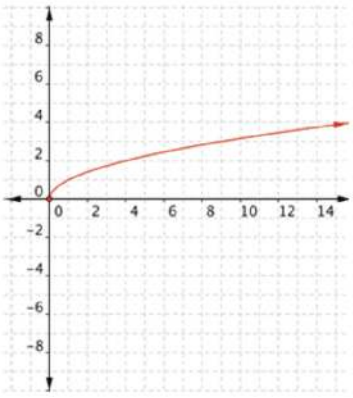
Radical Functions

You can also graph radical functions (such as square root functions) by choosing values for x and finding points that will be on the graph. Again, it's helpful to have some idea about what the graph will look like.

Think about the basic square root function, $f(x) = \sqrt{x}$. Let's take a look at a table of values for x and y and then graph the function. (Notice that all the values for x in the table are perfect squares. Since you are taking the square root of x, using perfect squares makes more sense than just finding the square roots of 0 ,1, 2 ,3 ,4, etc.)

x	$f(x)$
0	0
1	1
4	2
9	3
16	4

Recall that x can never be negative here because the square root of a negative number would be imaginary, and imaginary numbers cannot be graphed. There are also no values for x that will result in y being a negative number. Take a look at the graph.

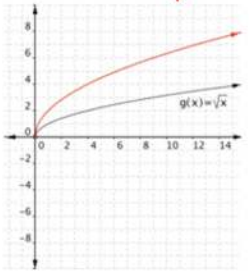


As with parabolas, multiplying and adding numbers makes some changes, but the basic shape is still the same. Here are some examples.

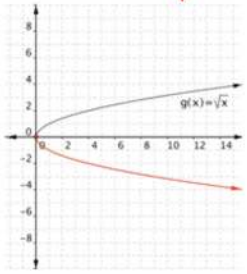
Multiplying $\sqrt{x}$ by a positive value changes the width of the half-parabola. Multiplying $\sqrt{x}$ by a negative number gives you the other half of a horizontal parabola.

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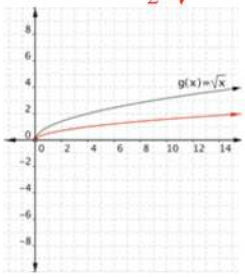
$f(x) = 2\sqrt{x}$



$f(x) = -\sqrt{x}$

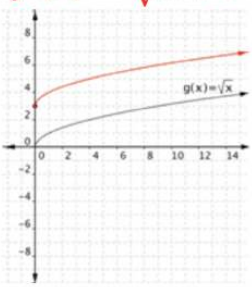


$f(x) = \frac{1}{2}\sqrt{x}$

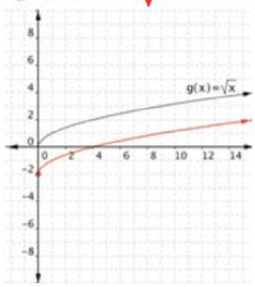


Adding a value *outside* the radical moves the graph up or down. Think about it as adding the value to the basic y value of $\sqrt{x}$, so a positive value added moves the graph up.

$f(x) = \sqrt{x} + 3$

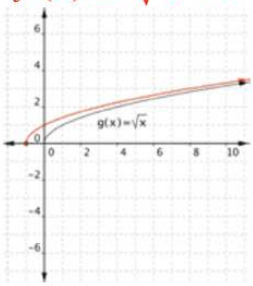


$f(x) = \sqrt{x} - 2$

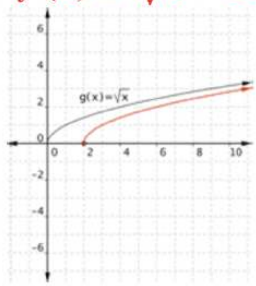


Adding a value *inside* the radical moves the graph left or right. Think about it as adding a value to x before you take the square root, so the y value gets moved to a different x value. For example, for $f(x) = \sqrt{x}$, the square root is 3 if $x = 9$. For $f(x) = \sqrt{x+1}$, the square root is 3 when $x+1$ is 9, so x is 8. Changing x to $x+1$ shifts the graph to the left by 1 unit (from 9 to 8). Changing x to $x-2$ shifts the graph to the right by 2 units.

$f(x) = \sqrt{x+1}$



$f(x) = \sqrt{x-2}$



Notice that as x gets greater, adding or subtracting a number inside the square root has less of an effect on the value of y!

Example

Graph $f(x) = -2 + \sqrt{x-1}$.

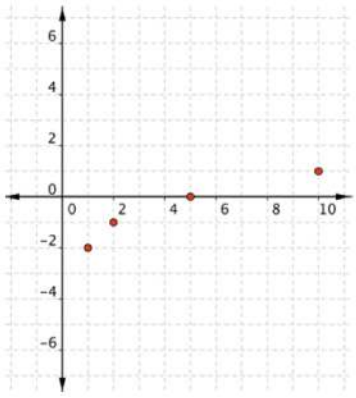
Solution

Inside the square root, you're subtracting 1, so the graph will move to the right 1 from the basic $f(x) = \sqrt{x}$ graph. You're also adding -2 outside the square root, so the graph will move down two from the basic $f(x) = \sqrt{x}$ graph.	Before making a table of values, look at the function equation to get a general idea what the graph should look like.

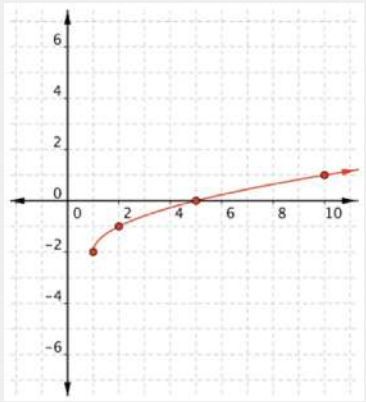
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x	$f(x)$
1	-2
2	-1
5	0
10	1

Create a table of values. Choose values that will make your calculations easy. You want $x-1$ to be a perfect square, such as 0, 1, 4, 9, and so on, so you can take the square root.
Since values of x less than 1 makes the value inside the square root negative, there will be no points on the coordinate graph to the left of $x=1$.
There is no need to choose x values less than 1 for your table!



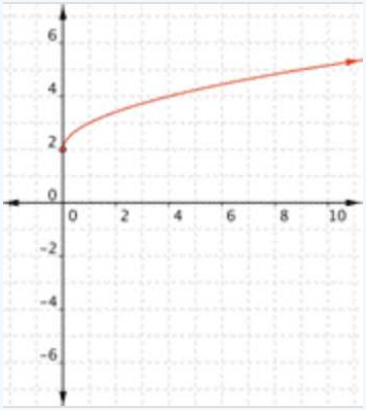
Use the table pairs to plot points.



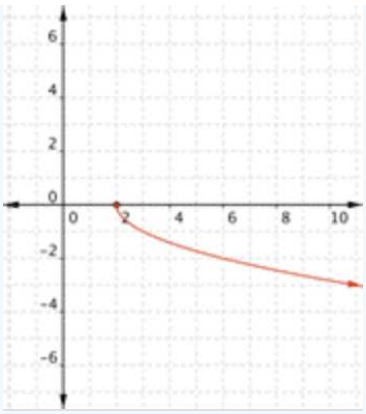
Connect the points as best you can, using a *smooth curve*.

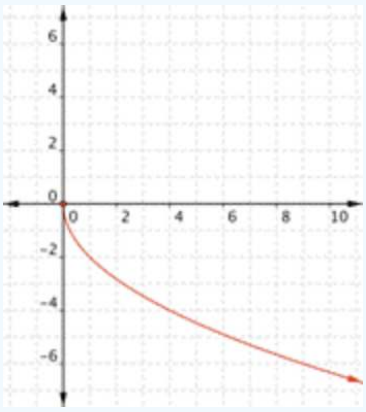
Exercise

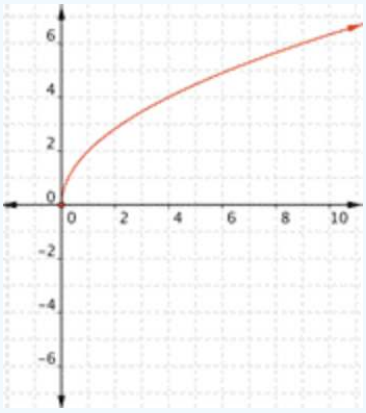
Which of the following is a graph of $f(x)=-2\sqrt{x}$?

A. 

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B. 

C. 

D. 

Answer

A. Incorrect.

Recall that the graph of $f(x) = -\sqrt{x}$ is entirely under the x-axis, in Quadrant IV. The graph of $f(x) = -2\sqrt{x}$ will be similar but wider. The correct answer is Graph C.

B. Incorrect.

In the function $f(x) = -2\sqrt{x}$, $f(0) = 0$. This point is not represented on this graph. Find a graph that looks similar to Graph B, but which goes through the point $(0, 0)$. The correct answer is Graph C.

C. Correct.

The graph of $f(x) = -2\sqrt{x}$ will be similar to the graph of $f(x) = -\sqrt{x}$, although it will be a bit wider.

D. Incorrect.

This is the graph of $f(x) = 2\sqrt{x}$. The graph of $f(x) = -2\sqrt{x}$ will be below the x-axis. The correct answer is Graph C.

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Summary

Creating a graph of a function is one way to understand the relationship between the inputs and outputs of that function. Creating a graph can be done by choosing values for x, finding the corresponding y values, and plotting them. However, it helps to understand the basic shape of the function. Knowing the effect of changes to the basic function equation is also helpful.

One common shape you will see is a parabola. Parabolas have the equation $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$. The value of a determines the width and the direction of the parabola, while the vertex depends on the values of a, b, and c. The vertex is $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

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Page 2.6: Finding Information About Functions From Graphs

Learning Objectives

- Use a graph to find domain and range.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graphing calculator to describe behavior of functions.

We have been learning a lot about functions in this course. Understanding their unique features and how they behave is helpful when determining which functions will be the best models when describing real-life situations. In this section, we will focus on what information we can gain about functions from a graph.

Finding Domain and Range from Graphs

We have learned how to find the domain of functions algebraically. In many cases, find range algebraically is not so simple as there are no systematic approaches since the output behavior of functions can vary a lot depending on the equation of the functions. Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see (as noted by arrows on the ends of the graph), the domain and range may be greater than the visible values shown in the coordinate system. See Figure $Page 2.6.1$.

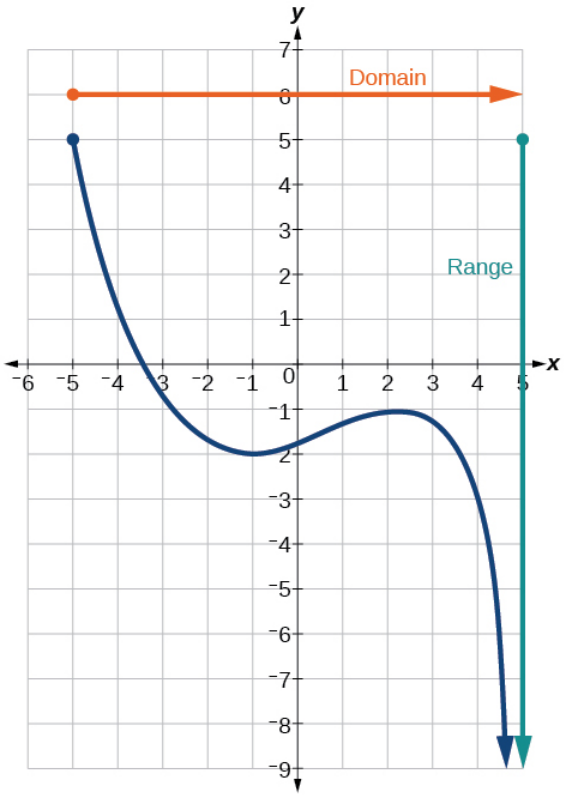


Figure $Page 2.6.1$: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example $Page 2.6.1$

Find the domain and range of the function f whose graph is shown in Figure $Page 2.6.2$.

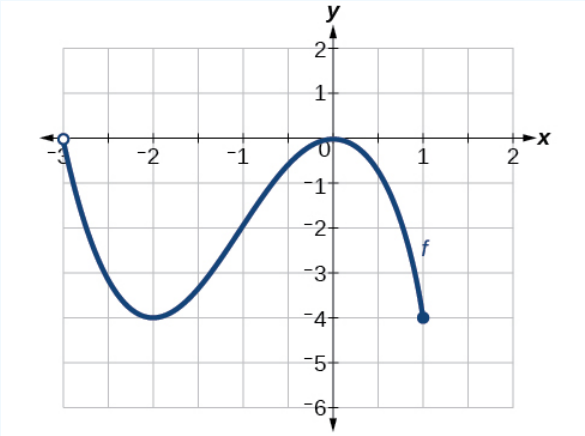


Figure $Page 2.6.2$: Graph of a function from (-3, 1].

Solution

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4 and it appears there is a point at $(0, 0)$ so the range is $[-4, 0]$.

Example $Page 2.6.2$

Find the domain and range of the function f whose graph is shown in Figure $Page 2.6.3$.

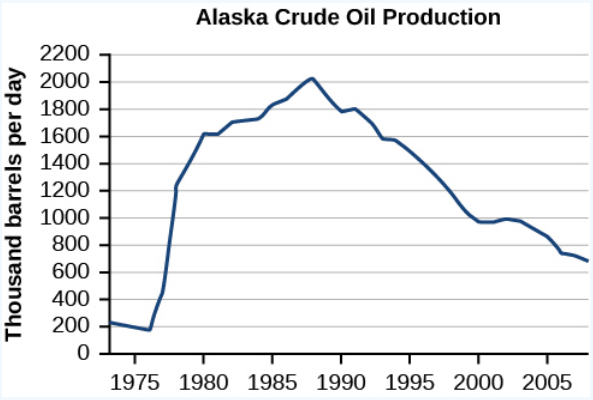


Figure $Page 2.6.3$: Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

Solution

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \le t \le 2008$ and the range as approximately $180 \le b \le 2010$.

In interval notation, the domain is $[1973, 2008]$ and the range is about $[180, 2010]$ For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Using a Graphing Calculator to Find Domain and Range

We’ve learned how to find the domain and range of a function by looking at a graph. If we define a function by means of an equation, such as $f(x) = \sqrt{4 - x}$, then we should be able to capture the domain and range of f from its graph, provided, of course, that we can draw the graph of f. If we don't know what a graph looks like, we can use graphing technology.

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Example $Page 2.6.3$

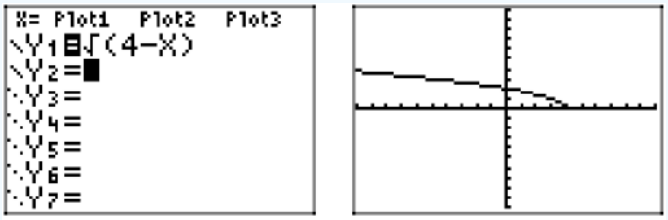
Use a graphing calculator to estimate the domain and range using interval notation for the function defined by

$$f(x) = \sqrt{4 - x}$$

.

Solution

Load the expression defining f into the Y= menu, as shown in Figure $Page 2.6.4$(a). Select 6:ZStandard from the ZOOM menu to produce the graph of f shown in Figure $Page 2.6.5$(b).



(a) (b)

Figure $Page 2.6.4$. Sketching the graph of $f(x) = \sqrt{4 - x}$.

Next, project each point on the graph of f onto the x-axis, as shown in Figure $Page 2.6.4$(b) to estimate the domain. Note that we've made two assumptions about the graph of f.

1. At the left end of the graph in Figure $Page 2.6.4$(b), we assume that the graph of f continues upward and to the left indefinitely. Hence, the "shadow" or projection onto the x-axis will move indefinitely to the left. This is indicated by attaching an arrowhead to the left-hand end of the region that "lies in shadow" on the x-axis, as shown in Figure $Page 2.6.4$(b).
2. We also assume that the right end of the graph ends at the point $(4, 0)$. This accounts for the "filled dot" when this point on the graph of f is projected onto the x-axis.

Note that the "shadow" or projection onto the x axis in Figure $Page 2.6.4$(b) includes all values of x less than or equal to 4. Thus, the domain of f is

$$\text{Domain of } f = (-\infty, 4] = \{x : x \leq 4\}$$

We can also see this result by considering how f is defined.

$$f(x) = \sqrt{4 - x}$$

Recall that we earlier defined the domain of f as the set of "permissible" x-values. In this case, it is impossible to take the square root of a negative number, so we must be careful selecting the x-values we use in this rule. Note that $x = 4$ is allowable, as

$$f(0) = \sqrt{4 - 4} = \sqrt{0} = 0$$

However, numbers larger than 4 cannot be used in this rule. For example, consider what happens when we attempt to use $x = 5$.

$$f(x) = \sqrt{4 - 5} = \sqrt{-1}$$

We'll leave it to our readers to test other values of x that are less than 4. They will also produce real answers when they are input into the rule $f(x) = \sqrt{4 - x}$. Note that this also verifies our earlier conjecture that the "shadow" or projection shown in Figure $Page 2.6.4$(b) continues indefinitely to the left.

Instead of "guessing and checking," we can speed up the analysis of the domain of $f(x) = \sqrt{4 - x}$ by noting that the expression under the radical must not be a negative number. Hence, $4 - x$ must either be greater than or equal to zero. This argument produces an inequality that is easily solved for x.

$$4 - x \geq 0$$
$$-x \geq -4$$
$$x \leq 4$$

This last result verifies that the domain of f is all values of x that are less than or equal to 4, which is in complete agreement with the "shadow" or projection onto the x-axis shown in Figure $Page 2.6.4$(b).

Simliary, to determine the range of f, we project each point on the graph of f onto the y-axis. We estimate the range as

$$\text{Range of } f = [0, \infty)$$

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. Figure $Page 2.6.5$ shows examples of increasing and decreasing intervals on a function.

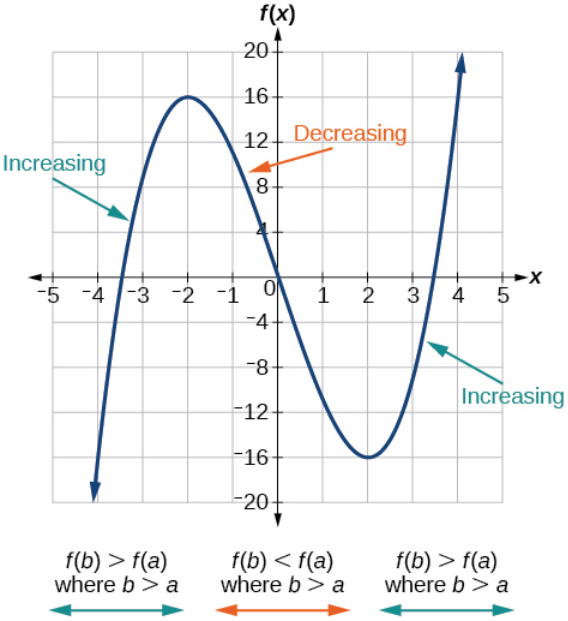


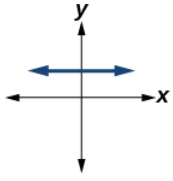
Figure $Page 2.6.5$: The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 2)$.

Increasing and Decreasing Function Behavior

- A function f is **increasing** on an open interval if for every a and b, whenever $a < b$, we have that $f(a) < f(b)$.
- A function f is **decreasing** on an open interval if for every a and b, whenever $a < b$, we have that $f(a) > f(b)$.
- A function f is **constant** on an open interval otherwise.

Increasing and Decreasing Behavior for Basic Functions

We will now return to our basic toolkit functions and discuss their graphical behavior in Figure $Page 2.6.6$. Note that this list includes some other functions that were not discussed in the previous section.

Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function	Increasing	

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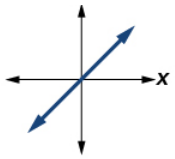
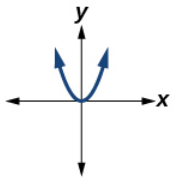
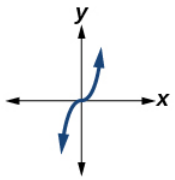
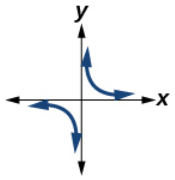
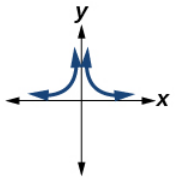
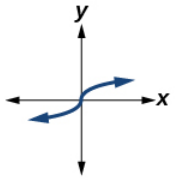
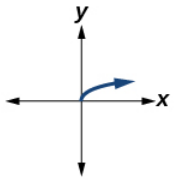
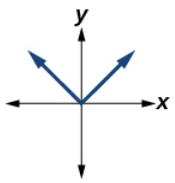
$f(x) = x$		
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \lvert x \rvert$	Increasing on $(0, \infty)$ Decreasing on $(\infty, 0)$	

Figure $Page 2.6.6$

Local Extrema

While some functions are increasing (or decreasing) over their entire domain, many are not. A **continuous function** is a function whose graph can be drawn in one movement without having to lift your pencil. If a function is continuous, a value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function

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changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is "local minima." Together, local maxima and minima are called **local extrema** of the function. Local extrema may are be referred to as **relative extrema**.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of local extrema, because any given local extrema as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown in Figure $Page 2.6.7$, the local maximum is 16, and it occurs at $x = -2$. The local minimum is −16 and it occurs at $x = 2$.

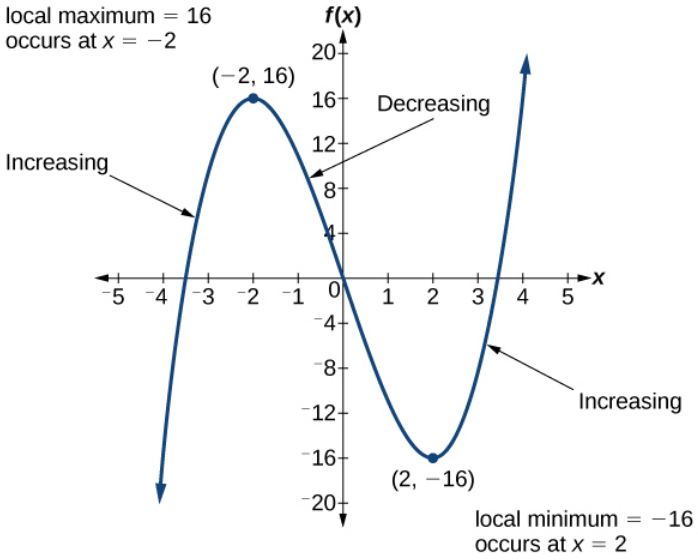


Figure $Page 2.6.7$: Graph of a polynomial that shows the increasing and decreasing intervals and local maximum.maximum

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure $Page 2.6.8$ illustrates these ideas for a local maximum.

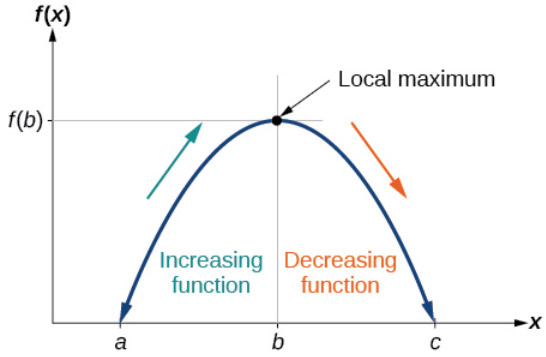


Figure $Page 2.6.8$: Definition of a local maximum

These observations lead us to a formal definition of local extrema.

Local Extrema

- A function f has a local maximum at a point b in an open interval (a, c) if $f(b)$ is greater than or equal to $f(x)$ for every point x (x does not equal b) in the interval.
- A function f has a local minimum at a point b in (a, c) if $f(b)$ is less than or equal to $f(x)$ for every x (x does not equal b) in the interval.

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Example $Page2.6.4$

Given the function $p(t)$ in Figure $Page2.6.9$, identify the intervals on which the function appears to be increasing.

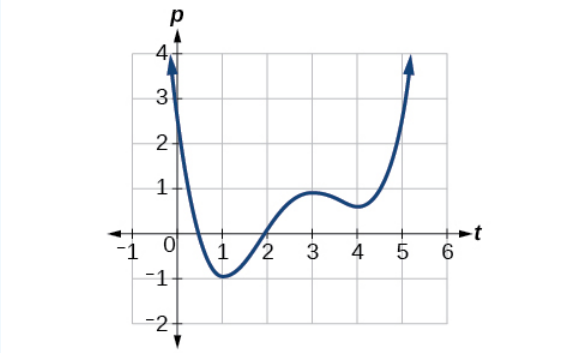


Figure $Page2.6.9$: Graph of a polynomial.

Solution

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

Analysis

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

Example $Page2.6.5$

Graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

Solution

Using technology, we find that the graph of the function looks like that in Figure $Page2.6.10$. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$

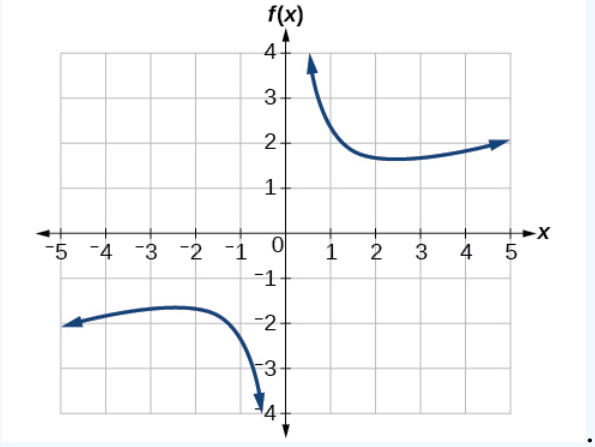
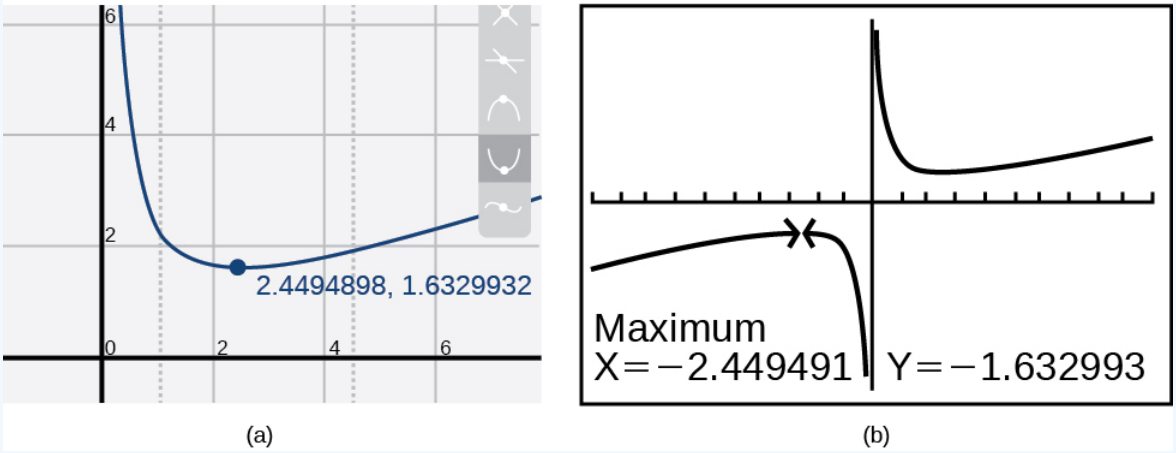


Figure $Page2.6.10$: Graph of a rational function.

Analysis

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Figure $Page2.6.11$ provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

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(a) (b)

Figure $Page2.6.11$: Graph of the reciprocal function on a graphing calculator.

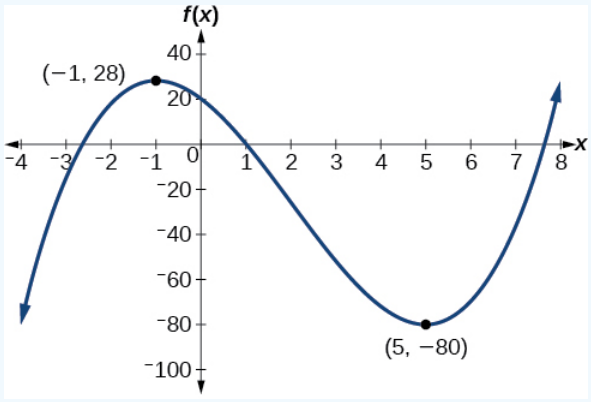
Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

You Try $Page2.6.1$

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ in a graphing calculator to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Solution

The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.



Graph of a polynomial with a local maximum at (-1, 28) and local minimum at (5, -80).

Example $Page2.6.6$

For the function f whose graph is shown in Figure $Page2.6.12$, find all local maxima and minima.

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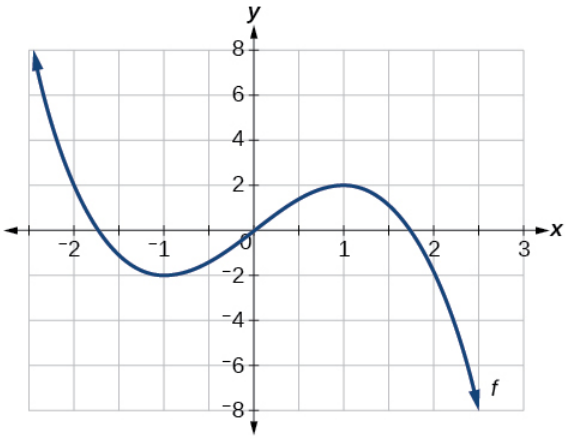


Figure $Page 2.6.12$: Graph of a polynomial.

Solution

Observe the graph of f. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$.The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is -2.

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Page 2.7: Piecewise Defined Functions

Piecewise Defined Functions

Piecewise defined functions are used in many real-world phenomena (e.g. postal rates and income tax formulas) are modeled by such functions. It is important that we are familiar with them and know how to evaluate them.

Consider the absolute value function $f(x) = |x|$.

The domain is real numbers and the range is values greater than or equal to 0.

If we input 0 or any positive value, the output is the same value as the input ($f(0) = |0| = 0$, $f(3) = |3| = 3$, $f(7.2) = |7.2| = 7.2$, ...).

$f(x) = x$ if $x \geq 0$

If we input a negative value the sign of the output must change from negative to positive ($f(-1) = |-1| = 1$, $f(-4.5) = |-4.5| = 4.5$, ...).

$f(x) = -x$ if $x < 0$, since multiplying a negative value by -1 makes it positive.

Since this requires two different processes or pieces, the absolute value function is often called the most basic piecewise defined function.

Piecewise Function

A **piecewise function** is a function in which the formula used depends upon the domain the input lies in. We notate this idea like:

$$f(x) = \begin{cases} \text{formula 1, if domain value satisfies given criteria 1} \\ \text{formula 2, if domain value satisfies given criteria 2} \\ \text{formula 3, if domain value satisfies given criteria 3} \end{cases}$$

Note that piecewise defined functions are not rescricted to 3 formulas, they may contain 2 or more.

Example $Page 2.7.10$

A museum charges $5 per person for a guided tour with a group of 1 to 9 people, or a fixed $50 fee for 10 or more people in the group. Set up a function relating the number of people, n, to the cost, C.

Solution

To set up this function, two different formulas would be needed. $C = 5n$ would work for n values under 10, and $C = 50$ would work for values of n ten or greater. Notating this:

$$C(n) = \begin{cases} 5n & \text{if } 0 < \text{n} < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

Example $Page 2.7.11$

For the function $f(x) = \begin{cases} x+6, & \text{if } x < 1 \\ 3 - x^2, & \text{if } x \geq 1 \end{cases}$, find the following:

1. $f(0)$
2. $f(1)$
3. $f(3)$

Solution

1. Since the input is 0, we input into the top formula, $x + 6$. So, $f(0) = 0 + 6 = 6$
2. Since the input is 1, we input into the top formula, $x + 6$. So, $f(1) = 1 + 6 = 7$
3. Since the input is 3 we input into the bottom formula, $3 - x^2$. So, $f(0) = 0 + 6 = 6$

Example $Page 2.7.12$

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if} \quad 0 < g < 2 \\ 25 + 10(g-2) & \text{if} \quad g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data, and the cost of using 4 gigabytes of data.

Solution

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which piece of domain our input falls in. Since 1.5 is less than 2, we use the first formula, giving $C(1.5)$ = \$25.

The find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we'll use the second formula. $C(4)$ = 25 + 10(4 - 2) = \$45.

Example $Page 2.7.13$

The height h in feet of a model rocket above the ground t seconds after lift-off is given by

$$h(t) = \begin{cases} -5t^2 + 100t, & \text{if} \quad 0 \leq t \leq 20 \\ 0, & \text{if} \quad t > 20 \end{cases}$$

1. Find and interpret $h(10)$ and $h(60)$.
2. Solve $h(t) = 375$ and interpret your answers.

Solution

1. We first note that the independent variable here is t, chosen because it represents time. Secondly, the function is broken up into two rules: one formula for values of t between 0 and 20 inclusive, and another for values of t greater than 20. Since $t = 10$ satisfies the inequality $0 \leq t \leq 20$, we use the first formula listed, $h(t) = -5t^2 + 100t$, to find $h(10)$. We get $h(10) = -5(10)^2 + 100(10) = 500$. Since t represents the number of seconds since lift-off and $h(t)$ is the height above the ground in feet, the equation $h(10) = 500$ means that 10 seconds after lift-off, the model rocket is 500 feet above the ground. To find $h(60)$, we note that $t = 60$ satisfies $t > 20$, so we use the rule $h(t) = 0$. This function returns a value of 0 regardless of what value is substituted in for t, so $h(60) = 0$. This means that 60 seconds after lift-off, the rocket is 0 feet above the ground; in other words, a minute after lift-off, the rocket has already returned to Earth.
2. Since the function h is defined in pieces, we need to solve $h(t) = 375$ in pieces. For $0 \leq t \leq 20$, $h(t) = -5t^2 + 100t$, so for these values of t, we solve $-5t^2 + 100t = 375$. Rearranging terms, we get $5t^2 - 100t + 375 = 0$, and factoring gives $5(t-5)(t-15) = 0$. Our answers are $t = 5$ and $t = 15$, and since both of these values of t lie between 0 and 20, we keep both solutions. For $t > 20$, $h(t) = 0$, and in this case, there are no solutions to $0 = 375$. In terms of the model rocket, solving $h(t) = 375$ corresponds to finding when, if ever, the rocket reaches 375 feet above the ground. Our two answers, $t = 5$ and $t = 15$ correspond to the rocket reaching this altitude twice - once 5 seconds after launch, and again 15 seconds after launch.

Graphing Piecewise Functions

Previously, we learned how to evaluate piecewise defined functions. Next, let's focus on how to graph them.

Given a piecewise function, sketch a graph.

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Note: It is important to not just plot some points when graphing piecewise functions.

Example $Page2.7.8$: Graphing a Piecewise-Defined Function

Sketch a graph of the following piecewise-defined function:

$$f(x) = \begin{cases} x+3, & \text{if } x < 1 \\ (x-2)^2, & \text{if } x \geq 1 \end{cases}$$

Solution

Graph the linear function $y = x+3$ on the interval $(-\infty, 1)$ and graph the quadratic function $y = (x-2)^2$ on the interval $[1, \infty)$. Since the value of the function at $x = 1$ is given by the formula $f(x) = (x-2)^2$, we see that $f(1) = 1$. To indicate this on the graph, we draw a closed circle at the point $(1, 1)$. The value of the function is given by $f(x) = x+3$ for all $x < 1$, but not at $x = 1$. To indicate this on the graph, we draw an open circle at $(1, 4)$.

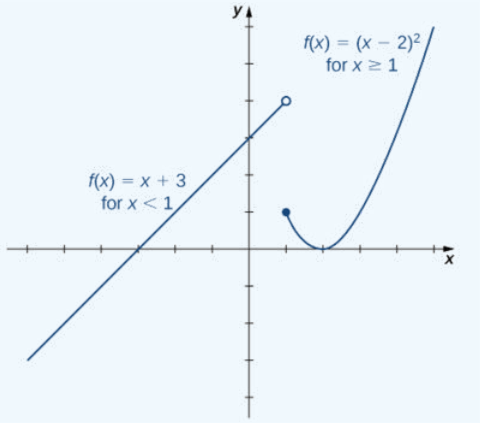
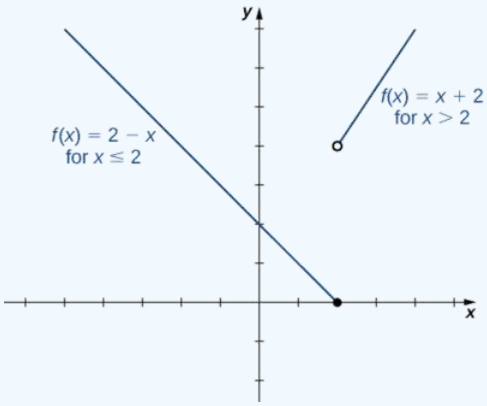


Figure $Page2.7.8$: This piecewise-defined function is linear for $x < 1$ and quadratic for $x \geq 1$.

2) Sketch a graph of the function

$$f(x) = \begin{cases} 2-x, & \text{if } x \leq 2 \\ x+2, & \text{if } x > 2 \end{cases}.$$

Solution:



Example $Page2.7.9$: Parking Fees Described by a Piecewise-Defined Function

In a big city, drivers are charged variable rates for parking in a parking garage. They are charged \$10 for the first hour or any part of the first hour and an additional \$2 for each hour or part thereof up to a maximum of \$30 for the day. The parking garage is open from 6 a.m. to 12 midnight.

a. Write a piecewise-defined function that describes the cost C to park in the parking garage as a function of hours parked x.
b. Sketch a graph of this function $C(x)$.

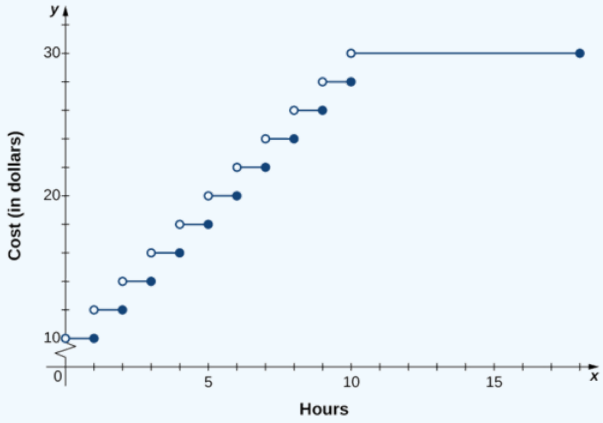
Solution

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1.Since the parking garage is open 18 hours each day, the domain for this function is $\{x \mid 0 < x \leq 18\}$. The cost to park a car at this parking garage can be described piecewise by the function

$$C(x) = \begin{cases} 10, & \text{for } 0 < x \leq 1 \\ 12, & \text{for } 1 < x \leq 2 \\ 14, & \text{for } 2 < x \leq 3 \\ 16, & \text{for } 3 < x \leq 4 \\ \vdots & \\ 30, & \text{for } 10 < x \leq 18 \end{cases}.$$

2.The graph of the function consists of several horizontal line segments.



Exercise $Page 2.7.6$

The cost of mailing a letter is a function of the weight of the letter. Suppose the cost of mailing a letter is $49¢$ for the first ounce and $21¢$ for each additional ounce. Write a piecewise-defined function describing the cost C as a function of the weight x for $0 < x \leq 3$, where C is measured in cents and x is measured in ounces.

Hint

The piecewise-defined function is constant on the intervals $(0, 1]$, $(1, 2]$, ...

Answer

$$C(x) = \begin{cases} 49, 0 < x \leq 1 \\ 70, 1 < x \leq 2 \\ 91, 2 < x \leq 3 \end{cases}$$

Example $Page 2.7.9$

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 3 & \text{if } 1 < x \leq 2 \\ x & \text{if } x > 2 \end{cases}$$

Solution

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the piece we need for the indicated domain of the piecewise function we are graphing. At the endpoints of the domain interval, draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure $Page 2.7.27$ shows the three components of the piecewise function graphed on separate coordinate systems.

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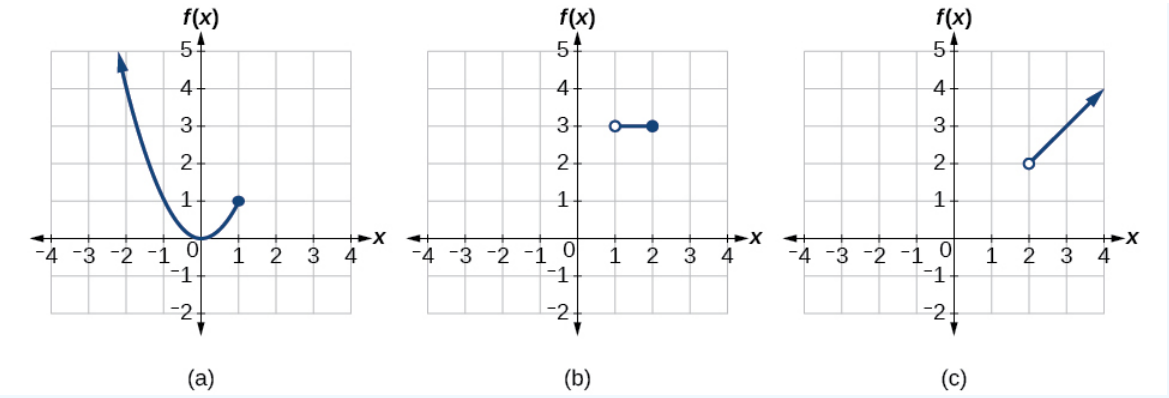


Figure $Page2.7.27$: Graph of each part of the piece-wise function f(x)

(a) $f(x) = x^2$ if $x \leq 1$; (b) $f(x) = 3$ if $1 < x \leq 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we then combine them in the same coordinate plane to form the graph of the piecewise function. See Figure $Page2.7.28$.

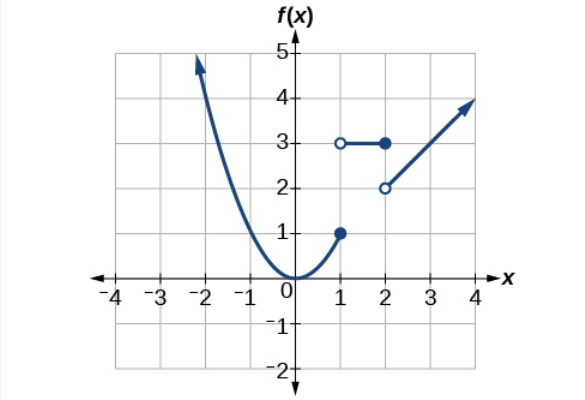


Figure $Page2.7.28$: Graph of the entire function.

Analysis

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

? You Try $Page2.7.10$

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Answer

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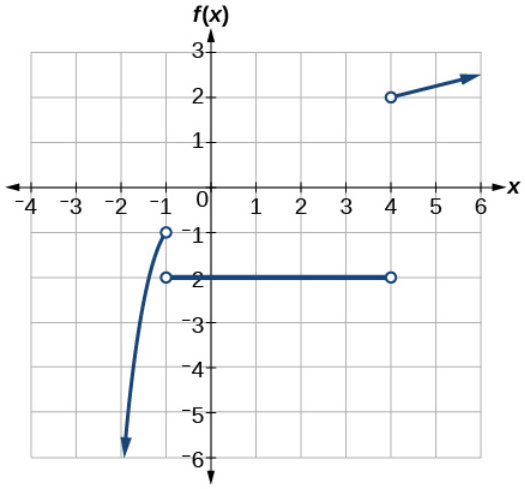


Figure $Page 2.7.29$

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CHAPTER OVERVIEW

Chapter 3: Day 3

Page 3.1: Difference Quotient
Page 3.2: Transformations of Functions
Page 3.3: More on Linear Functions
Page 3.4: Absolute Value Functions
Page 3.5: More on Quadratics

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Page 3.1: Difference Quotient

A big part of calculus is being able to understand the notation when it comes to dealing with functions. For this reason, we will spend some time on discussing how to think about our notation.

Suppose we have the function $f:\mathbb{R}\to\mathbb{R}$ specified by $f(x)=x^2$. To recap a few things, this notation is telling us that the domain is $\mathbb{R}$ meaning that the x values can be any real number. When we plug in these x values into f, the outputs are also real numbers and hence our codomain is $\mathbb{R}$.

Allow us to consider the following exercise.

? Exercise $Page3.1.1$

Suppose $f:\mathbb{R}\to\mathbb{R}$ specified by $f(x)=x^2$. Find the following:

1) $f(1)$

2) $f(2)$

3) $f(3)$

4) $f(a)$

5) $f(b)$

6) $f(a+h)$

7) $\frac{f(a+h)-f(a)}{h}$

Answer

1) The notation $f(1)$ is telling us that wherever we see x in the function $f(x)=x^2$, we should replace the x by a 1. Doing so yields $f(1)=1^2=1$

2) The notation $f(2)$ is telling us that wherever we see x in the function $f(x)=x^2$, we should replace the x by a 2. Doing so yields $f(2)=2^2=4$

3) The notation $f(3)$ is telling us that wherever we see x in the function $f(x)=x^2$, we should replace the x by a 3. Doing so yields $f(3)=3^2=9$

4) The notation $f(a)$ is telling us that wherever we see x in the function $f(x)=x^2$, we should replace the x by an a. Doing so yields $f(a)=a^2$.

5) The notation $f(b)$ is telling us that wherever we see x in the function $f(x)=x^2$, we should replace the x by a b. Doing so yields $f(b)=b^2$.

6) The notation $f(a+h)$ is telling us that wherever we see x in the function $f(x)=x^2$, we should replace the x by an $a+h$. Doing so yields $f(a+h)=(a+h)^2=a^2+2ah+h^2$.

7) Putting Parts 4 and 6 together yields:

$$\begin{aligned}\frac{f(a+h)-f(a)}{h} &= \frac{(a+h)^2-a^2}{h}\\ &= \frac{a^2+2ah+h^2-a^2}{h}\\ &= \frac{2ah+h^2}{h}\\ &= 2a+h\end{aligned}$$

The quantity we considered in Part 7 has a special name - the difference quotient.

Definition: The Difference Quotient

Definition: Given a function f, we refer to $\dfrac{f(a+h)-f(a)}{h}$ as the **difference quotient.**

More will be said about the difference quotient in the coming days but for now, allow us to compute a few more difference quotients.

Exercise $Page 3.1.1$

Compute the difference quotient of $f(x)=x$.

Answer

1.

Exercise $Page 3.1.1$

Compute the difference quotient of $f(x)=x^3-x$.

Answer

$3a^2+3ah+h^2-1$.

Exercise $Page 3.1.1$

Compute the difference quotient of $f(x)=\sqrt{x}$. Hint: to simplify, rationalize the numerator!

Answer

$\frac{1}{\sqrt{a+h}-\sqrt{a}}$.

Exercise $Page 3.1.1$

Compute the difference quotient of $f(x)=\dfrac{1}{x}$.

Answer

$-\dfrac{1}{(a+h)(a)}$.

Exercise $Page 3.1.1$

Compute the difference quotient of $f(x)=\dfrac{1-x}{2+x}$.

Answer

$\dfrac{-3}{(2+a+h)(2+a)}$.

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Page 3.2: Transformations of Functions

Learning Objectives

- Identify transformations of functions.
- Graph functions using transformations.
- Find functions that model transformed graphs.
- Determine whether a function is even, odd, or neither from its graph.

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.



Figure $Page 3.2.1$: (credit: "Misko"/Flickr)

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the basic functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + c$, the function $f(x)$ is shifted vertically c units. See Figure $Page 3.2.2$ for an example.

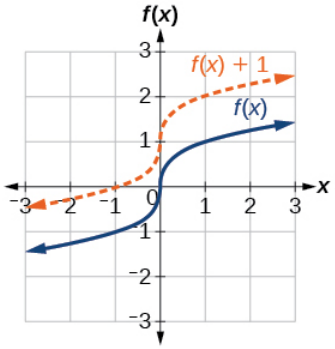


Figure $Page 3.2.2$: Vertical shift by $c = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + c$ is equivalent to $y + c$. Every unit of y is replaced by $y + c$, so the y-value increases or decreases depending on the value of c. The result is a shift upward or downward.

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Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + c$, where c is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by c units. If c is positive, the graph will shift up. If c is negative, the graph will shift down.

Example $Page 3.2.1$

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure $Page 3.2.3$ shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

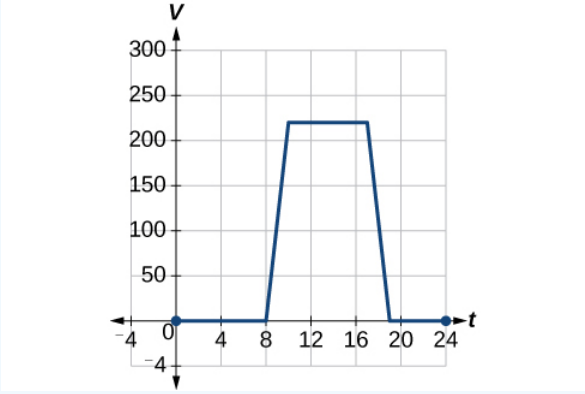


Figure $Page 3.2.3$

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure $Page 3.2.4$.

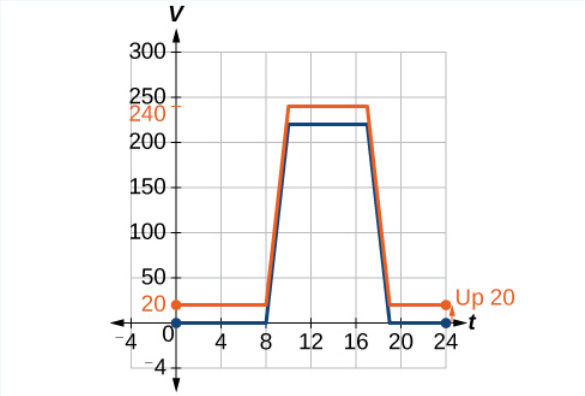


Figure $Page 3.2.4$

Notice that in Figure $Page 3.2.4$, for each input value, the output value has increased by 20, so if we call the new function $S(t)$,we could write

$$S(t) = V(t) + 20$$

This notation tells us that, for any value of t,$S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table $Page 3.2.1$.

Table $Page 3.2.1$

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0

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$S(t)$	20	20	240	240	20	20

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure $Page 3.2.5$.

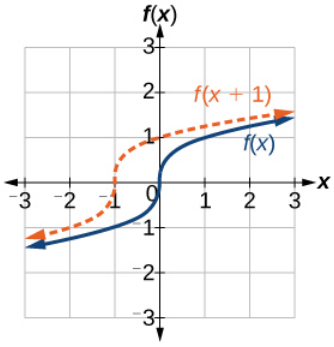


Figure $Page 3.2.5$: Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = +1$ shifts the graph to the left, that is, towards negative values of x.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Horizontal Shift

Given a function f, a new function $g(x) = f(x-c)$, where c is a constant, is a **horizontal shift** of the function f. If c is positive, the graph will shift right. If c is negative, the graph will shift left.

Example $Page 3.2.2$

Returning to our building airflow example from Figure $Page 3.2.3$, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$

$$F(t) = \text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure $Page 3.2.6$. Notice also that the vents first opened to 220ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $c = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$.

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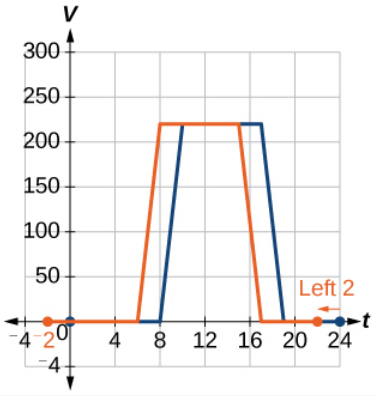


Figure $Page 3.2.6$

Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F: F(t)=V(t+2)$.

Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table $Page 3.2.2$.

Table $Page 3.2.2$

t	0	8	10	17	19	24
$t+2$	-2	6	8	15	17	22
$V(t)$	0	0	220	220	0	0
$S(t)$	0	0	220	220	0	0

Example $Page 3.2.3$

A graph of $f(x)$ is shown in Figure $Page 3.2.7$ which represents a transformation of the basic function $y=x^2$. Find an equation for $f(x)$.

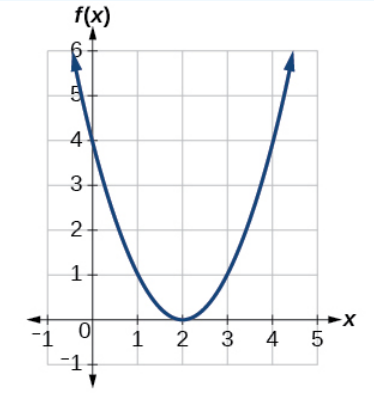


Figure $Page 3.2.7$: Graph of a parabola

Solution

Notice that the graph is identical in shape to the $f(x)=x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0,0)$, but now the vertex is at $(2,0)$. The graph is the basic quadratic function shifted 2 units to the right, so

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$$g(x) = f(x-2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x-2) \\ g(x) &= f(x-2) = (x-2)^2 \end{aligned}$$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x-2) = f(0) = 0$. For this to work, we will need to subtract 2 units from our input values.

Example $Page 3.2.4$

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)+10$ and $G(m+10)$

Solution

$G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

You Try $Page 3.2.1$

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x+2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Answer

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

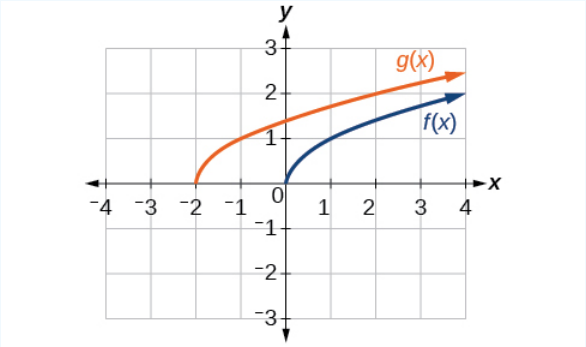


Figure $Page 3.2.8$

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the y-coordinates of points and shift the function up or down. Horizontal shifts are inside changes that affect the x-coordinates of points and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

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Example $Page 3.2.5$

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x+1) - 3$.

Solution

The function f is our basic absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure $Page 3.2.9$.

Let us follow one point of the graph of $f(x) = |x|$.

- The point $(0, 0)$ is transformed first by shifting left 1 unit:$(0, 0) \rightarrow (-1, 0)$
- The point $(-1, 0)$ is transformed next by shifting down 3 units:$(-1, 0) \rightarrow (-1, -3)$

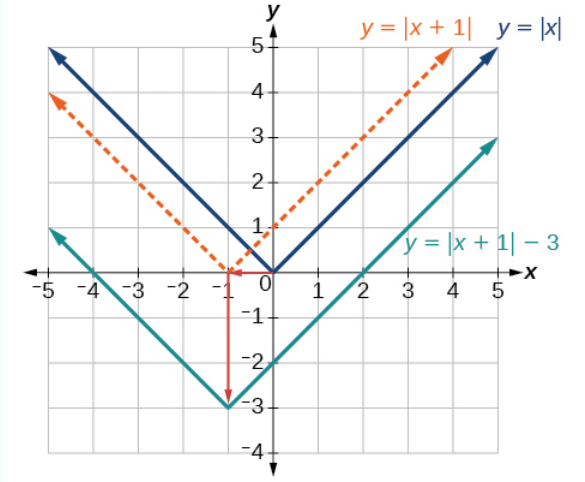


Figure $Page 3.2.9$: Graph of an absolute function, $y = |x|$, and how it was transformed to $y = |x+1| - 3$

Figure $Page 3.2.10$ shows the graph of h.

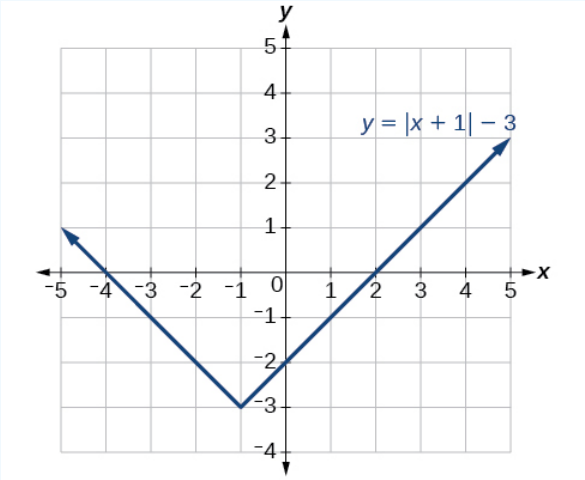


Figure $Page 3.2.10$: The final function $y = |x+1| - 3$.

You Try $Page 3.2.2$

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x-2) + 4$.

Answer

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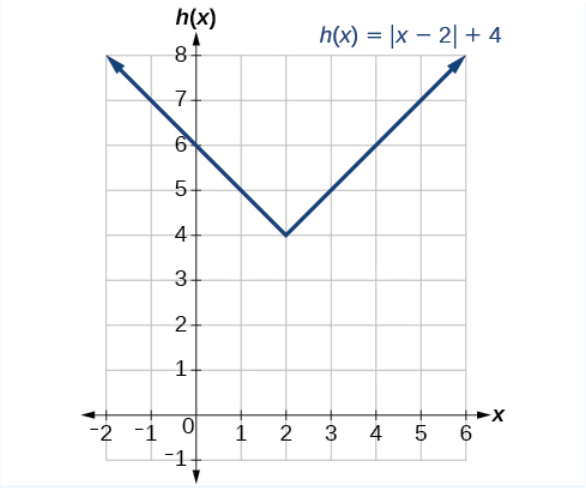


Figure $Page 3.2.11$

Example $Page 3.2.6$

Write an equation for the function, $h(x)$ that models the graph shown in Figure $Page 3.2.12$, which is a transformation of the basic square root function.

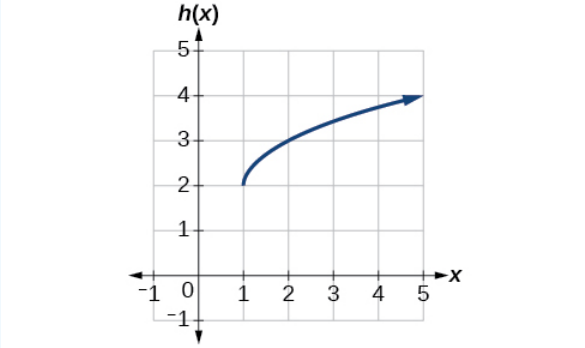


Figure $Page 3.2.12$: Graph of a square root function transposed right one unit and up 2.

Solution

The graph of the basic function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1)+2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1}+2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure $Page 3.2.13$.

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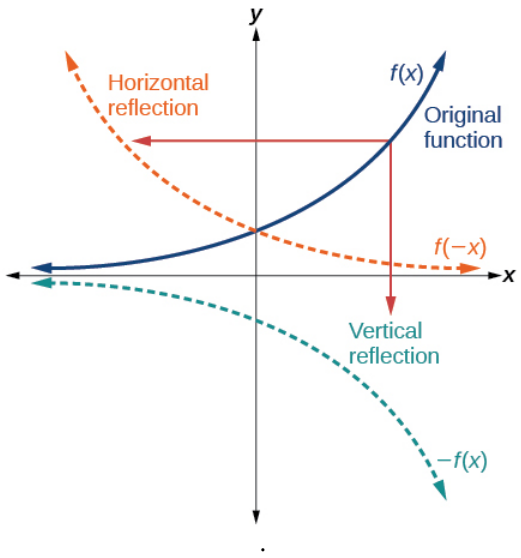


Figure $Page 3.2.13$: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. As a result, vertical reflections are also called **x-axis reflections**. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis. As a result, horizontal reflections are also called **y-axis reflections**.

Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

Example $Page 3.2.7$

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure $Page 3.2.14$.

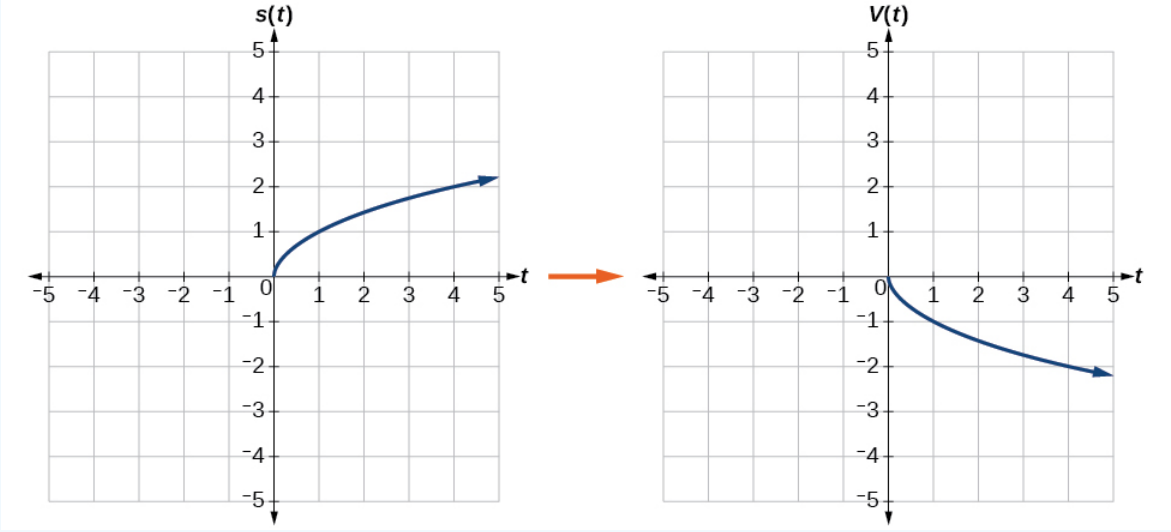


Figure $Page 3.2.14$: Graph of the vertical reflection of the square root function.

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Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure $Page 3.2.15$.

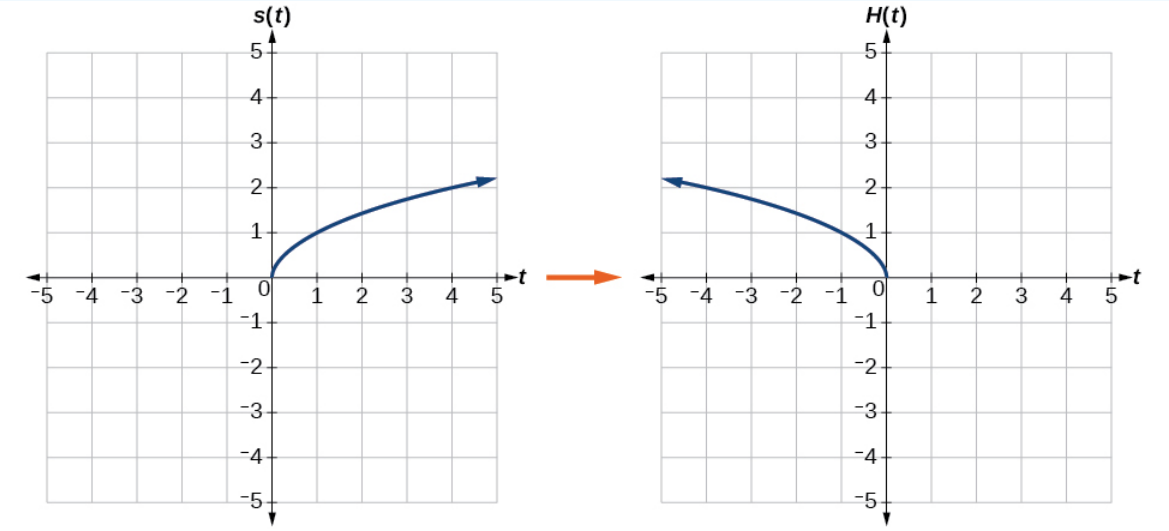


Figure $Page 3.2.15$: Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

You Try $Page 3.2.3$

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

Answer

a.

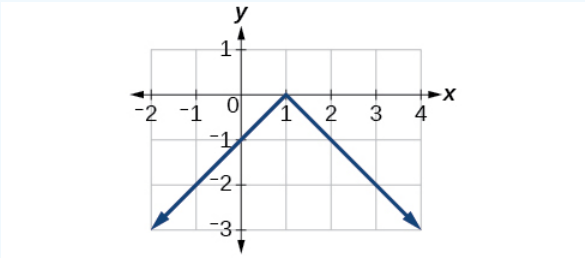


Figure $Page 3.2.16$: Graph of a vertically reflected absolute function.

The function that describes this graph is $g(x) = -|x - 1|$

b.

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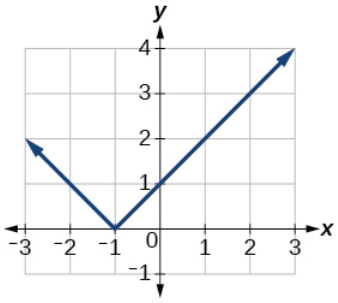


Figure $Page 3.2.17$: Graph of an absolute function translated one unit left.

The function that describes this graph is $h(x) = |-(x+1)|$ or $h(x) = |-x-1|$

Example $Page 3.2.8$

A function $f(x)$ is given as Table $Page 3.2.3$. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table $Page 3.2.3$

x	2	4	6	8
$f(x)$	1	3	7	11

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table $Page 3.2.4$.

Table $Page 3.2.4$

x	2	4	6	8
$g(x)$	-1	-3	-7	-11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table $Page 3.2.5$.

Table $Page 3.2.5$

x	-2	-4	-6	-8
$h(x)$	1	3	7	11

You Try $Page 3.2.4$

Given the basic function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$.

Answer

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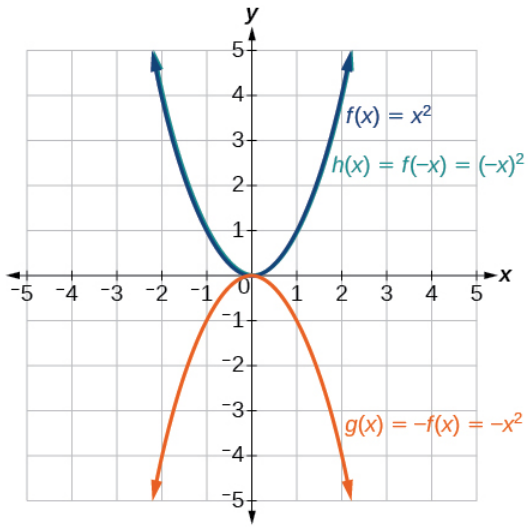


Figure $Page 3.2.18$: Graph of x^2 and its reflections.

Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure $Page 3.2.19$ shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

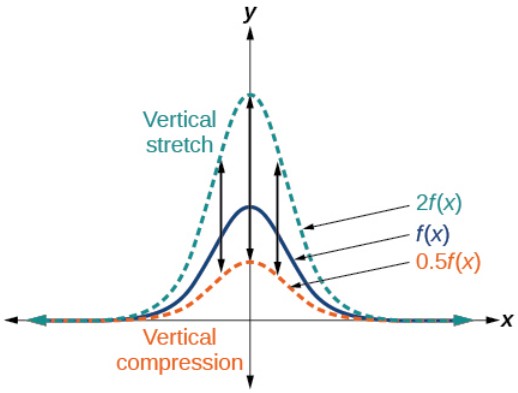


Figure $Page 3.2.19$: Vertical stretch and compression

Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched vertically.
- If $0 < a < 1$, then the graph will be compressed vertically.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

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Example $Page 3.2.9$

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure $Page 3.2.20$.

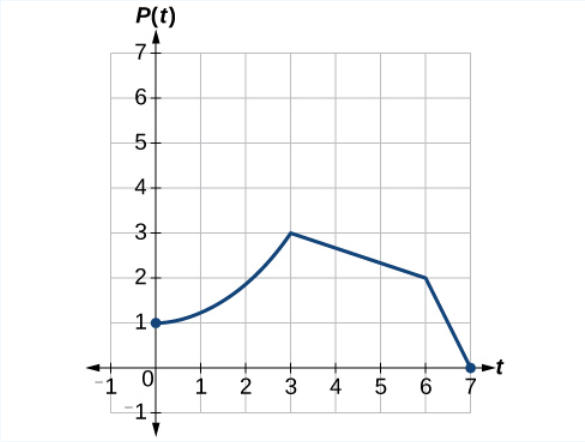


Figure $Page 3.2.20$: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure $Page 3.2.21$.

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \rightarrow (0, 2)$$

$$(3, 3) \rightarrow (3, 6)$$

$$(6, 2) \rightarrow (6, 4)$$

$$(7, 0) \rightarrow (7, 0)$$

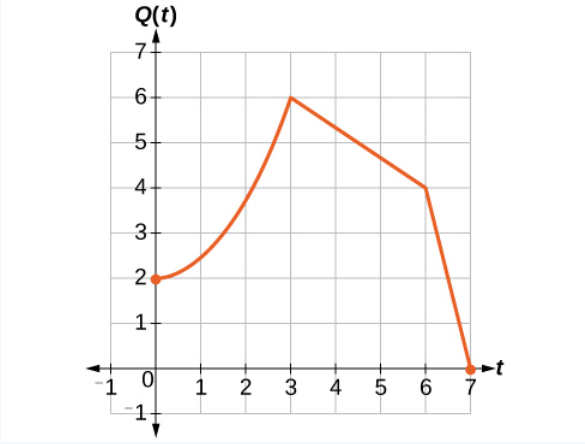


Figure $Page 3.2.21$: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t,

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stay the same while the output values are twice as large as before.

Example $Page 3.2.10$

Write an equation for the function, $g(x)$ that models the graph shown in Figure $Page 3.2.22$, which is a transformation of the basic cubic function.

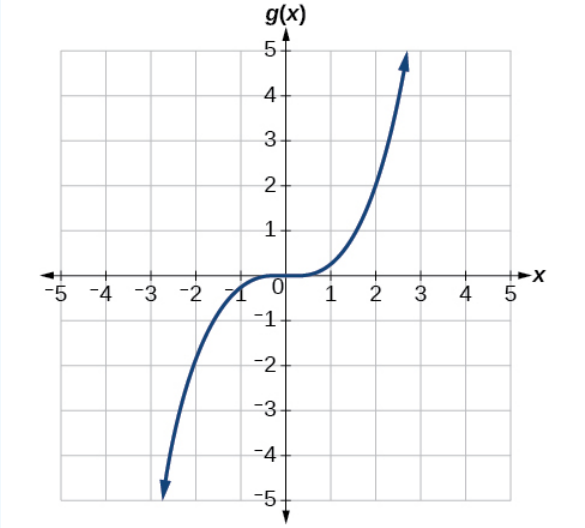


Figure $Page 3.2.22$: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4} f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4} f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4} f(x) = \frac{1}{4} x^3.$$

You Try $Page 3.2.5$

Write an equation for a function that is a stretch of the basic cubic function vertically by a factor of 4, shifted right 5 units, and then shifted down by 2 units.

Answer

$g(x) = 4(x-5)^3 - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

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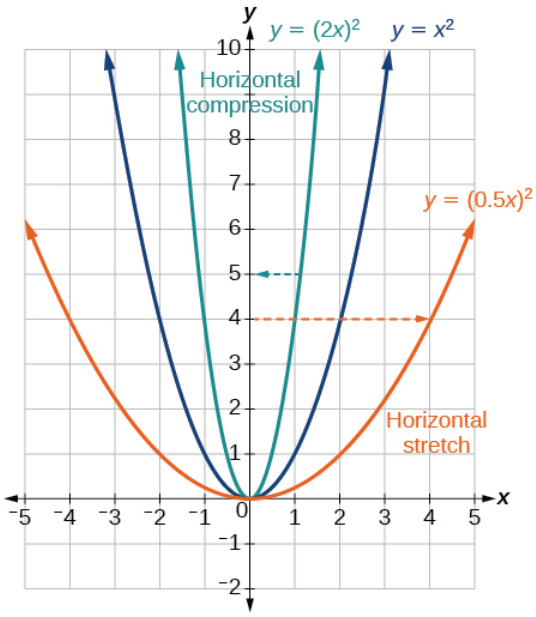


Figure $Page 3.2.23$: Graph of the vertical stretch and compression of x^2.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure $Page 3.2.23$. The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed horizontally by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched horizontally by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

Example $Page 3.2.11$

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$\begin{aligned} R(1) &= P(2), \\ R(2) &= P(4), \qquad \text{and in general,} \\ R(t) &= P(2t). \end{aligned}$$

See Figure $Page 3.2.24$ for a graphical comparison of the original population and the compressed population.

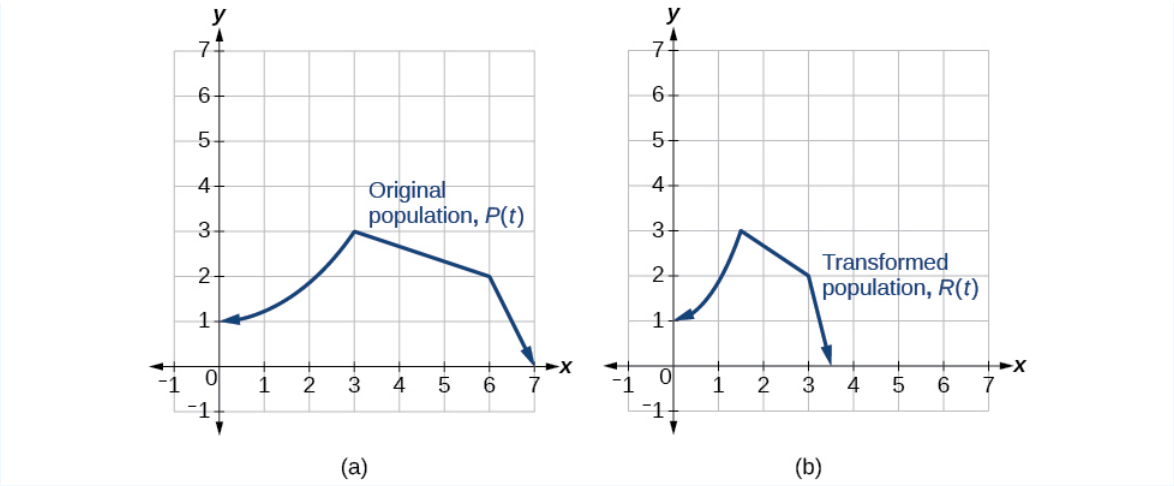


Figure $Page 3.2.24$: (a) Original population graph (b) Compressed population graph

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example $Page 3.2.12$

Use transformations to describe the function $g(x)$ in terms of $f(x)$ in Figure $Page 3.2.25$.

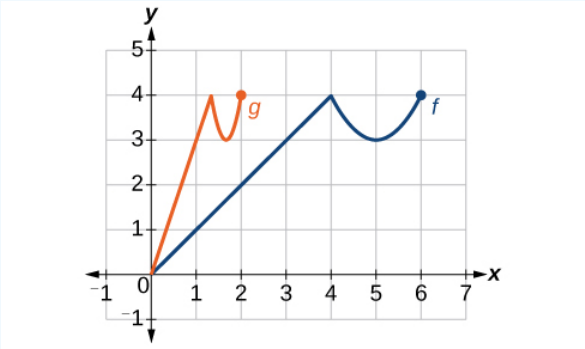


Figure $Page 3.2.25$: Graph of $f(x)$ being vertically compressed to $g(x)$.

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

You Try $Page 3.2.6$

Write a formula for the basic square root function horizontally stretched by a factor of 3.

Answer

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

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Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about sometimes. When we write $g(x)=f(2x+6)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. A way to identify the transformations is to factor inside the functionfirst to rewrite the function in a form that we can identify all the transformations, $g(x)=f(2x+6)=f(2(x+3)$. Function $g(x)$ is a horizontal compression of $f(x)$ by 2 and a horizontal shifting of $f(x)$ to the left by 3.

Let's work through an example.

$$f(x)=(3x-12)^2$$

We can factor out a 3.

$$f(x)=(3(x-4))^2$$

Now we can more clearly observe a horizontal shift to the right 4 units and a horizontal compression by 3. Factoring in this way allows us to horizontally compress first and then shift horizontally.

Combining Transformations

- When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h. If $b<0$ there is also a horizontal reflection over the y-axis.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h. If $b<0$ there is also a horizontal reflection over the y-axis.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical reflections are performed first.

Example $Page 3.2.13$

Use the graph of $f(x)$ in Figure $Page 3.2.26$ to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$.

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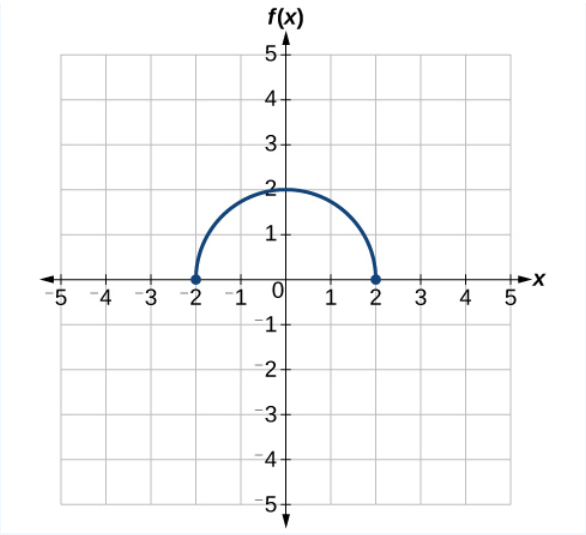


Figure $Page 3.2.26$: Graph of a half-circle.

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure $Page 3.2.27$.

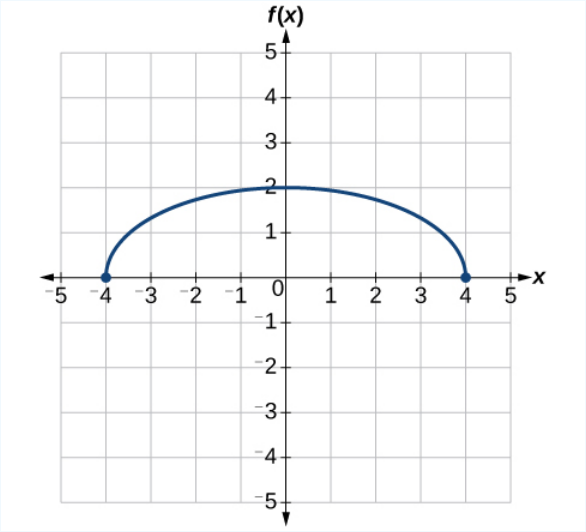


Figure $Page 3.2.27$: Graph of a vertically stretch half-circle.

Next, we horizontally shift left by 2 units, as indicated by $x+2$. See Figure $Page 3.2.28$.

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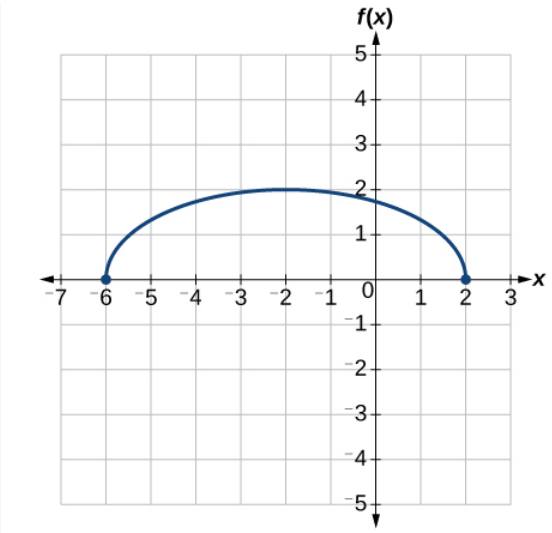


Figure $Page 3.2.28$: Graph of a vertically stretch and translated half-circle.

Last, we vertically shift down by 3 to complete our sketch, as indicated by the −3 on the outside of the function. See Figure $Page 3.2.29$.

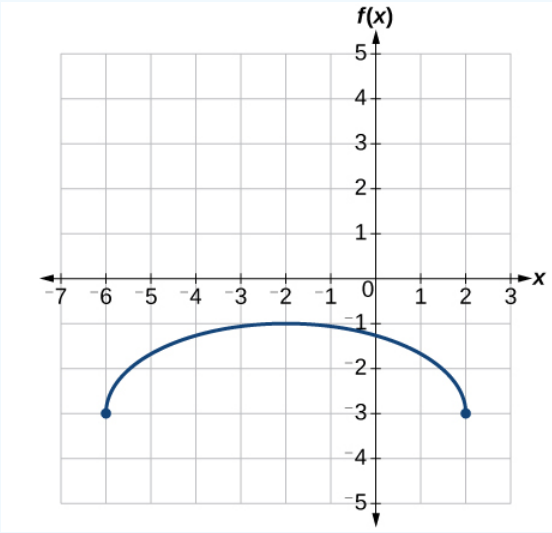


Figure $Page 3.2.29$: Graph of a vertically stretch and translated half-circle.

Example $Page 3.2.14$

Given the function $r(x) = \sqrt{x-3} - 5$,

1. Identify the transformations and use them to sketch the graph.
2. Based on what you found in part 1, how many x-intercepts do you expect there to be for this function? Explain.
3. Find the x-intercept(s) of $r(x)$.

Solution

1. The basic function shifts right 3 units and down 5 units.

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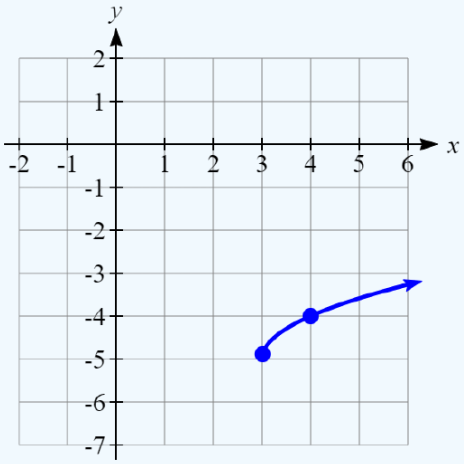


Figure $Page 3.2.30$: Graph of $r(x)$

2. It is expected that there is one x-intercept based on the transformations and the shape of the square root function. The x-intercept should have a fairly large x-coordinate, likely larger than 10, since this square root function increases fairly slowly.
3. Solving for the x-intercept:

$$\begin{aligned} 0 &= \sqrt{x-3}-5 \\ 5 &= \sqrt{x-3} \\ (5)^2 &= (\sqrt{x-3})^2 \\ 25 &= x-3 \\ 28 &= x \end{aligned}$$

The x-intercept is $(28, 0)$.

Example $Page 3.2.15$

Given the function $p(x) = -5(x-4)^2 + 2$,

1. Identify the transformations and use them to sketch the graph.
2. Based on what you found in part 1, how many x-intercepts do you expect there to be for this function? Explain.
3. Now, use your knowledge of solving equations to find the x-intercept(s) of $p(x)$ algebraically, if they exist.

Solution

1. The basic function shifts right 4 units and up 2 units. There is an x-axis reflection and a vertical stretch by 5.

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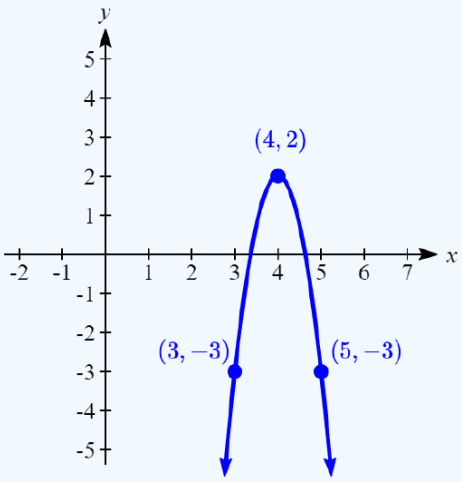


Figure $Page 3.2.31$: Graph of $p(x)$

1. Based on the transformations and shape of the basic graph, it is expected that there should be two x-intercepts with x-coordinates within the intervals $(3,4)$ and $(4,5)$.
2. Solving for the x-intercepts:

$$\begin{aligned} 0 &= -5(x-4)^2+2 \\ -2 &= -5(x-4)^2 \\ \frac{2}{5} &= (x-4)^2 \\ \pm\sqrt{\frac{2}{5}} &= \sqrt{(x-4)^2} \\ \pm\sqrt{\frac{2}{5}} &= x-4 \\ 4\pm\sqrt{\frac{2}{5}} &= x \end{aligned}$$

The x-intercepts are $\left(4-\sqrt{\frac{2}{5}},0\right)$ and $\left(4+\sqrt{\frac{2}{5}},0\right)$.

Note: $\left(4-\sqrt{\frac{2}{5}},0\right)\approx(3.37,0)$ and $\left(4+\sqrt{\frac{2}{5}},0\right)\approx(4.63,0)$ which follows what was expected visually.

? You Try $Page 3.2.7$

Given the function $k(x)=4(x+2)^3-3$:

1. Based on your knowledge of transformations, how many x-intercepts do you expect to have and where do you anticipate the x-intercept(s) to be within the coordinate system? Explain.
2. Now, use your knowledge of solving equations to find the x-intercept(s) of $k(x)$algebraically, if they exist.

Answer

1. This is a cubic function with a vertical stretch by 4 with a horizontal shift left 2 units and vertical shift down 3 units. Based on this knowledge, there should be one x-intercept with a x-coordinate within the interval $(-2,-1)$.
2. The x-intercept is $\left(-2+\sqrt[3]{\frac{3}{4}},0\right)$.

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Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the basic functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph, as shown in Figure $Page 3.2.32$.

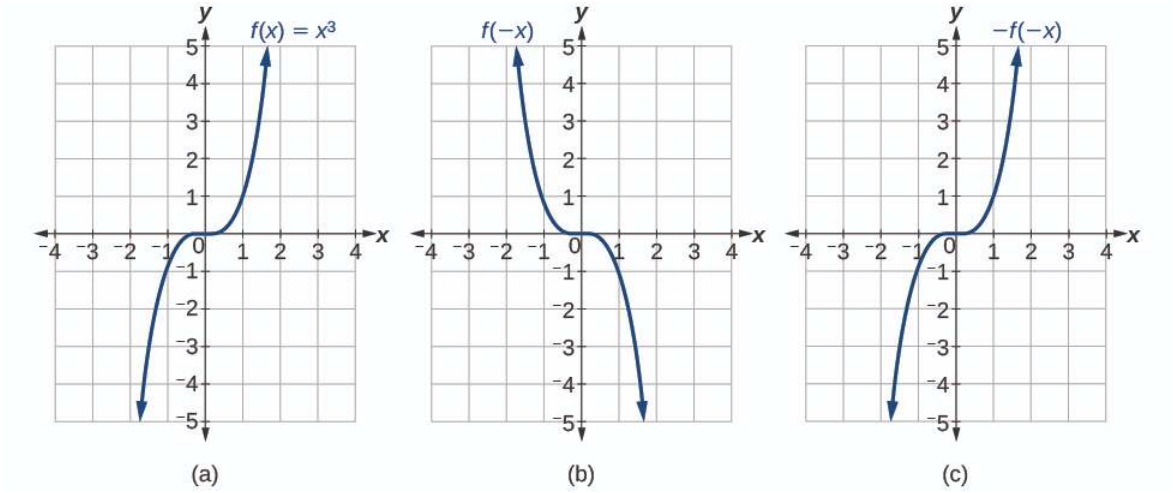


Figure $Page 3.2.32$: (a) The cubic basic function (b) Horizontal reflection of the cubic basic function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Even and Odd Functions

A function is called an **even function** if for every input x

$f(x) = f(-x)$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input x

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin.

Example $Page 3.2.16$

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

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Analysis

Consider the graph of f in Figure $Page 3.2.33$. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

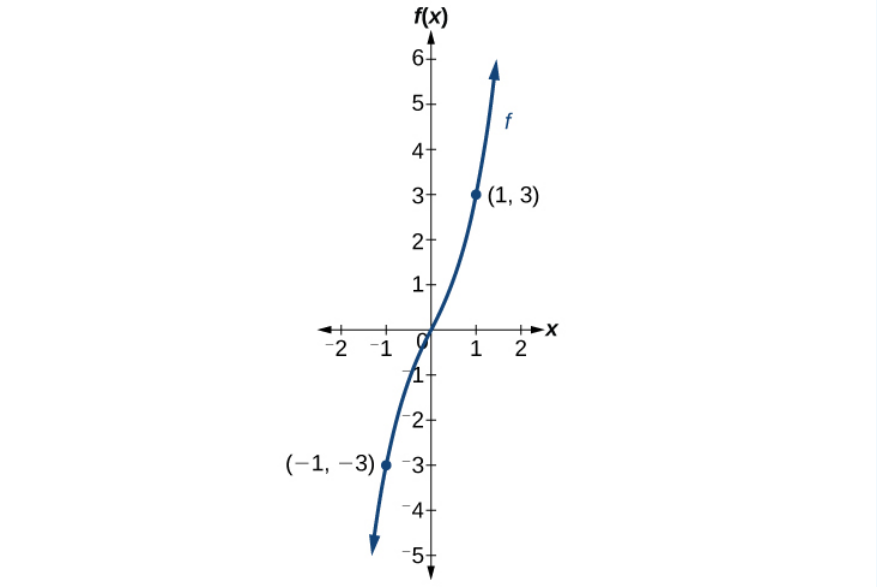


Figure $Page 3.2.33$: Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.

You Try $Page 3.2.8$

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Answer

even

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Page 3.3: More on Linear Functions

At this point in time, we should hopefully feel confident in working with functions. The goal now is to discuss linear functions in more depth. Although we did touch upon linear functions in Section 2.5, there is a lot more to note!

Recall that a linear function is a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = mx + b$ or $y = mx + b$. We refer to the form $y = mx + b$ is as the slope-intercept form (can you explain why?).

? Exercise $Page 3.3.1$

As a quick review exercise, take a few moments to graph a linear functions that meets the following conditions.

1) When the slope is positive and y-intercept is positive.

2) When the slope is positive and the y-intercept is negative.

3) When the slope is positive and the y-intercept is zero.

4) When the slope is negative and the y-intercept is positive.

5) When the slope is negative and the y-intercept is negative.

6) When the slope is negative and the y-intercept is zero.

7) When the slope is zero and the y-intercept is positive.

8) When the slope is zero and the y-intercept is negative.

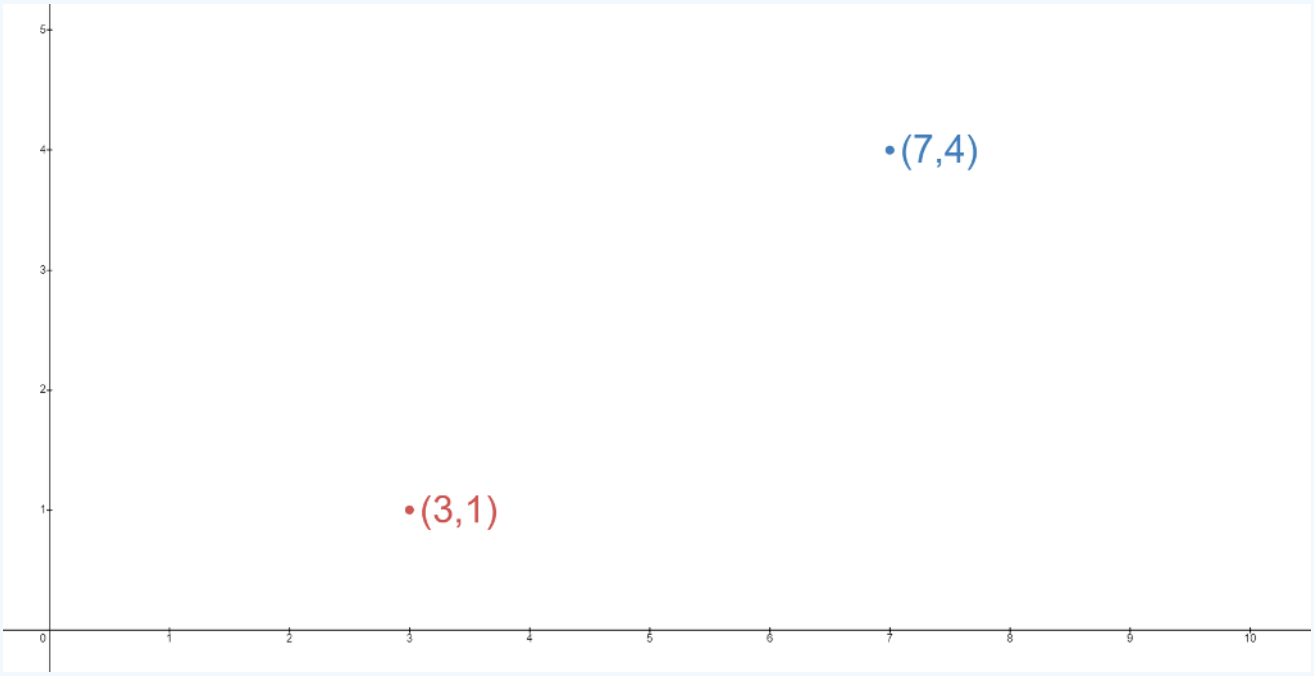
9) When the slope is zero and the y-intercept is zero.

Finding the Equation of a Line Given Two Points

? Exercise $Page 3.3.1$

Consider two points on the xy-plane as shown below:

How many linear lines can you draw which connects these two points?



Answer

We see that there is only one such linear line that connects these two points.

The question we will now consider is given two points, how can we find this single, unique, linear line, $y = mx + b$, that connects the two points? We see that to find the line, we must determine the value for m and the value for b.

Equation of a Linear Line

To find the linear line connecting two points, $(x_1, y_1), (x_2, y_2)$ we first:

1) Find the slope, m by computing $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{rise}{run} = \dfrac{\Delta y}{\Delta x}$.

2) With the value of m now known, we plug in one of the points (x_1, y_1) or (x_2, y_2) into $y = mx + b$ to find the value of b.

Exercise $Page 3.3.1$

Find the equation of the linear line that passes through the points $(3, 1)$ and $(7, 4)$.

Answer

1) We first compute m:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{7 - 3} = \frac{3}{4}$$

2) We now know that $m = \frac{3}{4}$ and we shall select any one of the given points, it does not matter which one we select. Suppose we selected $(3, 1)$. Plugging in m and the point $(3, 1)$ into the equation $y = mx + b$ yields:

$$y = mx + b$$

$$1 = \frac{3}{4}(3) + b$$

$$1 = \frac{9}{4} + b$$

$$-\frac{5}{4} = b$$

Hence $y = \frac{3}{4}x - \frac{5}{4}$.

To check our answer, we plug in the x-coordinate of both points to make sure we obtain the appropriate value of y.

Exercise $Page 3.3.1$

Find the equation of the line which passes through the points:

1) $(2, 1)$ and $(-1, -5)$.

2) $(-5, 7)$ and $(5, 9)$.

3) $(0, 3)$ and $(4, 15)$.

4) $(2, 7)$ and $(4, 7)$.

Answer

1) $y = 2x - 3$

2) $y = \frac{1}{5}x + 8$

3) $y = 3x + 3$

4) $y = 7$

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An important feature regarding linear lines is that given a linear line, regardless of which two (distinct) points are chosen, the slope will *always* be the same.

Finding the Equation of a Line Given a Point and a Slope

In the above discussion, we said that we can determine the equation of the linear line given two points on the line. However, we can also determine the equation of a linear line given a point and a slope. Essentially, this is just step 2 we are performing in the above recipe.

Example $Page 3.3.1$

Find the equation of the line passing through the point $(-1, 4)$ and whose slope is 10.

Solution 1

Since the equation of a line is given by $y = mx + b$ and we know that $m = 10$, then we have the following:

$$\begin{aligned} y &= mx + b \\ \rightarrow 4 &= 10(-1) + b \\ 4 &= -10 + b \\ 14 &= b \end{aligned}$$

Hence $y = 10x + 14$.

Alternatively, we can use what is called the point-slope form of a line which we will often seen in calculus.

Definition: Point-Slope Form

Definition: The **point-slope** form of a linear line is given by the following equation: $y - y_1 = m(x - x_1)$.

Although many students have seen the above equation, they are often unsure as to where it comes from. Imagine that we only have one point (x_1, y_1) and m. Recall that the slope formula is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Since we only have one point, allow us to drop the second index that appears in the above formula. Doing so gives us:

$$m = \frac{y - y_1}{x - x_1}$$

Multiplying both sides by $x - x_1$ yields

$$y - y_1 = m(x - x_1)$$

which is the point-slope formula.

Allow us to use the point-slope formula to answer the same question above:

Example $Page 3.3.1$

Find the equation of the line passing through the point $(-1, 4)$ and whose slope is 10.

Answer 2

From the point-slope formula, we have

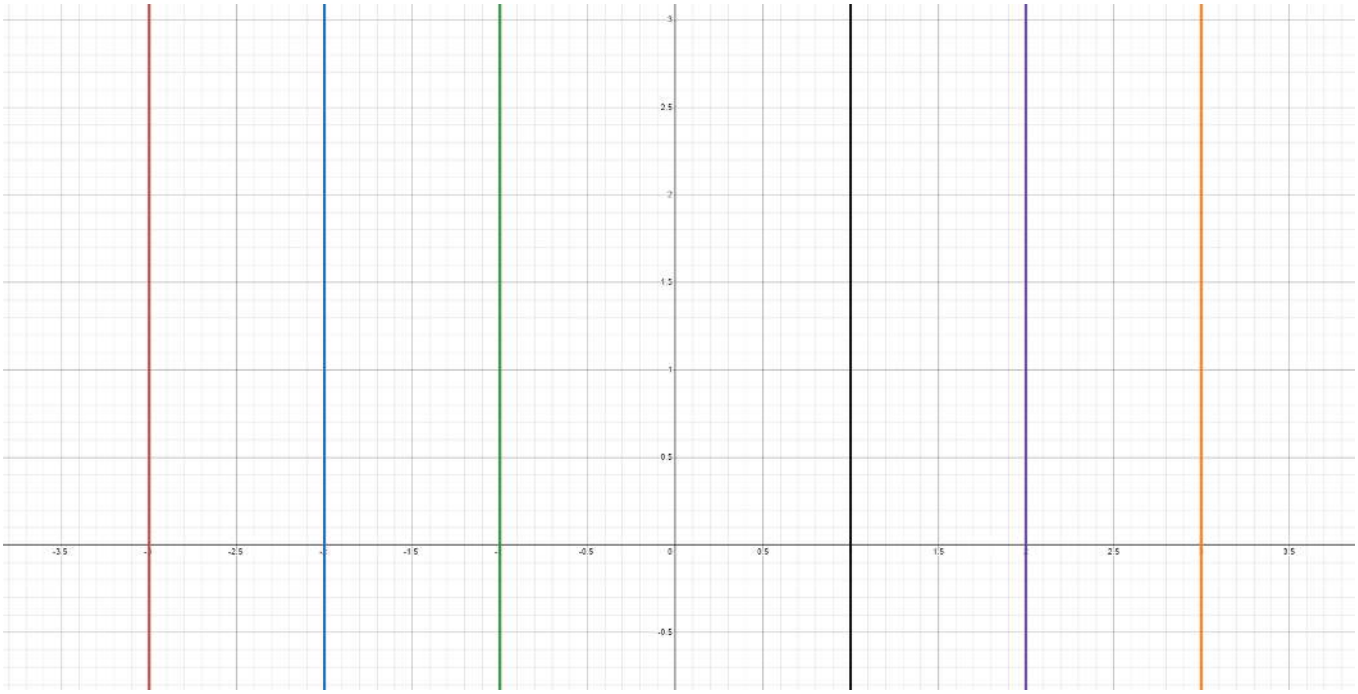
$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-4 &= 10\Big(x-(-1)\Big) \\ y-4 &= 10(x+1) \\ y-4 &= 10x+10 \\ y-4 &= 10x+10 \\ y &= 10x+14 \end{aligned}$$

Hence $y = 10x + 14$.

Finally, we end this section by discussing a particular type of line which is **not** a function. Recall that the line determined by $y = c$ is indeed a function since this is a horizontal line that passes through c on the y-axis. However, it's counterpart, which is a vertical line, is *not* a function.

Vertical Lines

The graph of the equation $x = a$ is a vertical line which passes through a on the x-axis.



Exercise $Page 3.3.1$

Select any two points on the line $x = 2$ and compute the slope. What do you notice?

Answer

Suppose we selected the points $(2, 1)$ and $(2, 5)$. We see that if we were to use the slope formula, we would obtain:

$$m = \frac{5-1}{2-2} = \frac{y_2-y_1}{x_2-x_1} = \frac{4}{0} = \text{not defined}$$

The same is true for any two points on a vertical line and so we note that the slope of any vertical line is not defined. (Note that some people say that the slope is infinite and the reason for this statement will be explained on Day 7).

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Page 3.4: Absolute Value Functions

Learning Objectives

- Graph an absolute value function.
- Solve an absolute value equation.

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy, some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right, at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate **absolute value functions**.



Figure $Page 3.4.1$: Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)

Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the **absolute value** function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

Absolute Value Function

The absolute value function can be defined as a piecewise function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \qquad \text{(Page 3.4.1)}$$

Example $Page 3.4.1$: Determine a Number within a Prescribed Distance

Describe all values x within or including a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as the one in , to represent the condition to be satisfied.

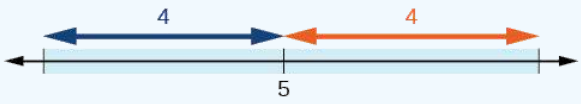


Figure $Page 3.4.2$: Number line describing the difference of the distance of 4 away from 5.

The distance from x to 5 can be represented using the absolute value as $|x-5|$. We want the values of x that satisfy the condition $|x-5| \leq 4$.

Analysis

Note that

$$\begin{aligned} -4 \leq x-5 \qquad & x-5 \leq 4 \\ 1 \leq x \qquad & x \leq 9 \end{aligned}$$

So $|x-5| \leq 4$ is equivalent to $1 \leq x \leq 9$.

However, mathematicians generally prefer absolute value notation.

? Exercise $Page 3.4.1$

Describe all values x within a distance of 3 from the number 2.

Answer

$|x-2| \leq 3$

✓ Example $Page 3.4.2$**: Resistance of a Resistor**

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece, even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often ±1%, ±5%, or ±10%.

Suppose we have a resistor rated at 680 ohms, ±5%. Use the absolute value function to express the range of possible values of the actual resistance.

Solution

5% of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance R in ohms,

$$|R-680| \leq 34$$

? Exercise $Page 3.4.2$

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.

Answer

Using the variable p for passing, $|p-80| \leq 20$

Graphing an Absolute Value Function

The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin in Figure $Page 3.4.3$.

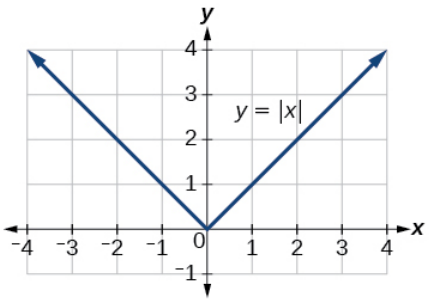


Figure $Page 3.4.3$: Graph of an absolute function.

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Figure $Page3.4.3$ shows the graph of $y = 2|x-3| + 4$. The graph of $y = |x|$ has been shifted right 3 units, vertically stretched by a factor of 2, and shifted up 4 units. This means that the corner point is located at $(3, 4)$ for this transformed function.

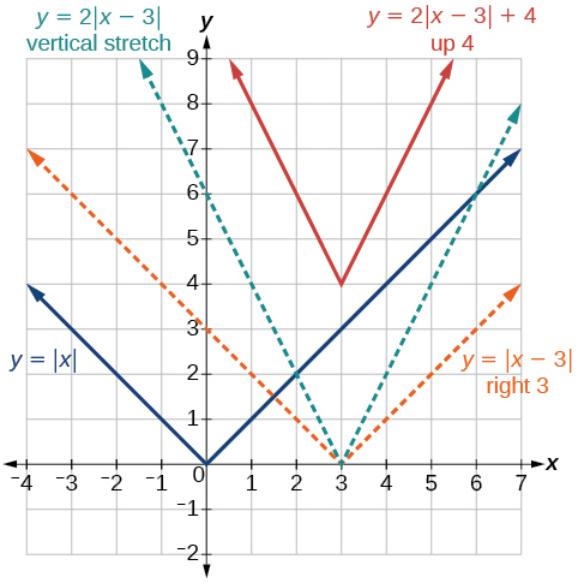


Figure $Page3.4.4$: Graph of the different types of transformations for an absolute function.

Example $Page3.4.3$: Writing an Equation for an Absolute Value Function

Write an equation for the function graphed in Figure $Page3.4.5$.

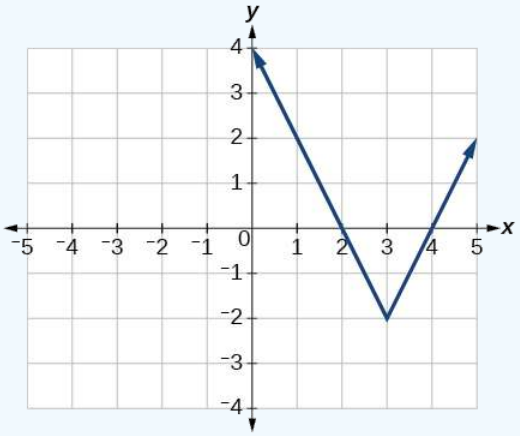


Figure $Page3.4.5$: Graph of an absolute function.

Solution

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function. See Figure $Page3.4.6$.

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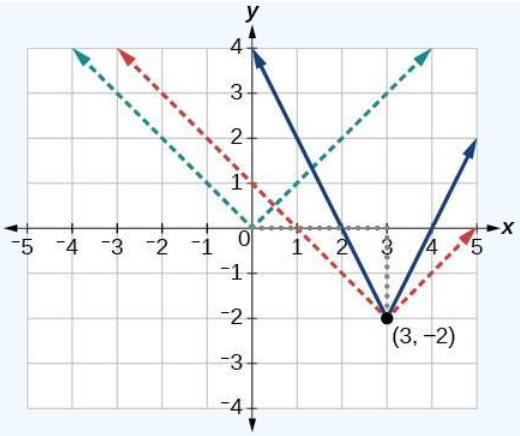


Figure $Page 3.4.6$: Graph of two transformations for an absolute function at $(3, -2)$.

We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance as shown in Figure $Page 3.4.7$.

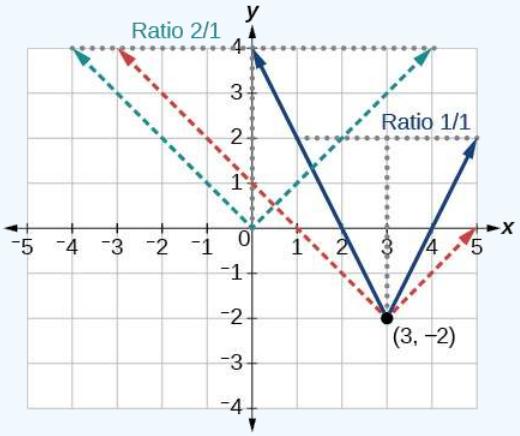


Figure $Page 3.4.7$: Graph of two transformations for an absolute function at $(3, -2)$ and the ratios between the two different transformations.

From this information we can write the equation

$$\begin{aligned} f(x) &= 2|x-3|-2, && \text{treating the stretch as a vertial stretch, or} \\ f(x) &= |2(x-3)|-2, && \text{treating the stretch as a horizontal compression.} \end{aligned}$$

Analysis

Note that these equations are algebraically equivalent—the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.

Q & A

If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?

Answer

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for x and $f(x)$.

$$f(x) = a|x-3|-2$$

Now substituting in the point $(1, 2)$

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$$2 = a|1-3|-2$$
$$4 = 2a$$
$$a = 2$$

? Exercise $Page 3.4.3$

Write the equation for the absolute value function that is horizontally shifted left 2 units, is vertically flipped, and vertically shifted up 3 units.

Answer

$f(x) = -|x+2|+3$

Q & A

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis?

Answer

Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.

No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis, depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points (Figure $Page 3.4.8$).

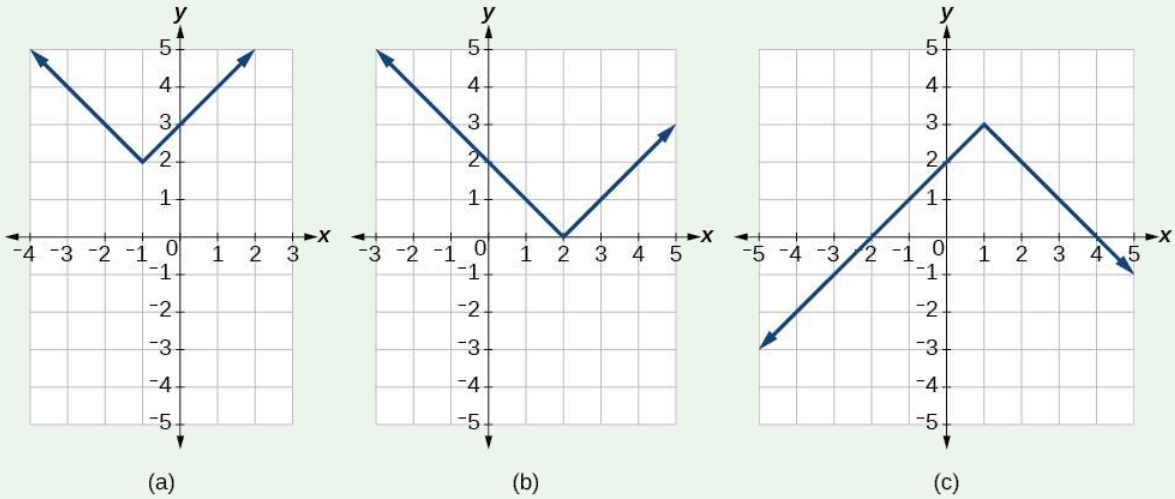


Figure $Page 3.4.8$: (a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

Solving an Absolute Value Equation

Now that we can graph an absolute value function, we will learn how to solve an absolute value equation. To solve an equation such as $8 = |2x-6|$, we notice that the absolute value will be equal to 8 if the quantity inside the absolute value is 8 or -8. This leads to two different equations we can solve independently.

$$2x-6 = 8 \quad \text{or} \quad 2x-6 = -8$$
$$2x = 14 \qquad 2x = -2$$
$$x = 7 \qquad x = -1$$

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

An **absolute value equation** is an equation in which the unknown variable appears in absolute value bars. For example,

$$|x| = 4,$$
$$|2x-1| = 3,$$
$$|5x+2| - 4 = 9.$$

Solutions to Absolute Value Equations

For real numbers A and B, an equation of the form $|A| = B$, with $B \geq 0$, will have solutions when $A = B$ or $A = -B$. If $B < 0$, the equation $|A| = B$ has no solution.

How To ...

Given the formula for an absolute value function, find the horizontal intercepts of its graph.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $-A = B$, assuming $B > 0$.
3. Solve for x.

Example $Page 3.4.4$: Finding the Zeros of an Absolute Value Function

For the function $f(x) = |4x+1| - 7$, find the values of x such that $f(x) = 0$.

Solution

$0 = \|4x+1\| - 7$		Substitute 0 for f(x).
$7 = \|4x+1\|$		Isolate the absolute value on one side of the equation.
$7 = 4x+1$	or $-7 = 4x+1$	Break into two separate equations and solve.
$6 = 4x$	$-8 = 4x$	
$x = \frac{6}{4} = 1.5$	$x = \frac{-8}{4} = -2$	

The function outputs 0 when $x = 1.5$ or $x = -2$ (Figure $Page 3.4.9$).

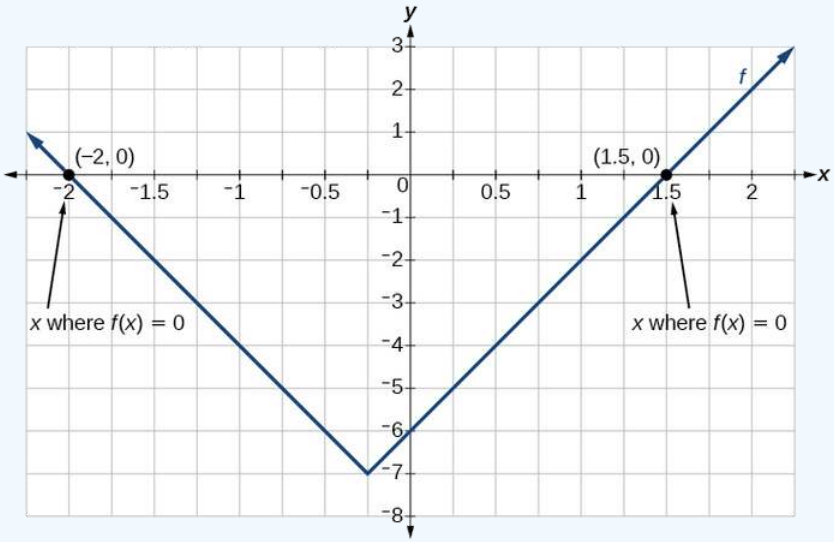


Figure $Page 3.4.9$: Graph of an absolute function with x-intercepts at -2 and 1.5.

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Exercise $Page3.4.4$

For the function $f(x) = |2x - 1| - 3$, find the values of x such that $f(x) = 0$.

Solution

$x = -1$ or $x = 2$

Q & A

Should we always expect two answers when solving $|A| = B$?

Answer

No. We may find one, two, or even no answers. For example, there is no solution to $2 + |3x - 5| = 1$.

How To ...

Given an absolute value equation, solve it.

1. Isolate the absolute value term.
2. Use $|A| = B$ to write $A = B$ or $A = -B$.
3. Solve for x.

Example $Page3.4.5$: Solving an Absolute Value Equation

Solve $1 = 4|x - 2| + 2$.

Solution

Isolating the absolute value on one side of the equation gives the following.

$$\begin{aligned} 1 &= 4|x - 2| + 2 \\ -1 &= 4|x - 2| \\ -\frac{1}{4} &= |x - 2| \end{aligned}$$

The absolute value always returns a positive value, so it is impossible for the absolute value to equal a negative value. At this point, we notice that this equation has no solutions.

Q & A

In Example $Page3.4.5$, if $f(x) = 1$ and $g(x) = 4|x - 2| + 2$ were graphed on the same set of axes, would the graphs intersect?

Answer

No. The graphs of f and g would not intersect, as shown in Figure $Page3.4.10$. This confirms, graphically, that the equation $1 = 4|x - 2| + 2$ has no solution.

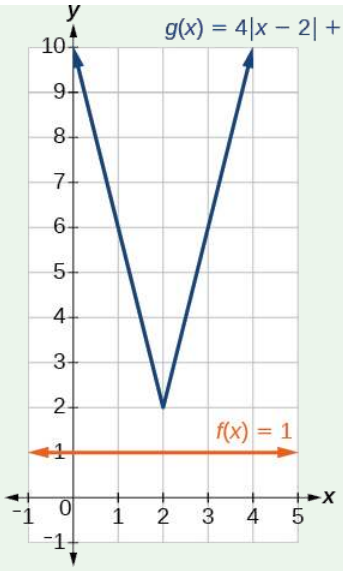


Figure $Page 3.4.10$: Graph of $g(x) = 4|x-2|+2$ and $f(x) = 1$.

Find where the graph of the function $f(x) = -|x+2|+3$ intersects the horizontal and vertical axes.

$f(0) = 1$, so the graph intersects the vertical axis at $(0, 1)$. $f(x) = 0$ when $x = -5$ and $x = 1$ so the graph intersects the horizontal axis at $(-5, 0)$ and $(1, 0)$.

Solving an Absolute Value Inequality

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values. We would use an **absolute value inequality** to solve such an equation. An absolute value inequality is an equation of the form

$$|A| < B,\ |A| \leq B, |A| > B,$$

or

$$|A| \geq B,$$

where an expression A (and possibly but not usually B) depends on a variable x. Solving the inequality means finding the set of all x that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two functions. The advantage of the algebraic approach is it yields solutions that may be difficult to read from the graph.

For example, we know that all numbers within 200 units of 0 may be expressed as

$$|x| < 200$$

or

$$-200 < x < 200$$

Suppose we want to know all possible returns on an investment if we could earn some amount of money within $200 of $600. We can solve algebraically for the set of values x such that the distance between x and 600 is less than 200. We represent the distance between x and 600 as $|x-600|$.

$$|x-600| < 200 \qquad (\text{Page } 3.4.2)$$

or

$$-200 < x-600 < 200 \qquad (\text{Page } 3.4.3)$$

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$$-200+600 < x-600+600 < 200+600$$
$$400 < x < 800$$

This means our returns would be between $400 and $800.

Sometimes an absolute value inequality problem will be presented to us in terms of a shifted and/or stretched or compressed absolute value function, where we must determine for which values of the input the function's output will be negative or positive.

How To ...

Given an absolute value inequality of the form $|x-A| \leq B$ for real numbers a and b where b is positive, solve the absolute value inequality algebraically.

1. Find boundary points by solving $|x-A| = B$.
2. Test intervals created by the boundary points to determine where $|x-A| \leq B$.
3. Write the interval or union of intervals satisfying the inequality in interval, inequality, or set-builder notation.

Example $Page 3.4.6$: Solving an Absolute Value Inequality

Solve $|x-5| \leq 4$.

Solution

With both approaches, we will need to know first where the corresponding equality is true. In this case we first will find where $|x-5| = 4$. We do this because the absolute value is a function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve $|x-5| = 4$.

$$\begin{aligned} x-5 &= 4 \quad \text{or} \quad x = 9 \\ x-5 &= -4 \quad\quad\;\; x = 1 \end{aligned}$$

After determining that the absolute value is equal to 4 at $x = 1$ and $x = 9$, we know the graph can change only from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:

$$x < 1,\ 1 < x < 9,\ \text{and}\ x > 9.$$

To determine when the function is less than 4, we could choose a value in each interval and see if the output is less than or greater than 4, as shown in Table $Page 3.4.1$.

Table $Page 3.4.1$

Interval test x	$f(x)$	< 4 or > 4	
$x < 1$	0	$\|0-5\| = 5$	Greater than
$1 < x < 9$	6	$\|6-5\| = 1$	Less than
$x > 9$	11	$\|11-5\| = 6$	Greater than

Because $1 \leq x \leq 9$ is the only interval in which the output at the test value is less than 4, we can conclude that the solution to $|x-5| \leq 4$ is $1 \leq x \leq 9$, or $[1, 9]$.

To use a graph, we can sketch the function $f(x) = |x-5|$. To help us see where the outputs are 4, the line $g(x) = 4$ could also be sketched as in Figure $Page 3.4.11$.

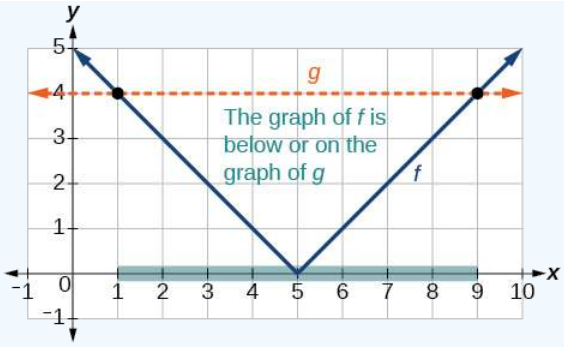


Figure $Page 3.4.11$: Graph to find the points satisfying an absolute value inequality.

We can see the following:

- The output values of the absolute value are equal to 4 at $x = 1$ and $x = 9$.
- The graph of f is below the graph of g on $1 < x < 9$. This means the output values of $f(x)$ are less than the output values of $g(x)$.
- The absolute value is less than or equal to 4 between these two points, when $1 \leq x \leq 9$. In interval notation, this would be the interval $[1, 9]$.

Analysis

For absolute value inequalities,

$$\begin{aligned} |x - A| < C, \qquad & |x - A| > C, \\ -C < x - A < C, \quad & x - A < -C \text{ or } x - A > C. \end{aligned}$$

The $<$ or $>$ symbol may be replaced by $\leq$ or $\geq$.

So, for this example, we could use this alternative approach.

$$\begin{aligned} |x - 5| &\leq 4 \\ -4 \leq x - 5 &\leq 4 && \text{Rewrite by removing the absolute value bars.} \\ -4 + 5 \leq x - 5 + 5 &\leq 4 + 5 && \text{Isolate the x.} \\ 1 \leq x &\leq 9 \end{aligned}$$

Exercise $Page 3.4.5$

Solve $|x + 2| \leq 6$.

Answer

$-8 \leq x \leq 4$

How To ...

Given an absolute value function, solve for the set of inputs where the output is positive (or negative).

1. Set the function equal to zero, and solve for the boundary points of the solution set.
2. Use test points or a graph to determine where the function's output is positive or negative.

Example $Page 3.4.7$: Using a Graphical Approach to Solve Absolute Value Inequalities

Given the function $f(x) = -\frac{1}{2}|4x - 5| + 3$, determine the x-values for which the function values are negative.

Solution

We are trying to determine where $f(x) < 0$, which is when $-\frac{1}{2}|4x - 5| + 3 < 0$. We begin by isolating the absolute value.

$$-\frac{1}{2}|4x-5| < -3 \quad \text{Multiply both sides by } -2, \text{ and reverse the inequality.}$$
$$|4x-5| > 6$$

Next we solve for the equality $|4x-5|=6$.

$$4x-5=6 \qquad 4x-5=-6$$
$$4x-6=6$$

or

$$4x=-1$$
$$x=\frac{11}{4} \qquad x=-\frac{1}{4}$$

Now, we can examine the graph of f to observe where the output is negative. We will observe where the branches are below the x-axis. Notice that it is not even important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x=-\frac{1}{4}$ and $x=\frac{11}{4}$ and that the graph has been reflected vertically. See Figure $Page 3.4.12$.

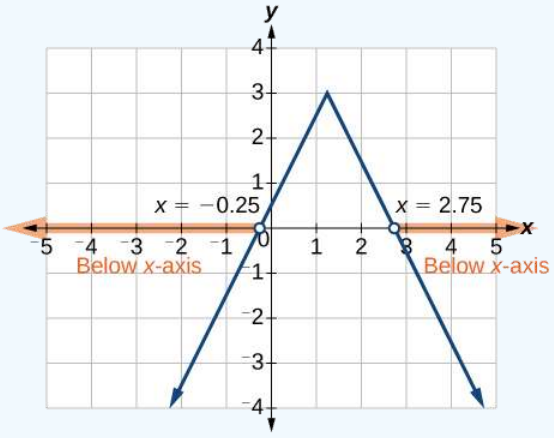


Figure $Page 3.4.12$: Graph of an absolute function with x-intercepts at -0.25 and 2.75.]

We observe that the graph of the function is below the x-axis left of $x=-\frac{1}{4}$ and right of $x=\frac{11}{4}$. This means the function values are negative to the left of the first horizontal intercept at $x=-\frac{1}{4}$, and negative to the right of the second intercept at $x=\frac{11}{4}$. This gives us the solution to the inequality.

$$x<-\frac{1}{4} \text{ or } x>1\frac{1}{4}$$

In interval notation, this would be $(-\infty,-0.25)\cup(2.75,\infty)$.

Exercise $Page 3.4.6$

Solve $-2|k-4|\leq -6$.

Answer

$k\leq 1$ or $k\geq 7$; in interval notation, this would be $(-\infty,1]\cup[7,\infty)$

Key Concepts

- The absolute value function is commonly used to measure distances between points.
- Applied problems, such as ranges of possible values, can also be solved using the absolute value function.
- The graph of the absolute value function resembles a letter V. It has a corner point at which the graph changes direction.
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable.
- An absolute value equation may have one solution, two solutions, or no solutions.

- An absolute value inequality is similar to an absolute value equation but takes the form | A |<B, | A |≤B, | A |>B, or | A |≥B.It can be solved by determining the boundaries of the solution set and then testing which segments are in the set.
- Absolute value inequalities can also be solved graphically.

Glossary

absolute value equation
an equation of the form $|A| = B$, with $B \geq 0$; it will have solutions when $A = B$ or $A = -B$

absolute value inequality
a relationship in the form $|A| < B$, $|A| \leq B$, $|A| > B$, or $|A| \geq B$

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Page 3.5: More on Quadratics

Learning Objectives

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.

Curved antennas, such as the ones shown in Figure $Page 3.5.1$, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

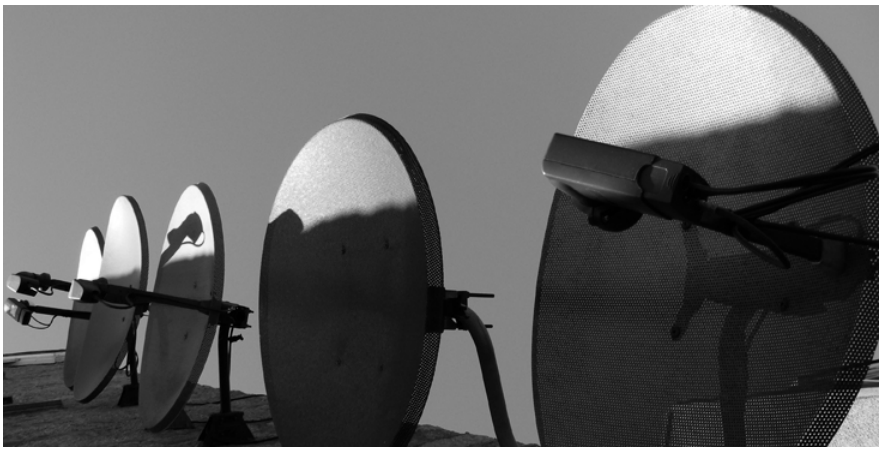


Figure $Page 3.5.1$: *An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)*

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure $Page 3.5.2$.

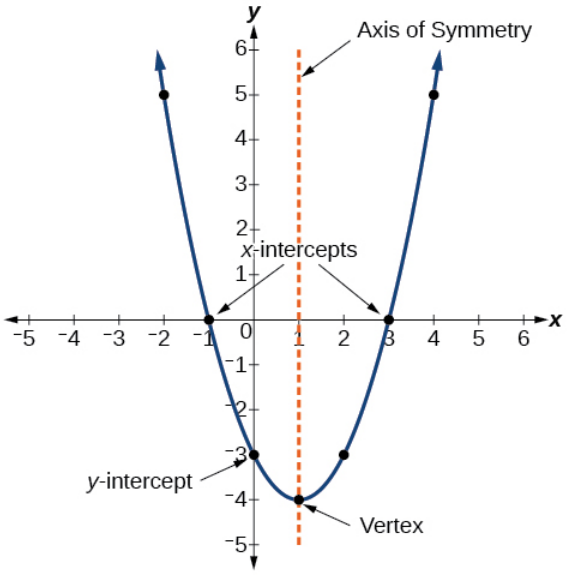


Figure $Page 3.5.2$: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are.

The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of x at which $y = 0$.

Example $Page 3.5.1$: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure $Page 3.5.3$.

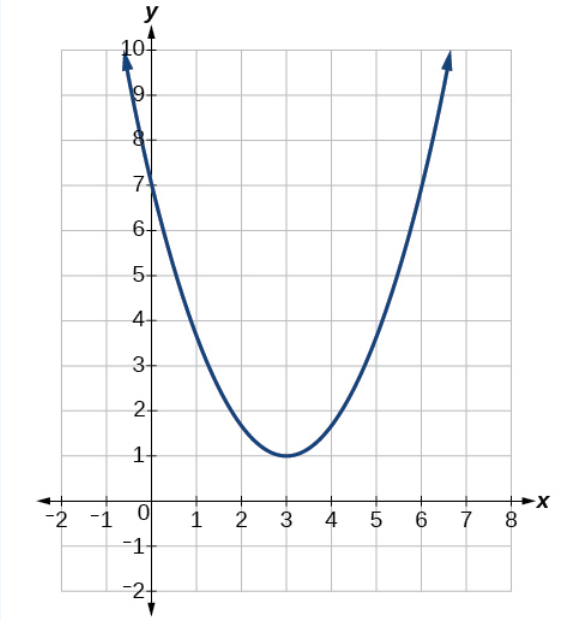


Figure $Page 3.5.3$.

Solution

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$ so this is the y-intercept.

Understanding How the Graphs of Parabolas are Related to Their Quadratic Functions

The **general form of a quadratic function** presents the function in the form

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$$f(x) = ax^2 + bx + c \qquad \text{(Page 3.5.1)}$$

where a, b, and c are real numbers and $a \neq 0$. If $a > 0$, the parabola opens upward. If $a < 0$, the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by $x = -\frac{b}{2a}$. If we use the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to solve $ax^2 + bx + c = 0$ for the x-intercepts, or zeros, we find the value of x halfway between them is always $x = -\frac{b}{2a}$, the equation for the axis of symmetry.

Figure $Page 3.5.4$ represents the graph of the quadratic function written in general form as $y = x^2 + 4x + 3$. In this form, $a = 1$, $b = 4$, and $c = 3$. Because $a > 0$, the parabola opens upward. The axis of symmetry is $x = -\frac{4}{2(1)} = -2$. This also makes sense because we can see from the graph that the vertical line $x = -2$ divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, $(-2, -1)$. The x-intercepts, those points where the parabola crosses the x-axis, occur at $(-3, 0)$ and $(-1, 0)$.

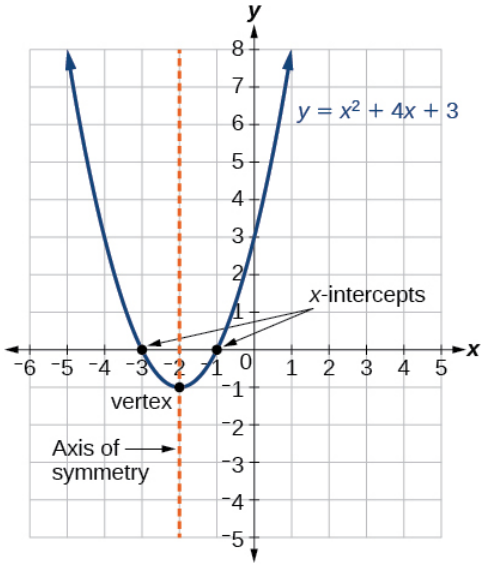


Figure $Page 3.5.4$: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are for the function $y = x^2 + 4x + 3$.

The **standard form of a quadratic function** presents the function in the form

$$f(x) = a(x - h)^2 + k \qquad \text{(Page 3.5.2)}$$

where (h, k) is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the **vertex form of a quadratic function**.

As with the general form, if $a > 0$, the parabola opens upward and the vertex is a minimum. If $a < 0$, the parabola opens downward, and the vertex is a maximum. Figure $Page 3.5.5$ represents the graph of the quadratic function written in standard form as $y = -3(x + 2)^2 + 4$. Since $x - h = x + 2$ in this example, $h = -2$. In this form, $a = -3$, $h = -2$, and $k = 4$. Because $a < 0$, the parabola opens downward. The vertex is at $(-2, 4)$.

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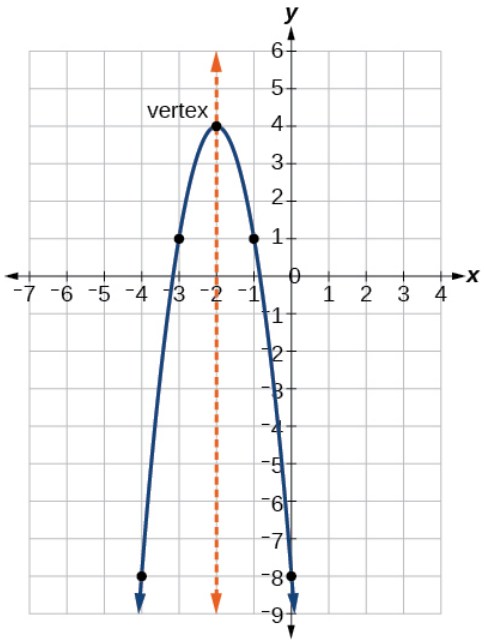


Figure $Page 3.5.5$: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are for the function $y=-3(x+2)^2+4$.

The standard form is useful for determining how the graph is transformed from the graph of $y=x^2$. Figure $Page 3.5.6$ is the graph of this basic function.

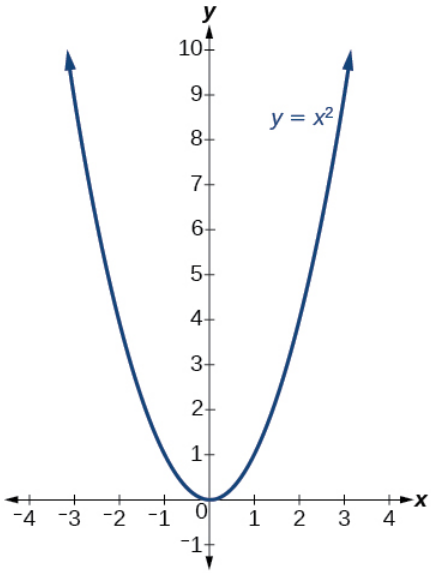


Figure $Page 3.5.6$: Graph of $y=x^2$.

If $k>0$, the graph shifts upward, whereas if $k<0$, the graph shifts downward. In Figure $Page 3.5.5$, $k>0$, so the graph is shifted 4 units upward. If $h>0$, the graph shifts toward the right and if $h<0$, the graph shifts to the left. In Figure $Page 3.5.5$, $h<0$, so the graph is shifted 2 units to the left. The magnitude of a indicates the stretch of the graph. If $|a|>1$, the point associated with a particular x-value shifts farther from the x-axis, so the graph appears to become narrower, and there is a vertical stretch. But if $|a|<1$, the point associated with a particular x-value shifts closer to the x-axis, so the graph appears to become wider, but in fact there is a vertical compression. In Figure $Page 3.5.5$, $|a|>1$, so the graph becomes narrower.

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.

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$$\begin{aligned} a(x-h)^2+k &= ax^2+bx+c \\ ax^2-2ahx+(ah^2+k) &= ax^2+bx+c \end{aligned}$$

For the linear terms to be equal, the coefficients must be equal.

$$-2ah=b, \text{ so } h=-\frac{b}{2a}.$$

This is the **axis of symmetry** we defined earlier. Setting the constant terms equal:

$$\begin{aligned} ah^2+k &= c \\ k &= c-ah^2 \\ &= c-a\cdot\left(-\frac{b}{2a}\right)^2 \\ &= c-\frac{b^2}{4a} \end{aligned}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $f(h)=k$.

Definitions: Forms of Quadratic Functions

A quadratic function is a function of degree two. The graph of a *quadratic function* is a parabola.

- The *general form* of a quadratic function is $f(x)=ax^2+bx+c$ where a, b, and c are real numbers and $a\neq 0$.
- The standard form of a quadratic function is $f(x)=a(x-h)^2+k$.
- The vertex (h,k) is located at

$$h=-\frac{b}{2a},\ k=f(h)=f(\frac{-b}{2a}).$$ (Page 3.5.3)

HOWTO: Write a quadratic function in a general form

Given a graph of a quadratic function, write the equation of the function in general form.

1. Identify the horizontal shift of the parabola; this value is h. Identify the vertical shift of the parabola; this value is k.
2. Substitute the values of the horizontal and vertical shift for h and k. in the function $f(x)=a(x-h)^2+k$.
3. Substitute the values of any point, other than the vertex, on the graph of the parabola for x and $f(x)$.
4. Solve for the stretch factor, $|a|$.
5. If the parabola opens up, $a>0$. If the parabola opens down, $a<0$ since this means the graph was reflected about the x-axis.
6. Expand and simplify to write in general form.

Example $Page 3.5.2$: Writing the Equation of a Quadratic Function from the Graph

Write an equation for the quadratic function g in Figure $Page 3.5.7$ as a transformation of $f(x)=x^2$, and then expand the formula, and simplify terms to write the equation in general form.

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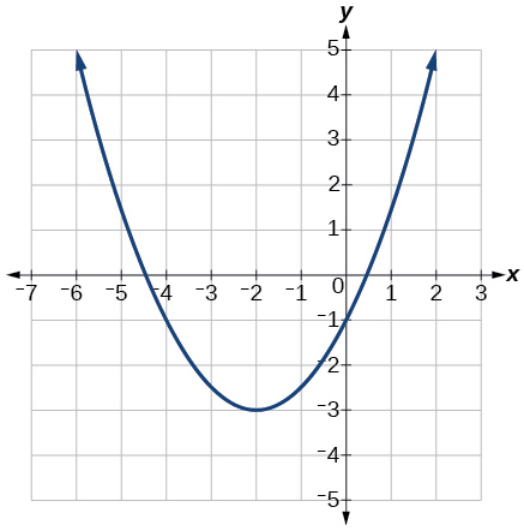


Figure $Page 3.5.7$: Graph of a parabola with its vertex at $(-2, -3)$.

Solution

We can see the graph of g is the graph of $f(x) = x^2$ shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$.

Substituting the coordinates of a point on the curve, such as $(0, -1)$, we can solve for the stretch factor.

$$-1 = a(0+2)^2 - 3 \quad (\text{Page } 3.5.4)$$
$$2 = 4a \quad (\text{Page } 3.5.5)$$
$$a = \frac{1}{2} \quad (\text{Page } 3.5.6)$$

In standard form, the algebraic model for this graph is $g(x) = \frac{1}{2}(x+2)^2 - 3$.

To write this in general polynomial form, we can expand the formula and simplify terms.

$$g(x) = \frac{1}{2}(x+2)^2 - 3 \quad (\text{Page } 3.5.7)$$
$$= \frac{1}{2}(x+2)(x+2) - 3 \quad (\text{Page } 3.5.8)$$
$$= \frac{1}{2}(x^2 + 4x + 4) - 3 \quad (\text{Page } 3.5.9)$$
$$= \frac{1}{2}x^2 + 2x + 2 - 3 \quad (\text{Page } 3.5.10)$$
$$= \frac{1}{2}x^2 + 2x - 1 \quad (\text{Page } 3.5.11)$$

Notice that the horizontal and vertical shifts of the basic graph of the quadratic function determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

Analysis

We can check our work using the table feature on a graphing utility. First enter $\mathrm{Y1} = \frac{1}{2}(\mathrm{x}+2)^2 - 3$. Next, select TBLSET, then use $\mathrm{TblStart} = -6$ and $\Delta\mathrm{Tbl} = 2$, and select TABLE. See Table $Page 3.5.1$

Table $Page 3.5.1$

x	-6	-4	-2	0	2
y	-5	-1	-3	-1	5

The ordered pairs in the table correspond to points on the graph.

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Exercise $Page3.5.2$

A coordinate grid has been superimposed over the quadratic path of a basketball in Figure $Page3.5.8$. Find an equation for the path of the ball. Does the shooter make the basket?

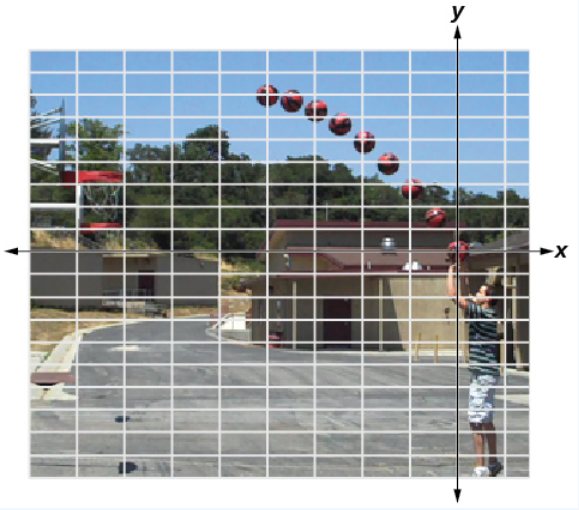


Figure $Page3.5.8$: *Stop motioned picture of a boy throwing a basketball into a hoop to show the parabolic curve it makes. (credit: modification of work by Dan Meyer)*

Answer

The path passes through the origin and has vertex at $(-4, 7)$, so $h(x) = -\frac{7}{16}(x+4)^2 + 7$. To make the shot, $h(-7.5)$ would need to be about 4 but $h(-7.5) \approx 1.64$; he doesn't make it.

How To: Given a quadratic function in general form, find the vertex of the parabola.

1. Identify a, b, and c.
2. Find h, the x-coordinate of the vertex, by substituting a and b into $h = -\frac{b}{2a}$.
3. Find k, the y-coordinate of the vertex, by evaluating $k = f(h) = f\left(-\frac{b}{2a}\right)$.

Example $Page3.5.3$**: Finding the Vertex of a Quadratic Function**

Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic in standard form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} \quad (Page\ 3.5.12)$$
$$= -\frac{-6}{2(2)} \quad (Page\ 3.5.13)$$
$$= \frac{6}{4} \quad (Page\ 3.5.14)$$
$$= \frac{3}{2} \quad (Page\ 3.5.15)$$

The vertical coordinate of the vertex will be at

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$$k = f(h) \quad \text{(Page 3.5.16)}$$
$$= f\left(\frac{3}{2}\right) \quad \text{(Page 3.5.17)}$$
$$= 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 7 \quad \text{(Page 3.5.18)}$$
$$= \frac{5}{2} \quad \text{(Page 3.5.19)}$$

Rewriting into standard form, the stretch factor will be the same as the a in the original quadratic.

$$f(x) = ax^2 + bx + c \quad \text{(Page 3.5.20)}$$
$$f(x) = 2x^2 - 6x + 7$$

Using the vertex to determine the shifts,

$$f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2} \quad \text{(Page 3.5.21)}$$

Analysis

One reason we may want to identify the vertex of the parabola is that this point will inform us what the maximum or minimum value of the function is, (k),and where it occurs, (h).

Exercise $Page 3.5.3$

Given the equation $g(x) = 13 + x^2 - 6x$, write the equation in general form and then in standard form.

Answer

$g(x) = x^2 - 6x + 13$ in general form; $g(x) = (x-3)^2 + 4$ in standard form.

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

Definition: Domain and Range of a Quadratic Function

The domain of any **quadratic function** is all real numbers.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $f(x) \geq f\left(-\frac{b}{2a}\right)$, or $[f(-\frac{b}{2a}), \infty)$; the range of a quadratic function written in general form with a negative a value is $f(x) \leq f(-\frac{b}{2a})$, or $(-\infty, f(-\frac{b}{2a})]$.

The range of a quadratic function written in standard form $f(x) = a(x-h)^2 + k$ with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

How To: Given a quadratic function, find the domain and range.

1. Identify the domain of any quadratic function as all real numbers.
2. Determine whether a is positive or negative. If a is positive, the parabola has a minimum. If a is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, k.
4. If the parabola has a minimum, the range is given by $f(x) \geq k$, or $[k, \infty)$. If the parabola has a maximum, the range is given by $f(x) \leq k$, or $(-\infty, k]$.

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Example $Page3.5.4$: Finding the Domain and Range of a Quadratic Function

Find the domain and range of $f(x) = -5x^2 + 9x - 1$.

Solution

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$h = -\frac{b}{2a} \quad (\text{Page } 3.5.22)$$
$$= -\frac{9}{2(-5)} \quad (\text{Page } 3.5.23)$$
$$= \frac{9}{10} \quad (\text{Page } 3.5.24)$$

The maximum value is given by $f(h)$.

$$f(\frac{9}{10}) = 5(\frac{9}{10})^2 + 9(\frac{9}{10}) - 1 \quad (\text{Page } 3.5.25)$$
$$= \frac{61}{20} \quad (\text{Page } 3.5.26)$$

The range is $f(x) \leq \frac{61}{20}$, or $\left(-\infty, \frac{61}{20}\right]$.

Exercise $Page3.5.4$

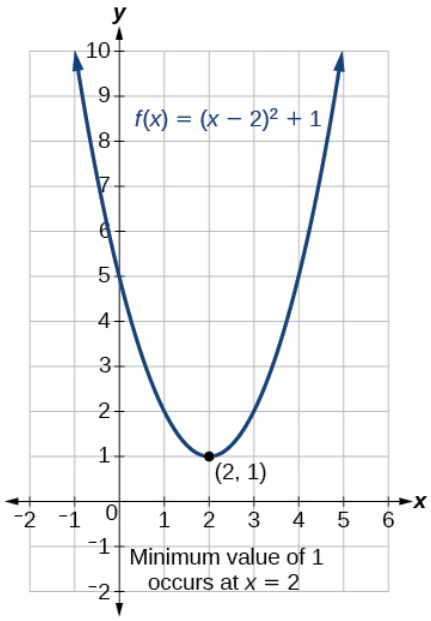
Find the domain and range of $f(x) = 2\left(x - \frac{4}{7}\right)^2 + \frac{8}{11}$.

Answer

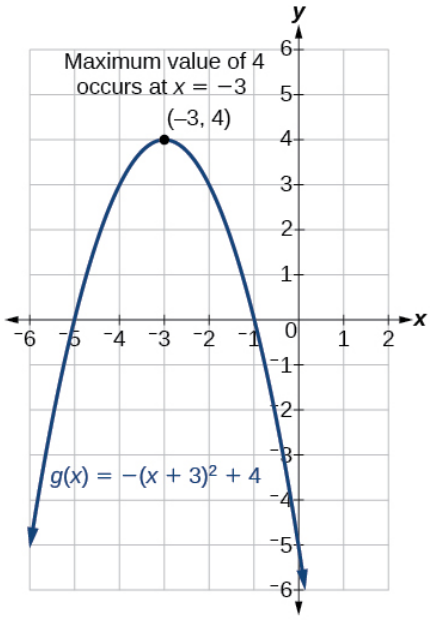
The domain is all real numbers. The range is $f(x) \geq \frac{8}{11}$, or $\left[\frac{8}{11}, \infty\right)$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the **parabola**. We can see the maximum and minimum values in Figure $Page3.5.9$.



(a)



(b)

Figure $Page 3.5.9$: Minimum and maximum of two quadratic functions.

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Example $Page 3.5.5$: Finding the Maximum Value of a Quadratic Function

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
b. What dimensions should she make her garden to maximize the enclosed area?

Solution

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Let's use a diagram such as Figure $Page 3.5.10$ to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

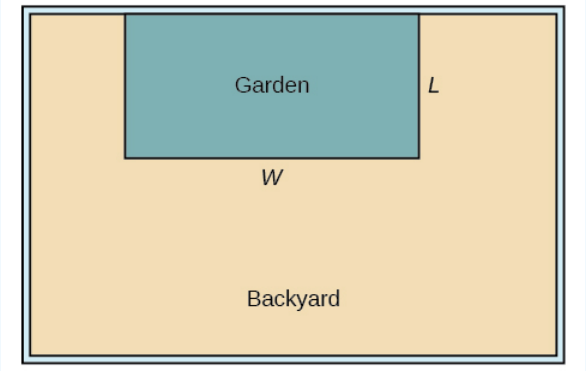


Figure $Page 3.5.10$: Diagram of the garden and the backyard.

a. We know we have only 80 feet of fence available, and $L+W+L=80$, or more simply, $2L+W=80$. This allows us to represent the width, W, in terms of L.

$$W=80-2L \qquad \text{(Page 3.5.27)}$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A = LW = L(80-2L) \qquad \text{(Page 3.5.28)}$$
$$A(L) = 80L-2L^2 \qquad \text{(Page 3.5.29)}$$

This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is

$$A(L)=-2L^2+80L \qquad \text{(Page 3.5.30)}$$

.

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a=-2$, $b=80$, and $c=0$.

To find the vertex:

$$h=-\frac{80}{2(-2)} \qquad k=A(20) \qquad \text{(Page 3.5.31)}$$
$$=20 \qquad \text{and} \qquad =80(20)-2(20)^2 \qquad \text{(Page 3.5.32)}$$
$$=800 \qquad \text{(Page 3.5.33)}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L=20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure $Page 3.5.11$.

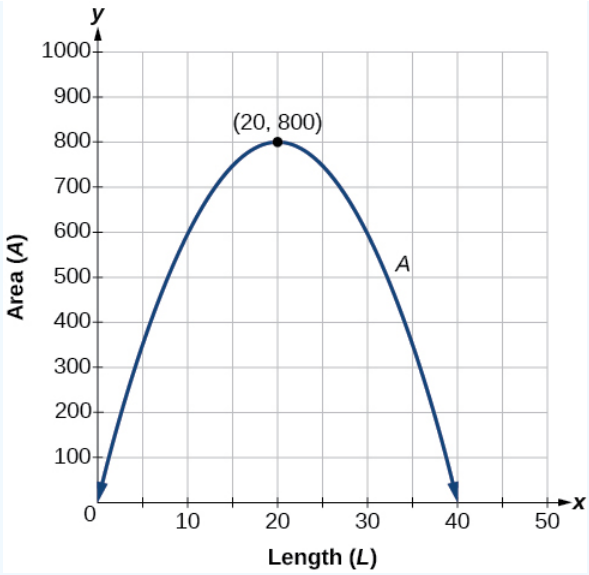


Figure $Page 3.5.11$: Graph of the parabolic function $A(L) = -2L^2 + 80L$

How To: Given an application involving revenue, use a quadratic equation to find the maximum.

1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.
3. Determine the y-value of the vertex.

Example $Page 3.5.6$: Finding Maximum Revenue

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of $30. Market research has suggested that if the owners raise the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, p for price per subscription and Q for quantity, giving us the equation $\text{Revenue} = pQ$.

Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $p = 30$ and $Q = 84,000$. We also know that if the price rises to $32, the newspaper would lose 5,000 subscribers, giving a second pair of values, $p = 32$ and $Q = 79,000$. From this we can find a linear equation relating the two quantities. The slope will be

$$m = \frac{79,000 - 84,000}{32 - 30} \quad \text{(Page 3.5.34)}$$

$$= -\frac{5,000}{2} \quad \text{(Page 3.5.35)}$$

$$= -2,500 \quad \text{(Page 3.5.36)}$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the y-intercept.

$$Q = -2500p + b \quad \text{Substitute in the point } Q = 84,000 \text{ and } p = 30 \quad \text{(Page 3.5.37)}$$

$$84,000 = -2500(30) + b \quad \text{Solve for } b \quad \text{(Page 3.5.38)}$$

$$b = 159,000 \quad \text{(Page 3.5.39)}$$

This gives us the linear equation $Q = -2,500p + 159,000$ relating cost and subscribers. We now return to our revenue equation.

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$$\text{Revenue} = pQ \quad \text{(Page 3.5.40)}$$
$$\text{Revenue} = p(-2,500p + 159,000) \quad \text{(Page 3.5.41)}$$
$$\text{Revenue} = -2,500p^2 + 159,000p \quad \text{(Page 3.5.42)}$$

We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.

$$h = -\frac{159,000}{2(-2,500)} \quad \text{(Page 3.5.43)}$$
$$= 31.8 \quad \text{(Page 3.5.44)}$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we evaluate the revenue function.

$$\text{maximum revenue} = -2,500(31.8)^2 + 159,000(31.8) \quad \text{(Page 3.5.45)}$$
$$= 2,528,100 \quad \text{(Page 3.5.46)}$$

Analysis

This could also be solved by graphing the quadratic as in Figure $Page 3.5.12$. We can see the maximum revenue on a graph of the quadratic function.

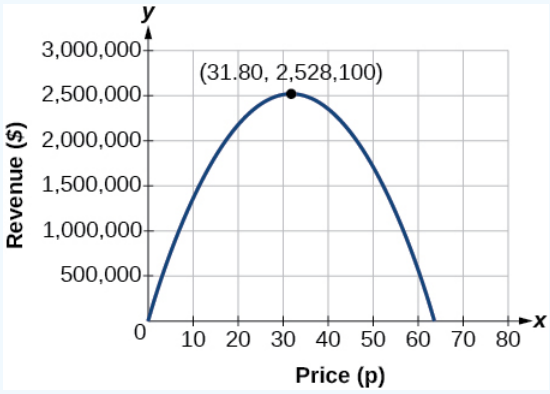


Figure $Page 3.5.12$: Graph of the parabolic function

Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure $Page 3.5.13$ that the number of x-intercepts can vary depending upon the location of the graph.

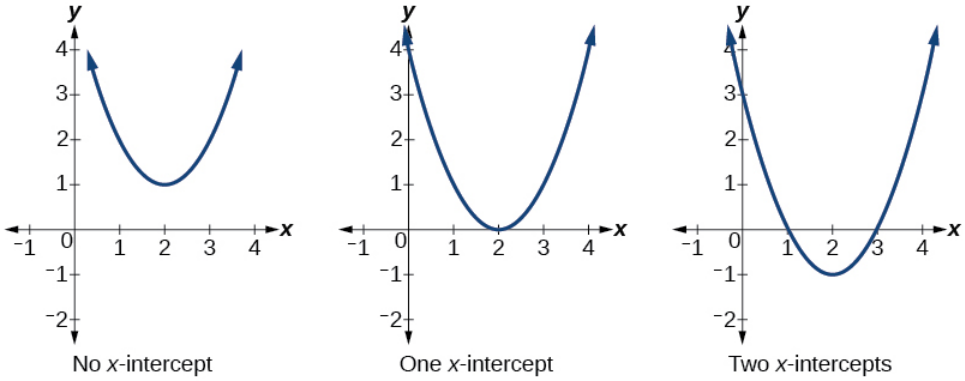


Figure $Page 3.5.13$: Number of x-intercepts of a parabola.

Given a quadratic function $f(x)$, find the y- and x-intercepts.

1. Evaluate $f(0)$ to find the y-intercept.
2. Solve the quadratic equation $f(x)=0$ to find the x-intercepts.

Example $Page3.5.7$: Finding the y- and x-Intercepts of a Parabola

Find the y- and x-intercepts of the quadratic $f(x)=3x^2+5x-2$.

Solution

We find the y-intercept by evaluating $f(0)$.

$$\begin{aligned} f(0) &= 3(0)^2+5(0)-2 && (Page\ 3.5.47)\\ &= -2 && (Page\ 3.5.48)\end{aligned}$$

So the y-intercept is at $(0,-2)$.

For the x-intercepts, we find all solutions of $f(x)=0$.

$$0=3x^2+5x-2 \qquad (Page\ 3.5.49)$$

In this case, the quadratic can be factored easily, providing the simplest method for solution.

$$0=(3x-1)(x+2) \qquad (Page\ 3.5.50)$$

$$\begin{aligned} 0 &= 3x-1 \qquad\qquad 0 = x+2 && (Page\ 3.5.51)\\ x &= \frac{1}{3} \qquad \text{or} \qquad x=-2 && (Page\ 3.5.52)\end{aligned}$$

So the x-intercepts are at $(\frac{1}{3},0)$ and $(-2,0)$.

Analysis

By graphing the function, we can confirm that the graph crosses the y-axis at $(0,-2)$. We can also confirm that the graph crosses the x-axis at $\left(\frac{1}{3},0\right)$ and $(-2,0)$. See Figure $Page3.5.14$.

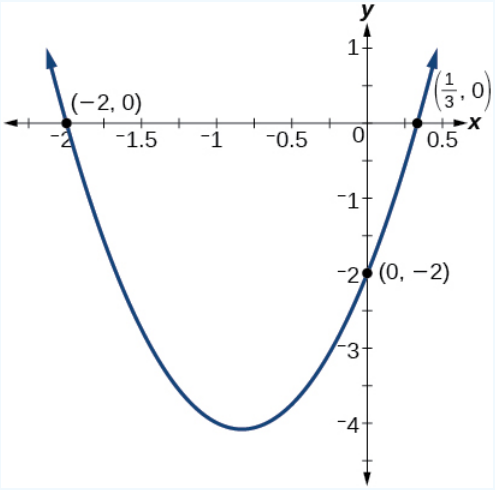


Figure $Page3.5.14$: Graph of a parabola.

Rewriting Quadratics in Standard Form

In Example $Page3.5.7$, the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

How to: Given a quadratic function, find the x-intercepts by rewriting in standard form.

1. Substitute a and b into $h=-\frac{b}{2a}$.
2. Substitute $x=h$ into the general form of the quadratic function to find k.
3. Rewrite the quadratic in standard form using h and k.

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4. Solve for when the output of the function will be zero to find the x-intercepts.

Example $Page3.5.8$: Finding the x-Intercepts of a Parabola

Find the x-intercepts of the quadratic function $f(x) = 2x^2 + 4x - 4$.

Solution

We begin by solving for when the output will be zero.

$$0 = 2x^2 + 4x - 4$$

Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.

$$f(x) = a(x-h)^2 + k$$

We know that $a = 2$. Then we solve for h and k.

$$\begin{aligned} h &= -\frac{b}{2a} & k &= f(-1) \\ &= -\frac{4}{2(2)} & &= 2(-1)^2 + 4(-1) - 4 \\ &= -1 & &= -6 \end{aligned}$$

So now we can rewrite in standard form.

$$f(x) = 2(x+1)^2 - 6$$

We can now solve for when the output will be zero.

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$.

Analysis

We can check our work by graphing the given function on a graphing utility and observing the x-intercepts. See Figure $Page3.5.15$.

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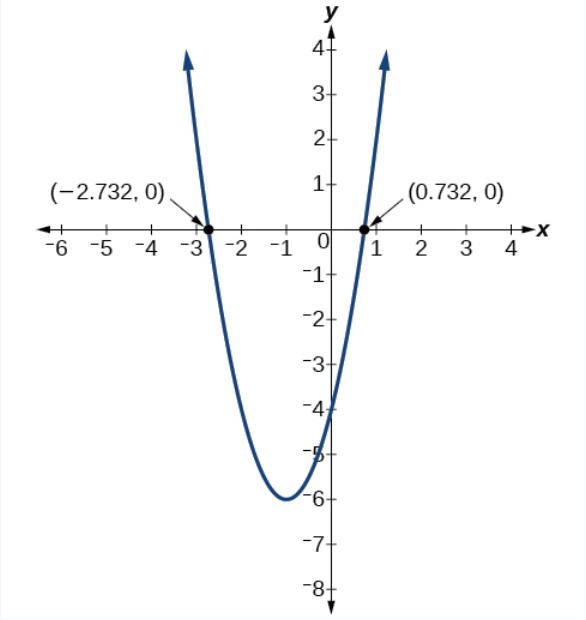


Figure $Page 3.5.15$: Graph of a parabola which has the following x-intercepts: $(-2.732, 0)$ and $(0.732, 0)$.

Exercise $Page 3.5.1$

In Try It $Page 3.5.1$, we found the standard and general form for the function $g(x) = 13 + x^2 - 6x$. Now find the y- and x-intercepts (if any).

Answer

y-intercept at $(0, 13)$, No x-intercepts

Example $Page 3.5.9$: Solving a Quadratic Equation with the Quadratic Formula

Solve $x^2 + x + 2 = 0$.

Solution

Let's begin by writing the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

When applying the quadratic formula, we identify the coefficients a, b and c. For the equation $x^2 + x + 2 = 0$, we have $a = 1$, $b = 1$, and $c = 2$. Substituting these values into the formula we have:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (2)}}{2 \cdot 1} \\ &= \frac{-1 \pm \sqrt{1 - 8}}{2} \\ &= \frac{-1 \pm \sqrt{-7}}{2} \\ &= \frac{-1 \pm i\sqrt{7}}{2} \end{aligned}$$

The solutions to the equation are $x = \frac{-1 + i\sqrt{7}}{2}$ and $x = \frac{-1 - i\sqrt{7}}{2}$ or $x = -\frac{1}{2} + \frac{i\sqrt{7}}{2}$ and $x = \frac{-1}{2} - \frac{i\sqrt{7}}{2}$.

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Example $Page 3.5.10$: Applying the Vertex and x-Intercepts of a Parabola

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

When does the ball reach the maximum height?
What is the maximum height of the ball?
When does the ball hit the ground?

The ball reaches the maximum height at the vertex of the parabola.

$$h = -\frac{80}{2(-16)} \quad (\text{Page } 3.5.53)$$
$$= \frac{80}{32} \quad (\text{Page } 3.5.54)$$
$$= \frac{5}{2} \quad (\text{Page } 3.5.55)$$
$$= 2.5 \quad (\text{Page } 3.5.56)$$

The ball reaches a maximum height after 2.5 seconds.

To find the maximum height, find the y-coordinate of the vertex of the parabola.

$$k = H\left(-\frac{b}{2a}\right) \quad (\text{Page } 3.5.57)$$
$$= H(2.5) \quad (\text{Page } 3.5.58)$$
$$= -16(2.5)^2 + 80(2.5) + 40 \quad (\text{Page } 3.5.59)$$
$$= 140 \quad (\text{Page } 3.5.60)$$

The ball reaches a maximum height of 140 feet.

To find when the ball hits the ground, we need to determine when the height is zero, $H(t) = 0$.

We use the quadratic formula.

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} \quad (\text{Page } 3.5.61)$$
$$= \frac{-80 \pm \sqrt{8960}}{-32} \quad (\text{Page } 3.5.62)$$

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458 \quad (\text{Page } 3.5.63)$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds. See Figure $Page 3.5.16$.

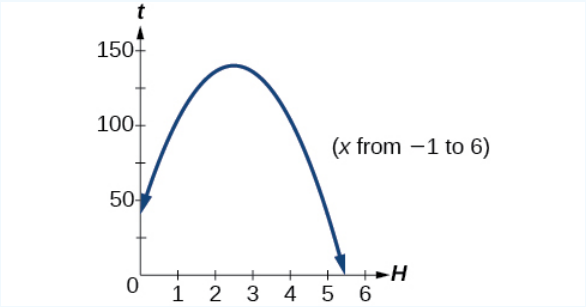


Figure $Page 3.5.16$

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Page 3.5.5: A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation $H(t) = -16t^2 + 96t + 112$.

a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Solution

a. 3 seconds b. 256 feet c. 7 seconds

Key Equations

- general form of a quadratic function: $f(x) = ax^2 + bx + c$
- the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- standard form of a quadratic function: $f(x) = a(x - h)^2 + k$

Key Concepts

- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex. The zeros, or x-intercepts, are the points at which the parabola crosses the x-axis. The y-intercept is the point at which the parabola crosses the y-axis.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph.
- The vertex can be found from an equation representing a quadratic function. .
- The domain of a quadratic function is all real numbers. The range varies with the function.
- A quadratic function's minimum or maximum value is given by the y-value of the vertex.
- The minimum or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue.
- Some quadratic equations must be solved by using the quadratic formula.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems.

Glossary

axis of symmetry
a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by $x = -\frac{b}{2a}$.

general form of a quadratic function
the function that describes a parabola, written in the form $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers and a≠0.

standard form of a quadratic function
the function that describes a parabola, written in the form $f(x) = a(x - h)^2 + k$, where (h, k) is the vertex.

vertex
the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function

vertex form of a quadratic function
another name for the standard form of a quadratic function

zeros
in a given function, the values of x at which $y = 0$, also called roots

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CHAPTER OVERVIEW

Chapter 4: Day 4

Page 4.1: Introduction to Function Composition
Page 4.2: Composition of Functions
Page 4.3: Rational Functions and Asymptotes
Page 4.4: Average Rate of Change
Page 4.5: Inverse Functions

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Page 4.1: Introduction to Function Composition

PREVIEW ACTIVITY $Page 4.1.1$: Constructing a New Function

Let $A = \{a, b, c, d\}$, $B = \{p, q, r\}$, and $C = \{s, t, u, v\}$. The arrow diagram in Figure 6.6 shows two functions: $f : A \to B$ and $g : B \to C$. Notice that if $x \in A$, then $f(x) \in B$. Since $f(x) \in B$, we can apply the function g to $f(x)$, and we obtain $g(f(x))$, which is an element of C.

Using this process, determine $g(f(a))$, $g(f(b))$, $g(f(c))$, and $g(f(d))$. Then explain how we can use this information to define a function from A to C.

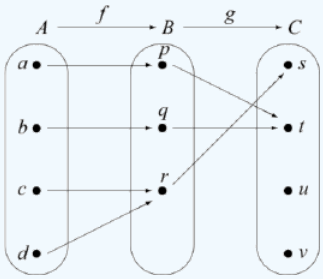


Figure 6.6: Arrow Diagram Showing Two Functions

PREVIEW ACTIVITY $Page 4.1.1$: Verbal Descriptions of Functions

The outputs of most real functions we have studied in previous mathematics courses have been determined by mathematical expressions. In many cases, it is possible to use these expressions to give step-by-step verbal descriptions of how to compute the outputs. For example, if

$$f : \mathbb{R} \to \mathbb{R} \text{ is defined by } f(x) = (3x + 2)^3,$$

we could describe how to compute the outputs as follows:

Step	Verbal Description	Symbolic Result
1	Choose an input.	x
2	Multiply by 3.	$3x$
3	Add 2.	$3x + 2$
4	Cube the result.	$(3x + 3)^3$

Complete step-by-step verbal descriptions for each of the following functions.

1. $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sqrt{3x^2 + 2}$, for each $x \in \mathbb{R}$.
2. $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \sin(3x^2 + 2)$, for each $x \in \mathbb{R}$.
3. $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = e^{3x^2+2}$, for each $x \in \mathbb{R}$.
4. $G : \mathbb{R} \to \mathbb{R}$ by $G(x) = \ln(x^4 + 3)$, for each $x \in \mathbb{R}$.
5. $k : \mathbb{R} \to \mathbb{R}$ by $k(x) = \sqrt[3]{\dfrac{\sin(4x + 3)}{x^2 + 1}}$, for each $x \in \mathbb{R}$.

Composition of Functions

There are several ways to combine two existing functions to create a new function. For example, in calculus, we learned how to form the product and quotient of two functions and then how to use the product rule to determine the derivative of a product of two functions and the quotient rule to determine the derivative of the quotient of two functions. The chain rule in calculus was used to determine the derivative of the composition of two functions, and in this section, we will focus only on the composition of two functions. We will then consider some results about the compositions of injections and surjections.

The basic idea of function composition is that when possible, the output of a function f is used as the input of a function g. This can be referred to as "f followed by g" and is called the composition of f and g. In previous mathematics courses, we used this idea to determine a formula for the composition of two real functions.

For example, if

$$f(x) = 3x^2 + 2 \text{ and } g(x) = sinx$$

then we can compute $g(f(x))$ as follows:

$$\begin{aligned} g(f(x)) &= g(3x^2+2) \\ &= sin(3x^2+2). \end{aligned} \qquad \text{(Page 4.1.1)}$$

In this case, $f(x)$, the output of the function f, was used as the input for the function g. We now give the formal definition of the composition of two functions.

Definition: composite function

Let A, B, and C be nonempty sets, and let $f : A \to B$ and $g : B \to C$ be functions. The *composition* of f and g is the function $g \circ f : A \to C$ defined by

$$(g \circ f)(x) = g(f(x))$$

for all $x \in A$. We often refer to the function $g \circ f$ as a *composite function.*

It is helpful to think of composite function $g \circ f$ as "**f followed by g**". We then refer to f as the **inner function** and g as the **outer function**.

Composition and Arrow Diagrams

The concept of the composition of two functions can be illustrated with arrow diagrams when the domain and codomain of the functions are small, finite sets. Although the term "composition" was not used then, this was done in Preview Activity $Page 4.1.1$, and another example is given here.

Let $A = \{a, b, c, d\}$, $B = \{p, q, r\}$, and $C = \{s, t, u, v\}$. The arrow diagram in Figure 6.7 shows two functions: $f : A \to B$ and $g : B \to C$.

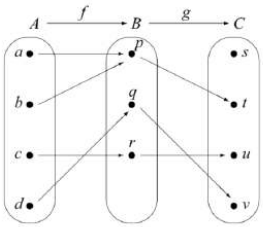


Figure 6.7: Arrow Diagram for Two Functions

If we follow the arrows from the set A to the set C, we will use the outputs of f as inputs of g, and get the arrow diagram from A to C shown in Figure 6.8. This diagram represents the composition of f followed by g.

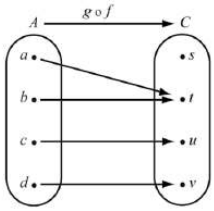


Figure 6.8: Arrow Diagram for $g \circ f : A \to C$

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Page 4.2: Composition of Functions

Learning Objectives

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, $Cost = C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the **cost function** at that temperature. We would write $C(T(5))$.

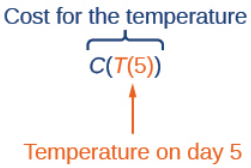


Figure $Page4.2.1$: Explanation of $C(T(5))$, which is the cost for the temperature and $T(5)$ is the temperature on day 5.

By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions $f(x)$ and $g(x)$ with real number outputs, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by the relations.

$$(f+g)(x) = f(x)+g(x)$$
$$(f-g)(x) = f(x)-g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Example $Page4.2.1$: Performing Algebraic Operations on Functions

Find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$, given $f(x)=x-1$ and $g(x)=x^2-1$. Are they the same function?

Solution

Begin by writing the general form, and then substitute the given functions.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ (g-f)(x) &= x^2-1-(x-1)\\ &= x^2-x\\ &= x(x-1)\end{aligned}$$

$$\left(\frac{g}{f}\right)(x) = g(x)f(x)$$

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{x^2-1}{x-1}\\ &= \frac{(x+1)(x-1)}{x-1}\\ &= x+1\end{aligned}$$

No, the functions are not the same.

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ is necessary because when $x=1$, the denominator is equal to 0, which makes the function undefined.

Exercise $Page4.2.1$

Find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.

$$f(x) = x-1$$

and

$$g(x) = x^2-1$$

Are they the same function?

Answer

$(fg)(x) = f(x)g(x) = (x-1)(x2-1) = x^3-x^2-x+1$

$(f-g)(x) = f(x)-g(x) = (x-1)-(x^2-1) = x-x^2$

No, the functions are not the same.

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Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of **combining functions** so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f \circ g(x) = f(g(x)) \qquad \text{(Page 4.2.1)}$$

We read the left-hand side as"f composed with g at x," and the right-hand side as"f of g of x."The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

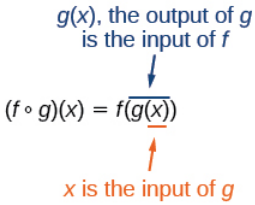


Figure $Page 4.2.2$: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)) \qquad \text{(Page 4.2.2)}$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

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Example $Page4.2.2$: Determining whether Composition of Functions is Commutative

Using the functions provided, find $f(g(x))$ and $g(f(x))$. Determine whether the composition of the functions is **commutative**.

$$f(x)=2x+1 \quad g(x)=3-x$$

Solution

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned} f(g(x)) &= 2(3-x)+1 \\ &= 6-2x+1 \\ &= 7-2x \end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned} g(f(x)) &= 3-(2x+1) \\ &= 3-2x-1 \\ &= 2-2x \end{aligned}$$

We find that $g(f(x))\neq f(g(x))$, so the operation of function composition is not commutative.

Example $Page4.2.3$: Interpreting Composite Functions

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $c(s(3))$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t=3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example $Page4.2.4$: Investigating the Order of Function Composition

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y=f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

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Question/Answer

Are there any situations where $f(g(y))$ and $g(f(x))$ would both be meaningful or useful expressions?

Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order

Exercise $Page 4.2.2$

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Answer

A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G(a(F))$ does not make sense.

Evaluating Composite Functions

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case, we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

Evaluating Composite Functions Using Tables

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example $Page 4.2.5$: Using a Table to Evaluate a Composite Function

Using Table $Page 4.2.1$, evaluate $f(g(3))$ and $g(f(3))$.

Table $Page 4.2.1$

x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

Solution

To evaluate $f(g(3))$, we start from the inside with the input value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g : $g(3) = 2$. We can then use that result as the input to the function f, so $g(3)$ is replaced by 2 and we get $f(2)$. Then, using the table that defines the function f, we find that $f(2) = 8$.

$$g(3) = 2$$

$$f(g(3)) = f(2) = 8$$

To evaluate $g(f(3))$, we first evaluate the inside expression $f(3)$ using the first table: $f(3) = 3$. Then, using the table for g, we can evaluate

$$g(f(3)) = g(3) = 2$$

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Table $Page4.2.2$ shows the composite functions $f \circ g$ and $g \circ f$ as tables.

Table $Page4.2.2$

x	$g(x)$	$f(g(x))$	$f(x)$	$g(f(x))$
3	2	8	3	2

Exercise $Page4.2.3$

Using Table $Page4.2.1$, evaluate $f(g(1))$ and $g(f(4))$.

Answer

$f(g(1)) = f(3) = 3$ and $g(f(4)) = g(1) = 3$

Evaluating Composite Functions Using Graphs

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

How To ...

Given a composite function and graphs of its individual functions, evaluate it using the information provided by the graphs.

1. Locate the given input to the inner function on the x-axis of its graph.
2. Read off the output of the inner function from the y-axis of its graph.
3. Locate the inner function output on the x-axis of the graph of the outer function.
4. Read the output of the outer function from the y-axis of its graph. This is the output of the composite function.

Example $Page4.2.6$: Using a Graph to Evaluate a Composite Function

Using Figure $Page4.2.3$, evaluate $f(g(1))$.

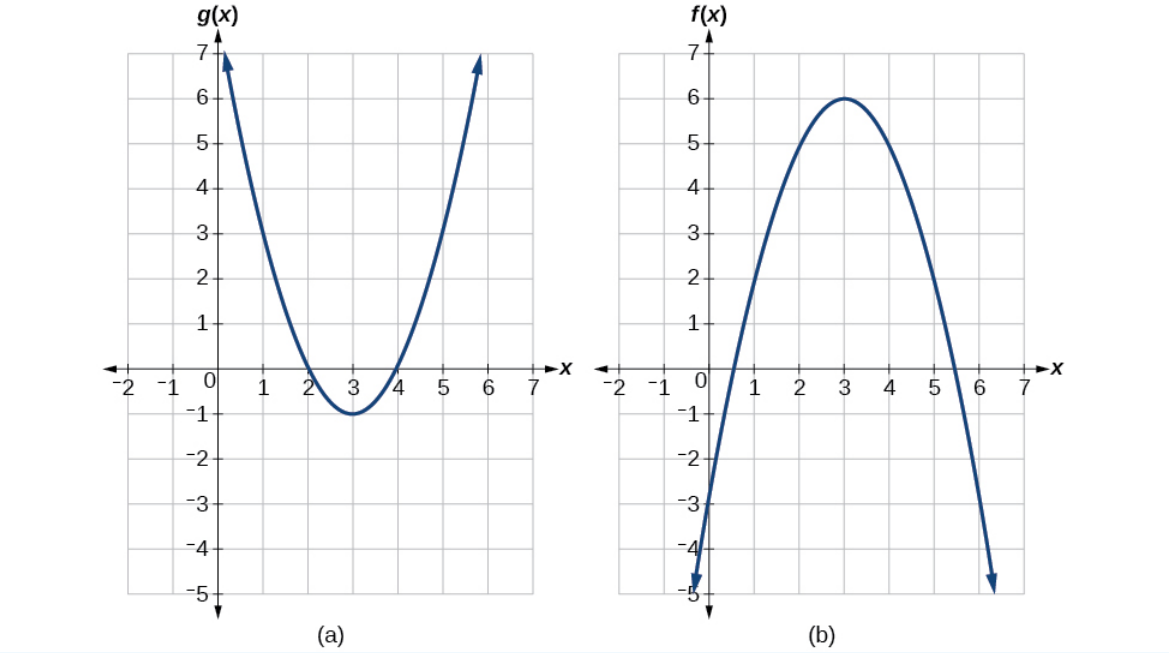


Figure $Page4.2.3$: Two graphs of a positive and negative parabola.

Solution

To evaluate $f(g(1))$, we start with the inside evaluation. See Figure $Page4.2.4$.

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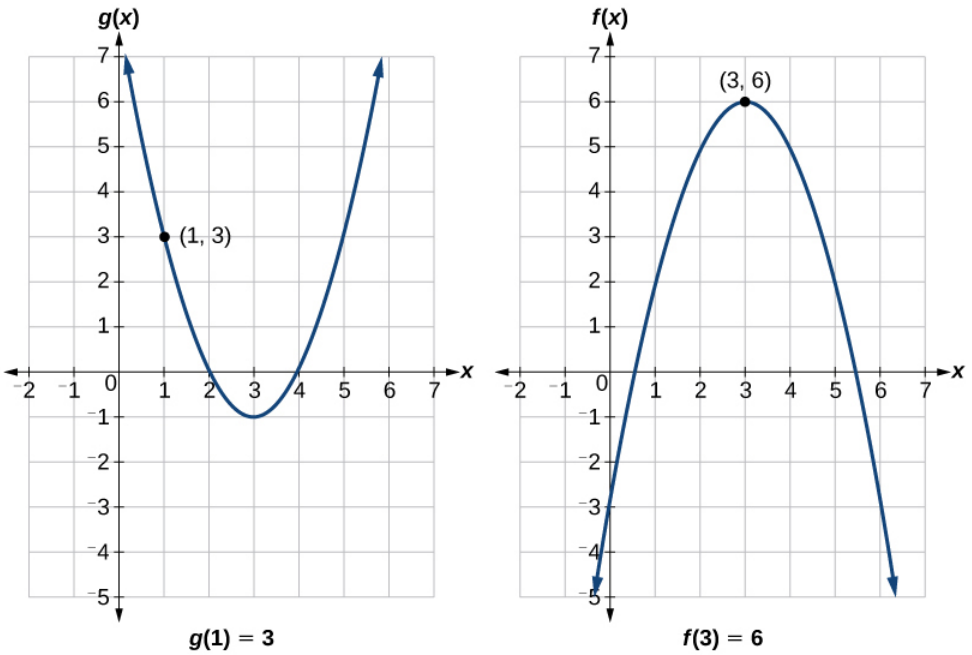


Figure $Page 4.2.4$: Two graphs of a positive parabola $g(x)$ and a negative parabola $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$f(g(1)) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $f(g(1)) = 6$.

Analysis

Figure $Page 4.2.5$ shows how we can mark the graphs with arrows to trace the path from the input value to the output value.

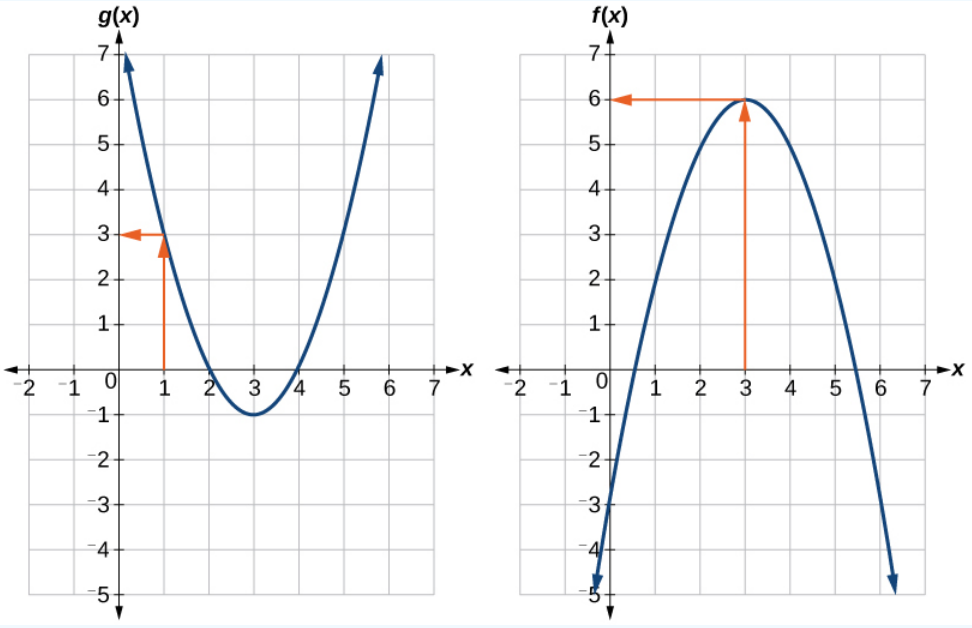


Figure $Page 4.2.5$: Two graphs of a positive and negative parabola.

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Exercise $Page4.2.4$

Using Figure $Page4.2.3$, evaluate $g(f(2))$.

Answer

$g(f(2)) = g(5) = 3$

Evaluating Composite Functions Using Formulas

When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the inner function, which may be a numerical value, a variable name, or a more complicated expression.

While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition $f(g(x))$. To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like $f(t) = t^2 - t$, we substitute the value inside the parentheses into the formula wherever we see the input variable.

How To...

Given a formula for a composite function, evaluate the function.

1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

Example $Page4.2.7$**: Evaluating a Composition of Functions Expressed as Formulas with a Numerical Input**

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$h(1) = 3(1) + 2$$
$$h(1) = 5$$

Then $f(h(1)) = f(5)$, so we evaluate $f(t)$ at an input of 5.

$$f(h(1)) = f(5)$$
$$f(h(1)) = 5^2 - 5$$
$$f(h(1)) = 20$$

Analysis

It makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

Exercise $Page4.2.5$

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate

a. $h(f(2))$
b. $h(f(-2))$

Answer a

8

Answer b

20

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Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to **decompose a composite function**, so we may choose the decomposition that appears to be most expedient.

Example $Page4.2.9$: Decomposing a Function

Write $f(x)=\sqrt{5-x^2}$ as the composition of two functions.

Solution

We are looking for two functions, g and h, so $f(x)=g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5-x^2$ is the inside of the square root. We could then decompose the function as

$$h(x)=5-x^2 \text{ and } g(x)=\sqrt{x}$$

We can check our answer by recomposing the functions.

$$g(h(x))=g(5-x^2)=\sqrt{5-x^2}$$

Exercise $Page4.2.7$

Write $f(x)=\dfrac{4}{3-\sqrt{4+x^2}}$ as the composition of two functions.

Answer

Possible answers:

$g(x)=\sqrt{4+x^2}$

$h(x)=\dfrac{4}{3-x}$

$f=h\circ g$

Access these online resources for additional instruction and practice with composite functions.

- Composite Functions (http://openstaxcollege.org/l/compfunction)
- Composite Function Notation Application (http://openstaxcollege.org/l/compfuncnot)
- Composite Functions Using Graphs (http://openstaxcollege.org/l/compfuncgraph)
- Decompose Functions (http://openstaxcollege.org/l/decompfunction)
- Composite Function Values (http://openstaxcollege.org/l/compfuncvalue)

Key Equation

- Composite function $(f\circ g)(x)=f(g(x))$

Key Concepts

- We can perform algebraic operations on functions. See Example.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function. See Example and Example.
- The order of function composition must be considered when interpreting the meaning of composite functions. See Example.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table. See Example.
- A composite function can be evaluated from a graph. See Example.

- A composite function can be evaluated from a formula. See Example.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function. See Example and Example.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way. See Example.

Glossary

composite function

the new function formed by function composition, when the output of one function is used as the input of another

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Page 4.3: Rational Functions and Asymptotes

Learning Objectives

- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, x, produced. This is given by the equation $C(x) = 15,000x - 0.1x^2 + 1000$. If we want to know the average cost for producing x items, we would divide the cost function by the number of items, x. The average cost function, which yields the average cost per item for x items produced, is

$$f(x) = \frac{15,000x - 0.1x^2 + 1000}{x}$$

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

Using Arrow Notation

We have seen the graphs of the basic **reciprocal function** and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure $Page4.3.1$, and notice some of their features.

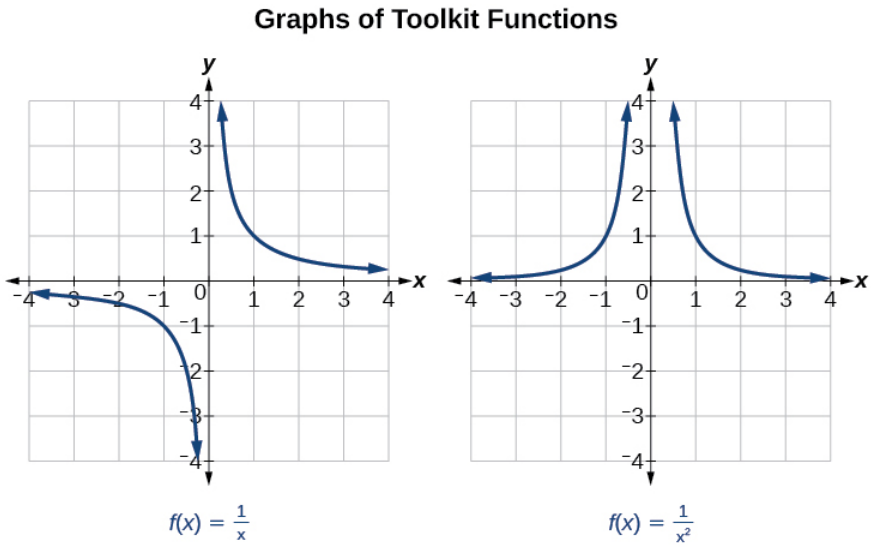


Figure $Page4.3.1$

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

- On the left branch of the graph, the curve approaches the x-axis $(y = 0)$ as $x \to -\infty$.
- As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
- Finally, on the right branch of the graph, the curves approaches the x-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that x or $f(x)$ is approaching a particular value (Table $Page4.3.1$).

Table $Page4.3.1$

Symbol	Meaning

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Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table $Page4.3.2$.

Table $Page4.3.2$

x	–0.1	–0.01	–0.001	–0.0001
$f(x) = \frac{1}{x}$	–10	–100	–1000	–10,000

We write in arrow notation

$$\text{as } x \to 0^-, f(x) \to -\infty$$

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table $Page4.3.3$.

Table $Page4.3.3$

x	0.1	0.01	0.001	0.0001
$f(x) = \frac{1}{x}$	10	100	1000	10,000

We write in arrow notation

$$\text{As } x \to 0^+, f(x) \to \infty \text{ .}$$

See Figure $Page4.3.2$.

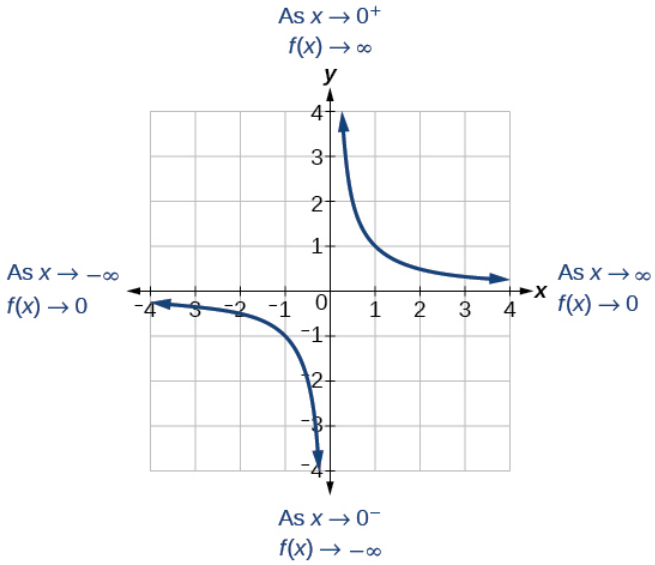


Figure $Page 4.3.2$.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero (Figure $Page 4.3.3$).

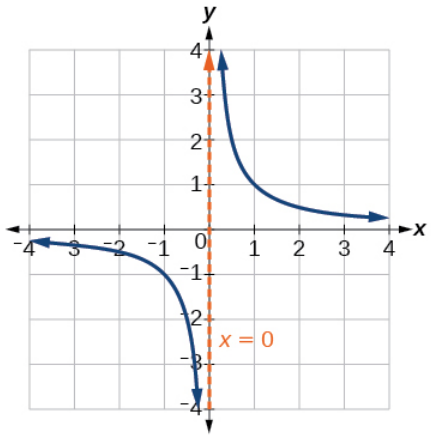


Figure $Page 4.3.3$.

Definition: VERTICAL ASYMPTOTE

A vertical asymptote of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a. We write

As $x \to a$, $f(x) \to \infty$, or as $x \to a$, $f(x) \to -\infty$.

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. As the values of x approach negative infinity, the function values approach 0 (Figure $Page 4.3.4$). Symbolically, using arrow notation

As $x \to \infty$, $f(x) \to 0$,and as $x \to -\infty$, $f(x) \to 0$.

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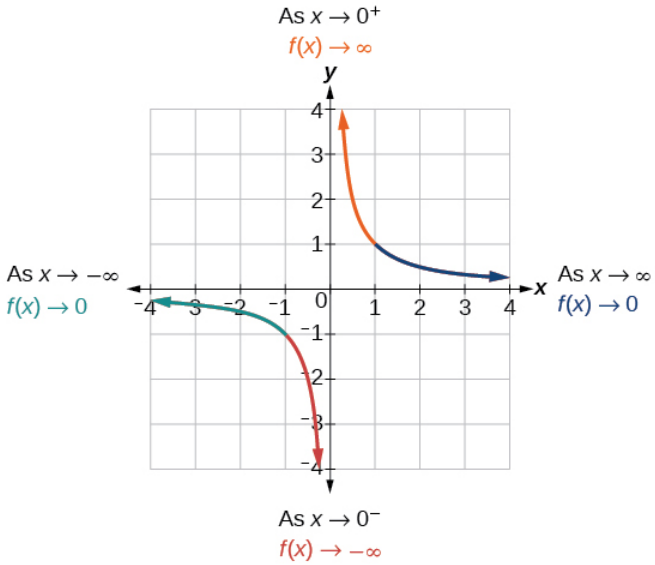


Figure $Page 4.3.4$.

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure $Page 4.3.5$.

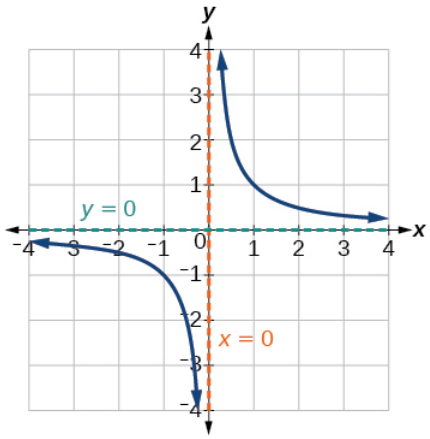


Figure $Page 4.3.5$.

Definition: HORIZONTAL ASYMPTOTE

A *horizontal asymptote* of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \to \infty \text{ or } x \to -\infty \text{ , } f(x) \to b.$$

Example $Page 4.3.1$: Using Arrow Notation.

Use arrow notation to describe the end behavior and local behavior of the function graphed in Figure $Page 4.3.6$.

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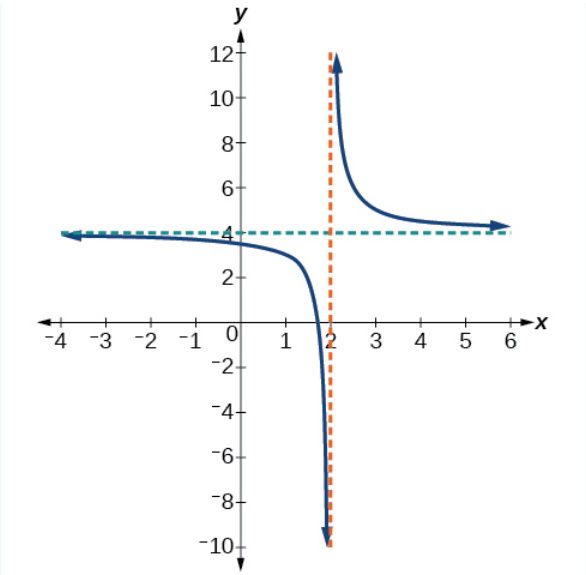


Figure $Page 4.3.6$.

Solution

Notice that the graph is showing a vertical asymptote at $x = 2$, which tells us that the function is undefined at $x = 2$.

$$\text{As } x \to 2^{-},\ f(x) \to -\infty, \text{ and as } x \to 2^{+},\ f(x) \to \infty.$$

And as the inputs decrease without bound, the graph appears to be leveling off at output values of 4, indicating a horizontal asymptote at $y = 4$. As the inputs increase without bound, the graph levels off at 4.

$$\text{As } x \to \infty,\ f(x) \to 4 \text{ and as } x \to -\infty,\ f(x) \to 4.$$

Exercises $Page 4.3.1$

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.

Answer

End behavior: as $x \to \pm\infty$, $f(x) \to 0$;

Local behavior: as $x \to 0$, $f(x) \to \infty$ (there are no x- or y-intercepts)

Example $Page 4.3.2$: Using Transformations to Graph a Rational Function.

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution

Shifting the graph left 2 and up 3 would result in the function

$$f(x) = \frac{1}{x+2} + 3 \qquad (\text{Page } 4.3.1)$$

or equivalently, by giving the terms a common denominator,

$$f(x) = \frac{3x+7}{x+2} \qquad (\text{Page } 4.3.2)$$

The graph of the shifted function is displayed in Figure $Page 4.3.7$.

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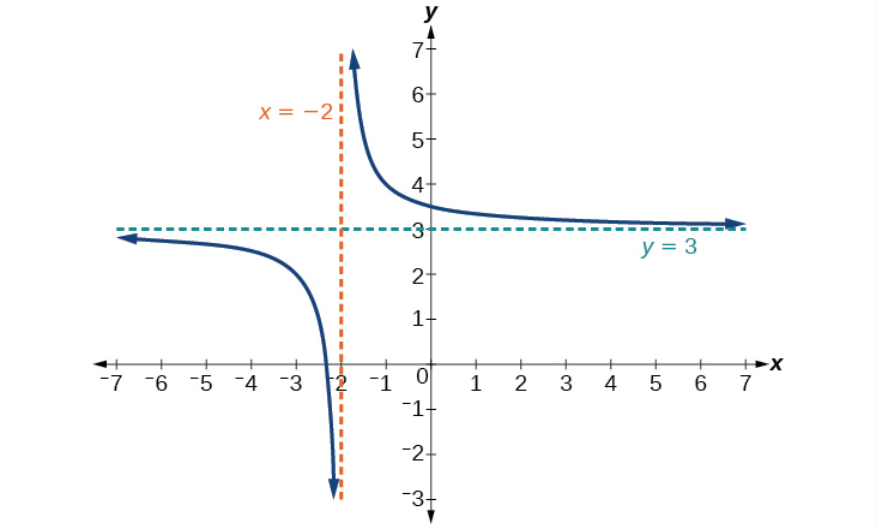


Figure $Page4.3.7$.

Notice that this function is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

$$\text{As } x \to -2^{-}\text{ , } f(x) \to -\infty\text{ , and as } x \to -2^{+}\text{ , } f(x) \to \infty\text{ .}$$

As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

$$\text{As } x \to \pm\infty\text{ , } f(x) \to 3\text{ .}$$

Analysis

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

Exercise $Page4.3.2$

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Solution

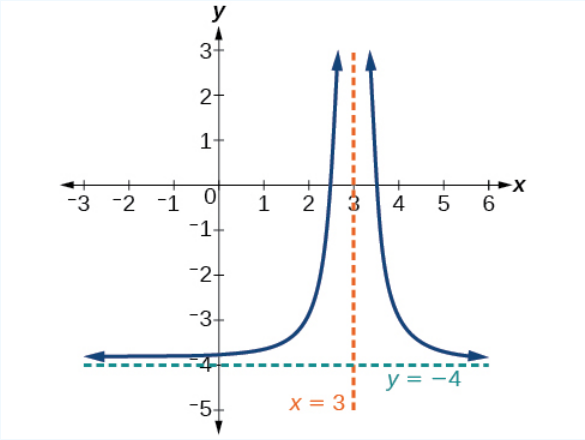


Figure $Page4.3.8$.

The function and the asymptotes are shifted 3 units right and 4 units down. As $x \to 3$, $f(x) \to \infty$, and as $x \to \pm\infty$, $f(x) \to -4$.

The function is $f(x) = \frac{1}{(x-3)^2} - 4$.

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Solving Applied Problems Involving Rational Functions

In Example $Page4.3.2$, we shifted a toolkit function in a way that resulted in the function $f(x)=\frac{3x+7}{x+2}$. This is an example of a rational function. A *rational function* is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

Definition: RATIONAL FUNCTION

A ***rational function*** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x)=\frac{P(x)}{Q(x)}=\frac{a_p x^p+a_{p-1}x^{p-1}+\ldots+a_1 x+a_0}{b_q x^q+b_{q-1}x^{q-1}+\ldots+b_1 x+b_0},\ Q(x)\neq 0 \qquad (Page\ 4.3.3)$$

Example $Page4.3.3$: Solving an Applied Problem Involving a Rational Function

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

Solution

Let t be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

water: $W(t)=100+10t$ in gallons
sugar: $S(t)=5+1t$ in pounds

The concentration, C, will be the ratio of pounds of sugar to gallons of water

$$C(t)=\frac{5+t}{100+10t} \qquad (Page\ 4.3.4)$$

The concentration after 12 minutes is given by evaluating $C(t)$ at $t=12$.

$$C(12)=\frac{5+12}{100+10(12)} \qquad (Page\ 4.3.5)$$

$$=\frac{17}{220} \qquad (Page\ 4.3.6)$$

This means the concentration is 17 pounds of sugar to 220 gallons of water.

At the beginning, the concentration is

$$C(0)=\frac{5+0}{100+10(0)} \qquad (Page\ 4.3.7)$$

$$=\frac{1}{20} \qquad (Page\ 4.3.8)$$

Since $\frac{17}{220}\approx 0.08>\frac{1}{20}=0.05$, the concentration is greater after 12 minutes than at the beginning.

Analysis

To find the horizontal asymptote, divide the leading coefficient in the numerator by the leading coefficient in the denominator:

$$\frac{1}{10}=0.1 \qquad (Page\ 4.3.9)$$

Notice the horizontal asymptote is $y=0.1$. This means the concentration, C, the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

Exercise $Page4.3.3$

There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

Answer

$\frac{12}{11}$

Finding the Domains of Rational Functions

A *vertical asymptote* represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Definition: DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

How To: Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Example $Page4.3.4$**: Finding the Domain of a Rational Function**

Find the domain of $f(x) = \dfrac{x+3}{x^2-9}$.

Solution

Begin by setting the denominator equal to zero and solving.

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

The denominator is equal to zero when $x = \pm 3$. The domain of the function is all real numbers except $x = \pm 3$.

Analysis

A graph of this function, as shown in Figure $Page4.3.9$, confirms that the function is not defined when $x = \pm 3$.

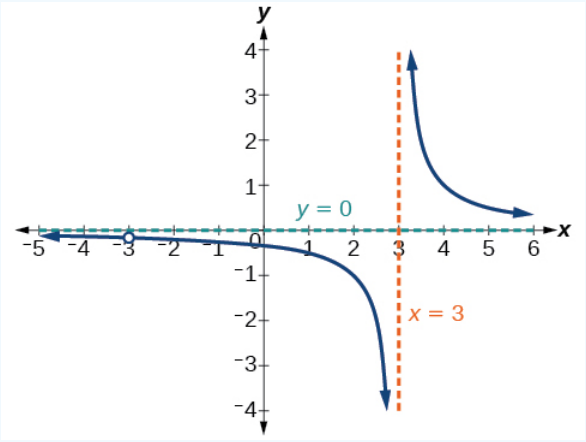


Figure $Page4.3.9$.

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There is a vertical asymptote at $x = 3$ and a hole in the graph at $x = -3$. We will discuss these types of holes in greater detail later in this section.

Exercise $Page 4.3.4$

Find the domain of $f(x) = \dfrac{4x}{5(x-1)(x-5)}$.

Answer

The domain is all real numbers except $x = 1$ and $x = 5$.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

How To: Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or "holes."

Example $Page 4.3.5$**: Identifying Vertical Asymptotes**

Find the vertical asymptotes of the graph of $k(x) = \dfrac{5+2x^2}{2-x-x^2}$.

Solution

First, factor the numerator and denominator.

$$k(x) = \frac{5+2x^2}{2-x-x^2}$$
$$= \frac{5+2x^2}{(2+x)(1-x)}$$

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$(2+x)(1-x) = 0$$
$$x = -2,\ x = 1$$

Neither $x = -2$ nor $x = 1$ are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph in Figure $Page 4.3.10$ confirms the location of the two vertical asymptotes.

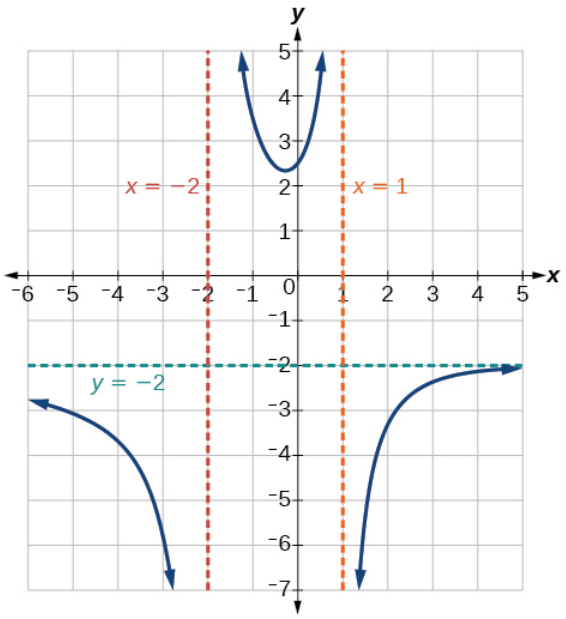


Figure $Page$4.3.10.

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x)=\dfrac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

$$f(x)=\frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x=-1$, is the location of the removable discontinuity. Notice also that $(x-3)$ is not a factor in both the numerator and denominator. The zero of this factor, $x=3$, is the vertical asymptote. See Figure $Page$4.3.11. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

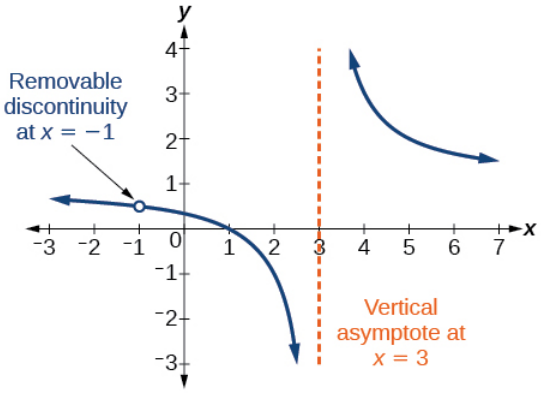


Figure $Page$4.3.11.

REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x=a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the

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multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Example $Page4.3.6$: Identifying Vertical Asymptotes and Removable Discontinuities for a Graph

Find the vertical asymptotes and removable discontinuities of the graph of $k(x)=\dfrac{x-2}{x^2-4}$.

Solution

Factor the numerator and the denominator.

$$k(x)=\frac{x-2}{(x-2)(x+2)}$$

Notice that there is a common factor in the numerator and the denominator, $x-2$. The zero for this factor is $x=2$. This is the location of the removable discontinuity.

Notice that there is a factor in the denominator that is not in the numerator, $x+2$. The zero for this factor is $x=-2$. The vertical asymptote is $x=-2$. See Figure $Page4.3.12$.

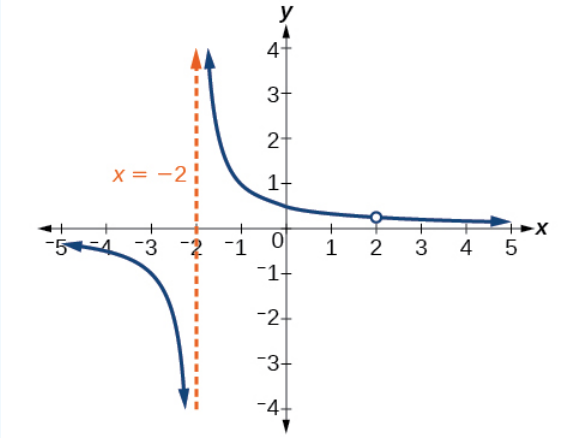


Figure $Page4.3.12$.

The graph of this function will have the vertical asymptote at $x=-2$, but at $x=2$ the graph will have a hole.

Exercise $Page4.3.5$

Find the vertical asymptotes and removable discontinuities of the graph of $f(x)=\dfrac{x^2-25}{x^3-6x^2+5x}$.

Answer

Removable discontinuity at $x=5$.

Vertical asymptotes: $x=0$, $x=1$.

Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator > degree of the numerator, there is a **horizontal asymptote** at $y=0$.

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Example: $f(x)=\dfrac{4x+2}{x^2+4x-5}$

In this case, the end behavior is $f(x)\approx\dfrac{4x}{x^2}=\dfrac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x)=\dfrac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y=0$. See **Figure** $Page4.3.13$. Note that this graph crosses the horizontal asymptote.

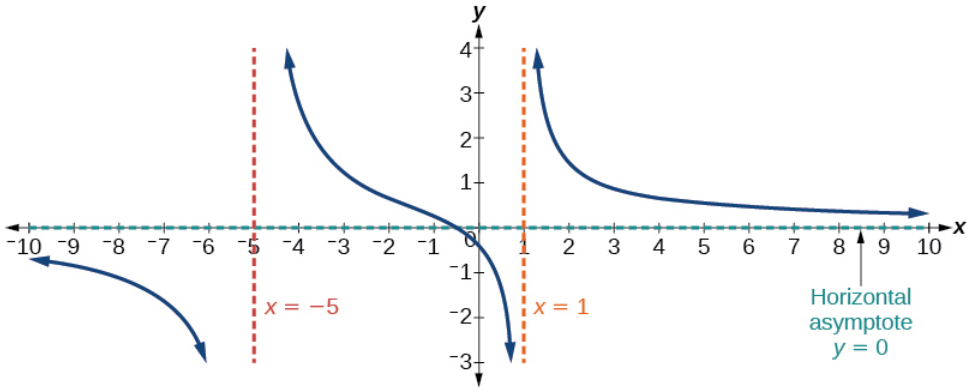


Figure $Page4.3.13$: Horizontal asymptote $y=0$ when $f(x)=\dfrac{p(x)}{q(x)}$, $q(x)\neq 0$ where degree of p < degree of q.

Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

Example: $f(x)=\dfrac{3x^2-2x+1}{x-1}$

In this case, the end behavior is $f(x)\approx\dfrac{3x^2}{x}=3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x)=3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x)=3x$ looks like a diagonal line, and since f will behave similarly to g, it will approach a line close to $y=3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\dfrac{3x^2-2x+1}{x-1}$. The quotient is $3x+1$, and the remainder is 2. The slant asymptote is the graph of the line $g(x)=3x+1$. See Figure $Page4.3.14$.

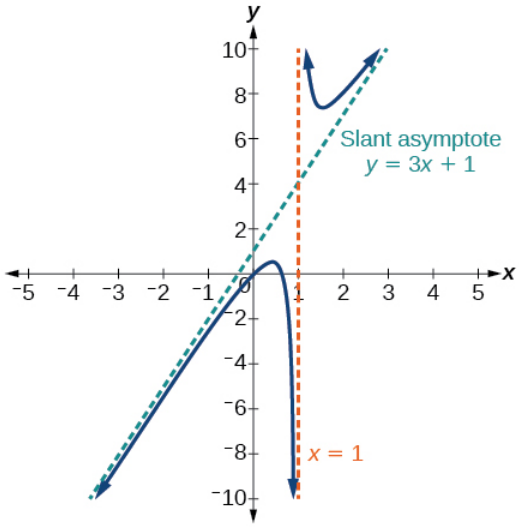


Figure $Page4.3.14$: Slant asymptote when $f(x)=\dfrac{p(x)}{q(x)}$, $q(x)\neq 0$ where degree of p >degree of q by 1.

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Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \frac{a_n}{b_n}$, where a_n and b_n are respectively the leading coefficients of $p(x)$ and $q(x)$ for $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$.

Example: $f(x) = \frac{3x^2 + 2}{x^2 + 4x - 5}$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$. See Figure $Page4.3.15$. Note that this graph crosses the horizontal asymptote.

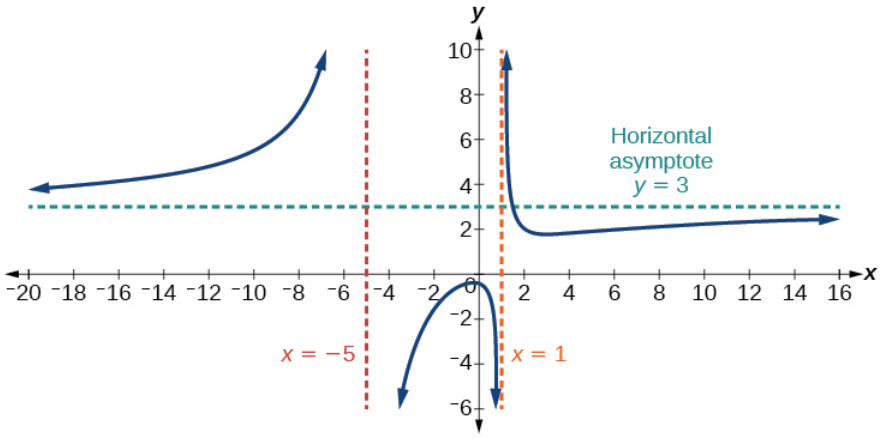


Figure $Page4.3.15$: Horizontal asymptote when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of p = degree of q.

Notice that, while the graph of a rational function will never cross a **vertical asymptote**, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the **end behavior** of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty, f(x) \to \infty$$

HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

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Example $Page4.3.7$: Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$
3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$

Solution

For these solutions, we will use $f(x) = \dfrac{p(x)}{q(x)},\ q(x) \neq 0$.

1. $g(x) = \dfrac{6x^3 - 10x}{2x^3 + 5x^2}$: The degree of $p =$ degree of $q = 3$, so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y = \frac{6}{2}$ or $y = 3$.
2. $h(x) = \dfrac{x^2 - 4x + 1}{x + 2}$: The degree of $p = 2$ and degree of $q = 1$. Since $p > q$ by 1, there is a slant asymptote found at $\dfrac{x^2 - 4x + 1}{x + 2}$.

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

3. $k(x) = \dfrac{x^2 + 4x}{x^3 - 8}$: The degree of $p = 2 <$ degree of $q = 3$, so there is a horizontal asymptote $y = 0$.

Example $Page4.3.8$ Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t) = \dfrac{5 + t}{100 + 10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is 10t, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t \to \infty,\ C(t) \to \tfrac{1}{10}$$

This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example $Page4.3.9$: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

Solution

First, note that this function has no common factors, so there are no potential removable discontinuities.

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The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x = 1, -2$,and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$. See Figure $Page 4.3.16$.

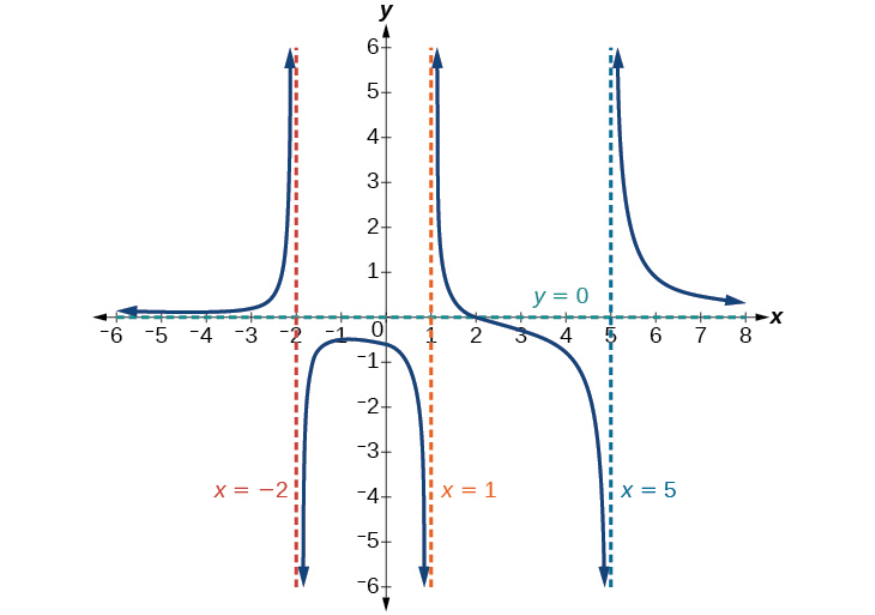


Figure $Page 4.3.16$.

Exercise $Page 4.3.6$

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Answer

Vertical asymptotes at $x = 2$ and $x = -3$

horizontal asymptote at $y = 4$.

INTERCEPTS OF RATIONAL FUNCTIONS

A *rational function* will have a y-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a y-intercept if the function is not defined at zero.

Likewise, a rational function will have x-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

Example $Page 4.3.10$: Finding the Intercepts of a Rational Function

Find the intercepts of $f(x) = \dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$.

Solution

We can find the y-intercept by evaluating the function at zero

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$$f(0)=\frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)}$$
$$=-\frac{6}{10}$$
$$=-\frac{3}{5}$$
$$=-0.6$$

The x-intercepts will occur when the function is equal to zero:

$$0=\frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}\text{ This is zero when the numerator is zero.}$$

$$0=(x-2)(x+3)$$

$$x=2, x=-3$$

The y-intercept is $(0,-0.6)$, the x-intercepts are $(2,0)$ and $(-3,0)$. See Figure $Page 4.3.17$.

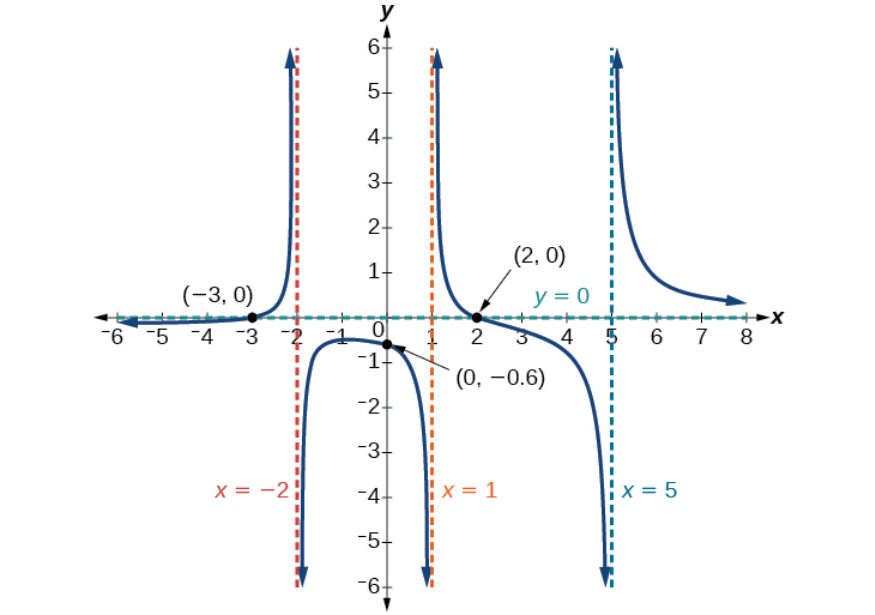


Figure $Page 4.3.17$.

Exercise $Page 4.3.7$

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the x- and y-intercepts and the horizontal and vertical asymptotes.

Answer

For the transformed reciprocal squared function, we find the rational form.

$$f(x)=\frac{1}{(x-3)^2}-4=\frac{1-4(x-3)^2}{(x-3)^2}=\frac{1-4(x^2-6x+9)}{(x-3)(x-3)}=\frac{-4x^2+24x-35}{x^2-6x+9}$$

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Because the numerator is the same degree as the denominator we know that as $x \to \pm\infty$, $f(x) \to -4$; so $y = -4$ is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is $x = 3$, because as $x \to 3$, $f(x) \to \infty$. We then set the numerator equal to 0 and find the x-intercepts are at $(2.5, 0)$ and $(3.5, 0)$. Finally, we evaluate the function at 0 and find the y-intercept to be at $(0, -\frac{35}{9})$.

Graphing Rational Functions

In Example $Page4.3.10$, we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure $Page4.3.18$.

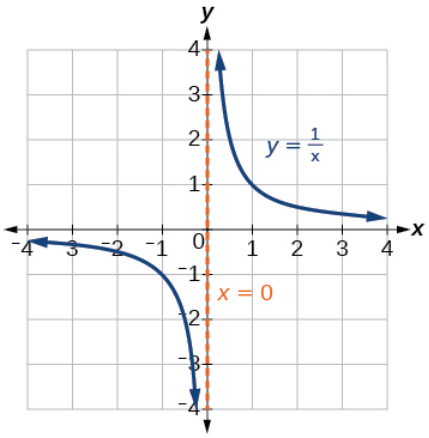


Figure $Page4.3.18$.

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure $Page4.3.19$.

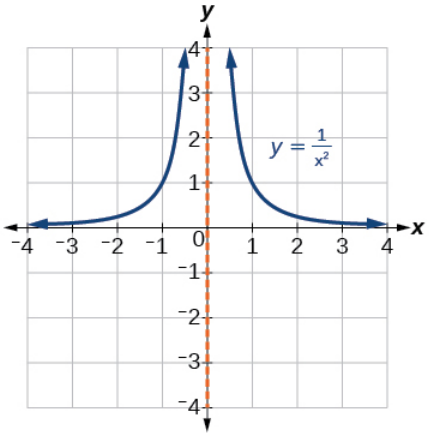


Figure $Page4.3.19$.

For example, the graph of $f(x) = \dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure $Page4.3.20$.

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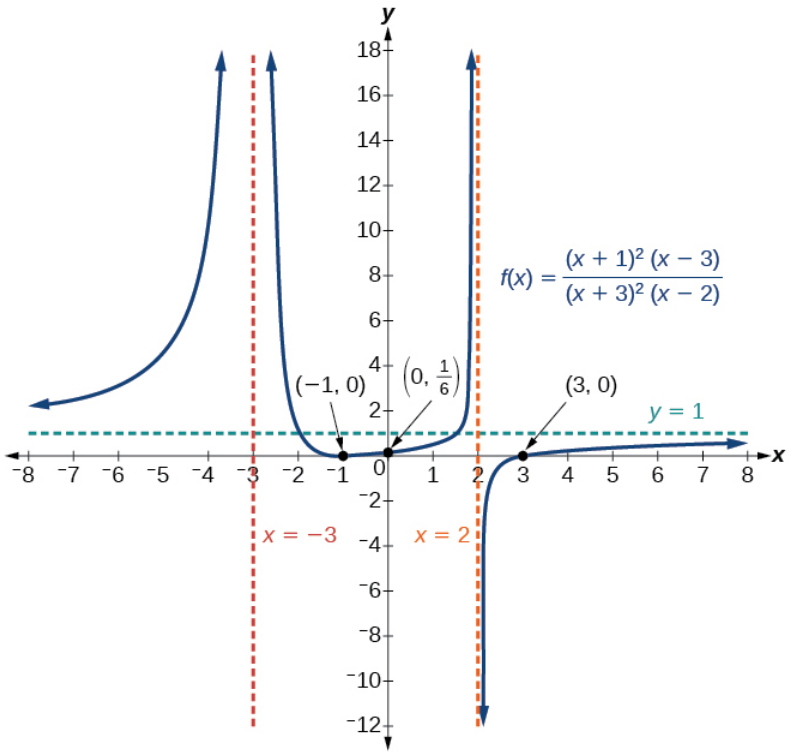


Figure $Page 4.3.20$.

- At the x-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the x-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \frac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x) = \frac{1}{x}$.

Howto: Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

Example $Page 4.3.11$: Graphing a Rational Function

Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution

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We can start by noting that the function is already factored, saving us a step.

Next, we will find the intercepts. Evaluating the function at zero gives the y-intercept:

$$f(0)=\frac{(0+2)(0-3)}{(0+1)^2(0-2)}=3$$

To find the x-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find x-intercepts at $x=-2$ and $x=3$. At each, the behavior will be linear (multiplicity 1), with the graph passing through the intercept.

We have a y-intercept at $(0,3)$ and x-intercepts at $(-2,0)$ and $(3,0)$.

To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when $x+1=0$ and when $x-2=0$, giving us vertical asymptotes at $x=-1$ and $x=2$.

There are no common factors in the numerator and denominator. This means there are no removable discontinuities.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y=0$.

To sketch the graph, we might start by plotting the three intercepts. Since the graph has no x-intercepts between the vertical asymptotes, and the y-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph as shown in Figure $Page 4.3.21$.

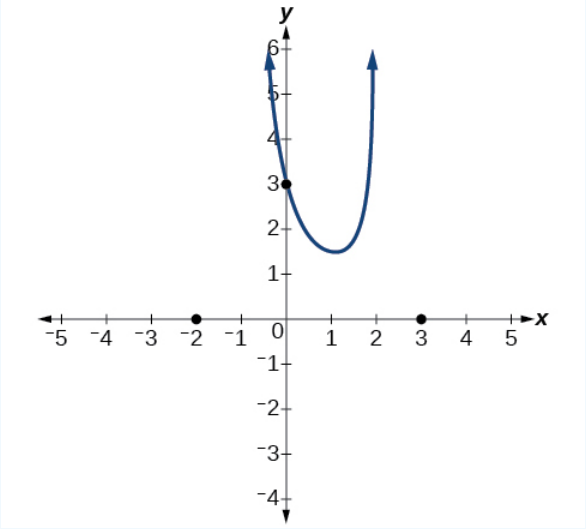


Figure $Page 4.3.21$.

The factor associated with the vertical asymptote at $x=-1$ was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.

For the vertical asymptote at $x=2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. See Figure $Page 4.3.22$. After passing through the x-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.

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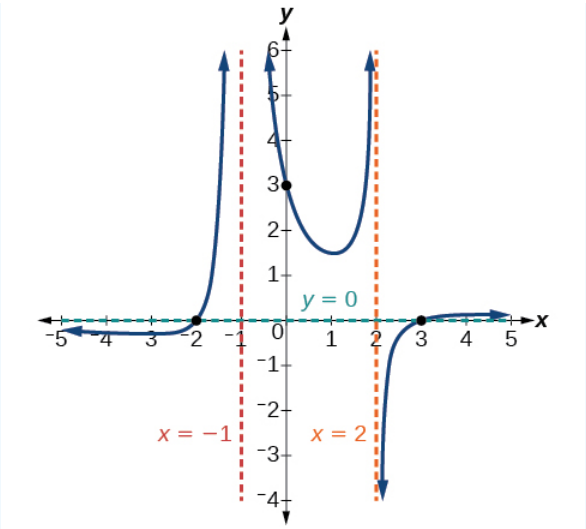


Figure $Page4.3.22$.

Exercise $Page4.3.8$

Given the function $f(x) = \frac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Answer

Horizontal asymptote at $y = \frac{1}{2}$. Vertical asymptotes at $x = 1$ and $x = 3$. y-intercept at $(0, \frac{4}{3})$.

x-intercepts at $(2, 0)$ and $(-2, 0)$. $(-2, 0)$ is a zero with multiplicity 2, and the graph bounces off the x-axis at this point. $(2, 0)$ is a single zero and the graph crosses the axis at this point.

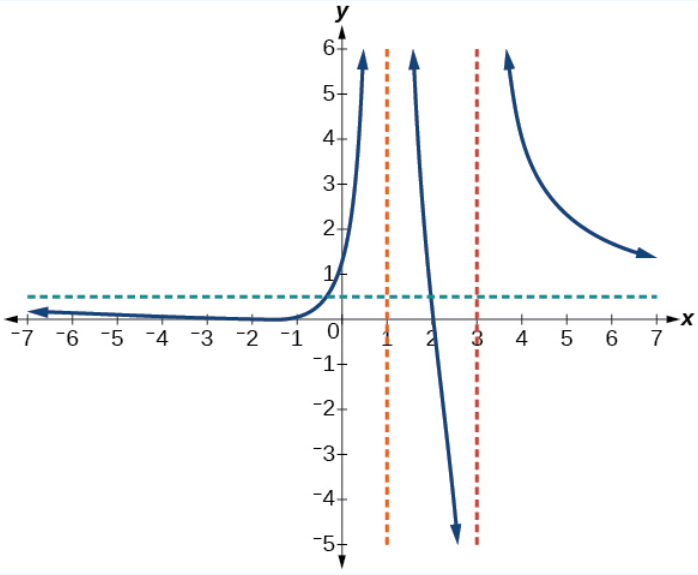


Figure $Page4.3.23$.

Writing Rational Functions

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an *x*-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a

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numerator of a function whose graph will pass through a set of *x*-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

WRITING RATIONAL FUNCTIONS FROM INTERCEPTS AND ASYMPTOTES

If a **rational function** has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any v_j, then the function can be written in the form:

$$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the x-intercept or by the horizontal asymptote if it is nonzero.

Given a graph of a rational function, write the function.

1. Determine the factors of the numerator. Examine the behavior of the graph at the *x*-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the "simplest" function with small multiplicities—such as 1 or 3—but may be difficult for larger multiplicities—such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

Example $Page4.3.12$: Writing a Rational Function from Intercepts and Asymptotes

Write an equation for the rational function shown in Figure $Page4.3.24$.

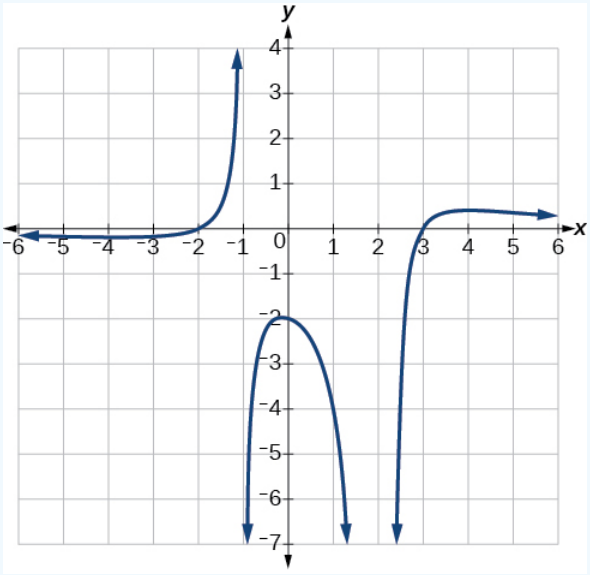


Figure $Page4.3.24$.

Solution

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote. See Figure $Page4.3.25$.

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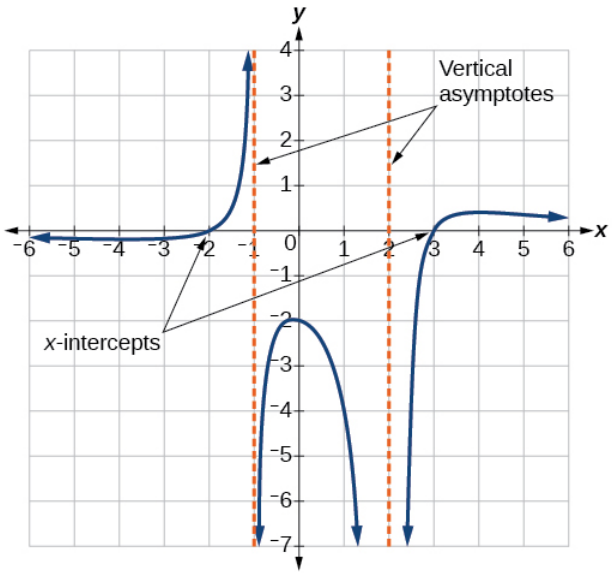


Figure $Page4.3.25$.

We can use this information to write a function of the form

$$f(x)=a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the *y*-intercept $(0,-2)$.

$$-2=a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$
$$-2=a\frac{-6}{4}$$
$$a=\frac{-8}{-6}=\frac{4}{3}$$

This gives us a final function of $f(x)=\dfrac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

Media

Access these online resources for additional instruction and practice with rational functions.

- Graphing Rational Functions
- Find the Equation of a Rational Function
- Determining Vertical and Horizontal Asymptotes
- Find the Intercepts, Asymptotes, and Hole of a Rational Function

Key Equations

Rational Function	$f(x)=\dfrac{P(x)}{Q(x)}=\dfrac{a_px^p+a_{p-1}x^{p-1}+\ldots+a_1x+a_0}{b_qx^q+b_{q-1}x^{q-1}+\ldots+b_1x+b_0},\ Q(x)\neq 0$

Key Concepts

- We can use arrow notation to describe local behavior and end behavior of the toolkit functions $f(x)=\frac{1}{x}$ and $f(x)=\frac{1}{x^2}$. See Example $Page4.3.1$.
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote. See Example.

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- Application problems involving rates and concentrations often involve rational functions. See Example.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero. See Example.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero. See Example.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero. See Example.
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions. See Example, Example, Example, and Example.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior. See Example.
- If a rational function has x-intercepts at $x = x_1, x_2, \ldots, x_n$, vertical asymptotes at $x = v_1, v_2, \ldots, v_m$, and no $x_i =$ any v_j, then the function can be written in the form

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Page 4.4: Average Rate of Change

Learning Objectives

- Find the average rate of change of a function.

Since functions represent how an output quantity varies with an input quantity, it is natural to ask about the rate at which the values of the function are changing. For example, the function $C(t)$ below gives the average cost, in dollars, of a gallon of gasoline t years after 2000.

t	2	3	4	5	6	7	8	9
$C(t)$	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

If we were interested in how the gas prices had changed between 2002 and 2009, we could compute that the cost per gallon had increased from \$1.47 to \$2.14, an increase of \$0.67. While this is interesting, it might be more useful to look at how much the price changed *per year*. You are probably noticing that the price didn't change the same amount each year, so we would be finding the **average rate of change** over a specified amount of time.

The gas price increased by \$0.67 from 2002 to 2009, over 7 years, for an average of $\frac{\$0.67}{7 years} \approx 0.096$ dollars per year. On average, the price of gas increased by about 9.6 cents each year.

Rate of Change

A **rate of change** describes how the output quantity changes in relation to the input quantity. The units on a rate of change are output units per input units.

Average Rate of Change

The **average rate of change** between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\text{Average rate of change} = \frac{\text{Change in Output}}{\text{Change in Input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example $Page 4.4.1$

Using the cost-of-gas function from earlier, find the average rate of change between 2007 and 2009.

Solution

From the table, in 2007 the cost of gas was \$2.64. In 2009 the cost was \$2.14.

The input (years) has changed by 2. The output has changed by \$2.14 - \$2.64 = -0.50.

The average rate of change is then $\frac{-\$0.50}{2 years}$ = -0.25 dollars per year

You Try $Page 4.4.1$

Using the same cost-of-gas function, find the average rate of change between 2003 and 2008

Answer

$\frac{\$3.01 - \$1.69}{5\ years} = \frac{\$1.32}{5\ years} = 0.264$ dollars per year.

Notice that in the last example the change of output was *negative* since the output value of the function had decreased. Correspondingly, the average rate of change is negative.

Example $Page 4.4.2$

Given the function $g(t)$ shown below in Figure $Page 4.4.1$, find the average rate of change on the interval [0, 3].

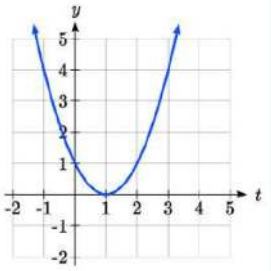


Figure $Page 4.4.1$: Graph of $g(t)$

Solution

At $t = 0$, the graph shows $g(0) = 1$

At $t = 3$, the graph shows $g(3) = 4$

The output has changed by 3 while the input has changed by 3, giving an average rate of change of:

$$\frac{4-1}{3-0} = \frac{3}{3} = 1$$

Example $Page 4.4.3$

On a road trip, after picking up your friend who lives 10 miles away, you decide to record your distance from home over time. Find your average speed over the first 6 hours.

t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Solution

Here, your average speed is the average rate of change.

You traveled 282 miles in 6 hours, for an average speed of

$$\frac{292-10}{6-0} = \frac{282}{6} = 47 \text{ miles per hour}$$

We can more formally state the average rate of change calculation using function notation.

Average Rate of Change Using Function Notation

Given a function $f(x)$, the average rate of change on the interval [a, b] is

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{f(b)-f(a)}{b-a}$$

Example $Page 4.4.4$

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval [2, 4].

Solution

We can start by computing the function values at each endpoint of the interval

$$f(2) = 2^2 - \frac{1}{2} = 4 - \frac{1}{2} = \frac{7}{2}$$

$$f(4) = 4^2 - \frac{1}{4} = 16 - \frac{1}{4} = \frac{63}{4}$$

Now computing the average rate of change

$$\text{Average rate of change} = \frac{f(4)-f(2)}{4-2} = \frac{\frac{63}{4} - \frac{7}{2}}{4-2} = \frac{\frac{49}{4}}{2} = \frac{49}{8}$$

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? You Try *Page*4.4.2

Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval [1, 9].

Answer

Average rate of change = $\dfrac{f(9)-f(1)}{9-1} = \dfrac{(9-2\sqrt{9})-(1-2\sqrt{1})}{9-1} = \dfrac{(3)-(-1)}{9-1} = \dfrac{4}{8} = \dfrac{1}{2}$

✓ Example *Page*4.4.5

The magnetic force F, measured in Newtons, between two magnets is related to the distance between the magnets d, in centimeters, by the formula $F(d) = \dfrac{2}{d^2}$. Find the average rate of change of force if the distance between the magnets is increased from 2 cm to 6 cm.

Solution

We are computing the average rate of change of $F(d) = \dfrac{2}{d^2}$ on the interval [2, 6].

Average rate of change = $\dfrac{F(6)-F(2)}{6-2}$

Evaluating the function

$$\frac{F(6)-F(2)}{6-2} = \frac{\frac{2}{6^2}-\frac{2}{2^2}}{6-2}$$

Simplifying

$$= \frac{\frac{2}{36}-\frac{2}{4}}{4}$$

Combining the numerator terms

$$= \frac{\frac{-16}{36}}{4}$$

Simplifying further

$$= \frac{-1}{9}$$

Newtons per centimeter

This tells us the magnetic force decreases, on average, by 1/9 Newtons per centimeter over this interval.

✓ Example *Page*4.4.6

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. Your answer will be an expression involving a.

Solution

Using the average rate of change formula

$$\frac{g(a)-g(0)}{a-0}$$

Evaluating the function

$$\frac{(a^2+3a+1)-(0^2+3(0)+1)}{a-0}$$

Simplifying

$$\frac{a^2+3a+1-1}{a}$$

Simplifying further, and factoring

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$$\frac{a(a+3)}{a}$$

Cancelling the common factor a

$$a+3$$

This result tells us the average rate of change between t = 0 and any other point t = a. For example, on the interval [0, 5], the average rate of change would be 5+3 = 8.

? You Try $Page4.4.3$

Find the average rate of change of $f(x)=x^3+2$ on the interval $[a, a+h]$.

Answer

$$\frac{f(a+h)-f(a)}{(a+h)-a}=\frac{((a+h)^3+2)-(a^3+2)}{h}=\frac{a^3+3a^2h+3ah^2+h^3+2-a^3-2}{h}=\frac{3a^2h+3ah^2+h^3}{h}=\frac{h(3a^2+3ah+h^2)}{h}=3a^2+3ah+h^2$$

Caution!

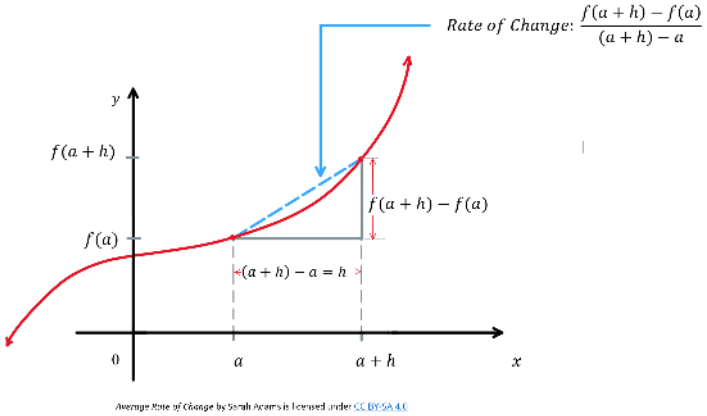
Do remember to watch out for common arithmetic errors: $(a+h)^3 \neq a^3+h^3$.

Instead, expand the product to help ensure the multiplication is done correctly: $(a+h)^3=(a+h)(a+h)(a+h)$

You may be wondering why we would want to find the average rate of change on an interval such as $[a, a+h]$. The value "a" represents an arbitrary x-coordinate for a point. The value "a+h" represents another arbitrary x-coordinate of another point. The value "h" represents the horizontal distance between these two points. The average rate of change on this interval, $[a, a+h]$, is

$$\frac{f(a+h)-f(a)}{(a+h)-a}$$

which is visually the slope of a **secant line** passing through these two points on the graph of the function, as shown in Figure $Page4.4.2$ below. In calculus, you will investigate what happens to the slope of this secant line as you change the distance "h" between the two points. By shrinking this distance we can describe instantaneous rate of change which is a very useful topic in many real-life situations. Stay tuned for more about this in your future calculus course!



Average Rate of Change by Sarah Adams is licensed under CC BY-SA 4.0

Figure $Page4.4.2$: Rate of Change Image with a and h (Sarah Adams: CC-BY-SA 4.0)

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Page 4.5: Inverse Functions

Learning Objectives

- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure $Page 4.5.1$ provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

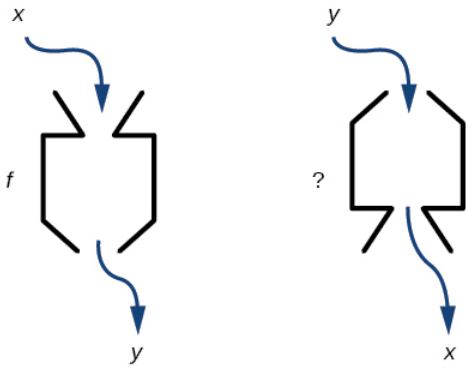


Figure $Page 4.5.1$: *Can a function "machine" operate in reverse?*

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the **Celsius** scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees **Fahrenheit** to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32) \qquad (\text{Page } 4.5.1)$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24^{\circ} \qquad (\text{Page } 4.5.2)$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure $Page 4.5.2$ for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

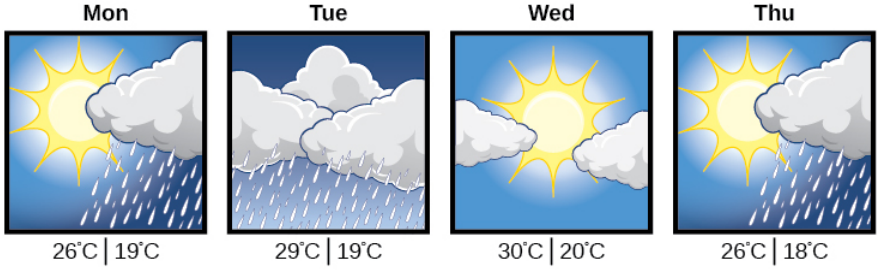


Figure $Page 4.5.2$: A forecast of Monday's through Thursday's weather.

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At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$26 = \frac{5}{9}(F-32) \quad \text{(Page 4.5.3)}$$

$$26 \cdot \frac{9}{5} = F-32 \quad \text{(Page 4.5.4)}$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79 \quad \text{(Page 4.5.5)}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x \quad \text{(Page 4.5.6)}$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a **reciprocal**, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x \quad \text{(Page 4.5.7)}$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x \quad \text{(Page 4.5.8)}$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2,-8)$, $(0,0)$, and $(2,8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8,-2)$, $(0,0)$, and $(8,2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Definition: Inverse Function

For any **one-to-one function** $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)} \quad \text{(Page 4.5.9)}$$

and not all functions have inverses.

Example $Page4.5.1$: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2)=4$ and $f(5)=12$, what are the corresponding input and output values for the inverse function?

Solution

The inverse function reverses the input and output quantities, so if

$$f(2)=4, \text{ then } f^{-1}(4)=2; \qquad \text{(Page 4.5.10)}$$
$$f(5)=12, \text{ then } f^{-1}(12)=5$$

.

Alternatively, if we want to name the inverse function g, then $g(4)=2$ and $g(12)=5$.

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See Table $Page4.5.1$.

Table $Page4.5.1$

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Exercise $Page4.5.1$

Given that $h^{-1}(6)=2$, what are the corresponding input and output values of the original function h?

Answer

$h(2)=6$

How To: Given two functions $f(x)$ and $g(x)$, test whether the functions are inverses of each other.

1. Determine whether $f(g(x))=x$ or $g(f(x))=x$.
2. If both statements are true, then $g=f^{-1}$ and $f=g^{-1}$. If either statement is false, then both are false, and $g\neq f^{-1}$ and $f\neq g^{-1}$.

Example $Page4.5.2$: Testing Inverse Relationships Algebraically

If $f(x)=\frac{1}{x+2}$ and $g(x)=\frac{1}{x}-2$, is $g=f^{-1}$?

Solution

$$g(f(x))=\frac{1}{(\frac{1}{x+2})-2} \qquad \text{(Page 4.5.11)}$$
$$=x+2-2 \qquad \text{(Page 4.5.12)}$$
$$=x \qquad \text{(Page 4.5.13)}$$

so

$$g=f^{-1} \text{ and } f=g^{-1} \qquad \text{(Page 4.5.14)}$$

This is enough to answer yes to the question, but we can also verify the other formula.

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$$f(g(x)) = \frac{1}{\frac{1}{x} - 2 + 2} \quad \text{(Page 4.5.15)}$$

$$= \frac{1}{\frac{1}{x}} \quad \text{(Page 4.5.16)}$$

$$= x \quad \text{(Page 4.5.17)}$$

Analysis

Notice the inverse operations are in reverse order of the operations from the original function.

Exercise $Page 4.5.2$

If $f(x) = x^3 - 4$ and $g(x) = \sqrt[3]{x+4}$, is $g = f^{-1}$?

Answer

Yes

Example $Page 4.5.3$**: Determining Inverse Relationships for Power Functions**

If $f(x) = x^3$ (the cube function) and $g(x) = \frac{1}{3}x$, is $g = f^{-1}$?

Solution

$$f(g(x)) = \frac{x^3}{27} \neq x \quad \text{(Page 4.5.18)}$$

No, the functions are not inverses.

Analysis

The correct inverse to the cube is, of course, the cube root $\sqrt[3]{x} = x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

Exercise $Page 4.5.3$

If $f(x) = (x-1)^3$ and $g(x) = \sqrt[3]{x} + 1$, is $g = f^{-1}$?

Answer

Yes

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure $Page 4.5.3$.

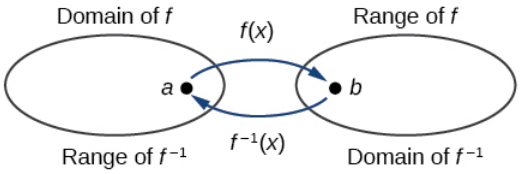


Figure $Page 4.5.3$***:*** *Domain and range of a function and its inverse.*

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs

3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its range limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of $f^{-1} = [1, \infty)$.
- The domain of f^{-1} = range of $f = [0, \infty)$.

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Note: Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

How To: Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Example $Page4.5.4$: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table $Page4.5.2$. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Table $Page4.5.2$

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	Cube Root	Square Root	Absolute Value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Solution

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$,where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Analysis

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure $Page4.5.4$. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the

horizontal line test, then in that restricted domain, it can have an inverse.

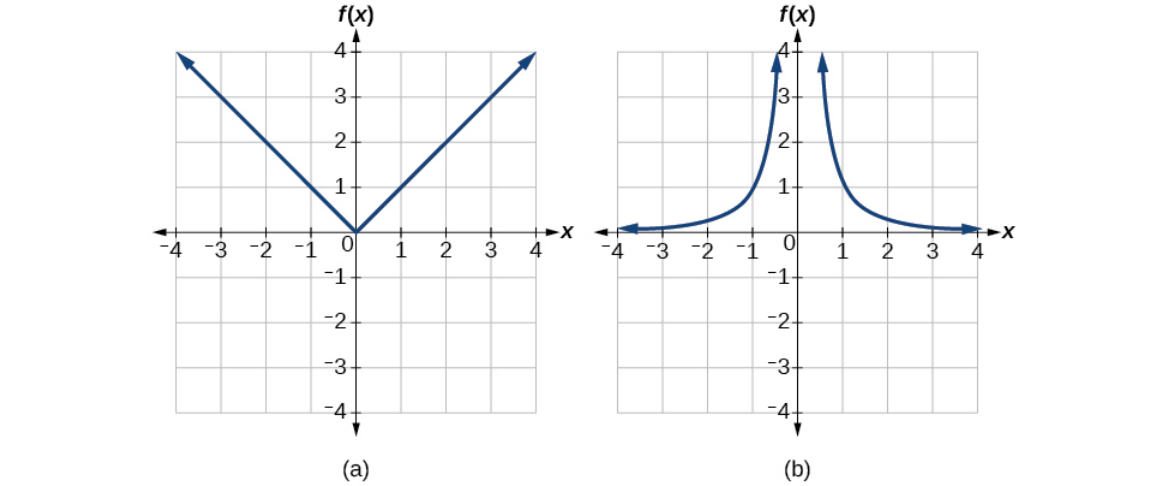


Figure Page4.5.4: *(a) Absolute value (b) Reciprocal squared*

Page4.5.4: The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Solution

The domain of function f^{-1} is $(-\infty, -2)$ and the range of function f^{-1} is $(1, \infty)$.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example Page4.5.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table Page4.5.3, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$

Table Page4.5.3

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

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Exercise $Page4.5.5$

Using Table $Page4.5.4$, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

Table $Page4.5.4$

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Answer

$f(60) = 50$. In 60 minutes, 50 miles are traveled.
$f^{-1}(60) = 70$. To travel 60 miles, it will take 70 minutes.

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

Example $Page4.5.6$: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure $Page4.5.5$. Find $g(3)$ and $g^{-1}(3)$.
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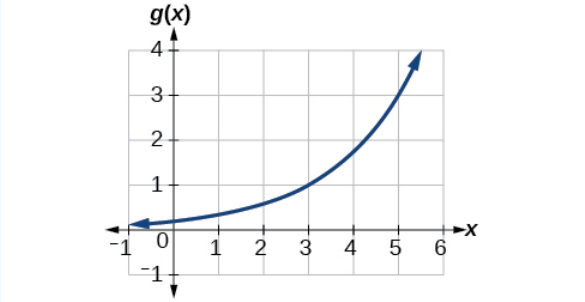


Figure $Page4.5.5$: Graph of $g(x)$

Solution

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure $Page4.5.6$.

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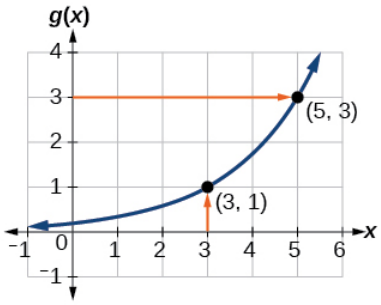


Figure $Page 4.5.6$: Graph of $g(x)$.

Exercise $Page 4.5.6$

Using the graph in Figure $Page 4.5.6$, (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

Answer a

3

Answer b

5.6

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, y as a function of x— we can often find the inverse function by solving to obtain x as a function of y.

How To: Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x
3. Interchange x and y.

Example $Page 4.5.7$: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32) \qquad (Page\ 4.5.19)$$

Solution

$$C = \frac{5}{9}(F - 32) \qquad (Page\ 4.5.20)$$

$$C \cdot \frac{9}{5} = F - 32 \qquad (Page\ 4.5.21)$$

$$F = \frac{9}{5}C + 32 \qquad (Page\ 4.5.22)$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32) \qquad (Page\ 4.5.23)$$

,

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32. \qquad (Page\ 4.5.24)$$

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In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Exercise $Page 4.5.7$

Solve for x in terms of y given $y = \frac{1}{3}(x-5)$

Answer

$x = 3y+5$

Example $Page 4.5.8$: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3}+4$.

Solution

$$y = \frac{2}{x-3+4} \quad \text{Set up an equation.} \quad (\text{Page } 4.5.25)$$
$$y-4 = \frac{2}{x-3} \quad \text{Subtract 4 from both sides.} \quad (\text{Page } 4.5.26)$$
$$x-3 = \frac{2}{y-4} \quad \text{Multiply both sides by x}-3 \text{ and divide by y}-4. \quad (\text{Page } 4.5.27)$$
$$x = \frac{2}{y-4}+3 \quad \text{Add 3 to both sides.} \quad (\text{Page } 4.5.28)$$

So $f^{-1}(y) = \frac{2}{y-4}+3$ or $f^{-1}(x) = \frac{2}{x-4}+3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table $Page 4.5.5$.

Table $Page 4.5.5$

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example $Page 4.5.9$: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2+\sqrt{x-4}$.

Solution

$$y = 2+\sqrt{x-4} \quad (\text{Page } 4.5.29)$$
$$(y-2)^2 = x-4 \quad (\text{Page } 4.5.30)$$
$$x = (y-2)^2+4 \quad (\text{Page } 4.5.31)$$

So $f^{-1}(x) = (x-2)^2+4$.

The domain of f is $[4,\infty)$. Notice that the range of f is $[2,\infty)$, so this means that the domain of the inverse function f^{-1} is also $[2,\infty)$

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2,\infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Exercise $Page4.5.8$

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Answer

$f^{-1}(x) = (2-x)^2$; domain of f: $[0, \infty)$; domain of f^{-1}: $(-\infty, 2]$

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure $Page4.5.7$.

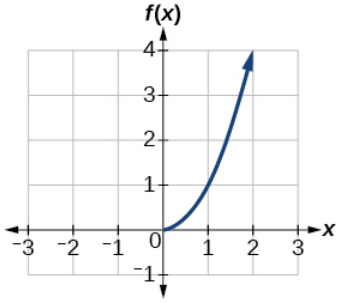


Figure $Page4.5.7$: *Quadratic function with domain restricted to* $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure $Page4.5.8$.

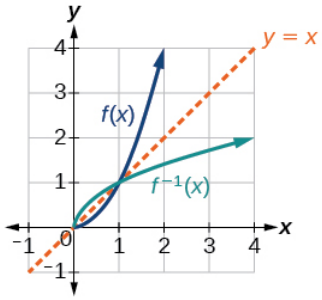


Figure $Page4.5.8$: *Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example $Page4.5.10$: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure $Page4.5.9$, sketch a graph of $f^{-1}(x)$.

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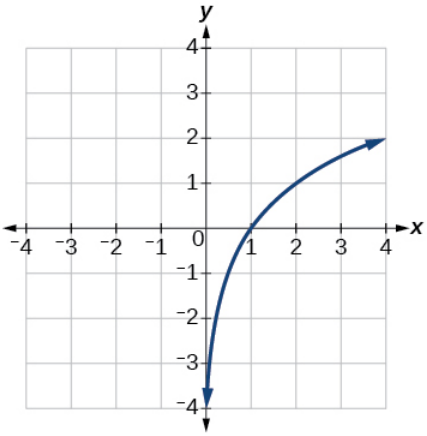


Figure $Page4.5.9$: Graph of $f^(-1)(x)$.

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure $Page4.5.10$.

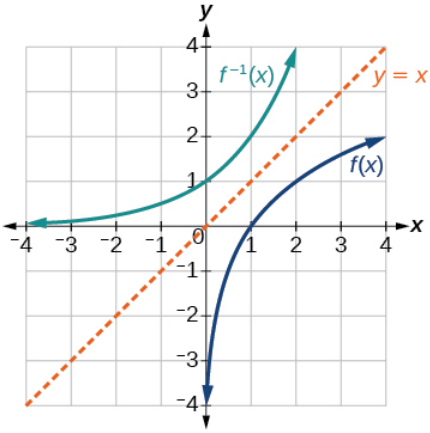


Figure $Page4.5.10$: The function and its inverse, showing reflection about the identity line

Exercise $Page4.5.1$

Draw graphs of the functions f and f^{-1} from Example $Page4.5.8$.

Answer

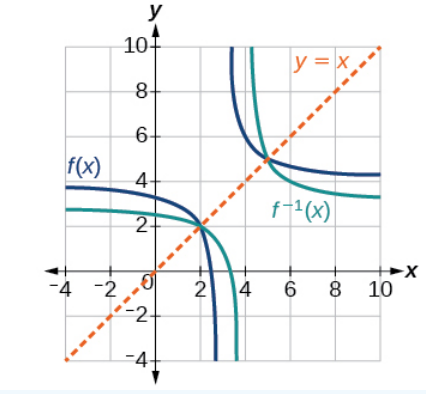


Figure $Page4.5.11$: Graph of $f(x)$ and $f^(-1)(x)$.

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Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x \qquad \text{(Page 4.5.32)}$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

Key Concepts

- If $g(x)$ is the inverse of $f(x)$, then $g(f(x)) = f(g(x)) = x$.
- Each of the toolkit functions has an inverse.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse.
- The inverse of a function can be determined at specific points on its graph.
- To find the inverse of a formula, solve the equation $y = f(x)$ for x as a function of y. Then exchange the labels x and y.
- The graph of an inverse function is the reflection of the graph of the original function across the line $y = x$.

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CHAPTER OVERVIEW

Chapter 5: Day 5

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Page 5.2: Graphing Exponential Functions
Page 5.3: Logarithmic Functions
Page 5.4: Graphing Logarithmic Functions
Page 5.5: Laws of Logarithms
Page 5.6: Exponential and Logarithmic Equations

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Page 5.1: Exponential Functions

Learning Objectives

- Evaluate exponential functions.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about $.2\%$ each year. If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.

Exponential Growth and Decay

What exactly does it mean to *grow exponentially*? For us to gain a clear understanding of exponential growth, let us compare exponential growth with linear growth. We will construct two functions. The first function, $f(x) = 2^x$, is exponential. The second function, $g(x) = 2x$, is linear. We will start with an input of 0, and increase each input by 1. The corresponding outputs are shown in Table $Page5.1.1$. From Table $Page5.1.1$, we can see that for the exponential function, $f(x)$ that as we increase the input by 1 that the output doubles. The linear function, $g(x)$, is a constant increase by 2. From Table $Page5.1.1$ we can conclude that exponential growth dwarfs linear growth.

- *Exponential growth* refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- *Linear growth* refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

Table $Page5.1.1$

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

For exponential growth, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

The general form of an **exponential function** is $f(x) = ab^x$, where a is any nonzero number, b is a positive real number not equal to 1.

- If $b > 1$,the function grows at a rate proportional to its size. This is called exponential growth.
- If $0 < b < 1$, the function decays at a rate proportional to its size. This is called exponential decay.

Exponential growth refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.

Exponential decay refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time .

Let's look at the function $f(x) = 2^x$ from our example. We will create a table (Table $Page5.1.2$) to determine the corresponding outputs over an interval in the domain from -3 to 3.

Table $Page5.1.2$

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$2^{-3} = \frac{1}{8}$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

Let us examine the graph of f by plotting the ordered pairs from Table $Page5.1.2$ and then make a few observations in Figure $Page5.1.1$.

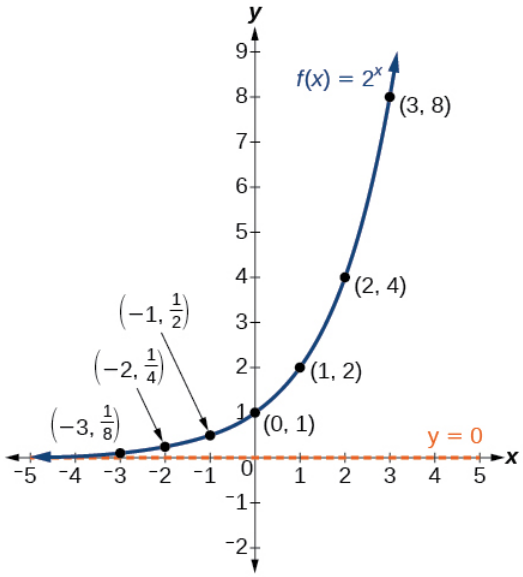


Figure $Page5.1.1$

Let's define the behavior of the graph of the exponential function $f(x) = 2^x$ and highlight some its key characteristics.

- the domain is $(-\infty, \infty)$,
- the range is $(0, \infty)$,
- as $x \to \infty$, $f(x) \to \infty$,
- as $x \to -\infty$, $f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the *y*-intercept is $(0, 1)$.

Exponential Functions

For any real number x, an **exponential function** is a function with the form

$$f(x) = ab^x$$

where

- a is a non-zero real number called the initial value and
- b is any positive real number such that $b \neq 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a > 0$.
- The range of f is all negative real numbers if $a < 0$.
- The y-intercept is $(0, a)$,and the horizontal asymptote is $y = 0$.

b is called the base of the function and x is called the exponent.

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Note that an exponential function does not intersect its horizontal asymptote. This is because there is no value for x that we can input into the function that would yield an output of 0, $ab^x \neq 0$.

Example $Page 5.1.1$

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution

By definition, an exponential function has a constant base and a variable exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is variable and the exponent is constant.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

$f(x)$ and $h(x)$ are exponential functions.

You Try $Page 5.1.1$

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Answer

$g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions. The others are not exponential functions.

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base, b ,to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{\frac{1}{2}} = \sqrt{-9}$, which is not a real number.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1:

Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x.

Evaluating Exponential Functions

To evaluate an exponential function with the form $f(x) = b^x$, we simply substitute x with the given value, and calculate the resulting power. For example:

Let $f(x) = 2^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 2^x \\ f(3) &= 2^3 \qquad \text{Substitute } x = 3 \\ &= 8 \qquad \text{Evaluate the power} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x) = 30(2)^x$. What is $f(3)$?

$$
\begin{aligned}
f(x) &= 30(2)^x \\
f(3) &= 30(2)^3 \qquad \text{Substitute } x = 3 \\
&= 30(8) \qquad \text{Simplify the power first} \\
&= 240 \qquad \text{Multiply}
\end{aligned}
$$

Caution!

Remember to follow order of operations correctly. Exponent operations must be performed before multiplication.

$$f(3) = 30(2)^3 \neq 60^3 = 216,000$$

Example $Page 5.1.2$

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

Solution

To estimate the population in 2031, we evaluate the models for $t = 18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549$$

There will be about 1.549 billion people in India in the year 2031.

Example $Page 5.1.3$

Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (polonium-210 in this case) each day. If you begin with 100 mg of Bismuth-210, how much remains after one week?

Solution

With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate. Our initial quantity is a = 100 mg, and our growth rate will be negative 13%, since we are decreasing: r = -0.13. This gives the equation:

$$Q(d) = 100(1 - 0.13)^d = 100(0.87)^d$$

This can also be explained by recognizing that if 13% decays, then 87 % remains.

After one week, 7 days, the quantity remaining would be

$$Q(7) = 100(0.87)^7 = 37.73 \text{ mg of Bismuth-210 remains.}$$

Example $Page 5.1.4$

$T(q)$ represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region measured quarterly since January 1, 2016, interpret all the parts of the equation $T(2) = 86(1.64)^2 = 231.3056$

Solution

Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan. 1, 2016 this region had 86,000 Android smart phone contracts. Since $b = 1 + r = 1.64$, we know that every quarter the number of smart phone contracts grows by 64%. $T(2) = 231.3056$ means that in the 2nd quarter (or at the end of the second quarter) there were approximately 231,306 Android smart phone contracts.

Compound Interest

A certificate of deposit (CD) is a type of savings account offered by banks, typically offering a higher interest rate in return for a fixed length of time you will leave your money invested. Typically bank accounts and other savings instruments in which earnings are reinvested, such as mutual funds and retirement accounts, utilize compound interest. The term *compounding* comes from the behavior that interest is earned not on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, annual percentage rate r, and number of compounding periods in a year n:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe Table $Page 5.1.3$, which shows the result of investing $\$1,000$ at 10% APR for one year. Notice how the value of the account increases as the compounding frequency increases.

Table $Page 5.1.3$

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

Compound Interest

Compound interest can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where

- $A(t)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Note that sometime rates are given without stating that they are annual percentage rates. In most textbooks, including this one, it is implied that the rate is an annual percentage rate unless explicitly stated otherwise.

Example $Page 5.1.5$

If you invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution

Since we are starting with $3000, $a = 3000$

Our interest rate is 3%, so $r = 0.03$

Since we are compounding quarterly, we are compounding 4 times per year, so $k = 4$

We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$$A(10) = 3000\left(1 + \frac{0.03}{4}\right)^{4(10)} = \$4045.05$$

The account will be worth $4045.05 in 10 years.

(*note that we round to the nearest cent when working with money)*

Example $Page 5.1.6$

A 529 plan is a college savings plan in which a relative can invest money to pay for a child's later college tuition, and the account grows tax free. If Lily wants to set up a 529 account for her new granddaughter, wants the account to grow to $40,000 over 18 years, and she believes the account will earn 6% compounded semi-annually (twice a year), how much will Lily need to invest in the account now?

Solution

Since the account is earning 6%, $r = 0.06$

Since interest is compounded twice a year, $k = 2$

In this problem, we don't know how much we are starting with, so we will be solving for a, the initial amount needed. We do know we want the end amount to be $40,000, so we will be looking for the value of a so that $A(18) = 40,000$.

$$40,000 = A(18) = a\left(1 + \frac{0.06}{2}\right)^{2(18)}$$
$$40,000 = a(2.8983)$$
$$a = \frac{40,000}{2.8983} \approx \$13,801$$

Lily will need to invest $13,801 to have $40,000 in 18 years.

Euler's Number and The Natural Exponential Function

As we saw earlier, the amount we earn increases as we increase the compounding frequency. The table, though, shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to believe that although increasing the frequency of compounding will increase our result, there is an upper limit to this process.

To see this, let us examine the value of $1 invested at 100% interest for 1 year.

Frequency	Value
Annual	$2
Quarterly	$2.441406
Monthly	$2.613035
Daily	$2.714567
Hourly	$2.718127
Once per minute	$2.718279
Once per second	$2.718282

These values do indeed appear to be approaching an upper limit. This value ends up being so important that it gets represented by its own letter, much like how π represents a number. This number is denoted by $\boldsymbol{e}$. The famous mathematician, Leonard Euler, was the first person to use this symbol to describe this number in 1731.

Euler's number: e

e is the letter used to represent the value that $\left(1+\frac{1}{k}\right)^k$ approaches as k gets big.

$$e \approx 2.718282$$

Because e is often used as the base of an exponential, most scientific and graphing calculators have a button that can calculate powers of e, usually labeled e^x. Some computer software instead defines a function $exp(x)$, where $exp(x) = e^x$.

Because e arises when the time between compounds becomes very small, e allows us to define **continuous growth** and allows us to define a new toolkit function, $f(x) = e^x$.

Continuous Growth Formula

Continuous Growth can be calculated using the formula

$$f(x) = ae^{rx}$$

where

a is the starting amount

r is the continuous growth rate

Euler's number is commonly used when describing quantities that change more or less continuously in nature, like chemical reactions, growth of large populations, and radioactive decay. The **natural exponential function** is an exponential function with a base e.

The Natural Exponential Function

The **natural exponential function** is the exponential function defined as

$$f(x) = e^x$$

Example $Page 5.1.7$

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon-222 decay to in 3 days?

Solution

Since we are given a continuous decay rate, we use the continuous growth formula. Since the substance is decaying, we know the growth rate will be negative: r = -0.173

$f(3) = 100e^{-0.173(3)} \approx 59.512$ mg of Radon-222 will remain.

You Try $Page 5.1.2$

Interpret the following: $S(t) = 20e^{0.12t}$ if $S(t)$ represents the growth of a substance in grams, and time is measured in days.

Answer

An initial substance weighing 20g is growing at a continuous rate of 12% per day.

Example $Page 5.1.8$

If $1000 is invested in an account earning 10% compounded continuously, find the value after 1 year.

Solution

Here, the continuous growth rate is 10%, so r = 0.10. We start with $1000, so a = 1000.

To find the value after 1 year,

$$f(1) = 1000e^{0.10(1)} \approx \$1105.17$$

Notice this is a \$105.17 increase for the year. As a percent increase, this is $\frac{105.17}{1000} = 0.10517 = 10.517\%$ increase over the original \$1000.

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Page 5.2: Graphing Exponential Functions

Learning Objectives

- Identify basic graphs of exponential functions and sketch their graphs.
- Identify transformations of exponential functions.
- Graph exponential functions.
- Find equations of exponential functions that model graphical information.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table $Page 5.2.1$ change as the input increases by 1.

Table $Page 5.2.1$

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x) = ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

Figure $Page 5.2.1$ shows the exponential growth function $f(x) = 2^x$.

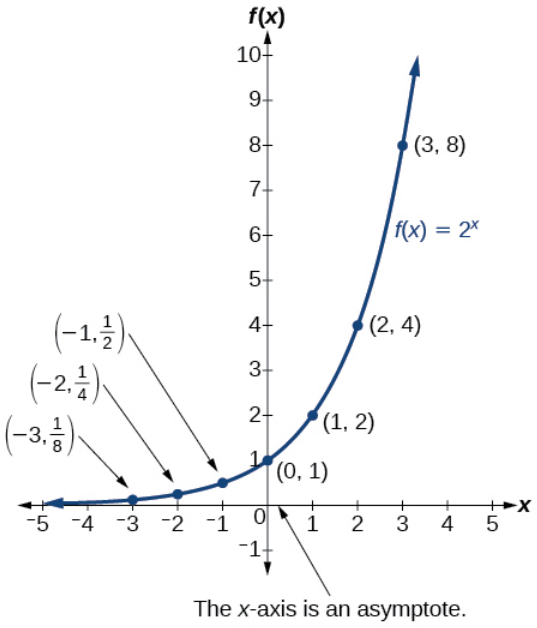


Figure $Page 5.2.1$: Notice that the graph gets close to the x-axis, but never touches it.

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table $Page 5.2.2$ change as the input increases by 1.

Table $Page 5.2.2$

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

Figure $Page 5.2.2$ shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

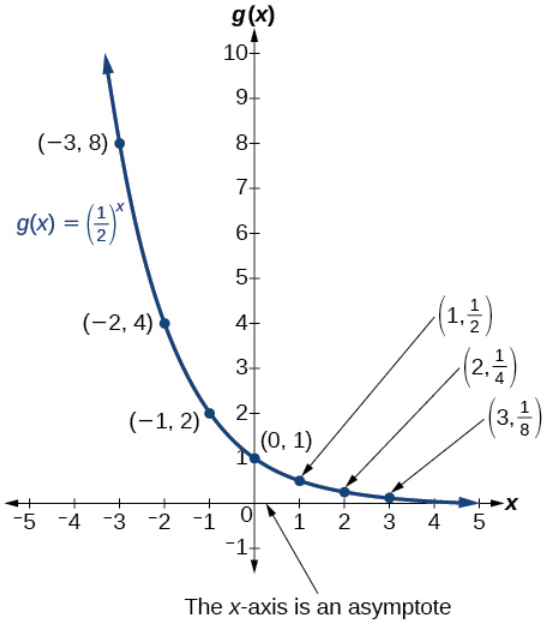


Figure $Page 5.2.2$

The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

Characteristics of Basic Exponential Functions

An exponential function of the form $f(x) = b^x$, $b > 0$, $b \neq 1$, has the following characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x-intercept: none
- y-intercept: $(0, 1)$

- point at $(1, b)$
- increasing if $b > 1$
- decreasing if $b < 1$

Figure $Page 5.2.3$ compares the graphs of exponential growth and decay functions.

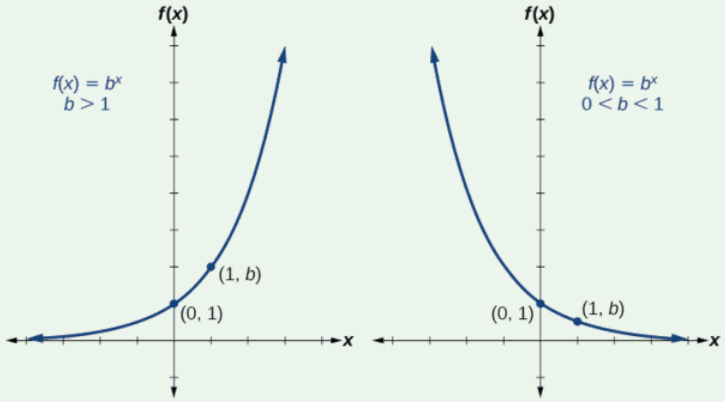


Figure $Page 5.2.3$

Example $Page 5.2.1$

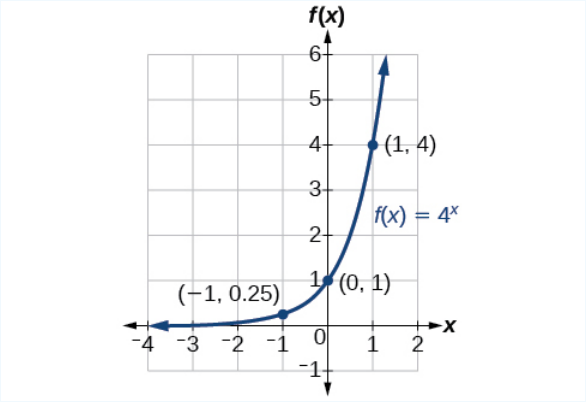
Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points for the graph.

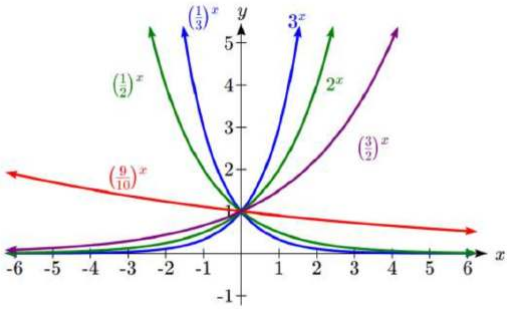
- Since $b = 4$ is greater than one, we know the function is increasing. The left tail of the graph will approach the horizontal asymptote, $y = 0$, and the right tail will increase without bound.
- The y-intercept is $(0, 1)$.
- The point $(1, 4)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



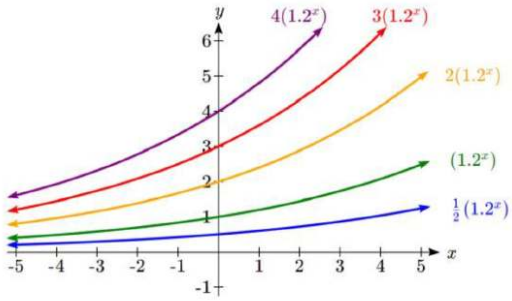
To get a better feeling for the effect of a and b on the graph of $f(x) = ab^x$, examine the sets of graphs below. The first set shows various graphs, where a remains the same and we only change the value for b.

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Notice that the closer the value of b is to 1, the less steep the graph will be. The growth/decay is faster the further b is from 1.

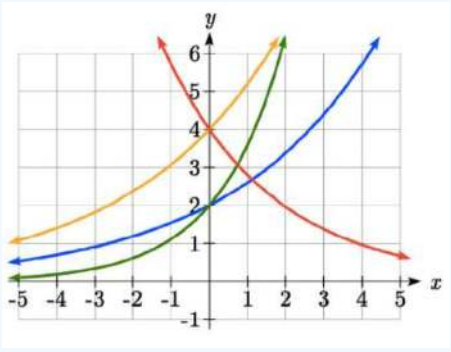
In the next set of graphs, a is altered and our value for b remains the same.



Notice that changing the value for a changes the y-intercept. Since a is multiplying the b^x term, a acts as a vertical stretch factor, not as a shift. Notice also that the end behavior for all of these functions is the same because the growth factor did not change and none of these a values introduced a vertical flip.

Example $Page 5.2.2$

Match each equation with its graph.



$$f(x) = 2(1.3)^x$$

$$g(x) = 2(1.8)^x$$

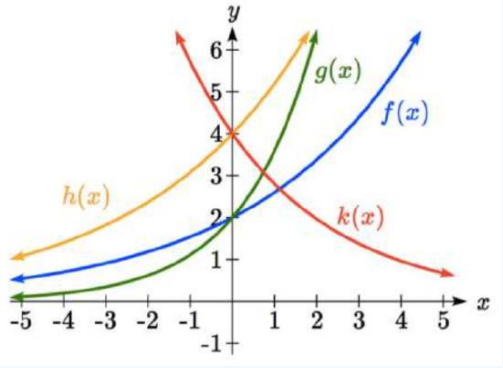
$$h(x) = 4(1.3)^x$$

$$k(x) = 4(0.7)^x$$

Solution

The graph of $k(x)$ is the easiest to identify, since it is the only equation with a growth factor less than one, which will produce a decreasing graph. The graph of $h(x)$ can be identified as the only growing exponential function with a vertical intercept at (0,4). The graphs of $f(x)$ and $g(x)$ both have a vertical intercept at (0,2), but since $g(x)$ has a larger growth factor, we can identify it as the graph increasing faster.

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Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other basic functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the basic function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the basic function $f(x) = b^x$, giving us a vertical shiftd d units in the same direction as the sign. For example, if we begin by graphing a basic function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure $Page 5.2.4$.

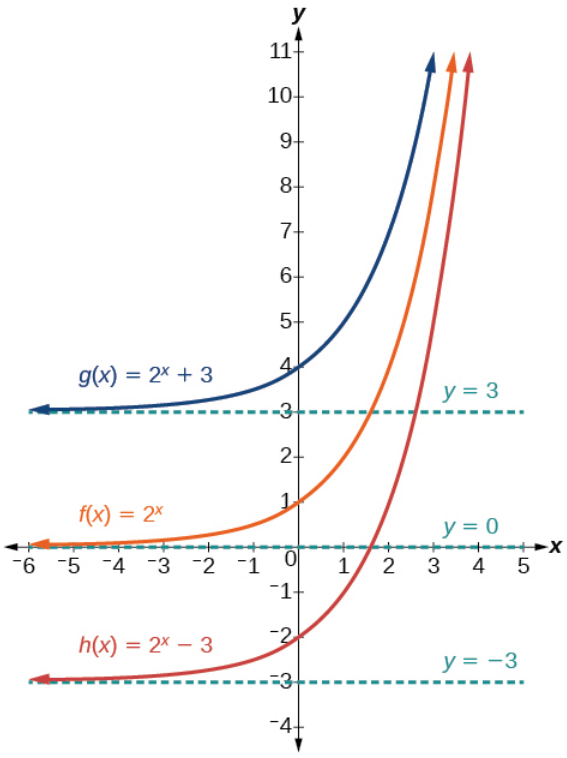


Figure $Page 5.2.4$

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The y-intercept shifts up 3 units to $(0, 4)$.

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- The asymptote shifts up 3 units to $y = 3$.
- The range becomes $(3, \infty)$.

- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The *y*-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the basic function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the basic function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure $Page 5.2.5$.

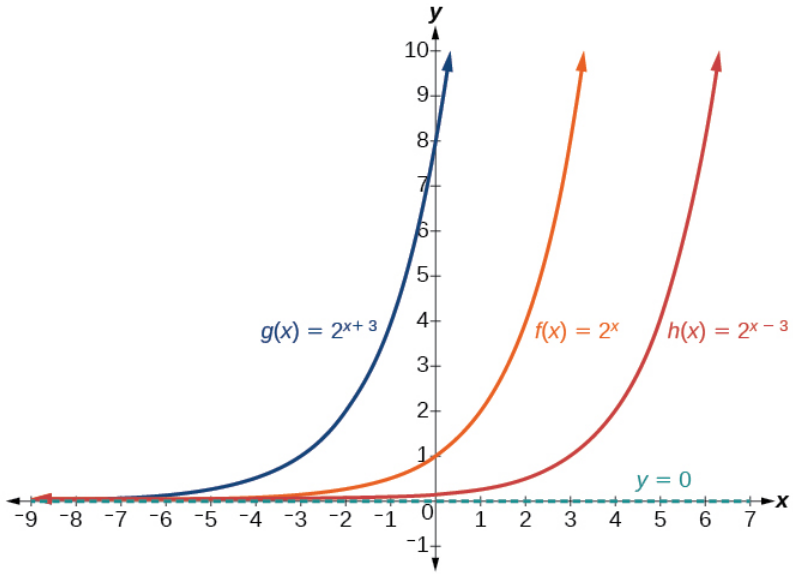


Figure $Page 5.2.5$

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$,remains unchanged.
- The asymptote, $y = 0$,remains unchanged.
- The *y*-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$, the y-intercept of the basic function shifts to (-3,1). The y-intercept of $g(x)$ is $(0, 8)$ which can be found by evaluating the function at $x = 0$.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$, the y-intercept of the basic function shifts to (3,1). The y-intercept of $h(x)$ is $(0, \frac{1}{8})$.

Example $Page 5.2.3$

Identify the transformations and use them to sketch the graph of $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

Solution

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

The basic function is $y = 2^x$. The graph will shift left 1 unit and down 3 units.

Shifting left 1 unit and down 3 units results in the y-intercept of the basic graph shifting to $(-1, -2)$. The point $(1, 2)$ on the basic graph shifts to $(0, -1)$. The horizontal asymptote $y = -3$.

Using these key points, asymptote, and the shape of $f(x) = 2^x$, shift the whole graph left 1 units and down 3 units.

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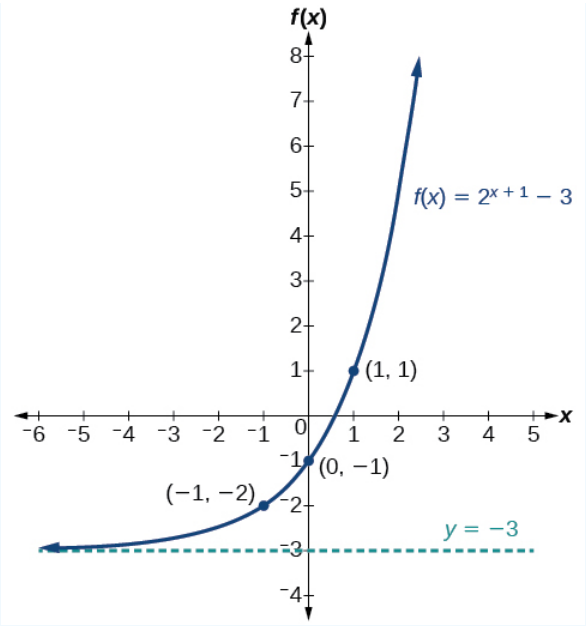


Figure $Page 5.2.6$

The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

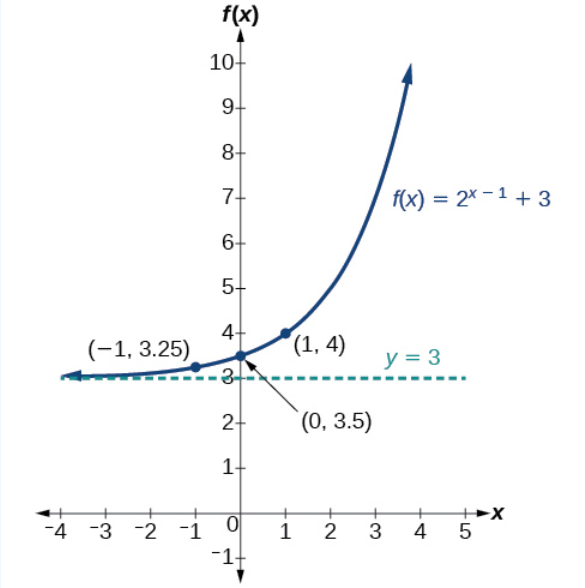
You Try $Page 5.2.1$

Identify the transformations and use them to sketch the graph of $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

Answer

The basic function is $y = 2^x$. The graph will shift right 1 unit and up 3 units.

The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.



Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the x-axis or the y-axis. When we multiply the basic function $f(x) = b^x$ by -1, we get a reflection about the x-axis. When we multiply the input by -1, we get a reflection

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about the y-axis. The reflection about the *x*-axis, $g(x) = -2^x$, is shown on the left side of Figure $Page 5.2.7$, and the reflection about the y-axis $h(x) = 2^{-x}$, is shown on the right side of Figure $Page 5.2.7$.

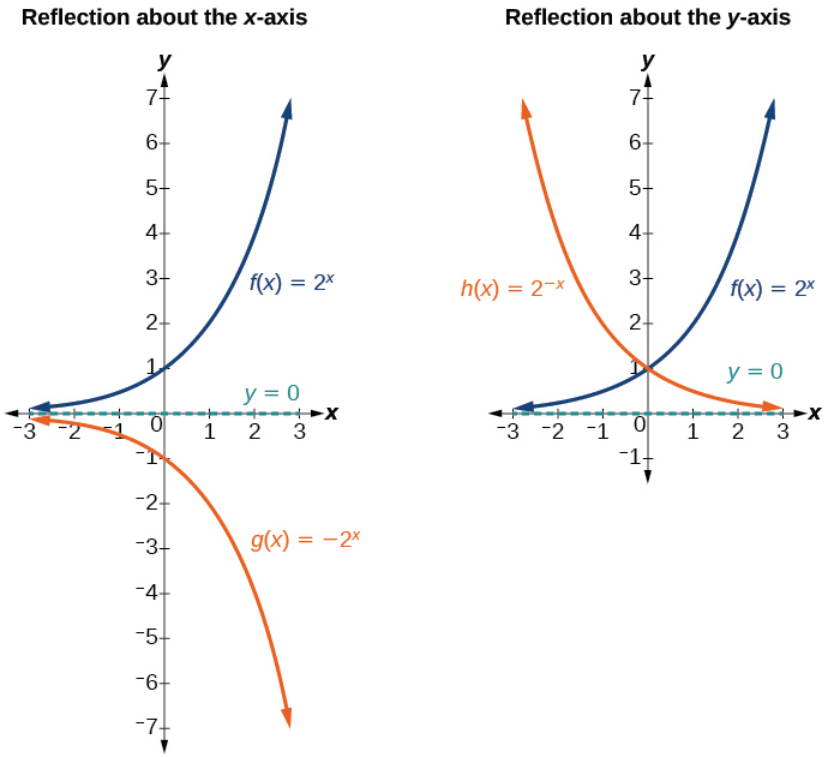


Figure $Page 5.2.7$: (a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the y-axis.

Example $Page 5.2.4$

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = \left(\frac{1}{4}\right)^x$ about the x-axis. State its domain, range, and asymptote.

Solution

Since we want to reflect the basic function $f(x) = \left(\frac{1}{4}\right)^x$ about the x-axis, we multiply $f(x)$ by -1 to get, $g(x) = -\left(\frac{1}{4}\right)^x$.

Next we create a table of points as in Table $Page 5.2.3$.

Table $Page 5.2.3$

x	-3	-2	-1	0	1	2	3
$g(x) = -(\frac{1}{4})^x$	-64	-16	-4	-1	-0.25	$\mathbf{-0.0625}$	$\mathbf{-0.0156}$

Plot the y-intercept, $(0, -1)$, along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

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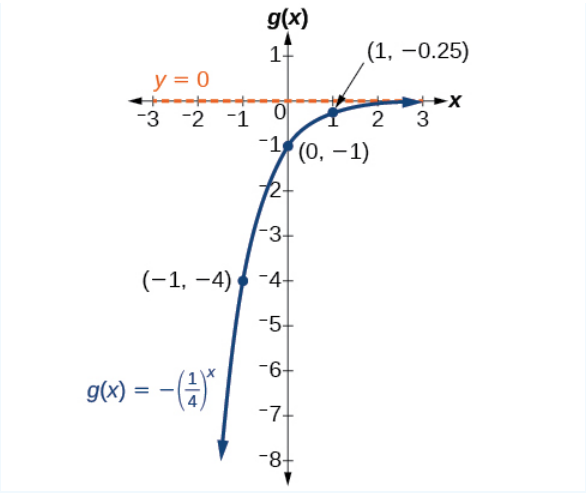


Figure $Page 5.2.8$

The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

Example $Page 5.2.5$

Identify the transformations and use them to sketch the graph of $h(x) = 3^{-x+2} - 4$. State domain, range, and asymptote.

Solution

The basic function is $y = 3^x$. Since x is negative, we must first factor so we can identify the transformations.

$$\begin{aligned} f(x) &= 3^{-x+2} - 4 \\ &= 3^{-(x-2)} - 4 \end{aligned}$$

There is a y-axis reflection, a horizontal shift to the right 2 units, and a vertical shift down 4 units. The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = -4$.

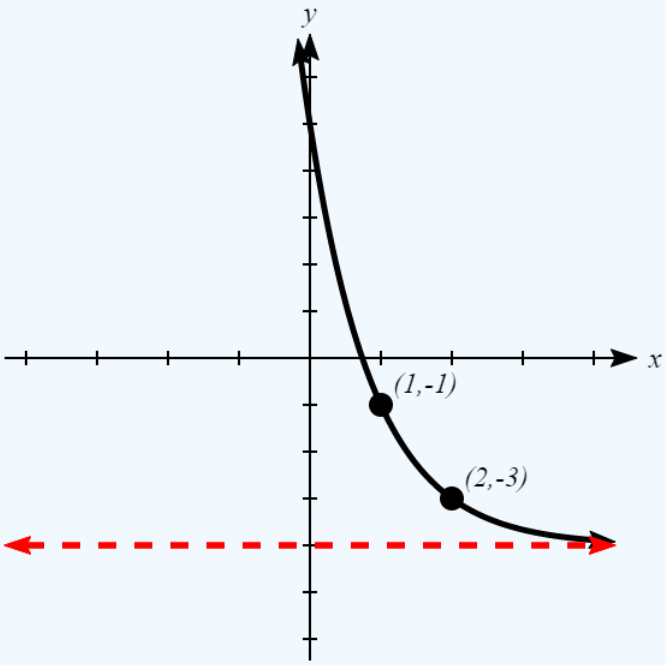


Figure $Page 5.2.9$: Graph of Transformed Exponential Function h(x).

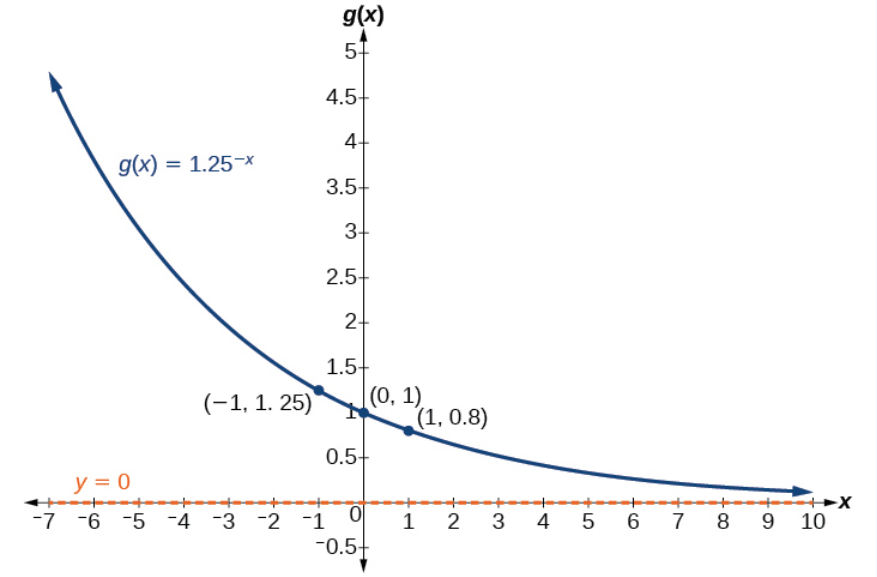
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? You Try $Page 5.2.2$

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the y-axis. State its domain, range, and asymptote.

Answer

The equation of the reflected graph is $y == 1.25^{-}x$; the domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a vertical stretch or compression occurs when we multiply the basic function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the basic function $f(x) = 2^x$,we can then graph the stretch, using $a = 3$,to get $g(x) = 3(2)^x$ as shown on the left in Figure $Page 5.2.10$, and the compression, using $a = \frac{1}{3}$,to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure $Page 5.2.10$.

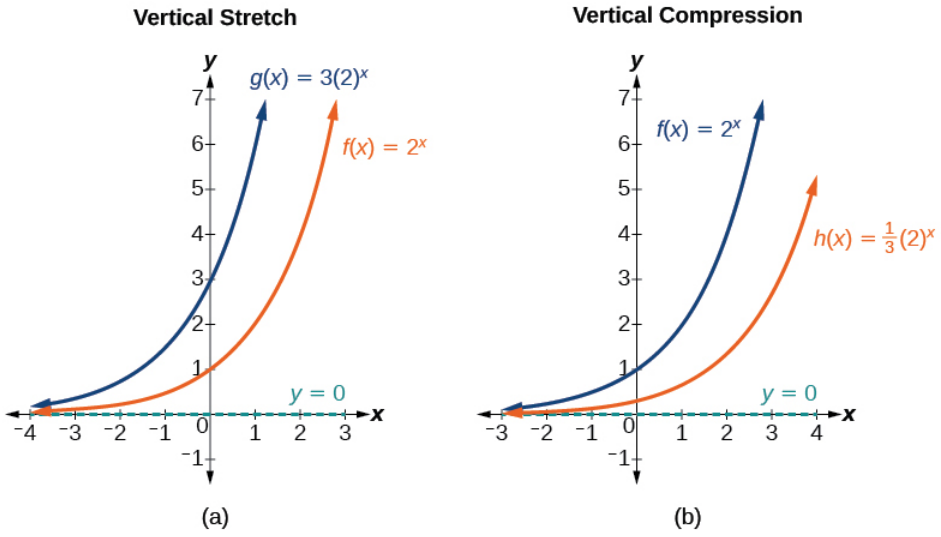


Figure $Page 5.2.10$: (a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

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Example $Page5.2.6$

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ this is a decreasing exponential function. The left end of the graph will increase without bound as x decreases, and the right tail will approach the x-axis as x increases.
- Since $a = 4$, the graph of $f(x) = 4\left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.
- Create a table of points as shown in Table $Page5.2.4$.

Table $Page5.2.4$

x	-3	-2	-1	0	1	2	3
$f(x) = 4(\frac{1}{2})^x$	32	16	8	4	2	1	0.5

- Plot the y-intercept, $(0, 4)$, along with two other points. We can use $(-1, 8)$ and $(1, 2)$. You may plot other points so your graph is more accurate also.

Draw a smooth curve connecting the points, as shown in Figure $Page5.2.11$.

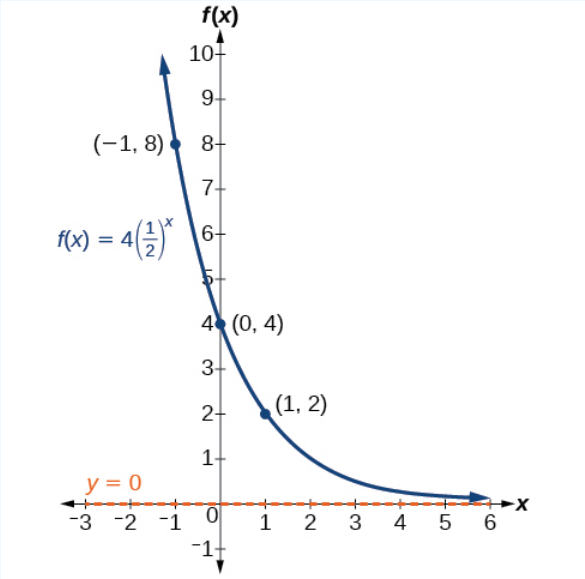


Figure $Page5.2.11$

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

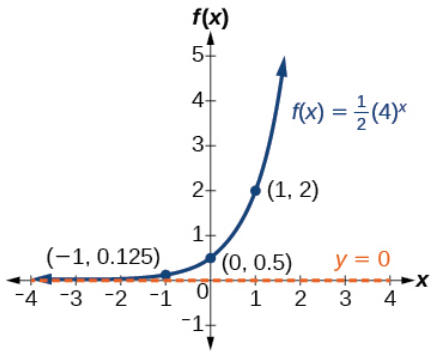
You Try $Page5.2.3$

Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

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Summary of Transformations of Exponential Functions

Now that we have worked with each type of transformation for the exponential function, we can summarize them to arrive at the general equation for translating exponential functions.

Transformations of Exponential Functions

A transformation of an exponential function has the form

$$f(x) = ab^{x+c} + d$$

where the basic function, $y = b^x$, $b > 1$, is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- shifted vertically d units.
- reflected about the x-axis when $a < 0$.

For $f(x) = b^{-x}$, the graph of the basic function is reflected about the y-axis.

Remember that the order for transformations follow the order of operations. This means that you start with stretch/compressions and reflections first, then shifts.

You Try $Page 5.2.4$

What is the domain, range, asymptote, and end behavior of $y = 8 - e^{-x+7}$?

Answer

The domain is $(-\infty, \infty)$, the range is $(-\infty, 8)$, the vertical asymptote is at $x = 8$, as $x \to -\infty$, $y \to -\infty$, and as $x \to \infty$, $y \to 8$.

Finding Equations of Exponential Functions that Model Graphical Information

Example $Page 5.2.7$

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the y-axis, and then shifted up 4 units.

Solution

We want to find an equation of the general form $f(x) = ab^{x+c} + d$. We use the description provided to find a, b, c, and d.

- We are given the basic function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of 2, so $a = 2$.
- There is no horizontal shift so $c = 0$.

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- The function is reflected about the *y*-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$$\begin{aligned} f(x) &= ab^{x+c} + d \\ &= 2e^{-x+0} + 4 \\ &= 2e^{-x} + 4 \end{aligned}$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

? You Try $Page 5.2.5$

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the *x*-axis and then shifted down 2 units.

Answer

$f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, 2)$; the horizontal asymptote is $y = 2$.

✔ Example $Page 5.2.8$

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$. Round to the nearest fourth decimal place.

Solution

Because we are not given the base for the function, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = ab^{-2}$
- Substituting $(2, 1)$ gives $1 = ab^2$

Use the first equation to solve for a in terms of b:

$$\begin{aligned} 6 &= ab^{-2} \\ 6 &= \frac{a}{b^2} && \text{Use properties of exponents to rewrite with positive expoents} \\ 6b^2 &= a && \text{Multiply by common denominator} \end{aligned}$$

Substitute a in the second equation, and solve for b:

$$\begin{aligned} 1 &= ab^2 \\ 1 &= 6b^2b^2 && \text{Substitute a} \\ 1 &= 6b^4 && \text{Now, solve for b} \\ \frac{1}{6} &= b^4 \\ \left(\frac{1}{6}\right)^{\frac{1}{4}} &= \left(b^4\right)^{\frac{1}{4}} \\ b &= \left(\frac{1}{6}\right)^{\frac{1}{4}} && \text{Round 4 decimal places rewrite the denominator} \\ b &\approx 0.6389 \end{aligned}$$

Use the value of b in the first equation to solve for the value of a:

$$\begin{aligned} a &= 6b^2 \\ &\approx 6(0.6389)^2 \\ &\approx 2.4492 \end{aligned}$$

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Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in Figure $Page 5.2.12$ passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

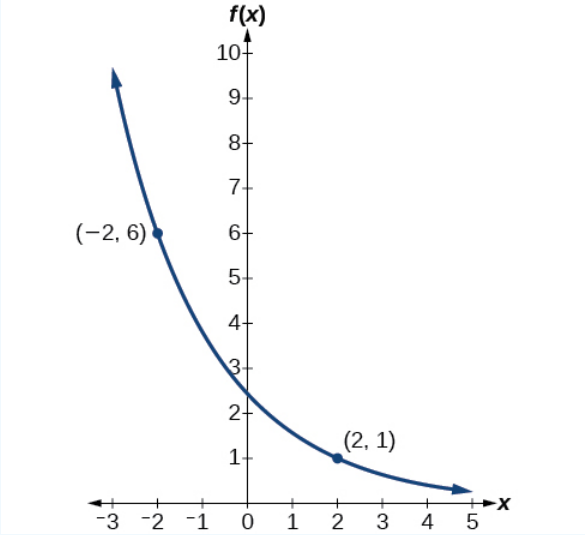


Figure $Page 5.2.12$: The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.

You Try $Page 5.2.6$

Given the two points $(1, 3)$ and $(2, 4.5)$, find the equation of the exponential function that passes through these two points.

Answer

$f(x) = 2(1.5)^x$

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Page 5.3: Logarithmic Functions

Learning Objectives

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.
- Finding inverses of exponential and logarithmic functions.

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, like those shown in Figure $Page5.3.1$. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.



Figure $Page5.3.1$: Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce).

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is

$$10^{8-4} = 10^4 = 10,000$$

times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in Figure $Page5.3.2$, to better estimate the solution.

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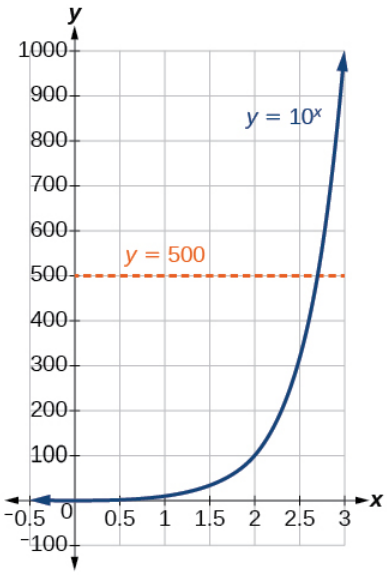


Figure $Page 5.3.2$

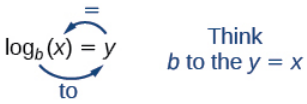
Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure $Page 5.3.2$ passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, "The logarithm with base b of x is equal to y," or, simplified, "log base b of x is y." We can also say, "b raised to the power of y is x," because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1$$

Note that the base b is always positive.



Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:

Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log^b(x)$ and $y = b^x$ are inverse functions.

Logarithmic Function

A logarithm base b of a positive number x satisfies the following definition.

For $x > 0, b > 0, b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x$$

where,

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- we read $\log_b(x)$ as, “the logarithm with base b of x” or the “log base b of x."
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

We call the input, x, the *argument* of the function.

Example $Page 5.3.1$

Write the following logarithmic equations in exponential form.

a. $\log_3(9) = 2$
b. $\log_6(\sqrt{6}) = \frac{1}{2}$
c. $\log_5(1) = 0$
d. $\log_{10}(1,000,000) = 6$

Solution

First, identify the values of b, y,and x. Then, write the equation in the form $b^y = x$.

a. $\log_3(9) = 2$

Here, $b = 3$, $y = 2$,and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to

$$3^2 = 9$$

b. $\log_6(\sqrt{6}) = \frac{1}{2}$

Here, $b = 6$, $y = \frac{1}{2}$,and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to

$$6^{\frac{1}{2}} = \sqrt{6}$$

c. $\log_5(1) = 0$

Here, $b = 5$, $y = 0$,and $x = 1$. Therefore, the equation $\log_5(1) = 0$ is equivalent to

$$5^0 = 1$$

You Try $Page 5.3.1$

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$
b. $\log_5(25) = 2$

Answer a

$\log_{10}(1,000,000) = 6$is equivalent to $10^6 = 1,000,000$

Answer b

$\log_5(25) = 2$ is equivalent to $5^2 = 25$

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic equations by rewriting them in exponential form.

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Example $Page 5.3.2$

Solve $\log_4(x) = 2$ for x.

Solution

By rewriting this expression as an exponential, $4^2 = x$, so $x = 16$.

You Try $Page 5.3.2$

Solve $\log_2(t) = 5$ for t.

Answer

$t = 32$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b,exponent x,and output y. Then we write $x = \log_b(y)$.

Example $Page 5.3.3$

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $4^2 = 16$
c. $10^{-4} = \dfrac{1}{10,000}$

Solution

First, identify the values of b, y,and x. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$

Here, $b = 2$, $x = 3$,and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $4^2 = 16$

Here, $b = 4$, $x = 2$,and $y = 16$. Therefore, the equation $4^2 = 16$ is equivalent to $\log_4(16) = 2$.

c. $10^{-4} = \dfrac{1}{10,000}$

Here, $b = 10$, $x = -4$,and $y = \dfrac{1}{10,000}$. Therefore, the equation $10^{-4} = \dfrac{1}{10,000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10,000}\right) = -4$.

You Try $Page 5.3.3$

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \dfrac{1}{2}$

Answer a

$3^2 = 9$ is equivalent to $\log_3(9) = 2$

Answer b

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$5^3 = 125$ is equivalent to $\log_5(125) = 3$

Answer c

$2^{-1} = \frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right) = -1$

By establishing the relationship between exponential and logarithmic functions, we can now solve basic exponential equations also.

Example $Page 5.3.4$

Solve $2^x = 10$ for x.

Solution

By rewriting this expression as a logarithm, we get $x = \log_2(10)$.

You Try $Page 5.3.4$

Solve $3^m = 11$ for m.

Answer

$m = \log_3(11)$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$?" We know $2^2 = 4$ and $3^2 = 9$, so

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Therefore,

$$\log_{\frac{2}{3}}\left(\frac{4}{9}\right) = 2.$$

Example $Page 5.3.5$

Evaluate the following without using a calculator:

1. $\log_4(64)$
2. $\log_3\left(\frac{1}{27}\right)$
3. $\log_5(-3)$

Solution

1. First let's set the expression equal to y, $y = \log_4(64)$, then we can rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must 4 be raised in order to get 64?"

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We know

$$4^3 = 64$$

So

$$4^y = 4^3$$

Therefore,

$$\log_4(64) = 3$$

2. First, set the expression equal to a variable, $y = \log_3\left(\frac{1}{27}\right)$, Then rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$.

   Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

   We know $3^3 = 27$,but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

   Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

3. Logarithms are only defined for positive values so $y = \log_5(-3)$ is undefined.

? You Try *Page*5.3.5

Evaluate the following without using a calculator:

1. $\log_{121}(11)$
2. $\log_2\left(\frac{1}{32}\right)$

Answer

1. $\log_{121}(11) = \frac{1}{2}$
2. $\log_2\left(\frac{1}{32}\right) = -5$

Properties of Logarithms

We saw above that it was convenient to convert to exponential form to evaluate logarithms. Recall that $\log_4(64) = 3$. We know that $4^3 = 64$, so we substitute this in, we get

$$\log_4(64) = \log_4(4^3) = 3$$

This gives us our first property for logarithms:

$$\log_b(b^x) = x$$

Next, consider $x = \log_3 1$. Converting to exponential form, $3^x = 1$. From properties of exponents we know $x = 0$. We can conclude that \({\log}_31\=0). This bring sus to our second property for logarithms:

$$\log_b 1 = 0$$

Similarly, if we convert $x = \log_b b$ to exponetial form we will find that $x = \log_b b = 1$. This is our third property for logarithms:

$$\log_b b = 1$$

To evaluate $x = 4_4^{\log}(9)$, we can rewrite in logarithmic form as $\log_4 9 = \log_4 x$. This means $x = 9$. This give us the following property:

$b^{\log_b x} = x$ where $x > 0$

Properties of Logarithms

- $log_b(b^x) = x$
- $\log_b 1 = 0$
- $\log_b b = 1$
- $b^{\log_b x} = x$ where $x > 0$

Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base 10 logarithm a **common logarithm**. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

Common Logarithm

A common logarithm is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x$$

We read $\log(x)$ as, "the logarithm with base 10 of x " or "log base 10 of x."

The logarithm y is the exponent to which 10 must be raised to get x.

Example $Page 5.3.6$

Evaluate $y = \log(1000)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know

$$10^3 = 1000$$

Therefore, $\log(1000) = 3$.

You Try $Page 5.3.6$

Evaluate $y = \log(1,000,000)$ without using a calculator.

Answer

$\log(1,000,000) = 6$

Example $Page 5.3.7$

Evaluate $y = log(321)$ and round to four decimal places using a calculator.

Solution

Rounding to four decimal places, $\log(321) \approx 2.5065$

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$must be between $\log(100)$and $\log(1000)$. This gives us the following:

$$100 < 321 < 1000$$

$$2 < 2.5065 < 3$$

When using a calculator when approximating logarithms, it is important to make sure you input everything properly including having parentheses in the corect place.

Example $Page 5.3.8$

Use your calculator to approximate $log3 - 4$ to four decimal places.

Solution

Remember that sometimes parentheses are not written, $log3 - 4$ is equivalent to $log(3) - 4$.

$log3 - 4 \approx -3.5229$

You Try $Page 5.3.7$

Evaluate $y = \log(123)$and round to four decimal places using a calculator.

Answer

$\log(123) \approx 2.0899$

Example $Page 5.3.9$

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. Rounding to three decimal places, what was the difference in magnitudes?

Solution

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$

$$\log(500) = x \text{ Use the definition of the common log.}$$

Next we evaluate the logarithm using a calculator:

To the nearest thousandth, $\log(500) \approx 2.699$

The difference in magnitudes was about 2.699.

You Try $Page 5.3.8$

The amount of energy released from one earthquake was $8,500$times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Answer

The difference in magnitudes was about 3.929

Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation,$\ln(x)$. Most values of $\ln(x)$ can be found

only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the ln key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

Natural Logarithm

A natural logarithm is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

Example $Page 5.3.10$

Approximate $y = \ln(500)$ to four decimal places using a calculator.

Solution

Rounding to four decimal places, $\ln(500) \approx 6.2146$

You Try $Page 5.3.9$

Approximate $\ln(-500)$ to four decimal places using a calculator.

Answer

It is not possible to take the logarithm of a negative number so $\ln(-500)$ is undefined.

You Try $Page 5.3.10$

Solve $\ln(a) = 7$ for a.

Answer

$a = e^7$

Note that this answer is exact as e is an irrational number. To represent as a decimal approximation when asked, use your calculator to approximate.

Note

There a couple of things worth mentioning from what we have learned about logarithms in this section.

- Logarithms are functions, $f(x) = \log_b(x)$. We call the input, x, the *argument* of the function.
- When evaluating, sometimes we do not write parentheses around the argument. Example:

$$\log_7(4) \text{ is equivalent to } \log_7 4$$

- Common logarithms are written as $y = \log(x)$. All other logarithms, excluding the natural logarithm must have the base written or else it is assumed to be a common logarithm
- The natural logarithm is written as $y = \ln(x)$. This is the only logarithm with a function name, ln, that is different than all other logarithms.

Let's end with some examples of using the relationship between exponential and logarithmic functions to find inverse functions.

Example $Page 5.3.11$

Given $J(x) = \log_3(x-2)+4$, find J^{-1}

Solution

$$\begin{aligned} J(x) &= \log_3(x-2)+4 \\ y &= \log_3(x-2)+4 \\ x &= \log_3(y-2)+4 \\ x-4 &= \log_3(y-2) \\ 3^{x-4} &= y-2 \\ 3^{x-4}+2 &= y \end{aligned}$$

Hence $J^{-1} = 3^{x-4}+2$

Notice that $J(x)$ is a basic logarithm function of base 3 that has a horizontal shift to the right 2 units and a vertical shift up 4 units. J^{-1} has a horizontal shift to the right 4 units and a vertical shift up 2 units. When working with transformed functions, the inverse will have its transformations reversed between the horizontal and vertical directions. Remember that the domain and range of functions and their inverse are also reversed.

You Try $Page 5.3.11$

Find the inverse of $V(x) = 5 - e^{x+3}$

Answer

$y = \ln(5-x) - 3$

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Page 5.4: Graphing Logarithmic Functions

Learning Objectives

- Identify the domain of logarithmic functions.
- Identify transformations of logarithmic functions.
- Graph logarithmic functions.
- Find equations of logarithmic functions that model graphical information.

Earlier in this chapter, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5%, compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? Figure $Page 5.4.1$ shows this point on the logarithmic graph.

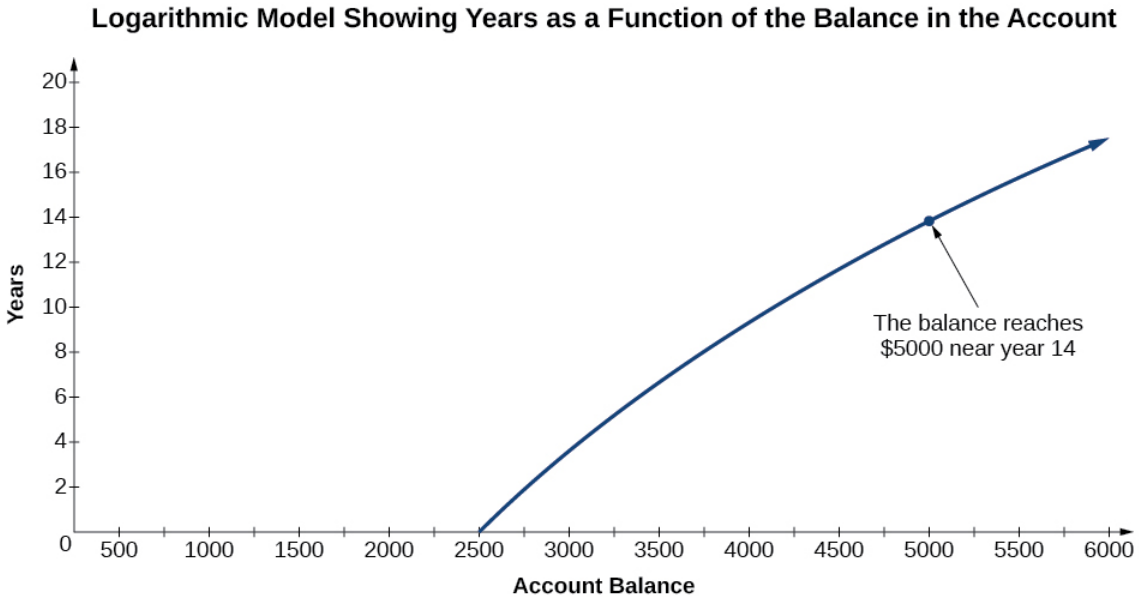


Figure $Page 5.4.1$

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.
- The range of y is $(0, \infty)$.

In the last section, we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

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- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the basic function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other basic functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the basic function without loss of shape.

We've seen that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the basic function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$, is greater than zero. To find the domain, we set up an inequality and solve for x:

$$\begin{aligned} 2x - 3 &> 0 && \text{Show the argument greater than zero} \\ 2x &> 3 && \text{Add 3} \\ x &> \frac{3}{2} && \text{Divide by 2} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $\left(\frac{3}{2}, \infty\right)$.

Example $Page 5.4.1$

Find the domain of the following functions:

1. $f(x) = \log_2(x + 3)$
2. $g(t) = 4 - \log(5 - 2t)$

Solution

1. The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$\begin{aligned} x + 3 &> 0 && \text{The input must be positive} \\ x &> -3 && \text{Subtract 3} \end{aligned}$$

   The domain of $f(x) = \log_2(x + 3)$ is $(-3, \infty)$.

2. This function is defined when $5 - 2t > 0$. Solving this inequality,

$$\begin{aligned} 5 - 2t &> 0 && \text{The input must be positive} \\ -2t &> -5 && \text{Subtract 5} \\ t &< \frac{5}{2} && \text{Divide by -2 and switch the inequality} \end{aligned}$$

   The domain of $g(t) = 4 - \log(5 - 2t)$ is $\left(-\infty, \frac{5}{2}\right)$.

Example $Page 5.4.2$

What is the domain of $h(x) = \log_4(x^2 - 4x + 3) + 2$?

Solution

This function is defined when $x^2 - 4x + 3 > 0$. Solving this inequality using a sign chart,

$$x^2 - 4x + 3 > 0$$

$$(x - 3)(x - 1) > 0$$

	(-∞,1)	(1,3)	(3,∞)
(x-3)	-	-	+
(x-1)	-	+	+
h(x)	+	-	+

Figure $Page 5.4.2$: Sign Chart for h(x)

The domain is $(-3, \infty)$

? You Try $Page 5.4.1$

What is the domain of $f(x) = \log(3x + 7) + 2$?

Answer

$(-/dfrac 7 3, \infty)$

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the basic function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the basic function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table $Page 5.4.1$.

Table $Page 5.4.1$

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from Table $Page 5.4.1$, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See Table $Page 5.4.2$.

Table $Page 5.4.2$

$f(x) = 2^x$	$\left(\mathbf{-3, \frac{1}{8}}\right)$	$\left(\mathbf{-2, \frac{1}{4}}\right)$	$\left(\mathbf{-1, \frac{1}{2}}\right)$	$(0, 1)$	$(1, 2)$	$(2, 4)$	$(3, 8)$
$\boldsymbol{g(x) = \log_2(x)}$	$\left(\frac{1}{8}, -3\right)$	$\left(\frac{1}{4}, -2\right)$	$\left(\frac{1}{2}, -1\right)$	$(1, 0)$	$(2, 1)$	$(4, 2)$	$(8, 3)$

As we'd expect, the x- and y-coordinates are reversed for the inverse functions. Figure $Page 5.4.3$ shows the graph of f and g.

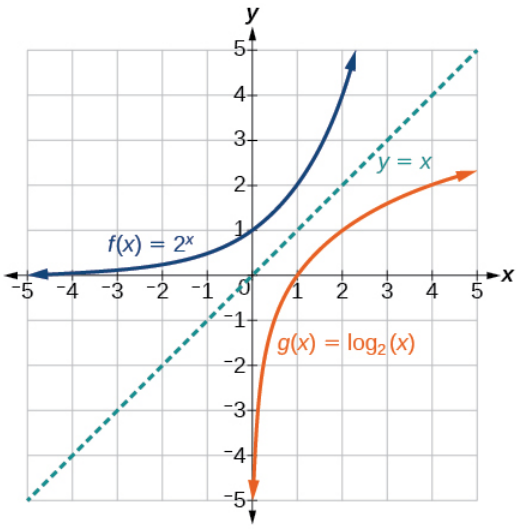


Figure $Page 5.4.3$: Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a y-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an x- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

Characteristics of the Graph of the Basic Function, $f(x) = log_b(x)$

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- x-intercept: $(1, 0)$
- y-intercept: none
- point at $(b, 1)$
- increasing if $b > 1$
- decreasing if $0 < b < 1$

Figure $Page 5.4.4$ compares the graphs of logarithmic functions with $b > 1$ and $0 < b < 1$

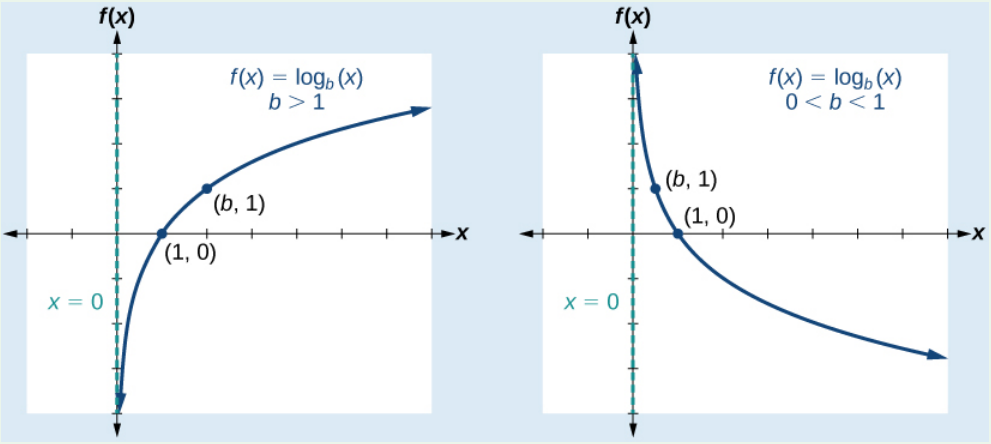


Figure $Page 5.4.4$

Figure $Page 5.4.5$ shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (Note: recall that the function $\ln(x)$ has base $e \approx 2.718$.)

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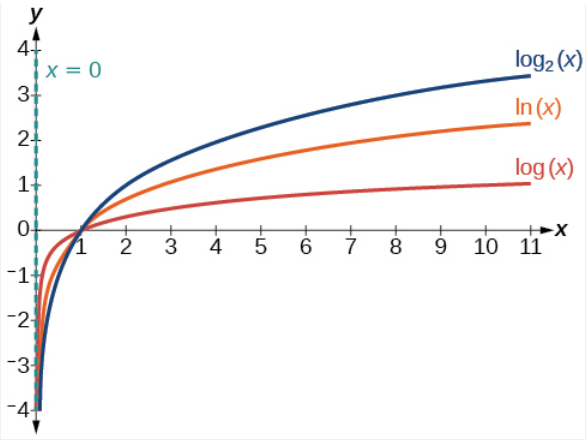


Figure $Page 5.4.5$: The graphs of three logarithmic functions with different bases, all greater than 1.

Example $Page 5.4.3$

Sketch the graph of $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote, $x = 0$, and the right tail will increase slowly without bound.
- The x-intercept is $(1, 0)$.
- The point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see Figure $Page 5.4.6$).

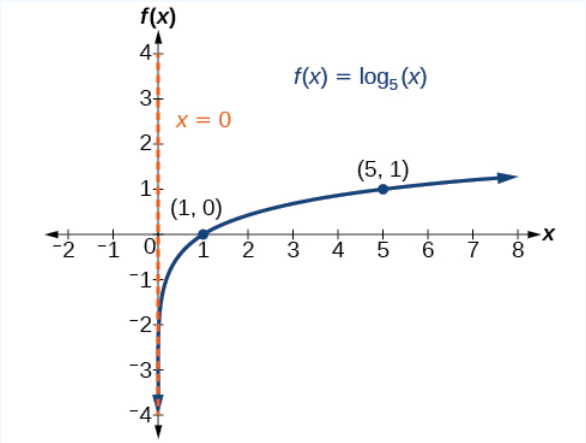


Figure $Page 5.4.6$

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other basic functions. We can shift, stretch, compress, and reflect the basic function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift

When a constant c is added to the input of the basic function $f(x) = \log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the basic function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure $Page 5.4.7$.

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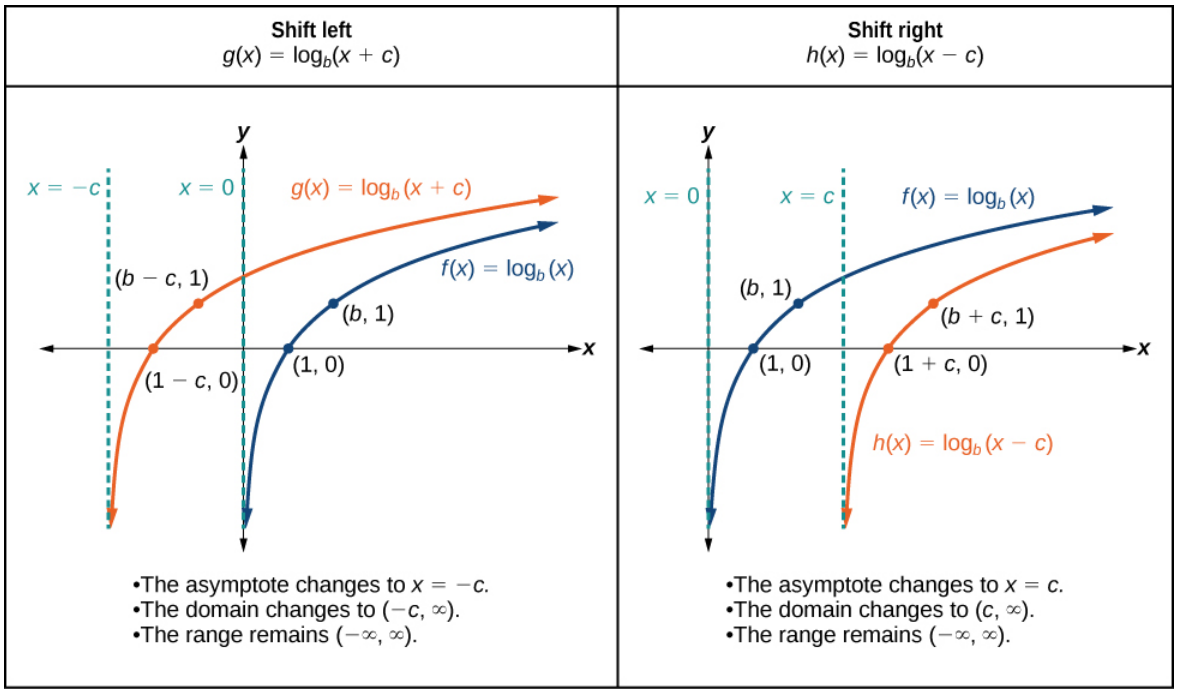


Figure $Page 5.4.7$

Example $Page 5.4.4$

Sketch the horizontal shift $f(x) = \log_3(x-2)$ alongside its basic function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = \log_3(x-2)$, we notice $x + (-2) = x - 2$.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the basic function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$, $(3, 0)$,and $(5, 1)$.

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

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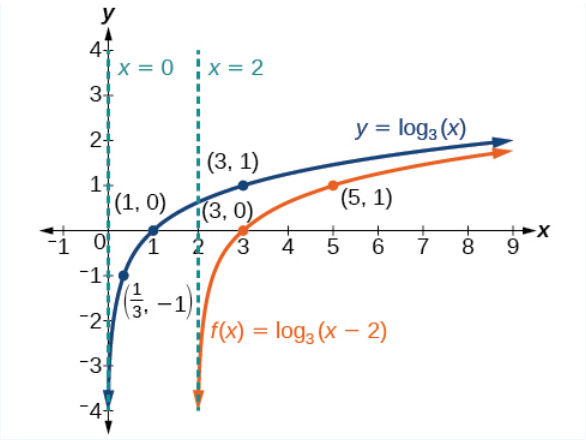


Figure $Page 5.4.8$

? You Try $Page 5.4.2$

Sketch a graph of $f(x) = \log_3(x+4)$ alongside its basic function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Answer

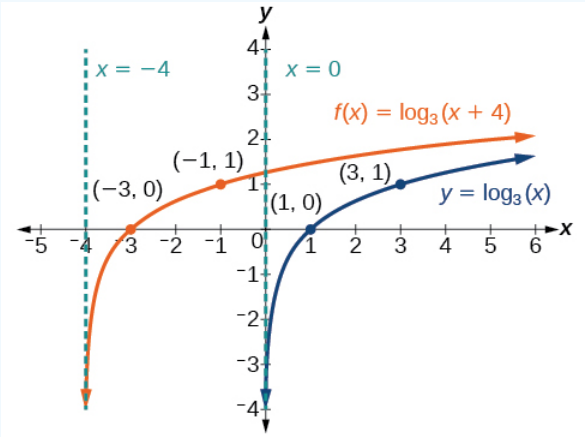


Figure $Page 5.4.9$

The domain is $(-4, \infty)$,the range $(-\infty, \infty)$,and the asymptote $x = -4$.

Graphing a Vertical Shift

When a constant d is added to the basic function $f(x) = \log_b(x)$,the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the basic function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$ .See Figure $Page 5.4.10$.

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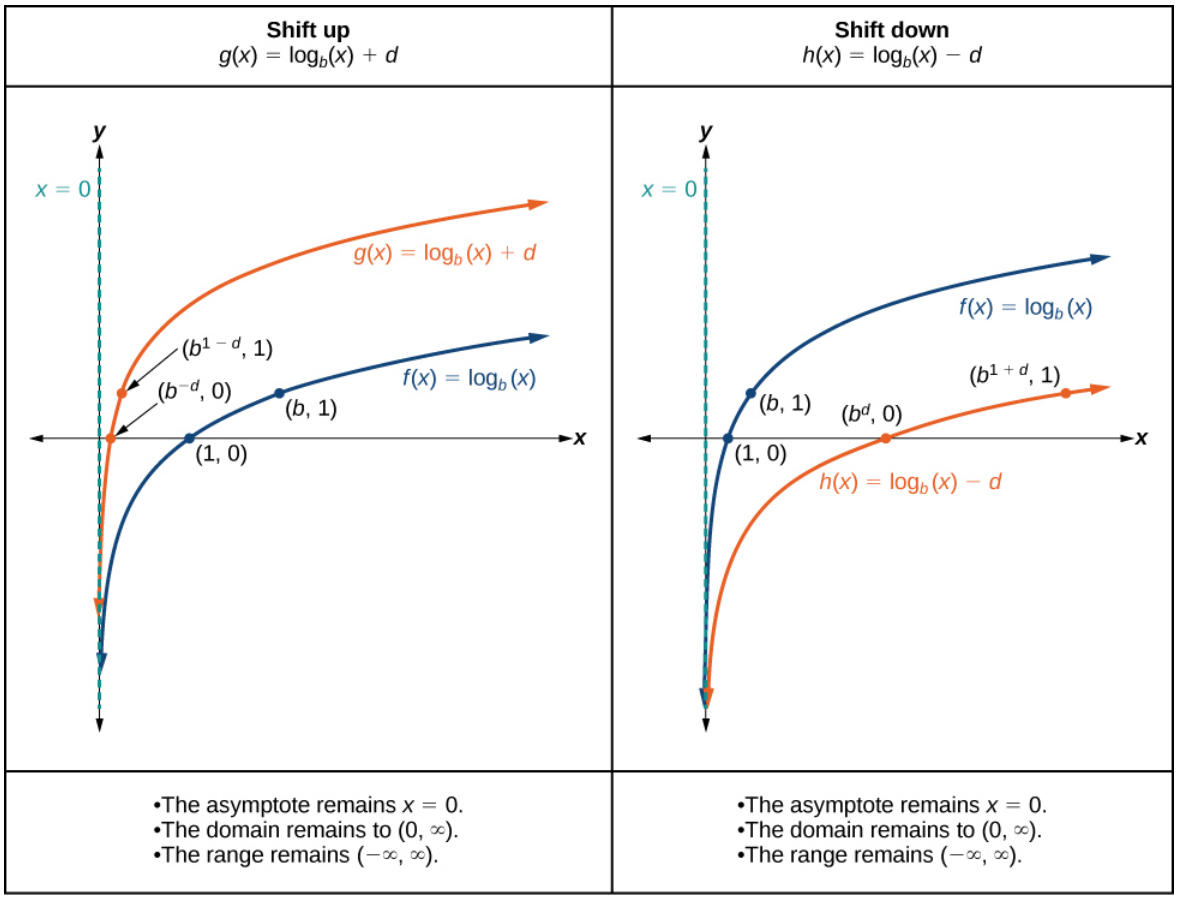


Figure $Page 5.4.10$

Example $Page 5.4.5$

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its basic function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = \log_3(x) - 2$,we will notice $d = -2$. Thus $d < 0$.

This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider the three key points from the basic function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by subtracting 2 from the y coordinates.

Label the points $\left(\frac{1}{3}, -3\right)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

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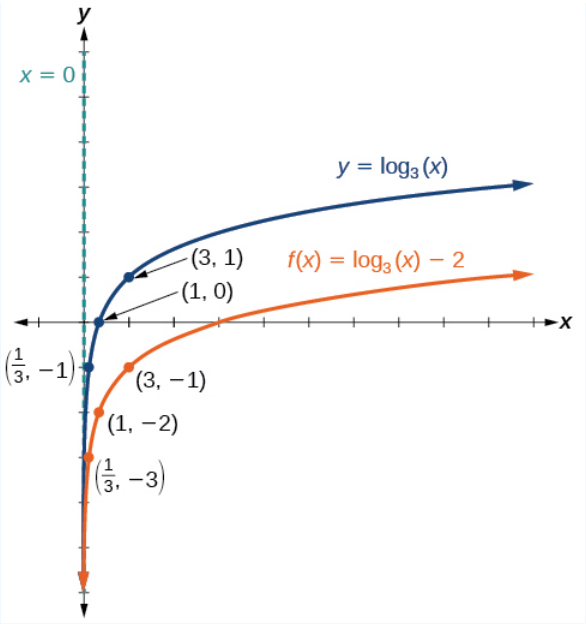


Figure $Page 5.4.11$

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Example $Page 5.4.6$

Identify the transformations and use them to sketch the graph of $L(x) = 2 + \log_3(x+1)$. State the domain, range, and asymptote.

Solution

We can rewrite $L(x) = 2 + \log_3(x+1)$ as $L(x) = \log_3(x+1) + 2$ which can be helpful for identifying transformations.

The basic function is $y = \log_3(x)$. The graph will shift left 1 unit and up 2 units.

Shifting left 1 unit and up 2 units results in the x-intercept of the basic graph shifting to $(0, 2)$. The point $(3, 1)$ on the basic graph shifts to $(2, 3)$. The horizontal asymptote $x = -1$.

Using these key points, asymptote, and the shape of $y = \log_3(x)$, shift the whole graph left 1 units and down 3 units.

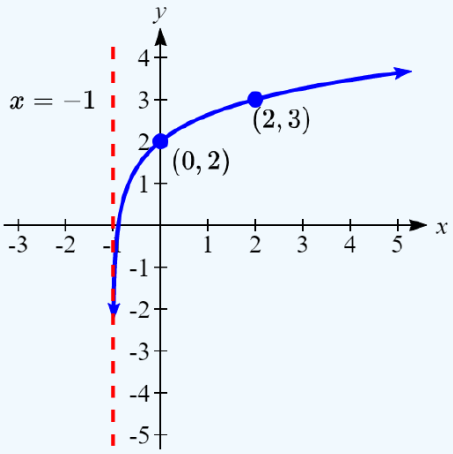


Figure $Page 5.4.12$: Graph of Transformed Logarithmic Function L(x)

The domain is $(-1, \infty)$, the range is $(-\infty, \infty)$, and the horizontal asymptote is $x = -1$

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You Try $Page 5.4.3$

Identify the transformations and use them to sketch the graph of $m(x) = \log_5(x-4) - 3$. State the domain, range, and asymptote.

Answer

The transformations are shift right 4 units and shift down 3 units.

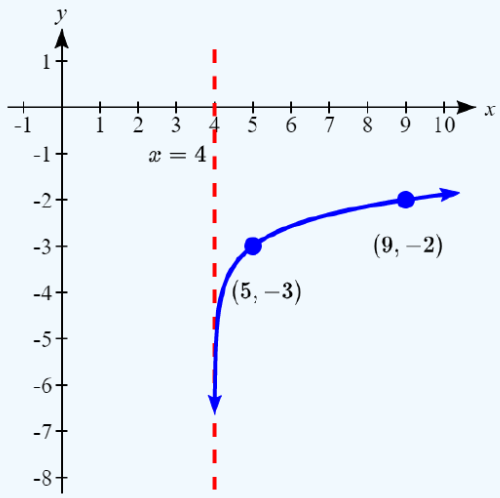


Figure $Page 5.4.13$: Graph of Transformed Logarithmic Function m(x)

The domain is $(4, \infty)$, the range is $(-\infty, \infty)$, and the horizontal asymptote is $x = 4$

Caution!

Remember that the input for a logarithm is called the argument. When the argument is a single term, it is a common practice to not write parentheses. Example:

$$f(x) = \log_4(x) \text{ is equivalent to } f(x) = \log_4 x$$

However, when the argument is more than one term, one must put parentheses around it. Example:

$$y = \ln(x+3) \text{ is not equivalent to } y = \ln x + 3$$

The function on the left is a natural logarithmic function shifted left 3 units, whereas, the function on the right is a natural logarithmic function shifted up 3 units.

In graphing calculators, one needs to make sure functions are inputted properly also.

Graphing Reflections of $f(x) = log_b(x)$

When the basic function $f(x) = \log_b(x)$ is multiplied by -1,the result is a reflection about the x-axis. When the *input* is multiplied by -1,the result is a reflection about the y-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the basic function $f(x) = \log_b(x)$ alongside the reflection about the x-axis, $g(x) = -\log_b(x)$ and the reflection about the y-axis, $h(x) = \log_b(-x)$.

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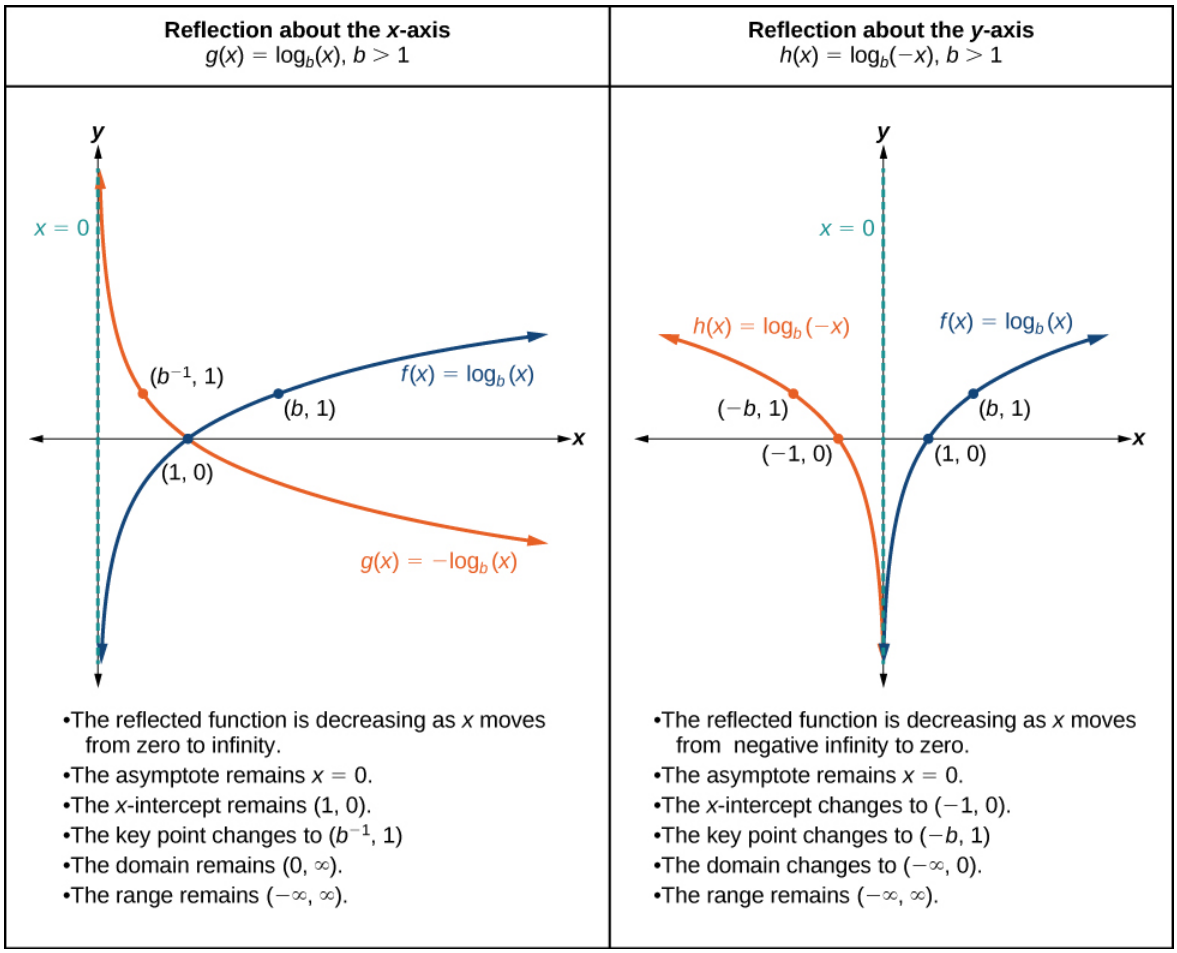


Figure $Page 5.4.14$

Example $Page 5.4.7$

Sketch a graph of $f(x) = \log(-x)$ alongside its basic function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Before graphing $f(x) = log(-x)$, $f(x) = log(-x)$,identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the basic function is increasing. Since the *input* value is multiplied by -1, $f(x)$ is a reflection of the basic graph about the y-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The x-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

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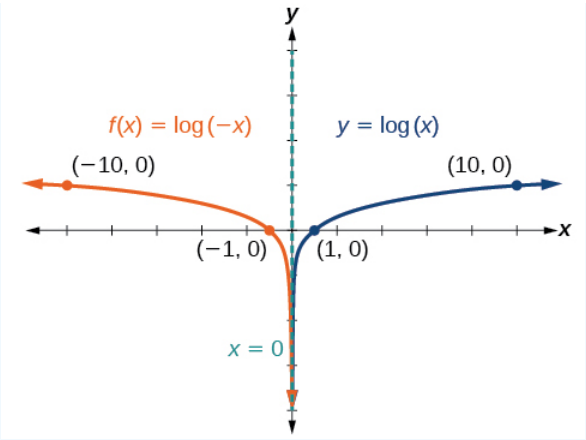


Figure $Page 5.4.15$

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

? You Try $Page 5.4.4$

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Answer

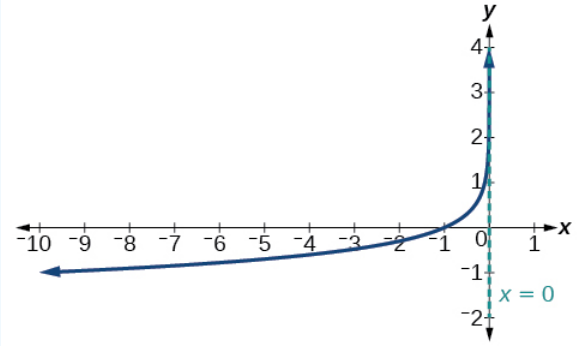


Figure $Page 5.4.16$

The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Graphing Stretches and Compressions of $y = log_b(x)$

When the basic function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the basic function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$. See Figure $Page 5.4.17$.

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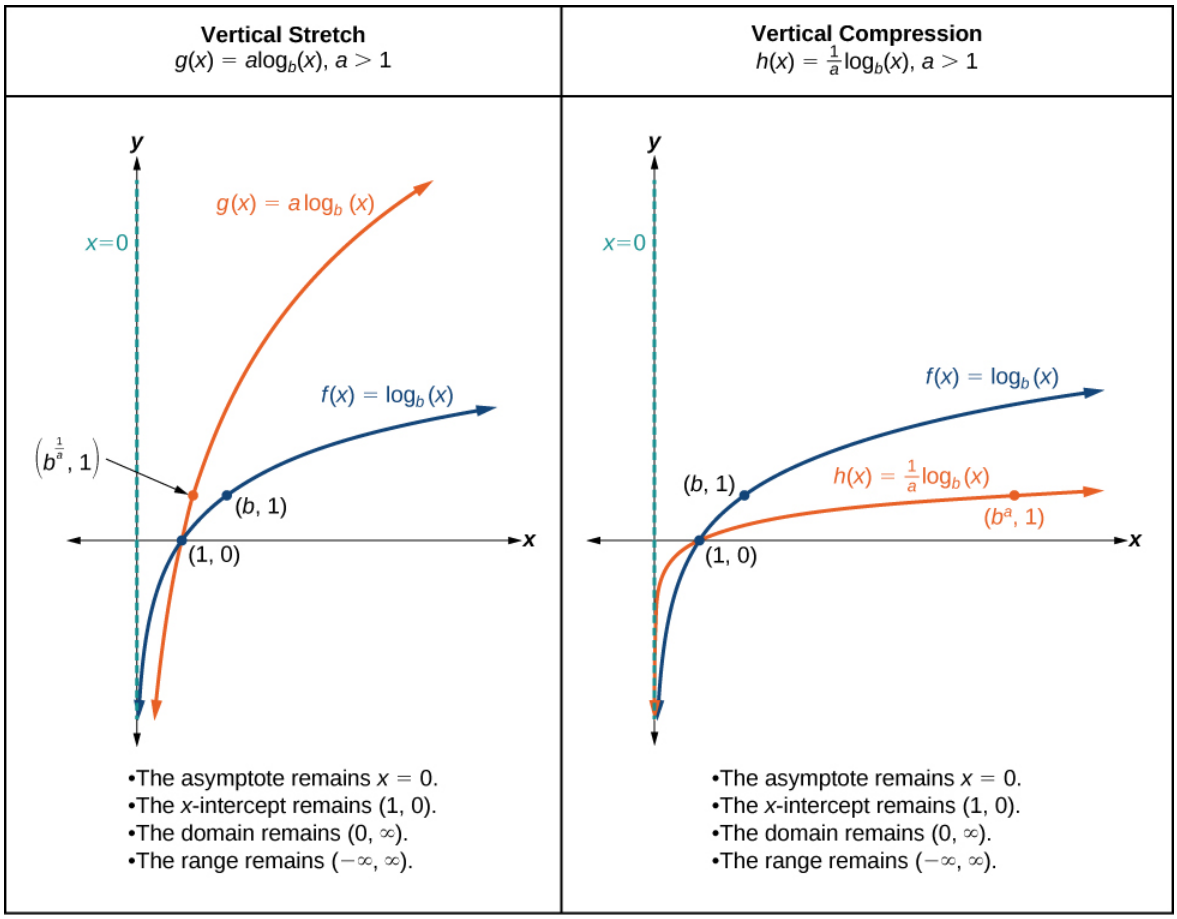


Figure $Page 5.4.17$

Example $Page 5.4.8$

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its basic function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = 2\log_4(x)$, we will notice $a = 2$.

This means we will stretch the function $f(x) = \log_4(x)$ by a factor of 2.

The vertical asymptote is $x = 0$.

Consider the three key points from the basic function, $\left(\frac{1}{4}, -1\right)$, $(1, 0)$, and $(4, 1)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $\left(\frac{1}{4}, -2\right)$, $(1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$. See Figure $Page 5.4.18$.

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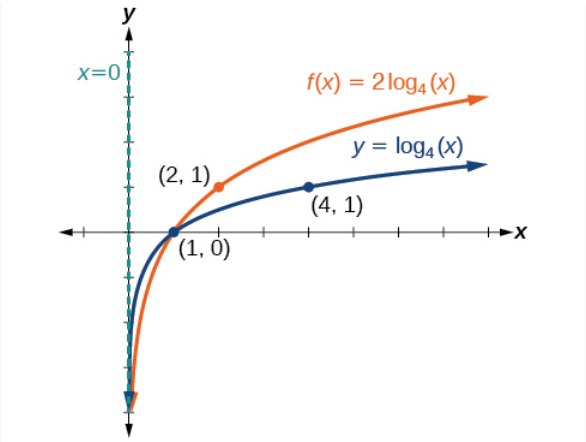


Figure $Page 5.4.18$

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Example $Page 5.4.9$

Sketch a graph of $f(x) = 5\log(x+2)$. State the domain, range, and asymptote.

Solution

Remember: what happens inside basicheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in Figure $Page 5.4.19$. The vertical asymptote will be shifted to $x = -2$. The x-intercept will be $(-1, 0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the x-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

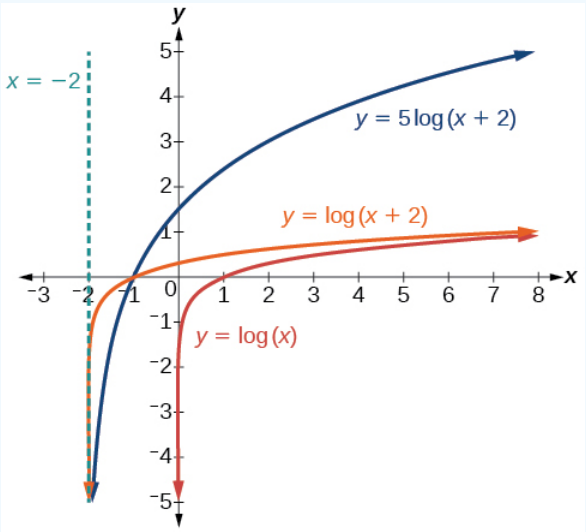


Figure $Page 5.4.19$

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = -2$.

You Try $Page 5.4.5$

Sketch a graph of the function $f(x) = 3\log(x-2)+1$. State the domain, range, and asymptote.

Answer

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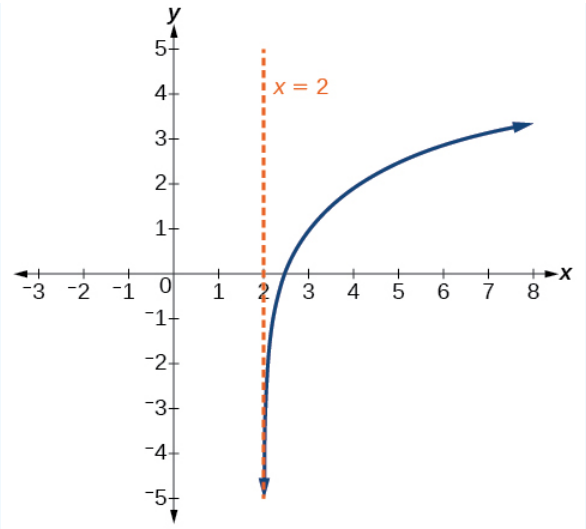


Figure $Page 5.4.20$

The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

Summarizing Transformations of the Logarithmic Function

Now that we have worked with each type of transformation for the logarithmic function, we can summarize each to arrive at the general equation for translating exponential functions.

Transformations of Logarithmic Functions

All transformations of the basic logarithmic function, $y = \log_b(x)$, have the form

$$f(x) = a\log_b(x + c) + d$$

where the basic function, $y = \log_b(x)$, $b > 1$,is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- reflected about the x-axis when $a < 0$.

For $f(x) = \log(-x)$, the graph of the basic function is reflected about the y-axis.

Example $Page 5.4.10$

What is the vertical asymptote and domain of $f(x) = -2\log_3(x + 4) + 5$?

Solution

The vertical asymptote is at $x = -4$.

Analysis

The coefficient, the base, and the upward transformation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to$x = -4$. The domain is $(-4, \infty)$

You Try $Page 5.4.6$

What is the vertical asymptote and domain of of $f(x) = 3 + \ln(-x + 1)$?

Answer

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The vertical asymptote is $x = 1$. The domain is $(-\infty, 1)$

Finding Equations of Logarithmic Functions that Model Graphical Information

Example $Page 5.4.11$

Write an equation of the natural logarithmic function that has no vertical stretch or compression that is graphed in Figure $Page 5.4.21$.

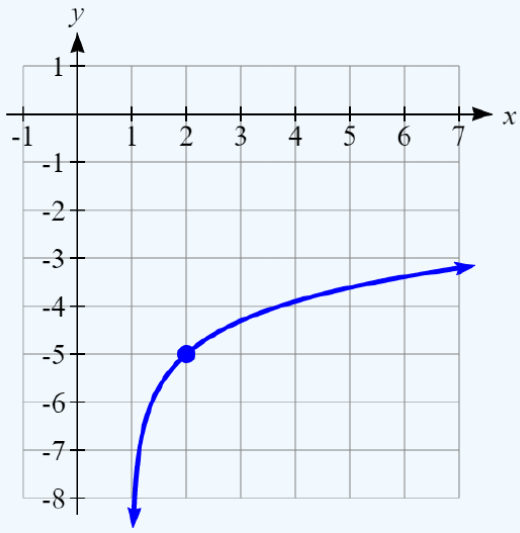


Figure $Page 5.4.21$: Graph of a Logarithmic Function

Solution

The vertical asymptote is at $x = 1$. There is a point at $(2, -5)$ that is one unit to the right of the asymptote which indicates this is the transformed x-intercept (*remember that for basic functions that the vertical asymptote is at* $x = 0$ *and the x-intercept is at* $(1, 0)$). This tells us the basic graph was shifted right 1 units and down 5 units. The equation for the graph is $y = \ln(x - 1) - 5$

You Try $Page 5.4.7$

Write an equation of the base 3 logarithmic function that is vertically stretched by 4, is reflected across the y-axis, and then shifted up 6 units.

Answer

$y = 4\log_3(-x) + 6$

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Page 5.5: Laws of Logarithms

Learning Objectives

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

In chemistry, the pH scale is used as a measure of the acidity or alkalinity of a substance. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 at $25^{\circ}C$
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where $[\mathrm{H}^+]$ is the concentration of hydrogen ions in the solution

$$\begin{aligned} pH &= -\log([\mathrm{H}^+]) \\ &= \log\left(\frac{1}{[\mathrm{H}^+]}\right) \end{aligned}$$

These two equations forpH look different but, in fact, are equivalent. This is one of the logarithm properties we will examine in this section.



Figure $Page$5.5.1: The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan).

Recall that the logarithmic and exponential functions “undo” each other. This means that logarithms have similar properties to exponents. First, let's recall the properties that we have learned thus far:

- $log_b(b^x) = x$
- $b^{\log_b x} = x$ where $x > 0$
- $\log_b 1 = 0$

- $\log_b b = 1$

The first two properties listed above are called *inverse properties*.

Recall that we use the product rule of exponents to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N) .$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for M and N} \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents} \\ &= m + n && \text{Apply the inverse property of logs} \\ &= \log_b(M) + \log_b(N) && \text{Substitute for m and n} \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

The Product Rule for Logarithms

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0$$

We can use this rule as well as others we will discuss shortly, to expand a single logarithm into sums and differences of simpler logarithms or we can use this rule to combine logarithms into a single logarithm. This is a useful things to be able to do as it will allow us a way to solve certain kings of logarithmic equations as well as determine behavior of logarithmic functions. As you will see in calculus, these rules are also very useful for making the process of taking derivatives and integrating much easier. Let's now look at some examples of applying this rule.

Example $Page 5.5.1$

Expand $\log_5(6a)$.

Solution

Notice that the argument, $6a$, is a product. Using the product rule, we can expand to write this as a sum of logarithms,

$$\log_5(6a) = \log_5(6) + \log_5(a)$$

Remember that is is a common practice to write single term arguments without parentheses. For the last example, it would be acceptable to also write the answer as $log_5 6 + \log_5 a$. However, it would not be correct to write $\log_5(6a)$ as $\log_5 6a$ since the argument is a product.

$$\begin{aligned} \log_5 6a &= \log_5(6)(a) && \text{a product of log base 5 of 6 which is multiplied by a} \\ &= a \log_5 6 \end{aligned}$$

When we have a product of a logarithm with a single term, it is common practice to write the logarithm last.

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Caution!

A common mistake is to not put parentheses around arguments of products.

$$\log_b(mn) \neq \log_b mn$$

Example $Page 5.5.2$

Expand $\log_3(30x(3x+4))$.

Solution

We begin by factoring the argument completely, expressing 30 as a product of primes.

$$\log_3(30x(3x+4)) = \log_3(2 \cdot 3 \cdot 5 \cdot x \cdot (3x+4))$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30x(3x+4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x+4)$$

We discussed earlier that it is acceptable to write a logarithm of a single term argument without parentheses. This is not the case with an argument that is a product or an argument that is a sum or difference.

You Try $Page 5.5.1$

Expand $\log_b(8k)$.

Answer

3{\log}_b2+{\log}_bk\)

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the quotient rule of exponents to combine the quotient of exponents by subtracting: $x^{\frac{a}{b}} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms.

The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number x and positive real numbers M, N, and b, b, where $b \neq 1$, we will show

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)\ .$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned}
\log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{Substitute for M and N} \\
&= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents} \\
&= m-n && \text{Apply the inverse property of logs} \\
&= \log_b(M) - \log_b(N) && \text{Substitute for m and n}
\end{aligned}$$

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Example $Page 5.5.3$

Expand $\log_7\left(\frac{3}{x}\right)$.

Solution

Notice that the argument is a quotient. Using the quotient rule are are able to write this is as a difference of logarithms,

$$\log_7\left(\frac{3}{x}\right) = \log_7(3) - \log_7(x)$$

A note about notation here. In the previous example, the argument is a quotient. Notice that the logarithm is aligned with the fraction bar in the argument. Unlike with arguments that are products, sometimes you will see arguments that are quotients written without parentheses, $\log_7\left(\frac{3}{x}\right) = \log_7 \frac{3}{x}$. Logarithms are functions and functions have inputs. For the statement, $\log_7 \frac{3}{x}$, to make sense the argument must be the single term $\frac{3}{x}$.

Example $Page 5.5.4$

Expand $log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

Solution

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x))$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\begin{aligned}\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3)+\log_2(5)+\log_2(x)+\log_2(x-1)] - [\log_2(3x+4)+\log_2(2-x)] \\ &= \log_2(3)+\log_2(5)+\log_2(x)+\log_2(x-1)-\log_2(3x+4)-\log_2(2-x)\end{aligned}$$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

You Try $Page 5.5.2$

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Answer

$\log_3(x+3) - \log_3(x-1) - \log_3(x-2)$

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\begin{aligned}\log_b(x^2) &= \log_b(x \cdot x) \\ &= \log_b x + \log_b x \\ &= 2\log_b x\end{aligned}$$

Notice that we used the product rule for logarithms to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \quad \sqrt{3} = 3^{\frac{1}{2}} \quad \frac{1}{e} = e^{-1}$$

The Power Rule for Logarithms

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b M$$

Note that since M^n is a single term that $\log_b(M^n) = \log_b M^n$.

Example $Page 5.5.5$

Expand $\log_2 x^5$.

Solution

The argument is already written as a power, so we identify the exponent, 5, and the base, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2 x$$

Notice in the previous example there were no parentheses around the argument. The argument is a single term so it is ok to do this.

You Try $Page 5.5.3$

Expand $\ln x^2$.

Answer

$2\ln x$

Example $Page 5.5.6$

Expand $\log_3(25)$ using the power rule for logs.

Solution

Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(25) = 2\log_3(5)$$

You Try $Page 5.5.4$

Expand $\ln\left(\frac{1}{x^2}\right)$.

Answer

$-2\ln(x)$

(remember to simplify the answer, $\ln 1 = 0$)

Now that we are getting good with expanding logarithms, let's look at using this rule in the reverse order.

Example $Page 5.5.7$

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Solution

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

You Try $Page 5.5.5$

Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Answer

$\log_3 16$

Expanding Logarithmic Expressions Using Multiple Rules

Taken together, the product rule, quotient rule, and power rule are often called **Laws of Logarithms**. Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1}) \\ &= \log_b(A) + \log_b(C^{-1}) \\ &= \log_b A + (-1)\log_b C \\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Example $Page 5.5.8$

Expand $\ln\left(\frac{x^4 y}{7}\right)$.

Solution

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4 y}{7}\right) = \ln(x^4 y) - \ln(7)$$

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Then seeing the product in the first term, we use the product rule:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

So, $ln\left(\frac{x^4y}{7}\right) = 4\ln(x) + \ln(y) - \ln(7)$

? You Try $Page 5.5.6$

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

Answer

$2\log x + 3\log y - 4\log z$

✓ Example $Page 5.5.9$

Expand $\log\sqrt{x}$.

Solution

$$\log\sqrt{x} = \log\left(x^{\frac{1}{2}}\right)$$
$$= \frac{1}{2}\log x$$

? You Try $Page 5.5.7$

Expand $\ln(\sqrt[3]{x^2})$.

Answer

$\frac{2}{3}\ln x$

✓ Example $Page 5.5.10$

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

Solution

We can expand by applying the Product and Quotient Rules.

$$\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) = \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \quad \text{Apply the Quotient Rule}$$
$$= \log_6 2^6 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \quad \text{Simplify by writing } 64 \text{ as } 2^6$$
$$= 6\log_6 2 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1) \quad \text{Apply the Power Rule}$$

? You Try $Page 5.5.8$

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Expand $\ln\left(\dfrac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$.

Answer

$\dfrac{1}{2}\ln(x-1)+\ln(2x+1)-\ln(x+3)-\ln(x-3)$

Condensing Logarithmic Expressions Using Multiple Rules

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

Example $Page5.5.11$

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

Solution

Using the product and quotient rules

$$\log_3(5)+\log_3(8)=\log_3(5\cdot 8)=\log_3(40)$$

This reduces our original expression to

$$\log_3(40)-\log_3(2)$$

Then, using the quotient rule

$$\log_3(40)-\log_3(2)=\log_3\left(\frac{40}{2}\right)=\log_3(20)$$

You Try $Page5.5.9$

Write $\log 3-\log 4+\log 5-\log 6$ as a single logarithm.

Answer

$\log\left(\dfrac{5}{8}\right)$ after reducing the fraction to lowest terms.

Example $Page5.5.12$

Condense $\log_2(x^2)+\dfrac{1}{2}\log_2(x-1)-3\log_2(x+3)^2$ into a single logarithm.

Solution

We apply the power rule first:

$$\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2(x+3)^2=\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2(x+3)^6$$

Next we apply the product rule to the sum:

$$\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2(x+3)^6=\log_2(x^2\sqrt{x-1})-\log_2(x+3)^6$$

Finally, we apply the quotient rule to the difference:

$$\log_2(x^2\sqrt{x-1})-\log_2(x+3)^6=\log_2\frac{x^2\sqrt{x-1}}{(x+3)^6}$$

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Example $Page5.5.13$

Rewrite $2\log x - 4\log(x+5) + \log(3x+5)$ as a single logarithm.

Solution

We apply the power rule first:

$$2\log x - 4\log(x+5) +$$
$$log(3x+5) = \log(x^2) - \log\left(x+5\right)^4 + \log((3x+5)^{x^{-1}})$$

Next we rearrange and apply the product rule to the sum:

$$\begin{aligned}\log(x^2) - \log\left(x+5\right)^4 + \log((3x+5)) &= \log(x^2) + \log((3x+5) - \log\left(x+5\right)^4 \\ &= \log(x^2(3x+5)) - \log\left(x+5\right)^4 \qquad \text{Apply the product rule} \\ &= \log\frac{x^2(3x+5)}{(x+5)^4} \qquad \text{Apply the quotient rule}\end{aligned}$$

You Try $Page5.5.10$

Rewrite $\log(5) + 0.5\log(x) - \log(7x-1) + 3\log(x-1)$ as a single logarithm.

Answer

$$\log\frac{5\sqrt{x}(x-1)^3}{(7x-1)}$$

You Try $Page5.5.11$

Condense $4(3\log(x) + \log(x+5) - \log(2x+3))$.

Answer

$\log\dfrac{x^{12}(x+5)^4}{(2x+3)^4}$; this answer could also be written $\log\left(\dfrac{x^3(x+5)}{(2x+3)}\right)^4$

Example $Page5.5.14$

In chemistry, pH is defined as $pH = -\log[H+]$ where $H+$ is the concentration of hydrogen ions in a liquid. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH = -\log(2C)$$

Using the product rule of logs

$$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$,the new pH is

$$pH = P - \log(2) \approx P - 0.301$$

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You Try $Page 5.5.12$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Answer

The pH increases by about 0.301.

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base;. When using a calculator to approximate a logarithm, we would change them to common or natural logs.

To derive the change-of-base formula, we use the one-to-one property and power rule for logarithms.

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M$. By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\begin{aligned} \log_n(b^y) &= \log_n M && \text{Apply the one-to-one property} \\ y\log_n b &= \log_n M && \text{Apply the power rule for logarithms} \\ y &= \frac{\log_n M}{\log_n b} && \text{Isolate y} \\ \log_b M &= \frac{\log_n M}{\log_n b} && \text{Substitute for y} \end{aligned}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\begin{aligned} \log_5 36 &= \frac{\log(36)}{\log(5)} && \text{Apply the change of base formula using base 10} \\ &\approx 2.2266 && \text{Use a calculator to evaluate to 4 decimal places} \end{aligned}$$

The Change-of-Base Formula

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b}$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b}$$

and

$$\log_b M = \frac{\log M}{\log b}$$

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Example $Page5.5.15$

Change $\log_5 3$ to a quotient of natural logarithms.

Solution

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_b M = \frac{\ln M}{\ln b}$$

$$\log_5 3 = \frac{\ln 3}{\ln 5}$$

Example $Page5.5.16$

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator and round to four decimal places.

Solution

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$$\log_2 10 = \frac{\ln 10}{\ln 2} \qquad \text{Apply the change of base formula using base } e$$

$$\approx 3.3219 \qquad \text{Use a calculator to evaluate to 4 decimal places}$$

You Try $Page5.5.13$

Evaluate $\log_5(100)$ using the change-of-base formula. and round to three decimal places.

Answer

$$\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$$

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Page 5.6: Exponential and Logarithmic Equations

Learning Objectives

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.



Figure $Page 5.6.1$: Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the "rabbit plague." (credit: Richard Taylor, Flickr)

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential and logarithmic equations.

Exponential Equations

Exponential equations are equations that contain variables in the exponent.

Some examples of exponential equations:

$$2^x = 2^{3x-7} \qquad\qquad 8^{t-3} + 4 = 7$$

There are two main techniques to solving exponential equations. The first technique is for solving equations that involve like bases. The second technique is for solving equations that we are unable to write with the same base.

Using Like Bases to Solve Exponential Equations

Consider the equation $3^{4x-7} = 3^{2x}$. Notice that the bases are the same. To have equivalency, the exponents must be equal: $4x - 7 = 2x$. Solving this, we get $x = \frac{7}{2}$.

The **one-to-one property of exponential functions** tells us that, for any real numbers b, S, and T, where $b > 0$, $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T.$$

Example $Page 5.6.1$

Solve $2^{x-1} = 2^{2x-4}$

Solution

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$2^{x-1} = 2^{2x-4}$	The common base is 2
$x - 1 = 2x - 4$	By the one-to-one property the exponents must be equal
$x = 3$	Solve for x

? You Try $Page 5.6.1$

Solve $5^{2x} = 5^{3x+2}$

Answer

$x = -2$

Sometimes the *common base* for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for x:

$16 = 2^{x-5}$	
$2^4 = 2^{x-5}$	Rewrite 16 as a power with base 2
$4 = x - 5$	Apply the one-to-one property of exponents
$9 = x$	Add 5 to both sides

✓ Example $Page 5.6.2$

Solve $8^{x+2} = 16^{x+1}$

Solution

$8^{x+2} = 16^{x+1}$	
$(2^3)^{x+2} = (2^4)^{x+1}$	Write 8 and 16 as powers of 2
$2^{3x+6} = 2^{4x+4}$	To take a power of a power, multiply exponents
$3x + 6 = 4x + 4$	Use the one-to-one property to set the exponents equal
$x = 2$	Solve for x

? You Try $Page 5.6.2$

Solve $5^{2x} = 25^{3x+2}$

Answer

$x = -1$

✓ Example $Page 5.6.3$

Solve $2^{5x} = \sqrt{2}$

Solution

$2^{5x} = 2^{\frac{1}{2}}$	Write the square root of 2 as a power of 2
$5x = \frac{1}{2}$	Use the one-to-one property
$x = \frac{1}{10}$	Solve for x

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? You Try $Page 5.6.3$

Solve $5^x = \sqrt{5}$

Answer

$$x = \frac{1}{2}$$

Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. The **one-to-one property for logarithms** tells us that, for real numbers $a > 0$ and $b > 0$, $\log(a) = \log(b)$ is equivalent to $a = b$. This means that we may apply logarithms with the same base on both sides of an exponential equation. It is most common to apply the natural logarithm on both sides but any logarithms with the same base applied to both sides may be used.

✓ Example $Page 5.6.4$

Solve $5^{x+2} = 4^x$

Solution

$5^{x+2} = 4^x$	There is no easy way to get the powers to have the same base
$\ln 5^{x+2} = \ln 4^x$	Take ln of both sides
$(x+2)\ln 5 = x \ln 4$	Use laws of logs
$x \ln 5 + 2 \ln 5 = x \ln 4$	Use the distributive law
$x \ln 5 - x \ln 4 = -2 \ln 5$	Get terms containing x on one side, terms without x on the other
$x(\ln 5 - \ln 4) = -2 \ln 5$	On the left hand side, factor out an x
$x \ln\left(\frac{5}{4}\right) = \ln\left(\frac{1}{25}\right)$	Use the laws of logs
$x = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}$	Divide by the coefficient of x
$x = \frac{-\ln 25}{\ln 5 - \ln 4}$	simplify using Laws of Logarithms

✓ Example $Page 5.6.5$

Solve $2^x = 3^x$

Solution

$\ln 2^x = \ln 3^x$	Take ln of both sides
$x \ln 2 = x \ln 3$	move exponents to the front
$x \ln 2 - x \ln 3 = 0$	get like terms on one side
$x(\ln 2 - \ln 3) = 0$	factor out x
$x = \frac{0}{\ln 2 - \ln 3}$	divide both sides by $\ln 2 - \ln 3$
$x = 0$	

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? You Try $Page 5.6.4$

Solve $2^x = 3^{x+1}$

Answer

$$x = \frac{\ln 3}{\ln 2 - \ln 3\left(\frac{2}{3}\right)} \text{ or } x = \frac{\ln 3}{\ln 2 - \ln 3}$$

✓ Example $Page 5.6.6$

Solve $5^x = 11$. Find the exact answer and then approximate it to three decimal places.

Solution:

$$5^x = 11$$

Since the exponential is isolated, take the logarithm of both sides.

$$\log 5^x = \log 11$$

Use the Power Property to get the x as a factor, not an exponent.

$$x \log 5 = \log 11$$

Solve for x. Find the exact answer.

$$x = \frac{\log 11}{\log 5}$$

Approximate the answer.

$$x \approx 1.490$$

Since $5^1 = 5$ and $5^2 = 25$, it makes sense that $5^{1.490} \approx 11$.

Note in the previous example that $y = 5^x$ is a basic exponential function. We know what the graph looks like and how the function values will start to become large quite quickly for larger x-values since this is an increasing exponential function of base 5, it seems reasonable that the function value of 11 would occur at a smaller x-value. Also, knowing that there is are points on the graph at $(0, 1)$ and at $(1, 5)$, this would indicate that the function would equal 11 for an x-value that is is just a bit bigger than 1.

? You Try $Page 5.6.5$

Solve $7^x = 43$. Find the exact answer and then approximate it to three decimal places.

Answer

$$x = \frac{\log 43}{\log 7} \approx 1.933$$

One common type of exponential equations are those with base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

✓ Example $Page 5.6.7$

Solve $100 = 20e^{2t}$

Solution

$$
\begin{aligned}
100 &= 20e^{2t} \\
5 &= e^{2t} && \text{Divide by the coefficient of the power} \\
\ln 5 &= 2t && \text{Take ln of both sides. Use the fact that } ln(x) \text{ and } e^x \text{ are inverse functions} \\
t &= \frac{\ln 5}{2} && \text{Divide by the coefficient of t}
\end{aligned}
$$

Analysis

Using laws of logs, we can also write this answer in the form $t = \ln \sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

? You Try *Page* 5.6.6

Solve $3e^{0.5t} = 11$

Answer

$t = 2\ln\left(\frac{11}{3}\right)$ or $\ln\left(\frac{11}{3}\right)^2$

✓ Example *Page* 5.6.8

Solve $4e^{2x} + 5 = 12$

Solution

$$
\begin{aligned}
4e^{2x} + 5 &= 12 \\
4e^{2x} &= 7 && \text{Combine like terms} \\
e^{2x} &= \frac{7}{4} && \text{Divide by the coefficient of the power} \\
2x &= \ln\left(\frac{7}{4}\right) && \text{Take ln of both sides} \\
x &= \frac{1}{2}\ln\left(\frac{7}{4}\right) && \text{Solve for x}
\end{aligned}
$$

? You Try *Page* 5.6.7

Solve $3 + e^{2t} = 7e^{2t}$

Answer

$t = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln(2)$

✓ Example *Page* 5.6.9

Solve $3e^{x+2} = 24$. Find the exact answer and then approximate it to three decimal places.

Solution:

$$3e^{x+2} = 24$$

Isolate the exponential by dividing both sides by 3.

$$e^{x+2} = 8$$

Take the natural logarithm of both sides.

$$\ln e^{x+2} = \ln 8$$

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Use the Power Property to get the x as a factor, not an exponent.

$$(x+2)\ln e = \ln 8$$

Use the property $\ln e = 1$ to simplify.

$$x+2 = \ln 8$$

Solve the equation. Find the exact answer.

$$x = \ln 8 - 2$$

Approximate the answer.

$$x \approx 0.079$$

? You Try $Page 5.6.8$

Solve $5e^{2x} = 25$. Find the exact answer and then approximate it to three decimal places.

Answer

$x = \frac{\ln 5}{2} \approx 0.805$

You may be wondering if all exponential equations have solutions. The answer is not all exponential equations have solutions.

The equation $3^{x+1} = -2$ has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Figure $Page 5.6.2$ shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

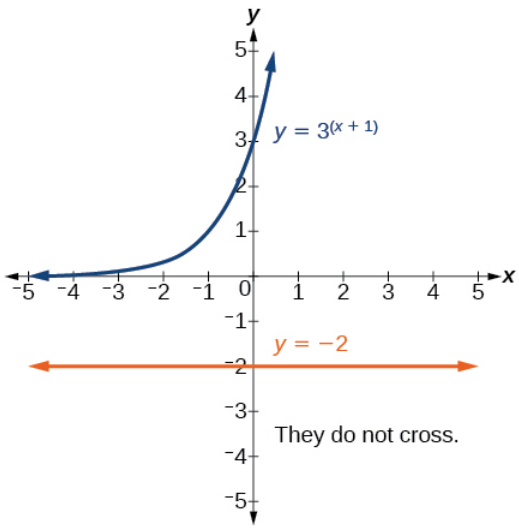


Figure $Page 5.6.2$

Another example of an equation that has no solution is $2 = -3e^t$.

Figure $Page 5.6.3$ shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

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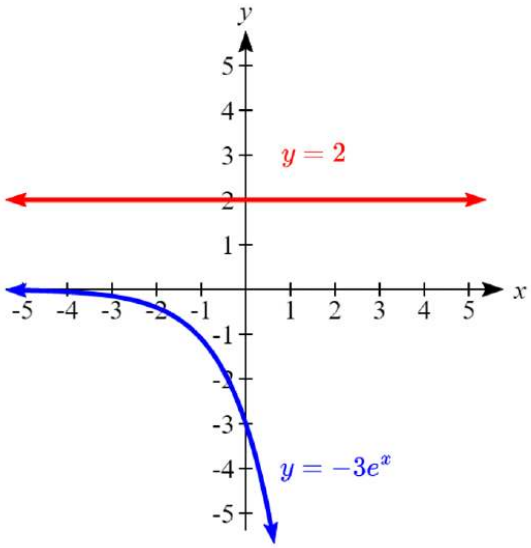


Figure $Page 5.6.3$: Graph shows that the functions do not intersect

Checking for Extraneous Solutions When Solving Exponential Equations

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

Example $Page 5.6.10$

Solve $e^{2x} - e^x = 56$

Solution

$$\begin{aligned} e^{2x} - e^x &= 56 \\ e^{2x} - e^x - 56 &= 0 \qquad \text{Get one side of the equation equal to zero since this equation is of quadratic form} \\ (e^x + 7)(e^x - 8) &= 0 \qquad \text{Factor by the FOIL method} \end{aligned}$$

Solving for each factor,

$$\begin{aligned} e^x + 7 = 0 \quad &\text{or} \quad e^x - 8 = 0 \\ e^x = -7 \quad &\text{or} \quad e^x = 8 \end{aligned}$$

$e^x = -7$ has no solution since $x = \ln(-7)$ is undefined. Solving $e^x = 8$,

$$x = \ln 8$$

You Try $Page 5.6.9$

Solve $e^{2x} = e^x + 2$

Answer

$x = \ln 2$

Logarithmic Equations

Logarithmic equations are equations that contain logarithms with variables.

Some examples of logarithmic equations:

$$\log_5 x = 8 \qquad \ln(t) - \ln(t-2) = \ln 11$$

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There are two main techniques to solving logarithmic equations. The first technique is for solving equations that contain only logarithmic terms with the same base. The second technique is for solving equations that contain logarithmic terms and constants.

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{ then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for x,and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x:

$$\begin{aligned}
\log(3x-2) - \log(2) &= \log(x+4) && \\
\log\left(\frac{3x-2}{2}\right) &= \log(x+4) && \text{Apply the quotient rule of logarithms} \\
\frac{3x-2}{2} &= x+4 && \text{Apply the one to one property of a logarithm} \\
3x-2 &= 2x+8 && \text{Multiply both sides of the equation by 2} \\
x &= 10 && \text{Subtract 2x and add 2}
\end{aligned}$$

To check the result, substitute $x = 10$ into $\log(3x-2) - \log(2) = \log(x+4)$.

$$\begin{aligned}
\log(3(10)-2) - \log(2) &= \log((10)+4) \\
\log(28) - \log(2) &= \log(14) \\
\log\left(\frac{28}{2}\right) &= \log(14) && \text{The solution checks}
\end{aligned}$$

Example $Page 5.6.11$

Solve $\ln(x^2) = \ln(2x+3)$

Solution

$$\begin{aligned}
\ln(x^2) &= \ln(2x+3) && \\
x^2 &= 2x+3 && \text{Use the one-to-one property of the logarithm} \\
x^2 - 2x - 3 &= 0 && \text{Get zero on one side before factoring} \\
(x-3)(x+1) &= 0 && \text{Factor}
\end{aligned}$$

Solving for each factor,

$$x - 3 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = 3 \quad \text{or} \quad x = -1$$

Analysis

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There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

? You Try *Page*5.6.10

Solve $\ln(x^2) = \ln 1$

Answer

$x = 1$ or $x = -1$

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x:

$$\begin{aligned} \log_2(2) + \log_2(3x-5) &= 3 && \\ \log_2(2(3x-5)) &= 3 && \text{Apply the product rule of logarithms} \\ \log_2(6x-10) &= 3 && \text{Distribute} \\ 2^3 &= 6x-10 && \text{Convert to exponential form} \\ 8 &= 6x-10 && \text{Calculate } 2^3 \\ 18 &= 6x && \text{Add 10 to both sides} \\ x &= 3 && \text{Divide by 6} \end{aligned}$$

✓ Example *Page*5.6.12

Solve $2\ln x + 3 = 7$

Solution

$$\begin{aligned} 2\ln x + 3 &= 7 && \\ 2\ln x &= 4 && \text{Subtract 3} \\ \ln x &= 2 && \text{Divide by 2} \\ x &= e^2 && \text{Rewrite in exponential form} \end{aligned}$$

? You Try *Page*5.6.11

Solve $6 + \ln x = 10$

Answer

$x = e^4$

✓ Example *Page*5.6.13

Solve $2\ln(6x) = 7$

Solution

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$$
\begin{aligned}
2\ln(6x) &= 7 \\
\ln(6x) &= \frac{7}{2} && \text{Divide by 2} \\
6x &= e^{\left(\frac{7}{2}\right)} && \text{convert to exponential form ln} \\
x &= \frac{1}{6}e^{\left(\frac{7}{2}\right)} && \text{Divide by 6}
\end{aligned}
$$

? You Try $Page5.6.12$

Solve $2\ln(x+1) = 10$

Answer

$x = e^5 - 1$

✓ Example $Page5.6.14$

Solve $\ln x = 3$

Solution

$$
\begin{aligned}
\ln x &= 3 \\
x &= e^3
\end{aligned}
$$

Figure $Page5.6.4$ represents the graph of the equation. On the graph, the x-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855$.

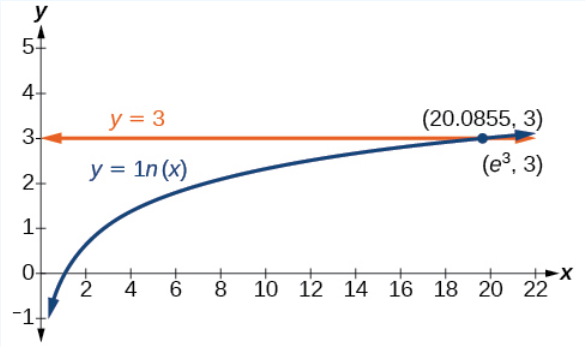


Figure $Page5.6.4$: The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3)$, which is approximately $(20.0855, 3)$.

? You Try $Page5.6.13$

Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

Answer

$x \approx 9.97$

Checking for Extraneous Solutions When Solving Logarithmic Equations

It is possible to have extraneous solutions when solving logarithmic equations since the domain of a logarithm is $(0, \infty)$. Check your answers to see if they are extraneous or not.

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Example $Page5.6.15$

Solve $\ln x + \ln(2x-5) = \ln 3$

Solution

$$\begin{aligned} \ln x + \ln(2x-5) &= \ln 3 \\ \ln(x(2x-5)) &= \ln 3 \\ \ln(2x^2-5x) &= \ln 3 \\ 2x^2-5x &= 3 \\ 2x^2-5x-3 &= 0 \\ (2x+1)(x-3) &= 0 \\ 2x+1=0 \qquad x-3 &= 0 \\ x = \frac{1}{2} \qquad x &= 3 \end{aligned}$$

Example $Page5.6.16$

Solve: $\log_4(x+6) - \log_4(2x+5) = -\log_4 x$

Solution:

$$\log_4(x+6) - \log_4(2x+5) = -\log_4 x$$

Use the Quotient Property on the left side and the Power Property on the right.

$$\log_4\left(\frac{x+6}{2x+5}\right) = \log_4 x^{-1}$$

Rewrite $x^{-1} = \frac{1}{x}$.

$$\log_4\left(\frac{x+6}{2x+5}\right) = \log_4 \frac{1}{x}$$

Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.

$$\frac{x+6}{2x+5} = \frac{1}{x}$$

Solve the rational equation.

$$x(x+6) = 2x+5$$

Distribute.

$$x^2+6x = 2x+5$$

Write in standard form.

$$x^2+4x-5 = 0$$

Factor.

$$(x+5)(x-1) = 0$$

Use the Zero-Product-Property.

$$x+5=0, \quad x-1=0$$

Solve each equation.

$$x = \cancel{-5}, \quad x = 1$$

Check.

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We leave the check for you.

Example $Page 5.6.17$

Solve: $2\log_5 x = \log_5 81$

Solution:

$$2\log_5 x = \log_5 81$$

Use the Power Property.

$$\log_5 x^2 = \log_5 81$$

Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.

$$x^2 = 81$$

Solve using the Square Root Property.

$$x = \pm 9$$

We eliminate $x = -9$ as we cannot take the logarithm of a negative number.

$$x = 9,\ \cancel{x = -9}$$

Check. $x = 9$

$$2\log_5 x = \log_5 81$$
$$2\log_5 9 \stackrel{?}{=} \log_5 81$$
$$\log_5 9^2 \stackrel{?}{=} \log_5 81$$
$$\log_5 81 = \log_5 81$$

You Try $Page 5.6.14$

Solve: $2\log_3 x = \log_3 36$

Answer

$x = 6$

You Try $Page 5.6.15$

Solve: $\log(x+2) - \log(4x+3) = -\log x$

Answer

$x = 3$

You Try $Page 5.6.16$

Solve: $\log(x-2) - \log(4x+16) = \log\dfrac{1}{x}$

Answer

$x = 8$

Solving Applied Problems

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic

equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

Example $Page 5.6.18$

Suppose $\$2000$ is invested in an account which offers 7.125% compounded monthly.

1. Express the amount A in the account as a function of the term of the investment t in years.
2. How much is in the account after 5 years?
3. How long will it take for the initial investment to double?
4. Find and interpret the average rate of change of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year.

Solution

1. Substituting $P = 2000$, $r = 0.07125$, and $n = 12$ (since interest is compounded *monthly* into the compound interest formula yields
$$A(t) = 2000\left(1 + \frac{0.07125}{12}\right)^{12t} = 2000(1.0059375)^{12t}$$
2. Since t represents the length of the investment in years, we substitute $t = 5$ into $A(t)$ to find
$$A(5) = 2000(1.0059375)^{12(5)} \approx 2852.92$$
After 5 years, we have approximately $\$2852.92$
3. Our initial investment is $\$2000$, so to find the time it takes this to double, we need to find t when $A(t) = 4000$. We get $2000(1.0059375)^{12t} = 4000$ or $(1.0059375)^{12t} = 2$ Taking natural logs of both sides, we get
$$t = \frac{\ln(2)}{12\ln(1.0059375)} \approx 9.75.$$
Hence, it takes approximately 9 years 9 months for the investment to double.
4. To find the average rate of change of A from the end of the fourth year to the end of the fifth year, we compute
$$\frac{A(5) - A(4)}{5 - 4} \approx 195.63$$
Similarly, the average rate of change of A from the end of the thirty-fourth year to the end of the thirty-fifth year is
$$\frac{A(35) - A(34)}{35 - 34} \approx 1648.21$$
This means that the value of the investment is increasing at a rate of approximately $\$195.63$ per year between the end of the fourth and fifth years, while that rate jumps to $\$1648.21$ per year between the end of the thirty-fourth and thirty-fifth years. So, not only is it true that the longer you wait, the more money you have, but also the longer you wait, the faster the money increases. (in fact, the rate of increase of the amount in the account is exponential as well. This is the quality that really defines exponential functions and we refer the reader to a course in Calculus.

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CHAPTER OVERVIEW

Chapter 6: Day 6

Page 6.1: The Pythagorean Theorem
Page 6.2: Trigonometry Preview - Circles
Page 6.3: Angles - Radians and Degrees
Page 6.4: Unit Circle - Sine and Cosine Functions
Page 6.5: Graphs of the Sine and Cosine Functions
Page 6.6: Right Triangle Trigonometry
Page 6.7: The Other Trigonometric Functions
Page 6.8: Graphs of the Other Trigonometric Functions
Page 6.9: Arcs, Angles, and Trig Function Values
Page 6.10: Inverse Trigonometric Functions

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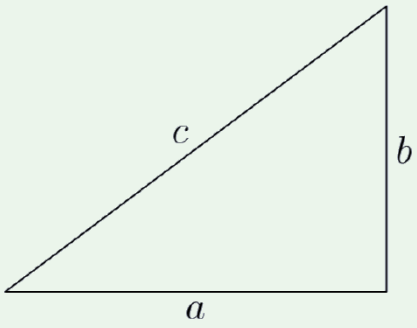
Page 6.1: The Pythagorean Theorem

Pythagoras was a Greek mathematician and philosopher, born on the island of Samos (ca. 582 BC). He founded a number of schools, one in particular in a town in southern Italy called Crotone, whose members eventually became known as the Pythagoreans. The inner circle at the school, the Mathematikoi, lived at the school, rid themselves of all personal possessions, were vegetarians, and observed a strict vow of silence. They studied mathematics, philosophy, and music, and held the belief that numbers constitute the true nature of things, giving numbers a mystical or even spiritual quality.

Today, nothing is known of Pythagoras's writings, perhaps due to the secrecy and silence of the Pythagorean society. However, one of the most famous theorems in all of mathematics does bear his name, the Pythagorean Theorem.

PYTHAGOREAN THEOREM

Let c represent the length of the **hypotenuse**, the side of a right triangle directly opposite the right angle (a right angle measures 90º) of the triangle. The remaining sides of the right triangle are called the **legs** of the right triangle, whose lengths are designated by the letters a and b.



The relationship involving the legs and hypotenuse of the right triangle, given by

$$a^2 + b^2 = c^2 \qquad \text{(Page 6.1.1)}$$

is called the **Pythagorean Theorem**.

Note that the Pythagorean Theorem can only be applied to right triangles. Let's look at a simple application of the Pythagorean Theorem (Equation Page 6.1.1).

? Example *Page*6.1.2

Given that the length of one leg of a right triangle is 4 centimeters and the hypotenuse has length 8 centimeters, find the length of the second leg.

Let's begin by sketching and labeling a right triangle with the given information. We will let *x* represent the length of the missing leg.

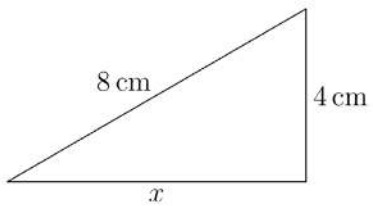


Figure 1. A sketch make things a bit easier

Here is an important piece of advice.

TIP $Page 6.1.3$: Hypotenuse

The hypotenuse is the longest side of the right triangle. It is located directly opposite the right angle of the triangle. Most importantly, it is the quantity that is **isolated** by itself in the Pythagorean Theorem (Equation Page 6.1.1). Always isolate the quantity representing the hypotenuse on one side of the equation. The "legs" go on the other side of the equation.

So, taking the tip to heart, and noting the lengths of the legs and hypotenuse in **Figure 1**, we write

$$4^2 + x^2 = 8^2 \ .$$

Square, then isolate x on one side of the equation.

$$16 + x^2 = 64$$

$$x^2 = 48$$

Normally, we would take plus or minus the square root in solving this equation, but *x* represents the length of a leg, which must be a positive number. Hence, we take just the positive square root of 48.

$$x = \sqrt{48}$$

Of course, place your answer in simple radical form.

$$x = \sqrt{16}\sqrt{3}$$

$$x = 4\sqrt{3}$$

If need be, you can use your graphing calculator to approximate this length. To the nearest hundredth of a centimeter, $x \approx 6.93$ centimeters.

Proof of the Pythagorean Theorem

It is not known whether Pythagoras was the first to provide a proof of the Pythagorean Theorem. Many mathematical historians think not. Indeed, it is not even known if Pythagoras crafted a proof of the theorem that bears his name, let alone was the first to provide a proof.

There is evidence that the ancient Babylonians were aware of the Pythagorean Theorem over a 1000 years before the time of Pythagoras. A clay tablet, now referred to as Plimpton 322 (see **Figure 2**), contains examples of Pythagorean Triples, sets of three numbers that satisfy the Pythagorean Theorem (such as 3, 4, 5).



Figure 2. A photograph of Plimpton 322.

One of the earliest recorded proofs of the Pythagorean Theorem dates from the Han dynasty (206 BC to AD 220), and is recorded in the Chou Pei Suan Ching (see **Figure 3**). You can see that this figure specifically addresses the case of the 3, 4, 5 right triangle. Mathematical historians are divided as to whether or not the image was meant to be part of a general proof or was just devised to address this specific case. There is also disagreement over whether the proof was provided by a more modern commentator or dates back further in time.

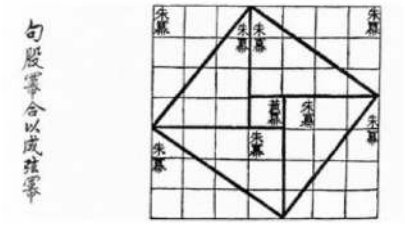


Figure 3. A figure from the Chou Pei Suan Ching

However, **Figure 3** does suggest a path we might take on the road to a proof of the Pythagorean Theorem. Start with an arbitrary right triangle having legs of lengths a and b, and hypotenuse having length c, as shown in **Figure 4**(a).

Next, make four copies of the triangle shown in **Figure 4**(a), then rotate and translate them into place as shown in **Figure 4**(b). Note that this forms a big square that is c units on a side.

Further, the position of the triangles in **Figure 4**(b) allows for the formation of a smaller, unshaded square in the middle of the larger square. It is not hard to calculate the length of the side of this smaller square. Simply subtract the length of the smaller leg from the larger leg of the original triangle. Thus, the side of the smaller square has length b − a.

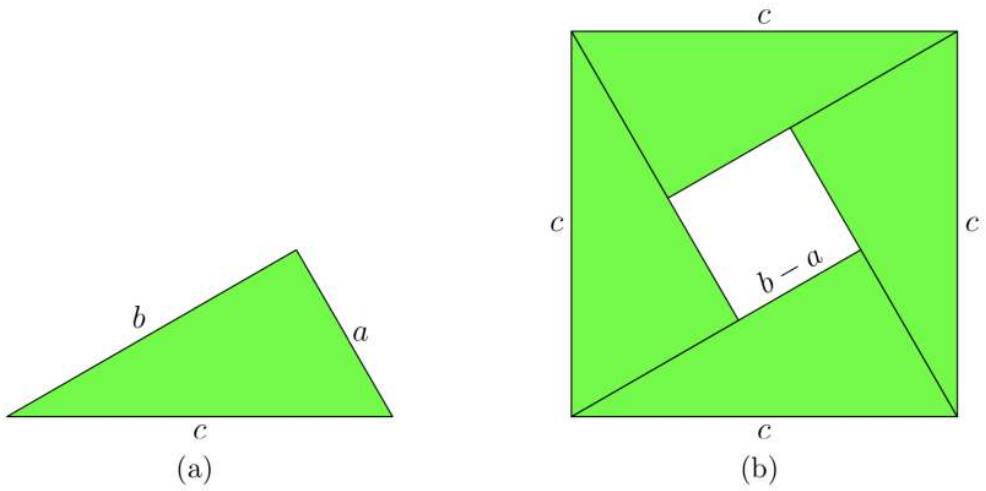


Figure 4. Proof of the Pythagorean Theorem.

Now, we will calculate the area of the large square in **Figure 4**(b) in two separate ways.

- First, the large square in **Figure 4**(b) has a side of length c. Therefore, the area of the large square is

$$Area = c^2 .$$

- Secondly, the large square in **Figure 4**(b) is made up of 4 triangles of the same size and one smaller square having a side of length b−a. We can calculate the area of the large square by summing the area of the 4 triangles and the smaller square.

  1. The area of the smaller square is $(b-a)^2$.
  2. The area of each triangle is $\frac{ab}{2}$. Hence, the area of four triangles of equal size is four times this number;

  i.e., $4(\frac{ab}{2})$. Thus, the area of the large square is

  Area = Area of small square + 4· Area of triangle

  $=(b-a)^2+4(\frac{ab}{2})$.

  We calculated the area of the larger square twice. The first time we got c^2; the second time we got $(b-a)^2+4(\frac{ab}{2})$. Therefore, these two quantities must be equal.

$$c^2 = (b-a)^2+4(\frac{ab}{2}) .$$

  Expand the binomial and simplify.

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$$c^2 = b^2 - 2ab + a^2 + 2ab$$
$$c^2 = b^2 + a^2$$

That is,

$$a^2 + b^2 = c^2 \ ,$$

and the Pythagorean Theorem is proven.

Applications of the Pythagorean Theorem

In this section we will look at a few applications of the Pythagorean Theorem, one of the most applied theorems in all of mathematics. Just ask your local carpenter.

The ancient Egyptians would take a rope with 12 equally spaced knots like that shown in **Figure 5**, and use it to square corners of their buildings. The tool was instrumental in the construction of the pyramids.

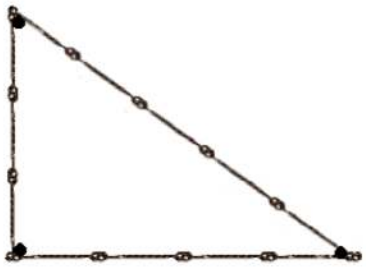


Figure 5. A basic 3-4-5 right triangle for squaring corners.

The Pythagorean theorem is also useful in surveying, cartography, and navigation, to name a few possibilities.

Let's look at a few examples of the Pythagorean Theorem in action.

? Example $Page 6.1.4$

One leg of a right triangle is 7 meters longer than the other leg. The length of the hypotenuse is 13 meters. Find the lengths of all sides of the right triangle.

Let *x* represent the length of one leg of the right triangle. Because the second leg is 7 meters longer than the first leg, the length of the second leg can be represented by the expression *x* + 7, as shown in **Figure 6**, where we've also labeled the length of the hypotenuse (13 meters).

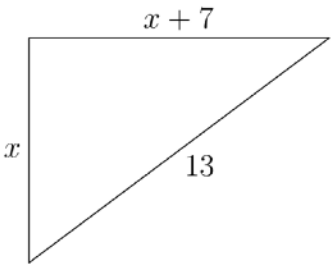


Figure 6. The second leg is 7 meters longer than the first.

Remember to isolate the length of the hypotenuse on one side of the equation representing the Pythagorean Theorem. That is,

$$x^2 + (x+7)^2 = 13^2 \ .$$

Note that the legs go on one side of the equation, the hypotenuse on the other. Square and simplify. Remember to use the squaring a binomial pattern.

$$x^2 + x^2 + 14x + 49 = 169$$
$$2x^2 + 14x + 49 = 169$$

This equation is nonlinear, so make one side zero by subtracting 169 from both sides of the equation.

$$2x^2 + 14x + 49 - 169 = 0$$

$$2x^2 + 14x - 120 = 0$$

Note that each term on the left-hand side of the equation is divisible by 2. Divide both sides of the equation by 2.

$$x^2 + 7x - 60 = 0$$

Let's use the quadratic formula with a = 1, b = 7, and c = −60.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-7 \pm \sqrt{7^2 - 4(1)(-60)}}{2(1)}$$

Simplify.

$$x = \frac{-7 \pm \sqrt{289}}{2}$$

Note that 289 is a perfect square ($17^2 = 289$). Thus,

$$x = \frac{-7 \pm 17}{2}.$$

Thus, we have two solutions,

x = 5 or x = −12.

Because length must be a positive number, we eliminate −12 from consideration. Thus, the length of the first leg is x = 5 meters. The length of the second leg is x+7, or 12 meters.

Check. Checking is an easy matter. The legs are 5 and 12 meters, respectively, and the hypotenuse is 13 meters. Note that the second leg is 7 meters longer than the first. Also,

$$5^2 + 12^2 = 25 + 144 = 169\ ,$$

which is the square of 13.

The integral sides of the triangle in the previous example, 5, 12, and 13, are an example of a Pythagorean Triple.

PYTHAGOREAN TRIPLE

A set of positive integers a, b, and c, is called a Pythagorean Triple if they satisfy the Pythagorean Theorem; that is, if $a^2 + b^2 = c^2$.

If the greatest common factor of a, b, and c is 1, then the triple (a, b, c) is called a **primitive** Pythagorean Triple.

Thus, for example, the Pythagorean Triple (5, 12, 13) is primitive. Let's look at another example.

Example $Page 6.1.5$

If (a,b,c) is a Pythagorean Triple, show that any positive integral multiple is also a Pythagorean Triple.

Thus, if the positive integers (a, b, c) is a Pythagorean Triple, we must show that(ka, kb, kc), where k is a positive integer, is also a Pythagorean Triple.

However, we know that

$$a^2 + b^2 = c^2\ .$$

Multiply both sides of this equation by k^2.

$$k^2a^2 + k^2b^2 = k^2c^2$$

This last result can be written

$$(ka)^2 + (kb)^2 = (kc)^2\ .$$

Hence, (ka, kb, kc) is a Pythagorean Triple.

Hence, because (3, 4, 5) is a Pythagorean Triple, you can double everything to get another triple (6, 8, 10). Note that $6^2+8^2=10^2$ is easily checked. Similarly, tripling gives another triple (9, 12, 15), and so on.

In **Example 5**, we showed that (5, 12, 13) was a triple, so we can take multiples to generate other Pythagorean Triples, such as (10, 24, 26) or (15, 36, 39), and so on.

Formulae for generating Pythagorean Triples have been know since antiquity.

? Example $Page 6.1.6$

The following formula for generating Pythagorean Triples was published in Euclid's (325–265 BC) Elements, one of the most successful textbooks in the history of mathematics. If m and n are positive integers with m > n, show

$a=m^2-n^2$,

b = 2mn, (7)

$c=m^2+n^2$,

generates Pythagorean Triples.

We need only show that the formulae for *a*, *b*, and *c* satisfy the Pythagorean Theorem. With that is mind, let's first compute \ (*a*^2+*b*^2\).

$$a^2+b^2=(m^2-n^2)^2+(2mn)^2$$
$$= m^4-2m^2n^2+n^4+4m^2n^2$$
$$= m^4+2m^2n^2+n^4$$

On the other hand,

$$c^2=(m^2+n^2)^2$$
$$= m^4+2m^2n^2+n^4 .$$

Hence, $a^2+b^2=c^2$, and the expressions for a, b, and c form a Pythagorean Triple.

It is both interesting and fun to generate Pythagorean Triples with the formulae from **Example 6**. Choose m = 4 and n = 2, then

$$a=m^2-n^2=(4)^2-(2)^2=12 \ ,$$
$$b=2mn=2(4)(2)=16 ,$$
$$c=m^2+n^2=(4)^2+(2)^2=20 \ .$$

It is easy to check that the triple (12, 16, 20) will satisfy $12^2+16^2=20^2$. Indeed, note that this triple is a multiple of the basic (3, 4, 5) triple, so it must also be a Pythagorean Triple.

It can also be shown that if m and n are relatively prime, and are not both odd or both even, then the formulae in **Example 6** will generate a **primitive** Pythagorean Triple. For example, choose m = 5 and n = 2. Note that the greatest common divisor of m = 5 and n = 2 is one, so m and n are relatively prime. Moreover, m is odd while n is even. These values of m and n generate

$$a=m^2-n^2=(5)^2-(2)^2=21 \ ,$$
$$b=2mn=2(5)(2)=20 ,$$
$$c=m^2+n^2=(5)^2+(2)^2=29 \ .$$

Note that

$$21^2+20^2=441+400=841=29^2 .$$

Hence, (21, 20, 29) is a Pythagorean Triple. Moreover, the greatest common divisor of 21, 20, and 29 is one, so (21, 20, 29) is primitive.
The practical applications of the Pythagorean Theorem are numerous.

? Example $Page 6.1.8$

A painter leans a 20 foot ladder against the wall of a house. The base of the ladder is on level ground 5 feet from the wall of the house. How high up the wall of the house will the ladder reach?

Consider the triangle in **Figure 7**. The hypotenuse of the triangle represents the ladder and has length 20 feet. The base of the triangle represents the distance of the base of the ladder from the wall of the house and is 5 feet in length. The vertical leg of the triangle is the distance the ladder reaches up the wall and the quantity we wish to determine.

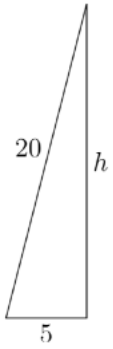


Figure 7. A ladder leans against the wall of a house.

Applying the Pythagorean Theorem,

$$5^2 + h^2 = 20^2 \ .$$

Again, note that the square of the length of the hypotenuse is the quantity that is isolated on one side of the equation.

Next, square, then isolate the term containing h on one side of the equation by subtracting 25 from both sides of the resulting equation.

$$25 + h^2 = 400$$

$$h^2 = 375$$

We need only extract the positive square root.

$$h = \sqrt{375}$$

We could place the solution in simple form, that is, $h = 5\sqrt{15}$, but the nature of the problem warrants a decimal approximation. Using a calculator and rounding to the nearest tenth of a foot,

$$h \approx 19.4.$$

Thus, the ladder reaches about 19.4 feet up the wall.

The Distance Formula

We often need to calculate the distance between two points P and Q in the plane. Indeed, this is such a frequently recurring need, we'd like to develop a formula that will quickly calculate the distance between the given points P and Q. Such a formula is the goal of this last section.

Let P(x1, y1) and Q(x2, y2) be two arbitrary points in the plane, as shown in **Figure 8**(a) and let d represent the distance between the two points.

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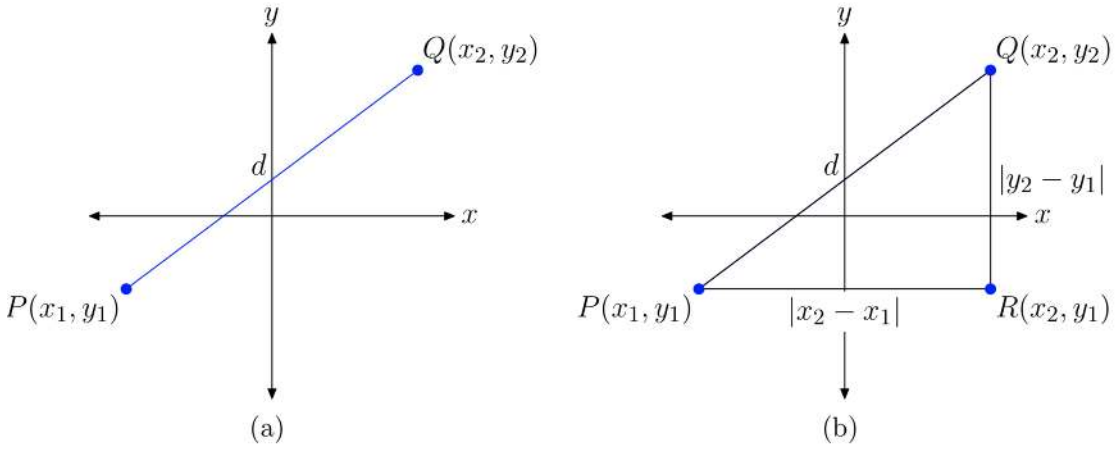


Figure 8. Finding the distance between the points *P* and *Q*.

To find the distance d, first draw the right triangle △PQR, with legs parallel to the axes, as shown in **Figure 8**(b). Next, we need to find the lengths of the legs of the right triangle △PQR.

- The distance between P and R is found by subtracting the x coordinate of P from the x-coordinate of R and taking the absolute value of the result. That is, the distance between P and R is $|x_2 - x_1|$.
- The distance between R and Q is found by subtracting the y-coordinate of R from the y-coordinate of Q and taking the absolute value of the result. That is, the distance between R and Q is $|y_2 - y_1|$.

We can now use the Pythagorean Theorem to calculate d. Thus,

$$d^2 = (|x_2 - x_1|)^2 + (|y_2 - y_1|)^2 \quad .$$

However, for any real number a,

$$(|a|)^2 = |a| \cdot |a| = |a^2| = a^2 \ ,$$

because a2 is nonnegative. Hence, $(|x_2 - x_1|)^2 = (x_2 - x_1)^2 and (|y_2 - y_1|)^2 = (y_2 - y_1)^2$ and we can write

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 \quad .$$

Taking the positive square root leads to the Distance Formula.

THE DISTANCE FORMULA

Let P (x1, y1) and Q(x2, y2) be two arbitrary points in the plane. The distance d between the points P and Q is given by the formula

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. (9)

The direction of subtraction is unimportant. Because you square the result of the subtraction, you get the same response regardless of the direction of subtraction (e.g.\((5 − 2)^2 = (2 − 5)^2\)). Thus, it doesn't matter which point you designate as the point P, nor does it matter which point you designate as the point *Q*. Simply subtract *x*- coordinates and square, subtract *y*-coordinates and square, add, then take the square root.

Let's look at an example.

? Example *Page*6.1.10

Find the distance between the points P (−4, −2) and Q (4, 4).

It helps the intuition if we draw a picture, as we have in **Figure 9**. One can now take a compass and open it to the distance between points P and Q. Then you can place your compass on the horizontal axis (or any horizontal gridline) to estimate the distance between the points P and Q. We did that on our graph paper and estimate the distance $d \approx 10$.

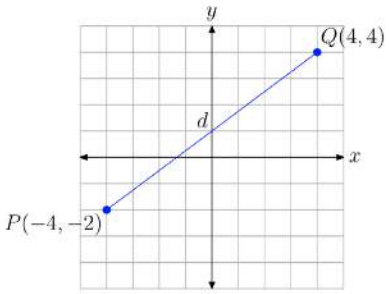


Figure 9. Graphing the distance between P (−4, −2) and Q (4, 4).

Let's now use the distance formula to obtain an exact value for the distance d. With (x_1, y_1) = P (−4, −2) and (x_2, y_2) = Q (4, 4),

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(4 - (-4))^2 + (4 - (-2))^2}$

$= \sqrt{8^2 + 6^2}$

$= \sqrt{64 + 36}$

$= \sqrt{100}$

= 10.

It's not often that your exact result agrees with your approximation, so never worry if you're off by just a little bit.

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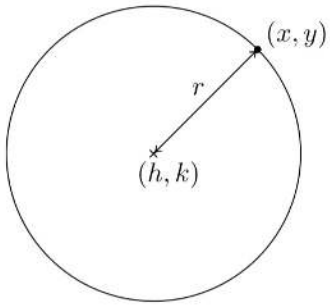
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Page 6.2: Trigonometry Preview - Circles

Recall from Geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

Circle

A **circle** with center (h,k) and radius $r>0$ is the set of all points (x,y) in the plane whose distance to (h,k) is r.



From the picture, we see that a point (x,y) is on the circle if and only if its distance to (h,k) is r. We express this relationship algebraically using the Distance Formula, as

$$r=\sqrt{(x-h)^2+(y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r>0$) which gives us the standard equation of a circle.

The Standard Form of a Circle

The standard form of the equation of a circle with center (h,k) and radius $r>0$ is

$$(x-h)^2+(y-k)^2=r^2.$$

Example $Page 6.2.1$

Write the standard equation of the circle with center (−2,3) and radius 5.

Solution.

Here, $(h,k)=(-2,3)$ and $r=5$, so we get

$$\begin{aligned}(x-(-2))^2+(y-3)^2 &= (5)^2\\ (x+2)^2+(y-3)^2 &= 25\end{aligned}$$

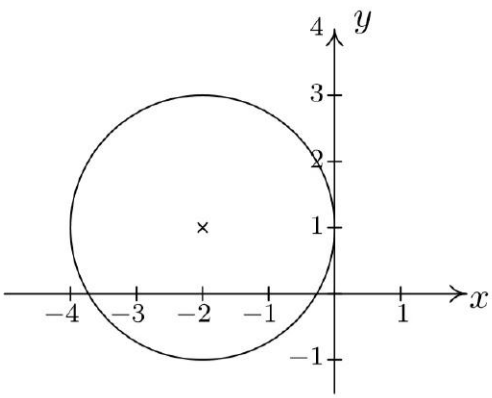
Example $Page 6.2.2$

Graph $(x+2)^2+(y-1)^2=4$. Find the center and radius.

Solution

From the standard form of a circle, we have that $x+2$ is $x-h$, so $h=-2$ and $y-1$ is $y-k$ so $k=1$. This tells us that our center is $(-2,1)$. Furthermore, $r^2=4$, so $r=2$. Thus we have a circle centered at $(-2,1)$ with a radius of 2. Graphing gives us

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If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2+(y-1)^2=4$, we'd be contending with $x^2+4x+y^2-2y+1=0$. If we're given such an equation, we can complete the square to rewrite the equation in standard form for a circle.

Example $Page 6.2.3$

Complete the square to find the center and radius of $3x^2-6x+3y^2+4y-4=0$.

Solution

$$\begin{aligned} 3x^2-6x+3y^2+4y-4 &= 0 \\ 3x^2-6x+3y^2+4y &= 4 && \text{add 4 to both sides} \\ 3\left(x^2-2x\right)+3\left(y^2+\frac{4}{3}y\right) &= 4 && \text{factor out leading coefficients} \\ 3\left(x^2-2x+\underline{1}\right)+3\left(y^2+\frac{4}{3}y+\underline{\underline{\frac{4}{9}}}\right) &= 4+3\underline{(1)}+3\underline{\underline{\left(\frac{4}{9}\right)}} && \text{complete the square in } x, y \\ 3(x-1)^2+3\left(y+\frac{2}{3}\right)^2 &= \frac{25}{3} && \text{factor} \\ (x-1)^2+\left(y+\frac{2}{3}\right)^2 &= \frac{25}{9} && \text{divide both sides by 3} \end{aligned}$$

From the standard form of a circle, we identify $x-1$ as $x-h$, so $h=1$, and $y+\frac{2}{3}$ as $y-k$, so $k=-\frac{2}{3}$. Hence, the center is $(h,k)=\left(1,-\frac{2}{3}\right)$. Furthermore, we see that $r^2=\frac{25}{9}$ so the radius is $r=\frac{5}{3}$.

It is possible to obtain equations like $(x-3)^2+(y+1)^2=0$ or $(x-3)^2+(y+1)^2=-1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations.

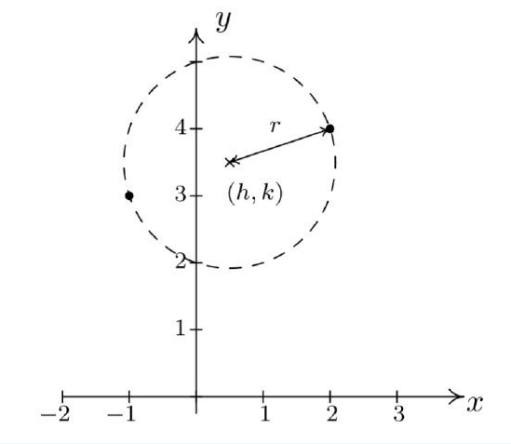
Example $Page 6.2.4$

Write the standard equation of the circle which has $(-1,3)$ and $(2,4)$ as the endpoints of a diameter.

Solution

We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields

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Since the given points are endpoints of a diameter, we know their midpoint (h,k) is the center of the circle. Equation ??? gives us

$$\begin{aligned}(h,k) &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\\ &= \left(\frac{-1+2}{2}, \frac{3+4}{2}\right)\\ &= \left(\frac{1}{2}, \frac{7}{2}\right)\end{aligned}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}\\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ &= \frac{1}{2}\sqrt{3^2+1^2}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2 = \frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2 = \frac{10}{4}$

We close this section with a very important circle that you will learn more about in trigonometry: the **Unit Circle**.

The Unit Circle

The **Unit Circle** is the circle centered at $(0,0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2+y^2=1$.

Example $Page 6.2.5$

Find the points on the unit circle with y-coordinate $\frac{\sqrt{3}}{2}$.

Solution

We replace y with $\frac{\sqrt{3}}{2}$ in the equation $x^2+y^2=1$ to get

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$$\begin{aligned} x^2+y^2 &= 1 \\ x^2+\left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\ \frac{3}{4}+x^2 &= 1 \\ x^2 &= \frac{1}{4} \\ x &= \pm\sqrt{\frac{1}{4}} \\ x &= \pm\frac{1}{2} \end{aligned}$$

Our final answers are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

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Page 6.3: Angles - Radians and Degrees

Learning Objectives

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a *ray*. A ray consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its *endpoint* and any other point on it. The ray in Figure $Page 6.3.1$ can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

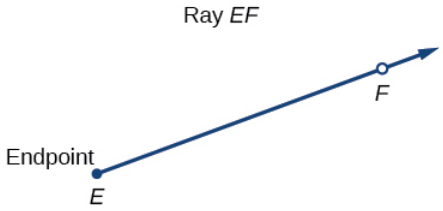


Figure $Page 6.3.1$

An *angle* is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in Figure $Page 6.3.2$ is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.

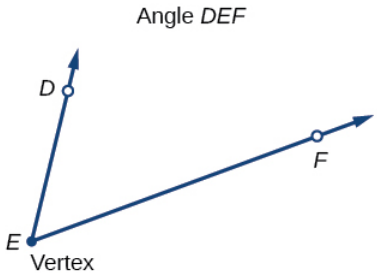


Figure $Page 6.3.2$

Greek letters are often used as variables for the measure of an angle. Table $Page 6.3.1$ is a list of Greek letters commonly used to represent angles, and a sample angle is shown in Figure $Page 6.3.3$.

Table $Page 6.3.1$

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

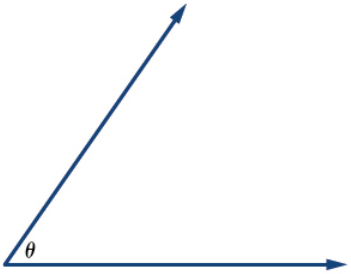


***Figure* $Page6.3.3$**: *Angle theta, shown as* $\angle\theta$

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure $Page6.3.4$.

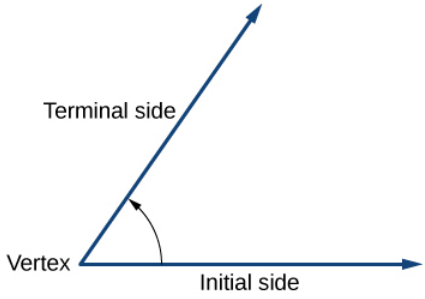


Figure $Page6.3.4$

As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit “degrees” after the number, or include the degree symbol °. For example, 90 degrees = 90°.

To formalize our work, we will begin by drawing angles on an *x*-*y* coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive *x*-axis. See Figure $Page6.3.5$.

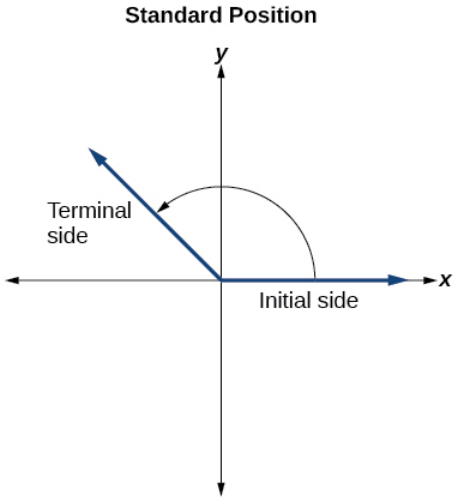


Figure $Page6.3.5$

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

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Drawing an angle in standard position always starts the same way—draw the initial side along the positive *x*-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90^\circ}{360^\circ} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive *x*-axis. To draw a 360° angle, we calculate that $\frac{360^\circ}{360^\circ} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive *x*-axis. In this case, the initial side and the terminal side overlap. See Figure $Page 6.3.6$.

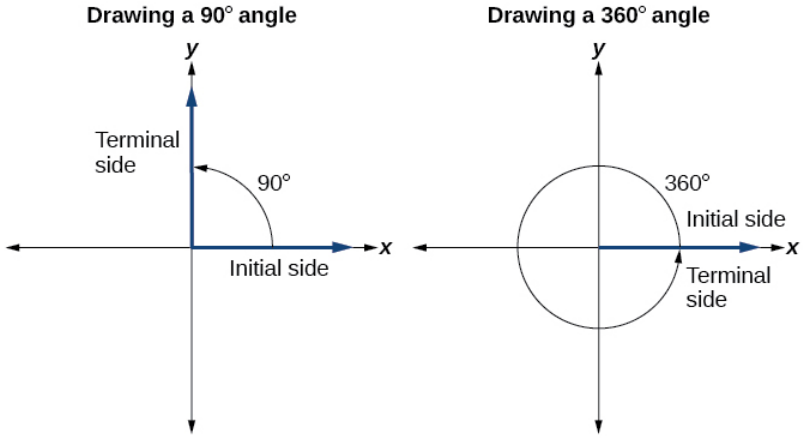


Figure $Page 6.3.6$

Since we define an angle in **standard position** by its initial side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270° or 360°. See Figure $Page 6.3.7$.

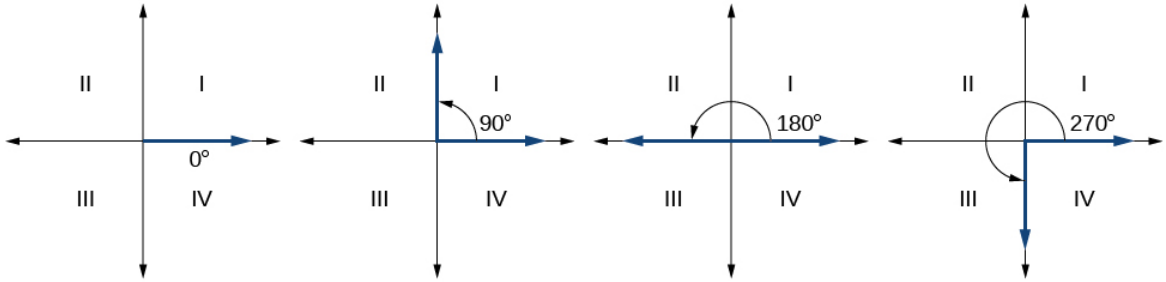


Figure $Page 6.3.7$: Quadrantal angles are angles in standard position whose terminal side lies along an axis. Examples are shown.

QUADRANTAL ANGLES

Quadrantal angles are angles in standard position whose terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.

Given an angle measure in degrees, draw the angle in standard position

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive *x*-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example $Page 6.3.1$: Drawing an Angle in Standard Position Measured in Degrees

1. Sketch an angle of 30° in standard position.
2. Sketch an angle of −135° in standard position.

Solution

1. Divide the angle measure by 360°.

$$\frac{30^\circ}{360^\circ} = \frac{1}{12} \qquad \text{(Page 6.3.1)}$$

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To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}(\frac{1}{4}) \qquad \text{(Page 6.3.2)}$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in Figure *Page*6.3.8.

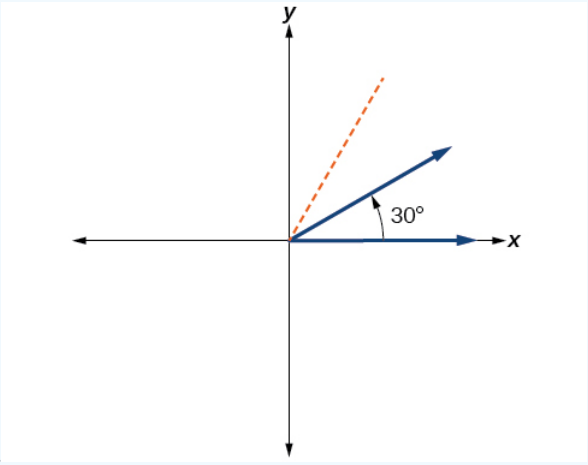


***Figure Page*6.3.8**

2. Divide the angle measure by 360°.

$$\frac{-135^\circ}{360^\circ} = -\frac{3}{8} \qquad \text{(Page 6.3.3)}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}(\frac{1}{4}) \qquad \text{(Page 6.3.4)}$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in Figure *Page*6.3.9.

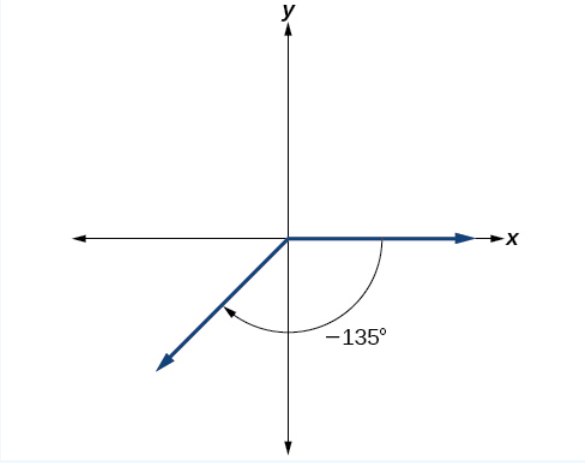


***Figure Page*6.3.9**

*Page*6.3.1:

Show an angle of 240° on a circle in standard position.

Solution

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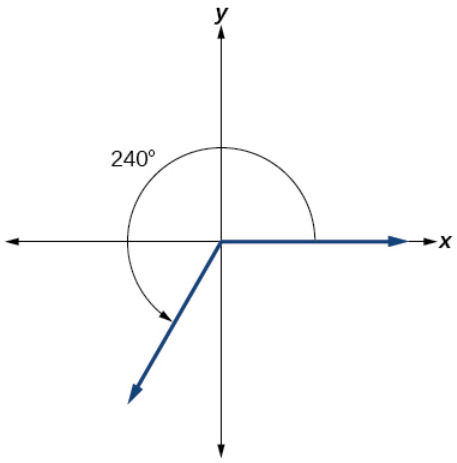


Figure $Page6.3.10$

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An **arc** may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the **circumference** of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in Figure $Page6.3.11$.

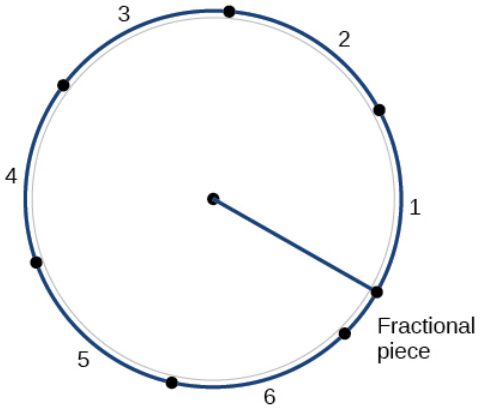


Figure $Page6.3.11$

This brings us to our new angle measure. One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians } = 360^\circ \qquad (\text{Page } 6.3.5)$$

$$\pi \text{ radians } = \frac{360^\circ}{2} = 180^\circ \qquad (\text{Page } 6.3.6)$$

$$1 \text{ radian } = \frac{180^\circ}{\pi} \approx 57.3^\circ \qquad (\text{Page } 6.3.7)$$

See Figure $Page6.3.12$. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

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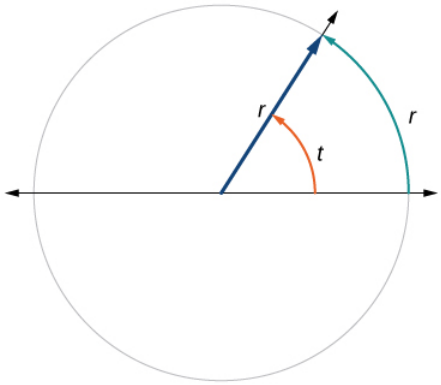


Figure $Page 6.3.12$: The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the **radian measure**, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s s to the radius r. See Figure $Page 6.3.13$.

$$s = r\theta \qquad \text{(Page 6.3.8)}$$

$$\theta = \frac{s}{r} \qquad \text{(Page 6.3.9)}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$ radian.

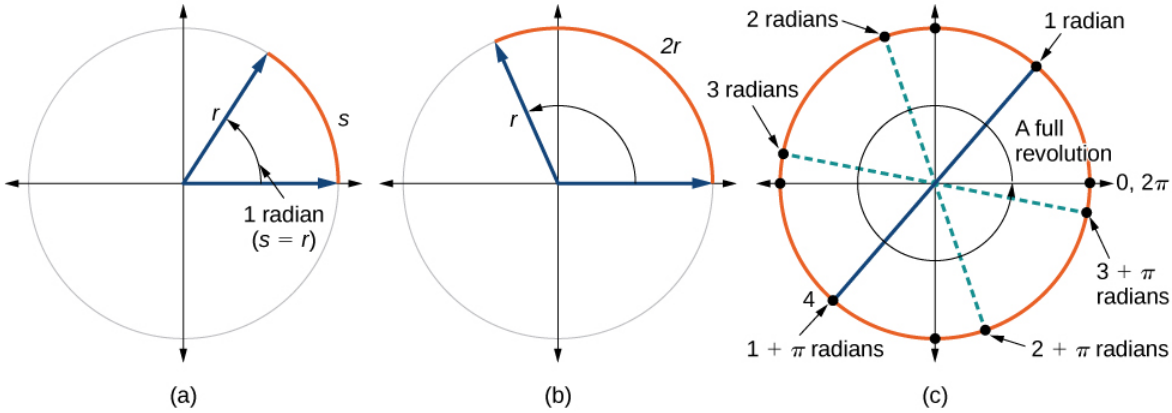


Figure $Page 6.3.13$: (a) In an angle of 1 radian, the arc length s equals the radius r. (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$,where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45° angle on the two circles, as in Figure $Page 6.3.14$.

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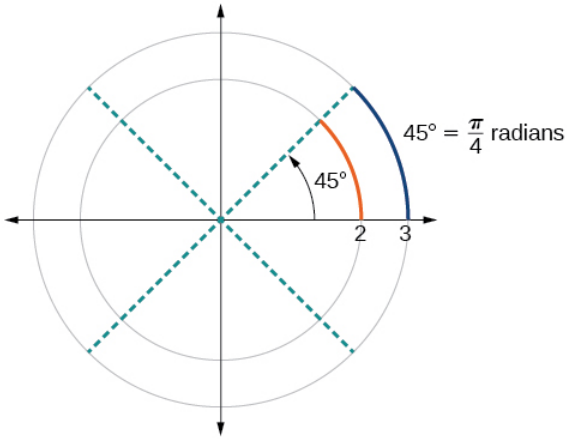


Figure $Page6.3.14$: A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi \qquad (\text{Page } 6.3.10)$$

$$\text{Larger circle: } \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi \qquad (\text{Page } 6.3.11)$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

RADIANS

One **radian** is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.

The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

A measure of 1 radian looks to be about 60°. Is that correct?

Yes. It is approximately 57.3°. Because 2π radians equals 360°, 1 radian equals $\frac{360^\circ}{2\pi} \approx 57.3^\circ$.

Using Radians

Because **radian** measure is the ratio of two lengths, it is a unitless measure. For example, in Figure $Page6.3.14$, suppose the radius were 2 inches and the distance along the arc were also 2 inches. When we calculate the radian measure of the angle, the "inches" cancel, and we have a result without units. Therefore, it is not necessary to write the label "radians" after a radian measure, and if we see an angle that is not labeled with "degrees" or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the **unit circle** (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$\begin{aligned} 1\text{rotation} &= 360^\circ &&= 2\pi \quad \text{radians} \\ \frac{1}{2}\text{rotation} &= 180^\circ &&= \pi \quad \text{radians} \\ \frac{1}{4}\text{rotation} &= 90^\circ &&= \frac{\pi}{2} \quad \text{radians} \end{aligned} \qquad (\text{Page } 6.3.12)$$

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Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in Figure $Page 6.3.15$. Memorizing these angles will be very useful as we study the properties associated with angles.

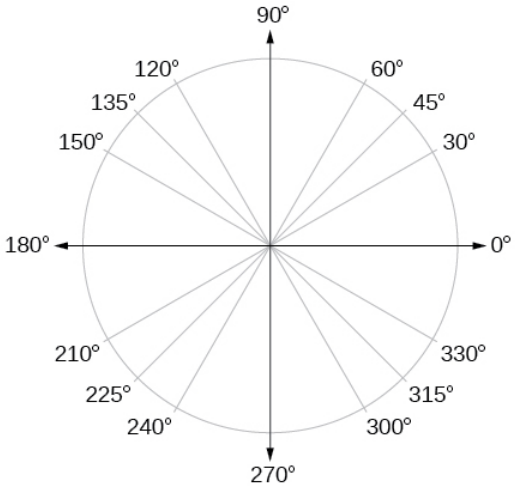


Figure $Page 6.3.15$: Commonly encountered angles measured in degrees

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in Figure $Page 6.3.15$, which are shown in Figure $Page 6.3.16$. Be sure you can verify each of these measures.

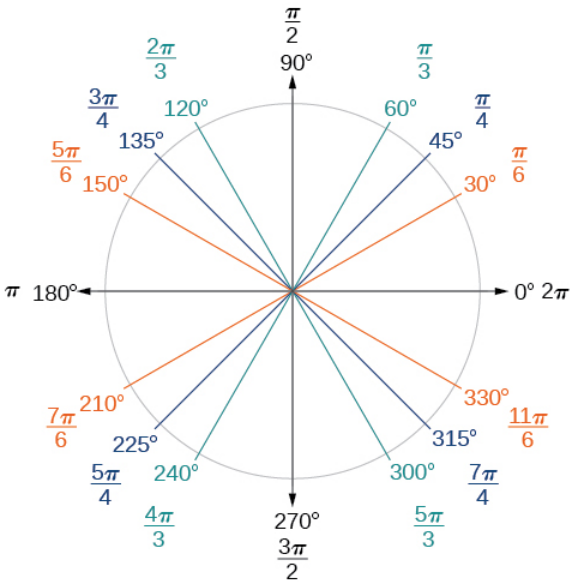


Figure $Page 6.3.16$: Commonly encountered angles measured in radians

Example $Page 6.3.2$: Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Solution

For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$$1 \text{ rotation} = 2\pi r \qquad \text{(Page 6.3.13)}$$

So,

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$$s = \frac{1}{3}(2\pi r) \quad \text{(Page 6.3.14)}$$

$$= \frac{2\pi r}{3} \quad \text{(Page 6.3.15)}$$

The radian measure would be the arc length divided by the radius.

$$\text{radian measure} = \frac{\frac{2\pi r}{3}}{r} \quad \text{(Page 6.3.16)}$$

$$= \frac{2\pi r}{3r} \quad \text{(Page 6.3.17)}$$

$$= \frac{2\pi}{3} \quad \text{(Page 6.3.18)}$$

Page 6.3.2

Find the radian measure of three-fourths of a full rotation.

Solution

$$\frac{3\pi}{2} \quad \text{(Page 6.3.19)}$$

Converting between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \quad \text{(Page 6.3.20)}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{Radians}{\pi} \quad \text{(Page 6.3.21)}$$

CONVERTING BETWEEN RADIANS AND DEGREES

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \quad \text{(Page 6.3.22)}$$

Example *Page* 6.3.3: Converting Radians to Degrees

Convert each radian measure to degrees.

1. $\frac{\pi}{6}$
2. 3

Solution

Because we are given radians and we want degrees, we should set up a proportion and solve it.

1. We use the proportion, substituting the given information.

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$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \quad \text{(Page 6.3.23)}$$

$$\frac{\theta}{180} = \frac{\frac{\pi}{6}}{\pi} \quad \text{(Page 6.3.24)}$$

$$\theta = \frac{180}{6} \quad \text{(Page 6.3.25)}$$

$$\theta = 30^\circ \quad \text{(Page 6.3.26)}$$

2. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \quad \text{(Page 6.3.27)}$$

$$\frac{\theta}{180} = \frac{3}{\pi} \quad \text{(Page 6.3.28)}$$

$$\theta = \frac{3(180)}{\pi} \quad \text{(Page 6.3.29)}$$

$$\theta \approx 172^\circ \quad \text{(Page 6.3.30)}$$

Page 6.3.3

Convert $-\frac{3\pi}{4}$ radians to degrees.

Solution

$-135°$

Example *Page* 6.3.4: Converting Degrees to Radians

Convert 15 degrees to radians.

Solution

In this example, we start with degrees and want radians, so we again set up a proportion and solve it, but we substitute the given information into a different part of the proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \quad \text{(Page 6.3.31)}$$

$$\frac{15}{180} = \frac{\theta^R}{\pi} \quad \text{(Page 6.3.32)}$$

$$\frac{15\pi}{180} = \theta^R \quad \text{(Page 6.3.33)}$$

$$\frac{\pi}{12} = \theta^R \quad \text{(Page 6.3.34)}$$

Analysis

Another way to think about this problem is by remembering that $30^\circ = \frac{\pi}{6}$. Because $15^\circ = \frac{1}{2}(30^\circ)$, ,we can find that $\frac{1}{2}(\frac{\pi}{6})$ is $\frac{\pi}{12}$.

Page 6.3.4

Convert 126° to radians.

Solution

$\frac{7\pi}{10}$

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Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at Figure $Page 6.3.17$. The angle of 140° is a **positive angle**, measured counterclockwise. The angle of –220° is a **negative angle**, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are **coterminal** angles. Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.

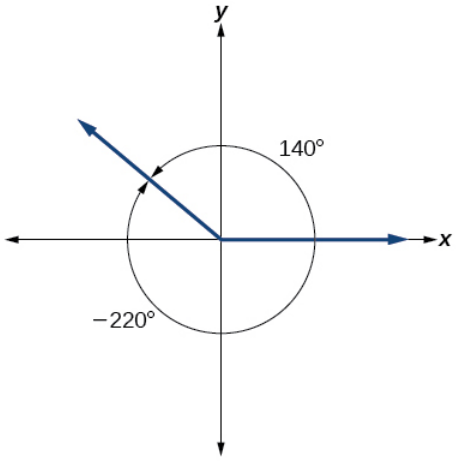


Figure $Page 6.3.17$: An angle of 140° and an angle of –220° are coterminal angles.

Any angle has infinitely many **coterminal angles** because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is −260°. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See Figure $Page 6.3.18$ for examples of reference angles for angles in different quadrants.

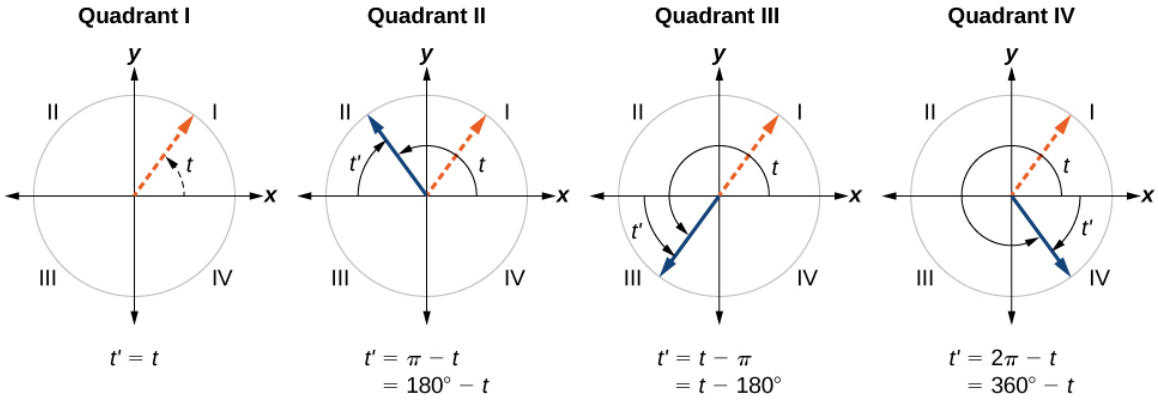


Figure $Page 6.3.18$

COTERMINAL AND REFERENCE ANGLES

- *Coterminal angles* are two angles in standard position that have the same terminal side.
- An angle's *reference angle* is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

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Given an angle greater than 360°, find a coterminal angle between 0° and 360°

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example $Page6.3.5$: Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0^\circ \le \theta < 360^\circ$.

Solution

An angle with measure 800° is coterminal with an angle with measure 800 − 360 = 440°, but 440° is still greater than 360°, so we subtract 360° again to find another coterminal angle: 440 − 360 = 80°.

The angle $\theta = 80^\circ$ is coterminal with 800°. To put it another way, 800° equals 80° plus two full rotations, as shown in Figure $Page6.3.19$.

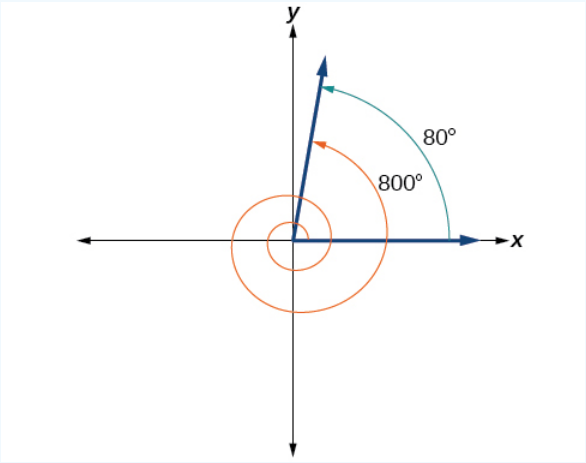


Figure $Page6.3.19$

$Page6.3.5$

Find an angle α that is coterminal with an angle measuring 870°, where $0^\circ \le \alpha < 360^\circ$.

Solution

$\alpha = 150^\circ$

Given an angle with measure less than 0°, find a coterminal angle having a measure between 0° and 360°.

1. Add 360° to the given angle.
2. If the result is still less than 0°, add 360° again until the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example $Page6.3.6$: Finding an Angle Coterminal with an Angle Measuring Less Than 0°

Show the angle with measure −45° on a circle and find a positive coterminal angle α such that 0° ≤ α < 360°.

Solution

Since 45° is half of 90°, we can start at the positive horizontal axis and measure clockwise half of a 90° angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360°, we can find a positive coterminal angle here by adding 360°:

$$-45^\circ + 360^\circ = 315^\circ \qquad (\text{Page } 6.3.35)$$

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We can then show the angle on a circle, as in Figure $Page 6.3.20$.

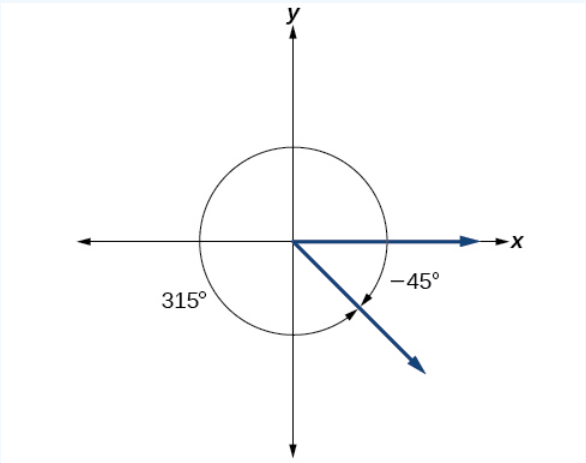


Figure $Page 6.3.20$

$Page 6.3.6$

Find an angle β that is coterminal with an angle measuring −300° such that $0^\circ \le \beta < 360^\circ$.

Solution

$\beta = 60^\circ$

Finding Coterminal Angles Measured in Radians

We can find **coterminal** angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Example $Page 6.3.7$: Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \le \beta < 2\pi$.

Solution

When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} \quad (\text{Page } 6.3.36)$$

$$= \frac{11\pi}{4} \quad (\text{Page } 6.3.37)$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} \quad (\text{Page } 6.3.38)$$

$$= \frac{3\pi}{4} \quad (\text{Page } 6.3.39)$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in Figure $Page 6.3.21$.

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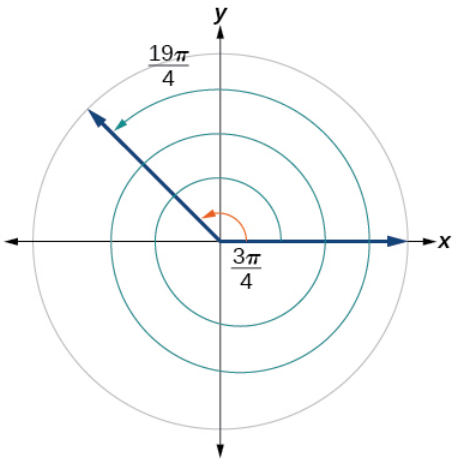


Figure $Page6.3.21$

$Page6.3.7$

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

Solution

$\frac{7\pi}{6}$

Determining the Length of an Arc

Recall that the **radian measure** θ of an angle was defined as the ratio of the **arc length** s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

ARC LENGTH ON A CIRCLE

In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians, shown in Figure $Page6.3.22$, is

$$s = r\theta \qquad (\text{Page } 6.3.40)$$

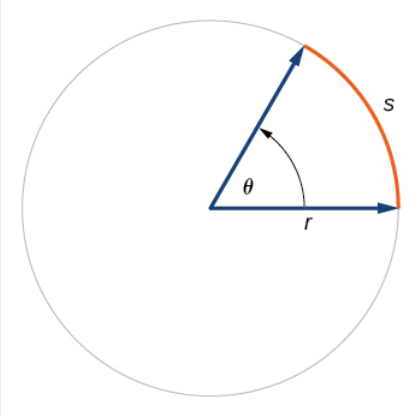


Figure $Page6.3.22$

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of θ : $s = r\theta$.

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Example $Page6.3.8$: Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Solution

1. Let's begin by finding the circumference of Mercury's orbit.

$$\begin{aligned} C &= 2\pi r && \text{(Page 6.3.41)} \\ &= 2\pi(36 \text{ million miles}) && \text{(Page 6.3.42)} \\ &\approx 226 \text{ million miles} && \text{(Page 6.3.43)} \end{aligned}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled:

$$(0.0114)226 \text{ million miles} = 2.58 \text{ million miles} \qquad \text{(Page 6.3.44)}$$

2. Now, we convert to radians:

$$\begin{aligned} \text{radian} &= \frac{\text{arc length}}{\text{radius}} && \text{(Page 6.3.45)} \\ &= \frac{2.58 \text{ million miles}}{36 \text{ million miles}} && \text{(Page 6.3.46)} \\ &= 0.0717 && \text{(Page 6.3.47)} \end{aligned}$$

$Page6.3.8$

Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Solution

$$\frac{215\pi}{18} = 37.525 \text{ units} \qquad \text{(Page 6.3.48)}$$

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a **sector of a circle**. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A = \pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\begin{aligned} \text{Area of sector} &= (\frac{\theta}{2\pi})\pi r^2 && \text{(Page 6.3.49)} \\ &= \frac{\theta\pi r^2}{2\pi} && \text{(Page 6.3.50)} \\ &= \frac{1}{2}\theta r^2 && \text{(Page 6.3.51)} \end{aligned}$$

AREA OF A SECTOR

The area of a **sector of a circle** with radius r subtended by an angle θ, measured in radians, is

$$A = \frac{1}{2}\theta r^2 \qquad \text{(Page 6.3.52)}$$

See Figure $Page6.3.23$.

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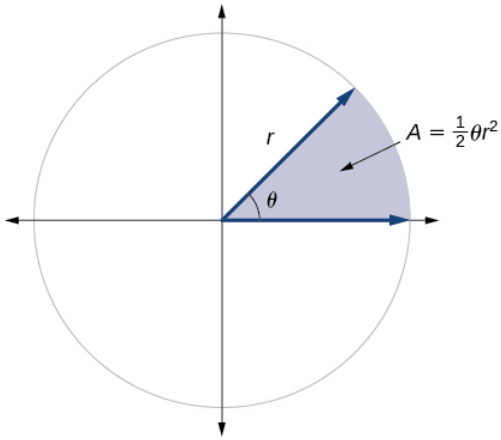


Figure $Page 6.3.23$: The area of the sector equals half the square of the radius times the central angle measured in radians.

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r : $A = \frac{1}{2}\theta r^2$.

Example $Page 6.3.9$: Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in Figure $Page 6.3.24$. What is the area of the sector of grass the sprinkler waters?

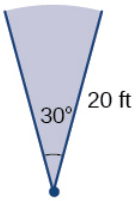


Figure $Page 6.3.24$: The sprinkler sprays 20 ft within an arc of 30°.

Solution

First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$30 \text{ degrees} = 30 \cdot \frac{\pi}{180} \quad \text{(Page 6.3.53)}$$

$$= \frac{\pi}{6} \text{ radians} \quad \text{(Page 6.3.54)}$$

The area of the sector is then

$$\text{Area} = \frac{1}{2}\left(\frac{\pi}{6}\right)(20)^2 \quad \text{(Page 6.3.55)}$$

$$\approx 104.72 \quad \text{(Page 6.3.56)}$$

So the area is about 104.72 ft^2.

$Page 6.3.9$

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

Solution

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Use Linear and Angular Speed to Describe Motion on a Circular Path

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its **displacement**) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10π inches, every second. So the linear speed of the point is 10π in./s. The equation for linear speed is as follows where v is linear speed, s is displacement, and t is time.

$$v = \frac{s}{t} \qquad \text{(Page 6.3.57)}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, angular speed is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}} = 90$ degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega = \frac{\theta}{t} \qquad \text{(Page 6.3.58)}$$

Combining the definition of angular speed with the arc length equation, $s = r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arc length equation gives:

$$s = r\theta \qquad \text{(Page 6.3.59)}$$
$$= r\omega t \qquad \text{(Page 6.3.60)}$$

Substituting this into the linear speed equation gives:

$$v = \frac{s}{t} \qquad = \frac{r\omega t}{t} = r\omega \qquad \text{(Page 6.3.61)}$$

ANGULAR AND LINEAR SPEED

As a point moves along a circle of radius r, its **angular speed**, ω, is the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t} \qquad \text{(Page 6.3.62)}$$

The **linear speed.** v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$$v = \frac{s}{t} \qquad \text{(Page 6.3.63)}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v = r\omega \qquad \text{(Page 6.3.64)}$$

This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

Given the amount of angle rotation and the time elapsed, calculate the angular speed

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

Example $Page6.3.10$: Finding Angular Speed

A water wheel, shown in Figure $Page6.3.25$, completes 1 rotation every 5 seconds. Find the angular speed in radians per second.

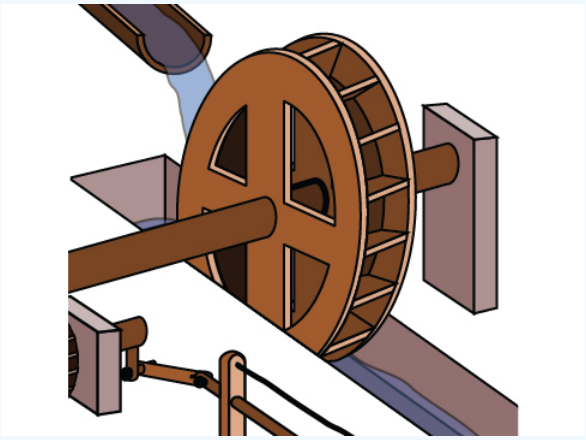


Figure $Page6.3.25$

Solution

The wheel completes 1 rotation, or passes through an angle of 2π radians in 5 seconds, so the angular speed would be $\omega = \frac{2\pi}{5} \approx 1.257$ radians per second.

$Page6.3.10$

An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

Solution

$-\frac{3\pi}{2}$ rad/s

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega = \frac{\theta}{t}$.
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply $v = r\omega$.

Example $Page6.3.11$: Finding a Linear Speed

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Solution

Here, we have an angular speed and need to find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road.

We begin by converting from rotations per minute to radians per minute. It can be helpful to utilize the units to make this conversion:

$$180\frac{\cancel{\text{rotations}}}{\text{minute}} \cdot \frac{2\pi \text{ radians}}{\cancel{\text{rotation}}} = 360\pi\frac{\text{radians}}{\text{minute}} \qquad (\text{Page } 6.3.65)$$

Using the formula from above along with the radius of the wheels, we can find the linear speed:

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$$v = (14 \text{ inches})(360\pi \frac{\text{radians}}{\text{minute}}) \quad \text{(Page 6.3.66)}$$

$$= 5040\pi \frac{\text{inches}}{\text{minute}} \quad \text{(Page 6.3.67)}$$

Remember that radians are a unitless measure, so it is not necessary to include them.unitless measure, so it is not necessary to include them.

Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour.

$$5040\pi \frac{\cancel{\text{inches}}}{\cancel{\text{minute}}} \cdot \frac{1 \ \cancel{\text{feet}}}{12 \ \cancel{\text{inches}}} \cdot \frac{1 \ \cancel{\text{mile}}}{5280 \ \cancel{\text{feet}}} \cdot \frac{60 \ \cancel{\text{minutes}}}{1 \text{ hour}} \approx 14.99 \text{ miles per hour (mph)} \quad \text{(Page 6.3.68)}$$

Page 6.3.11

A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

Solution

1655 kilometers per hour

Media

Access these online resources for additional instruction and practice with angles, arc length, and areas of sectors.

- Angles in Standard Position
- Angle of Rotation
- Coterminal Angles
- Determining Coterminal Angles
- Positive and Negative Coterminal Angles
- Radian Measure
- Coterminal Angles in Radians
- Arc Length and Area of a Sector

Key Equations

arc length	$s = r\theta$
area of a sector	$A = \frac{1}{2}\theta r^2$
angular speed	$\omega = \frac{\theta}{t}$
linear speed	$v = \frac{s}{t}$
linear speed related to angular speed	$v = r\omega$

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive *x*-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive *x*-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See **Example**.
- In addition to degrees, the measure of an angle can be described in radians. See **Example**.
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta^R}{\pi}$. See Example and Example.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting 360° or 2π. See Example and Example.

- Coterminal angles can be found using radians just as they are for degrees. See **Example**.
- The length of a circular arc is a fraction of the circumference of the entire circle. See **Example**.
- The area of sector is a fraction of the area of the entire circle. See **Example**.
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time. See **Example**.
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time. See **Example**.

Glossary

angle
the union of two rays having a common endpoint

angular speed
the angle through which a rotating object travels in a unit of time

arc length
the length of the curve formed by an arc

area of a sector
area of a portion of a circle bordered by two radii and the intercepted arc; the fraction $\frac{\theta}{2\pi}$ multiplied by the area of the entire circle

coterminal angles
description of positive and negative angles in standard position sharing the same terminal side

degree
a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

initial side
the side of an angle from which rotation begins

linear speed
the distance along a straight path a rotating object travels in a unit of time; determined by the arc length

measure of an angle
the amount of rotation from the initial side to the terminal side

negative angle
description of an angle measured clockwise from the positive x-axis

positive angle
description of an angle measured counterclockwise from the positive x-axis

quadrantal angle
an angle whose terminal side lies on an axis

radian measure
the ratio of the arc length formed by an angle divided by the radius of the circle

radian
the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

ray
one point on a line and all points extending in one direction from that point; one side of an angle

reference angle

the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

standard position

the position of an angle having the vertex at the origin and the initial side along the positive *x*-axis

terminal side

the side of an angle at which rotation ends

vertex

the common endpoint of two rays that form an angle

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Page 6.4: Unit Circle - Sine and Cosine Functions

Learning Objectives

- Find function values for the sine and cosine of $30°$ or $(\frac{\pi}{6})$,$45°$ or $(\frac{\pi}{4})$,and $60°$ or $(\frac{\pi}{3})$.
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.



Figure $Page6.4.1$: The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in Figure $Page6.4.2$. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a **unit circle**, $s = t$.

Recall that the *x*- and *y*-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

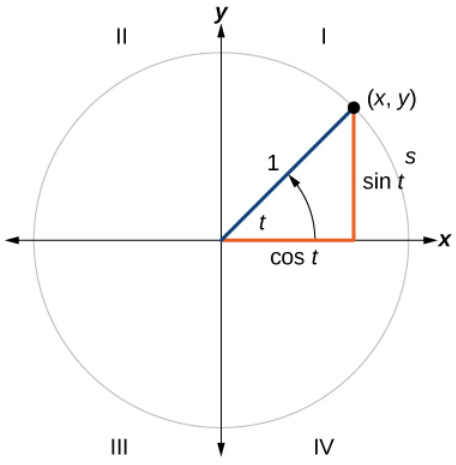


Figure $Page6.4.2$: Unit circle where the central angle is t radians

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UNIT CIRCLE

A unit circle has a center at $(0, 0)$ and radius 1. The length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions

Now that we have our unit circle labeled, we can learn how the (x, y) coordinates relate to the **arc length** and **angle**. The **sine function** relates a real number t to the y-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t. In Figure $Page 6.4.3$, the sine is equal to y. Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the y-coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x-value of the endpoint on the unit circle of an arc of length t. In Figure $Page 6.4.1$, the cosine is equal to x.

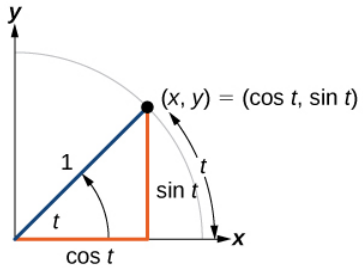


Figure $Page 6.4.3$

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

SINE AND COSINE FUNCTIONS

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t, then

$$\cos t = x \qquad \text{(Page 6.4.1)}$$
$$\sin t = y \qquad \text{(Page 6.4.2)}$$

How To: Given a point $P(x, y)$ on the unit circle corresponding to an angle of t, find the sine and cosine

1. The sine of t is equal to the y-coordinate of point P: $\sin t = y$.
2. The cosine of t is equal to the x-coordinate of point P: $\cos t = x$.

Example $Page 6.4.1$: Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t, as shown in Figure $Page 6.4.4$. Find $\cos(t)$ and $\sin(t)$.

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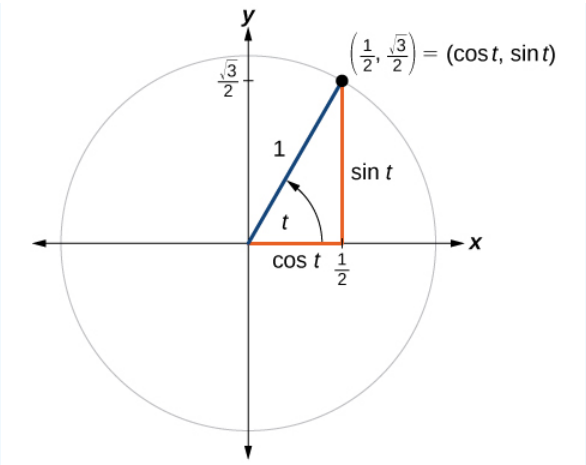


Figure $Page 6.4.4$

Solution

We know that $\cos t$ is the x-coordinate of the corresponding point on the unit circle and $\sin t$ is the y-coordinate of the corresponding point on the unit circle. So:

$$x = \cos t = \frac{1}{2}$$
$$y = \sin t = \frac{\sqrt{3}}{2}$$

Try It $Page 6.4.1$

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in Figure $Page 6.4.5$. Find $\cos t$ and $\sin t$.

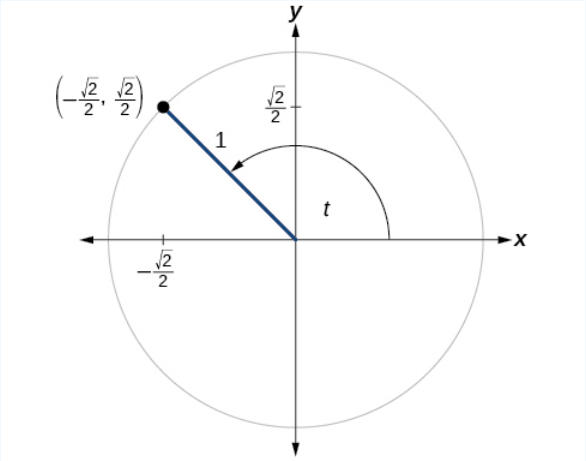


Figure $Page 6.4.5$

Answer

$$\cos(t) = -\frac{\sqrt{2}}{2}, \sin(t) = \frac{\sqrt{2}}{2}$$

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Finding Sines and Cosines of Angles on an Axis

For quadrantral angles, the corresponding point on the unit circle falls on the x- or y-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Example $Page 6.4.2$: Calculating Sines and Cosines along an Axis

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.

Solution

Moving 90° counterclockwise around the unit circle from the positive x-axis brings us to the top of the circle, where the (x, y) coordinates are (0, 1), as shown in Figure $Page 6.4.6$.

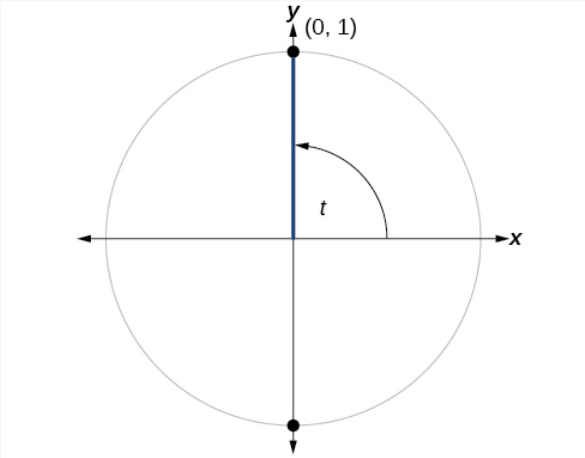


Figure $Page 6.4.6$

Using our definitions of cosine and sine,

$$x = \cos t = \cos(90^\circ) = 0$$
$$y = \sin t = \sin(90^\circ) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

Try It $Page 6.4.2$

Find cosine and sine of the angle π.

Answer

$\cos(\pi) = -1,\ \sin(\pi) = 0$

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See Figure $Page 6.4.7$.

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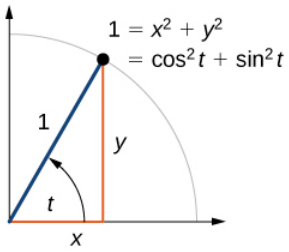


Figure $Page 6.4.7$

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1 \qquad (\text{Page } 6.4.3)$$

How To: Given the sine of some angle t and its quadrant location, find the cosine of t

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

Example $Page 6.4.3$: Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution

If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See Figure $Page 6.4.8$.

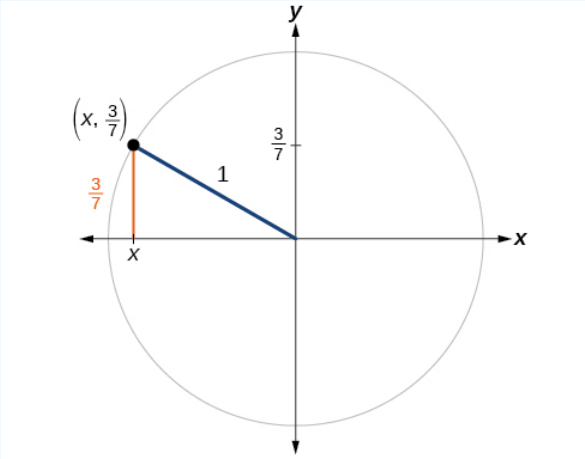


Figure $Page 6.4.8$

Substituting the known value for sine into the Pythagorean Identity,

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$$\begin{aligned}\cos^2(t)+\sin^2(t) &= 1\\ \cos^2(t)+\frac{9}{49} &= 1\\ \cos^2(t) &= \frac{40}{49}\\ \cos(t) &= \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7}\end{aligned}$$

Because the angle is in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative. So

$$\cos(t) = -\frac{2\sqrt{10}}{7}$$

Try It $Page 6.4.3$

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Answer

$$\sin(t) = -\frac{7}{25}$$

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the **Pythagorean Identity** and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in Figure $Page 6.4.9$. A $45^\circ - 45^\circ - 90^\circ$ triangle is an isosceles triangle, so the x- and y-coordinates of the corresponding point on the circle are the same. Because the x- and y-values are the same, the sine and cosine values will also be equal.

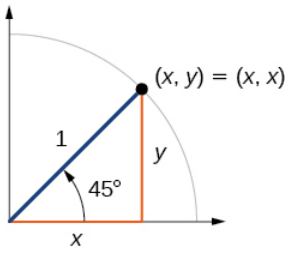


Figure $Page 6.4.9$

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first **quadrantal angle**. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y (with $y = x$), and a radius = 1. See Figure $Page 6.4.10$.

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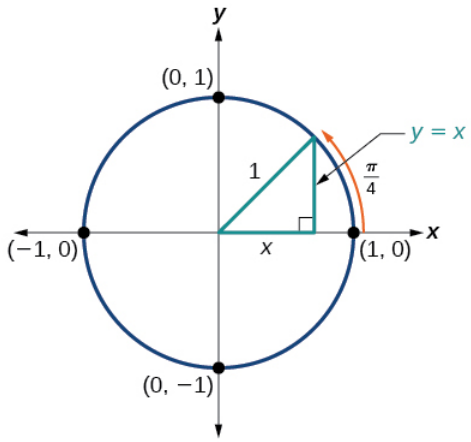


Figure $Page$6.4.10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

Substituting $y = x$, we get

$$x^2 + x^2 = 1$$

Combining like terms we get

$$2x^2 = 1$$

And solving for x, we get

$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$(x, y) = (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$$
$$\cos t = \frac{1}{\sqrt{2}},\ \sin t = \frac{1}{\sqrt{2}}$$

If we then rationalize the denominators, we get

$$\cos t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{2}}{2}$$
$$\sin t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{2}}{2}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

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Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30°, or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of 30°, and another at an angle of -30°, as shown in Figure $Page 6.4.11$. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60°, as shown in Figure $Page 6.4.12$.

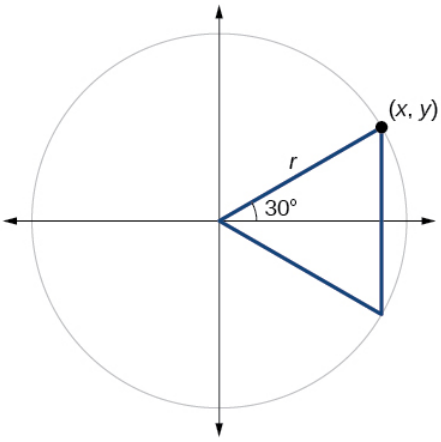


Figure $Page 6.4.11$

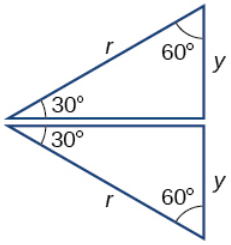


Figure $Page 6.4.12$

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

And since $r = 1$ in our **unit circle**,

$$\begin{aligned} \sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1) \\ &= \frac{1}{2} \end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\begin{aligned} \cos^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{6}\right) &= 1 \\ \cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 &= 1 \\ \cos^2\left(\frac{\pi}{6}\right) &= \frac{3}{4} && \text{Use the square root property.} \\ \cos\left(\frac{\pi}{6}\right) &= \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} && \text{Since } y \text{ is positive, choose the positive root.} \end{aligned}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30° are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t = \frac{\pi}{3}$ (60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in Figure $Page 6.4.13$. Angle A has measure 60°. 60°. At point B, we draw an angle ABC with measure of 60°. We know the angles in a triangle sum to 180°, so the

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measure of angle C is also 60°. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

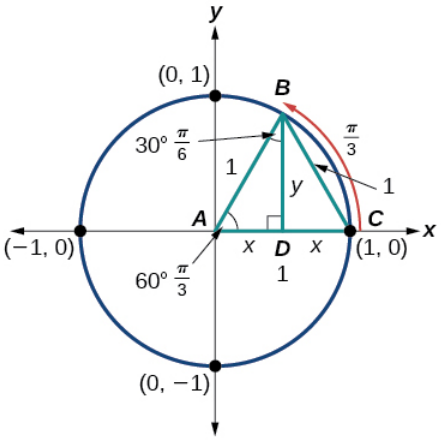


Figure $Page$6.4.13

The measure of angle ABD is 30°. So, if double, angle ABC is 60°. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x-coordinate of point B, which is at the intersection of the 60° angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2+y^2=1$$

Substituting $x=\frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2+y^2=1$$

Solving for y, we get

$$\begin{aligned}\frac{1}{4}+y^2&=1\\ y^2&=1-\frac{1}{4}\\ y^2&=\frac{3}{4}\\ y&=\pm\frac{\sqrt{3}}{2}\end{aligned}$$

Since $t=\frac{\pi}{3}$ has the terminal side in quadrant I where the y-coordinate is positive, we choose $y=\frac{\sqrt{3}}{2}$, the positive value.

At $t=\frac{\pi}{3}$ (60°), the (x,y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$(x,y)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$$

$$x=\frac{1}{2},\ y=\frac{\sqrt{3}}{2}$$

$$\cos t=\frac{1}{2},\ \sin t=\frac{\sqrt{3}}{2}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. Table $Page$6.4.1 summarizes these values.

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Table $Page 6.4.1$

Angle	0	$\frac{\pi}{6}$, or 30	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Figure $Page 6.4.14$ shows the common angles in the first quadrant of the unit circle.

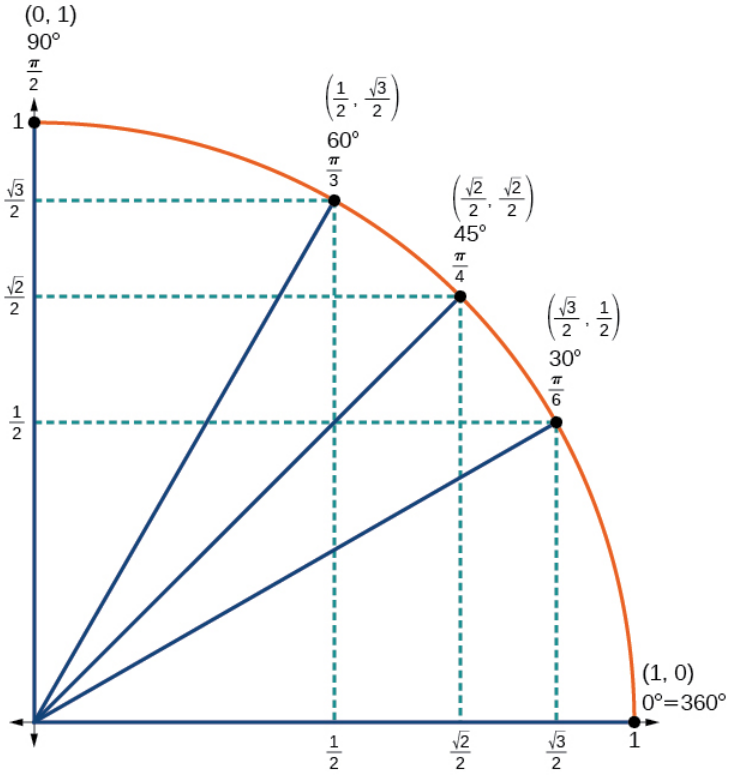


Figure $Page 6.4.14$

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the **special angles**, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To: Given an angle in radians, use a graphing calculator to find the cosine

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Example $Page 6.4.4$: Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

Solution

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Enter the following keystrokes:

$$\text{COS}(5 \times \pi \div 3) \text{ ENTER}$$

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of 20°, for example, by including the conversion factor to radians as part of the input:

$$\text{SIN}(20 \times \pi \div 180) \text{ ENTER}$$

Try It $Page 6.4.4$

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Answer

approximately 0.866025403

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the **unit circle**, as shown in Figure $Page 6.4.15$. The bounds of the x-coordinate are $[-1, 1]$. The bounds of the y-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

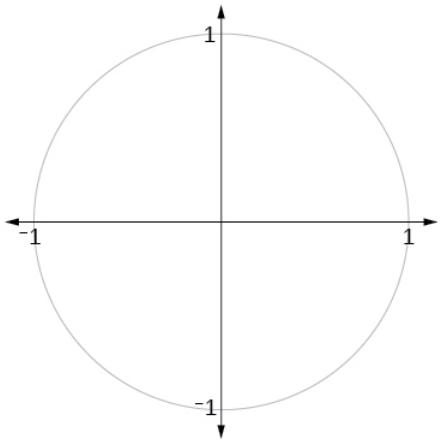


Figure $Page 6.4.15$

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y-coordinate on the unit circle, the other angle with the same sine will share the same y-value, but have the opposite x-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x-value but will have the opposite y-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in Figure $Page 6.4.16$, angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\sin(t) = \sin(\alpha) \quad \text{and} \quad \cos(t) = -\cos(\alpha)$$
$$\sin(t) = -\sin(\beta) \quad \text{and} \quad \cos(t) = \cos(\beta)$$

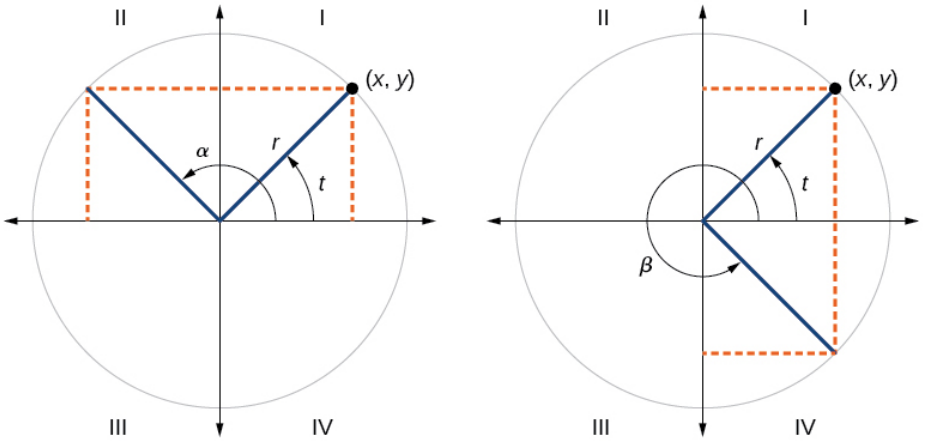


Figure $Page 6.4.16$

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90°, or 0 and $\frac{\pi}{2}$ radians. As we can see from Figure $Page 6.4.17$, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

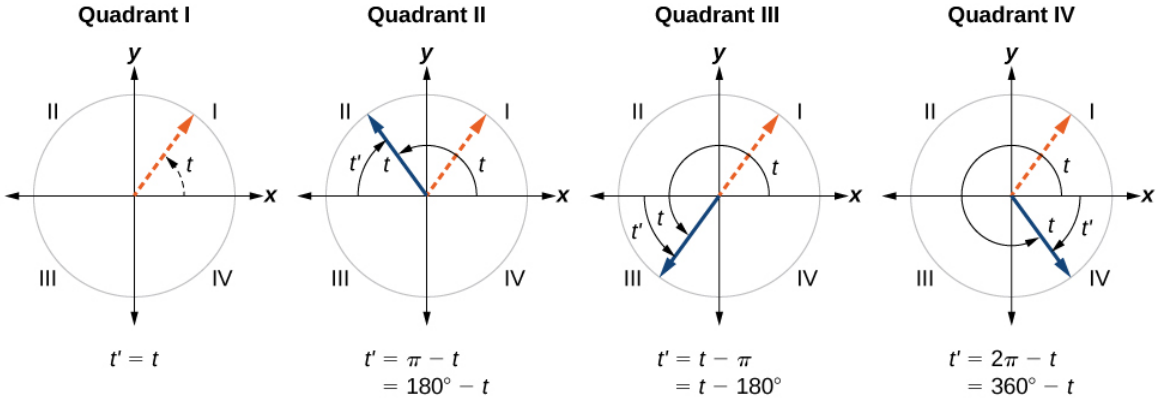


Figure $Page 6.4.17$

How To: Given an angle between 0 and 2π, find its reference angle

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180^\circ - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360^\circ - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Example $Page 6.4.5$: Finding a Reference Angle

Find the reference angle of 225° as shown in Figure $Page 6.4.18$.

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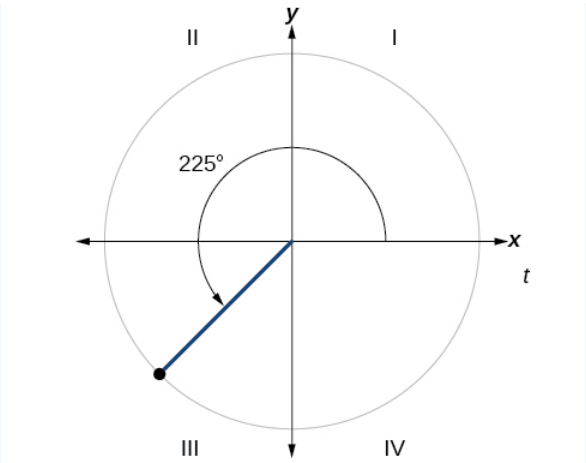


Figure $Page 6.4.18$

Solution

Because 225° is in the third quadrant, the reference angle is

$$|(180^\circ - 225^\circ)| = |-45^\circ| = 45^\circ$$

Try It $Page 6.4.5$

Find the reference angle of $\frac{5\pi}{3}$.

Answer

(\dfrac{π}{3}\)

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a **negative angle**. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the **reference angle** of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

USING REFERENCE ANGLES TO FIND COSINE AND SINE

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To: Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

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Example $Page 6.4.6$: Using Reference Angles to Find Sine and Cosine

1. Using a reference angle, find the exact value of $\cos(150^\circ)$ and $\sin(150^\circ)$.
2. Using the reference angle, find $\cos\ \frac{5\pi}{4}$ and $\sin\ \frac{5\pi}{4}$.

Solution

This tells us that 150° has the same sine and cosine values as 30°, except for the sign. We know that

Since 150° is in the second quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. The y-coordinate is positive, so the sine value is positive.

$\frac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\left|\frac{5\pi}{4}-\pi\right|=\frac{\pi}{4}$. The cosine and sine of $\frac{\pi}{4}$ are both $\frac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$$\cos\ \frac{5\pi}{4}=-\frac{\sqrt{2}}{2}\ \text{and}\ \sin\ \frac{5\pi}{4}=-\frac{\sqrt{2}}{2}$$

Try It $Page 6.4.6$

1. Use the reference angle of 315° to find $\cos(315^\circ)$ and $\sin(315^\circ)$.
2. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos\left(-\frac{\pi}{6}\right)$ and $\sin\left(-\frac{\pi}{6}\right)$

Answer

1. $\cos(315^\circ)=\frac{\sqrt{2}}{2}, \sin(315^\circ)=\frac{-\sqrt{2}}{2}$
2. $\cos\left(-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}, \sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in Figure $Page 6.4.19$. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

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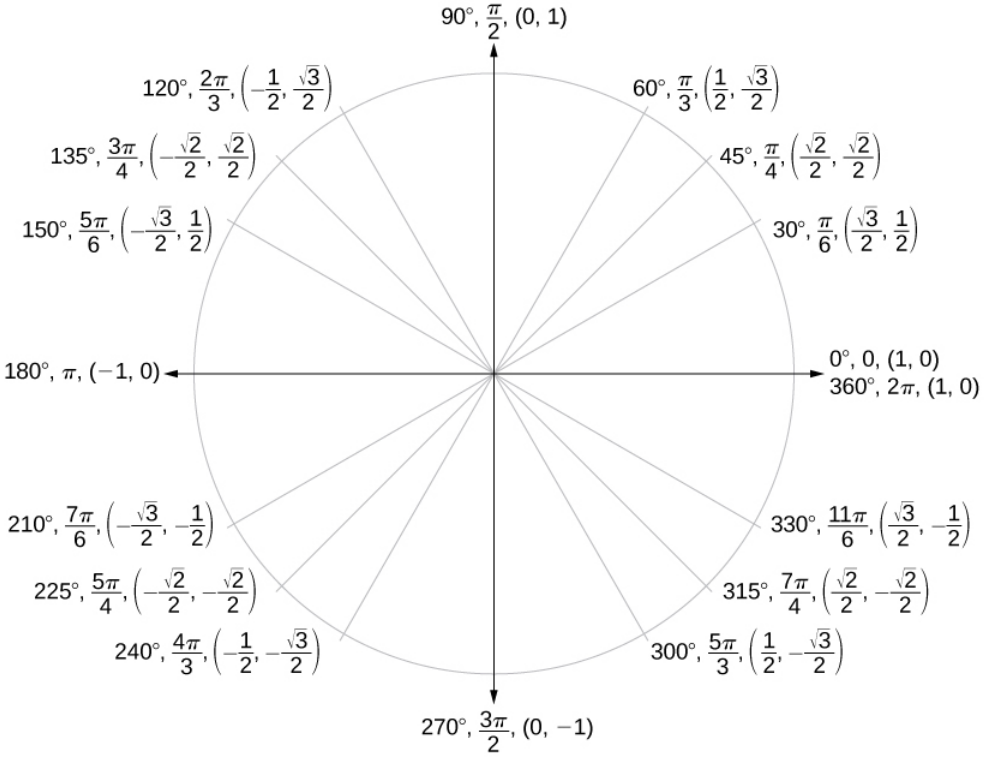


Figure $Page 6.4.19$: Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the **identities**

$$x = \cos t$$
$$y = \sin t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the **reference angle**, and give them the signs corresponding to the y- and x-values of the quadrant.

How To: Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point

1. Find the reference angle by measuring the smallest angle to the x-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example $Page 6.4.7$: Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π.

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{3}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

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We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$
$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos\,\theta$ and $y = \sin\,\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It *Page* 6.4.7

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Answer

$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

media

Access these online resources for additional instruction and practice with sine and cosine functions.

- Trigonometric Functions Using the Unit Circle
- Sine and Cosine from the Unit Circle
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four
- Trigonometric Functions Using Reference Angles

Key Equations

Cosine	$\cos\,t = x$
Sine	$\sin\,t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x-value of the endpoint. See Example.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See Example.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See Example.
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See Example.
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the x- and y-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis. See Example.

- Reference angles can be used to find the sine and cosine of the original angle. See Example.
- Reference angles can also be used to find the coordinates of a point on a circle. See Example.

Glossary

cosine function

the x-value of the point on a unit circle corresponding to a given angle

Pythagorean Identity

a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

sine function

the y-value of the point on a unit circle corresponding to a given angle

unit circle

a circle with a center at $(0, 0)$ and radius 1.

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Page 6.5: Graphs of the Sine and Cosine Functions

Learning Objectives

- Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
- Use phase shifts of sine and cosine curves.

In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x- and y-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. Table $Page 6.5.1$ lists some of the values for the sine function on a unit circle.

Table $Page 6.5.1$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Plotting the points from the table and continuing along the x-axis gives the shape of the sine function. See Figure $Page 6.5.2$.

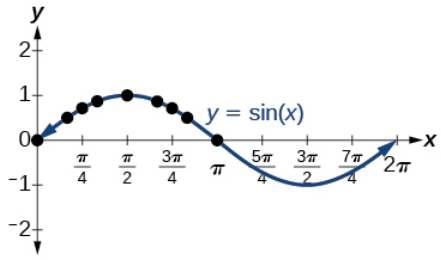


Figure $Page 6.5.2$: The sine function

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See Figure $Page 6.5.3$.

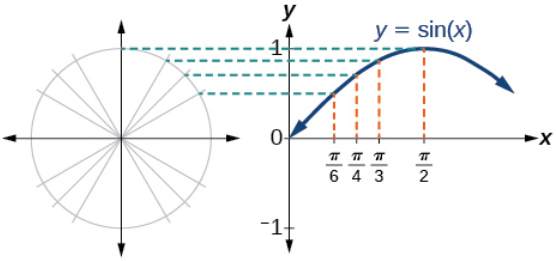


Figure $Page 6.5.3$: Plotting values of the sine function

Now let's take a similar look at the **cosine function**. Again, we can create a table of values and use them to sketch a graph. Table $Page 6.5.2$ lists some of the values for the cosine function on a unit circle.

Table $Page 6.5.2$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

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As with the sine function, we can plots points to create a graph of the cosine function as in Figure $Page 6.5.4$.

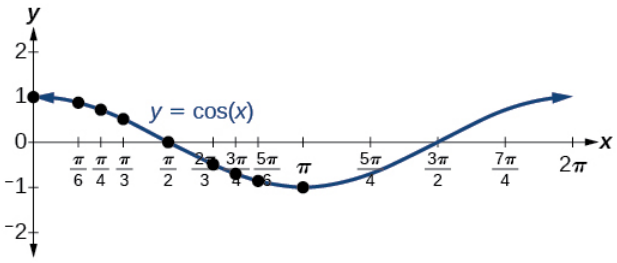


Figure $Page 6.5.4$: The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A **periodic function** is a function for which a specific **horizontal shift**, P, results in a function equal to the original function: $f(x+P) = f(x)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P > 0$ the **period** of the function. Figure $Page 6.5.5$ shows several periods of the sine and cosine functions.

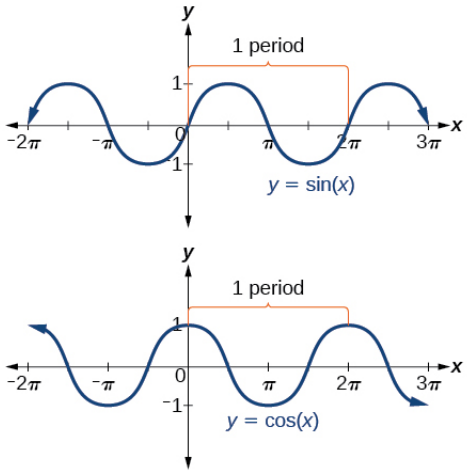


Figure $Page 6.5.5$

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in Figure $Page 6.5.6$, the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin\ x$. Now we can clearly see this property from the graph.

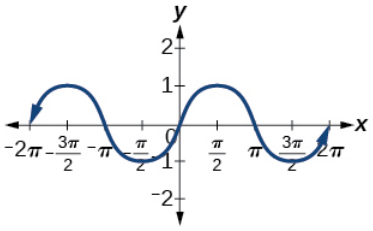


Figure $Page 6.5.6$: Odd symmetry of the sine function

Figure $Page 6.5.7$ shows that the cosine function is symmetric about the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos\ x$.

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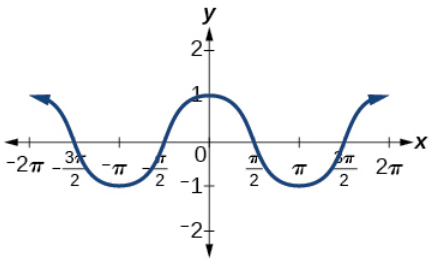


Figure $Page 6.5.7$: Even symmetry of the cosine function

CHARACTERISTICS OF SINE AND COSINE FUNCTIONS

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos\ x$ is symmetric about they- y-axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or **cosine function** is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D \qquad (\text{Page } 6.5.1)$$

and

$$y = A\cos(Bx - C) + D \qquad (\text{Page } 6.5.2)$$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \dfrac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π,which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π and the graph is compressed. If $f(x) = \sin\left(\dfrac{x}{2}\right)$, then $B = \dfrac{1}{2}$, so the period is 4π and the graph is stretched. Notice in Figure $Page 6.5.8$ how the period is indirectly related to $|B|$.

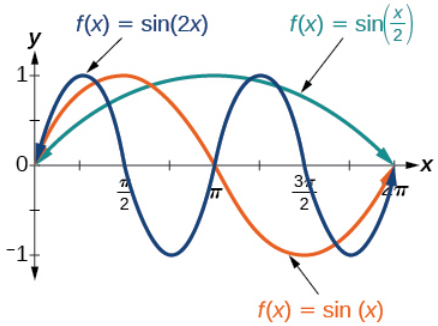


Figure $Page 6.5.8$

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PERIOD OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

- $y = A\sin(Bx)$
- $y = A\cos(Bx)$

The period is $\dfrac{2\pi}{|B|}$.

Example $Page 6.5.1$: Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x) = \sin\left(\dfrac{\pi}{6}x\right)$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx)$.

In the given equation, $B = \dfrac{\pi}{6}$, so the period will be

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{6}} \\ &= 2\pi \cdot \frac{6}{\pi} \\ &= 12 \end{aligned}$$

Exercise $Page 6.5.1$

Determine the period of the function $g(x) = \cos(\frac{x}{3})$.

Answer

6π

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical **midline** of the graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the *x*-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4sinx$ is twice the amplitude of

$$f(x) = 2\sin x$$

If $|A| < 1$, the function is compressed. Figure $Page 6.5.9$ compares several sine functions with different amplitudes.

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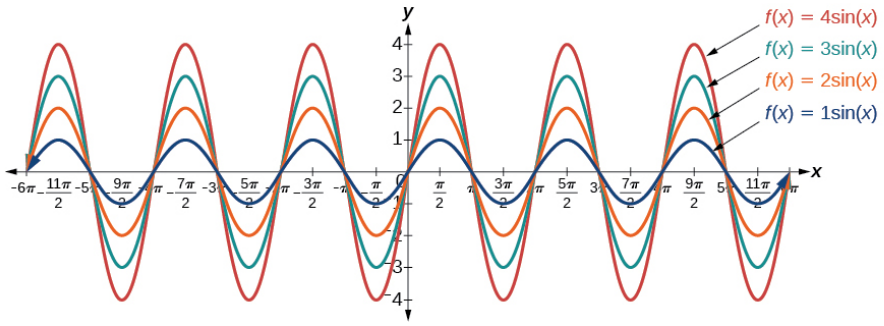


Figure $Page 6.5.9$

AMPLITUDE OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A\sin(Bx) \text{ and } y = A\cos(Bx) \qquad (\text{Page } 6.5.3)$$

The amplitude is A, and the vertical height from the midline is $|A|$. In addition, notice in the example that

$$|A| = amplitude = \frac{1}{2}\,|\,maximum - minimum| \qquad (\text{Page } 6.5.4)$$

Example $Page 6.5.2$: Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Solution

Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis

The negative value of A results in a reflection across the x-axis of the sine function, as shown in Figure $Page 6.5.10$.

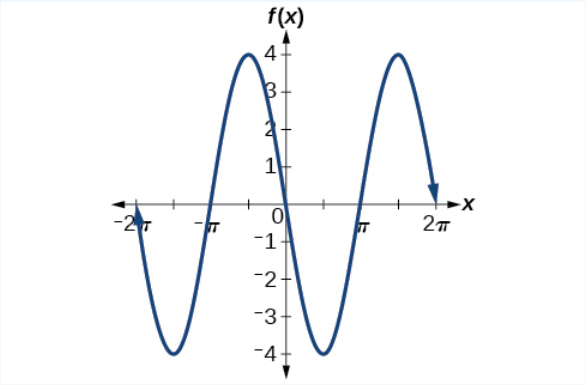


Figure $Page 6.5.10$

Exercise $Page 6.5.2$

What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2}\sin(x)$? Is the function stretched or compressed vertically?

Answer

$\frac{1}{2}$ compressed

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Analyzing Graphs of Variations of $y = \sin\ x$ and $y = \cos\ x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y = A\sin(Bx - C) + D \quad \text{and} \quad y = A\cos(Bx - C) + D \qquad \text{(Page 6.5.5)}$$

or

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \quad \text{and} \quad y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D \qquad \text{(Page 6.5.6)}$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. Figure $Page 6.5.11$ shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is more than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

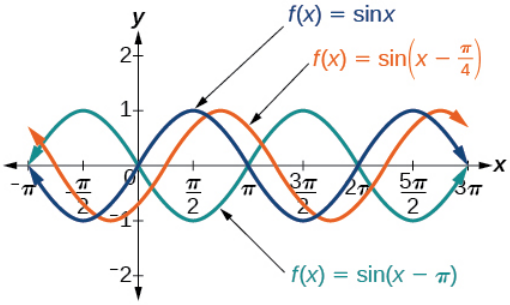


Figure $Page 6.5.11$

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See Figure $Page 6.5.12$. The function $y = \cos(x) + D$ has its midline at $y = D$.

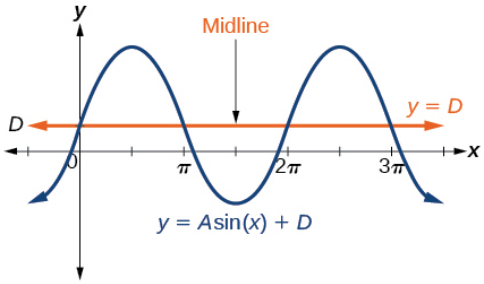


Figure $Page 6.5.12$

Any value of D other than zero shifts the graph up or down. Figure $Page 6.5.13$ compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

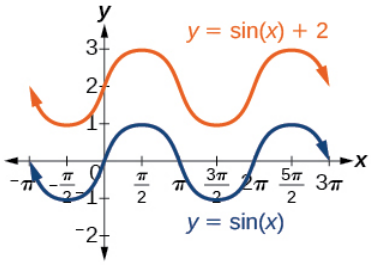


Figure $Page 6.5.13$

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VARIATIONS OF SINE AND COSINE FUNCTIONS

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{D}$ is the **phase shift** and D is the **vertical shift.**

Example $Page 6.5.3$: Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

In the given equation, notice that $B = 1$ and $C = -\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned} \frac{C}{B} &= -\frac{\frac{\pi}{6}}{1} \\ &= -\frac{\pi}{6} \end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f(x) = \sin(x + \frac{\pi}{6}) - 2$ can be rewritten as $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2$. If the value of C is negative, the shift is to the left.

Exercise $Page 6.5.3$

Determine the direction and magnitude of the phase shift for $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$.

Answer

$\frac{\pi}{2}$; right

Example $Page 6.5.4$: Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f(x) = \cos(x) - 3$.

Solution

Let's begin by comparing the equation to the general form $y = A\cos(Bx - C) + D$.

In the given equation, $D = -3$ so the shift is 3 units downward.

Exercise $Page 6.5.4$

Determine the direction and magnitude of the vertical shift for $f(x) = 3\sin(x) + 2$.

Answer

2 units up

How to: Given a sinusoidal function in the form $f(x) = A\sin(Bx - C) + D$,identify the midline, amplitude, period, and phase shift

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.

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4. Determine the midline as $y = D$.

Example $Page6.5.5$: Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

Next, $B = 2$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\dfrac{C}{B} = \dfrac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

Analysis

Inspecting the graph, we can determine that the period is π, the midline is $y = 1$, and the amplitude is 3. See Figure $Page6.5.14$.

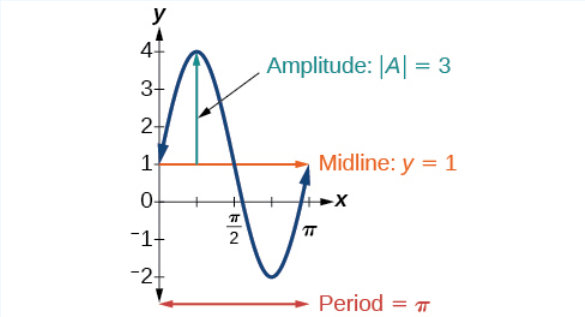


Figure $Page6.5.14$

Exercise $Page6.5.5$

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

Answer

midline: $y = 0$; amplitude: $|A| = \frac{1}{2}$; period: $P = \frac{2\pi}{|B|} = 6\pi$; phase shift: $\frac{C}{B} = \pi$

Example $Page6.5.6$: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in Figure $Page6.5.15$.

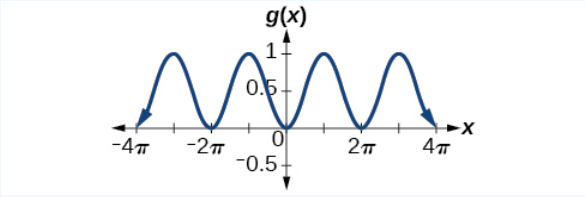


Figure $Page6.5.15$

Solution

To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A\sin(Bx - C) + D$$

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$$y = A\cos(Bx - C) + D$$

The graph could represent either a sine or a **cosine function** that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5$. Also, the graph is reflected about the x-axis so that $A = -0.5$.

The graph is not horizontally stretched or compressed, so $B = 1$; and the graph is not shifted horizontally, so $C = 0$.

Putting this all together,

$$g(x) = -0.5\cos(x) + 0.5$$

? Exercise $Page 6.5.6$

Determine the formula for the sine function in Figure $Page 6.5.16$.

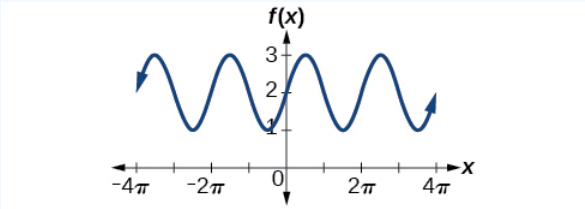


Figure $Page 6.5.16$

Answer

$f(x) = \sin(x) + 2$

✓ Example $Page 6.5.7$: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in Figure $Page 6.5.17$.

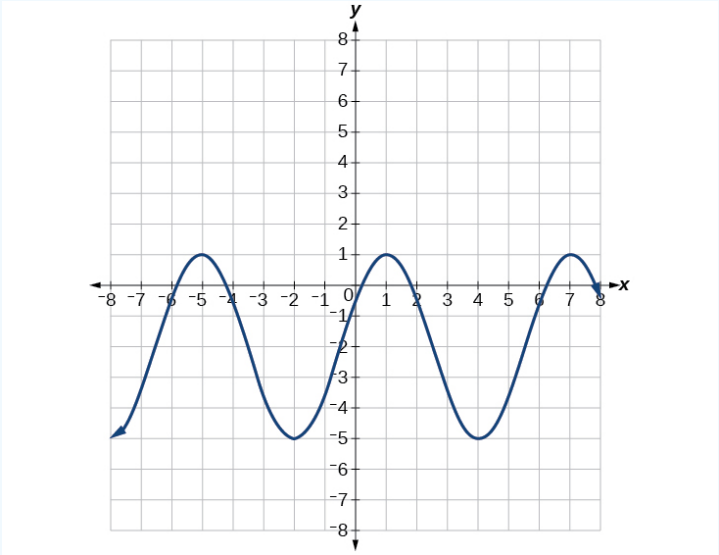


Figure $Page 6.5.17$

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Solution

With the highest value at 1 and the lowest value at -5, the midline will be halfway between at -2. So $D=-2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A|=3$.

The period of the graph is 6, which can be measured from the peak at $x=1$ to the next peak at $x=7$,or from the distance between the lowest points. Therefore, $P=\dfrac{2\pi}{|B|}=6$. Using the positive value for B,we find that

$$\begin{aligned} B &= \frac{2\pi}{P} \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

So far, our equation is either $y=3\sin\left(\dfrac{\pi}{3}x-C\right)-2$ or $y=3\cos\left(\dfrac{\pi}{3}x-C\right)-2$.For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes

$$\begin{aligned} y &= 3\cos\left(\frac{\pi}{3}x-\frac{\pi}{3}\right)-2 \qquad \text{or} \\ y &= -3\cos\left(\frac{\pi}{3}x+\frac{2\pi}{3}\right)-2 \end{aligned}$$

Again, these functions are equivalent, so both yield the same graph.

? Exercise $Page 6.5.7$

Write a formula for the function graphed in Figure $Page 6.5.18$.

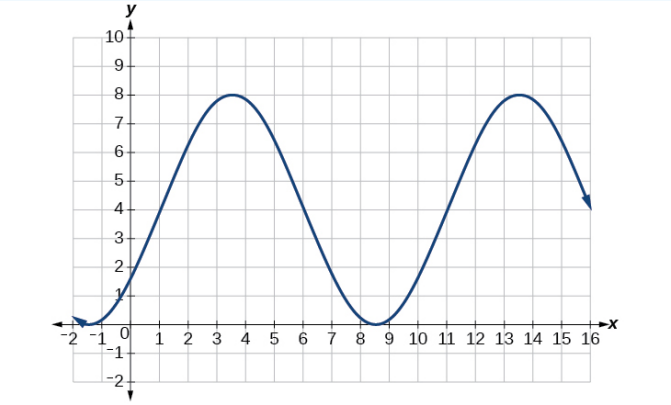


Figure $Page 6.5.18$

Answer

two possibilities: $y=4\sin\left(\dfrac{\pi}{5}x-\dfrac{\pi}{5}\right)+4$ or $y=-4\sin\left(\dfrac{\pi}{5}x+\dfrac{4\pi}{5}\right)+4$

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Graphing Variations of $y = \sin\ x$ and $y = \cos\ x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

Given the function $y = A\sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \dfrac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \dfrac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the *x*-axis at $x = \dfrac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \dfrac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the *x*-axis at $x = \dfrac{\pi}{2|B|}$.

Example $Page 6.5.8$: Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\dfrac{\pi x}{2}\right)$.

Solution

Let's begin by comparing the equation to the form $y = A\sin(Bx)$.

- *Step 1.* We can see from the equation that $A = -2$, so the amplitude is 2.

  $|A| = 2$

- *Step 2.* The equation shows that $B = \dfrac{\pi}{2}$, so the period is

$$\begin{aligned} P &= \frac{2\pi}{\frac{\pi}{2}} \\ &= 2\pi \cdot \frac{2}{\pi} \\ &= 4 \end{aligned}$$

- *Step 3.* Because A is negative, the graph descends as we move to the right of the origin.
- *Step 4.* The x-intercepts are at the beginning of one period, $x = 0$, the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline, at $x = 1$, and a local maximum will occur at 2 units above the midline, at $x = 3$. Figure $Page 6.5.19$ shows the graph of the function.

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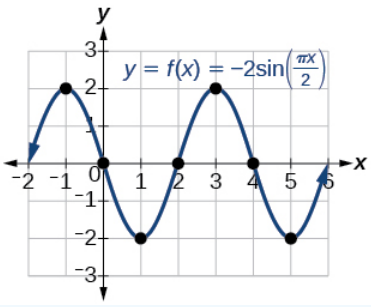


Figure $Page 6.5.19$

Exercise $Page 6.5.8$

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.

Answer

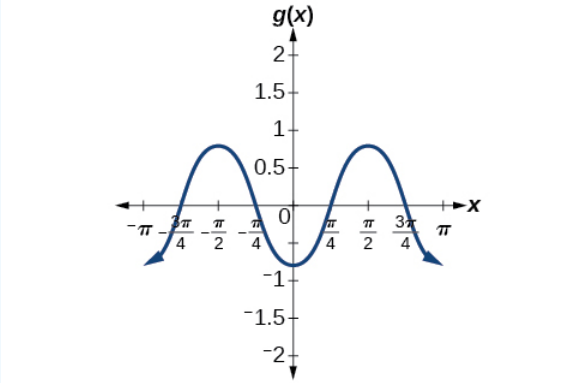


Figure $Page 6.5.20$

midline: $y = 0$; amplitude: $|A| = 0.8$; period: $P = \dfrac{2\pi}{|B|} = \pi$; phase shift: $\dfrac{C}{B} = 0$ or none

How to: Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \dfrac{2\pi}{|B|}$.
4. Identify the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $f(x) = A\sin(Bx)$ shifted to the right or left by $\dfrac{C}{B}$ and up or down by D.

Example $Page 6.5.9$: Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3\sin\left(\dfrac{\pi}{4x} - \dfrac{\pi}{4}\right)$.

Solution

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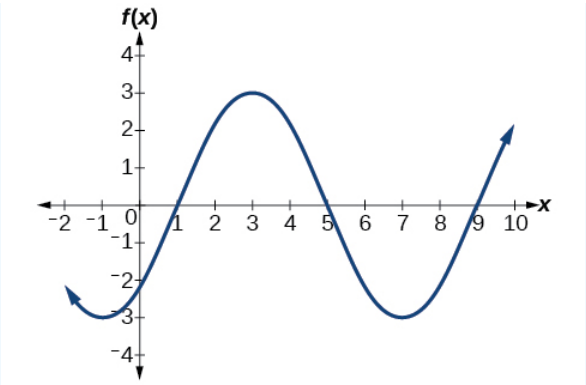


Figure $Page 6.5.21$: A horizontally compressed, vertically stretched, and horizontally shifted sinusoid

- *Step 1.* The function is already written in general form: $f(x) = 3\sin\left(\frac{\pi}{4x} - \frac{\pi}{4}\right)$ .This graph will have the shape of a sine function, starting at the midline and increasing to the right.
- *Step 2.* $|A| = |3| = 3$. The amplitude is 3.
- *Step 3.* Since $|B| = |\frac{\pi}{4}| = \frac{\pi}{4}$, we determine the period as follows.

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{4}} \\ &= 2\pi \cdot \frac{4}{\pi} \\ &= 8 \end{aligned}$$

The period is 8.

- *Step 4.* Since $C = \frac{\pi}{4}$, the phase shift is

$$\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1 \qquad \text{(Page 6.5.7)}$$

.

The phase shift is 1 unit.

- *Step 5.* Figure $Page 6.5.21$ shows the graph of the function.

? Exercise $Page 6.5.9$

Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Answer

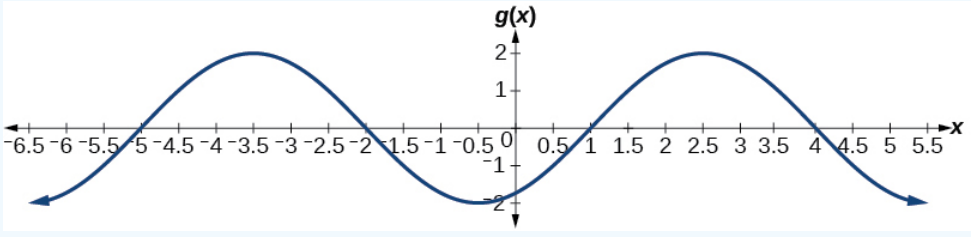


Figure $Page 6.5.22$

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midline: $y = 0$; amplitude: $|A| = 2$; period: $P = \dfrac{2\pi}{|B|} = 6$; phase shift: $\dfrac{C}{B} = -\dfrac{1}{2}$

Example $Page 6.5.10$: Identifying the Properties of a Sinusoidal Function

Given $y = -2cos\left(\dfrac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution

Begin by comparing the equation to the general form.

$$y = A\cos(Bx - C) + D$$

- *Step 1.* The function is already written in general form.
- *Step 2.* Since $A = -2$, the amplitude is $|A| = 2$.
- *Step 3.* $|B| = \dfrac{\pi}{2}$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{\frac{pi}{2}} = 2\pi \cdot \dfrac{2}{\pi} = 4$. The period is 4.
- *Step 4.* $C = -\pi$,so we calculate the phase shift as $\dfrac{C}{B} = -\dfrac{\pi}{\frac{\pi}{2}} = -\pi \cdot \dfrac{2}{\pi} = -2$. The phase shift is -2.
- *Step 5.* $D = 3$,so the midline is $y = 3$, and the vertical shift is up 3.

Since A is negative, the graph of the cosine function has been reflected about the x-axis.

Figure $Page 6.5.23$ shows one cycle of the graph of the function.

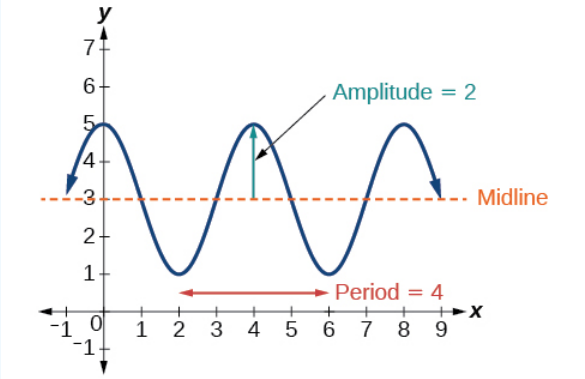


Figure $Page 6.5.23$

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter,**circular motion** can be modeled using either the sine or **cosine function**.

Example $Page 6.5.11$: Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution

Recall that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in Figure $Page 6.5.24$.

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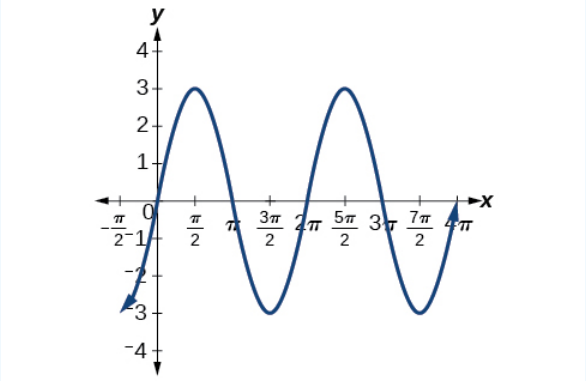


Figure $Page 6.5.24$

Analysis

Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi$,...Because the outputs of the graph will now oscillate between -3 and 3, the amplitude of the sine wave is 3.

? Exercise $Page 6.5.10$

What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Answer

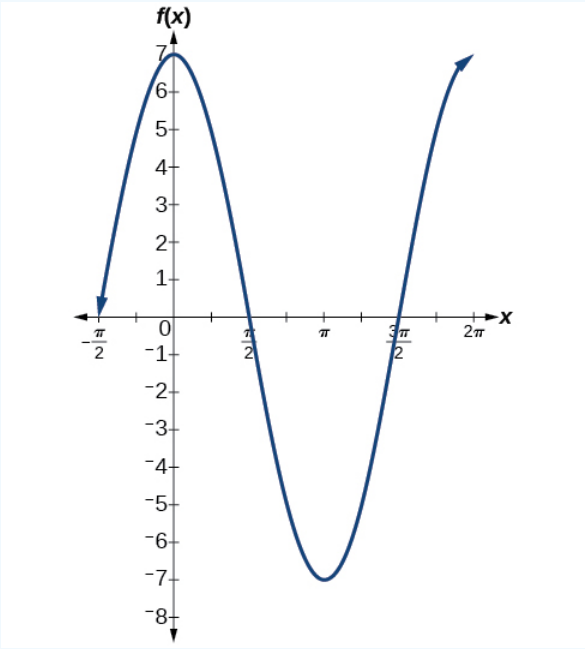


Figure $Page 6.5.25$

✔ Example $Page 6.5.12$: Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in Figure $Page 6.5.26$. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

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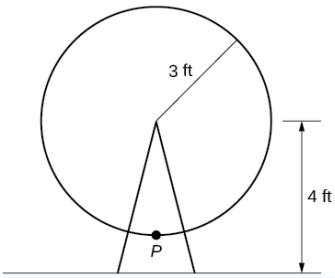


Figure $Page 6.5.26$

Solution

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in Figure $Page 6.5.27$.

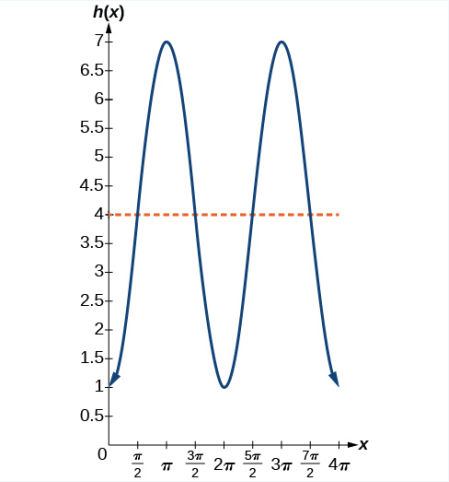


Figure $Page 6.5.27$

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a **sine function** starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

? Exercise $Page 6.5.11$

A weight is attached to a spring that is then hung from a board, as shown in Figure $Page 6.5.28$. As the spring oscillates up and down, the position y of the weight relative to the board ranges from -1 in. (at time $x = 0$) to -7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.

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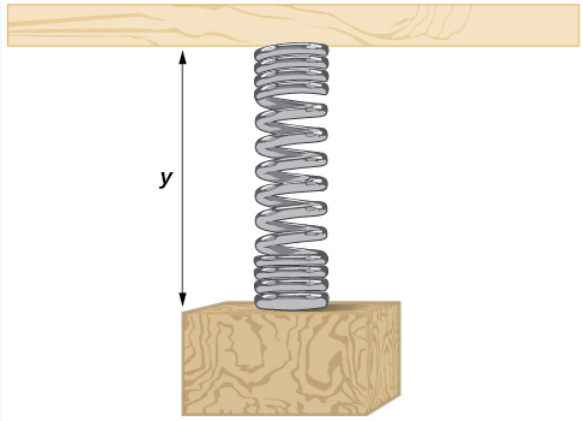


Figure $Page 6.5.28$

Answer

$y = 3\cos(x) - 4$

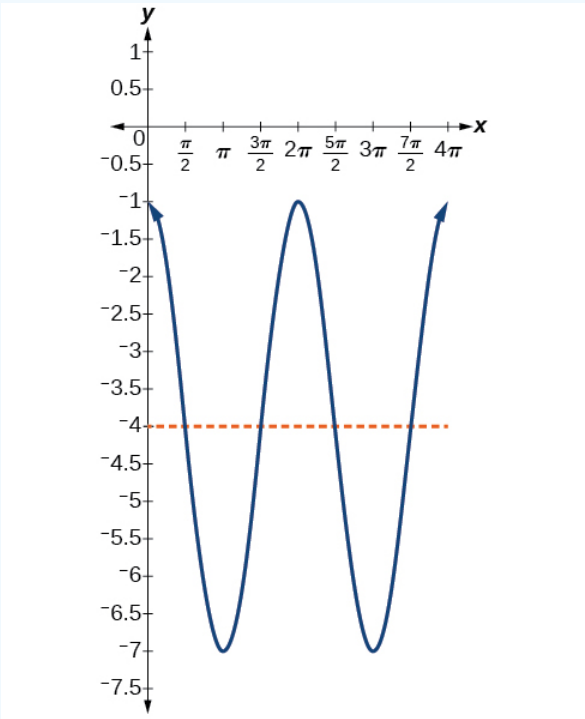


Figure $Page 6.5.29$

Example $Page 6.5.13$: Determining a Rider's Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

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Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period:30, so $B = \dfrac{2\pi}{30} = \dfrac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

Media

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine
- Translations of Sine and Cosine
- Graphing Sine and Cosine Transformations
- Graphing the Sine Function

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$ $f(x) = A\cos(Bx - C) + D$

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the *y*-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \dfrac{2\pi}{|B|}$. See Example $Page 6.5.1$.
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See Example $Page 6.5.2$.
- The value $\dfrac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See Example $Page 6.5.3$.
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See Example $Page 6.5.4$.
- Combinations of variations of sinusoidal functions can be detected from an equation. See Example $Page 6.5.5$.
- The equation for a sinusoidal function can be determined from a graph. See Example $Page 6.5.6$ and Example $Page 6.5.7$.
- A function can be graphed by identifying its amplitude and period. See Example $Page 6.5.8$ and Example $Page 6.5.9$.
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See Example $Page 6.5.10$.
- Sinusoidal functions can be used to solve real-world problems. See Example $Page 6.5.11$, Example $Page 6.5.12$, and Example $Page 6.5.13$.

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Page 6.6: Right Triangle Trigonometry

Learning Objectives

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30°(\frac{\pi}{6})$,$45°(\frac{\pi}{4})$,and $60°(\frac{\pi}{3})$.
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

We have previously defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\cos t = x$$
$$\sin t = y$$

In this section, we will see another way to define trigonometric functions using properties of *right triangles*.

Using Right Triangles to Evaluate Trigonometric Functions

In earlier sections, we used a unit circle to define the *trigonometric functions*. In this section, we will extend those definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure $Page 6.6.1$ shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the *x*-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

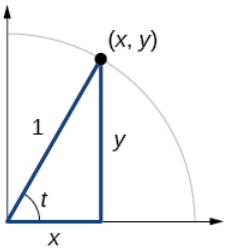


Figure $Page 6.6.1$

We know

$$\cos t = \frac{x}{1} = x \qquad (\text{Page } 6.6.1)$$

Likewise, we know

$$\sin t = \frac{y}{1} = y \qquad (\text{Page } 6.6.2)$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x,we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means "next to.") Instead of y,we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1,we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure $Page 6.6.2$.

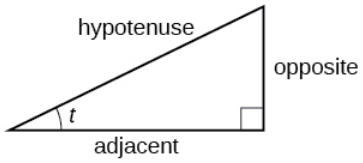


Figure $Page 6.6.2$: The sides of a right triangle in relation to angle t.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{(Page 6.6.3)}$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \text{(Page 6.6.4)}$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} \quad \text{(Page 6.6.5)}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

how to: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example $Page 6.6.1$: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure $Page 6.6.3$, find the value of $\cos\alpha$.

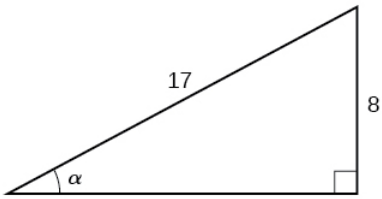


Figure $Page 6.6.3$

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so via Equation Page 6.6.4:

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{15}{17}$$

Exercise $Page 6.6.1$

Given the triangle shown in Figure $Page 6.6.4$, find the value of $\sin t$.

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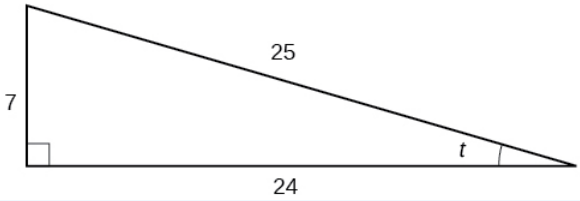


Figure $Page 6.6.4$

Answer

$\frac{7}{25}$

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in Figure $Page 6.6.5$. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

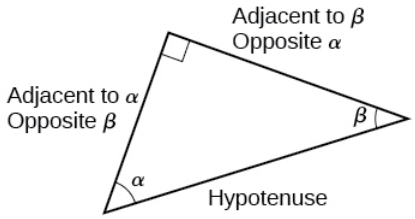


Figure $Page 6.6.5$: The side adjacent to one angle is opposite the other.

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

how to: Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example $Page 6.6.2$: Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in Figure $Page 6.6.6$, evaluate $\sin\alpha, \cos\alpha, \tan\alpha, \sec\alpha, \csc\alpha,$ and $\cot\alpha$.

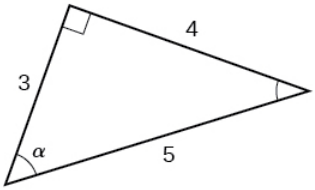


Figure $Page 6.6.6$

Solution

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$$\sin\alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos\alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan\alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$
$$\sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$
$$\csc\alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$
$$\cot\alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Exercise $Page 6.6.2$

Using the triangle shown in Figure $Page 6.6.7$, evaluate $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.

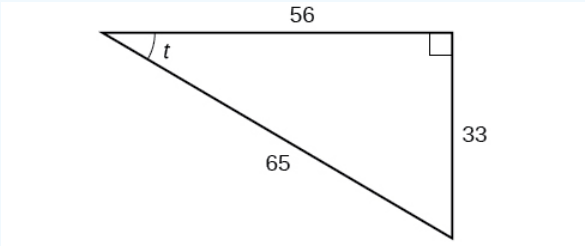


Figure $Page 6.6.7$

Answer

$$\sin t = \frac{33}{65}, \cos t = \frac{56}{65}, \tan t = \frac{33}{56},$$

$$\sec t = \frac{65}{56}, \csc t = \frac{65}{33}, \cot t = \frac{56}{33}$$

Finding Trigonometric Functions of Special Angles Using Side Lengths

We have already discussed the trigonometric functions as they relate to the **special angles** on the unit circle. Now, we can use those relationships to evaluate triangles that contain those special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of $30^\circ, 60^\circ$, and 45°, however, remember that when dealing with right triangles, we are limited to angles between 0° and 90°.

Suppose we have a $30^\circ, 60^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45^\circ, 45^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure $Page 6.6.8$.

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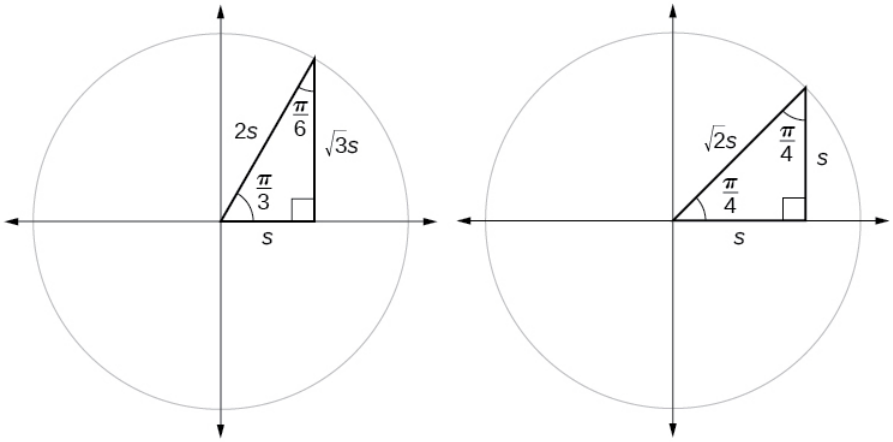


Figure $Page 6.6.8$: Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure $Page 6.6.8$ for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example $Page 6.6.3$: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin(\frac{\pi}{3}) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\cos(\frac{\pi}{3}) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$
$$\tan(\frac{\pi}{3}) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$
$$\sec(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$
$$\csc(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot(\frac{\pi}{3}) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise $Page 6.6.3$

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$ using side lengths.

Answer

$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{\pi}{4}) = 1,$
$\sec(\frac{\pi}{4}) = \sqrt{2}, \csc(\frac{\pi}{4}) = \sqrt{2}, \cot(\frac{\pi}{4}) = 1$

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the

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sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure $Page 6.6.9$).

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

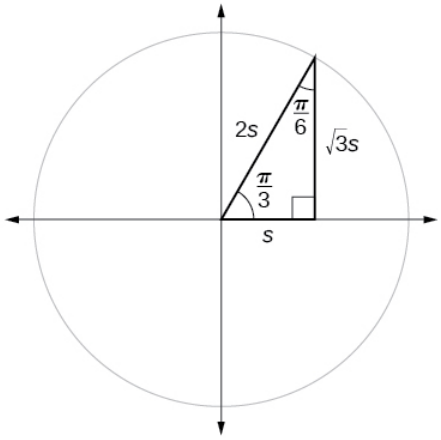
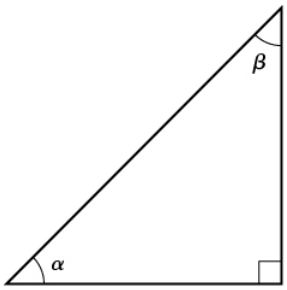


Figure $Page 6.6.9$: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from Figure $Page 6.6.9$, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure $Page 6.6.10$.



$\sin\alpha = \cos\beta$
$\sin\beta = \cos\alpha$

Figure $Page 6.6.10$: Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

COFUNCTION IDENTITIES

The **cofunction identities** in radians are listed in Table $Page 6.6.1$.

Table $Page 6.6.1$

$\cos t = \sin(\frac{\pi}{2} - t)$	$\sin t = \cos(\frac{\pi}{2} - t)$

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$\tan t = \cot(\frac{\pi}{2} - t)$	$\cot t = \tan(\frac{\pi}{2} - t)$
$\sec t = \csc(\frac{\pi}{2} - t)$	$\csc t = \sec(\frac{\pi}{2} - t)$

how to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example $Page 6.6.4$: Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $(\cos\frac{\pi}{2} - t)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t = \cos(\frac{\pi}{2} - t).$$

So

$$\cos(\frac{\pi}{2} - t) = \frac{5}{12}.$$

Exercise $Page 6.6.4$

If $\csc(\frac{\pi}{6}) = 2$, find $\sec(\frac{\pi}{3})$.

Solution

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

how to: Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example $Page 6.6.5$: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in Figure $Page 6.6.11$.

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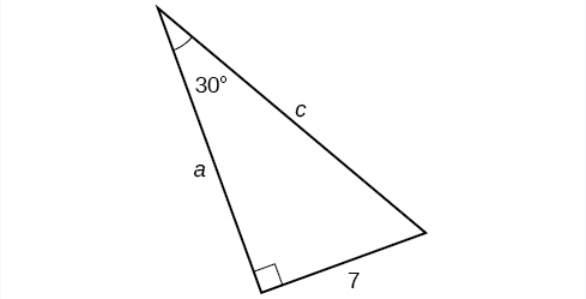


Figure $Page 6.6.11$

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30^\circ) = \frac{7}{a}$$

We rearrange to solve for a.

$$a = \frac{7}{\tan(30^\circ)} \quad \text{(Page 6.6.6)}$$
$$= 12.1 \quad \text{(Page 6.6.7)}$$

We can use the sine to find the hypotenuse.

$$\sin(30^\circ) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$c = \frac{7}{\sin(30^\circ)} = 14$$

Exercise $Page 6.6.5$:

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Answer

adjacent $= 10$; opposite $= 10\sqrt{3}$; missing angle is $\frac{\pi}{6}$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure $Page 6.6.12$.

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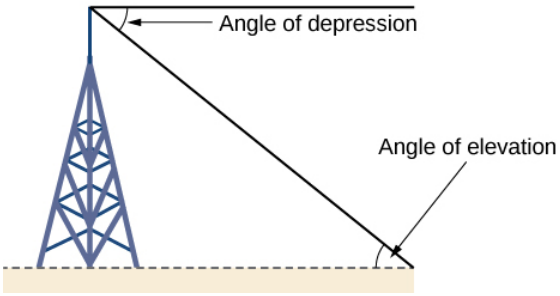


Figure $Page 6.6.12$

how to: Given a tall object, measure its height indirectly

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example $Page 6.6.6$: Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° 57° between a line of sight to the top of the tree and the ground, as shown in Figure $Page 6.6.13$. Find the height of the tree.

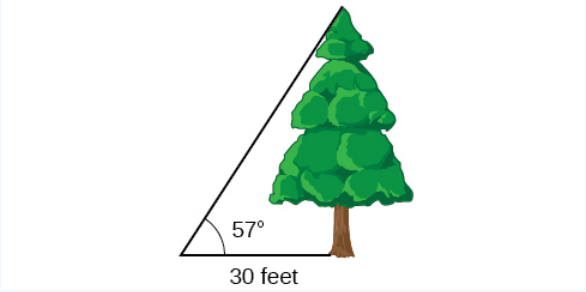


Figure $Page 6.6.13$

Solution

We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57°, letting h be the unknown height.

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan(57^\circ) = \frac{h}{30} \quad \text{Solve for } h. \qquad (\text{Page } 6.6.8)$$

$$h = 30\tan(57^\circ) \quad \text{Multiply.}$$

$$h \approx 46.2 \quad \text{Use a calculator.}$$

The tree is approximately 46 feet tall.

Exercise $Page 6.6.6$:

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Answer

https://math.libretexts.org/@go/page/132728

About 52 ft

media:

Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator
- Finding Trig Functions Using a Right Triangle
- Relate Trig Functions to Sides of a Right Triangle
- Determine Six Trig Functions from a Triangle
- Determine Length of Right Triangle Side

Visit this website for additional practice questions from Learningpod.

Key Equations

Cofunction Identities

$$\begin{aligned}\cos t &= \sin\left(\frac{\pi}{2} - t\right)\\ \sin t &= \cos\left(\frac{\pi}{2} - t\right)\\ \tan t &= \cot\left(\frac{\pi}{2} - t\right)\\ \cot t &= \tan\left(\frac{\pi}{2} - t\right)\\ \sec t &= \csc\left(\frac{\pi}{2} - t\right)\\ \csc t &= \sec\left(\frac{\pi}{2} - t\right)\end{aligned}$$

Key Concepts

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See Example.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See Example.
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See Example.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See Example.
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See Example.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See Example.

Glossary

adjacent side

in a right triangle, the side between a given angle and the right angle

angle of depression

the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation

the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side

in a right triangle, the side most distant from a given angle

hypotenuse

the side of a right triangle opposite the right angle

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Page 6.7: The Other Trigonometric Functions

Learning Objectives

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

To define the remaining functions, we will once again draw a unit circle with a point (x, y) corresponding to an angle of t,as shown in Figure $Page 6.7.1$. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

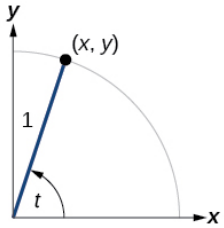


Figure $Page 6.7.1$

The first function we will define is the tangent. The **tangent** of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle. In Figure $Page 6.7.1$, the tangent of angle t is equal to $\frac{y}{x}, x \neq 0$. Because the *y*-value is equal to the sine of t, and the *x*-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}, \cos t \neq 0$. The tangent function is abbreviated as tan. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In Figure $Page 6.7.1$, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}, x \neq 0$. The secant function is abbreviated as sec.
- The **cotangent** function is the reciprocal of the tangent function. In Figure $Page 6.7.1$, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as cot.
- The **cosecant** function is the reciprocal of the sine function. In Figure $Page 6.7.1$, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as csc.

TANGENT, SECANT, COSECANT, AND COTANGENT FUNCTIONS

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0 \quad \text{(Page 6.7.1)}$$

$$\sec t = \frac{1}{x}, x \neq 0 \quad \text{(Page 6.7.2)}$$

$$\csc t = \frac{1}{y}, y \neq 0 \quad \text{(Page 6.7.3)}$$

$$\cot t = \frac{x}{y}, y \neq 0 \quad \text{(Page 6.7.4)}$$

Example $Page 6.7.1$: Finding Trigonometric Functions from a Point on the Unit Circle

The point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the unit circle, as shown in Figure $Page 6.7.2$. Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.

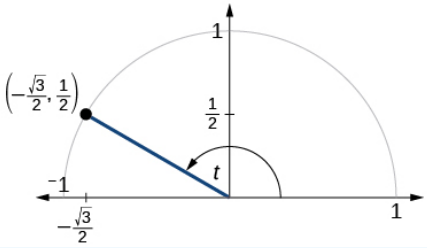


Figure $Page 6.7.2$

Solution

Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

$$\sin t = y = \frac{1}{2}$$
$$\cos t = x = -\frac{\sqrt{3}}{2}$$
$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$
$$\sec t = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$
$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$
$$\cot t = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

Exercise $Page 6.7.1$:

The point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is on the unit circle, as shown in Figure $Page 6.7.3$. Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.

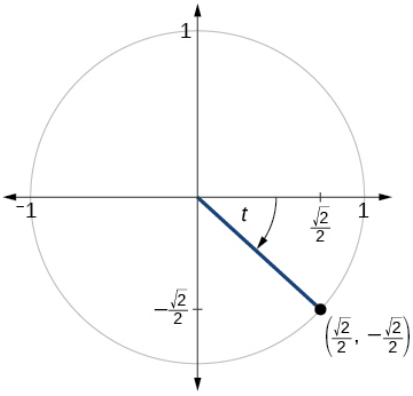


Figure $Page 6.7.3$

Solution

$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = -1, \sec t = \sqrt{2}, \csc t = -\sqrt{2}, \cot t = -1$

Example $Page 6.7.2$: Finding the Trigonometric Functions of an Angle

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{6}$.

Solution

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We have previously used the properties of equilateral triangles to demonstrate that $\sin\frac{\pi}{6}=\frac{1}{2}$ and $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$. We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\begin{aligned}\tan\frac{\pi}{6}&=\frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}}\\&=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}\\\sec\frac{\pi}{6}&=\frac{1}{\cos\frac{\pi}{6}}\\&=\frac{1}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}\\\csc\frac{\pi}{6}&=\frac{1}{\sin\frac{\pi}{6}}=\frac{1}{\frac{1}{2}}=2\\\cot\frac{\pi}{6}&=\frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}\\&=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3}\end{aligned}$$

Exercise $Page 6.7.2$:

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t=\frac{\pi}{3}$.

Solution

$$\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2} \quad \text{(Page 6.7.5)}$$
$$\cos\frac{\pi}{3}=\frac{1}{2} \quad \text{(Page 6.7.6)}$$
$$\tan\frac{\pi}{3}=\sqrt{3} \quad \text{(Page 6.7.7)}$$
$$\sec\frac{\pi}{3}=2 \quad \text{(Page 6.7.8)}$$
$$\csc\frac{\pi}{3}=\frac{2\sqrt{3}}{3} \quad \text{(Page 6.7.9)}$$
$$\cot\frac{\pi}{3}=\frac{\sqrt{3}}{3} \quad \text{(Page 6.7.10)}$$

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x x equal to the cosine and y y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in Table $Page 6.7.1$.

Table $Page 6.7.1$

Angle	**0**	$\frac{\pi}{6}$, or **30**$^\circ$	$\frac{\pi}{4}$, or **45**$^\circ$	$\frac{\pi}{3}$, or **60**$^\circ$	$\frac{\pi}{2}$, or **90**$^\circ$
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

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Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the **reference angle** formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by *x*- and *y*-values in the original quadrant. Figure $Page 6.7.4$ shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **all** of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **sine** and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **tangent** and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **cosine** and its reciprocal function, secant, are positive.

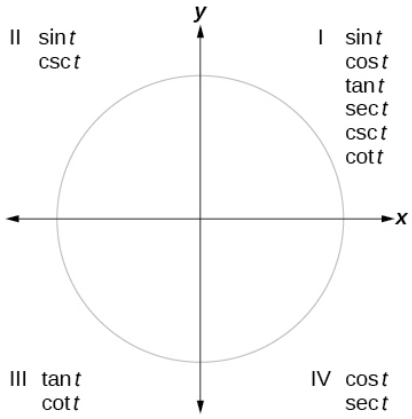


Figure $Page 6.7.4$

HOWTO: Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example $Page 6.7.3$: Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution

The angle between this angle's terminal side and the *x*-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\cos(-\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}, \sin(-\frac{5\pi}{6}) = -\frac{1}{2}, \tan(-\frac{5\pi}{6}) = \frac{\sqrt{3}}{3} \quad \text{(Page 6.7.11)}$$

$$\sec(-\frac{5\pi}{6}) = -\frac{2\sqrt{3}}{3}, \csc(-\frac{5\pi}{6}) = -2, \cot(-\frac{5\pi}{6}) = \sqrt{3} \quad \text{(Page 6.7.12)}$$

Exercise $Page 6.7.3$

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Solution

$\sin(-\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{-7\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{-7\pi}{4}) = 1,$

$\sec(\frac{-7\pi}{4}) = \sqrt{2}, \csc(\frac{-7\pi}{4}) = \sqrt{2}, \cot(\frac{-7\pi}{4}) = 1$

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

Consider the function $f(x) = x^2$, shown in Figure $Page 6.7.5$. The graph of the function is symmetrical about the *y*-axis. All along the curve, any two points with opposite *x*-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2, (-5)^2 = (5)^2$, and so on. So $f(x) = x^2$ is an **even function**, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.

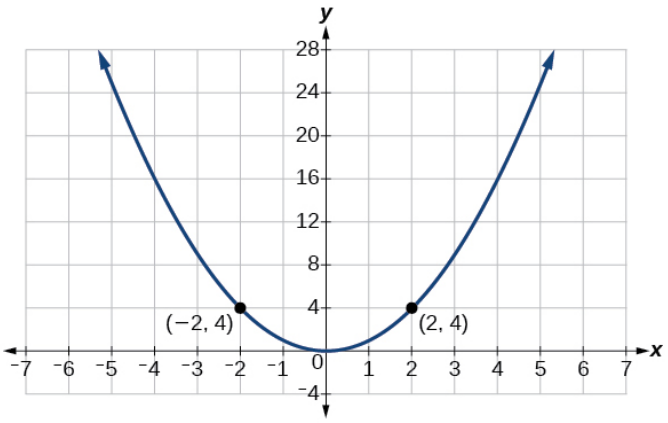


Figure $Page 6.7.5$: The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in Figure $Page 6.7.6$. The graph is not symmetrical about the *y*-axis. All along the graph, any two points with opposite *x*-values also have opposite *y*-values. So $f(x) = x^3$ is an **odd function**, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.

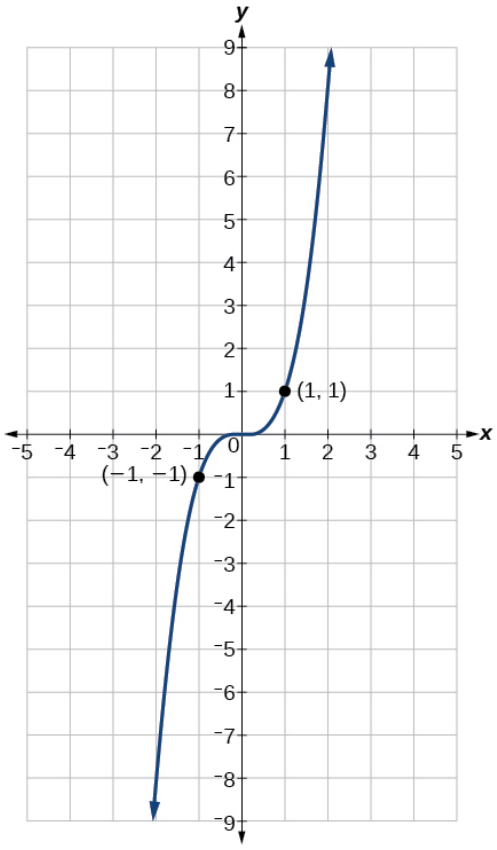


Figure $Page 6.7.6$: The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a **unit circle** with a positive and a negative angle, as in Figure $Page 6.7.7$. The sine of the positive angle is y. The sine of the negative angle is $-y$. The **sine function**, then, is an odd function. We can

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test each of the six trigonometric functions in this fashion. The results are shown in Table $Page 6.7.2$.

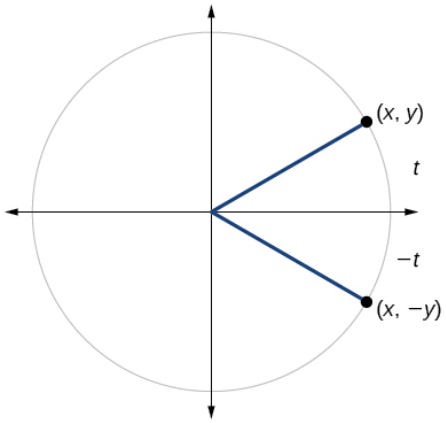


Figure $Page 6.7.7$

Table $Page 6.7.2$

$\sin t = y$ (Page 6.7.13)	$\cos t = x$ (Page 6.7.16)	$\tan(t) = \frac{y}{x}$ (Page 6.7
$\sin(-t) = -y$ (Page 6.7.14)	$\cos(-t) = x$ (Page 6.7.17)	$\tan(-t) = -\frac{y}{x}$ (Page 6.7
$\sin t \neq sin(-t)$ (Page 6.7.15)	$\cos t = \cos(-t)$ (Page 6.7.18)	$\tan t \neq \tan(-t)$ (Page 6.7
$\sec t = \frac{1}{x}$ (Page 6.7.22)	$\csc t = \frac{1}{y}$ (Page 6.7.25)	$\cot t = \frac{x}{y}$ (Page 6.7
$\sec(-t) = \frac{1}{x}$ (Page 6.7.23)	$\csc(-t) = \frac{1}{-y}$ (Page 6.7.26)	$\cot(-t) = \frac{x}{-y}$ (Page 6.7
$\sec t = \sec(-t)$ (Page 6.7.24)	$\csc t \neq \csc(-t)$ (Page 6.7.27)	$\cot t \neq \cot(-t)$ (Page 6.7

EVEN AND ODD TRIGONOMETRIC FUNCTIONS

- An **even function** is one in which $f(-x) = f(x)$.
- An **odd function** is one in which $f(-x) = -f(x)$.

Cosine and secant are even:

$$\cos(-t) = \cos t \quad \text{(Page 6.7.31)}$$
$$\sec(-t) = \sec t \quad \text{(Page 6.7.32)}$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t \quad \text{(Page 6.7.33)}$$
$$\tan(-t) = -\tan t \quad \text{(Page 6.7.34)}$$
$$\csc(-t) = -\csc t \quad \text{(Page 6.7.35)}$$
$$\cot(-t) = -\cot t \quad \text{(Page 6.7.36)}$$

Example $Page 6.7.4$: Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Solution

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Exercise $Page 6.7.4$:

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Solution

$-\sqrt{3}$

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

FUNDAMENTAL IDENTITIES

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \quad \text{(Page 6.7.37)}$$

$$\sec t = \frac{1}{\cos t} \quad \text{(Page 6.7.38)}$$

$$\csc t = \frac{1}{\sin t} \quad \text{(Page 6.7.39)}$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t} \quad \text{(Page 6.7.40)}$$

Example $Page 6.7.5$: Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45^\circ) = \frac{\sqrt{2}}{2}, \cos(45^\circ) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45^\circ)$.

b. Given $\sin(\frac{5\pi}{6}) = \frac{1}{2}, \cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$, evaluate $\sec(\frac{5\pi}{6})$.

Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.
$$\begin{aligned}\tan(45^\circ) &= \frac{\sin(45^\circ)}{\cos(45^\circ)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1\end{aligned}$$

b.
$$\begin{aligned}\sec(\frac{5\pi}{6}) &= \frac{1}{\cos(\frac{5\pi}{6})} \\ &= \frac{1}{-\frac{\sqrt{3}}{2}} \\ &= \frac{-2\sqrt{3}}{1} \\ &= \frac{-2}{\sqrt{3}} \\ &= -\frac{2\sqrt{3}}{3}\end{aligned}$$

Exercise $Page 6.7.5$

Evaluate $\csc(\frac{7\pi}{6})$.

Solution

-2

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Example $Page 6.7.6$: Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Solution

We can simplify this by rewriting both functions in terms of sine and cosine.

$$\begin{aligned}\frac{\sec t}{\tan t} &= \frac{1/\cos t}{\sin t/\cos t} && \text{To divide the functions, we multiply by the reciprocal.}\\ &= \frac{1}{\cos t}\frac{\cos t}{\sin t} && \text{Divide out the cosines.}\\ &= \frac{1}{\sin t} && \text{Simplify and use the identity.}\\ &= \csc t\end{aligned} \quad (\text{Page } 6.7.41)$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$,we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t$$

Exercise $Page 6.7.6$

Simplify $(\tan t)(\cos t)$.

Solution

$\sin t$

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the **Pythagorean Identity**, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \quad (\text{Page } 6.7.42)$$

$$1 + \tan^2 t = \sec^2 t \quad (\text{Page } 6.7.43)$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \quad (\text{Page } 6.7.44)$$

$$\cot^2 t + 1 = \csc^2 t \quad (\text{Page } 6.7.45)$$

ALTERNATE FORMS OF THE PYTHAGOREAN IDENTITY

$$1 + \tan^2 t = \sec^2 t \quad (\text{Page } 6.7.46)$$

$$\cot^2 t + 1 = \csc^2 t \quad (\text{Page } 6.7.47)$$

Example $Page 6.7.7$: Using Identities to Relate Trigonometric Functions

If cos(t)=1213 cos(t)=1213 and t t is in quadrant IV, as shown in Figure $Page 6.7.8$, find the values of the other five trigonometric functions.

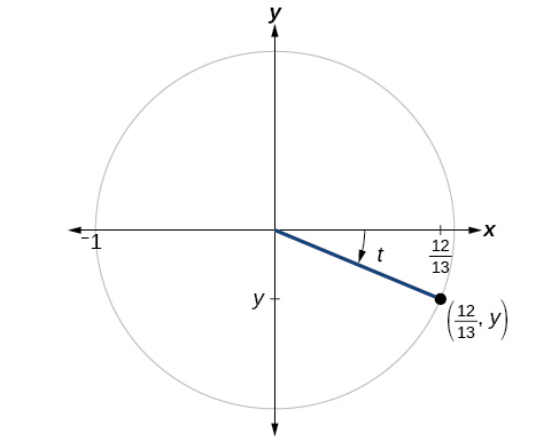


Figure $Page 6.7.8$

Solution

We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$(\frac{12}{13})^2 + \sin^2 t = 1 \quad \text{(Page 6.7.48)}$$

$$\sin^2 t = 1 - (\frac{12}{13})^2 \quad \text{(Page 6.7.49)}$$

$$\sin^2 t = 1 - \frac{144}{169} \quad \text{(Page 6.7.50)}$$

$$\sin^2 t = \frac{25}{169} \quad \text{(Page 6.7.51)}$$

$$\sin t = \pm\sqrt{\frac{25}{169}} \quad \text{(Page 6.7.52)}$$

$$\sin t = \pm\frac{\sqrt{25}}{\sqrt{169}} \quad \text{(Page 6.7.53)}$$

$$\sin t = \pm\frac{5}{13} \quad \text{(Page 6.7.54)}$$

The sign of the sine depends on the *y*-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the *y*-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12} \quad \text{(Page 6.7.55)}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{12}{13}} = \frac{13}{12} \quad \text{(Page 6.7.56)}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5} \quad \text{(Page 6.7.57)}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5} \quad \text{(Page 6.7.58)}$$

Exercise $Page 6.7.7$:

If $\sec(t) = -\frac{17}{8}$ and $0 < t < \pi$, find the values of the other five functions.

Solution

$\cos t = -\frac{8}{17}, \sin t = \frac{15}{17}, \tan t = -\frac{15}{8}$

$\csc t = \frac{17}{15}, \cot t = -\frac{8}{15}$

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As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a **periodic function**. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π,will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x+4)=f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

PERIOD OF A FUNCTION

The **period** P of a repeating function f f is the number representing the interval such that $f(x+P)=f(x)$ for any value of x.

The period of the cosine, sine, secant, and cosecant functions is 2π.

The period of the tangent and cotangent functions is π.

Example $Page 6.7.8$: Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on Figure $Page 6.7.9$.

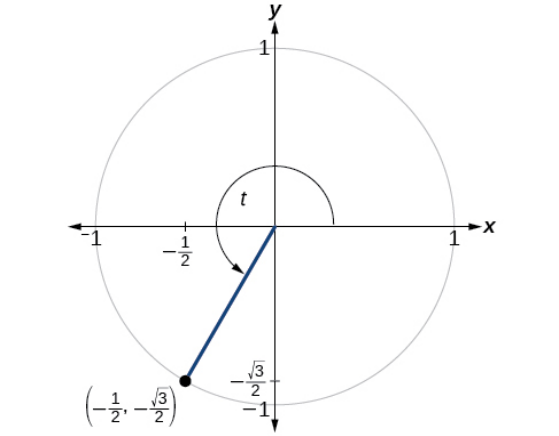


Figure $Page 6.7.9$

Solution

$$\sin t = y = -\frac{\sqrt{3}}{2}$$
$$\cos t = x = -\frac{1}{2}$$
$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$
$$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2$$
$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$
$$\cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

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Exercise $Page 6.7.8$

Find the values of the six trigonometric functions of angle t based on Figure $Page 6.7.10$.

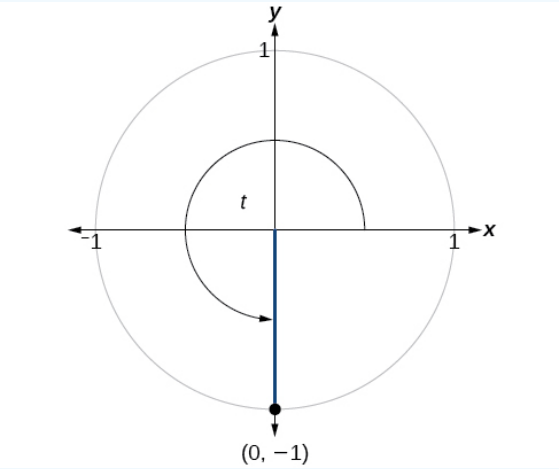


Figure $Page 6.7.10$

Solution

$$\sin t = -1, \cos t = 0, \tan t = \text{Undefined} \quad (Page\ 6.7.59)$$

$$(Page\ 6.7.60)$$

$$sect = \text{Undefined}, \csc t = -1, \cot t = 0 \quad (Page\ 6.7.61)$$

Example $Page 6.7.9$: Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$,find $\sec(t), \csc(t), \tan(t), \cot(t)$.

Solution

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2 \quad (Page\ 6.7.62)$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} - \frac{2\sqrt{3}}{3} \quad (Page\ 6.7.63)$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \quad (Page\ 6.7.64)$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3} \quad (Page\ 6.7.65)$$

Exercise $Page 6.7.9$:

If $\sin(t) = \frac{\sqrt{2}}{2}$ and $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sec(t), \csc(t), \tan(t)$, and $\cot(t)$.

Solution

$\sec t = \sqrt{2}, \csc t = \sqrt{2}, \tan t = 1, \cot t = 1$

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

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If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of 30°, we could press

$$(\text{for a scientific calculator}): \frac{1}{30 \times \frac{\pi}{180}} \text{COS} \qquad \text{(Page 6.7.66)}$$

or

$$(\text{for a graphing calculator}): \frac{1}{\cos(\frac{30\pi}{180})} \qquad \text{(Page 6.7.67)}$$

how to: Given an angle measure in radians, use a scientific calculator to find the cosecant

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

how to: Given an angle measure in radians, use a graphing utility/calculator to find the cosecant

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Press the SIN key.
4. Enter the value of the angle inside parentheses.
5. Press the ENTER key.

Example $Page 6.7.10$: Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Solution

For a scientific calculator, enter information as follows:

$$1/(5 \times \pi/7)\text{SIN} = \qquad \text{(Page 6.7.68)}$$

$$\csc(\frac{5\pi}{7}) \approx 1.279 \qquad \text{(Page 6.7.69)}$$

Exercise $Page 6.7.10$:

Evaluate the cotangent of $-\frac{\pi}{8}$.

≈ -2.414

media

Access these online resources for additional instruction and practice with other trigonometric functions.

- Determining Trig Function Values
- More Examples of Determining Trig Functions
- Pythagorean Identities
- Trig Functions on a Calculator

Key Equations

Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$
Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Key Concepts

- The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See Example.
- Trigonometric functions can also be found from an angle. See Example.
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See Example.
- A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See Example.
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See Example and Example.
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See Example.
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f f is the smallest interval such that $f(x + P) = f(x)$ for any value of x.
- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See Example.

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Glossary

cosecant

the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$

cotangent

the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

identities

statements that are true for all values of the input on which they are defined

period

the smallest interval P of a repeating function f such that $f(x + P) = f(x)$

secant

the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$

tangent

the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

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Page 6.8: Graphs of the Other Trigonometric Functions

Learning Objectives

- Analyze the graph of $y = \tan x$.
- Graph variations of $y = \tan x$.
- Analyze the graphs of $y = \sec x$ and $y = \csc x$.
- Graph variations of $y = \sec x$ and $y = \csc x$.
- Analyze the graph of $y = \cot x$.
- Graph variations of $y = \cot x$.

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

$$\tan x = \frac{\sin x}{\cos x} \qquad \text{(Page 6.8.1)}$$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{aligned}
\tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Definition of tangent} \\
&= \frac{-\sin x}{\cos x} && \text{Sine is an odd function, cosine is even} \\
&= -\frac{\sin x}{\cos x} && \text{The quotient of an odd and an even function is odd} \\
&= -\tan x && \text{Definition of tangent}
\end{aligned}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in Table $Page6.8.1$.

Table $Page6.8.1$

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in Table $Page6.8.2$.

Table $Page6.8.2$

x	1.3	1.5	1.55	1.56
$\tan x$	3.6	14.1	48.1	92.6

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As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in Table $Page 6.8.3$.

Table $Page 6.8.3$

x	−1.3	−1.5	−1.55	−1.56
$\tan x$	−3.6	−14.1	−48.1	−92.6

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos x = 0$. For example, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. Figure $Page 6.8.1$ represents the graph of $y = \tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

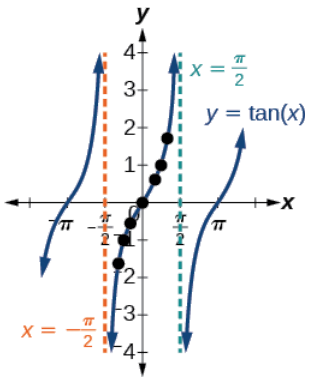


Figure $Page 6.8.1$: Graph of the tangent function

Graphing Variations of $y = \tan x$

As with the sine and cosine functions, the *tangent function* can be described by a general equation.

$$y = A\tan(Bx)$$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

FEATURES OF THE GRAPH OF $Y = A\tan(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \dfrac{\pi}{|B|}$.
- The domain is all real numbers x,where $x \neq \dfrac{\pi}{2|B|} + \dfrac{\pi}{|B|}k$ such that k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{2|B|} + \dfrac{\pi}{|B|}k$ where k is an integer.
- $y = A\tan(Bx)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the **tangent function** to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x) = A\tan(Bx)$. We focus on a single **period** of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is

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then the interval $\left(-\frac{P}{2}, \frac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\frac{P}{2}$ where $P = \frac{\pi}{B}$. On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the graph will come up from the left asymptote at $x = -\frac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x = \frac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right) = A\tan\left(B\frac{P}{4}\right) = A\tan\left(B\frac{\pi}{4B}\right) = A$$

because $\tan\left(\frac{\pi}{4}\right) = 1$.

Howto: Given the function $f(x) = A\tan(Bx)$, graph one period.

1. Identify the stretching factor, $|A|$.
2. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
3. Draw vertical asymptotes at $x = -\frac{P}{2}$ and $x = \frac{P}{2}$.
4. For $A > 0$, the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for $A < 0$).
5. Plot reference points at $\left(\frac{P}{4}, A\right)$, $(0, 0)$, and $\left(-\frac{P}{4}, -A\right)$, and draw the graph through these points.

Example $Page 6.8.1$: Sketching a Compressed Tangent

Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

Solution

First, we identify A and B.

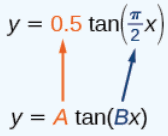


Figure $Page 6.8.2$

Because $A = 0.5$ and $B = \frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}} = 2$, so the asymptotes are at $x = \pm 1$. At a quarter period from the origin, we have

$$\begin{aligned} f(0.5) &= 0.5\tan\left(\frac{0.5\pi}{2}\right) \\ &= 0.5\tan\left(\frac{\pi}{4}\right) \\ &= 0.5 \end{aligned}$$

This means the curve must pass through the points $(0.5, 0.5)$, $(0, 0)$,and $(-0.5, -0.5)$. The only inflection point is at the origin. Figure $Page 6.8.3$ shows the graph of one period of the function.

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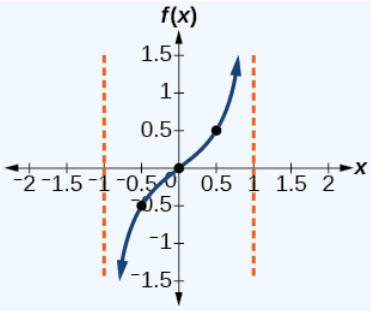


Figure $Page 6.8.3$

? Exercise $Page 6.8.1$

Sketch a graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$.

Answer

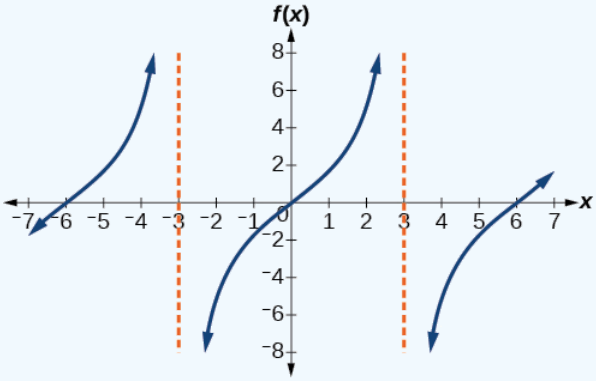


Figure $Page 6.8.4$

Graphing One Period of a Shifted Tangent Function

Now that we can graph a **tangent function** that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.

$$f(x) = A\tan(Bx - C) + D$$

The graph of a transformed tangent function is different from the basic tangent function $\tan x$ in several ways:

FEATURES OF THE GRAPH OF $Y = A\tan(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an odd integer.
- There is no amplitude.
- $y = A\tan(Bx)$ is and odd function because it is the qoutient of odd and even functions(sin and cosine perspectively).

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Howto: Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the **stretching/compressing** factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Example $Page 6.8.2$: Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.

Solution

- *Step 1.* The function is already written in the form $y = A\tan(Bx - C) + D$.
- *Step 2.*\(A=−2\), so the stretching factor is $|A| = 2$.
- *Step 3.* \(B=\pi\), so the period is $P = \dfrac{\pi}{|B|} = \dfrac{\pi}{pi} = 1$.
- *Step 4.* \(C=−\pi\), so the phase shift is $CB = \dfrac{-\pi}{\pi} = -1$.
- *Step 5-7.* The asymptotes are at $x = -\dfrac{3}{2}$ and $x = -\dfrac{1}{2}$ and the three recommended reference points are $(-1.25, 1)$, $(-1, -1)$, and $(-0.75, -3)$. The graph is shown in Figure $Page 6.8.5$.

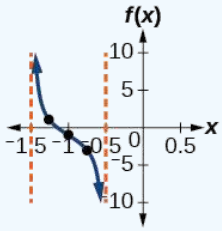


Figure $Page 6.8.5$

Analysis

Note that this is a decreasing function because $A < 0$.

Exercise $Page 6.8.2$

How would the graph in Example $Page 6.8.2$ look different if we made $A = 2$ instead of -2?

Answer

It would be reflected across the line $y = -1$, becoming an increasing function.

Howto: Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period P from the spacing between successive vertical asymptotes or *x*-intercepts.
2. Write $f(x) = A\tan\left(\dfrac{\pi}{P}x\right)$.
3. Determine a convenient point $(x, f(x))$ on the given graph and use it to determine A.

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Example $Page 6.8.3$: Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in Figure $Page 6.8.6$.

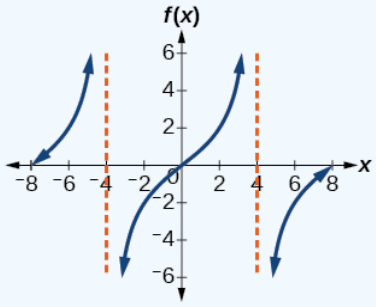


Figure $Page 6.8.6$: A stretched tangent function

Solution

The graph has the shape of a tangent function.

- *Step 1.* One cycle extends from -4 to 4, so the period is $P = 8$. Since $P = \frac{\pi}{|B|}$, we have $B = \frac{\pi}{P} = \frac{\pi}{8}$.
- *Step 2.* The equation must have the form $f(x) = A\tan\left(\frac{\pi}{8}x\right)$.
- *Step 3.* To find the vertical stretch A,we can use the point $(2, 2)$.

$$\begin{aligned} 2 &= A\tan\left(\frac{\pi}{8}\cdot 2\right) \\ &= A\tan\left(\frac{\pi}{4}\right) \end{aligned}$$

Because $\tan\left(\frac{\pi}{4}\right) = 1$, $A = 2$.

This function would have a formula $f(x) = 2\tan\left(\frac{\pi}{8}x\right)$.

Exercise $Page 6.8.3$

Find a formula for the function in Figure $Page 6.8.7$.

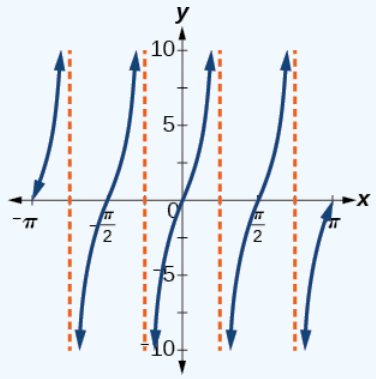


Figure $Page 6.8.7$

Answer

$g(x) = 4\tan(2x)$

Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

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The secant was defined by the reciprocal identity $sec\, x = \dfrac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\dfrac{\pi}{2}$, $\dfrac{3\pi}{2}$ etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See Figure $Page 6.8.8$. The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the x-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

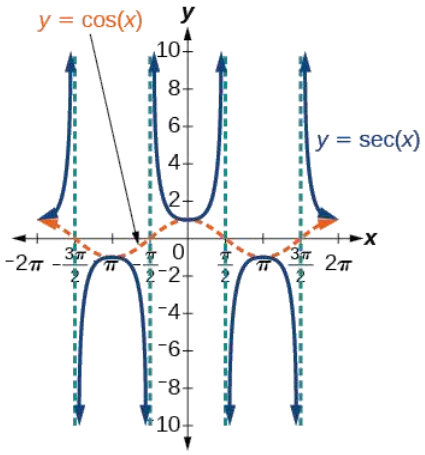


Figure $Page 6.8.8$: Graph of the secant function, $f(x) = \sec x = \dfrac{1}{\cos x}$

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

FEATURES OF THE GRAPH OF $Y = A\sec(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

Similar to the secant, the **cosecant** is defined by the reciprocal identity $\csc x = \dfrac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See Figure $Page 6.8.7$. The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the **cosecant function** decreases.

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The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in Figure $Page 6.8.9$, is similar to the graph of secant.

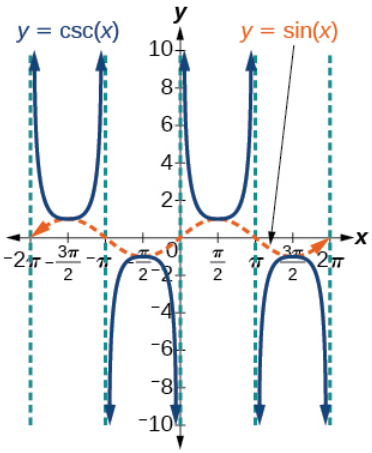


Figure $Page 6.8.9$: The graph of the cosecant function, $f(x) = \csc x = \frac{1}{\sin x}$

FEATURES OF THE GRAPH OF $Y = A\csc(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

Graphing Variations of $y = \sec x$ and $y = \csc x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions.The equations become the following.

$$y = A\sec(Bx - C) + D \qquad (\text{Page } 6.8.2)$$

$$y = A\csc(Bx - C) + D \qquad (\text{Page } 6.8.3)$$

FEATURES OF THE GRAPH OF $Y = A\sec(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- There is no amplitude.

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- $y = A\sec(Bx)$ is an even function because cosine is an even function.

FEATURES OF THE GRAPH OF $Y = A\csc(Bx - C) + D$

1. The stretching factor is $|A|$.
2. The period is $\dfrac{2\pi}{|B|}$.
3. The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an integer.
4. The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
5. The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
6. There is no amplitude.
7. $y = A\csc(Bx)$ is an odd function because sine is an odd function.

HOWTO: Given a function of the form $y = A\sec(Bx)$, graph one period

1. Express the function given in the form $y = A\sec(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Sketch the graph of $y = A\cos(Bx)$.
5. Use the reciprocal relationship between $y = \cos\ x$ and $y = \sec\ x$ to draw the graph of $y = A\sec(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example $Page 6.8.4$: Graphing a Variation of the Secant Function

Graph one period of $f(x) = 2.5\sec(0.4x)$

Solution

- *Step 1.* The given function is already written in the general form, $y = A\sec(Bx)$.
- *Step 2.* $A = 2.5$ so the stretching factor is 2.5.
- *Step 3.* $B = 0.4$ so $P = \dfrac{2\pi}{0.4} = 5\pi$. The period is 5π units.
- *Step 4.* Sketch the graph of the function $g(x) = 2.5\cos(0.4x)$.
- *Step 5.* Use the reciprocal relationship of the cosine and secant functions to draw the cosecant function.
- *Steps 6–7.* Sketch two asymptotes at $x = 1.25\pi$ and $x = 3.75\pi$. We can use two reference points, the local minimum at $(0, 2.5)$ and the local maximum at $(2.5\pi, -2.5)$ Figure $Page 6.8.10$ shows the graph.

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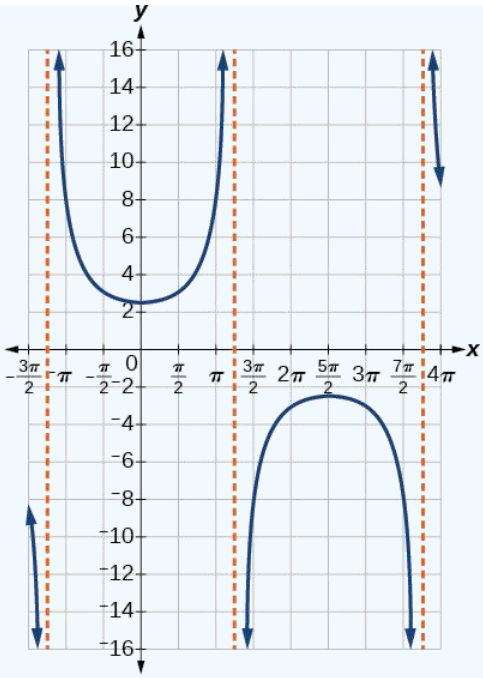


Figure $Page 6.8.10$

Exercise $Page 6.8.4$

Graph one period of $f(x) = -2.5\sec(0.4x)$.

Answer

This is a vertical reflection of the preceding graph because A is negative.

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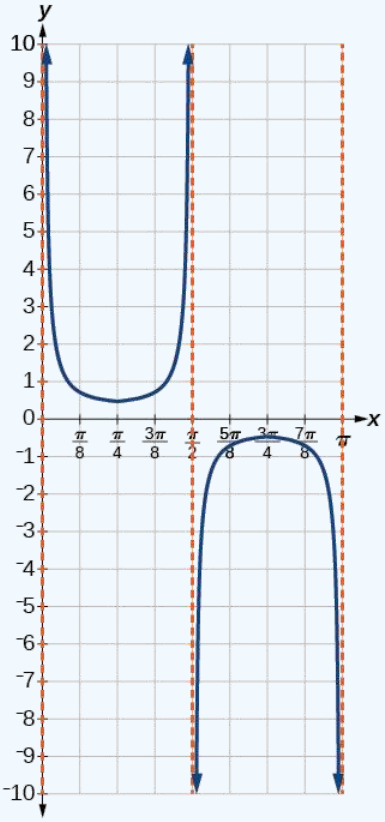


Figure $Page 6.8.11$

Q&A: Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f(x) = A\sec(Bx - C) + D$ is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.

Howto: Given a function of the form $f(x) = A\sec(Bx - C) + D$, graph one period.

1. Express the function given in the form $y = A\sec(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\dfrac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\sec(Bx)$. but shift it to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$, where k is an odd integer.

Example $Page 6.8.5$: Graphing a Variation of the Secant Function

Graph one period of $y = 4\sec\left(\dfrac{\pi}{3x} - \dfrac{\pi}{2}\right) + 1$.

Solution

- *Step 1.* Express the function given in the form $y = 4\sec\left(\dfrac{\pi}{3x} - \dfrac{\pi}{2}\right) + 1$.
- *Step 2.* The stretching/compressing factor is $|A| = 4$.
- *Step 3.* The period is

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$$\begin{aligned}\frac{2\pi}{|B|} &= \frac{2\pi}{\frac{\pi}{3}} \\ &= 2\pi \cdot \frac{3}{\pi} \\ &= 6\end{aligned}$$

- *Step 4.* The phase shift is

$$\begin{aligned}\frac{C}{B} &= \frac{\frac{\pi}{2}}{\frac{\pi}{3}} \\ &= \frac{\pi}{2} \cdot \frac{3}{\pi} \\ &= 1.5\end{aligned}$$

- *Step 5.* Draw the graph of $y = A\sec(Bx)$, but shift it to the right by $\frac{C}{B} = 1.5$ and up by $D = 6$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 3$, and $x = 6$. There is a local minimum at $(1.5, 5)$ and a local maximum at $(4.5, -3)$. Figure $Page 6.8.12$ shows the graph.

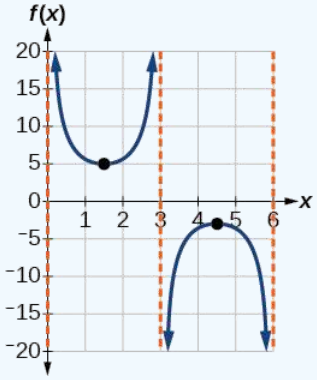


Figure $Page 6.8.12$

? Exercise $Page 6.8.5$

Graph one period of $f(x) = -6\sec(4x + 2) - 8$.

Answer

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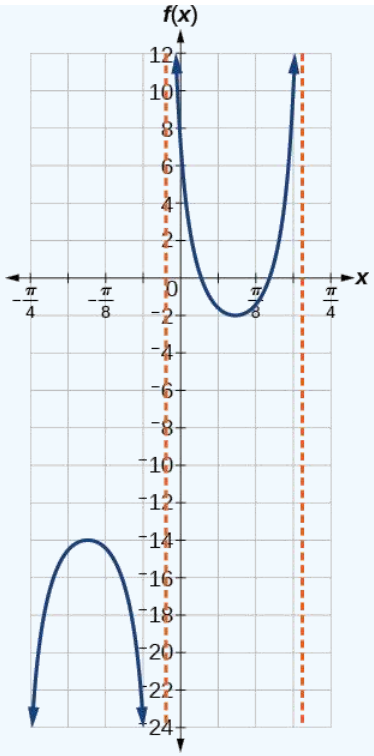


Figure $Page 6.8.13$

Q&A: The domain of $\csc\ x$ was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc(Bx - C) + D$ be $x \neq \dfrac{C + k\pi}{B}$?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

Howto: Given a function of the form $y = A\csc(Bx)$, graph one period.

1. Express the function given in the form $y = A\csc(Bx)$.
2. $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Draw the graph of $y = A\sin(Bx)$.
5. Use the reciprocal relationship between $y = sin\ x$ and $y = \csc\ x$ to draw the graph of $y = A\csc(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example $Page 6.8.6$: Graphing a Variation of the Cosecant Function

Graph one period of $f(x) = -3\csc(4x)$.

Solution

- *Step 1.* The given function is already written in the general form, $y = A\csc(Bx)$.
- *Step 2.* $|A| = |-3| = 3$,so the stretching factor is 3.
- *Step 3.* $B = 4$,so $P = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$. The period is $\dfrac{\pi}{2}$ units.
- *Step 4.* Sketch the graph of the function $g(x) = -3\sin(4x)$.
- *Step 5.* Use the reciprocal relationship of the sine and cosecant functions to draw the cosecant function.

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- *Steps 6–7.* Sketch three asymptotes at $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$. We can use two reference points, the local maximum at $\left(\frac{\pi}{8}, -3\right)$ and the local minimum at $\left(\frac{3\pi}{8}, 3\right)$. Figure $Page 6.8.14$ shows the graph.

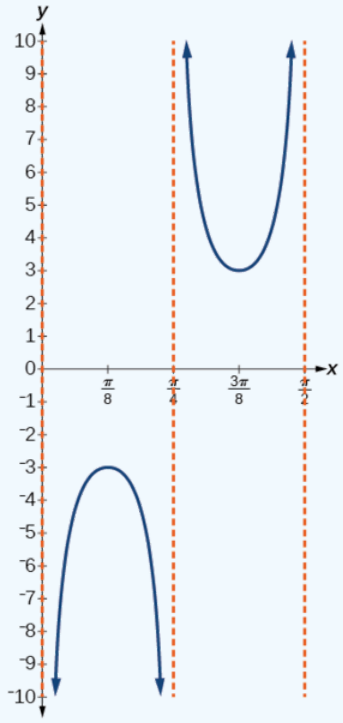


Figure $Page 6.8.14$

Exercise $Page 6.8.6$

Graph one period of $f(x) = 0.5 \csc(2x)$.

Answer

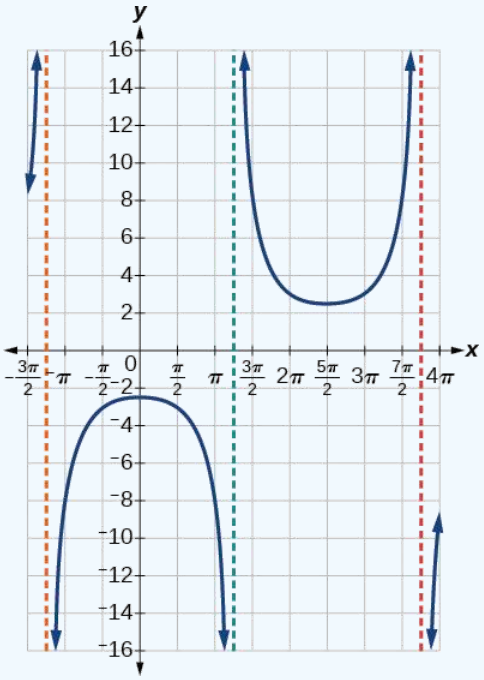


Figure $Page 6.8.15$

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Howto: Given a function of the form $f(x) = A\csc(Bx - C) + D$, graph one period

1. Express the function given in the form $y = A\csc(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\csc(Bx)$ but shift it to the right by and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.

Example $Page6.8.7$: Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$. What are the domain and range of this function?

Solution

- *Step 1.* Express the function given in the form $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$.
- *Step 2.* Identify the stretching/compressing factor, $|A| = 2$.
- *Step 3.* The period is $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$.
- *Step 4.* The phase shift is $\frac{0}{\frac{\pi}{2}} = 0$.
- *Step 5.* Draw the graph of $y = A\csc(Bx)$ but shift it up $D = 1$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 2$, $x = 4$.

The graph for this function is shown in Figure $Page6.8.16$.

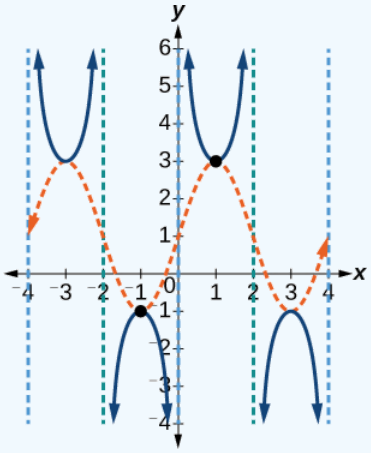


Figure $Page6.8.16$: A transformed cosecant function

Analysis

The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$,shown as the orange dashed wave.

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Exercise $Page 6.8.7$

Given the graph of $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1$ shown in Figure $Page 6.8.17$, sketch the graph of $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$ on the same axes.

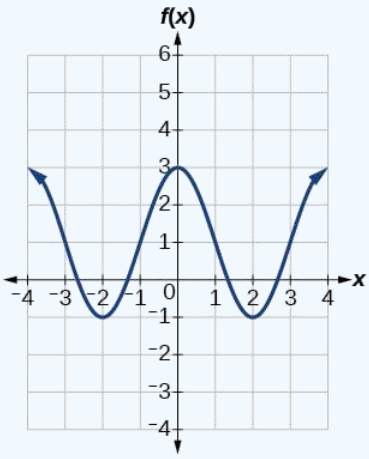


Figure $Page 6.8.17$

Answer

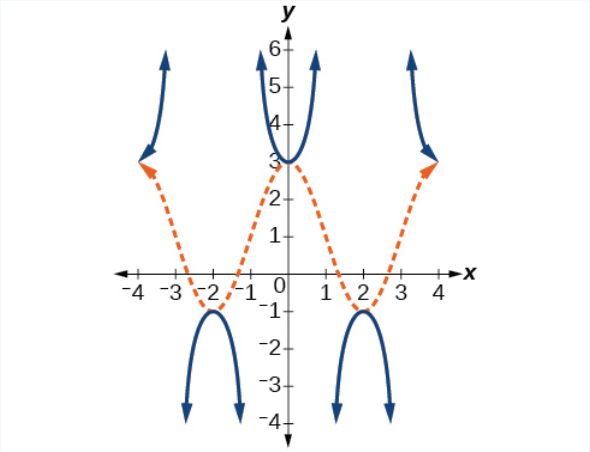


Figure $Page 6.8.18$

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $cot\, x = \frac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See Figure $Page 6.8.19$. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

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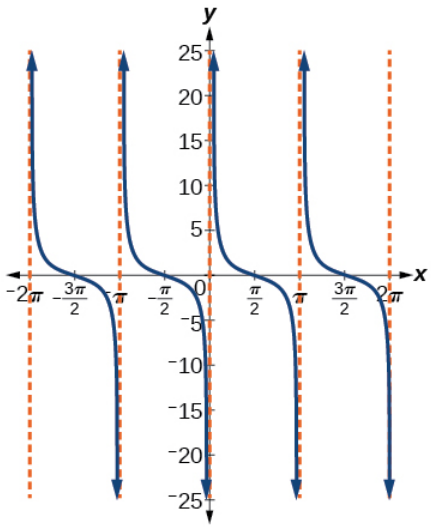


Figure $Page 6.8.19$: The cotangent function

FEATURES OF THE GRAPH OF $Y = A\cot(BX)$

- The stretching factor is $|A|$.
- The period is $P = \dfrac{\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Graphing Variations of $y = \cot x$

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A\cot(Bx - C) + D \qquad (\text{Page } 6.8.4)$$

PROPERTIES OF THE GRAPH OF $Y = A\cot(Bx - c) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{\pi}{|B|}$
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
- There is no amplitude.
- $y = A\cot(Bx)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively)

Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx)$,graph one period.

1. Express the function in the form $f(x) = A\cot(Bx)$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \dfrac{\pi}{|B|}$.
4. Draw the graph of $y = A\tan(Bx)$.
5. Plot any two reference points.

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6. Use the reciprocal relationship between tangent and cotangent to draw the graph of $y = Acot(Bx)$.
7. Sketch the asymptotes.

Example $Page6.8.8$: Graphing Variations of the Cotangent Function

Determine the stretching factor, period, and phase shift of $y = 3\cot(4x)$, and then sketch a graph.

Solution

- *Step 1.* Expressing the function in the form $f(x) = A\cot(Bx)$ gives $f(x) = 3\cot(4x)$.
- *Step 2.* The stretching factor is $|A| = 3$.
- *Step 3.* The period is $P = \frac{\pi}{4}$.
- *Step 4.* Sketch the graph of $y = 3\tan(4x)$.
- *Step 5.* Plot two reference points. Two such points are $\left(\frac{\pi}{16}, 3\right)$ and $\left(\frac{3\pi}{16}, -3\right)$.
- *Step 6.* Use the reciprocal relationship to draw $y = 3\cot(4x)$.
- *Step 7.* Sketch the asymptotes, $x = 0$, $x = \frac{\pi}{4}$.

The orange graph in Figure $Page6.8.20$ shows $y = 3\tan(4x)$ and the blue graph shows $y = 3\cot(4x)$.

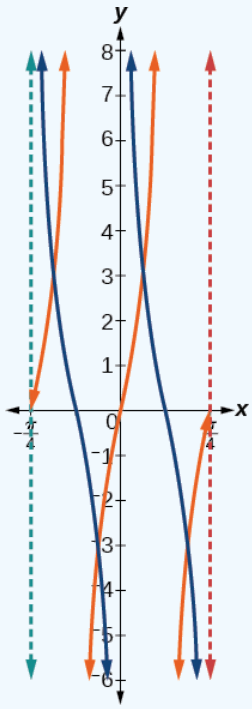


Figure $Page6.8.20$

Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx - C) + D$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the asymptotes $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
7. Plot any three reference points and draw the graph through these points.

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Example $Page6.8.9$: Graphing a Modified Cotangent

Sketch a graph of one period of the function $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

Solution

- *Step 1.* The function is already written in the general form $f(x) = A\cot(Bx - C) + D$.
- *Step 2.* $A = 4$,so the stretching factor is 4.
- *Step 3.* $B = \frac{\pi}{8}$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8$.
- *Step 4.* $C = \frac{\pi}{2}$,so the phase shift is $CB = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$.
- *Step 5.* We draw $f(x) = 4\tan\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 6-7.* Three points we can use to guide the graph are $(6, 2)$, $(8, -2)$, and $(10, -6)$. We use the reciprocal relationship of tangent and cotangent to draw $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 8.* The vertical asymptotes are $x = 4$ and $x = 12$.

The graph is shown in Figure $Page6.8.21$.

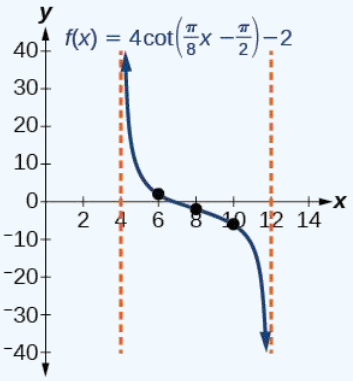


Figure $Page6.8.21$: One period of a modified cotangent function

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the **tangent function**.

Example $Page6.8.10$: Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan(\frac{\pi}{4}t)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.
b. Graph on the interval $[0, 5]$.
c. Evaluate $f(1)$ and discuss the function's value at that input.

Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the stretching factor and $\frac{\pi}{B}$ is the period.

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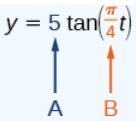


Figure $Page 6.8.22$

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is $\frac{\pi}{\frac{\pi}{4}} = \frac{\pi}{1} \cdot \frac{4}{\pi} = 4$. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the stretching factor and period. See Figure $Page 6.8.23$

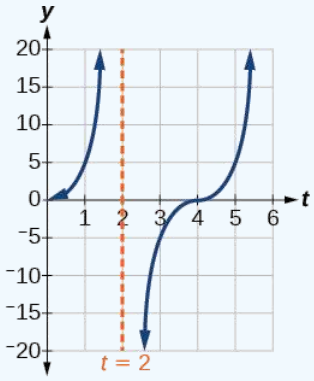


Figure $Page 6.8.23$

c. period: $f(1) = 5\tan(\frac{\pi}{4}(1)) = 5(1) = 5$; after 1 second, the beam of has moved 5 ft from the spot across from the police car.

Media

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent
- Graphing Cosecant and Secant
- Graphing the Cotangent

Key Equations

Shifted, compressed, and/or stretched tangent function	$y = A\tan(Bx - C) + D$
Shifted, compressed, and/or stretched secant function	$y = A\sec(Bx - C) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc(Bx - C) + D$
Shifted, compressed, and/or stretched cotangent function	$y = A\cot(Bx - C) + D$

Key Concepts

- The tangent function has period π.
- $f(x) = A\tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift. See Example $Page 6.8.1$, Example $Page 6.8.2$, and Example $Page 6.8.3$.
- The secant and cosecant are both periodic functions with a period of 2π. $f(x) = A\sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph. See Example $Page 6.8.4$ and Example $Page 6.8.5$.
- $f(x) = A\csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See Example $Page 6.8.6$ and Example $Page 6.8.7$.
- The cotangent function has period π and vertical asymptotes at $0, \pm\pi, \pm 2\pi$,....
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.

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- The cotangent is zero at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$,....
- $f(x) = A\cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift. See Example $Page 6.8.8$ and Example $Page 6.8.9$.
- Real-world scenarios can be solved using graphs of trigonometric functions. See Example $Page 6.8.10$.

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Page 6.9: Arcs, Angles, and Trig Function Values

Learning Objectives

After studying this section, we should understand the concepts motivated by the questions below and be able to write precise, coherent answers to these questions:

- How do we measure angles using degrees?
- What do we mean by the radian measure of an angle? How is the radian measure of an angle related to the length of an arc on the unit circle?
- Why is radian measure important?
- How do we convert from radians to degrees and from degrees to radians?
- How do we determine the values of both x and y when parameters on the domain are restricted?
- What trigonometric functions are positive in the first, second, third and fourth quadrants?
- How do we use one trigonometric function to represent or describe another?

The ancient civilization known as Babylonia was a cultural region based in southern Mesopotamia, which is present-day Iraq. Babylonia emerged as an independent state around 1894 BCE. The Babylonians developed a system of mathematics that was based on a sexigesimal (base 60) number system. This was the origin of the modern day usage of 60 minutes in an hour, 60 seconds in a minute, and 360 degrees in a circle.

Many historians now believe that for the ancient Babylonians, the year consisted of 360 days, which is not a bad approximation given the crudeness of the ancient astronomical tools. As a consequence, they divided the circle into 360 equal length arcs, which gave them a unit angle that was 1/360 of a circle or what we now know as a degree. Even though there are 365.24 days in a year, the Babylonian unit angle is still used as the basis for measuring angles in a circle. Figure $Page 6.9.1$ shows a circle divided up into 6 angles of 60 degrees each, which is also something that fit nicely with the Babylonian base-60 number system.

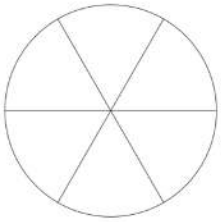


Figure $Page 6.9.1$: A circle with six 60-degree angles.

We often denote a line that is drawn through 2 points A and B by $\overleftrightarrow{AB}$. The portion of the line $\overleftrightarrow{AB}$ that starts at the point A and continues indefinitely in the direction point of point B is called **ray AB** and is denoted by $\overrightarrow{AB}$. The point A is the **initial point** of ray $\overrightarrow{AB}$. An **angle** is formed by rotating a ray about its endpoint. The ray in its initial position is called the **initial side** of the angle, and the position of the ray after it has been rotated is called the **terminal side** of the ray. The endpoint of the ray is called the **vertex of the angle.**

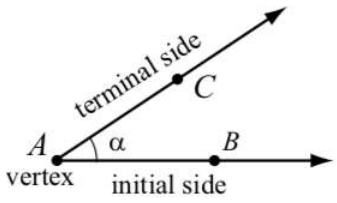


Figure $Page 6.9.2$: An angle including some notation.

Figure $Page 6.9.2$ shows the ray $\overrightarrow{AB}$ rotated about the point A to form an angle. The terminal side of the angle is the ray $\overrightarrow{AC}$. We often refer to this as angle BAC, which is abbreviated as $\angle BAC$. We can also refer to this angle as angle CAB or $\angle CAB$. If we

want to use a single letter for this angle, we often use a Greek letter such as α(alpha). We then just say the angleα. Other Greek letters that are often used are β(beta), γ(gamma), θ(theta), ϕ(phi), and ρ(rho).

Arcs and Angles

To define the trigonometric functions in terms of angles, we will make a simple connection between angles and arcs by using the standard position of an angle. When the vertex of an angle is at the origin in the xy-plane and the initial side lies along the positive x-axis, we see that the angle is in **standard position**. The terminal side of the angle is then in one of the four quadrants or lies along one of the axes. When the terminal side is in one of the four quadrants, the terminal side determines the so-called quadrant designation of the angle. See Figure $Page 6.9.3$.

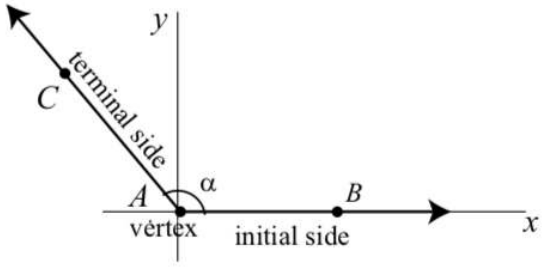


Figure $Page 6.9.3$: Standard position of an angle in the second quadrant.

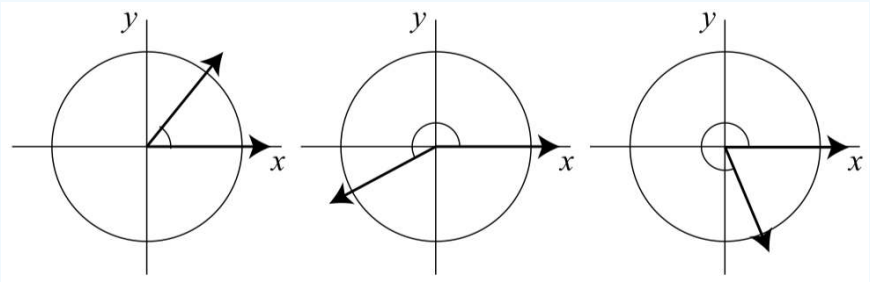
Exercise $Page 6.9.1$

Draw an angle in standard position in

a. the first quadrant;
b. the third quadrant; and
c. the fourth quadrant.

Answer

These graphs show positive angles in standard position. The one on the left has its terminal point in the first quadrant, the one in the middle has its terminal point in the third quadrant, and the one on the right has its terminal point in the fourth quadrant.



If an angle is in standard position, then the point where the terminal side of the angle intersects the unit circle marks the terminal point of an arc as shown in Figure 1.11. Similarly, the terminal point of an arc on the unit circle determines a ray through the origin and that point, which in turn defines an angle in standard position. In this case we say that the angle is *subtended* by the arc. So there is a natural correspondence between arcs on the unit circle and angles in standard position. Because of this correspondence, we can also define the trigonometric terminal side functions in terms of angles as well as arcs. Before we do this, however, we need to discuss two different ways to measure angles.

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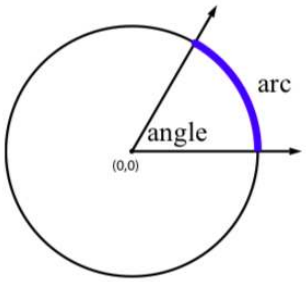


Figure $Page 6.9.4$: An arc and its corresponding angle.

Degrees Versus Radians - A Review (Because we ALL need one!)

There are two ways we will measure angles – in degrees and radians. When we measure the length of an arc, the measurement has a dimension (the length, be it inches, centimeters, or something else). As mentioned in the introduction, the Babylonians divided the circle into 360 regions. So one complete wrap around a circle is 360 degrees, denoted 360°. The unit measure of 1° is an angle that is 1/360 of the central angle of a circle. Figure $Page 6.9.1$ shows 6 angles of 60° each. The degree $^\circ$ is a dimension, just like a length. So to compare an angle measured in degrees to an arc measured with some kind of length, we need to connect the dimensions. We can do that with the radian measure of an angle.

Radians will be useful in that a radian is a dimensionless measurement. We want to connect angle measurements to arc measurements, and to do so we will directly define an angle of 1 radian to be an angle subtended by an arc of length 1 (the length of the radius) on the unit circle as shown in Figure $Page 6.9.5$.

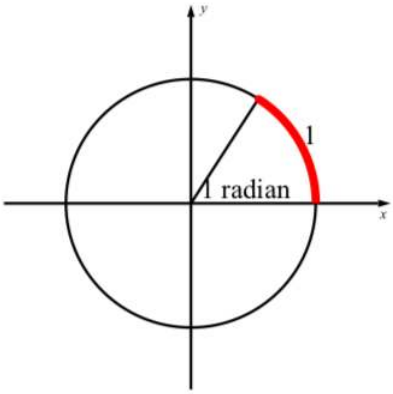


Figure $Page 6.9.5$: One radian.

Definition: Radian

An angle of **one radian** is the angle in standard position on the unit circle that is subtended by an arc of length 1 (in the positive direction).

This directly connects angles measured in radians to arcs in that we associate a real number with both the arc and the angle. So an angle of 2 radians cuts off an arc of length 2 on the unit circle, an angle of 3 radians cuts of an arc of length 3 on the unit circle, and so on. Figure 1.13 shows the terminal sides of angles with measures of 0 radians, 1 radian, 2 radians, 3 radians, 4 radians, 5 radians, and 6 radians. Notice that $2\pi \approx 6.2832$ and so $6 < 2\pi$ as shown in Figure $Page 6.9.6$.

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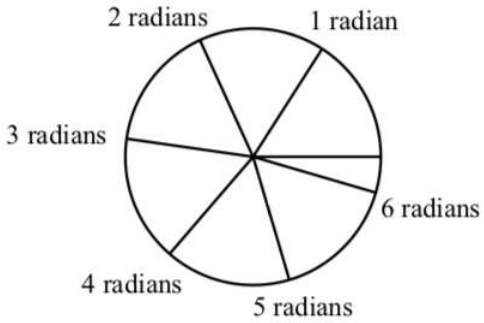


Figure $Page 6.9.6$: Angles with Radian Measure 1, 2, 3, 4, 5, and 6

We can also have angles whose radian measure is negative just like we have arcs with a negative length. The idea is simply to measure in the negative (clockwise) direction around the unit circle. So an angle whose measure is -1 radian is the angle in standard position on the unit circle that is subtended by an arc of length 1 in the negative (clockwise) direction.

So in general, an angle (in standard position) of t radians will correspond to an arc of length t on the unit circle. This allows us to discuss the sine and cosine of an angle measured in radians. That is, when we think of $\sin(t)$ and $\cos(t)$, we can consider t to be:

- a real number;
- the length of an arc with initial point $(1, 0)$ on the unit circle;
- the radian measure of an angle in standard position.

When we draw a picture of an angle in standard position, we often draw a small arc near the vertex from the initial side to the terminal side as shown in Figure $Page 6.9.7$, which shows an angle whose measure is $\frac{3}{4}\pi$ radians.

Example $Page 6.9.1$

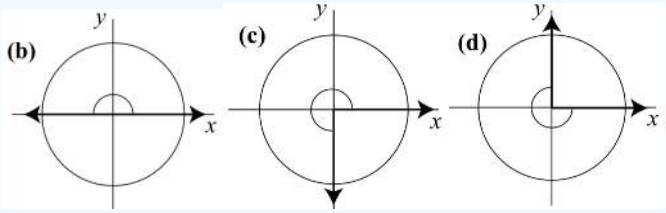
1. Draw a terminal ray in the quadrant defined by the given θ restrictions; measuring from standard position with a radian measure:

Let $\sin(\theta$

a. $\frac{\pi}{2}$
b. π radians.
c. $\frac{3\pi}{2}$ radians.
d. $\frac{-3\pi}{2}$ radians.

2. What is the degree measure of each of the angles in part (1)?

Answer



a. 90° (Page 6.9.1)

b. 180° (Page 6.9.2)

c. 270° (Page 6.9.3)

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d. -270° (Page 6.9.4)

Conversion Between Radians and Degrees

Radian measure is the preferred measure of angles in mathematics for many reasons, the main one being that a radian has no dimensions. However, to effectively use radians, we will want to be able to convert angle measurements between radians and degrees.

Recall that one wrap of the unit circle corresponds to an arc of length 2π, and an arc of length 2π on the unit circle corresponds to an angle of 2π radians. An angle of 360° is also an angle that wraps once around the unit circle, so an angle of 360° is equivalent to an angle of 2π radians, or

- each degree is $\frac{\pi}{180}$ radians,
- each radian is $\frac{180}{\pi}$ degrees.

Notice that 1 radian is then $\frac{180}{\pi} \approx 57.3^\circ$, so a radian is quite large compared to a degree. These relationships allow us to quickly convert between degrees and radians.

Example $Page 6.9.1$

- If an angle has a degree measure of 35 degrees, then its radian measure can be calculated as follows:

$$35\ degrees \times \frac{\pi\ radians}{180\ degrees} = \frac{35\pi}{180}\ radians \quad \text{(Page 6.9.5)}$$

Rewriting this fraction, we see that an angle with a measure of 35 degrees has a radian measure of $\frac{7\pi}{36}$ radians.

- If an angle has a radian measure of $\frac{3\pi}{10}$ radians, then its degree measure can be calculated as follows:

$$\frac{3\pi}{10}\ radians \times \frac{180\ degrees}{\pi\ radians} = \frac{540}{10}\ degrees \quad \text{(Page 6.9.6)}$$

So an angle with a radian measure of $\frac{3\pi}{10}$ has an angle measure of 54°.

IMPORTANT NOTE

Since a degree is a dimension, we MUST include the degree mark $^\circ$ whenever we write the degree measure of an angle. A radian has no dimension so there is no dimension mark to go along with it. Consequently, if we write 2 for the measure of an angle we understand that the angle is measured in radians. If we really mean an angle of 2 degrees, then we *must* write 2°.

Exercise $Page 6.9.3$

Complete the following table to convert from degrees to radians and vice versa.

Answer

Angle in Radians	Angle in degrees
0	0°
$\frac{\pi}{6}$	30°
$\frac{\pi}{4}$	45°
$\frac{\pi}{3}$	60°

Angle in Radians	Angle in degrees
$\frac{\pi}{2}$	90°
$\frac{7\pi}{6}$	210°
$\frac{5\pi}{4}$	225°
$\frac{4\pi}{3}$	240°
$\frac{3\pi}{2}$	270°
$\frac{5\pi}{3}$	300°
$\frac{2\pi}{3}$	120°
$\frac{4\pi}{3}$	135°
$\frac{5\pi}{6}$	150°
π	180°
$\frac{7\pi}{4}$	315°
$\frac{11\pi}{6}$	330°
2π	360°

Calculators and the Trigonometric Functions

We have now seen that when we think of $\sin(t)$ or $\cos(t)$, we can think of (t) as a real number, the length of an arc, or the radian measure of an angle. We will see how to determine the exact values of the cosine and sine functions for a few special arcs (or angles). For example, we will see that $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$. However, the definition of cosine and sine as coordinates of points on the unit circle makes it difficult to find exact values for these functions *except at very special arcs*. While exact values are always best, technology plays an important role in allowing us to approximate the values of the circular (or trigonometric)functions. Most hand-held calculators, calculators in phone or tablet apps, and online calculators have a cosine key and a sine key that you can use to approximate values of these functions, but we must keep in mind that the calculator only provides an *approximation* of the value, not the exact value (except for a small collection of arcs). In addition, most calculators will approximate the sine and cosine of angles. Remember to read homework questions carefully - if it says anything about an exact number, it probably does NOT want a calculator approximation.

Angle in radians	Angle in degrees
0	0°
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	
$\frac{\pi}{2}$	90°

	120°
$\frac{3\pi}{4}$	135°
	150°
	180°
$\frac{7\pi}{6}$	
$\frac{5\pi}{4}$	
$\frac{4\pi}{3}$	
$\frac{3\pi}{2}$	270°
	300°
	315°
	330°
2π	360°

Table 1.1: Conversions between radians and degrees.

To do this, the calculator has two modes for angles: Radian and Degree. Because of the correspondence between real numbers, length of arcs, and radian measures of angles, for now, we will always put our calculators in radian mode. In fact, we have seen that an angle measured in radians subtends an arc of that radian measure along the unit circle. **So the cosine or sine of an angle measure in radians is the same thing as the cosine or sine of a real number when that real number is interpreted as the length of an arc along the unit circle.** (When we study the trigonometry of triangles, we will use the degree mode. In general, if a questions or application is given in degrees the solution is usually also given in degrees. Similarly, when working in radians the solution would also be given in radians, unless you have specifically been asked to convert from one to the other form.

Exercise $Page 6.9.4$

We can use the Geogebra Applet called *Terminal Points of Arcs on the Unit Circle* at http://gvsu.edu/s/JY to approximate the values of the cosine and sine functions at certain values. For example, we could find the following:

- $\cos(1) \approx 0.5403, \sin(1) \approx 0.8415$.
- $\cos(2) \approx -0.4161, \sin(2) \approx 0.9093$.
- $\cos(-4) \approx -0.6536, \sin(-4) \approx 0.7568$.
- $\cos(-15) \approx -0.7597, \sin(-15) \approx -0.6503$.

Use a calculator to determine these values of the cosine and sine functions and compare the values to the ones above. Are they the same? How are they different?

Answer

Using a calculator, we obtain the following results correct to ten decimal places:

$\cos(1) \approx 0.5403023059, \sin(1) \approx 0.8414709848 \cos(1) \approx 0 : 4161468365, \sin(1) \approx 0.9092974268$

The difference between these values and those obtained previously, is that these values are correct to 10 decimal places (and the others are correct to 4 decimal places). If we round off each of the values above to 4 decimal places, we get the same results.

- $\cos(1) \approx 0 : 6536436209, \sin(1) \approx 0.7568024953$
- $\cos(1) \approx 0 : 7596879129, \sin(1) \approx 0.6502878402$

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Exercise $Page 6.9.5$

When working with sine and cosine we are reminded that these two functions can be used to generate the remaining 4 functions - tangent, secant, cosecant and cotangent.

Each of these trigonometric functions are positive in different quadrants of the unit circle. Now that we can calculate the values of sine and cosine at ANY terminal point along a unit circle, we now need to be able to do the same for the remaining 4 functions.

- In quadrant I **A**LL trigonometric functions are positive.
- In quadrant II the **S**INE function is positive.
- In quadrant III the **T**ANGENT function is positive.
- In quadrant IV the **C**OSINE function is positive.

Using the first letter from each of these quadrants and putting it into an easy-to-remember phrase can help you remember where functions are positive.

Here are some common phrases: "**A**ll **S**tudents **T**ake **C**alculus" "**A** **S**mart **T**rig **C**lass" or maybe you can think up a better one!

a. In what quadrant does the terminal ray reside if $sin(t) < 0$ and $cos(t) < 0$?
b. In what quadrant does the terminal ray reside if $tan(t) < 0$ and $cos(t) > 0$?
c. In what quadrant does the terminal ray reside if $sec(t) < 0$ and $tan(t) > 0$?
d. In what quadrant does the terminal ray reside if $csc(t) > 0$ and $cot(t) < 0$?

Answer

a. Sine is negative in quadrant III and IV, while cosine is negative in quadrant II and III. Therefore the overlap must be in **quadrant II** where both functions are negative.
b. Recall that tangent has an identity where $tan(t) = \dfrac{sin(t)}{cos(t)}$ is negative in quadrant II and IV, while cosine is positive in quadrant I and IV. Therefore the overlap must be in **quadrant IV** where these parameters are both met.
c. Recall that secant has an identity where $sec(t) = \dfrac{1}{cos(t)}$ this implies that the same quadrants will be positive for secant as for cosine. Secant is negative in quadrants II and III, while tangent is positive in quadrant I and III. Therefore the overlap must be in **quadrant III** where these parameters are both met.
d. Recall that cosecant has an identity where $csc(t) = \dfrac{1}{sin(t)}$ this implies that the same quadrants will be positive for cosecant as for sine. Recall that cotangent is the reciprocal of tangent and has an identity where $cot(t) = \dfrac{cos(t)}{sin(t)}$ is negative in quadrant II and IV. Cosecant is positive in quadrants I and II, while cotangent is negative in quadrant II and IV. Therefore, the overlap must be in **quadrant II** where these parameters are both met.

Exercise $Page 6.9.6$

Sometimes it will be important to write a given trigonometric function in terms of another.

The Pythagorean Theorem is $sin^2(\theta)+cos^2(\theta)=1$.

Dividing this initial identity by $sin^2(\theta)$ will generate another form for Pythagorean Theorem.

Dividing this initial identity by $cos^2(\theta)$ will generate a third form.

Use these three different forms to find the functions listed in terms of another function. Use the quadrant to determine if the function is positive or negative.

Remember from the previous exercise that the following are true:

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- In quadrant I **A**LL trigonometric functions are positive.
- In quadrant II the **S**INE function is positive.
- In quadrant III the **T**ANGENT function is positive.
- In quadrant IV the **C**OSINE function is positive.

Use the Pythagorean Theorem and the phrase you have memorized to find the following trigonometric functions:

a. Write the $tan\theta$ in terms of $cos\theta$ for θ in Quadrant II.
b. Write the $sin\theta$ in terms of $cos\theta$ for θ in Quadrant III.
c. Write the $tan\theta$ in terms of $sec\theta$ for θ in Quadrant I.
d. Write the $sec\theta$ in terms of $cos\theta$ for θ in Quadrant IV.

Answer

a. We need to find a way to write $tan\theta$ equal to something on the right with only cosine functions included. If we divide $sin^2\theta+cos^2\theta=1$ by $cos^2\theta$ we will have a form for the Pythagorean Theorem in terms of tangent and cosine.

$$\frac{sin^2\theta}{cos^2\theta}+\frac{cos^2\theta}{cos^2\theta}=\frac{1}{cos^2\theta} \qquad \text{(Page 6.9.7)}$$

Now, solving for tangent we should have:

$$tan^2\theta=\frac{1}{cos^2\theta}-\frac{cos^2\theta}{cos^2\theta} \qquad \text{(Page 6.9.8)}$$

Simplify and take the square root of both sides. Remember, tangent is negative in Quadrant II, therefore our answer should be:

$$tan\theta=-\frac{\sqrt{1-cos^2\theta}}{cos\theta} \qquad \text{(Page 6.9.9)}$$

b. $$sin\theta=-\sqrt{1-cos^2\theta} \qquad \text{(Page 6.9.10)}$$

c. $$tan\theta=\sqrt{sec^2\theta-1} \qquad \text{(Page 6.9.11)}$$

d. $$tan\theta=-\frac{1}{\sqrt{1-cos^2\theta}} \qquad \text{(Page 6.9.12)}$$

Exercise $Page 6.9.7$

Throughout trigonometry - even in Calculus, Linear Algebra and some Differential Equations classes - we will be asked to find the terminal point given limited information and sometimes restrictions on your domain or quadrant. Use the Pythagorean Theorem to find the missing sides and recognize when the function will be positive or negative based on the quadrant. It is usually helpful to first restrict the quadrant to the one in which the values are true, then calculate the exact value for the remaining five functions based on the given information.

a. Given $sin\theta=-\frac{4}{5}$ find the remaining five values for the remaining five functions if θ is in Quadrant IV.
b. Given $cos\theta=-\frac{5}{6}$ find the remaining five values for the remaining five functions if $cot\theta>0$.
c. Given $csc\theta=3$ find the remaining five values for the remaining five functions in Quadrant I.
d. Given $cos\theta=\frac{7}{13}$ find the remaining five values for the remaining five functions if $sin\theta<0$.
e. Given $sec\theta=-\frac{1}{2}$ find the remaining five values for the remaining five functions if $csc\theta<0$.
f. Given $cos\theta=-\frac{2}{11}$ find the remaining five values for the remaining five functions if$tan\theta<0$.

Answer

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a. Rotating through to a terminal ray in Quadrant IV, we now can draw a triangle using the terminal ray as the hypotenuse of the triangle. Use a reference angle off the x axis to orient the opposite and adjacent sides. The missing side is the x-direction length. Using the Pythagorean Theorem the sides of the triangle should be: Opposite is -4, Adjacent is 3 and Hypotenuse is 5. Now, using the identities,

$$sin\theta = \frac{opposite}{hypotenuse} = \frac{-4}{5} \quad \text{(Page 6.9.13)}$$

$$cos\theta = \frac{adjacent}{hypotenuse} = \frac{3}{5} \quad \text{(Page 6.9.14)}$$

and

$$tan\theta = \frac{opposite}{adjacent} = \frac{sin\theta}{cos\theta} = \frac{-4}{3} \quad \text{(Page 6.9.15)}$$

For the remaining three trig functions we can use what we found previously, just as reciprocals:

$$csc\theta = \frac{hypotenuse}{opposite} = \frac{5}{-4} \quad \text{(Page 6.9.16)}$$

$$sec\theta = \frac{hypotenuse}{adjacent} = \frac{5}{3} \quad \text{(Page 6.9.17)}$$

$$cot\theta = \frac{adjacent}{opposite} = \frac{cos\theta}{sin\theta} = \frac{3}{-4} \quad \text{(Page 6.9.18)}$$

NOTE:From here on the answers will be simplified such that we bring the negative sign in front of the fraction and it is understood that it was associated with the negative side of the triangle. It is more formal to write it in front of the fraction instead of with the numerator or denominator.

b. Cotangent is only positive in Quadrants I and III. While Cosine is only negative in Quadrants II and III. Therefore, we know our values for all six trig functions must exist in Quadrant III!

$$sin\theta = \frac{opposite}{hypotenuse} = -\frac{\sqrt{11}}{6} \quad \text{(Page 6.9.19)}$$

$$cos\theta = -\frac{5}{6} \quad \text{(Page 6.9.20)}$$

$$tan\theta = \frac{opposite}{adjacent} = \frac{sin\theta}{cos\theta} = \frac{\sqrt{11}}{5} \quad \text{(Page 6.9.21)}$$

For the remaining three trig functions we can use what we found previously, just as reciprocals:

$$csc\theta = \frac{hypotenuse}{opposite} = -\frac{6\sqrt{11}}{11} \quad \text{(Page 6.9.22)}$$

$$sec\theta = \frac{hypotenuse}{adjacent} = -\frac{6}{5} \quad \text{(Page 6.9.23)}$$

$$cot\theta = \frac{adjacent}{opposite} = \frac{cos\theta}{sin\theta} = \frac{5\sqrt{11}}{11} \quad \text{(Page 6.9.24)}$$

c.

$$sin\theta = \frac{1}{3} \quad \text{(Page 6.9.25)}$$

$$cos\theta = \frac{2\sqrt{2}}{3} \quad \text{(Page 6.9.26)}$$

$$tan\theta = \frac{\sqrt{2}}{4} \quad \text{(Page 6.9.27)}$$

$$csc\theta = 3 \quad \text{(Page 6.9.28)}$$

$$sec\theta = \frac{3\sqrt{2}}{4} \quad \text{(Page 6.9.29)}$$

$$cot\theta = 2\sqrt{2} \quad \text{(Page 6.9.30)}$$

d.

$$sin\theta = -\frac{2\sqrt{30}}{13} \quad \text{(Page 6.9.31)}$$

$$cos\theta = \frac{7}{13} \quad \text{(Page 6.9.32)}$$

$$tan\theta = 2\frac{2\sqrt{30}}{7} \quad \text{(Page 6.9.33)}$$

$$csc\theta = -\frac{13\sqrt{30}}{60} \quad \text{(Page 6.9.34)}$$

$$sec\theta = \frac{13}{7} \quad \text{(Page 6.9.35)}$$

$$cot\theta = -\frac{7\sqrt{30}}{60} \quad \text{(Page 6.9.36)}$$

e. Algebraically, we can calculate the value of cosine because $sec\theta = -\frac{1}{2}$. Therefore, cosine is -2. If we try to calculate the values for the remaining 5 functions we notice that the hypotenuse is less than the adjacent side. This should be a warning sign that the values do not exist. Geometrically/Graphically, we could plot or graph the function $f(x) = sec\theta$ simultaneously with $f(x) = -\frac{1}{2}$ to see that there is never a point at which the two functions intersect. This should lead us to believe that there is NO SOLUTION, or a solution DOES NOT EXIST (sometimes annotated as DNE).

f.

$$sin\theta = \frac{3\sqrt{13}}{11} \quad \text{(Page 6.9.37)}$$

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$$cos\theta = -\frac{2}{1} \quad \text{(Page 6.9.38)}$$

$$tan\theta = -\frac{3\sqrt{13}}{2} \quad \text{(Page 6.9.39)}$$

$$csc\theta = \frac{11\sqrt{13}}{39} \quad \text{(Page 6.9.40)}$$

$$sec\theta = -\frac{11}{2} \quad \text{(Page 6.9.41)}$$

$$cot\theta = -\frac{2\sqrt{13}}{39} \quad \text{(Page 6.9.42)}$$

Section Summary

In this section, we studied the following important concepts and ideas:

- An **angle** is formed by rotating a ray about its endpoint. The ray in its initial position is called the **initial side** of the angle, and the position of the ray after it has been rotated is called the **terminal side** of the ray. The endpoint of the ray is called the **vertex of the angle**.
- When the vertex of an angle is at the origin in the xy-plane and the initial side lies along the positive x-axis, we see that the angle is in **standard position**.
- There are two ways to measure angles. For degree measure, one complete wrap around a circle is 360 degrees, denoted 360 $^\circ$. The unit measure of 1° is an angle that is 1=360 of the central angle of a circle. An angle of **one radian** is the angle in standard position on the unit circle that is subtended by an arc of length 1 (in the positive direction).
- We convert the measure of an angle from degrees to radians by using the fact that each degree is radians. We convert the measure of an angle from 180 radians to degrees by using the fact that each radian is 180 degrees.
- We looked at the strategies to use when finding the range values for all 6 trigonometric functions.
- We imposed restriction on the domain and quadrant, then found the missing trig function value. This will come in handy in the following sections!

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Page 6.10: Inverse Trigonometric Functions

Learning Objectives

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in Figure $Page 6.10.1$.

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

Figure $Page 6.10.1$

For example, if $f(x) = \sin\ x$, then we would write $f^{-1}(x) = \sin^{-1} x$. Be aware that $\sin^{-1} x$ does not mean $\dfrac{1}{\sin\ x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$, then $\dfrac{\pi}{6} = \sin^{-1}\left(\dfrac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\dfrac{\pi}{4}\right) = 1$, then $\dfrac{\pi}{4} = \tan^{-1}(1)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the **domain** of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. Figure $Page 6.10.2$ shows the graph of the sine function limited to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ and the graph of the cosine function limited to $[0, \pi]$.

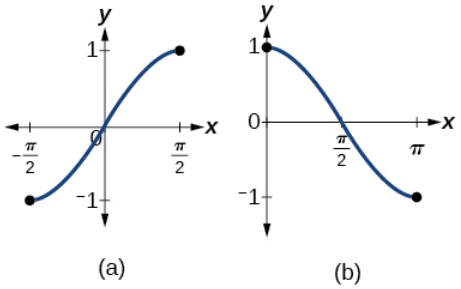


Figure $Page 6.10.2$: (a) Sine function on a restricted domain of $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$; (b) Cosine function on a restricted domain of $[0, \pi]$

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Figure $Page6.10.3$ shows the graph of the tangent function limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

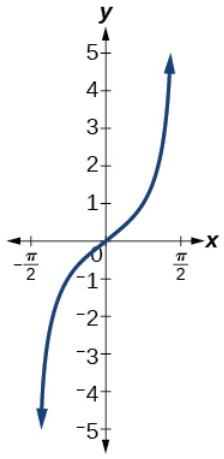


Figure $Page6.10.3$: Tangent function on a restricted domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one **vertical asymptote** to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin\ y$. The inverse sine function is sometimes called the **arcsine** function, and notated $\arcsin\ x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos\ y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos\ x$.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The inverse tangent function $y = \tan^{-1} x$ means $x = \tan\ y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan\ x$.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

The graphs of the inverse functions are shown in Figures $Page6.10.4$ - $Page6.10.6$. Notice that the output of each of these inverse functions is a *number,* an angle in radian measure. We see that $\sin^{-1} x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1} x$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

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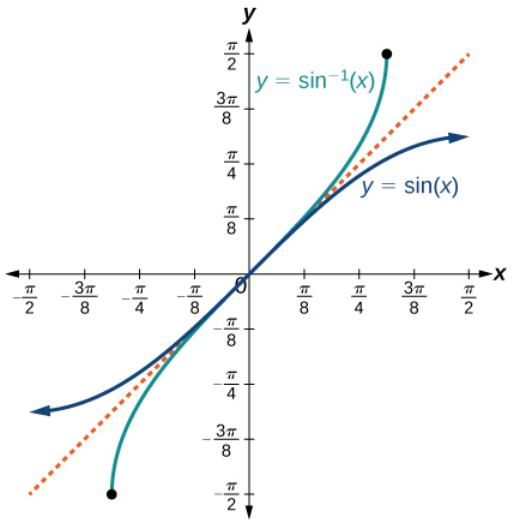


Figure $Page$6.10.4: The sine function and inverse sine (or arcsine) function

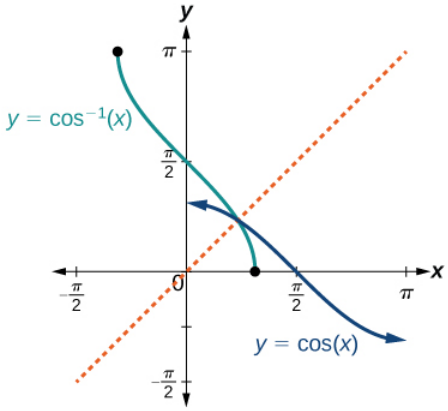


Figure $Page$6.10.5: The cosine function and inverse cosine (or arccosine) function

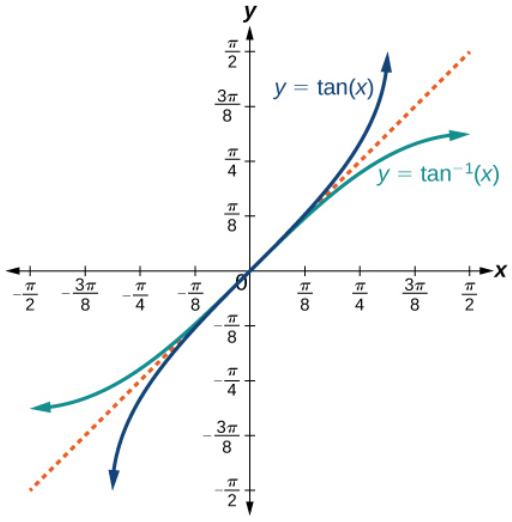


Figure $Page$6.10.6: The tangent function and inverse tangent (or arctangent) function

RELATIONS FOR INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

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For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$,then $\tan^{-1} x = y$.

Example $Page 6.10.1$: Writing a Relation for an Inverse Function

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

Solution

Use the relation for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and $y = \frac{5\pi}{12}$.

$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$

Exercise $Page 6.10.1$

Given $\cos(0.5) \approx 0.8776$,write a relation involving the inverse cosine.

Answer

$\arccos(0.8776) \approx 0.5$

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}(30°)$, $\frac{\pi}{4}(45°)$, and $\frac{\pi}{3}(60°)$, and their reflections into other quadrants.

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x,depending on which corresponds to the given inverse function.

Example $Page 6.10.2$: Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$

b. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

c. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

d. $\tan^{-1}(1)$

Solution

a. Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$. In other words, what angle x would satisfy $\sin(x) = \frac{1}{2}$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Remember that the inverse is a

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function, so for each input, we will get exactly one output.

b. To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$.

c. To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0,\pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$.

d. Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1)=\frac{\pi}{4}$.

? Exercise $Page 6.10.2$

Evaluate each of the following.

a. $\sin^{-1}(-1)$
b. $\tan^{-1}(-1)$
c. $\cos^{-1}(-1)$
d. $\cos^{-1}\left(\frac{1}{2}\right)$

Answer a

$-\frac{\pi}{2}$

Answer b

$-\frac{\pi}{4}$

Answer c

π

Answer d

$\frac{\pi}{3}$

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate **inverse trigonometric functions** that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, SIN-1, ARCSIN, or ASIN.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use θ as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

✓ Example $Page 6.10.3$**: Evaluating the Inverse Sine on a Calculator**

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Solution

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Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97) \approx 1.3252$. In degree mode, $\sin^{-1}(0.97) \approx 75.93^\circ$. Note that in calculus and beyond we will use radians in almost all cases.

? Exercise $Page 6.10.3$

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Answer

1.9823 or 113.578°

Given two sides of a right triangle like the one shown in Figure 8.4.7, find an angle.

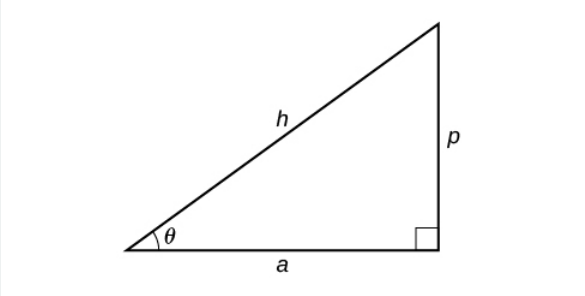


Figure $Page 6.10.7$

1. If one given side is the hypotenuse of length h and the side of length a adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$.
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

✓ Example $Page 6.10.4$**: Applying the Inverse Cosine to a Right Triangle**

Solve the triangle in Figure $Page 6.10.8$ for the angle θ.

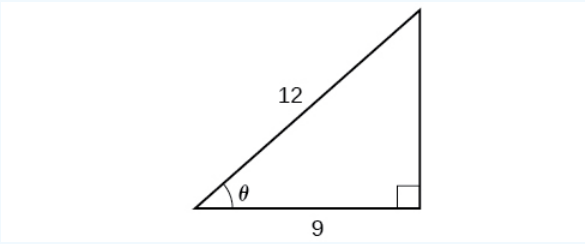


Figure $Page 6.10.8$

Solution

Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos\theta = \frac{9}{12}$$
$$\theta = \cos^{-1}\left(\frac{9}{12}\right) \quad \text{Apply definition of the inverse}$$
$$\theta \approx 0.7227 \quad \text{or about } 41.4096^\circ \text{ Evaluate}$$

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? Exercise $Page 6.10.4$

Solve the triangle in Figure $Page 6.10.9$ for the angle θ.

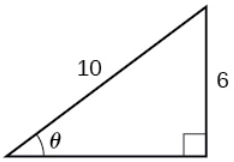


Figure $Page 6.10.9$

Answer

$\sin^{-1}(0.6) = 36.87^\circ = 0.6435$radians

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set{ $\sin(x)$,$\cos(x)$,$\tan(x)$ } and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function,$f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

COMPOSITIONS OF A TRIGONOMETRIC FUNCTION AND ITS INVERSE

$$\begin{aligned}
\sin(\sin^{-1}x) &= x \quad && \text{for } -1 \le x \le 1 \\
\cos(\cos^{-1}x) &= x \quad && \text{for } -1 \le x \le 1 \\
\tan(\tan^{-1}x) &= x \quad && \text{for } -\infty < x < \infty \\
\sin^{-1}(\sin x) &= x \quad && \text{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\
\cos^{-1}(\cos x) &= x \quad && \text{only for } 0 \le x \le \pi \\
\tan^{-1}(\tan x) &= x \quad && \text{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{aligned}$$

Q&A

Is it correct that $\sin^{-1}(\sin x) = x$?

No. This equation is correct ifx x belongs to the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but sine is defined for all real input values, and for x outside the restricted interval, the equation is not correct because its inverse always returns a value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The situation is similar for cosine and tangent and their inverses. For example, $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain off f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

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Example $Page6.10.5$: Using Inverse Trigonometric Functions

Evaluate the following:

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$
b. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$
c. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$
d. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Solution

a. $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)=\frac{\pi}{3}$.
b. $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, but $sin\left(\frac{2\pi}{3}\right)=sin\left(\frac{\pi}{3}\right)$, so $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)=\frac{\pi}{3}$.
c. $\frac{2\pi}{3}$ is in $[0,\pi]$, so $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)=\frac{2\pi}{3}$.
d. $-\frac{\pi}{3}$ is not in $[0,\pi]$, but $\cos\left(-\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)$ because cosine is an even function. $\frac{\pi}{3}$ is in $[0,\pi]$, so $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)=\frac{\pi}{3}$.

Exercise $Page6.10.5$

Evaluate $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$.

Answer

$\frac{\pi}{8};\frac{2\pi}{9}$

Evaluating Compositions of the Form $f^{-1}(g(x))$

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form $f^{-1}(g(x))$. For special values of x,we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is θ, making the other $\frac{\pi}{2}-\theta$.Consider the sine and cosine of each angle of the right triangle in Figure $Page6.10.10$.

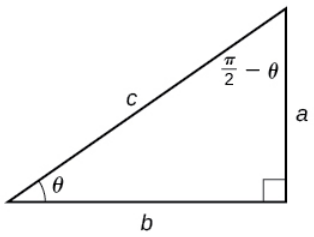


Figure $Page6.10.10$: Right triangle illustrating the cofunction relationships

Because $\cos\theta=\frac{b}{c}=sin\left(\frac{\pi}{2}-\theta\right)$, we have $\sin^{-1}(\cos\theta)=\frac{\pi}{2}-\theta$ if $0\le\theta\le\pi$. If θ is not in this domain, then we need to find another angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$.Similarly, $\sin\theta=\frac{a}{c}=\cos\left(\frac{\pi}{2}-\theta\right)$, so $\cos^{-1}(\sin\theta)=\frac{\pi}{2}-\theta$ if $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

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Given functions of the form $\sin^{-1}(\cos x)$ and $\cos^{-1}(\sin x)$, evaluate them.

1. If x is in $[0, \pi]$, then $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$.
2. If x is not in $[0, \pi]$, then find another angle y in $[0, \pi]$ such that $\cos y = \cos x$.

$$\sin^{-1}(\cos x) = \frac{\pi}{2} - y \qquad \text{(Page 6.10.1)}$$

3. If x is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.
4. If x is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then find another angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \sin x$.

$$\cos^{-1}(\sin x) = \frac{\pi}{2} - y \qquad \text{(Page 6.10.2)}$$

Example $Page 6.10.6$: Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

a. by direct evaluation.
b. by the method described previously.

Solution

a. Here, we can directly evaluate the inside of the composition.

$$\begin{aligned}\cos\left(\frac{13\pi}{6}\right) &= \cos\left(\frac{\pi}{6} + 2\pi\right) \\ &= \cos\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

Now, we can evaluate the inverse function as we did earlier.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \qquad \text{(Page 6.10.3)}$$

b. We have $x = \frac{13\pi}{6}$, $y = \frac{\pi}{6}$, and

$$\begin{aligned}\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) &= \frac{\pi}{2} - \frac{\pi}{6} \\ &= \frac{\pi}{3}\end{aligned}$$

Exercise $Page 6.10.6$

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Answer

$\frac{3\pi}{4}$

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of

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Pythagoras's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the inverse trigonometric functions to find compositions involving algebraic expressions.

Example $Page 6.10.7$: Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Solution

Beginning with the inside, we can say there is some angle such that $\theta = \cos^{-1}\left(\frac{4}{5}\right)$, which means $\cos\theta = \frac{4}{5}$, and we are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 && \text{Use our known value for cosine} \\ \sin^2\theta + \left(\frac{4}{5}\right)^2 &= 1 && \text{Solve for sine} \\ \sin^2\theta &= 1 - \frac{16}{25} \\ \sin\theta &= \pm\frac{9}{25} \\ &= \pm\frac{3}{5} \end{aligned}$$

Since $\theta = \cos^{-1}\left(\frac{4}{5}\right)$ is in quadrant I, $\sin\theta$ must be positive, so the solution is 35. See Figure $Page 6.10.11$.

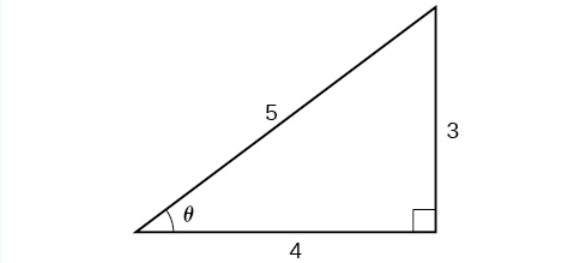


Figure $Page 6.10.11$: Right triangle illustrating that if $\cos\theta = \frac{4}{5}$, then $\sin\theta = \frac{3}{5}$

We know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive; therefore $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\theta = \frac{3}{5}$.

Exercise $Page 6.10.7$

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Answer

$\frac{12}{13}$

Example $Page 6.10.8$: Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Solution

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While we could use a similar technique as in Example $Page 6.10.6$, we will demonstrate a different technique here. From the inside, we know there is an angle such that $\tan\theta = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle, as shown in Figure $Page 6.10.12$.

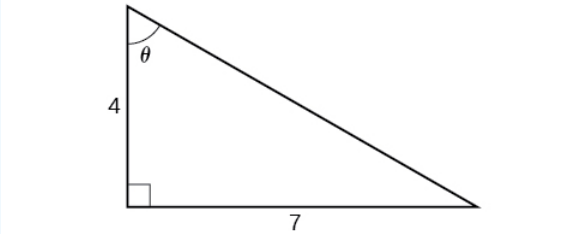


Figure $Page 6.10.12$: A right triangle with two sides known

Using the Pythagorean Theorem, we can find the hypotenuse of this triangle.

$$\begin{aligned} 4^2 + 7^2 &= hypotenuse^2 \\ hypotenuse &= \sqrt{65} \end{aligned}$$

Now, we can evaluate the sine of the angle as the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{7}{\sqrt{65}}$$

This gives us our desired composition.

$$\begin{aligned} \sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) &= \sin\theta \\ &= \frac{7}{\sqrt{65}} \\ &= \frac{7\sqrt{65}}{65} \end{aligned}$$

? Exercise $Page 6.10.8$

Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

Answer

$\frac{4\sqrt{2}}{9}$

✓ Example $Page 6.10.9$: Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \le x \le 3$.

Solution

We know there is an angle θ such that $\sin\theta = \frac{x}{3}$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 && \text{Use the Pythagorean Theorem} \\ \left(\frac{x}{3}\right)^2 + \cos^2\theta &= 1 && \text{Solve for cosine} \\ \cos^2\theta &= 1 - \frac{x^2}{9} \\ \cos\theta &= \pm\sqrt{\frac{9-x^2}{9}} \\ &= \pm\sqrt{\frac{9-x^2}{3}} \end{aligned}$$

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Because we know that the inverse sine must give an angle on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we can deduce that the cosine of that angle must be positive.

$cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \sqrt{\frac{9-x^2}{3}}$

? Exercise $Page 6.10.9$

Find a simplified expression for $\sin(\tan^{-1}(4x))$ for $-\frac{1}{4} \leq x \leq \frac{1}{4}$.

Answer

$$\frac{4x}{\sqrt{16x^2+1}}$$

Media

Access this online resource for additional instruction and practice with inverse trigonometric functions.

- Evaluate Expressions Involving Inverse Trigonometric Functions

Visit this website for additional practice questions from Learningpod.

Key Concepts

- An inverse function is one that "undoes" another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
- For any trigonometric function $f(x)$, if $x = f^{-1}(y)$, then $f(x) = y$. However, $f(x) = y$ only implies $x = f^{-1}(y)$ if x is in the restricted domain of f. See Example $Page 6.10.1$.
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4} = \tan^{-1}(1)$ and $\frac{\pi}{6} = \sin^{-1}(\frac{1}{2})$ .See Example $Page 6.10.2$.
- A calculator will return an angle within the restricted domain of the original trigonometric function. See Example $Page 6.10.3$.
- Inverse functions allow us to find an angle when given two sides of a right triangle. See Example $Page 6.10.4$.
- In function composition, if the inside function is an inverse trigonometric function, then there are exact expressions; for example,$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$. See Example $Page 6.10.5$.
- If the inside function is a trigonometric function, then the only possible combinations are $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ if $0 \leq x \leq \pi$ and $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$ if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. See Example $Page 6.10.6$ and Example $Page 6.10.7$.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See Example $Page 6.10.8$.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides. See Example $Page 6.10.9$.

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CHAPTER OVERVIEW

Chapter 7: Day 7 (A Preview of Differential Calculus)

Page 7.1: Finding Limits - Numerical and Graphical Approaches
Page 7.2: Finding Limits - Properties of Limits
Page 7.3: Continuity
Page 7.4: Derivatives

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Page 7.1: Finding Limits - Numerical and Graphical Approaches

Intuitively, we know what a **limit** is. A car can go only so fast and no faster. A trash can might hold 33 gallons and no more. It is natural for measured amounts to have limits. What, for instance, is the limit to the height of a woman? The tallest woman on record was Jinlian Zeng from China, who was 8 ft 1 in.[1] Is this the limit of the height to which women can grow? Perhaps not, but there is likely a limit that we might describe in inches if we were able to determine what it was.

To put it mathematically, the function whose input is a woman and whose output is a measured height in inches has a limit. In this section, we will examine numerical and graphical approaches to identifying limits.

Understanding Limit Notation

We have seen how a **sequence** can have a limit, a value that the sequence of terms moves toward as the nu mber of terms increases. For example, the terms of the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$$

gets closer and closer to 0. A sequence is one type of function, but functions that are not sequences can also have limits. We can describe the behavior of the function as the input values get close to a specific value. If the limit of a function $f(x) = L$, then as the input x gets closer and closer to a, the output y-coordinate gets closer and closer to L. We say that the output "approaches" L.

Figure $Page 7.1.1$ provides a visual representation of the mathematical concept of limit. As the input value x approaches a, the output value $f(x)$ approaches L.

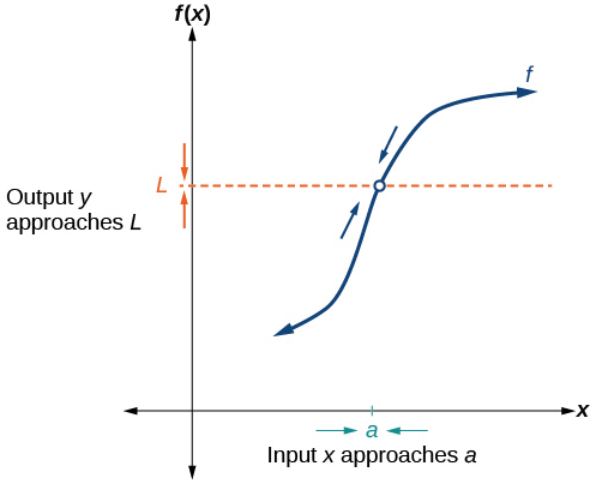


Figure $Page 7.1.1$: The output (y--coordinate) approaches L as the input (x-coordinate) approaches a.

We write the equation of a limit as

$$\lim_{x \to a} f(x) = L.$$

This notation indicates that as x approaches a both from the left of $x = a$ and the right of $x = a$, the output value approaches L.

Consider the function

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}. \qquad \text{(Page 7.1.1)}$$

We can factor the function in Equation Page 7.1.1 as shown.

$$f(x) = \frac{\cancel{(x-7)}(x+1)}{\cancel{x-7}} \qquad \text{Cancel like factors in numerator and denominator.}$$

$$f(x) = x + 1, x \neq 7 \qquad \text{Simplify.}$$

Notice that x cannot be 7, or we would be dividing by 0, so 7 is not in the domain of the function in Equation Page 7.1.1. To avoid changing the function when we simplify, we set the same condition, $x \neq 7$, for the simplified function. We can represent the function graphically as shown in Figure $Page 7.1.2$.

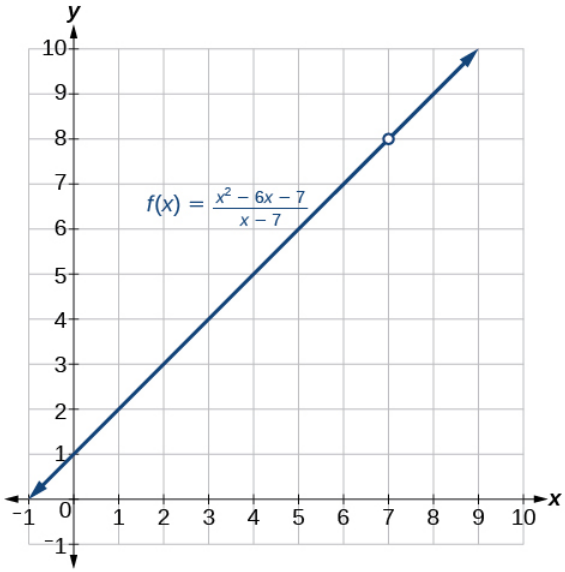


Figure $Page 7.1.2$: Because 7 is not allowed as an input, there is no point at $x = 7$.

What happens at $x = 7$ is completely different from what happens at points close to $x = 7$ on either side. The notation

$$\lim_{x \to 7} f(x) = 8$$

indicates that as the input x approaches 7 from either the left or the right, the output approaches 8. The output can get as close to 8 as we like if the input is sufficiently near 7.

What happens at $x = 7$? When $x = 7$, there is no corresponding output. We write this as

$$f(7) \text{does not exist.}$$

This notation indicates that 7 is not in the domain of the function. We had already indicated this when we wrote the function as

$$f(x) = x + 1, x \neq 7.$$

Notice that the limit of a function can exist even when $f(x)$ is not defined at $x = a$. Much of our subsequent work will be determining limits of functions as x nears a, even though the output at $x = a$ does not exist.

THE LIMIT OF A FUNCTION

A quantity L is the limit of a function $f(x)$ as x approaches a if, as the input values of x approach a (but do not equal a),the corresponding output values of $f(x)$ get closer to L. Note that the value of the limit is not affected by the output value of $f(x)$ at a. Both a and L must be real numbers. We write it as

$$\lim_{x \to a} f(x) = L$$

Example $Page 7.1.1$: Understanding the Limit of a Function

For the following limit, define a, $f(x)$, and L.

$$\lim_{x \to 2}(3x + 5) = 11$$

Solution

First, we recognize the notation of a limit. If the limit exists, as x approaches a, we write

$$\lim_{x \to a} f(x) = L.$$

We are given

$$\lim_{x \to 2}(3x + 5) = 11.$$

This means that $a = 2, f(x) = 3x + 5,$ and $L = 11.$

Analysis

Recall that $y = 3x + 5$ is a line with no breaks. As the input values approach 2, the output values will get close to 11. This may be phrased with the equation $\lim_{x \to 2}(3x + 5) = 11,$ which means that as x nears 2 (but is not exactly 2), the output of the function $f(x) = 3x + 5$ gets as close as we want to $3(2) + 5,$ or 11, which is the limit L, as we take values of x sufficiently near 2 but not at $x = 2.$

Exercise $Page 7.1.1$

For the following limit, define a, $f(x)$, and L.

$$\lim_{x \to 5}(2x^2 - 4) = 46$$

Solution

$a = 5, f(x) = 2x^2 - 4,$ and $L = 46.$

Understanding Left-Hand Limits and Right-Hand Limits

We can approach the input of a function from either side of a value—from the left or the right. Figure $Page 7.1.3$ shows the values of

$$f(x) = x + 1, x \neq 7$$

as described earlier and depicted in Figure $Page 7.1.3$.

	Values of x approach 7 from the left ($x < 7$)			$x = 7$	Values of x approach 7 from the right ($x > 7$)		
x	6.9	6.99	6.999	7	7.001	7.01	7.1
$f(x)$	7.9	7.99	7.999	Undefined	8.001	8.01	8.1
	Values of output approach the limit, 8				Values of output approach the limit, 8		

Figure $Page 7.1.3$ are $6.9, 6.99,$ and 6.999. The corresponding outputs are $7.9, 7.99,$ and 7.999. These values are getting closer to 8. The limit of values of $f(x)$ as x approaches from the left is known as the left-hand limit. For this function, 8 is the left-hand limit of the function $f(x) = x + 1, x \neq 7$ as x approaches 7.

Values described as "from the right" are greater than the input value 7 and would therefore appear to the right of the value on a number line. The input values that approach 7 from the right in Figure $Page 7.1.3$ are $7.1, 7.01,$ and 7.001. The corresponding outputs are $8.1, 8.01,$ and 8.001. These values are getting closer to 8. The limit of values of $f(x)$ as x approaches from the right is known as the right-hand limit. For this function, 8 is also the right-hand limit of the function $f(x) = x + 1, x \neq 7$ as x approaches 7.

Figure $Page 7.1.3$ shows that we can get the output of the function within a distance of 0.1 from 8 by using an input within a distance of 0.1 from 7. In other words, we need an input x within the interval $6.9 < x < 7.1$ to produce an output value of $f(x)$ within the interval $7.9 < f(x) < 8.1$.

We also see that we can get output values of $f(x)$ successively closer to 8 by selecting input values closer to 7. In fact, we can obtain output values within any specified interval if we choose appropriate input values.

Figure $Page 7.1.4$ provides a visual representation of the left- and right-hand limits of the function. From the graph of $f(x)$, we observe the output can get infinitesimally close to $L = 8$ as x approaches 7 from the left and as x approaches 7 from the right.

To indicate the left-hand limit, we write

$$\lim_{x \to 7^-} f(x) = 8.$$

To indicate the right-hand limit, we write

$$\lim_{x \to 7^+} f(x) = 8.$$

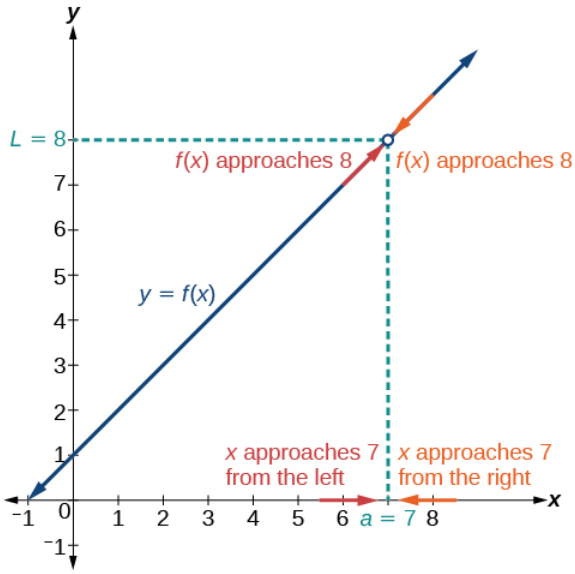


Figure $Page 7.1.4$: The left- and right-hand limits are the same for this function.

LEFT- AND RIGHT-HAND LIMITS

The left-hand limit of a function $f(x)$ as x approaches a from the left is equal to L, denoted by

$$\lim_{x \to a^-} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a such that $x < a$ and $x \neq a$.

The right-hand limit of a function $f(x)$, as x approaches a from the right, is equal to L, denoted by

$$\lim_{x \to a^+} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a but greater than a. Both a and L are real numbers.

Understanding Two-Sided Limits

In the previous example, the left-hand limit and right-hand limit as x approaches a are equal. If the left- and right-hand limits are equal, we say that the function $f(x)$ has a two-sided limit as x approaches a. More commonly, we simply refer to a **two-sided limit** as a limit. If the left-hand limit does not equal the right-hand limit, or if one of them does not exist, we say the limit does not exist.

A

The limit of a function $f(x)$, as x approaches a, is equal to L, that is,

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

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In other words, the left-hand limit of a function $f(x)$ as x approaches a is equal to the right-hand limit of the same function as x approaches a. If such a limit exists, we refer to the limit as a two-sided limit. Otherwise we say the limit does not exist.

Finding a Limit Using a Graph

To visually determine if a limit exists as x approaches a, we observe the graph of the function when x is very near to $x = a$. In Figure $Page 7.1.5$ we observe the behavior of the graph on both sides of a.

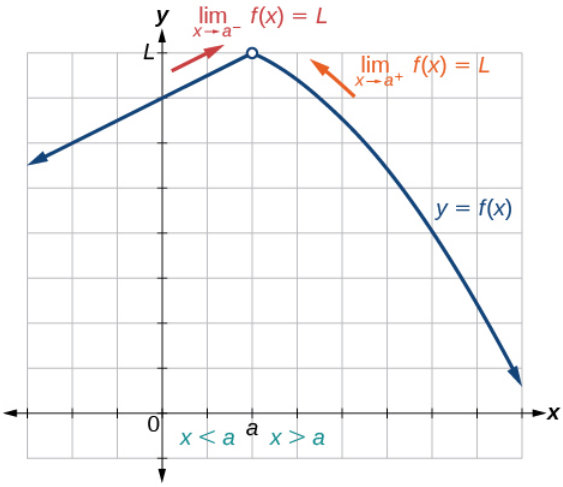


Figure $Page 7.1.5$

To determine if a left-hand limit exists, we observe the branch of the graph to the left of $x = a$, but near $x = a$. This is where $x < a$. We see that the outputs are getting close to some real number L so there is a left-hand limit.

To determine if a right-hand limit exists, observe the branch of the graph to the right of $x = a$, but near $x = a$. This is where $x > a$. We see that the outputs are getting close to some real number L, so there is a right-hand limit.

If the left-hand limit and the right-hand limit are the same, as they are in Figure $Page 7.1.5$, then we know that the function has a two-sided limit. Normally, when we refer to a "limit," we mean a two-sided limit, unless we call it a one-sided limit.

Finally, we can look for an output value for the function $f(x)$ when the input value x is equal to a. The coordinate pair of the point would be $(a, f(a))$. If such a point exists, then $f(a)$ has a value. If the point does not exist, as in Figure $Page 7.1.5$, then we say that $f(a)$ does not exist.

HOW TO: Given a function $f(x)$, use a graph to find the limits and a function value as x approaches a.

1. Examine the graph to determine whether a left-hand limit exists.
2. Examine the graph to determine whether a right-hand limit exists.
3. If the two one-sided limits exist and are equal, then there is a two-sided limit—what we normally call a "limit."
4. If there is a point at $x = a$, then $f(a)$ is the corresponding function value.

Example $Page 7.1.2$: Finding a Limit Using a Graph

Determine the following limits and function value for the function f shown in Figure $Page 7.1.6$.

a. $\lim_{x \to 2^-} f(x)$
b. $\lim_{x \to 2^+} f(x)$
c. $\lim_{x \to 2} f(x)$
d. $f(2)$

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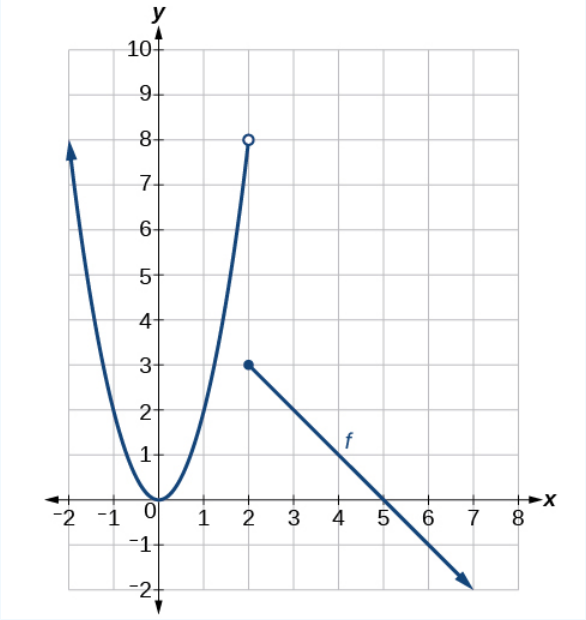


Figure $Page 7.1.6$

Determine the following limits and function value for the function f f shown in Figure.

i. $\lim_{x\to 2^-} f(x)$
ii. $\lim_{x\to 2^+} f(x)$
iii. $\lim_{x\to 2} f(x)$
iv. $f(2)$

a. Looking at Figure:
   i. $\lim_{x\to 2^-} f(x) = 8$; when $x < 2$,but infinitesimally close to 2, the output values get close to $y = 8$.
   ii. $\lim_{x\to 2^+} f(x) = 3$; when $x > 2$,but infinitesimally close to 2, the output values approach $y = 3$.
   iii. $\lim_{x\to 2} f(x)$ does not exist because $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$; the left and right-hand limits are not equal.
   iv. $f(2) = 3$ because the graph of the function f passes through the point $(2, f(2))$ or $(2, 3)$.

b. Looking at Figure:
   i. $\lim_{x\to 2^-} f(x) = 8$; when $x < 2$ but infinitesimally close to 2, the output values approach $y = 8$.
   ii. $\lim_{x\to 2^+} f(x) = 8$; when $x > 2$ but infinitesimally close to 2, the output values approach $y = 8$.
   iii. $\lim_{x\to 2} f(x) = 8$; because $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = 8;$ the left and right-hand limits are equal.
   iv. $f(2) = 4$ because the graph of the function f passes through the point $(2, f(2))$ or $(2, 4)$.

Exercise $Page 7.1.2$:

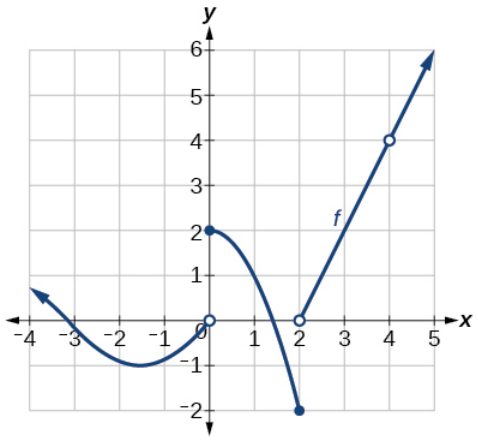
Using the graph of the function y=f(x) y=f(x) shown in Figure, estimate the following limits.

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a. $\lim_{x \to 0^-} f(x)$ b. $\lim_{x \to 0^+} f(x)$ c. $\lim_{x \to 0} f(x)$

d. $\lim_{x \to 2^-} f(x)$ e. $\lim_{x \to 2^+} f(x)$ f. $\lim_{x \to 2} f(x)$

g. $\lim_{x \to 4^-} f(x)$ h. $\lim_{x \to 4^+} f(x)$ i. $\lim_{x \to 4} f(x)$



Solution

a. 0; b. 2; c. does not exist; d.−2; −2; e. 0; f. does not exist; g. 4; h. 4; i. 4

Finding a Limit Using a Table

Creating a table is a way to determine limits using numeric information. We create a table of values in which the input values of x approach a from both sides. Then we determine if the output values get closer and closer to some real value, the limit L.

Let's consider an example using the following function:

$$\lim_{x \to 5} \left(\frac{x^3 - 125}{x - 5} \right)$$

To create the table, we evaluate the function at values close to $x = 5$. We use some input values less than 5 and some values greater than 5 as in Figure. The table values show that when $x > 5$ but nearing 5, the corresponding output gets close to 75. When $x > 5$ but nearing 5, the corresponding output also gets close to 75.

x	4.9	4.99	4.999	5	5.001	5.01	5.1
f(x)	73.51	74.8501	74.985001	Undefined	75.015001	75.1501	76.51

$\lim_{x \to 5^-} f(x) = 75$ $\lim_{x \to 5^+} f(x) = 75$

Because

$$\lim_{x \to 5^-} f(x) = 75 = \lim_{x \to 5^+} f(x),$$

then

$$\lim_{x \to 5} f(x) = 75.$$

Remember that $f(5)$ does not exist.

How to: Given a function f,use a table to find the limit as x approaches a and the value of $f(a)$, if it exists.

1. Choose several input values that approach a a from both the left and right. Record them in a table.
2. Evaluate the function at each input value. Record them in the table.
3. Determine if the table values indicate a left-hand limit and a right-hand limit.
4. If the left-hand and right-hand limits exist and are equal, there is a two-sided limit.
5. Replace x with a to find the value of $f(a)$.

Example $Page7.1.3$: Finding a Limit Using a Table

Numerically estimate the limit of the following expression by setting up a table of values on both sides of the limit.

$$\lim_{x\to 0}\left(\frac{5\sin(x)}{3x}\right)$$

Solution

We can estimate the value of a limit, if it exists, by evaluating the function at values near $x=0$. We cannot find a function value for $x=0$ directly because the result would have a denominator equal to 0, and thus would be undefined.

$$f(x)=\frac{5\sin(x)}{3x}$$

We create Figure by choosing several input values close to $x=0$, with half of them less than $x=0$ and half of them greater than $x=0$. Note that we need to be sure we are using radian mode. We evaluate the function at each input value to complete the table.

The table values indicate that when $x<0$ but approaching 0, the corresponding output nears $\frac{5}{3}$.

When $x>0$ but approaching 0, the corresponding output also nears $\frac{5}{3}$.

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	1.66389	1.666639	1.666666	Undefined	1.666666	1.666639	1.66389

$\lim_{x\to 0^-} f(x)=\frac{5}{3}$ $\lim_{x\to 0^+} f(x)=\frac{5}{3}$

Because

$$\lim_{x\to 0^-} f(x)=\frac{5}{3}=\lim_{x\to 0^+} f(x),$$

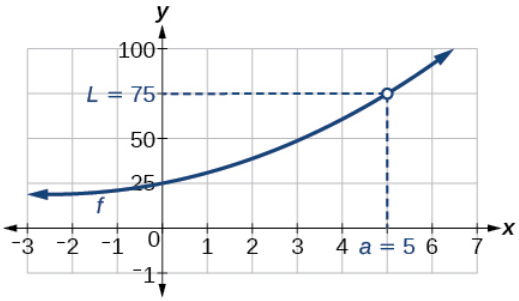
then

$$\lim_{x\to 0} f(x)=\frac{5}{3}.$$

Q & A: Is it possible to check our answer using a graphing utility?

Yes. We previously used a table to find a limit of 75 for the function $f(x)=\frac{x^3-125}{x-5}$ as x approaches 5. To check, we graph the function on a viewing window as shown in Figure. A graphical check shows both branches of the graph of the function get close to the output 75 as x nears 5. Furthermore, we can use the 'trace' feature of a graphing calculator. By appraoching $x=5$ we may numerically observe the corresponding outputs getting close to 75.

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Exercise $Page 7.1.3$:

Numerically estimate the limit of the following function by making a table:

$$\lim_{x \to 0} \left(\frac{20 \sin(x)}{4x} \right)$$

Solution

$$\lim_{x \to 0} \left(\frac{20 \sin(x)}{4x} \right) = 5$$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	4.9916708	4.9999167	4.9999992	Error	4.9999992	4.9999167	4.9916708

$\lim_{x \to 0^-} \left(\frac{20 \sin(x)}{4x} \right) \longrightarrow 5$ $\qquad$ $5 \longleftarrow \lim_{x \to 0^+} \left(\frac{20 \sin(x)}{4x} \right)$

Q & A

Is one method for determining a limit better than the other?

No. Both methods have advantages. Graphing allows for quick inspection. Tables can be used when graphical utilities aren't available, and they can be calculated to a higher precision than could be seen with an unaided eye inspecting a graph.

Example $Page 7.1.4$: Using a Graphing Utility to Determine a Limit

With the use of a graphing utility, if possible, determine the left- and right-hand limits of the following function as x approaches 0. If the function has a limit as x approaches 0, state it. If not, discuss why there is no limit.

$$f(x) = 3 \sin(\frac{\pi}{x})$$

Solution

We can use a graphing utility to investigate the behavior of the graph close to $x = 0$. Centering around $x = 0$, we choose two viewing windows such that the second one is zoomed in closer to $x = 0$ than the first one. The result would resemble Figure for $[-2, 2]$ by $[-3, 3]$.

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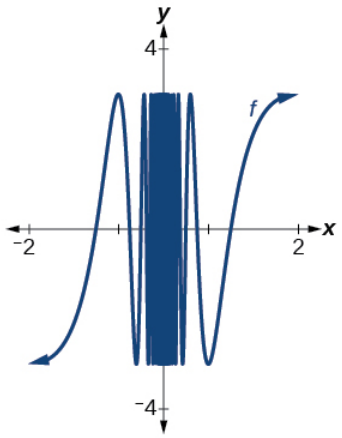


Figure_12_01_012">Figure for$[-0.1, 0.1]$ by $[-3, 3]$.

Even closer to zero, we are even less able to distinguish any limits.

The closer we get to 0, the greater the swings in the output values are. That is not the behavior of a function with either a left-hand limit or a right-hand limit. And if there is no left-hand limit or right-hand limit, there certainly is no limit to the function $f(x)$ as x approaches 0.

We write

$$\lim_{x \to 0^-} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

$$\lim_{x \to 0^+} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

$$\lim_{x \to 0} \left(3 \sin\left(\frac{\pi}{x}\right)\right) \quad \text{does not exist.}$$

Exercise $Page 7.1.4$:

Numerically estimate the following limit: $\lim_{x \to 0}(\sin(\frac{2}{x}))$.

Solution

does not exist

Access these online resources for additional instruction and practice with finding limits.

- Introduction to Limits
- Formal Definition of a Limit

Key Concepts

- A function has a limit if the output values approach some value L as the input values approach some quantity a. a. See Example.
- A shorthand notation is used to describe the limit of a function according to the form $\lim_{x \to a} f(x) = L$, which indicates that as x approaches a, both from the left of $x = a$ and the right of $x = a$, the output value gets close to L.
- A function has a left-hand limit if $f(x)$ approaches L as x approaches a a where $x < a$. A function has a right-hand limit if $f(x)$ approaches L as x approaches a where $x > a$.
- A two-sided limit exists if the left-hand limit and the right-hand limit of a function are the same. A function is said to have a limit if it has a two-sided limit.
- A graph provides a visual method of determining the limit of a function.
- If the function has a limit as x approaches a, the branches of the graph will approach the same $y-$ coordinate near $x = a$ from the left and the right. See Example.

- A table can be used to determine if a function has a limit. The table should show input values that approach a from both directions so that the resulting output values can be evaluated. If the output values approach some number, the function has a limit. See Example.
- A graphing utility can also be used to find a limit. See Example.

Footnotes

- 1 https://en.Wikipedia.org/wiki/Human_height and http://en.Wikipedia.org/wiki/List_of_tallest_people

Glossary

left-hand limit

the limit of values of $f(x)$ as x approaches from a the left, denoted

$$\lim_{x \to a^-} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a a such that $x < a$ and $x \neq a$.. Both a and L are real numbers.

limit

when it exists, the value, L,that the output of a function $f(x)$ approaches as the input x gets closer and closer to a but does not equal a. The value of the output, $f(x)$, can get as close to L as we choose to make it by using input values of x sufficiently near to $x = a$,but not necessarily at $x = a$. Both a and L are real numbers, and L is denoted

$$\lim_{x \to a} f(x) = L.$$

right-hand limit

the limit of values of $f(x)$ as x approaches a from the right, denoted

$$\lim_{x \to a^+} f(x) = L.$$

The values of $f(x)$ can get as close to the limit L as we like by taking values of x sufficiently close to a where $x > a,$ and $x \neq a.$ Both a and L are real numbers.

two-sided limit

the limit of a function

$$f(x),$$

as x approaches a,is equal to L,that is,

$$\lim_{x \to a} f(x) = L$$

if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

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Page 7.2: Finding Limits - Properties of Limits

Consider the *rational function*

$$f(x) = \frac{x^2 - 6x - 7}{x - 7}$$

The function can be factored as follows:

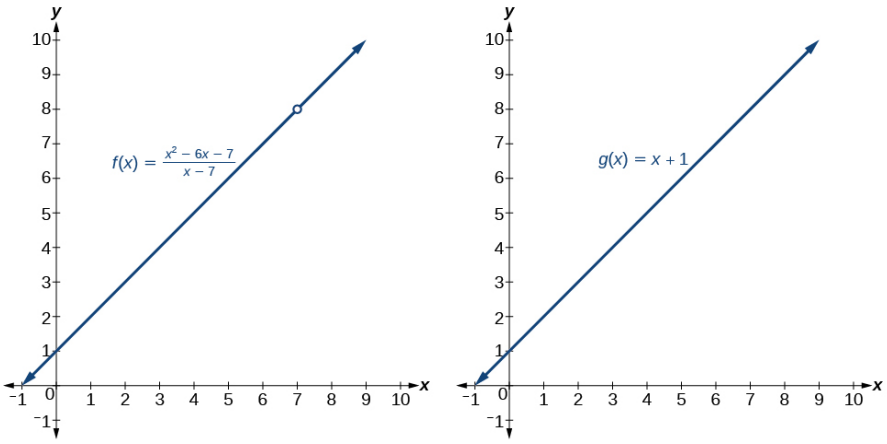
$$f(x) = \frac{\cancel{(x-7)}(x+1)}{\cancel{x-7}}$$

which gives us

$$f(x) = x + 1, x \neq 7.$$

Does this mean the function $f(x)$ is the same as the function $g(x) = x + 1$?

The answer is no. Function $f(x)$ does not have $x = 7$ in its domain, but $g(x)$ does. Graphically, we observe there is a hole in the graph of $f(x)$ at $x = 7$, as shown in Figure and no such hole in the graph of $g(x)$, as shown in Figure.



(left) The graph of function f contains a break at $x = 7$ and is therefore not continuous at $x = 7$. (Right)The graph of function g is continuous.

So, do these two different functions also have different limits as x approaches 7? Not necessarily. Remember, in determining a limit of a function as x approaches a, what matters is whether the output approaches a real number as we get close to $x = a$. The existence of a limit does not depend on what happens when x equals a.

Look again at Figure and Figure. Notice that in both graphs, as x approaches 7, the output values approach 8. This means

$$\lim_{x \to 7} f(x) = \lim_{x \to 7} g(x).$$

Remember that when determining a limit, the concern is what occurs near $x = a$, not at $x = a$. In this section, we will use a variety of methods, such as rewriting functions by factoring, to evaluate the limit. These methods will give us formal verification for what we formerly accomplished by intuition.

Finding the Limit of a Sum, a Difference, and a Product

Graphing a function or exploring a table of values to determine a limit can be cumbersome and time-consuming. When possible, it is more efficient to use the **properties of limits**, which is a collection of theorems for finding limits.

Knowing the properties of limits allows us to compute limits directly. We can add, subtract, multiply, and divide the limits of functions as if we were performing the operations on the functions themselves to find the limit of the result. Similarly, we can find the limit of a function raised to a power by raising the limit to that power. We can also find the limit of the root of a function by taking the root of the limit. Using these operations on limits, we can find the limits of more complex functions by finding the limits of their simpler component functions.

properties of limits

Let a, k, A, and B represent real numbers, and f and g be functions, such that $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$. For limits that exist and are finite, the properties of limits are summarized in Table

Constant, k	$\lim_{x \to a} k = k$
Constant times a function	$\lim_{x \to a}[k \cdot f(x)] = k \lim_{x \to a} f(x) = kA$
Sum of functions	$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{xtoa} g(x) = A + B$
Difference of functions	$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = A - B$
Product of functions	$\lim_{x \to a}[f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = A \cdot B$
Quotient of functions	$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{A}{B}, B \neq 0$

Function raised to an exponent	$\lim_{x\to a}[f(x)]^n = [\lim_{x\to\infty} f(x)]^n = A^n$, where n is a positive integer
nth root of a function, where n is a positive integer	$\lim_{x\to a} f(x)\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a}[f(x)]} = \sqrt[n]{A}$
Polynomial function	$\lim_{x\to a} p(x) = p(a)$

Example $Page 7.2.1$: Evaluating the Limit of a Function Algebraically

Evaluate

$$\lim_{x\to 3}(2x+5).$$

Solution

$$\begin{aligned}\lim_{x\to 3}(2x+5) &= \lim_{x\to 3}(2x)+\lim_{x\to 3}(5) && \text{Sum of functions property} && (\text{Page } 7.2.1)\\ &= 2\lim_{x\to 3}(x)+\lim_{x\to 3}(5) && \text{Constant times a function property} && (\text{Page } 7.2.2)\\ &= 2(3)+5 && \text{Evaluate} && (\text{Page } 7.2.3)\\ &= 11 && && (\text{Page } 7.2.4)\end{aligned}$$

Exercise $Page 7.2.1$:

Evaluate the following limit:

$$\lim_{x\to -12}(-2x+2).$$

Solution

26

Finding the Limit of a Polynomial

Not all functions or their limits involve simple addition, subtraction, or multiplication. Some may include polynomials. Recall that a polynomial is an expression consisting of the sum of two or more terms, each of which consists of a constant and a variable raised to a nonnegative integral power. To find the limit of a polynomial function, we can find the limits of the individual terms of the function, and then add them together. Also, the limit of a polynomial function as x approaches a is equivalent to simply evaluating the function for a.

how to: Given a function containing a polynomial, find its limit

1. Use the properties of limits to break up the polynomial into individual terms.
2. Find the limits of the individual terms.
3. Add the limits together.
4. Alternatively, evaluate the function for a.

Example $Page 7.2.1$: Evaluating the Limit of a Function Algebraically

Evaluate

$$\lim_{x\to 3}(5x^2).$$

Solution

$$\begin{aligned}\lim_{x\to 3}(5x^2) &= 5\lim_{x\to 3}(x^2) && \text{Constant times a function property} && (\text{Page } 7.2.5)\\ &= 5(3^2) && \text{Function raised to an exponent property} && (\text{Page } 7.2.6)\\ &= 45 && && (\text{Page } 7.2.7)\end{aligned}$$

Exercise $Page 7.2.1$:

Evaluate

$$\lim_{x\to 4}(x^3-5).$$

Solution

59

Example $Page 7.2.2$: Evaluating the Limit of a Polynomial Algebraically

Evaluate

$$\lim_{x\to 5}(2x^3-3x+1).$$

Solution

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$$\begin{aligned}\lim_{x\to 5}(2x^3-3x+1) &= \lim_{x\to 5}(2x3)-\lim_{x\to 5}(3x)+\lim_{x\to 5}(1) && \text{Sum of functions} && \text{(Page 7.2.8)}\\ &= 2\lim_{x\to 5}(x^3)-3\lim_{x\to 5}(x)+\lim_{x\to 5}(1) && \text{Constant times a function} && \text{(Page 7.2.9)}\\ &= 2(5^3)-3(5)+1 && \text{Function raised to an exponent} && \text{(Page 7.2.10)}\\ &= 236 && \text{Evaluate} && \text{(Page 7.2.11)}\end{aligned}$$

Exercise $Page 7.2.2$:

Evaluate the following limit:

$$\lim_{x\to -1}(x^4-4x^3+5).$$

Solution

10

Finding the Limit of a Power or a Root

When a limit includes a power or a root, we need another property to help us evaluate it. The square of the limit of a function equals the limit of the square of the function; the same goes for higher powers. Likewise, the square root of the limit of a function equals the limit of the square root of the function; the same holds true for higher roots.

Example $Page 7.2.3$: Evaluating a Limit of a Power

Evaluate

$$\lim_{x\to 2}(3x+1)^5.$$

Solution

We will take the limit of the function as x approaches 2 and raise the result to the 5th power.

$$\begin{aligned}\lim_{x\to 2}(3x+1)^5 &= (\lim_{x\to 2}(3x+1))^5 && \text{(Page 7.2.12)}\\ &= (3(2)+1)^5 && \text{(Page 7.2.13)}\\ &= 7^5 && \text{(Page 7.2.14)}\\ &= 16,807 && \text{(Page 7.2.15)}\end{aligned}$$

Exercise $Page 7.2.3$:

Evaluate the following limit: $\lim_{x\to -4}(10x+36)^3$.

Solution

-64

Q & A: If we can't directly apply the properties of a limit, for example in $\lim_{x\to 2}(\frac{x^2+6x+8}{x-2})$, can we still determine the limit of the function as x approaches a?

Yes. Some functions may be algebraically rearranged so that one can evaluate the limit of a simplified equivalent form of the function.

Finding the Limit of a Quotient

Finding the limit of a function expressed as a quotient can be more complicated. We often need to rewrite the function algebraically before applying the properties of a limit. If the denominator evaluates to 0 when we apply the properties of a limit directly, we must rewrite the quotient in a different form. One approach is to write the quotient in factored form and simplify.

Example $Page 7.2.4$: Evaluating the Limit of a Quotient by Factoring

Evaluate

$$\lim_{x\to 2}(\frac{x^2-6x+8}{x-2}).$$

Solution

Factor where possible, and simplify.

$$\begin{aligned}\lim_{x\to 2}(\frac{x^2-6x+8}{x-2}) &= \lim_{x\to 2}(\frac{(x-2)(x-4)}{x-2}) && \text{Factor the numerator.} && \text{(Page 7.2.16)}\\ &= \lim_{x\to 2}(\frac{\cancel{(x-2)}(x-4)}{\cancel{x-2}}) && \text{Cancel the common factors.} && \text{(Page 7.2.17)}\\ &= \lim_{x\to 2}(x-4) && \text{Evaluate.} && \text{(Page 7.2.18)}\\ &= 2-4=-2 && && \text{(Page 7.2.19)}\end{aligned}$$

Analysis

When the limit of a rational function cannot be evaluated directly, factored forms of the numerator and denominator may simplify to a result that can be evaluated.

Notice, the function

$$f(x)=\frac{x^2-6x+8}{x-2}$$

is equivalent to the function

$$f(x)=x-4, x\neq 2.$$

Notice that the limit exists even though the function is not defined at $x=2$.

Exercise $Page 7.2.4$

Evaluate the following limit:

$$\lim_{x\to 7}\left(\frac{x^2-11x+28}{7-x}\right).$$

Solution

-3

Example $Page 7.2.5$: Evaluating the Limit of a Quotient by Finding the LCD

Evaluate

$$\lim_{x\to 5}\left(\frac{\frac{1}{x}-\frac{1}{5}}{x-5}\right).$$

Solution

Find the LCD for the denominators of the two terms in the numerator, and convert both fractions to have the LCD as their denominator.

Analysis

When determining the limit of a rational function that has terms added or subtracted in either the numerator or denominator, the first step is to find the common denominator of the added or subtracted terms; then, convert both terms to have that denominator, or simplify the rational function by multiplying numerator and denominator by the least common denominator. Then check to see if the resulting numerator and denominator have any common factors.

Exercise $Page 7.2.5$:

Evaluate

$$\lim_{x\to -5}\left(\frac{\frac{1}{5}+\frac{1}{x}}{10+2x}\right).$$

Solution

$-\frac{1}{50}$

how to: Given a limit of a function containing a root, use a conjugate to evaluate

1. If the quotient as given is not in indeterminate $(\frac{0}{0})$ form, evaluate directly.
2. Otherwise, rewrite the sum (or difference) of two quotients as a single quotient, using the **least common denominator (LCD).**
3. If the numerator includes a root, rationalize the numerator; multiply the numerator and denominator by the **conjugate** of the numerator. Recall that $a\pm\sqrt{b}$ are conjugates.
4. Simplify.
5. Evaluate the resulting limit.

Example $Page 7.2.6$: Evaluating a Limit Containing a Root Using a Conjugate

Evaluate

$$\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right).$$

Solution

$$\lim_{x\to 0}\left(\frac{\sqrt{25-x}-5}{x}\right)=\lim_{x\to 0}\left(\frac{(\sqrt{25-x}-5)}{x}\cdot\frac{(\sqrt{25-x}+5)}{(\sqrt{25-x}+5)}\right)$$ Multiply numerator and denominator by the conjugate. (Page 7.2.20)

$$=\lim_{x\to 0}\left(\frac{(25-x)-25}{x(\sqrt{25-x}+5)}\right)$$ Multiply: $(\sqrt{25-x}-5)\cdot(\sqrt{25-x}+5)=(25-x)-25$. (Page 7.2.21)

$$=\lim_{x\to 0}\left(\frac{-\cancel{x}}{\cancel{x}(25-x+5)}\right)$$ Combine like terms. (Page 7.2.22)

$$=\lim_{x\to 0}\left(\frac{-\cancel{x}}{\cancel{x}(\sqrt{25-x}+5)}\right)$$ Simplify $\frac{-x}{x}=-1$. (Page 7.2.23)

$$=\frac{-1}{\sqrt{25-0}+5}$$ Evaluate. (Page 7.2.24)

$$=\frac{-1}{5+5}=-\frac{1}{10}$$ (Page 7.2.25)

Analysis

When determining a **limit** of a function with a root as one of two terms where we cannot evaluate directly, think about multiplying the numerator and denominator by the conjugate of the terms.

Exercise $Page 7.2.6$

Evaluate the following limit: $\lim_{h\to 0}\left(\frac{\sqrt{16-h}-4}{h}\right)$.

Solution

$-\frac{1}{8}$

Example $Page 7.2.7$: Evaluating the Limit of a Quotient of a Function by Factoring

Evaluate

$$\lim_{x\to 4}\left(\frac{4-x}{\sqrt{x}-2}\right).$$

Solution

$$\lim_{x\to 4}\left(\frac{4-x}{\sqrt{x}-2}\right)=\lim_{x\to 4}\left(\frac{(2+\sqrt{x})(2-x)}{\sqrt{x}-2}\right)$$ Factor. (Page 7.2.26)

$$=\lim_{x\to 4}\left(\frac{(2+\sqrt{x})(\,2-\cancel{\sqrt{x}})}{-\,(2-\cancel{\sqrt{x}})}\right)$$ Factor -1 out of the denominator. Simplify. (Page 7.2.27)

$$=\lim_{x\to 4}-(2+x)$$ Evaluate. (Page 7.2.28)

$$=-(2+\sqrt{4})$$ (Page 7.2.29)

$$=-4$$ (Page 7.2.30)

Analysis

Multiplying by a conjugate would expand the numerator; look instead for factors in the numerator. Four is a perfect square so that the numerator is in the form

$$a^2-b^2$$

and may be factored as

$$(a+b)(a-b).$$

Exercise $Page 7.2.7$

Evaluate the following limit:

$$\lim_{x\to 3}\left(\frac{x-3}{\sqrt{x}-\sqrt{3}}\right).$$

Solution

$2\sqrt{3}$

how to: Given a quotient with absolute values, evaluate its limit

1. Try factoring or finding the LCD.
2. If the **limit** cannot be found, choose several values close to and on either side of the input where the function is undefined.
3. Use the numeric evidence to estimate the limits on both sides.

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Example $Page 7.2.8$: Evaluating the Limit of a Quotient with Absolute Values

Evaluate

$$\lim_{x \to 7} \frac{|x-7|}{x-7}.$$

Solution

The function is undefined at $x = 7$, so we will try values close to 7 from the left and the right.

Left-hand limit:

$$\frac{|6.9-7|}{6.9-7} = \frac{|6.99-7|}{6.99-7} = \frac{|6.999-7|}{6.999-7} = -1$$

Right-hand limit:

$$\frac{|7.1-7|}{7.1-7} = \frac{|7.01-7|}{7.01-7} = \frac{|7.001-7|}{7.001-7} = 1$$

Since the left- and right-hand limits are not equal, there is no limit.

Exercise $Page 7.2.8$

Evaluate

$$\lim_{x \to 6^+} \frac{6-x}{|x-6|}.$$

Solution

Key Concepts

- The properties of limits can be used to perform operations on the limits of functions rather than the functions themselves. See Example.
- The limit of a polynomial function can be found by finding the sum of the limits of the individual terms. See Example and Example.
- The limit of a function that has been raised to a power equals the same power of the limit of the function. Another method is direct substitution. See Example.
- The limit of the root of a function equals the corresponding root of the limit of the function.
- One way to find the limit of a function expressed as a quotient is to write the quotient in factored form and simplify. See Example.
- Another method of finding the limit of a complex fraction is to find the LCD. See Example.
- A limit containing a function containing a root may be evaluated using a conjugate. See Example.
- The limits of some functions expressed as quotients can be found by factoring. See Example.
- One way to evaluate the limit of a quotient containing absolute values is by using numeric evidence. Setting it up piecewise can also be useful. See Example.

Glossary

properties of limits

a collection of theorems for finding limits of functions by performing mathematical operations on the limits

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Page 7.3: Continuity

Arizona is known for its dry heat. On a particular day, the temperature might rise as high as $118^\circ F$ and drop down only to a brisk $95^\circ F$. Figure $Page 7.3.1$ shows the function T, where the output of $T(x)$ is the temperature in Fahrenheit degrees and the input x is the time of day, using a 24-hour clock on a particular summer day.

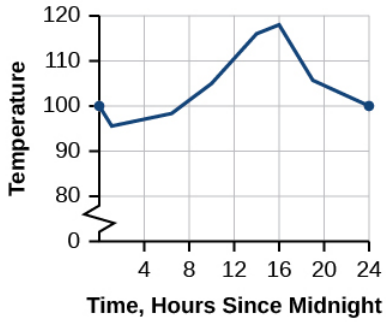


Figure $Page 7.3.1$: *Temperature as a function of time forms a continuous function.*

When we analyze this graph, we notice a specific characteristic. There are no breaks in the graph. We could trace the graph without picking up our pencil. This single observation tells us a great deal about the function. In this section, we will investigate functions with and without breaks.

Determining Whether a Function Is Continuous at a Number

Let's consider a specific example of temperature in terms of date and location, such as June 27, 2013, in Phoenix, AZ. The graph in Figure $Page 7.3.1$ indicates that, at 2 a.m., the temperature was $96^\circ F$. By 2 p.m. the temperature had risen to $116^\circ F$, and by 4 p.m. it was $118^\circ F$. Sometime between 2 a.m. and 4 p.m., the temperature outside must have been exactly $110.5^\circ F$. In fact, any temperature between $96^\circ F$ and $118^\circ F$ occurred at some point that day. This means all real numbers in the output between $96^\circ F$ and $118^\circ F$ are generated at some point by the function according to the intermediate value theorem,

Look again at Figure $Page 7.3.1$. There are no breaks in the function's graph for this 24-hour period. At no point did the temperature cease to exist, nor was there a point at which the temperature jumped instantaneously by several degrees. A function that has no holes or breaks in its graph is known as a continuous function. Temperature as a function of time is an example of a continuous function.

If temperature represents a continuous function, what kind of function would not be continuous? Consider an example of dollars expressed as a function of hours of parking. Let's create the function D, where $D(x)$ is the output representing cost in dollars for parking x number of hours (Figure $Page 7.3.2$).

Suppose a parking garage charges $4.00 per hour or fraction of an hour, with a $25 per day maximum charge. Park for two hours and five minutes and the charge is $12. Park an additional hour and the charge is $16. We can never be charged $13, $14, or $15. There are real numbers between 12 and 16 that the function never outputs. There are breaks in the function's graph for this 24-hour period, points at which the price of parking jumps instantaneously by several dollars.

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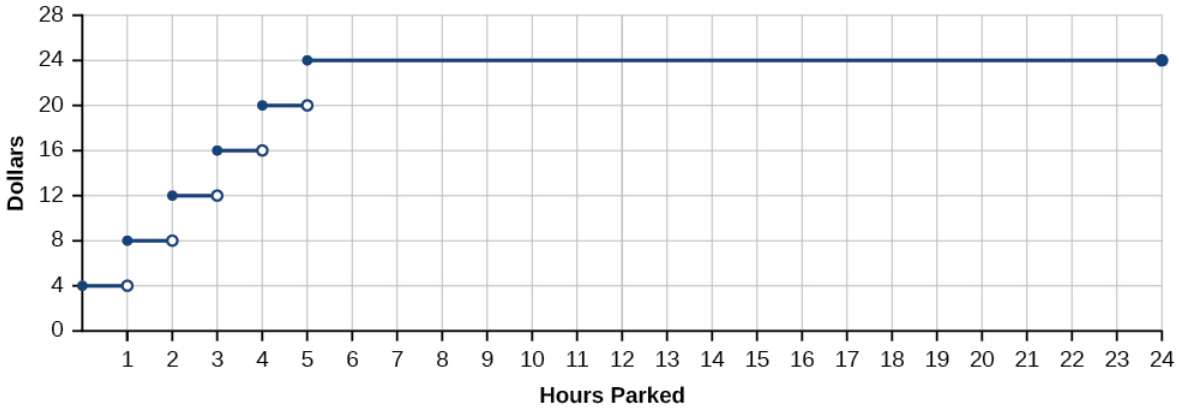


Figure $Page7.3.2$: Parking-garage charges form a discontinuous function.

A function that remains level for an interval and then jumps instantaneously to a higher value is called a **stepwise** function. This function is an example.

A function that has any hole or break in its graph is known as a **discontinuous function**. A stepwise function, such as parking-garage charges as a function of hours parked, is an example of a discontinuous function.

So how can we decide if a function is continuous at a particular number? We can check three different conditions. Let's use the function $y = f(x)$ represented in Figure as an example.

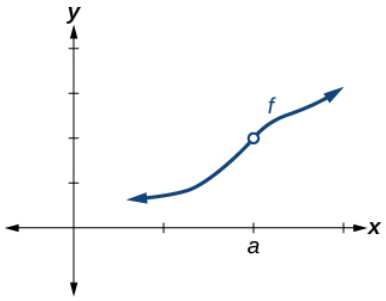


Figure_12_03_004">Figure.

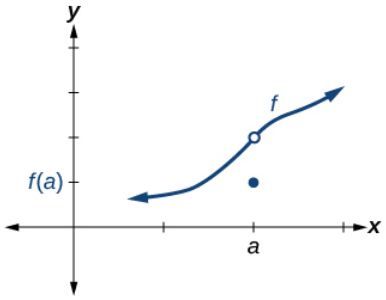


Figure_12_03_003">Figure approaches $x = a$ from the left and right, the same *y*-coordinate is approached. Therefore, Condition 2 is satisfied. However, there could still be a hole in the graph at $x = a$.

Condition 3 According to Condition 3, the corresponding y y coordinate at $x = a$ fills in the hole in the graph of f. This is written $\lim_{x \to a} f(x) = f(a)$.

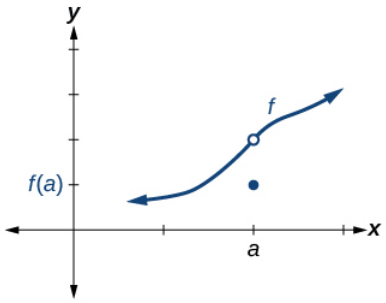
Satisfying all three conditions means that the function is continuous. All three conditions are satisfied for the function represented in Figure so the function is continuous as $x = a$.

All three conditions are satisfied. The function is continuous at $x = a$.

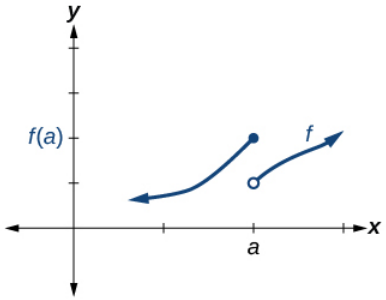
Figure through Figure provide several examples of graphs of functions that are not continuous at $x = a$ and the condition or conditions that fail.

Condition 2 is satisfied. Conditions 1 and 3 both fail.

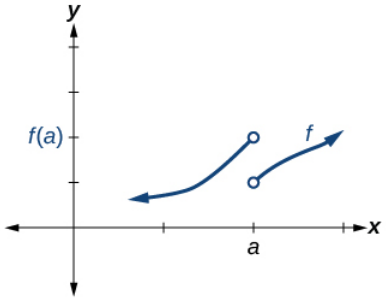
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Conditions 1 and 2 are both satisfied. Condition 3 fails.



Condition 1 is satisfied. Conditions 2 and 3 fail.



Conditions 1, 2, and 3 all fail.

Definition of continuity

A function $f(x)$ is **continuous** at $x = a$ provided all three of the following conditions hold true:

- Condition 1: $f(a)$ exists.
- Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
- Condition 3: $\lim_{x \to a} f(x) = f(a)$

If a function $f(x)$ is not continuous at $x = a$,the function is **discontinuous** at $x = a$.

Identifying a Jump Discontinuity

Discontinuity can occur in different ways. We saw in the previous section that a function could have a left-hand limit and a right-hand limit even if they are not equal. If the left- and right-hand limits exist but are different, the graph "jumps" at $x = a$. The function is said to have a jump discontinuity.

As an example, look at the graph of the function $y = f(x)$ in Figure. Notice as x approaches a how the output approaches different values from the left and from the right.

Graph of a function with a jump discontinuity.

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JUMP DISCONTINUITY

A function $f(x)$ has a jump discontinuity at $x=a$ if the left- and right-hand limits both exist but are not equal: $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$

Identifying Removable Discontinuity

Some functions have a discontinuity, but it is possible to redefine the function at that point to make it continuous. This type of function is said to have a removable discontinuity. Let's look at the function $y=f(x)$ represented by the graph in Figure. The function has a limit. However, there is a hole at $x=a$. The hole can be filled by extending the domain to include the input $x=a$ and defining the corresponding output of the function at that value as the limit of the function at $x=a$.

Graph of function f with a removable discontinuity at $x=a$.

removable discontinuity

A function f(x) f(x) has a removable discontinuity at $x=a$ if the limit, $\lim\limits_{x\to a} f(x)$, exists, but either

1. $f(a)$ does not exist *or*
2. $f(a)$, the value of the function at $x=a$ does not equal the limit, $f(a) \neq \lim\limits_{x\to a} f(x)$.

Example $Page 7.3.1$: Identifying Discontinuities

Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x)=\frac{x^2-2x-15}{x-5}$

b. $$g(x)=\begin{cases} x+1, & x<2 \\ -x, & x\geq 2 \end{cases} \qquad (\text{Page } 7.3.1)$$

---

a. Notice that the function is defined everywhere except at $x=5$.

Thus, $f(5)$ does not exist, Condition 2 is not satisfied. Since Condition 1 is satisfied, the limit as x approaches 5 is 8, and Condition 2 is not satisfied.This means there is a removable discontinuity at $x=5$.

b. Condition 2 is satisfied because $g(2)=-2$.

Notice that the function is a **piecewise function**, and for each piece, the function is defined everywhere on its domain. Let's examine Condition 1 by determining the left- and right-hand limits as x approaches 2.

Left-hand limit: $\lim\limits_{x\to 2^-}(x+1)=2+1=3$. The left-hand limit exists.

Right-hand limit: $\lim\limits_{x\to 2^+}(-x)=-2$. The right-hand limit exists. But

$$\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x). \qquad (\text{Page } 7.3.2)$$

So, $\lim\limits_{x\to 2} f(x)$ does not exist, and Condition 2 fails: There is no removable discontinuity. However, since both left- and right-hand limits exist but are not equal, the conditions are satisfied for a jump discontinuity at $x=2$.

Exercise $Page 7.3.1$:

Identify all discontinuities for the following functions as either a jump or a removable discontinuity.

a. $f(x)=\frac{x^2-6x}{x-6}$

b. $g(x)=\begin{cases} \sqrt{x}, & 0\leq x<4 \\ 2x, & x\geq 4 \end{cases}$

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a. removable discontinuity at $x = 6$;
b. jump discontinuity at $x = 4$

Recognizing Continuous and Discontinuous Real-Number Functions

Many of the functions we have encountered in earlier chapters are continuous everywhere. They never have a hole in them, and they never jump from one value to the next. For all of these functions, the limit of $f(x)$ as x approaches a a is the same as the value of $f(x)$ when $x = a$. So $\lim_{x \to a} f(x) = f(a)$. There are some functions that are continuous everywhere and some that are only continuous where they are defined on their domain because they are not defined for all real numbers.

EXAMPLES OF CONTINUOUS FUNCTIONS

The following functions are continuous everywhere:

Polynomial functions	Ex: $f(x) = x^4 - 9x^2$
Exponential functions	Ex: $f(x) = 4^{x+2} - 5$
Sine functions	Ex: $f(x) = \sin(2x) - 4$
Cosine functions	Ex: $f(x) = -\cos(x + \frac{\pi}{3})$

The following functions are continuous everywhere they are defined on their domain:

Logarithmic functions	Ex: $f(x) = 2\ln(x), x > 0$
Tangent functions	Ex: $f(x) = \tan(x) + 2, x \neq \frac{\pi}{2} + k\pi, k$ is an integer
Rational functions	Ex: $f(x) = \frac{x^2 - 25}{x - 7}, x \neq 7$

how to:Given a function $f(x)$, determine if the function is continuous at $x = a$.

1. Check Condition 1: $f(a)$ exists.
2. Check Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
3. Check Condition 3: $\lim_{x \to a} f(x) = f(a)$.
4. If all three conditions are satisfied, the function is continuous at $x = a$. If any one of the conditions is not satisfied, the function is not continuous at $x = a$.

Example $Page 7.3.2$: Determining Whether a Piecewise Function is Continuous at a Given Number

Determine whether the function $f(x) = \begin{cases} 4x, & x \leq 3 \\ 8 + x, & x > 3 \end{cases}$ is continuous at

a. $x = 3$
b. $x = \frac{8}{3}$

To determine if the function f is continuous at $x = a$, we will determine if the three conditions of continuity are satisfied at $x = a$.

a. Condition 1: Does $f(a)$ exist?

$$f(3) = 4(3) = 12 \quad \text{(Page 7.3.3)}$$
$$\Rightarrow \text{Condition 1 is satisfied.} \quad \text{(Page 7.3.4)}$$

Condition 2: Does $\lim_{x \to 3} f(x)$ exist?

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To the left of $x=3$, $f(x)=4x$; to the right of $x=3$, $f(x)=8+x$. We need to evaluate the left- and right-hand limits as x approaches 1.

- Left-hand limit: $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-} 4(3)=12$
- Right-hand limit: $\lim\limits_{x\to 3^+} f(x)=\lim\limits_{x\to 3^+}(8+x)=8+3=11$

Because $\lim\limits_{x\to 1^-} f(x)\neq \lim\limits_{x\to 1^+} f(x)$, $\lim\limits_{x\to 1} f(x)$ does not exist.

$$\Rightarrow \text{Condition 2 fails.} \qquad \text{(Page 7.3.5)}$$

There is no need to proceed further. Condition 2 fails at $x=3$. If any of the conditions of continuity are not satisfied at $x=3$, the function $f(x)$ is not continuous at $x=3$.

b. $x=\frac{8}{3}$

Condition 1: Does $f(\frac{8}{3})$ exist?

$$f(\frac{8}{3})=4(\frac{8}{3})=\frac{32}{3} \qquad \text{(Page 7.3.6)}$$

$$\Rightarrow \text{Condition 1 is satisfied.} \qquad \text{(Page 7.3.7)}$$

Condition 2: Does $\lim\limits_{x\to\frac{8}{3}} f(x)$ exist?

To the left of $x=\frac{8}{3}$, $f(x)=4x$; to the right of $x=\frac{8}{3}$, $f(x)=8+x$. We need to evaluate the left- and right-hand limits as x approaches $\frac{8}{3}$.

- Left-hand limit: $\lim\limits_{x\to\frac{8}{3}^-} f(x)=\lim\limits_{x\to\frac{8}{3}^-} 4(\frac{8}{3})=\frac{32}{3}$
- Right-hand limit: $\lim\limits_{x\to\frac{8}{3}^+} f(x)=\lim\limits_{x\to\frac{8}{3}^+}(8+x)=8+\frac{8}{3}=\frac{32}{3}$

Because $\lim\limits_{x\to\frac{8}{3}} f(x)$ exists,

$$\Rightarrow \text{Condition 2 is satisfied.} \qquad \text{(Page 7.3.8)}$$

Condition 3: Is $f(\frac{8}{3})=\lim\limits_{x\to\frac{8}{3}} f(x)$?

$$f(\frac{32}{3})=\frac{32}{3}=\lim_{x\to\frac{8}{3}} f(x) \qquad \text{(Page 7.3.9)}$$

$$\Rightarrow \text{Condition 3 is satisfied.} \qquad \text{(Page 7.3.10)}$$

Because all three conditions of continuity are satisfied at $x=\frac{8}{3}$, the function $f(x)$ is continuous at $x=\frac{8}{3}$.

Exercise $Page 7.3.2$:

Determine whether the function $f(x)=\begin{cases}\frac{1}{x}, & x\leq 2\\ 9x-11.5, & x>2\end{cases}$ is continuous at $x=2$.

---

yes

Example $Page 7.3.3$: Determining Whether a Rational Function is Continuous at a Given Number

Determine whether the function $f(x)=\frac{x^2-25}{x-5}$ is continuous at $x=5$.

To determine if the function f is continuous at $x=5$, we will determine if the three conditions of continuity are satisfied at $x=5$.

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Condition 1:

$$f(5) \text{ does not exist.} \quad \text{(Page 7.3.11)}$$
$$\Rightarrow \text{Condition 1 fails.} \quad \text{(Page 7.3.12)}$$

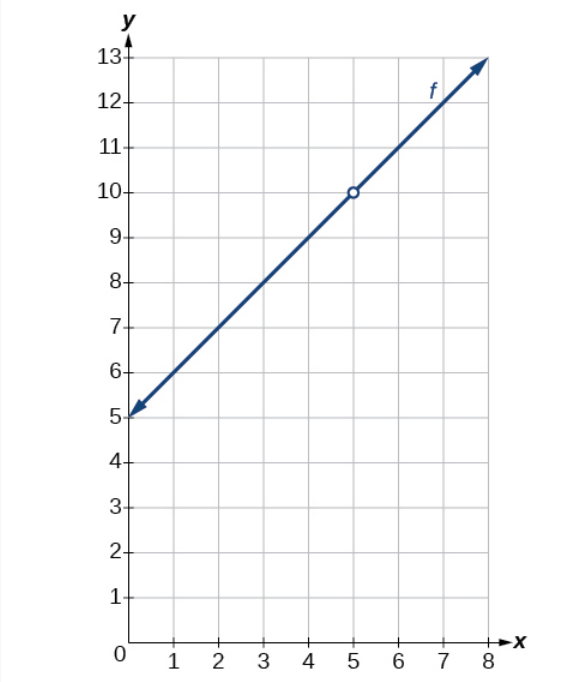
There is no need to proceed further. Condition 2 fails at $x = 5$. If any of the conditions of continuity are not satisfied at $x = 5$,the function f f is not continuous at $x = 5$.

Analysis

See Figure. Notice that for Condition 2 we have

$$\lim_{x\to 5} \frac{x^2 - 25}{x - 5} = \lim_{x\to 3} \frac{\cancel{(x-5)}(x+5)}{\cancel{x-5}} \quad \text{(Page 7.3.13)}$$
$$= \lim_{x\to 5}(x+5) \quad \text{(Page 7.3.14)}$$
$$= 5 + 5 = 10 \quad \text{(Page 7.3.15)}$$
$$\Rightarrow \text{Condition 2 is satisfied.} \quad \text{(Page 7.3.16)}$$

At x=5, x=5,there exists a removable discontinuity. See Figure.



Exercise $Page 7.3.3$:

Determine whether the function $f(x) = \frac{9-x^2}{x^2-3x}$ is continuous at $x = 3$. If not, state the type of discontinuity.

No, the function is not continuous at $x = 3$. There exists a removable discontinuity at $x = 3$.

Determining the Input Values for Which a Function Is Discontinuous

Now that we can identify continuous functions, jump discontinuities, and removable discontinuities, we will look at more complex functions to find discontinuities. Here, we will analyze a piecewise function to determine if any real numbers exist where the function is not continuous. A **piecewise function** may have discontinuities at the boundary points of the function as well as within the functions that make it up.

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To determine the real numbers for which a piecewise function composed of **polynomial** functions is not continuous, recall that polynomial functions themselves are continuous on the set of real numbers. Any discontinuity would be at the boundary points. So we need to explore the three conditions of continuity at the boundary points of the piecewise function.

how to: Given a piecewise function, determine whether it is continuous at the boundary points

1. For each boundary point a of the piecewise function, determine the left- and right-hand limits as x approaches a, as well as the function value at a.
2. Check each condition for each value to determine if all three conditions are satisfied.
3. Determine whether each value satisfies condition 1: $f(a)$ exists.
4. Determine whether each value satisfies condition 2: $\lim_{x\to a} f(x)$ exists.
5. Determine whether each value satisfies condition 3: $\lim_{x\to a} f(x) = f(a)$.
6. If all three conditions are satisfied, the function is continuous at $x = a$. If any one of the conditions fails, the function is not continuous at $x = a$.

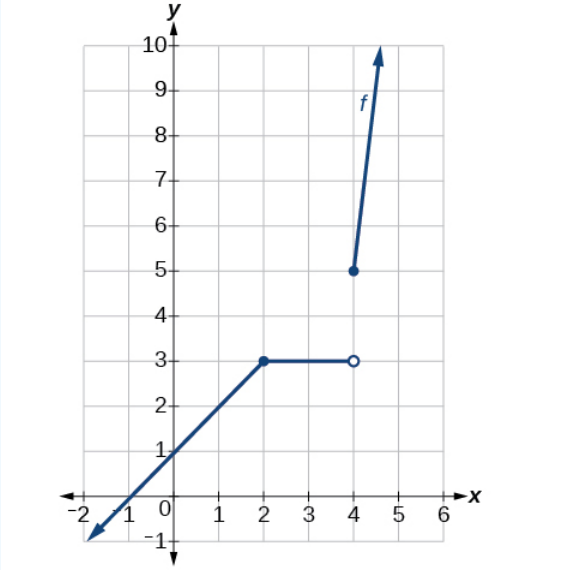
Example $Page 7.3.4$: Determining the Input Values for Which a Piecewise Function Is Discontinuous

Determine whether the function f f is discontinuous for any real numbers.

$$fx = \begin{cases} x+1, & x < 2 \\ 3, & 2 \le x < 4 \\ x^2 - 11, & x \ge 4 \end{cases} \qquad \text{(Page 7.3.17)}$$

Analysis

See Figure. At $x = 4$,there exists a jump discontinuity. Notice that the function is continuous at $x = 2$.



Graph is continuous at $x = 2$ but shows a jump discontinuity at $x = 4$.

Exercise $Page 7.3.4$:

Determine where the function $f(x) = \begin{cases} \frac{\pi x}{4}, & x < 2 \\ \frac{\pi}{x}, & 2 \le x \le 6 \\ 2\pi x, & x > 6 \end{cases}$ is discontinuous.

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$x = 6$

Determining Whether a Function Is Continuous

To determine whether a **piecewise function** is continuous or discontinuous, in addition to checking the boundary points, we must also check whether each of the functions that make up the piecewise function is continuous.

how to: Given a piecewise function, determine whether it is continuous.

1. Determine whether each component function of the piecewise function is continuous. If there are discontinuities, do they occur within the domain where that component function is applied?
2. For each boundary point $x = a$ of the piecewise function, determine if each of the three conditions hold.

Example $Page 7.3.5$: Determining Whether a Piecewise Function Is Continuous

Determine whether the function below is continuous. If it is not, state the location and type of each discontinuity.

$$fx = \begin{cases} \sin(x), & x < 0 \\ x^3, & x > 0 \end{cases} \quad \text{(Page 7.3.18)}$$

The two functions composing this piecewise function are $f(x) = \sin(x)$ on $x < 0$ and $f(x) = x^3$ on $x > 0$. The sine function and all polynomial functions are continuous everywhere. Any discontinuities would be at the boundary point,

At $x = 0$,let us check the three conditions of continuity.

Condition 1:

$$f(0) \text{ does not exist.} \quad \text{(Page 7.3.19)}$$
$$\Rightarrow \text{Condition 1 fails.} \quad \text{(Page 7.3.20)}$$

Because all three conditions are not satisfied at $x = 0$, the function $f(x)$ is discontinuous at $x = 0$.

Analysis

See Figure. There exists a removable discontinuity at $x = 0$; $\lim_{x \to 0} f(x) = 0$, thus the limit exists and is finite, but $f(a)$ does not exist.

Function has removable discontinuity at 0.

Media

Access these online resources for additional instruction and practice with continuity.

- Continuity at a Point
- Continuity at a Point: Concept Check

Key Concepts

- A continuous function can be represented by a graph without holes or breaks.
- A function whose graph has holes is a discontinuous function.
- A function is continuous at a particular number if three conditions are met:
  - Condition 1: $f(a)$ exists.
  - Condition 2: $\lim_{x \to a} f(x)$ exists at $x = a$.
  - Condition 3: $\lim_{x \to a} f(x) = f(a)$.
- A function has a jump discontinuity if the left- and right-hand limits are different, causing the graph to “jump.”
- A function has a removable discontinuity if it can be redefined at its discontinuous point to make it continuous. See Example.
- Some functions, such as polynomial functions, are continuous everywhere. Other functions, such as logarithmic functions, are continuous on their domain. See Example and Example.

- For a piecewise function to be continuous each piece must be continuous on its part of the domain and the function as a whole must be continuous at the boundaries. See Example and Example.

Glossary

continuous function

a function that has no holes or breaks in its graph

discontinuous function

a function that is not continuous at $x = a$

jump discontinuity

a point of discontinuity in a function $f(x)$ at $x = a$ where both the left and right-hand limits exist, but $\lim\limits_{x \to a^-} f(x) \neq \lim\limits_{x \to a^+} f(x)$

removable discontinuity

a point of discontinuity in a function $f(x)$ where the function is discontinuous, but can be redefined to make it continuous

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Page 7.4: Derivatives

The average teen in the United States opens a refrigerator door an estimated 25 times per day. Supposedly, this average is up from 10 years ago when the average teenager opened a refrigerator door 20 times per day[1]. It is estimated that a television is on in a home 6.75 hours per day, whereas parents spend an estimated 5.5 minutes per day having a meaningful conversation with their children. These averages, too, are not the same as they were 10 years ago, when the television was on an estimated 6 hours per day in the typical household, and parents spent 12 minutes per day in meaningful conversation with their kids. What do these scenarios have in common? The functions representing them have changed over time. In this section, we will consider methods of computing such changes over time.

Finding the Average Rate of Change of a Function

The functions describing the examples above involve a change over time. Change divided by time is one example of a rate. The rates of change in the previous examples are each different. In other words, some changed faster than others. If we were to graph the functions, we could compare the rates by determining the slopes of the graphs.

A **tangent line** to a curve is a line that intersects the curve at only a single point but does not cross it there. (The tangent line may intersect the curve at another point away from the point of interest.) If we zoom in on a curve at that point, the curve appears linear, and the **slope of the curve** at that point is close to the slope of the tangent line at that point.

Figure $Page 7.4.1$ represents the function $f(x) = x^3 - 4x$. We can see the slope at various points along the curve.

- slope at $x = -2$ is 8
- slope at $x = -1$ is –1
- slope at $x = 2$ is 8

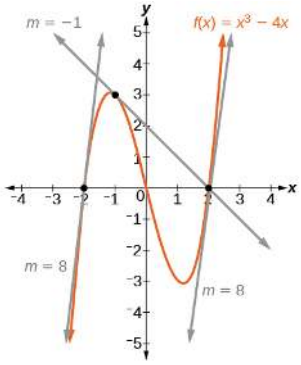


Figure $Page 7.4.1$: Graph showing tangents to curve at –2, –1, and 2.

Let's imagine a point on the curve of function f at $x = a$ as shown in Figure $Page 7.4.1$. The coordinates of the point are $(a, f(a))$. Connect this point with a second point on the curve a little to the right of $x = a$, with an x-value increased by some small real number h. The coordinates of this second point are $(a+h, f(a+h))$ for some positive-value h.

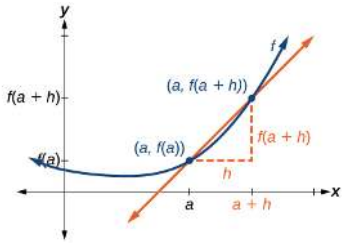


Figure $Page 7.4.2$: Connecting point a with a point just beyond allows us to measure a slope close to that of a tangent line at $x = a$.

We can calculate the slope of the line connecting the two points $(a, f(a))$ and $(a+h, f(a+h))$, called a **secant line**, by applying the slope formula,

$$\text{slope} = \frac{\text{change in y}}{\text{change in x}} \tag{Page 7.4.1}$$

We use the notation m_{sec} to represent the slope of the secant line connecting two points.

$$m_{sec} = \frac{f(a+h) - f(a)}{(a+h) - (a)} \tag{Page 7.4.2}$$

$$= \frac{f(a+h) - f(a)}{\cancel{a}+h-\cancel{a}} \tag{Page 7.4.3}$$

The slope m_{sec} equals the *average rate of change* between two points $(a, f(a))$ and $(a+h, f(a+h))$.

$$m_{sec} = \frac{f(a+h) - f(a)}{h} \tag{Page 7.4.4}$$

the AVERAGE RATE OF CHANGE BETWEEN TWO POINTS ON A CURVE

The **average rate of change** (AROC) between two points $(a, f(a))$ and $(a+h, f(a+h))$ on the curve of f is the slope of the line connecting the two points and is given by

$$\text{AROC} = \frac{f(a+h) - f(a)}{h} \tag{Page 7.4.5}$$

Example $Page 7.4.1$: Finding the Average Rate of Change

Find the average rate of change connecting the points $(2, -6)$ and $(-1, 5)$.

Solution

We know the average rate of change connecting two points may be given by

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$$\text{AROC} = \frac{f(a+h)-f(a)}{h} \tag{Page 7.4.6}$$

If one point is $(2, -6)$, or $(2, f(2))$, then $f(2) = -6$.

The value h is the displacement from 2 to -1, which equals $-1 - 2 = -3$.

For the other point, $f(a+h)$ is the y-coordinate at $a+h$, which is $2+(-3)$ or -1, so $f(a+h) = f(-1) = 5$.

$$\text{AROC} = \frac{f(a+h)-f(a)}{h} \tag{Page 7.4.7}$$

$$= \frac{5-(-6)}{-3} \tag{Page 7.4.8}$$

$$= \frac{11}{-3} \tag{Page 7.4.9}$$

$$= -\frac{11}{3} \tag{Page 7.4.10}$$

? Exercise $Page 7.4.1$

Find the average rate of change connecting the points $(-5, 1.5)$ and $(-2.5, 9)$

Solution

3

Understanding the Instantaneous Rate of Change

Now that we can find the average rate of change, suppose we make h in Figure $Page 7.4.2$ smaller and smaller. Then $a+h$ will approach a as h gets smaller, getting closer and closer to 0. Likewise, the second point $(a+h, f(a+h))$ will approach the first point, $(a, f(a))$. As a consequence, the connecting line between the two points, called the secant line, will get closer and closer to being a tangent to the function at $x = a$, and the slope of the secant line will get closer and closer to the slope of the tangent at $x = a$ (Figure $Page 7.4.3$).

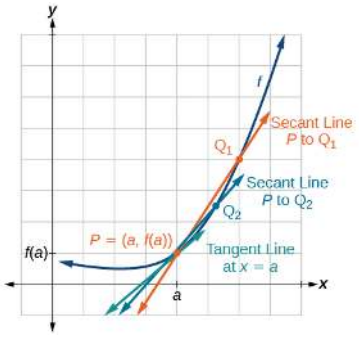


Figure $Page 7.4.3$: The connecting line between two points moves closer to being a tangent line at $x = a$.

Because we are looking for the slope of the tangent at $x = a$, we can think of the measure of the slope of the curve of a function f at a given point as the rate of change at a particular instant. We call this slope the **instantaneous rate of change**, or the **derivative** of the function at $x = a$. Both can be found by finding the limit of the slope of a line connecting the point at $x = a$ with a second point infinitesimally close along the curve. For a function f both the instantaneous rate of change of the function and the derivative of the function at $x = a$ are written as $f'(a)$, and we can define them as a two-sided limit that has the same value whether approached from the left or the right.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \tag{Page 7.4.11}$$

The expression by which the limit is found is known as the **difference quotient**.

DEFINITION OF INSTANTANEOUS RATE OF CHANGE AND DERIVATIVE

The derivative, or instantaneous rate of change, of a function f at $x = a$, is given by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} \tag{Page 7.4.12}$$

The expression $\frac{f(a+h)-f(a)}{h}$ is called the difference quotient.

We use the difference quotient to evaluate the limit of the rate of change of the function as h approaches 0.

Derivatives: Interpretations and Notation

The derivative of a function can be interpreted in different ways. It can be observed as the behavior of a graph of the function or calculated as a numerical rate of change of the function.

- The derivative of a function $f(x)$ at a point $x = a$ is the slope of the tangent line to the curve $f(x)$ at $x = a$. The derivative of $f(x)$ at $x = a$ is written $f'(a)$.
- The derivative $f'(a)$ measures how the curve changes at the point $(a, f(a))$.
- The derivative $f'(a)$ may be thought of as the instantaneous rate of change of the function $f(x)$ at $x = a$.
- If a function measures distance as a function of time, then the derivative measures the instantaneous **velocity** at time $t = a$.

NOTATIONS FOR THE DERIVATIVE

The equation of the derivative of a function $f(x)$ is written as $y' = f'(x)$, where $y = f(x)$. The notation $f'(x)$ is read as "f prime of x." Alternate notations for the derivative include the following:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) \tag{Page 7.4.13}$$

The expression $f'(x)$ is now a function of x; this function gives the slope of the curve $y = f(x)$ at any value of x. The derivative of a function $f(x)$ at a point $x = a$ is denoted $f'(a)$.

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how to: Given a function f, find the derivative by applying the definition of the derivative.

1. Calculate $f(a+h)$.
2. Calculate $f(a)$.
3. Substitute and simplify $\frac{f(a+h)-f(a)}{h}$.
4. Evaluate the limit if it exists: $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

Example $Page 7.4.1$: Finding the Derivative of a Polynomial Function

Find the derivative of the function $f(x)=x^2-3x+5$ at $x=a$.

Solution

We have:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \qquad \text{Definition of a derivative} \qquad \text{(Page 7.4.14)}$$

Substitute $f(a+h)=(a+h)^2-3(a+h)+5$ and $f(a)=a^2-3a+5$.

$$f'(a)=\lim_{h\to 0}\frac{(a+h)(a+h)-3(a+h)+5-(a^2-3a+5)}{h} \qquad \text{(Page 7.4.15)}$$

$$=\lim_{h\to 0}\frac{a^2+2ah+h^2-3a-3h+5-a^2+3a-5}{h} \qquad \text{Evaluate to remove parentheses.} \qquad \text{(Page 7.4.16)}$$

$$=\lim_{h\to 0}\frac{\cancel{a^2}+2ah+h^2-\cancel{3a}-3h+\cancel{5}-\cancel{a^2}+\cancel{3a}-\cancel{5}}{h} \qquad \text{Simplify.} \qquad \text{(Page 7.4.17)}$$

$$=\lim_{h\to 0}\frac{2ah+h^2-3h}{h} \qquad \text{Factor out an } h. \qquad \text{(Page 7.4.18)}$$

$$=2a+0-3 \qquad \text{Evaluate the limit.} \qquad \text{(Page 7.4.19)}$$

$$=2a-3 \qquad \text{(Page 7.4.20)}$$

Exercise $Page 7.4.1$

Find the derivative of the function $f(x)=3x^2+7x$ at $x=a$

Solution

$f'(a)=6a+7$

Finding Derivatives of Rational Functions

To find the derivative of a rational function, we will sometimes simplify the expression using algebraic techniques we have already learned.

Example $Page 7.4.1$: Finding the Derivative of a Rational Function

Find the derivative of the function $f(x)=\frac{3+x}{2-x}$ at $x=a$.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \qquad \text{(Page 7.4.21)}$$

$$=\lim_{h\to 0}\frac{\frac{3+(a+h)}{2-(a+h)}-(\frac{3+a}{2-a})}{h} \qquad \text{Substitute } f(a+h) \text{ and } f(a) \qquad \text{(Page 7.4.22)}$$

$$=\lim_{h\to 0}\frac{(2-(a+h))(2-a)[\frac{3+(a+h)}{2-(a+h)}-(\frac{3+a}{2-a})]}{(2-(a+h))(2-a)(h)} \qquad \text{Multiply numerator and denominator by } (2-(a+h))(2-a) \qquad \text{(Page 7.4.23)}$$

$$=\lim_{h\to 0}\frac{(\cancel{2-(a+h)})(2-a)(\frac{3+(a+h)}{\cancel{(2-(a+h))}})-(2-(a+h))(\cancel{2-a})(\frac{3+a}{\cancel{2-a}})}{(2-(a+h))(2-a)(h)} \qquad \text{Distribute} \qquad \text{(Page 7.4.24)}$$

$$=\lim_{h\to 0}\frac{6-3a+2a-a^2+2h-ah-6+3a+3h-2a+a^2+ah}{(2-(a+h))(2-a)(h)} \qquad \text{Multiply} \qquad \text{(Page 7.4.25)}$$

$$=\lim_{h\to 0}\frac{5\,\cancel{h}}{(2-(a+h))(2-a)(\cancel{h})} \qquad \text{Combine like terms} \qquad \text{(Page 7.4.26)}$$

$$=\lim_{h\to 0}\frac{5}{(2-(a+h))(2-a)} \qquad \text{Cancel like factors} \qquad \text{(Page 7.4.27)}$$

$$=\frac{5}{(2-(a+0))(2-a)}=\frac{5}{(2-a)(2-a)}=\frac{5}{(2-a)^2} \qquad \text{Evaluate the limit} \qquad \text{(Page 7.4.28)}$$

Exercise $Page 7.4.1$:

Find the derivative of the function $f(x)=\frac{10x+11}{5x+4}$ at $x=a$.

$$f'(a)=\frac{-15}{(5a+4)^2} \qquad \text{(Page 7.4.29)}$$

Finding Derivatives of Functions with Roots

To find derivatives of functions with roots, we use the methods we have learned to find limits of functions with roots, including multiplying by a conjugate.

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Example $Page 7.4.1$: Finding the Derivative of a Function with a Root

Find the derivative of the function $f(x)=4\sqrt{x}$ at $x=36$.

We have

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad \text{(Page 7.4.30)}$$

$$=\lim_{h\to 0}\frac{4\sqrt{a+h}-4\sqrt{a}}{h} \quad \text{Substitute } f(a+h) \text{ and } f(a) \quad \text{(Page 7.4.31)}$$

Multiply the numerator and denominator by the conjugate: $\frac{4\sqrt{a+h}+4\sqrt{a}}{4\sqrt{a+h}+4\sqrt{a}}$.

$$f'(a)=\lim_{h\to 0}\left(\frac{4\sqrt{a+h}-4\sqrt{a}}{h}\right)\cdot\left(\frac{4\sqrt{a+h}+4\sqrt{a}}{4\sqrt{a+h}+4\sqrt{a}}\right) \quad \text{(Page 7.4.32)}$$

$$=\lim_{h\to 0}\left(\frac{16(a+h)-16a}{h4(\sqrt{a+h}+4\sqrt{a})}\right) \quad \text{Multiply.} \quad \text{(Page 7.4.33)}$$

$$=\lim_{h\to 0}\left(\frac{\cancel{16a}+16h-\cancel{16a}}{h4(\sqrt{a+h}+4\sqrt{a})}\right) \quad \text{Distribute and combine like terms.} \quad \text{(Page 7.4.34)}$$

$$=\lim_{h\to 0}\left(\frac{16\,\cancel{h}}{\cancel{h}(4\sqrt{a+h}+4\sqrt{a})}\right) \quad \text{Simplify.} \quad \text{(Page 7.4.35)}$$

$$=\lim_{h\to 0}\left(\frac{16}{4\sqrt{a+h}+4\sqrt{a}}\right) \quad \text{Evaluate the limit by letting } h=0. \quad \text{(Page 7.4.36)}$$

$$=\frac{16}{8\sqrt{a}}=\frac{2}{\sqrt{a}} \quad \text{(Page 7.4.37)}$$

$$f'(36)=\frac{2}{\sqrt{36}} \quad \text{Evaluate the derivative at } x=36. \quad \text{(Page 7.4.38)}$$

$$=\frac{2}{6} \quad \text{(Page 7.4.39)}$$

$$=\frac{1}{3} \quad \text{(Page 7.4.40)}$$

Exercise $Page 7.4.1$:

Find the derivative of the function $f(x)=9\sqrt{x}$ at $x=9$.

$\frac{3}{2}$

Finding Instantaneous Rates of Change

Many applications of the derivative involve determining the rate of change at a given instant of a function with the independent variable time—which is why the term *instantaneous* is used. Consider the height of a ball tossed upward with an initial velocity of 64 feet per second, given by $s(t)=-16t^2+64t+6$, where t is measured in seconds and $s(t)$ is measured in feet. We know the path is that of a parabola. The derivative will tell us how the height is changing at any given point in time. The height of the ball is shown in Figure as a function of time. In physics, we call this the "*s*-*t* graph."

Example $Page 7.4.1$: Finding the Instantaneous Rate of Change

Using the function above, $s(t)=-16t^2+64t+6$,what is the instantaneous velocity of the ball at 1 second and 3 seconds into its flight?

The velocity at $t=1$ and $t=3$ is the instantaneous rate of change of distance per time, or velocity. Notice that the initial height is 6 feet. To find the instantaneous velocity, we find the **derivative** and evaluate it at $t=1$ and $t=3$:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad \text{(Page 7.4.41)}$$

$$=\lim_{h\to 0}\frac{-16(t+h)^2+64(t+h)+6-(-16t^2+64t+6)}{h} \quad \text{Substitute } s(t+h) \text{ and } s(t). \quad \text{(Page 7.4.42)}$$

$$=\lim_{h\to 0}\frac{-16t^2-32ht-h^2+64t+64h+6+16t^2-64t-6}{h} \quad \text{Distribute} \quad \text{(Page 7.4.43)}$$

$$=\lim_{h\to 0}\frac{-32ht-h^2+64h}{h} \quad \text{Simplify.} \quad \text{(Page 7.4.44)}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(-32t-h+64)}{\cancel{h}} \quad \text{Factor the numerator.} \quad \text{(Page 7.4.45)}$$

$$=\lim_{h\to 0}-32t-h+64 \quad \text{Cancel out the common factor} h. \quad \text{(Page 7.4.46)}$$

$$s'(t)=-32t+64 \quad \text{Evaluate the limit by letting} h=0. \quad \text{(Page 7.4.47)}$$

For any value of t, $s'(t)$ tells us the velocity at that value of t.

Evaluate $t=1$ and $t=3$.

$$s'(1)=-32(1)+64=32 \quad \text{(Page 7.4.48)}$$

$$s'(3)=-32(3)+64=-32 \quad \text{(Page 7.4.49)}$$

The velocity of the ball after 1 second is 32 feet per second, as it is on the way up.

The velocity of the ball after 3 seconds is −32 feet per second, as it is on the way down.

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? Exercise $Page 7.4.1$:

The position of the ball is given by $s(t) = -16t^2 + 64t + 6$. What is its velocity 2 seconds into flight?

—

0

Using Graphs to Find Instantaneous Rates of Change

We can estimate an instantaneous rate of change at $x = a$ by observing the slope of the curve of the function $f(x)$ at $x = a$. We do this by drawing a line tangent to the function at $x = a$ and finding its slope.

how to: Given a graph of a function $f(x)$, find the instantaneous rate of change of the function at $x = a$.

1. Locate $x = a$ on the graph of the function $f(x)$.
2. Draw a tangent line, a line that goes through $x = a$ at a and at no other point in that section of the curve. Extend the line far enough to calculate its slope as

$$\frac{\text{change in } y}{\text{change in } x.} \qquad (\text{Page } 7.4.50)$$

✓ Example $Page 7.4.1$: Estimating the Derivative at a Point on the Graph of a Function

From the graph of the function $y = f(x)$ presented in Figure, estimate each of the following:

$f(0); f(2); f'(0); f'(2)$

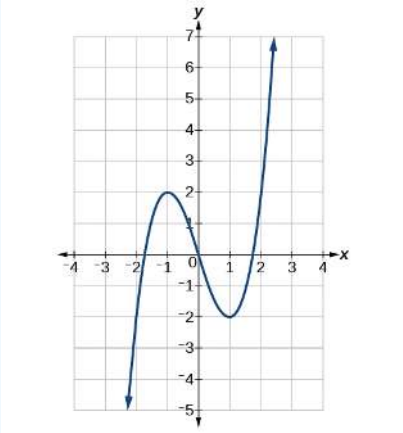
To find the functional value, $f(a)$, find the y-coordinate at $x = a$.

To find the **derivative** at $x = a$, $f\,'(a)$, draw a tangent line at $x = a$, and estimate the slope of that tangent line. See Figure.

- $f(0)$ is the y-coordinate at $x = 0$. The point has coordinates $(0, 1)$, thus $f(0) = 1$.
- $f(2)$ is the y-coordinate at $x = 2$. The point has coordinates $(2, 1)$, thus $f(2) = 1$.
- $f\,'(0)$ is found by estimating the slope of the tangent line to the curve at $x = 0$. The tangent line to the curve at $x = 0$ appears horizontal. Horizontal lines have a slope of 0, thus $f\,'(0) = 0$.
- $f\,'(2)$ is found by estimating the slope of the tangent line to the curve at $x = 2$. Observe the path of the tangent line to the curve at $x = 2$. As the x value moves one unit to the right, the y value moves up four units to another point on the line. Thus, the slope is 4, so $f\,'(2) = 4$.

? Exercise $Page 7.4.1$:

Using the graph of the function $f(x) = x^3 - 3x$ shown in Figure, estimate: $f(1), f\,'(1), f(0)$ and $f\,'(0)$.



−2,−2,0, 0, −3

Using Instantaneous Rates of Change to Solve Real-World Problems

Another way to interpret an instantaneous rate of change at $x = a$ is to observe the function in a real-world context. The unit for the derivative of a function $f(x)$ is

$$\frac{\text{output units}}{\text{input unit}} \qquad (\text{Page } 7.4.51)$$

Such a unit shows by how many units the output changes for each one-unit change of input. The instantaneous rate of change at a given instant shows the same thing: the units of change of output per one-unit change of input.

One example of an instantaneous rate of change is a marginal cost. For example, suppose the production cost for a company to produce x items is given by $C(x)$, in thousands of dollars. The derivative function tells us how the cost is changing for any value of x in the domain of the function. In other words, $C\,'(x)$ is interpreted as a **marginal cost**, the additional cost in thousands of dollars of producing one more item when x items have been produced. For example, $C\,'(11)$ is the approximate additional cost in thousands of dollars of producing the 12th item after 11 items have been produced. $C\,'(11) = 2.50$ means that when 11 items have been produced, producing the 12th item would increase the total cost by approximately $2,500.00.

✓ Example $Page 7.4.1$: Finding a Marginal Cost

The cost in dollars of producing x laptop computers in dollars is $f(x) = x^2 - 100x$. At the point where 200 computers have been produced, what is the approximate cost of producing the 201^{st}unit?

—

If $f(x) = x^2 - 100x$ describes the cost of producing x computers, $f'(x)$ will describe the marginal cost. We need to find the derivative. For purposes of calculating the derivative, we can use the following functions:

$$f(a+b) = (x+h)^2 - 100(x+h) \quad \text{(Page 7.4.52)}$$

$$f(a) = a^2 - 100a \quad \text{(Page 7.4.53)}$$

$$f'(x) = \frac{f(a+h) - f(a)}{h} \quad \text{Formula for a derivative} \quad \text{(Page 7.4.54)}$$

$$= \frac{(x+h)^2 - 100(x+h) - (x^2 - 100x)}{h} \quad \text{(Page 7.4.55)}$$

Substitute $f(a+h)$ and $f(a)$. (Page 7.4.56)

$$= \frac{x^2 + 2xh + h^2 - 100x - 100h - x^2 + 100x}{h} \quad \text{Multiply polynomials, distribute.} \quad \text{(Page 7.4.57)}$$

$$= 2\text{xh+h\^2-100h} \quad \text{Collect like terms.} \quad \text{(Page 7.4.58)}$$

$$= \frac{\cancel{h}(2x + h - 100)}{\cancel{h}} \quad \text{Factor and cancel like terms.} \quad \text{(Page 7.4.59)}$$

$$= 2x + h - 100 \quad \text{Simplify.} \quad \text{(Page 7.4.60)}$$

$$= 2x - 100 \quad \text{Evaluate when } h = 0. \quad \text{(Page 7.4.61)}$$

$$f'(x) = 2x - 100 \quad \text{Formula for marginal cost} \quad \text{(Page 7.4.62)}$$

$$f'(200) = 2(200) - 100 = 300 \quad \text{Evaluate for 200 units.} \quad \text{(Page 7.4.63)}$$

The marginal cost of producing the 201st unit will be approximately \$300.

Example *Page*7.4.1:Interpreting a Derivative in Context

A car leaves an intersection. The distance it travels in miles is given by the function $f(t)$, where t represents hours. Explain the following notations:

$f(0) = 0 f'(1) = 60 f(1) = 70 f(2.5) = 150$

First we need to evaluate the function $f(t)$ and the derivative of the function $f'(t)$, and distinguish between the two. When we evaluate the function $f(t)$, we are finding the distance the car has traveled in t hours. When we evaluate the derivative f'(t), f'(t), we are finding the speed of the car after t hours.

a. $f(0) = 0$ means that in zero hours, the car has traveled zero miles.
b. $f'(1) = 60$ means that one hour into the trip, the car is traveling 60 miles per hour.
c. $f(1) = 70$ means that one hour into the trip, the car has traveled 70 miles. At some point during the first hour, then, the car must have been traveling faster than it was at the 1-hour mark.
d. $f(2.5) = 150$ means that two hours and thirty minutes into the trip, the car has traveled 150 miles.

Exercise *Page*7.4.1

A runner runs along a straight east-west road. The function $f(t)$ gives how many feet eastward of her starting point she is after t seconds. Interpret each of the following as it relates to the runner.

$f(0) = 0; f(10) = 150; f'(10) = 15; f'(20) = -10; f(40) = -100$

---

a. After zero seconds, she has traveled 0 feet.
b. After 10 seconds, she has traveled 150 feet east.
c. After 10 seconds, she is moving eastward at a rate of 15 ft/sec.
d. After 20 seconds, she is moving westward at a rate of 10 ft/sec.
e. After 40 seconds, she is 100 feet westward of her starting point.

Finding Points Where a Function's Derivative Does Not Exist

To understand where a function's derivative does not exist, we need to recall what normally happens when a function $f(x)$ has a derivative at $x = a$. Suppose we use a graphing utility to zoom in on $x = a$. If the function $f(x)$ is differentiable, that is, if it is a function that can be differentiated, then the closer one zooms in, the more closely the graph approaches a straight line. This characteristic is called *linearity*.

Look at the graph in Figure. The closer we zoom in on the point, the more linear the curve appears.

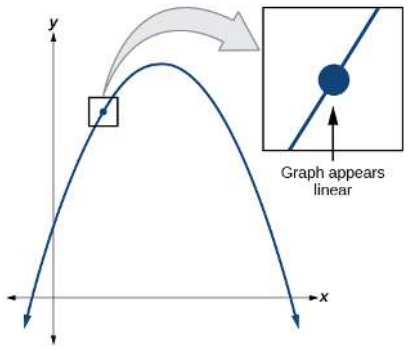


Figure_12_04_009">Figure, the graph does not approach a straight line. No matter how close we zoom in, the graph maintains its sharp corner.

Graph of the function $f(x) = |x|$,with *x*-axis from –0.1 to 0.1 and *y*-axis from –0.1 to 0.1.

What are the characteristics of a graph that is not differentiable at a point? Here are some examples in which function $f(x)$ is not differentiable at $x = a$.

In Figure, we see the graph of

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$$f(x) = \begin{cases} x^2, & x \leq 2 \\ 8 - x, & x > 2. \end{cases} . \qquad \text{(Page 7.4.64)}$$

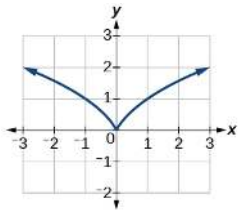
Notice that, as x approaches 2 from the left, the left-hand limit may be observed to be 4, while as x approaches 2 from the right, the right-hand limit may be observed to be 6. We see that it has a discontinuity at $x = 2$.

The graph of $f(x)$ has a discontinuity at $x = 2$.

In Figure, we see the graph of $f(x) = |x|$. We see that the graph has a corner point at $x = 0$.

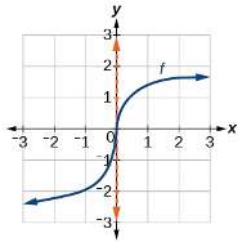
The graph of $f(x) = |x|$ has a corner point at $x = 0$.

In Figure, we see that the graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$. A cusp has a unique feature. Moving away from the cusp, both the left-hand and right-hand limits approach either infinity or negative infinity. Notice the tangent lines as x approaches 0 from both the left and the right appear to get increasingly steeper, but one has a negative slope, the other has a positive slope.



The graph of $f(x) = x^{\frac{2}{3}}$ has a cusp at $x = 0$.

In Figure, we see that the graph of $f(x) = x^{frac13}$ has a vertical tangent at $x = 0$. Recall that vertical tangents are vertical lines, so where a vertical tangent exists, the slope of the line is undefined. This is why the derivative, which measures the slope, does not exist there.



The graph of $f(x) = x^{\frac{1}{3}}$ has a vertical tangent at $x = 0$.

differentiability

A function $f(x)$ is differentiable at $x = a$ if the derivative exists at $x = a$,which means that $f'(a)$ exists.

There are four cases for which a function $f(x)$ is not differentiable at a point $x = a$.

1. When there is a discontinuity at $x = a$.
2. When there is a corner point at $x = a$.
3. When there is a cusp at $x = a$.
4. Any other time when there is a vertical tangent at $x = a$.

Example $Page7.4.1$: Determining Where a Function Is Continuous and Differentiable from a Graph

Using Figure, determine where the function is

a. continuous
b. discontinuous
c. differentiable
d. not differentiable

At the points where the graph is discontinuous or not differentiable, state why.

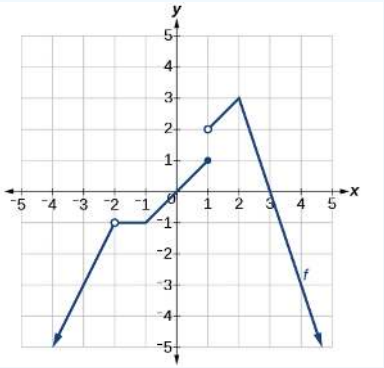


Figure_12_04_016">Figure.

Three intervals where the function is continuous

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The graph of is differentiable on $(-\infty,-2)\cup(-2,-1)\cup(-1,1)\cup(1,2)\cup(2,\infty)$. The graph of $f(x)$ is not differentiable at $x=-2$ because it is a point of discontinuity, at $x=-1$ because of a sharp corner, at $x=1$ because it is a point of discontinuity, and at $x=2$ because of a sharp corner. See Figure.

Five intervals where the function is differentiable

? Exercise $Page7.4.1$:

Determine where the function $y=f(x)$ shown in Figure is continuous and differentiable from the graph.

The graph of f is continuous on $(-\infty,1)\cup(1,3)\cup(3,\infty)$. The graph of f f is discontinuous at $x=1$ and $x=3$. The graph of f is differentiable on $(-\infty,1)\cup(1,3)\cup(3,\infty)$. The graph of f is not differentiable at $x=1$ and $x=3$.

Finding an Equation of a Line Tangent to the Graph of a Function

The equation of a tangent line to a curve of the function $f(x)$ at $x=a$ is derived from the point-slope form of a line, $y=m(x-x_1)+y_1$. The slope of the line is the slope of the curve at $x=a$ and is therefore equal to $f'(a)$, the derivative of $f(x)$ at $x=a$. The coordinate pair of the point on the line at $x=a$ is $(a,f(a))$.

If we substitute into the point-slope form, we have

The equation of the tangent line is

$$y=f'(a)(x-a)+f(a) \qquad \text{(Page 7.4.65)}$$

F

The equation of a line tangent to the curve of a function f at a point $x=a$ is

$$y=f'(a)(x-a)+f(a) \qquad \text{(Page 7.4.66)}$$

how to: Given a function f, find the equation of a line tangent to the function at $x=a$.

1. Find the derivative of $f(x)$ at $x=a$ using $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.
2. Evaluate the function at $x=a$. This is $f(a)$.
3. Substitute $(a,f(a))$ and $f'(a)$ into $y=f'(a)(x-a)+f(a)$.
4. Write the equation of the tangent line in the form $y=mx+b$.

✓ Example $Page7.4.1$: Finding the Equation of a Line Tangent to a Function at a Point

Find the equation of a line tangent to the curve $f(x)=x^2-4x$ at $x=3$.

—

Using:

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \qquad \text{(Page 7.4.67)}$$

Substitute $f(a+h)=(a+h)^2-4(a+h)$ and $f(a)=a^2-4a$.

$$f'(a)=\lim_{h\to 0}\frac{(a+h)(a+h)-4(a+h)-(a2-4a)}{h} \qquad \text{(Page 7.4.68)}$$

$$=\lim_{h\to 0}\frac{a^2+2ah+h^2-4a-4h-a^2+4a}{h} \quad \text{Remove parentheses.} \qquad \text{(Page 7.4.69)}$$

$$=\lim_{h\to 0}\frac{\cancel{a^2}+2ah+h^2-\cancel{4a}-4h-\cancel{a^2}+\cancel{4a}}{h} \quad \text{Combine like terms.} \qquad \text{(Page 7.4.70)}$$

$$=\lim_{h\to 0}\frac{2ah+h^2-4h}{h} \qquad \text{(Page 7.4.71)}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2a+h-4)}{h} \quad \text{Factor out } h. \qquad \text{(Page 7.4.72)}$$

$$=2a+0-4 \qquad \text{(Page 7.4.73)}$$

$$f'(a)=2a-4 \quad \text{Evaluate the limit.} \qquad \text{(Page 7.4.74)}$$

$$f'(3)=2(3)-4=2 \qquad \text{(Page 7.4.75)}$$

Equation of tangent line at $x=3$:

$$y=f'(a)(x-a)+f(a) \qquad \text{(Page 7.4.76)}$$

$$y=f'(3)(x-3)+f(3) \qquad \text{(Page 7.4.77)}$$

$$y=2(x-3)+(-3) \qquad \text{(Page 7.4.78)}$$

$$y=2x-9 \qquad \text{(Page 7.4.79)}$$

Analysis

We can use a graphing utility to graph the function and the tangent line. In so doing, we can observe the point of tangency at $x=3$ as shown in Figure.

Graph confirms the point of tangency at $x=3$.

? Exercise $Page7.4.1$:

Find the equation of a tangent line to the curve of the function $f(x)=5x^2-x+4$ at $x=2$.

—

$y=19x-16$

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Finding the Instantaneous Speed of a Particle

If a function measures position versus time, the derivative measures displacement versus time, or the speed of the object. A change in speed or direction relative to a change in time is known as ***velocity***. The velocity at a given instant is known as **instantaneous velocity**.

In trying to find the speed or velocity of an object at a given instant, we seem to encounter a contradiction. We normally define speed as the distance traveled divided by the elapsed time. But in an instant, no distance is traveled, and no time elapses. How will we divide zero by zero? The use of a derivative solves this problem. A derivative allows us to say that even while the object's velocity is constantly changing, it has a certain velocity at a given instant. That means that if the object traveled at that exact velocity for a unit of time, it would travel the specified distance.

INSTANTANEOUS VELOCITY

Let the function $s(t)$ represent the position of an object at time t. The **instantaneous velocity** or velocity of the object at time $t=a$ is given by

$$s'(a)=\lim_{h\to 0}\frac{s(a+h)-s(a)}{h} \qquad \text{(Page 7.4.80)}$$

Example $Page 7.4.1$: Finding the Instantaneous Velocity

A ball is tossed upward from a height of 200 feet with an initial velocity of 36 ft/sec. If the height of the ball in feet after t seconds is given by $s(t)=-16t^2+36t+200$, find the instantaneous velocity of the ball at $t=2$.

First, we must find the derivative $s'(t)$. Then we evaluate the derivative at $t=2$, using $s(a+h)=-16(a+h)^2+36(a+h)+200$ and $s(a)=-16a^2+36a+200$.

$$s'(a)=\lim_{h\to 0}\frac{s(a+h)-s(a)}{h} \qquad \text{(Page 7.4.81)}$$

$$=\lim_{h\to 0}\frac{-16(a+h)^2+36(a+h)+200-(-16a^2+36a+200)}{h} \qquad \text{(Page 7.4.82)}$$

$$=\lim_{h\to 0}\frac{-16(a^2+2ah+h^2)+36(a+h)+200-(-16a^2+36a+200)}{h} \qquad \text{(Page 7.4.83)}$$

$$=\lim_{h\to 0}\frac{-16a^2-32ah-16h^2+36a+36h+200+16a^2-36a-200}{h} \qquad \text{(Page 7.4.84)}$$

$$=\lim_{h\to 0}\frac{\cancel{-16a^2}-32ah-16h^2+\cancel{36a}+36h+\cancel{200}+\cancel{16a^2}-\cancel{36a}-\cancel{200}}{h} \qquad \text{(Page 7.4.85)}$$

$$=\lim_{h\to 0}\frac{-32ah-16h^2+36h}{h} \qquad \text{(Page 7.4.86)}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(-32a-16h+36)}{\cancel{h}} \qquad \text{(Page 7.4.87)}$$

$$=\lim_{h\to 0}(-32a-16h+36) \qquad \text{(Page 7.4.88)}$$

$$=-32a-16\cdot 0+36 \qquad \text{(Page 7.4.89)}$$

$$s'(a)=-32a+36 \qquad \text{(Page 7.4.90)}$$

$$s'(2)=-32(2)+36 \qquad \text{(Page 7.4.91)}$$

$$=-28 \qquad \text{(Page 7.4.92)}$$

Analysis

This result means that at time $t=2$ seconds, the ball is dropping at a rate of 28 ft/sec.

Exercise $Page 7.4.1$:

A fireworks rocket is shot upward out of a pit 12 ft below the ground at a velocity of 60 ft/sec. Its height in feet after t seconds is given by $s=-16t^2+60t-12$. What is its instantaneous velocity after 4 seconds?

–68 ft/sec, it is dropping back to Earth at a rate of 68 ft/s.

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CHAPTER OVERVIEW

Chapter 8: Day 8 (A Preview of Integral Calculus)

Page 8.1: Summation and Areas

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Page 8.1: Summation and Areas

Archimedes was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the **method of exhaustion**, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

Here, on our last day, we will develop techniques to approximate the area between a curve, defined by a function $f(x)$, and the x-axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Let's start by introducing some notation to make the calculations easier. We then consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

Sigma (Summation) Notation

Summations appear quiet frequently throughout calculus and so allow us to motivate this idea.

Notice that we typically never write $3+3+3+3$ since there is a more compact or shorter way to denote this: $3(4)$.

And similarly, we never write $5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5\times5$ since there is often a more efficient way to denote this: 5^{20}.

Likewise, there is a nice and compact way to denote a sum of numbers. Why is this important? As mentioned, we will use the area of known objects to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ, sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20.$$

We could probably skip writing a couple of terms and write

$$1+2+3+4+\cdots+19+20,$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i$$

which is much more compact. Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i$$

where a_i describes the terms to be added, and the i is called the $index$. Each term is evaluated, then we sum all the values, beginning with the value when $i=1$ and ending with the value when $i=n$.

Definition: Summation

Definition:

$$\sum_{i=1}^{n} a_i = a_1+a_2+a_3+\ldots+a_n$$

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For example, an expression like $\sum_{i=2}^{7} s_i$ is interpreted as $s_2+s_3+s_4+s_5+s_6+s_7$. Note that the index is used only to keep track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a **dummy variable**. We can use any letter we like for the index. Typically, mathematicians use i, j, k, m, and n for indices.

Let's try a couple of examples of using sigma notation.

? Exercise $Page 8.1.1$

Expand each summation:

1) $\sum_{i=1}^{6} i$

2) $\sum_{i=1}^{7} 2i$

3) $\sum_{i=1}^{8} i^2$

4) $\sum_{i=1}^{10} (3i+2)$

5) $\sum_{i=1}^{10} 3$

Answer

1) $\sum_{i=1}^{6} i = 1+2+3+4+5+6$

2) $\sum_{i=1}^{7} 2i = 2+4+6+8+10+12+14$

3) $\sum_{i=1}^{8} i^2 = 1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2$

4) $\sum_{i=1}^{10} (3i+2) = 5+8+11+14+17+20+23+26+29+32$

5) $\sum_{i=1}^{10} 3 = 3+3+3+3+3+3+3+3+3+3$

Notice in our definition, we said

$$\sum_{i=1}^{n} a_i = a_1+a_2+a_3+\ldots+a_n$$

and we called i our index of summation. In the above examples, our index of summation always started at 1 but this is not always the case. For instance we may be interested in studying

$$\sum_{i=2}^{6} i \qquad \text{(Page 8.1.1)}$$

which means we first substitute in when $i=2$ and we stop at $i=6$.

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Exercise $Page 8.1.1$

Expand each summation:

1) $\sum_{i=2}^{6} i$

2) $\sum_{i=3}^{7} 2i$

3) $\sum_{i=4}^{8} i^2$

4) $\sum_{i=5}^{10} (3i+2)$

5) $\sum_{i=6}^{10} 3$

Answer

1) $\sum_{i=1}^{6} i = 2+3+4+5+6$

2) $\sum_{i=3}^{7} 2i = 6+8+10+12+14$

3) $\sum_{i=4}^{8} i^2 = 4^2+5^2+6^2+7^2+8^2$

4) $\sum_{i=5}^{10} (3i+2) = 17+20+23+26+29+32$

5) $\sum_{i=6}^{10} 3 = 3+3+3+3+3$

In summary, we have the following:

Definition: Summation

Definition:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \ldots + a_n$$

Exercise $Page 8.1.1$

Expand the following:

1) $\sum_{i=1}^{5} 2^i$

2) $\sum_{j=1}^{5} 2^j$

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3) $\sum_{k=1}^{6} 2^{k-1}$

Answer

All of the summations above are equal to $2^0+2^1+2^2+2^3+2^4+2^5$. We see that there are multiple ways to denote the same sum.

Example $Page8.1.1$: Using Sigma Notation

a. Write in sigma notation and evaluate the sum of terms 3^i for $i=1,2,3,4,5$.
b. Write the sum in sigma notation:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}.$$

Answer

a. Write

$$\sum_{i=1}^{5} 3^i = 3+3^2+3^3+3^4+3^5 = 363.$$

b. The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

The properties associated with the summation process are given in the following rule.

Rule: Properties of Sigma Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \le m \le n$.

i. $\sum_{i=1}^{n} c = nc$
ii. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
iii. $\sum_{i=1}^{n} (a_i+b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
iv. $\sum_{i=1}^{n} (a_i-b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
v. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

Proof

We prove properties (ii.) and (iii.) here, and leave proof of the other properties to the Exercises.

(ii.) We have

$$\sum_{i=1}^{n} ca_i = ca_1+ca_2+ca_3+\cdots+ca_n = c(a_1+a_2+a_3+\cdots+a_n) = c\sum_{i=1}^{n} a_i.$$

(iii.) We have

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$$\sum_{i=1}^{n}(a_i+b_i) = (a_1+b_1)+(a_2+b_2)+(a_3+b_3)+\cdots+(a_n+b_n) \quad \text{(Page 8.1.2)}$$

$$= (a_1+a_2+a_3+\cdots+a_n)+(b_1+b_2+b_3+\cdots+b_n) \quad \text{(Page 8.1.3)}$$

$$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i. \quad \text{(Page 8.1.4)}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we use them in the next set of examples.

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2}. \quad \text{(Page 8.1.5)}$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}. \quad \text{(Page 8.1.6)}$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}. \quad \text{(Page 8.1.7)}$$

Example $Page 8.1.2$: Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

a. The sum of the terms $(i-3)^2$ for $i = 1, 2, \ldots, 200$.
b. The sum of the terms $(i^3 - i^2)$ for $i = 1, 2, 3, 4, 5, 6$

Solution

a. Multiplying out $(i-3)^2$, we can break the expression into three terms.

$$\begin{aligned}
\sum_{i=1}^{200}(i-3)^2 &= \sum_{i=1}^{200}(i^2-6i+9) \\
&= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \\
&= \sum_{i=1}^{200} i^2 - 6\sum_{i=1}^{200} i + \sum_{i=1}^{200} 9 \\
&= \frac{200(200+1)(400+1)}{6} - 6\left[\frac{200(200+1)}{2}\right] + 9(200) \\
&= 2,686,700 - 120,600 + 1800 \\
&= 2,567,900
\end{aligned}$$

b. Use sigma notation property iv. and the rules for the sum of squared terms and the sum of cubed terms.

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$$\begin{aligned}\sum_{i=1}^{6}(i^3-i^2) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2 \\ &= \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(2(6)+1)}{6} \\ &= \frac{1764}{4} - \frac{546}{6} \\ &= 350\end{aligned}$$

? Exercise $Page 8.1.2$

Find the sum of the values of $4+3i$ for $i = 1, 2, \ldots, 100$.

Hint

Use the properties of sigma notation to solve the problem.

Answer

$15,550$

✓ Example $Page 8.1.3$**: Finding the Sum of the Function Values**

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10$.

Solution

Using Equation Page 8.1.7, we have

$$\sum_{i=0}^{10} i^3 = \frac{(10)^2(10+1)^2}{4} = \frac{100(121)}{4} = 3025$$

? Exercise $Page 8.1.3$

Evaluate the sum indicated by the notation $\sum_{k=1}^{20}(2k+1)$.

Hint

Use the rule on sum and powers of integers (Equations Page 8.1.5-Page 8.1.7).

Answer

440

? Exercise $Page 8.1.1$

Evaluate

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i}{n}\right)^2\right]$$

Answer

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$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{n}\left(\frac{i}{n}\right)^2\right]=\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{1}{n}\frac{i^2}{n^2}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{n^3}\sum_{i=1}^{n}i^2\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{6}\frac{n(n+1)(2n+1)}{n^3}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{6}\times\frac{1}{n}\times\frac{n+1}{n}\times\frac{2n+1}{n}\right]$$

$$=\frac{1}{6}\times 1\times 1\times 2$$

$$=\frac{1}{3}$$

? Exercise $Page 8.1.1$

Evaluate

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\frac{81i^3}{n^4}-\frac{54i}{n^2}\right)\right]$$

Answer

$$\begin{aligned}\lim_{n\to\infty}\left[\sum_{i=1}^{n}\left(\frac{81i^3}{n^4}-\frac{54i}{n^2}\right)\right]&=\lim_{n\to\infty}\left[\sum_{i=1}^{n}\frac{81i^3}{n^4}-\sum_{i=1}^{n}\frac{54i}{n^2}\right]\\&=\lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n}i^3-\frac{54}{n^2}\sum_{i=1}^{n}i\right]\\&=\lim_{n\to\infty}\left[\frac{81}{n^4}\left(\frac{n(n+1)}{2}\right)^2-\frac{54}{n^2}\frac{n(n+1)}{2}\right]\\&=\lim_{n\to\infty}\left[\frac{81}{n^4}\cdot\frac{n^2(n+1)^2}{2^2}-\frac{54}{n^2}\frac{n(n+1)}{2}\right]\\&=\lim_{n\to\infty}\left[\frac{81}{4}\frac{n^2(n+1)^2}{n^4}-\frac{54}{2}\frac{n(n+1)}{n^2}\right]\\&=\lim_{n\to\infty}\left[\frac{81}{4}\frac{n\cdot n}{n}\frac{n}{n}\frac{(n+1)}{n}\cdot\frac{(n+1)}{n}-\frac{54}{2}\cdot\frac{n}{n}\frac{(n+1)}{n}\right]\\&=\frac{81}{4}-\frac{54}{2}\\&=\frac{81}{4}-\frac{108}{4}\\&=-\frac{27}{4}=-6.75\end{aligned}$$

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Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by $f(x)$ above, the x-axis below, the line $x = a$ on the left, and the line $x = b$ on the right (Figure $Page 8.1.1$).

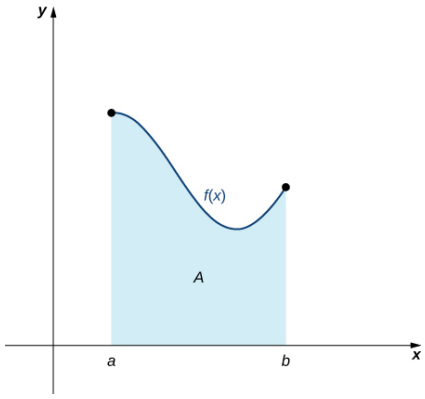


Figure $Page 8.1.1$: An area (shaded region) bounded by the curve $f(x)$ at top, the x-axis at bottom, the line $x = a$ to the left, and the line $x = b$ at right.

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\dfrac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b,$ and

$$x_i - x_{i-1} = \frac{b-a}{n}$$

for $i = 1, 2, 3, \ldots, n$.

We denote the width of each subinterval with the notation Δx, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x$$

for $i = 1, 2, 3, \ldots, n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

Definition: Partitions

A set of points $P = x_i$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** (or uniform partition) of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

Rule: Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for A (Figure $Page 8.1.2$). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

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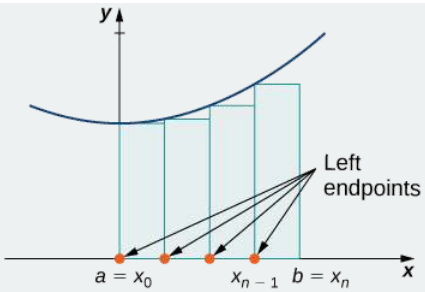


Figure $Page 8.1.2$: In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

Rule: Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\,\Delta x$ and the approximation for A is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

The notation R_n indicates this is a **right-endpoint approximation** for A (Figure $Page 8.1.3$).

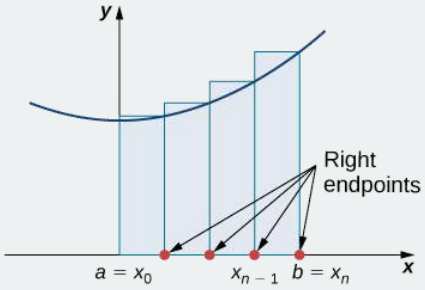


Figure $Page 8.1.3$: In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in Figure $Page 8.1.2$.

The graphs in Figure $Page 8.1.4$ represent the curve $f(x) = \dfrac{x^2}{2}$. In Figure $Page 8.1.4b$ we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width 0.5. Thus, $\Delta x = 0.5$. We then form six rectangles by drawing vertical lines perpendicular to x_{i-1}, the left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1, 2, 3, 4, 5, 6$. The intervals are $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$. We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between $f(x)$ and the x-axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation. Thus,

$$\begin{aligned} A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\ &= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375 \text{ units}^2 \end{aligned}$$

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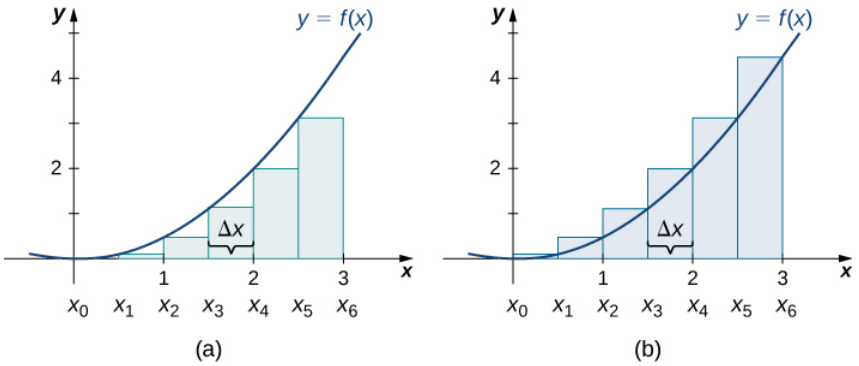


Figure $Page 8.1.4$: Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

In Figure $Page 8.1.4b$, we draw vertical lines perpendicular to x_i such that x_i is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1, 2, 3, 4, 5, 6$. We multiply each $f(x_i)$ by Δx to find the rectangular areas, and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

$$\begin{aligned} A \approx R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\ &= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\ &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875 \text{ units}^2. \end{aligned}$$

Example $Page 8.1.4$: Approximating the Area Under a Curve

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \dfrac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$ are shown in Figure $Page 8.1.5$. Using a left-endpoint approximation, the heights are $f(0) = 0$, $f(0.5) = 0.25$, $f(1) = 1$, and $f(1.5) = 2.25$. Then,

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\ &= 1.75 \text{ units}^2 \end{aligned}$$

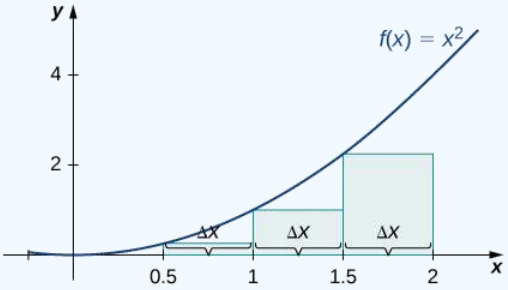


Figure $Page 8.1.5$: The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in Figure $Page 8.1.6$. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

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$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\ &= 3.75 \text{ units}^2 \end{aligned}$$

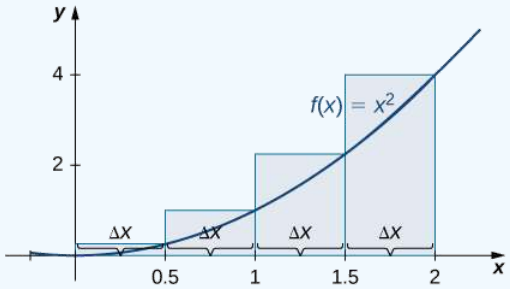


Figure $Page 8.1.6$: The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is 1.75 units^2; the right-endpoint approximation is 3.75 units^2.

? Exercise $Page 8.1.4$

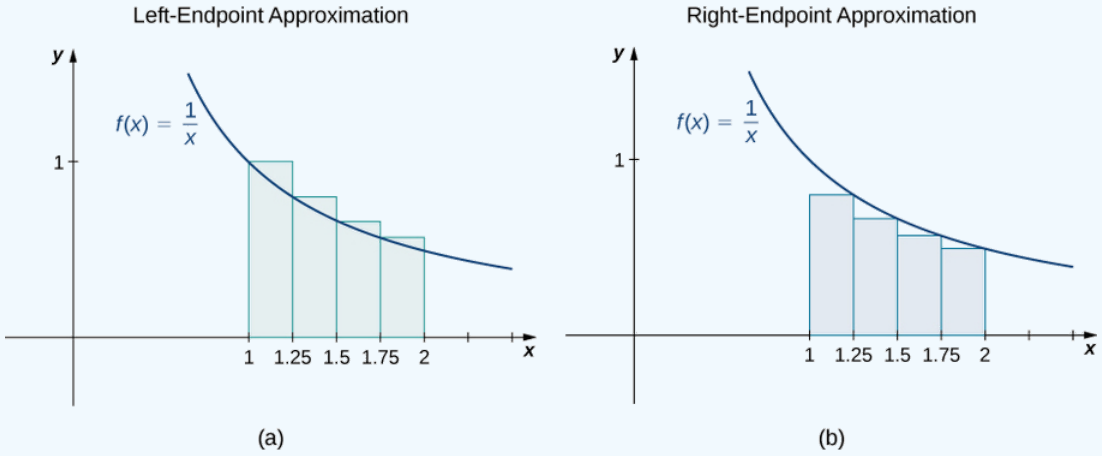
Sketch left-endpoint and right-endpoint approximations for $f(x) = \dfrac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

Hint

Follow the solving strategy in Example $Page 8.1.4$ step-by-step.

Answer

The left-endpoint approximation is 0.7595 units^2. The right-endpoint approximation is 0.6345 units^2. See the below Media.



Looking at Figure $Page 8.1.4$ and the graphs in Example $Page 8.1.4$, we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, it seems logical that if we increase the number of points in our partition, our estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing n, first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals, of the same curved region. Figure $Page 8.1.7$ shows the area of the region under the curve $f(x) = (x-1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

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$$\Delta x = \frac{2-0}{4} = \frac{1}{2}.$$

The area is approximated by the summed areas of the rectangles, or

$$L_4 = f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)0.5 = 7.5 \text{ units}^2$$

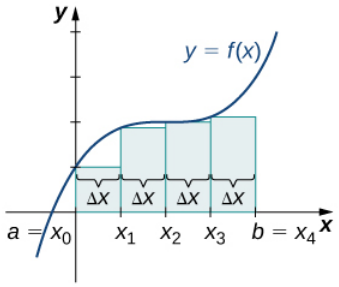


Figure $Page 8.1.7$: With a left-endpoint approximation and dividing the region from a to b into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.

Figure $Page 8.1.8$ shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in Figure $Page 8.1.7$ with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is area under the curve we are unable to include using our approximation. The area of the rectangles is

$$L_8 = f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) = 7.75 \text{ units}^2$$

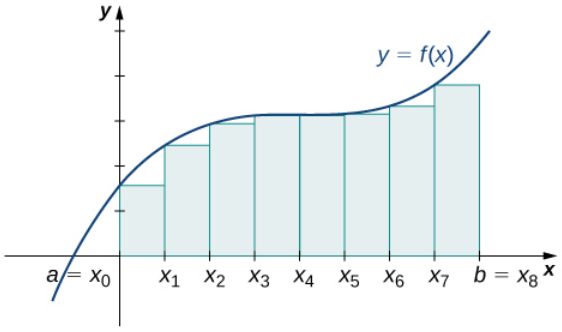


Figure $Page 8.1.8$: The region under the curve is divided into $n = 8$ rectangular areas of equal width for a left-endpoint approximation.

The graph in Figure $Page 8.1.9$ shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

$$L_{32} = f(0)(0.0625) + f(0.0625)(0.0625) + f(0.125)(0.0625) + \cdots + f(1.9375)(0.0625) = 7.9375 \text{ units}^2.$$

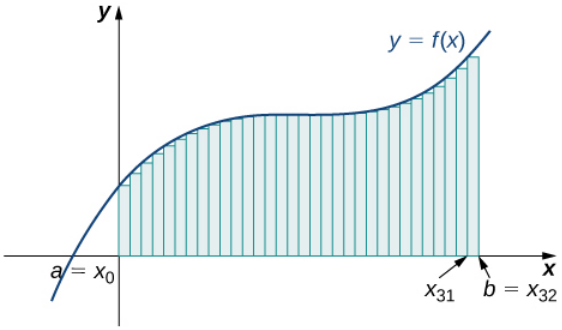


Figure $Page 8.1.9$: Here, 32 rectangles are inscribed under the curve for a left-endpoint approximation.

We can carry out a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles (Figure $Page 8.1.10$), yields an area

$$R_4 = f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) = 8.5 \text{ units}^2.$$

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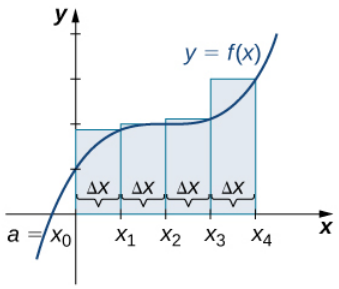


Figure $Page 8.1.10$: Now we divide the area under the curve into four equal subintervals for a right-endpoint approximation.

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \dfrac{2-0}{8} = 0.25$. The graph is shown in Figure $Page 8.1.11$. The area is

$$R_8 = f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) = 8.25 \text{ units}^2$$

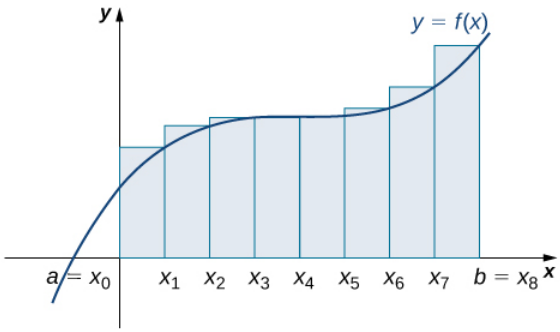


Figure $Page 8.1.11$: Here we use right-endpoint approximation for a region divided into eight equal subintervals.

Last, the right-endpoint approximation with $n = 32$ is close to the actual area (Figure $Page 8.1.12$). The area is approximately

$$R_{32} = f(0.0625)(0.0625) + f(0.125)(0.0625) + f(0.1875)(0.0625) + \cdots + f(2)(0.0625) = 8.0625 \text{ units}^2$$

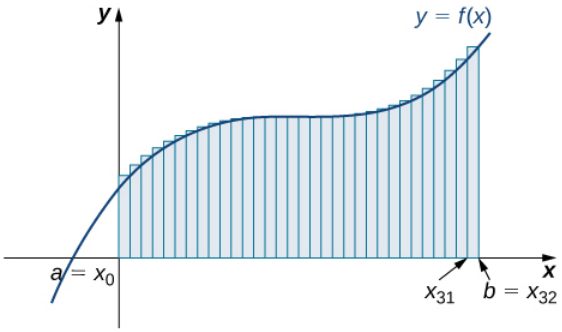


Figure $Page 8.1.12$: The region is divided into 32 equal subintervals for a right-endpoint approximation.

Based on these figures and calculations, it appears we are on the right track; the rectangles appear to approximate the area under the curve better as n gets larger. Furthermore, as n increases, both the left-endpoint and right-endpoint approximations appear to approach an area of 8 square units. Table $Page 8.1.15$ shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as n gets larger and larger is very important, and we now explore this idea in more detail.

Table $Page 8.1.15$: Converging Values of Left- and Right-Endpoint Approximations as n Increases

Value of n	Approximate Area L_n	Approximate Area R_n
$n = 4$	7.5	8.5
$n = 8$	7.75	8.25

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Value of n	Approximate Area L_n	Approximate Area R_n
$n = 32$	7.94	8.06

Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point x_i^* in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Definition: Riemann sum

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be any partition of $[a, b]$. Let Δx_i be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1},\ x_i]$. A Riemann sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

At this point, we'll choose a regular partition P, as we have in our examples above. This forces all Δx_i to be equal to $\Delta x = \dfrac{b-a}{n}$ for any natural number of intervals n.

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Definition: Area Under the Curve

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\displaystyle\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ be a Riemann sum for $f(x)$ with a regular partition P. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

See a graphical demonstration of the construction of a Riemann sum.

Some subtleties here are worth discussing. First, note that taking the limit of a sum is a little different from taking the limit of a function $f(x)$ as x goes to infinity. Limits of sums are discussed in detail in the chapter on Sequences and Series; however, for now we can assume that the computational techniques we used to compute limits of functions can also be used to calculate limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of x_i^*. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique (in other words, it does not depend on the choice of x_i^*).

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for x_i^*. Although any choice for x_i^* gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high

(overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for x_i^* to guarantee one result or the other.

If we want an overestimate, for example, we can choose x_i^* such that for $i = 1, 2, 3, \ldots, n, f(x_i^*) \geq f(x)$ for all $x \in [x_i - 1, x_i]$. In other words, we choose x_i^* so that for $i = 1, 2, 3, \ldots, n, f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select x_i^* in this way, then the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose $x * i$ so that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

Example $Page8.1.5$: Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

Solution

With $n = 4$ over the interval $[1, 2]$, $\Delta x = \frac{1}{4}$. We can list the intervals as $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, and $[1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, Figure shows that a lower sum is obtained by using the right endpoints.

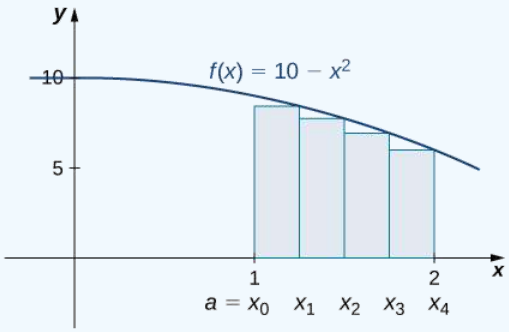


Figure $Page8.1.13$: The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the x-axis on $[1, 2]$, and it shows a lower sum.

The Riemann sum is

$$\begin{aligned}\sum_{k=1}^{4}(10 - x^2)(0.25) &= 0.25[10 - (1.25)^2 + 10 - (1.5)^2 + 10 - (1.75)^2 + 10 - (2)^2] \\ &= 0.25[8.4375 + 7.75 + 6.9375 + 6] \\ &= 7.28 \text{ units}^2.\end{aligned}$$

The area of 7.28 units^2 is a lower sum and an underestimate.

Exercise $Page8.1.5$

a. Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$.
b. Sketch the approximation.

Hint

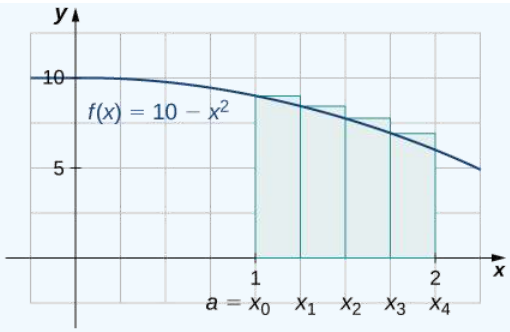
$f(x)$ is decreasing on $[1, 2]$, so the maximum function values occur at the left endpoints of the subintervals.

Answer

a. Upper sum=8.0313 units^2.

b.

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Example $Page 8.1.6$: Finding Lower and Upper Sums for $f(x) = \sin x$

Find a lower sum for $f(x) = \sin x$ over the interval $[a, b] = \left[0, \frac{\pi}{2}\right]$; let $n = 6$.

Solution

Let's first look at the graph in Figure $Page 8.1.14$ to get a better idea of the area of interest.

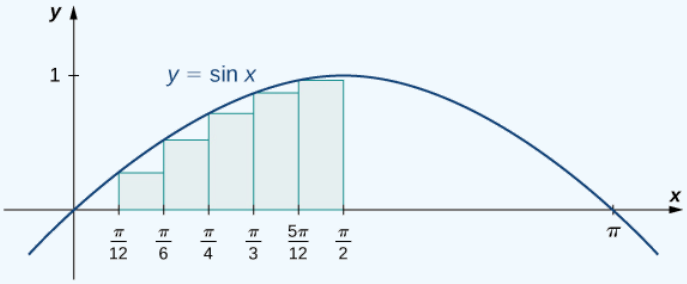


Figure $Page 8.1.14$: The graph of $y = \sin x$ is divided into six regions: $\Delta x = \frac{\pi/2}{6} = \frac{\pi}{12}$.

The intervals are $\left[0, \frac{\pi}{12}\right]$, $\left[\frac{\pi}{12}, \frac{\pi}{6}\right]$, $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$, $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$, $\left[\frac{\pi}{3}, \frac{5\pi}{12}\right]$, and $\left[\frac{5\pi}{12}, \frac{\pi}{2}\right]$. Note that $f(x) = \sin x$ is increasing on the interval $\left[0, \frac{\pi}{2}\right]$, so a left-endpoint approximation gives us the lower sum. A left-endpoint approximation is the Riemann sum $\sum_{i=0}^{5} \sin x_i \left(\frac{\pi}{12}\right)$. We have

$$A \approx \sin(0)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{12}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{6}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{4}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{3}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{5\pi}{12}\right)\left(\tfrac{\pi}{12}\right) \approx 0.863 \text{ units}^2.$$

Exercise $Page 8.1.6$

Using the function $f(x) = \sin x$ over the interval $\left[0, \frac{\pi}{2}\right]$, find an upper sum; let $n = 6$.

Hint

Follow the steps from Example $Page 8.1.6$.

Answer

$A \approx 1.125 \text{ units}^2$

We end off our bootcamp with the following example:

Exercise $Page 8.1.1$

Find the area underneath the graph of $f(x) = 4 - x^2$ over the interval $[1, 2]$.

Answer

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$$A = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\Delta x\right] \qquad \Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i)\left(\frac{1}{n}\right)\right] \qquad x_i = a + i\Delta x = 1 + i\left(\frac{1}{n}\right) = 1 + \frac{i}{n}$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} f\left(1+\frac{i}{n}\right)\left(\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \left(4-\left(1+\frac{i}{n}\right)^2\right)\left(\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \left(3-\frac{2i}{n}-\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \left(\frac{3}{n}-\frac{2i}{n}-\frac{i^2}{n^3}\right)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{i=1}^{n} \frac{3}{n}-\sum_{i=1}^{n}\frac{2i}{n^2}-\sum_{i=1}^{n}\frac{i^2}{n^3}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n} 3-\frac{2}{n^2}\sum_{i=1}^{n} i-\frac{1}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}(3n)-\frac{2}{n^2}\left(\frac{n(n+1)}{2}\right)-\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty}\left[3-\frac{2}{2}\frac{n}{n}\frac{n+1}{n}-\frac{1}{6}\frac{n}{n}\frac{n+1}{n}\frac{2n+1}{n}\right]$$

$$= 3-1(1)(1)-\frac{1}{6}(1)(1)(2)$$

$$= \frac{5}{3}$$

Allow us to make one final observation. It can be shown that the derivative of $F(x) = 4x - \frac{1}{3}x^3$ is precisely $f(x) = 4 - x^2$. Notice that $F(2) - F(1) = \frac{16}{3} - \frac{11}{3} = \frac{5}{3}$.

Is this merely a coincendence? The answer is no and will be further explored in your calculus courses. Thank you!

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