MATH C100: LIBERAL ARTS MATHEMATICS



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This text was compiled on 06/10/2025

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Detailed Licensing

Licensing

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CHAPTER OVERVIEW

1: Mathematics and Problem-Solving

We are surrounded by information—information in various forms such as numbers, charts, graphs, figures, sentences, songs, speeches, and conversations. Much of the information comes through media including television, magazines, and online communications. Many of the things we hear and read are true while many are not. How are we to determine what is true and what is not? What is truth anyway? Are there ways to detect false information? What are we to watch out for?

One of the primary goals of this book is to answer some of these questions through logic and mathematics. This is particularly important in the Internet-dominated, media-centered, and information-driven age in which we find ourselves living today. With new technologies, it is becoming much easier for businesses and organizations to send out deceptive messages and misleading advertisements, and they can target specific individuals based on their online activities. Throughout this book you will find data, numbers, and statements that can be presented honestly or dishonestly in a deceptive manner. They may be numbers, percentages, odds, statistics, or written sentences.

In this first chapter we study logic, both correct use of logic and logical fallacies. Before we get into the formal side of logic, however, we first look at some types of fallacies presented as English sentences. In other words, we will see some bad examples of logical arguments presented in common language. Then, in the following sections we will look carefully into some formal components of logic like propositions, truth tables, and logical equivalences.

1.1: Fallacies in Common Language
1.2: Propositions and Logic
1.3: Truth Tables
1.4: Analyzing Arguments
1.5: Problem Solving

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1.1: Fallacies in Common Language

In the previous discussion, we saw that logical arguments can be invalid when the premises are not true, when the premises are not sufficient to guarantee the conclusion, or when there are invalid chains in logic. There are a number of other ways in which arguments can be invalid, a sampling of which are given here.

Ad hominem (Personal Attack)

An ad hominem argument attacks the person making the argument, ignoring the argument itself.

Example 1.1.1

"Jane says that whales aren't fish, but she's only in the second grade, so she can't be right."

Solution

Here the argument is attacking Jane, not the validity of her claim, so this is an ad hominem argument.

Example 1.1.2

"Jane says that whales aren't fish, but everyone knows that they're really mammals. She's so stupid."

Solution

This certainly isn't very nice, but it is not ad hominem since a valid counterargument is made along with the personal insult.

Appeal to ignorance

This type of argument assumes something is true because it hasn't been proven false.

Example 1.1.3

"Nobody has proven that photo isn't of Bigfoot, so it must be Bigfoot." "I know there are aliens in space because science has not yet eliminated the possibility"

Appeal to authority

These arguments attempt to use the authority of a person to prove a claim. While often authority can provide strength to an argument, problems can occur when the person's opinion is not shared by other experts, or when the authority is irrelevant to the claim.

Example 1.1.4

"A diet high in bacon can be healthy; Doctor Atkins said so."

Solution

Here, an appeal to the authority of a doctor is used for the argument. This generally would provide strength to the argument, except that the opinion that eating a diet high in saturated fat runs counter to general medical opinion. More supporting evidence would be needed to justify this claim.

Example 1.1.5

"Jennifer Hudson lost weight with Weight Watchers, so their program must work."

Solution

Here, there is an appeal to the authority of a celebrity. While her experience does provide evidence, it provides no more than any other person's experience would.

Appeal to consequence

An appeal to consequence concludes that a premise is true or false based on whether the consequences are desirable or not.

Example 1.1.6

"Humans will travel faster than light: faster-than-light travel would be beneficial for space travel."

False dilemma (Limited Choice)

A false dilemma argument falsely frames an argument as an "either or" choice, without allowing for additional options.

Example 1.1.7

"Either those lights in the sky were an airplane or aliens. There are no airplanes scheduled for tonight, so it must be aliens."

Solution

This argument ignores the possibility that the lights could be something other than an airplane or aliens.

Circular reasoning

Circular reasoning is an argument that relies on the conclusion being true for the premise to be true.

Example 1.1.8

"I shouldn't have gotten a C in that class; I'm an A student!"

Solution

In this argument, the student is claiming that because he is an A student, he shouldn't have gotten a C. But because he got a C, he is not an A student.

Post hoc (false cause)

A post hoc argument claims that because two things happened sequentially, then the first must have caused the second.

Example 1.1.9

"Today I wore a red shirt, and my football team won! I need to wear a red shirt every time they play to make sure they keep winning."

Straw man

A straw man argument involves misrepresenting the argument in a less favorable way to make it easier to attack.

Example 1.1.10

"Senator Jones has proposed reducing military funding by 10%. Apparently he wants to leave us defenseless against attacks by terrorists"

Solution

Here the arguer has represented a 10% funding cut as equivalent to leaving us defenseless, making it easier to attack Senator Jones' position.

Correlation implies causation

Similar to post hoc, but without the requirement of sequence, this fallacy assumes that just because two things are related one must have caused the other. Often there is a third variable not considered.

Example 1.1.11

"Months with high ice cream sales also have a high rate of deaths by drowning. Therefore, ice cream must be causing people to drown."

Solution

This argument is implying a causal relation, when really both are more likely dependent on the weather; that ice cream and drowning are both more likely during warm summer months.

Try it Now 20

Identify the logical fallacy in each of the arguments

a. Only an untrustworthy person would run for office. The fact that politicians are untrustworthy is proof of this.
b. Since the 1950s, both the atmospheric carbon dioxide level and obesity levels have increased sharply. Hence, atmospheric carbon dioxide causes obesity.
c. The oven was working fine until you started using it, so you must have broken it.
d. You can't give me a D in the class because I can't afford to retake it.
e. The senator wants to increase support for food stamps. He wants to take the taxpayers' hard-earned money and give it away to lazy people. This isn't fair, so we shouldn't do it.

It may be difficult to identify one particular fallacy for an argument. Consider this argument: "Emma Watson says she's a feminist, but she posed for these racy pictures. I'm a feminist, and no self-respecting feminist would do that." Could this be ad hominem, saying that Emma Watson has no self-respect? Could it be appealing to authority because the person making the argument claims to be a feminist? Could it be a false dilemma because the argument assumes that a woman is either a feminist or not, with no gray area in between?

Reference

1. References (1)

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1.2: Propositions and Logic

Logic is, basically, the study of valid reasoning. When searching the internet, we use Boolean logic – terms like “and” and “or” – to help us find specific web pages that fit in the sets we are interested in. After exploring this form of logic, we will look at logical arguments and how we can determine the validity of a claim.

Boolean Logic

We can often classify items as belonging to sets. If you went the library to search for a book and they asked you to express your search using unions, intersections, and complements of sets, that would feel a little strange. Instead, we typically using words like “and”, “or”, and “not” to connect our keywords together to form a search. These words, which form the basis of Boolean logic, are directly related to our set operations. (Boolean logic was developed by the 19th-century English mathematician George Boole.)

Boolean Logic

Boolean logic combines multiple statements that are either true or false into an expression that is either true or false.

In connection to sets, a search is true if the element is part of the set.

Suppose M is the set of all mystery books, and C is the set of all comedy books. If we search for “mystery”, we are looking for all the books that are an element of the set M; the search is true for books that are in the set.

When we search for “mystery and comedy”, we are looking for a book that is an element of both sets, in the intersection. If we were to search for “mystery or comedy”, we are looking for a book that is a mystery, a comedy, or both, which is the union of the sets. If we searched for “not comedy”, we are looking for any book in the library that is not a comedy, the complement of the set C.

Connection to Set Operations

A and B elements in the intersection $A \cap B$

A or B elements in the union $A \cup B$

not A elements in the complement A^c

Notice here that *or* is not exclusive. This is a difference between the Boolean logic use of the word and common everyday use. When your significant other asks “do you want to go to the park or the movies?” they usually are proposing an exclusive choice – one option or the other, but not both. In Boolean logic, the *or* is not exclusive – more like being asked at a restaurant “would you like fries *or* a drink with that?” Answering “both, please” is an acceptable answer.

Example 1.2.1

Suppose we are searching a library database for Mexican universities. Express a reasonable search using Boolean logic.

Solution

We could start with the search “Mexico and university”, but would be likely to find results for the U.S. state New Mexico. To account for this, we could revise our search to read:

Mexico *and* university *not* “New Mexico”

In most internet search engines, it is not necessary to include the word *and*; the search engine assumes that if you provide two keywords you are looking for both. In Google’s search, the keyword *or* has be capitalized as OR, and a negative sign in front of a word is used to indicate *not*. Quotes around a phrase indicate that the entire phrase should be looked for. The search from the previous example on Google could be written:

Mexico university -“New Mexico”

Example 1.2.2

Describe the numbers that meet the condition: even and less than 10 and greater than 0

Solution

The numbers that satisfy all three requirements are {2, 4, 6, 8}

Sometimes statements made in English can be ambiguous. For this reason, Boolean logic uses parentheses to show precedent, just like in algebraic order of operations.

Example 1.2.3

Describe the numbers that meet the condition: odd number and less than 20 and greater than 0 and (multiple of 3 or multiple of 5)

Solution

The first three conditions limit us to the set {1, 3, 5, 7, 9, 11, 13, 15, 17, 19}

The last grouped conditions tell us to find elements of this set that are also either a multiple of 3 or a multiple of 5. This leaves us with the set {3, 5, 9, 15}

Notice that we would have gotten a very different result if we had written (odd number and less than 20 and greater than 0 and multiple of 3) or multiple of 5

The first grouped set of conditions would give {3, 9, 15}. When combined with the last condition, though, this set expands without limits:

{3, 5, 9, 15, 20, 25, 30, 35, 40, 45, …}

Example 1.2.4

The English phrase “Go to the store and buy me eggs and bagels or cereal” is ambiguous; it is not clear whether the requestors is asking for eggs always along with either bagels or cereal, or whether they’re asking for either the combination of eggs and bagels, or just cereal.

Solution

For this reason, using parentheses clarifies the intent:

Eggs and (bagels or cereal) means Option 1: Eggs and bagels, Option 2: Eggs and cereal

(Eggs and bagels) or cereal means Option 1: Eggs and bagels, Option 2: Cereal

Be aware that when a string of conditions is written without grouping symbols, it is often interpreted from the left to right, resulting in the latter interpretation.

Conditional Statements

Beyond searching, Boolean logic is commonly used in spreadsheet applications like Excel to do conditional calculations. A statement is something that is either true or false. A statement like $3 < 5$ is true; a statement like “a rat is a fish” is false. A statement like “$x < 5$” is true for some values of x and false for others. When an action or outcome depends on the value of a statement, it forms a conditional.

Statements and Conditionals

A statement is either true or false. A conditional is a compound statement of the form “if p then q” or “if p then q, else s”.

Example 1.2.5

In common language, an example of a conditional statement would be “If it is raining, then we’ll go to the mall. Otherwise we’ll go for a hike.”

Solution

The statement “If it is raining” is the condition – this may be true or false for any given day. If the condition is true, then we will follow the first course of action and go to the mall. If the condition is false, then we will use the alternative and go for a hike.

Example 1.2.6

As mentioned earlier, conditional statements are commonly used in spreadsheet applications like Excel or Google Sheets.

In Excel, you can enter an expression like =IF(A1<2000, A1+1, A1*2)

Solution

Notice that after the IF, there are three parts. The first part is the condition, and the second two are calculations. Excel will look at the value in cell A1 and compare it to 2000. If that condition is true, then the first calculation is used, and 1 is added to the value of A1 and the result is stored. If the condition is false, then the second calculation is used, and A1 is multiplied by 2 and the result is stored.

In other words, this statement is equivalent to saying “If the value of A1 is less than 2000, then add 1 to the value in A1. Otherwise, multiply A1 by 2”.

Example 1.2.7

The expression =IF(A1>5, 2*A1, 3*A1) is used. Find the result if A1 is 3, and the result if A1 is 8.

Solution

This is equivalent to saying

If A1 > 5, then calculate 2*A1. Otherwise, calculate 3*A1

If A1 is 3, then the condition is false, since 3 > 5 is not true, so we do the alternate action, and multiply by 3, giving 3*3 = 9

If A1 is 8, then the condition is true, since 8 > 5, so we multiply the value by 2, giving 2*8=16

Example 1.2.8

An accountant needs to withhold 15% of income for taxes if the income is below $30,000, and 20% of income if the income is $30,000 or more. Write an expression that would calculate the amount to withhold.

Solution

Our conditional needs to compare the value to 30,000. If the income is less than 30,000, we need to calculate 15% of the income: 0.15*income. If the income is more than 30,000, we need to calculate 20% of the income: 0.20*income.

In words we could write “If income < 30,000, then multiply by 0.15, otherwise multiply by 0.20”. In Excel, we would write:

=IF(A1<30000, 0.15*A1, 0.20*A1)

As we did earlier, we can create more complex conditions by using the operators *and*, *or*, and *not* to join simpler conditions together.

Example 1.2.9

A parent might say to their child “if you clean your room and take out the garbage, then you can have ice cream.”

Here, there are two simpler conditions:

1) The child cleaning her room

2) The child taking out the garbage

Solution

Since these conditions were joined with *and*, the combined conditional will be true only if both simpler conditions are true; if either chore is not completed, then the parent's condition is not met.

Notice that if the parent had said "if you clean your room or take out the garbage, then you can have ice cream", then the child would need to complete only one chore to meet the condition.

Suppose you wanted to have something happen when a certain value is between 100 and 300. To create the condition "A1 < 300 and A1 > 100" in Excel, you would need to enter "AND(A1<300, A1>100)". Likewise, for the condition "A1=4 or A1=6" you would enter "OR(A1=4, A1=6)"

Example 1.2.10

In a spreadsheet, cell A1 contains annual income, and A2 contains number of dependents.

A certain tax credit applies if someone with no dependents earns less than $10,000, or if someone with dependents earns less than $20,000. Write a rule that describes this.

Solution

There are two ways the rule is met:

income is less than 10,000 and dependents is 0, or

income is less than 20,000 and dependents is not 0.

Informally, we could write these as

(A1 < 10000 and A2 = 0) or (A1 < 20000 and A2 > 0)

In Excel's format, we'd write

IF(OR(AND(A1<10000, A2=0), AND(A1<20000, A2>0)), "you qualify", "you don't qualify")

Quantified Statements

Words that describe an entire set, such as "all", "every", or "none", are called universal quantifiers because that set could be considered a universal set. In contrast, words or phrases such as "some", "one", or "at least one" are called existential quantifiers because they describe the existence of at least one element in a set.

Quantifiers

A **universal quantifier** states that an entire set of things share a characteristic.

An **existential quantifier** states that a set contains at least one element.

Something interesting happens when we negate – or state the opposite of – a quantified statement.

Example 1.2.11

Suppose your friend says "Everybody cheats on their taxes." What is the minimum amount of evidence you would need to prove your friend wrong?

Solution

To show that it is not true that everybody cheats on their taxes, all you need is one person who does not cheat on their taxes. It would be perfectly fine to produce more people who do not cheat, but one counterexample is all you need.

It is important to note that you do not need to show that absolutely nobody cheats on their taxes.

Example 1.2.12

Suppose your friend says “One of these six cartons of milk is leaking.” What is the minimum amount of evidence you would need to prove your friend wrong?

Solution

In this case, you would need to check all six cartons and show that none of them is leaking. You cannot disprove your friend’s statement by checking only one of the cartons.

When we negate a statement with a universal quantifier, we get a statement with an existential quantifier, and vice-versa.

Negating a quantified statement

The negation of “all A are B” is “at least one A is not B”.

The negation of “no A are B” is “at least one A is B”.

The negation of “at least one A is B” is “no A are B”.

The negation of “at least one A is not B” is “all A are B”.

Example 1.2.13

“Somebody brought a flashlight.” Write the negation of this statement.

Solution

The negation is “Nobody brought a flashlight.”

Example 1.2.14

“There are no prime numbers that are even.” Write the negation of this statement.

Solution

The negation is “At least one prime number is even.”

Try it Now 1

1. Write the negation of “All Icelandic children learn English in school.”

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1.3: Truth Tables

Before we focus on truth tables, we're going to introduce some symbols that are commonly used for *and*, *or*, and *not*.

Symbols

The symbol $\bigwedge$ is used for and: A and B is notated A $\bigwedge$ B

The symbol $\bigvee$ is used for or: A or B is notated A $\bigvee$ B

The symbol ~ is used for not: not A is notated ~A

You can remember the first two symbols by relating them to the shapes for the union and intersection. A $\bigwedge$ B would be the elements that exist in both sets, in A $\bigcap$ B. Likewise, A $\bigvee$ B would be the elements that exist in either set, in A $\bigcup$ B. When we are working with sets, we use the rounded version of the symbols; when we are working with statements, we use the pointy version.

Example 1.3.1

Translate each statement into symbolic notation. Let P represent "I like Pepsi" and let C represent "I like Coke".

a. I like Pepsi or I like Coke.
b. I like Pepsi and I like Coke.
c. I do not like Pepsi.
d. It is not the case that I like Pepsi or Coke.
e. I like Pepsi and I do not like Coke.

Solution

a. P $\bigvee$ C
b. P $\bigwedge$ C
c. ~P
d. ~(P $\bigvee$ C)
e. P $\bigwedge$ ~C

As you can see, we can use parentheses to organize more complicated statements.

Try it Now 2

1. Translate "We have carrots or we will not make soup" into symbols. Let C represent "we have carrots" and let S represent "we will make soup".

Because complex Boolean statements can get tricky to think about, we can create a truth table to keep track of what truth values for the simple statements make the complex statement true and false.

Truth table

A table showing what the resulting truth value of a complex statement is for all the possible truth values for the simple statements.

Example 1.3.2

Suppose you're picking out a new couch, and your significant other says "get a sectional or something with a chaise".

Solution

This is a complex statement made of two simpler conditions: "is a sectional", and "has a chaise". For simplicity, let's use S to designate "is a sectional", and C to designate "has a chaise".

A truth table for this situation would look like this:

Table 1.3.1

S	C	S or C

S	C	S or C
T	T	T
T	F	T
F	T	T
F	F	F

In the table, T is used for true, and F for false. In the first row, if S is true and C is also true, then the complex statement "S or C" is true. This would be a sectional that also has a chaise, which meets our desire. (Remember that *or* in logic is not exclusive; if the couch has both features, it meets the condition.)

In the previous example about the couch, the truth table was really just summarizing what we already know about how the or statement work. The truth tables for the basic *and*, *or*, and *not* statements are shown below.

Table 1.3.2: Conjunction

A	B	A ⋀ B
T	T	T
T	F	F
F	T	F
F	F	F

Table 1.3.3: Disjunction

A	B	A ⋁ B
T	T	T
T	F	T
F	T	T
F	F	F

Table 1.3.4: Negation

A	~A
T	F
F	T

Truth tables really become useful when we analyze more complex Boolean statements.

Example 1.3.3

When we create the truth table, we need to list all the possible truth value combinations for A and B. Notice how the first column contains 2 Ts followed by 2 Fs, and the second column alternates T, F, T, F. This pattern ensures that all 4 combinations are considered.

Table 1.3.5

A	B
T	T
T	F

A	B
F	T
F	F

After creating columns with those initial values, we create a third column for the expression ~B. Now we will temporarily ignore the column for A and write the truth values for ~B.

Table 1.3.6

A	B	~B
T	T	F
T	F	T
F	T	F
F	F	T

Next we can find the truth values of $A \bigvee {\sim}B$, using the first and third columns.

Table 1.3.7

A	B	~B	A ⋁ ~B
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

The truth table shows that A ⋁ ~B is true in three cases and false in one case. If you're wondering what the point of this is, suppose it is the last day of the baseball season and two teams, who are not playing each other, are competing for the final playoff spot. Anaheim will make the playoffs if it wins its game or if Boston does not win its game. (Anaheim owns the tie-breaker; if both teams win, or if both teams lose, then Anaheim gets the playoff spot.) If A = Anaheim wins its game and B = Boston wins its game, then A ⋁ ~B represents the situation "Anaheim wins its game or Boston does not win its game". The truth table shows us the different scenarios related to Anaheim making the playoffs. In the first row, Anaheim wins its game and Boston wins its game, so it is true that Anaheim makes the playoffs. In the second row, Anaheim wins and Boston does not win, so it is true that Anaheim makes the playoffs. In the third row, Anaheim does not win its game and Boston wins its game, so it is false that Anaheim makes the playoffs. In the fourth row, Anaheim does not win and Boston does not win, so it is true that Anaheim makes the playoffs.

Try it Now 3

1. Create a truth table for this statement: (~A) ⋀B

Example 1.3.4

Create a truth table for the statement A ⋀ ~(B ⋁ C).

Solution

It helps to work from the inside out when creating a truth table, and to create columns in the table for intermediate operations. We start by listing all the possible truth value combinations for A, B, and C. Notice how the first column contains 4 Ts followed by 4 Fs, the second column contains 2 Ts, 2 Fs, then repeats, and the last column alternates T, F, T, F... This pattern ensures that all 8 combinations are considered. After creating columns with those initial values, we create a fourth column for the innermost expression, B ⋁ C. Now we will temporarily ignore the column for A and focus on B and C, writing the truth values for B ⋁ C.

Table 1.3.8

A	B	C	B ⋁ C
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

Table 1.3.9

A	B	C
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Next we can find the negation of B ⋁ C, working off the B ⋁ C column we just created. (Ignore the first three columns and simply negate the values in the B ⋁ C column.)

Table 1.3.10

A	B	C	B ⋁ C	~(B ⋁ C)
T	T	T	T	F
T	T	F	T	F
T	F	T	T	F
T	F	F	F	T
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	T

Finally, we find the values of A and ~(B ⋁ C). (Ignore the second, third, and fourth columns.)

Table 1.3.11

A	B	C	B ⋁ C	~(B ⋁ C)	A ⋀ ~(B ⋁ C)

A	B	C	B ⋁ C	~(B ⋁ C)	A ⋀ ~(B ⋁ C)
T	T	T	T	F	F
T	T	F	T	F	F
T	F	T	T	F	F
T	F	F	F	T	T
F	T	T	T	F	F
F	T	F	T	F	F
F	F	T	T	F	F
F	F	F	F	T	F

It turns out that this complex expression is true in only one case: when A is true, B is false, and C is false. To illustrate this situation, suppose that Anaheim will make the playoffs if: (1) Anaheim wins, and (2) neither Boston nor Cleveland wins. TFF is the only scenario in which Anaheim will make the playoffs.

Try it Now 4

1. Create a truth table for this statement: (~A ⋀B) ⋁ ~B

Truth Tables: Conditional, Biconditional

We discussed conditional statements earlier, in which we take an action based on the value of the condition. We are now going to look at another version of a conditional, sometimes called an implication, which states that the second part must logically follow from the first.

Conditional

A conditional is a logical compound statement in which a statement p, called the antecedent, implies a statement q, called the consequent.

A conditional is written as p → q and is translated as “if p, then q”.

Example 1.3.5

The English statement “If it is raining, then there are clouds is the sky” is a conditional statement. It makes sense because if the antecedent “it is raining” is true, then the consequent “there are clouds in the sky” must also be true.

Notice that the statement tells us nothing of what to expect if it is not raining; there might be clouds in the sky, or there might not. If the antecedent is false, then the consquent becomes irrelevant.

Example 1.3.6

Suppose you order a team jersey online on Tuesday and want to receive it by Friday so you can wear it to Saturday’s game. The website says that if you pay for expedited shipping, you will receive the jersey by Friday. In what situation is the website telling a lie?

Solution

There are four possible outcomes:

1) You pay for expedited shipping and receive the jersey by Friday

2) You pay for expedited shipping and don’t receive the jersey by Friday

3) You don’t pay for expedited shipping and receive the jersey by Friday

4) You don’t pay for expedited shipping and don’t receive the jersey by Friday

Only one of these outcomes proves that the website was lying: the second outcome in which you pay for expedited shipping but don't receive the jersey by Friday. The first outcome is exactly what was promised, so there's no problem with that. The third outcome is not a lie because the website never said what would happen if you didn't pay for expedited shipping; maybe the jersey would arrive by Friday whether you paid for expedited shipping or not. The fourth outcome is not a lie because, again, the website didn't make any promises about when the jersey would arrive if you didn't pay for expedited shipping.

It may seem strange that the third outcome in the previous example, in which the first part is false but the second part is true, is not a lie. Remember, though, that if the antecedent is false, we cannot make any judgment about the consequent. The website never said that paying for expedited shipping was the only way to receive the jersey by Friday.

Example 1.3.7

A friend tells you "If you upload that picture to Facebook, you'll lose your job." Under what conditions can you say that your friend was wrong?

Solution

There are four possible outcomes:

1) You upload the picture and lose your job

2) You upload the picture and don't lose your job

3) You don't upload the picture and lose your job

4) You don't upload the picture and don't lose your job

There is only one possible case in which you can say your friend was wrong: the second outcome in which you upload the picture but still keep your job. In the last two cases, your friend didn't say anything about what would happen if you didn't upload the picture, so you can't say that their statement was wrong. Even if you didn't upload the picture and lost your job anyway, your friend never said that you were guaranteed to keep your job if you didn't upload the picture; you might lose your job for missing a shift or punching your boss instead.

In traditional logic, a conditional is considered true as long as there are no cases in which the antecedent is true and the consequent is false.

Truth table for the conditional

Table 1.3.12

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Again, if the antecedent p is false, we cannot prove that the statement is a lie, so the result of the third and fourth rows is true.

Example 1.3.8

Construct a truth table for the statement (m ⋀ ~p) → r

Solution

We start by constructing a truth table with 8 rows to cover all possible scenarios. Next, we can focus on the antecedent, m ⋀ ~p.

Table 1.3.13

m	p	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Table 1.3.14

m	p	r	~p
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	T

Table 1.3.15

m	p	r	~p	m ⋀ ~p
T	T	T	F	F
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	F	F
F	T	F	F	F
F	F	T	T	F
F	F	F	T	F

Now we can create a column for the conditional. Because it can be confusing to keep track of all the **Ts** and **Fs**, why don't we copy the column for r to the right of the column for **m ⋀ ~p** ? This makes it a lot easier to read the conditional from left to right.

Table 1.3.16

m	p	r	~p	m ⋀ ~p	r	(m ⋀ ~p) → r
T	T	T	F	F	T	T

m	p	r	~p	m ⋀ ~p	r	(m ⋀ ~p) → r
T	T	F	F	F	F	T
T	F	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	T	F	F	F	F	T
F	F	T	T	F	T	T
F	F	F	T	F	F	T

When m is true, p is false, and r is false—the fourth row of the table—then the antecedent m ⋀ ~p will be true but the consequent false, resulting in an invalid conditional; every other case gives a valid conditional.

If you want a real-life situation that could be modeled by (m ⋀ ~p) → r, consider this: let m = we order meatballs, p = we order pasta, and r = Rob is happy. The statement (m ⋀ ~p) → r is "if we order meatballs and don't order pasta, then Rob is happy". If m is true (we order meatballs), p is false (we don't order pasta), and r is false (Rob is not happy), then the statement is false, because we satisfied the antecedent but Rob did not satisfy the consequent.

For any conditional, there are three related statements, the converse, the inverse, and the contrapositive.

Related Statements

The original conditional is "if p, then q" p → q

The converse is "if q, then p" q → p

The inverse is "if not p, then not q" ~p →~ q

The contrapositive is "if not q, then not p" q → ~p

Example 1.3.9

Consider again the conditional "If it is raining, then there are clouds in the sky." It seems reasonable to assume that this is true.

The converse would be "If there are clouds in the sky, then it is raining." This is not always true.

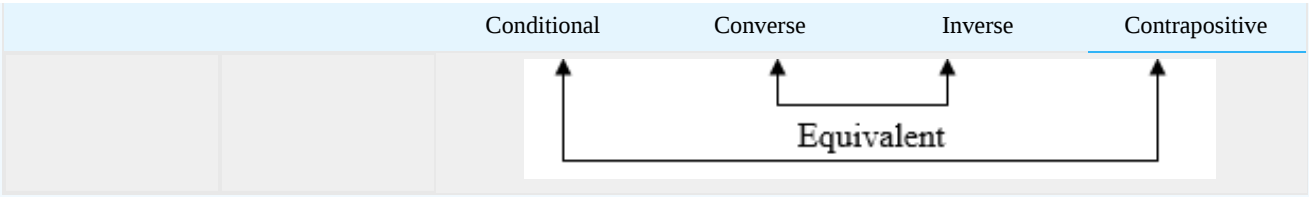
The inverse would be "If it is not raining, then there are not clouds in the sky." Likewise, this is not always true.

The contrapositive would be "If there are not clouds in the sky, then it is not raining." This statement is true, and is equivalent to the original conditional.

Looking at truth tables, we can see that the original conditional and the contrapositive are logically equivalent, and that the converse and inverse are logically equivalent.

Table 1.3.17

		Conditional	Converse	Inverse	Contrapositive
p	q	p → q	q → p	~p → ~q	~q → ~p
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T



Equivalence

A conditional statement and its contrapositive are logically equivalent.

The converse and inverse of a conditional statement are logically equivalent.

In other words, the original statement and the contrapositive must agree with each other; they must both be true, or they must both be false. Similarly, the converse and the inverse must agree with each other; they must both be true, or they must both be false.

Be aware that symbolic logic cannot represent the English language perfectly. For example, we may need to change the verb tense to show that one thing occurred before another.

Example 1.3.10

Suppose this statement is true: "If I eat this giant cookie, then I will feel sick." Which of the following statements must also be true?

a. If I feel sick, then I ate that giant cookie.
b. If I don't eat this giant cookie, then I won't feel sick.
c. If I don't feel sick, then I didn't eat that giant cookie.

Solution

a. This is the converse, which is not necessarily true. I could feel sick for some other reason, such as drinking sour milk.
b. This is the inverse, which is not necessarily true. Again, I could feel sick for some other reason; avoiding the cookie doesn't guarantee that I won't feel sick.
c. This is the contrapositive, which is true, but we have to think somewhat backwards to explain it. If I ate the cookie, I would feel sick, but since I don't feel sick, I must not have eaten the cookie.

Notice again that the original statement and the contrapositive have the same truth value (both are true), and the converse and the inverse have the same truth value (both are false).

Try it Now 5

1. "If you microwave salmon in the staff kitchen, then I will be mad at you." If this statement is true, which of the following statements must also be true?
   a. If you don't microwave salmon in the staff kitchen, then I won't be mad at you.
   b. If I am not mad at you, then you didn't microwave salmon in the staff kitchen.
   c. If I am mad at you, then you microwaved salmon in the staff kitchen.

Consider the statement "If you park here, then you will get a ticket." What set of conditions would prove this statement false?

a. You don't park here and you get a ticket.
b. You don't park here and you don't get a ticket.
c. You park here and you don't get a ticket.

The first two statements are irrelevant because we don't know what will happen if you park somewhere else. The third statement, however, contradicts the conditional statement "If you park here, then you will get a ticket" because you parked here but didn't get a ticket. This example demonstrates a general rule; the negation of a conditional can be written as a conjunction: "It is not the case that if you park here, then you will get a ticket" is equivalent to "You park here and you do not get a ticket."

The Negation of a Conditional

The negation of a conditional statement is logically equivalent to a conjunction of the antecedent and the negation of the consequent.

~ (p → q) is equivalent to p ⋀ ~q

Example 1.3.11

Which of the following statements is equivalent to the negation of "If you don't grease the pan, then the food will stick to it"?

a. I didn't grease the pan and the food didn't stick to it.
b. I didn't grease the pan and the food stuck to it.
c. I greased the pan and the food didn't stick to it.

Solution

a. This is correct; it is the conjunction of the antecedent and the negation of the consequent. To disprove that not greasing the pan will cause the food to stick, I have to not grease the pan and have the food not stick.
b. This is essentially the original statement with no negation; the "if…then" has been replaced by "and".
c. This essentially agrees with the original statement and cannot disprove it.

Try it Now 6

1. "If you go swimming less than an hour after eating lunch, then you will get cramps." Which of the following statements is equivalent to the negation of this statement?
   a. I went swimming more than an hour after eating lunch and I got cramps.
   b. I went swimming less than an hour after eating lunch and I didn't get cramps.
   c. I went swimming more than an hour after eating lunch and I didn't get cramps.

In everyday life, we often have a stronger meaning in mind when we use a conditional statement. Consider "If you submit your hours today, then you will be paid next Friday." What the payroll rep really means is "If you submit your hours today, then you will be paid next Friday, and if you don't submit your hours today, then you won't be paid next Friday." The conditional statement if *t*, *then p* also includes the inverse of the statement: *if not t, then not p*. A more compact way to express this statement is "You will be paid next Friday *if and only if* you submit your timesheet today." A statement of this form is called a **biconditional.**

Biconditional

A biconditional is a logical conditional statement in which the antecedent and consequent are interchangeable.

A biconditional is written as p ↔ q and is translated as "p if and only if q".

Because a biconditional statement p ↔ q is equivalent to (p → q) ⋀ (q → p), we may think of it as a conditional statement combined with its converse: *if p, then q* and *if q, then p*. The double-headed arrow shows that the conditional statement goes from left to right and from right to left. A biconditional is considered true as long as the antecedent and the consequent have the same truth value; that is, they are either both true or both false.

Truth table for the biconditional

Table 1.3.18

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Notice that the fourth row, where both components are false, is true; if you don't submit your timesheet and you don't get paid, the person from payroll told you the truth.

Example 1.3.12

Suppose this statement is true: "The garbage truck comes down my street if and only if it is Thursday morning." Which of the following statements could be true?

a. It is noon on Thursday and the garbage truck did not come down my street this morning.
b. It is Monday and the garbage truck is coming down my street.
c. It is Wednesday at 11:59 PM and the garbage truck did not come down my street today.

Solution

a. This cannot be true. This is like the second row of the truth table; it is true that I just experienced Thursday morning, but it is false that the garbage truck came.
b. This cannot be true. This is like the third row of the truth table; it is false that it is Thursday, but it is true that the garbage truck came.
c. This could be true. This is like the fourth row of the truth table; it is false that it is Thursday, but it is also false that the garbage truck came, so everything worked out like it should.

Try it Now 7

1. Suppose this statement is true: "I wear my running shoes if and only if I am exercising." Determine whether each of the following statements must be true or false.
   a. I am exercising and I am not wearing my running shoes.
   b. I am wearing my running shoes and I am not exercising.
   c. I am not exercising and I am not wearing my running shoes.

Example 1.3.13

Create a truth table for the statement $(A \vee B) \leftrightarrow \sim C$

Solution

Whenever we have three component statements, we start by listing all the possible truth value combinations for A, B, and C. After creating those three columns, we can create a fourth column for the antecedent, $A \vee B$. Now we will temporarily ignore the column for C and focus on A and B, writing the truth values for $A \vee B$.

Table 1.3.19

A	B	C
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

Table 1.3.20

A	B	C	$A \vee B$

A	B	C	A ⋁ B
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

Next we can create a column for the negation of C. (Ignore the A ⋁ B column and simply negate the values in the C column.)

Table 1.3.21

A	B	C	A ⋁ B	~C
T	T	T	T	F
T	T	F	T	T
T	F	T	T	F
T	F	F	T	T
F	T	T	T	F
F	T	F	T	T
F	F	T	F	F
F	F	F	F	T

Finally, we find the truth values of (A ⋁ B) ↔ ~C. Remember, a biconditional is true when the truth value of the two parts match, but it is false when the truth values do not match.

Table 1.3.22

A	B	C	A ⋁ B	~C	(A ⋁ B) ↔ ~C
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	T	F	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	F	T	F

To illustrate this situation, suppose your boss needs you to do either project A or project B (or both, if you have the time). If you do one of the projects, you will not get a crummy review (C is for crummy). So (A ⋁ B) ↔ ~C means “You will not get a crummy review if and only if you do project A or project B.” Looking at a few of the rows of the truth table, we can see how this works out. In the first row, A, B, and C are all true: you did both projects and got a crummy review, which is not what your

boss told you would happen! That is why the final result of the first row is false. In the fourth row, A is true, B is false, and C is false: you did project A and did not get a crummy review. This is what your boss said would happen, so the final result of this row is true. And in the eighth row, A, B, and C are all false: you didn't do either project and did not get a crummy review. This is not what your boss said would happen, so the final result of this row is false. (Even though you may be happy that your boss didn't follow through on the threat, the truth table shows that your boss lied about what would happen.)

De Morgan's Laws

A contemporary of Boole's, Augustus De Morgan, formalized two rules of logic that had previously been known informally. They allow us to rewrite the negation of a conjunction as a disjunction, and vice-versa.

For example, suppose you want to schedule a meeting with two colleagues at 4:30 PM on Friday, and you need both of them to be available at that time. What situation would make it impossible to have the meeting? It is NOT the case that colleague a is available AND colleague *b* is available: ~ (a ∧ b). This situation is equivalent to either colleague a NOT being available OR colleague b NOT being available: ~a ∨ ~b.

De Morgan's Laws

The negation of a conjunction is equivalent to the disjunction of the negation of the statements making up the conjunction. To negate an "and" statement, negate each part and change the "and" to "or".

$$\sim (p \wedge q) \text{ is equivalent to } \sim p \wedge \sim q$$

The negation of a disjunction is equivalent to the conjunction of the negation of the statements making up the disjunction. To negate an "or" statement, negate each part and change the "or" to "and".

$$\sim (p \vee q) \text{ is equivalent to } \sim p \vee \sim q$$

Example 1.3.14

For Valentine's Day, you did not get your sweetie flowers or candy: Which of the following statements is logically equivalent?

a. You did not get them flowers or did not get them candy.
b. You did not get them flowers and did not get them candy.
c. You got them flowers or got them candy.

Solution

a. This statement does not go far enough; it leaves open the possibility that you got them one of the two things.
b. This statement is equivalent to the original; ~(f ˇ c) is equivalent to ~f ˆ~c.
c. This statement says that you got them something, but we know that you did not.

Try it Now 8

1. To serve as the President of the US, a person must have been born in the US, must be at least 35 years old, and must have lived in the US for at least 14 years. What minimum set of conditions would disqualify someone from serving as President?

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1.4: Analyzing Arguments

A logical argument is a claim that a set of premises support a conclusion. There are two general types of arguments: inductive and deductive arguments.

Argument types

An **inductive** argument uses a collection of specific examples as its premises and uses them to propose a general conclusion.

A **deductive** argument uses a collection of general statements as its premises and uses them to propose a specific situation as the conclusion.

Example 1.4.1

The argument "when I went to the store last week I forgot my purse, and when I went today I forgot my purse. I always forget my purse when I go the store" is an inductive argument.

The premises are:

I forgot my purse last week

I forgot my purse today

The conclusion is:

I always forget my purse

Notice that the premises are specific situations, while the conclusion is a general statement. In this case, this is a fairly weak argument, since it is based on only two instances.

Example 1.4.2

The argument "every day for the past year, a plane flies over my house at 2:00 P.M. A plane will fly over my house every day at 2:00 P.M." is a stronger inductive argument, since it is based on a larger set of evidence. While it is not necessarily true—the airline may have cancelled its afternoon flight—it is probably a safe bet.

Evaluating Inductive Arguments

An inductive argument is never able to prove the conclusion true, but it can provide either weak or strong evidence to suggest that it may be true.

Many scientific theories, such as the big bang theory, can never be proven. Instead, they are inductive arguments supported by a wide variety of evidence. Usually in science, an idea is considered a hypothesis until it has been well tested, at which point it graduates to being considered a theory. Common scientific theories, like Newton's theory of gravity, have all stood up to years of testing and evidence, though sometimes they need to be adjusted based on new evidence, such as when Einstein proposed the theory of general relativity.

A deductive argument is more clearly valid or not, which makes it easier to evaluate.

Evaluating deductive arguments

A deductive argument is considered valid if, assuming that all the premises are true, the conclusion follows logically from those premises. In other words, when the premises are all true, the conclusion must be true.

Evaluating Deductive Arguments with Euler (Venn) Diagrams

We can interpret a deductive argument visually with an Euler diagram, which is essentially the same thing as a Venn diagram. This can make it easier to determine whether the argument is valid or invalid.

Example 1.4.3

Consider the deductive argument "All cats are mammals and a tiger is a cat, so a tiger is a mammal." Is this argument valid?

Solution

The premises are:

All cats are mammals.

A tiger is a cat.

The conclusion is:

A tiger is a mammal.

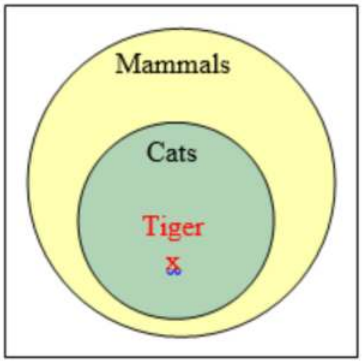


Figure 1.4.1

Both the premises are true. To see that the premises must logically lead to the conclusion, we can use a Venn diagram. From the first premise, we draw the set of cats as a subset of the set of mammals. From the second premise, we are told that a tiger is contained within the set of cats. From that, we can see in the Venn diagram that the tiger must also be inside the set of mammals, so the conclusion is valid.

Analyzing arguments with Euler diagrams

To analyze an argument with an Euler diagram:

1. Draw an Euler diagram based on the premises of the argument
2. The argument is invalid if there is a way to draw the diagram that makes the conclusion false
3. The argument is valid if the diagram cannot be drawn to make the conclusion false
4. If the premises are insufficient to determine the location of an element or a set mentioned in the conclusion, then the argument is invalid.

Try it Now 8

Determine the validity of this argument:

Premise: All cats are scared of vacuum cleaners.

Premise: Max is a cat.

Conclusion: Max is scared of vacuum cleaners.

Example 1.4.4

Premise: All firefighters know CPR.

Premise: Jill knows CPR.

Conclusion: Jill is a firefighter.

Solution

From the first premise, we know that firefighters all lie inside the set of those who know CPR. (Firefighters are a subset of people who know CPR.) From the second premise, we know that Jill is a member of that larger set, but we do not have enough information to know whether she also is a member of the smaller subset that is firefighters.

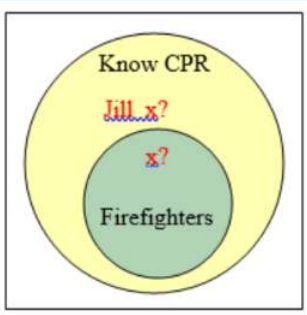


Figure 1.4.2

Since the conclusion does not necessarily follow from the premises, this is an invalid argument. It's possible that Jill is a firefighter, but the structure of the argument doesn't allow us to conclude that she definitely is.

It is important to note that whether or not Jill is actually a firefighter is not important in evaluating the validity of the argument; we are concerned with whether the premises are enough to prove the conclusion.

Try it Now 10

Determine the validity of this argument:

Premise: All bicycles have two wheels.

Premise: This Harley-Davidson has two wheels.

Conclusion: This Harley-Davidson is a bicycle.

Try it Now 11

Determine the validity of this argument:

Premise: No cows are purple.

Premise: Fido is not a cow.

Conclusion: Fido is purple.

In addition to these categorical style premises of the form "all ___", "some ____", and "no ____", it is also common to see premises that are conditionals.

Example 1.4.5

Premise: If you live in Seattle, you live in Washington.

Premise: Marcus does not live in Seattle.

Conclusion: Marcus does not live in Washington.

Solution

From the first premise, we know that the set of people who live in Seattle is inside the set of those who live in Washington. From the second premise, we know that Marcus does not lie in the Seattle set, but we have insufficient information to know whether Marcus lives in Washington or not. This is an invalid argument.



Figure 1.4.3

Try it Now 12

Determine the validity of this argument:

Premise: If you have lipstick on your collar, then you are cheating on me.

Premise: If you are cheating on me, then I will divorce you.

Premise: You do not have lipstick on your collar.

Conclusion: I will not divorce you.

Evaluating Deductive Arguments with Truth Tables

Arguments can also be analyzed using truth tables, although this can be a lot of work.

Analyzing arguments using truth tables

To analyze an argument with a truth table:

1. Represent each of the premises symbolically.
2. Create a conditional statement, joining all the premises to form the antecedent and using the conclusion as the consequent.
3. Create a truth table for the statement. If it is always true, then the argument is valid.

Example 1.4.6

Consider the argument

Premise: If you bought bread, then you went to the store.

Premise: You bought bread.

Conclusion: You went to the store.

Solution

While this example is fairly obviously a valid argument, we can analyze it using a truth table by representing each of the premises symbolically. We can then form a conditional statement showing that the premises together imply the conclusion. If the truth table is a tautology (always true), then the argument is valid.

We'll let b represent "you bought bread" and s represent "you went to the store". Then the argument becomes:

Premise: $b \rightarrow s$

Premise: b

Conclusion: s

To test the validity, we look at whether the combination of both premises implies the conclusion; is it true that $[(b \rightarrow s) \bigwedge b] \rightarrow s$?

Table 1.4.1

b	s	b → s
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.4.2

b	s	b → s	(b → s) ⋀ b
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

Table 1.4.3

b	s	b → s	(b → s) ⋀ b	[(b → s) ⋀ b] → s
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the truth table for [(b → s) ⋀ b] → s is always true, this is a valid argument.

Try it Now 13

Determine whether the argument is valid:

Premise: If I have a shovel, I can dig a hole.

Premise: I dug a hole.

Conclusion: Therefore, I had a shovel.

Example 1.4.7

Premise: If I go to the mall, then I'll buy new jeans.

Premise: If I buy new jeans, I'll buy a shirt to go with it.

Conclusion: If I go to the mall, I'll buy a shirt.

Solution

Let m = I go to the mall, j = I buy jeans, and s = I buy a shirt.

The premises and conclusion can be stated as:

Premise: $m \rightarrow j$

Premise: $j \rightarrow s$

Conclusion: $m \rightarrow s$

We can construct a truth table for [(m → j) ⋀ (j → s)] → (m → s). Try to recreate each step and see how the truth table was constructed.

Table 1.4.4

m	j	s	m → j	j → s	(m → j) ⋀ (j → s)	m → s	[(m → j) ⋀ (j → s)] → (m → s)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From the final column of the truth table, we can see this is a valid argument.

Forms of Valid Arguments

Rather than making a truth table for every argument, we may be able to recognize certain common forms of arguments that are valid (or invalid). If we can determine that an argument fits one of the common forms, we can immediately state whether it is valid or invalid.

The Law of Detachment (Modus Ponens)

The law of detachment applies when a conditional and its antecedent are given as premises, and the consequent is the conclusion. The general form is:

Premise: p → q

Premise: p

Conclusion: q

The Latin name, modus ponens, translates to “mode that affirms”.

Example 1.4.8

Recall this argument from an earlier example:

Premise: If you bought bread, then you went to the store.

Premise: You bought bread.

Conclusion: You went to the store.

Solution

In symbolic form:

Premise: $b \rightarrow s$

Premise: b

Conclusion: s

This argument has the structure described by the law of detachment. (The second premise and the conclusion are simply the two parts of the first premise detached from each other.) Instead of making a truth table, we can say that this argument is valid by stating that it satisfies the law of detachment.

The Law of Contraposition (Modus Tollens)

The law of contraposition applies when a conditional and the negation of its consequent are given as premises, and the negation of its antecedent is the conclusion. The general form is:

Premise: $p \rightarrow q$

Premise: $\sim q$

Conclusion: $\sim p$

The Latin name, modus tollens, translates to “mode that denies”.

Notice that the second premise and the conclusion look like the contrapositive of the first premise, $\sim q \rightarrow \sim p$, but they have been detached. You can think of the law of contraposition as a combination of the law of detachment and the fact that the contrapositive is logically equivalent to the original statement.

Example 1.4.9

Premise: If I drop my phone into the swimming pool, my phone will be ruined.

Premise: My phone isn’t ruined.

Conclusion: I didn’t drop my phone into the swimming pool.

Solution

If we let d = I drop the phone in the pool and r = the phone is ruined, then we can represent the argument this way:

Premise $d \rightarrow r$

Premise $\sim r$

Conclusion: $\sim d$

The form of this argument matches what we need to invoke the law of contraposition, so it is a valid argument.

Try it Now 14

Is this argument valid?

> Premise: If you brushed your teeth before bed, then your toothbrush will be wet.
>
> Premise: Your toothbrush is dry.
>
> Conclusion: You didn’t brush your teeth before bed.

The Transitive Property (Hypothetical Syllogism)

The transitive property has as its premises a series of conditionals, where the consequent of one is the antecedent of the next. The conclusion is a conditional with the same antecedent as the first premise and the same consequent as the final premise. The general form is:

Premise: $p \rightarrow q$

Premise: $q \rightarrow r$

Conclusion: $p \rightarrow r$

The earlier example about buying a shirt at the mall is an example illustrating the transitive property. It describes a chain reaction: if the first thing happens, then the second thing happens, and if the second thing happens, then the third thing happens. Therefore, if we want to ignore the second thing, we can say that if the first thing happens, then we know the third thing will happen. We don’t have to mention the part about buying jeans; we can simply say that the first event leads to the final event. We could even have more than two premises; as long as they form a chain reaction, the transitive property will give us a valid argument.

Example 1.4.10

Premise: If a soccer player commits a reckless foul, she will receive a yellow card.

Premise: If Hayley receives a yellow card, she will be suspended for the next match.

Conclusion: If Hayley commits a reckless foul, she will be suspended for the next match.

Solution

If we let r = committing a reckless foul, y = receiving a yellow card, and s = being suspended, then our argument looks like this:

Premise $r \rightarrow y$

Premise $y \rightarrow s$

Conclusion: $r \rightarrow s$

This argument has the exact structure required to use the transitive property, so it is a valid argument.

Try it Now 15

Is this argument valid?

Premise: If the old lady swallows a fly, she will swallow a spider.

Premise: If the old lady swallows a spider, she will swallow a bird.

Premise: If the old lady swallows a bird, she will swallow a cat.

Premise: If the old lady swallows a cat, she will swallow a dog.

Premise: If the old lady swallows a dog, she will swallow a goat.

Premise: If the old lady swallows a goat, she will swallow a cow.

Premise: If the old lady swallows a cow, she will swallow a horse.

Premise: If the old lady swallows a horse, she will die, of course.

Conclusion: If the old lady swallows a fly, she will die, of course.

Disjunctive Syllogism

In a disjunctive syllogism, the premises consist of an or statement and the negation of one of the options. The conclusion is the other option. The general form is:

Premise: p $\bigvee$ q

Premise: ~p

Conclusion: q

The order of the two parts of the disjunction isn't important. In other words, we could have the premises p $\bigvee$ q and ~q, and the conclusion p.

Example 1.4.11

Premise: I can either drive or take the train.

Premise: I refuse to drive.

Conclusion: I will take the train.

Solution

If we let d = I drive and t = I take the train, then the symbolic representation of the argument is:

Premise $d \bigvee t$

Premise $\sim d$

Conclusion: t

This argument is valid because it has the form of a disjunctive syllogism. I have two choices, and one of them is not going to happen, so the other one must happen.

Try it Now 16

Is this argument valid?

Premise: Alison was required to write a 10-page paper or give a 5-minute speech.

Premise: Alison did not give a 5-minute speech.

Conclusion: Alison wrote a 10-page paper.

Keep in mind that, when you are determining the validity of an argument, you must assume that the premises are true. If you don't agree with one of the premises, you need to keep your personal opinion out of it. Your job is to pretend that the premises are true and then determine whether they force you to accept the conclusion. You may attack the premises in a court of law or a political discussion, of course, but here we are focusing on the structure of the arguments, not the truth of what they actually say.

We have just looked at four forms of valid arguments; there are two common forms that represent invalid arguments, which are also called fallacies.

The Fallacy of the Converse

The fallacy of the converse arises when a conditional and its consequent are given as premises, and the antecedent is the conclusion. The general form is:

Premise: p → q

Premise: q

Conclusion: p

Notice that the second premise and the conclusion look like the converse of the first premise, $q \rightarrow p$, but they have been detached. The fallacy of the converse incorrectly tries to assert that the converse of a statement is equivalent to that statement.

Example 1.4.12

Premise: If I drink coffee after noon, then I have a hard time falling asleep that night.

Premise: I had a hard time falling asleep last night.

Conclusion: I drank coffee after noon yesterday.

Solution

If we let c = I drink coffee after noon and h = I have a hard time falling asleep, then our argument looks like this:

Premise $c \rightarrow h$

Premise h

Conclusion: c

This argument uses converse reasoning, so it is an invalid argument. There could be plenty of other reasons why I couldn't fall asleep: I could be worried about money, my neighbors might have been setting off fireworks, …

Try it Now 17

Is this argument valid?

Premise: If you pull that fire alarm, you will get in big trouble.

Premise: You got in big trouble.

Conclusion: You must have pulled the fire alarm.

The Fallacy of the Inverse

The fallacy of the inverse occurs when a conditional and the negation of its antecedent are given as premises, and the negation of the consequent is the conclusion. The general form is:

Premise: p → q

Premise: ~p

Conclusion: ~q

Again, notice that the second premise and the conclusion look like the inverse of the first premise, ~p → ~q, but they have been detached. The fallacy of the inverse incorrectly tries to assert that the inverse of a statement is equivalent to that statement.

Example 1.4.13

Premise: If you listen to the Grateful Dead, then you are a hippie.

Premise: Sky doesn't listen to the Grateful Dead.

Conclusion: Sky is not a hippie.

Solution

If we let g = listen to the Grateful Dead and *h* = is a hippie, then this is the argument:

Premise $g \rightarrow h$

Premise $\sim g$

Conclusion: $\sim h$

This argument is invalid because it uses inverse reasoning. The first premise does not imply that all hippies listen to the Grateful Dead; there could be some hippies who listen to Phish instead.

Try it Now 18

Is this argument valid?

Premise: If a hockey player trips an opponent, he will be assessed a 2-minute penalty.

Premise: Alexei did not trip an opponent.

Conclusion: Alexei will not be assessed a 2-minute penalty.

Of course, arguments are not limited to these six basic forms; some arguments have more premises, or premises that need to be rearranged before you can see what is really happening. There are plenty of other forms of arguments that are invalid. If an argument doesn't seem to fit the pattern of any of these common forms, though, you may want to use a Venn diagram or a truth table instead.

Lewis Carroll, author of Alice's Adventures in Wonderland, was a math and logic teacher, and wrote two books on logic. In them, he would propose premises as a puzzle, to be connected using syllogisms. The following example is one such puzzle.

Example 1.4.14

Solve the puzzle. In other words, find a logical conclusion from these premises.

All babies are illogical.

Nobody is despised who can manage a crocodile.

Illogical persons are despised.

Solution

Let *b* = is a baby, *d* = is despised, *i* = is illogical, and m = can manage a crocodile.

Then we can write the premises as:

$b \rightarrow i$

$m \rightarrow \sim d$

$i \rightarrow d$

Writing the second premise correctly can be a challenge; it can be rephrased as “If you can manage a crocodile, then you are not despised.”

Using the transitive property with the first and third premises, we can conclude that b → d; that all babies are despised. Using the contrapositive of the second premise, d → ~m, we can then use the transitive property with b → d to conclude that b → ~m; that babies cannot manage crocodiles. While it is silly, this is a logical conclusion from the given premises.

Example 1.4.15

Premise: If I work hard, I’ll get a raise.

Premise: If I get a raise, I’ll buy a boat.

Conclusion: If I don’t buy a boat, I must not have worked hard.

Solution

If we let h = working hard, r = getting a raise, and b = buying a boat, then we can represent our argument symbolically:

Premise $h \rightarrow r$

Premise $r \rightarrow b$

Conclusion: $\sim b \rightarrow \sim h$

Using the transitive property with the two premises, we can conclude that h → b; if I work hard, then I will buy a boat. When we learned about the contrapositive, we saw that the conditional statement h → b is equivalent to ~b → ~h. Therefore, the conclusion is indeed a logical syllogism derived from the premises.

Try it Now 19

Is this argument valid?

Premise: If I go to the party, I’ll be really tired tomorrow.

Premise: If I go to the party, I’ll get to see friends.

Conclusion: If I don’t see friends, I won’t be tired tomorrow.

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1.5: Problem Solving

Unlike exercises, there is never a simple recipe for solving a problem. You can get better and better at solving problems, both by building up your background knowledge and by simply practicing. As you solve more problems (and learn how other people solved them), you learn strategies and techniques that can be useful. But no single strategy works every time.

Pólya's How to Solve It

George Pólya was a great champion in the field of *teaching* effective problem solving skills. He was born in Hungary in 1887, received his Ph.D. at the University of Budapest, and was a professor at Stanford University (among other universities). He wrote many mathematical papers along with three books, most famously, "How to Solve it." Pólya died at the age 98 in 1985.

In 1945, Pólya published the short book *How to Solve It*, which gave a four-step method for solving mathematical problems:

1. First, you have to understand the problem.
2. After understanding, then make a plan.
3. Carry out the plan.
4. Look back on your work. How could it be better?

This is all well and good, but how do you actually do these steps?!?! Steps 1. and 2. are particularly mysterious! How do you "make a plan?" That is where you need some tools in your toolbox, and some experience to draw upon.

Much has been written since 1945 to explain these steps in more detail, but the truth is that they are more art than science. This is where math becomes a creative endeavor (and where it becomes so much fun). We will articulate some useful problem solving strategies, but no such list will ever be complete. This is really just a start to help you on your way. The best way to become a skilled problem solver is to learn the background material well, and then to solve a lot of problems!

Problem Solving Strategy 1 (Guess and Test)

Make a guess and test to see if it satisfies the demands of the problem. If it doesn't, alter the guess appropriately and check again. Keep doing this until you find a solution.

Example 1.5.1

Mr. Jones has a total of 25 chickens and cows on his farm. How many of each does he have if all together there are 76 feet?

Solution

Step 1: Understanding the problem

We are given in the problem that there are 25 chickens and cows.

All together there are 76 feet.

Chickens have 2 feet and cows have 4 feet.

We are trying to determine how many cows and how many chickens Mr. Jones has on his farm.

Step 2: Devise a plan

Going to use Guess and test along with making a tab

Many times the strategy below is used with guess and test.

Make a table and look for a pattern:

Procedure: Make a table reflecting the data in the problem. If done in an orderly way, such a table will often reveal patterns and relationships that suggest how the problem can be solved.

Step 3: Carry out the plan:

Table 1.5.1

Chickens	Cows	Number of chicken feet	Number of cow feet	Total number of feet
20	5	40	20	60

Chickens	Cows	Number of chicken feet	Number of cow feet	Total number of feet
21	4	42	16	58
Notice we are going in the wrong direction! The total number of feet is decreasing!				
19	6	38	24	62
Better! The total number of feet are increasing!				
15	10	30	40	70
12	13	24	52	76

Step 4: Looking back:

Check: 12 + 13 = 25 heads

24 + 52 = 76 feet.

We have found a solution to this problem. I could use this strategy when there are a limited number of possible answers and when two items are the same but they have one characteristic that is different.

Check in question 1:

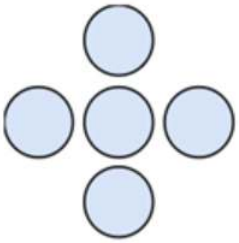


Figure 1.5.1

Place the digits 8, 10, 11, 12, and 13 in the circles to make the sums across and vertically equal 31.

Check in question 2:

Old McDonald has 250 chickens and goats in the barnyard. Altogether there are 760 feet. How many of each animal does he have? Make sure you use Polya's 4 problem-solving steps.

Problem Solving Strategy 2 (Using a variable to find the sum of a sequence.)

Gauss's strategy for sequences:

- last term = (fixed number) (n -1) + first term

The fix number is the amount each term is increasing or decreasing by. *"n"* is the number of terms you have. You can use this formula to find the last term in the sequence or the number of terms you have in a sequence.

Example 1.5.2

2, 5, 8, ... Find the 200th term.

Solution

Last term = 3(200-1) +2

Last term is 599.

Check in question 3:

Find the 320th term of 7, 10, 13, 16 …

Problem Solving Strategy 3 (Working Backwards)

This is considered a strategy in many schools. If you are given an answer, and the steps that were taken to arrive at that answer, you should be able to determine the starting point.

Go Deeper! *Working Backwards*

1. Click here to watch a Ted Talk that explains the benefits of working backwards.

Example 1.5.3

Karen is thinking of a number. If you double it and subtract 7, you obtain 11. What is Karen's number?

Solution

1. We start with 11 and work backwards.

2. The opposite of subtraction is addition. We will add 7 to 11. We are now at 18.

3. The opposite of doubling something is dividing by 2. 18/2 = 9

4. This should be our answer. Looking back:

9 x 2 = 18 -7 = 11

5. We have the right answer.

Check in question 4:

Christina is thinking of a number.

If you multiply her number by 93, add 6, and divide by 3, you obtain 436. What is her number? Solve this problem by working backwards. (5 points)

Problem Solving Strategy 4 (Looking for a Pattern)

Definition: Sequence

A sequence is a pattern involving an ordered arrangement of numbers.

We first need to find a pattern.

Ask yourself as you search for a pattern – are the numbers growing steadily larger? Steadily smaller? How is each number related?

Example 1.5.4

1, 4, 7, 10, 13… Find the next 2 numbers.

Solution

The pattern is each number is increasing by 3. The next two numbers would be 16 and 19.

Example 1.5.5

1, 4, 9, 16 … Find the next 2 numbers.

Solution

It looks like each number is a perfect square. $1^2 = 1$, $2^2 = 4$

$3^2 = 9$

$4^2 = 16$

So the next numbers would be

$5^2 = 25$

$6^2 = 36$

Example 1.5.6

10, 7, 4, 1, -2… Find the next 2 numbers.

Solution

In this sequence, the numbers are decreasing by 3. So the next 2 numbers would be

-2 -3 = -5

-5 – 3 = -8

Example 1.5.7

1, 2, 4, 8 …Ffind the next two numbers.

Solution

This example is a little bit harder. The numbers are increasing but not by a constant. Maybe a factor?

1 x 2 = 2

2 x 2 = 4

4 x 2 =8

So each number is being multiplied by 2.

8 x 2 = 16

16 x 2 = 32

Problem Solving Strategy 5 (Make a List)

Example 1.5.8

Can perfect squares end in a 2 or a 3?

Solution

List all the squares of the numbers 1 to 20.

1 4 9 16 25 36 49 64 81 100 121 144 169 196 225 256 289 324 361 400.

Now look at the number in the ones digits. Notice they are 0, 1, 4, 5, 6, or 9. Notice none of the perfect squares end in 2, 3, 7, or 8. This list suggests that perfect squares cannot end in a 2, 3, 7 or 8. But note that this is just an observation. To answer this question, one would need a mathematically rigorous proof.

Example 1.5.9

How many different amounts of money can you have in your pocket if you have only three coins including only dimes and quarters?

Solution

Quarter's dimes

0 3 30 cents

1 2 45 cents

2 1 60 cents

3 0 75 cents

Check in question 5:

How many ways can you make change for 23 cents using only pennies, nickels, and dimes? (10 points)

Problem Solving Strategy 6 (Process of Elimination)

This strategy can be used when there is only one possible solution.

Example 1.5.10

I'm thinking of a number.

- The number is odd.
- It is more than 1 but less than 100.
- It is greater than 20.
- It is less than 5 times 7.
- The sum of the digits is 7.
- It is evenly divisible by 5.

Solution

- We know it is an odd number between 1 and 100.
- It is greater than 20 but less than 35
  - 21, 23, 25, 27, 29, 31, 33, 35. These are the possibilities.
- The sum of the digits is 7
  - 21 (2+1=3) No
  - 23 (2+3 = 5) No
  - 25 (2 + 5= 7) Yes

Using the same process we see there are no other numbers that meet this criteria. Also we notice 25 is divisible by 5. By using the strategy elimination, we have found our answer.

Reference

1. References (2)

Exercises

Fallacies in Common Language

For each of the following statements, name the type of logical fallacy being used.

1. If you don't want to drive from Boston to New York, then you will have to take the train.
2. Every time I go to Dodger Stadium, the Dodgers win. I should go there for every game.
3. New England Patriots quarterback Tom Brady likes his footballs slightly underinflated. The "Cheatriots" have a history of bending or breaking the rules, so Brady must have told the equipment manager to make sure that the footballs were underinflated.
4. What you are doing is clearly illegal because it's against the law.
5. The county supervisor voted against the new education tax. He must not believe in education.
6. "Apples a day keeps doctors away." No one has said apples are bad, so this old saying must be true.
7. Wine has to be good for your health because… I mean, can you imagine a life without wine?
8. Studies show that slightly overweight senior citizens live longer than underweight ones. The extra weight must make people live longer.
9. Whenever our smoke detector beeps, my kids eat cereal for dinner. The loud beeping sound must make them want to eat cereal for some reason.

10. There is a scientist who works at a really good university, and he says there is no strong evidence for climate change, especially global warming. Some politicians also question climate change. So I don't really believe it.
11. My neighbor cheats on his tax returns. I don't believe anything he says.
12. A: "Don't fight over small things. Just let them go."

    B: "What exactly are 'small things'? How do I know what is small and what isn't?"

    A: "Well, small things are things you really don't want to fight over."

Propositions and Logic

13. List the set of integers that satisfy the following statement: A positive multiple of 5 and not a multiple of 2
14. List the set of integers that satisfy the following statement: Greater than 12 and less than or equal to 18
15. List the set of integers that satisfy the following statement: Even number less than 10 or odd number between 12 and 10
16. You qualify for a special discount if you are either

- a full-time student in the state of California or
- at least 18 and your income is less than $20,000 a year.

For each person below, determine if the person qualifies for this discount. If more information is needed, indicate that.

a. A 17-year-old full-time student at a California community college with no job

b. A 28-year-old man earning $50,000 a year

c. A 60-year-old grandmother who does not work and does not go to school

d. A boy in first grade

e. A mother making $18,000 a year and not enrolled in any college

f. A 22-year-old full-time student at Arizona State University with no job

g. A 18-year-old earning exactly $20,000 a year while attending UCLA part-time

17. Write the negation: Everyone failed the quiz today.
18. Write the negation: Someone in the car needs to use the restroom.
19. How can you prove this statement wrong?

    "Everyone who ate at that restaurant got sick."
20. How can you prove this statement wrong?

    "There was someone who ate at that restaurant and got sick."
21. How can you prove this statement wrong?

    "There is no baseball player who can excel at both pitching and hitting. Everyone must choose one or the other."
22. How can you prove this statement wrong?

    "Every student must have an ID number before registering for classes."

Truth Tables

23. Translate each statement from symbolic notation into English sentences. Let A represent "Elvis is alive" and let G represent "Elvis gained weight."

a. $A \bigvee G$

b. $\sim(A \bigwedge G)$

c. $G \rightarrow \sim A$

d. $A \leftrightarrow \sim G$

e. $A \bigwedge \sim G$

f. $\sim(A \bigvee G)$

24. Create a truth table for each statement below.

a. $A \bigwedge \sim B$

b. $\sim(\sim A \bigvee B)$

c. $(A \bigwedge B) \rightarrow C$

d. $(A \bigvee B) \rightarrow \sim C$

25. Complete the truth table for $(A \bigvee B) \bigwedge \sim(A \bigwedge B)$.

A	B	A ⋁ B	A ⋀ B	~(A ⋀ B)	(A ⋁ B) ⋀ ~(A ⋀ B)
T	T				
T	F				
F	T				
F	F				

26. We have been studying the *inclusive* or, which allows both *A* and *B* to be true. The *exclusive or* does not allow both to be true; it translates to "either *A* or *B*, but not both." For each situation, decide whether the "or" is most likely exclusive or inclusive.

a. An entrée at a restaurant includes soup or a salad.

b. You should bring an umbrella or a raincoat with you.

c. We can keep driving on I-5 or get on I-405 at the next exit.

d. Use Gate 1 if you are at least 35 years old, or Gate 2 if you are younger.

e. You should save this document on your computer or a flash drive.

f. I am not sure if my pregnant wife is going to have a boy or a girl.

27. Consider the statement "If you are under age 17, then you cannot attend this movie."

a. Write the converse.

b. Write the inverse.

c. Write the contrapositive.

28. Consider the statement "If you have a house in Beverly Hills, you are rich."

a. Write the converse.

b. Write the inverse.

c. Write the contrapositive.

29. Assume that the statement "If you swear, then you will get your mouth washed out with soap" is true. Which of the following statements must also be true?

a. If you don't swear, then you won't get your mouth washed out with soap.

b. If you don't get your mouth washed out with soap, then you didn't swear.

c. If you get your mouth washed out with soap, then you swore.

30. Write the negation: If Luke faces Vader, then Obi-Wan cannot interfere.

31. Write the negation: If you look both ways before crossing the street, then you will not get hit by a car.

32. Write the negation: If you weren't talking, then you wouldn't have missed the instructions.

33. Write the negation: If you score a goal now, we will win.
34. Assume that the biconditional statement “You will play in the game if and only if you attend all practices this week” is true. Which of the following situations could NOT happen?
    a. You attended all practices this week and didn’t play in the game.
    b. You didn’t attend all practices this week and played in the game.
    c. You didn’t attend all practices this week and didn’t play in the game.
35. Use De Morgan’s Laws to rewrite the disjunction as a conjunction: It is not true that Tina likes Sprite or 7-Up.
36. Use De Morgan’s Laws to rewrite the disjunction as a conjunction: It is not true that the father or the mother of that child will be required to testify.
37. Use De Morgan’s Laws to rewrite the conjunction as a disjunction: It is not the case that both the House and the Senate passed the bill.
38. Use De Morgan’s Laws to rewrite the conjunction as a disjunction: It is not the case that you need a dated receipt and your credit card to return this item.

Analyzing Arguments

39. Determine whether each of the following is an inductive or deductive argument:
    a. The new medicine works. We tried it on 100 patients, and all of them were cured.
    b. Every student has an ID number. Sandra is a student, so she has an ID number.
    c. Every angle of a rectangle is 90 degrees. A soccer field (pitch) is rectangular, so every corner is 90 degrees.
    d. Everything that goes up comes down. I throw a ball up. It must come down.
    e. Every time it rains, my grass grows fast. Rain speeds up the growth of grass.
    f. Sports makes a person strong. My daughter plays basketball. She will be strong.
40. Analyze the validity of the argument.

    Everyone who gets a degree in science will get a good job. I got a science degree. Therefore, I will get a good job.
41. Analyze the validity of the argument.

    If someone turns off the switch, the lights will not be on. The lights are off. Therefore, someone must have turned off the switch.
42. Suppose the statement: “If today is Dec. 25, then the store is closed.”
    a. “Today is Dec. 25. Thus, the store is closed.” Which property was used? Is this valid?
    b. “The store is not closed. So today is not Dec. 25.” Which property was used? Is this valid?
    c. “The store is closed. Therefore, today must be Dec. 25.” Which property was used? Is this valid?
    d. “Today is not Dec. 25. Thus, the store is open.” Which property was used? Is this valid?

For the following questions, use a Venn diagram or a truth table to determine the validity.

43. If a person is on this reality show, they must be self-absorbed. Laura is not self-absorbed. Therefore, Laura cannot be on this reality show.
44. If you are a triathlete, then you have outstanding endurance. LeBron James is not a triathlete. Therefore, LeBron does not have outstanding endurance.
45. Jamie must scrub the toilets or hose down the garbage cans. Jamie refuses to scrub the toilets. Therefore, Jamie will hose down the garbage cans.
46. Some of these kids are rude. Jimmy is one of these kids. Therefore, Jimmy is rude!
47. Every student brought a pencil or a pen. Marcie brought a pencil. Therefore, Marcie did not bring a pen.

48. If a creature is a chimpanzee, then it is a primate. If a creature is a primate, then it is a mammal. Bobo is a mammal. Therefore, Bobo is a chimpanzee.

49. Every cripsee is a domwow. Mekep is not a domwow. Therefore, Mekep is not a cripsee. (This sentence has a lot of made-up words, but it is still possible to check for the validity of the argument. This is a good practice for abstract thinking.)

50. Whoever dephels a kipoc will be bopied. I did not dephel any kipoc. Therefore, I will not be bopied. (Again, you do not need to know the meaning of each word to do this exercise.)

Problem Solving

For the following exercises, apply any problem-solving strategies and your critical-thinking skills to solve various types of problems. There is single formula or procedure to follow. Be flexible and consider all possibilities.

51. There are 13 postage stamps on the table. Some are 20-cent stamps while others are 45-cent stamps. The total postage value of these stamps is $4.10.
    a. If they were all 20-cent stamps, would 13 of them add up to $4.10?
    b. If there were five 20-cent stamps, would these stamps add up to $4.10?
    c. How about ten 20-cent stamps? OK, you probably got some idea now.
    d. How many 20-cent stamps are there?
    e. Can you think of another way to solve the problem?
    f. What would you say to a friend of yours who tries to help you out by writing a system of two equations to solve this problem?

52. Find the next two terms of each of the following sequences and explain why.
    a. 9, 7, 5, 3, …
    b. 0, 1, 4, 9, 16, 25, …
    c. 3, 6, 12, 24, …
    d. 0, 1, 3, 6, 10, 15, …
    e. 5, 7, 5, 5, 7, 5, 5, 7, 5, 5, …
    f. 1, 1, 2, 3, 5, 8, 13, 21, … (Fibonacci Sequence) We will study it later.

53. A man bought an old car for $2000. He fixed it up and sold it for $2,500. But he missed it, so he brought it back for $3,200. Later, he sold it for $4,000. How much did he make in these transactions?
    a. Is it $4,000 - $2,000 = $2,000?
    b. He made $500 on the first sale and then $800 on the second. But he lost $700 in between when he re-purchased it. So is it $800 - $700 = $100?
    c. Is it $500 + $800 = $1,300?

54. See the pattern below. Can you make a conjecture? Is it true? Can you prove it?

    1=1 (=12).

    1+3=4 (=22).

    1+3+5=9 (=32).

    1+3+5+7=16 (=42).

55. A heart surgeon is about to perform a medical procedure on a boy. The surgeon told the nurses, “I want you to know that this is my son. I am operating on my own child today.” Everyone there knew that the boy was the surgeon’s son. But the surgeon was not the boy’s father. How can this be true?

56. What can “H.D.” in this little story be?

"H.D. sat on a wall. H.D. had a great fall. All the president's horses and all the president's men couldn't put H.D. together again."

57. You have decided to work out at the gym at least twice a week, but never on two consecutive days. If you are to keep the same schedule every week, list all possible days of the week) you can exercise at the gym.
58. A man must be married to have a mother-in-law but can be single and have a brother-in-law. Explain why.
59. Your sprinklers are set to water your plants at 6 am every morning during summer, when the daylight saving time is in effect. Once the time goes back to regular time (in November, say from PDT to PST), what time do your sprinklers start? (Hint: it's either 5 am or 7 am.) What is the best way to explain this to your friend?
60. You are to visit a friend who lives 300 miles away. You drive to his house early in the morning, averaging 60 mph, but you return in the afternoon, when the freeway is jammed, at an average speed of 40 mph. Is the overall average speed 50 mph? Why or why not?
61. You have 10 identical pairs of black socks and 9 identical parts of white socks in your drawer, except each pair is not "paired up," i.e., each sock is in the drawer, separated from all others. The room is completely dark, and you cannot see which socks you are taking out. How many socks do you have to pull out if you want to be sure that you get
    a. A pair of white socks?
    b. A pair of black socks?
    c. Any matching pair?
62. A divorced 49-year-old man with a 25-year-old son marries a young woman whose mother is a widow. The 25-year-old son marries the widow and have a baby girl.
    a. How is that baby related to the 49-year-old man? (The baby is his son's daughter.)
    b. How else is that baby related to the 49-year-old man? (The baby is his wife's mother's daughter.)
    c. Does that baby have a step-brother? If so, under what conditions?
    d. Who is that baby's step-sister?
    e. Combine your answers to describe how the 49-year-old man is related to the baby.
63. A person can be ½ Chinese and ½ Italian. Are the following cases possible?
    a. ½ Japanese, ¼ Russian, ¼ Irish
    b. 3/8 Scottish, 3/8 Vietnamese, ¼ Spanish
    c. 3/8 Mexican, ¼ Turkish, and ½ Norwegian
    d. 1/6 French, ½ Canadian, 1/3 Brazilian
    e. ½ Armenian, ¼ German, ½ Swedish
64. John says, "I don't have any brothers, sisters, step-brothers, or step-sisters. See that tall woman? Her father is my mother's child." Who is the tall woman?

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CHAPTER OVERVIEW

2: Mathematics and Numbers

In previous math courses, you've no doubt run into the infamous "word problems." Unfortunately, these problems rarely resemble the type of problems we actually encounter in everyday life. In math books, you usually are told exactly which formula or procedure to use and are given exactly the information you need to answer the question. In real life, problem solving requires identifying an appropriate formula or procedure and determining what information you will need (and won't need) to answer the question.

In this chapter, we will review several basic but powerful ideas: percents, rates, proportions, conversions, and estimation. We will then focus on the problem solving process and explore how to use these ideas to solve problems where we don't have perfect information. We will then conclude the chapter by discussing deceptive and misleading use of numbers.

2.1: Estimating the Answer
2.2: Really Big and Really Small Numbers
2.3: Deceptive and Misleading Numbers
2.4: Percentages and Proportions
2.5: Conversions

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2.1: Estimating the Answer

We now bring together the mathematical tools we've reviewed and use them to approach more complex problems. In many problems, it is tempting to take the given information, plug it into whatever formulas you have handy, and hope that the result is what you were supposed to find. Chances are that this approach has served you well in other math classes.

This approach may not work well with real life problems. Instead, problem solving is best approached by first starting at the end: identifying exactly what you are looking for. From there, you then work backwards, asking "what information and procedures will I need to find this?" Very few interesting questions can be answered in one mathematical step; often you will need to chain together a solution pathway, a series of steps that will allow you to answer the question.

Here is another problem solving process, similar to Polya's method presented in Chapter 1. This time, the emphasis is in determining what you need to know and estimating the answer.

1. Identify the question you're trying to answer.
2. Work backwards, identifying the information you will need and the relationships you will use to answer that question.
3. Continue working backwards, creating a solution pathway.
4. If you are missing necessary information, look it up or estimate it. If you have unnecessary information, ignore it.
5. Solve the problem, following your solution pathway.

In most problems we work, we will be approximating a solution because we will not have perfect information. We will begin with a few examples where we will be able to approximate the solution using basic knowledge from our lives.

Example 2.1.1

How many times does your heart beat in a year?

Solution

This question is asking for the rate of heart beats per year. Since a year is a long time to measure heart beats for, if we knew the rate of heart beats per minute, we could scale that quantity up to a year. So the information we need to answer this question is heart beats per minute. This is something you can easily measure by counting your pulse while watching a clock for a minute.

Suppose you count 80 beats in a minute. To convert this beats per year:

$$\frac{80 \text{ beats}}{1 \text{ minute}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} \cdot \frac{24 \text{ hours}}{1 \text{ day}} \cdot \frac{365 \text{ days}}{1 \text{ year}} = 42,048,000 \text{ beats per year}$$

Example 2.1.2

How thick is a single sheet of paper? How much does it weigh?

Solution

While you might have a sheet of paper handy, trying to measure it would be tricky. Instead we might imagine a stack of paper and then scale the thickness and weight to a single sheet. If you've ever bought paper for a printer or copier, you probably bought a ream, which contains 500 sheets. We could estimate that a ream of paper is about 2 inches thick and weighs about 5 pounds. Scaling these down,

$$\frac{2 \text{ inches}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.004 \text{ inches per sheet}$$

$$\frac{5 \text{ pounds}}{\text{ream}} \cdot \frac{1 \text{ ream}}{500 \text{ pages}} = 0.01 \text{ pounds per sheet, or 0.16 ounces per sheet.}$$

Example 2.1.3

A recipe for zucchini muffins states that it yields 12 muffins, with 250 calories per muffin. You instead decide to make mini-muffins, and the recipe yields 20 muffins. If you eat 4, how many calories will you consume?

Solution

There are several possible solution pathways to answer this question. We will explore one.

To answer the question of how many calories 4 mini-muffins will contain, we would want to know the number of calories in each mini-muffin. To find the calories in each mini-muffin, we could first find the total calories for the entire recipe and then divide it by the number of mini-muffins produced. To find the total calories for the recipe we could multiply the calories per standard muffin by the number per muffin. Notice that this produces a multi-step solution pathway. It is often easier to solve a problem in small steps rather than trying to find a way to jump directly from the given information to the solution.

We can now execute our plan:

$12 \text{ muffins} \cdot \dfrac{250 \text{ calories}}{\text{muffin}} = 3000$ calories for thew whole recipe

$\dfrac{3000 \text{ calories}}{20 \text{ mini } - \text{ muffins}}$ gives 150 calories per mini-muffin

$4 \text{ mini muffins} \cdot \dfrac{150 \text{ calories}}{\text{mini } - \text{ muffin}}$ totals 600 calories consumed.

Example 2.1.4

You need to replace the boards on your deck. About how much will the materials cost?

Solution

There are two approaches we could take to this problem: 1) estimate the number of boards we will need and find the cost per board, or 2) estimate the area of the deck and find the approximate cost per square foot for deck boards. We will take the latter approach.

For this solution pathway, we will be able to answer the question if we know the cost per square foot for decking boards and the square footage of the deck. To find the cost per square foot for decking boards, we could compute the area of a single board and divide it into the cost for that board. We can compute the square footage of the deck using geometric formulas. So first we need information: the dimensions of the deck and the cost and dimensions of a single deck board.

Suppose that measuring the deck, it is rectangular, measuring 16 ft by 24 ft, for a total area of 384 ft^2.

From a visit to the local home store, you find that an 8 foot by 4 inch cedar deck board costs about $7.50. The area of this board, doing the necessary conversion from inches to feet, is:

$8 \text{ feet } \cdot 4 \text{ inches} \cdot \dfrac{1 \text{ foot}}{12 \text{ inches}} = 2.667 \text{ ft}^2$.

The cost per square foot is then $\dfrac{\$7.50}{2.667 \text{ ft}^2} = \$2.8125 \text{ per ft}^2$.

This will allow us to estimate the material cost for the whole 384 ft^2 deck

$\$384 \text{ ft}^2 \cdot \dfrac{\$2.8125}{\text{ft}^2} = \$1080$ total cost.

Of course, this cost estimate assumes that there is no waste, which is rarely the case. It is common to add at least 10% to the cost estimate to account for waste.

Example 2.1.5

Is it worth buying a Hyundai Sonata hybrid instead the regular Hyundai Sonata?

Solution

To make this decision, we must first decide what our basis for comparison will be. For the purposes of this example, we'll focus on fuel and purchase costs, but environmental impacts and maintenance costs are other factors a buyer might consider.

It might be interesting to compare the cost of gas to run both cars for a year. To determine this, we will need to know the miles per gallon both cars get, as well as the number of miles we expect to drive in a year. From that information, we can find the number of gallons required from a year. Using the price of gas per gallon, we can find the running cost.

From Hyundai's website, the 2013 Sonata will get 24 miles per gallon (mpg) in the city and 35 mpg on the highway. The hybrid will get 35 mpg in the city, and 40 mpg on the highway.

An average driver drives about 12,000 miles a year. Suppose that you expect to drive about 75% of that in the city, so 9,000 city miles a year, and 3,000 highway miles a year.

We can then find the number of gallons each car would require for the year.

Sonata:

$$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{24 \text{ city miles}} + 3000 \text{ highway miles} \cdot \frac{1 \text{ gallon}}{35 \text{ highway miles}} = 460.7 \text{ gallons}$$

Hybrid:

$$9000 \text{ city miles} \cdot \frac{1 \text{ gallon}}{35 \text{ city miles}} + 3000 \text{ highway miles} \cdot \frac{1 \text{ gallon}}{40 \text{ highway miles}} = 332.1 \text{ gallons}$$

If gas in your area averages about $3.50 per gallon, we can use that to find the running cost:

Sonata: $460.7 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1612.45$

Hybrid: $332.1 \text{ gallons} \cdot \frac{\$3.50}{\text{gallon}} = \$1162.35$

The hybrid will save $450.10 a year. The gas costs for the hybrid are about $\frac{\$450.10}{1612.45} = 0.279 = 27.9\%$ lower than the costs for the standard Sonata.

While both the absolute and relative comparisons are useful here, they still make it hard to answer the original question, since "is it worth it" implies there is some tradeoff for the gas savings. Indeed, the hybrid Sonata costs about $25,850, compared to the base model for the regular Sonata, at $20,895.

To better answer the "is it worth it" question, we might explore how long it will take the gas savings to make up for the additional initial cost. The hybrid costs $4965 more. With gas savings of $451.10 a year, it will take about 11 years for the gas savings to make up for the higher initial costs.

We can conclude that if you expect to own the car 11 years, the hybrid is indeed worth it. If you plan to own the car for less than 11 years, it may still be worth it, since the resale value of the hybrid may be higher, or for other non-monetary reasons. This is a case where math can help guide your decision, but it can't make the decision for you.

Try it Now 6

If traveling from Los Angeles to Las Vegas for a three-day conference, does it make more sense to drive or fly?

Extension: Taxes

Governments collect taxes to pay for the services they provide. In the United States, federal income taxes help fund the military, the environmental protection agency, and thousands of other programs. Property taxes help fund schools. Gasoline taxes help pay for road improvements. While very few people enjoy paying taxes, they are necessary to pay for the services we all depend upon.

Taxes can be computed in a variety of ways, but are typically computed as a percentage of a sale, of one's income, or of one's assets.

Example 2.1.6

The sales tax rate in a city is 9.3%. How much sales tax will you pay on a $140 purchase?

Solution

The sales tax will be 9.3% of $140. To compute this, we multiply $140 by the percent written as a decimal: $140(0.093) = $13.02.

> *When taxes are not given as a fixed percentage rate, sometimes it is necessary to calculate the effective rate.*

Effective rate

The effective tax rate is the equivalent percent rate of the tax paid out of the dollar amount the tax is based on.

Example 2.1.7

Joan paid $3,200 in property taxes on her house valued at $215,000 last year. What is the effective tax rate?

Solution

We can compute the equivalent percentage: 3200/215000 = 0.01488, or about 1.49% effective rate.

Taxes are often referred to as progressive, regressive, or flat.

Tax categories

A **flat tax**, or proportional tax, charges a constant percentage rate.

A **progressive tax** increases the percent rate as the base amount increases.

A **regressive tax** decreases the percent rate as the base amount increases.

Example 2.1.8

The United States federal income tax on earned wages is an example of a progressive tax. People with a higher wage income pay a higher percent tax on their income.

Solution

For a single person in 2011, adjusted gross income (income after deductions) under $8,500 was taxed at 10%. Income over $8,500 but under $34,500 was taxed at 15%.

A person earning $10,000 would pay 10% on the portion of their income under $8,500, and 15% on the income over $8,500, so they'd pay:

8500(0.10) = 850 10% of $8500
1500(0.15) = 225 15% of the remaining $1500 of income
Total tax: = $1075

The effective tax rate paid is 1075/10000 = 10.75%

A person earning $30,000 would also pay 10% on the portion of their income under $8,500, and 15% on the income over $8,500, so they'd pay:

8500(0.10) = 850 10% of $8500
21500(0.15) = 3225 15% of the remaining $21500 of income
Total tax: = $4075

The effective tax rate paid is 4075/30000 = 13.58%.

Notice that the effective rate has increased with income, showing this is a progressive tax.

Example 2.1.9

A gasoline tax is a flat tax when considered in terms of consumption, a tax of, say, $0.30 per gallon is proportional to the amount of gasoline purchased. Someone buying 10 gallons of gas at $4 a gallon would pay $3 in tax, which is $3/$40 = 7.5%. Someone buying 30 gallons of gas at $4 a gallon would pay $9 in tax, which is $9/$120 = 7.5%, the same effective rate.

However, in terms of income, a gasoline tax is often considered a regressive tax. It is likely that someone earning $30,000 a year and someone earning $60,000 a year will drive about the same amount. If both pay $60 in gasoline taxes over a year, the

person earning $30,000 has paid 0.2% of their income, while the person earning $60,000 has paid 0.1% of their income in gas taxes.

Try it Now 7

A sales tax is a fixed percentage tax on a person's purchases. Is this a flat, progressive, or regressive tax?

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2.2: Really Big and Really Small Numbers

Globally, there is no absolute agreement on what to call large numbers. ‘One million’ is expressed as 1,000,000, but Americans express ‘one billion’ as 1,000,000,000. This is the same as 10^9, or one followed by nine zeros. In the UK, however, 10^9 may be called ‘one thousand million.’ So, in the UK, it may be 10^{12} that represents ‘one billion’ but for Americans, that would mean ‘one trillion.’ (It should be noted some British people are beginning to adopt the American naming system.)

Now, in India, they may use commas after two zeros as opposed to three when counting higher than 1,000. So, the Indian system calls 100,000 (one hundred thousand) **one lakh** and 10,000,000 (ten million) is called **one crore.**

In many Asian countries, large numbers have names based on 10,000s rather than 1,000s. Yes, large and small numbers can be quite confusing.

The following table describes the standardized and scientific names in the American system for very large and very small numbers. Understanding your properties of exponents and exponent tables will help you interpret this. (Note: This table uses metric prefixes. EX: 1000 meters = 1 kilometer).

Table 2.2.1: Big Numbers

Number	Name	Metric Prefix
$10^1 = 10$	Ten	deka-
$10^2 = 100$	Hundred	hecto-
$10^3 = 1000$	Thousand	kilo-
$10^4 = 10,000$	Ten Thousand	myria-
$10^6 = 1,000,000$	One Million	Mega-
$10^9 = 1,000,000,000$	One Billion	Giga-
10^{12}	One Trillion	Tera-
10^{15}	One Quadrillion	Peta-
10^{18}	One Quintillion	Exa-

By the way, 10^{100}, a one followed by 100 zeros is actually named “googol.” Note that this spelling is different from the name of a large tech company.

Table 2.2.2: Small Numbers

Number	Name	Metric Prefix
$10^{-1} = 0.1$	One Tenth	deci-
$10^{-2} = 0.01$	One Hundredth	centi-
$10^{-3} = 0.001$	One Thousandth	milli-
$10^{-6} = 0.000001$	One Millionth	micro-
$10^{-9} = 0.000000001$	One Billionth	nano-
10^{-12}	One Trillionth	pico-
10^{-15}	One Quadrillionth	femto-
10^{-18}	One Quintillionth	atto-

Reference

1. References (4)

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2.3: Deceptive and Misleading Numbers

Although mathematics covers a whole lot more than numbers, it is still true that arithmetic, including the basic operations on numbers such as integers, fractions, decimals, and percentages, remains one of the most practical and fundamental applications of mathematics that you encounter every day. Many decisions are made, both in large and small scales, based on numbers, figures, statistics, and all sorts of data. Based on various numbers, companies may decide where to build a distribution center or a shopping mall, and government agencies may decide to implement certain policies. Different political parties may present different numbers to push their own agenda or to silence their opponents. Advertisers know that figures can be very effective in persuading consumers. In fact, many of us may not even be aware of how we make our shopping decisions based on numbers presented to us in a variety of forms. This is especially true in the Internet era in which we live since advertisements can be targeted based on our online activities and digital footprints.

It is precisely because of this fact that numbers and figures are frequently manipulated. In this brief section we will look at how certain numbers can be presented to us in misleading or deceptive ways to give certain impressions or to change our mind.

Relative Changes and Differences

As we have seen earlier, percentages can be calculated in different ways to produce different numbers. For instance, suppose the annual tuition is $50,000 at College A and $20,000 at College B. In terms of absolute difference, you can state, truthfully, that the tuition is $30,000 higher at College A than at College B. In terms of relative difference, however, each of the following four statements is true:

- The tuition at College A is 250% of the tuition at College B since

$$50,000 \div 20,000 = 2.50$$

- The tuition at College A is 150% more than at College B since

$$(50,000 - 20,000) \div 20,000 = 1.50$$

- The tuition at College B is 40% of the tuition at College A since

$$20,000 \div 50,000 = 0.40$$

- The tuition at College B is 60% cheaper than at College A since

$$(50,000 - 20,000) \div 50,000 = 0.60$$

Here, the first two statements use the tuition at College B as the reference number while the last two use the tuition at College A as the reference number. Note the words like “of,” “more than,” and “cheaper than.” The amount that follows “of” and “than” is the reference number.

Now, 250, 150, 40, and 60 are very different numbers, and yet, each of these numbers can be used to represent the tuition difference between these two colleges. If you are trying to persuade someone to go to College B based on the tuition, you could say, “Hey, College A charges you 250% as much as College B.” And *that* sounds dramatic.

Similar calculations can be carried out in comparing many things including prices, index numbers, crime rates, populations, and averages.

Example 2.3.1

According to Wikipedia.org, the population estimates (in 2018) for some states are as follows (rounded to the nearest thousand):

California: 39,557,000

Michigan: 9,996,000

Virginia: 8,518,000

Mississippi: 2,987,000

Wyoming: 578,000

a. The population of Michigan is what percent less than the population of California?
b. The population of California is what percent more than the population of Michigan?
c. The population of Michigan is what percent of the population of California?
d. The population of California is what percent of the population of Michigan?

Solution

a. $(39,557,000-9,996,000)\div 39,557,000=0.7473$Therefore, the population of Michigan is about 74.7% less than the population of California.
b. $(39,557,000-9,996,000)\div 9,996,000=2.9573$Therefore, California has about 295.7% more people than Michigan does.
c. $9,996,000\div 39,557,000=0.2527$Therefore, the population of Michigan is about 25.3% of that of California.
d. $39,557,000\div 9,996,000=3.957$Therefore, the population of California is about 396% of that of Michigan.

Of course, they all imply the same fact: California has about 4 times as many people as Michigan.

Try it Now 1

1. Make the same comparison as above (finding four percentage figures) between
   a. Virginia and Mississippi
   b. Michigan and Wyoming
2. As of 2018, California is the most populous state in the union while Wyoming is the least populous. The population of Wyoming is about what percent of that of California? How many times is the population of California as compared to that of Wyoming?

Ever-Changing Reference Numbers

We have seen this earlier also, but here is a very common mistake we need to pay attention to. One thing you may want to remember is this: you cannot simply add percentages when a percentage change is followed by another. This is not a difficult concept, but many people are unaware of this. Here are a few examples.

Example 2.3.2

The number of students at a city college increased 20% one year, and the number again increased by 30% the following year. Does this mean the two-year increase was 50%?

Solution

These problems can be simplified if you start with a fixed number. For example, let's assume that the college had 1000 students two years ago. Then, in one year, with a 20% increase, the college had 1200 students. But then, in the second year, the college added 30% of 1200, which is 360. With the addition of 360 students over 1200, the college has 1560 students after two years. That represents an increase of 560 over 1000, which amounts to a 56% (not 50%) increase. (By the way, it turns out that the original number is irrelevant. Regardless of the number you start with, a 20% increase followed by a 30% increase would *always* result in a total increase by 56% since 1+0.21+0.3=1.56.

Example 2.3.3

The value of a certain stock jumped up by 50% one day, only to lose 50% the following day. Does it mean that the stock is back to the original value after the two-day period?

Solution

Again, let us consider a specific number to start with. Suppose the stock was valued at $100 two days ago. After jumping up by 50%, the stock was valued at $150 yesterday. Then, after a 50% loss (and 50% of $150 is $75), the stock is now only worth $75, meaning that it lost 25% of the value in the two-day period.

Why do these things happen? It's because the reference number for the percentage has changed when the value changed. Again, remember that percentages cannot be simply added in these cases.

A change in the reference number can happen in the simplest of cases, as illustrated by the following example.

Example 2.3.4

a. A local theme park just increased the parking fee by 25%, and now it is $20. How much was the parking fee before?

b. A dining room set just went on sale and is now available at $1400, which is 30% off the regular price. What is the regular price?

Solution

Many people make the error in assuming that $200 is the reference number for 25% in (a) and, similarly, $1400 is the reference number for 30% in (b). However, careful reading suggests that this is not correct.

In (a), the new parking fee is $20 while we do not yet know what the fee was before. In other words, the problem does NOT state the reference number explicitly. However, we DO know that whatever it was, adding 25% of that old parking fee results in the new fee of $20. Now, the old parking fee is of course 100% of itself, so $20 represents 125% of the old parking fee. Hence, if PF represents the old parking fee, we have

$$PF \times 1.25 = 20$$

$$PF = 20 \div 1.25 = 16$$

Therefore, the parking fee used to be $16. If you make the common mistake of finding 25% of $20, that would be $5, and deducting $5 from $20 would give you $15. But this is clearly incorrect because adding 25% to $15 would result in $18.75 (as shown below), not $20.

$$15 \times (1 + .25) = 18.75$$

For (2), the calculation is similar except we subtract (not add) 30% (0.3) from 1. Whatever the regular price of the dining room set was, a 30% discount means that you are still paying 70% of the original price (1-0.3=0.7). Denoting the regular price by RP, we have

$$RP \times (1 - .3) = RP \times 0.7 = 1400$$

$$RP = 1400 \div 0.7 = 2000$$

So the original price of the dining room set was $2000. This makes sense because 30% of $2000 is $600, which is the discount you are getting when you pay $1400 rather than $2000 for it.

Percentages of Percentages

This was mentioned in one example in Section 2.1, but we discuss it again here with an emphasis on avoiding confusion.

Suppose the mayor of your city has worked very hard to fight poverty in your city and, as a result, the poverty rate in your city has gone down from 16% to 12% over the last four years. How much of a drop does that represent?

Of course, you can say that the poverty rate has gone down by 4% (16% – 12%), right? Well, that would be a little ambiguous, and here is why. Generally, when we use percentages to express a relative change, we use the earlier number (in this case, 16%) as the reference number. Therefore, the relative change in this case is

$$(12 - 16) \div 16 = 0.25,$$

representing a 25% drop in the poverty *rate*.

So what are we saying? Is 4% equal to 25%? Not really. A more accurate way of describing this is as follows: “The 4% drop represents 25% of the reference number, which is 16%” (because 4 is 25% of 16). Yes, we are dealing with a percentage of a percentage. These numbers are particularly misleading and quite confusing. For that reason, when we refer to the *absolute* difference in two percentages (such as the 4% drop from 16% to 12%), we refer to that as “points.” In other words, here we can say that the poverty rate has dropped by 4 *points* (as opposed to the more confusing “4 percent”). 25% then refers to the *relative* difference from 16% to 12%.

Example 2.3.5

a. The approval rate of a senator has gone up from 44% to 49%. How many points has the rate increased by? What percentage has the approval rate increased by?
b. The fatality rate of a certain disease has gone down from 22% to 10%. How many points has the rate gone down by? What percentage drop does this represent?

Solution

a. The rate has gone up by 5 *points*. However, one can also say that the rate has gone up by about 11.4% since

$$(49-44)\div 44=0.1136$$

b. The rate has dropped by 12 *points*. Also, the rate has gone down by about 54.5% (a little more than half) because

$$(10-22)\div 22=-0.5454\ldots$$

This number is negative, representing a decrease, not an increase.

There are a few other things to remember when you are presented with numbers or other quantitative information. In addition to these percentage-related statements, watch out for things like the following:

- What is the unit? 20,000,000,000 bytes may sound really big, but this is the same as 20 GB. Similarly, 15,000 Rupiahs (the currency of Indonesia) may sound like a lot of money, but it is worth only about a dollar in the U.S. (as of June, 2019).
- What is the reference number? Many things could happen with a small sample size. For instance, it may not be all that unusual if a basketball player makes 1 out of 2 free throws (50%) in a given game, but it is a completely different story if the same player makes only 100 out of 200 free throws (still 50%) over a long period of time. In other words, the same percentage could mean something quite different if the denominator (the reference number) is large (or small).
- What is the absolute difference? How big or small is the original number? For example, according to CDC (Center for Disease Control) data, the number of measles cases reported in the U.S. more than *tripled* (grew by more than 200%) in 2013 (the number went up from 55 in 2012 to 187 in 2013). That sounded quite alarming and dramatic (and, in a sense, it was); however, those were very small numbers compared to the dire situation the country is facing in 2019, where the number exceeded 1,000 just in the first six months of the year. Sometimes the percentages do not tell the whole story, and you may want to ask more questions.

There are many other ways in which numbers can deceive. Some of those will be discussed in later chapters on finance, statistics, growth, and politics. What has been presented in this section is just a small portion of number manipulation. The message is clear: we must all be numerically literate and quantitatively intelligent to make wise decisions in this data-driven world.

Exercises

1. You invested some money in a start-up company, and now your investment has earned 60% and is worth $6,720. How much was your original investment?
2. A college football ticket can be purchased at $28 after a 20% student discount is applied. How much is the ticket without the student discount?
3. The unemployment rate in a certain small town has changed from 4% to 7% during the last ten years. The rate went up by how many percentage points? By what percent?
4. Suppose that the survival rate of a certain form of cancer increased from 33% to 59%. Express this change in terms of points and percentage as above.
5. Out of 230 racers who started the marathon, 212 completed the race, 14 gave up, and 4 were disqualified. What percentage did not complete the marathon?
6. Patrick left an $8 tip on a $50 restaurant bill. What percent tip is that?
7. Ireland has a 23% VAT (value-added tax, similar to a sales tax). How much will the VAT be on a purchase of a €250 item?

8. Employees in 2012 paid 4.2% of their gross wages towards social security (FICA tax), while employers paid another 6.2%. How much will someone earning $45,000 a year pay towards social security out of their gross wages?
9. A project on Kickstarter.com was aiming to raise $15,000 for a precision coffee press. They ended up with 714 supporters, raising 557% of their goal. How much did they raise?
10. Another project on Kickstarter for an iPad stylus raised 1,253% of their goal, raising a total of $313,490 from 7,511 supporters. What was their original goal?
11. The population of a town increased from 3,250 in 2008 to 4,300 in 2010. Find the absolute and relative (percent) increase.
12. The number of CDs sold in 2010 was 114 million, down from 147 million the previous year.[1] Find the absolute and relative (percent) decrease.
13. A company wants to decrease their energy use by 15%.
    a. If their electric bill is currently $2,200 a month, what will their bill be if they're successful?
    b. If their next bill is $1,700 a month, were they successful? Why or why not?
14. A store is hoping an advertising campaign will increase their number of customers by 30%. They currently have about 80 customers a day.
    a. How many customers will they have if their campaign is successful?
    b. If they increase to 120 customers a day, were they successful? Why or why not?
15. An article reports "attendance dropped 6% this year, to 300." What was the attendance before the drop?
16. An article reports "sales have grown by 30% this year, to $200 million." What were sales before the growth?
17. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    a. "16.3% of Americans are without health insurance" [2]
    b. "only 55.9% of adults receive employer provided health insurance" [3]
18. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    a. "We mark up the wholesale price by 33% to come up with the retail price"
    b. "The store has a 25% profit margin"
19. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    a. "Every year since 1950, the number of American children gunned down has doubled."
    b. "The number of child gunshot deaths has doubled from 1950 to 1994."
20. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things? [4]
    a. "75 percent of the federal health care law's taxes would be paid by those earning less than $120,000 a year"
    b. "76 percent of those who would pay the penalty [health care law's taxes] for not having insurance in 2016 would earn under $120,000"
21. Are these two claims equivalent, in conflict, or not comparable because they're talking about different things?
    a. "The school levy is only a 0.1% increase of the property tax rate."
    b. "This new levy is a 12% tax hike, raising our total rate to $9.33 per $1000 of value."
22. Are the values compared in this statement comparable or not comparable? "Guns have murdered more Americans here at home in recent years than have died on the battlefields of Iraq and Afghanistan. In support of the two wars, more than 6,500 American soldiers have lost their lives. During the same period, however, guns have been used to murder about 100,000 people on American soil" [5]
23. A high school currently has a 30% dropout rate. They've been tasked to decrease that rate by 20%. Find the equivalent percentage point drop.
24. A politician's support grew from 42% by 3 percentage points to 45%. What percent (relative) change is this?

25. Marcy has a 70% average in her class going into the final exam. She says "I need to get a 100% on this final so I can raise my score to 85%." Is she correct?
26. Suppose you have one quart of water/juice mix that is 50% juice, and you add 2 quarts of juice. What percent juice is the final mix?
27. Find a unit rate: You bought 10 pounds of potatoes for $4.
28. Find a unit rate: Joel ran 1500 meters in 4 minutes, 45 seconds.
29. Solve: $\frac{2}{5} = \frac{6}{x}$
30. Solve: $\frac{n}{5} = \frac{16}{20}$
31. A crepe recipe calls for 2 eggs, 1 cup of flour, and 1 cup of milk. How much flour would you need if you use 5 eggs?
32. An 8 ft length of 4 inch wide crown molding costs $14. How much will it cost to buy 40 ft of crown molding?
33. Four 3-megawatt wind turbines can supply enough electricity to power 3000 homes. How many turbines would be required to power 55,000 homes?
34. A highway had a landslide, where 3,000 cubic yards of material fell on the road, requiring 200 dump truck loads to clear. On another highway, a slide left 40,000 cubic yards on the road. How many dump truck loads would be needed to clear this slide?
35. Convert 8 feet to inches.
36. Convert 6 kilograms to grams.
37. A wire costs $2 per meter. How much will 3 kilometers of wire cost?
38. Sugar contains 15 calories per teaspoon. How many calories are in 1 cup of sugar?
39. A car is driving at 100 kilometers per hour. How far does it travel in 2 seconds?
40. A chain weighs 10 pounds per foot. How many ounces will 4 inches weigh?
41. The table below gives data on three movies. Gross earnings is the amount of money the movie brings in. Compare the net earnings (money made after expenses) for the three movies. [6]

Table 2.3.1

Movie	Release Date	Budget	Gross earnings
Saw	10/29/2004	$1,200,000	$103,096,345
Titanic	12/19/1997	$200,000,000	$1,842,879,955
Jurassic Park	6/11/1993	$63,000,000	$923,863,984

42. For the movies in the previous problem, which provided the best return on investment?
43. Your chocolate milk mix says to use 4 scoops of mix for 2 cups of milk. After pouring in the milk, you start adding the mix, but get distracted and accidentally put in 5 scoops of mix. How can you adjust the mix if:
    a. There is still room in the cup?
    b. The cup is already full?
44. A recipe for sabayon calls for 2 egg yolks, 3 tablespoons of sugar, and ¼ cup of white wine. After cracking the eggs, you start measuring the sugar, but accidentally put in 4 tablespoons of sugar. How can you compensate?
45. The Deepwater Horizon oil spill resulted in 4.9 million barrels of oil spilling into the Gulf of Mexico. Each barrel of oil can be processed into about 19 gallons of gasoline. How many cars could this have fueled for a year? Assume an average car gets 20 miles to the gallon, and drives about 12,000 miles in a year.
46. The store is selling lemons at 2 for $1. Each yields about 2 tablespoons of juice. How much will it cost to buy enough lemons to make a 9-inch lemon pie requiring ½ cup of lemon juice?

47. A piece of paper can be made into a cylinder in two ways: by joining the short sides together, or by joining the long sides together.[7] Which cylinder would hold more? How much more?

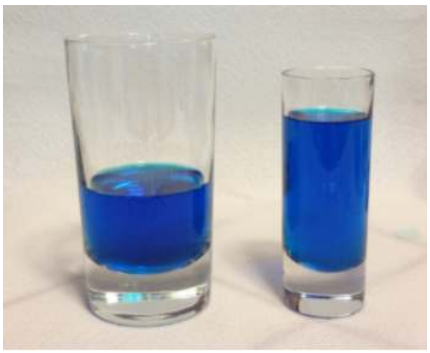


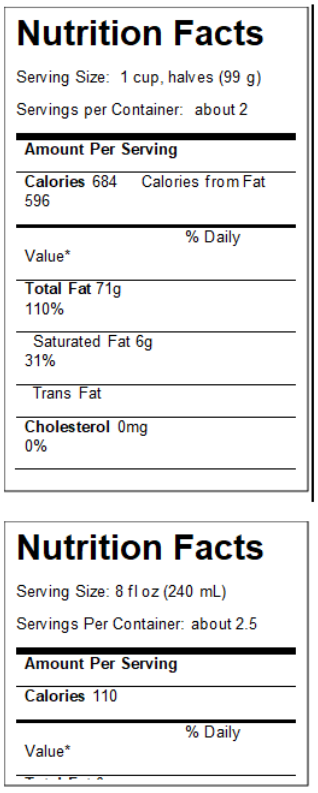
Figure 2.3.1

48. Which of these glasses contains more liquid? How much more?
49. The flash of lightning travels at the speed of light, which is about 186,000 miles per second. The sound of lightning (thunder) travels at the speed of sound, which is about 750 miles per hour.
    a. If you see a flash of lightning, then hear the thunder 4 seconds later, how far away is the lightning?
    b. Now let's generalize that result. Suppose it takes n seconds to hear the thunder after a flash of lightning. How far away is the lightning, in terms of n?
50. Sound travels about 750 miles per hour. If you stand in a parking lot near a building and sound a horn, you will hear an echo.
    a. Suppose it takes about ½ a second to hear the echo. How far away is the building? [8]
    b. Now let's generalize that result. Suppose it takes n seconds to hear the echo. How far away is the building, in terms of n?
51. It takes an air pump 5 minutes to fill a twin sized air mattress (39 by 8.75 by 75 inches). How long will it take to fill a queen sized mattress (60 by 8.75 by 80 inches)?
52. It takes your garden hose 20 seconds to fill your 2-gallon watering can. How long will it take to fill
    a. An inflatable pool measuring 3 feet wide, 8 feet long, and 1 foot deep.[9]
    b. A circular inflatable pool 13 feet in diameter and 3 feet deep.[10]
53. You want to put a 2" thick layer of topsoil for a new 20'x30' garden. The dirt store sells by the cubic yards. How many cubic yards will you need to order?
54. A box of Jell-O costs $0.50, and makes 2 cups. How much would it cost to fill a swimming pool 4 feet deep, 8 feet wide, and 12 feet long with Jell-O? (1 cubic foot is about 7.5 gallons)
55. You read online that a 15 ft by 20 ft brick patio would cost about $2,275 to have professionally installed. Estimate the cost of having a 18 by 22 ft brick patio installed.
56. I was at the store, and saw two sizes of avocados being sold. The regular size sold for $0.88 each, while the jumbo ones sold for $1.68 each. Which is the better deal?



Figure 2.3.2

58. The grocery store has bulk pecans on sale, which is great since you're planning on making 10 pecan pies for a wedding. Your recipe calls for 1¾ cups pecans per pie. However, in the bulk section there's only a scale available, not a measuring cup. You run over to the baking aisle and find a bag of pecans, and look at the nutrition label to gather some info. How many pounds of pecans should you buy?



Nutrition Facts

Serving Size: 1 cup, halves (99 g)

Servings per Container: about 2

Amount Per Serving	
Calories 684	Calories from Fat 596
	% Daily Value*
Total Fat 71g	110%
Saturated Fat 6g	31%
Trans Fat	
Cholesterol 0mg	0%

Nutrition Facts

Serving Size: 8 fl oz (240 mL)

Servings Per Container: about 2.5

Amount Per Serving	
Calories 110	
	% Daily Value*

Figure 2.3.3

59. Soda is often sold in 20 ounce bottles. The nutrition label for one of these bottles is shown to the right. A packet of sugar (the kind they have at restaurants for your coffee or tea) typically contain 4 grams of sugar in the U.S. Drinking a 20 oz soda is equivalent to eating how many packets of sugar?[11]

Some More Things to Ponder…

60. Suppose you hear someone say, "The number of homicides in my school district has quadrupled over the last two years." What does this mean? What possible scenarios can you think of? Is there anything else you would want to know regarding this claim?

61. Is it possible for the price of a concert ticket to increase by 150%? To decrease by 150%? Why or why not?

62. The sun is about 93 million miles from the earth while the moon is about 240,000 miles away (on the average). What percentages can you use to compare these two distances?

63. Your friend, who lives in a typical suburban neighborhood in your town, just says, “The average household income for last year in my neighborhood was $12 million.” Can this be true? What else would you want to know? What could have happened (assuming this statement is true)?
64. The population of the U.S. is about 309,975,000, covering a land area of 3,717,000 square miles. The population of India is about 1,184,639,000, covering a land area of 1,269,000 square miles. Compare the population densities of the two countries.
65. The GDP (Gross Domestic Product) of China was $5,739 billion in 2010, and the GDP of Sweden was $435 billion. The population of China is about 1,347 million, while the population of Sweden is about 9.5 million. Compare the GDP per capita of the two countries.
66. In June 2012, Twitter was reporting 400 million tweets per day. Each tweet can consist of up to 140 characters (letter, numbers, etc.). Create a comparison to help understand the amount of tweets in a year by imagining each character was a drop of water and comparing to filling something up.
67. The photo sharing site Flickr had 2.7 billion photos in June 2012. Create a comparison to understand this number by assuming each picture is about 2 megabytes in size, and comparing to the data stored on other media like DVDs, iPods, or flash drives.
68. If your favorite shoes are on sale with a 30% discount, and if you get an additional employee discount of 10%, can you get the shoes at 40% off? Why or why not?
69. How would you react if your boss tells you the following? “I will have to give you a 50% pay cut now, but I will give you a permanent 60% pay raise next week.” (Start with some fixed pay now and see what happens. Then explain why this is counter-intuitive.)
70. The Walden University had 47,456 students in 2010, while Kaplan University had 77,966 students. Complete the following statements:
    a. Kaplan’s enrollment was ___% larger than Walden’s.
    b. Walden’s enrollment was ___% smaller than Kaplan’s.
    c. Walden’s enrollment was ___% of Kaplan’s.
71. In the 2012 Olympics, Usain Bolt ran the 100m dash in 9.63 seconds. Jim Hines won the 1968 Olympic gold with a time of 9.95 seconds.
    a. Bolt’s time was ___% faster than Hines’.
    b. Hine’ time was ___% slower than Bolt’s.
    c. Hine’ time was ___% of Bolt’s.
72. A store has clearance items that have been marked down by 60%. They are having a sale, advertising an additional 30% off clearance items. What percent of the original price do you end up paying?
73. Which is better: having a stock that goes up 30% on Monday than drops 30% on Tuesday, or a stock that drops 30% on Monday and goes up 30% on Tuesday? In each case, what is the net percent gain or loss?
74. Define a “millionaire.” Does it depend on what currency is used?
75. You are concerned that your friend is taking too much ibuprofen for her headache, but she says, “I never take any more than 1.3 grams of the stuff. Some people take like 900 mg.” What would you say to her?
76. Which is harder? Walking 1,000,000 inches or walking 20 miles? Does one “sound” harder than the other?

Reference

1. http://www.cnn.com/2010/SHOWBIZ/Musi...les/index.html
2. http://www.cnn.com/2012/06/27/politi...are/index.html
3. http://www.politico.com/news/stories/0712/78134.html
4. http://factcheck.org/2012/07/twistin...th-care-taxes/
5. www.northjersey.com/news/opin...tml?c=y&page=2
6. www.northjersey.com/news/opin...tml?c=y&page=2
7. vimeo.com/42501010

8. vimeo.com/40377128
9. http://www.youtube.com/watch?v=DIkwefReHZc
10. http://www.youtube.com/watch?v=p9SABH7Yg9M
11. http://www.youtube.com/watch?v=62JMfv0tf3Q

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2.4: Percentages and Proportions

Percents

In the 2004 vice-presidential debates, John Edwards claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Dick Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties. Who is correct? How can we make sense of these numbers?

Percent literally means “per 100,” or “parts per hundred.” When we write 40%, this is equivalent to the fraction 40100 or the decimal 0.40. Notice that 80 out of 200 and 10 out of 25 are also 40%, since

$$\frac{80}{200} = \frac{10}{25} = \frac{40}{100}$$

Example 2.4.1

243 people out of 400 state that they like dogs. What percent is this?

Solution

$\frac{243}{400} = 0.6075 = \frac{60.75}{100}$ This is 60.75%.

Notice that the percent can be found from the equivalent decimal by moving the decimal point two places to the right.

Example 2.4.2

Write each as a percent:

a. $\frac{1}{4}$
b. 0.02
c. 2.35

Solution

a. $\frac{1}{4} = 0.25 = 25\%$
b. 0.02 = 2%
c. 2.35 = 235%

Percents

If we have a part that is some *percent of a whole*, then

$\text{Percent} = \frac{\text{part}}{\text{whole}}$, or equivalently, $\text{part} = \text{percent} \cdot \text{whole}$

To do the calculations, we write the percent as a decimal.

Example 2.4.3

The sales tax in a town is 9.4%. How much tax will you pay on a $140 purchase?

Solution

Here, $140 is the whole, and we want to find 9.4% *of* $140. We start by writing the percent as a decimal by moving the decimal point two places to the left (which is equivalent to dividing by 100). We can then compute:

tax = 0.094 (140) = $13.16 in tax.

Example 2.4.4

In the news, you hear "tuition is expected to increase by 7% next year." If tuition this year was $1200 per quarter, what will it be next year?

Solution

The tuition next year will be the current tuition plus an additional 7%, so it will be 107% of this year's tuition:

$1200(1.07) = $1284.

Alternatively, we could have first calculated 7% of $1200: $1200(0.07) = $84.

Notice this is *not* the expected tuition for next year (we could only wish). Instead, this is the expected *increase*, so to calculate the expected tuition, we'll need to add this change to the previous year's tuition:

$1200 + $84 = $1284.

Try it Now 1

A TV originally priced at $799 is on sale for 30% off. There is then a 9.2% sales tax. Find the price after including the discount and sales tax.

Example 2.4.5

The value of a car dropped from $7400 to $6800 over the last year. What percent decrease is this?

Solution

To compute the percent change, we first need to find the dollar value change: $6800-$7400 = -$600. Often we will take the absolute value of this amount, which is called the **absolute change**:

$|-600\,| = 600$

Since we are computing the decrease relative to the starting value, we compute this percent out of $7400:

$\frac{600}{7400} = 0.081 = 8.1\%$ decrease. This is called a **relative change**.

Absolute and Relative Change

Given two quantities,

$$\text{Absolute change} = |\text{ending quantity} - \text{starting quantity}| \tag{2.4.1}$$

$$\text{Relative change} = \frac{\text{absolute change}}{\text{staring quantity}} \tag{2.4.2}$$

Absolute change has the same units as the original quantity and relative change gives a percent change.

The starting quantity is called the **base** of the percent change.

The base of a percent is very important. For example, while Nixon was president, it was argued that marijuana was a "gateway" drug, claiming that 80% of marijuana smokers went on to use harder drugs like cocaine. The problem is, this isn't true. The true claim is that 80% of harder drug users first smoked marijuana. The difference is one of base: 80% of marijuana smokers using hard drugs, vs. 80% of hard drug users having smoked marijuana. These numbers are not equivalent. As it turns out, only one in 2,400 marijuana users actually go on to use harder drugs.[2]

Example 2.4.6

There are about 75 QFC supermarkets in the U.S. Albertsons has about 215 stores. Compare the size of the two companies.

Solution

When we make comparisons, we must ask first whether an absolute or relative comparison. The absolute difference is 215 – 75 = 140. From this, we could say "Albertsons has 140 more stores than QFC." However, if you wrote this in an article or paper,

that number does not mean much. The relative difference may be more meaningful. There are two different relative changes we could calculate, depending on which store we use as the base:

Using QFC as the base, $\frac{140}{75} = 1.867$.

This tells us Albertsons is 186.7% larger than QFC.

Using Albertsons as the base, $\frac{140}{215} = 0.651$.

This tells us QFC is 65.1% smaller than Albertsons.

Notice both of these are showing percent *differences*. We could also calculate the size of Albertsons relative to QFC: $\frac{215}{75} = 2.867$, which tells us Albertsons is 2.867 times the size of QFC. Likewise, we could calculate the size of QFC relative to Albertsons: $\frac{75}{215} = 0.349$, which tells us that QFC is 34.9% of the size of Albertsons.

Example 2.4.7

Suppose a stock drops in value by 60% one week, then increases in value the next week by 75%. Is the value higher or lower than where it started?

Solution

To answer this question, suppose the value started at \$100. After one week, the value dropped by 60%:

\$100 - \$100(0.60) = \$100 - \$60 = \$40.

In the next week, notice that base of the percent has changed to the new value, \$40. Computing the 75% increase:

\$40 + \$40(0.75) = \$40 + \$30 = \$70.

In the end, the stock is still \$30 lower, or $\frac{\$30}{\$100} = 30\%$ lower, valued than it started.

Try it Now 2

The U.S. federal debt at the end of 2001 was \$5.77 trillion, and grew to \$6.20 trillion by the end of 2002. At the end of 2005 it was \$7.91 trillion, and grew to \$8.45 trillion by the end of 2006[3]. Calculate the absolute and relative increase for 2001-2002 and 2005-2006. Which year saw a larger increase in federal debt?

Example 2.4.8

A Seattle Times article on high school graduation rates reported "The number of schools graduating 60 percent or fewer students in four years – sometimes referred to as "dropout factories" – decreased by 17 during that time period. The number of kids attending schools with such low graduation rates was cut in half."

a. Is the "decrease by 17" number a useful comparison?
b. Considering the last sentence, can we conclude that the number of "dropout factories" was originally 34?

Solution

a. This number is hard to evaluate, since we have no basis for judging whether this is a larger or small change. If the number of "dropout factories" dropped from 20 to 3, that'd be a very significant change, but if the number dropped from 217 to 200, that'd be less of an improvement.
b. The last sentence provides relative change which helps put the first sentence in perspective. We can estimate that the number of "dropout factories" was probably previously around 34. However, it's possible that students simply moved schools rather than the school improving, so that estimate might not be fully accurate.

Example 2.4.9

In the 2004 vice-presidential debates, Edwards' claimed that US forces have suffered "90% of the coalition casualties" in Iraq. Cheney disputed this, saying that in fact Iraqi security forces and coalition allies "have taken almost 50 percent" of the casualties. Who is correct?

Solution

Without more information, it is hard for us to judge who is correct, but we can easily conclude that these two percents are talking about different things, so one does not necessarily contradict the other. Edwards' claim was a percent with coalition forces as the base of the percent, while Cheney's claim was a percent with both coalition and Iraqi security forces as the base of the percent. It turns out both statistics are in fact fairly accurate.

Try it Now 3

In the 2012 presidential elections, one candidate argued that "the president's plan will cut $716 billion from Medicare, leading to fewer services for seniors," while the other candidate rebuts that "our plan does not cut current spending and actually expands benefits for seniors, while implementing cost saving measures." Are these claims in conflict, in agreement, or not comparable because they're talking about different things?

We'll wrap up our review of percents with a couple cautions. First, when talking about a change of quantities that are already measured in percents, we have to be careful in how we describe the change.

Example 2.4.10

A politician's support increases from 40% of voters to 50% of voters. Describe the change.

Solution

We could describe this using an absolute change: $|50\% - 40\%| = 10\%$. Notice that since the original quantities were percents, this change also has the units of percent. In this case, it is best to describe this as an increase of 10 **percentage points**.

In contrast, we could compute the percent change: $\frac{10\%}{40\%} = 0.25 = 25\%$ increase. This is the relative change, and we'd say the politician's support has increased by 25%.

Lastly, a caution against averaging percents.

Example 2.4.11

A basketball player scores on 40% of two-point field goal attempts and on 30% of three-point field goal attempts. Find the player's overall field goal percentage.

Solution

It is very tempting to average these values, and claim the overall average is 35%, but this is likely not correct, since most players make many more two-point attempts than three-point attempts. We don't actually have enough information to answer the question. Suppose the player attempted 200 two-point field goals and 100 three-point field goals. Then they made 200(0.40) = 80 two-point shots and 100(0.30) = 30 three-point shots. Overall, they made 110 shots out of 300, for a $\frac{110}{300} = 0.367 = 36.7\%$ overall field goal percentage.

Proportions and Rates

If you wanted to power the city of Seattle using wind power, how many windmills would you need to install? Questions like these can be answered using rates and proportions.

Rates

A rate is the ratio (fraction) of two quantities.

A **unit rate** is a rate with a denominator of one.

Example 2.4.12

Your car can drive 300 miles on a tank of 15 gallons. Express this as a rate.

Solution

Expressed as a rate, $\frac{300 \text{ miles}}{15 \text{ gallons}}$ We can divide to find a unit rate: $\frac{20 \text{ miles}}{1 \text{ gallon}}$, which we could also write as $20\frac{\text{miles}}{\text{gallon}}$, or just 20 miles per gallon

Proportion Equation

A proportion equation is an equation showing the equivalence of two rates or ratios.

A key to solving a proportion equation is "cross multiplication." When two rations are equal to each other, two products obtained by cross multiplication are equal.

Example 2.4.13

Solve the proportion $\frac{5}{3} = \frac{x}{6}$ for the unknown value x.

Solution

This proportion is asking us to find a fraction with denominator 6 that is equivalent to the fraction $\frac{5}{3}$. We can solve this by cross multiplication, giving

$$\begin{aligned} 3x &= 5 \cdot 6 \\ 3x &= 30 \\ x &= 10 \end{aligned}$$

Example 2.4.14

A map scale indicates that ½ inch on the map corresponds with 3 real miles. How many miles apart are two cities that are $2\frac{1}{4}$ inches apart on the map?

Solution

We can set up a proportion by setting equal two $\frac{\text{map inches}}{\text{real miles}}$ rates, and introducing a variable, x, to represent the unknown quantity – the mile distance between the cities.

$$\frac{\frac{1}{2} \text{ Map inch}}{3 \text{ miles}} = \frac{2\frac{1}{4} \text{ map inch}}{x \text{ miles}}$$

By cross multiplication, we obtain: (rewriting the mixed number $2\frac{1}{4}$ as $\frac{9}{4}$)

$$\frac{1}{2}x = 3 \cdot \left(\frac{3}{9}\right)$$
$$\frac{1}{2}x = \frac{27}{4}, \text{ Multiply both sides by } 2$$
$$x = \frac{27}{4} \cdot 2$$
$$x = \frac{27}{2}$$
$$x = 13\frac{1}{2} \text{ miles}$$

Reference

1. References (3)
2. http://tvtropes.org/pmwiki/pmwiki.php/Main/LiesDamnedLiesAndStatistics
3. www.whitehouse.gov/sites/defa...s/hist07z1.xls

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2.5: Conversions

Many proportion problems can also be solved using **dimensional analysis**, the process of multiplying a quantity by rates to change the units. This process is also used in conversion problems.

Example 2.5.1

Your car can drive 300 miles on a tank of 15 gallons. How far can it drive on 40 gallons?

Solution

We could certainly answer this question using a proportion: $\frac{300 \text{ miles}}{15 \text{ gallons}} = \frac{x \text{ miles}}{40 \text{ gallons}}$

However, we earlier found that 300 miles on 15 gallons gives a rate of 20 miles per gallon. If we multiply the given 40 gallon quantity by this rate, the *gallons* unit "cancels" and we're left with a number of miles:

$$40 \text{ gallons} \cdot \frac{20 \text{ miles}}{\text{gallon}} = \frac{40 \text{ gallons}}{1} \cdot \frac{20 \text{ miles}}{\text{gallon}} = 800 \text{ miles}$$

If, instead, we were asked "how many gallons are needed to drive 50 miles?" we could answer this question by inverting the 20 mile per gallon rate so that the *miles* unit cancels and we're left with gallons:

$$50 \text{ miles} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ miles}}{1} \cdot \frac{1 \text{ gallon}}{20 \text{ miles}} = \frac{50 \text{ gallons}}{20} = 2.5 \text{ gallons}$$

Unit Conversions

Here are some unit conversions for reference.

For everyday use, you may want to download a free app for unit conversions.

Length

1 foot (ft) = 12 inches (in) 1 yard (yd) = 3 feet (ft)

1 mile = 5,280 feet

1000 millimeters (mm) = 1 meter (m) 100 centimeters (cm) = 1 meter

1000 meters (m) = 1 kilometer (km) 2.54 centimeters (cm) = 1 inch

Weight and Mass

1 pound (lb) = 16 ounces (oz) 1 ton = 2000 pounds

1000 milligrams (mg) = 1 gram (g) 1000 grams = 1kilogram (kg)

1 kilogram = 2.2 pounds (on earth)

Capacity (Volume)

1 cup = 8 fluid ounces (fl oz)* 1 pint = 2 cups

1 quart = 2 pints = 4 cups 1 gallon = 4 quarts = 16 cups

1000 milliliters (ml) = 1 liter (L)

*Fluid ounces are a capacity (volume) measurement for liquids. 1 fluid ounce ≈ 1 ounce (weight) for water only.

Example 2.5.2

A bicycle is traveling at 15 miles per hour. How many feet will it cover in 20 seconds?

Solution

To answer this question, we need to convert 20 seconds into feet. If we know the speed of the bicycle in feet per second, this question would be simpler. Since we don't, we will need to do additional unit conversions. We will need to know that 5280 ft =

1 mile. We might start by converting the speed into feet per second.

$$15\frac{\text{miles}}{\text{hr}} = \frac{15 \text{ miles}}{1 \text{ hr}} \cdot \frac{5280 \text{ ft}}{1 \text{ mile}} = 79,200\frac{\text{ft}}{\text{hr}} = \frac{79,200 \text{ ft}}{\text{hr}} \cdot \frac{1 \text{ hr}}{60 \text{ min}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} = 22\frac{\text{ft}}{\text{sec}}$$

$$22\frac{\text{ft}}{\text{sec}} \cdot 20 \text{ sec} = 400 \text{ ft}$$

Try it Now 4

A 1000 foot spool of bare 12-gauge copper wire weighs 19.8 pounds. How much will 18 inches of the wire weigh, in ounces?

Notice that with the miles per gallon example, if we double the miles driven, we double the gas used. Likewise, with the map distance example, if the map distance doubles, the real-life distance doubles. This is a key feature of proportional relationships, and one we must confirm before assuming two things are related proportionally.

Example 2.5.3

Suppose you're tiling the floor of a 10 ft by 10 ft room, and find that 100 tiles will be needed. How many tiles will be needed to tile the floor of a 20 ft by 20 ft room?

Solution

In this case, while the width the room has doubled, the area has quadrupled. Since the number of tiles needed corresponds with the area of the floor, not the width, 400 tiles will be needed. We could find this using a proportion based on the areas of the rooms:

$$\frac{100 \text{ tiles}}{100 \text{ ft}^2} = \frac{n \text{ tiles}}{400 \text{ ft}^2}$$

Other quantities just don't scale proportionally at all.

Example 2.5.4

Suppose a small company spends $1000 on an advertising campaign, and gains 100 new customers from it. How many new customers should they expect if they spend $10,000?

Solution

While it is tempting to say that they will gain 1000 new customers, it is likely that additional advertising will be less effective than the initial advertising. For example, if the company is a hot tub store, there are likely only a fixed number of people interested in buying a hot tub, so there might not even be 1000 people in the town who would be potential customers.

Sometimes when working with rates, proportions, and percents, the process can be made more challenging by the magnitude of the numbers involved. Sometimes, large numbers are just difficult to comprehend.

Example 2.5.5

Compare the 2010 U.S. military budget of $683.7 billion to other quantities.

Solution

Here we have a very large number, about $683,700,000,000 written out. Of course, imagining a billion dollars is very difficult, so it can help to compare it to other quantities.

If that amount of money was used to pay the salaries of the 1.4 million Walmart employees in the U.S., each would earn over $488,000.

There are about 300 million people in the U.S. The military budget is about $2,200 per person.

If you were to put $683.7 billion in $100 bills, and count out 1 per second, it would take 216 years to finish counting it.

Example 2.5.6

Compare the electricity consumption per capita in China to the rate in Japan.

Solution

To address this question, we will first need data. From the CIA[1] website we can find the electricity consumption in 2011 for China was 4,693,000,000,000 KWH (kilowatt-hours), or 4.693 trillion KWH, while the consumption for Japan was 859,700,000,000, or 859.7 billion KWH. To find the rate per capita (per person), we will also need the population of the two countries. From the World Bank[2], we can find the population of China is 1,344,130,000, or 1.344 billion, and the population of Japan is 127,817,277, or 127.8 million.

Computing the consumption per capita for each country:

China: $\frac{4,693,000,000,000 \text{ KWH}}{1,344,130,000 \text{ people}} \approx 3491.5 \text{ KWH per person}$

Japan: $\frac{859,700,000,000 \text{ KWH}}{127,817,277 \text{ people}} \approx 6726 \text{ KWH per person}$

While China uses more than 5 times the electricity of Japan overall, because the population of Japan is so much smaller, it turns out Japan uses almost twice the electricity per person compared to China.

Geometry

Geometric shapes, as well as area and volumes, can often be important in problem solving.

Example 2.5.7

You are curious how tall a tree is but don't have any way to climb it. Describe a method for determining the height.

Solution

There are several approaches we could take. We'll use one based on triangles, which requires that it's a sunny day. Suppose the tree is casting a shadow, say 15 ft long. I can then have a friend help me measure my own shadow. Suppose I am 6 ft tall and cast a 2 ft shadow. Since the triangle formed by the tree and its shadow has the same angles as the triangle formed by me and my shadow, these triangles are called **similar triangles,** and their sides will scale proportionally. In other words, the ratio of height to width will be the same in both triangles. Using this, we can find the height of the tree, which we'll denote by h:

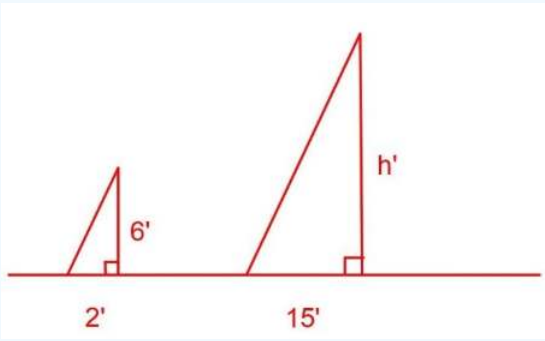


Figure 2.5.1

$$\frac{6 \text{ ft tall}}{2 \text{ ft shadow}} = \frac{h \text{ ft tall}}{15 \text{ ft shadow}}$$

By cross multiplication, we get 2h=90. Thus, h=45(ft).

The tree is about 45 ft tall.

It may be helpful to recall some formulas for areas and volumes of a few basic shapes.

Areas

Rectangle

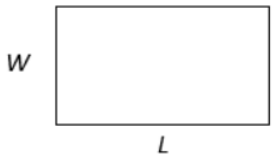


Figure 2.5.2

Area: $L \cdot W$

Perimeter: $2L + 2W$

Circle, radius r

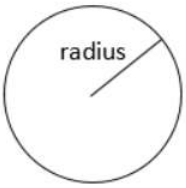


Figure 2.5.3

Area: πr^2

Circumference: $2\pi r$

Volumes

Rectangular Box

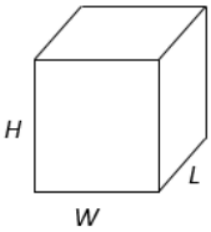


Figure 2.5.4

Volume: $L \cdot W \cdot H$

Cylinder

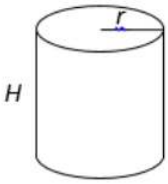


Figure 2.5.5

Volume: $\pi r^2 H$

Example 2.5.8

If a 12 inch diameter pizza requires 10 ounces of dough, how much dough is needed for a 16 inch pizza?

Solution

To answer this question, we need to consider how the weight of the dough will scale. The weight will be based on the volume of the dough. However, since both pizzas will be about the same thickness, the weight will scale with the area of the top of the pizza. We can find the area of each pizza using the formula for area of a circle, $= \pi r^2$:

A 12" pizza has radius 6 inches, so the area will be $\pi 6^2$ = about 113 square inches.

A 16" pizza has radius 8 inches, so the area will be $\pi 6^2$ = about 201 square inches.

Notice that if both pizzas were 1 inch thick, the volumes would be 113 in^3 and 201 in^3 respectively, which are at the same ratio as the areas. As mentioned earlier, since the thickness is the same for both pizzas, we can safely ignore it.

We can now set up a proportion to find the weight of the dough for a 16" pizza:

$$\frac{10 \text{ ounces}}{113 \text{in}^2} = \frac{x \text{ ounces}}{201 \text{in}^2}$$

Multiply both sides by 201.

$x = 201 \cdot \frac{10}{113} =$ about 17.8 ounces of dough for a 16" pizza.

It is interesting to note that while the diameter is $\frac{16}{12} = 1.33$ times as large, the dough required, which scales with area, is $1.33^2 = 1.78$ times as large.

Example 2.5.9

A company makes regular and jumbo marshmallows. The regular marshmallow has 25 calories. How many calories will the jumbo marshmallow have?

Solution

We would expect the calories to scale with volume. Since the marshmallows have cylindrical shapes, we can use that formula to find the volume. From the grid in the image, we can estimate the radius and height of each marshmallow.



Figure 2.5.6: Photo courtesy Christopher Danielson

The regular marshmallow appears to have a diameter of about 3.5 units, giving a radius of 1.75 units, and a height of about 3.5 units. The volume is about $\pi(1.75)^2(3.5) = 33.7$ units ^{3}.

The jumbo marshmallow appears to have a diameter of about 5.5 units, giving a radius of 2.75 units, and a height of about 5 units. The volume is about $\pi(2.75)^2(5) = 118.8$ units ^{3}.

We could now set up a proportion, or use rates. The regular marshmallow has 25 calories for 33.7 cubic units of volume. The jumbo marshmallow will have: $118.8 \text{ units}^3 \cdot \frac{25 \text{ calories}}{33.7 \text{ units}^3} = \$88.1 \text{ calories}$

It is interesting to note that while the diameter and height are about 1.5 times as large for the jumbo marshmallow, the volume and calories are about $1.5^3 = 3.375$ times as large.

Try it Now 5

A website says that you'll need 48 fifty-pound bags of sand to fill a sandbox that measure 8ft by 8ft by 1ft. How many bags would you need for a sandbox 6ft by 4ft by 1ft?

Reference

1. www.cia.gov/library/publicat.../2042rank.html
2. http://data.worldbank.org/indicator/SP.POP.TOTL

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CHAPTER OVERVIEW

3: Mathematics and Finance

3.1: Simple and Compound Interest
3.2: Annuities and Loans
3.3: Continuous Compounding
3.4: Federal Budgets and National Debt

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3.1: Simple and Compound Interest

We have to work with money every day. Balancing your checkbook or calculating your monthly expenditures on espresso requires only arithmetic, but we need more complex mathematics when we start saving, plan for retirement, or need a loan.

Simple Interest

Discussing interest starts with the principal, or amount your account starts with. This could be a starting investment or the starting amount of a loan. Interest, in its most simple form, is calculated as a percent of the principal. For example, if you borrowed $100 from a friend and agree to repay it with 5% interest, then the amount of interest you would pay would just be 5% of 100: $100(0.05) = $5. The total amount you would repay would be $105, the original principal plus the interest.

Simple One-time Interest

$I = P_0 r$

$A = P_0 + I = P_0 + P_0 r = P_0(1 + r)$

I is the interest

A is the end amount: principal plus interest

P_0 is the principle (starting amount)

r is the interest rate (in decimal form. Example: 5%=0.05)

Example 3.1.1

A Friend asks to borrow $300 and agrees to repay it in 30 days with 3% interest. How much interest will you earn?

Solution

P_0=$300 The principal

r=0.03 3% rate

I=$300(0.03)=9 You will earn $9 interest

One-time simple interest is only common for extremely short-term loans. For longer term loans, it is common for interest to be paid on a daily, monthly, quarterly, or annual basis. In that case, interest would be earned regularly. For example, bonds are essentially a loan made to the bond issuer (a company or government) by you, the bond holder. In return for the loan, the issuer agrees to pay interest, often annually. Bonds have a maturity date, at which time the issuer pays back the original bond value.

Example 3.1.2

Suppose your city is building a new park, and issues bonds to raise the money to build it. You obtain a $1,000 bond that pays 5% interest annually that matures in 5 years. How much interest will you earn?

Solution

Each year, you would earn 5% interest: $1000(0.05) = $50 in interest. So over the course of five years, you would earn a total of $250 in interest. When the bond matures, you would receive back the $1,000 you originally paid, leaving you with a total of $1,250.

We can generalize this idea of simple interest over time.

Simple Interest over Time

$I = P_0 rt$

$A = P_0 + I = P_0 + P_0 rt = P_0(1 + rt)$

I is the interest

A is the end amount: principle plus interest

P_0 is the principal (starting amount)

r is the interest rate in decimal form

t is time

The units of measurement (year, months, etc.) for the time should match the time period for the interest rate.

APR - Annual Percentage Rate

Interest rates are usually given as an annual percentage rate (APR), the total interest that will be paid in the year. If the interest is paid in smaller time increments, the APR will be divided up.

For example, a 6% APR paid monthly would be divided into twelve 0.5% payments.

A 4% annual rate paid quarterly would be divided into four 1% payments.

Example 3.1.3

Treasury Notes (T-notes) are bonds issued by the federal government to cover its expenses. Suppose you obtain a $1000 T-note with a 4% annual rate, paid semi-annually, with a maturity in 4 years. How much interest will you earn?

Solution

Since interest is being paid semi-annually (twice a year), the 4% interest will be divided into two 2% payments.

P_0= $1000 The principal

r=0.02 2% rate per half-year

t=8 4 years= 8 half-years

I=$1000(.02)(8) = $160 You will earn $160 interest total over the four years.

Try it Now 1

A loan company charges $30 interest for a one month loan of $500. Find the annual interest rate they are charging.

Compound Interest

With simple interest, we were assuming that we pocketed the interest when we received it. In a standard bank account, any interest we earn is automatically added to our balance, and we earn interest on that interest in future years. This reinvestment of interest is called compounding.

Suppose that we deposit $1000 in a bank account offering 3% interest, compounded monthly. How will our money grow?

The 3% interest is an annual percentage rate (APR). (Note that this does NOT mean you get 3% every month, as explained below. That would be too good to be true.) Since interest is being paid monthly, each month, we will earn $3\% \div 12 = 0.25\%$ per month.

In the first month,

P_0=$1000

r=0.0025 (0.25%)

I=$1000 (0.0025)=$2.50 (Interest for the first month)

A=$1000+$250=$1002.50

In the first month, we will earn $2.50 in interest, raising our account balance to $1002.50.

In the second month,

P_0=$1002.50

I= $1002.50 (0.0025)=$2.51 (rounded)

A = \$1002.50+\$2.51=\$1005.01

Notice that in the second month we earned more interest than we did in the first month. This is because we earned interest not only on the original \$1000 we deposited, but we also earned interest on the \$2.50 of interest we earned the first month. This is the key advantage that compounding of interest gives us.

Calculating out a few more months:

Table 3.1.1

Month	Starting balance	Interest earned	Ending Balance
1	1000.00	2.50	1002.50
2	1002.50	2.51	1005.01
3	1005.01	2.51	1007.52
4	1007.52	2.52	1010.04
5	1010.04	2.53	1012.57
6	1012.57	2.53	1015.10
7	1015.10	2.54	1017.64
8	1017.64	2.54	1020.18
9	1020.18	2.55	1022.73
10	1022.73	2.56	1025.29
11	1025.29	2.56	1027.85
12	1027.85	2.57	1030.42

To find an equation to represent this, if P_m represents the amount of money after m months, then we could write the recursive equation:

P_0=\$1000

$P_m = (1+0.0025)P_{m-1}$. Note that adding 0.25% to P amounts to calculating 100.25% of P.

This is a type of exponential growth to be studied in a later chapter. Let's go through the steps to build an explicit equation for the growth:

$P_0 = \$1000$

$P_1 = 1.0025P_0 = 1.0025(1000)$

$P_2 = 1.0025P_1 = 1.0025(1.0025(1000)) = 1.0025^2(1000)$

$P_3 = 1.0025P_2 = 1.0025\left(1.0025^2(1000)\right) = 1.0025^3(1000)$

$P_4 = 1.0025P_3 = 1.0025\left(1.0025^3(1000)\right) = 1.0025^4(1000)$

Observing a pattern, we could conclude

$P_m = (1.0025)^m(\$1000)$

Notice that the \$1000 in the equation was P_0, the starting amount. We found 1.0025 by adding one to the growth rate divided by 12, since we were compounding 12 times per year.

Generalizing our result, we could write

$$P_m = P_0\left(1+\frac{r}{k}\right)^m$$

In this formula:

m is the number of compounding periods (months in our example)

r is the annual percentage rate (APR)

k is the number of compounds per year.

While this formula works fine, it is more common to use a formula that involves the number of years rather than the number of compounding periods. If N is the number of years, then Nk is the total number of compounding periods. Making this change gives us the standard formula for compound interest.

Compound Interest

$$P_N = P_0\left(1+\frac{r}{k}\right)^{Nk}$$

P_N is the balance in the account after N years.

P_0 is the starting balance of the account (also called initial deposit, or principal)

r is the annual interest rate (APR) in decimal form

k is the number of compounding periods in one year.

If the compounding is done annually (once a year), $k = 1$.

If the compounding is done quarterly, $k = 4$.

If the compounding is done monthly, $k = 12$.

If the compounding is done daily, $k = 365$.

The most important thing to remember about using this formula is that it assumes that we put money in the account ***once*** and let it sit there earning interest.

Example 3.1.4

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit $3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

Solution

In this example,

P_0=$3000 the initial deposit

r=0.06 6% annual rate

k=12 12 months in 1 year

N=20 since we're looking for how much we'll have after 20 years

So, $P_{20} = 3000\left(1+\frac{0.06}{12}\right)^{20\times 12} = \$9930.61$ (round your answer to the nearest penny)

Table 3.1.2

Years	Simple Interest ($15 per month)	6% compounded monthly = 0.5% each month.
5	$3900	$4046.55
10	$4800	$5458.19
15	$5700	$7362.28
20	$6600	$9930.61
25	$7500	$13394.91

Years	Simple Interest ($15 per month)	6% compounded monthly = 0.5% each month.
30	$8400	$18067.73
35	$9300	$24370.65

Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

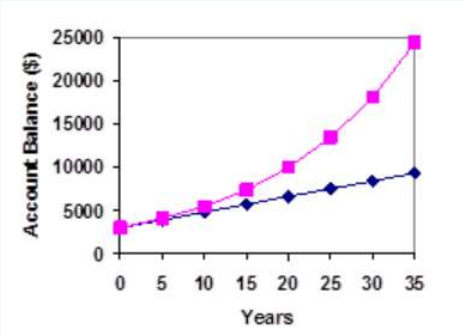


Figure 3.1.1

As you can see, over a long period of time, compounding makes a large difference in the account balance. You may want to keep this in mind as a main difference between linear growth and exponential growth. We will see this in Chapter 6 again.

Evaluating exponents on the calculator

When we need to calculate something like 5^3 it is easy enough to just multiply $5 \cdot 5 \cdot 5 = 125$ But when we need to calculate something like 1.005^{240}, it would be very tedious to calculate this by multiplying 1.005 by itself 240 times! So to make things easier, we can harness the power of our scientific calculators.

Most scientific calculators have a button for exponents. It is typically either labeled like:
$\wedge$, y^x, or x^y

To evaluate 1.005^{240} we'd type $1.005 \wedge 240$, or $1.005 y^x 240$ Try it out- you should get something around 3.3102044758

Example 3.1.5

You know that you will need $40,000 for your child's education in 18 years. If your account earns 4% compounded quarterly, how much would you need to deposit now to reach your goal?

Solution

In this example,

We're looking for P_0

r=0.04 4%

k=4 4 quarters in 1 year

N=18 Since we know the balance in 18 years

P_{18}=$40,000 The amount we have in 18 years

In this case, we're going to have to set up the equation, and solve for P_0.

$$40000 = P_0\left(1 + \frac{0.04}{4}\right)^{18\times 4}$$
$$40000 = P_0(2.0471)$$
$$P_0 = \frac{40000}{2.0471} = \$19539.84$$

So, you would need to deposit $19,539.84 now to have $40,000 in 18 years.

Using your Calculator

In many cases, you can avoid rounding completely by how you enter things in your calculator. For example, in the example above, we needed to calculate

$$P_{30} = 1000\left(1+\frac{0.05}{12}\right)^{12\times 30}$$

We can quickly calculate $12 \times 30 = 360$, giving $P_{30} = 1000\left(1+\frac{0.05}{12}\right)^{360}$

Now we can use the calculator.

Table 3.1.3

Type this	Calculator shows
$0.05 \div 12 =$	0.00416666666667
$+1 =$	1.00416666666667
$y^x 360 =$	4.46774431400613
$\times 1000 =$	4467.74431400613

Using your Calculator (continued)

The previous steps were assuming you have a "one operation at a time" calculator; a more advanced calculator will often allow you to type in the entire expression to be evaluated.

If you have a calculator like this, you will probably just need to enter:

$$1000 \times (1 + 0.05 \div 12) y^x 360 =$$

Reference

1. References (5)

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3.2: Annuities and Loans

Annuities

For most of us, we aren't able to put a large sum of money in the bank today. Instead, we save for the future by depositing a smaller amount of money from each paycheck into the bank. This idea is called a savings annuity. Most retirement plans like 401k plans or IRA plans are examples of savings annuities.

An annuity can be described recursively in a fairly simple way. Recall that basic compound interest follows from the relationship

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1}$$

For a savings annuity, we simply need to add a deposit, d, to the account with each compounding period:

$$P_m = \left(1 + \frac{r}{k}\right) P_{m-1} + d$$

Taking this equation from recursive form to explicit form is a bit trickier than with compound interest. It will be easiest to see by working with an example rather than working in general.

Suppose we will deposit $100 each month into an account paying 6% interest. We assume that the account is compounded with the same frequency as we make deposits unless stated otherwise. In this example:

r=0.06(6%)

k=12 (12 compounds/deposits per year)

d=$100 (our deposit per month)

Writing out the recursive equation gives

$$P_m = \left(1 + \frac{0.06}{12}\right) P_{m-1} + 100 = (1.005)P_{m-1} + 100$$

Assuming we start with an empty account, we can begin using this relationship:

$P_0 = 0$

$P_1 = (1.005)P_0 + 100 = 100$

$P_2 = (1.005)P_1 + 100 = (1.005)(100) + 100 = 100(1.005) + 100$

$P_3 = (1.005)P_2 + 100 = (1.005)(100(1.005) + 100) + 100 = 100(1.005)2 + 100(1.005) + 100$

Continuing this pattern, after m deposits, we'd have saved:

$$P_m = 100(1.005)^{m-1} + 100(1.005)^{m-2} + \ldots + 100(1.005) + 100$$

In other words, after m months, the first deposit will have earned compound interest for $m-1$ months. The second deposit will have earned interest for $m-2$ months. Last month's deposit would have earned only one month worth of interest. The most recent deposit will have earned no interest yet.

This equation leaves a lot to be desired, though – it doesn't make calculating the ending balance any easier! Fortunately, however, there is a simpler way to add up all of these amounts in a formula, which comes from the idea called "geometric series." Here is the formula:

Annuity Formula

$$P_N = \frac{d\left(\left(1 + \frac{r}{k}\right)^{Nk} - 1\right)}{\left(\frac{r}{k}\right)}$$

P_N is the balance in the account after N years.

d is the regular deposit (the amount you deposit each year, each month, etc.)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

If the compounding frequency is not explicitly stated, assume there are the same number of compounds in a year as there are deposits made in a year.

For example, if the compounding frequency isn't stated:

If you make your deposits every month, use monthly compounding, k=12.

If you make your deposits every year, use yearly compounding, k=1.

If you make your deposits every quarter, use quarterly compounding, k=4.

Etc.

When do you use this?

Annuities assume that you put money in the account on a regular schedule (every month, year, quarter, etc.) and let it sit there earning interest.

In the previous section, the compound interest formula assumed that you put money in the account once and let it sit there earning interest.

Remember, an annuity usually has many deposits (of the same amount).

Example 3.2.1

A traditional individual retirement account (IRA) is a special type of retirement account in which the money you invest is exempt from income taxes until you withdraw it. If you deposit $100 each month into an IRA earning 6% interest, how much will you have in the account after 20 years?

Solution

In this example,

d=$100

r=0.06

k=12

N=20

Putting this into the equation:

$$P_{20} = \frac{100\left(\left(1+\frac{0.06}{12}\right)^{20(12)}-1\right)}{\left(\frac{0.06}{12}\right)}$$

$$P_{20} = \frac{100\left((1.005)^{240}-1\right)}{(0.005)}$$ (Here, 0.005=0.5% is the interest rate per period, and 240 is the total number of periods)

$$P_{20} = \frac{100(3.310-1)}{(0.005)}$$

$$P_{20} = \frac{100(2.310)}{(0.005)} = \$46200$$

The account will grow to $46,200 after 20 years.

Notice that you deposited into the account a total of $24,000 ($100 a month for 240 months). The difference between what you end up with and how much you put in is the interest earned. In this case it is $46,200 - $24,000 = $22,200.

Example 3.2.2

You want to have $200,000 in your account when you retire in 30 years. Your retirement account earns 8% interest. How much do you need to deposit each month to meet your retirement goal?

Solution

In this example,

We're looking for d.

r=0.08 8% annual rate

k=12 since we're depositing monthly

N=30 30 years

P_{30}= $200,000 The amount we want 30 years

In this case, we're going to have to set up the equation, and solve for d.

$$200,000 = \frac{d\left(1+\frac{0.08}{12}\right)^{30(12)}-1}{\frac{0.08}{12}}$$

$$200,000 = \frac{d\left((1.00667)^{360}-1\right)}{(0.00667)}$$ The right-hand side turns out to be $1491.57d$

$$d = 200,0001491.57 = \$134.09$$

So you would need to deposit $134.09 each month to have $200,000 in 30 years if your account earns 8% interest.

Try it Now 2

A more conservative investment account pays 3% interest. If you deposit $5 a day into this account, how much will you have after 10 years? How much is from interest?

Loans

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

Loans Formula

$$P_0 = \frac{d\left(1-\left(1+\frac{r}{k}\right)^{-Nk}\right)}{\left(\frac{r}{k}\right)}$$

P_0 is the balance in the account at the beginning (the principal, or amount of the loan).

d is your loan payment (the amount of each payment, made monthly, annually, etc.)

r is the annual interest rate in decimal form.

k is the number of compounding periods in one year.

N is the length of the loan, in years.

Like before, the compounding frequency is not always explicitly given, but is determined by how often you make payments.

When do you use this?

The loan formula assumes that you make loan payments **on a regular schedule (every month, year, quarter, etc.)** and are paying interest on the loan.

The compound interest formula in Section 3.1 assumes just one deposit. An annuity assumes regular deposits, and a loan assumes regular payments.

Example 3.2.3

You can afford $200 per month as a car payment. If you can get an auto loan at 3% interest for 60 months (5 years), how expensive of a car can you afford? In other words, what amount loan can you pay off with $200 per month?

Solution

In this example,

d=$200 The monthly loan payment

r=0.03 3% annual rate

k=12 since we're doing monthly payments, we'll compound monthly

N=5 since we're making monthly payments for 5 years

We're looking for P_0, the starting amount of the loan.

$$P_0 = \frac{200\left(1-\left(1+\frac{0.03}{12}\right)^{-5(12)}\right)}{\left(\frac{0.03}{12}\right)}$$

$$P_0 = \frac{200\left(1-(1.0025)^{-60}\right)}{(0.0025)}$$

$$P_0 = \frac{200(1-0.861)}{(0.0025)} = \$11,120$$

You can afford a $11,120 loan. (You may have to give up buying your dream car for now.)

You will pay a total of $12,000 ($200 per month for 60 months) to the loan company. The difference between the amount you pay and the amount of the loan is the interest paid. In this case, you're paying $12,000-$11,120 = $880 interest total.

Example 3.2.4

You want to take out a $140,000 mortgage (home loan). The interest rate on the loan is 6%, and the loan is for 30 years. How much will your monthly payments be?

Solution

In this example,

We're looking for d.

r=0.06 6% annual rate

k=12 Since we're paying monthly

N=30 30 years

P_0=$140,000 the starting loan amount

In this case, we're going to have to set up the equation, and solve for d.

$$140,000 = \frac{d\left(1-\left(1+\frac{0.06}{12}\right)^{-30(12)}\right)}{\left(\frac{0.06}{12}\right)}$$
$$140,000 = \frac{d\left(1-(1.005)^{-360}\right)}{(0.005)}$$
$$140,000 = d(166.792)$$
$$d = \frac{140,000}{166.792} = \$839.37$$

You will make payments of $839.37

You're paying a total of $302,173.20 to the loan company: $839.37 per month for 360 months. You are paying a total of $302,173.20 - $140,000 = $162,173.20 in interest over the life of the loan. Yes, over the next 30 years, you will be spending more money on the interest than on the house you are about to buy. This is the reality!

Try it Now 3

Janine bought $3,000 of new furniture on credit. Because her credit score isn't very good, the store is charging her a fairly high interest rate on the loan: 16%. If she agreed to pay off the furniture over 2 years, how much will she have to pay each month?

Remaining Loan Balance

With loans, it is often desirable to determine what the remaining loan balance will be after some number of years. For example, if you purchase a home and plan to sell it in five years, you might want to know how much of the loan balance you will have paid off and how much you have to pay from the sale.

To determine the remaining loan balance after some number of years, we first need to know the loan payments, if we don't already know them. Remember that only a portion of your loan payments go towards the loan balance; a portion is going to go towards interest. For example, if your payments were $1,000 a month, after a year you will not have paid off $12,000 of the loan balance.

To determine the remaining loan balance, we can think "how much loan will these loan payments be able to pay off in the remaining time on the loan?"

Example 3.2.5

If a mortgage at a 6% interest rate has payments of $1,000 a month, how much will the loan balance be 10 years from the end the loan?

Solution

To determine this, we are looking for the amount of the loan that can be paid off by $1,000 a month payments in 10 years. In other words, we're looking for P_0 when

d=$1,000 The monthly loan payment.

r=0.06 6% annual rate

k=12 since we're doing monthly payments,

N=10 since we're making monthly payments for 10 more years

$$P_0 = \frac{1000\left(1-\left(1+\frac{0.06}{12}\right)^{-10(12)}\right)}{\frac{0.06}{12}}$$

$$P_0 = \frac{1000\left(1-(1.005)^{-120}\right)}{(0.005)}$$

$$P_0 = \frac{1000(1-0.5496)}{(0.005)} = \$90,073.45$$

The loan balance with 10 years remaining on the loan will be $90,073.45

Answering remaining balance questions requires two steps:

1. Calculating the monthly payments on the loan
2. Calculating the remaining loan balance based on the remaining time on the loan

Example 3.2.6

A couple purchases a home with a $180,000 mortgage at 4% for 30 years with monthly payments. What will the remaining balance on their mortgage be after 5 years? (The answer may be depressing.)

Solution

First we will calculate their monthly payments.

We're looking for d.

r=0.04 4% annual rate

k=12 since they're paying monthly

N=30 30 years

P_0=$180,000 the starting loan amount

We set up the equation and solve for d.

$$180,000 = \frac{d\left(1-\left(1+\frac{0.04}{12}\right)^{-30(12)}\right)}{\left(\frac{0.04}{12}\right)}$$

$$180,000 = \frac{d\left(1-(1.00333)^{-360}\right)}{(0.00333)}$$

$$180,000 = d(209,562)$$

$$d = \frac{180,000}{209.562} = \$858.93$$

Now that we know the monthly payments, we can determine the remaining balance. We want the remaining balance after 5 years, when 25 years will be remaining on the loan, so we calculate the loan balance that will be paid off with the monthly payments over those 25 years.

d=$858.93 The monthly loan payment we calculated above.

r=0.04 4% annual rate

k=12 since they're doing monthly payments

N=25 since they'd be making monthly payments over those 25

$$P_0 = \frac{858.93\left(1-\left(1+\frac{0.04}{12}\right)^{-25(12)}\right)}{\left(\frac{0.04}{12}\right)}$$

$$P_0 = \frac{858.93\left(1-(1.00333)^{-300}\right)}{(0.00333)}$$

$$P_0 = \frac{858.93(1-0.369)}{(0.00333)} \approx \$162,758$$

The loan balance after 5 years, with 25 years remaining on the loan, will be $162,758

Over that 5 years, the couple has paid off $180,000 - $162,758 = $17,242 of the loan balance. They have paid $858.93 a month for 5 years (60 months), for a total of $51,535.80, so $51,535.80 - $17,242 = $34,292.80 of what they have paid so far has been interest.

Which Equation to use?

When presented with a finance problem (on an exam or in real life), you're usually not told what type of problem it is or which equation to use. Here are some hints on deciding which equation to use based on the wording of the problem.

The easiest types of problem to identify are loans. Loan problems almost always include words like: "loan", "amortize" (the fancy word for loans), "finance (a car)", or "mortgage" (a home loan). Look for these words. If they're there, you're probably looking at a loan problem. To make sure, see if you're given what your monthly (or annual) payment is, or if you're trying to find a monthly payment.

If the problem is not a loan, the next question you want to ask is: "Am I putting money in an account and letting it sit, or am I making regular (monthly/annually/quarterly) payments or withdrawals?" If you're letting the money sit in the account with nothing but interest changing the balance, then you're looking at a compound interest problem. The exception would be bonds and other investments where the interest is not reinvested; in those cases you're looking at simple interest.

If you're making regular payments or withdrawals, the next questions is: "Am I putting money into the account, or am I pulling money out?" If you're putting money into the account on a regular basis (monthly/annually/quarterly), then you're looking at a basic annuity problem. In basic annuities, you are saving money. Usually in an annuity problem, your account starts empty and has money in the future.

Remember, the most important part of answering any kind of question, money or otherwise, is first to correctly identify what the question is really asking, and then to determine what approach will best allow you to solve the problem.

Try it Now 4

For each of the following scenarios, determine if it is a compound interest problem with one deposit, a savings annuity problem, or a loans problem. Then solve each problem.

a. Paul wants to buy a new car. Rather than take out a loan, he decides to save $200 a month in an account earning 3% interest compounded monthly. How much will he have saved up after 3 years?
b. Keisha is managing investments for a non-profit company. They want to invest some money in an account earning 5% interest compounded annually with the goal to have $30,000 in the account in 6 years. How much should Keisha deposit into the account?
c. Miao is going to finance new office equipment at a 2% rate over a 4 year term. If she can afford monthly payments of $100, how much new equipment can she buy?
d. How much would you need to save every month in an account earning 4% interest to have $5,000 saved up in two years?

Reference

1. References (6)

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3.3: Continuous Compounding

We continue the discussion that was started in Section 3.1: Simple and Compound Interest.

In a nutshell:

1. You invest some money (the ***principal***) in a bank which pays interest.
2. After a specified time (the ***compounding period***), you add in the interest that has been earned on the principal.
3. After another compounding period, you add in the interest earned on the principal plus accumulated interest.

Repeat.

Compound interest, by definition, is interest calculated on ***the principal amount together with accumulated interest***. Interest can be added in at different fixed intervals: annually, monthly, weekly, daily, and so on. Adding in interest more and more often produces a slightly better accumulation. (Most banks add in interest daily.)

What happens as you increase the number of times that interest is added in each year?

Your earnings will get bigger. How much bigger? Will your earnings increase without bound?

Equivalently, what happens as you add in interest over shorter and shorter periods of time?

In other words, what happens as you let the compounding period approach zero?

You'll end up with more money. How much more? Will you get rich?

The amount of money produced under these situations does NOT increase without bound! (Wishful thinking, but doesn't happen.)

Instead, the irrational number ‘e’ emerges to describe the limiting amount earned.

The resulting formula is called the ***Continuous Compounding Formula*** and is the subject of this section.

A quick review of the Compound Interest Formula

1. Put P dollars (the ***principal***) in a bank.
2. Assume the bank offers an annual interest rate r. For example, 3% per year corresponds to r=0.03.
3. Add in the earned interest n equally-spaced times each year. In other words, assume there are n ***compounding periods*** per year. For example, adding in interest monthly corresponds to n=12. Adding in interest daily corresponds to n=365
4. Continue this process for t years.
5. The accumulation (principal plus interest) is given by the variable A in the Compound Interest Formula:
   $A = P\left(1+\frac{r}{n}\right)^{nt}$

Letting n go to infinity in the Compound Interest Formula

As n gets large, $\frac{r}{n}$ gets small. That is, as $n \to \infty$, we have $\frac{r}{n} \to 0$.

As $\frac{r}{n}$ approaches zero, $1+\frac{r}{n}$ approaches 1. That is, as $\frac{r}{n} \to 0$, we have $1+\frac{r}{n} \to 1$

As n gets big, nt gets big. That is, as $n \to \infty$, we have $nt \to \infty$

So, as we add interest in more and more frequently (let $n \to \infty$), here's what happens in the Compound Interest Formula:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

We're looking at a '1’ form—which is just a shorthand for saying that the base approaches 1, and the exponent approaches .
But, here's the important question:

As the base and exponent travel together along their journeys to 1 and , respectively, what number (if any) does the ***entire expression*** approach?

A discussion of the '1^∞' form

There are lots of things that make the '1^∞' form both interesting and difficult to analyze. All of the following are true:

- 1 to any finite power equals 1. For example, $1^{1,000,000,000} = 1$
- When a number a bit less than 1 is raised to higher and higher powers, it approaches zero. For example, $\left(\frac{99}{100}\right)^n$ approaches 0 as n approaches infinity. (This is an exponential function with base between zero and one.)
- When a number a bit more than 1 is raised to higher and higher powers, it approaches infinity. For example, $\left(\frac{101}{100}\right)^n$ approaches as n approaches infinity. (This is an exponential function with base greater than one.)

Depending on precisely ***how*** the base approaches 1, and ***how*** the exponent approaches infinity, the form 1^∞ can approach different numbers!

'1^∞' is an example of what is called (in calculus) an ***indeterminate form***. An ***indeterminate form*** occurs when ***you can't figure out what is happening without further analysis***. That is, each occurrence of must be investigated separately. For us, the investigation will involve the following definition of the irrational number e

$$\left(1+\frac{1}{n}\right)^n \to\to\to\to\to\to e \approx 2.718 \quad \text{as } n \to \infty$$

Continuous Compounding

Letting $n \to \infty$ in the Compound Interest Formula, $A = P\left(1+\frac{r}{n}\right)^{nt}$ yields the Continuous Compounding Formula:

$$A = Pe^{rt}$$

Roughly, ***continuous compounding*** describes interest being added in the ***instant*** it is earned.

Example 3.3.1

Suppose that $1000 is invested at 3% annual interest. What is the accumulation after ten years if compounded monthly, daily, and continuously?

Solution

Compounded monthly:

$$A = P\left(1+\frac{r}{n}\right)^{nt} = 1000\left(1+\frac{0.03}{12}\right)^{12\cdot 10} = \$1,349.35$$

Compounded daily:

$$A = 1000\left(1+\frac{0.03}{365}\right)^{365\cdot 10} = \$1,349.84$$

Compounded continuously:

$$A = Pe^{rt} = 1000e^{0.03\cdot 10} = \$1349.86$$

Not much difference! You won't get rich if your bank decides to compound continuously!

Reference

1. References (7)

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3.4: Federal Budgets and National Debt

Figure 3.4.1 shows the pattern of annual federal budget deficits and surpluses, back to 1930, as a share of GDP. When the line is above the horizontal axis, the budget is in surplus. When the line is below the horizontal axis, a budget deficit occurred. Clearly, the biggest deficits as a share of GDP during this time were incurred to finance World War II. Deficits were also large during the 1930s, the 1980s, the early 1990s, and most recently during the 2008-2009 recession.

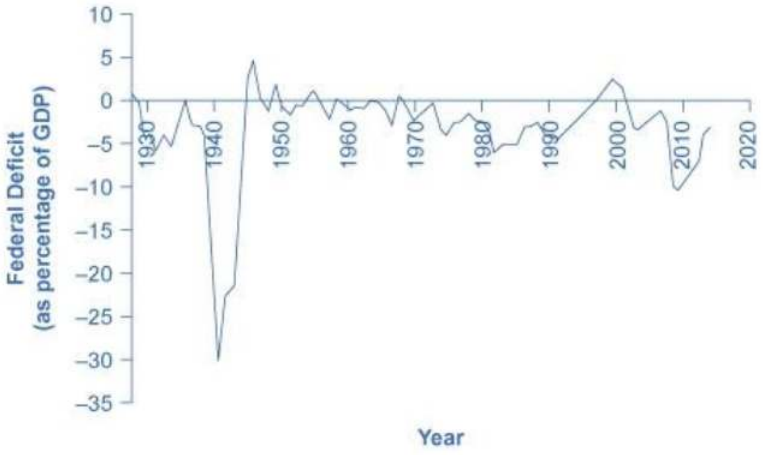


Figure 3.4.1: Pattern of Federal Budget Deficits and Surpluses, 1929–2014. The federal government has run budget deficits for decades. The budget was briefly in surplus in the late 1990s, before heading into deficit again in the first decade of the 2000s—and especially deep deficits in the 2008-2009 recession. (Source: Federal Reserve Bank of St. Louis (FRED). Image by Federal Reserve Bank of St. Louis

Debt/GDP Ratio

Another useful way to view the budget deficit is through the prism of accumulated debt rather than annual deficits. The **national debt** refers to the total amount that the government has borrowed over time. In contrast, the budget deficit refers to how much the government has borrowed in one particular year. Figure 3.4.2 shows the ratio of debt/GDP since 1940. Until the 1970s, the debt/GDP ratio revealed a fairly clear pattern of federal borrowing. The government ran up large deficits and raised the debt/GDP ratio in World War II, but from the 1950s to the 1970s the government ran either surpluses or relatively small deficits, and so the debt/GDP ratio drifted down. Large deficits in the 1980s and early 1990s caused the ratio to rise sharply. When budget surpluses arrived from 1998 to 2001, the debt/GDP ratio declined substantially. The budget deficits starting in 2002 then tugged the debt/GDP ratio higher—with a big jump when the recession took hold in 2008–2009.

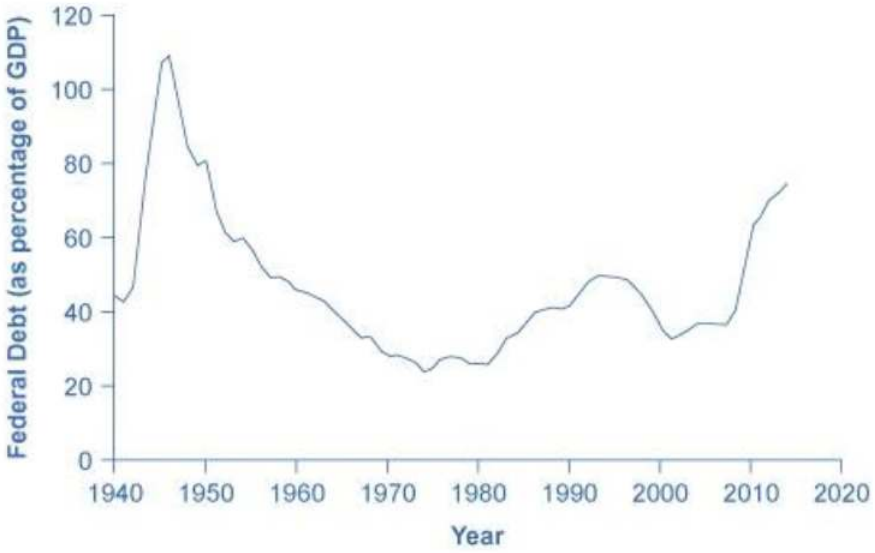


Figure 3.4.2: Federal Debt as a Percentage of GDP, 1942–2014. Federal debt is the sum of annual budget deficits and surpluses. Annual deficits do not always mean that the debt/GDP ratio is rising. During the 1960s and 1970s, the government often ran small deficits, but since the debt was growing more slowly than the economy, the debt/GDP ratio was declining over this time. In the 2008–2009 recession, the debt/GDP ratio rose sharply. (Source: Economic Report of the President, Table B-20. Image by Executive office of the President Council of Economic Advisers is in the public domain.

Debt VS. Defecit?

The deficit is not the debt. The difference between the deficit and the debt lies in the time frame. The federal deficit (or surplus) refers to what happens with the federal government budget each year. The public (or Federal government) debt is accumulated over time; it is the sum of all past deficits and surpluses. If you borrow $10,000 per year for each of the four years of college, you might say that your annual deficit was $10,000, but your accumulated debt over the four years is $40,000.

One year's federal budget deficit causes the federal government to sell Treasury bonds to make up the difference between spending programs and tax revenues. The dollar value of all the outstanding Treasury bonds on which the federal government owes money is equal to the national debt.

The Path from Deficits to Surpluses to Deficits

Why did the budget deficits suddenly turn to surpluses from 1998 to 2001 and why did the surpluses return to deficits in 2002? Why did the deficit become so large after 2007? Figure 3.4.3 suggests some answers. The graph combines the earlier information on total federal spending and taxes in a single graph, but focuses on the federal budget since 1990.

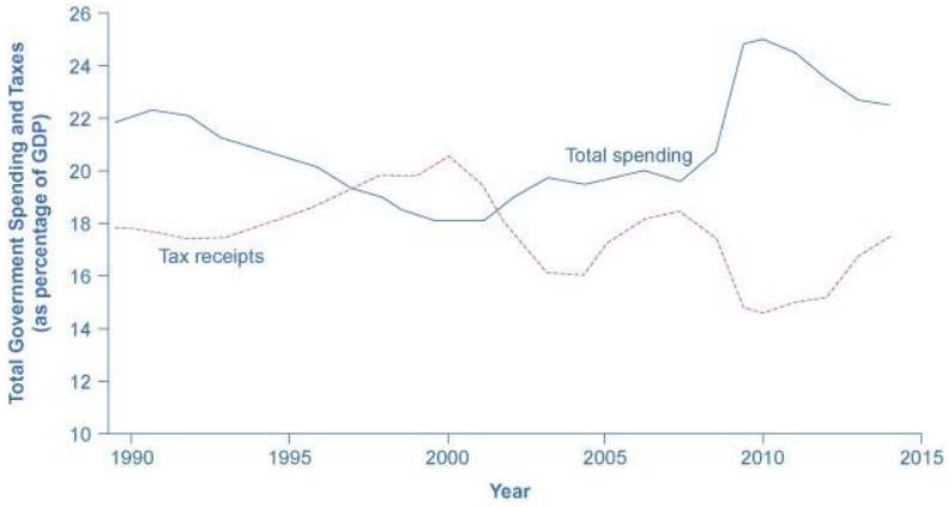


Figure 3.4.3: Total Government Spending and Taxes as a Share of GDP, 1990–2014. When government spending exceeds taxes, the gap is the budget deficit. When taxes exceed spending, the gap is a budget surplus. The recessionary period starting in late 2007 saw higher spending and lower taxes, combining to create a large deficit in 2009. (Source: Economic Report of the President, Tables B-21 and B-1, Image by Executive office of the President Council of Economic Advisers is in the public domain.

Government spending as a share of GDP declined steadily through the 1990s. The biggest single reason was that defense spending declined from 5.2% of GDP in 1990 to 3.0% in 2000, but interest payments by the federal government also fell by about 1.0% of GDP. However, federal tax collections increased substantially in the later 1990s, jumping from 18.1% of GDP in 1994 to 20.8% in 2000. Powerful economic growth in the late 1990s fueled the boom in taxes. Personal income taxes rise as income goes up; payroll taxes rise as jobs and payrolls go up; corporate income taxes rise as profits go up. At the same time, government spending on transfer payments such as unemployment benefits, foods stamps, and welfare declined with more people working.

This sharp increase in tax revenues and decrease in expenditures on transfer payments was largely unexpected even by experienced budget analysts, and so budget surpluses came as a surprise. However, in the early 2000s, many of these factors started running in reverse. Tax revenues sagged, due largely to the recession that started in March 2001, which reduced revenues. Congress enacted a series of tax cuts and President George W. Bush signed them into law, starting in 2001. In addition, government spending swelled due to increases in defense, healthcare, education, Social Security, and support programs for those who were hurt by the recession and the slow growth that followed. Deficits returned. When the severe recession hit in late 2007, spending climbed and tax collections fell to historically unusual levels, resulting in enormous deficits.

Longer-term U.S. budget forecasts, a decade or more into the future, predict enormous deficits. The higher deficits during the 2008-2009 recession have repercussions, and the demographics will be challenging. The primary reason is the “baby boom”—the exceptionally high birthrates that began in 1946, right after World War II, and lasted for about two decades. Starting in 2010, the front edge of the baby boom generation began to reach age 65, and in the next two decades, the proportion of Americans over the age of 65 will increase substantially. The current level of the payroll taxes that support Social Security and Medicare will fall well short of the projected expenses of these programs, as the following Clear It Up feature shows; thus, the forecast is for large budget deficits. A decision to collect more revenue to support these programs or to decrease benefit levels would alter this long-term forecast.

What is the Long-Term Budget Outlook for Social Security and Medicare?

In 1946, just one in every thirteen Americans was over age 65. By 2000, it was one in eight. By 2030, one American in five will be over age 65. Two enormous U.S. federal programs focus on the elderly—Social Security and Medicare. The growing numbers of elderly Americans will increase spending on these programs, as well as on Medicaid. The current payroll tax levied on workers, which supports all of Social Security and the hospitalization insurance part of Medicare, will not be enough to cover the expected costs, so what are the options?

Long-term projections from the Congressional Budget Office in 2009 are that Medicare and Social Security spending combined will rise from 8.3% of GDP in 2009 to about 13% by 2035 and about 20% in 2080. If this rise in spending occurs, without any corresponding rise in tax collections, then some mix of changes must occur: (1) taxes will need to increase dramatically; (2) other

spending will need to be cut dramatically; (3) the retirement age and/or age receiving Medicare benefits will need to increase, or (4) the federal government will need to run extremely large budget deficits.

Some proposals suggest removing the cap on wages subject to the payroll tax, so that those with very high incomes would have to pay the tax on the entire amount of their wages. Other proposals suggest moving Social Security and Medicare from systems in which workers pay for retirees toward programs that set up accounts where workers save funds over their lifetimes and then draw out after retirement to pay for healthcare.

The United States is not alone in this problem. Providing the promised level of retirement and health benefits to a growing proportion of elderly with a falling proportion of workers is an even more severe problem in many European nations and in Japan. How to pay promised levels of benefits to the elderly will be a difficult public policy decision.

Exercises

1. A friend lends you $200 for a week, which you agree to repay with 5% one-time interest. How much will you have to repay?
2. Suppose you obtain a $3,000 T-note with a 3% annual rate, paid quarterly, with maturity in 5 years. How much interest will you earn?
3. A T-bill is a type of bond that is sold at a discount over the face value. For example, suppose you buy a 13-week T-bill with a face value of $10,000 for $9,800. This means that in 13 weeks, the government will give you the face value, earning you $200. What annual interest rate have you earned?
4. Suppose you are looking to buy a $5000 face value 26-week T-bill. If you want to earn at least 1% annual interest, what is the most you should pay for the T-bill?
5. You deposit $300 in an account earning 5% interest compounded annually. How much will you have in the account in 10 years?
6. How much will $1000 deposited in an account earning 7% interest compounded annually be worth in 20 years?
7. You deposit $2000 in an account earning 3% interest compounded monthly
    a. How much will you have in the account in 20 years?
    b. How much interest will you earn?
8. You deposit $10,000 in an account earning 4% interest compounded monthly.
    a. How much will you have in the account in 25 years?
    b. How much interest will you earn?
9. How much would you need to deposit in an account now in order to have $6,000 in the account in 8 years? Assume the account earns 6% interest compounded monthly.
10. How much would you need to deposit in an account now in order to have $20,000 in the account in 4 years? Assume the account earns 5% interest.
11. You deposit $200 each month into an account earning 3% interest compounded monthly.
    a. How much will you have in the account in 30 years?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
12. You deposit $1000 each year into an account earning 8% compounded annually.
    a. How much will you have in the account in 10 years?
    b. How much total money will you put into the account?
    c. How much total interest will you earn?
13. Jose has determined he needs to have $800,000 for retirement in 30 years. His account earns 6% interest.
    a. How much would he need to deposit in the account each month?
    b. How much total money will he put into the account?
    c. How much total interest will he earn?
14. You wish to have $3000 in 2 years to buy a fancy new stereo system. How much should you deposit each quarter into an account paying 8% compounded quarterly?
15. You want to be able to withdraw $30,000 each year for 25 years. Your account earns 8% interest.
    a. How much do you need in your account at the beginning?
    b. How much total money will you pull out of the account?
    c. How much of that money is interest?

16. How much money will I need to have at retirement so I can withdraw $60,000 a year for 20 years from an account earning 8% compounded annually?
    a. How much do you need in your account at the beginning?
    b. How much total money will you pull out of the account?
    c. How much of that money is interest?
17. You have $500,000 saved for retirement. Your account earns 6% interest. How much will you be able to pull out each month if you want to be able to take withdrawals for 20 years?
18. Loren already knows that she will have $500,000 when she retires. If she sets up a payout annuity for 30 years in an account paying 10% interest, how much could the annuity provide each month?
19. You can afford a $700 per month mortgage payment. You've found a 30 year loan at 5% interest.
    a. How big of a loan can you afford?
    b. How much total money will you pay the loan company?
    c. How much of that money is interest?
20. Marie can afford a $250 per month car payment. She's found a 5 year loan at 7% interest.
    a. How expensive of a car can she afford?
    b. How much total money will she pay the loan company?
    c. How much of that money is interest?
21. You want to buy a $25,000 car. The company is offering a 2% interest rate for 48 months (4 years). What will your monthly payments be?
22. You decide finance a $12,000 car at 3% compounded monthly for 4 years. What will your monthly payments be? How much interest will you pay over the life of the loan?
23. You want to buy a $200,000 home. You plan to pay 10% as a down payment and take out a 30 year loan for the rest.
    a. How much is the loan amount going to be?
    b. What will your monthly payments be if the interest rate is 5%?
    c. What will your monthly payments be if the interest rate is 6%?
24. Lynn bought a $300,000 house, paying 10% down, and financing the rest at 6% interest for 30 years.
    a. Find her monthly payments
    b. How much interest will she pay over the life of the loan?
25. Emile bought a car for $24,000 three years ago. The loan had a 5 year term at 3% interest rate, making monthly payments. How much does he still owe on the car?
26. A friend bought a house 15 years ago, taking out a $120,000 mortgage at 6% for 30 years, making monthly payments. How much does she still owe on the mortgage?
27. Suppose you invest $50 a month for 5 years into an account earning 8% compounded monthly. After 5 years, you leave the money, without making additional deposits, in the account for another 25 years. How much will you have in the end?
28. Suppose you put off making investments for the first 5 years, and instead made deposits of $50 a month for 25 years into an account earning 8% compounded monthly. How much will you have in the end?
29. Mike plans to make contributions to his retirement account for 15 years. After the last contribution, he will start withdrawing $10,000 a quarter for 10 years. Assuming Mike's account earns 8% compounded quarterly, how large must his quarterly contributions be during the first 15 years, in order to accomplish his goal?
30. Kendra wants to be able to make withdrawals of $60,000 a year for 30 years after retiring in 35 years. How much will she have to save each year up until retirement if her account earns 7% interest?
31. You have $2,000 to invest and want it to grow to $3,000 in two years. What interest rate would you need to find to make this possible?
32. You have $5,000 to invest and want it to grow to $20,000 in ten years. What interest rate would you need to find to make this possible?
33. You plan to save $600 a month for the next 30 years for retirement. What interest rate would you need to have $1,000,000 at retirement?
34. You really want to buy a used car for $11,000 but can only afford $200 a month. What interest rate would you need to find to be able to afford the car, assuming the loan is for 60 months?
35. Suppose you just discovered that your uncle, who died 100 years ago, had left you $2000 as a gift so you can attend college.

a. If your uncle invested that money at 4% annually for 100 years, how much would you have today?
b. Repeat Part (a), using 4% monthly instead.
c. Repeat Part (a), using 4% continuously instead. Compare your answers.

36. If $15,000 is invested in an account earning 3% interest continuously, how much is the investment worth after 20 years?
37. You want to invest some money so that in 36 years, you will have $100,000. Supposing that you have an account earning interest at 5.2% continuously, how much do you have to invest today?
38. You have just found a great investment opportunity, assuring you that the money will be compounded continuously at 9%. How much money do you have to set aside now so that in 10 years you will have $1,000,000?
39. Briefly describe the main difference between a “deficit” and a “debt.”
40. Give a simple definition for “federal deficit.”
41. In FY 2017 (October 1, 2017 to September 30, 2018), the U.S. Federal Government had a total revenue of about $3.33 trillion and total expenditures of $4.11 trillion. The GDP for that year was $20.23 trillion.
42. How much was the federal deficit for FY 2017?
43. The national debt as of the end of FY 2016 was $672. Using (a), calculate the national debt as of the end of FY 2017.
44. Find the debt/GDP ratio at the end of FY 2017.
45. Find the deficit/GDP ratio for FY 2017.
46. In FY 1987, the national deficit was $150 million. By what percentage did the deficit change between 1987 and 2017 (30 years)?
47. So far (as of 2019) there has been just one year when the national deficit exceed $1.4 trillion. Using online resources, find out what year that was and why the deficit was so high.
48. What was the last fiscal year when there was a national surplus instead of a national debt?
49. The last few paragraphs of the text of this chapter suggest three potential solutions to the crisis of Medicare and Social Security. Discuss the pros and cons of each possible solution listed. What other factors need to be known to decide which is best? The three potential solutions are as follows:
50. Dramatic tax increase
51. Dramatic spending cut in other areas
52. Mandatory age increase for retirement/Medicare eligibility

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CHAPTER OVERVIEW

4: Probability and Odds

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be experimental in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin, it would turn up heads about half of the time. When a weather reporter says “there is a 10% chance of rain tomorrow,” she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be subjective in nature, an educated guess. If someone asked you the probability that the Los Angeles Dodgers would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, if the same two teams were to play each other repeatedly under identical conditions, the Dodgers would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with theoretical probability, which is defined as follows: Suppose there is a situation with n equally likely possible outcomes and that m of those n outcomes correspond to a particular event; then the probability of that event is defined as m/n.

4.1: Basic Concepts
4.2: Conditional Probability
4.3: Counting
4.4: Expected Value

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4.1: Basic Concepts

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

Events and Outcomes

Events and Outcomes

- The result of an experiment is called an **outcome**.
- An **event** is any particular outcome or group of outcomes.
- A **simple event** is an event that cannot be broken down further.
- The **sample space** is the set of all possible simple events.

Example 4.1.1

If we roll a standard 6-sided die, describe the sample space and some simple events.

Solution

The sample space is the set of all possible simple events: {1,2,3,4,5,6}

Some examples of simple events:

- We roll a 1
- We roll a 5

Some compound events:

- We roll a number bigger than 4
- We roll an even number

Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event E using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event E}}{\text{Total number of equally likely outcomes}}$$

Example 4.1.2

If we roll a 6-sided die, calculate

a. P(rolling a 1)
b. P(rolling a number bigger than 4)

Solution

Recall that the sample space is {1,2,3,4,5,6}

a. There is one outcome corresponding to “rolling a 1”, so the probability is $\frac{1}{6}$
b. There are two outcomes bigger than a 4, so the probability is $\frac{2}{6} = \frac{1}{3}$

Probabilities are essentially fractions and can be reduced to lower terms like fractions.

Example 4.1.3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

Solution

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\frac{14}{20} = \frac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Exercise 4.1.1

At some random moment, you look at your clock and note the minutes reading.

a. What is the probability of the minutes reading 15?
b. What is the probability the minutes reading 15 or less?

Cards

A standard deck of 52 playing cards consists of four suits (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different rank: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

Example 4.1.4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

Solution

There are 52 cards in the deck and 4 Aces so $P(\text{Ace}) = \frac{4}{52} = \frac{1}{13} \approx 0.0769$

We can also think also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 (if there are no outcomes that correspond with the event). The largest possible probability is 1 (if all possible outcomes correspond with the event).

Certain and Impossible events

An impossible event has a probability of 0.

A certain event has a probability of 1.

The probability of any event must be $0 \le P(E) \le 1$

If you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is P(six) = $\frac{1}{6}$. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is P(not a six) = $\frac{5}{6}$. Notice that

$$P(\text{six}) + P(\text{ not a six }) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

This is not a coincidence. Consider a generic situation with n possible outcomes and an event E that corresponds to m of these outcomes. Then the remaining $n - m$ outcomes correspond to E not happening, thus.

$$P(\text{not} E) = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

Complement of an Event

The complement of an event is the event "E doesn't happen".

The notation $\overline{E}$ is used for the complement of event E.

We can compute the probability of the complement using $P(\overline{E}) = 1 - P(E)$

Notice also that $P(E) = 1 - P(\overline{E})$

Example 4.1.5

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

Solution

There are 13 hearts in the deck, so $P(\text{ heart }) = \frac{13}{52} = \frac{1}{4}$.

The probability of not drawing a heart is the complement:

$$P(\text{ not heart }) = 1 - P(\text{ heart }) = 1 - \frac{1}{4} = \frac{3}{4}$$

Independent events

Example 4.1.6

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

Solution

We could list all possible outcomes: $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

Notice there are $2 \cdot 6 = 12$ total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$

The prior example was looking at two independent events.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example 4.1.7

Are these events independent?

a. A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.
b. The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).
c. You draw a card from a deck, then draw a second card without replacing the first.

Solution

a. The probability that a head comes up on the second toss is 1/2 regardless of whether or not a head came up on the first toss, so these events are independent.
b. These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.
c. The probability of the second card being red depends on whether the first card is red or not, so these events are not independent. Remember, if you draw one red card, now the stack has one fewer red card, affecting the probability.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

$P(A \text{ and } B)$ for independent events

If events A and B are independent, then the probability of both A and B occurring is: $P(A \text{ and } B) = P(A) \cdot P(B)$

Where $P(A \text{ and } B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring.

If you look back at the coin and die examples from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example 4.1.8

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

Solution

The probability of choosing a white pair of socks is $\frac{6}{10}$

The probability of choosing a white tee shirt is $\frac{3}{7}$

The probability of both being white is $\frac{6}{10} \cdot \frac{3}{7} = \frac{18}{70} = \frac{9}{35}$

Try it Now 2

A card is pulled from a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

The previous examples looked at the probability of both events occurring. Now we will look at the probability of either event occurring.

Example 4.1.9

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Solution

Here, there are still 12 possible outcomes: $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are $H1, H2, H3, H4, H5, H6, T6$), so the probability is $\frac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12}$, which is not the correct probability. Looking at the outcomes we can see why: the outcome $H6$ would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$. If we subtract out this double count, we have the correct probability: $\frac{8}{12} - \frac{1}{12} = \frac{7}{12}$

$P(A \text{ or } B)$

The probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example 4.1.10

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

Solution

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$$P(\text{ King or Queen }) = \frac{8}{52}$$

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{ King or Queen }) = 0$. Using our probability rule, we could have said:

$$P(\text{ King or Queen }) = P(\text{ King }) + P(\text{ Queen }) - P(\text{ King and Queen }) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52}$$

In the last example, the events were **mutually exclusive**, so $P(\text{A or } B) = P(A) + P(B)$.

Example 4.1.11

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Solution

Half the cards are red, so $P(\text{ red }) = \frac{26}{52}$

There are four kings, so $P(\text{ King }) = \frac{4}{52}$

There are two red kings, so $P(\text{ Red and King }) = \frac{2}{52}$

We can then calculate

$$P(\text{ Red or King }) = P(\text{ Red }) + P(\text{ King }) - P(\text{ Red and King }) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52}$$

Try it Now 3

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what is the probability that at least one is white?

Example 4.1.12

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

a. Has a red car *and* got a speeding ticket.
b. Has a red car *or* got a speeding ticket.

Solution

Table 4.1.1

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

a. We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is $\frac{15}{665} \approx 0.0226$

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

b. We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\frac{195}{665} \approx 0.2932$

We also could have found this probability by:

$$P(\text{ had a red car }) + P(\text{ got a speeding ticket }) - P(\text{ had a red car and got a speeding ticket }) = \frac{150}{665} + \frac{60}{665} - \frac{15}{665} = \frac{195}{665}$$

Reference

1. References (8)

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4.2: Conditional Probability

Often, it is required to compute the probability of an event given that another event has occurred.

Example 4.2.1

What is the probability that two cards drawn at random from a deck of playing cards will both be Aces?

Solution

It might seem that you could use the formula for the probability of two independent events and simply multiply $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$. This would be incorrect, however, because the two events are not independent. If the first card drawn is an Ace, then the probability that the second card is also an Ace would be lower because there would only be three Aces left in the deck.

Once the first card chosen is an Ace, the probability that the second card chosen is also an Ace is called the conditional probability of drawing an Ace. In this case the "condition" is that the first card is an Ace. Symbolically, we write this as:

$$P(\text{Ace on second draw} \mid \text{an Ace on the first draw})$$

The vertical bar "|" is read as "given," so the above expression is short for "The probability that an Ace is drawn on the second draw given that an Ace was drawn on the first draw." What is this probability? After an Ace is drawn on the first draw, there are 3 Aces out of 51 total cards left. This means that the conditional probability of drawing an Ace after one Ace has already been drawn is $\frac{3}{51} = \frac{1}{17}$.

Thus, the probability of both cards being Aces is $\frac{4}{52} \cdot \frac{3}{51} = \frac{12}{2652} = \frac{1}{221}$

Conditional Probability

The probability the event B occurs, given that event A has happened, is represented as $P(B \mid A)$. This is read as “the probability of B given A"

Then, the probability that A and B both occur is this conditional probability MULTIPLIED by the probability that A occurs, i.e., $P(A \text{ and } B) = P(A)P(B \mid A)$

Example 4.2.2

Find the probability that a die rolled shows a 6, given that a flipped coin shows a head.

Solution

These are two independent events, so the probability of the die rolling a 6 is, $\frac{1}{6}$ regardless of the result of the coin flip.

Example 4.2.3

The table below shows the number of survey subjects who have received a speeding ticket in the last year and the color of their car. Find the probability that a randomly chosen person:

a. Has a speeding ticket *given* they have a red car
b. Has a red car *given* they have a speeding ticket

Solution

Table 4.2.1

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515

	Speeding ticket	No speeding ticket	Total
Total	60	605	665

a. Since we know the person has a red car, we are only considering the 150 people in the first row of the table. Of those, 15 have a speeding ticket, so

$$P(\text{ ticket } \mid \text{ red car }) = \frac{15}{150} = \frac{1}{10} = 0.1$$

b. Since we know the person has a speeding ticket, we are only considering the 60 people in the first column of the table. Of those, 15 have a red car, so

$$P(\text{ red car } \mid \text{ ticket }) = \frac{15}{60} = \frac{1}{4} = 0.25$$

Notice from the last example that $P(B|A)$ is not equal to $P(A|B)$.

These kinds of conditional probabilities are what insurance companies use to determine your insurance rates. They look at the conditional probability of you having accident, given your age, your car, your car color, your driving history, etc., and price your policy based on that likelihood.

Example 4.2.4

If you pull 2 cards out of a deck, what is the probability that both are spades?

Solution

The probability that the first card is a spade is $\frac{13}{52} = \frac{1}{4}$

The probability that the second card is a spade, given the first was a spade, is $\frac{12}{51}$, since there is one less spade in the deck, and one less total cards. Note $\frac{12}{51} = \frac{4}{17}$

The probability that both cards are spades is $\frac{1}{4} \cdot \frac{4}{17} = \frac{1}{17} \approx 0.0588$

Example 4.2.5

If you draw two cards from a deck, what is the probability that you will get the Ace of Diamonds and a black card?

Solution

You can satisfy this condition by having Case A or Case B, as follows:

Case A) you can get the Ace of Diamonds first and then a black card or

Case B) you can get a black card first and then the Ace of Diamonds.

Let's calculate the probability of Case A:

The probability that the first card is the Ace of Diamonds is $\frac{1}{52}$. The probability that the second card is black given that the first card is the Ace of Diamonds is $\frac{26}{51}$ because 26 of the remaining 51 cards are black. The probability is therefore $\frac{1}{52} \cdot \frac{26}{51} = \frac{1}{102}$

Now for Case B:

The probability that the first card is black is $\frac{26}{52} = \frac{1}{2}$. The probability that the second card is the Ace of Diamonds given that the first card is black is $\frac{1}{51}$. The probability of Case B is therefore $\frac{1}{2} \cdot \frac{1}{51} = \frac{1}{102}$, the same as the probability of Case 1.

Recall that the probability of A or B is $P(A)+P(B)-P(\text{ A and B })$. In this problem, $P(\text{ A and B })=0$ since it is clear that Case A and Case B cannot both occur. Therefore, the probability of Case A or Case B is $\frac{1}{102}+\frac{1}{102}=\frac{2}{102}=\frac{1}{51}$. The probability that you will get the Ace of Diamonds and a black card when drawing two cards from a deck is $\frac{1}{51}$.

Try it Now 4

In your drawer you have 10 pairs of socks, 6 of which are white. If you reach in and randomly grab two pairs of socks, what is the probability that both are white?

Example 4.2.6

A home pregnancy test was given to women, then pregnancy was verified through blood tests. The following table shows the home pregnancy test results. Find

a. $P(\text{ not pregnant } \mid \text{ positive test result })$
b. $P(\text{ positive test result } \mid \text{ not pregnant })$

Solution

Table 4.2.2

	Positive test	Negative test	Total
Pregnant	70	4	74
Not Pregnant	5	14	19
Total	75	18	93

a. Since we know the test result was positive, we're limited to the 75 women in the first column, of which 5 were not pregnant.

$$P(\text{ not pregnant } \mid \text{ positive test result })=\frac{5}{75}\approx 0.067$$

b. Since we know the woman is not pregnant, we are limited to the 19 women in the second row, of which 5 had a positive test.

$$P(\text{ positive test result } \mid \text{ not pregnant })=\frac{5}{19}\approx 0.263$$

The second result is what is usually called a false positive: A positive result when the woman is not actually pregnant. False positives deserve close attention; due to its importance in everyday life, elaborate on this idea below.

Let us now concentrate on the more complex conditional probability problems we began looking at above.

Example 4.2.7

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it), but the false positive rate is 5% (that is, about 5% of people who take the test will test positive even though they do not have the disease). Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease?

Solution

There are two ways to approach the solution to this problem. One involves an important result in probability theory called Bayes' theorem, which we will discuss a bit later, but for now we will use an alternative and, we hope, much more intuitive approach.

Let's break down the information in the problem piece by piece.

Suppose a certain disease has an incidence rate of 0.1% (that is, it afflicts 0.1% of the population). The percentage 0.1% can be converted to a decimal number by moving the decimal place two places to the left, to get 0.001. In turn, 0.001 can be rewritten as a fraction: 1/1000. This tells us that about 1 in every 1000 people has the disease. (If we wanted, we could write $P(\text{disease}) = 0.001$)

A test has been devised to detect this disease. The test does not produce false negatives (that is, anyone who has the disease will test positive for it). This part is fairly straightforward: everyone who has the disease will test positive, or alternatively everyone who tests negative does not have the disease. (We could also say $P(\text{ positive } \mid \text{ disease }) = 1$)

The false positive rate is 5% (that is, about 5% of people who take the test will test positive even though they do not have the disease). This is even more straightforward. Another way of looking at it is that out of every 100 people who are tested and do not have the disease, 5 will test positive even though they do not have the disease. (We could also say that $P(\text{ positive } \mid \text{ no disease }) = 0.05$

Suppose a randomly selected person takes the test and tests positive. What is the probability that this person actually has the disease? Here we want to compute $P(\text{ disease } \mid \text{ positive })$. We already know that $P(\text{ positive } \mid \text{ disease }) = 1$, but remember that conditional probabilities are not equal if the conditions are switched.

Rather than thinking in terms of all these probabilities we have developed, let's create a hypothetical situation and apply the facts as set out above. First, suppose we randomly select 1000 people and administer the test. How many do we expect to have the disease? Since about 1/1000 of all people are afflicted with the disease, 1/1000 of 1000 people is 1. (Now you know why we chose 1000.) Only 1 of 1000 test subjects actually has the disease; the other 999 do not.

We also know that 5% of all people who do not have the disease will test positive. There are 999 disease-free people, so we would expect (0.05)(999)=49.95 (so, about 50) people to test positive who do not have the disease.

Now back to the original question, computing $P(\text{ disease } \mid \text{ positive })$. There are 51 people who test positive in our example (the one unfortunate person who actually has the disease, plus the 50 people who tested positive but don't). Only one of these people has the disease, so

$$P(\text{ disease } \mid \text{ positive }) \approx \frac{1}{51} \approx 0.0196$$

or less than 2%. Does this surprise you? This means that of all people who test positive, over 98% *do not have the disease.*

The answer we got was slightly approximate, since we rounded 49.95 to 50. We could redo the problem with 100,000 test subjects, 100 of whom would have the disease and (0.05)(99,900)=4995 test positive but do not have the disease, so the exact probability of having the disease if you test positive is

$$P(\text{ disease } \mid \text{ positive }) \approx \frac{100}{5095} \approx 0.0196$$

which is pretty much the same answer.

But back to the surprising result. *Of all people who test positive, over 98% do not have the disease*. If your guess for the probability a person who tests positive has the disease was wildly different from the right answer (2%), don't feel bad. The exact same problem was posed to doctors and medical students at the Harvard Medical School 25 years ago and the results revealed in a 1978 *New England Journal of Medicine* article. Only about 18% of the participants got the right answer. Most of the rest thought the answer was closer to 95% (perhaps they were misled by the false positive rate of 5%).

So at least you should feel a little better that a bunch of doctors didn't get the right answer either (assuming you thought the answer was much higher). But the significance of this finding and similar results from other studies in the intervening years lies not in making math students feel better but in the possibly catastrophic consequences it might have for patient care. If a doctor thinks the chances that a positive test result nearly guarantees that a patient has a disease, they might begin an unnecessary and possibly harmful treatment regimen on a healthy patient. Or worse, as in the early days of the AIDS crisis when being HIV-positive was often equated with a death sentence, the patient might take a drastic action and commit suicide.

As we have seen in this hypothetical example, the most responsible course of action for treating a patient who tests positive would be to counsel the patient that they most likely do *not* have the disease and to order further, more reliable, tests to verify the diagnosis.

One of the reasons that the doctors and medical students in the study did so poorly is that such problems, when presented in the types of statistics courses that medical students often take, are solved by use of Bayes' theorem, which is stated as follows:

Bayes' Theorem

$$P(A \mid B) = \frac{P(A)P(B \mid A)}{P(A)P(B \mid A) + P(\bar{A})P(B \mid \bar{A})}$$

In our earlier example, this translates to

$$P(\text{ disease } \mid \text{ positive }) = \frac{P(\text{ disease })P(\text{ positive } \mid \text{ disease })}{P(\text{ disease })P(\text{ positive } \mid \text{ disease }) + P(\text{ no disease })P(\text{ positive } \mid \text{ no disease })}$$

Plugging in the numbers gives

$$P(\text{ disease } \mid \text{ positive }) = \frac{(0.001)(1)}{(0.001)(1) + (0.999)(0.05)} = 0.0196$$

which is exactly the same answer as our original solution.

The problem is that you (or the typical medical student, or even the typical math professor) are much more likely to be able to remember the original solution than to remember Bayes' theorem. Psychologists, such as Gerd Gigerenzer, author of *Calculated Risks: How to Know When Numbers Deceive You*, have advocated that the method involved in the original solution (which Gigerenzer calls the method of "natural frequencies") be employed in place of Bayes' Theorem. Gigerenzer performed a study and found that those educated in the natural frequency method were able to recall it far longer than those who were taught Bayes' theorem. When one considers the possible life-and-death consequences associated with such calculations it seems wise to heed his advice.

Example 4.2.8

A certain disease has an incidence rate of 2%. If the false negative rate is 10% and the false positive rate is 1%, compute the probability that a person who tests positive actually has the disease.

Solution

Imagine 10,000 people who are tested. Of these 10,000, 200 will have the disease; 10% of them, or 20, will test negative and the remaining 180 will test positive. Of the 9800 who do not have the disease, 98 will test positive. So of the 278 total people who test positive, 180 will have the disease. Thus

$$P(\text{ disease } \mid \text{ positive }) = \frac{180}{278} \approx 0.647$$

so about 65% of the people who test positive will have the disease.

Using Bayes theorem directly would give the same result:

$$P(\text{ disease } \mid \text{ positive }) = \frac{(0.02)(0.90)}{(0.02)(0.90) + (0.98)(0.01)} = \frac{0.018}{0.0278} \approx 0.647$$

Try it Now 5

A certain disease has an incidence rate of 0.5%. If there are no false negatives and if the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.

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4.3: Counting

Counting? You already know how to count or you wouldn't be taking a college-level math class, right? Well yes, but what we'll really be investigating here are ways of counting efficiently. When we get to the probability situations a bit later in this chapter we will need to count some very large numbers, like the number of possible winning lottery tickets. One way to do this would be to write down every possible set of numbers that might show up on a lottery ticket, but believe me: you don't want to do this.

Basic Counting

We will start, however, with some more reasonable sorts of counting problems in order to develop the ideas that we will soon need.

Example 4.3.1

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad or breadsticks) and five choices for a main course (hamburger, sandwich, quiche, fajita or pizza). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution 1: One way to solve this problem would be to systematically list each possible meal:

soup + hamburger soup + sandwich soup + quiche

soup + fajita soup + pizza salad + hamburger

salad + sandwich salad + quiche salad + fajita

salad + pizza breadsticks + hamburger breadsticks + sandwich

breadsticks + quiche breadsticks + fajita breadsticks + pizza

Assuming that we did this systematically and that we neither missed any possibilities nor listed any possibility more than once, the answer would be 15. Thus you could go to the restaurant 15 nights in a row and have a different meal each night.

Solution 2: Another way to solve this problem would be to list all the possibilities in a table:

Table 4.3.1

	Hamburger	Sandwich	Quiche	Fajita	pizza
Soup	Soup+burger				
Salad	Salad+burger				
Bread	etc				

In each of the cells in the table we could list the corresponding meal: soup + hamburger in the upper left corner, salad + hamburger below it, etc. But if we didn't really care what the possible meals are, only how many possible meals there are, we could just count the number of cells and arrive at an answer of 15, which matches our answer from the first solution. (It's always good when you solve a problem two different ways and get the same answer!)

Solution 3: We already have two perfectly good solutions. Why do we need a third? The first method was not very systematic, and we might easily have made an omission. The second method was better, but suppose that in addition to the appetizer and the main course we further complicated the problem by adding desserts to the menu: we've used the rows of the table for the appetizers and the columns for the main courses—where will the desserts go? We would need a third dimension, and since drawing 3-D tables on a 2-D page or computer screen isn't terribly easy, we need a better way in case we have three categories to choose from instead of just two.

So, back to the problem in the example. What else can we do? Let's draw a **tree diagram**:

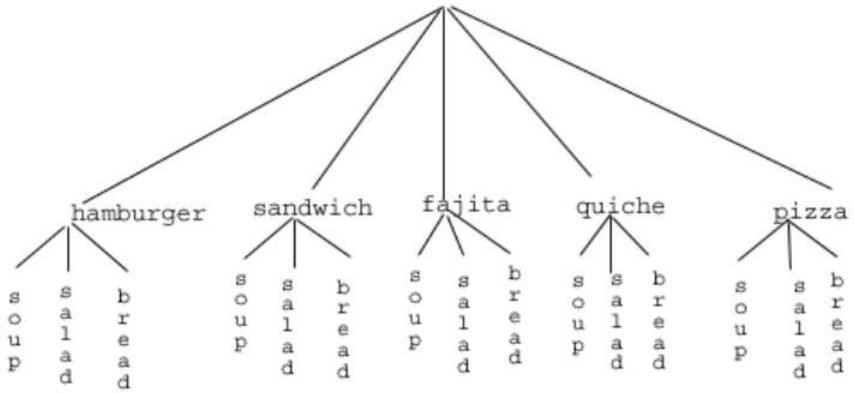


Figure 4.3.1: Tree diagram

This is called a "tree" diagram because at each stage we branch out, like the branches on a tree. In this case, we first drew five branches (one for each main course) and then for each of those branches we drew three more branches (one for each appetizer). We count the number of branches at the final level and get (surprise, surprise!) 15.

If we wanted, we could instead draw three branches at the first stage for the three appetizers and then five branches (one for each main course) branching out of each of those three branches.

OK, so now we know how to count possibilities using tables and tree diagrams. These methods will continue to be useful in certain cases, but imagine a game where you have two decks of cards (with 52 cards in each deck) and you select one card from each deck. Would you really want to draw a table or tree diagram to determine the number of outcomes of this game?

Let's go back to the previous example that involved selecting a meal from three appetizers and five main courses, and look at the second solution that used a table. Notice that one way to count the number of possible meals is simply to number each of the appropriate cells in the table, as we have done above. But another way to count the number of cells in the table would be multiply the number of rows (3) by the number of columns (5) to get 15. Notice that we could have arrived at the same result without making a table at all by simply multiplying the number of choices for the appetizer (3) by the number of choices for the main course (5). We generalize this technique as the basic counting rule:

Basic Counting Rule

If we are asked to choose one item from each of two separate categories where there are m items in the first category and n items in the second category, then the total number of available choices is $m \cdot n$.

This is sometimes called the multiplication rule.

Example 4.3.2

There are 21 novels and 18 volumes of poetry on a reading list for a college English course. How many different ways can a student select one novel and one volume of poetry to read during the quarter?

Solution

There are 21 choices from the first category and 18 for the second, so there are $21 \cdot 18 = 378$ possibilities.

The Basic Counting Rule can be extended when there are more than two categories by applying it repeatedly, as we see in the next example.

Example 4.3.3

Suppose at a particular restaurant you have three choices for an appetizer (soup, salad, or breadsticks), five choices for a main course (hamburger, sandwich, quiche, fajita, or pasta) and two choices for dessert (pie or ice cream). If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Solution

There are 3 choices for an appetizer, 5 for the main course and 2 for dessert, so there are

$3 \cdot 5 \cdot 2 = 30$ possibilities.

Example 4.3.4

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

Solution

There are 3 questions. Each question has 2 possible answers (true or false), so the quiz may be answered in $2 \cdot 2 \cdot 2 = 8$ different ways. Recall that another way to write $2 \cdot 2 \cdot 2$ is 2^3, which is much more compact.

Try it Now 6

Suppose at a particular restaurant you have eight choices for an appetizer, eleven choices for a main course and five choices for dessert. If you are allowed to choose exactly one item from each category for your meal, how many different meal options do you have?

Permutations

In this section we will develop an even faster way to solve some of the problems we have already learned to solve by other means. Let's start with a couple of examples.

Example 4.3.5

How many different ways can the letters of the word MATH be rearranged to form a four-letter code word?

Solution

This problem is a bit different. Instead of choosing one item from each of several different categories, we are repeatedly choosing items from the *same* category (the category is: the letters of the word MATH) and each time we choose an item we *do not replace* it, so there is one fewer choice at the next stage: we have 4 choices for the first letter (say we choose A), then 3 choices for the second (M, T and H; say we choose H), then 2 choices for the next letter (M and T; say we choose M) and only one choice at the last stage (T). Thus there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to spell a code word with the letters MATH.

In this example, we needed to calculate $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. This calculation shows up often in mathematics and is called the factorial, and is notated $n!$

Definition: Factorial

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$$

Example 4.3.6

How many ways can five different door prizes be distributed among five people?

Solution

There are 5 choices of prize for the first person, 4 choices for the second, and so on. The number of ways the prizes can be distributed will be $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways.

Now we will consider some slightly different examples.

Example 4.3.7

A charity benefit is attended by 25 people and three gift certificates are given away as door prizes: one gift certificate is in the amount of $100, the second is worth $25 and the third is worth $10. Assuming that no person receives more than one prize, how many different ways can the three gift certificates be awarded?

Solution

Using the Basic Counting Rule, there are 25 choices for the person who receives the $100 certificate, 24 remaining choices for the $25 certificate and 23 choices for the $10 certificate, so there are 25 · 24 · 23 = 13,800 ways in which the prizes can be awarded.

Example 4.3.8

Eight sprinters have made it to the Olympic finals in the 100-meter race. In how many different ways can the gold, silver and bronze medals be awarded?

Solution

Using the Basic Counting Rule, there are 8 choices for the gold medal winner, 7 remaining choices for the silver, and 6 for the bronze, so there are 8 · 7 · 6 = 336 ways the three medals can be awarded to the 8 runners.

Note that in these preceding examples, the gift certificates and the Olympic medals were awarded *without replacement*; that is, once we have chosen a winner of the first door prize or the gold medal, they are not eligible for the other prizes. Thus, at each succeeding stage of the solution there is one fewer choice (25, then 24, then 23 in the first example; 8, then 7, then 6 in the second). Contrast this with the situation of a multiple choice test, where there might be five possible answers — A, B, C, D or E — for each question on the test.

Note also that *the order of selection was important* in each example: for the three door prizes, being chosen first means that you receive substantially more money; in the Olympics example, coming in first means that you get the gold medal instead of the silver or bronze. In each case, if we had chosen the same three people in a different order, there might have been a different person who received the $100 prize or a different gold medalist. (Contrast this with the situation where we might draw three names out of a hat to each receive a $10 gift certificate; in this case the order of selection is not important since each of the three people receives the same prize. Situations where the order is not important will be discussed later.)

We can generalize the situation in the two examples above to any problem *without replacement* where the *order of selection is important*. If we are arranging in order r items out of n possibilities (instead of 3 out of 25 or 3 out of 8 as in the previous examples), the number of possible arrangements will be given by:

$$n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

If you don't see why $(n—r+1)$ is the right number to use for the last factor, just think back to the first example in this section, where we calculated 25 · 24 · 23 to get 13,800. In this case $n = 25$ and $r = 3$, so $n—r+1 = 25—3+1 = 23$, which is exactly the right number for the final factor.

Now, why would we want to use this complicated formula when it's actually easier to use the Basic Counting Rule, as we did in the first two examples? Well, we won't actually use this formula all that often. We only developed it so that we could attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* and where the *order of selection is important*. In this situation we write:

Permutations

$$_nP_r = n \cdot (n-1) \cdot (n-2) \cdots (n-r+1)$$

We say that there are $_nP_r$ permutations of the size r that may be selected from among n choices without replacement when order matters. It turns out that we can express this result more simply using factorials.

$$_nP_r = \frac{n!}{(n-r)!}$$

In practicality, we usually use technology rather than factorials or repeated multiplication to compute permutations.

Example 4.3.9

I have nine paintings and have room to display only four of them at a time on my wall in a certain order. How many different ways could I do this?

Solution

Since we are choosing 4 paintings out of 9 without replacement where the order of selection is important there are ${}_9P_4 = 9 \cdot 8 \cdot 7 \cdot 6 = 3,024$ permutations.

Example 4.3.10

How many ways can a four-person executive committee (president, vice-president, secretary, treasurer) be selected from a 16-member board of directors of a non-profit organization?

Solution

We want to choose 4 people out of 16 without replacement and where the order of selection is important. So the answer is ${}_{16}P_4 = 16 \cdot 15 \cdot 14 \cdot 13 = 43,680$

Try it Now 7

How many 5-character passwords can be made using the letters A through Z

a. if repeats are allowed
b. if no repeats are allowed

Combinations

Above we considered the situation where we chose r items out of n possibilities without replacement and where the order of selection was important. We now consider a similar situation in which the order of selection is not important.

Example 4.3.11

A charity benefit is attended by 25 people at which three $50 gift certificates are given away as door prizes. Assuming no person receives more than one prize, how many different ways can the gift certificates be awarded?

Solution

Using the Basic Counting Rule, there are 25 choices for the first person, 24 remaining choices for the second person and 23 for the third, so there are 25 · 24 · 23 = 13,800 ways to choose three people. Suppose for a moment that Abe is chosen first, Bea second, and Cindy third; this is one of the 13,800 possible outcomes. Another way to award the prizes would be to choose Abe first, Cindy second, and Bea third; this is another of the 13,800 possible outcomes. But either way Abe, Bea and Cindy each get $50, so the order in which we select them really doesn't matter. In how many different orders can Abe, Bea, and Cindy be selected? It turns out there are 6:

ABC ACB BAC BCA CAB CBA

How can we be sure that we have counted them all? We are really just choosing 3 people out of 3, so there are 3 · 2 · 1 = 6 ways to do this; we didn't really need to list them all, we can just use permutations!

So, out of the 13,800 ways to select 3 people out of 25, six of them involve Abe, Bea and Cindy. The same argument works for any other group of three people (say Abe, Bea, and David or Frank, Gloria, and Hildy) so each three-person group is counted *six times*. Thus the 13,800 figure is six times too big. The number of distinct three-person groups will be 13,800/6 = 2300.

We can generalize the situation in this example above to any problem of choosing a collection of items *without replacement* where the *order of selection is **not** important*. If we are choosing r items out of n possibilities (instead of 3 out of 25 as in the previous examples), the number of possible choices will be given by $\frac{{}_nP_r}{{}_rP_r}$, and we could use this formula for computation. However this situation arises so frequently that we attach a special notation and a special definition to this situation where we are choosing r items out of n possibilities *without replacement* where the *order of selection is **not** important.*

Definition: Combinations

$$ {}_nC_r = \frac{{}_nP_r}{{}_rP_r} $$

We say that there are ${}_nC_r$ combinations of size r that may be selected from among n choices without replacement where order doesn't matter.

We can also write the combinations formula in terms of factorials:

$$ {}_nC_r = \frac{n!}{(n-r)!r!} $$

Example 4.3.12

A group of four students is to be chosen from a 35-member class to represent the class on the student council. How many ways can this be done?

Solution

Since we are choosing 4 people out of 35 without replacement where the order of selection is not important, there are ${}_{35}C_4 = \frac{35 \cdot 54 \cdot 33 \cdot 32}{4 \cdot 3 \cdot 2 \cdot 1} = 52,360$ combinations.

Try it Now 8

The United States Senate Appropriations Committee consists of 29 members; the Defense Subcommittee of the Appropriations Committee consists of 19 members. Disregarding party affiliation or any special seats on the Subcommittee, how many different 19-member subcommittees may be chosen from among the 29 Senators on the Appropriations Committee?

In the preceding Try it Now problem we assumed that the 19 members of the Defense Subcommittee were chosen without regard to party affiliation. In reality this would never happen: if Republicans are in the majority they would never let a majority of Democrats sit on (and thus control) any subcommittee. (The same, of course, would be true if the Democrats were in control.) So let's consider the problem again, in a slightly more complicated form.

Example 4.3.13

The United States Senate Appropriations Committee consists of 29 members, 15 Republicans and 14 Democrats. The Defense Subcommittee consists of 19 members, 10 Republicans and 9 Democrats. How many different ways can the members of the Defense Subcommittee be chosen from among the 29 Senators on the Appropriations Committee?

Solution

In this case we need to choose 10 of the 15 Republicans and 9 of the 14 Democrats. There are ${}_{15}C_{10} = 3003$ ways to choose the 10 Republicans and ${}_{14}C_9 = 2002$ ways to choose the 9 Democrats. But now what? How do we finish the problem?

Suppose we listed all of the possible 10-member Republican groups on 3003 slips of red paper and all of the possible 9-member Democratic groups on 2002 slips of blue paper. How many ways can we choose one red slip and one blue slip? This is a job for the Basic Counting Rule! We are simply making one choice from the first category and one choice from the second category, just like in the restaurant menu problems from earlier.

There must be $3003 \cdot 2002 = 6,012,006$possible ways of selecting the members of the Defense Subcommittee.

Probability using Permutations and Combinations

We can use permutations and combinations to help us answer more complex probability questions

Example 4.3.14

A 4-digit PIN number is selected. What is the probability that there are no repeated digits?

Solution

There are 10 possible values for each digit of the PIN (namely: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9), so there are $10 \cdot 10 \cdot 10 \cdot 10 = 10^4 = 10000$ total possible PIN numbers.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute $10 \cdot 9 \cdot 8 \cdot 7$, or notice that this is the same as the permutation ${}_{10}P_4 = 5040$

The probability of no repeated digits is the number of 4 digit PIN numbers with no repeated digits divided by the total number of 4 digit PIN numbers. This probability is $\frac{{}_{10}P_4}{10^4} = \frac{5040}{10000} = 0.504$

Example 4.3.15

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. In this lottery, the order the numbers are drawn in doesn't matter. Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

Solution

In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player's ticket could match the six numbers drawn from the machine. Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is ${}_{48}C_6 = 12,271,512$. Of these possible outcomes, only one would match all six numbers on the player's ticket, so the probability of winning the grand prize is:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815 \text{ (Less than 1 in 10 million)}$$

Example 4.3.16

In the state lottery from the previous example, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of $1,000. Compute the probability that you win the second prize if you purchase a single lottery ticket.

Solution

As above, the number of possible outcomes of the lottery drawing is ${}_{48}C_5 = 12,271,512$. In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers; in other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by ${}_6C_5 = 6$ and the number of ways to choose 1 out of the 42 losing numbers is given by ${}_{42}C_1 = 42$. Thus the number of favorable outcomes is then given by the Basic Counting Rule: ${}_6C_5 \cdot {}_{42}C_1 = 6 \cdot 42 = 252$. So the probability of winning the second prize is:

$$\frac{({}_6C_5)({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205 \text{ (About 205 in 10 million)}$$

Try it Now 9

A multiple-choice question on an economics quiz contains 10 questions with five possible answers each. Compute the probability of randomly guessing the answers and getting 9 questions correct.

Example 4.3.17

Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

Solution

In many card games (such as poker) the order in which the cards are drawn is not important; in the problems that follow, we will assume that this is the case unless otherwise stated. Thus we use combinations to compute the possible number of 5-card hands, ${}_{52}C_5$. This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck. Since there are four Aces and we want exactly one of them, there will be ${}_4C_1$ ways to select one Ace; since there are 48 non-Aces and we want 4 of them, there will be ${}_{48}C_4$ ways to select the four non-Aces. Now we use the Basic Counting Rule to calculate that there will be ${}_4C_1 \cdot {}_{48}C_4$ ways to choose one ace and four non-Aces.

Putting this all together, we have

$$P(\text{one Ace}) = \frac{({}_4C_1)\,({}_{48}C_4)}{{}_{52}C_5} = \frac{778320}{2598960} \approx 0.299$$

Example 4.3.18

Compute the probability of randomly drawing five cards from a deck and getting exactly two Aces.

Solution

The solution is similar to the previous example, except now we are choosing 2 Aces out of 4 and 3 non-Aces out of 48; the denominator remains the same:

$$P(\text{two Aces}) = \frac{({}_4C_2)\,({}_{48}C_3)}{{}_{52}C_5} = \frac{103776}{2598960} \approx 0.0399$$

It is useful to note that these card problems are remarkably similar to the lottery problems discussed earlier.

Try it Now 10

Compute the probability of randomly drawing five cards from a deck of cards and getting three Aces and two Kings.

Birthday Problem

Let's take a pause to consider a famous problem in probability theory:

Suppose you have a room full of 30 people. What is the probability that there is at least one shared birthday?

Take a guess at the answer to the above problem. Was your guess fairly low, like around 10%? That seems to be the intuitive answer (30/365, perhaps?). Let's see if we should listen to our intuition. Let's start with a simpler problem, however.

Example 4.3.19

Suppose three people are in a room. What is the probability that there is at least one shared birthday among these three people?

Solution

There are a lot of ways there could be at least one shared birthday. Fortunately there is an easier way. We ask ourselves "What is the alternative to having at least one shared birthday?" In this case, the alternative is that there are **no** shared birthdays. The alternative to "at least one" is having **none.** In other words, since this is a complementary event,

$$P(\text{at least one}) = 1 - P(\text{none})$$

We will start, then, by computing the probability that there is no shared birthday. Let's imagine that you are one of these three people. Your birthday can be anything without conflict, so there are 365 choices out of 365 for your birthday. What is the probability that the second person does not share your birthday? There are 365 days in the year (let's ignore leap years) and removing your birthday from contention, there are 364 choices that will guarantee that you do not share a birthday with this

person, so the probability that the second person does not share your birthday is 364/365. Now we move to the third person. What is the probability that this third person does not have the same birthday as either you or the second person? There are 363 days that will not duplicate your birthday or the second person's, so the probability that the third person does not share a birthday with the first two is 363/365.

We want the second person not to share a birthday with you *and* the third person not to share a birthday with the first two people, so we use the multiplication rule:

$$P(\text{ no shared birthday }) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \approx 0.9918$$

and then subtract from 1 to get

$$P(\text{shared birthday }) = 1 - P(\text{no shared birthday}) = 1 - 0.9918 = 0.0082$$

This is a pretty small number, so maybe it makes sense that the answer to our original problem will be small. Let's make our group a bit bigger.

Example 4.3.20

Suppose five people are in a room. What is the probability that there is at least one shared birthday among these five people?

Solution

Continuing the pattern of the previous example, the answer should be

$$P(\text{shared birthday}) = 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \approx 0.0271$$

Note that we could rewrite this more compactly as

$$P(\text{ shared birthday }) = 1 - \frac{{}_{365}P_5}{365^5} \approx 0.0271$$

which makes it a bit easier to type into a calculator or computer, and this suggests a nice formula as we continue to expand the population of our group.

Example 4.3.21

Suppose 30 people are in a room. What is the probability that there is at least one shared birthday among these 30 people?

Solution

Here we can calculate

$$P(\text{ shared birthday }) = 1 - \frac{{}_{365}P_{30}}{365^{30}} \approx 0.706$$

which gives us the surprising result that when you are in a room with 30 people there is a 70% chance that there will be at least one shared birthday!

If you like to bet, and if you can convince 30 people to reveal their birthdays, you might be able to win some money by betting a friend that there will be at least two people with the same birthday in the room anytime you are in a room of 30 or more people. (Of course, you would need to make sure your friend hasn't studied probability!) You wouldn't be guaranteed to win, but you should win more than half the time.

This is one of many results in probability theory that is counterintuitive; that is, it goes against our gut instincts. If you still don't believe the math, you can carry out a simulation. Just so you won't have to go around rounding up groups of 30 people, someone has kindly developed a Java applet so that you can conduct a computer simulation. Go to this web page: https://mste.illinois.edu/activity/birthday/

If you keep track of the number of times that there is a repeated birthday, you should get a repeated birthday about 7 out of every 10 times you run the simulation.

Try it Now 11

Suppose 10 people are in a room. What is the probability that there is at least one shared birthday among these 10 people?

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4.4: Expected Value

Expected value is perhaps the most useful probability concept we will discuss. It has many applications, from insurance policies to making financial decisions, and it's one thing that the casinos and government agencies that run gambling operations and lotteries hope most people never learn about.

Example 4.4.1

In the casino game roulette, a wheel with 38 spaces (18 red, 18 black, and 2 green) is spun. In one possible bet, the player bets $1 on a single number. If that number is spun on the wheel, then they receive $36 (their original $1 + $35). Otherwise, they lose their $1. On average, how much money should a player expect to win or lose if they play this game repeatedly?

Solution

Suppose you bet $1 on each of the 38 spaces on the wheel, for a total of $38 bet. When the winning number is spun, you are paid $36 on that number. While you won on that one number, overall you've lost $2. On a per-space basis, you have "won" -$2/$38 ≈ -$0.053. In other words, on average you lose 5.3 cents per space you bet on.

We call this average gain or loss the expected value of playing roulette. Notice that no one ever loses exactly 5.3 cents: most people (in fact, about 37 out of every 38) lose $1 and a very few people (about 1 person out of every 38) gain $35 (the $36 they win minus the $1 they spent to play the game).

There is another way to compute expected value without imagining what would happen if we play every possible space. There are 38 possible outcomes when the wheel spins, so the probability of winning is $\frac{1}{38}$. The complement, the probability of losing, is $\frac{37}{38}$.

Summarizing these along with the values, we get this table:

Table 4.4.1

Outcome	Probability of Outcome
$35	$\frac{1}{38}$
-$1	$\frac{37}{38}$

Notice that if we multiply each outcome by its corresponding probability we get $\$35 \cdot \frac{1}{38} = 0.9211$ and $-\$1 \cdot \frac{37}{38} = -0.9737$, and if we add these numbers we get $0.9211 + (-0.9737) \approx -0.053$ which is the expected value we computed above.

Expected Value

Expected Value is the average gain or loss of an event if the procedure is repeated many times.

We can compute the expected value by multiplying each outcome by the probability of that outcome, then adding up the products

Try it Now 12

You purchase a raffle ticket to help out a charity. The raffle ticket costs $5. The charity is selling 2000 tickets. One of them will be drawn and the person holding the ticket will be given a prize worth $4000. Compute the expected value for this raffle.

Example 4.4.2

In a certain state's lottery, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins $1,000,000. If they match 5 numbers, then win $1,000. It costs $1 to buy a ticket. Find the expected value.

Solution

Earlier, we calculated the probability of matching all 6 numbers and the probability of matching 5 numbers:

$$\frac{{}_6C_6}{{}_{48}C_6} = \frac{1}{12271512} \approx 0.0000000815 \text{ for all 6 numbers,}$$
$$\frac{({}_6C_5)\,({}_{42}C_1)}{{}_{48}C_6} = \frac{252}{12271512} \approx 0.0000205 \text{ for 5 numbers.}$$

Our probabilities and outcome values are:

Table 4.4.2

Outcome	Probability of outcome
$999,999	$\frac{1}{12271512}$
$999	$\frac{252}{12271512}$
-$1	$1-\frac{253}{12271512}=\frac{12271259}{12271512}$

The expected value, then is:

$$(\$999,999)\cdot\frac{1}{12271512}+(\$999)\cdot\frac{252}{12271512}+(-\$1)\cdot\frac{12271259}{12271512}\approx -\$0.898$$

On average, one can expect to lose about 90 cents on a lottery ticket. Of course, most players will lose $1.

In general, if the expected value of a game is negative, it is not a good idea to play the game, since on average you will lose money. It would be better to play a game with a positive expected value (good luck trying to find one!), although keep in mind that even if the average winnings are positive, it could be the case that most people lose money and one very fortunate individual wins a great deal of money. If the expected value of a game is 0, we call it a fair game, since neither side has an advantage.

Not surprisingly, the expected value for casino games is negative for the player and is positive for the casino. It must be positive, or they would go out of business. Players just need to keep in mind that when they play a game repeatedly, they should expect to lose money because their expected value is negative. That is fine so long as you enjoy playing the game and think it is worth the cost. But it would be wrong to expect to come out ahead.

Try it Now 13

A friend offers to play a game, in which you roll 3 standard 6-sided dice. If all the dice roll different values, you give him $1. If any two dice match values, you get $2. What is the expected value of this game? Would you play?

Expected value also has applications outside of gambling. Expected value is very common in making insurance decisions.

Example 4.4.3

A 40-year-old man in the U.S. has a 0.242% risk of dying during the next year . An insurance company charges $275 for a life-insurance policy that pays a $100,000 death benefit. What is the expected value for the person buying the insurance?

Solution

The probabilities and outcomes are:

Table 4.4.3

Outcome	Probability of outcome
$100,000-$275=$99,725	0.00242
$275	1-0.00242=0.99758

The expected value is ($99,725)(0.00242) + (-$275)(0.99758) = -$33.

Not surprisingly, the expected value is negative; the insurance company can only afford to offer policies if they, on average, make money on each policy. They can afford to pay out the occasional benefit because they offer enough policies that those benefit payouts are balanced by the rest of the insured people.

For people buying the insurance, there is a negative expected value, but there is a security that comes from insurance that is worth that cost.

Odds and Risks as they relate to Probability

The **odds** of an event ("odds", always plural) occurring are the probability (e.g**. risk**) that this event will occur divided by the **probability** that the event will not occur. It can also be expressed as the probability that an event will occur divided by "1 minus the probability that the event will occur".

$$\text{Odds of event } = \frac{P}{1-P}$$

This probability measure is popular in the world of gambling. If we compute the number of people putting money on one horse winning and the number of people putting money on the horse not winning (i.e. putting money on other horses) we can compute the odds of winning. For example among 3100 persons gambling on horses, 100 persons put money on horse "A" to win and 3000 do not (they bet on other horses). The odds of winning are then 1/30 (100/3100 divided by 3000/3100, which can be simplified as 100/3000, or 1 / 30). In fact in gambling the odds of not winning are preferred and expressed as a ratio X/1. In our example, 30/1, or in words "thirty to one". This means that for every dollar that you bet, you will receive 30 if you win.

While odds are not the same as expected value, we often hear both of these terms as they relate to chance and probability. One way to remember the difference is this: odds are a ratio while expected value is a weighted sum as explained above. Odds are very similar to probability, but be aware of the difference. For instance, if you only have 10% chance of winning a game, the odds against you are 9 to 1 while the probability of your not winning would be 9/10 (which, of course, is 90%).

Exercises

1. A ball is drawn randomly from a jar that contains 6 red balls, 2 white balls, and 5 yellow balls. Find the probability of the given event.
   a. A red ball is drawn
   b. A white ball is drawn
2. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. What is the probability of drawing one slip of paper from the hat at random and getting:
   a. A consonant
   b. A vowel
3. A group of people were asked if they had run a red light in the last year. 150 responded "yes", and 185 responded "no". Find the probability that if a person is chosen at random, they have run a red light in the last year.
4. In a survey, 205 people indicated they prefer cats, 160 indicated they prefer dots, and 40 indicated they don’t enjoy either pet. Find the probability that if a person is chosen at random, they prefer cats.
5. Compute the probability of tossing a six-sided die (with sides numbered 1 through 6) and getting a 5.
6. Compute the probability of tossing a six-sided die and getting a 7.
7. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female.

Table 4.4.4

	A	B	C	Total

	A	B	C	Total
Male	8	18	13	39
Female	10	4	12	26
Total	18	22	25	65

8. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person had no credit cards.

Table 4.4.5

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

9. Compute the probability of tossing a six-sided die and getting an even number.
10. Compute the probability of tossing a six-sided die and getting a number less than 3.
11. If you pick one card at random from a standard deck of cards, what is the probability it will be a King?
12. If you pick one card at random from a standard deck of cards, what is the probability it will be a Diamond?
13. Compute the probability of rolling a 12-sided die and getting a number other than 8.
14. If you pick one card at random from a standard deck of cards, what is the probability it is not the Ace of Spades?
15. Referring to the grade table from question #7, what is the probability that a student chosen at random did NOT earn a C?
16. Referring to the credit card table from question #8, what is the probability that a person chosen at random has at least one credit card?
17. A six-sided die is rolled twice. What is the probability of showing a 6 on both rolls?
18. A fair coin is flipped twice. What is the probability of showing heads on both flips?
19. A die is rolled twice. What is the probability of showing a 5 on the first roll and an even number on the second roll?
20. Suppose that 21% of people own dogs. If you pick two people at random, what is the probability that they both own a dog?
21. Suppose a jar contains 17 red marbles and 32 blue marbles. If you reach in the jar and pull out 2 marbles at random, find the probability that both are red.
22. Suppose you write each letter of the alphabet on a different slip of paper and put the slips into a hat. If you pull out two slips at random, find the probability that both are vowels.
23. Bert and Ernie each have a well-shuffled standard deck of 52 cards. They each draw one card from their own deck. Compute the probability that:
    a. Bert and Ernie both draw an Ace.
    b. Bert draws an Ace but Ernie does not.
    c. neither Bert nor Ernie draws an Ace.
    d. Bert and Ernie both draw a heart.
    e. Bert gets a card that is not a Jack and Ernie draws a card that is not a heart.
24. Bert has a well-shuffled standard deck of 52 cards, from which he draws one card; Ernie has a 12-sided die, which he rolls at the same time Bert draws a card. Compute the probability that:
    a. Bert gets a Jack and Ernie rolls a five.
    b. Bert gets a heart and Ernie rolls a number less than six.
    c. Bert gets a face card (Jack, Queen or King) and Ernie rolls an even number.
    d. Bert gets a red card and Ernie rolls a fifteen.

e. Bert gets a card that is not a Jack and Ernie rolls a number that is not twelve.

25. Compute the probability of drawing a King from a deck of cards and then drawing a Queen.
26. Compute the probability of drawing two spades from a deck of cards.
27. A math class consists of 25 students, 14 female and 11 male. Two students are selected at random to participate in a probability experiment. Compute the probability that
    a. a male is selected, then a female.
    b. a female is selected, then a male.
    c. two males are selected.
    d. two females are selected.
    e. no males are selected.
28. A math class consists of 25 students, 14 female and 11 male. Three students are selected at random to participate in a probability experiment. Compute the probability that
    a. a male is selected, then two females.
    b. a female is selected, then two males.
    c. two females are selected, then one male.
    d. three males are selected.
    e. three females are selected.
29. Giving a test to a group of students, the grades and gender are summarized below. If one student was chosen at random, find the probability that the student was female and earned an A.

Table 4.4.6

	A	B	C	Total
Male	8	18	13	33
Female	10	4	12	48
Total	18	22	25	81

30. The table below shows the number of credit cards owned by a group of individuals. If one person was chosen at random, find the probability that the person was male and had two or more credit cards.

Table 4.4.7

	Zero	One	Two or more	Total
Male	9	5	19	33
Female	18	10	20	48
Total	27	15	39	81

31. A jar contains 6 red marbles numbered 1 to 6 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is red or odd-numbered.
32. A jar contains 4 red marbles numbered 1 to 4 and 10 blue marbles numbered 1 to 10. A marble is drawn at random from the jar. Find the probability the marble is blue or even-numbered.
33. Referring to the table from #29, find the probability that a student chosen at random is female or earned a B.
34. Referring to the table from #30, find the probability that a person chosen at random is male or has no credit cards.
35. Compute the probability of drawing the King of hearts or a Queen from a deck of cards.
36. Compute the probability of drawing a King or a heart from a deck of cards.
37. A jar contains 5 red marbles numbered 1 to 5 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
    a. Even-numbered given that the marble is red.
    b. Red given that the marble is even-numbered.

38. A jar contains 4 red marbles numbered 1 to 4 and 8 blue marbles numbered 1 to 8. A marble is drawn at random from the jar. Find the probability the marble is
    a. Odd-numbered given that the marble is blue.
    b. Blue given that the marble is odd-numbered.
39. Compute the probability of flipping a coin and getting heads, given that the previous flip was tails.
40. Find the probability of rolling a “1” on a fair die, given that the last 3 rolls were all ones.
41. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student speaks French, given that the student is female.
42. Suppose a math class contains 25 students, 14 females (three of whom speak French) and 11 males (two of whom speak French). Compute the probability that a randomly selected student is male, given that the student speaks French.
43. A certain virus infects one in every 400 people. A test used to detect the virus in a person is positive 90% of the time if the person has the virus and 10% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".
    a. Find the probability that a person has the virus given that they have tested positive, i.e. find P(A | B).
    b. Find the probability that a person does not have the virus given that they test negative, i.e. find P(not A | not B).
44. A certain virus infects one in every 2000 people. A test used to detect the virus in a person is positive 96% of the time if the person has the virus and 4% of the time if the person does not have the virus. Let A be the event "the person is infected" and B be the event "the person tests positive".
    a. Find the probability that a person has the virus given that they have tested positive, i.e. find P(A | B).
    b. Find the probability that a person does not have the virus given that they test negative, i.e. find P(not A | not B).
45. A certain disease has an incidence rate of 0.3%. If the false negative rate is 6% and the false positive rate is 4%, compute the probability that a person who tests positive actually has the disease.
46. A certain disease has an incidence rate of 0.1%. If the false negative rate is 8% and the false positive rate is 3%, compute the probability that a person who tests positive actually has the disease.
47. A certain group of symptom-free women between the ages of 40 and 50 are randomly selected to participate in mammography screening. The incidence rate of breast cancer among such women is 0.8%. The false negative rate for the mammogram is 10%. The false positive rate is 7%. If a the mammogram results for a particular woman are positive (indicating that she has breast cancer), what is the probability that she actually has breast cancer?
48. About 0.01% of men with no known risk behavior are infected with HIV. The false negative rate for the standard HIV test 0.01% and the false positive rate is also 0.01%. If a randomly selected man with no known risk behavior tests positive for HIV, what is the probability that he is actually infected with HIV?
49. A boy owns 2 pairs of pants, 3 shirts, 8 ties, and 2 jackets. How many different outfits can he wear to school if he must wear one of each item?
50. At a restaurant you can choose from 3 appetizers, 8 entrees, and 2 desserts. How many different three-course meals can you have?
51. How many three-letter "words" can be made from 4 letters "FGHI" if
    a. repetition of letters is allowed
    b. repetition of letters is not allowed
52. How many four-letter "words" can be made from 6 letters "AEBWDP" if
    a. repetition of letters is allowed
    b. repetition of letters is not allowed
53. All of the license plates in a particular state feature three letters followed by three digits (e.g. ABC 123). How many different license plate numbers are available to the state's Department of Motor Vehicles?
54. A computer password must be eight characters long. How many passwords are possible if only the 26 letters of the alphabet are allowed?
55. A pianist plans to play 4 pieces at a recital. In how many ways can she arrange these pieces in the program?
56. In how many ways can first, second, and third prizes be awarded in a contest with 210 contestants?
57. Seven Olympic sprinters are eligible to compete in the 4 x 100 m relay race for the USA Olympic team. How many four-person relay teams can be selected from among the seven athletes?

58. A computer user has downloaded 25 songs using an online file-sharing program and wants to create a CD-R with ten songs to use in his portable CD player. If the order that the songs are placed on the CD-R is important to him, how many different CD-Rs could he make from the 25 songs available to him?
59. In western music, an octave is divided into 12 pitches. For the film Close Encounters of the Third Kind, director Steven Spielberg asked composer John Williams to write a five-note theme, which aliens would use to communicate with people on Earth. Disregarding rhythm and octave changes, how many five-note themes are possible if no note is repeated?
60. In the early twentieth century, proponents of the Second Viennese School of musical composition (including Arnold Schönberg, Anton Webern and Alban Berg) devised the twelve-tone technique, which utilized a tone row consisting of all 12 pitches from the chromatic scale in any order, but with not pitches repeated in the row. Disregarding rhythm and octave changes, how many tone rows are possible?
61. In how many ways can 4 pizza toppings be chosen from 12 available toppings?
62. At a baby shower 17 guests are in attendance and 5 of them are randomly selected to receive a door prize. If all 5 prizes are identical, in how many ways can the prizes be awarded?
63. In the 6/50 lottery game, a player picks six numbers from 1 to 50. How many different choices does the player have if order doesn't matter?
64. In a lottery daily game, a player picks three numbers from 0 to 9. How many different choices does the player have if order doesn't matter?
65. A jury pool consists of 27 people. How many different ways can 11 people be chosen to serve on a jury and one additional person be chosen to serve as the jury foreman?
66. The United States Senate Committee on Commerce, Science, and Transportation consists of 23 members, 12 Republicans and 11 Democrats. The Surface Transportation and Merchant Marine Subcommittee consists of 8 Republicans and 7 Democrats. How many ways can members of the Subcommittee be chosen from the Committee?
67. You own 16 CDs. You want to randomly arrange 5 of them in a CD rack. What is the probability that the rack ends up in alphabetical order?
68. A jury pool consists of 27 people, 14 men and 13 women. Compute the probability that a randomly selected jury of 12 people is all male.
69. In a lottery game, a player picks six numbers from 1 to 48. If 5 of the 6 numbers match those drawn, they player wins second prize. What is the probability of winning this prize?
70. In a lottery game, a player picks six numbers from 1 to 48. If 4 of the 6 numbers match those drawn, they player wins third prize. What is the probability of winning this prize?
71. Compute the probability that a 5-card poker hand is dealt to you that contains all hearts.
72. Compute the probability that a 5-card poker hand is dealt to you that contains four Aces.
73. A bag contains 3 gold marbles, 6 silver marbles, and 28 black marbles. Someone offers to play this game: You randomly select on marble from the bag. If it is gold, you win $3. If it is silver, you win $2. If it is black, you lose $1. What is your expected value if you play this game?
74. A friend devises a game that is played by rolling a single six-sided die once. If you roll a 6, he pays you $3; if you roll a 5, he pays you nothing; if you roll a number less than 5, you pay him $1. Compute the expected value for this game. Should you play this game?
75. In a lottery game, a player picks six numbers from 1 to 23. If the player matches all six numbers, they win 30,000 dollars. Otherwise, they lose $1. Find the expected value of this game.
76. A game is played by picking two cards from a deck. If they are the same value, then you win $5, otherwise you lose $1. What is the expected value of this game?
77. A company estimates that 0.7% of their products will fail after the original warranty period but within 2 years of the purchase, with a replacement cost of $350. If they offer a 2 year extended warranty for $48, what is the company's expected value of each warranty sold?
78. An insurance company estimates the probability of an earthquake in the next year to be 0.0013. The average damage done by an earthquake it estimates to be $60,000. If the company offers earthquake insurance for $100, what is their expected value of the policy?

Some of these questions were adapted from puzzles at mindyourdecisions.com.

79. A small college has been accused of gender bias in its admissions to graduate programs.

a. Out of 500 men who applied, 255 were accepted. Out of 700 women who applied, 240 were accepted. Find the acceptance rate for each gender. Does this suggest bias?
b. The college then looked at each of the two departments with graduate programs, and found the data below. Compute the acceptance rate within each department by gender. Does this suggest bias?
c. Looking at our results from Parts *a* and *b*, what can you conclude? Is there gender bias in this college's admissions? If so, in which direction?

Table 4.4.8

Department	Men		Women	
	Applied	Admitted	Applied	Admitted
Dept A	400	240	100	90
Dept B	100	15	600	150

80. A bet on "black" in Roulette has a probability of 18/38 of winning. If you win, you double your money. You can bet anywhere from $1 to $100 on each spin.
    a. Suppose you have $10, and are going to play until you go broke or have $20. What is your best strategy for playing?
    b. Suppose you have $10, and are going to play until you go broke or have $30. What is your best strategy for playing?
81. Your friend proposes a game: You flip a coin. If it's heads, you win $1. If it's tails, you lose $1. However, you are worried the coin might not be fair coin. How could you change the game to make the game fair, without replacing the coin?
82. Fifty people are in a line. The first person in the line to have a birthday matching someone in front of them will win a prize. Of course, this means the first person in the line has no chance of winning. Which person has the highest likelihood of winning?
83. Three people put their names in a hat, then each draw a name, as part of a randomized gift exchange. What is the probability that no one draws their own name? What about with four people?
84. How many different "words" can be formed by using all the letters of each of the following words exactly once?
    a. "ALICE"
    b. "APPLE"
85. How many different "words" can be formed by using all the letters of each of the following words exactly once?
    a. "TRUMPS"
    b. "TEETER"
86. *The Monty Hall problem* is named for the host of the game show *Let's make a Deal*. In this game, there would be three doors, behind one of which there was a prize. The contestant was asked to choose one of the doors. Monty Hall would then open one of the other doors to show there was no prize there. The contestant was then asked if they wanted to stay with their original door, or switch to the other unopened door. Is it better to stay or switch, or does it matter?
87. Suppose you have two coins, where one is a fair coin, and the other coin comes up heads 70% of the time. What is the probability you have the fair coin given each of the following outcomes from a series of flips?
    a. 5 Heads and 0 Tails
    b. 8 Heads and 3 Tails
    c. 10 Heads and 10 Tails
    d. 3 Heads and 8 Tails
88. Suppose you have six coins, where five are fair coins, and one coin comes up heads 80% of the time. What is the probability you have a fair coin given each of the following outcomes from a series of flips?
    a. 5 Heads and 0 Tails
    b. 8 Heads and 3 Tails
    c. 10 Heads and 10 Tails
    d. 3 Heads and 8 Tails
89. In this problem, we will explore probabilities from a series of events.
    a. If you flip 20 coins, how many would you expect to come up "heads", on average? Would you expect every flip of 20 coins to come up with exactly that many heads?
    b. If you were to flip 20 coins, what would you consider a "usual" result? An "unusual" result?

c. Flip 20 coins (or one coin 20 times) and record how many come up “heads”. Repeat this experiment 9 more times. Collect the data from the entire class.
d. When flipping 20 coins, what is the theoretic probability of flipping 20 heads?
e. Based on the class’s experimental data, what appears to be the probability of flipping 10 heads out of 20 coins?
f. The formula ${}_nC_x p^x (1-p)^{n-x}$ will compute the probability of an event with probability p occurring x times out of n, such as flipping x heads out of n coins where the probability of heads is $p = ½$. Use this to compute the theoretic probability of flipping 10 heads out of 20 coins.
g. If you were to flip 20 coins, based on the class’ experimental data, what range of values would you consider a “usual” result? What is the combined probability of these results? What would you consider an “unusual” result? What is the combined probability of these results?
h. We’ll now consider a simplification of a case from the 1960s. In the area, about 26% of the jury eligible population was black. In the court case, there were 100 men on the juror panel, of which 8 were black. Does this provide evidence of racial bias in jury selection?

90. When three coins are tossed, the probability of getting three tails is 1/8. Suppose you get $6 if you get three tails and lose $2 otherwise. Calculate the expected value (this represents your average gain per game).
91. A die is tossed in a game. Suppose you win $10 if you get a six and lose $2 otherwise. Calculate the expected value of this game.
92. You agree to play this game in which the dealer tosses a die. You lose $2 if you get a one, two, or three. You lose $5 if you get a four or a five. But you win $20 if you get a six. Would you play the game? Answer this by computing the expected value per game.
93. Repeat the previous problem, except now that you have to pay $1 to play the game. Would you keep playing?
94. A 50-50 raffle is a way to raise money (often used by non-profit organizations). Tickets are sold at an event for a certain price, and the total proceeds are divided evenly (50-50) between one lucky winner and the host organization. Suppose a ticket costs $1, and 700 tickets were sold.
    a. Find the expected value per ticket purchased. In other words, for each ticket you purchase, how much do you expected to lose/donate?
    b. Suppose you are the one who bought all 700 tickets sold. How much do you win? Is that amount consistent with your answer for Part (a)?

Reference

1. References (9)

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CHAPTER OVERVIEW

5: Data and Statistics

5.1: Basics of Statistics
5.2: Describing Data
5.3: Numerical Measures of Center and Variation
5.4: Normal Distribution

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5.1: Basics of Statistics

Like most people, you probably feel that it is important to "take control of your life." But what does this mean? Partly it means being able to properly evaluate the data and claims that bombard you every day. If you cannot distinguish good from faulty reasoning, then you are vulnerable to manipulation and to decisions that are not in your best interest. Statistics provides tools that you need in order to react intelligently to information you hear or read. In this sense, statistics is one of the most important things that you can study.

Here are some claims that you may have heard on several occasions. (We are not saying that each one of these claims is true!)

- 4 out of 5 dentists recommend Dentyne.
- Almost 85% of lung cancers in men and 45% in women are tobacco-related.
- Condoms are effective 94% of the time.
- Native Americans are significantly more likely to be hit crossing the streets than are people of other ethnicities.
- People tend to be more persuasive when they look others directly in the eye and speak loudly and quickly.
- Women make 75 cents to every dollar a man makes when they work the same job.
- A surprising new study shows that eating egg whites can increase one's life span.
- People predict that it is very unlikely there will ever be another baseball player with a batting average over 400.
- There is an 70% chance that in a room full of 30 people that at least two people will share the same birthday.
- 79.48% of all statistics are made up on the spot.

Before going further, however, two distinct meanings of the word “statistics” should be clarified. First, “statistics” could mean the academic discipline, a field of study, involving the collection, presentation, interpretation, and use of data. When used in this sense, the word is considered singular (though it looks plural). Second, in a narrower sense, a number collected in a survey or study is referred to as a “statistic.” For instance, the average height of a sample of 100 people would be a (one) statistic. This is a singular word (of course, two or more of these numbers would be referred to as “statistics”). Be aware of the difference when you encounter this word.

All of these claims are statistical in character. We suspect that some of them sound familiar; if not, we bet that you have heard other claims like them. Notice how diverse the examples are; they come from psychology, health, law, sports, business, etc. Indeed, data and data-interpretation show up in discourse from virtually every facet of contemporary life.

Statistics are often presented in an effort to add credibility to an argument or advice. You can see this by paying attention to television advertisements. Many of the numbers thrown about in this way do not represent careful statistical analysis. They can be misleading and push you into decisions that you might find cause to regret. For these reasons, learning about statistics is a long step towards taking control of your life. (It is not, of course, the only step needed for this purpose.) This chapter will help you learn statistical essentials. It will make you into an intelligent consumer of statistical claims.

You can take the first step right away. To be an intelligent consumer of statistics, your first reflex must be to question the statistics that you encounter. The British Prime Minister Benjamin Disraeli famously said, "There are three kinds of lies -- lies, damned lies, and statistics." This quote reminds us why it is so important to understand statistics. So let us invite you to reform your statistical habits from now on. No longer will you blindly accept numbers or findings. Instead, you will begin to think about the numbers, their sources, and most importantly, the procedures used to generate them.

We have put the emphasis on defending ourselves against fraudulent claims wrapped up as statistics. Just as important as detecting the deceptive use of statistics is the appreciation of the proper use of statistics. You must also learn to recognize statistical evidence that supports a stated conclusion. When a research team is testing a new treatment for a disease, statistics allows them to conclude based on a relatively small trial that there is good evidence their drug is effective. Statistics allowed prosecutors in the 1950’s and 60’s to demonstrate racial bias existed in jury panels. Statistics are all around you, sometimes used well, sometimes not. We must learn how to distinguish the two cases.

Populations and samples

Before we begin gathering and analyzing data, we need to characterize the population we are studying. If we want to study the amount of money spent on textbooks by a typical first-year college student, our population might be all first-year students at your college. Or it might be:

- All first-year community college students in the state of California.
- All first-year students at public colleges and universities in the state of California.
- All first-year students at all colleges and universities in the state of California.
- All first-year students at all colleges and universities in the entire United States.
- And so on.

Definition: Population

The population of a study is the group the collected data is intended to describe.

Sometimes the intended population is called the target population since the collected data might not actually be representative of the intended population if we design our study badly.

Why is it important to specify the population? We might get different answers to our question as we vary the population we are studying. First-year students at a large university might take slightly more diverse courses than those at your college, and some of these courses may require less popular textbooks that cost more; or, on the other hand, the University Bookstore might have a larger pool of used textbooks, reducing the cost of these books to the students. Whichever the case (and it is likely that some combination of these and other factors are in play), the data we gather from your college will probably not be the same as that from a large university. Particularly when conveying our results to others, we want to be clear about the population we are describing with our data.

Example 5.1.1

A newspaper website contains a poll asking people their opinion on a recent news article. What is the population?

Solution

While the target (intended) population may have been all people, the real population of the survey is readers of the website.

If we were able to gather data on every member of our population, say the average (we will define "average" more carefully in a subsequent section) amount of money spent on textbooks by each first-year student at your college during the 2009-2010 academic year, the resulting number would be called a **parameter**.

Definition: Parameter

A parameter is a value (average, percentage, etc.) calculated using all the data from a population

We seldom see parameters, however, since surveying an entire population is usually very time-consuming and expensive unless the population is very small or we already have the data collected.

Definition: Census

A survey of an entire population is called a census.

You are probably familiar with two common censuses: the official government Census that attempts to count the population of the U.S. every ten years, and voting, which asks the opinion of all eligible voters in a district. The first of these demonstrates one additional problem with a census: the difficulty in finding and getting participation from everyone in a large population, which can bias, or skew, the results.

There are occasionally times when a census is appropriate, usually when the population is fairly small. For example, if the manager of Starbucks wanted to know the average number of hours her employees worked last week, she should be able to pull up payroll records or ask each employee directly.

Since surveying an entire population is often impractical, we usually select a sample to study;

Definition: Sample

A sample is a smaller subset of the entire population, ideally one that is fairly representative of the whole population.

We will discuss sampling methods in greater detail in a later section. For now, let us assume that samples are chosen in an appropriate manner. If we survey a sample, say 100 first-year students at your college, and find the average amount of money spent by these students on textbooks, the resulting number is called a statistic.

Definition: Statistic

A statistic is a value (average, percentage, etc.) calculated using the data from a sample.

Example 5.1.2

A researcher wanted to know how citizens of Los Angeles felt about a voter initiative. To study this, she goes to a large shopping mall and randomly selects 500 shoppers and asks them their opinion. 60% indicate they are supportive of the initiative. What is the sample and population? Is the 60% value a parameter or a statistic?

Solution

The sample is the 500 shoppers questioned. The population is less clear. While the intended population of this survey was LA citizens, the effective population was mall shoppers. There is no reason to assume that the 500 shoppers questioned would be representative of all LA citizens.

The 60% value was based on the sample, so it is a statistic. (Note that this is singular).

Try it Now 1

The U.S. federal debt at the end of 2001 was $5.77 trillion and grew to $6.20 trillion by the end of 2002. At the end of 2005 it was $7.91 trillion and grew to $8.45 trillion by the end of 2006.[2] Calculate the absolute and relative increase for 2001-2002 and 2005-2006. Which year saw a larger increase in federal debt?

Try it Now 2

A college reports that the average age of their students is 28 years old. Is this a statistic or a parameter?

Categorizing data

Once we have gathered data, we might wish to classify it. Roughly speaking, data can be classified as categorical data or quantitative data.

Definition: Quantitative and Categorical Data

Categorical (qualitative) data are pieces of information that allow us to classify the objects under investigation into various categories.

Quantitative data are responses that are numerical in nature and with which we can perform meaningful arithmetic calculations.

Example 5.1.3

We might conduct a survey to determine the name of the favorite movie that each person in a math class saw in a movie theater.

Solution

When we conduct such a survey, the responses would look like: *Finding Nemo*, *The Avengers*, *or Dunkirk*. We might count the number of people who give each answer, but the answers themselves do not have any numerical values: we cannot perform computations with an answer like "*Finding Nemo*." This would be categorical data.

Example 5.1.4

A survey could ask the number of movies they have seen in a movie theater in the past 12 months (0, 1, 2, 3, 4, ...)

Solution

This would be quantitative data.

Other examples of quantitative data would be the running time of the movie you saw most recently (104 minutes, 137 minutes, 104 minutes, ...) or the amount of money you paid for a movie ticket the last time you went to a movie theater ($5.50, $7.75, $9, ...).

Sometimes, determining whether or not data is categorical or quantitative can be a bit trickier.

Example 5.1.5

Suppose we gather respondents' ZIP codes in a survey to track their geographical location.

ZIP codes are numbers, but we can't do any meaningful mathematical calculations with them (it doesn't make sense to say that 98036 is "twice" 49018 — that's like saying that Lynnwood, WA is "twice" Battle Creek, MI, which doesn't make sense at all), so ZIP codes are really categorical data.

Example 5.1.6

A survey about the movie you most recently attended includes the question "How would you rate the movie you just saw?" with these possible answers:

1. it was awful
2. it was just OK
3. I liked it
4. it was great
5. best movie ever!

Solution

Again, there are numbers associated with the responses, but we can't really do any calculations with them: a movie that rates a 4 is not necessarily twice as good as a movie that rates a 2, whatever that means; if two people see the movie and one of them thinks it stinks and the other thinks it's the best ever it doesn't necessarily make sense to say that "on average they liked it."

However, it DOES make sense to discuss the average rating and other statistics in this case. For instance, a movie whose average rating is a 1.8 is clearly different from one whose average is a 4.2. Also, it would mean something if the maximum rating of another movie was a 2. So, in a sense, this data can be considered quantitative.

As we study movie-going habits and preferences, we shouldn't forget to specify the population under consideration. If we survey 3-7 year-olds the runaway favorite might be *Finding Nemo*. 13-17 year-olds might prefer *The Avengers*. And 33-37 year-olds might prefer...well, *Finding Nemo*.

Try it Now 3

Classify each measurement as categorical or quantitative

a. Eye color of a group of people
b. Daily high temperature of a city over several weeks
c. Annual income

Sampling methods

As we mentioned in a previous section, the first thing we should do before conducting a survey is to identify the population that we want to study. Suppose we are hired by a politician to determine the amount of support he has among the electorate should he decide to run for another term. What population should we study? Every person in the district? Not every person is eligible to vote,

and regardless of how strongly someone likes or dislikes the candidate, they don't have much to do with him being re-elected if they are not able to vote.

What about eligible voters in the district? That might be better, but if someone is eligible to vote but does not register by the deadline, they won't have any say in the election either. What about registered voters? Many people are registered but choose not to vote. What about "likely voters?"

This is the criteria used in much political polling, but it is sometimes difficult to define a "likely voter." Is it someone who voted in the last election? In the last general election? In the last presidential election? Should we consider someone who just turned 18 a "likely voter?" They weren't eligible to vote in the past, so how do we judge the likelihood that they will vote in the next election?

In November 1998, former professional wrestler Jesse "The Body" Ventura was elected governor of Minnesota. Up until right before the election, most polls showed he had little chance of winning. There were several contributing factors to the polls not reflecting the actual intent of the electorate:

- Ventura was running on a third-party ticket, and most polling methods are better suited to a two-candidate race.
- Many respondents to polls may have been embarrassed to tell pollsters that they were planning to vote for a professional wrestler.
- The mere fact that the polls showed Ventura had little chance of winning might have prompted some people to vote for him in protest to send a message to the major-party candidates.

But one of the major contributing factors was that Ventura recruited a substantial amount of support from young people, particularly college students, who had never voted before and who registered specifically to vote in the gubernatorial election. The polls did not deem these young people likely voters (since in most cases young people have a lower rate of voter registration and a turnout rate for elections) and so the polling samples were subject to **sampling bias**: they omitted a portion of the electorate that was weighted in favor of the winning candidate.

Definition: Sampling Bias (selection bias)

A sampling bias is the bias caused when the sample chosen does not fairly represent the population being studied. The sampling method, therefore, must be carefully considered to avoid this type of bias.

So even identifying the population can be a difficult job, but once we have identified the population, how do we choose an appropriate sample? Remember, although we would prefer to survey all members of the population, this is usually impractical unless the population is very small, so we choose a sample. There are many ways to sample a population, but there is one goal we need to keep in mind: we would like the sample to be *representative of the population*.

Returning to our hypothetical job as a political pollster, we would not anticipate very accurate results if we drew all of our samples from among the customers at a Starbucks, nor would we expect that a sample drawn entirely from the membership list of the local Elks club would provide a useful picture of district-wide support for our candidate.

One way to ensure that the sample has a reasonable chance of mirroring the population is to employ *randomness*. The most basic random method is simple random sampling.

Definition: Simple Random Sample

A random sample is one in which each member of the population has an equal probability of being chosen. A simple random sample is one in which every member of the population and any group of members has an equal probability of being chosen.

Example 5.1.7

If we could somehow identify all likely voters in the state, put each of their names on a piece of paper, toss the slips into a (very large) hat and draw 1000 slips out of the hat, we would have a simple random sample.

Solution

In practice, computers are better suited for this sort of endeavor than millions of slips of paper and extremely large headgear.

It is always possible, however, that even a random sample might end up not being totally representative of the population. If we repeatedly take samples of 1000 people from among the population of likely voters in the state, some of these samples

might tend to have a slightly higher percentage of Democrats (or Republicans) than does the general population; some samples might include more older people and some samples might include more younger people; etc. In most cases, this **sampling variability** is not significant.

Definition: Sampling Variability

The natural variation of samples is called sampling variability.

This is unavoidable and expected in random sampling, and in most cases it is not an issue

To help account for variability, pollsters might instead use a **stratified sample**.

Definition: Stratified Sampling

In stratified sampling, a population is divided into a number of subgroups (or strata). Random samples are then taken from each subgroup with sample sizes proportional to the size of the subgroup in the population.

Example 5.1.8

Suppose in a particular state that previous data indicated that the electorate was comprised of 39% Democrats, 37% Republicans, and 24% independents. In a sample of 1000 people, they would then expect to get about 390 Democrats, 370 Republicans, and 240 independents. To accomplish this, they could randomly select 390 people from among those voters known to be Democrats, 370 from those known to be Republicans, and 240 from those with no party affiliation.

Stratified sampling can also be used to select a sample with people in desired age groups, a specified mix ratio of males and females, etc. A variation on this technique is called quota sampling.

Definition: Quota Sampling

Quota sampling is a variation on stratified sampling, wherein samples are collected in each subgroup until the desired quota is met.

Example 5.1.9

Suppose the pollsters call people at random, but once they have met their quota of 390 Democrats, they only gather people who do not identify themselves as a Democrat.

You may have had the experience of being called by a telephone pollster who started by asking you your age, income, etc. and then thanked you for your time and hung up before asking any "real" questions. Most likely, they already had contacted enough people in your demographic group and were looking for people who were older or younger, richer or poorer, etc. Quota sampling is usually a bit easier than stratified sampling but also does not ensure the same level of randomness.

Another sampling method is cluster sampling, in which the population is divided into groups, and one or more groups are randomly selected to be in the sample.

Definition: Cluster Sampling

In cluster sampling, the population is divided into subgroups (clusters), and a set of subgroups is selected to be in the sample.

Example 5.1.10

If the college wanted to survey students, since students are already divided into classes, they could randomly select 10 classes and give the survey to all the students in those classes. This would be cluster sampling.

Other sampling methods include **systematic sampling**.

Definition: Systemic Sampling

In systematic sampling, every n^{th} member of the population is selected to be in the sample.

Example 5.1.11

To select a sample using systematic sampling, a pollster calls every 100th name in the phone book.

Systematic sampling is not as random as a simple random sample (if your name is Albert Aardvark and your sister Alexis Aardvark is right after you in the phone book, there is no way you could both end up in the sample), but it can yield acceptable samples.

Perhaps the worst types of sampling methods are convenience samples and voluntary response samples.

Definition: Convenience Sampling and Voluntary Response Sampling

Convenience sampling is the method of choosing a sample based on what is convenient or easy.

Voluntary response sampling is allowing the sample to volunteer.

Example 5.1.12

A pollster stands on a street corner and interviews the first 100 people who agree to speak to him. This is a convenience sample.

Example 5.1.13

A website has a survey asking readers to give their opinion on a tax proposal. This is a self-selected sample, or voluntary response sample, in which respondents volunteer to participate.

Usually voluntary response samples are skewed towards people who have a particularly strong opinion about the subject of the survey or who just have way too much time on their hands and enjoy taking surveys.

Try it Now 4

In each case, indicate what sampling method was used:

a. Every 4th person in the class was selected
b. A sample was selected to contain 25 men and 35 women
c. Viewers of a new show are asked to vote on the show's website
d. A website randomly selects 50 of their customers to send a satisfaction survey to
e. To survey voters in a town, a polling company randomly selects 10 city blocks and interviews everyone who lives on those blocks.

How to mess things up before you start

There are number of ways that a study can be ruined before you even start collecting data. The first we have already explored – sampling or selection bias, which is when the sample is not representative of the population. One example of this is voluntary response bias, which is bias introduced by collecting data from only those who volunteer to participate. This is not the only potential source of bias.

Note

Sampling bias – when the sample is not representative of the population

Voluntary response bias – the sampling bias that often occurs when the sample is volunteers. (This is also called "participation bias.")

Self-interest study – bias that can occur when the researchers have an interest in the outcome

Response bias – when the responder gives inaccurate responses for any reason

Perceived lack of anonymity – when the responder fears giving an honest answer might negatively affect them

Loaded questions – when the question wording influences the responses

Non-response bias – when people refusing to participate in the study can influence the validity of the outcome

Example 5.1.14

Consider a recent study which found that chewing gum may raise math grades in teenagers. This study was conducted by the Wrigley Science Institute, a branch of Wrigley, a chewing gum company. This is an example of a **self-interest study**; one in which the researches have a vested interest in the outcome of the study. While this does not necessarily ensure that the study was biased, it certainly suggests that we should subject the study to extra scrutiny.

Example 5.1.15

A survey asks people "when was the last time you visited your doctor?" This might suffer from response bias, since many people might not remember exactly when they last saw a doctor and give inaccurate responses.

Sources of response bias may be innocent, such as bad memory, or as intentional as pressuring by the pollster. Consider, for example, how many voting initiative petitions people sign without even reading them.

Example 5.1.16

A survey asks participants a question about their interactions with members of other races. Here, a perceived lack of anonymity could influence the outcome. The respondent might not want to be perceived as racist even if they are and give an untruthful answer.

Example 5.1.17

An employer puts out a survey asking their employees if they have a drug abuse problem and need treatment help. Here, answering truthfully might have consequences; responses might not be accurate if the employees do not feel their responses are anonymous or fear retribution from their employer.

Example 5.1.18

A survey asks "Do you support funding research of alternative energy sources to reduce our reliance on high-polluting fossil fuels?" This is an example of a loaded or leading question – questions whose wording leads the respondent towards an answer.

Loaded questions can occur intentionally by pollsters with an agenda or accidentally through poor question wording. Also a concern is question order, where the order of questions changes the results. A psychology researcher provides an example:

"My favorite finding is this: we did a study where we asked students, 'How satisfied are you with your life? How often do you have a date?' The two answers were not statistically related - you would conclude that there is no relationship between dating frequency and life satisfaction. But when we reversed the order and asked, 'How often do you have a date? How satisfied are you with your life?' the statistical relationship was a strong one. You would now conclude that there is nothing as important in a student's life as dating frequency."

Example 5.1.19

A telephone poll asks the question "Do you often have time to relax and read a book?", and 50% of the people called refused to answer the survey. It is unlikely that the results will be representative of the entire population. This is an example of non-response bias, introduced by people refusing to participate in a study or dropping out of an experiment. When people refuse to participate, we can no longer be so certain that our sample is representative of the population.

Try it Now 5

In each situation, identify a potential source of bias:

1. A survey asks how many sexual partners a person has had in the last year.
2. A radio station asks readers to phone in their choice in a daily poll.
3. A substitute teacher wants to know how students in the class did on their last test. The teacher asks the 10 students sitting in the front row to state their latest test score.
4. High school students are asked if they have consumed alcohol in the last two weeks.
5. The Beef Council releases a study stating that consuming red meat poses little cardiovascular risk.
6. A poll asks "Do you support a new transportation tax, or would you prefer to see our public transportation system fall apart?"

Experiments

So far, we have primarily discussed observational studies – studies in which conclusions would be drawn from observations of a sample or the population. In some cases these observations might be unsolicited, such as studying the percentage of cars that turn right at a red light even when there is a "no turn on red" sign. In other cases the observations are solicited, like in a survey or a poll.

In contrast, it is common to use experiments when exploring how subjects react to an outside influence. In an experiment, some kind of treatment is applied to the subjects, and the results are measured and recorded.

Definition: Observational Studies and Experiments

An observational study is a study based on observations or measurements.

An experiment is a study in which the effects of a treatment are measured.

Here are some examples of experiments:

Example 5.1.20

a. A pharmaceutical company tests a new medicine for treating Alzheimer's disease by administering the drug to 50 elderly patients with recent diagnoses. The treatment here is the new drug.
b. A gym tests out a new weight loss program by enlisting 30 volunteers to try out the program. The treatment here is the new program.
c. You test a new kitchen cleaner by buying a bottle and cleaning your kitchen. The new cleaner is the treatment.
d. A psychology researcher explores the effect of music on temperament by measuring people's temperament while listening to different types of music. The music is the treatment.

Try it Now 6

Is each scenario describing an observational study or an experiment?

a. The weights of 30 randomly selected people are measured
b. Subjects are asked to do 20 jumping jacks, and then their heart rates are measured
c. Twenty coffee drinkers and twenty tea drinkers are given a concentration test

When conducting experiments, it is essential to isolate the treatment being tested.

Example 5.1.21

Suppose a middle school (junior high) finds that their students are not scoring well on the state's standardized math test. They decide to run an experiment to see if an alternate curriculum would improve scores. To run the test, they hire a math specialist to come in and teach a class using the new curriculum. To their delight, they see an improvement in test scores.

The difficulty with this scenario is that it is not clear whether the curriculum is responsible for the improvement or the improvement is due to a math specialist teaching the class. This is called **confounding** – when it is not clear which factor or factors caused the observed effect. Confounding is the downfall of many experiments though sometimes it is hidden.

Definition: Confounding

Confounding occurs when there are two potential variables that could have caused the outcome and it is not possible to determine which actually caused the result.

Example 5.1.22

A drug company study about a weight loss pill might report that people lost an average of 8 pounds while using their new drug. However, in the fine print you find a statement saying that participants were encouraged to also diet and exercise. It is not clear in this case whether the weight loss is due to the pill, to diet and exercise, or a combination of both. In this case confounding has occurred.

Example 5.1.23

Researchers conduct an experiment to determine whether students will perform better on an arithmetic test if they listen to music during the test. They first give the student a test without music and then give a similar test while the student listens to music. In this case, the student might perform better on the second test, regardless of the music, simply because it was the second test and they were warmed up.

There are a number of measures that can be introduced to help reduce the likelihood of confounding. The primary measure is to use a control group.

Definition: Control Group

When using a control group, the participants are divided into two or more groups, typically a control group and a treatment group. The treatment group receives the treatment being tested; the control group does not receive the treatment.

Ideally, the groups are otherwise as similar as possible, isolating the treatment as the only potential source of difference between the groups. For this reason, the method of dividing groups is important. Some researchers attempt to ensure that the groups have similar characteristics (same number of females, same number of people over 50, etc.), but it is nearly impossible to control for every characteristic. Because of this, random assignment is very commonly used.

Example 5.1.24

To determine if a two-day prep course would help high school students improve their scores on the SAT test, a group of students was randomly divided into two subgroups. The first group, the treatment group, was given a two-day prep course. The second group, the control group, was not given the prep course. Afterwards, both groups were given the SAT.

Example 5.1.25

A company testing a new plant food grows two crops of plants in adjacent fields, the treatment group receiving the new plant food and the control group not. The crop yield would then be compared. By growing them at the same time in adjacent fields, they are controlling for weather and other confounding factors.

Sometimes not giving the control group anything does not completely control for confounding variables. For example, suppose a medicine study is testing a new headache pill by giving the treatment group the pill and the control group nothing. If the treatment group showed improvement, we would not know whether it was due to the medicine in the pill or a response to have taken any pill. This is called a **placebo effect.**

Definition: Placebo Effect

The placebo effect refers to the effectiveness of a treatment influenced by the patient's perception of how effective they think the treatment will be, so a result might be seen even if the treatment is ineffectual.

Example 5.1.26

A study found that when doing painful dental tooth extractions, patients who were told they were receiving a strong painkiller while actually receiving a saltwater injection found as much pain relief as patients receiving a dose of morphine.

To control for the placebo effect, a placebo, or dummy treatment, is often given to the control group. This way, both groups are truly identical except for the specific treatment given.

Definition: Placebo and Placebo controlled Experiments

A **placebo** is a dummy treatment given to control for the placebo effect.

An experiment that gives the control group a placebo is called a **placebo controlled experiment**.

Example 5.1.27

a. In a study for a new medicine that is dispensed in a pill form, a sugar pill could be used as a placebo.
b. In a study on the effect of alcohol on memory, a non-alcoholic beer might be given to the control group as a placebo.
c. In a study of a frozen meal diet plan, the treatment group would receive the diet food, and the control could be given standard frozen meals stripped of their original packaging.

In some cases, it is more appropriate to compare to a conventional treatment than a placebo. For example, in a cancer research study, it would not be ethical to deny any treatment to the control group or to give a placebo treatment. In this case, the currently acceptable medicine would be given to the control group, sometimes called a comparison group in this case. In our SAT test example, the non-treatment group would most likely be encouraged to study on their own, rather than be asked to not study at all, to provide a meaningful comparison.

When using a placebo, it would defeat the purpose if the participant knew they were receiving the placebo.

Definition: Blind studies

A **blind study** is one in which the participant does not know whether or not they are receiving the treatment or a placebo.

A **double-blind study** is one in which those interacting with the participants don't know who is in the treatment group and who is in the group receiving a placebo.

Example 5.1.28

In a study about anti-depression medicine, you would not want the psychological evaluator to know whether the patient is in the treatment or control group either, as it might influence their evaluation, so the experiment should be conducted as a double-blind study.

It should be noted that not every experiment needs a control group.

Example 5.1.29

If a researcher is testing whether a new fabric can withstand fire, she simply needs to torch multiple samples of the fabric – there is no need for a control group.

Try it Now 7

To test a new lie detector, two groups of subjects are given the new test. One group is asked to answer all the questions truthfully, and the second group is asked to lie on one set of questions. The person administering the lie detector test does not know what group each subject is in.

Does this experiment have a control group? Is it blind, double-blind, or neither?

Reference

1. References (10)
2. www.whitehouse.gov/sites/defa...s/hist07z1.xls

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5.2: Describing Data

Once we have collected data from surveys or experiments, we need to summarize and present the data in a way that will be meaningful to the reader. We will begin with graphical presentations of data and then explore numerical summaries of data.

Presenting Categorical Data Graphically

Categorical, or qualitative, data are pieces of information that allow us to classify the objects under investigation into various categories. We usually begin working with categorical data by summarizing the data into a **frequency table**.

Definition: Frequency Table

A frequency table is a table with two columns. One column lists the categories, and another for the frequencies with which the items in the categories occur (how many items fit into each category).

Example 5.2.1

An insurance company determines vehicle insurance premiums based on known risk factors. If a person is considered a higher risk, their premiums will be higher. One potential factor is the color of your car. The insurance company believes that people with some color cars are more likely to get in accidents. To research this, they examine police reports for recent total-loss collisions. The data is summarized in the frequency table below.

Table 5.2.1

Color	Frequency
Blue	25
Green	52
Red	41
White	36
Black	39
Grey	23

Sometimes we need an even more intuitive way of displaying data. This is where charts and graphs come in. There are many, many ways of displaying data graphically, but we will concentrate on one very useful type of graph called a bar graph. Here, we will work with bar graphs that display categorical data; later, we will be devoted to bar graphs that display quantitative data.

Definition: Bar Graph

A bar graph is a graph that displays a bar for each category with the length of each bar indicating the frequency of that category.

Example 5.2.2

Using our car data from above, note the highest frequency is 52, so our vertical axis needs to go from 0 to 52, but we might as well use 0 to 55, so that we can put a hash mark every 5 units:

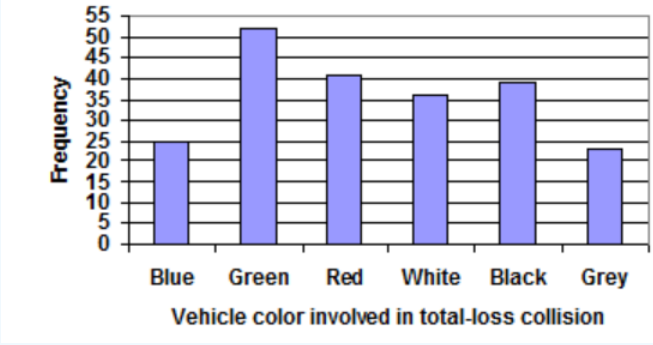


Figure 5.2.1: Car Data

Notice that the height of each bar is determined by the frequency of the corresponding color. The horizontal gridlines are a nice touch, but not necessary. In practice, you will find it useful to draw bar graphs using graph paper, so the gridlines will already be in place, or using technology. Instead of gridlines, we might also list the frequencies at the top of each bar, like this:

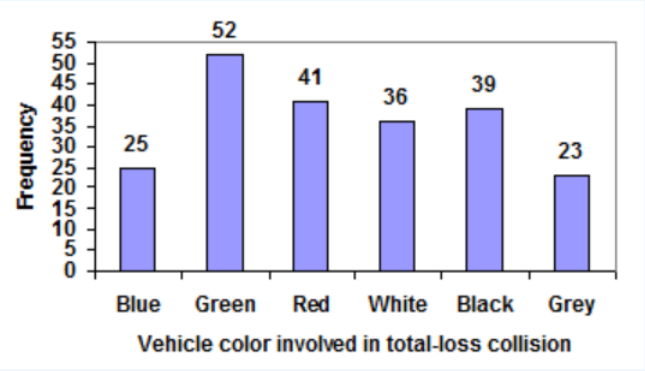


Figure 5.2.2: Car Data (Frequency listed at the top of each bar)

In this case, our chart might benefit from being reordered from largest to smallest frequency values. This arrangement can make it easier to compare similar values in the chart even without gridlines. When we arrange the categories in decreasing frequency order like this, it is called a **Pareto chart.**

Definition: Pareto Chart

A pareto chart is a bar graph ordered from highest to lowest frequency.

Example 5.2.3

Transforming our bar graph from earlier into a Pareto chart, we get:

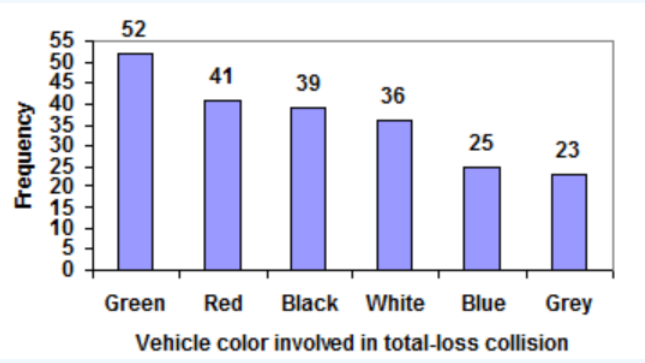


Figure 5.2.3: Car Data

Example 5.2.4

In a survey, adults were asked whether they personally worried about a variety of environmental concerns. The numbers (out of 1012 surveyed) who indicated that they worried "a great deal" about some selected concerns are summarized below.

Table 5.2.2

Environmental Issue	Frequency
Pollution of drinking water	597
Contamination of soil and water by toxic waste	526
Air pollution	455
Global warming	354

This data could be shown graphically in a bar graph:

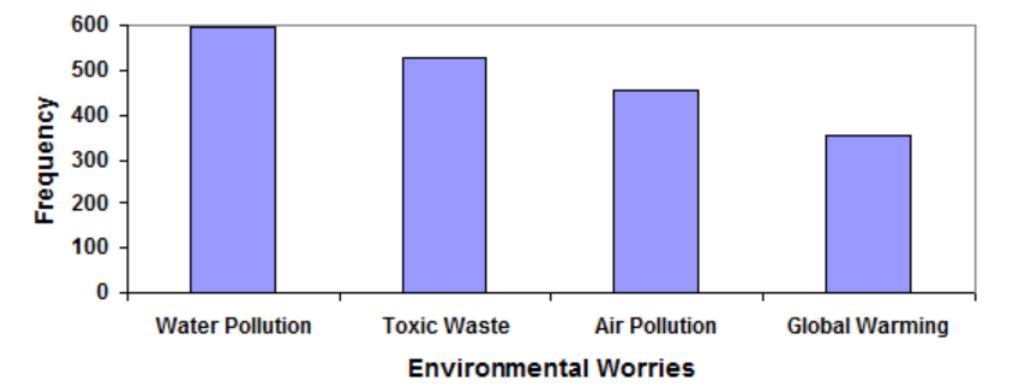


Figure 5.2.4: Environmental worries data

To show relative sizes, it is common to use a pie chart.

Definition: Pie Chart

A pie chart is a circle with wedges cut of varying sizes marked out like slices of pie or pizza. The relative sizes of the wedges correspond to the relative frequencies of the categories.

Example 5.2.5

For our vehicle color data, a pie chart might look like this:

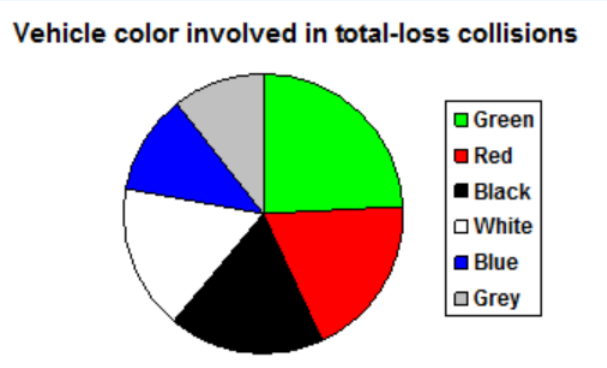


Figure 5.2.5: Pie chart of Vehicle color involved in total-loss collisions

Pie charts can often benefit from including frequencies or relative frequencies (percents) in the chart next to the pie slices. Often having the category names next to the pie slices also makes the chart clearer.

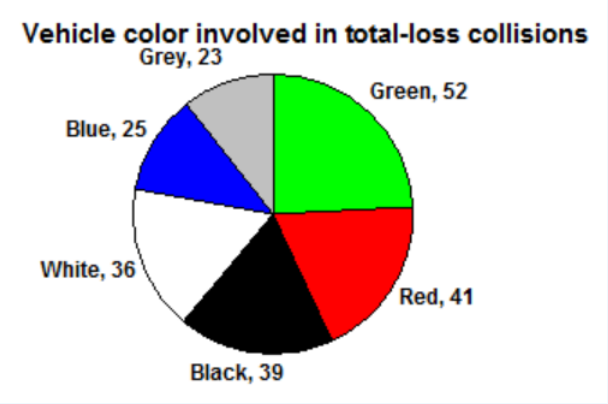


Figure 5.2.6: Frequencies (percents) in the chart next to the pie slices

Example 5.2.6

The pie chart below shows the percentage of voters supporting each candidate running for a local senate seat.

If there are 20,000 voters in the district, the pie chart shows that about 11% of those, about 2,200 voters, support Reeves.

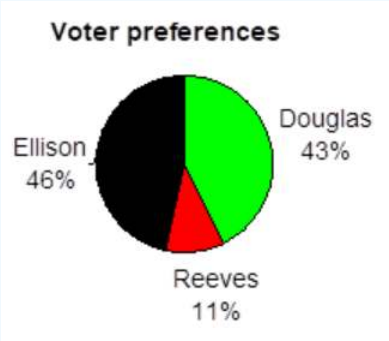


Figure 5.2.7: Voter preferences

Pie charts look nice but are harder to draw by hand than bar charts since we would need to compute the angle each wedge cuts out of the circle and then measure the angle with a protractor. Computers are much better suited to drawing pie charts. Common software programs like Microsoft Word or Excel, OpenOffice.org Write or Calc, or Google Docs are able to create bar graphs, pie charts, and other graph types. There are also numerous online tools that can create graphs.

Try it Now 1

Create a bar graph and a pie chart to illustrate the grades on a history exam below. A: 12 student, B: 19 students, C: 14 students, D: 4 students, F: 5 students.

Don't get fancy with graphs! People sometimes add features to graphs that don't help to convey their information. For example, 3-dimensional bar charts like the one shown below are usually not as effective as their two-dimensional counterparts.

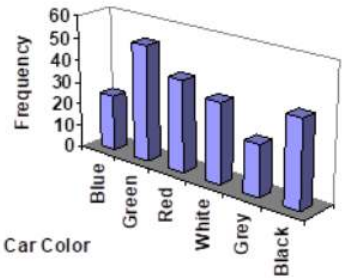


Figure 5.2.8: 3-dimensional bar chart

Here is another way that fanciness can lead to trouble. Instead of plain bars, it is tempting to substitute meaningful images. This type of graph is called a **pictogram.**

Definition: Pictogram

A pictogram is a statistical graphic in which the size of the picture is intended to represent the frequencies or size of the values being represented.

Example 5.2.7

A labor union might produce the graph below to show the difference between the average manager salary and the average worker salary.

Looking at the picture, it would be reasonable to guess that the manager salaries is 4 times as large as the worker salaries – the area of the bag looks about 4 times as large. However, the manager salaries are in fact only twice as large as worker salaries, which were reflected in the picture by making the manager bag twice as tall.



Figure 5.2.9: Salary representations

Another distortion in bar charts results from setting the baseline to a value other than zero. The baseline is the bottom of the vertical axis, representing the least number of cases that could have occurred in a category. Normally, this number should be zero.

Example 5.2.8

Compare the two graphs below showing support for same-sex marriage rights from a poll taken in December 2008. The difference in the vertical scale on the first graph suggests a different story than the true differences in percentages; the second graph makes it look like three times as many people oppose marriage rights as support it.

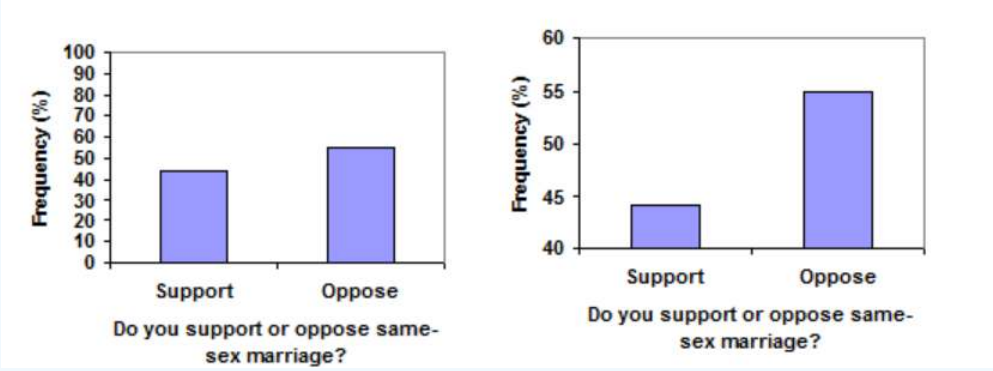


Figure 5.2.10: Same-sex marriage graphs

Try it Now 2

A poll was taken asking people if they agreed with the positions of the 4 candidates for a county office. Does the pie chart present a good representation of this data? Explain.

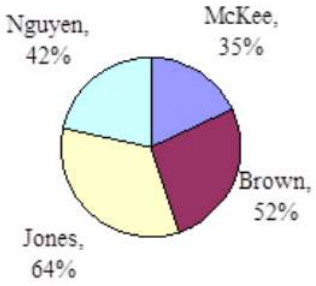


Figure 5.2.11: Pie chart

Presenting Quantitative Data Graphically

Quantitative, or numerical, data can also be summarized into frequency tables.

Example 5.2.9

A teacher records scores on a 20-point quiz for the 30 students in his class. The scores are:

19, 20, 18, 18, 17, 18, 19, 17, 20, 18, 20, 16, 20, 15, 17, 12, 18, 19, 18, 19, 17, 20, 18, 16, 15, 18, 20, 5, 0, 0

Table 5.2.3

Score	Frequency
0	2
5	1
12	1
15	2
16	2
17	4
18	8
19	4
20	6

Using this table, it would be possible to create a standard bar chart from this summary as we did for categorical data:

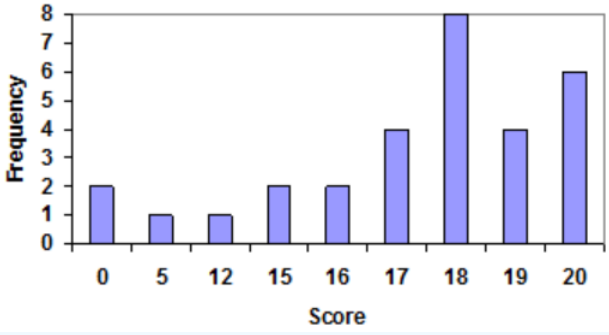


Figure 5.2.12: Standard bar chart from the above summary

However, since the scores are numerical values, this chart doesn't really make sense; the first and second bars are five values apart, while the later bars are only one value apart. It would be better to treat the horizontal axis as a number line. This type of graph is called a **histogram.**

Definition: Histogram

A histogram is like a bar graph, but the horizontal axis is a number line.

Example 5.2.10

For the values above, a histogram would look like:

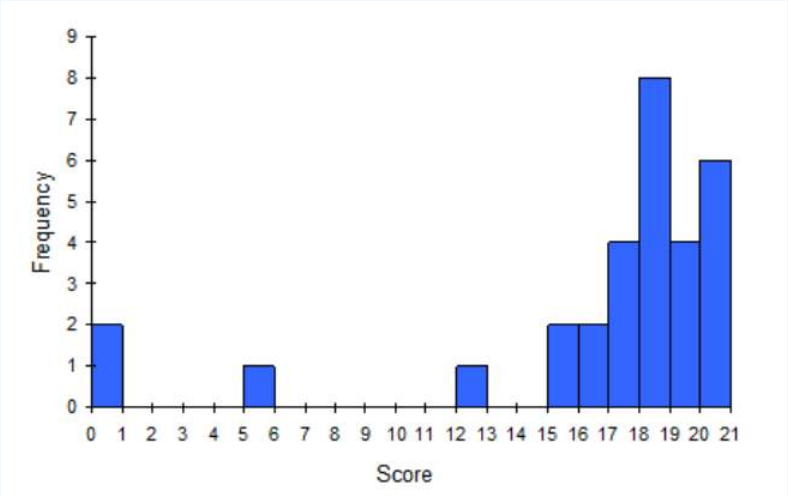


Figure 5.2.13: Histogram for above values

Notice that in the histogram, a bar represents values on the horizontal axis from that on the left hand-side of the bar up to, but not including, the value on the right hand side of the bar. So, for instance, the fact that there are four occurrences of 17 in the dataset is indicated by the bar, with height 4, between 17 and 18 in this histograph. Some people choose to have bars start at ½ values to avoid this ambiguity.

Unfortunately, not a lot of common software packages can correctly graph a histogram. About the best you can do in Excel or Word is a bar graph with no gap between the bars and spacing added to simulate a numerical horizontal axis.

If we have a large number of widely varying data values, creating a frequency table that lists every possible value as a category would lead to an exceptionally long frequency table and probably would not reveal any patterns. For this reason, it is common with quantitative data to group data into **class intervals.**

Definition: Class Intervals

Class intervals are groupings of the data. In general, we define class intervals so that:

- Each interval is equal in size. For example, if the first class contains values from 120-129, the second class should include values from 130-139.
- We have somewhere between 5 and 20 classes, typically, depending upon the number of data we're working with.

Example 5.2.11

Suppose that we have collected weights from 100 male subjects as part of a nutrition study. For our weight data, we have values ranging from a low of 121 pounds to a high of 263 pounds, giving a total span of 263-121 = 142. We could create 7 intervals with a width of around 20 or 14 intervals with a width of around 10, or somewhere in between. Often we have to experiment with a few possibilities to find something that represents the data well. Let us try using an interval width of 15. We could start at 121, or at 120 since it is a nice round number.

Table 5.2.4

Interval	Frequency
120-134	4

Interval	Frequency
135-149	14
150-164	16
165-179	28
180-194	12
195-209	8
210-224	7
225-239	6
240-254	2
255-269	3

A histogram of this data would look like:

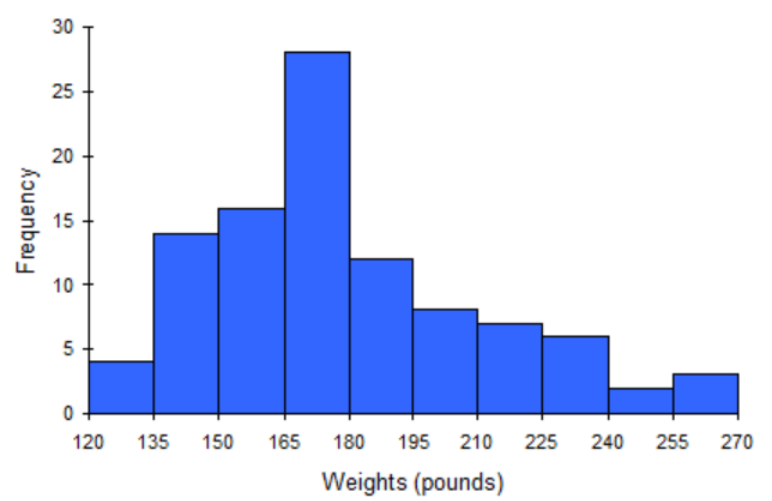


Figure 5.2.14: Histogram of the above data

In many software packages, you can create a graph similar to a histogram by putting the class intervals as the labels on a bar chart.

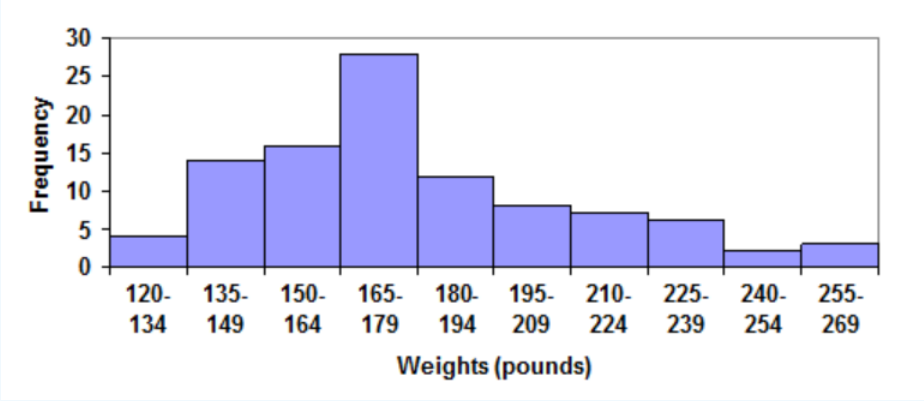


Figure 5.2.15: Histogram created by putting class intervals on bar chart

Other graph types such as pie charts are possible for quantitative data. The usefulness of different graph types will vary depending upon the number of intervals and the type of data being represented. For example, a pie chart of our weight data is difficult to read because of the quantity of intervals we used.

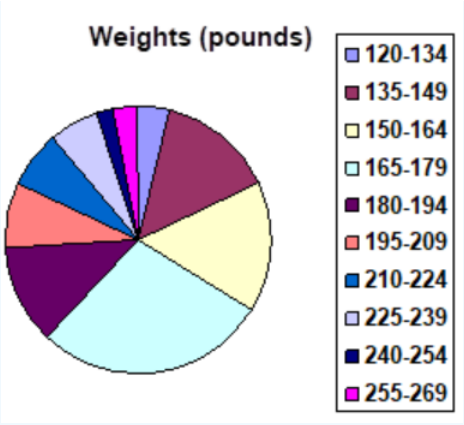


Figure 5.2.16: Pie chart of our weight data

Try it Now 3

The total cost of textbooks for the term was collected from 36 students. Create a histogram for this data.

$140 $160 $160 $165 $180 $220 $235 $240 $250 $260 $280 $285

$285 $285 $290 $300 $300 $305 $310 $310 $315 $315 $320 $320

$330 $340 $345 $350 $355 $360 $360 $380 $395 $420 $460 $460

When collecting data to compare two groups, it is desirable to create a graph that compares quantities.

Example 5.2.12

The data below came from a task in which the goal is to move a computer mouse to a target on the screen as fast as possible. On 20 of the trials, the target was a small rectangle; on the other 20, the target was a large rectangle. Time to reach the target was recorded on each trial.

Table 5.2.5

Interval (milliseconds)	Frequency (small target)	Frequency (large target)
300-399	0	0
400-499	1	5
500-599	3	10
600-699	6	5
700-799	5	0
800-899	4	0
900-999	0	0
1000-1099	1	0
1100-1199	0	0

One option to represent this data would be a comparative histogram or bar chart in which bars for the small target group and large target group are placed next to each other.

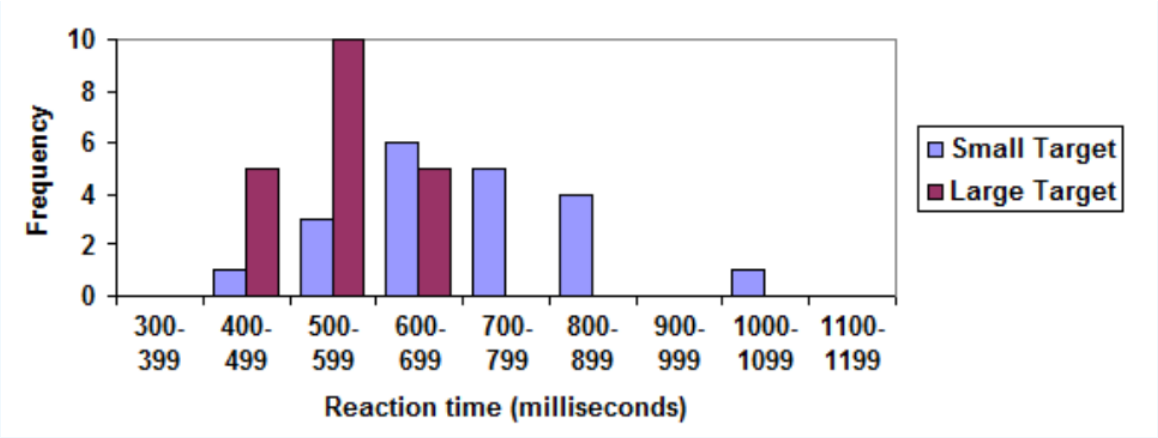


Figure 5.2.17: A comparative histogram or bar chart

Definition: Frequency Polygon

An alternative representation is a frequency polygon. A frequency polygon starts out like a histogram, but instead of drawing a bar, a point is placed in the midpoint of each interval at height equal to the frequency. Typically the points are connected with straight lines to emphasize the distribution of the data.

Example 5.2.13

This graph makes it easier to see that reaction times were generally shorter for the larger target and that the reaction times for the smaller target were more spread out.

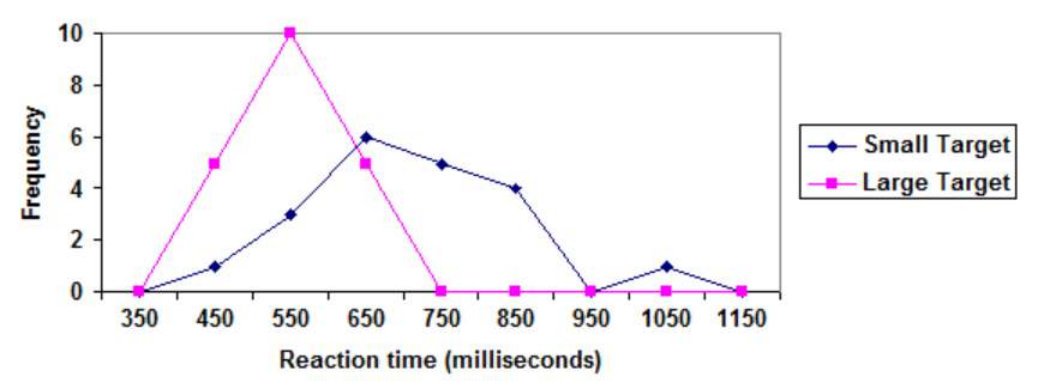


Figure 5.2.18: Graph

Reference

1. References (11)

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5.3: Numerical Measures of Center and Variation

It is often desirable to use a few numbers to summarize a distribution. One important aspect of a distribution is where its center is located. Measures of central tendency are discussed first. A second aspect of a distribution is how spread out it is. In other words, how much the data in the distribution vary from one another. The second part of this section describes measures of variation.

Measures of Central Tendency

Let's begin by trying to find the most "typical" value of a data set.

Note that we just used the word "typical" although in many cases you might think of using the word "average." We need to be careful with the word "average" as it means different things to different people in different contexts. One of the most common uses of the word "average" is what mathematicians and statisticians call the “arithmetic mean”, or just plain old “mean” for short. "Arithmetic mean" sounds rather fancy, but you have likely calculated a mean many times without realizing it; the mean is what most people think of when they use the word "average.”

Mean

The mean of a set of data is the sum of the data values divided by the number of values.

Example 5.3.1

Marci’s exam scores for her last math class were 79, 86, 82, 94. Find the mean.

Solution

The mean of these values would be:

79+86+82+944=85.25

Example 5.3.2

The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2000 season are shown below.

37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 21, 20

20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6

What is the mean number of touchdown passes thrown in the NFL in the 2000 season?

Solution

Adding these values, we get 634 total TDs. Dividing by 31, the number of data values, we get 634/31 = 20.4516. It would be appropriate to round this to 20.5.

It would be correct for us to report that “The mean number of touchdown passes thrown in the NFL in the 2000 season was 20.5 passes,” but it is not uncommon to see the more casual word “average” used in place of “mean.”

Try it Now 4

The prices of a jar of peanut butter at 5 stores were $3.29, $3.59, $3.79, $3.75, and $3.99. Find the mean price.

Example 5.3.3

One hundred families in a particular neighborhood are asked their annual household income, to the nearest $5 thousand dollars. The results are summarized in a frequency table below.

Table 5.3.1

Income (thousands of dollars)	Frequency
15	6

Income (thousands of dollars)	Frequency
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Calculate the mean.

Solution

Calculating the mean by hand could get tricky if we try to type in all 100 values:

$$\frac{\overbrace{15+\cdots 15}^{\text{6 terms}}+\overbrace{20+\cdots 20}^{\text{8 terms}}+\overbrace{25+\cdots 25}^{\text{11 terms}}+\cdots}{100}$$

We could calculate this more easily by noticing that adding 15 to itself six times is the same as 90. Using this simplification, we get:

$$\frac{15\cdot 6+20\cdot 8+25\cdot 11+30\cdot 17+35\cdot 19+40\cdot 20+45\cdot 12+50\cdot 7}{100}=\frac{3390}{100}=33.9$$

The mean household income of our sample is 33.9 thousand dollars ($33,900).

Example 5.3.4

Extending off the last example, suppose a new family moves into the neighborhood example that has a household income of $5 million ($5000 thousand).

Solution

Adding this to our sample, our mean is now:

$$\frac{15\cdot 6+20\cdot 8+25\cdot 11+30\cdot 17+35\cdot 19+40\cdot 20+45\cdot 12+50\cdot 7+5000\cdot 1}{101}=\frac{8390}{101}=83.069$$

While 83.1 thousand dollars ($83,069) is the correct mean household income, it no longer represents a “typical” value.

An extreme value, much bigger or much smaller than most other values in a given dataset, is called an “outlier”. An outlier can affect the mean.

Imagine the data values on a see-saw or balance scale. The mean is the value that keeps the data in balance, like in the picture below.



Figure 5.3.1

If we graph our household data, the $5 million data value is so far out to the right that the mean has to adjust up to keep things in balance.

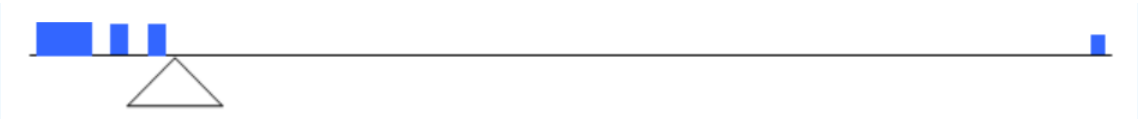


Figure 5.3.2

In situations like this, where one value is much bigger or smaller than most other value in the data, there is a better measure of center than the mean.

Median

The median of a set of data is the value in the middle when the data is in order.

To find the median, begin by listing the data in order from smallest to largest, or largest to smallest.

If the number of data values, N, is odd, then the median is the middle data value. This can be found by rounding $N/2$ up to the next whole number.

If the number of data values is even, there is no one middle value, so we find the mean of the two middle values (values $N/2$ and $N/2+1$).

Example 5.3.5

Returning to the football touchdown data, we would start by listing the data in order. Luckily, it was already in decreasing order, so we can work with it without needing to reorder it first.

37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 21, 20

20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6

Find the median touchdown pass.

Solution

Since there are 31 data values, an odd number, the median will be the middle number, the 16th data value (31/2 = 15.5, round up to 16, leaving 15 values below and 15 above). The 16th data value is 20, so the median number of touchdown passes in the 2000 season was 20 passes. Notice that for this data, the median is fairly close to the mean we calculated earlier, 20.5.

Example 5.3.6

Find the median of these quiz scores: 5, 10, 8, 6, 4, 8, 2, 5, 7, 7

Solution

We start by listing the data in order: 2, 4, 5, 5, 6, 7, 7, 8, 8, 10

Since there are 10 data values, an even number, there is no one middle number. So we find the mean of the two middle numbers, 6 and 7, and get (6+7)/2 = 6.5.

The median quiz score was 6.5.

Try it Now 5

The prices of a jar of peanut butter at 5 stores were $3.29, $3.59, $3.79, $3.75, and $3.99. Find the median price.

Example 5.3.7

Let us return now to our original household income data.

Table 5.3.2

Income (thousands of dollars)	Frequency
15	6

Income (thousands of dollars)	Frequency
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Find the mean and median income.

Solution

Here we have 100 data values. If we didn't already know that, we could find it by adding the frequencies. Since 100 is an even number, we need to find the mean of the middle two data values - the 50th and 51st data values. To find these, we start counting up from the bottom:

There are 6 data values of $15, so Values 1 to 6 are $15 thousand

The next 8 data values are $20, so Values 7 to (6+8)=14 are $20 thousand

The next 11 data values are $25, so Values 15 to (14+11)=25 are $25 thousand

The next 17 data values are $30, so Values 26 to (25+17)=42 are $30 thousand

The next 19 data values are $35, so Values 43 to (42+19)=61 are $35 thousand

From this we can tell that values 50 and 51 will be $35 thousand, and the mean of these two values is $35 thousand. The median income in this neighborhood is $35 thousand.

Example 5.3.8

If we add in the new neighbor with a $5 million household income, how does the median change?

Solution

If we add in the new neighbor with a $5 million household income, then there will be 101 data values, and the 51st value will be the median. As we discovered in the last example, the 51st value is $35 thousand. Notice that the new neighbor did not affect the median in this case. The median is not swayed as much by outliers as the mean is.

This is the main reason the median, not the mean, is used to represent certain data, such as the average household income or home price.

In addition to the mean and the median, there is one other common measurement of the "typical" value of a data set: the **mode.**

Mode

The mode is the element of the data set that occurs most frequently.

The mode is fairly useless with data like weights or heights where there are a large number of possible values. The mode is most commonly used for categorical data, for which median and mean cannot be computed.

Example 5.3.9

In our vehicle color survey, we collected the data

Table 5.3.3

Color	Frequency
Blue	3
Green	5
Red	4
White	3
Black	2
Grey	3

What is the mode?

Solution

For this data, Green is the mode, since it is the data value that occurred the most frequently.

It is possible for a data set to have more than one mode (if several categories have the same frequency), or no modes (if every category occurs only once.)

Try it Now 6

Reviewers were asked to rate a product on a scale of 1 to 5. Find

a. The mean rating
b. The median rating
c. The mode rating

Table 5.3.4

Rating	Frequency
1	4
2	8
3	7
4	3
5	1

Measures of Variation

Consider these three sets of quiz scores:

Section A: 5, 5, 5, 5, 5, 5, 5, 5, 5, 5

Section B: 0, 0, 0, 0, 0, 10, 10, 10, 10, 10

Section C: 4, 4, 4, 5, 5, 5, 5, 6, 6, 6

All three of these sets of data have a mean of 5 and median of 5, yet the sets of scores are clearly quite different. In section A, everyone had the same score; in section B half the class got no points and the other half got a perfect score, assuming this was a 10-point quiz. Section C was not as consistent as section A, but not as widely varied as section B.

In addition to the mean and median, which are measures of the "typical" or "middle" value, we also need a measure of how "spread out" or varied each data set is.

There are several ways to measure this "spread" of the data. The first is the simplest and is called the **range.**

Range

The range is the difference between the maximum value and the minimum value of the data.

Example 5.3.10

Using the quiz scores from above, find the range for each section.

Solution

For section A, the range is 0 since both maximum and minimum are 5 and 5 – 5 = 0.

For section B, the range is 10 since 10 – 0 = 10.

For section C, the range is 2 since 6 – 4 = 2.

In the last example, the range seems to be revealing how spread out the data is.

However, suppose we add a fourth section, Section D, with scores 0, 5, 5, 5, 5, 5, 5, 5, 5, 10.

This section also has a mean and median of 5. The range is 10, yet this data set is quite different from Section B. To better illuminate the differences, we'll have to turn to more sophisticated measures of variation.

Standard Deviation

The standard deviation is a measure of variation based on measuring how far each data value deviates, or is different, from the mean. A few important characteristics:

- Standard deviation is always positive. Standard deviation will be zero only if all the data values are equal and will get larger as the data spreads out.
- Standard deviation has the same unit as the original data.
- Standard deviation, like the mean, can be highly influenced by outliers.

Standard deviation is rather complicated. Let me explain below, step by step.

Using the data from section D (Example 5.3.10), we could compute for each data value the difference between the data value and the mean:

Table 5.3.5

Data value	Deviation: Data value - Mean
0	0-5 = -5
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
5	5-5 = 0
10	10-5 = 5

We would like to get an idea of the "average" deviation from the mean, but if we find the average of the values in the second column, the negative and positive values cancel each other out, resulting in the average deviation of 0 (this will always happen). To prevent this we square every value in the second column:

Table 5.3.6

Data value	Deviation: Data value - Mean	Deviation squared
0	0-5 = -5	$(-5)^2 = 25$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
5	5-5 = 0	$0^2 = 0$
10	10-5 = 5	$(5)^2 = 25$

We then add the squared deviations up to get 25 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 25 = 50. Ordinarily we would then divide by the number of scores, n (in this case, 10), to find the mean of the squares of the deviations. But we only do this if the data set represents a population; if the data set represents a sample (as it almost always does), we instead divide by $n-1$ (in this case, 10-1=9).

The quotient (by n or by $n-1$) is called the "variance" of the data set.

So in our example, we would have 50/10 = 5 if section D represents a population and 50/9 = about 5.56 if section D represents a sample. These values (5 and 5.56) are called, respectively, the **population variance** and the **sample variance** for section D.

We are almost there—one more step, and we will find the standard deviation.

Variance can be a useful statistical concept, but note that the unit of variance in this instance would be points-squared since we squared all of the deviations. What are points-squared? Good question. We would rather deal with the units we started with (points in this case), so to convert back we take the square root and get:

Population standard deviation=$\sqrt{\frac{50}{10}} = \sqrt{5} \approx 2.2$

or

sample standard deviation=$\sqrt{\frac{50}{9}} \approx 2.4$

If we are unsure whether the data set is a sample or a population, we will usually assume it is a sample, and we will round answers to one more decimal place than the original data as we have done above.

As you can see, it takes multiple steps to calculate the standard deviation of a dataset. For most real-life applications, the calculations are performed using technology. Here is a brief summary.

To compute Standard Deviation

1. Find the deviation of each data value from the mean. In other words, subtract the mean from the data value.
2. Square each deviation.
3. Add the squared deviations.
4. Divide by n, the number of data values, if the data represents a whole population; divide by $n-1$ if the data is from a sample.
5. Compute the square root of the result.

Example 5.3.11

Compute the standard deviation for Section B above

Solution

Computing the standard deviation for Section B above, we first calculate that the mean is 5. Using a table can help keep track of your computations for the standard deviation:

Table 5.3.7

Data value	Deviation: Data value - Mean	Deviation squared
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
0	0-5 = -5	$(-5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$
10	10-5 = 5	$(5)^2 = 25$

Assuming this data represents a population, we will add the squared deviations, divide by 10, the number of data values, and compute the square root:

$$\sqrt{\frac{25+25+25+25+25+25+25+25+25+25}{10}} = \sqrt{\frac{250}{10}} = 5$$

Notice that the standard deviation of this data set is much larger than that of section D since the data in this set is more spread out.

For comparison, the standard deviations of all four sections are as follows:

Table 5.3.8

Section	Standard deviation
Section A: 5, 5, 5, 5, 5, 5, 5, 5, 5, 5	Standard deviation: 0
Section B: 0, 0, 0, 0, 0, 10, 10, 10, 10, 10	Standard deviation: 5
Section C: 4, 4, 4, 5, 5, 5, 5, 6, 6, 6	Standard deviation: 0.8
Section D: 0, 5, 5, 5, 5, 5, 5, 5, 5, 10	Standard deviation: 2.2

Try it Now 7

The prices of a jar of peanut butter at 5 stores were: $3.29, $3.59, $3.79, $3.75, and $3.99. Find the standard deviation of the prices.

Where standard deviation is a measure of variation based on the mean, quartiles are based on the median.

Quartiles

Quartiles are values that divide the data in quarters.

The first quartile (Q1) is the value so that 25% of the data values are below it; the third quartile (Q3) is the value so that 75% of the data values are below it. You may have guessed that the second quartile is the same as the median, since the median is the value so that 50% of the data values are below it.

This divides the data into quarters; 25% of the data is between the minimum and Q1, 25% is between Q1 and the median, 25% is between the median and Q3, and 25% is between Q3 and the maximum value

While quartiles are not a 1-number summary of variation like standard deviation, the quartiles are used with the median, minimum, and maximum values to form a 5-number summary of the data.

Five-number Summary

The five-number summary takes this form:

Minimum, Q1, Median, Q3, Maximum.

To find the first quartile, we need to find the data value so that 25% of the data is below it.

But note that this is the same as the "median" of the first (lower) half of the dataset. So an easy way to find the first quartile is to split the original data set in half and find the median of the lower half. The third quartile is similar; find the median of the second (higher) half of the data set.

Warning: The precise definitions and method of finding quartiles vary slightly from book to book. You may see slightly different explanations in other resources.

Example 5.3.12

Suppose we have measured 9 females and their heights (in inches), sorted from smallest to largest are:

59, 60, 62, 64, 66, 67, 69, 70, 72

Find Quartiles.

Solution

Note the median is 66.

The lower half, {59, 60, 62, 64}, has the median of 61, and the higher half, {67, 69, 70, 72}, has the median of 69.5.

So Q1 is 61, Q2 is 66, and Q3 is 69.5.

Example 5.3.13

Suppose we had measured 8 females and their heights (in inches), sorted from smallest to largest are:

59, 60, 62, 64, 66, 67, 69, 70

Find Quartiles.

Solution

Note the median in this case is 65, the mean of 64 and 66.

Then the lower half is the same as in Example 25 (its median is 61). The higher half now has the median 68.

So, Q1 is 61, Q2 is 65, and Q3 is 68.

The 5-number summary combines the first and third quartile with the minimum, median, and maximum values.

Example 5.3.14

For the 9 female sample, the median is 66, the minimum is 59, and the maximum is 72. The 5- number summary is: 59, 61, 66, 69.5, 72.

For the 8 female sample, the median is 65, the minimum is 59, and the maximum is 70, so the 5- number summary would be: 59, 61, 65, 68, 70.

Example 5.3.15

Returning to our quiz score data. In each case, there are 10 values, so the median is the mean of the 5th and 6th values. The first quartile will be the 3rd data value, and the third quartile will be the 8th data value. Creating the five-number summaries:

Table 5.3.9

Section and data	5-number summary
Section A: 5, 5, 5, 5, 5, 5, 5, 5, 5, 5	5, 5, 5, 5, 5
Section B: 0, 0, 0, 0, 0, 10, 10, 10, 10, 10	0, 0, 5, 10, 10
Section C: 4, 4, 4, 5, 5, 5, 5, 6, 6, 6	4, 4, 5, 6, 6
Section D: 0, 5, 5, 5, 5, 5, 5, 5, 5, 10	0, 5, 5, 5, 10

Of course, with a relatively small data set, finding a five-number summary is a bit silly since the summary contains almost as many values as the original data.

Try it Now 8

The total cost of textbooks for the term was collected from 36 students. Find the 5-number summary of this data.

$140 $160 $160 $165 $180 $220 $235 $240 $250 $260 $280 $285

$285 $285 $290 $300 $300 $305 $310 $310 $315 $315 $320 $320

$330 $340 $345 $350 $355 $360 $360 $380 $395 $420 $460 $460

Example 5.3.16

Returning to the household income data from earlier, create the 5-number summary.

Table 5.3.10

Income (thousands of dollars)	Frequency
15	6
20	8
25	11
30	17
35	19
40	20
45	12
50	7

Solution

By adding the frequencies, we can see there are 100 data values represented in the table. In Example 20, we found the median was $35 thousand. We can see in the table that the minimum income is $15 thousand, and the maximum is $50 thousand.

Counting up in the data as we did before,

There are 6 data values of $15, so Values 1 to 6 are $15 thousand

The next 8 data values are $20, so Values 7 to (6+8)=14 are $20 thousand

The next 11 data values are $25, so Values 15 to (14+11)=25 are $25 thousand

The next 17 data values are $30, so Values 26 to (25+17)=42 are $30 thousand

The lower half has 50 values, so Q1 is the mean of the 25th and the 26th values.

The 25th data value is $25 thousand, and the 26th data value is $30 thousand, so Q1 will be the mean of these: (25 + 30)/2 = $27.5 thousand.

Similarly, Q3 will be the mean of the 75th and 76th data values. Continuing our counting from earlier,

The next 19 data values are $35, so Values 43 to (42+19)=61 are $35 thousand

The next 20 data values are $40, so Values 61 to (61+20)=81 are $40 thousand

Both the 75th and 76th data values lie in this group, so Q3 will be $40 thousand.

Putting these values together into a five-number summary, we get: 15, 27.5, 35, 40, 50.

Note that the 5-number summary divides the data into four intervals, each of which will contain about 25% of the data. In the previous example, that means about 25% of households have income between $40 thousand and $50 thousand.

For visualizing data, there is a graphical representation of a 5-number summary called a box plot, or box and whisker graph.

Box Plot

A box plot is a graphical representation of a 5-number summary.

To create a box plot, a number line is first drawn. A box is drawn from the first quartile to the third quartile, and a line is drawn through the box at the median. “Whiskers” are extended out to the minimum and maximum values.

Example 5.3.17

Draw a box plot below based on the 5-number summary:

59, 62, 66, 69, 72.

Solution

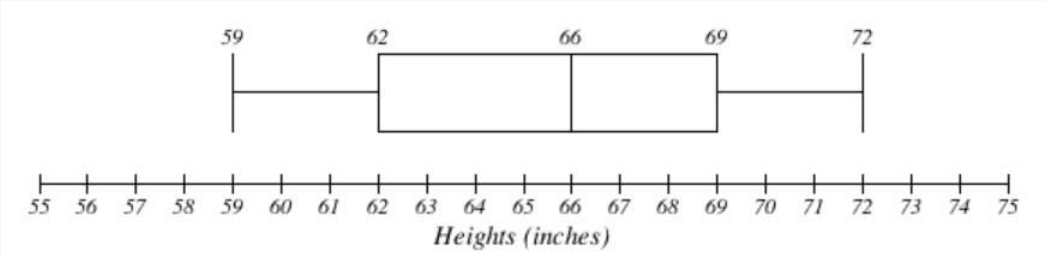


Figure 5.3.3: Box Plot

Example 5.3.18

Draw a box plot below based on the household income data with 5 number summary:

15, 27.5, 35, 40, 50

Solution



Figure 5.3.4: Box Plot

Try it Now 9

Create a boxplot based on the textbook price data from the last Try it Now.

Box plots are particularly useful for comparing data from two populations.

Example 5.3.19

The box plot of service times for two fast-food restaurants is shown below.

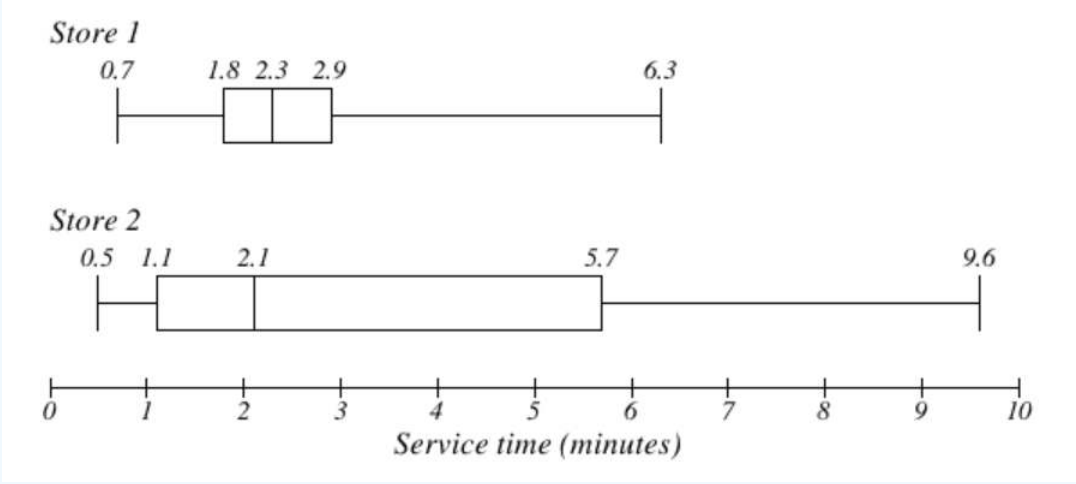


Figure 5.3.5: Box Plot

While store 2 had a slightly shorter median service time (2.1 minutes vs. 2.3 minutes), store 2 is less consistent, with a wider spread of the data.

At store 1, 75% of customers were served within 2.9 minutes, while at store 2, 75% of customers were served within 5.7 minutes.

Which store should you go to in a hurry?

Solution

That depends upon your opinions about luck – 25% of customers at store 2 had to wait between 5.7 and 9.6 minutes.

Example 5.3.20

The boxplot below is based on the birth weights of infants with severe idiopathic respiratory distress syndrome (SIRDS). The boxplot is separated to show the birth weights of infants who survived and those that did not.

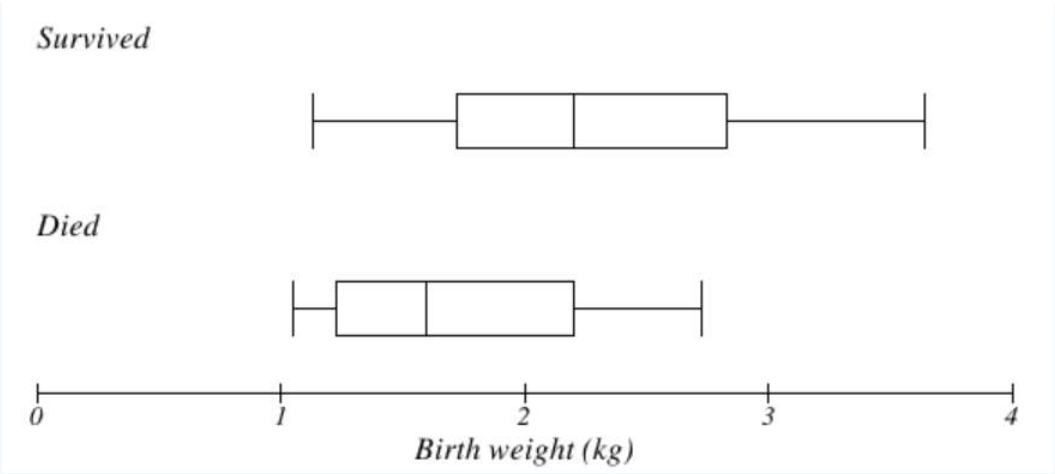


Figure 5.3.6: Box Plot

Compare the two groups to determine if birth weight is quite likely linked to survival of infants with SIRDS.

Solution

Comparing the two groups, the boxplot reveals that the birth weights of the infants that died appear to be, overall, smaller than the weights of infants that survived. In fact, we can see that the median birth weight of infants that survived is the same as the third quartile of the infants that died.

Similarly, we can see that the first quartile of the survivors is larger than the median weight of those that died, meaning that over 75% of the survivors had a birth weight larger than the median birth weight of those that died.

Looking at the maximum value for those that died and the third quartile of the survivors, we can see that over 25% of the survivors had birth weights higher than the heaviest infant that died.

The box plot gives us a quick, albeit informal, way to determine that birth weight is quite likely linked to survival of infants with SIRDS.

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5.4: Normal Distribution

In the previous sections you learned that sampling is an important tool for determining the characteristics of a population. Although the parameters of the population (mean, standard deviation, etc.) were unknown, random sampling was used to yield reliable estimates of these values. The estimates were plotted on graphs to provide a visual representation of the distribution of the sample mean for various sample sizes. It is now time to define some properties of the sampling distribution of the sample mean and to examine what we can conclude about the entire population based on it.

You have probably seen a "bell-shaped curve" before. It turns out that many data sets, particularly when the sample size is sufficiently large, are distributed symmetrically, with most data values around the center (mean) and few outliers, resulting in a bell-shaped curve. This is called a "normal distribution." For instance, the weights of all the people in a certain country are likely distributed normally, and so are their heights. Normal distributions can describe all sorts of data involving people's cholesterol level, blood pressure, duration of pregnancy, IQ (intelligence quotient) score, credit score, and many more. For instance, the pregnancy (gestation) period for humans is known to have an average of 266 days (38 weeks) with a standard deviation of 16 days. The average IQ score is defined as 100 with a standard deviation of 15. Your SAT and other standardized test scores are defined in similar ways.

But what does it mean, and how does this information help us?

When data values are distributed normally, given its mean and standard deviation, we can then standardize the normal distribution and calculate various percentages related to scores/values. For example, given that the IQ scores have a mean of 100 and a standard deviation of 15, one can conclude that only about 2.3% of the population have an IQ of 130 or higher.

Here is how this works. First, let us define the "Standard Normal Distribution."

Definition: Standard normal distribution

The standard normal distribution is the normal distribution (bell-shaped curve) with 0 as the mean and 1 as the standard deviation.

It turns out that there is a simple way to "convert" any normal distribution to the standard one. It involves what is known as the "z-score."

The $z-$score measures how many standard deviations a score is away from the mean. The $z-$score of a term x in a population distribution whose mean is μ and whose standard deviation σ is given by:

$$z = \frac{x - \mu}{\sigma}$$

Since σ is always positive, z will be positive when x is greater than μ and negative when x is less than μ. A $z-$score of zero means that the term has the same value as the mean.

Example 5.4.1

On a nationwide math test the mean was 65 and the standard deviation was 10. If Robert scored 81, what was his $z-$score?

Solution

$$z = \frac{x - \mu}{\sigma}$$
$$z = \frac{81 - 65}{10}$$
$$z = \frac{16}{10}$$
$$z = 1.6$$

This means Robert's score was 1.6 standard deviations above the mean.

Example 5.4.2

On a college entrance exam, the mean was 70 and the standard deviation was 8. If Helen's $z-$score was -1.5, what was her exam mark?

Solution

$$z = \frac{x - \mu}{\sigma}$$
$$\therefore z \cdot \sigma = x - \mu$$
$$x = \mu + z \cdot \sigma$$
$$x = (70) + (-1.5)(8)$$
$$x = 58$$

Now you will see how $z-$scores are used to determine the probability of an event.

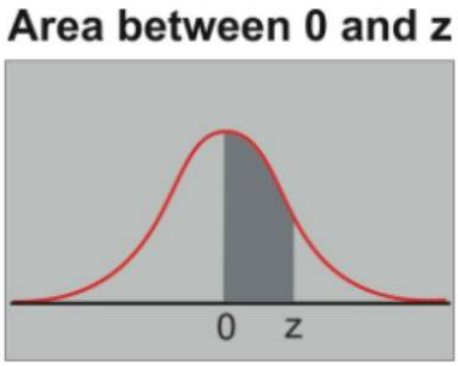


Figure 5.4.1

Table 5.4.1

z-value	Probability (area)
0.00	0.0000
0.50	0.1915
1.00	0.3413
1.50	0.4332
2.00	0.4772
2.50	0.4938
3.00	0.4987 (almost 50%)

The graph above shows the standard normal distribution (with the mean 0). Each decimal number in the table represents the probability (percentage) that a data value is between 0 and the corresponding z-value. For instance, about 34% of all data values lie between z=0 (mean) and z=1.0. Using symmetry, then, we can conclude that about 68% (twice 34%) of all data values are within 1 standard deviation of the mean (z between -1 and 1). The standard normal distribution has been extensively studied using advanced mathematics, and the probability can be calculated for any z-score. This table just shows a few of those numbers. It is known, for instance, that 99.74% of values lie within 3 standard deviations of the mean, and virtually everything is within 4 standard deviations of the mean.

Example 5.4.3

Suppose that the cholesterol levels are normally distributed with a mean of 160 (mg/dl) with a standard deviation of 40 (mg/dl).

a. What is the z-score corresponding to the cholesterol level of 200 (mg/dl)? How about 260 (mg/dl)?
b. What percent of people have a cholesterol level between 160 and 200? How about 260 or higher?

Solution

a. Note that $\frac{200-160}{40}=1$, so the z-score for 200 (mg/dl) is 1.0. Similarly, $\frac{260-160}{40}=2.5$, so the z-score for 260 (mg/dl) is 2.5.
b. Between 160 and 200 would correspond to the z-score between 0 and 1. According to the table, this amounts to 0.3413, or about 34%. In other words, about 34% of all the people have a cholesterol level between 160 and 200. However, 260 mg/dl corresponds to z=2.5. The table shows that 49.38% of the people belong to the interval between z=0 and z=2.5. Adding the 50% of those who are below average (z less than 0), this gives us 99.38% below z=2.5. Since 100-99.38=0.62 (%), only 0.62%, i.e., less than 1% of the population, has a cholesterol level 260 or higher. This person should go see the doctor immediately.

Example 5.4.4

Suppose that IQ scores are distributed with a mean of 100 and a standard deviation of 15. What are the probabilities that one's IQ score is

1. 100 or above?
2. 115 or above?
3. 130 or above?
4. 145 or above?

Solution

a. 100 is the average, so by symmetry, exactly 50% of the population has an IQ score of 100 or better.
b. 115 is one standard deviation above the mean, i.e., z = 1.0. So, by the table, 34.13% of the population has an IQ score between 100 and 115. Since 50% is supposed to be above the average of 100 (by symmetry), this means 50 - 34.13 = 15.87 (%) has an IQ score above 115.
c. Similarly, 130 corresponds to z = 2.0. The table has .4772, and 50 - 47.72 = 2.28 (%). In other words, less than 3 out of 100 people have an IQ score higher than 130.
d. Again, 145 corresponds to z = 3.0, and the table shows .4987. 50-49.87=0.13 (%). This implies about 1 in 1000 has an IQ of 145 or higher.

It should be noted that various details of IQ testing have changed from time to time, and one's IQ score also varies over time. Many controversies exist for IQ scores, and some even consider IQ scores unreliable. Interested readers can search for and find a lot of information on the Internet.

Exercises

1. A political scientist surveys 28 of the current 106 representatives in a state's congress. Of them, 14 said they were supporting a new education bill, 12 said there were not supporting the bill, and 2 were undecided.
   a. What is the population of this survey?
   b. What is the size of the population?
   c. What is the size of the sample?
   d. Give the sample statistic for the proportion of voters surveyed who said they were supporting the education bill.
   e. Based on this sample, we might expect how many of the representatives to support the education bill?
2. The city of Raleigh has 9500 registered voters. There are two candidates for city council in an upcoming election: Brown and Feliz. The day before the election, a telephone poll of 350 randomly selected registered voters was conducted. 112 said they'd vote for Brown, 207 said they'd vote for Feliz, and 31 were undecided.

a. What is the population of this survey?
b. What is the size of the population?
c. What is the size of the sample?
d. Give the sample statistic for the proportion of voters surveyed who said they'd vote for Brown.
e. Based on this sample, we might expect how many of the 9500 voters to vote for Brown?

3. Identify the most relevant source of bias in this situation: A survey asks the following: Should the mall prohibit loud and annoying rock music in clothing stores catering to teenagers?
4. Identify the most relevant source of bias in this situation: To determine opinions on voter support for a downtown renovation project, a surveyor randomly questions people working in downtown businesses.
5. Identify the most relevant source of bias in this situation: A survey asks people to report their actual income and the income they reported on their IRS tax form.
6. Identify the most relevant source of bias in this situation: A survey randomly calls people from the phone book and asks them to answer a long series of questions.
7. Identify the most relevant source of bias in this situation: A survey asks the following: Should the death penalty be permitted if innocent people might die?
8. Identify the most relevant source of bias in this situation: A study seeks to investigate whether a new pain medication is safe to market to the public. They test by randomly selecting 300 men from a set of volunteers.
9. In a study, you ask the subjects their age in years. Is this data qualitative or quantitative?
10. In a study, you ask the subjects their gender. Is this data qualitative or quantitative?
11. Does this describe an observational study or an experiment: The temperature on randomly selected days throughout the year was measured.
12. Does this describe an observational study or an experiment? A group of students are told to listen to music while taking a test and their results are compared to a group not listening to music.
13. In a study, the sample is chosen by separating all cars by size, and selecting 10 of each size grouping. What is the sampling method?
14. In a study, the sample is chosen by writing everyone’s name on a playing card, shuffling the deck, then choosing the top 20 cards. What is the sampling method?
15. A team of researchers is testing the effectiveness of a new HPV vaccine. They randomly divide the subjects into two groups. Group 1 receives new HPV vaccine, and Group 2 receives the existing HPV vaccine. The patients in the study do not know which group they are in.
    a. Which is the treatment group?
    b. Which is the control group (if there is one)?
    c. Is this study blind, double-blind, or neither?
    d. Is this best described as an experiment, a controlled experiment, or a placebo controlled experiment?
16. For the clinical trials of a weight loss drug containing Garcinia cambogia the subjects were randomly divided into two groups. The first received an inert pill along with an exercise and diet plan, while the second received the test medicine along with the same exercise and diet plan. The patients do not know which group they are in, nor do the fitness and nutrition advisors.
    a. Which is the treatment group?
    b. Which is the control group (if there is one)?
    c. Is this study blind, double-blind, or neither?
    d. Is this best described as an experiment, a controlled experiment, or a placebo controlled experiment?
17. A teacher wishes to know whether the males in his/her class have more conservative attitudes than the females. A questionnaire is distributed assessing attitudes.
    a. Is this a sampling or a census?
    b. Is this an observational study or an experiment?
    c. Are there any possible sources of bias in this study?
18. A study is conducted to determine whether people learn better with spaced or massed practice. Subjects volunteer from an introductory psychology class. At the beginning of the semester 12 subjects volunteer and are assigned to the massed-practice group. At the end of the semester 12 subjects volunteer and are assigned to the spaced-practice condition.
    a. Is this a sampling or a census?
    b. Is this an observational study or an experiment?

c. This study involves two kinds of non-random sampling: (1) Subjects are not randomly sampled from some specified population and (2) Subjects are not randomly assigned to groups. Which problem is more serious? What effect on the results does each have?

19. A farmer believes that playing Barry Manilow songs to his peas will increase their yield. Describe a controlled experiment the farmer could use to test his theory.
20. A sports psychologist believes that people are more likely to be extroverted as adults if they played team sports as children. Describe two possible studies to test this theory. Design one as an observational study and the other as an experiment. Which is more practical?
21. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 50 AIDS patients from the start of treatment until their deaths.
    a. What is the population of this study?
    b. List two reasons why the data may differ.
    c. Can you tell if one researcher is correct and the other one is incorrect? Why?
    d. Would you expect the data to be identical? Why or why not?
    e. If the first researcher collected her data by randomly selecting 40 states, then selecting 1 person from each of those states. What sampling method is that?
    f. If the second researcher collected his data by choosing 40 patients he knew. What sampling method would that researcher have used? What concerns would you have about this data set, based upon the data collection method?
22. Find a newspaper or magazine article, or the online equivalent, describing the results of a recent study (the results of a poll are not sufficient). Give a summary of the study's findings, then analyze whether the article provided enough information to determine the validity of the conclusions. If not, produce a list of things that are missing from the article that would help you determine the validity of the study. Look for the things discussed in the text: population, sample, randomness, blind, control, placebos, etc.
23. The table below shows scores on a Math test.
    a. Complete the frequency table for the Math test scores
    b. Construct a histogram of the data
    c. Construct a pie chart of the data

80	50	50	90	70	70	100	60	70	80	70	50
90	100	80	70	30	80	80	70	100	60	60	50

24. A group of adults where asked how many cars they had in their household
    a. Complete the frequency table for the car number data
    b. Construct a histogram of the data
    c. Construct a pie chart of the data

1	4	2	2	1	2	3	3	1	4	2	2
1	2	1	3	2	2	1	2	1	1	1	2

25. A group of adults were asked how many children they have in their families. The bar graph to the right shows the number of adults who indicated each number of children.
    a. How many adults where questioned?
    b. What percentage of the adults questioned had 0 children?

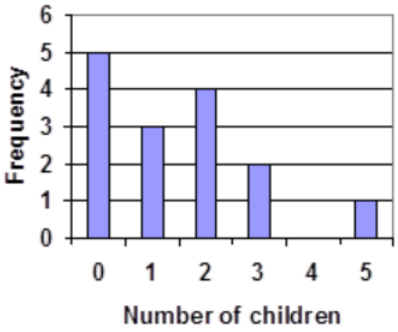


Figure 5.4.2

26. Jasmine was interested in how many days it would take an order from Netflix to arrive at her door. The graph below shows the data she collected.
    a. How many movies did she order?
    b. What percentage of the movies arrived in one day?



Figure 5.4.3

27. The bar graph below shows the percentage of students who received each letter grade on their last English paper. The class contains 20 students. What number of students earned an A on their paper?

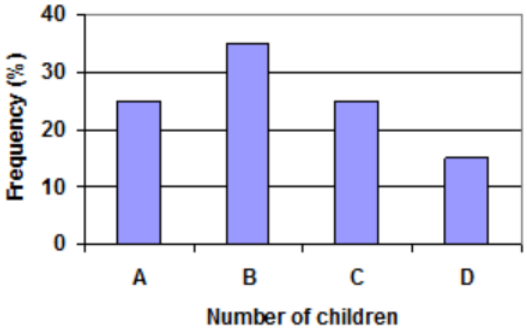


Figure 5.4.4

28. Kori categorized her spending for this month into four categories: Rent, Food, Fun, and Other. The percents she spent in each category are pictured here. If she spent a total of $2600 this month, how much did she spend on rent?

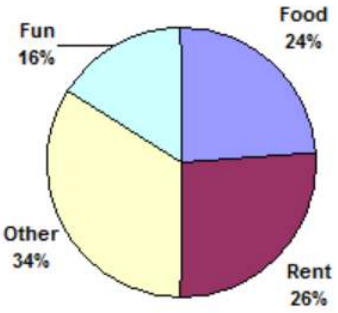


Figure 5.4.5

29. A group of diners were asked how much they would pay for a meal. Their responses were: $7.50, $8.25, $9.00, $8.00, $7.25, $7.50, $8.00, $7.00.
    a. Find the mean
    b. Find the median
    c. Write the 5-number summary for this data
    d. Find the standard deviation of this data
30. You recorded the time in seconds it took for 8 participants to solve a puzzle. The times were: 15.2, 18.8, 19.3, 19.7, 20.2, 21.8, 22.1, 29.4.
    a. Find the mean
    b. Find the median
    c. Write the 5-number summary for this data
    d. Find the standard deviation of this data
31. Refer back to the histogram from question #3.
    a. Compute the mean number of children for the group surveyed
    b. Compute the median number of children for the group surveyed
    c. Write the 5-number summary for this data.
    d. Create box plot.
32. Refer back to the histogram from question #4.
    a. Compute the mean number of shipping days
    b. Compute the median number of shipping days
    c. Write the 5-number summary for this data.
    d. Create box plot.
33. The box plot below shows salaries for Actuaries and CPAs. Kendra makes the median salary for an Actuary. Kelsey makes the first quartile salary for a CPA. Who makes more money? How much more?

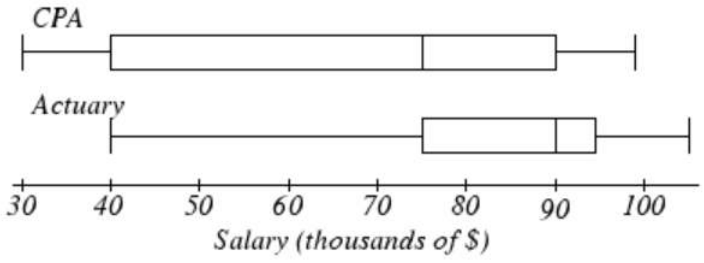


Figure 5.4.6

34. Referring to the boxplot above, what percentage of actuaries makes more than the median salary of a CPA?

Exploration

35. Studies are often done by pharmaceutical companies to determine the effectiveness of a treatment program. Suppose that a new AIDS antibody drug is currently under study. It is given to patients once the AIDS symptoms have revealed themselves. Of interest

is the average length of time in months patients live once starting the treatment. Two researchers each follow a different set of 40 AIDS patients from the start of treatment until their deaths. The following data (in months) are collected.

Researcher 1:

3; 4; 11; 15; 16; 17; 22; 44; 37; 16; 14; 24; 25; 15; 26; 27; 33; 29; 35; 44; 13; 21; 22; 10; 12; 8; 40; 32; 26; 27; 31; 34; 29; 17; 8; 24; 18; 47; 33; 34

Researcher 2:

3; 14; 11; 5; 16; 17; 28; 41; 31; 18; 14; 14; 26; 25; 21; 22; 31; 2; 35; 44; 23; 21; 21; 16; 12; 18; 41; 22; 16; 25; 33; 34; 29; 13; 18; 24; 23; 42; 33; 29

a. Create comparative histograms of the data
b. Create comparative boxplots of the data

36. A graph appears below showing the number of adults and children who prefer each type of soda. There were 130 adults and kids surveyed. Discuss some ways in which the graph below could be improved.

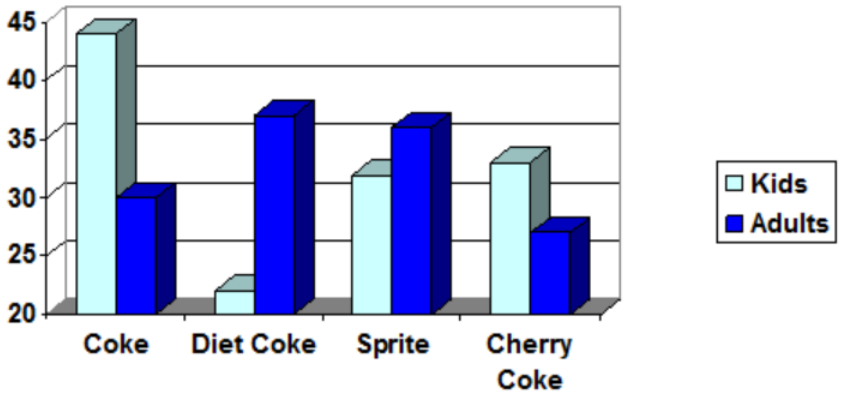


Figure 5.4.7

37. Make up three data sets with 5 numbers each that have:
    a. the same mean but different standard deviations.
    b. the same mean but different medians.
    c. the same median but different means.
38. A sample of 30 distance scores measured in yards has a mean of 7, a variance of 16, and a standard deviation of 4.
    a. You want to convert all your distances from yards to feet, so you multiply each score in the sample by 3. What are the new mean, median, variance, and standard deviation?
    b. You then decide that you only want to look at the distance past a certain point. Thus, after multiplying the original scores by 3, you decide to subtract 4 feet from each of the scores. Now what are the new mean, median, variance, and standard deviation?
39. Suppose that human pregnancies have an average duration of 270 days and standard deviation of 14 days. Find the probability that a pregnancy lasts for
    a. Less than 270 days
    b. Less than 263 days
    c. More than 277 days
40. Suppose that in a youth basketball league, the average points for a team was 63 per game, with standard deviation 5. What is the probability that a team ends a game with
    a. More than 68 points
    b. Less than 53 points
    c. More than 73 points

41. Suppose that the batting averages in baseball are normally distributed so that the mean is .258 and the standard deviation is 0.04. What is the probability that a player has a batting average of
    a. More than .258?
    b. More than .298?
    c. More than .318?
42. In a certain country, people's heights are normally distributed with a mean of 66 inches and a standard deviation of 3 inches. What is the probability that a person in that country is
    a. Shorter than 60 inches?
    b. Taller than 72 inches?
    c. Shorter than 57 inches?

Reference

1. References (12)

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CHAPTER OVERVIEW

6: Growth and Decay

6.1: Growth Models

6.2: Logarithmic Scales in Natural Sciences

6.3: The pH Scale

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6.1: Growth Models

Populations of people, animals, and items are growing all around us. By understanding how things grow, we can better understand what to expect in the future. In this chapter, we focus on time-dependent change.

Linear Growth

Marco is a collector of antique soda bottles. His collection currently contains 437 bottles. Every year, he budgets enough money to buy 32 new bottles. Can we determine how many bottles he will have in 5 years, and how long it will take for his collection to reach 1000 bottles?

While both of these questions you could probably solve without an equation or formal mathematics, we are going to formalize our approach to this problem to provide a means to answer more complicated questions.

Suppose that P_n represents the number, or population, of bottles Marco has after n years. So P_0 would represent the number of bottles now, P_1 would represent the number of bottles after 1 year, P_2 would represent the number of bottles after 2 years, and so on. We could describe how Marco's bottle collection is changing using:

$$\begin{aligned} P_0 &= 437 \\ P_n &= P_{n-1} + 32 \end{aligned}$$

This is called a **recursive relationship**. A recursive relationship is a formula which relates the next value in a sequence to the previous values. Here, the number of bottles in year n can be found by adding 32 to the number of bottles in the previous year, P_{n-1}. Using this relationship, we could calculate:

$$\begin{aligned} P_1 &= P_0 + 32 = 437 + 32 = 469 \\ P_2 &= P_1 + 32 = 469 + 32 = 501 \\ P_3 &= P_2 + 32 = 501 + 32 = 533 \\ P_4 &= P_3 + 32 = 533 + 32 = 565 \\ P_5 &= P_4 + 32 = 565 + 32 = 597 \end{aligned}$$

We have answered the question of how many bottles Marco will have in 5 years. However, solving how long it will take for his collection to reach 1000 bottles would require a lot more calculations.

While recursive relationships are excellent for describing simply and cleanly how a quantity is changing, they are not convenient for making predictions or solving problems that stretch far into the future. For that, a closed or explicit form for the relationship is preferred. An explicit equation allows us to calculate P_n directly, without needing to know P_{n-1}. While you may already be able to guess the **explicit equation**, let us derive it from the recursive formula. We can do so by selectively not simplifying as we go:

$$\begin{aligned} P_1 &= 437 + 32 && = 437 + 1(32) \\ P_2 &= P_1 + 32 = 437 + 32 + 32 && = 437 + 2(32) \\ P_3 &= P_2 + 32 = (437 + 2(32)) + 32 && = 437 + 3(32) \\ P_4 &= P_3 + 32 = (437 + 3(32)) && = 437 + 4(32) \end{aligned}$$

You can probably see the pattern now, and generalize that

$$P_n = 437 + n(32) = 437 + 32n$$

Using this equation, we can calculate how many bottles he'll have after 5 years:

$$P_5 = 437 + 32(5) = 437 + 160 = 597$$

We can now also solve for when the collection will reach 1000 bottles by substituting in 1000 for P_n and solving for n

$$\begin{aligned} 1000 &= 437 + 32n \\ 563 &= 32n \\ n &= 563/32 = 17.59 \end{aligned}$$

So Marco will reach 1000 bottles in 18 years.

In the previous example, Marco's collection grew by the *same number* of bottles every year. This constant change is the defining characteristic of linear growth. Plotting the values we calculated for Marco's collection, we can see the values form a straight line, the shape of linear growth.

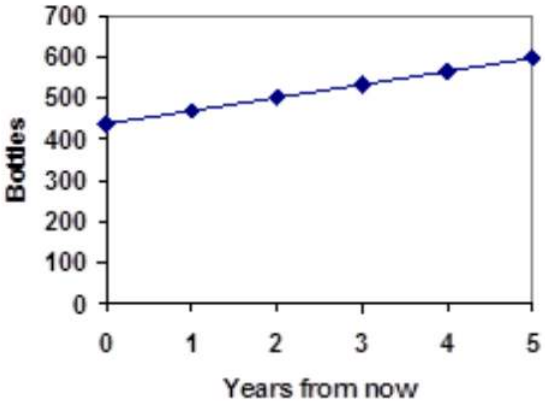


Figure 6.1.1: Marco's collection linear growth graph

Definition: Linear Growth

If a quantity starts at size P_0 and grows by d every time period, then the quantity after n time periods can be determined as follows:

$$P_n = P_0 + d \cdot n$$

In this equation, d represents the common difference, the amount that the population changes each time n increases by 1.

Connection to prior learning: slope and intercept

You may recognize the common difference, d, in our linear equation as slope. In fact, the entire explicit equation should look familiar – it is the same linear equation you learned in algebra, probably stated as $y = mx + b$.

In the standard algebraic equation $y = mx + b$, recall$b$ was the y-intercept, or the y value when x was zero. In the form of the equation here, we are using P_0 to represent that initial amount.

In the $y = mx + b$ equation, recall that m was the slope. You might remember this as "rise over run" or the change in y divided by the change in x. Either way, it represents the same thing as the common difference, d, we are using, the amount the output P_n changes when the input n increases by 1.

The equations $y = mx + b$ and $P_n = P_0 + d \cdot n$ mean the same thing and can be used the same ways. We're just writing it somewhat differently.

Example 6.1.1

The population of elk in a national forest was measured to be 12,000 in 2003 and was measured again to be 15,000 in 2007. If the population continues to grow linearly at this rate, what will the elk population be in 2014?

Solution

To begin, we need to define how we're going to measure n. Remember that P_0 is the population when $n = 0$, so we probably don't want to literally use the year 0. Since we already know the population in 2003, let us define $n = 0$ to be the year 2003. Then

$$P_0 = 12,000$$

Next we need to find d. Remember d is the growth per time period, in this case growth per year. Between the two measurements, the population grew by 15,000-12,000 = 3,000, but it took 2007-2003 = 4 years to grow that much. To find the growth per year, we can divide: 3000 elk / 4 years = 750 elk in 1 year.

Alternatively, you can use the slope formula from algebra to determine the common difference, noting that the population is the output of the formula, and time is the input.

$$d = \text{slope} = \frac{15,000 - 12,000}{2007 - 2003} = \frac{3000}{4} = 750$$

We can now write our equation.

$$P_n = 12,000 + 750n$$

To answer the question, we need to first note that the year 2014 will be $n = 11$, since 2014 is 11 years after 2003.

$$P_{11} = 12,000 + 750(11) = 20,250 \text{ elk}$$

Example 6.1.2

Gasoline consumption in the US has been increasing steadily. Consumption data from 1992 to 2004 is shown below. Find a model for this data and use it to predict consumption in 2023. If the trend continues, when will consumption reach 200 billion gallons?

Table 6.1.1

Year	'92	'93	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billion of gallons)	110	111	113	116	118	119	123	125	126	128	131	133	136

Solution

Plotting this data, it appears to have an approximately linear relationship:

While there are more advanced statistical techniques that can be used to find an equation to model the data, to get an idea of what is happening, we can find an equation by using two pieces of the data – perhaps the data from 1993 and 2003.

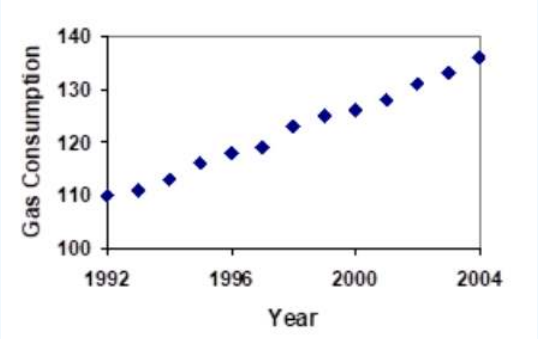


Figure 6.1.1: Gasoline consumption graph

Letting $n = 0$ correspond with 1993 would give $P_0 = 111$ billion gallons.

To find d, we need to know how much the gas consumption increased each year, on average. From 1993 to 2003 the gas consumption increased from 111 billion gallons to 133 billion gallons, a total change of 133 – 111 = 22 billion gallons, over 10 years.

This gives us an average change of 22 billion gallons / 10 year = 2.2 billion gallons per year.

Equivalently,

$$d = \text{slope} = \frac{\text{change in output}}{\text{change in input}} = \frac{133-111}{10-0} = \frac{22}{10} = 2.2 \text{ billion gallons per year}$$

$$P_n = 111 + 2.2n$$

Calculating values using the explicit form and plotting them with the original data shows how well our model fits the data.

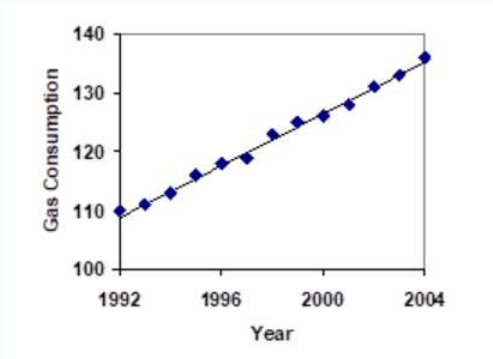


Figure 6.1.2: Calculating values using the explicit form and plotting them with the original data

We can now use our model to make predictions about the future, assuming that the previous trend continues unchanged. To predict the gasoline consumption in 2023:

n = 30 (2023-1993=30 years later)

$$P_{23} = 111 + 2.2(30) = 177$$

Our model predicts that the US will consume 177 billion gallons of gasoline in 2023 if the current trend continues.

To find when the consumption will reach 200 billion gallons, we would set $P_n = 200$, and solve for n:

$$\begin{aligned} P_n &= 200 && \text{Replace } P_n \text{ with our model} \\ 111 + 2.2n &= 200 && \text{Subtract 111 from both sides} \\ 2.2n &= 89 && \text{Divide both sides by 2.2} \\ n &= 40.4545 \end{aligned}$$

This tells us that consumption will reach 200 billion about 41 years after 1993, which would be in the year 2034.

By the way, the actual U.S. consumption in 2018 was 143 billion while this formula would give 166 billion. So, in reality, the growth rate appears to be going down a little.

Example 6.1.3

The cost, in dollars, of a gym membership for n months can be described by the explicit equation $P_n = 70 + 30n$. What does this equation tell us?

Solution

The value for P_0 in this equation is 70, so the initial starting cost is \$70. This tells us that there must be an initiation or start-up fee of \$70 to join the gym.

The value for d in the equation is 30, so the cost increases by \$30 each month. This tells us that the monthly membership fee for the gym is \$30 a month. This is a pretty good deal.

Try it Now 1

The number of stay-at-home fathers in Canada has been growing steadily. While the trend is not perfectly linear, it is fairly linear. Use the data from 1976 and 2010 to find an explicit formula for the number of stay-at-home fathers; then use it to predict the number in 2020.

Table 6.1.2

Year	Number of stay-at-home fathers

Year	Number of stay-at-home fathers
1976	20,610
1984	28,725
1991	43,530
2000	47,665
2010	53,555

When good models go bad

When using mathematical models to predict future behavior, it is important to keep in mind that very few trends will continue indefinitely.

Example 6.1.4

Suppose a four-year old boy is currently 39 inches tall, and you are told to expect him to grow 2.5 inches a year. Setup a growth model.

Solution

We can set up a growth model, with $n = 0$ corresponding to 4 years old.

$$P_n = 39 + 2.5n$$

So at 6 years old, we would expect him to be

$P_2 = 39 + 2.5(2) = 44$ inches tall

Most mathematical model will break down eventually. Certainly, we shouldn't expect this boy to continue to grow at the same rate all his life. If he did, at age 50 he would be

$P_{46} = 39 + 2.5(46) = 154$ inches tall = 12.8 feet tall!

Exponential Growth

Suppose that every year, only 10% of the fish in a lake have surviving offspring. Assuming that none of the adult fish dies, if there were 100 fish in the lake last year, there would now be 110 fish. If there were 1000 fish in the lake last year, there would now be 1100 fish. Absent of any inhibiting factors, populations of people and animals tend to grow by a percent of the existing population each year. Whenever a quantity grows by a fixed percentage rather than a fixed amount, the growth is exponential, like a compound interest.

Suppose our lake began with 1000 fish and 10% of the fish have surviving offspring each year. Since we start with 1000 fish, $P_0 = 1000$. How do we calculate P_1? The new population will be the old population plus an additional 10%. Symbolically:

$$P_1 = P_0 + 0.10P_0 \tag{6.1.1}$$

Notice this could be condensed to a shorter form by factoring:

$$P_1 = P_0 + 0.10P_0 = 1P_0 + 0.10P_0 = (1 + 0.10)P_0 = 1.10P_0 \tag{6.1.2}$$

While 10% is the **growth rate**, 1.10 is the **growth multiplier**. Notice that 1.10 can be thought of as "the original 100% plus an additional 10%", which is 110% (of the original population).

For our fish population,

$$P_0 = 1.10(1000) = 1100 \tag{6.1.3}$$

We could then calculate the population in later years:

$$P_2 = 1.10P_1 = 1.10(1100) = 1210 \tag{6.1.4}$$

$$P_3 = 1.10P_2 = 1.10(1210) = 1331 \tag{6.1.5}$$

Notice that in the first year, the population grew by 100 fish, in the second year, the population grew by 110 fish, and in the third year the population grew by 121 fish. While there is a constant *percentage* growth, the actual increase in number of fish is increasing each year.

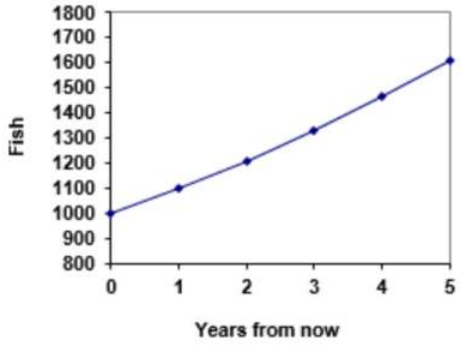


Figure 6.1.3: Fish growth graph

Graphing these values we see that this growth doesn't quite appear linear.

To get a better picture of how this percentage-based growth affects things, we need an explicit form, so we can quickly calculate values further out in the future.

Like we did for the linear model, we will start building from the recursive equation:

$$P_1 = 1.10P_0 = 1.10(1000)$$
$$P_2 = 1.10P_1 = 1.10(1.10(1000)) = 1.10^2(1000)$$
$$P_3 = 1.10P_2 = 1.10\left(1.10^2(1000)\right) = 1.00^3(1000)$$
$$P_4 = 1.10P_3 = 1.10\left(1.10^3(1000)\right) = 1.10^4(1000)$$

Observing a pattern, we can generalize the explicit form to be:

$P_n = 1.10^n(1000)$ or equivalently, $P_n = 1000\left(1.10^n\right)$. Note that this is exactly like the compound interest formula with $r = 10\%$ annually.

From this, we can quickly calculate the number of fish in 10, 20, or 30 years:

$$P_{10} = 1.10^{10}(1000) = 2594$$
$$P_{20} = 1.10^{20}(1000) = 6727$$
$$P_{30} = 1.10^{30}(1000) = 17,449$$

Adding these values to our graph reveals a shape that is definitely not linear. If our fish population had been growing linearly, by 100 fish each year, the population would have only reached 4000 in 30 years compared to almost 18,000 with this percent-based growth, called **exponential growth**.

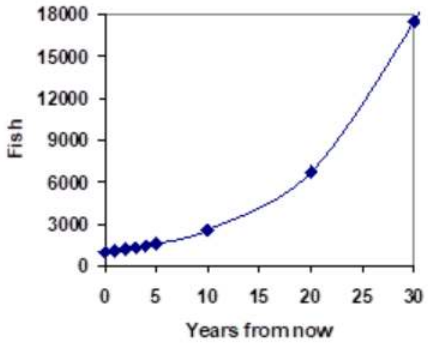


Figure 6.1.4: Fish population exponential growth graph

In exponential growth, the population grows proportional to the size of the population, so as the population gets larger, the same percent growth will yield a larger numeric growth.

Definition: Exponential Growth

If a quantity starts at size P_0 and grows by $R\%$ (written as a decimal, r) every time period, then the quantity after n time periods can be determined as follows:

$$P_n = P_0(1+r)^n$$

We call r the growth rate.

Example 6.1.5

Between 2007 and 2008, Olympia, WA grew almost 3% to a population of 245 thousand people. If this growth rate was to continue, what would the population of Olympia be in 2014?

Solution

As we did before, we first need to define what year will correspond to $n = 0$. Since we know the population in 2008, it would make sense to have 2008 correspond to $n = 0$, so $P_0 = 245,000$. The year 2014 would then be $n = 6$.

We know the growth rate is 3%, giving $r = 0.03$.

Using the explicit form:

$$P_6 = (1+0.03)^6(245,000) = 1.19405(245,000) = 292,542.25$$

The model predicts that in 2014, Olympia would have a population of about 293 thousand people.

Evaluating Exponents on the Calculator

To evaluate expressions like $(1.03)^6$, it will be easier to use a calculator than multiply 1.03 by itself six times. Most scientific calculators have a button for exponents. It is typically labeled:

$$\wedge, \quad y^x, \text{ or } \quad x^y$$

To evaluate 1.036 we'd type 1.03 ^ 6, or 1.03 y^x 6.

Try it out. You should get an answer around 1.1940523.

Try it Now 2

India is the second most populous country in the world, with a population in 2008 of about 1.14 billion people. The population is growing by about 1.34% each year. If this trend continues, what will India's population grow to by 2020?

Example 6.1.6

A friend is using the equation $P_n = 4600(1.072)^n$ to predict the annual tuition at a local college. She says the formula is based on years after 2010. What does this equation tell us?

Solution

In the equation, $P_0 = 4600$, which is the starting value of the tuition when $n = 0$. This tells us that the tuition in 2010 was \$4,600.

The growth multiplier is 1.072, so the growth rate is 0.072, or 7.2%. This tells us that the tuition is expected to grow by 7.2% each year.

Putting this together, we could say that the tuition in 2010 was \$4,600 and is expected to grow by 7.2% each year.

Example 6.1.7

In 1990, the residential energy use in the US was responsible for 962 million metric tons of carbon dioxide emissions. By the year 2000, that number had risen to 1182 million metric tons. If the emissions grow exponentially and continue at the same rate, what will the emissions grow to by 2050?

Solution

Similar to before, we will correspond $n = 0$ with 1990, as that is the year for the first piece of data we have. That will make $P_0 = 962$ (million metric tons of CO_2). In this problem, we are not given the growth rate, but instead are given that $P_{10} = 1182$.

When $n = 10$, the explicit equation looks like:

$$P_{10} = (1 + r)^{10} P_0$$

We know the value for P_0, so we can put that into the equation:

$$P_{10} = (1 + r)^{10} 962$$

We also know that $P_0 = 1182$, so substituting that in, we get

$$1182 = (1 + r)^{10} 962$$

We can now solve this equation for the growth rate, r. Start by dividing by 962.

$$\frac{1182}{962} = (1 + r)^{10} \quad \text{Take the 10th root of both sides}$$

$$\sqrt[10]{\frac{1182}{962}} = 1 + r \quad \text{Subtract 1 from both sides}$$

$$r = \sqrt[10]{\frac{1182}{962}} - 1 = 0.0208 = 2.08\%$$

So if the emissions are growing exponentially, they are growing by about 2.08% per year. We can now predict the emissions in 2050 by finding P_{60}.

$P_{60} = (1 + 0.0208)^{60} 962 = 3308.4$ million metric tons of CO_2 in 2050 (That's 3.3 billion metric tons!)

Rounding

As a note on rounding, notice that if we had rounded the growth rate to 2.1%, our calculation for the emissions in 2050 would have been 3347. Rounding to 2% would have changed our result to 3156. A very small difference in the growth rates gets magnified greatly in exponential growth. For this reason, it is recommended to round the growth rate as little as possible.

Evaluating Roots on the Calculator

In the previous example, we had to calculate the 10th root of a number. This is different from taking the basic square root, $\sqrt{}$. Many scientific calculators have a button for general roots. It is typically labeled like

$$\sqrt[n]{}, \sqrt[x]{}, \text{ or } \sqrt[y]{x}$$

To evaluate the 3rd root of 8, for example, we'd either type $3\sqrt[x]{}8$, or $8\sqrt[x]{}3$, depending on the calculator. Try it on yours to see which to use – you should get an answer of 2. If your calculator does not have a general root button, all is not lost. You can instead use the property of exponents which states that $\sqrt[n]{a} = a^1/n$. So, to compute the 3rd root of 8, you could use your calculator's exponent key to evaluate $8^{1/3}$. To do this, type: $8y^x(1 \div 3)$ The parentheses tell the calculator to divide 1/3 before doing the exponent.

Try it Now 3

The number of users on a social networking site was 45 thousand in February when they officially went public and grew to 60 thousand by October. If the site is growing exponentially, and growth continues at the same rate, how many users should they

expect two years after they went public?

Example 6.1.8

Looking back at the last example, for the sake of comparison, what would the carbon emissions be in 2050 if emissions grow linearly at the same rate?

Solution

Again we will get $n = 0$ correspond with 1990, giving $P_0 = 962$. To find d, we could take the same approach as earlier, noting that the emissions increased by 220 million metric tons in 10 years, giving a common difference of 22 million metric tons each year.

Alternatively, we could use an approach similar to that which we used to find the exponential equation. When $n = 10$, the explicit linear equation looks like:

$$P_{10} = P_0 + 10d$$

We know the value for P_0, so we can put that into the equation:

$$P_{10} = 962 + 10d$$

Since we know that $P_{10} = 1182$, substituting that in we get

$$1182 = 962 + 10d$$

We can now solve this equation for the common difference, d.

$$1182 - 962 = 10d$$
$$220 = 10d$$
$$d = 22$$

This tells us that if the emissions are changing linearly, they are growing by 22 million metric tons each year. Predicting the emissions in 2050,

$P_{60} = 962 + 22(60) = 2282$ million metric tons.

You will notice that this number is substantially smaller than the prediction from the exponential growth model. Calculating and plotting more values helps illustrate the differences.

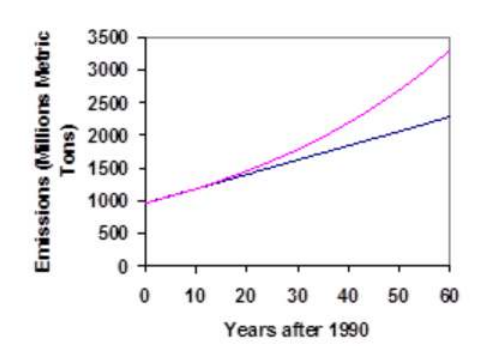


Figure 6.1.5: Predicting carbon emissions growth

So how do we know which growth model to use when working with data? There are two approaches which should be used together whenever possible:

1. Find more than two pieces of data. Plot the values, and look for a trend. Does the data appear to be changing like a line, or do the values appear to be curving upwards?
2. Consider the factors contributing to the data. Are they things you would expect to change linearly or exponentially? For example, in the case of carbon emissions, we could expect that, absent other factors, they would be tied closely to population values, which tend to change exponentially.

Reference

1. References (13)

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6.2: Logarithmic Scales in Natural Sciences



Figure 6.2.1: Image by Daniel Pierce is licensed by CC 2.0

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes[1]. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings,[2] like those shown in the image above. Even though both caused substantial damage, the earthquake in Japan was 100 times as strong as the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale[3] whereas the Japanese earthquake registered a 9.0.[4]

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is 108-4=104=10,000 times as great! In this section, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ increasing. We can examine a graph, below, to better estimate the solution.

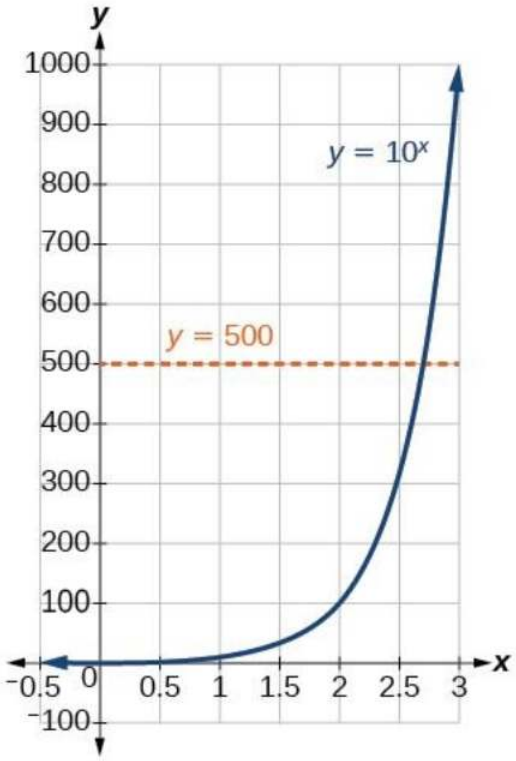


Figure 6.2.2: Graph to better estimate the solution.

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base-10 logarithm a **common logarithm**. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

Definition: Common Logarithm

A common logarithm is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$

$y = \log(x)$ is equivalent to $10^y = x$

We read $\log(x)$ as, "the logarithm with base 10 of x" or "log base 10 of x."

The logarithm y is the exponent to which 10 must be raised to get x.

How to

Given a common logarithm of the form $y = \log(x)$, evaluate it mentally.

1. Rewrite the argument x as a power of 10 : $10^y = x$.
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get y?

Example 6.2.1: Finding value of a common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know

$$10^3 = 1000$$
$$\therefore \log(1000) = 3$$

Try it Now 4

Evaluate $y = \log(1,000,000)$

How to

Given a common logarithm with the form $y = \log(x)$, evaluate it using a calculator.

1. Press [LOG].
2. Enter the value given for x followed by [ )].
3. Press [ENTER].

Example 6.2.2: Finding value of a common Logarithm using a Calculator

Evaluate $y = \log(321)$ to four decimal places using a calculator.

Solution

- Press **[LOG]**.
- Enter 321, followed by **[ )]**.
- Press **[ENTER]**.

Rounding to four decimal places, $\log(321) \approx 2.5065$

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$ must be between $\log(100)$ and $\log(1000)$ This gives us the following:

$$100 < 321 < 1000$$
$$2 < 205065 < 3$$

Try it Now 5

Evaluate $y = \log(123)$ to four decimal places using a calculator.

Example 6.2.3

The 1994 Northridge Earthquake in Southern California registered a 6.7 on the Richter scale. Exactly a year later, the 1995 Kobe Earthquake in Japan registered a 6.9. Compare the amount of energy released from these two quakes.

Solution

$$10^{6.9-6.7} = 10^{0.2} = 1.585$$

The Kobe Earthquake released about 1.56 times the energy (about 56% more) released by the Northridge quake.

Example 6.2.4: Rewriting and solving a real-world exponential model

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Solution

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$
$$\log(500) = x \quad \text{Use the definition of the common log.}$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500, followed by **[ )]**.
- Press **[ENTER]**.
- To the nearest thousandth, $\log(500) \approx 2.699$

The difference in magnitudes was about 2.699. This means that the bigger earthquake has the magnitude about 2.7 more than the magnitude of the smaller earthquake.

Try it Now 6

The amount of energy released from one earthquake was 8,500 times as great as the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Reference

1. References (14)

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6.3: The pH Scale

You may have heard of pH levels in chemistry. pH is a measure of how acidic or basic a solution is. The pH scale is actually a logarithmic scale, similar to the Richter scale studied in the previous section. In chemistry, the level of acidity depends on the molar concentration of hydrogen ions in a given solution. The more hydrogen ions there are, the more acidic the solution is. But the concentrations are really, really small fractions, like 1/10,000,000 for pure water. Because this denominator ranges drastically, it is convenient to use a logarithmic scale.

pH is a logarithmic function of $[H^+]$, where $[H^+]$ is the concentration of the hydrogen ion $[H^+]$.

$$pH = -\log[H^+] \tag{6.3.1}$$

pH is usually between 0 and 14. Knowing the dependence of pH on $[\ H^+\ ]$, we can summarize as follows:

- If pH < 7, then the solution is acidic.
- If pH = 7, then the solution is neutral.
- If pH > 7, then the solution is basic.

This is known as the pH scale. You can use pH to make a quick determination of whether a given water based solution is acidic, basic, or neutral.

Example 6.3.1

Label each solution as acidic, basic, or neutral based only on the stated pH.

a. milk of magnesia, pH = 10.5
b. pure water, pH = 7
c. wine, pH = 3.0

Solution

a. With a pH greater than 7, milk of magnesia is basic. (Milk of magnesia is largely $Mg(OH)_2$.)
b. Pure water, with a pH of 7, is neutral.
c. With a pH of less than 7, wine is acidic.

Example 6.3.2

Identify each substance as acidic, basic, or neutral based only on the stated pH.

a. human blood, pH = 7.4
b. household ammonia, pH = 11.0
c. cherries, pH = 3.6

Solution

a. basic
b. basic (very)
c. acidic

The table below gives the typical pH values of some common substances. Note that several food items are on the list, and most of them are acidic.

Table 6.3.1: Typical pH Values of Various Substances*

Substance	pH
stomach acid	1.7
lemon juice	2.2
vinegar	2.9
soda	3.0

Substance	pH
wine	3.5
coffee, black	5.0
milk	6.9
pure water	7.0
blood	7.4
seawater	8.5
milk of magnesia	10.5
ammonia solution	12.5
1.0 M NaOH	14.0
*Actual values may vary depending on conditions.	

pH is a *logarithmic* scale, a solution that has a pH of 1.0 has 10 times the $[H^+]$ as a solution with a pH of 2.0, which in turn has 10 times the $[H^+]$ as a solution with a pH of 3.0 and so forth.

Using the definition of pH, it is also possible to calculate $[H^+]$ from pH and vice versa. The general formula for determining $[H^+]$ from pH is as follows:

$$\left[H^+\right] = 10^{-pH}$$

You need to determine how to evaluate the above expression on your calculator. Ask your instructor if you have any questions. The other issue that concerns us here is significant figures. Because the number(s) before the decimal point in a logarithm relate to the power on 10, the number of digits *after* the decimal point is what determines the number of significant figures in the final answer.

Example 6.3.2

What is $[H^+]$ for an acid solution whose pH is 4.88?

Solution

We need to evaluate the expression

$$\left[H^+\right] = 10^{-4.88}$$

Depending on the calculator you use, the method for solving this problem will vary. In some cases, the "−4.88" is entered and a "10^x" key is pressed; for other calculators, the sequence of keystrokes is reversed. In any case, the correct numerical answer is as follows:

$$\left[H^+\right] = 10^{-4.88} = 0.0001318\ldots \approx 1.32 \times 10^{-5}$$

Exercises

1. Marko currently has 20 tulips in his yard. Each year he plants 5 more.
   a. Write a recursive formula for the number of tulips Marko has
   b. Write an explicit formula for the number of tulips Marko has
2. Pam is a Disc Jockey. Every week she buys 3 new albums to keep her collection current. She currently owns 450 albums.
   1. Write a recursive formula for the number of albums Pam has
   2. Write an explicit formula for the number of albums Pam has
3. A store's sales (in thousands of dollars) grow according to the recursive rule $P_n = P_{n-1} + 15$, with initial population $P_0 = 40$.
   a. Calculate P_1 and P_2
   b. Find an explicit formula for P_n

c. Use your formula to predict the store's sales in 10 years
d. When will the store's sales exceed $100,000?

4. The number of houses in a town has been growing according to the recursive rule $P_n = P_{n-1} + 30$, with initial population $P_0 = 200$.
   a. Calculate P_1 and P_2
   b. Find an explicit formula for P_n
   c. Use your formula to predict the number of houses in 10 years
   d. When will the number of houses reach 400 houses?
5. A population of beetles is growing according to a linear growth model. The initial population (week 0) was $P_0 = 3$, and the population after 8 weeks is $P_8 = 67$.
   a. Find an explicit formula for the beetle population in week n.
   b. After how many weeks will the beetle population reach 187?
6. The number of streetlights in a town is growing linearly. Four months ago ($n = 0$) there were 130 lights. Now ($n = 4$) there are 146 lights. If this trend continues,
   a. Find an explicit formula for the number of lights in month n.
   b. How many months will it take to reach 200 lights?
7. Tacoma's population in 2000 was about 200 thousand, and had been growing by about 9% each year.
   a. Write a recursive formula for the population of Tacoma
   b. Write an explicit formula for the population of Tacoma
   c. If this trend continues, what will Tacoma's population be in 2016?
   d. When does this model predict Tacoma's population to exceed 400 thousand?
8. Portland's population in 2007 was about 568 thousand, and had been growing by about 1.1% each year.
   a. Write a recursive formula for the population of Portland
   b. Write an explicit formula for the population of Portland
   c. If this trend continues, what will Portland's population be in 2016?
   d. If this trend continues, when will Portland's population reach 700 thousand?
9. Diseases tend to spread according to the exponential growth model. In the early days of AIDS, the growth rate was around 190%. In 1983, about 1700 people in the U.S. died of AIDS. If the trend had continued unchecked, how many people would have died from AIDS in 2005?
10. The population of the world in 1987 was 5 billion and the annual growth rate was estimated at 2 percent per year. Assuming that the world population follows an exponential growth model, find the projected world population in 2015.
11. A bacteria culture is started with 300 bacteria. After 4 hours, the population has grown to 500 bacteria. If the population grows exponentially,
    a. Write a recursive formula for the number of bacteria
    b. Write an explicit formula for the number of bacteria
    c. If this trend continues, how many bacteria will there be in 1 day?
    d. How long does it take for the culture to triple in size?
12. A native wolf species has been reintroduced into a national forest. Originally 200 wolves were transplanted. After 3 years, the population had grown to 270 wolves. If the population grows exponentially,
    a. Write a recursive formula for the number of wolves
    b. Write an explicit formula for the number of wolves
    c. If this trend continues, how many wolves will there be in 10 years?
    d. If this trend continues, how long will it take the population to grow to 1000 wolves?
13. One hundred trout are seeded into a lake. Absent constraint, their population will grow by 70% a year. The lake can sustain a maximum of 2000 trout. Using the logistic growth model,
    a. Write a recursive formula for the number of trout
    b. Calculate the number of trout after 1 year and after 2 years.
14. Ten blackberry plants started growing in my yard. Absent constraint, blackberries will spread by 200% a month. My yard can only sustain about 50 plants. Using the logistic growth model,

a. Write a recursive formula for the number of blackberry plants in my yard
b. Calculate the number of plants after 1, 2, and 3 months

15. In 1968, the U.S. minimum wage was $1.60 per hour. In 1976, the minimum wage was $2.30 per hour. Assume the minimum wage grows according to an exponential model where n represents the time in years after 1960.
    a. Find an explicit formula for the minimum wage.
    b. What does the model predict for the minimum wage in 1960?
    c. If the minimum wage was $5.15 in 1996, is this above, below or equal to what the model predicts?
16. The population of a small town can be described by the equation $P_n = 4000 + 70n$, where n is the number of years after 2005. Explain in words what this equation tells us about how the population is changing.
17. The population of a small town can be described by the equation $P_n = 4000(1.04)^n$, where n is the number of years after 2005. Explain in words what this equation tells us about how the population is changing.

Most of the examples in the text examined growing quantities, but linear and exponential equations can also describe decreasing quantities, as the next few problems will explore.

18. A new truck costs $32,000. The car's value will depreciate over time, which means it will lose value. For tax purposes, depreciation is usually calculated linearly. If the truck is worth $24,500 after three years, write an explicit formula for the value of the car after n years.
19. Inflation causes things to cost more, and for our money to buy less (hence your grandparents saying, "In my day, you could buy a cup of coffee for a nickel"). Suppose inflation decreases the value of money by 5% each year. In other words, if you have $1 this year, next year it will only buy you $0.95 worth of stuff. How much will $100 buy you in 20 years?
20. Suppose that you have a bowl of 500 M&M candies, and each day you eat ¼ of the candies you have. Is the number of candies left changing linearly or exponentially? Write an equation to model the number of candies left after n days.
21. A warm object in a cooler room will decrease in temperature exponentially, approaching the room temperature according to the formula $T_n = a(1-r)^n + T_r$ where T_n is the temperature after n minutes, r is the rate at which temperature is changing, a is a constant, and T_r is the temperature of the room. Forensic investigators can use this to predict the time of death of a homicide victim. Suppose that when a body was discovered ($n = 0$) it was 85 degrees. After 20 minutes, the temperature was measured again to be 80 degrees. The body was in a 70 degree room.
    a. Use the given information with the formula provided to find a formula for the temperature of the body.
    b. When did the victim die, if the body started at 98.6 degrees?
22. Recursive equations can be very handy for modeling complicated situations for which explicit equations would be hard to interpret. As an example, consider a lake in which 2000 fish currently reside. The fish population grows by 10% each year, but every year 100 fish are harvested from the lake by people fishing.
    a. Write a recursive equation for the number of fish in the lake after n years.
    b. Calculate the population after 1 and 2 years. Does the population appear to be increasing or decreasing?
    c. What is the maximum number of fish that could be harvested each year without causing the fish population to decrease in the long run?
23. The number of Starbucks stores grew after first opened. The number of stores from 1990-2007, as reported on their corporate website,[2] is shown below.
    a. Carefully plot the data. Does is appear to be changing linearly or exponentially?
    b. Try finding an equation to model the data by picking two points to work from. How well does the equation model the data?
    c. Try using an equation of the form $P_n = P_0 n^k$, where k is a constant, to model the data. This type of model is called a Power model. Compare your results to the results from part b. *Note: to use this model, you will need to have 1990 correspond with $n = 1$ rather than $n = 0$.*

Table 6.3.2: Starbucks stores 1997-2007

Year	Number of Starbucks stores
1990	84
1991	116
1992	165

Year	Number of Starbucks stores
1993	272
1994	425
1995	677
1996	1015
1997	1412
1998	1886
1999	2498
2000	3501
2001	4709
2002	5886
2003	7225
2004	8569
2005	10241
2006	12440
2007	15756

24. Thomas Malthus was an economist who put forth the principle that population grows based on an exponential growth model, while food and resources grow based on a linear growth model. Based on this, Malthus predicted that eventually demand for food and resources would out outgrow supply, with doom-and-gloom consequences. Do some research about Malthus to answer these questions.
    a. What societal changes did Malthus propose to avoid the doom-and-gloom outcome he was predicting?
    b. Why do you think his predictions did not occur?
    c. What are the similarities and differences between Malthus's theory and the logistic growth model?
25. What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^y = x$ and $\log_b x = y$ for $b > 0, b \neq 1$
26. How is the logarithmic function $f(x) = \log_b x$ related to the exponential function $g(x) = b^x$ What is the result of composing these two functions?
    a. How can the logarithmic equation $\log_b x = y$ be solved using the properties of exponents?
27. Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?
    a. Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b, and how does the notation differ?

For the following exercises, rewrite each equation in exponential form.

28. $\log_4(q) = m$
    a. $\log_a(b) = c$
29. $\log_{16}(y) = x$
    a. $\log_x(64) = y$
30. $\log_{16}(y) = x$
    a. $\log_x(64) = y$
31. $\log_y(x) = -11$
    a. $\log_{15}(a) = b$

32. $\log_y(137) = x$
    a. $\log_{13}(142) = a$
33. $\log(v) = t$
    a. $\ln(w) = n$

For the following exercises, rewrite each equation in logarithmic form.

34. The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$, where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2 seconds. What will the resulting exposure index be?
    a. Suppose the light meter on a camera indicates an EI of -2, and the desired exposure time is 16 seconds. What should the f-stop setting be?
35. Describe, using your own words, what happens to the energy of an earthquake each time the magnitude on the Richter scale increases by 1.
36. Remember that $\log(M)$ is the exponent so that 10 to that exponent is M. Given this fact, find
    a. $\log(1000)$
    b. $\log(10)$
    c. $\log(10,000,000)$
37. As mentioned in the text, the 1994 Northridge Earthquake registered a 6.7 on the Richter scale. In July, 2019, there was a major earthquake in Ridgecrest, California, with a magnitude of 7.1. How much bigger was the Ridgecrest quake compared to the Northridge Earthquake 25 years before (in terms of the energy released)? Use a calculator.
38. The pH level of lemon juice is 2.2 while the pH level of a typical wine is 3.5. They are both acidic, but one is much more acid than the other.
    a. Which one is more acidic?
    b. Compare the concentration of hydrogen ions between these two liquids. (How many times as much?) Hint: use the same method applied in the previous section to compare two earthquakes. The mathematics is the same.
39. Many orange farmers pay attention to the pH level of their oranges. If an orange produces a juice with a pH level of 4.0, how much is the concentration of hydrogen ions in that juice compared a lemon juice with a pH level of 2.2?

Reference

1. References (15)
2. www.starbucks.com/aboutus/Company_Timeline.pdf retrieved May 2009

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CHAPTER OVERVIEW

7: Mathematics and the Arts

7.1: Projective Geometry
7.2: The Golden Ratio and Fibonacci Sequence
7.3: Musical Scales
7.4: Fractals

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7.1: Projective Geometry

Before drawing a 3-dimensional object on a flat surface, such as a sketch book or a canvas, one needs to learn some principles of perspective. This skill, based on projective geometry and developed during the Renaissance, allows the artist to draw things more realistically. All of us have seen, for instance, pictures of railroad tracks and straight highways where the two lines, though parallel in space, appear to converge to a point, referred to as the principal vanishing point.

Perhaps you have seen the famous art piece, *The Last Supper*, by Leonardo da Vinci (1452—1519). In that picture, the parallel lines on the ceiling are drawn in such a way that they, if extended, would converge in the middle of the picture, directly behind the head of Christ.

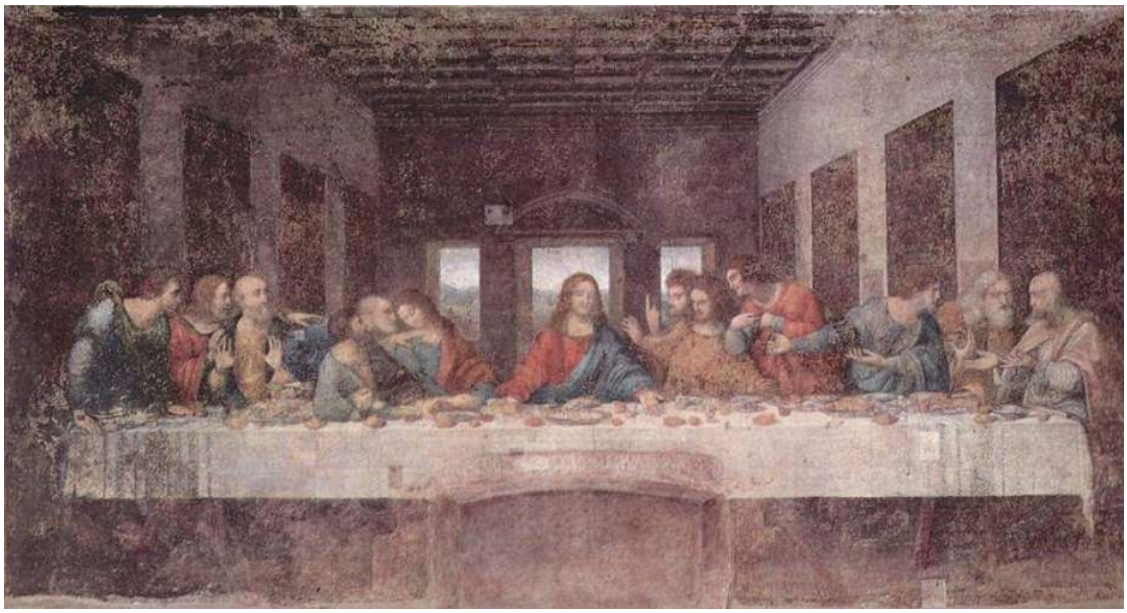


Figure 7.1.1: The Last Supper, by Leonardo da Vinci is in the public domain

During the 15th Century, many more artists started to use perspective, practicing it by sketching polyhedral and buildings, while exploring the mathematical basis and rules for such a technique. It was no surprise that da Vinci was also an accomplished mathematician, scientist, and engineer. Interested readers can look up other painters who contributed to the geometric theory of perspective, such as Filippo Brunelleschi (1377—1446), Leon Battista Alberti (1404—1472), Paolo Uccello (1397—1475), Albrecht Dürer (1471—1528), and Wenzel Jamnitzer (1508—1585).

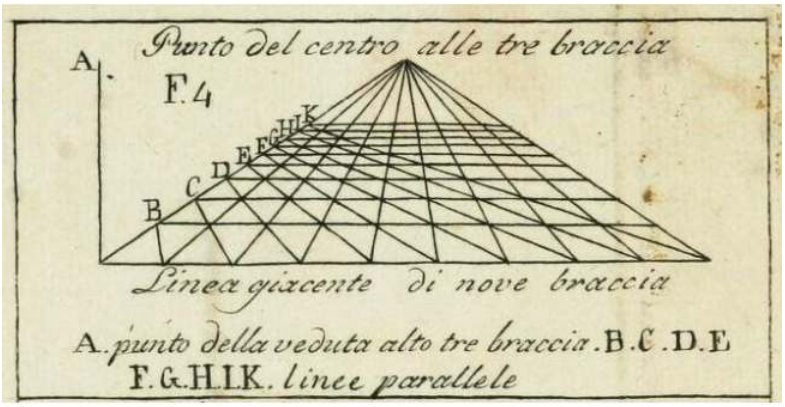


Figure 7.1.2: Leon B. Alberti's study on perspective is in the public domain.

In particular, Alberti is famous for the concept of "visual pyramid," which gives us the foundation for perspective, as seen in this picture:

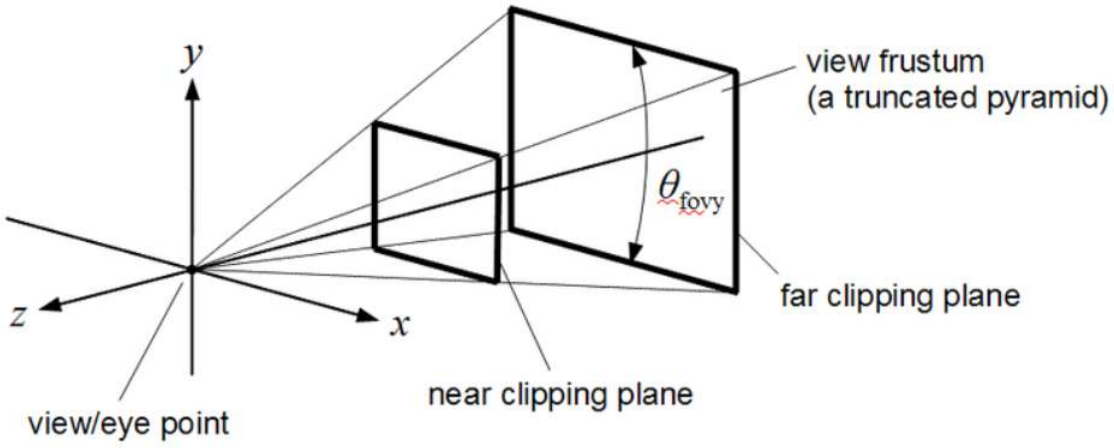


Figure 7.1.3: Visual pyramid (CC-BY-SA image by Martin Kraus)

Without going into all the details developed over the centuries, we now state some basic principles of perspective.



Figure 7.1.4: Image by Moajjem Hossain is licensed by CC-BY-SA-4.0

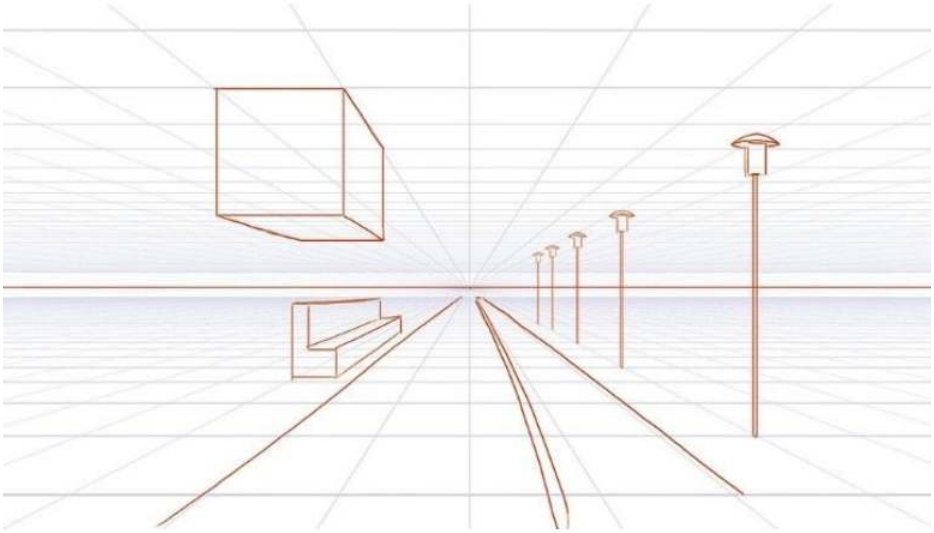


Figure 7.1.5: Image by mellowchu

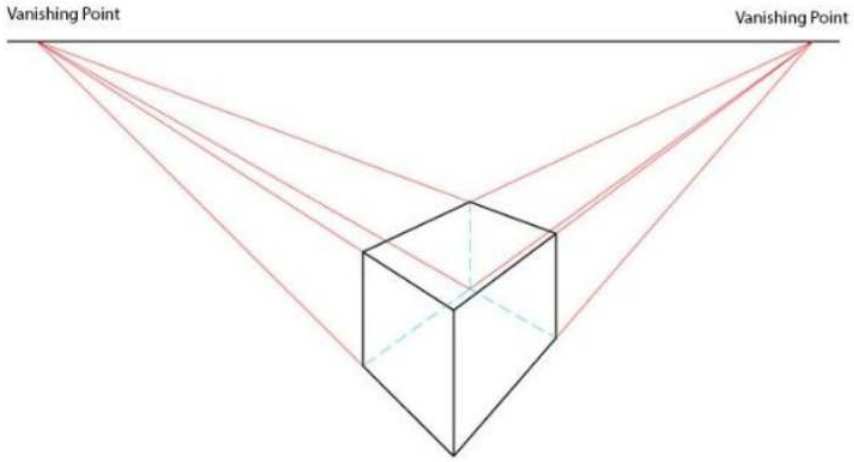


Figure 7.1.6: Image by Wikimedia Commons is licensed by CC-3.0

Principles of Perspective

1. As the first two figures show, all lines perpendicular to the canvas (the plane on which the objectives are drawn) are parallel in space and will converge at a point, called the “principal vanishing point.”
2. All lines parallel to the canvas, such as railroad crossties (wooden pieces below the railroad track) and horizontal edges of the floor tiles, will not converge on the canvas but stay parallel and horizontal on the canvas.
3. All other lines parallel in space will converge at their vanishing points, as shown on the third picture, and all of the vanishing points are on the horizontal line (which also contains the principal vanishing point).

Knowing these three principles, or rules, of perspective will help you draw objectives in 3-dimensional space in a much more realistic way.

Reference

1. References (16)

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7.2: The Golden Ratio and Fibonacci Sequence

In this section, we will discuss a very special number called the Golden Ratio. It is an irrational number, slightly bigger than 1.6, and it has (somewhat surprisingly) had huge significance in the world of science, art and music. It was also discovered that this number has an amazing connection with what is called the Fibonacci Sequence, originally studied in the context of biology centuries ago. This unexpected link among algebra, biology, and the arts suggests the mathematical unity of the world and is sometimes discussed in philosophy as well.

Golden Ratio

With one number a and another smaller number b, the ratio of the two numbers is found by dividing them. Their ratio is a/b. Another ratio is found by adding the two numbers together $a+b$ and dividing this by the larger number a. The new ratio is $(a+b)/a$. If these two ratios are equal to the same number, then that number is called the Golden Ratio. The Greek letter φ (phi) is usually used to denote the Golden Ratio.

For example, if $b=1$ and $a/b=\varphi$, then $a=\varphi$. The second ratio $(a+b)/a$ is then $(\varphi+1)/\varphi$. Because these two ratios are equal, this is true:

$$\varphi=\frac{\varphi+1}{\varphi}$$

(This equation has two solutions, but only the positive solution is referred to as the Golden Ratio φ).

One way to write this number is

$$\varphi=\frac{1+\sqrt{5}}{2}$$

$\sqrt{5}$ is the positive number which, when multiplied by itself, makes 5 : $\sqrt{5}\times\sqrt{5}=5$.

The Golden Ratio is an irrational number. If a person tries to write the decimal representation of it, it will never stop and never make a pattern, but it will start this way: 1.6180339887... An interesting thing about this number is that you can subtract 1 from it or divide 1 by it, and the result will be the same.

$$\varphi-1=1.6180339887\ldots-1=0.6180339887$$

$$1/\varphi=\frac{1}{1.6180339887}=0.6180339887$$

Golden rectangle

If the length of a rectangle divided by its width is equal to the Golden Ratio, then the rectangle is called a "golden rectangle." If a square is cut off from one end of a golden rectangle, then the other end is a new golden rectangle. In the picture, the big rectangle (blue and pink together) is a golden rectangle because $a/b=\varphi$. The blue part (B) is a square. The pink part by itself (A) is another golden rectangle because $b/(a-b)=\varphi$.

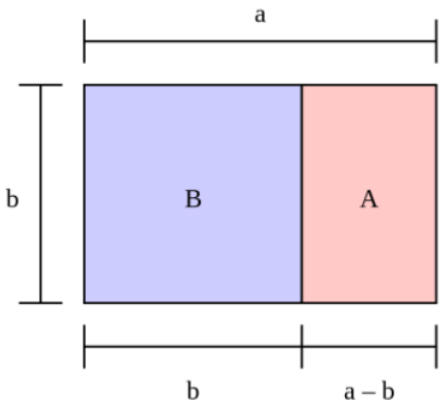


Figure 7.2.1: Image by Peter John Acklam is licensed by CC-3.0

Assume that $\varphi = \frac{a}{b}$, and φ is the positive solution to $\varphi^2 - \varphi - 1 = 0$. Then, $\frac{a^2}{b^2} - \frac{a}{b} - \frac{b}{b} = 0$. Multiply by b^2, $a^2 - ab - b^2 = 0$. So, $a^2 - ab = b^2$. Thus, $a(a-b) = b^2$. We then get $\frac{a}{b} = \frac{b}{a-b}$. Both sides are φ.

Fibonacci Sequence

The Fibonacci sequence is a list of numbers. Start with 1, 1, and then you can find the next number in the list by adding the last two numbers together. The resulting (infinite) sequence is called the Fibonacci Sequence. Since we start with 1, 1, the next number is 1+1=2. We now have 1, 1, 2. The next number is 1+2=3. We now have 1, 1, 2, 3. The next number is 2+3=5. The next one is 3+5=8, and so on. Each of these numbers is called a Fibonacci number. Originally, Fibonacci (Leonardo of Pisa, who lived some 800 years ago) came up with this sequence to study rabbit populations! He probably had no idea what would happen when you divide each Fibonacci number by the previous one, as seen below.

Table 7.2.1:

Fibonacci number	divided by the one before	ratio
1		
1	1/1	= 1.0000
2	2/1	= 2.0000
3	3/2	= 1.5000
5	5/3	= 1.6667
8	8/5	= 1.6000
13	13/8	= 1.6250
21	21/13	= 1.6154...
34	34/21	= 1.6190...
55	55/34	= 1.6177...
89	89/55	= 1.6182...
...	...	...
		= 1.6180...

Here is a very surprising fact:

Note

The ratio of two consecutive Fibonacci numbers approaches the Golden Ratio.

It turns out that Fibonacci numbers show up quite often in nature. Some examples are the pattern of leaves on a stem, the parts of a pineapple, the flowering of artichoke, the uncurling of a fern and the arrangement of a pine cone. The Fibonacci numbers are also found in the family tree of honeybees.

Meanwhile, many artists and music researchers have studied artistic works in which the Golden Ratio plays an integral role. These include the works of Michelangelo, Da Vinci, and Mozart. Interested readers can find many resources and videos online. Perhaps it is not surprising that numbers like 3, 5, 8, and 13 are rather important in music theory; just take a quick look at the piano keys!

Math in the Real World -

Ctrl click here to watch a short video that explores the golden ratio in nature, art, and society.

Reference

1. References (17)

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7.3: Musical Scales

Introduction

The first thing musicians must do before they can play together is "tune.” For musicians in the standard Western music tradition, this means agreeing on exactly what pitch (what frequency) is an "A,” what is a "B flat," and so on. Other cultures not only have different note names and different scales, they may even have different notes - different pitches - based on a different tuning system. In fact, the modern Western tuning system, which is called equal temperament, replaced (relatively recently) other tuning systems that were once popular in Europe. All tuning systems are based on the physics of sound. But they all are also affected by the history of their music traditions, as well as by the tuning peculiarities of the instruments used in those traditions.

Many musical instruments, including pianos, guitars, and violins, make sound when strings are plucked, hit, or set in vibration in some other way. This vibration causes the air to move, creating an audible wave, which eventually reaches our ears. We call this wave “sound.” The pitch—how high or low the sound is—is determined by various factors such as the length of the string, its thickness, and its tension. A guitarist may play an open string, say G, and then place a finger on a fret to create sound with a higher pitch, say G#, A, or B. What is happening is that the player is shortening the vibrating part of the string to produce a higher pitch. This is based on the fundamental fact that shorter (or skinner or tighter) strings produce higher pitches. Similar facts are true with woodwind, percussion, and other types of instruments in general: the smaller the vibrating portion gets, the higher the sound becomes. That is why, for instance, a tuba is bigger than a French horn, and a contrabass flute is longer than a soprano flute.

In this section, we focus our attention on string vibrations. In particular, we look at the relation between the pitch and the string length, assuming that the material, thickness, and tightness of the string remain constant.

Suppose we pluck a string of a certain length and create a wave, a vibration of the air. The string will move up and down in oscillation. Each wave, whether it is a simple water wave, sound, earthquake, radio wave, microwave, or light wave, has a frequency, which measures how many times the oscillation occurs in a given time period, such as one second. For sound, the frequency is measured by “cycles per second” (abbreviated “cps”), sometimes called “hertz” (denoted “Hz”). Frequency is what determines the pitch of the sound generated by the vibration.

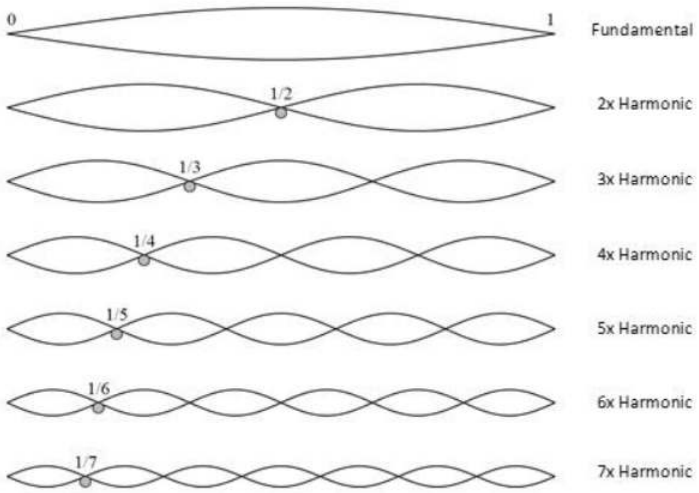


Figure 7.3.1: Image on StackExchange Physics is licensed by CC-3.0

When a string is plucked and set in motion, the entire string vibrates, as shown in the picture above. The frequency created by the whole string is referred to as the “fundamental frequency” of the string. It generates the lowest sound the string can possibly make. This is equivalent to playing an open string on the guitar. If the guitar is tuned in the standard way, the sixth string (the thickest one, located on the top) is the E string, meaning that its fundamental frequency corresponds to the pitch we call an “E.” This is the lowest sound a guitar can play.

Now, if we take that same string, press the midpoint of it and pluck it, only half of the string would vibrate, shortening the wave length to half and doubling the frequency. This, of course, results in a higher pitch. But how much higher? When the frequency is doubled, we call the interval between those two pitches an “octave” (meaning 8). For instance, if the open string has a frequency of

440 cps, then the half string would have a frequency of 880 cps, and that sound is called an octave higher than the first. One can, of course, keep making the vibrating portion shorter and shorter to create higher and higher sounds.

Pythagoras: Numbers, Music, and Harmony

Although music has existed in different cultures for millennia, it was Pythagoras, an ancient Greek mathematician and philosopher, who explicitly linked sound (pitch) with frequency (number), laying a foundation of music theory based on mathematics. It was known to Pythagoras that two notes (sounds with different frequencies) sound nice together (harmonious, pleasant) when the ratio of the two frequencies is a simple fraction. For instance, two pitches that are an octave apart have the ratio of 2 to 1 as described above. 2/1 (or its reciprocal ½) is a simple fraction, and these two notes sound nice if played simultaneously. In fact, if two people sing a song one octave apart, it could sound almost like a unison. These simple fractions give rise to well-known interval names in music theory:

2 = 2.00: octave

5/3 = 1.666…: major sixth

3/2 = 1.50: perfect fifth

4/3 = 1.333…: perfect fourth

5/4 = 1.25: major third

Pythagoras once said, ***"There is geometry in the humming of the strings, there is music in the spacing of the spheres."***

These simple ratios became a foundation for various musical scales and musical chords. For this reason, sometimes Pythagoras is called the father of music theory. In fact, as a religious and philosophical leader, he taught that ratios create harmony which forms beauty in nature; therefore, ratios are divine. His followers, called the Pythagoreans, thus did not believe the existence of numbers that are not ratios of integers. It is then quite ironic that Pythagoras himself later proved that square root of 2 (the length of the hypotenuse of a right isosceles triangle whose two sides are 1 unit long) is irrational, contradicting his own teaching that every number is rational. (By the way, this number, 2, came about by a very theorem named after him—the Pythagorean Theorem!)

Regardless of this devastating discovery, numerical ratios played a very significant role in the composition of musical scales. A **scale** is an ordered list of music tones that covers an octave; for instance, today the standard C-major scale consists of 8 notes starting with a C and ending with the next high C. I am certain that readers who are piano students have practiced these scales thousands of times (and your family may be quite tired of hearing them).

Throughout history, there have been many musical scales in various cultures, traditions, genres, and generations. Many authors have documented a wide variety of scales and methods of tuning instruments. Interested readers can look up topics such as Pythagorean intonation, mean-tone system, and temperaments. For simplicity, we now focus on the current music scale structure, consisting of 12 notes and their frequencies. This scale may not have been so widespread (or so standard) until J. S. Bach around the beginning of the 18th century.

Here are some facts about today's standard musical scale:

- A scale begins and ends with two pitches with the same name, one octave apart (meaning that the frequency of the last note is exactly twice that of the first note).
- This whole interval, an octave, is divided into 12 half-steps, in such a way that the frequency ratio of any consecutive half-steps is the same (described as "equal distance apart").

Let's take a look at these requirements mathematically now.

Suppose P_0 represents the frequency of the first note in a scale. Every time you move half-step to the next note, there must be a fixed ratio increase, i.e., growth by a fixed percentage, say, r (percentage written as a decimal). So the second note has the frequency

$$P_1 = P_0(1+r) \tag{7.3.1}$$

Then, to go to the next note by moving half-step up, we need to multiply this new number by the same multiplier, giving us

$$P_2 = P_1(1+r) = P_0(1+r)(1+r) = P_0(1+r)^2 \tag{7.3.2}$$

Repeating, we get

$$P_3 = P_0(1+r)^3 \tag{7.3.3}$$

$$P_4 = P_0(1+r)^4 \tag{7.3.4}$$

and so on. This look quite familiar, correct? This is what we saw in exponential growth. This is what we saw in compounding interest. Yes, it turns out that the mathematics that rules musical scales is the same as the mathematics that describes compound interest in finance and any exponential (fixed-percentage) growth in natural or social science!

So what is this magic number r, the common ratio that separates every half-step on the musical scale? Remember that after 12 half-steps, we have traveled an octave, doubling the original frequency. In other words, this is the number r so that the following equation is satisfied:

$$P_{12} = P_0(1+r)^{12} = 2P_0$$

If this equation were to appear in the context of finance, then this r would be the annual interest rate such that the original principal is doubled after 12 years. After dividing both sides by P0 and taking the 12th root of both sides, we get

$$1+r = \sqrt[12]{2} \approx 1.059463$$

We finally conclude that the fixed ratio between any two half-steps is approximately 0.059463, which is close to 5.946%. So, if you get this interest rate in a bank account compounding annually, you will double your balance in exactly 12 years. If you start your musical scale with a note with a certain frequency, after 12 half-steps, you will double your frequency, resulting in the note exactly one octave higher. (Think of the first two notes in the song “Somewhere Over the Rainbow.”) Below is a graph showing what happens to your bank balance if you begin with the original principal of $440 with the annual rate 5.946% for 24 years; your money doubles twice, i.e., quadruples to $1,760. This is like hitting a note two octaves higher.

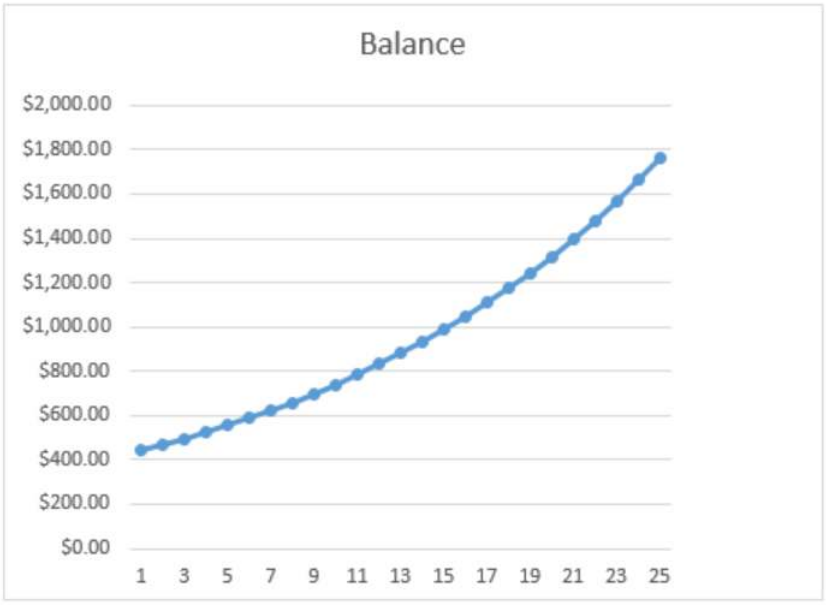


Figure 7.3.2: Graph of bank balance if you begin with the original principal of $440 with the annual rate 5.946% for 24 years

Why did I start this graph with $440? Well, in standard tuning, the middle A is set to 440 cps. Tuning forks are designed to vibrate at this frequency (if you know what a tuning fork is). Electronic tuners allow you to adjust this a bit, depending on the music you play (or how you want your music to sound). Increasing this frequency, 440 cps, by $r = 0.059463$ each time (for 12 times), you will get the following notes in the standard A scale. For some notes, the ratio of the frequency and 440 is also shown.

Table 7.3.1

Note	Frequency (Hz)	Ratio	
A	440.00	1	
A#	466.16		
B	493.88	1.122462	

Note	Frequency (Hz)	Ratio	
C	523.25	1.189207	
C#	554.37	1.259921	approx. 5:4
D	587.33	1.33484	approx. 4:3
D#	622.25		
E	659.26	1.498307	approx. 3:2
F	698.46		
F#	739.99		
G	783.99		
G#	830.61		
A	880.00	2	exactly 2:1

Take a look, for instance, at C#, with a frequency ratio of 1.2599, which is approximately 5:4. This note in the key of A major is called “major third”; D is called “perfect fourth,” and E is “perfect fifth.” Because of these simple (approximate) ratios, the notes A, C#, and E sound beautiful together. The set of these three notes is called the “A major” chord. All harmony is based on these simple ratios.

But each ratio is irrational and not exactly a simple ratio, right? This is the unfortunate result of equal temperament (devised in the 18th century); the common ratio r is the (not-so-secret) number that would allow a piece of music to sound good in every key, but the trade-off of temperament is that all ratios are now approximate, except for octaves, and thus, strictly speaking, “out of tune.” However, given the difficulty in tuning big instruments like the piano, this was the preferred compromise. The alternative is to have some ratios perfect while others are way off simple ratios, resulting in some music perhaps sounding great in one key but sounding terrible in another. Equal temperament is thus widely accepted and regarded as standard now. In other words, one irrational number, $\sqrt[12]{2}$, seems to control the harmonious world of music.

Just one more thing… I mentioned that the A major chord consists of A, C#, and E. If you change that middle note and lower it by half-step, you get the notes A, C (natural), and E, which together form the A minor chord. Play these chords and hear the difference. The mood shifts dramatically, right? Most people can hear the difference between a song written in a major key and a minor key. In fact, it would make people laugh if you take a popular song written in a minor key and play it in the corresponding major key. That huge difference, we now know, is generated by one little note dropped by one half-step, i.e., a frequency difference of about 5.946%. Such a small ratio, yet such a big difference!

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7.4: Fractals

Fractals are mathematical sets, usually obtained through recursion, that exhibit interesting dimensional properties. We'll explore what that sentence means through the rest of this section. For now, we can begin with the idea of self-similarity, a characteristic of most fractals.

Definition: Self-similarity

A shape is self-similar when it looks essentially the same from a distance as it does closer up.

Self-similarity can often be found in nature. In the Romanesco broccoli pictured below, if we zoom in on part of the image, the piece remaining looks similar to the whole.



Figure 7.4.1: Romanesco broccoli. (Image by AVM is licensed by CC 3.0)

Likewise, in the fern frond below, one piece of the frond looks similar to the whole.



Figure 7.4.2: Fern frond. (Image by Carolyn Jewel is licensed by CC 2.0)

Similarly, if we zoom in on the coastline of Portugal, each zoom reveals previously hidden detail, and the coastline, while not identical to the view from further way, does exhibit similar characteristics.



Figure 7.4.3: Coastline of Portugal. (Image by OpenStreetMap is licensed by CC 3.0)

Iterated Fractals

This self-similar behavior can be replicated through recursion: repeating a process over and over.

Example 7.4.1

Suppose that we start with a filled-in triangle. We connect the midpoints of each side and remove the middle triangle. We then repeat this process.

Solution

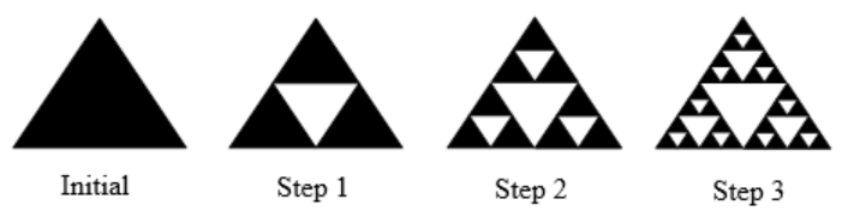


Figure 7.4.4

If we repeat this process, the shape that emerges is called the Sierpinski gasket. Notice that it exhibits self-similarity – any piece of the gasket will look identical to the whole. In fact, we can say that the Sierpinski gasket contains three copies of itself, each half as tall and wide as the original. Of course, each of those copies also contains three copies of itself.

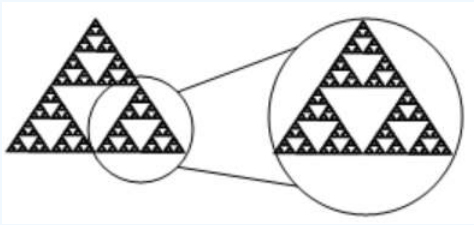


Figure 7.4.5

We can construct other fractals using a similar approach. To formalize this a bit, we're going to introduce the idea of initiators and generators.

Definition: Initiators and Generators

An **initiator** is a starting shape.

A **generator** is an arranged collection of scaled copies of the initiator.

To generate fractals from initiators and generators, we follow a simple rule:

Fractal Generation Rule

At each step, replace every copy of the initiator with a scaled copy of the generator, rotating as necessary.

This process is easiest to understand through example.

Example 7.4.2

Use the initiator and generator shown to create the iterated fractal.

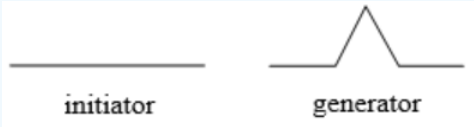


Figure 7.4.6

Solution

This tells us to, at each step, replace each line segment with the spiked shape shown in the generator. Notice that the generator itself is made up of 4 copies of the initiator. In step 1, the single line segment in the initiator is replaced with the generator. For step 2, each of the four line segments of step 1 is replaced with a scaled copy of the generator:

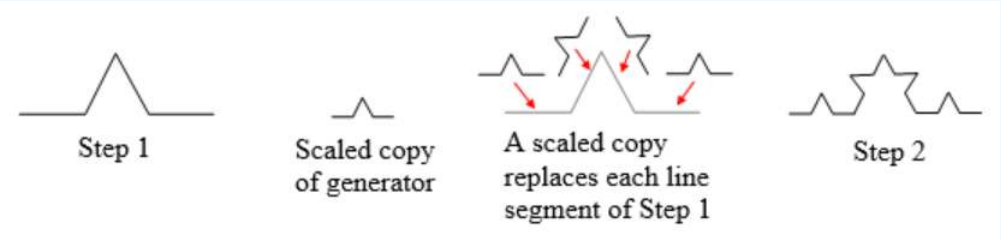


Figure 7.4.7

This process is repeated to form Step 3. Again, each line segment is replaced with a scaled copy of the generator.

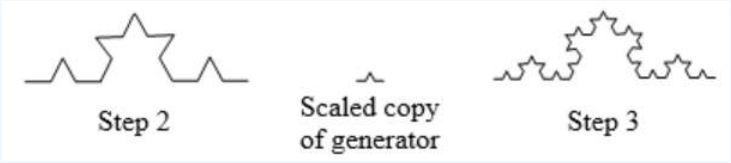


Figure 7.4.8

Notice that since Step 0 only had 1 line segment, Step 1 only required one copy of Step 0.

Since Step 1 had 4 line segments, Step 2 required 4 copies of the generator.

Step 2 then had 16 line segments, so Step 3 required 16 copies of the generator.

Step 4, then, would require 16*4 = 64 copies of the generator.

The shape resulting from iterating this process is called the Koch curve, named for Helge von Koch, who first explored it in 1904.

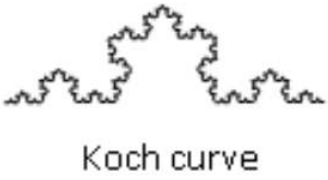


Figure 7.4.9: Koch curve

Notice that the Sierpinski gasket can also be described using the initiator-generator approach

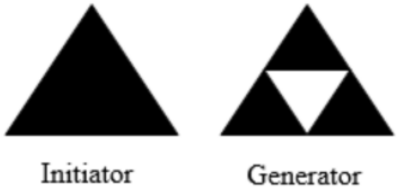


Figure 7.4.10: Sierpinski gasket

Example 7.4.3

Use the initiator and generator below; however, only iterate on the “branches.” Sketch several steps of the iteration.

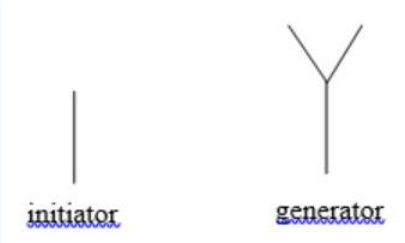


Figure 7.4.11

Solution

We begin by replacing the initiator with the generator. We then replace each “branch” of Step 1 with a scaled copy of the generator to create Step 2.

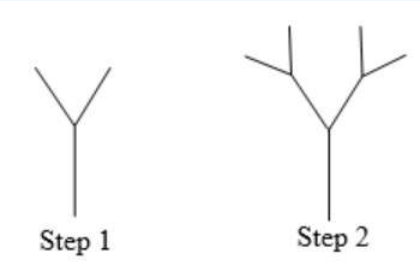


Figure 7.4.12

We can repeat this process to create later steps. Repeating this process can create intricate tree shapes.

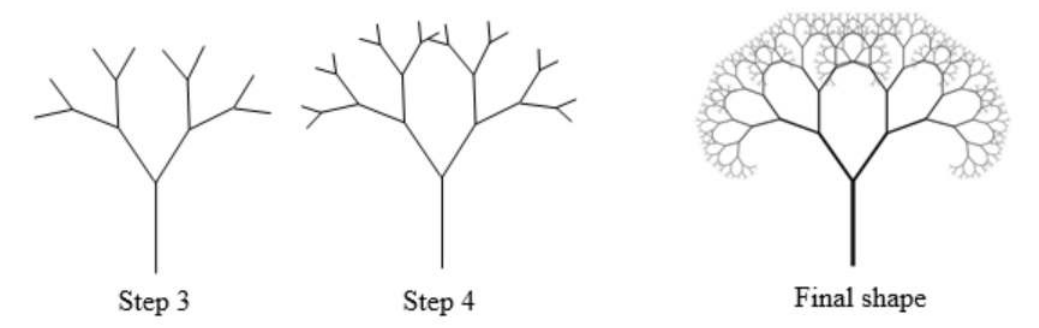


Figure 7.4.13: Intricate tree shapes. (Image by Manuel Noah Angeja is licensed by CC 2.0)

Try it Now 1

Use the initiator and generator shown to produce the next two stages:

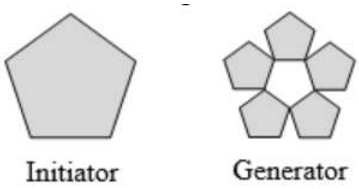


Figure 7.4.14

Using iteration processes like those above can create a variety of beautiful images evocative of nature.

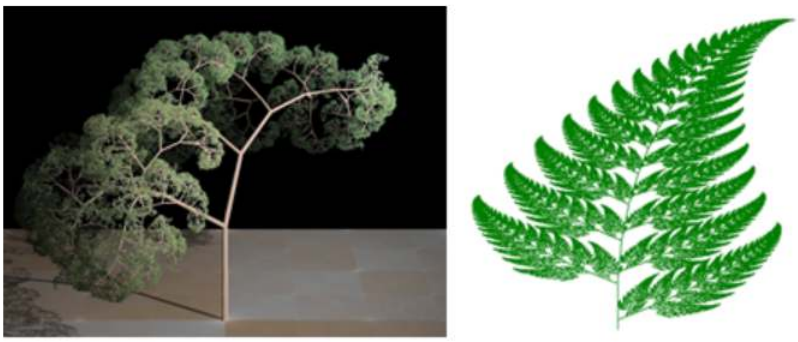


Figure 7.4.15: Image by Solkoll is in the public domain/Image by DSP-user is licensed by CC 3.0

More natural shapes can be created by adding in randomness to the steps.

Example 7.4.4

Create a variation on the Sierpinski gasket by randomly skewing the corner points each time an iteration is made.

Solution

Suppose we start with the triangle below. We begin, as before, by removing the middle triangle. We then add in some randomness.

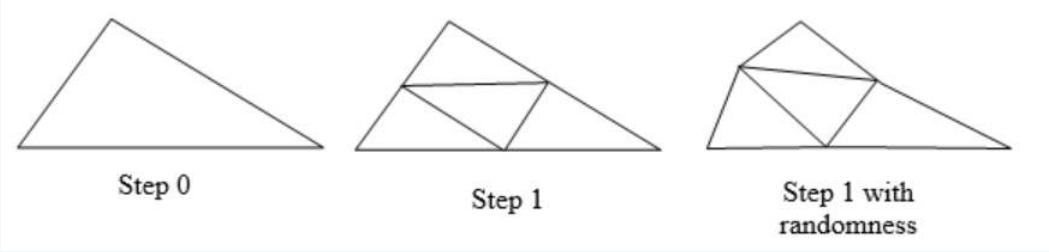


Figure 7.4.16

We then repeat this process.

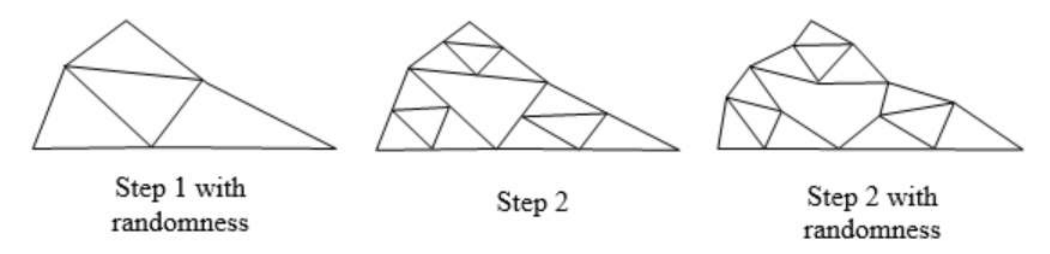


Figure 7.4.17

Continuing this process can create mountain-like structures.

The landscape below was created using fractals first and then colored and textured.

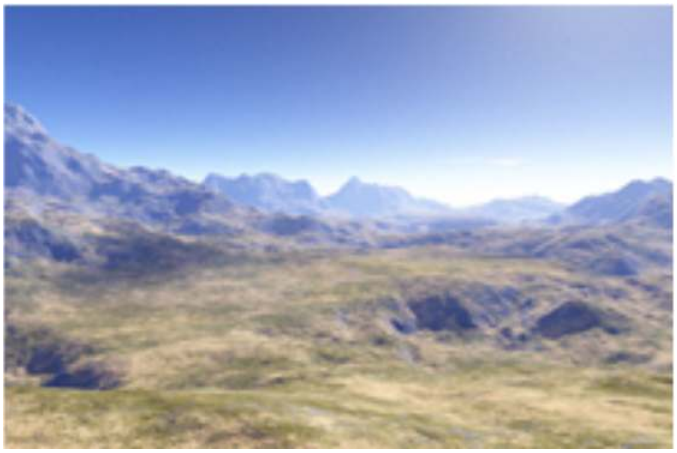


Figure 7.4.18: Image by The Ostrich is licensed by CC 3.0

Fractal Dimension

In addition to visual self-similarity, fractals exhibit other interesting properties. For example, notice that each step of the Sierpinski gasket iteration removes one quarter of the remaining area. If this process is continued indefinitely, we would end up essentially removing all the area, meaning we started with a 2-dimensional area and somehow end up with something less than that, but seemingly more than just a 1-dimensional line.

To explore this idea, we need to discuss dimension. Something like a line is 1-dimensional; it only has length. Any curve is 1-dimensional. Things like rectangles and circles are 2-dimensional since they have length and width, describing an area. Objects like boxes and cylinders have length, width, and height, describing a volume, and they are 3-dimensional.

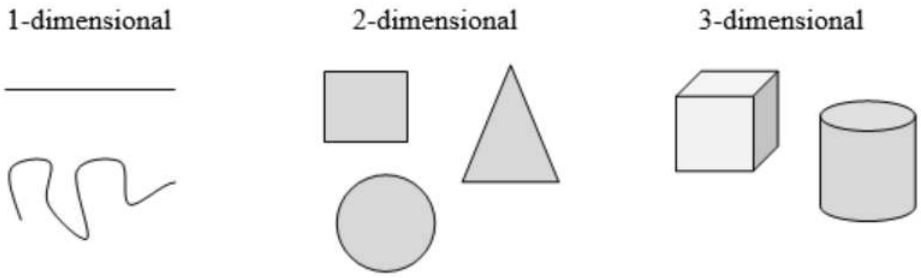


Figure 7.4.19: Dimensions

Certain rules apply for scaling objects, and these rules are related to their dimension.

If I had a line with length 1 and wanted scale its length by 2, I would need two copies of the original line. If I had a line of length 1 and wanted to scale its length by 3, I would need three copies of the original.



Figure 7.4.20

If I had a rectangle with length 2 and height 1 and wanted to scale its length and width by 2, I would need four copies of the original rectangle. If I wanted to scale the length and width by 3, I would need nine copies of the original rectangle.

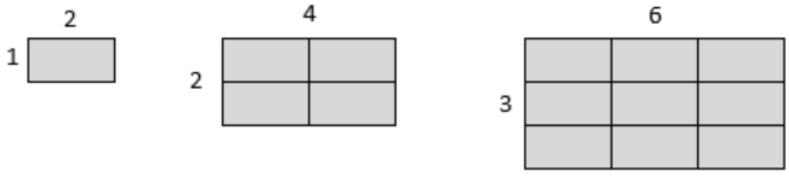


Figure 7.4.21

If I had a cubical box with sides of length 1 and wanted to scale its length, width, and height by 2, I would need eight copies of the original cube. If I wanted to scale the length, width and height by 3, I would need 27 copies of the original cube.

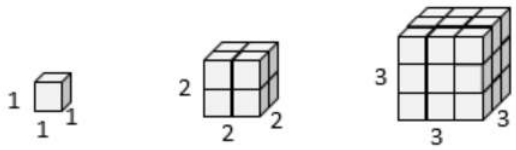


Figure 7.4.22

Notice that in the 1-dimensional case, copies needed = scale.

In the 2-dimensional case, copies needed = scale^2.

In the 3-dimensional case, copies needed = scale^3.

From these examples, we might infer a pattern.

In fact, in this section, we use this pattern to define the dimension of an object.

Scaling-Dimension Relation

To scale a D-dimensional shape by a scaling factor S, the number of copies C of the original shape needed will be given by:

$$\text{Copies} = \text{Scale}^{\text{Dimension}}, \text{ or } C = S^D$$

Example 7.4.5

Use the scaling-dimension relation to determine the dimension of the Sierpinski gasket.

Solution

Suppose we define the original gasket to have side length 1. The larger gasket shown is twice as wide and twice as tall, so has been scaled by a factor of 2.

Notice that to construct the larger gasket, 3 copies of the original gasket are needed.

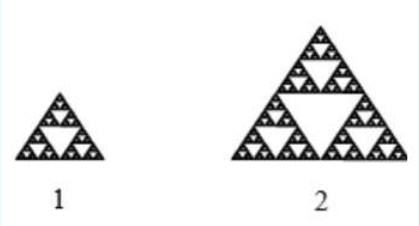


Figure 7.4.23

Using the scaling-dimension relation $C = S^D$, we obtain the equation $3 = 2^D$.

Since $2^1 = 2$ and $2^2 = 4$, we can immediately see that D is somewhere between 1 and 2; the gasket is more than a 1-dimensional shape, but we've taken away so much area its now less than 2-dimensional.

Solving the equation $3 = 2^D$ requires logarithms. If you studied logarithms earlier, you may recall how to solve this equation (if not, just skip to the box below and use that formula):

$$\begin{aligned} 3 &= 2^d \quad \text{Take the logarithm of both sides} \\ \log(3) &= \log\left(2^D\right) \quad \text{Use the exponent property of } \log s \\ \log(3) &= D\log(2) \quad \text{Divide by } \log(2) \\ D &= \frac{\log(3)}{\log(2)} \approx 1.585 \end{aligned}$$

The dimension of the gasket is about 1.585

Scaling-Dimension Relation, to find Dimension

To find the dimension D of a fractal, determine the scaling factor S and the number of copies C of the original shape needed, then use the formula

$$D = \frac{\log(C)}{\log(S)}$$

Try it Now 2

Use the initiator and generator shown to produce the next two stages.

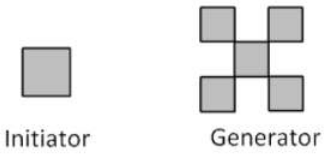


Figure 7.4.24

We will now briefly mention another type of fractal, defined by a different type of recursion. To understand this type, you will need to know operations on complex numbers (including the imaginary number i), certain functions on those numbers, and iteration. These topics are beyond the scope of this text, so we will not include the details.

Here is what is known as the Mandelbrot set, defined using the iteration of a certain complex-valued function. Note that this well-known set, discovered in 1980, also shows a fractal-like shape. The boundary of this shape exhibits quasi-self-similarity, in that portions look very similar to the whole. In addition to coloring the Mandelbrot set itself black, it is common to the color the points in the complex plane surrounding the set.

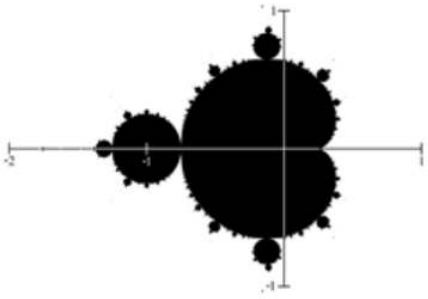


Figure 7.4.25: Image by Connelly Barnes is in the public domain

Again, these colorings have specific mathematical meanings, but you will need to learn more advanced mathematics to fully appreciate the significance of these colors. All you need to know here is that these colors were not just added to make the pictures prettier. Studying some complex functions using certain colors resulted in these pretty pictures.

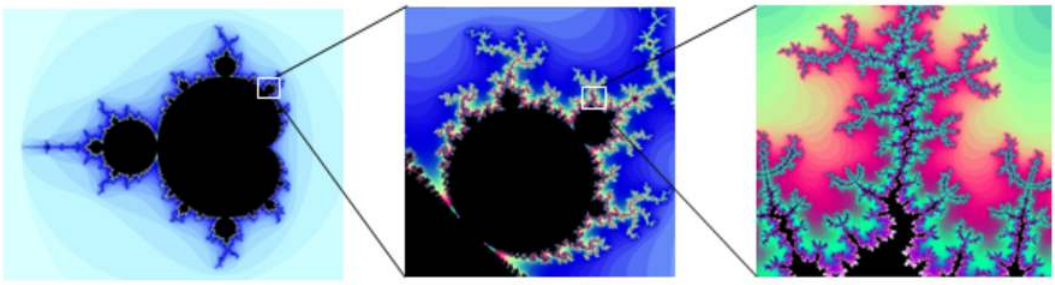


Figure 7.4.26: Series created using Scott's Mandelbrot Set Explorer, licensed by CC 3.0

The Mandelbrot set, for having a simple definition, exhibits immense complexity. Zooming in on other portions of the set yields fascinating swirling shapes.

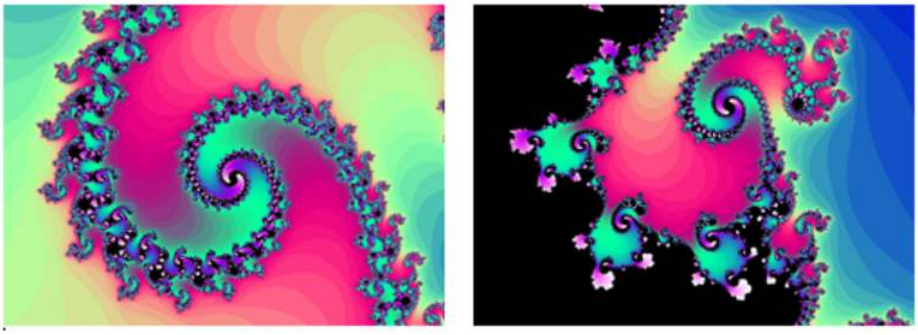


Figure 7.4.27: Series created using Scott's Mandelbrot Set Explorer, licensed by CC 3.0

Additional Resources

The Mandelbrot Explorer site, provides more details on the Mandelbrot set, including a nice visualization of Mandelbrot sequences.

Exercises

1. One side of a circular cylinder, a rectangular box, a triangular prism are shown below. Connect each vertex with the principal vanishing point to draw the cylinder, box, and prism.

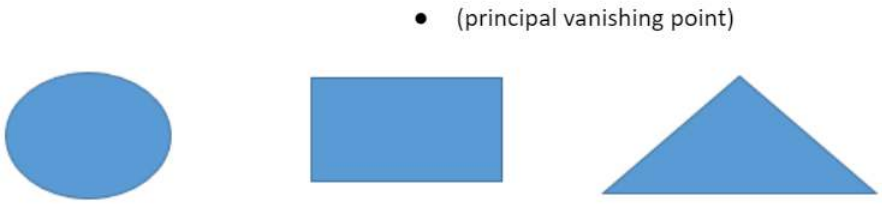


Figure 7.4.28

2. Suppose you are sitting at the end of a long hallway, where the ceiling has rectangular panels ,the floor has square tiles, and the two side walls have vertical lines every few feet. Draw the hallway from your perspective, with a principal vanishing point. Now, what happens if you stand up? How would your picture change? How about if you move to the left?
3. Do some online research to find which artists seem to have used the Golden Ratio in their works of art. Give some examples of their art pieces.
4. Do some online research to find which composers seem to have written music that reflects the Golden Ratio. Give some specific examples of their music works.
5. Do some online research on the Golden Rectangle. Why is this called “golden”? Where do we see Golden Rectangles around us?
6. The Fibonacci Sequence begins with 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, … List the next three Fibonacci numbers (the next three terms of this sequence).
7. Do some online research on the Fibonacci Sequence. Find its relationship to the biological world, such as pine cones, sunflowers, nautilus shells, and human bodies.
8. Do some online research on the history of Fibonacci and his famous sequence. Who was he? What country is he from? Why did he study these numbers in particular?
9. A guitarist tunes his first (thinnest) string at E = 330 cps (typically denoted “E4”). If his sixth (thickest) string is to be two octaves lower, what is the frequency of the sixth string?
10. We started our musical scale at A4 = 440 cps. Some people may start a scale at C5 = 520 cps. Find the frequencies of the thirteen notes that spans an octave (C, C#, D, D#, E, F, F#, G, G#, A, A#, B, C), starting at C = 520 cps.
11. If A = 440 cps, the real “perfect fifth” would be E, whose frequency would be exactly 3/2 of A. What frequency is that? Compare that with the ‘tempered” frequency of E.
12. If A = 440 cps, the real “perfect fourth” would be D, whose frequency would be exactly 4/3 of A. What frequency is that? Compare that with the ‘tempered” frequency of D.

13. The range known as “tenor” is often defined from Second B (B2) to Third G (G3), about one and a half octave. Suppose that Middle C (C4) is 262 cps (and C3 is half of it), find the frequencies that cover this range. (Disclaimer: tenor opera singers have a much wider range than this. This is only the range normally referred to as “tenor.”)
14. It is said that Mariah Carey can hit a note as low as F2 and as high as G7 (slightly more than five octaves). Find the frequencies that cover this range. (Assume “A4” is 440 cps. A3 is half of it. A2 is half of that frequency. F2 is even lower. The numerical portion increases by 1 for each octave, and a new number starts with a C. So, for instance, the sequence goes like this: A2, A2#, B2, C3, C3#, …)

Using the initiator and generator shown, draw the next two stages of the iterated fractal.

15.



Figure 7.4.29

16.

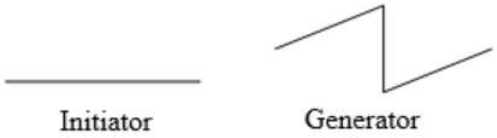


Figure 7.4.30

17.



Figure 7.4.31

18.



Figure 7.4.32

19.



Figure 7.4.33

20.

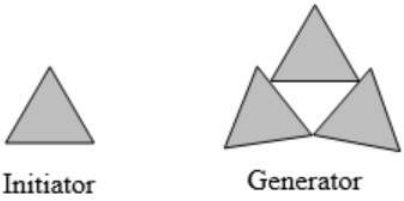


Figure 7.4.34

21. Create your own version of Sierpinski gasket with added randomness.
22. Create a version of the branching tree fractal from example #3 with added randomness.
23. Determine the fractal dimension of the Koch curve.
24. Determine the fractal dimension of the curve generated in exercise #1
25. Determine the fractal dimension of the Sierpinski carpet generated in exercise #5
26. Determine the fractal dimension of the Cantor set generated in exercise #4

Reference

1. References (18)

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CHAPTER OVERVIEW

8: Mathematics and Politics

One of the main goals of a course like Liberal Arts Mathematics is to help students become logically and quantitatively educated so that they can be well-informed, contributing members of the society. To that end, this chapter is particularly important as we will explore how mathematics is used in politics, especially in two crucial aspects—election and representation. We will investigate some fundamental facets of democracy as well as our pursuit of fairness and justice.

8.1: Apportionment
8.2: Apportionment of Legislative Districts
8.3: Voting Theory
8.4: Weighted Voting

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8.1: Apportionment

Apportionment is the problem of dividing up a fixed number of things among groups of different sizes. In politics, this takes the form of allocating a limited number of representatives amongst voters. This problem, presumably, is older than the United States, but the best-known ways to solve it have their origins in the problem of assigning each state an appropriate number of representatives in the new Congress when the country was formed. States also face this apportionment problem in defining how to draw districts for state representatives. The apportionment problem comes up in a variety of non-political areas too, though. Here is a simple question. Say your school district has 5 schools, some of which have a lot more students than others. The state budget allows the district to have 8 counselors. Assuming that each counselor can only serve one school, which school gets how many? Here, the "set" of 8 counselor positions is being divided up among the 5 schools. This sounds pretty realistic, right?

Apportionment Rules

1. The things being divided up can exist only in whole numbers.
2. We must use all of the things being divided up, and we cannot add any more.
3. Each group must get at least one of the things being divided up.
4. The number of things assigned to each group should be at least approximately proportional to the population of the group. (Exact proportionality isn't possible because of the whole number requirement, but we should try to be close. If Group A is larger than Group B, then Group B shouldn't get more of the things than Group A does.)

In terms of the apportionment of the United States House of Representatives, these rules imply:

1. We can only have whole representatives (a state can't have 3.4 representatives).
2. We can only use the (currently) 435 representatives available. If one state gets another representative, another state has to lose one.
3. Every state gets at least one representative.
4. The number of representatives each state gets should be approximately proportional to the state population. This way, the number of constituents each representative has should be approximately equal.

We will look at four ways of solving the apportionment problem. Three of them (Lowndes's method is the exception) have been used at various times to apportion the U.S. Congress although the method currently in use (the Huntington-Hill method) is significantly more complicated.

Hamilton's Method

Alexander Hamilton proposed the method that now bears his name. His method was approved by Congress in 1791 but was vetoed by President Washington. (This, by the way, was the very first presidential veto in the United States). It was later adopted in 1852 and used through 1911. He begins by determining, to several decimal places, how many things each group should get. Since he was interested in the question of Congressional representation, we'll use the language of states and representatives. He determines how many representatives each state should get by following these steps:

Hamilton's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the divisor.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the quota.

Since we can only allocate whole representatives, Hamilton resolves the whole number problem as follows:

3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). These are called the lower quotas. Add up the remaining whole numbers. This answer will always be less than or equal to the total number of representatives (and the "or equal to" part happens only in very specific circumstances that are incredibly unlikely to turn up).
4. Assuming that the total from Step 3 was less than the total number of representatives, assign the remaining representatives, one each, to the states whose decimal parts of the quota were largest, until the desired total is reached.

Note on rounding: Today we have technological advantages that Hamilton (and the others) couldn't even have imagined. Take advantage of them, and keep several decimal places.

Example 8.1.1

The state of Delaware has three counties: Kent, New Castle, and Sussex. The Delaware state House of Representatives has 41 members. If Delaware wants to divide this representation along county lines (which is not required, but let's pretend they do), let's use Hamilton's method to apportion them. The populations of the counties are as follows (from the 2010 Census):

Table 8.1.1

County	Population
Kent	162,310
New Castle	538,479
Sussex	197,145
Total	**897,934**

Solution

1. First, we determine the divisor: 897,934/41 = 21,900.82927
2. Now we determine each county's quota by dividing the county's population by the divisor:

Table 8.1.2

County	Population	Quota
Kent	162,310	7.4111
New Castle	538,479	24.5872
Sussex	197,145	9.0017
Total	**897,934**	

3. Removing the decimal parts of the quotas gives:

Table 8.1.3

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	**897,934**		

4. We need 41 representatives and this only gives 40. The remaining one goes to the county with the largest decimal part, which is New Castle:

Table 8.1.4

County	Population	Quota	Initial	Final
Kent	162,310	7.4111	7	7
New Castle	538,479	24.5872	24	25
Sussex	197,145	9.0017	9	9
Total	**897,934**		**40**	

Example 8.1.2

Use Hamilton's method to apportion the 75 seats of Rhode Island's House of Representatives among its five counties.

Table 8.1.5

County	Population
Bristol	49,875
Kent	166,158
Newport	82,888
Providence	626,667
Washington	126,979
Total	**1,052,567**

Solution

1. The divisor is 1,052,567/75 = 14,034.22667
2. Determine each county's quota by dividing its population by the divisor:

Table 8.1.6

County	Population	Quota
Bristol	49,875	3.5538
Kent	166,158	11.8395
Newport	82,888	5.9061
Providence	626,667	44.6528
Washington	126,979	9.0478
Total	**1,052,567**	

3. Remove the decimal part of each quota:

Table 8.1.7

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	**1,052,567**		**72**

4. We need 75 representatives and we only have 72, so we assign the remaining three, one each, to the three counties with the largest decimal parts, which are Newport, Kent, and Providence:

Table 8.1.8

County	Population	Quota	Initial	Final
Bristol	49,875	3.5538	3	3

County	Population	Quota	Initial	Final
Kent	166,158	10.8395	11	12
Bristol	49,875	3.2538	3	3
Providence	626,667	44.6528	44	45
Washington	126,979	9.0478	9	9
Newport	82,888	5.9061	5	6
Total	**1,052,567**		**72**	**75**

Note that even though Bristol County's decimal part is greater than .5, it isn't big enough to get an additional representative because three other counties have greater decimal parts.

Hamilton's method obeys something called the Quota Rule. The Quota Rule isn't a law of any sort, but it is just an idea that some people, including Hamilton, think is a good one.

Quota Rule

The Quota Rule says that the final number of representatives a state gets should be within one of that state's quota. Since we're dealing with whole numbers for our final answers, each state should either go up to the next whole number above its quota, or down to the next whole number below its quota.

Controversy

After seeing Hamilton's method, many people find that it makes sense. It is not that difficult to use, and they wonder why anyone would want another method. The problem is that Hamilton's method is subject to several paradoxes. Three of them happened, on separate occasions, when Hamilton's method was used to apportion the United States House of Representatives.

The **Alabama Paradox** is named for an incident that happened during the apportionment that took place after the 1880 census. (A similar incident happened ten years earlier involving the state of Rhode Island, but the paradox is named after Alabama.) The post-1880 apportionment had been completed, using Hamilton's method applied to the new population numbers from the census. Then it was decided that because of the country's growing population, the House of Representatives should be made larger. That meant that the apportionment would need to be done again, still using Hamilton's method and the same 1880 census numbers, but with more representatives. The assumption was that some states would gain another representative and others would stay with the same number they already had (since there weren't enough new representatives being added to give one more to every state). The paradox is that Alabama ended up losing a representative in the process even though no populations were changed and the total number of representatives increased.

The **New States Paradox** happened when Oklahoma became a state in 1907. Oklahoma had enough population to qualify for five representatives in Congress. Those five representatives would need to come from somewhere, though, so five states, presumably, would lose one representative each. That happened, but another unexpected thing also happened: Maine gained a representative (from New York)!

The **Population Paradox** happened between the apportionments after the census of 1900 and of 1910. In those ten years, Virginia's population grew at an average annual rate of 1.07%, while Maine's grew at an average annual rate of 0.67%. Virginia started with more people, grew at a faster rate, grew by more people, and ended up with more people than Maine. By itself, that doesn't mean that Virginia should gain representatives or Maine shouldn't, because there are lots of other states involved. But Virginia ended up losing a representative to Maine.

Jefferson's Method

Thomas Jefferson, who lived before any of these paradoxes, proposed a different method for apportionment. After Washington vetoed Hamilton's method, Jefferson's method was adopted and used in Congress from 1791 through 1842. Jefferson, of course, had political reasons for wanting his method to be used rather than Hamilton's. Primarily, his method favors larger states, and his own home state of Virginia was the largest in the country at the time. He would also argue that it's the ratio of people to representatives that is the critical thing, and apportionment methods should be based on that. But the paradoxes we saw also provide mathematical reasons for concluding that Hamilton's method isn't so good. While Jefferson's method might or might not be the best one to replace it, at least Jefferson provided an alternative.

The first steps of Jefferson's method are the same as Hamilton's method. He finds the same divisor and the same quota and cuts off the decimal parts in the same way, giving a total number of representatives that is less than the required total. The difference is in how Jefferson resolves that difference. He says that since we ended up with an answer that is too small, our divisor must have been too big. He changes the divisor by making it smaller, finding new quotas with the new divisor, cutting off the decimal parts, and looking at the new total, until we find a divisor that produces the required total.

Jefferson's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the standard divisor.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the quota.
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). These are the lower quotas. Add up the remaining whole numbers. This answer will always be less than or equal to the total number of representatives.
4. If the total from Step 3 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 3 is equal to the total number of representatives. The divisor we end up using is called the modified divisor or adjusted divisor

A careful reader should be able to spot the problem immediately. Jefferson's method does not state exactly how to reduce that devisor. Let's take a look at some examples.

Example 8.1.3

We'll return to Delaware and apply Jefferson's method (Example 8.1.1). We begin, as we did with Hamilton's method, by finding the quotas with the original divisor, 21,900.82927:

Table 8.1.9

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	**897,934**		**40**

Solution

We need 41 representatives, and this divisor gives only 40. We must reduce the divisor until we get 41 representatives. Let's try 21,500 as the divisor:

Table 8.1.10

County	Population	Quota	Initial
Kent	162,310	7.5493	7
New Castle	538,479	25.0455	25
Sussex	197,145	9.1695	9
Total	**897,934**		**41**

This gives us the required 41 representatives, so we're done. If we had fewer than 41, we'd need to reduce the divisor more. If we had more than 41, we'd need to choose a divisor less than the original but greater than the second choice.

Notice that with the new, lower divisor, the quota for New Castle County (the largest county in the state) increased by much more than those of Kent County or Sussex County.

Example 8.1.4

We'll apply Jefferson's method for Rhode Island (Example 8.1.2). The original divisor of 14,034.22667 gave these results:

Table 8.1.11

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	**1,052,567**		**72**

Solution

We need 75 representatives and we only have 72, so we need to use a smaller divisor. Let's try 13,500:

Table 8.1.11

County	Population	Quota	Initial
Bristol	49,875	3.6944	3
Kent	166,158	12.3080	12
Newport	82,888	6.1399	6
Providence	626,667	46.4198	46
Washington	126,979	9.4059	9
Total	**1,052,567**		**76**

We've gone too far. We need a divisor that's greater than 13,500 but less than 14,034.22667. Let's try 13,700:

Table 8.1.12

County	Population	Quota	Initial
Bristol	49,875	3.6405	3
Kent	166,158	12.1283	12
Newport	82,888	6.0502	6
Providence	626,667	45.7421	45
Washington	126,979	9.2685	9
Total	**1,052,567**		**75**

This works.

Notice, in comparison to Hamilton's method, that although the results were the same, they came about in a different way, and the outcome was almost different. Providence County (the largest) almost went up to 46 representatives before Kent (which is much smaller) got to 12. Although that didn't happen here, it can. Divisor-adjusting methods like Jefferson's are not guaranteed to follow the quota rule!

Webster's Method

Daniel Webster (1782-1852) proposed a method similar to Jefferson's in 1832. It was adopted by Congress in 1842 but was replaced by Hamilton's method in 1852. It was then adopted again in 1901. The difference is that Webster rounds the quotas to the nearest whole number rather than dropping the decimal parts. If that doesn't produce the desired results at the beginning, he says, like Jefferson, to adjust the divisor until it does. (In Jefferson's case, at least the first adjustment will always be to make the divisor smaller. That is not always the case with Webster's method.)

Webster's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the standard divisor.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the quota.
3. Round (up or down) all the quotas to the nearest whole number. Add up the remaining whole numbers.
4. If the total from Step 3 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. If the total from step 3 was larger than the total number of representatives, increase the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 3 is equal to the total number of representatives. The divisor we end up using is called the modified divisor or adjusted divisor

Example 8.1.5

Again, Delaware (Example 8.1.1), with an initial divisor of 21,900.82927:

Solution

Table 8.1.13

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	25 (Rounded Up)
Sussex	197,145	9.0017	9
Total	**897,934**		**41**

This gives the required total, so we're done.

Example 8.1.6

Again, Rhode Island (Example 8.1.2), with an initial divisor of 14,034.22667:

Solution

Table 8.1.14

County	Population	Quota	Initial
Bristol	49,875	3.5538	4
Kent	166,158	11.8395	12
Newport	82,888	5.9061	6
Providence	626,667	44.6528	45
Washington	126,979	9.0478	9
Total	**1,052,567**		**76**

This is too many, so we need to increase the divisor. Let's try 14,100:

Table 8.1.15

County	Population	Quota	Initial
Bristol	49,875	3.5372	4
Kent	166,158	11.7843	12
Newport	82,888	5.8786	6
Providence	626,667	44.4445	44
Washington	126,979	9.0056	9
Total	**1,052,567**		**75**

This works, so we're done.

Like Jefferson's method, Webster's method carries a bias in favor of states with large populations, but rounding the quotas to the nearest whole number greatly reduces this bias. (Notice that Providence County, the largest, is the one that gets a representative trimmed because of the increased quota.) Also like Jefferson's method, Webster's method does not always follow the quota rule, but it follows the quota rule much more often than Jefferson's method does. (In fact, if Webster's method had been applied to every apportionment of Congress in all of American history, it would have followed the quota rule every single time.)

In 1980, two mathematicians, Peyton Young and Mike Balinski, proved what we now call the Balinski-Young Impossibility Theorem.

Balinski-Young Impossibility Theorem

Any apportionment method which always follows the quota rule will be subject to the possibility of paradoxes like the Alabama, New States, or Population Paradoxes. In other words, we can choose a method that avoids those paradoxes, but only if we are willing to give up the guarantee of following the quota rule.

Huntington-Hill Method

In 1920, no new apportionment was done because Congress couldn't agree on the method to be used. They appointed a committee of mathematicians to investigate, and they recommended the Huntington-Hill Method. They continued to use Webster's method in 1931, but after a second report recommending Huntington-Hill, it was adopted in 1941 and is the current method of apportionment used in Congress.

The Huntington-Hill Method is similar to Webster's method, but attempts to minimize the percent differences of how many people each representative will represent.

Huntington-Hill Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the standard divisor.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the quota.
3. Cut off the decimal part of the quota to obtain the lower quota, which we'll call n. Compute $\sqrt{n(n+1)}$, which is the geometric mean of the lower quota and one value higher.
4. If the quota is larger than the geometric mean, round up the quota; if the quota is smaller than the geometric mean, round down the quota. Add up the resulting whole numbers to get the initial allocation.
5. If the total from Step 4 was less than the total number of representatives, reduce the divisor and recalculate the quota and allocation. If the total from step 4 was larger than the total number of representatives, increase the divisor and recalculate the quota and allocation. Continue doing this until the total in Step 4 is equal to the total number of representatives. The divisor we end up using is called the modified divisor or adjusted divisor.

Example 8.1.7

Again, Delaware (Example 8.1.1), with an initial divisor of 21,900.82927:

Solution

Table 8.1.16

County	Population	Quota	Lower Quota	Geom Mean	Initial
Kent	162,310	7.4111	7	7.48	7
New Castle	538,479	24.5872	24	24.49	25 (Rounded Up)
Sussex	197,145	9.0017	9	9.49	9
Total	**897,934**				**41**

This gives the required total, so we're done.

Example 8.1.8

Again, Rhode Island (Example 8.1.2), with an initial divisor of 14,034.22667:

Solution

Table 8.1.17

County	Population	Quota	Lower Quota	Geom Mean	Initial
Bristol	49,875	3.5538	3	3.46	4
Kent	166,158	11.8395	11	11.49	12
Newport	82,888	5.9061	5	5.48	6
Providence	626,667	44.6528	44	44.50	45
Washington	126,979	9.0478	9	9.49	9
Total	**1,052,567**				**76**

This is too many, so we need to increase the divisor. Let's try 14,100:

Table 8.1.18

County	Population	Quota	Lower Quota	Geom Mean	Initial
Bristol	49,875	3.5538	3	3.46	4
Kent	166,158	11.8395	11	11.49	12
Newport	82,888	5.9061	5	5.48	6
Providence	626,667	44.6528	44	44.50	44
Washington	126,979	9.0478	9	9.49	9
Total	**1,052,567**				**75**

This works, so we're done.

In both these cases, the apportionment produced by the Huntington-Hill method was the same as those from Webster's method.

Example 8.1.9

Consider a small country with 5 states, two of which are much larger than the others. We need to apportion 70 representatives. We will apportion using both Webster's method and the Huntington-Hill method.

Table 8.1.19

State	Population
A	300,500
B	200,000
C	50,000
D	38,000
E	21,500

Solution

1. The total population is 610,000. Dividing this by the 70 representatives gives the divisor: 8714.286

2. Dividing each state's population by the divisor gives the quotas

Table 8.1.20

State	Population	Quota
A	300,500	34.48361
B	200,000	22.95082
C	50,000	5.737705
D	38,000	4.360656
E	21,500	2.467213

Webster's Method

1. Using Webster's method, we round each quota to the nearest whole number

Table 8.1.21

State	Population	Quota	Initial
A	300,500	34.48361	34
B	200,000	22.95082	23
C	50,000	5.737705	6
D	38,000	4.360656	4
E	21,500	2.467213	2

2. Adding these up, they only total 69 representatives, so we adjust the divisor down. Adjusting the divisor down to 8700 gives an updated allocation totaling 70 representatives

Table 8.1.22

State	Population	Quota	Initial
A	300,500	34.54023	35
B	200,000	22.98851	23
C	50,000	5.74126	6

State	Population	Quota	Initial
D	38,000	4.367816	4
E	21,500	2.471264	2

Huntington-Hill Method

3. Using the Huntington-Hill method, we round down to find the lower quota, then calculate the geometric mean based on each lower quota. If the quota is less than the geometric mean, we round down; if the quota is more than the geometric mean, we round up.

Table 8.1.23

State	Population	Quota	Lower Quota	Geom Mean	Initial
A	300,500	34.48361	34	34.49638	34
B	200,000	22.95082	22	22.49444	23
C	50,000	5.737705	5	5.477226	6
D	38,000	4.360656	4	4.472136	4
E	21,500	2.467213	2	2.44949	3

These allocations add up to 70, so we're done.

Notice that this allocation is different from what was produced by Webster's method. In this case, state E got the extra seat instead of state A.

The use of geometric mean actually favors smaller states, which was exactly what Huntington and Hill were trying to achieve.

Lowndes' Method

William Lowndes (1782-1822) was a Congressman from South Carolina who proposed a method of apportionment that was more favorable to smaller states. Unlike the methods of Hamilton, Jefferson, and Webster, Lowndes's method has never been used to apportion Congress.

Lowndes believed that an additional representative was much more valuable to a small state than to a large one. If a state already has 20 or 30 representatives, getting one more doesn't matter very much. But if it only has 2 or 3, one more is a big deal, and he felt that the additional representatives should go where they could make the most difference.

Like Hamilton's method, Lowndes's method follows the quota rule. In fact, it arrives at the same quotas as Hamilton and the rest, and like Hamilton and Jefferson, it drops the decimal parts. But in deciding where the remaining representatives should go, we divide the decimal part of each state's quota by the whole number part (so that the same decimal part with a smaller whole number is worth more because it matters more to that state).

Lowndes's Method

1. Determine how many people each representative should represent. Do this by dividing the total population of all the states by the total number of representatives. This answer is called the divisor.
2. Divide each state's population by the divisor to determine how many representatives it should have. Record this answer to several decimal places. This answer is called the quota.
3. Cut off all the decimal parts of all the quotas (but don't forget what the decimals were). Add up the remaining whole numbers.
4. Assuming that the total from Step 3 was less than the total number of representatives, divide the decimal part of each state's quota by the whole number part. Assign the remaining representatives, one each, to the states whose ratio of decimal part to whole part were largest, until the desired total is reached.

Example 8.1.10

We'll do Delaware again. We begin in the same way as with Hamilton's method:

Table 8.1.24

County	Population	Quota	Initial
Kent	162,310	7.4111	7
New Castle	538,479	24.5872	24
Sussex	197,145	9.0017	9
Total	**897,934**		**40**

Solution

We need one more representative. To find out which county should get it, Lowndes says to divide each county's decimal part by its whole number part, with the largest result getting the extra representative:

Kent: $0.4111/7 \approx 0.0587$

New Castle: $0.5872/24 \approx 0.0245$

Sussex: $0.0017/9 \approx 0.0002$

The largest of these is Kent's, so Kent gets the 41st representative:

Table 8.1.25

County	Population	Quota	Initial	Ratio	Final
Kent	162,310	7.4111	7	0.0587	8
New Castle	538,479	24.5872	24	0.0245	24
Sussex	197,145	9.0017	9	0.0002	9
Total	**897,934**		**40**		**41**

Example 8.1.11

Rhode Island, again beginning in the same way as Hamilton:

Table 8.1.26

County	Population	Quota	Initial
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	**1,052,567**		**72**

Solution

We divide each county's quota's decimal part by its whole number part to determine which three should get the remaining representatives:

Bristol: $0.5538/3 \approx \quad 0.1846$

Kent: $0.8395/11 \approx \quad 0.0763$

Newport: $0.9061/5 \approx \quad 0.1812$

Providence: $0.6528/44 \approx \quad 0.0148$

Washington: $0.0478/9 \approx \quad 0.0053$

The three largest of these are Bristol, Newport, and Kent, so they get the remaining three representatives:

Table 8.1.27

County	Population	Quota	Initial	Ratio	Final
Bristol	49,875	3.5538	3	0.1846	4
Kent	166,158	11.8395	11	0.0763	12
Newport	82,888	5.9061	5	0.1812	6
Providence	626,667	44.6528	44	0.0148	44
Washington	126,979	9.0478	9	0.0053	9
Total	**1,052,567**		**72**		**75**

As you can see, there is no “right answer” when it comes to choosing a method for apportionment. Each method has its virtues and favors different sized states.

Reference

1. References (19)

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8.2: Apportionment of Legislative Districts

In most states, there are a fixed number of representatives to the state legislature. Rather than apportioning each county a number of representatives, legislative districts are drawn so that each legislator represents a district. The apportionment process, then, involves the drawing of the legislative districts, with the goal of having each district include approximately the same number of constituents. Because of this goal, a geographically small city may have several representatives while a large rural region may be represented by one legislator.

When populations change, it becomes necessary to redistrict the regions each legislator represents (Incidentally, this also occurs for the regions that federal legislators represent). The process of redistricting is typically done by the legislature itself, so not surprisingly it is common to see **gerrymandering.**

Definition: Gerrymandering

Gerrymandering is an intentional act whereby legislative districts are drawn based on the political affiliation of the constituents to the advantage of those drawing the boundary.

Example 8.2.1

Consider three districts, simplified to the three boxes below. On the left there is a college area that typically votes Democratic. On the right is a rural area that typically votes Republican. The rest of the people are more evenly split. The middle district has been voting 50% Democratic and 50% Republican.

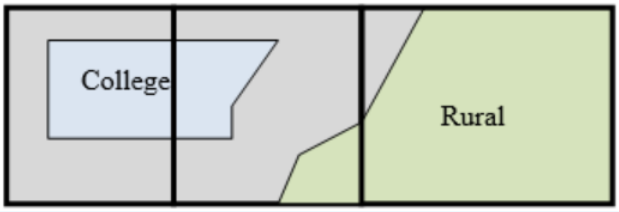


Figure 8.2.1

Solution

As part of a redistricting, a Democratic-led committee could redraw the boundaries so that the middle district includes less of the typically Republican voters, thereby making it more likely that their party will win in that district while increasing the Republican majority in the third district.



Figure 8.2.2

Gerrymandering is named after Elbridge Gerry (1744-1814). Gerrymandering works by wasting votes. It puts more votes of winners into the district they will win so the losers win in another district.

Suppose we have Group A and Group B both trying to win in a district. Group A has 40 votes and Group B has 50. In another district Group A knows Group B will win for sure. Group A changes the voting district so that 11 votes are moved to the other district. Now they will win 40 votes to 39.

Math in the Real World - *Using big data to Gerrymander*

Click here to watch a short video that explains the how big tech is making it easier to gerrymander voting districts. Why does the use of data technology in this way mean for democracy?

Example 8.2.1

The map below shows the 38th congressional district in California in 2004. This district was created through a bi-partisan committee of incumbent legislators. This gerrymandering leads to districts that are not competitive; the prevailing party almost always wins with a large margin.



Figure 8.2.3: Image from National Atlas of the United States is in the Public Domain

The map below shows the 4th congressional district in Illinois in 2004. This district was drawn to contain the two predominantly Hispanic areas of Chicago. The largely Puerto Rican area to the north and the southern Mexican areas are only connected in this districting by a piece of the highway to the west.

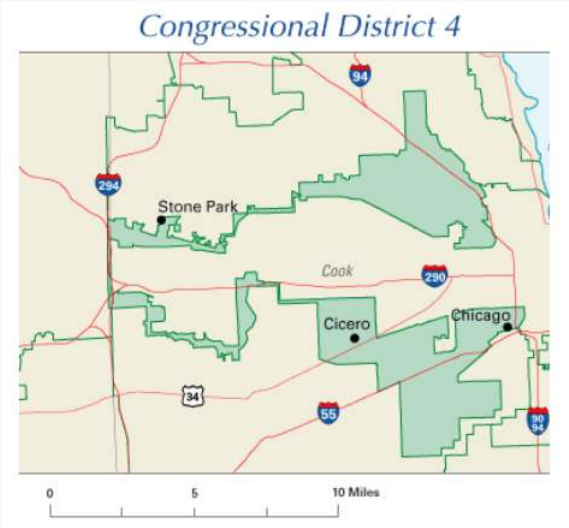


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8.3: Voting Theory

In many decision-making situations, it is necessary to gather the group consensus. This happens when a group of friends decides which movie to watch, when a company decides which product design to manufacture, and when a democratic country elects its leaders.

While the basic idea of voting is fairly universal, the method by which those votes are used to determine a winner can vary. Amongst a group of friends, you may decide upon a movie by voting for all the movies you're willing to watch, with the winner being the one with the greatest approval. A company might eliminate unpopular designs then revote on the remaining. A country might look for the candidate with the most votes.

In deciding upon a winner, there is always one main goal: to reflect the preferences of the people in the most fair way possible.

Preference Schedules

To begin, we're going to want more information than a traditional ballot normally provides. A traditional ballot usually asks you to pick your favorite from a list of choices. This ballot fails to provide any information on how a voter would rank the alternatives if their first choice was unsuccessful.

Definition: Preference Ballot

A preference ballot is a ballot in which the voter ranks the choices in order of preference.

Example 8.3.1

Ann	Marv	Alice	Eve	Omar	Lupe	Dave	Tish	Jim
A	O	H	A	O	H	O	H	A
H	H	A	H	H	A	H	A	H
O	A	O	O	A	O	A	O	O

Suppose this is the list of all preference ballots when these ten people were asked where they wanted to go for a conference: Anaheim (A), Hawaii (H), or Orlando (O). Each person orders three destinations from 1 to 3 based on their preference. There are a total of 3!=6 ways to order these three places, but only four permutations appear here.

These individual ballots are typically combined into one **preference schedule**, which shows the number of voters in the top row that voted for each option:

	1	3	3	3
1st	A	A	O	H
2nd	O	H	H	A
3rd	H	O	A	O

Notice that by totaling the vote counts across the top of the preference schedule we can recover the total number of votes cast: 1+3+3+3 = 10 total votes.

Plurality

The voting method we're most familiar with in the United States is the **plurality method**.

Plurality Method

In this method, the choice with the most first-preference votes is declared the winner. Ties are possible and would have to be settled through some sort of run-off vote.

This method is sometimes mistakenly called the majority method, or “majority rules,” but it is not necessary for a choice to have gained a majority of votes to win. A majority is over 50%; it is possible for a winner to have a **plurality** without having a majority.

Example 8.3.2

In our election from above, we had the preference table:

	1	3	3	3
1st	A	A	O	H
2nd	O	H	H	A
3rd	H	O	A	O

For the plurality method, we only care about the first choice options. Totaling them up:

Anaheim: 1+3 = 4 first-choice votes

Orlando: 3 first-choice votes

Hawaii: 3 first-choice votes

Anaheim is the winner using the plurality voting method.

Notice that Anaheim won with 4 out of 10 votes, 40% of the votes, which is a plurality of the votes, but not a majority.

Try it Now 1

Three candidates are running in an election for County Executive: Goings (G), McCarthy (M), and Bunney (B). The voting schedule is shown below. Which candidate wins under the plurality method?

	44	14	20	70	22	80	90
1st	G	G	G	M	M	B	B
2nd	M	B		G	B	M	
3rd	B	M		B	G	G	

Note: In the third column and last column, those voters only recorded a first-place vote, so we don’t know who their second and third choices would have been.

What’s Wrong with Plurality?

The election from Example 2 may seem totally clean, but there is a problem lurking that arises whenever there are three or more choices. Looking back at our preference table, how would our members vote if they only had two choices?

Anaheim vs Orlando: 7 out of the 10 would prefer Anaheim over Orlando

	1	3	3	3
1st	A	A	O	H
2nd	O	H	H	A
3rd	H	O	A	O

Anaheim vs Hawaii: 6 out of 10 would prefer Hawaii over Anaheim

	1	3	3	3
1st	A	A	O	H

2nd	O	H	H	A
3rd	H	O	A	O

This doesn't seem right, does it? Anaheim just won the election, yet 6 out of 10 voters, 60% of them, would have preferred Hawaii! That hardly seems fair. Marquis de Condorcet, a French philosopher, mathematician, and political scientist wrote about how this could happen in 1785, and for him we name our first **fairness criterion**.

Definition: Fairness Criteria

The fairness criteria are statements that should be true in a fair election.

Condorcet Criterion

If there is a choice that is preferred in every one-to-one comparison with the other choices, that choice should be the winner. We call this winner the Condorcet Winner, or Condorcet Candidate.

Example 8.3.3

In the election from Example 8.3.2, what choice is the Condorcet Winner?

Solution

We see above that Hawaii is preferred over Anaheim. Comparing Hawaii to Orlando, we can see 6 out of 10 would prefer Hawaii to Orlando.

	1	3	3	3
1st	A	A	O	H
2nd	O	H	H	A
3rd	H	O	A	O

Since Hawaii is preferred in a one-to-one comparison to both other choices, Hawaii is the Condorcet Winner.

Example 8.3.4

Consider a city council election in a district that is historically 60% Democratic voters and 40% Republican voters. Even though city council is technically a nonpartisan office, people generally know the affiliations of the candidates. In this election there are three candidates: Don and Key, both Democrats, and Elle, a Republican. A preference schedule for the votes looks as follows:

	342	214	298
1st	Elle	Don	Key
2nd	Don	Key	Don
3rd	Key	Elle	Elle

Who is the Condorcet winner?

Solution

We can see a total of 342 + 214 + 298 = 854 voters participated in this election. Computing percentage of first place votes:

Don: 214/854 = 25.1%

Key: 298/854 = 34.9%

Elle: 342/854 = 40.0%

So in this election, the Democratic voters split their vote over the two Democratic candidates, allowing the Republican candidate Elle to win under the plurality method with 40% of the vote.

Analyzing this election closer, we see that it violates the Condorcet Criterion. Analyzing the one-to-one comparisons:

Elle vs Don: 342 prefer Elle; 512 prefer Don: Don is preferred

Elle vs Key: 342 prefer Elle; 512 prefer Key: Key is preferred

Don vs Key: 556 prefer Don; 298 prefer Key: Don is preferred

So even though Don had the smallest number of first-place votes in the election, he is the Condorcet winner, being preferred in every one-to-one comparison with the other candidates.

Try it Now 2

Consider the election from Try it Now 1. Is there a Condorcet winner in this election?

	44	14	20	70	22	80	39
1st	G	G	G	M	M	B	B
2nd	M	B		G	B	M	
3rd	B	M		B	G	G	

Insincere Voting

Situations like the one in Example 4 above, when there are more than one candidate that share somewhat similar points of view, can lead to **insincere voting**. Insincere voting refers to a person casting a ballot counter to their actual preference for strategic purposes. In the case above, the democratic leadership might realize that Don and Key will split the vote, and encourage voters to vote for Key by officially endorsing him. Not wanting to see their party lose the election, as happened in the scenario above, Don's supporters might insincerely vote for Key, effectively voting against Elle.

Instant Runoff Voting

Instant Runoff Voting (IRV), also called Plurality with Elimination, is a modification of the plurality method that attempts to address the issue of insincere voting.

Instant Runoff Voting (IRV)

In IRV, voting is done with preference ballots, and a preference schedule is generated. The choice with the least first-place votes is then eliminated from the election, and any votes for that candidate are redistributed to the voters' next choice. This continues until a choice has a majority (over 50%).

This is similar to the idea of holding runoff elections, but since every voter's order of preference is recorded on the ballot, the runoff can be computed without requiring a second costly election.

This voting method is used in several political elections around the world, including election of members of the Australian House of Representatives, and was used for county positions in Pierce County, Washington until it was eliminated by voters in 2009. A version of IRV is used by the International Olympic Committee to select host nations.

Example 8.3.5

Consider the preference schedule below, in which a company's advertising team is voting on five different advertising slogans, called A, B, C, D, and E here for simplicity.

Initial votes

	3	4	4	6	2	1
1st	B	C	B	D	B	E

2nd	C	A	D	C	E	A
3rd	A	D	C	A	A	D
4th	D	B	A	E	C	B
5th	E	E	E	B	D	C

Which one of the A, B, C, D, E will be declared winner under IRV?

Solution

If this was a plurality election, note that B would be the winner with 9 first-choice votes, compared to 6 for D, 4 for C, and 1 for E.

There are total of 3+4+4+6+2+1 = 20 votes. A majority would be 11 votes. No one yet has a majority, so we proceed to elimination rounds.

Round 1: We make our first elimination. Choice A has the fewest first-place votes, so we remove that choice.

	3	4	4	6	2	1
1st	B	C	B	D	B	E
2nd	C		D	C	E	
3rd		D	C			D
4th	D	B		E	C	B
5th	E	E	E	B	D	C

We then shift everyone's choices up to fill the gaps. There is still no choice with a majority, so we eliminate again.

	3	4	4	6	2	1
1st	B	C	B	D	B	E
2nd	C	D	D	C	E	D
3rd	D	B	C	E	C	B
4th	E	E	E	B	D	C

Round 2: We make our second elimination. Choice E has the fewest first-place votes, so we remove that choice, shifting everyone's options to fill the gaps.

	3	4	4	6	2	1
1st	B	C	B	D	B	D
2nd	C	D	D	C	C	B
3rd	D	B	C	B	D	C

Notice that the first and fifth columns have the same preferences now, we can condense those down to one column.

	5	4	4	6	1
1st	B	C	B	D	D
2nd	C	D	D	C	B
3rd	D	B	C	B	C

Now B has 9 first-choice votes, C has 4 votes, and D has 7 votes. Still no majority, so we eliminate again.

Round 3: We make our third elimination. C has the fewest votes.

	5	4	4	6	1
1st	B	D	B	D	D
2nd	D	B	D	B	B

Condensing this down:

	9	11
1st	B	D
2nd	D	B

D has now gained a majority, and is declared the winner under IRV.

Try it Now 3

Consider again the election from Try it Now 1. Find the winner using IRV.

		44	14	20	70	22	80	39
	1st	G	G	G	M	M	B	B
	2nd	M	B		G	B	M	
	3rd	B	M		B	G	G	

What's Wrong with IRV?

Example 8.3.6

Let's return to our City Council Election

	342	214	298
1st	Elle	Don	Key
2nd	Don	Key	Don
3rd	Key	Elle	Elle

In this election, Don has the smallest number of first place votes, so Don is eliminated in the first round. The 214 people who voted for Don have their votes transferred to their second choice, Key.

Solution

	342	512
1st	Elle	Key
2nd	Key	Elle

So Key is the winner under the IRV method.

We can immediately notice that in this election, IRV violates the Condorcet Criterion since we determined earlier that Don was the Condorcet winner. On the other hand, the temptation has been removed for Don's supporters to vote for Key; they now know their vote will be transferred to Key, not simply discarded.

Example 8.3.7

Consider the voting system below.

	37	22	12	29
1st	Adams	Brown	Brown	Carter
2nd	Brown	Carter	Adams	Adams
3rd	Carter	Adams	Carter	Brown

Solution

In this election, Carter would be eliminated in the first round, and Adams would be the winner with 66 votes to 34 for Brown.

Now suppose that the results were announced, but election officials accidentally destroyed the ballots before they could be certified, and the votes had to be recast. Wanting to "jump on the bandwagon", 10 of the voters who had originally voted in the order Brown, Adams, Carter change their vote to favor the presumed winner, changing those votes to Adams, Brown, Carter.

	47	22	2	29
1st	Adams	Brown	Brown	Carter
2nd	Brown	Carter	Adams	Adams
3rd	Carter	Adams	Carter	Brown

In this re-vote, Brown will be eliminated in the first round, having the fewest first-place votes. After transferring votes, we find that Carter will win this election with 51 votes to Adams' 49 votes! Even though the only vote changes made *favored* Adams, the change ended up costing Adams the election. This doesn't seem right and introduces our second fairness criterion below.

Monotonocity Criterion

If voters change their votes to increase the preference for a candidate, it should not harm that candidate's chances of winning.

This criterion is violated by this election. Note that even though the criterion is violated in this particular election, it does not mean that IRV always violates the criterion; it just means that IRV has the potential to violate the criterion in certain elections.

Borda Count

Borda Count is another voting method, named for Jean-Charles de Borda, who developed the system in 1770.

Borda Count

In this method, points are assigned to candidates based on their ranking; 1 point for last choice, 2 points for second-to-last choice, and so on. The point values for all ballots are totaled, and the candidate with the largest point total is the winner.

Example 8.3.8

A group of managers are getting together for a conference. The members are coming from four cities: Seattle, Tacoma, Puyallup, and Olympia. Their approximate locations on a map are shown below.



Figure 8.3.1

The votes for where to hold the conference were as follows:

	51	25	10	14
1st	Seattle	Tacoma	Puyallup	Olympia
2nd	Tacoma	Puyallup	Tacoma	Tacoma
3rd	Olympia	Olympia	Olympia	Puyallup
4th	Puyallup	Seattle	Seattle	Seattle

Use Borda count method to find the winner of this vote.

Solution

In each of the 51 ballots ranking Seattle first, Puyallup will be given 1 point, Olympia 2 points, Tacoma 3 points, and Seattle 4 points. Multiplying the points per vote times the number of votes allows us to calculate points awarded:

	51	25	10	14
1st choice 4 points	Seattle 4·51 = 204	Tacoma 4·25 = 100	Puyallup 4·10 = 40	Olympia 4·14 = 56
2nd choice 3 points	Tacoma 3·51 = 153	Puyallup 3·25 = 75	Tacoma 3·10 = 30	Tacoma 3·14 = 42
3rd choice 2 points	Olympia 2·51 = 102	Olympia 2·25 = 50	Olympia 2·10 = 20	Puyallup 2·14 = 28
4th choice 1 point	Puyallup 1·51 = 51	Seattle 1·25 = 25	Seattle 1·10 = 10	Seattle 1·14 = 14

Adding up the points:

Seattle: 204 + 25 + 10 + 14 = 253 points

Tacoma: 153 + 100 + 30 + 42 = 325 points

Puyallup: 51 + 75 + 40 + 28 = 194 points

Olympia: 102 + 50 + 20 + 56 = 228 points

Under the Borda Count method, Tacoma is the winner of this vote.

Try it Now 4

Consider again the election from Try it Now 1. Find the winner using Borda Count. Since we have some incomplete preference ballots, for simplicity, give every unranked candidate 1 point, the points they would normally get for last place.

		44	14	20	70	22	80	39
	1st	G	G	G	M	M	B	B
	2nd	M	B		G	B	M	
	3rd	B	M		B	G	G	

What's Wrong with Borda Count?

You might have already noticed one potential flaw of the Borda Count from the previous example. In that example, Seattle had a majority of first-choice votes, yet lost the election! This seems odd and prompts our next fairness criterion:

Majority Criterion

If a choice has a majority of first-place votes, that choice should be the winner.

The election from Example 8 using the Borda Count violates the Majority Criterion. Notice also that this automatically means that the Condorcet Criterion will also be violated, as Seattle would have been preferred by 51% of voters in any head-to-head comparison.

Borda count is sometimes described as a consensus-based voting system since it can sometimes choose a more broadly acceptable option over the one with majority support. In the example above, Tacoma is probably the best compromise location. This is an approach different from plurality and instant runoff voting, which focuses on first-choice votes; Borda Count considers every voter's entire ranking to determine the outcome.

Because of this consensus behavior, Borda Count, or some variation of it, is commonly used in awarding sports awards. Variations are used to determine the Most Valuable Player in baseball, to rank teams in NCAA sports, and to award the Heisman trophy.

Copeland's Method (Pairwise Comparisons)

So far none of our voting methods have satisfied the Condorcet Criterion. The Copeland Method specifically attempts to satisfy the Condorcet Criterion by looking at pairwise (one-to-one) comparisons.

Copeland's Method

In this method, each pair of candidates is compared, using all preferences to determine which of the two is more preferred. The more preferred candidate is awarded 1 point. If there is a tie, each candidate is awarded ½ point. After all pairwise comparisons are made, the candidate with the most points, and hence the most pairwise wins, is declared the winner.

Variations of Copeland's Method are used in many professional organizations, including election of the Board of Trustees for the Wikimedia Foundation, which runs Wikipedia.

Example 8.3.9

Consider our vacation group example from the beginning of the chapter. Determine the winner using Copeland's Method.

	1	3	3	3
1st	A	A	O	H
2nd	O	H	H	A
3rd	H	O	A	O

Solution

We need to look at each pair of choices, and see which choice would win in a one-to-one comparison. You may recall we did this earlier when determining the Condorcet Winner. For example, comparing Hawaii vs Orlando, we see that 6 voters, those shaded below in the first table below, would prefer Hawaii to Orlando. Note that Hawaii doesn't have to be the voter's first choice – we're imagining that Anaheim wasn't an option. If it helps, you can imagine removing Anaheim, as in the second table below.

	1	3	3	3
1st	A	A	O	H
2nd	O	H	H	A
3rd	H	O	A	O

	1	3	3	3
1st			O	H
2nd	O	H	H	
3rd	H	O		O

Based on this, in the comparison of Hawaii vs Orlando, Hawaii wins and receives 1 point.

Comparing Anaheim to Orlando, the 1 voter in the first column clearly prefers Anaheim, as do the 3 voters in the second column. The 3 voters in the third column clearly prefer Orlando. The 3 voters in the last column prefer Hawaii as their first choice, but if they had to choose between Anaheim and Orlando, they'd choose Anaheim, their second choice overall. So, altogether 1+3+3=7 voters prefer Anaheim over Orlando, and 3 prefer Orlando over Anaheim. So, comparing Anaheim vs Orlando: 7 votes to 3 votes, Anaheim gets 1 point.

All together,

Hawaii vs Orlando: 6 votes to 4 votes: Hawaii gets 1 point

Anaheim vs Orlando: 7 votes to 3 votes: Anaheim gets 1 point

Hawaii vs Anaheim: 6 votes to 4 votes: Hawaii gets 1 point

Hawaii is the winner under Copeland's Method, having earned the most points.

Notice this process is consistent with our determination of a Condorcet Winner.

Example 8.3.10

	3	4	4	6	2	1
1st	B	C	B	D	B	E
2nd	C	A	D	C	E	A
3rd	A	D	C	A	A	D
4th	D	B	A	E	C	B
5th	E	E	E	B	D	C

Determine a winner using Copeland's Method.

Solution

With 5 candidates, there are 10 comparisons to make:

A vs B: 11 votes to 9 votes A gets 1 point

A vs C: 3 votes to 17 votes C gets 1 point

A vs D: 10 votes to 10 votes A gets ½ point, D gets ½ point

A vs E: 17 votes to 3 votes A gets 1 point

B vs C: 10 votes to 10 votes B gets ½ point, C gets ½ point

B vs D: 9 votes to 11 votes D gets 1 point

B vs E: 13 votes to 7 votes B gets 1 point

C vs D: 9 votes to 11 votes D gets 1 point

C vs E: 17 votes to 3 votes C gets 1 point

D vs E: 17 votes to 3 votes D gets 1 point

Totaling these up:

A gets 2½ points

B gets 1½ points

C gets 2½ points

D gets 3½ points

E gets 0 points

Using Copeland's Method, we declare D as the winner.

Notice that in this case, D is not a Condorcet Winner. While Copeland's method will also select a Condorcet Candidate as the winner, the method still works in cases where there is no Condorcet Winner.

Try it Now 5

Consider again the election from Try it Now 1. Find the winner using Copeland's method. Since we have some incomplete preference ballots, we'll have to adjust. For example, when comparing M to B, we'll ignore the 20 votes in the third column which do not rank either candidate.

	44	14	20	70	22	80	39
1st	G	G	G	M	M	B	B
2nd	M	B		G	B	M	
3rd	B	M		B	G	G	

What's Wrong with Copeland's Method?

As already noted, Copeland's Method does satisfy the Condorcet Criterion. It also satisfies the Majority Criterion and the Monotonicity Criterion. So is this the perfect method? Well, in a word, no.

Example 8.3.11

A committee is trying to award a scholarship to one of four students, Anna (A), Brian (B), Carlos (C), and Dimitry (D). The votes are shown below:

	5	5	6	4
1st	D	A	C	B
2nd	A	C	B	D
3rd	C	B	D	A
4th	B	D	A	C

Determine the winner.

Solution

Making the comparisons:

A vs B: 10 votes to 10 votes A gets ½ point, B gets ½ point

A vs C: 14 votes to 6 votes: A gets 1 point

A vs D: 5 votes to 15 votes: D gets 1 point

B vs C: 4 votes to 16 votes: C gets 1 point

B vs D: 15 votes to 5 votes: B gets 1 point

C vs D: 11 votes to 9 votes: C gets 1 point

Totaling:

A has 1 ½ points B has 1 ½ points

C has 2 points D has 1 point

So Carlos is awarded the scholarship. However, the committee then discovers that Dimitry was not eligible for the scholarship (he failed his last math class). Even though this seems like it shouldn't affect the outcome, the committee decides to recount the vote, removing Dimitry from consideration. This reduces the preference schedule:

	5	5	6	4
1st	A	A	C	B
2nd	C	C	B	A
3rd	B	B	A	C

A vs B: 10 votes to 10 votes A gets ½ point, B gets ½ point

A vs C: 14 votes to 6 votes A gets 1 point

B vs C: 4 votes to 16 votes C gets 1 point

Totaling:

A has 1 ½ points B has ½ point

C has 1 point

Suddenly Anna is the winner! This leads us to another fairness criterion.

The Independence of Irrelevant Alternatives (IIA) Criterion

If a non-winning choice is removed from the ballot, it should not change the winner of the election.

Equivalently, if choice A is preferred over choice B, introducing or removing a choice C should not cause B to be preferred over A.

In the election from Example 8.3.11, the IIA Criterion was violated.

This anecdote illustrating the IIA issue is attributed to Sidney Morgenbesser:

After finishing dinner, Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes the waitress returns and says that they also have cherry pie at which point Morgenbesser says "In that case I'll have the blueberry pie."

Another disadvantage of Copeland's Method is that it is fairly easy for the election to end in a tie. For this reason, Copeland's method is usually the first part of a more advanced method that uses more sophisticated methods for breaking ties and determining the winner when there is not a Condorcet Candidate.

So Where's the Fair Method?

At this point, you're probably asking why we keep looking at method after method just to point out that they are not fully fair. We must be holding out on the perfect method, right?

Unfortunately, no. A mathematical economist, Kenneth Arrow, was able to prove in 1949 that there is no voting method that will satisfy all the fairness criteria we have discussed.

Arrow's Impossibility Theorem

It is not possible for a voting method to satisfy every fairness criteria that we've discussed.

To see a very simple example of how difficult voting can be, consider the election below:

	5	5	5
1st	A	C	B
2nd	B	A	C
3rd	C	B	A

Notice that in this election:

10 people prefer A to B

10 people prefer B to C

10 people prefer C to A

No matter whom we choose as the winner, 2/3 of voters would prefer someone else! This scenario is dubbed **Condorcet's Voting Paradox** and demonstrates how voting preferences are not transitive (the facts that A is preferred over B, and B over C, do not mean A is preferred over C). In this election, there is no fair resolution.

It is because of this impossibility of a totally fair method that Plurality, IRV, Borda Count, Copeland's Method, and dozens of variants are all still used. Usually the decision of which method to use is based on what seems most fair for the situation in which it is being applied.

Approval Voting

Up until now, we've been considering voting methods that require ranking of candidates on a preference ballot. There is another method of voting that can be more appropriate in some decision making scenarios. With Approval Voting, the ballot asks you to mark all choices that you find acceptable. The results are tallied, and the option with the most approval is the winner.

Example 8.3.12

A group of friends is trying to decide upon a movie to watch. Three choices are provided, and each person is asked to mark with an "X" which movies they are willing to watch. The results are as follows:

	Bob	Ann	Marv	Alice	Eve	Omar	Lupe	Dave	Tish	Jim
Titanic		X	X			X		X		X
Scream	X		X	X		X	X		X	
The Matrix	X	X	X	X	X		X			X

Find the winner.

Solution

Totaling the results, we find

Titanic received 5 approvals

Scream received 6 approvals

The Matrix received 7 approvals.

In this vote, The Matrix would be the winner.

Try it Now 6

Our managers deciding on a conference location from earlier decide to use Approval voting. Their votes are tallied below. Find the winner using Approval voting.

	30	10	15	20	15	5	5
Seattle	X	X	X			X	
Tacoma	X		X	X	X	X	
Puyallup		X		X	X	X	
Olympia			X		X		X

What's Wrong with Approval Voting?

Approval voting can very easily violate the Majority Criterion.

Example 8.3.13

Consider the voting schedule:

	80	15	5
1st	A	B	C
2nd	B	C	B
3rd	C	A	A

Clearly A is the majority winner. Now suppose that this election was held using Approval Voting, and every voter marked approval of their top two candidates.

A would receive approval from 80 voters

B would receive approval from 100 voters

C would receive approval from 20 voters

B would be the winner. Some argue that Approval Voting tends to vote the least disliked choice, rather than the most liked candidate.

Additionally, Approval Voting is susceptible to strategic insincere voting, in which a voter does not vote their true preference to try to increase the chances of their choice winning. For example, in the movie example above, suppose Bob and Alice would much rather watch Scream. They remove The Matrix from their approval list, resulting in a different result.

	Bob	Ann	Marv	Alice	Eve	Omar	Lupe	Dave	Tish	Jim
Titanic		X	X			X		X		X
Scream	X		X	X		X	X		X	
The Matrix		X	X		X		X			X

Totaling the results, we find Titanic received 5 approvals, Scream received 6 approvals, and The Matrix received 5 approvals. By voting insincerely, Bob and Alice were able to sway the result in favor of their preference.

Voting in America

In American politics, there is a lot more to selecting our representatives than simply casting and counting ballots. The process of selecting the president is even more complicated. Instead, let's look at the process by which state congressional representatives and local politicians get elected.

For most offices, a sequence of two public votes is held: a primary election and the general election. For non-partisan offices like sheriff and judge, in which political party affiliation is not declared, the primary election is usually used to narrow the field of candidates.

Typically, the two candidates receiving the most votes in the primary will then move forward to the general election. While somewhat similar to instant runoff voting, this is actually an example of **sequential voting** - a process in which voters cast totally new ballots after each round of eliminations. Sequential voting has become quite common in television, where it is used in reality competition shows like *American Idol*.

Congressional, county, and city representatives are partisan offices, in which candidates usually declare themselves a member of a political party, like the Democrats, Republicans, the Green Party, or one of the many other smaller parties. As with non-partisan offices, a primary election is usually held to narrow down the field prior to the general election. Prior to the primary election, the candidate would have met with the political party leaders and gotten their approval to run under that party's affiliation.

In some states a **closed primary** is used, in which only voters who are members of the Democrat party can vote on the Democratic candidates, and similar for Republican voters. In other states, an **open primary** is used, in which any voter can pick the party whose primary they want to vote in. Other states use **caucuses**, which are basically meetings of the political parties, only open to party members. Closed primaries are often disliked by independent voters, who like the flexibility to change which party they are voting in. Open primaries do have the disadvantage that they allow raiding, in which a voter will vote in their non-preferred party's primary with the intent of selecting a weaker opponent for their preferred party's candidate.

Some states currently use a different method, called a **top 2 primary**, in which voters select from the candidates from all political parties on the primary; top two candidates, regardless of party affiliation, move on to the general election. While this method is liked by independent voters, it gives the political parties incentive to select a top candidate internally before the primary, so that two candidates will not split the party's vote.

Regardless of the primary type, the general election is the main election, open to all voters. Except in the case of the top 2 primary, the top candidate from each major political party would be included in the general election. While rules vary state-to-state, for independent or minor-party candidates to get listed on the ballot, they typically have to gather a certain number of signatures to petition for inclusion.

Reference

1. References (20)

Contributors and Attributions

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8.4: Weighted Voting

In a corporate shareholders meeting, each shareholders' vote counts proportional to the amount of shares they own. An individual with one share gets the equivalent of one vote, while someone with 100 shares gets the equivalent of 100 votes. This is called weighted voting, where each vote has some weight attached to it. Weighted voting is sometimes used to vote on candidates, but more commonly to decide "yes" or "no" on a proposal, sometimes called a motion. Weighted voting is applicable in corporate settings, as well as decision making in parliamentary governments and voting in the United Nations Security Council.

In weighted voting, we are most often interested in the power each voter has in influencing the outcome.

Beginnings

We'll begin with some basic vocabulary for weighted voting systems.

Vocabulary for Weighted Voting

Each individual or entity casting a vote is called a **player** in the election. They're often notated as $P_1, P_2, P_3, \ldots P_N$, where N is the total number of voters.

Each player is given a **weight**, which usually represents how many votes they get.

The quota is the minimum weight needed for the votes or weight needed for the proposal to be approved. A weighted voting system will often be represented in a shorthand form:

$$[q : w_1, w_2, w_3, \ldots, w_n]$$

In this form, q is the quota, $w_1$is the weight for player 1, and so on.

Example 8.4.1

In a small company, there are 4 shareholders. Mr. Smith has a 30% ownership stake in the company, Mr. Garcia has a 25% stake, Mrs. Hughes has a 25% stake, and Mrs. Lee has a 20% stake. They are trying to decide whether to open a new location. The company by-laws state that more than 50% of the ownership has to approve any decision like this. This could be represented by the weighted voting system:

$$[51 : 30, 25, 25, 20]$$

Here we have treated the percentage ownership as votes, so Mr. Smith gets the equivalent of 30 votes, having a 30% ownership stake. Since more than 50% is required to approve the decision, the quota is 51, the smallest whole number over 50.

In order to have a meaningful weighted voting system, it is necessary to put some limits on the quota.

Limits on the Quota

The quota must be more than ½ the total number of votes.

The quota can't be larger than the total number of votes.

Why? Consider the voting system $[q : 3, 2, 1]$

Here there are 6 total votes. If the quota was set at only 3, then player 1 could vote yes, players 2 and 3 could vote no, and both would reach quota, which doesn't lead to a decision being made. In order for only one decision to reach quota at a time, the quota must be at least half the total number of votes. If the quota was set to 7, then no group of voters could ever reach quota, and no decision can be made, so it doesn't make sense for the quota to be larger than the total number of voters.

Try it Now 1

In a committee there are four representatives from the management and three representatives from the workers' union. For a proposal to pass, four of the members must support it, including at least one member of the union. Find a voting system that can represent this situation.

A Look at Power

Consider the voting system [10: 11, 3, 2]. Notice that in this system, player 1 can reach quota without the support of any other player. When this happens, we say that player 1 is a dictator.

Definition: Dictator

A player will be a dictator if their weight is equal to or greater than the quota. The dictator can also block any proposal from passing; the other players cannot reach quota without the dictator.

In the voting system [8: 6, 3, 2], no player is a dictator. However, in this system, the quota can only be reached if player 1 is in support of the proposal; player 2 and 3 cannot reach quota without player 1's support. In this case, player 1 is said to have veto power. Notice that player 1 is not a dictator, since player 1 would still need player 2 or 3's support to reach quota.

Definition: Veto Power

A player has veto power if their support is necessary for the quota to be reached. It is possible for more than one player to have veto power, or for no player to have veto power.

With the system [10: 7, 6, 2], player 3 is said to be a dummy, meaning they have no influence in the outcome. The only way the quota can be met is with the support of both players 1 and 2 (both of which would have veto power here); the vote of player 3 cannot affect the outcome.

Definition: Dummy

A player is a dummy if their vote is never essential for a group to reach quota.

Example 8.4.2

In the voting system [16: 7, 6, 3, 3, 2], are any players dictators? Do any have veto power? Are any dummies?

Solution

No player can reach quota alone, so there are no dictators.

Without player 1, the rest of the players' weights add to 14, which doesn't reach quota, so player 1 has veto power. Likewise, without player 2, the rest of the players' weights add to 15, which doesn't reach quota, so player 2 also has veto power.

Since player 1 and 2 can reach quota with either player 3 or player 4's support, neither player 3 or player 4 have veto power. However they cannot reach quota with player 5's support alone, so player 5 has no influence on the outcome and is a dummy.

Try it Now 2

In the voting system $[q: 10, 5, 3]$, which players are dictators, have veto power, and are dummies if the quota is 10? 12? 16?

To better define power, we need to introduce the idea of a **coalition**. A coalition is a group of players voting the same way. In the example above, {P1, P2, P4} would represent the coalition of players 1, 2 and 4. This coalition has a combined weight of 7+6+3 = 16, which meets quota, so this would be a winning coalition.

A player is said to be **critical** in a coalition if them leaving the coalition would change it from a winning coalition to a losing coalition. In the coalition {P1, P2, P4}, every player is critical. In the coalition {P3, P4, P5}, no player is critical, since it wasn't a winning coalition to begin with. In the coalition {P1, P2, P3, P4, P5}, only players 1 and 2 are critical; any other player could leave the coalition and it would still meet quota.

Definition: Coalition

A coalition is any group of players voting the same way. A coalition is a winning coalition if the coalition has enough weight to meet quota.

Definition: Critical Player

A player is critical in a coalition if them leaving the coalition would change it from a winning coalition to a losing coalition.

Example 8.4.3

In the Scottish Parliament in 2009 there were 5 political parties: 47 representatives for the Scottish National Party, 46 for the Labour Party, 17 for the Conservative Party, 16 for the Liberal Democrats, and 2 for the Scottish Green Party. Typically all representatives from a party vote as a block, so the parliament can be treated like the weighted voting system:

[65: 47, 46, 17, 16, 2]

Solution

Consider the coalition {P1, P3, P4}. No two players alone could meet the quota, so all three players are critical in this coalition.

In the coalition {P1, P3, P4, P5}, any player except P1 could leave the coalition and it would still meet quota, so only P1 is critical in this coalition.

Notice that a player with veto power will be critical in every winning coalition, since removing their support would prevent a proposal from passing.

Likewise, a dummy will never be critical, since their support will never change a losing coalition to a winning one.

Dictators,veto, and Dummies and Critical Players

- A player is a **dictator** if the single-player coalition containing them is a winning coalition.
- A player has **veto power** if they are critical in every winning coalition.
- A player is a **dummy** if they are not critical in any winning coalition.

Calculating Power: Banzhaf Power Index

The Banzhaf power index was originally created in 1946 by Lionel Penrose, but was reintroduced by John Banzhaf in 1965. The power index is a numerical way of looking at power in a weighted voting situation.

Calculating the Banzhaf Power Index

To calculate the Banzhaf power index:

1. List all winning coalitions
2. In each coalition, identify the players who are critical
3. Count up how many times each player is critical
4. Convert these counts to fractions or decimals by dividing by the total times any player is critical

Example 8.4.4

Find the Banzhaf power index for the voting system [8: 6, 3, 2].

Solution

We start by listing all winning coalitions. If you aren't sure how to do this, you can list all coalitions, then eliminate the non-winning coalitions. No player is a dictator, so we'll only consider two and three player coalitions.

$\{P_1, P_2\}$ Total weight: 9. Meets quota.

$\{P_1, P_3\}$ Total weight: 8. Meets quota.

$\{P_2, P_3\}$ Total weight: 5. Does not meet quota.

$\{P_1, P_2, P_3\}$Total weight: 11. Meets quota.

Next we determine which players are critical in each winning coalition. In the winning two-player coalitions, both players are critical since no player can meet quota alone. Underlining the critical players to make it easier to count:

$\{\underline{P_1}, \underline{P_2}\}$

$\{\underline{P_1}, \underline{P_3}\}$

In the three-person coalition, either P2 or P3 could leave the coalition and the remaining players could still meet quota, so neither is critical. If P1 were to leave, the remaining players could not reach quota, so P1 is critical.

$\{P_1, P_2, P_3\}$

Altogether, P1 is critical 3 times, P2 is critical 1 time, and P3 is critical 1 time.

Converting to percents:

$P_1 = 3/5 = 60\%$
$P_2 = 1/5 = 20\%$
$P_3 = 1/5 = 20\%$

Example 8.4.5

Consider the voting system [16: 7, 6, 3, 3, 2]. Find the Banzhaf power index.

Solution

The winning coalitions are listed below, with the critical players underlined.

$\{P_1, P_2, P_3\}$
$\{P_1, P_2, P_4\}$
$\{P_1, P_2, P_3, P_4\}$
$\{P_1, P_2, P_3, P_5\}$
$\{P_1, P_2, P_4, P_5\}$
$\{P_1, P_2, P_3, P_4, P_5\}$

Counting up times that each player is critical:

- $P_1 = 6$
- $P_2 = 6$
- $P_3 = 2$
- $P_4 = 2$
- $P_5 = 0$

Total of all: 16

Divide each player's count by 16 to convert to fractions or percents:

$P_1 = 6/16 = 3/8 = 37.5\%$
$P_2 = 6/16 = 3/8 = 37.5\%$
$P_3 = 2/16 = 1/8 = 12.5\%$
$P_4 = 2/16 = 1/8 = 12.5\%$

The Banzhaf power index measures a player's ability to influence the outcome of the vote. Notice that player 5 has a power index of 0, indicating that there is no coalition in which they would be critical power and could influence the outcome. This means player 5 is a dummy, as we noted earlier.

Example 8.4.6

Revisiting the Scottish Parliament, with voting system [65: 47, 46, 17, 16, 2], the winning coalitions are listed, with the critical players underlined.

$\{\underline{P_1}, \underline{P_2}\}$
$\{\underline{P_1}, \underline{P_2}, P_3\}$ $\{\underline{P_1}, \underline{P_2}, P_4\}$
$\{\underline{P_1}, \underline{P_2}, P_5\}$ $\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$
$\{\underline{P_1}, \underline{P_3}, \underline{P_5}\}$ $\{\underline{P_1}, \underline{P_4}, \underline{P_5}\}$
$\{\underline{P_2}, \underline{P_3}, \underline{P_4}\}$ $\{\underline{P_2}, \underline{P_3}, \underline{P_5}\}$
$\{P_1, P_2, P_3, P_4\}$ $\{P_1, P_2, P_3, P_5\}$
$\{\underline{P_1}, P_2, P_4, P_5\}$ $\{\underline{P_1}, P_3, P_4, P_5\}$
$\{\underline{P_2}, P_3, P_4, P_5\}$
$\{P_1, P_2, P_3, P_4, P_5\}$

Counting up times that each player is critical:

District	Times critical	Power index
P_1 (Scottish National Party)	9	9/27 = 33.3%
P_2 (Labour Party)	7	7/27 = 25.9%
P_3 (Conservative Party)	5	5/27 = 18.5%
P_4 (Liberal Democrats Party)	3	3/27 = 11.1%
P_5 (Scottish Green Party)	3	3/27 = 11.1%

Interestingly, even though the Liberal Democrats party has only one less representative than the Conservative Party, and 14 more than the Scottish Green Party, their Banzhaf power index is the same as the Scottish Green Party's. In parliamentary governments, forming coalitions is an essential part of getting results, and a party's ability to help a coalition reach quota defines its influence.

Try it Now 3

Find the Banzhaf power index for the weighted voting system [36: 20, 17, 16, 3].

Example 8.4.7

Banzhaf used this index to argue that the weighted voting system used in the Nassau County Board of Supervisors in New York was unfair. The county was divided up into 6 districts, each getting voting weight proportional to the population in the district, as shown below. Calculate the power index for each district.

District	Weight
Hempstead #1	31
Hempstead #2	31
Oyster Bay	28
North Hempstead	21
Long Beach	2
Glen Cove	2

Solution

Translated into a weighted voting system, assuming a simple majority is needed for a proposal to pass:

[58: 31, 31, 28, 21, 2, 2]

Listing the winning coalitions and marking critical players:

{H1, H2} {H1, OB} {H2, OB} {H1, H2, NH} {H1, H2, LB} {H1, H2, GC} {H1, H2, NH, LB} {H1, H2, NH, GC} {H1, H2, LB, GC} {H1, H2, NH, LB. GC}	{H1, OB, NH} {H1, OB, LB} {H1, OB, GC} {H1, OB, NH, LB} {H1, OB, NH, GC} {H1, OB, LB, GC} {H1, OB, NH, LB. GC} {H2, OB, NH} {H2, OB, LB} {H2, OB, GC}	{H2, OB, NH, LB} {H2, OB, NH, GC} {H2, OB, LB, GC} {H2, OB, NH, LB, GC} {H1, H2, OB} {H1, H2, OB, NH} {H1, H2, OB, LB} {H1, H2, OB, GC} {H1, H2, OB, NH, LB} {H1, H2, OB, NH, GC} {H1, H2, OB, NH, LB, GC}

There are a lot of them! Counting up how many times each player is critical,

District	Times critical	Power index
Hempstead #1	16	16/48 = 1/3 = 33%
Hempstead #2	16	16/48 = 1/3 = 33%
Oyster Bay	16	16/48 = 1/3 = 33%
North Hempstead	0	0/48 = 0%
Long Beach	0	0/48 = 0%
Glen Cove	0	0/48 = 0%

It turns out that the three smaller districts are dummies. Any winning coalition requires two of the larger districts.

The weighted voting system that Americans are most familiar with is the Electoral College system used to elect the President. In the Electoral College, states are given a number of votes equal to the number of their congressional representatives (house + senate). Most states give all their electoral votes to the candidate that wins a majority in their state, turning the Electoral College into a weighted voting system, in which the states are the players. As I'm sure you can imagine, there are billions of possible winning coalitions, so the power index for the Electoral College has to be computed by a computer using approximation techniques.

Calculating Power: Shapley-Shubik Power Index

The **Shapley-Shubik** power index was introduced in 1954 by economists Lloyd Shapley and Martin Shubik, and provides a different approach for calculating power.

In situations like political alliances, the order in which players join an alliance could be considered the most important consideration. In particular, if a proposal is introduced, the player that joins the coalition and allows it to reach quota might be considered the most essential. The Shapley-Shubik power index counts how likely a player is to be **pivotal.** What does it mean for a player to be pivotal?

First, we need to change our approach to coalitions. Previously, the coalition $\{P_1, P_2\}$ and $\{P_2, P_1\}$ would be considered equivalent, since they contain the same players. We now need to consider the order in which players join the coalition. For that, we will consider sequential coalitions – coalitions that contain all the players in which the order players are listed reflect the order they joined the coalition. For example, the sequential coalition

<P2, P1, P3> would mean that P2 joined the coalition first, then P1, and finally P3. The angle brackets < > are used instead of curly brackets to distinguish sequential coalitions.

Definition: Pivotal Player

A sequential coalition lists the players in the order in which they joined the coalition.

A pivotal player is the player in a sequential coalition that changes a coalition from a losing coalition to a winning one. Notice there can only be one pivotal player in any sequential coalition.

Example 8.4.8

In the weighted voting system [8: 6, 4, 3, 2], which player is pivotal in the sequential coalition <P3, P2, P4, P1>?

Solution

The sequential coalition shows the order in which players joined the coalition. Consider the running totals as each player joins:

P_3 Total weight: 3 Not winning

P_3, P_2 Total weight: $3+4=7$ Not winning

P_3, P_2, P_4 Total weight: $3+4+2=9$ Winning

P_3, P_2, P_4, P_1 Total weight: $3+4+2+6=15$ Winning

Since the coalition becomes winning when P_4 joins, P_4 is the pivotal player in this coalition.

Calculating Shapley-Shubik Power Index

To calculate the Shapley-Shubik Power Index:

1. List all sequential coalitions
2. In each sequential coalition, determine the pivotal player
3. Count up how many times each player is pivotal
4. Convert these counts to fractions or decimals by dividing by the total number of sequential coalitions

How many sequential coalitions should we expect to have? If there are N players in the voting system, then there are N possibilities for the first player in the coalition, $N-1$ possibilities for the second player in the coalition, and so on. Combining these possibilities, the total number of coalitions would be:

$$N(N-1)(N-2)(3-N)\ldots(3)(2)(1)$$

This calculation is called a **factorial**, and is notated $N!$ The number of sequential coalitions with N players is $N!$

Example 8.4.9

How many sequential coalitions will there be in a voting system with 7 players?

Solution

There will be 7! sequential coalitions.

As you can see, computing the Shapley-Shubik power index by hand would be very difficult for voting systems that are not very small.

Example 8.4.10

Consider the weighted voting system [6: 4, 3, 2]. We will list all the sequential coalitions and identify the pivotal player. We will have 3! = 6 sequential coalitions. The coalitions are listed, and the pivotal player is underlined.

$< P_1, \underline{P_2}, P_3 >$ $< P_1, \underline{P_3}, P_2 >$ $< P_2, \underline{P_{1_2}} P_3 >$

$< P_2, P_3, \underline{P_1} >$ $< P_3, P_2, \underline{P_1} >$ $< P_3, \underline{P_1}, P_2 >$

Calculate the Shapley-Shubik Power Index.

Solution

P_1 is pivotal 4 times, P_2 is pivotal 1 time, and P_3 is pivotal 1 time.

Player	Times pivotal	Power index
P1	4	4/6 = 66.7%
P2	1	1/6 = 16.7%
P3	1	1/6 = 16.7%

For comparison, the Banzhaf power index for the same weighted voting system would be P1: 60%, P2: 20%, P3: 20%. While the Banzhaf power index and Shapley-Shubik power index are usually not terribly different, the two different approaches usually produce somewhat different results.

Try it Now 4

Find the Shapley-Shubik power index for the weighted voting system [36: 20, 17, 15].

Exercises

In exercises 1-8, determine the apportionment using

a. Hamilton's Method
b. Jefferson's Method
c. Webster's Method
d. Huntington-Hill Method
e. Lowndes' method

1. A college offers tutoring in Math, English, Chemistry, and Biology. The number of students enrolled in each subject is listed below. If the college can only afford to hire 15 tutors, determine how many tutors should be assigned to each subject.

   Math: 330 English: 265 Chemistry: 130 Biology: 70

2. Reapportion the previous problem if the college can hire 20 tutors.
3. The number of salespeople assigned to work during a shift is apportioned based on the average number of customers during that shift. Apportion 20 salespeople given the information below.

Shift	Morning	Midday	Afternoon	Evening
Average Number of Customers	95	305	435	515

4. Reapportion the previous problem if the store has 25 salespeople.
5. Three people invest in a treasure dive, each investing the amount listed below. The dive results in 36 gold coins. Apportion those coins to the investors.

   Alice: $7,600 Ben: $5,900 Carlos: $1,400

6. Reapportion the previous problem if 37 gold coins are recovered.
7. A small country consists of five states, whose populations are listed below. If the legislature has 119 seats, apportion the seats.

   A: 810,000 B: 473,000 C: 292,000 D: 594,000 E: 211,000

8. A small country consists of six states, whose populations are listed below. If the legislature has 200 seats, apportion the seats.

   A: 3,411 B: 2,421 C: 11,586 D: 4,494 E: 3,126 F: 4,962

9. A small country consists of three states, whose populations are listed below.

   A: 6,000 B: 6,000 C: 2,000

   a. If the legislature has 10 seats, use Hamilton's method to apportion the seats.
   b. If the legislature grows to 11 seats, use Hamilton's method to apportion the seats.

c. Which apportionment paradox does this illustrate?

10. A state with five counties has 50 seats in their legislature. Using Hamilton's method, apportion the seats based on the 2000 census, then again using the 2010 census. Which apportionment paradox does this illustrate?

County	2000 Population	2010 Population
Jefferson	60,000	60,000
Clay	31,200	31,200
Madison	69,200	72,400
Jackson	81,600	81,600
Franklin	118,000	118,400

11. A school district has two high schools: Lowell, serving 1715 students, and Fairview, serving 7364. The district could only afford to hire 13 guidance counselors.
    a. Determine how many counselors should be assigned to each school using Hamilton's method.
    b. The following year, the district expands to include a third school, serving 2989 students. Based on the divisor from above, how many additional counselors should be hired for the new school?
    c. After hiring that many new counselors, the district recalculates the reapportion using Hamilton's method. Determine the outcome.
    d. Does this situation illustrate any apportionment issues?
12. A small country consists of four states, whose populations are listed below. If the legislature has 116 seats, apportion the seats using Hamilton's method. Does this illustrate any apportionment issues?

    A: 33,700 B: 559,500 C: 141,300 D: 89,100

13. Explore and describe the similarities, differences, and interplay between weighted voting, fair division (if you've studied it yet), and apportionment.
14. In the methods discussed in the text, it was assumed that the number of seats being apportioned was fixed. Suppose instead that the number of seats could be adjusted slightly, perhaps 10% up or down. Create a method for apportioning that incorporates this additional freedom, and describe why you feel it is the best approach. Apply your method to the apportionment in Exercise 7.
15. Lowndes felt that small states deserved additional seats more than larger states. Suppose you were a legislator from a larger state, and write an argument refuting Lowndes.
16. Research how apportionment of legislative seats is done in other countries around the world. What are the similarities and differences compared to how the United States apportions congress?
17. Adams's method is similar to Jefferson's method, but rounds quotas up rather than down. This means we usually need a modified divisor that is smaller than the standard divisor. Rework problems 1-8 using Adam's method. Which other method are the results most similar to?
18. Consider the weighted voting system [47: 10,9,9,5,4,4,3,2,2]
    a. How many players are there?
    b. What is the total number (weight) of votes?
    c. What is the quota in this system?
19. Consider the weighted voting system [31: 10,10,8,7,6,4,1,1]
    a. How many players are there?
    b. What is the total number (weight) of votes?
    c. What is the quota in this system?
20. Consider the weighted voting system [q: 7,5,3,1,1]
    a. What is the smallest value that the quota q can take?
    b. What is the largest value that the quota q can take?
    c. What is the value of the quota if at least two-thirds of the votes are required to pass a motion?
21. Consider the weighted voting system [q: 10,9,8,8,8,6]

a. What is the smallest value that the quota q can take?
b. What is the largest value that the quota q can take?
c. What is the value of the quota if at least two-thirds of the votes are required to pass a motion?

22. Consider the weighted voting system [13: 13, 6, 4, 2]
    a. Identify the dictators, if any.
    b. Identify players with veto power, if any
    c. Identify dummies, if any.
23. Consider the weighted voting system [11: 9, 6, 3, 1]
    a. Identify the dictators, if any.
    b. Identify players with veto power, if any
    c. Identify dummies, if any.
24. Consider the weighted voting system [19: 13, 6, 4, 2]
    a. Identify the dictators, if any.
    b. Identify players with veto power, if any
    c. Identify dummies, if any.
25. Consider the weighted voting system [17: 9, 6, 3, 1]
    a. Identify the dictators, if any.
    b. Identify players with veto power, if any
    c. Identify dummies, if any.
26. Consider the weighted voting system [15: 11, 7, 5, 2]
    a. What is the weight of the coalition {P1,P2,P4}
    b. In the coalition {P1,P2,P4} which players are critical?
27. Consider the weighted voting system [17: 13, 9, 5, 2]
    a. What is the weight of the coalition {P1,P2,P3}
    b. In the coalition {P1,P2,P3} which players are critical?
28. Find the Banzhaf power distribution of the weighted voting system [27: 16, 12, 11, 3]
29. Find the Banzhaf power distribution of the weighted voting system [33: 18, 16, 15, 2]
30. Consider the weighted voting system [q: 15, 8, 3, 1] Find the Banzhaf power distribution of this weighted voting system,
    a. When the quota is 15
    b. When the quota is 16
    c. When the quota is 18
31. Consider the weighted voting system [q: 15, 8, 3, 1] Find the Banzhaf power distribution of this weighted voting system,
    a. When the quota is 19
    b. When the quota is 23
    c. When the quota is 26
32. Consider the weighted voting system [17: 13, 9, 5, 2]. In the sequential coalition <P3,P2,P1,P4> which player is pivotal?
33. Consider the weighted voting system [15: 13, 9, 5, 2]. In the sequential coalition <P1,P4,P2,P3> which player is pivotal?
34. Find the Shapley-Shubik power distribution for the system [24: 17, 13, 11]
35. Find the Shapley-Shubik power distribution for the system [25: 17, 13, 11]
36. Consider the weighted voting system [q: 7, 3, 1]
    a. Which values of q result in a dictator (list all possible values)
    b. What is the smallest value for q that results in exactly one player with veto power but no dictators?
    c. What is the smallest value for q that results in exactly two players with veto power?
37. Consider the weighted voting system [q: 9, 4, 2]
    a. Which values of q result in a dictator (list all possible values)
    b. What is the smallest value for q that results in exactly one player with veto power?
    c. What is the smallest value for q that results in exactly two players with veto power?
38. Using the Shapley-Shubik method, is it possible for a dummy to be pivotal?

39. If a specific weighted voting system requires a unanimous vote for a motion to pass:
    a. Which player will be pivotal in any sequential coalition?
    b. How many winning coalitions will there be?
40. Consider a weighted voting system with three players. If Player 1 is the only player with veto power, there are no dictators, and there are no dummies:
    a. Find the Banzhof power distribution.
    b. Find the Shapley-Shubik power distribution
41. Consider a weighted voting system with three players. If Players 1 and 2 have veto power but are not dictators, and Player 3 is a dummy:
    a. Find the Banzhof power distribution.
    b. Find the Shapley-Shubik power distribution
42. An executive board consists of a president (P) and three vice-presidents (V1,V2,V3). For a motion to pass it must have three yes votes, one of which must be the president's. Find a weighted voting system to represent this situation.
43. On a college's basketball team, the decision of whether a student is allowed to play is made by four people: the head coach and the three assistant coaches. To be allowed to play, the student needs approval from the head coach and at least one assistant coach. Find a weighted voting system to represent this situation.
44. In a corporation, the shareholders receive 1 vote for each share of stock they hold, which is usually based on the amount of money the invested in the company. Suppose a small corporation has two people who invested $30,000 each, two people who invested $20,000 each, and one person who invested $10,000. If they receive one share of stock for each $1000 invested, and any decisions require a majority vote, set up a weighted voting system to represent this corporation's shareholder votes.
45. A contract negotiations group consists of 4 workers and 3 managers. For a proposal to be accepted, a majority of workers and a majority of managers must approve of it. Calculate the Banzhaf power distribution for this situation. Who has more power: a worker or a manager?
46. The United Nations Security Council consists of 15 members, 10 of which are elected, and 5 of which are permanent members. For a resolution to pass, 9 members must support it, which must include all 5 of the permanent members. Set up a weighted voting system to represent the UN Security Council and calculate the Banzhaf power distribution.
47. In the U.S., the Electoral College is used in presidential elections. Each state is awarded a number of electors equal to the number of representatives (based on population) and senators (2 per state) they have in congress. Since most states award the winner of the popular vote in their state all their state's electoral votes, the Electoral College acts as a weighted voting system. To explore how the Electoral College works, we'll look at a mini-country with only 4 states. Here is the outcome of a hypothetical election:

State	Smalota	Medigan	Bigonia	Hugodo
Population	50,000	70,000	100,000	240,000
Votes for A	40,000	50,000	80,000	50,000
Votes for B	10,000	20,000	20,000	190,000

    a. If this country did not use an Electoral College, which candidate would win the election?
    b. Suppose that each state gets 1 electoral vote for every 10,000 people. Set up a weighted voting system for this scenario, calculate the Banzhaf power index for each state, then calculate the winner if each state awards all their electoral votes to the winner of the election in their state.
    c. Suppose that each state gets 1 electoral vote for every 10,000 people, plus an additional 2 votes. Set up a weighted voting system for this scenario, calculate the Banzhaf power index for each state, then calculate the winner if each state awards all their electoral votes to the winner of the election in their state.
    d. Suppose that each state gets 1 electoral vote for every 10,000 people, and awards them based on the number of people who voted for each candidate. Additionally, they get 2 votes that are awarded to the majority winner in the state. Calculate the winner under these conditions.
    e. Does it seem like an individual state has more power in the Electoral College under the vote distribution from part c or from part d?

f. Research the history behind the Electoral College to explore why the system was introduced instead of using a popular vote. Based on your research and experiences, state and defend your opinion on whether the Electoral College system is or is not fair.

48. The value of the Electoral College (see previous problem for an overview) in modern elections is often debated. Find an article or paper providing an argument for or against the Electoral College. Evaluate the source and summarize the article, then give your opinion of why you agree or disagree with the writer's point of view. If done in class, form groups and hold a debate.
49. To decide on a new website design, the designer asks people to rank three designs that have been created (labeled A, B, and C). The individual ballots are shown below. Create a preference table.

    ABC, ABC, ACB, BAC, BCA, BCA, ACB, CAB, CAB, BCA, ACB, ABC

50. To decide on a movie to watch, a group of friends all vote for one of the choices (labeled A, B, and C). The individual ballots are shown below. Create a preference table.

    CAB, CBA, BAC, BCA, CBA, ABC, ABC, CBA, BCA, CAB, CAB, BAC

51. The planning committee for a renewable energy trade show is trying to decide what city to hold their next show in. The votes are shown below.

Number of voters	9	19	11	8
1st choice	Buffalo	Atlanta	Chicago	Buffalo
2nd choice	Atlanta	Buffalo	Buffalo	Chicago
3rd choice	Chicago	Chicago	Atlanta	Atlanta

    a. How many voters voted in this election?
    b. How many votes are needed for a majority? A plurality?
    c. Find the winner under the plurality method.
    d. Find the winner under the Borda Count Method.
    e. Find the winner under the Instant Runoff Voting method.
    f. Find the winner under Copeland's method.

52. A non-profit agency is electing a new chair of the board. The votes are shown below.

Number of voters	11	5	10	3
1st choice	Atkins	Cortez	Burke	Atkins
2nd choice	Cortez	Burke	Cortez	Burke
3rd choice	Burke	Atkins	Atkins	Cortez

    a. How many voters voted in this election?
    b. How many votes are needed for a majority? A plurality?
    c. Find the winner under the plurality method.
    d. Find the winner under the Borda Count Method.
    e. Find the winner under the Instant Runoff Voting method.
    f. Find the winner under Copeland's method.

53. The student government is holding elections for president. There are four candidates (labeled A, B, C, and D for convenience). The preference schedule for the election is:

Number of voters	120	50	40	90	60	100
1st choice	C	B	D	A	A	D
2nd choice	D	C	A	C	D	B

3rd choice	B	A	B	B	C	A
4th choice	A	D	C	D	B	C

54. The homeowners association is deciding a new set of neighborhood standards for architecture, yard maintenance, etc. Four options have been proposed. The votes are:

Number of voters	8	9	11	7	7	5
1st choice	B	A	D	A	B	C
2nd choice	C	D	B	B	A	D
3rd choice	A	C	C	D	C	A
4th choice	D	B	A	C	D	B

a. How many voters voted in this election?
b. How many votes are needed for a majority? A plurality?
c. Find the winner under the plurality method.
d. Find the winner under the Borda Count Method.
e. Find the winner under the Instant Runoff Voting method.
f. Find the winner under Copeland's method.

55. Consider an election with 129 votes.
    a. If there are 4 candidates, what is the smallest number of votes that a plurality candidate could have?
    b. If there are 8 candidates, what is the smallest number of votes that a plurality candidate could have?
56. Consider an election with 953 votes.
    a. If there are 7 candidates, what is the smallest number of votes that a plurality candidate could have?
    b. If there are 8 candidates, what is the smallest number of votes that a plurality candidate could have?
57. Does this voting system having a Condorcet Candidate? If so, find it.

Number of voters	14	15	2
1st choice	A	C	B
2nd choice	B	B	C
3rd choice	C	A	A

58. Does this voting system having a Condorcet Candidate? If so, find it.

Number of voters	8	7	6
1st choice	A	C	B
2nd choice	B	B	C
3rd choice	C	A	A

59. The marketing committee at a company decides to vote on a new company logo. They decide to use approval voting. Their results are tallied below. Each column shows the number of voters with the particular approval vote. Which logo wins under approval voting?

Number of voters	8	7	6	3
A	X	X		

B	X		X	X
C		X	X	X

60. The downtown business association is electing a new chairperson, and decides to use approval voting. The tally is below, where each column shows the number of voters with the particular approval vote. Which candidate wins under approval voting?

Number of voters	8	7	6	3	4	2	5
A	X	X			X		X
B	X		X	X			X
C		X	X	X		X	
D	X		X		X	X	

61. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If for some reason the election had to be held again and C decided to drop out of the election, which caused B to become the winner, which is the primary fairness criterion violated in this election?
62. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If for some reason the election had to be held again and many people who had voted for C switched their preferences to favor A, which caused B to become the winner, which is the primary fairness criterion violated in this election?
63. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If in a head-to-head comparison a majority of people prefer B to A or C, which is the primary fairness criterion violated in this election?
64. An election resulted in Candidate A winning, with Candidate B coming in a close second, and candidate C being a distant third. If B had received a majority of first place votes, which is the primary fairness criterion violated in this election?
65. In the election shown below under the Plurality method, explain why voters in the third column might be inclined to vote insincerely. How could it affect the outcome of the election?

Number of voters	96	90	10
1st choice	A	B	C
2nd choice	B	A	B
3rd choice	C	C	A

66. In the election shown below under the Borda Count method, explain why voters in the second column might be inclined to vote insincerely. How could it affect the outcome of the election?

Number of voters	20	18
1st choice	A	B
2nd choice	B	A
3rd choice	C	C

67. Compare and contrast the motives of the insincere voters in the two questions above.
68. Consider a two party election with preferences shown below. Suppose a third candidate, C, entered the race, and a segment of voters sincerely voted for that third candidate, producing the preference schedule from #17 above. Explain how other voters might perceive candidate C.

Number of voters	96	100
1st choice	A	B

2nd choice	B	A

69. In question 18, we showed that the outcome of Borda Count can be manipulated if a group of individuals change their vote.
    a. Show that it is possible for a single voter to change the outcome under Borda Count if there are four candidates.
    b. Show that it is not possible for a single voter to change the outcome under Borda Count if there are three candidates.
70. Show that when there is a Condorcet winner in an election, it is impossible for a single voter to manipulate the vote to help a different candidate become a Condorcet winner.
71. The Pareto criterion is another fairness criterion that states: *If every voter prefers choice A to choice B, then B should not be the winner*. Explain why plurality, instant runoff, Borda count, and Copeland's method all satisfy the Pareto condition.
72. Sequential Pairwise voting is a method not commonly used for political elections, but sometimes used for shopping and games of pool. In this method, the choices are assigned an order of comparison, called an agenda. The first two choices are compared. The winner is then compared to the next choice on the agenda, and this continues until all choices have been compared against the winner of the previous comparison.
    a. Using the preference schedule below, apply Sequential Pairwise voting to determine the winner, using the agenda: A, B, C, D.

Number of voters	10	15	12
1st choice	C	A	B
2nd choice	A	B	D
3rd choice	B	D	C
4th choice	D	C	A

    b. Show that Sequential Pairwise voting can violate the Pareto criterion.
    c. Show that Sequential Pairwise voting can violate the Majority criterion.
73. The Coombs method is a variation of instant runoff voting. In Coombs method, the choice with the most last place votes is eliminated. Apply Coombs method to the preference schedules from questions 5 and 6.
74. Copeland's Method is designed to identify a Condorcet Candidate if there is one, and is considered a Condorcet Method. There are many Condorcet Methods, which vary primarily in how they deal with ties, which are very common when a Condorcet winner does not exist. Copeland's method does not have a tie-breaking procedure built-in. Research the Schulze method, another Condorcet method that is used by the Wikimedia foundation that runs Wikipedia, and give some examples of how it works.
75. The plurality method is used in most U.S. elections. Some people feel that Ross Perot in 1992 and Ralph Nader in 2000 changed what the outcome of the election would have been if they had not run. Research the outcomes of these elections and explain how each candidate could have affected the outcome of the elections (for the 2000 election, you may wish to focus on the count in Florida). Describe how an alternative voting method could have avoided this issue.
76. Instant Runoff Voting and Approval voting have supporters advocating that they be adopted in the United States and elsewhere to decide elections. Research comparisons between the two methods describing the advantages and disadvantages of each in practice. Summarize the comparisons, and form your own opinion about whether either method should be adopted.
77. In a primary system, a first vote is held with multiple candidates. In some states, each political party has its own primary. In Washington State, there is a "top two" primary, where all candidates are on the ballot and the top two candidates advance to the general election, regardless of party. Compare and contrast the top two primary with general election system to instant runoff voting, considering both differences in the methods, and practical differences like cost, campaigning, fairness, etc.
78. In a primary system, a first vote is held with multiple candidates. In some many states, where voters must declare a party to vote in the primary election, and they are only able to choose between candidates for their declared party. The top candidate from each party then advances to the general election. Compare and contrast this primary with general election system to instant runoff voting, considering both differences in the methods, and practical differences like cost, campaigning, fairness, etc.

79. Sometimes in a voting scenario it is desirable to rank the candidates, either to establish preference order between a set of choices, or because the election requires multiple winners. For example, a hiring committee may have 30 candidates apply, and need to select 6 to interview, so the voting by the committee would need to produce the top 6 candidates. Describe how Plurality, Instant Runoff Voting, Borda Count, and Copeland's Method could be extended to produce a ranked list of candidates.

Reference

1. References (21)

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CHAPTER OVERVIEW

9: Selected Topics

Throughout this book, we have presented various practical and useful topics in which mathematics plays a crucial role in solving problems. These include logical fallacies, abuse of percentages, compound interest, chances, statistics, exponential growth, perspective in art, harmony in music, and political applications. We have also seen the inspiring side of mathematics in topics like the Golden Ratio and fractals. Unfortunately, many of these "crowning achievements" in mathematics stay hidden from students because of the traditional emphasis on algebra-and calculus-based college mathematics curriculum. In this chapter, we will present a few selected topics featuring some of the "cool" topics that you can appreciate without taking advanced mathematics. We hope to add more topics in our future editions of the book.

9.1: Four Color Theorem

9.2: How Big is Infinity? Or Is it "Infinities"?

9.3: "Seven Bridges of Konigsberg"

9.4: Russell's Paradox

9.5: Non-Euclidean Geometry

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9.1: Four Color Theorem

The **Four-Color Theorem** states that in any plane surface with regions in it (people think of them as maps), the regions can be colored with no more than four colors in such a way that two regions that have a common border do not get the same color. They are called *adjacent* (next to each other) if they share a segment of the border, not just a point.[1]

This was one of the first theorems to be proved by a computer, using a *proof by exhaustion*. In proof by exhaustion, the conclusion is established by dividing it into cases and proving each one separately. The number of cases sometimes may be very large. For example, the first proof of the Four-Color Theorem was a proof by exhaustion with 1,936 cases (in 1976). This proof was controversial because most of the cases were checked by a computer program, not by traditional mathematical arguments. The shortest known proof of the Four-Color Theorem today still has over 600 cases.

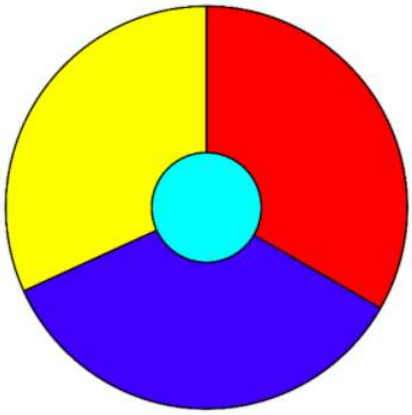


Figure 9.1.1: Three colors are not enough to color the map in the Figure. (Figure is in the public domain.)

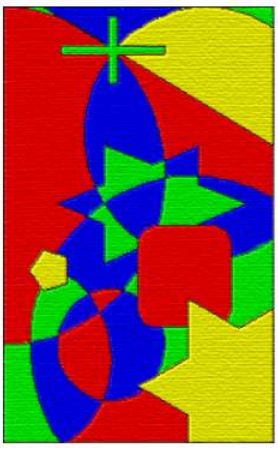


Figure 9.1.2: Figure illustrates the potential of a four-color map. (Figure is used under a CC BY-SA 3.0 license.)

Even though the problem was first presented in 1852 as a problem to color political maps of countries, mapmakers are not particularly interested in it. According to an article by the math historian Kenneth May , “Maps utilizing only four colors are rare, and those that do usually require only three. Books on cartography and the history of map-making do not mention the four-color property.”

Many simpler maps can be colored using three colors. The fourth color is required for some maps, such as one in which one region is surrounded by an odd number of others, which touch each other in a cycle. One such example is given in the image. The Five-Color Theorem states that five colors are enough to color a map. It has a short, elementary proof and was proved in the late 19th century (Heawood 1890). Proving that four colors suffice turned out to be significantly more difficult. A number of false proofs and false counterexamples have appeared since the first statement of the Four-Color Theorem at University College London in England.

To be able to correctly state and solve the problem, it is necessary to clarify some aspects. First, all points that belong to three or more countries must be ignored.

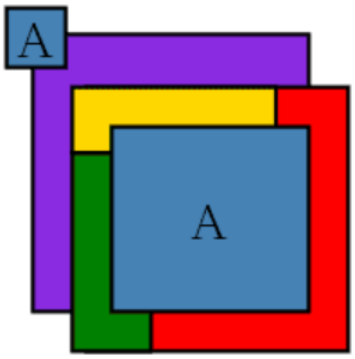


Figure 9.1.3: This map cannot be colored with four colors. (Image is used under a CC BY-SA 3.0 license.)

For the purpose of the theorem every "country" has to be a connected region, or contiguous. In the real world, this is not true: Alaska as part of the United States, Nakhchivan as part of Azerbaijan, and Kaliningrad as part of Russia are not contiguous. Because the territory of a particular country must be the same color, four colors may not be sufficient. For instance, consider a simplified map, such as the one shown above: in this map, the two regions labeled *A* belong to the same country and must be the same color. This map then requires five colors since the two *A* regions together are contiguous with four other regions, each of which is contiguous with all the others. If *A* consisted of three regions, six or more colors might be required. In this manner, it is possible to construct maps that require an arbitrarily high number of colors. A similar construction also applies if a single color is used for all bodies of water, as is usual on real maps.



Figure 9.1.4: Example of a map of Azerbaijan with non-contiguous regions. (Image is in the public domain.)

An easier-to-state version of the theorem uses graph theory. The set of regions of a map can be represented more abstractly as an undirected graph that has a vertex for each region and an edge for every pair of regions that share a boundary segment.[3] This graph is planar, meaning that these edges and vertices can be drawn on paper without any edges crossing each other. A planar graph can be formed from every map in this way. In graph-theoretic terminology, the Four-Color Theorem states that the vertices of every planar graph can be colored with at most four colors so that no two adjacent vertices receive the same color, or for short, "every planar graph is four-colorable" (Thomas 1998, p. 849; Wilson 2002).

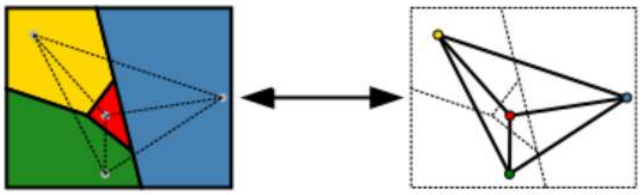


Figure 9.1.5: Diagram showing a map colored with four colors being transformed into a planer graph. (Image is in the public domain.)

Reference

1. References (22)

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9.2: How Big is Infinity? Or Is it “Infinities”?

The first numbers you learned as a child were probably whole numbers like 1, 2, 3, and so on. You learned basic quantitative facts like you have one nose, two ears, and ten fingers. So you learned to count before you learned how to add and subtract. You then grew up seeing a lot of objects like cars on the road, people in a concert, and stars in the sky and eventually learned, perhaps intuitively, that there is no end to this sequence of numbers {1, 2, 3, 4, 5, …}

In other words, you learned that there must be infinitely many numbers. But what exactly is infinity?

Whatever it is, it must be bigger than any number you can think of. And, in a sense, that’s not a bad definition. You may have seen the notation “∞" as the symbol for infinity; this symbol does NOT represent a number. That’s right. Let’s be clear: is not a number. We will come back to this notation later in this section.

Natural Numbers

The first set of numbers you learned, {1, 2, 3, 4, 5, …}, is called the set[1] of *natural numbers* and is often denoted by the symbol $\mathbb{N}$[2]. Clearly, this set $\mathbb{N}$ is an infinite set since it contains more than any finite number of objects, more than a billion, more than ten trillion, and so on. Now, you can easily see that the sum of any two natural numbers is also a natural number. Similarly, you can always take any two natural numbers and find the product, which is also a natural number. We call this property of $\mathbb{N}$ “closure.” In other words, $\mathbb{N}$ is “closed” under addition and multiplication.

But you have a problem when you try to subtract, say, 5 from 2. While both are natural numbers, the difference, 2 – 5, is not in $\mathbb{N}$. We say, therefore, that $\mathbb{N}$ is not closed under subtraction. For centuries, however, this did not cause any problem because this operation appeared to make no sense. After all, how can you take away five apples when you only have two? Or how can five people come out of a house which contains only two people?

Well, are there any applications for subtraction like 2 – 5? Of course there are. The temperature may go down by 5 degrees from 2 degrees (and the temperature just does not disappear). One can borrow $5 even though, for now, there may only be $2 in the account. So is there a way to make sense out of subtraction like this?

Integers

Of course, there is an easy way to fix this. Extend N to include numbers like 0, -1, -2, etc. We can then define a new set of numbers as follows:

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$$

Each of these numbers is called an *integer*. Of course, every natural number is an integer. So we now have two infinite sets, $\mathbb{N}$ and $\mathbb{Z}$. Obviously, $\mathbb{Z}$ contains just about twice as many numbers as $\mathbb{N}$, so perhaps $\mathbb{Z}$ is “more infinite” than $\mathbb{N}$, or are they both “just as infinite”? (This question itself sounds weird, doesn’t it?)

$\mathbb{Z}$ is closed under addition, subtraction, and multiplication. This means that the sum, difference, and product of any two integers are all integers, members of $\mathbb{Z}$. However, it is not closed under division because, for instance, 3 and 4 are both integers, but 3÷4 is not.

Rational Numbers

This gives rise to yet another infinite set, the set of fractions where both the numerator (top) and the denominator (bottom) are integers (and we assume that the denominator is not zero). This set, denoted with the symbol $\mathbb{Q}$, is called the set of rational numbers (the term “rational” here comes from the word “ratio”). Any integer divided by any non-zero integer is a rational number —including all of the following:

$$\frac{1}{2}, \frac{-3}{5}, \frac{2}{-7}, \frac{-1}{-3}, \frac{5}{1}, \frac{-6}{2}$$

Of course, the negative sign can be put on the top, bottom, or in the front of the fraction. Two negative signs cancel each other, and some of these fractions can be reduced. Note that every integer can be written with 1 in the denominator, so every integer is also a rational number. In other words, $\mathbb{Q}$ is an infinite set which contains the entire set of integers (which contains the entire set of natural numbers). Thus, the size of the set $\mathbb{Q}$ is bigger than the size of $\mathbb{N}$ or $\mathbb{Z}$, right? Or are they equally infinite?

An interesting (and perhaps confusing) property of rational numbers is that these numbers appear to show up “almost everywhere.” If you draw a number line (a straight line where each point represents a number), fractions appear almost everywhere while

integers are “spread out” at an interval of 1.

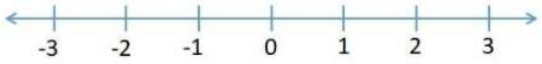


Figure 9.2.1: Number Line

For instance, between 0 and 1, there are no integers, but there are many fractions, i.e., rational numbers. As a matter of fact, between any two fractions, there are infinitely many fractions! Because of this property, we call $\mathbb{Q}$ a “dense set.” Thus, certainly $\mathbb{Q}$ appears to be a much, much larger set than the set of natural numbers or integers. But does this then mean that there are *levels* of infinity? Would it make sense to say that some infinite sets are bigger than other infinite sets?

The answer is mind-boggling (just as the question may be), but we are not quite there yet.

Before we get to the answer (and more questions), we will need at least one more infinite set. It is the set of real numbers, denoted $\mathbb{R}$. Roughly speaking, a *real number* is a number represented by a point on the number line. For each point on the line, there is a real number, and vice-versa. Because each point on the line also represents the distance between that point and 0 (the origin), every real number can be thought of as the length of a line segment. Although a formal definition of a real number is quite difficult and is beyond the scope of this book, this idea of the number line gives us a good understanding of the set $\mathbb{R}$, which is obviously infinite as it contains all of $\mathbb{Q}$.

Irrational Numbers

Based on the paragraphs above, you may be under the impression that $\mathbb{R}$ is the same set as $\mathbb{Q}$, which seems to occupy virtually every point on the line already. Indeed, this was the thought of ancient Greek mathematicians before Pythagoras; they believed that every point on the line can be represented by a ratio of two integers. This idea went well with their philosophy that the world of beauty consists of ratios, and all they needed was ratios of integers, that is, $\mathbb{Q}$. Pythagoras even used ratios and rational numbers to explain musical harmony and chords and is often referred to as the father of music theory (our modern notions like “perfect fifth” and “octave” in music are based on simple rational numbers). But it was also Pythagoras himself who proved that some real numbers (lengths) cannot be represented as a ratio of two integers. For example, imagine the right isosceles triangle (triangle with 45-, 45-, and 90-degree angles) where each of the two legs (the congruent sides) has length 1.

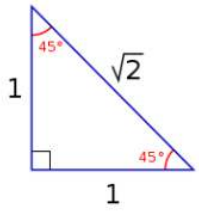


Figure 9.2.2: Right Isosceles triangle

The hypotenuse, according to the theorem that bears Pythagoras’ name, has length 2. Pythagoras proved that no matter how hard one tries, it is impossible to represent 2 as the ratio of two integers, thus being forced to admit that there are real numbers (lengths of line segments) that are not rational numbers. These are now called *irrational* numbers. They include not only 2, 3, 5 and many other square roots, but also numbers like (the ratio between the circumference and the diameter of any circle) and e (the number we saw in the section on finance).

The set of real numbers, $\mathbb{R}$, then contains both the set of rational numbers $\mathbb{Q}$ and the set of irrational numbers, both on the number line, of course. Every point on the line is either a rational number or an irrational number. Once again, it is clear that $\mathbb{R}$ contains the entire set $\mathbb{Q}$ and much more.

Thus we now have four infinite sets, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, with each containing the previous set. Let us review:

$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$: the set of natural numbers

$\mathbb{Z} = 0, \pm 1, \pm 2, \pm 3, \ldots$: the set of integers

$\mathbb{Q} = \left\{ \frac{p}{q} \right\}$, where p and q are integers and q is not 0: the set of rational numbers

$\mathbb{R}$, rational and irrational numbers: the set of real numbers

Georg Cantor and Set Theory

The idea of sets is so intuitive that mathematicians took it for granted for centuries. No one really paid much attention to this fundamental concept, at least formally, until Georg Cantor in the late 1800s. Cantor is one of the most interesting characters in the history of mathematics. He decided to take on the challenging problems involving infinity though many mathematicians had stayed away from infinity,

calling it as "belonging to the realm of the divine." Obviously, everyone knew that there are many infinite sets, but Cantor came up with a way to compare their sizes. The size of a set S is called the *cardinality* of S. For finite sets, the cardinality is simply the number of elements in that set. For instance, the set $S = \{3, 5, 6, 7\}$ and the set $T = \{P, R, S, T\}$ both have four elements, so the cardinality of both of these sets is 4. For each finite natural number n, there are sets with that cardinality (a simple example is the set $1, 2, 3, \ldots, n$). But how do you find the cardinality of an infinite set? If all of the infinite sets are of the same size, then there is just one infinite cardinality, and that would be the end of the story. However, as Cantor soon discovered, this was *not* the case.

The main tool Cantor used to measure and compare the sizes of various sets is "pairing." In mathematics, this is called a one-to-one correspondence. It is a way to pair up elements of two different sets (say *A* and *B*) in such a way that every element of *A* is paired up with exactly one element of *B* with nothing left over on each side. For example, let's say *A* is the set of five seats, labeled X1, X2, X3, X4, and X5, and *B* is the group of five people, Ann, Bob, Carl, Dee, and Ernie. If the five people sit in these seats, in the respective order, there is a natural pairing, or a one-to-one correspondence, in the following way:

Ann → X1

Bob → X2

Carl → X3

Dee → X4

Ernie → X5

No one is left standing, and every seat is occupied. When there is a one-to-one correspondence between two sets, you can clearly see that the two sets are of the same size (in this case, both *A* and *B* have the cardinality 5). Conversely, if two sets are of the same size, then there should be a one-to-one correspondence. Hence, we have the following:

Definition: Cardinality

Two sets have the same size (cardinality) if and only if there is a one-to-one correspondence between the two sets.

With a one-to-one correspondence, if you know the size of one set, you can determine the size of the other set without actually counting the elements. For instance, Dodger Stadium has 56,000 seats. If every person is sitting in one of these seats and if each seat is occupied by exactly one person, you know that there are exactly 56,000 seats in the stadium. So a one-to-one correspondence can be used to measure and compare the sizes of two sets.

This does not sound so hard, right? Well, that's true between finite sets, but things can get a little tricky when infinite sets are involved. This is where Cantor made some ground-breaking discoveries at the end of the 19th century. Equipped with this definition, he found a way to define a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Z}$ even though, at a first glance, you may think $\mathbb{N}$ is about half of $\mathbb{Z}$ (because the natural numbers make up only the positive "half" of the integers). Here is one such one-to-one correspondence:

$1 \to 0, 2 \to 1, 3 \to -1, 4 \to 2, 5 \to -2$, etc.

Note that the even numbers are paired up with positive numbers in $\mathbb{Z}$ while the odd numbers are paired up with negative numbers in $\mathbb{Z}$ without any duplication and with no elements of $\mathbb{Z}$ left out.

Therefore, despite our intuition, $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality, i.e., they are "equally infinite," or these two sets are of the same size. In fact, it is not difficult to see that all of the following proper subsets of $\mathbb{N}$ are just "as big as" $\mathbb{N}$ itself:

$A_1 = 2, 4, 6, 8, 10, \ldots$ (the set of positive natural numbers)

$A_2 = 100, 200, 300, 400, \ldots$ (the set of positive multiples of 100)

$A_3 = 2, 4, 8, 16, 32 \ldots$ (the set of positive powers of 2)

This is strange because a part can be equal (in size) to the whole. In fact, A_2 contains every 100th natural number (someone could say 1% of $\mathbb{N}$), but they are both equally infinite in the sense that their cardinalities are the same. This may lead one to believe that all infinite sets have the same cardinality and thus "equally big."

But what about the dense set $\mathbb{Q}$? This set, the set of rational numbers, contains infinitely many fractions between any two real numbers (between any two distinct points on the line, there are infinitely many rational numbers). This must be much bigger than $\mathbb{N}$ or $\mathbb{Z}$, right?

Wrong!

Again, as counter-intuitive as it may be, Cantor proved that there is indeed a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{Q}$, thus proving that these two sets are just "as infinite" as each other, i.e., they share the same cardinality. Yes, between 0 and 2, there is only one natural number, namely 1, while there are infinitely many rational numbers. In fact, there are infinitely many rational numbers between 0.999 and 1.001. Yet, the set of natural numbers is just as big as the set of rational numbers, and this fact has been mathematically verified.

You may not even be surprised that all of these sets are of the same size because, after all, they are all infinite, correct? Well, it turns out that is not true because here is another thing Cantor was able to show: $\mathbb{R}$ is actually strictly bigger than $\mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$. He prove this with an original, creative method called *diagonalization*. Essentially, he began by assuming that there is a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{R}$ and then showed that this assumption would lead to a contradiction (because one can always come up with a real number left over after the pairing). Hence, we now know this truth:

Fact

The cardinality of $\mathbb{R}$ is strictly bigger than the cardinality of $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$. In other words, there are different levels of infinity.

Any cardinality that is not finite is called *transfinite*. Today, we denote the transfinite cardinality of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ as 0(where is the first letter in the Hebrew alphabet and is pronounced 'Aleph'). This is the first transfinite cardinality; in other words, you can say 0 is the "smallest infinity" though it may sound like an oxymoron and your friends and family may wonder if you are feeling okay. But wait. There is more.

It turns out that the cardinality of $\mathbb{R}$ can be denoted as 20 (the number of subsets of $\mathbb{N}$), which, as stated above, is strictly bigger than 0. Furthermore, it was discovered that you can always take 2 to any aleph number (or any cardinality) to obtain a strictly bigger cardinality. By repeating this process indefinitely, one can obtain (wait for this…) *infinite* levels of infinite numbers. Here, then, is the conclusion:

Fact

There are infinitely many levels of infinities.

The story does not end here, either. Cantor also went on to prove many other theorems about transfinite cardinalities. At one point, after proving another surprising result, he famously wrote to his friend Dedekind, "I see it, but I don't believe it." Despite his outstanding contributions to mathematics, Cantor suffered from depression as his work on infinite cardinalities was often ridiculed and criticized. Even under those circumstances, however, he continued his work on mathematics. He introduced what has become known as the "Continuum Hypothesis," which would (among many things) imply that there is no level of infinity between 0 and 20. This would mean that there are no sets strictly bigger than $\mathbb{N}$ but strictly smaller than $\mathbb{R}$. Is it true, or is it false? (It must be true or false, right?) Cantor actually made it his lifetime goal to prove this hypothesis and spent much of his energy in late years of his life trying to solve this problem. However, he passed away in 1918 in a mental institution without solving it. Little did he know then that a completely surprising ending would eventually put this fascinating problem to rest—but not for another 45 years. If you are interested in how this story ends (or in the life of Georg Cantor), you are encouraged to do a little research on the Internet or in the library to find out more.

Oh, one more thing… earlier the symbol was mentioned. This symbol is used to represent that which is bigger than any real number (on the real line); it does not represent any number. In particular, this symbol does not express any cardinality. It simply denotes the idea of "letting a real number x get bigger and bigger without bound."

Challenging Exercises:

1. Verify that each of the following numbers is rational. Then, use a calculator or a computer to express each as a decimal number. What do you notice about all of these decimal numbers?

$$\frac{-2}{3} \quad \frac{3}{40} \quad \frac{2}{7} \quad \frac{12}{4} \quad -\frac{1}{32}$$

2. Classify the following numbers as natural numbers, integers, rational numbers, irrational numbers, or real numbers.

$$0.33 \quad 262 \quad -1\frac{4}{5} \quad \frac{51}{3} \quad \frac{\pi}{2} \quad -5 \quad 0 \quad 0.11111\ldots$$

3. Try to pair up the elements of the two sets below to define a one-to-one correspondence. Describe your pairing so that each element is paired up with exactly one element and nothing is left behind.

$$A = \{1, 2, 3, 4, 5, \ldots\}$$
$$B = \{4, 8, 12, 16, 20, \ldots\}$$

4. Determine if each of the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ is closed under addition and multiplication.
5. It is mentioned in this section that $\sqrt{2}$, π, and e are not rational. What other irrational numbers are there? Try to find ways to generate more irrational numbers (we know there are infinitely many of them).

Reference

1. The term “set” is extremely difficult to define formally; here, we simply mean any collection of objects (like numbers).
2. Some mathematicians, particularly those whose areas of expertise include set theory, include 0 as a natural number and simply call the set {1, 2, 3, 4, 5, …} the “set of positive integers.”

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9.3: “Seven Bridges of Konigsberg”

Through the city of Königsberg in Russia flowed the Pregel River. In this river were two large islands, which were part of the city. Joining the mainland either side of the river and those two islands there stood seven bridges.

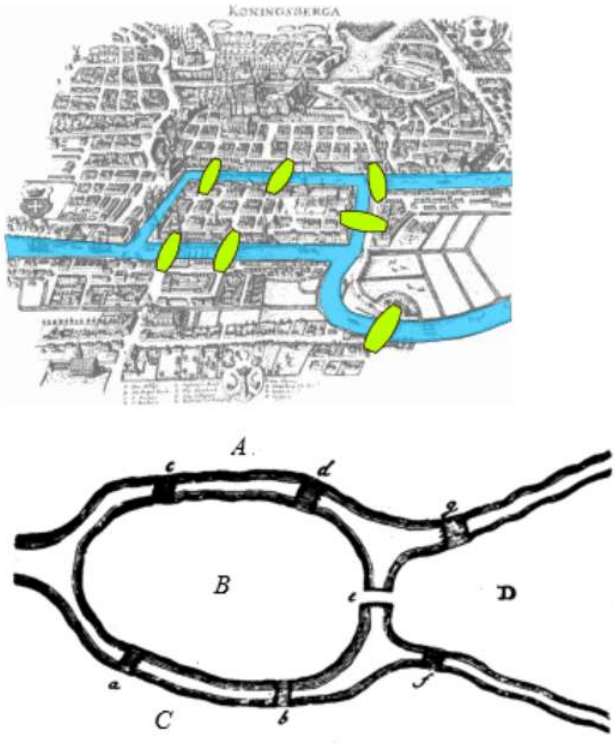


Figure 9.3.1: Image is used under a CC-BY 3.0 license/Image by Leonhard Euler is in the public domain.

It was a popular exercise among the citizens to take a pleasure stroll across the bridges. The question naturally arose: was it possible to cross each bridge once and once only during the course of a single walk? More to the point: could this be done such that the walkers end up back at their starting point?

Solution

Leonhard Euler, a mathematician, solved the problem in 1735. This led to the beginning of graph theory and topology. Euler realized that the shape of the land and the positions of the bridges were irrelevant. What was important is the relationships between the land and the bridges. We can represent each of the landmasses as the vertices and the bridges as the edges of a multigraph:

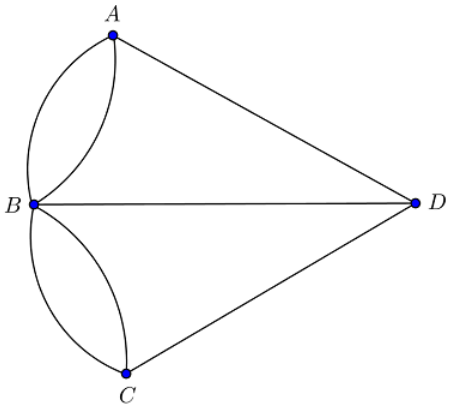


Figure 9.3.2: Graph G: Konigsberg Bridges Graph. Image is used under a CC-BY 3.0 license.

The question now evolves into: does this graph allow the construction of an Euler Path?

An **Euler path** is a trail *T* that passes through every edge of *G* exactly once. An Euler circuit is an Euler path that begins and ends at the same vertex (a loop).

Suppose you start at some vertex, say D, and end your trip at another, say A. Let's say from D you sue the middle edge to reach B. You have to keep going, so you pick another edge from B to some other vertex. So, every time you enter B and leave B, you need two edges. When you come back to B, again you'll have to leave B, so you need two more edges. In other words, B must have an even number of edges. In fact, every vertex needs an even number of edges, except at the starting point and the ending point.

In graph theory, the number of edges coming out of a vertex is called the "degree" (or "valence") of the vertex and is denoted with "deg." In this graph, deg (A)=3, deg (D)=3, deg (C)=3, and deg (B)=5. Remember, only the starting vertex and the ending vertex can have an odd degree. All other vertices are points to pass through, so they must have an even degree. Clearly, then, this graph G cannot have an Euler path.

In other words, one cannot take a stroll around Konigsberg crossing each bridge exactly once.

Now, if, in addition to crossing each bridge exactly once, you want to return to the starting vertex, then it is easy to see that EVERY vertex must have an even degree.

To summarize, here are the requirements so that an Euler path and an Euler circuit can exist for graph G:

1. G has an Euler path if exactly two vertices have an odd degree. An Euler path must begin and end at these two vertices.
2. G has an Euler circuit if and only if every vertex has an even degree.

Reference

1. References (23)

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9.4: Russell's Paradox

Russell's Paradox is a well-known logical paradox involving self-reference. It is a little tricky, so you may want to read this carefully and slowly. If you have a list of lists that do not list themselves, then that list must list itself, because it doesn't contain itself. However, if it lists itself, it then contains itself, meaning it cannot list itself. This makes logical usages of lists of lists that don't contain themselves somewhat difficult. Did you get it? Probably not. Let me explain this using a couple of examples below.

Since this introduction defines Russell's Paradox with lists, let's begin by making up a few lists.

List 1: Apples, California, James, Canada, sun, basketball

List 2: Adam, computer, beef, List 2, dachshund, washing machine

List 3: Boys, tables, anger, List 1

Note that List 2 has itself ("List 2") as one of the items on the list. List 3 has "List 1" as an item, but not itself ("List 3"). Good so far?

OK, now, let's create a new list, List L, which consists of all lists that do not list themselves. List 2 lists itself as an item, so List 2 is excluded from List L. The other two are included since they do not include themselves as an item. So, we have

List L: List 1, List 3, ….

Here is Russell's question: is List L an item under itself? In other words, among all the items in List L (such as List 1 and List 2), do you expect List L to appear?

If List L appears as an item under itself, then List L DOES include itself, but remember what List L was? It was supposed to be the list of all lists that do NOT list themselves. So clearly this cannot happen. Therefore, List L cannot appear as an item under itself. Right?

Well, then suppose List L does not appear as an item under itself. Then, again by the very definition of List L (it is supposed to be the list of all lists that do NOT list themselves), so List L must appear as an item under itself.

So here is a quick summary so far: If List L appears as an item under itself, then it cannot appear as an item under itself. If, on the other hand, List L does NOT appear as an item under itself, then by definition it must appear as an item under itself.

That, my friend, is a paradox. This is Russell's Paradox.

There is another version of this paradox which may be a bit easier to understand. This is often called "The barber's dilemma." Suppose there is a large group of men, one of whom is a barber. This particular barber was ordered to shave those men who do not shave themselves and ONLY those men who do not shave themselves. Got it? Think of dividing the group into two subgroups: Group A, the "shavers" and Group B, the "non-shavers." The barber shaves only those in Group B. So if you are one of the men (not the barber), and if you do not shave yourself (you would be in Group B), the barber will shave you. If you shave yourself (you would be in Group A), the barber will not shave you. Easy, right?

The question is this: Does the barber shave himself or not? If he does shave himself, he would be in Group A, so he does not shave himself. If he does not shave himself, he would be in Group B, so he must shave himself. Yes, this guy has a serious dilemma.

You probably noticed the similarity between the two explanations above. This is a paradox generated by self-referencing.

There are many other forms of this paradox, and much has been written about it. See if you can come up with your own story or example that illustrates Russell's Paradox.

Think About it . . . **Russell's Paradox**

Does the above explanation seem a bit confusing? Sometimes abstract concepts can take a minute to click. Watch this video on Russel's Paradox to further your understanding of the concept. Is this paradox helpful in understanding mathematics? Why or why not?

This paradox was originally devised in 1901 by Bertrand Russell, probably the only mathematician who won the Nobel Prize in Literature. Interested readers can find many online resources on the exciting life of Russell, a philosopher, mathematician, author, and one of the most brilliant people who ever lived.

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9.5: Non-Euclidean Geometry

In your geometry class, you probably learned that the sum of the three angles in any triangle is 180 degrees. This is a well-known theorem in geometry—more specifically, “plane” or “Euclidean” geometry, which has served as the foundation for all learning since the time of the Greek civilization (In fact, the word “mathematics” comes from the Greek word meaning “lessons” or “things to be learned.”) But it turns out that if you go away from the “plane,” then plane geometry may not work. This gives rise to non-Euclidean geometry.

An example of Non-Euclidian geometry can be seen by drawing lines on a sphere or other round object; straight lines that are parallel at the equator can meet at the poles. This “triangle” has an angle sum of 90+90+50=230 degrees!

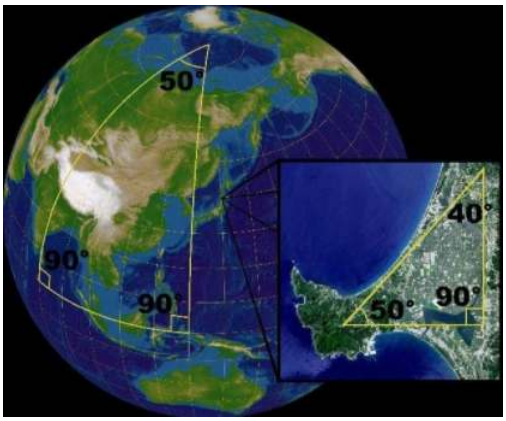


Figure 9.5.1: On a sphere, the sum of the angles of a triangle is not equal to 180°. The surface of a sphere is not a Euclidean plane, but locally the laws of the Euclidean geometry are good approximations. In a small triangle on the face of the earth, the sum of the angles is very nearly 180°. Image is used under a CC BY-SA 3.0 license.

It is called "Non-Euclidean" because it is different from Euclidean geometry, which was developed by an ancient Greek mathematician called Euclid.

Some History…

The birth of non-Euclidean geometry was REALLY a big deal. It was truly a ground-shaking event, not only in the history of mathematics and but also in philosophy. You see, Euclid (who lived over 2300 years ago!) wrote a textbook that was so popular that practically every educated person in the world used it to study geometry for the next 2000 plus years. This subject, “Euclidean geometry” (the type of geometry you studied in high school), was so popular and dominant that no one, for over two millennia, doubted its truthfulness, questioned its authority, or thought of coming up with an alternative. Well, almost no one. The conception and arrival of non-Euclidean geometry involved three mathematicians—one very famous and two completely unknown. The story is worthy of a movie or a play. It is that dramatic. It teaches us that

- Common sense could be the greatest obstacle to finding truth.
- What stands the “test of time” may not be absolutely true.
- What many experts feel is offensive and repugnant may actually be true.

Here, a very abbreviated version of the story is presented. The reader is encouraged to find out more by doing a search under “non-Euclidean geometry.”

Euclid, who lived around 300 B.C., is best known for his book *The Elements*, a 13- volume masterpiece laying the foundations of geometry (and some number theory as well). This book may be the most widely read treatise in world history because no other books have been read longer or by more people, with the exception of the Bible. It was the standard book in geometry for over 2000 years, and there are over 1000 editions of the book in hundreds of languages.

It’s too bad that the notions of copyrights and intellectual properties did not exist back then. He could have been quite wealthy all the royalties he could have earned (except he would not have cared—there is a well-known story of Euclid embarrassing and humiliating one of his students who wanted to know what he would gain by learning geometry).

What made this book so good?

Well, for one thing, it was the first book that laid the foundation of deductive logic—to prove general statements (called propositions) by definitions, general assumptions, and already known propositions. If you remember your high school geometry, you may recall memorizing postulates (general assumptions) and proving theorems based on known properties and other theorems. That whole thing—which is the fundamental structure of mathematics—was first established by Euclid.

Another amazing accomplishment of Euclid was that he proved tons of propositions—465 to be exact—based on a very small number of assumptions. In fact, he started with only five axioms. What is an axiom? It is a (self-evident) statement assumed true without proof. Just as you cannot define every word you use (because each definition uses other words, each of which also needs to be defined using other words), you cannot prove everything; some statements must be assumed true at the beginning. (And they'd better be obvious to everyone so that no one would question them.) Here are the first four:

Definition: Euclid's Axioms

1. A line can be drawn through any two points.
2. A line can be extended indefinitely in both directions
3. A circle can be drawn with any center point and any radius.
4. All right angles are congruent (equal measure of 90 degrees).

They sound obvious, right? Join the club. Over 2200 years passed without anyone seriously challenging these.

But the last of his five axioms was much longer and complex than the other four, and it seems that even Euclid himself hesitated using it. It is his fifth axiom (thus often called "Euclid's Fifth," like "Beethoven's Fifth"):

Euclid's Fifth Axiom (Parallel Postulate)

Given a line L and a point P that is not on the line, there is one and only one line through P parallel to L.

(Originally this statement was more like this: "If two lines both crossing another line form two interior angles on the same side whose sum is less than two right angles (180 degrees), then the two lines, when extended indefinitely on that side, will eventually meet.")

The original statement is weird, right? This is *exactly* what caused the controversy and, eventually, a revolution.

Now, using the example at the beginning of this section (the "triangle" on the sphere), you may be able to see that this postulate is not true on the sphere. You saw the two vertical lines (both perpendicular to the equator) are parallel at the equator but end up meeting each other at the North Pole. In fact, on the sphere, there are no parallel lines. "Yeah, but these are not lines," you may say. That's understandable. You must define what a line is carefully here. But anyway, people did not figure out for a long, long time that spherical geometry is one of the models of non-Euclidean geometry. People had that much faith in Euclid.

Because this one particular axiom was so odd, many mathematicians wanted to clean up the system—more specifically, they wanted to prove that Axiom 5 follows directly from the first four axioms so they could eliminate it altogether.

But they all failed. Some of the most intelligent people who ever lived could not do it. It wasn't just a few folks either. Ptolemy tried it in the 2nd century, and Girolamo Saccheri in the 18th century, and many others in between. In fact, so many had attempted this that in 1763, G. S. Glugel wrote a paper on how NOT to prove it.

Let's review the problem again. What were they trying to do? Remember that axioms (postulates) are assumptions we begin with (and accept as truths), perhaps what Thomas Jefferson (who, as a political philosopher and an architect, knew Euclid's textbook extremely well) referred to as "self-evident" truths. Everyone was fine with Axioms 1 through 4. Axiom 5, on the other hand, was quite different, lengthy, and awkward. There were over 400 statements proved by Euclid based on these five axioms. Many of them depended on Axiom 5. If someone could prove Axiom 5 from the first four axioms, then we could simply take Axiom 5 as another proposition, and all of those 400+ propositions would still be true, based on four axioms. That would be really cool.

By the way, among the more than 400 propositions were quite obvious statements like the following:

- The sum of the three angles of any triangle equals 180 degrees.
- There are bigger and smaller triangles with the same set of three angle measures (called similar triangles).
- There are 4-sided polygons with four right angles (we simply call them rectangles).

This may be hard to believe, but there were mathematicians who had dedicated their lives to solve this problem—and failed. Among them was Wolfgang Bolyai, a long-time friend and classmate of Carl F. Gauss, one of the most celebrated mathematicians of all times. After years of frustration, Wolfgang Bolyai wrote to his son John (Janos), telling him not to waste a lifetime as he himself had. These were the words of Wolfgang to his son:

> I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone. . . . I saw that no man can reach the bottom of the night. . . . I have traveled past all reefs on this infernal Dead Sea and have always come back with broken mast and torn sail. . . . I thoughtlessly risked my life and happiness.

Pretty depressing, right? Well, what would a boy do when his father tells him NOT to do something? Yes, that's exactly what the son Janos did. The young, unknown student Janos Bolyai worked really hard on this problem, and within a few years he wrote back to his father:

> "I have discovered such wonderful things that I was amazed…. Out of nothing I have created a strange new universe."

A young crazy dreamer, correct? Maybe he was. In fact, he did *not* solve the problem. He did *not* prove Axiom 5 using the first four. In fact, he did exactly the opposite; he assumed that Axiom 5 was false and pursued what difference it would make. The difference turned out to be huge. The three "obvious" facts listed above would be all false in this "strange new universe" Janos Bolyai had created. Now, that was a bold statement. Who would question the existence of a rectangle? (Remember, though, this was all theoretical, so it would be possible to imagine such a world which is consistent with the four axioms plus the negation of the fifth.)

Recognizing his son's marvelous work (maybe as a proud and biased dad), Wolfgang Bolyai decided to publish Janos' results in 1832—but not as a book or a paper, but as an appendix to his own book he was publishing. Remember he had a famous friend? Wolfgang sent a copy of this book to Prof. Gauss, wondering what kind of reaction Gauss would have. Perhaps Bolyai was hoping that Gauss would be so impressed that he would praise the work of his son Janos.

The response? Well, in a sense, Gauss was impressed. But this is what Gauss said about the "strange new universe" created by young Janos Bolyai: "To praise it would amount to praising myself."

What?? Gauss was saying that he had secretly been working on the same thing and came up with pretty much the same results as Janos Bolyai, except Gauss had stuffed all the notes in his drawer, hoping that perhaps someone would publish them after his death. He was afraid of ruining his reputation because it was such a crazy idea. Who would challenge the truthfulness of Axiom 5, which the entire world had accepted as absolute truth for over 2000 years? Some people claim that Gauss was especially afraid of being ridiculed by the philosophical group that had been following Immanuel Kant (1724—1804), a leading philosopher who had scoffed at anyone challenging Euclid's axioms.

So here we have a brilliant, well-known German mathematician trying to avoid criticism, his old Hungarian friend with a brilliant son, and that inquisitive, incredibly smart son who had developed a crazy new idea. Oh, by the way, what do you think happened to this son Janos? After receiving the reply from Gauss, this gifted young man stopped studying mathematics altogether and never published anything else, except those few pages known as "the most important appendix in the history of the world." The fact that the whole academic world pretty much ignored the appendix did not help the situation either.

Meanwhile, over in Russia (yes, in a totally different country), there was another person who was independently working on the same idea, coming up with pretty much the same results. His name is Nikolai Lobachevsky (1792—1856). Without electronic communications or social media, this Russian mathematician had no idea what Gauss and Bolyai were up to. In 1829, two years before Janos Bolyai's appendix was published, Lobachevsky had published his work on non-Euclidean geometry in a little-known journal, written in Russian, published at Kazan University. Bolyai and Guass did not read Russian. Even if they had, they would not have known about this journal. But because of this publication, Lobachevsky became the first person to ever publish non-Euclidean geometry.

What did Lobachevsky say (in Russian) that was so controversial? Well, in a sense, he basically showed the world that no one would be able to prove Axiom 5 from the first four axioms because it is possible to create a new and consistent world in which Axiom 5 is false. In that world ("imaginary" in Lobachevsky's word), similar triangles would not exist, and neither do rectangles.

Well, Lobachevsky did not have a famous friend to write to. All he had was a bunch of traditional, old-fashioned colleagues and scholars who thought he had gone insane. Lobachevsky was relentlessly criticized, mocked, and rejected by the academic world. His new "imaginary" geometry represented the "shamelessness of false new inventions" to them. Oh, the news eventually reached

Gauss in Germany. While recognizing Lobachevsky's work as great (because it was just like his own), Gauss still did not support or endorse this new idea in fear of criticism.

Lobachevsky was fired from the university (perhaps due to his failing health, perhaps due to his crazy new idea), lost a child, became blind, and died in poverty. Along with Janos Bolyai, the two obscure mathematicians who had (correctly) challenged the 2000-year-old teaching and reached the surprising apex—the existence of non-Euclidean geometry—ended up leading a tragic life. Perhaps Janos' father Wolfgang was right when he wrote of the "broken mast" and "torn sail."

So why does all this matter? Isn't that just a theoretical world, totally separate from the real world where we live? Is Euclidean geometry true or false? The answer to this last question is "It depends." If you assume all five axioms of Euclid, you get plane geometry, just as you learned in high school. If you decide not to assume Euclid's Fifth, then… well, you get non-Euclidean geometry. One example of this is spherical geometry—the entire navigation theory is based on this. Another example is hyperbolic geometry. When Einstein was studying the structure of our universe, he needed non-Euclidean geometry. In fact, our modern astronomy would not exist without non-Euclidean geometry. Only those crazy dreamers like young Bolyai and Lobachevsky could help pioneer our space explorations. What appears ridiculous and repugnant may not be so crazy after all.

Imagine a small village where, based on observation, everyone believes that the earth is flat. You and I could sit back and laugh at those people because, well, the earth is not a Euclidean plane; it models spherical (non-Euclidean) geometry. In addition, some research today shows that it is possible that our universe could have curvature, meaning that we may be living in a non-Euclidean universe. Imagine that! A small group of ignorant human beings who think Euclidean geometry is absolute while living on a non-Euclidean planet, which revoles around in a non-Euclidean universe. That would be a true irony.

Math in the Real World – *The 5th Postulate*

1. Watch this video to get an overview of Euclidean Geometry and the 5th postulate.
2. Watch this video to understand the basis of Non-Euclidean Geometry.

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CHAPTER OVERVIEW

10: Appendix

10.1: Appendix A - Solution to Select "Try it Now" Questions
10.2: Appendix B - Solutions to Selected Exercises
10.3: References

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10.1: Appendix A - Solution to Select "Try it Now" Questions

1: Mathematics and Problem-Solving

1. At least one Icelandic child did not learn English in school.
2. C ⋁ ~S
3.

A	B	~A	~A ⋀ B
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

4.

A	B	~A	~A ⋀ B	~B	(~A ⋀ B) ⋁ ~B
T	T	F	F	F	F
T	F	F	F	T	T
F	T	T	T	F	T
F	F	T	F	T	T

5. Choice b is correct because it is the contrapositive of the original statement.
6. Choice b is equivalent to the negation; it keeps the first part the same and negates the second part.
7. Choices a & b are false; c is true.
8. Failing to meet just one of the three conditions is all it takes to be disqualified. A person is disqualified if they were not born in the US, or are not at least 35 years old, or have not lived in the US for at least 14 years. The key word here is “or” instead of “and”.
9. Valid. Cats are a subset of creatures that are scared by vacuum cleaners. Max is in the set of cats, so he must also be in the set of creatures that are scared by vacuum cleaners.
10. Invalid. The set of bicycles is a subset of the set of vehicles with two wheels; the Harley-Davidson is in the set of two-wheeled vehicles but not necessarily in the smaller circle.
11. Invalid. Since no cows are purple, we know there is no overlap between the set of cows and the set of purple things. We know Fido is not in the cow set, but that is not enough to conclude that Fido is in the purple things set.
12. Invalid. Lipstick on your collar is a subset of scenarios in which you are cheating, and cheating is a subset of the scenarios in which I will divorce you. Although it is wonderful that you don’t have lipstick on your collar, you could still be cheating on me, and I will divorce you. In fact, even if you aren’t cheating on me, I might divorce you for another reason. You’d better shape up.
13. Let S = have a shovel, D = dig a hole.
    The first premise is equivalent to $S \rightarrow D$. The second premise is D. The conclusion is S.
    We are testing $[(S \rightarrow D) \bigwedge D] \rightarrow S$

S	D	S→D	(S→D) ⋀ D	[(S→D) ⋀ D] → S
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

    This is not a tautology, so this is an invalid argument.

14. Let b = brushed teeth and w = toothbrush is wet.
    Premise: $b \rightarrow w$

Premise: $\sim w$
Conclusion: $\sim b$
This argument is valid by the Law of Contraposition.

15. This argument is valid by the Transitive Property, which can involve more than two premises, as long as they continue the chain reaction. The premises $f \to s$, $s \to b$, $b \to c$, $c \to d$, $d \to g$, $g \to w$, $w \to h$, $h \to x$ can be reduced to $f \to x$. (Because we had already used c and d, we decided to use w for cow and x for death.) If the old lady swallows the fly, she will eventually eat a horse and die.
16. Let p = wrote a paper and s = gave a speech.
    Premise: $p \bigvee s$
    Premise: $\sim s$
    Conclusion: p
    This argument is valid by Disjunctive Syllogism. Alison had to do one or the other; she didn't choose the speech, so she must have chosen the paper.
17. Let f = pulled fire alarm and t = got in big trouble.
    Premise: $f \to t$
    Premise: t
    Conclusion: f

    This argument is invalid because it has the form of the Fallacy of the Converse. The young rascal may have gotten in trouble for any number of reasons besides pulling the fire alarm.
18. Let t = tripped and p = got a penalty.
    Premise: $t \to p$
    Premise: $\sim t$
    Conclusion: $\sim p$
    This argument is invalid because it has the form of the Fallacy of the Inverse. Alexei may have gotten a penalty for an infraction other than tripping.
19. Let p = go to party, t = be tired, and f = see friends.
    Premise: $p \to t$
    Premise: $p \to f$
    Conclusion: $\sim f \to \sim t$
    We could try to rewrite the second premise using the contrapositive to state $\sim f \to \sim p$, but that does not allow us to form a syllogism. If I don't see friends, then I didn't go the party, but that is not sufficient to claim I won't be tired tomorrow. Maybe I stayed up all night watching movies.
    (Explanation continued on next page)

    A Venn diagram can help, if we set it up correctly. The "party" circle must be completely contained within the intersection of the other circles. We know that I am somewhere outside the "friends" circle, but we cannot determine whether I am in the "tired" circle. All we really know for sure is that I didn't go to the party.

20. a. Circular
    b. Correlation does not imply causation
    c. Post hoc
    d. Appeal to consequence
    e. Straw man

2: Mathematics and Numbers

1. The sale price is \$799(0.70) = \$559.30. After tax, the price is \$559.30(1.092) = \$610.76
2. 2001-2002: Absolute change: \$0.43 trillion. Relative change: 7.45%

   2005-2006: Absolute change: \$0.54 trillion. Relative change: 6.83%

   2005-2006 saw a larger absolute increase, but a smaller relative increase.

3. Without more information, it is hard to judge these arguments. This is compounded by the complexity of Medicare. As it turns out, the \$716 billion is not a cut in current spending, but a cut in future increases in spending, largely reducing future growth in

health care payments. In this case, at least the numerical claims in both statements could be considered at least partially true. Here is one source of more information if you're interested: http://factcheck.org/2012/08/a-campaign-full-of-mediscare/

4. $18\text{inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} \cdot \frac{19.8 \text{ pounds}}{1000 \text{ feet}} \cdot \frac{16 \text{ ounces}}{1\text{pound}} \approx \$0.475\text{ounces}$
5. The original sandbox has volume 64 ft^3. The smaller sandbox has volume $24ft^3$. $\frac{48 \text{ bags}}{64ft^3} = \frac{x \text{ bags}}{24ft^3}$ results in $x = 18$ bags.
6. There is not enough information provided to answer the question, so we will have to make some assumptions, and look up some values.

   Assumptions:

   a) We own a car. Suppose it gets 24 miles to the gallon. We will only consider gas cost.

   b) We will not need to rent a car in Spokane, but will need to get a taxi from the airport to the conference hotel downtown and back.

   c) We can get someone to drop us off at the airport, so we don't need to consider airport parking.

   d) We will not consider whether we will lose money by having to take time off work to drive.

   Values looked up (your values may be different)

   a) Flight cost: $184

   b) Taxi cost: $25 each way (estimate, according to hotel website)

   c) Driving distance: 280 miles each way

   d) Gas cost: $3.79 a gallon

   Cost for flying: $184 flight cost + $50 in taxi fares = $234.

   Cost for driving: 560 miles round trip will require 23.3 gallons of gas, costing $88.31.

   Based on these assumptions, driving is cheaper. However, our assumption that we only include gas cost may not be a good one. Tax law allows you deduct $0.55 (in 2012) for each mile driven, a value that accounts for gas as well as a portion of the car cost, insurance, maintenance, etc. Based on this number, the cost of driving would be $319.

Taxes

1. While sales tax is a flat percentage rate, it is often considered a regressive tax for the same reasons as the gasoline tax.

3: Mathematics and Finance

1. I=$30 of interest

   P_0 = $500

   r= unknown

   t= 1 month

   Using $I = P_0rt$, we get $30 = 500 \cdot r \cdot 1$. Solving, we get r=0.06, or 6%. Since the time was monthly, this is monthly interest. The annual rate would be 12 times this: 72% interest.

2. d=$5 The daily deposit

   r=0.03 3% annual rate

   k=365 since we're doing daily deposits, we'll compound daily

   N=10 we want the amount after 10 years

   $$P_{10} = \frac{5\left(\left(1+\frac{0.03}{365}\right)^{365\times10} - 1\right)}{\frac{0.03}{365}} = \$21,282.07$$

   We would have deposited a total of $\$5 \cdot 365 \cdot 10 = \$18,250$, so $3,032.07 is from interest

3. d= unknown

   r=0.16 16% annual rate

   k=12 since we're making monthly payments

   N=2 2 years to repay

   P_0=3,000 we're starting with a $3000 loan

   $$3,000 = \frac{d\left(1-\left(1+\frac{0.16}{12}\right)^{-2\times 12}\right)}{\frac{0.16}{12}}$$

   Solving for d gives $146.89 as monthly payments.

   In total, she will pay $3,525.36 to the store, meaning she will pay $525.36 in interest over the two years.

4. a. This is a savings annuity problem. He will have saved up $7,524.11.

   b. This is a compound interest problem. She would need to deposit $22,386.46.

   c. This is a loans problem. She can buy $4,609.33 of new equipment.

   d. This is a savings annuity problem. You would need to save $200.46 each month.

4: Probability and Odds

1. There are 60 possible readings, from 00 to 59. a. $\frac{1}{60}$ b. $\frac{16}{60}$ (counting 00 through 15)
2. Since the second draw is made after replacing the first card, these events are independent. The probability of an ace on each draw is $\frac{4}{52}=\frac{1}{13}$, so the probability of an Ace on both draws is $\frac{1}{13}\cdot\frac{1}{13}=\frac{1}{169}$
3. P(white sock and white tee) = $\frac{6}{10}\cdot\frac{3}{7}=\frac{9}{35}$

   P(white sock or white tee) = $\frac{6}{10}+\frac{3}{7}-\frac{9}{35}=\frac{27}{35}$
4. a. $\frac{6}{10}+\frac{3}{7}-\frac{9}{35}=\frac{27}{35}$
5. Out of 100,000 people, 500 would have the disease. Of those, all 500 would test positive. Of the 99,500 without the disease, 2,985 would falsely test positive and the other 96,515 would test negative.

   $$P(\text{ disease } \mid \text{ positive }) = \frac{500}{500+2985} \approx 14.3\%$$
6. $8\cdot 11\cdot 5 = 440$ menu combinations
7. There are 26 characters. a. $26^5 = 11,881,376$ b. ${}_{26}\mathrm{P}_5 = 26\cdot 25\cdot 24\cdot 23\cdot 22 = 7,893,600$
8. Order does not matter. ${}_{29}C_{19} = 20,030,010$ possible subcommittees
9. There are $5^{10} = 9,765,625$ different ways the exam can be answered. There are 10 possible locations for the one missed question, and in each of those locations there are 4 wrong answers, so there are 36 ways the test could be answered with one wrong answer.

   $$P(9 \text{ answers correct }) = \frac{40}{5^{10}} \approx 0.0000041 \text{ chance}$$
10. $P(\text{ three Aces and two Kings }) = \frac{({}_4C_3)({}_4C_2)}{{}_{52}C_5} = \frac{24}{2598960} \approx 0.0000092$
11. $P(\text{ shared birthday }) = 1 - \frac{{}_{365}P_{10}}{365^{10}} \approx 0.117$
12. $(\$3,995)\cdot\frac{1}{2000}+(-\$5)\cdot\frac{1999}{2000} \approx -\$3.00$
13. Suppose you roll the first die. The probability the second will be different is $\frac{5}{6}$. The probability that the third roll is different than the previous two is $\frac{4}{6}$, so the probability that the three dice are different is $\frac{5}{6}\cdot\frac{4}{6}=\frac{20}{36}$. The probability that two dice will

match is the complement, $1-\frac{20}{36}=\frac{16}{36}$.

The expected value is: $(\$2)\cdot\frac{16}{36}+(-\$1)\cdot\frac{20}{36}=\frac{12}{36}\approx\$0.33$ Yes, it is in your advantage to play. On average, you'd win $0.33 per play.

5: Data and Statistics

5.1: Basics of Statistics

1. The sample is the 20 fish caught. The population is all fish in the lake. The sample may be somewhat unrepresentative of the population since not all fish may be large enough to catch the bait.
2. This is a parameter, since the college would have access to data on all students (the population)
3. a. Categorical. b. Quantitative c. Quantitative
4. a. Systematic

   b. Stratified or Quota

   c. Voluntary response

   d. Simple random

   e. Cluster
5. a. Response bias – historically, men are likely to over-report, and women are likely to under-report to this question.

   b. Voluntary response bias – the sample is self-selected

   c. Sampling bias – the sample may not be representative of the whole class

   d. Lack of anonymity

   e. Self-interest study

   f. Loaded question
6. a. Observational study

   b. Experiment; the treatment is the jumping jacks

   c. Experiment; the treatments are coffee and tea
7. The truth-telling group could be considered the control group, but really both groups are treatment groups here, since it is important for the lie detector to be able to correctly identify lies, and also not identify truth telling as lying. This study is blind, since the person running the test does not know what group each subject is in.

5.2: Describing Data

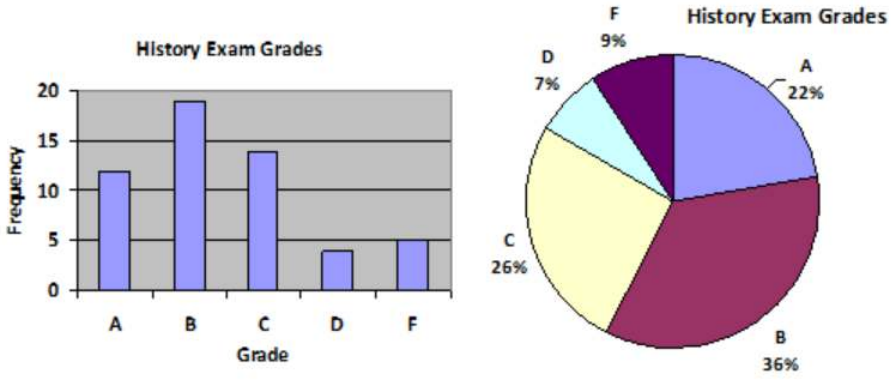
1. 

   Figure 10.1.1
2. While the pie chart accurately depicts the relative size of the people agreeing with each candidate, the chart is confusing, since usually percents on a pie chart represent the percentage of the pie the slice represents.
3. Using a class intervals of size 55, we can group our data into six intervals:

Cost Interval	Frequency
$140-194	5
$195-249	3
$250-304	9
$305-359	12
$360-414	4
$415-469	3

4. Adding the prices and dividing by 5 we get the mean price: $3.682
5. First we put the data in order: $3.29, $3.59, $3.75, $3.79, $3.99. Since there are an odd number of data, the median will be the middle value, $3.75.
6. There are 23 ratings.
   a. The mean is $\frac{1 \cdot 4 + 2 \cdot 8 + 3 \cdot 7 + 4 \cdot 3 + 5 \cdot 1}{23} \approx 2.5$
   b. There are 23 data values, so the median will be the 12th data value. Ratings of 1 are the first 4 values, while a rating of 2 are the next 8 values, so the 12th value will be a rating of 2. The median is 2.
   c. The mode is the most frequent rating. The mode rating is 2.
7. Earlier we found the mean of the data was $3.682

data value	deviation: data value - mean	deviation squared
3.29	3.29 – 3.682 = -0.391	0.153664
3.59	3.59 – 3.682 = -0.092	0.008464
3.79	3.79 – 3.682 = 0.108	0.011664
3.75	3.75 – 3.682 = 0.068	0.004624
3.99	3.99 – 3.682 = 0.308	0.094864

   This data is from a sample, so we will add the squared deviations, divide by 4, the number of data values minus 1, and compute the square root:

$$\sqrt{\frac{0.153664 + 0.008464 + 0.011664 + 0.004624 + 0.094864}{4}} \approx 0.261$$

8. The data is already in order, so we don't need to sort it first.

   The minimum value is $140 and the maximum is $460.

   There are 36 data values so $n = 36$. $n/2 = 18$, which is a whole number, so the median is the mean of the 18th and 19th data values, $305 and $310. The median is $307.50.

   To find the first quartile, we calculate the locator, $L = 0.25(36) = 9$. Since this is a whole number, we know Q1 is the mean of the 9th and 10th data values, $250 and $260. Q1 = $255.

   To find the third quartile, we calculate the locator, $L = 0.75(36) = 27$. Since this is a whole number, we know Q3 is the mean of the 27th and 28th data values, $345 and $350. Q3 = $347.50.

   The 5 number summary of this data is: $140, $255, $307.50, $347.50, $460

9. Boxplot of textbook costs

6: Growth and Decay

6.1: Growth Models

1. Letting $n = 0$ correspond with 1976, then $P_0 = 20,610$.

   From 1976 to 2010 the number of stay-at-home fathers increased by

   53,555 – 20,610 = 32,945

   This happened over 34 years, giving a common different d of 32,945 / 34 = 969.

   $P_n = 20,610 + 969n$

   Predicting for 2020, we use $n = 44$

   $P_{44} = 20,610 + 969(44) = 63,246$ stay-at-home fathers in 2020.

2. Using $n = 0$ corresponding with 2008,

   $P_{12} = (1 + 0.0134)^{12}(1.14) =$ about 1.337 billion people in 2020

3. Here we will measure *n* in months rather than years, with $n = 0$ corresponding to the February when they went public. This gives $P_0 = 45$ thousand. October is 8 months later, so $P_8 = 60$.

   $P_8 = (1 + r)^8 P_0$
   $60 = (1 + r)^8 45$
   $\frac{60}{45} = (1 + r)^8$
   $\frac{\sqrt{60}}{45} = 1 + r$ or 3.66%
   $r = \sqrt[8]{\frac{60}{45}} - 1 = 0.0366$

   The general explicit equation is $P_n = (1.0366)^n 45$

   Predicting 24 months (2 years) after they went public:

   $P_{24} = (1.0366)^{24} 45 = 106.63$thousand users.

4. $1.14(1.0134)^n = 1.2\ n = 3.853$, which is during 2011

5. $P_1 = P_0 + 0.70\left(1 - \frac{P_0}{300}\right) P_0 = 20 + 0.70\left(1 - \frac{20}{300}\right) 20 = 33$

   $P_2 = 54$

   $P_3 = 85$

6.2: Logarithmic Scales in Natural Sciences

1. Letting $n = 0$ correspond with 1976, then $P_0 = 20,610$.

   From 1976 to 2010 the number of stay-at-home fathers increased by

   53,555 – 20,610 = 32,945

   This happened over 34 years, giving a common different *d* of 32,945 / 34 = 969.

   $P_n = 20,610 + 969n$

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3. Here we will measure n in months rather than years, with *n* = 0 corresponding to the February when they went public. This gives $P_0 = 45$ thousand. October is 8 months later, so $P_8 = 60$.

$P_8 = (1+r)^8 P_0$

$60 = (1+r)^8 45$

$\frac{60}{45} = (1+r)^8$

$\sqrt[8]{\frac{60}{45}} = 1+r$

$r = \sqrt[8]{\frac{60}{45}} - 1 = 0.0366 \text{ or } 3.66\%$

The general explicit equation is $P_n = (1.0366)^n 45$

Predicting 24 months (2 years) after they went public:

$P_{24} = (1.0366)^{24} 45 = 106.63$thousand users.

4. $\log(1,000,000) = 6$
5. $\log(123) \approx 2.0899$
6. The difference in magnitudes was about 3.929.

7: Mathematics and the Arts

7.4: Fractals

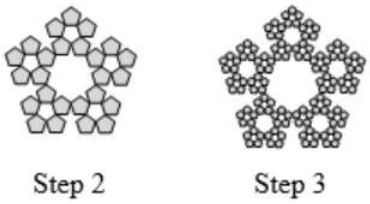
1. 

Figure 10.1.2

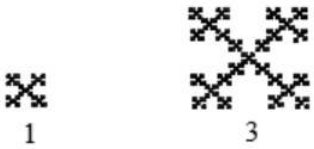
2. 

Figure 10.1.3

8: Mathematics and Politics

8.3: Voting Theory

1. Using plurality method:

   G gets 44+14+20 = 78 first-choice votes

   M gets 70+22 = 92 first-choice votes

   B gets 80+39 = 119 first-choice votes

   Bunney (B) wins under plurality method.

2. Determining the Condorcet Winner:

   G vs M: 44+14+20 = 78 prefer G, 70+22+80=172 prefer M: M preferred

   G vs B: 44+14+20+70=148 prefer G, 22+80+39 = 141 prefer B: G preferred

   M vs B: 44+70+22=136 prefer M, 14+80+39=133 prefer B: M preferred

   M is the Condorcet winner, based on the information we have.

3. Using IRV:

G has the fewest first-choice votes, so is eliminated first. The 20 voters who did not list a second choice do not get transferred - they simply get eliminated

	136	133
1st choice	M	B
2nd choice	B	M

McCarthy (M) now has a majority, and is declared the winner.

4. Using Borda Count:

   We give 1 point for 3rd place, 2 points for 2nd place, and 3 points for 1st place.

	44	14	20	70	22	80	39
1st choice	G 132 pt	G 42 pt	G 60 pt	M 210 pt	M 66 pt	B 240 pt	B 117 pt
2nd choice	M 88 pt	B 28 pt		G 140 pt	B 44 pt	M 160 pt	
3rd choice	B 44 pt	M 14 pt	M 20 pt B 20 pt	B 70 pt	G 22 pt	G 80 pt	M 39 pt G 39 pt

   G: 132+42+60+140+22+80+39 = 515 pts

   M: 88+14+20+210+66+160+39 = 597 pts

   B: 44+28+20+70+44+240+117 = 563 pts

   McCarthy (M) would be the winner using Borda Count.

5. Using Copeland's Method:

   Looking back at our work from Try it Now #2, we see

   G vs M: 44+14+20 = 78 prefer G, 70+22+80=172 prefer M: M preferred – 1 point

   G vs B: 44+14+20+70=148 prefer G, 22+80+39 = 141 prefer B: G preferred – 1 point

   M vs B: 44+70+22=136 prefer M, 14+80+39=133 prefer B: M preferred – 1 point

   M earns 2 points; G earns 1 point. M wins under Copeland's method.

6. Using Approval voting:

   Seattle has 30+10+15+5 = 60 approval votes

   Tacoma has 30+15+20+15+5 = 85 approval votes

   Puyallup has 10+20+25+5 = 50 approval votes

   Olympia has 15+15+5 = 35 approval votes

   Tacoma wins under this approval voting

8.4: Weighted Voting

1. If we represent the players as $M_1, M_2, M_3, M_4, U_1, U_2, U_3$, then we may be tempted to set up a system like [4: 1, 1, 1, 1, 1, 1, 1]. While this system would meet the first requirement that four members must support a proposal for it to pass, this does not satisfy the requirement that at least one member of the union must support it.

   To accomplish that, we might try increasing the voting weight of the union members: [5: 1, 1, 1, 1, 2, 2, 2]. The quota was set at 5 so that the four management members alone would not be able to reach quota without one of the union members. Unfortunately, now the three union members can reach quota alone. To fix this, three management members need to have more weight than two union members.

After trying several other guesses, we land on the system [13: 3, 3, 3, 3, 4, 4, 4]. Here, the four management members have combined weight of 12, so cannot reach quota. Likewise, the three union members have combined weight of 12, so cannot reach quota alone. But, as required, any group of 4 members that includes at least one union member will reach the quota of 13. For example, three management members and one union member have combined weight of 3+3+3+4=13, and reach quota.

2. In the voting system $[q : 10, 5, 3]$, if the quota is 10, then player 1 is a dictator since they can reach quota without the support of the other players. This makes the other two players automatically dummies.

   If the quota is 12, then player 1 is necessary to reach quota, so has veto power. Since at this point either player 2 or player 3 would allow player 1 to reach quota, neither player is a dummy, so they are regular players (not dictators, no veto power, and not a dummy).

   If the quota is 16, then no two players alone can reach quota, so all three players have veto power.

3. The voting system tells us that the quota is 36, that Player 1 has 20 votes (or equivalently, has a weight of 20), Player 2 has 17 votes, Player 3 has 16 votes, and Player 4 has 3 votes.

   A coalition is any group of one or more players. What we're looking for is winning coalitions - coalitions whose combined votes (weights) add to up to the quota or more. So the coalition $\{P_3, P_4\}$ is not a winning coalition because the combined weight is 16+3=19, which is below the quota.

   So we look at each possible combination of players and identify the winning ones:

   {P1, P2} (weight: 37) {P1, P3} (weight: 36)

   {P1, P2, P3} (weight: 53) {P1, P2, P4} (weight: 40)

   {P1, P3, P4} (weight: 39) {P1, P2, P3, P4} (weight: 56)

   {P2, P3, P4} (weight: 36)

   Now, in each coalition, we need to identify which players are critical. A player is critical if the coalition would no longer reach quota without that person. So, in the coalition {P1, P2}, both players are necessary to reach quota, so both are critical. However in the coalition {P1, P2, P3}, we can see from the earlier two coalitions that either P2 or P3 could leave the coalition and it would still reach quota. But if P1 left, it would not reach quota, so P1 is the only player critical in this coalition. We evaluate the rest of the coalitions similarly, giving us this (underlining the critical players)

   {P1, P2} {P1, P3}

   {P1, P2, P3} {P1, P2, P4}

   {P1, P3, P4} {P1, P2, P3, P4}

   {P2, P3, P4}

   Next we count how many times each player is critical:

   P1: 5 times

   P2: 3 times

   P3: 3 times

   P4: 1 time

   In total, there were 5+3+3+1 = 12 times anyone was critical, so we take our counts and turn them into fractions, giving us our Banzhaf power:

   P1: 5/12

   P2: 3/12 = 1/4

   P3: 3/12 = 1/4

   P4: 1/12

   [36: 20, 17, 15]

4. Listing all sequential coalitions and identifying the pivotal player:

< P1, P2, P3 > < P1, P3, P2 > < P2, P1, P3 >

< P2, P3, P1 > < P3, P2, P1 > < P3, P1, P2 >

P1 is pivotal 3 times, P2 is pivotal 3 times, and P3 is pivotal 0 times.

Player	Times pivotal	Power index
P1	3	3/6 = 50%
P2	3	3/6 = 50%
P3	0	0/6 = 0%

Contributors and Attributions

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10.1: Appendix A - Solution to Select "Try it Now" Questions is shared under a CC BY license and was authored, remixed, and/or curated by LibreTexts.

10.2: Appendix B - Solutions to Selected Exercises

1: Mathematics and Problem-Solving

1. 18/230 = 0.07826 = about 7.8%

3. €250(0.23) =€ 57.50 in VAT

5. $15000(5.57) = $83,550

7. absolute increase: 1050. Relative: 1050/3250 = 0.323 = 32.3% increase

9. a. 2200 – 2200(0.15) = 2200(0.85) = $1870

b. Yes, their goal was to decrease by at least 15%. They exceeded their goal.

11. Dropping by 6% is the same as keeping 94%. a(0.94) = 300. a = 319.15. Attendance was about 319 before the drop.

13. a. Kaplan's enrollment was 64.3% larger than Walden's. 30510

b. Walden's enrollment was 39.1% smaller than Kaplan's.

c. Walden's enrollment was 60.9% of Kaplan's.

15. If the original price was $100, the basic clearance price would be $100 – $100(0.60) = $40. The additional markdown would bring it to $40 - $40(0.30) = $28. This is 28% of the original price.

17. These are not comparable; "a" is using a base of all Americans and is talking about health insurance from any source, while "b" is using a base of adults and is talking specifically about health insurance provided by employers.

21. These statements are equivalent, if we assume the claim in "a" is a percentage point increase, not a relative change. Certainly these messages are phrased to convey different opinions of the levy. We are told the new rate will be $9.33 per $1000, which is 0.933% tax rate. If the original rate was 0.833% (0.1 percentage point lower), then this would indeed be a 12% relative increase.

23. 20% of 30% is 30%(0.20) = 6%, a 6 percentage point decrease.

25. Probably not, unless the final is worth 50% of the overall class grade. If the final was worth 25% of the overall grade, then a 100% would only raise her average to 77.5%

27. $4/10 pounds = $0.40 per pound (or 10 pounds/$4 = 2.5 pounds per dollar)

29. x = 15

31. 2.5 cups

33. 74 turbines

35. 96 inches

37. $6000

39. 55.6 meters

43. The population density of the US is 84 people per square mile. The density of India is about 933 people per square mile. The density of India is about 11 times greater than that of the U.S.

49. The oil in the spill could produce 93.1 million gallons of gasoline. Each car uses about 600 gallons a year. That would fuel 155,167 cars for a year.

53. An answer around 100-300 gallons would be reasonable

57. 156 million miles

59. The time it takes the light to reach you is so tiny for any reasonable distance that we can safely ignore it. 750 miles/hr is about 0.21 miles/sec. If the sound takes 4 seconds to reach you, the lightning is about 0.84 miles away. In general, the lightning will be 0.21*n* miles away, which is often approximated by dividing the number of seconds by 5.

61. About 8.2 minutes

63. Four cubic yards (or 3.7 if they sell partial cubic yards)

3: Mathematics and Finance

1. A = 200 + .05(200) = $210

3. I=200. t = 13/52 (13 weeks out of 52 in a year). P_0 = 9800

200 = 9800(r)(13/52) r = 0.0816 = 8.16% annual rate

5. $P_{10} = 300(1+.05/1)^{10(1)} = \$488.67$

7. a. $P_{20} = 2000(1+.03/12)^{20(12)} = \$3641.51$ in 20 years

b. 3641.51 – 2000 = $1641.51 in interest

9. $P_8 = P_0(1+.06/12)^{8(12)} = 6000$. $P_0 = \$3717.14$ would be needed

11. a. $P_{30} = \dfrac{200\left((1+0.03/12)^{30(12)}-1\right)}{0.03/12} = \$116,547.38$

b. 200(12)(30) = $72,000

c. $116,547.40 - $72,000 = $44,547.38 of interest

13. a. $P_{30} = 800,000 = \dfrac{d\left((1+0.06/12)^{30(12)}-1\right)}{0.06/12}$ d = $796.40 each month

b. $796.40(12)(30) = $286,704

c. $800,000 - $286,704 = $513,296 in interest

15. a. $P_0 = \dfrac{30000\left(1-(1+0.08/1)^{-25(1)}\right)}{0.08/1} = \$320,243.29$

b. 30000(25) = $750,000

c. $750,000 - $320,243.29 = $429,756.71

17. $P_0 = 500,000 = \dfrac{d\left(1-(1+0.06/12)^{-20(12)}\right)}{0.06/12}$ d = $3582.16 each month

19. a. $P_0 = \dfrac{700\left(1-(1+0.05/12)^{-30(12)}\right)}{0.05/12} =$ a $130,397.13 loan

b. 700(12)(30) = $252,000

c. $252,200 - $130,397.13 = $121,602.87 in interest

21. $P_0 = 25,000 = \dfrac{d\left(1-(1+0.02/12)^{-48}\right)}{0.02/12} = \$542.38$ a month

23. a. Down payment of 10% is $20,000, leaving $180,000 as the loan amount

b. $P_0 = 180,000 = \dfrac{d\left(1-(1+0.05/12)^{-30(12)}\right)}{0.05/12}$, d = $966.28 a month

c. $P_0 = 180,000 = \dfrac{d\left(1-(1+0.06/12)^{-30(12)}\right)}{0.06/12}$, d = $1079.19 a month

25. First we find the monthly payments:

$P_0 = 24,000 = \dfrac{d\left(1-(1+0.03/12)^{-5(12)}\right)}{0.03/12}$, d = $431.25

Remaining balance: $P_0 = \dfrac{431.25\left(1-(1+0.03/12)^{-2(12)}\right)}{0.03/12} = \$10,033.45$

27.
$$\begin{aligned} 6000(1+0.04/12)^{12N} &= 10000 \\ (1.00333)^{12N} &= 1.667 \\ \log\left((1.00333)^{12N}\right) &= \log(1.667) \\ 12N\log(1.00333) &= \log(1.667) \\ N &= \frac{\log(1.667)}{12\log(1.00333)} = \text{ about } 12.8 \text{ year} \end{aligned}$$

29.
$$\begin{aligned} 3000 &= \frac{60\left(1-(1+0.14/12)^{-12N}\right)}{0.14/12} \\ 3000(0.14/12) &= 60\left(1-(1.0117)^{-12N}\right) \\ \frac{3000(0.14/12)}{60} &= 0.5833 = 1-(1.0117)^{-12N} \\ 0.5833-1 &= -(1.0117)^{-12N} \\ -(0.5833-1) &= (1.0117)^{-12N} \\ \log(0.4167) &= \log\left((1.0117)^{-12N}\right) \\ \log(0.4167) &= -12N\log(1.0117) \\ N &= \frac{\log(0.4167)}{-12\log(1.0117)} = \text{ about } 6.3 \text{ years} \end{aligned}$$

31. First 5 years: $P_5 = \dfrac{50\left((1+0.08/12)^{5(12)}-1\right)}{0.08/12}$ = \$3673.84

Next 25 years: $3673.84(1+.08/12)^{25(12)} = \$26,966.65$

33. Working backwards, $P_0 = \dfrac{10000\left(1-(1+0.08/4)^{-10(4)}\right)}{0.08/4} = \$273,554.79$ needed at retirement. To end up with that amount of money, $273,554.70 = \dfrac{d\left((1+0.08/4)^{15(4)}-1\right)}{0.08/4}$. He'll need to contribute d = \$2398.52 a quarter.

5: Data and Statistics

1. a. Population is the current representatives in the state's congress

b. 106

c. the 28 representatives surveyed

d. 14 out of 28 = ½ = 50%

e. We might expect 50% of the 106 representatives = 53 representatives

3. This suffers from leading question bias

5. This question would likely suffer from a perceived lack of anonymity

7. This suffers from leading question bias

9. Quantitative

11. Observational study

13. Stratified sample

15. a. Group 1, receiving the vaccine

b. Group 2 is acting as a control group. They are not receiving the treatment (new vaccine).

c. The study is at least blind. We are not provided enough information to determine if it is double-blind.

d. This is a controlled experiment

17. a. Census

b. Observational study

6: Growth and Decay

1. a. $P_0 = 20$. $P_n = P_{n-1} + 5$

b. $P_n = 20 + 5n$

3. a. $P_1 = P_0 + 15 = 40 + 15 = 55$. $P_2 = 55 + 15 = 70$

b. $P_n = 40 + 15n$

c. $P_{10} = 40 + 15(10) = 190$ thousand dollars

d. $40 + 15n = 100$ when $n = 4$ years.

5. Grew 64 in 8 weeks: 8 per week

a. $P_n = 3 + 8n$

b. $187 = 3 + 8n$. $n = 23$ weeks

7. a. $P_0 = 200$ (thousand), $P_n = (1 + .09)P_{n-1}$ where n is years after 2000

b. $P_n = 200(1.09)^n$

c. $P_{16} = 200(1.09)^{16} = 794.061$(thousand) = 794,061

d. $200(1.09)^n = 400$ $n = \log(2)/\log(1.09) = 8.043$ In 2008.

9. Let $n = 0$ be 1983. Pn = 1700(2.9)n. 2005 is n=22. P22 = 1700(2.9)22 = 25,304,914,552,324 people. Clearly not realistic, but mathematically accurate.

11. If n is in hours, better to start with the explicit form. $P_0 = 300$. $P_4 = 500 = 300(1 + r)^4$

$500/300 = (1 + r)^4$. $1 + r = 1.136$. $r = 0.136$

a. $P_0 = 300$. $P_n = (1.136)P_{n-1}$

b. $P_n = 300(1.136)^n$

c. $P_{24} = 300(1.136)^{24} = 6400$ bacteria

d. $300(1.136)^n = 900$ $n = \log(3)/\log(1.136)$ =about 8.62 hours

13. a. $P_0 = 100$ $P_n = P_{n-1} + 0.70\,(1 - P_{n-1}/2000)\,P_{n-1}$

b. $P_1 = 100 + 0.70(1 - 100/2000)(100) = 166.5$

c. $P_2 = 166.5 + 0.70(1 - 166.5/2000)(166.5) = 273.3$

15. To find the growth rate, suppose $n = 0$ was 1968. Then P_0 would be 1.60 and $P_8 = 2.30 = 1.60(1 + r)^8$, $r = 0.0464$. Since we want $n = 0$ to correspond to 1960, then we don't know P_0, but P_8 would $1.60 = P_0(1.0464)^8$. $P_0 = 1.113$

a. $P_n = 1.113(1.0464)^n$

b. $P_0 = \$1.113$, or about \$1.11

c. 1996 would be $n = 36$. $P_{36} = 1.113(1.0464)^{36} = \$5.697$ Actual is slightly lower.

17. The population in the town was 4000 in 2005, and is growing by 4% per year.

25. A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given, the base b has the same value. The exponent, y, in the expression by can also be written as the logarithm, $\log_b x$, and the value of x is the result of raising b to the power of y.

26. Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$, and then properties of exponents can be applied to solve for x.

27. The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e. Rather than notating the natural logarithm as $\log_e(x)$, the notation used is $\ln(x)$.

28. $a^c = b$

29. $x^y = 64$

30. $x^y = 64$

31. $15^b = a$

32. $13^a = 142$

33. $e^n = w$

34. 2

8: Mathematics and Politics

Apportionment

1. a. Math: 6, English: 5, Chemistry: 3, Biology: 1

b. Math: 7, English: 5, Chemistry: 2, Biology: 1

c. Math: 6, English: 5, Chemistry: 3, Biology: 1

d. Math: 6, English: 5, Chemistry: 3, Biology: 1

e. Math: 6, English: 5, Chemistry: 2, Biology: 2

3. a. Morning: 1, Midday: 5, Afternoon: 6, Evening: 8

b. Morning: 1, Midday: 4, Afternoon: 7, Evening: 8

c. Morning: 1, Midday: 5, Afternoon: 6, Evening: 8

d. Morning: 1, Midday: 5, Afternoon: 6, Evening: 8

e. Morning: 2, Midday: 5, Afternoon: 6, Evening: 7

5. a. Alice: 18, Ben: 14, Carlos: 4

b. Alice: 19, Ben: 14, Carlos: 3

c. Alice: 19, Ben: 14, Carlos: 3

d. Alice: 19, Ben: 14, Carlos: 3

e. Alice: 18, Ben: 14, Carlos: 4

7. a. A: 40, B: 24, C: 15, D: 30, E: 10

b. A: 41, B: 24, C: 14, D: 30, E: 10

c. A: 40, B: 24, C: 15, D: 30, E: 10

d. A: 40, B: 24, C: 15, D: 30, E: 10

e. A: 40, B: 24, C: 15, D: 29, E: 11

Describing Data

1. a. Different tables are possible

Score	Frequency
30	1
40	0
50	4
60	3
70	6
80	5

Score	Frequency
90	2
100	3

b. This is technically a bar graph, not a histogram:

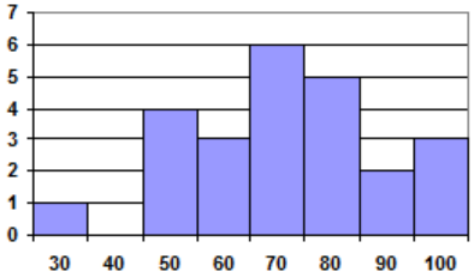


Figure 10.2.1

c.

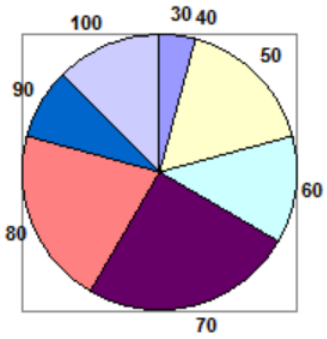


Figure 10.2.2

3. a. 5+3+4+2+1 = 15

b. 5/15 = 0.3333 = 33.33%

5. Bar is at 25%. 25% of 20 = 5 students earned an A

7. a. (7.25+8.25+9.00+8.00+7.25+7.50+8.00+7.00)/8 = $7.781

b. In order, 7.50 and 8.00 are in middle positions. Median = $7.75

c. 0.25*8 = 2. Q1 is average of 2nd and 3rd data values: $7.375

0.75*8 = 6. Q3 is average of 6th and 7th data values: $8.125

5-number summary: $7.00, $7.375, $7.75, $8.125, $9.00

d. 0.637

9. a. (5*0 + 3*1 + 4*2 + 2*3 + 1*5)/15 = 1.4667

b. Median is 8th data value: 1 child

c. 0.25*15 = 3.75. Q1 is 4th data value: 0 children

0.75*15 = 11.25. Q3 is 12th data value: 2 children

5-number summary: 0, 0, 1, 2, 5

d.

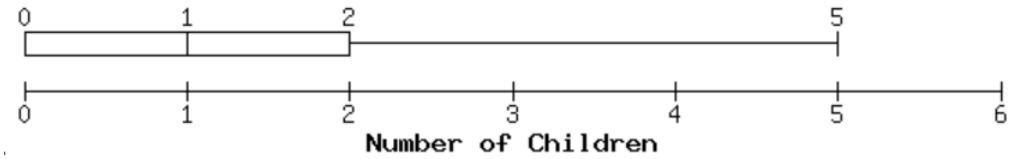


Figure 10.2.3

11. Kendra makes $90,000. Kelsey makes $40,000. Kendra makes $50,000 more.

Fractals

3.

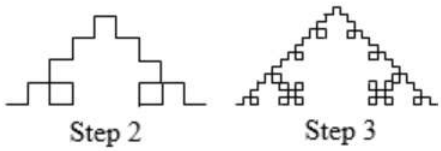


Figure 10.2.4

5.

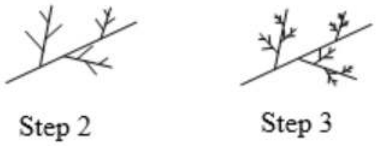


Figure 10.2.5

7.

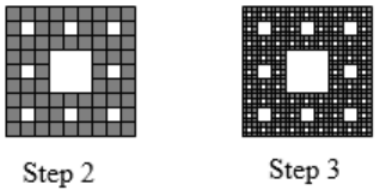


Figure 10.2.6

11. Four copies of the Koch curve are needed to create a curve scaled by 3.

$$D = \frac{\log(4)}{\log(3)} \approx 1.262$$



Figure 10.2.7

13. Eight copies of the shape are needed to make a copy scaled by 3.

Voting Theory

1.

Number of voters	3	3	1	3	2
1st choice	A	A	B	B	C

2nd choice	B	C	A	C	A
3rd choice	C	B	C	A	B

3. a. 9+19+11+8 = 47

b. 24 for majority; 16 for plurality (though a choice would need a minimum of 17 votes to actually win under the Plurality method)

c. Atlanta, with 19 first-choice votes

d. Atlanta 94, Buffalo 111, Chicago 77. Winner: Buffalo

e. Chicago eliminated, 11 votes go to Buffalo. Winner: Buffalo

f. A vs B: B. A vs C: A. B vs C: B. B gets 2 pts, A 1 pt. Buffalo wins.

5. a. 120+50+40+90+60+100 = 460

b. 231 for majority; 116 for plurality

c. A with 150 first choice votes

d. A 1140, B 1060, C 1160, D 1240. Winner: D

e. B eliminated, votes to C. D eliminated, votes to A. Winner: A

f. A vs B: B. A vs C: A. A vs D: D. B vs C: C. B vs D: D. C vs D: C

A 1pt, B 1pt, C 2pt, D 2pt. Tie between C and D.

Winner would probably be C since C was preferred over D

7. a. 33

b. 17

9. Yes, B

11. B, with 17 approvals

13. Independence of Irrelevant Alternatives Criterion

15. Condorcet Criterion

Weighted Voting

1. a. 9 players

b. 10+9+9+5+4+4+3+2+2 = 48

c. 47

3. a. 9, a majority of votes

b. 17, the total number of votes

c. 12, which is 2/3 of 17, rounded up

5. a. P1 is a dictator (can reach quota by themselves)

b. P1, since dictators also have veto power

c. P2, P3, P4

7. a. none

b. P1

c. none

9. a. 11+7+2 = 20

b. P1 and P2 are critical

11. Winning coalitions, with critical players underlined:

{P1,P2} {P1,P2,P3} {P1,P2,P4} {P1,P2,P3,P4} {P1,P3} {P1,P3,P4}

P1: 6 times, P2: 2 times, P3: 2 times, P4: 0 times. Total: 10 times

Power: P1: 6/10 = 60%, P2: 2/10 = 20%, P3: 2/10 = 20%, P4: 0/10 = 0%

13. a. {P1} {P1,P2} {P1,P3} {P1,P4} {P1,P2,P3} {P1,P2,P4} {P1,P3,P4} {P1,P2,P3,P4}

P1: 100%, P2: 0%, P3: 0%, P4: 0%

b. {P1,P2} {P1,P3} {P1,P4} {P1,P2,P3} {P1,P2,P4} {P1,P3,P4} {P1,P2,P3,P4}

P1: 7/10 = 70%, P2: 1/10 = 10%, P3: 1/10 = 10%, P4: 1/10 = 10%

c. {P1,P2} {P1,P3} {P1,P2,P3} {P1,P2,P4} {P1,P3,P4} {P1,P2,P3,P4}

P1: 6/10 = 60%, P2: 2/10 = 20%, P3: 2/10 = 20%, P4: 0/10 = 0%

15. P3 = 5. P3+P2 = 14. P3+P2+P1 = 27, reaching quota. P1 is critical.

17. Sequential coalitions with pivotal player underlined

<P1,P2,P3> <P1,P3,P2> <P2,P1,P3> <P2,P3,P1> <P3,P1,P2> <P3,P2,P1>

P1: 2/6 = 33.3%, P2: 2/6 = 33.3%, P3: 2/6 = 33.3%

19. a. 6, 7

b. 8, given P1 veto power

c. 9, given P1 and P2 veto power

21. If adding a player to a coalition could cause it to reach quota, that player would also be critical in that coalition, which means they are not a dummy. So a dummy cannot be pivotal.

23. We know P2+P3 can't reach quota, or else P1 wouldn't have veto power.

P1 can't reach quota alone.

P1+P2 and P1+P3 must reach quota or else P2/P3 would be dummy.

a. {P1,P2} {P1,P3} {P1,P2,P3}. P1: 3/5, P2: 1/5, P3: 1/5

b. <P1,P2,P3> <P1,P3,P2> <P2,P1,P3> <P2,P3,P1> <P3,P1,P2> <P3,P2,P1>

P1: 4/6, P2: 1/6, P3: 1/6

25. [4: 2, 1, 1, 1] is one of many possibilities

27. [56: 30, 30, 20, 20, 10]

29. [54: 10, 10, 10, 10, 10, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] is one of many possibilities

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17. Simple English Wikipedia's pages on the Golden Ratio/ Fibonacci number
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21. Math in Society: A survey of mathematics for the liberal arts major by David Lippman, Chapter on Weighted Voting
22. simple.Wikipedia.org/wiki/Four_color_theorem
23. simple.Wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg

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