MATH C185: CALCULUS II



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Michael Everett

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TABLE OF CONTENTS

Licensing

1: Calculus I Review

- 1.1: Derivatives of Exponential and Logarithmic Functions
- 1.2: Exercises on Derivatives of Logarithms and Exponential Functions
- 1.3: Derivatives of the Inverse Trigonometric Functions

2: Integration

- 2.1: Antiderivatives
  - 2.1E: Antiderivative and Indefinite Integral Exercises
- 2.2: Prelude to Integration
- 2.3: Approximating Areas
  - 2.3E: Approximating Areas (Exercises)
- 2.4: The Definite Integral
- 2.5: The Fundamental Theorem of Calculus
- 2.6: Integration Formulas and the Net Change Theorem
- 2.7: Substitution
- 2.8: 5.5E and 5.6E u-Substitution Exercises
- 2.9: Integrals Involving Exponential and Logarithmic Functions
- 2.10: Integrals Resulting in Inverse Trigonometric Functions and Related Integration Techniques
  - 2.10E: Exercises for Integrals Resulting in Inverse Trigonometric Functions
- 2.11: Chapter 5 Review Exercises

3: Applications of Integration

- 3.1: Prelude to Applications of Integration
- 3.2: Areas between Curves
  - 3.2E: Exercises for Section 6.1
- 3.3: Determining Volumes by Slicing
  - 3.3E: Exercises for Volumes of Common Cross-Section and Disk/Washer Method
- 3.4: Volumes of Revolution- The Shell Method
  - 3.4b: Volumes of Revolution- Cylindrical Shells OS
  - 3.4E: Exercises for the Shell Method
- 3.5: Arc Length and Surface Area
- 3.6: Arc Length of a Curve and Surface Area
- 3.7: Exercises for Section 6.4
- 3.8: Using Integration to Determine Work
- 3.9: More Physical Applications of Integration
- 3.10: Exercises on Work
- 3.11: Moments and Centers of Mass
  - 3.11E: Exercises for Section 6.6
- 3.12: Integrals, Exponential Functions, and Logarithms
  - 3.12E: Exercises for Section 6.7
- 3.13: Exponential Growth and Decay

- 3.13E: Exercises for Section 6.8
- 3.14: Calculus of the Hyperbolic Functions
  - 3.14E: Exercises for Section 6.9
- 3.15: Chapter 6 Review Exercises

4: Techniques of Integration

- 4.1: Prelude to Techniques of Integration
- 4.2: Integration by Parts
  - 4.2E: Exercises for Integration by Parts
- 4.3: Trigonometric Integrals
  - 4.3E: Exercises for Trigonometric Integrals
- 4.4: Trigonometric Substitution
  - 4.4E: Exercises for Trigonometric Substitution
- 4.5: Partial Fractions
  - 4.5E: Exercises for Integration by Partial Fractions
- 4.6: Other Strategies for Integration
  - 4.6E: Exercises for Section 7.5
- 4.7: Numerical Integration
  - 4.7E: Exercises for Numerical Integration
- 4.8: L'Hôpital's Rule
  - 4.8E: Exercises for L'Hôpital's Rule
- 4.9: Improper Integrals
  - 4.9E: Exercises for Improper Integrals
- 4.10: Chapter 7 Review Exercises

5: Introduction to Differential Equations

- 5.1: Prelude to Differential Equations
- 5.2: Basics of Differential Equations
  - 5.2E: Exercises for Basics of Differential Equations
- 5.3: Direction Fields and Numerical Methods
  - 5.3E: Exercises for Direction Fields and Numerical Methods
- 5.4: Separable Differential Equations
  - 5.4E: Exercises for Separable Differential Equations
- 5.5: The Logistic Equation
  - 5.5E: Exercises for the Logistic Equation
- 5.6: First-order Linear Equations
  - 5.6E: Exercises for Section 8.5
- 5.7: Chapter 8 Review Exercises

6: Sequences and Series

- 6.1: Prelude to Sequence and Series
- 6.2: Sequences
  - 6.2E: Exercises for Sequences
- 6.3: Infinite Series
  - 6.3E: Exercises for Infinite Series

- 6.4: The Divergence and Integral Tests
  - 6.4E: Exercises for Divergence and Integral Tests
- 6.5: Comparison Tests
  - 6.5E: Exercises for Comparison Test
- 6.6: Alternating Series
  - 6.6E: Exercises for Alternating Series
- 6.7: Ratio and Root Tests
  - 6.7E: Exercises for Ratio and Root Tests
- 6.8: Chapter 9 Review Exercises

7: Power Series

- 7.1: Prelude to Power Series
- 7.2: Power Series and Functions
  - 7.2E: Exercises for Section 10.1
- 7.3: Properties of Power Series
  - 7.3E: Exercises for Section 10.2
- 7.4: Taylor and Maclaurin Series
  - 7.4E: Exercises for Taylor Polynomials and Taylor Series
- 7.5: Working with Taylor Series
  - 7.5E: Exercises for Section 10.4
- 7.6: Chapter 10 Review Exercises

8: Appendices

- 8.1: Trigonometric Identities
- 8.2: Review of Derivative Rules
- 8.3: Table of Derivatives
- 8.4: Table of Integrals

Index

Glossary

Detailed Licensing

Detailed Licensing

Licensing

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CHAPTER OVERVIEW

1: Calculus I Review

1.1: Derivatives of Exponential and Logarithmic Functions

1.2: Exercises on Derivatives of Logarithms and Exponential Functions

1.3: Derivatives of the Inverse Trigonometric Functions

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1.1: Derivatives of Exponential and Logarithmic Functions

Learning Objectives

- Find the derivative of exponential functions.
- Find the derivative of logarithmic functions.
- Use logarithmic differentiation to determine the derivative of a function.

So far, we have learned how to differentiate a variety of functions, including trigonometric, inverse, and implicit functions. In this section, we explore derivatives of exponential and logarithmic functions. As we discussed in Introduction to Functions and Graphs, exponential functions play an important role in modeling population growth and the decay of radioactive materials. Logarithmic functions can help rescale large quantities and are particularly helpful for rewriting complicated expressions.

Derivative of the Exponential Function

Just as when we found the derivatives of other functions, we can find the derivatives of exponential and logarithmic functions using formulas. As we develop these formulas, we need to make certain basic assumptions. The proofs that these assumptions hold are beyond the scope of this course.

First of all, we begin with the assumption that the function $B(x) = b^x$, $b > 0$, is defined for every real number and is continuous. In previous courses, the values of exponential functions for all rational numbers were defined—beginning with the definition of b^n, where n is a positive integer—as the product of b multiplied by itself n times. Later, we defined $b^0 = 1, b^{-n} = \dfrac{1}{b^n}$, for a positive integer n, and $b^{s/t} = (\sqrt[t]{b})^s$ for positive integers s and t. These definitions leave open the question of the value of b^r where r is an arbitrary real number. By assuming the continuity of $B(x) = b^x, b > 0$, we may interpret b^r as $\lim\limits_{x \to r} b^x$ where the values of x as we take the limit are rational. For example, we may view 4^π as the number satisfying

$$4^3 < 4^\pi < 4^4, \quad 4^{3.1} < 4^\pi < 4^{3.2}, \quad 4^{3.14} < 4^\pi < 4^{3.15},$$
$$4^{3.141} < 4^\pi < 4^{3.142}, \quad 4^{3.1415} < 4^\pi < 4^{3.1416}, \quad \ldots.$$

As we see in the following table, $4^\pi \approx 77.88$.

x	4^x	x	4^x
4^3	64	$4^{3.141593}$	77.8802710486
$4^{3.1}$	73.5166947198	$4^{3.1416}$	77.8810268071
$4^{3.14}$	77.7084726013	$4^{3.142}$	77.9242251944
$4^{3.141}$	77.8162741237	$4^{3.15}$	78.7932424541
$4^{3.1415}$	77.8702309526	$4^{3.2}$	84.4485062895
$4^{3.14159}$	77.8799471543	4^4	256

Approximating a Value of 4^π

We also assume that for $B(x) = b^x$, $b > 0$, the value $B'(0)$ of the derivative exists. In this section, we show that by making this one additional assumption, it is possible to prove that the function $B(x)$ is differentiable everywhere.

We make one final assumption: that there is a unique value of $b > 0$ for which $B'(0) = 1$. We define e to be this unique value, as we did in Introduction to Functions and Graphs. Figure 1.1.1 provides graphs of the functions $y = 2^x$, $y = 3^x$, $y = 2.7^x$, and $y = 2.8^x$. A visual estimate of the slopes of the tangent lines to these functions at 0 provides evidence that the value of e lies somewhere between 2.7 and 2.8. The function $E(x) = e^x$ is called the **natural exponential function**. Its inverse, $L(x) = \log_e x = \ln x$ is called the **natural logarithmic function.**

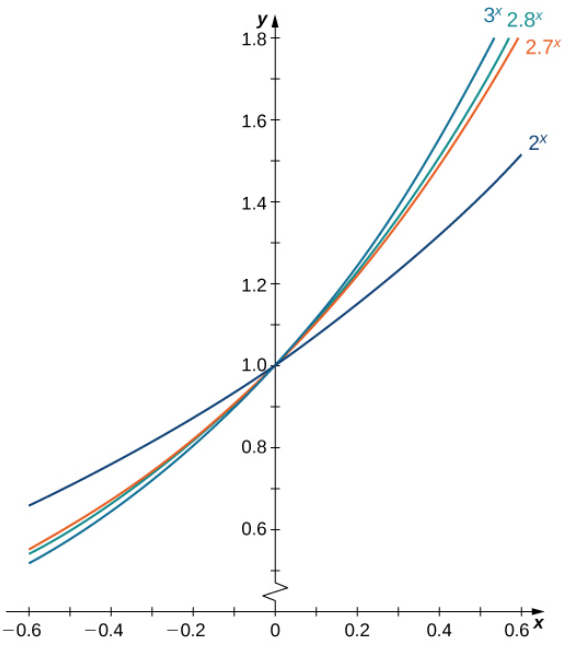


Figure 1.1.1: The graph of $E(x) = e^x$ is between $y = 2^x$ and $y = 3^x$.

For a better estimate of e, we may construct a table of estimates of $B'(0)$ for functions of the form $B(x) = b^x$. Before doing this, recall that

$$B'(0) = \lim_{x \to 0} \frac{b^x - b^0}{x - 0} = \lim_{x \to 0} \frac{b^x - 1}{x} \approx \frac{b^x - 1}{x}$$

for values of x very close to zero. For our estimates, we choose $x = 0.00001$ and $x = -0.00001$

to obtain the estimate

$$\frac{b^{-0.00001} - 1}{-0.00001} < B'(0) < \frac{b^{0.00001} - 1}{0.00001}.$$

See the following table.

Table : Estimating the value of e

b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$.	b	$\frac{b^{-0.00001}-1}{-0.00001} < B'(0) < \frac{b^{0.00001}-1}{0.00001}$.
2	$0.693145 < B'(0) < 0.69315$	2.7183	$1.000002 < B'(0) < 1.000012$
2.7	$0.993247 < B'(0) < 0.993257$	2.719	$1.000259 < B'(0) < 1.000269$
2.71	$0.996944 < B'(0) < 0.996954$	2.72	$1.000627 < B'(0) < 1.000637$
2.718	$0.999891 < B'(0) < 0.999901$	2.8	$1.029614 < B'(0) < 1.029625$
2.7182	$0.999965 < B'(0) < 0.999975$	3	$1.098606 < B'(0) < 1.098618$

The evidence from the table suggests that $2.7182 < e < 2.7183$.

The graph of $E(x) = e^x$ together with the line $y = x + 1$ are shown in Figure 1.1.2. This line is tangent to the graph of $E(x) = e^x$ at $x = 0$.

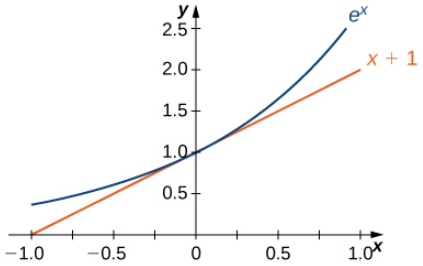


Figure 1.1.2: The tangent line to $E(x) = e^x$ at $x = 0$ has slope 1.

Now that we have laid out our basic assumptions, we begin our investigation by exploring the derivative of $B(x) = b^x$, $b > 0$. Recall that we have assumed that $B'(0)$ exists. By applying the limit definition to the derivative we conclude that

$$B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}$$

Turning to $B'(x)$, we obtain the following.

$$\begin{aligned}
B'(x) &= \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} && \text{Apply the limit definition of the derivative.} \\
&= \lim_{h \to 0} \frac{b^x b^h - b^x}{h} && \text{Note that } b^{x+h} = b^x b^h. \\
&= \lim_{h \to 0} \frac{b^x (b^h - 1)}{h} && \text{Factor out } b^x. \\
&= b^x \lim_{h \to 0} \frac{b^h - 1}{h} && \text{Apply a property of limits.} \\
&= b^x B'(0) && \text{Use } B'(0) = \lim_{h \to 0} \frac{b^{0+h} - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}.
\end{aligned}$$

We see that on the basis of the assumption that $B(x) = b^x$ is differentiable at 0, $B(x)$ is not only differentiable everywhere, but its derivative is

$$B'(x) = b^x B'(0).$$

For $E(x) = e^x$, $E'(0) = 1$. Thus, we have $E'(x) = e^x$. (The value of $B'(0)$ for an arbitrary function of the form $B(x) = b^x$, $b > 0$, will be derived later.)

Derivative of the Natural Exponential Function

Let $E(x) = e^x$ be the natural exponential function. Then

$$E'(x) = e^x.$$

In general,

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x)$$

Example 1.1.1: Derivative of an Exponential Function

Find the derivative of $f(x) = e^{\tan(2x)}$.

Solution:

Using the derivative formula and the chain rule,

$$f'(x) = e^{\tan(2x)} \frac{d}{dx}\left(\tan(2x)\right) = e^{\tan(2x)} \sec^2(2x) \cdot 2$$

Example 1.1.2: Combining Differentiation Rules

Find the derivative of $y = \dfrac{e^{x^2}}{x}$.

Solution

Use the derivative of the natural exponential function, the quotient rule, and the chain rule.

$$\begin{aligned}
y' &= \frac{(e^{x^2} \cdot 2)x \cdot x - 1 \cdot e^{x^2}}{x^2} && \text{Apply the quotient rule.} \\
&= \frac{e^{x^2}(2x^2 - 1)}{x^2} && \text{Simplify.}
\end{aligned}$$

Exercise 1.1.1

Find the derivative of $h(x) = xe^{2x}$.

Hint

Don't forget to use the product rule.

Answer

$h'(x) = e^{2x} + 2xe^{2x}$

Example 1.1.3: Applying the Natural Exponential Function

A colony of mosquitoes has an initial population of 1000. After t days, the population is given by $A(t) = 1000e^{0.3t}$. Show that the ratio of the rate of change of the population, $A'(t)$, to the population, $A(t)$ is constant.

Solution

First find $A'(t)$. By using the chain rule, we have $A'(t) = 300e^{0.3t}$. Thus, the ratio of the rate of change of the population to the population is given by

$$\frac{A'(t)}{A(t)} = \frac{300e^{0.3t}}{1000e^{0.3t}} = 0.3.$$

The ratio of the rate of change of the population to the population is the constant 0.3.

Exercise 1.1.2

If $A(t) = 1000e^{0.3t}$ describes the mosquito population after t days, as in the preceding example, what is the rate of change of $A(t)$ after 4 days?

Hint

Find $A'(4)$.

Answer

996

Derivative of the Logarithmic Function

Now that we have the derivative of the natural exponential function, we can use implicit differentiation to find the derivative of its inverse, the natural logarithmic function.

Definition: The Derivative of the Natural Logarithmic Function

If $x > 0$ and $y = \ln x$, then

$$\frac{dy}{dx} = \frac{1}{x}.$$

More generally, let $g(x)$ be a differentiable function. For all values of x for which $g'(x) > 0$, the derivative of $h(x) = \ln(g(x))$ is given by

$$h'(x) = \frac{1}{g(x)}g'(x).$$

Proof

If $x > 0$ and $y = \ln x$, then $e^y = x$. Differentiating both sides of this equation results in the equation

$$e^y \frac{dy}{dx} = 1.$$

Solving for $\frac{dy}{dx}$ yields

$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Finally, we substitute $x = e^y$ to obtain

$$\frac{dy}{dx} = \frac{1}{x}.$$

We may also derive this result by applying the inverse function theorem, as follows. Since $y = g(x) = \ln x$ is the inverse of $f(x) = e^x$, by applying the inverse function theorem we have

$$\frac{dy}{dx} = \frac{1}{f'(g(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Using this result and applying the chain rule to $h(x) = \ln(g(x))$ yields

$$h'(x) = \frac{1}{g(x)} g'(x). \tag{1.1.1}$$

□

The graph of $y = \ln x$ and its derivative $\frac{dy}{dx} = \frac{1}{x}$ are shown in Figure 1.1.3.

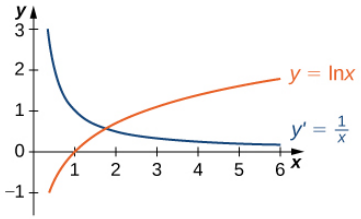


Figure 1.1.3: The function $y = \ln x$ is increasing on $(0, +\infty)$. Its derivative $y' = \frac{1}{x}$ is greater than zero on $(0, +\infty)$

Example 1.1.4: Taking a Derivative of a Natural Logarithm

Find the derivative of $f(x) = \ln(x^3 + 3x - 4)$.

Solution

Use Equation 1.1.1 directly.

$$f'(x) = \frac{1}{x^3 + 3x - 4} \cdot (3x^2 + 3) \qquad \text{Use } g(x) = x^3 + 3x - 4 \text{ in } h'(x) = \frac{1}{g(x)} g'(x).$$

$$= \frac{3x^2 + 3}{x^3 + 3x - 4} \qquad \text{Rewrite.}$$

Example 1.1.5: Using Properties of Logarithms in a Derivative

Find the derivative of $f(x) = \ln\left(\frac{x^2 \sin x}{2x + 1}\right)$.

Solution

At first glance, taking this derivative appears rather complicated. However, by using the properties of logarithms prior to finding the derivative, we can make the problem much simpler.

$$f(x) = \ln\left(\frac{x^2 \sin x}{2x + 1}\right) = 2\ln x + \ln(\sin x) - \ln(2x + 1) \qquad \text{Apply properties of logarithms.}$$

$$f'(x) = \frac{2}{x} + \cot x - \frac{2}{2x + 1} \qquad \text{Apply sum rule and } h'(x) = \frac{1}{g(x)} g'(x).$$

Exercise 1.1.3

Differentiate: $f(x) = \ln(3x + 2)^5$.

Hint

Use a property of logarithms to simplify before taking the derivative.

Answer

$$f'(x) = \frac{15}{3x+2}$$

Now that we can differentiate the natural logarithmic function, we can use this result to find the derivatives of $y = \log_b x$ and $y = b^x$ for $b > 0$, $b \neq 1$.

Derivatives of General Exponential and Logarithmic Functions

Let $b > 0, b \neq 1,$ and let $g(x)$ be a differentiable function.

i. If $y = \log_b x$, then

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

More generally, if $h(x) = \log_b(g(x))$, then for all values of x for which $g(x) > 0$,

$$h'(x) = \frac{g'(x)}{g(x) \ln b}. \tag{1.1.2}$$

ii. If $y = b^x$, then

$$\frac{dy}{dx} = b^x \ln b.$$

More generally, if $h(x) = b^{g(x)}$, then

$$h'(x) = b^{g(x)} g'(x) \ln b \tag{1.1.3}$$

Proof

If $y = \log_b x,$ then $b^y = x$. It follows that $\ln(b^y) = \ln x$. Thus $y \ln b = \ln x$. Solving for y, we have $y = \frac{\ln x}{\ln b}$. Differentiating and keeping in mind that $\ln b$ is a constant, we see that

$$\frac{dy}{dx} = \frac{1}{x \ln b}.$$

The derivative in Equation 1.1.2 now follows from the chain rule.

If $y = b^x$. then $\ln y = x \ln b$. Using implicit differentiation, again keeping in mind that $\ln b$ is constant, it follows that $\frac{1}{y}\frac{dy}{dx} = \ln b$. Solving for $\frac{dy}{dx}$ and substituting $y = b^x$, we see that

$$\frac{dy}{dx} = y \ln b = b^x \ln b.$$

The more general derivative (Equation 1.1.3) follows from the chain rule.

□

Example 1.1.6: Applying Derivative Formulas

Find the derivative of $h(x) = \frac{3^x}{3^x + 2}$.

Solution

Use the quotient rule and Note.

$$h'(x) = \frac{3^x \ln 3(3^x + 2) - 3^x \ln 3(3^x)}{(3^x + 2)^2} \qquad \text{Apply the quotient rule.}$$

$$= \frac{2 \cdot 3^x \ln 3}{(3x + 2)^2} \qquad \text{Simplify.}$$

Example 1.1.7: Finding the Slope of a Tangent Line

Find the slope of the line tangent to the graph of $y = \log_2(3x+1)$ at $x = 1$.

Solution

To find the slope, we must evaluate $\frac{dy}{dx}$ at $x = 1$. Using Equation 1.1.2, we see that

$$\frac{dy}{dx} = \frac{3}{(3x+1)\ln 2}.$$

By evaluating the derivative at $x = 1$, we see that the tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{4\ln 2} = \frac{3}{\ln 16}.$$

Exercise 1.1.4

Find the slope for the line tangent to $y = 3^x$ at $x = 2$.

Hint

Evaluate the derivative at $x = 2$.

Answer

$9\ln(3)$

Logarithmic Differentiation

At this point, we can take derivatives of functions of the form $y = (g(x))^n$ for certain values of n, as well as functions of the form $y = b^{g(x)}$, where $b > 0$ and $b \neq 1$. Unfortunately, we still do not know the derivatives of functions such as $y = x^x$ or $y = x^\pi$. These functions require a technique called **logarithmic differentiation**, which allows us to differentiate any function of the form $h(x) = g(x)^{f(x)}$. It can also be used to convert a very complex differentiation problem into a simpler one, such as finding the derivative of $y = \frac{x\sqrt{2x+1}}{e^x \sin^3 x}$. We outline this technique in the following problem-solving strategy.

Problem-Solving Strategy: Using Logarithmic Differentiation

1. To differentiate $y = h(x)$ using logarithmic differentiation, take the natural logarithm of both sides of the equation to obtain $\ln y = \ln(h(x))$.
2. Use properties of logarithms to expand $\ln(h(x))$ as much as possible.
3. Differentiate both sides of the equation. On the left we will have $\frac{1}{y}\frac{dy}{dx}$.
4. Multiply both sides of the equation by y to solve for $\frac{dy}{dx}$.
5. Replace y by $h(x)$.

Example 1.1.8: Using Logarithmic Differentiation

Find the derivative of $y = (2x^4+1)^{\tan x}$.

Solution

Use logarithmic differentiation to find this derivative.

$\ln y = \ln(2x^4+1)^{\tan x}$ — Step 1. Take the natural logarithm of both sides.

$\ln y = \tan x \ln(2x^4+1)$ — Step 2. Expand using properties of logarithms.

$\frac{1}{y}\frac{dy}{dx} = \sec^2 x \ln(2x^4+1) + \frac{8x^3}{2x^4+1} \cdot \tan x$ — Step 3. Differentiate both sides. Use the product rule on the right.

$\frac{dy}{dx} = y \cdot (\sec^2 x \ln(2x^4+1) + \frac{8x^3}{2x^4+1} \cdot \tan x)$ — Step 4. Multiply by y on both sides.

$\frac{dy}{dx} = (2x^4+1)^{\tan x}(\sec^2 x \ln(2x^4+1) + \frac{8x^3}{2x^4+1} \cdot \tan x)$ — Step 5. Substitute $y = (2x^4+1)^{\tan x}$.

Example 1.1.9: Extending the Power Rule

Find the derivative of $y = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Solution

This problem really makes use of the properties of logarithms and the differentiation rules given in this chapter.

$\ln y = \ln \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$	Step 1. Take the natural logarithm of both sides.
$\ln y = \ln x + \frac{1}{2}\ln(2x+1) - x \ln e - 3 \ln \sin x$	Step 2. Expand using properties of logarithms.
$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{1}{2x+1} - 1 - 3\dfrac{\cos x}{\sin x}$	Step 3. Differentiate both sides.
$\dfrac{dy}{dx} = y\left(\dfrac{1}{x} + \dfrac{1}{2x+1} - 1 - 3\cot x\right)$	Step 4. Multiply by y on both sides.
$\dfrac{dy}{dx} = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}\left(\dfrac{1}{x} + \dfrac{1}{2x+1} - 1 - 3\cot x\right)$	Step 5. Substitute $y = \dfrac{x\sqrt{2x+1}}{e^x \sin^3 x}$.

Exercise 1.1.5

Use logarithmic differentiation to find the derivative of $y = x^x$.

Hint

Follow the problem solving strategy.

Answer

Solution: $\dfrac{dy}{dx} = x^x(1 + \ln x)$

Exercise 1.1.6

Find the derivative of $y = (\tan x)^\pi$.

Hint

Use the power rule (since the exponent π is a constant) and the chain rule.

Answer

$y' = \pi(\tan x)^{\pi-1} \sec^2 x$

Key Concepts

- On the basis of the assumption that the exponential function $y = b^x$, $b > 0$ is continuous everywhere and differentiable at 0, this function is differentiable everywhere and there is a formula for its derivative.
- We can use a formula to find the derivative of $y = \ln x$, and the relationship $\log_b x = \dfrac{\ln x}{\ln b}$ allows us to extend our differentiation formulas to include logarithms with arbitrary bases.
- Logarithmic differentiation allows us to differentiate functions of the form $y = g(x)^{f(x)}$ or very complex functions by taking the natural logarithm of both sides and exploiting the properties of logarithms before differentiating.

Key Equations

- **Derivative of the natural exponential function**

  $$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)} g'(x)$$

- **Derivative of the natural logarithmic function**

$$\frac{d}{dx}\Big(\ln g(x)\Big) = \frac{1}{g(x)}g'(x)$$

- **Derivative of the general exponential function**

$$\frac{d}{dx}\Big(b^{g(x)}\Big) = b^{g(x)}g'(x)\ln b$$

- **Derivative of the general logarithmic function**

$$\frac{d}{dx}\Big(\log_b g(x)\Big) = \frac{g'(x)}{g(x)\ln b}$$

Glossary

logarithmic differentiation

is a technique that allows us to differentiate a function by first taking the natural logarithm of both sides of an equation, applying properties of logarithms to simplify the equation, and differentiating implicitly

---

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1.2: Exercises on Derivatives of Logarithms and Exponential Functions

For exercises 1 - 15, find $f'(x)$ for each function.

1) $f(x) = x^2 e^x$

Answer:

$f'(x) = 2xe^x + x^2 e^x$

2) $f(x) = \dfrac{e^{-x}}{x}$

3) $f(x) = e^{x^3 \ln x}$

Answer:

$f'(x) = (3x^2 \ln x + x^2) e^{x^3 \ln x}$

4) $f(x) = \sqrt{e^{2x} + 2x}$

5) $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Answer:

$f'(x) = \dfrac{4}{(e^x + e^{-x})^2}$

6) $f(x) = \dfrac{10^x}{\ln 10}$

7) $f(x) = 2^{4x} + 4x^2$

Answer:

$f'(x) = 2^{4x+2} \cdot \ln 2 + 8x$

8) $f(x) = 3^{\sin 3x}$

9) $f(x) = x^\pi \cdot \pi^x$

Answer:

$f'(x) = \pi x^{\pi - 1} \cdot \pi^x + x^\pi \cdot \pi^x \ln \pi$

10) $f(x) = \ln(4x^3 + x)$

11) $f(x) = \ln \sqrt{5x - 7}$

Answer:

$f'(x) = \dfrac{5}{2(5x - 7)}$

12) $f(x) = x^2 \ln 9x$

13) $f(x) = \log(\sec x)$

Answer:

$f'(x) = \dfrac{\tan x}{\ln 10}$

14) $f(x) = \log_7 (6x^4 + 3)^5$

15) $f(x) = 2^x \cdot \log_3 7^{x^2 - 4}$

Answer:

$$f'(x) = 2^x \cdot \ln 2 \cdot \log_3 7^{x^2-4} + 2^x \cdot \frac{2x \ln 7}{\ln 3}$$

For exercises 16 - 23, use logarithmic differentiation to find $\frac{dy}{dx}$.

16) $y = x^{\sqrt{x}}$

17) $y = (\sin 2x)^{4x}$

Answer:

$$\frac{dy}{dx} = (\sin 2x)^{4x}[4 \cdot \ln(\sin 2x) + 8x \cdot \cot 2x]$$

18) $y = (\ln x)^{\ln x}$

19) $y = x^{\log_2 x}$

Answer:

$$\frac{dy}{dx} = x^{\log_2 x} \cdot \frac{2 \ln x}{x \ln 2}$$

20) $y = (x^2 - 1)^{\ln x}$

21) $y = x^{\cot x}$

Answer:

$$\frac{dy}{dx} = x^{\cot x} \cdot [-\csc^2 x \cdot \ln x + \tfrac{\cot x}{x}]$$

22) $y = \dfrac{x+11}{\sqrt[3]{x^2-4}}$

Answer:

$$\frac{dy}{dx} == \frac{x+11}{\sqrt[3]{x^2-4}}\left[\frac{1}{x+11} - \frac{2x}{3\,(x^2+4)}\right]$$

23) $y = x^{-1/2}(x^2+3)^{2/3}(3x-4)^4$

Answer:

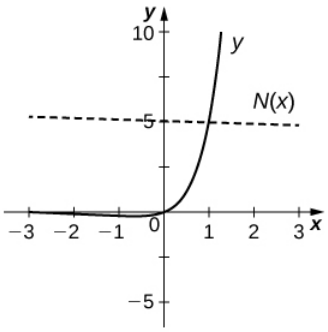
$$\frac{dy}{dx} = x^{-1/2}(x2+3)^{2/3}(3x-4)^4 \cdot \left[\frac{-1}{2x} + \frac{4x}{3(x^2+3)} + \frac{12}{3x-4}\right]$$

24) [T] Find an equation of the tangent line to the graph of $f(x) = 4xe^{(x^2-1)}$ at the point where $x = -1$. Graph both the function and the tangent line.

25) [T] Find the equation of the line that is normal to the graph of $f(x) = x \cdot 5^x$ at the point where $x = 1$. Graph both the function and the normal line.

Answer:

$$y = \tfrac{-1}{5+5\ln 5}x + (5 + \tfrac{1}{5+5\ln 5})$$



26) [T] Find the equation of the tangent line to the graph of $x^3 - x\ln y + y^3 = 2x + 5$ at the point where $x = 2$. (Hint: Use implicit differentiation to find $\frac{dy}{dx}$.) Graph both the curve and the tangent line.

27) Consider the function $y = x^{1/x}$ for $x > 0$.

a. Determine the points on the graph where the tangent line is horizontal.

b. Determine the points on the graph where $y' > 0$ and those where $y' < 0$.

Answer:

$a.\ x = e\ 2.718$
$b.\ (e, \infty),\ (0, e)$

28) The formula $I(t) = \frac{\sin t}{e^t}$ is the formula for a decaying alternating current.

a. Complete the following table with the appropriate values.

t	$\frac{\sin t}{e^t}$
0	(i)
$\pi/2$	(ii)
π	(iii)
$3\pi/2$	(vi)
2π	(v)
2π	(vi)
3π	(vii)
$7\pi/2$	(viii)
4π	(ix)

b. Using only the values in the table, determine where the tangent line to the graph of (I(t)\) is horizontal.

29) [T] The population of Toledo, Ohio, in 2000 was approximately 500,000. Assume the population is increasing at a rate of 5% per year.

a. Write the exponential function that relates the total population as a function of t.

b. Use a. to determine the rate at which the population is increasing in t years.

c. Use b. to determine the rate at which the population is increasing in 10 years

Answer:

a. $P = 500,000(1.05)^t$ individuals
b. $P'(t) = 24395 \cdot (1.05)^t$ individuals per year
c. $39,737$ individuals per year

30)[T] An isotope of the element erbium has a half-life of approximately 12 hours. Initially there are 9 grams of the isotope present.

a. Write the exponential function that relates the amount of substance remaining as a function of t, measured in hours.

b. Use a. to determine the rate at which the substance is decaying in t hours.

c. Use b. to determine the rate of decay at $t = 4$ hours.

31) [T] The number of cases of influenza in New York City from the beginning of 1960 to the beginning of 1961 is modeled by the function

$$N(t) = 5.3e^{0.093t^2 - 0.87t}, \quad (0 \leq t \leq 4),$$

where $N(t)$ gives the number of cases (in thousands) and t is measured in years, with $t = 0$ corresponding to the beginning of 1960.

a. Show work that evaluates $N(0)$ and $N(4)$. Briefly describe what these values indicate about the disease in New York City.

b. Show work that evaluates $N'(0)$ and $N'(3)$. Briefly describe what these values indicate about the disease in the United States.

a. At the beginning of 1960 there were 5.3 thousand cases of the disease in New York City. At the beginning of 1963 there were approximately 723 cases of the disease in the United States. b. At the beginning of 1960 the number of cases of the disease was decreasing at rate of -4.611 thousand per year; at the beginning of 1963, the number of cases of the disease was decreasing at a rate of -0.2808 thousand per year.

32) [T] The relative rate of change of a differentiable function $y = f(x)$ is given by $\frac{100 \cdot f'(x)}{f(x)}\%$. One model for population growth is a Gompertz growth function, given by $P(x) = ae^{-b \cdot e^{-cx}}$ where a, b, and c are constants.

a. Find the relative rate of change formula for the generic Gompertz function.

b. Use a. to find the relative rate of change of a population in $x = 20$ months when $a = 204, b = 0.0198$, and $c = 0.15$.

c. Briefly interpret what the result of b. means.

33) For the following exercises, use the population of New York City from 1790 to 1860, given in the following table.

Year since 1790	Population
0	33,131
10	60,515
20	96,373
30	123,706
40	202,300
50	312,710
60	515,547
70	813,669

New York City Population Over TimeSource: http://en.wikipedia.org/wiki/Largest..._United_States_by_population_by_decade

34) [T] Using a computer program or a calculator, fit a growth curve to the data of the form $p = ab^t$.

Answer:

$p = 35741(1.045)^t$

35) [T] Using the exponential best fit for the data, write a table containing the derivatives evaluated at each year.

36) [T] Using the exponential best fit for the data, write a table containing the second derivatives evaluated at each year.

Answer:

Year since 1790	P"
0	69.25
10	107.5
20	167.0
30	259.4
40	402.8
50	625.5
60	971.4
70	1508.5

37) [T] Using the tables of first and second derivatives and the best fit, answer the following questions:

a. Will the model be accurate in predicting the future population of New York City? Why or why not?

b. Estimate the population in 2010. Was the prediction correct from a.?

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1.3: Derivatives of the Inverse Trigonometric Functions

Determining the Derivatives of the Inverse Trigonometric Functions

Now let's determine the derivatives of the inverse trigonometric functions, $y = \arcsin x$, $y = \arccos x$, $y = \arctan x$, $y = \operatorname{arccot} x$, $y = \operatorname{arcsec} x$, and $y = \operatorname{arccsc} x$.

One way to do this that is particularly helpful in understanding how these derivatives are obtained is to use a combination of implicit differentiation and right triangles. An added benefit of this approach is that it will prepare you to be more successful in a future topic called trigonometric substitution.

Example 1.3.1: Finding the derivative of $y = \arcsin x$

Find the derivative of $y = \arcsin x$.

Solution:

To find the derivative of $y = \arcsin x$, we will first rewrite this equation in terms of its inverse form. That is,

$$\sin y = x \tag{1.3.1}$$

Now this equation shows that y can be considered an acute angle in a right triangle with a sine ratio of $\frac{x}{1}$. Since the sine ratio gives us the length of the opposite side over the length of the hypotenuse, this means that the opposite side has a length of x and the hypotenuse has a length of 1. See Figure 1.3.1.

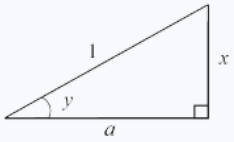


***Figure* 1.3.1**

Now let's differentiate Equation 1.3.1 implicitly with respect to x.

$$\cos y \cdot \frac{dy}{dx} = 1 \tag{1.3.2}$$

Then we solve this for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{1}{\cos y} \tag{1.3.3}$$

Looking at Figure 1.3.1, we see that $\cos y = a$. Now we use the Pythagorean Theorem to find an expression for a in terms of x using the other sides of the right triangle that we know.

Thus we have:

$$\begin{aligned} a^2 + x^2 &= 1^2 \\ a^2 &= 1 - x^2 \\ a &= \sqrt{1 - x^2} \end{aligned}$$

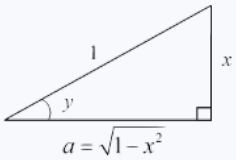


***Figure* 1.3.2**

Now using this result, we see that

$$\cos y = \sqrt{1 - x^2} \tag{1.3.4}$$

so

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}} \tag{1.3.5}$$

Thus we have found the derivative of $y = \arcsin x$,

$$\frac{d}{dx}\left(\arcsin x\right) = \frac{1}{\sqrt{1-x^2}} \tag{1.3.6}$$

Exercise 1.3.1

Use the same approach to determine the derivatives of $y = \arccos x$, $y = \arctan x$, and $y = \operatorname{arccot} x$.

Answer

$$\frac{d}{dx}\left(\arccos x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\arctan x\right) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\operatorname{arccot} x\right) = \frac{-1}{1+x^2}$$

Example 1.3.2: Finding the derivative of $y = \operatorname{arcsec} x$

Find the derivative of $y = \operatorname{arcsec} x$.

Solution:

To find the derivative of $y = \operatorname{arcsec} x$, we will first rewrite this equation in terms of its inverse form. That is,

$$\sec y = x \tag{1.3.7}$$

As before, let y be considered an acute angle in a right triangle with a secant ratio of $\frac{x}{1}$. Since the secant ratio is the reciprocal of the cosine ratio, it gives us the length of the hypotenuse over the length of the adjacent side, so this means that the hypotenuse has a length of x and the adjacent side has a length of 1. See Figure 1.3.3.

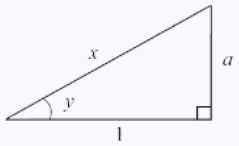


***Figure* 1.3.3**

Differentiating Equation 1.3.7 implicitly with respect to x, gives us:

$$\sec y \tan y \cdot \frac{dy}{dx} = 1 \tag{1.3.8}$$

Solving this for $\frac{dy}{dx}$, we get:

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \tag{1.3.9}$$

In order to find $\tan y$ in terms of x, we need to find the length of the opposite side, a, in terms of x. By the Pythagorean Theorem, we have:

$$1^2 + a^2 = x^2$$
$$a^2 = x^2 - 1$$
$$a = \sqrt{x^2 - 1}$$

Figure 1.3.4 shows the resulting right triangle.

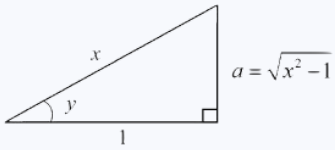


***Figure* 1.3.4**

From the right triangle in Figure 1.3.4, we can see that

$$\tan y = \sqrt{x^2 - 1}. \tag{1.3.10}$$

Since $\sec y = x$, it appears that

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2 - 1}}. \tag{1.3.11}$$

But this is not completely correct, at least not for negative values of x. Considering the graph of $y = \operatorname{arcsec} x$ in Figure 1.3.5, we see that its slope is always positive. But for negative values of x, the form of the derivative stated above would be negative (and clearly incorrect).

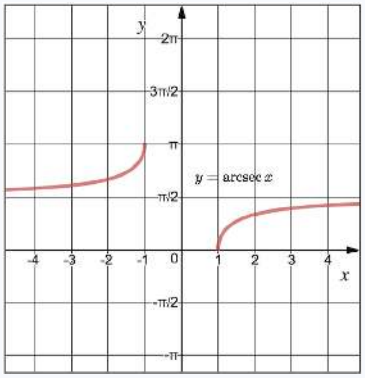


***Figure* 1.3.5**

As we'll prove below, the actual derivative formula for this function is:

$$\frac{d}{dx}\left(\operatorname{arcsec} x\right) = \frac{1}{|x|\sqrt{x^2 - 1}} \tag{1.3.12}$$

Consider the domain and range of the original function, $y = \operatorname{arcsec} x$:

$$\text{Domain: } (-\infty, -1] \cup [1, \infty) \quad \text{or} \quad |x| \geq 1 \tag{1.3.13}$$

$$\text{Range: } \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \quad \text{or} \quad 0 \leq y \leq \pi, y \neq \frac{\pi}{2} \tag{1.3.14}$$

Note that the domain of the derivative is a subset of the domain of the original function, excluding the endpoints, $x = -1$ and $x = 1$.

Now, let's rewrite $\dfrac{dy}{dx}$ as:

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{\cos y}{1} \cdot \frac{\cos y}{\sin y} = \frac{\cos^2 y}{\sin y} \tag{1.3.15}$$

We see that this function allows y to have all values between 0 and π, except for $y = \frac{\pi}{2}$, where the original expression $\frac{1}{\sec y \tan y}$ is undefined. This is a subset of the range of the original function $y = \operatorname{arcsec} x$. Note that in the derivative, y cannot take on the values of the endpoints of this interval, while these were part of the range of $y = \operatorname{arcsec} x$.

Now consider that for all values of y in this range between 0 and π, the expression for $\frac{dy}{dx}$ is positive, as both

$$\sin y > 0 \text{ and } \cos^2 y > 0 \text{ for all } y \in (0, \pi).$$

Therefore, we have that,

$$\frac{d}{dx}\left(\operatorname{arcsec} x\right) = \frac{1}{|x|\sqrt{x^2-1}}. \tag{1.3.16}$$

Exercise 1.3.2

Use the same approach to determine the derivative of $y = \operatorname{arccsc} x$.

Answer

$$\frac{d}{dx}\left(\operatorname{arccsc} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}.$$

Using the Chain Rule with Inverse Trigonometric Functions

Now let's see how to use the chain rule to find the derivatives of inverse trigonometric functions with more interesting functional arguments.

Example 1.3.3:

Find the derivatives for each of the following functions:

1. $y = \arcsin(x^2)$
2. $y = \arctan(x^3 + 1)$
3. $y = \operatorname{arcsec}(\ln|x|)$
4. $y = \arcsin(\cos x)$

Solution:

1. Using the chain rule, we see that:

$$\frac{d}{dx}\left(\arcsin(x^2)\right) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}\left(x^2\right) = \frac{2x}{\sqrt{1-x^4}} \tag{1.3.17}$$

2. Here we have:

$$\frac{d}{dx}\left(\arctan(x^3+1)\right) = \frac{1}{1+(x^3+1)^2} \cdot \frac{d}{dx}\left(x^3+1\right) = \frac{3x^2}{1+(x^3+1)^2} \tag{1.3.18}$$

Although it would likely be fine as it is, we can simplify it to obtain:

$$\frac{d}{dx}\left(\arctan(x^3+1)\right) = \frac{3x^2}{x^6+2x^3+2} \tag{1.3.19}$$

3. For $y = \operatorname{arcsec}(\ln|x|)$, we obtain:

$$\frac{d}{dx}\left(\operatorname{arcsec}(\ln|x|)\right) = \frac{1}{|\ln|x||\sqrt{(\ln|x|)^2-1}} \cdot \frac{d}{dx}\left(\ln|x|\right) = \frac{1}{x|\ln|x||\sqrt{(\ln|x|)^2-1}} \tag{1.3.20}$$

4. For $y = \arcsin(\cos x)$, we obtain:

$$\frac{d}{dx}\big(\arcsin(\cos x)\big) = \frac{1}{\sqrt{1-(\cos x)^2}} \cdot \frac{d}{dx}\big(\cos x\big) = \frac{-\sin x}{\sqrt{\sin^2 x}} \tag{1.3.21}$$

Note that it may look like the denominator should simplify to $\sin x$ and the entire derivative to $\frac{dy}{dx} = -1$. But this is not the case. Remember that

$$\sqrt{x^2} = |x|. \tag{1.3.22}$$

This means that what we actually obtain here is something more interesting:

$$\frac{d}{dx}\big(\arcsin(\cos x)\big) = \frac{-\sin x}{|\sin x|} \tag{1.3.23}$$

This function alternates between values of -1 and 1, when $\sin x$ is positive and negative, respectively.
See the graph of the function $y = \arcsin(\cos x)$ below in Figure 1.3.6.

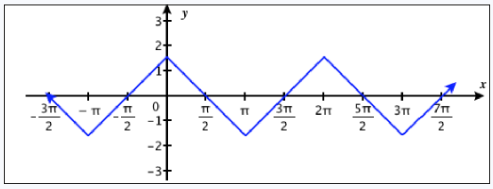


***Figure* 1.3.6:** *Graph of* $y = \arcsin(\cos x)$.

Derivative Formulas

In the same way that we can encapsulate the chain rule in the derivative of $\ln u$ as $\frac{d}{dx}\big(\ln u\big) = \frac{u'}{u}$, we can write formulas for the derivative of the inverse trigonometric functions that encapsulate the chain rule. Note that u represents a function of x in these formulas, and u' represents the derivative of u with respect to x.

$$\begin{aligned}
\frac{d}{dx}\big(\arcsin u\big) &= \frac{u'}{\sqrt{1-u^2}} & \frac{d}{dx}\big(\arccos u\big) &= \frac{-u'}{\sqrt{1-u^2}} \\
\frac{d}{dx}\big(\arctan u\big) &= \frac{u'}{1+u^2} & \frac{d}{dx}\big(\operatorname{arccot} u\big) &= \frac{-u'}{1+u^2} \\
\frac{d}{dx}\big(\operatorname{arcsec} u\big) &= \frac{u'}{|u|\sqrt{u^2-1}} & \frac{d}{dx}\big(\operatorname{arccsc} u\big) &= \frac{-u'}{|u|\sqrt{u^2-1}}
\end{aligned}$$

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CHAPTER OVERVIEW

2: Integration

2.1: Antiderivatives

2.1E: Antiderivative and Indefinite Integral Exercises

2.2: Prelude to Integration

2.3: Approximating Areas

2.3E: Approximating Areas (Exercises)

2.4: The Definite Integral

2.5: The Fundamental Theorem of Calculus

2.6: Integration Formulas and the Net Change Theorem

2.7: Substitution

2.8: 5.5E and 5.6E u-Substitution Exercises

2.9: Integrals Involving Exponential and Logarithmic Functions

2.10: Integrals Resulting in Inverse Trigonometric Functions and Related Integration Techniques

2.10E: Exercises for Integrals Resulting in Inverse Trigonometric Functions

2.11: Chapter 5 Review Exercises

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2.1: Antiderivatives

At this point, we have seen how to calculate derivatives of many functions and have been introduced to a variety of their applications. We now ask a question that turns this process around: Given a function f, how do we find a function with the derivative f and why would we be interested in such a function?

We answer the first part of this question by defining antiderivatives. The antiderivative of a function f is a function with a derivative f. Why are we interested in antiderivatives? The need for antiderivatives arises in many situations, and we look at various examples throughout the remainder of the text. Here we examine one specific example that involves rectilinear motion. In our examination in Derivatives of rectilinear motion, we showed that given a position function $s(t)$ of an object, then its velocity function $v(t)$ is the derivative of $s(t)$—that is, $v(t) = s'(t)$. Furthermore, the acceleration $a(t)$ is the derivative of the velocity $v(t)$—that is, $a(t) = v'(t) = s''(t)$. Now suppose we are given an acceleration function a, but not the velocity function v or the position function s. Since $a(t) = v'(t)$, determining the velocity function requires us to find an antiderivative of the acceleration function. Then, since $v(t) = s'(t)$, determining the position function requires us to find an antiderivative of the velocity function. Rectilinear motion is just one case in which the need for antiderivatives arises. We will see many more examples throughout the remainder of the text. For now, let's look at the terminology and notation for antiderivatives, and determine the antiderivatives for several types of functions. We examine various techniques for finding antiderivatives of more complicated functions later in the text (Introduction to Techniques of Integration).

The Reverse of Differentiation

At this point, we know how to find derivatives of various functions. We now ask the opposite question. Given a function f, how can we find a function with derivative f? If we can find a function F derivative f, we call F an antiderivative of f.

Definition: Antiderivative

A function F is an **antiderivative** of the function f if

$$F'(x) = f(x) \tag{2.1.1}$$

for all x in the domain of f.

Consider the function $f(x) = 2x$. Knowing the power rule of differentiation, we conclude that $F(x) = x^2$ is an antiderivative of f since $F'(x) = 2x$. Are there any other antiderivatives of f? Yes; since the derivative of any constant C is zero, $x^2 + C$ is also an antiderivative of $2x$. Therefore, $x^2 + 5$ and $x^2 - \sqrt{2}$ are also antiderivatives. Are there any others that are not of the form $x^2 + C$ for some constant C? The answer is no. From Corollary 2 of the Mean Value Theorem, we know that if F and G are differentiable functions such that $F'(x) = G'(x)$, then $F(x) - G(x) = C$ for some constant C. This fact leads to the following important theorem.

General Form of an Antiderivative

Let F be an antiderivative of f over an interval I. Then,

I. for each constant C, the function $F(x) + C$ is also an antiderivative of f over I;
II. if G is an antiderivative of f over I, there is a constant C for which $G(x) = F(x) + C$ over I.

In other words, the most general form of the antiderivative of f over I is $F(x) + C$.

We use this fact and our knowledge of derivatives to find all the antiderivatives for several functions.

Example 2.1.1: Finding Antiderivatives

For each of the following functions, find all antiderivatives.

a. $f(x) = 3x^2$
b. $f(x) = \dfrac{1}{x}$
c. $f(x) = \cos x$
d. $f(x) = e^x$

Solution:

a. Because

$$\frac{d}{dx}\left(x^3\right) = 3x^2$$

then $F(x) = x^3$ is an antiderivative of $3x^2$. Therefore, every antiderivative of $3x^2$ is of the form $x^3 + C$ for some constant C, and every function of the form $x^3 + C$ is an antiderivative of $3x^2$.

b. Let $f(x) = \ln|x|$. For $x > 0$, $f(x) = \ln(x)$ and

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

For $x < 0$, $f(x) = \ln(-x)$ and

$$\frac{d}{dx}(\ln(-x)) = -\frac{1}{-x} = \frac{1}{x}.$$

Therefore,

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}.$$

Thus, $F(x) = \ln|x|$ is an antiderivative of $\frac{1}{x}$. Therefore, every antiderivative of $\frac{1}{x}$ is of the form $\ln|x| + C$ for some constant C and every function of the form $\ln|x| + C$ is an antiderivative of $\frac{1}{x}$.

c. We have

$$\frac{d}{dx}(\sin x) = \cos x,$$

so $F(x) = \sin x$ is an antiderivative of $\cos x$. Therefore, every antiderivative of $\cos x$ is of the form $\sin x + C$ for some constant C and every function of the form $\sin x + C$ is an antiderivative of $\cos x$.

d. Since

$$\frac{d}{dx}(e^x) = e^x,$$

then $F(x) = e^x$ is an antiderivative of e^x. Therefore, every antiderivative of e^x is of the form $e^x + C$ for some constant C and every function of the form $e^x + C$ is an antiderivative of e^x.

Exercise 2.1.1

Find all antiderivatives of $f(x) = \sin x$.

Hint

What function has a derivative of $\sin x$?

Answer

$-\cos x + C$

Indefinite Integrals

We now look at the formal notation used to represent antiderivatives and examine some of their properties. These properties allow us to find antiderivatives of more complicated functions. Given a function f, we use the notation $f'(x)$ or $\frac{df}{dx}$ to denote the derivative of f. Here we introduce notation for antiderivatives. If F is an antiderivative of f, we say that $F(x) + C$ is the most general antiderivative of f and write

$$\int f(x)\,dx = F(x) + C. \tag{2.1.2}$$

The symbol $\int$ is called an integral sign, and $\int f(x)\,dx$ is called the indefinite integral of f.

Definition: Indefinite Integrals

Given a function f, the indefinite integral of f, denoted

$$\int f(x)\,dx, \tag{2.1.3}$$

is the most general antiderivative of f. If F is an antiderivative of f, then

$$\int f(x)\,dx = F(x) + C. \tag{2.1.4}$$

The expression $f(x)$ is called the integrand and the variable x is the variable of integration.

Given the terminology introduced in this definition, the act of finding the antiderivatives of a function f is usually referred to as integrating f.

For a function f and an antiderivative F, the functions $F(x)+C$, where C is any real number, is often referred to as the family of antiderivatives of f. For example, since x^2 is an antiderivative of $2x$ and any antiderivative of $2x$ is of the form x^2+C, we write

$$\int 2x\,dx = x^2 + C. \tag{2.1.5}$$

The collection of all functions of the form x^2+C, where C is any real number, is known as the family of antiderivatives of $2x$. Figure shows a graph of this family of antiderivatives.

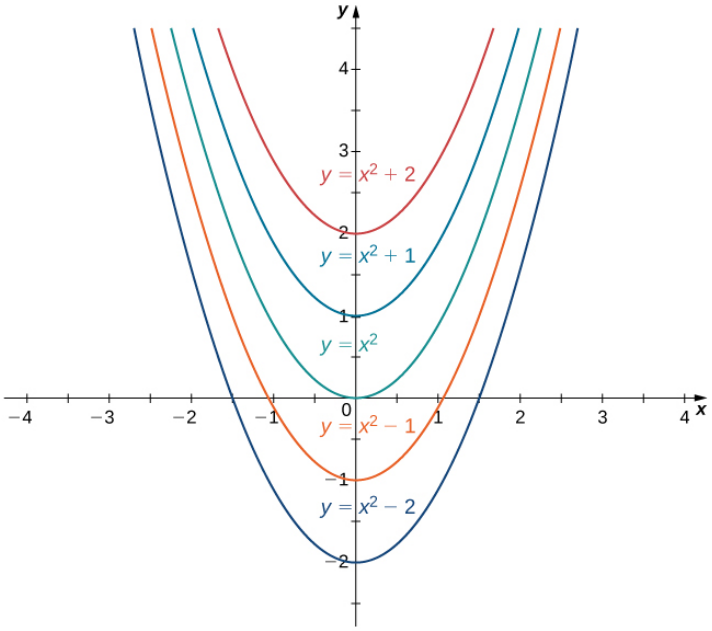


Figure* 2.1.1*: *The family of antiderivatives of $2x$ consists of all functions of the form x^2+C, where C is any real number.*

For some functions, evaluating indefinite integrals follows directly from properties of derivatives. For example, for $n \neq -1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,$$

which comes directly from

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = (n+1)\frac{x^n}{n+1} = x^n\ .$$

This fact is known as the power rule for integrals.

Power Rule for Integrals

For $n \neq -1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C. \tag{2.1.6}$$

Evaluating indefinite integrals for some other functions is also a straightforward calculation. The following table lists the indefinite integrals for several common functions. A more complete list appears in Appendix B.

Table : Integration Formulas

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(k) = 0$	$\int k\,dx = \int kx^0\,dx = kx + C$
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
$\frac{d}{dx}(\ln\lvert x\rvert) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

Differentiation Formula	Indefinite Integral
$\frac{d}{dx}(e^x) = e^x$	$\int e^x\,dx = e^x + C$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} = \arcsin x + C$
$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,dx = \arctan x + C$
$\frac{d}{dx}(\text{arcsec} x) = \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \text{arcsec}\lvert x\rvert + C$

From the definition of indefinite integral of f, we know

$$\int f(x)\,dx = F(x) + C \tag{2.1.7}$$

if and only if F is an antiderivative of f. Therefore, when claiming that

$$\int f(x)\,dx = F(x) + C \tag{2.1.8}$$

it is important to check whether this statement is correct by verifying that $F'(x) = f(x)$.

Example 2.1.2: Verifying an Indefinite Integral

Each of the following statements is of the form $\int f(x)\,dx = F(x) + C$. Verify that each statement is correct by showing that $F'(x) = f(x)$.

a. $\int (x + e^x)\,dx = \frac{x^2}{2} + e^x + C$

b. $\int xe^x\,dx = xe^x - e^x + C$

Solution:

a. Since

$$\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = x + e^x\ ,$$

the statement

$$\int (x + e^x)\,dx = \frac{x^2}{2} + e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a sum. Furthermore, $\frac{x^2}{2}$ and e^x are antiderivatives of x and e^x, respectively, and the sum of the antiderivatives is an antiderivative of the sum. We discuss this fact again later in this section.

b. Using the product rule, we see that

$$\frac{d}{dx}(xe^x - e^x + C) = e^x + xe^x - e^x = xe^x.$$

Therefore, the statement

$$\int xe^x\,dx = xe^x - e^x + C$$

is correct.

Note that we are verifying an indefinite integral for a product. The antiderivative $xe^x - e^x$ is not a product of the antiderivatives. Furthermore, the product of antiderivatives, $x^2e^x/2$ is not an antiderivative of xe^x since

$$\frac{d}{dx}\left(\frac{x^2e^x}{2}\right) = xe^x + \frac{x^2e^x}{2} \neq xe^x\,.$$

In general, the product of antiderivatives is not an antiderivative of a product.

Exercise 2.1.2

Verify that

$$\int x\cos x\;dx = x\sin x + \cos x + C.$$

Hint

Calculate

$$\frac{d}{dx}(x\sin x + \cos x + C).$$

Answer

$$\frac{d}{dx}(x\sin x + \cos x + C) = \sin x + x\cos x - \sin x = x\cos x$$

In Table, we listed the indefinite integrals for many elementary functions. Let's now turn our attention to evaluating indefinite integrals for more complicated functions. For example, consider finding an antiderivative of a sum $f + g$. In Example a. we showed that an antiderivative of the sum $x + e^x$ is given by the sum $\frac{x^2}{2} + e^x$—that is, an antiderivative of a sum is given by a sum of antiderivatives. This result was not specific to this example. In general, if F and G are antiderivatives of any functions f and g, respectively, then

$$\frac{d}{dx}(F(x) + G(x)) = F'(x) + G'(x) = f(x) + g(x).$$

Therefore, $F(x) + G(x)$ is an antiderivative of $f(x) + g(x)$ and we have

$$\int (f(x) + g(x))\,dx = F(x) + G(x) + C.$$

Similarly,

$$\int (f(x) - g(x))\,dx = F(x) - G(x) + C.$$

In addition, consider the task of finding an antiderivative of $kf(x)$, where k is any real number. Since

$$\frac{d}{dx}(kf(x)) = k\frac{d}{dx}F(x) = kF'(x)$$

for any real number k, we conclude that

$$\int kf(x)\,dx = kF(x) + C.$$

These properties are summarized next.

Properties of Indefinite Integrals

Let F and G be antiderivatives of f and g, respectively, and let k be any real number.

Sums and Differences

$$\int (f(x) \pm g(x))\, dx = F(x) \pm G(x) + C$$

Constant Multiples

$$\int kf(x)\, dx = kF(x) + C$$

From this theorem, we can evaluate any integral involving a sum, difference, or constant multiple of functions with antiderivatives that are known. Evaluating integrals involving products, quotients, or compositions is more complicated (see Exampleb. for an example involving an antiderivative of a product.) We look at and address integrals involving these more complicated functions in Introduction to Integration. In the next example, we examine how to use this theorem to calculate the indefinite integrals of several functions.

Example 2.1.3: Evaluating Indefinite Integrals

Evaluate each of the following indefinite integrals:

a. $\int (5x^3 - 7x^2 + 3x + 4)\, dx$

b. $\int \dfrac{x^2 + 4\sqrt[3]{x}}{x}\, dx$

c. $\int \dfrac{4}{1+x^2}\, dx$

d. $\int \tan x \cos x\, dx$

Solution:

a. Using Note, we can integrate each of the four terms in the integrand separately. We obtain

$$\int (5x^3 - 7x^2 + 3x + 4)\, dx = \int 5x^3\, dx - \int 7x^2\, dx + \int 3x\, dx + \int 4\, dx.$$

From the second part of Note, each coefficient can be written in front of the integral sign, which gives

$$\int 5x^3\, dx - \int 7x^2\, dx + \int 3x\, dx + \int 4\, dx = 5\int x^3\, dx - 7\int x^2\, dx + 3\int x\, dx + 4\int 1\, dx.$$

Using the power rule for integrals, we conclude that

$$\int (5x^3 - 7x^2 + 3x + 4)\, dx = \frac{5}{4}x^4 - \frac{7}{3}x^3 + \frac{3}{2}x^2 + 4x + C.$$

b. Rewrite the integrand as

$$\frac{x^2 + 4\sqrt[3]{x}}{x} = \frac{x^2}{x} + \frac{4\sqrt[3]{x}}{x} = 0.$$

Then, to evaluate the integral, integrate each of these terms separately. Using the power rule, we have

$$\begin{aligned}\int (x + \frac{4}{x^{2/3}})\, dx &= \int x\, dx + 4\int x^{-2/3}\, dx \\ &= \frac{1}{2}x^2 + 4\frac{1}{(\frac{-2}{3}) + 1}x^{(-2/3)+1} + C]) \\ &= \frac{1}{2}x^2 + 12x^{1/3} + C.\end{aligned}$$

c. Using Note, write the integral as

$$4\int \frac{1}{1+x^2}\, dx.$$

Then, use the fact that $\arctan(x)$ is an antiderivative of $\dfrac{1}{(1+x^2)}$ to conclude that

$$\int \frac{4}{1+x^2}\, dx = 4\arctan(x) + C.$$

d. Rewrite the integrand as

$$\tan x \cos x = \frac{\sin x}{\cos x}\cos x = \sin x.$$

Therefore,

$$\int \tan x \cos x \, dx = \int \sin x \, dx = -\cos x + C.$$

Exercise 2.1.3

Evaluate $\int (4x^3 - 5x^2 + x - 7)\, dx$.

Hint

Integrate each term in the integrand separately, making use of the power rule.

Answer

$x^4 - \frac{5}{3}x^3 + \frac{1}{2}x^2 - 7x + C$

Initial-Value Problems

We look at techniques for integrating a large variety of functions involving products, quotients, and compositions later in the text. Here we turn to one common use for antiderivatives that arises often in many applications: solving differential equations.

A differential equation is an equation that relates an unknown function and one or more of its derivatives. The equation

$$\frac{dy}{dx} = f(x)$$

is a simple example of a differential equation. Solving this equation means finding a function y with a derivative f. Therefore, the solutions of Equation are the antiderivatives of f. If F is one antiderivative of f, every function of the form $y = F(x) + C$ is a solution of that differential equation. For example, the solutions of

$$\frac{dy}{dx} = 6x^2$$

are given by

$$y = \int 6x^2 \, dx = 2x^3 + C \ .$$

Sometimes we are interested in determining whether a particular solution curve passes through a certain point (x_0, y_0) —that is, $y(x_0) = y_0$. The problem of finding a function y that satisfies a differential equation

$$\frac{dy}{dx} = f(x)$$

with the additional condition

$$y(x_0) = y_0$$

is an example of an initial-value problem. The condition $y(x_0) = y_0$ is known as an initial condition. For example, looking for a function y that satisfies the differential equation

$$\frac{dy}{dx} = 6x^2$$

and the initial condition

$$y(1) = 5$$

is an example of an initial-value problem. Since the solutions of the differential equation are $y = 2x^3 + C$, to find a function y that also satisfies the initial condition, we need to find C such that $y(1) = 2(1)^3 + C = 5$. From this equation, we see that $C = 3$, and we conclude that $y = 2x^3 + 3$ is the solution of this initial-value problem as shown in the following graph.

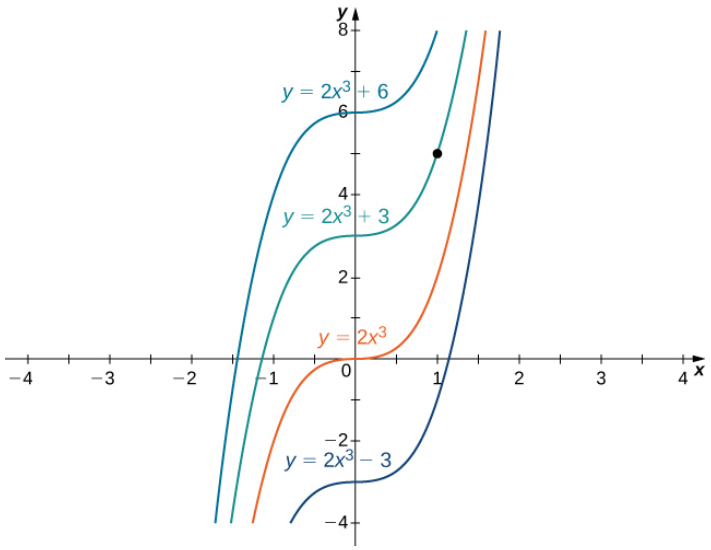


***Figure* 2.1.2:** *Some of the solution curves of the differential equation* $\frac{dy}{dx} = 6x^2$ *are displayed. The function* $y = 2x^3 + 3$ *satisfies the differential equation and the initial condition* $y(1) = 5$.

Example 2.1.4: Solving an Initial-Value Problem

Solve the initial-value problem

$$\frac{dy}{dx} = \sin x, y(0) = 5. \tag{2.1.9}$$

Solution

First we need to solve the differential equation. If $\frac{dy}{dx} = \sin x$, then

$$y = \int \sin(x)\, dx = -\cos x + C. \tag{2.1.10}$$

Next we need to look for a solution y that satisfies the initial condition. The initial condition y(0)=5 means we need a constant C such that $-\cos x + C = 5$. Therefore,

$$C = 5 + \cos(0) = 6. \tag{2.1.11}$$

The solution of the initial-value problem is $y = -\cos x + 6$.

Exercise 2.1.4

Solve the initial value problem $\frac{dy}{dx} = 3x^{-2}, y(1) = 2$.

Hint

Find all antiderivatives of $f(x) = 3x^{-2}$.

Answer

$y = -\frac{3}{x} + 5$

Initial-value problems arise in many applications. Next we consider a problem in which a driver applies the brakes in a car. We are interested in how long it takes for the car to stop. Recall that the velocity function $v(t)$ is the derivative of a position function $s(t)$, and the acceleration $a(t)$ is the derivative of the velocity function. In earlier examples in the text, we could calculate the velocity from the position and then compute the acceleration from the velocity. In the next example we work the other way around. Given an acceleration function, we calculate the velocity function. We then use the velocity function to determine the position function.

Example 2.1.5:

A car is traveling at the rate of 88 ft/sec (60 mph) when the brakes are applied. The car begins decelerating at a constant rate of 15 ft/sec^2.

a. How many seconds elapse before the car stops?
b. How far does the car travel during that time?

Solution

a. First we introduce variables for this problem. Let t be the time (in seconds) after the brakes are first applied. Let $a(t)$ be the acceleration of the car (in feet per seconds squared) at time t. Let $v(t)$ be the velocity of the car (in feet per second) at time t. Let $s(t)$ be the car's position (in feet) beyond the point where the brakes are applied at time t.

The car is traveling at a rate of 88 ft/sec. Therefore, the initial velocity is $v(0) = 88$ ft/sec. Since the car is decelerating, the acceleration is

$$a(t) = -15 \text{ ft}/\text{sec}^2.$$

The acceleration is the derivative of the velocity,

$$v'(t) = 15.$$

Therefore, we have an initial-value problem to solve:

$$v'(t) = -15, v(0) = 88.$$

Integrating, we find that

$$v(t) = -15t + C.$$

Since $v(0) = 88, C = 88.$ Thus, the velocity function is

$$v(t) = -15t + 88.$$

To find how long it takes for the car to stop, we need to find the time t such that the velocity is zero. Solving $-15t + 88 = 0,$ we obtain $t = \dfrac{88}{15}$ sec.

b. To find how far the car travels during this time, we need to find the position of the car after $\dfrac{88}{15}$ sec. We know the velocity $v(t)$ is the derivative of the position $s(t)$. Consider the initial position to be $s(0) = 0$. Therefore, we need to solve the initial-value problem

$$s'(t) = -15t + 88,\ s(0) = 0.$$

Integrating, we have

$$s(t) = -\frac{15}{2}t^2 + 88t + C.$$

Since $s(0) = 0$, the constant is $C = 0$. Therefore, the position function is

$$s(t) = -\frac{15}{2}t^2 + 88t.$$

After $t = \frac{88}{15}$ sec, the position is $s\left(\frac{88}{15}\right) \approx 258.133$ ft.

Exercise 2.1.5

Suppose the car is traveling at the rate of 44 ft/sec. How long does it take for the car to stop? How far will the car travel?

Hint

$v(t) = -15t + 44.$

Answer

$2.93 sec, 64.5$ft

Key Concepts

- If F is an antiderivative of f, then every antiderivative of f is of the form $F(x) + C$ for some constant C.
- Solving the initial-value problem

$$\frac{dy}{dx} = f(x), y(x_0) = y_0$$

- requires us first to find the set of antiderivatives of f and then to look for the particular antiderivative that also satisfies the initial condition.

Glossary

antiderivative

a function F such that $F'(x) = f(x)$ for all x in the domain of f is an antiderivative of f

indefinite integral

the most general antiderivative of $f(x)$ is the indefinite integral of f; we use the notation $\int f(x)\,dx$ to denote the indefinite integral of f

initial value problem

a problem that requires finding a function y that satisfies the differential equation $\frac{dy}{dx} = f(x)$ together with the initial condition $y(x_0) = y_0$

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2.1E: Antiderivative and Indefinite Integral Exercises

In exercises 1 - 20, find the antiderivative $F(x)$ of each function $f(x)$.

1) $f(x) = \dfrac{1}{x^2} + x$

2) $f(x) = e^x - 3x^2 + \sin x$

Answer

$F(x) = e^x - x^3 - \cos x + C$

3) $f(x) = e^x + 3x - x^2$

4) $f(x) = x - 1 + 4\sin(2x)$

Answer

$F(x) = \dfrac{x^2}{2} - x - 2\cos(2x) + C$

5) $f(x) = 5x^4 + 4x^5$

6) $f(x) = x + 12x^2$

Answer

$F(x) = \frac{1}{2}x^2 + 4x^3 + C$

7) $f(x) = \dfrac{1}{\sqrt{x}}$

8) $f(x) = (\sqrt{x})^3$

Answer

$F(x) = \frac{2}{5}(\sqrt{x})^5 + C$

9) $f(x) = x^{1/3} + (2x)^{1/3}$

10) $f(x) = \dfrac{x^{1/3}}{x^{2/3}}$

Answer

$F(x) = \frac{3}{2}x^{2/3} + C$

11) $f(x) = 2\sin(x) + \sin(2x)$

12) $f(x) = \sec^2 x + 1$

Answer

$F(x) = x + \tan x + C$

13) $f(x) = \sin x \cos x$

14) $f(x) = \sin^2(x)\cos(x)$

Answer

$F(x) = \frac{1}{3}\sin^3(x) + C$

15) $f(x) = 0$

16) $f(x) = \frac{1}{2}\csc^2 x + \dfrac{1}{x^2}$

Answer

$$F(x) = -\tfrac{1}{2}\cot x - \frac{1}{x} + C$$

17) $f(x) = \csc x \cot x + 3x$

18) $f(x) = 4\csc x \cot x - \sec x \tan x$

Answer

$$F(x) = -\sec x - 4\csc x + C$$

19) $f(x) = 8(\sec x)\big(\sec x - 4\tan x\big)$

20) $f(x) = \tfrac{1}{2}e^{-4x} + \sin x$

Answer

$$F(x) = -\tfrac{1}{8}e^{-4x} - \cos x + C$$

For exercises 21 - 29, evaluate the integral.

21) $\displaystyle\int (-1)\,dx$

22) $\displaystyle\int \sin x\,dx$

Answer

$$\int \sin x\,dx = -\cos x + C$$

23) $\displaystyle\int \left(4x + \sqrt{x}\right) dx$

24) $\displaystyle\int \frac{3x^2+2}{x^2}\,dx$

Answer

$$\int \frac{3x^2+2}{x^2}\,dx = 3x - \frac{2}{x} + C$$

25) $\displaystyle\int \left(\sec x \tan x + 4x\right) dx$

26) $\displaystyle\int \left(4\sqrt{x} + \sqrt[4]{x}\right) dx$

Answer

$$\int \left(4\sqrt{x} + \sqrt[4]{x}\right) dx = \frac{8}{3}x^{3/2} + \frac{4}{5}x^{5/4} + C$$

27) $\displaystyle\int \left(x^{-1/3} - x^{2/3}\right)\,dx$

28) $\displaystyle\int \frac{14x^3+2x+1}{x^3}\,dx$

Answer

$$\int \frac{14x^3+2x+1}{x^3}\,dx = 14x - \frac{2}{x} - \frac{1}{2x^2} + C$$

29) $\int \left(e^x + e^{-x}\right) dx$

In exercises 30 - 34, solve the initial value problem.

30) $f'(x) = x^{-3}, \quad f(1) = 1$

Answer

$$f(x) = -\frac{1}{2x^2} + \frac{3}{2}$$

31) $f'(x) = \sqrt{x} + x^2, \quad f(0) = 2$

32) $f'(x) = \cos x + \sec^2(x), \quad f(\frac{\pi}{4}) = 2 + \frac{\sqrt{2}}{2}$

Answer

$$f(x) = \sin x + \tan x + 1$$

33) $f'(x) = x^3 - 8x^2 + 16x + 1, \quad f(0) = 0$

34) $f'(x) = \frac{2}{x^2} - \frac{x^2}{2}, \quad f(1) = 0$

Answer

$$f(x) = -\tfrac{1}{6}x^3 - \frac{2}{x} + \frac{13}{6}$$

In exercises 35 - 39, find two possible functions f given the second- or third-order derivatives

35) $f''(x) = x^2 + 2$

36) $f''(x) = e^{-x}$

Answer

Answers may vary; one possible answer is $f(x) = e^{-x}$

37) $f''(x) = 1 + x$

38) $f'''(x) = \cos x$

Answer

Answers may vary; one possible answer is $f(x) = -\sin x$

39) $f'''(x) = 8e^{-2x} - \sin x$

40) A car is being driven at a rate of 40 mph when the brakes are applied. The car decelerates at a constant rate of $10\,\text{ft}/\text{sec}^2$. How long before the car stops?

Answer

5.867sec

41) In the preceding problem, calculate how far the car travels in the time it takes to stop.

42) You are merging onto the freeway, accelerating at a constant rate of $12\,\text{ft}/\text{sec}^2$. How long does it take you to reach merging speed at 60 mph?

Answer

7.333sec

43) Based on the previous problem, how far does the car travel to reach merging speed?

44) A car company wants to ensure its newest model can stop in 8 sec when traveling at 75 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

Answer

$13.75\,\text{ft}/\text{sec}^2$

45) A car company wants to ensure its newest model can stop in less than 450 ft when traveling at 60 mph. If we assume constant deceleration, find the value of deceleration that accomplishes this.

In exercises 46 - 51, find the antiderivative of the function, assuming $F(0)=0$.

46) [T] $f(x)=x^2+2$

Answer

$F(x)=\frac{1}{3}x^3+2x$

47) [T] $f(x)=4x-\sqrt{x}$

48) [T] $f(x)=\sin x+2x$

Answer

$F(x)=x^2-\cos x+1$

49) [T] $f(x)=e^x$

50) [T] $f(x)=\dfrac{1}{(x+1)^2}$

Answer

$F(x)=-\dfrac{1}{x+1}+1$

51) [T] $f(x)=e^{-2x}+3x^2$

In exercises 52 - 55, determine whether the statement is true or false. Either prove it is true or find a counterexample if it is false.

52) If $f(x)$ is the antiderivative of $v(x)$, then $2f(x)$ is the antiderivative of $2v(x)$.

Answer

True

53) If $f(x)$ is the antiderivative of $v(x)$, then $f(2x)$ is the antiderivative of $v(2x)$.

54) If $f(x)$ is the antiderivative of $v(x)$, then $f(x)+1$ is the antiderivative of $v(x)+1$.

Answer

False

55) If $f(x)$ is the antiderivative of $v(x)$, then $(f(x))^2$ is the antiderivative of $(v(x))^2$.

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2.2: Prelude to Integration

Iceboats are a common sight on the lakes of Wisconsin and Minnesota on winter weekends. Iceboats are similar to sailboats, but they are fitted with runners, or “skates,” and are designed to run over the ice, rather than on water. Iceboats can move very quickly, and many ice boating enthusiasts are drawn to the sport because of the speed. Top iceboat racers can attain speeds up to five times the wind speed. If we know how fast an iceboat is moving, we can use integration to determine how far it travels. We revisit this question later in the chapter (see [link]).



Figure 2.2.1: *Iceboating is a popular winter sport in parts of the northern United States and Europe. (credit: modification of work by Carter Brown, Flickr)*

Determining distance from velocity is just one of many applications of integration. In fact, integrals are used in a wide variety of mechanical and physical applications. In this chapter, we first introduce the theory behind integration and use integrals to calculate areas. From there, we develop the Fundamental Theorem of Calculus, which relates differentiation and integration. We then study some basic integration techniques and briefly examine some applications.

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2.3: Approximating Areas

Learning Objectives

- Use sigma (summation) notation to calculate sums and powers of integers.
- Use the sum of rectangular areas to approximate the area under a curve.
- Use Riemann sums to approximate area.

Archimedes was fascinated with calculating the areas of various shapes—in other words, the amount of space enclosed by the shape. He used a process that has come to be known as the **method of exhaustion**, which used smaller and smaller shapes, the areas of which could be calculated exactly, to fill an irregular region and thereby obtain closer and closer approximations to the total area. In this process, an area bounded by curves is filled with rectangles, triangles, and shapes with exact area formulas. These areas are then summed to approximate the area of the curved region.

In this section, we develop techniques to approximate the area between a curve, defined by a function $f(x)$, and the x-axis on a closed interval $[a, b]$. Like Archimedes, we first approximate the area under the curve using shapes of known area (namely, rectangles). By using smaller and smaller rectangles, we get closer and closer approximations to the area. Taking a limit allows us to calculate the exact area under the curve.

Let's start by introducing some notation to make the calculations easier. We then consider the case when $f(x)$ is continuous and nonnegative. Later in the chapter, we relax some of these restrictions and develop techniques that apply in more general cases.

Sigma (Summation) Notation

As mentioned, we will use shapes of known area to approximate the area of an irregular region bounded by curves. This process often requires adding up long strings of numbers. To make it easier to write down these lengthy sums, we look at some new notation here, called **sigma notation** (also known as **summation notation**). The Greek capital letter Σ, sigma, is used to express long sums of values in a compact form. For example, if we want to add all the integers from 1 to 20 without sigma notation, we have to write

$$1+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16+17+18+19+20.$$

We could probably skip writing a couple of terms and write

$$1+2+3+4+\cdots+19+20,$$

which is better, but still cumbersome. With sigma notation, we write this sum as

$$\sum_{i=1}^{20} i$$

which is much more compact. Typically, sigma notation is presented in the form

$$\sum_{i=1}^{n} a_i$$

where a_i describes the terms to be added, and the i is called the $index$. Each term is evaluated, then we sum all the values, beginning with the value when $i = 1$ and ending with the value when $i = n$. For example, an expression like $\sum_{i=2}^{7} s_i$ is interpreted as $s_2 + s_3 + s_4 + s_5 + s_6 + s_7$. Note that the index is used only to keep track of the terms to be added; it does not factor into the calculation of the sum itself. The index is therefore called a **dummy variable**. We can use any letter we like for the index. Typically, mathematicians use i, j, k, m, and n for indices.

Let's try a couple of examples of using sigma notation.

Example 2.3.1: Using Sigma Notation

a. Write in sigma notation and evaluate the sum of terms 3^i for $i = 1, 2, 3, 4, 5$.
b. Write the sum in sigma notation:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}.$$

Solution

a. Write

$$\sum_{i=1}^{5} 3^i = 3 + 3^2 + 3^3 + 3^4 + 3^5 = 363.$$

b. The denominator of each term is a perfect square. Using sigma notation, this sum can be written as $\sum_{i=1}^{5} \frac{1}{i^2}$.

Exercise 2.3.1

Write in sigma notation and evaluate the sum of terms 2^i for $i = 3, 4, 5, 6$.

Hint

Use the solving steps in Example 2.3.1as a guide.

Answer

$$\sum_{i=3}^{6} 2^i = 2^3 + 2^4 + 2^5 + 2^6 = 120$$

The properties associated with the summation process are given in the following rule.

Rule: Properties of Sigma Notation

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ represent two sequences of terms and let c be a constant. The following properties hold for all positive integers n and for integers m, with $1 \le m \le n$.

i. $\sum_{i=1}^{n} c = nc$
ii. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
iii. $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$
iv. $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$
v. $\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$

Proof

We prove properties (ii.) and (iii.) here, and leave proof of the other properties to the Exercises.

(ii.) We have

$$\sum_{i=1}^{n} ca_i = ca_1 + ca_2 + ca_3 + \cdots + ca_n = c(a_1 + a_2 + a_3 + \cdots + a_n) = c\sum_{i=1}^{n} a_i.$$

(iii.) We have

$$\sum_{i=1}^{n}(a_i+b_i) = (a_1+b_1)+(a_2+b_2)+(a_3+b_3)+\cdots+(a_n+b_n) \tag{2.3.1}$$

$$= (a_1+a_2+a_3+\cdots+a_n)+(b_1+b_2+b_3+\cdots+b_n) \tag{2.3.2}$$

$$= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i. \tag{2.3.3}$$

□

A few more formulas for frequently found functions simplify the summation process further. These are shown in the next rule, for **sums and powers of integers**, and we use them in the next set of examples.

Rule: Sums and Powers of Integers

1. The sum of n integers is given by

$$\sum_{i=1}^{n} i = 1+2+\cdots+n = \frac{n(n+1)}{2}. \tag{2.3.4}$$

2. The sum of consecutive integers squared is given by

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{2.3.5}$$

3. The sum of consecutive integers cubed is given by

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}. \tag{2.3.6}$$

Example 2.3.2: Evaluation Using Sigma Notation

Write using sigma notation and evaluate:

a. The sum of the terms $(i-3)^2$ for $i = 1, 2, \ldots, 200$.
b. The sum of the terms (i^3-i^2) for $i = 1, 2, 3, 4, 5, 6$

Solution

a. Multiplying out $(i-3)^2$, we can break the expression into three terms.

$$\begin{aligned}
\sum_{i=1}^{200}(i-3)^2 &= \sum_{i=1}^{200}(i^2-6i+9) \\
&= \sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} 6i + \sum_{i=1}^{200} 9 \\
&= \sum_{i=1}^{200} i^2 - 6\sum_{i=1}^{200} i + \sum_{i=1}^{200} 9 \\
&= \frac{200(200+1)(400+1)}{6} - 6\left[\frac{200(200+1)}{2}\right] + 9(200) \\
&= 2,686,700 - 120,600 + 1800 \\
&= 2,567,900
\end{aligned}$$

b. Use sigma notation property iv. and the rules for the sum of squared terms and the sum of cubed terms.

$$\begin{aligned}\sum_{i=1}^{6}(i^3-i^2) &= \sum_{i=1}^{6} i^3 - \sum_{i=1}^{6} i^2 \\ &= \frac{6^2(6+1)^2}{4} - \frac{6(6+1)(2(6)+1)}{6} \\ &= \frac{1764}{4} - \frac{546}{6} \\ &= 350\end{aligned}$$

Exercise 2.3.2

Find the sum of the values of $4+3i$ for $i = 1, 2, \ldots, 100$.

Hint

Use the properties of sigma notation to solve the problem.

Answer

$15,550$

Example 2.3.3: Finding the Sum of the Function Values

Find the sum of the values of $f(x) = x^3$ over the integers $1, 2, 3, \ldots, 10$.

Solution

Using Equation 2.3.6, we have

$$\sum_{i=0}^{10} i^3 = \frac{(10)^2(10+1)^2}{4} = \frac{100(121)}{4} = 3025$$

Exercise 2.3.3

Evaluate the sum indicated by the notation $\sum_{k=1}^{20}(2k+1)$.

Hint

Use the rule on sum and powers of integers (Equations 2.3.4–2.3.6).

Answer

440

Approximating Area

Now that we have the necessary notation, we return to the problem at hand: approximating the area under a curve. Let $f(x)$ be a continuous, nonnegative function defined on the closed interval $[a, b]$. We want to approximate the area A bounded by $f(x)$ above, the x-axis below, the line $x = a$ on the left, and the line $x = b$ on the right (Figure 2.3.1).

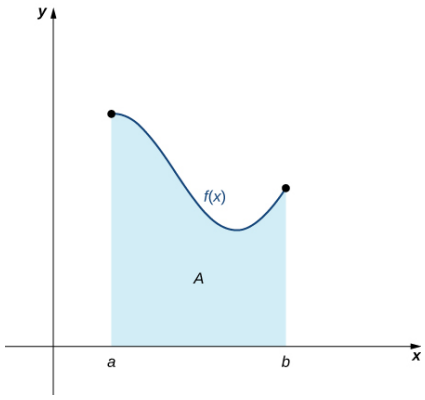


Figure 2.3.1: An area (shaded region) bounded by the curve $f(x)$ at top, the x-axis at bottom, the line $x = a$ to the left, and the line $x = b$ at right.

How do we approximate the area under this curve? The approach is a geometric one. By dividing a region into many small shapes that have known area formulas, we can sum these areas and obtain a reasonable estimate of the true area. We begin by dividing the interval $[a, b]$ into n subintervals of equal width, $\dfrac{b-a}{n}$. We do this by selecting equally spaced points $x_0, x_1, x_2, \ldots, x_n$ with $x_0 = a, x_n = b,$ and

$$x_i - x_{i-1} = \frac{b-a}{n}$$

for $i = 1, 2, 3, \ldots, n$.

We denote the width of each subinterval with the notation Δx, so $\Delta x = \frac{b-a}{n}$ and

$$x_i = x_0 + i\Delta x$$

for $i = 1, 2, 3, \ldots, n$. This notion of dividing an interval $[a, b]$ into subintervals by selecting points from within the interval is used quite often in approximating the area under a curve, so let's define some relevant terminology.

Definition: Partitions

A set of points $P = x_i$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a **partition** of $[a, b]$. If the subintervals all have the same width, the set of points forms a **regular partition** (or uniform partition) of the interval $[a, b]$.

We can use this regular partition as the basis of a method for estimating the area under the curve. We next examine two methods: the left-endpoint approximation and the right-endpoint approximation.

Rule: Left-Endpoint Approximation

On each subinterval $[x_{i-1}, x_i]$ (for $i = 1, 2, 3, \ldots, n$), construct a rectangle with width Δx and height equal to $f(x_{i-1})$, which is the function value at the left endpoint of the subinterval. Then the area of this rectangle is $f(x_{i-1})\Delta x$. Adding the areas of all these rectangles, we get an approximate value for A (Figure 2.3.2). We use the notation L_n to denote that this is a **left-endpoint approximation** of A using n subintervals.

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

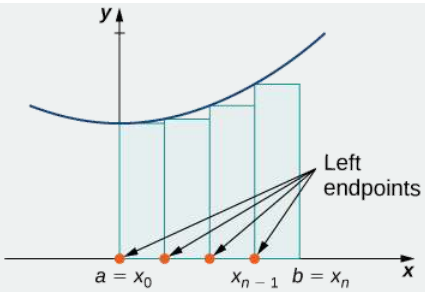


Figure 2.3.2: In the left-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the left of each subinterval.

The second method for approximating area under a curve is the right-endpoint approximation. It is almost the same as the left-endpoint approximation, but now the heights of the rectangles are determined by the function values at the right of each subinterval.

Rule: Right-Endpoint Approximation

Construct a rectangle on each subinterval $[x_{i-1}, x_i]$, only this time the height of the rectangle is determined by the function value $f(x_i)$ at the right endpoint of the subinterval. Then, the area of each rectangle is $f(x_i)\,\Delta x$ and the approximation for A is given by

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x.$$

The notation R_n indicates this is a **right-endpoint approximation** for A (Figure 2.3.3).

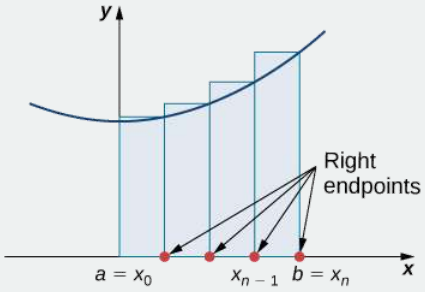


Figure 2.3.3: In the right-endpoint approximation of area under a curve, the height of each rectangle is determined by the function value at the right of each subinterval. Note that the right-endpoint approximation differs from the left-endpoint approximation in Figure 2.3.2.

The graphs in Figure 2.3.4 represent the curve $f(x) = \dfrac{x^2}{2}$. In Figure 2.3.4*b* we divide the region represented by the interval $[0, 3]$ into six subintervals, each of width 0.5. Thus, $\Delta x = 0.5$. We then form six rectangles by drawing vertical lines perpendicular to x_{i-1}, the left endpoint of each subinterval. We determine the height of each rectangle by calculating $f(x_{i-1})$ for $i = 1, 2, 3, 4, 5, 6$. The intervals are $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$. We find the area of each rectangle by multiplying the height by the width. Then, the sum of the rectangular areas approximates the area between $f(x)$ and the x-axis. When the left endpoints are used to calculate height, we have a left-endpoint approximation. Thus,

$$\begin{aligned} A \approx L_6 &= \sum_{i=1}^{6} f(x_{i-1})\Delta x = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x \\ &= f(0)0.5 + f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 \\ &= (0)0.5 + (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 \\ &= 0 + 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 \\ &= 3.4375 \text{ units}^2 \end{aligned}$$

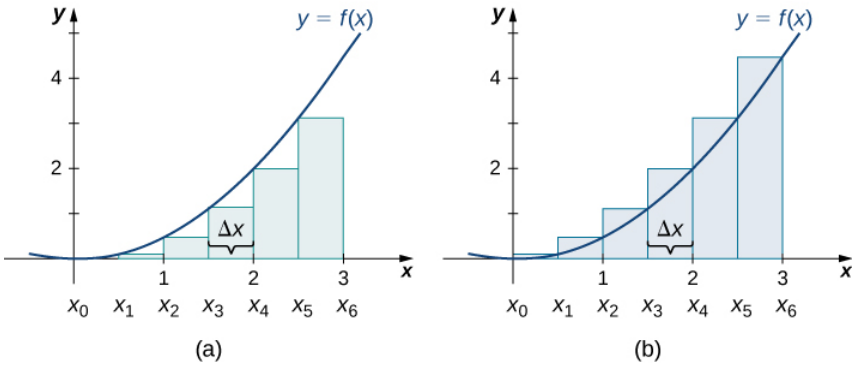


Figure 2.3.4: Methods of approximating the area under a curve by using (a) the left endpoints and (b) the right endpoints.

In Figure 2.3.4b, we draw vertical lines perpendicular to x_i such that x_i is the right endpoint of each subinterval, and calculate $f(x_i)$ for $i = 1, 2, 3, 4, 5, 6$. We multiply each $f(x_i)$ by Δx to find the rectangular areas, and then add them. This is a right-endpoint approximation of the area under $f(x)$. Thus,

$$\begin{aligned} A \approx R_6 &= \sum_{i=1}^{6} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x + f(x_5)\Delta x + f(x_6)\Delta x \\ &= f(0.5)0.5 + f(1)0.5 + f(1.5)0.5 + f(2)0.5 + f(2.5)0.5 + f(3)0.5 \\ &= (0.125)0.5 + (0.5)0.5 + (1.125)0.5 + (2)0.5 + (3.125)0.5 + (4.5)0.5 \\ &= 0.0625 + 0.25 + 0.5625 + 1 + 1.5625 + 2.25 \\ &= 5.6875 \text{ units}^2. \end{aligned}$$

Example 2.3.4: Approximating the Area Under a Curve

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Solution

First, divide the interval $[0, 2]$ into n equal subintervals. Using $n = 4$, $\Delta x = \dfrac{(2-0)}{4} = 0.5$. This is the width of each rectangle. The intervals $[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$ are shown in Figure 2.3.5. Using a left-endpoint approximation, the heights are $f(0) = 0$, $f(0.5) = 0.25$, $f(1) = 1$, and $f(1.5) = 2.25$. Then,

$$\begin{aligned} L_4 &= f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x \\ &= 0(0.5) + 0.25(0.5) + 1(0.5) + 2.25(0.5) \\ &= 1.75 \text{ units}^2 \end{aligned}$$

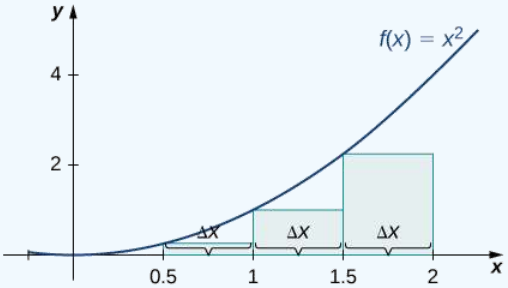


Figure 2.3.5: The graph shows the left-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The right-endpoint approximation is shown in Figure 2.3.6. The intervals are the same, $\Delta x = 0.5$, but now use the right endpoint to calculate the height of the rectangles. We have

$$\begin{aligned} R_4 &= f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x \\ &= 0.25(0.5) + 1(0.5) + 2.25(0.5) + 4(0.5) \\ &= 3.75 \text{ units}^2 \end{aligned}$$

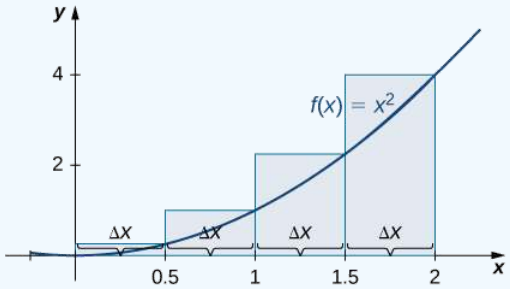


Figure 2.3.6: The graph shows the right-endpoint approximation of the area under $f(x) = x^2$ from 0 to 2.

The left-endpoint approximation is 1.75 units^2; the right-endpoint approximation is 3.75 units^2.

Exercise 2.3.4

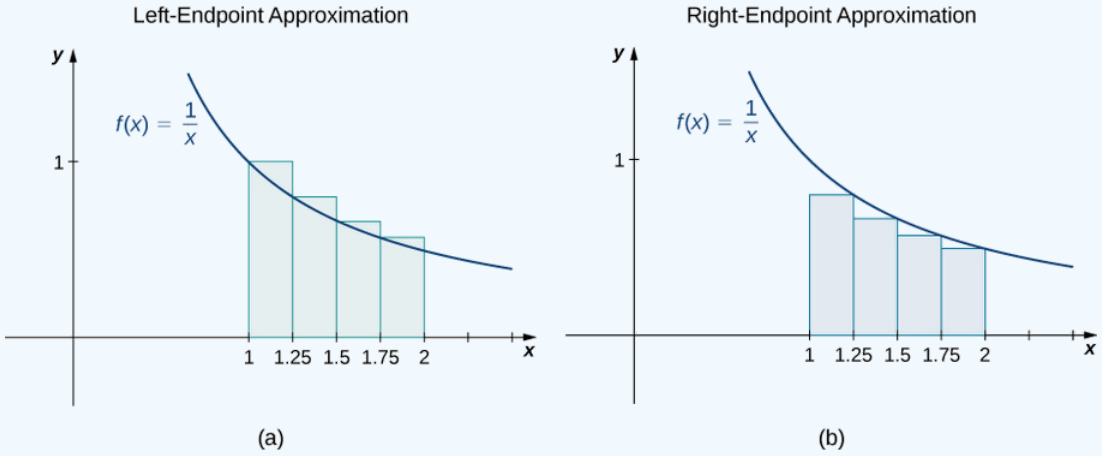
Sketch left-endpoint and right-endpoint approximations for $f(x) = \dfrac{1}{x}$ on $[1, 2]$; use $n = 4$. Approximate the area using both methods.

Hint

Follow the solving strategy in Example 2.3.4step-by-step.

Answer

The left-endpoint approximation is 0.7595 units^2. The right-endpoint approximation is 0.6345 units^2. See the below Media.



Looking at Figure 2.3.4 and the graphs in Example 2.3.4, we can see that when we use a small number of intervals, neither the left-endpoint approximation nor the right-endpoint approximation is a particularly accurate estimate of the area under the curve. However, it seems logical that if we increase the number of points in our partition, our estimate of A will improve. We will have more rectangles, but each rectangle will be thinner, so we will be able to fit the rectangles to the curve more precisely.

We can demonstrate the improved approximation obtained through smaller intervals with an example. Let's explore the idea of increasing n, first in a left-endpoint approximation with four rectangles, then eight rectangles, and finally 32 rectangles. Then, let's do the same thing in a right-endpoint approximation, using the same sets of intervals, of the same curved region. Figure 2.3.7 shows the area of the region under the curve $f(x) = (x-1)^3 + 4$ on the interval $[0, 2]$ using a left-endpoint approximation where $n = 4$. The width of each rectangle is

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}.$$

The area is approximated by the summed areas of the rectangles, or

$$L_4 = f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)0.5 = 7.5 \text{ units}^2$$

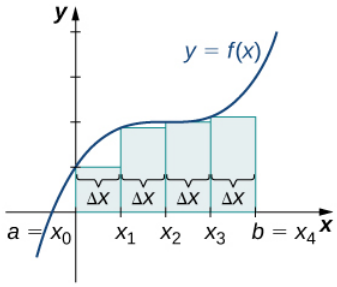


Figure 2.3.7: With a left-endpoint approximation and dividing the region from a to b into four equal intervals, the area under the curve is approximately equal to the sum of the areas of the rectangles.

Figure 2.3.8 shows the same curve divided into eight subintervals. Comparing the graph with four rectangles in Figure 2.3.7 with this graph with eight rectangles, we can see there appears to be less white space under the curve when $n = 8$. This white space is area under the curve we are unable to include using our approximation. The area of the rectangles is

$$L_8 = f(0)(0.25) + f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) = 7.75 \text{ units}^2$$

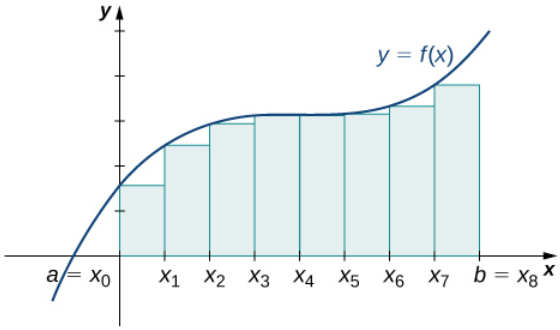


Figure 2.3.8: The region under the curve is divided into $n = 8$ rectangular areas of equal width for a left-endpoint approximation.

The graph in Figure 2.3.9 shows the same function with 32 rectangles inscribed under the curve. There appears to be little white space left. The area occupied by the rectangles is

$$L_{32} = f(0)(0.0625) + f(0.0625)(0.0625) + f(0.125)(0.0625) + \cdots + f(1.9375)(0.0625) = 7.9375 \text{ units}^2.$$

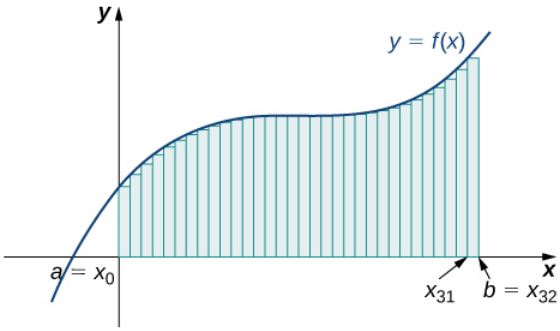


Figure 2.3.9: Here, 32 rectangles are inscribed under the curve for a left-endpoint approximation.

We can carry out a similar process for the right-endpoint approximation method. A right-endpoint approximation of the same curve, using four rectangles (Figure 2.3.10), yields an area

$$R_4 = f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) = 8.5 \text{ units}^2.$$

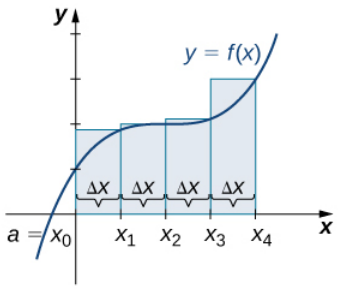


Figure 2.3.10: Now we divide the area under the curve into four equal subintervals for a right-endpoint approximation.

Dividing the region over the interval $[0, 2]$ into eight rectangles results in $\Delta x = \dfrac{2-0}{8} = 0.25$. The graph is shown in Figure 2.3.11 The area is

$$R_8 = f(0.25)(0.25) + f(0.5)(0.25) + f(0.75)(0.25) + f(1)(0.25) + f(1.25)(0.25) + f(1.5)(0.25) + f(1.75)(0.25) + f(2)(0.25) = 8.25 \text{ units}^2$$

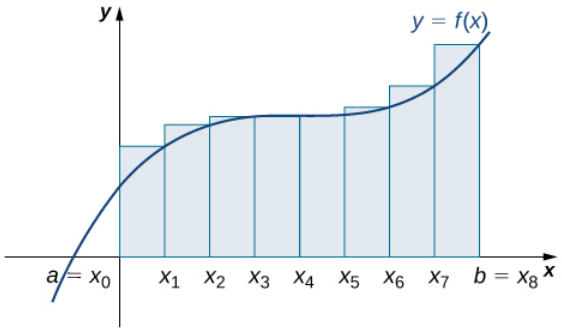


Figure 2.3.11: Here we use right-endpoint approximation for a region divided into eight equal subintervals.

Last, the right-endpoint approximation with $n = 32$ is close to the actual area (Figure 2.3.12). The area is approximately

$$R_{32} = f(0.0625)(0.0625) + f(0.125)(0.0625) + f(0.1875)(0.0625) + \cdots + f(2)(0.0625) = 8.0625 \text{ units}^2$$

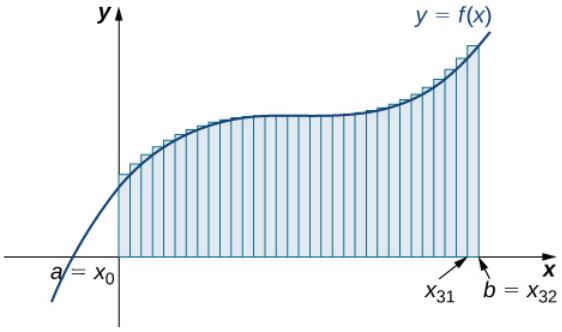


Figure 2.3.12: The region is divided into 32 equal subintervals for a right-endpoint approximation.

Based on these figures and calculations, it appears we are on the right track; the rectangles appear to approximate the area under the curve better as n gets larger. Furthermore, as n increases, both the left-endpoint and right-endpoint approximations appear to approach an area of 8 square units. Table 2.3.15 shows a numerical comparison of the left- and right-endpoint methods. The idea that the approximations of the area under the curve get better and better as n gets larger and larger is very important, and we now explore this idea in more detail.

Table 2.3.15: Converging Values of Left- and Right-Endpoint Approximations as n Increases

Value of n	Approximate Area L_n	Approximate Area R_n
$n = 4$	7.5	8.5
$n = 8$	7.75	8.25

Value of n	Approximate Area L_n	Approximate Area R_n
$n = 32$	7.94	8.06

Forming Riemann Sums

So far we have been using rectangles to approximate the area under a curve. The heights of these rectangles have been determined by evaluating the function at either the right or left endpoints of the subinterval $[x_{i-1}, x_i]$. In reality, there is no reason to restrict evaluation of the function to one of these two points only. We could evaluate the function at any point x_i^* in the subinterval $[x_{i-1}, x_i]$, and use $f(x_i^*)$ as the height of our rectangle. This gives us an estimate for the area of the form

$$A \approx \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

A sum of this form is called a Riemann sum, named for the 19th-century mathematician Bernhard Riemann, who developed the idea.

Definition: Riemann sum

Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be any partition of $[a, b]$. Let Δx_i be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1},\ x_i]$. A Riemann sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i.$$

At this point, we'll choose a regular partition P, as we have in our examples above. This forces all Δx_i to be equal to $\Delta x = \dfrac{b-a}{n}$ for any natural number of intervals n.

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Definition: Area Under the Curve

Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$ be a Riemann sum for $f(x)$ with a regular partition P. Then, the **area under the curve** $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

See a graphical demonstration of the construction of a Riemann sum.

Some subtleties here are worth discussing. First, note that taking the limit of a sum is a little different from taking the limit of a function $f(x)$ as x goes to infinity. Limits of sums are discussed in detail in the chapter on Sequences and Series; however, for now we can assume that the computational techniques we used to compute limits of functions can also be used to calculate limits of sums.

Second, we must consider what to do if the expression converges to different limits for different choices of x_i^*. Fortunately, this does not happen. Although the proof is beyond the scope of this text, it can be shown that if $f(x)$ is continuous on the closed interval $[a, b]$, then $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique (in other words, it does not depend on the choice of x_i^*).

We look at some examples shortly. But, before we do, let's take a moment and talk about some specific choices for x_i^*. Although any choice for x_i^* gives us an estimate of the area under the curve, we don't necessarily know whether that estimate is too high

(overestimate) or too low (underestimate). If it is important to know whether our estimate is high or low, we can select our value for x_i^* to guarantee one result or the other.

If we want an overestimate, for example, we can choose x_i^* such that for $i = 1, 2, 3, \ldots, n, f(x_i^*) \geq f(x)$ for all $x \in [x_i - 1, x_i]$. In other words, we choose x_i^* so that for $i = 1, 2, 3, \ldots, n, f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select x_i^* in this way, then the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose x_i^* so that for $i = 1, 2, 3, \ldots, n, f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

Example 2.3.5: Finding Lower and Upper Sums

Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.

Solution

With $n = 4$ over the interval $[1, 2]$, $\Delta x = \frac{1}{4}$. We can list the intervals as $[1, 1.25]$, $[1.25, 1.5]$, $[1.5, 1.75]$, and $[1.75, 2]$. Because the function is decreasing over the interval $[1, 2]$, Figure shows that a lower sum is obtained by using the right endpoints.

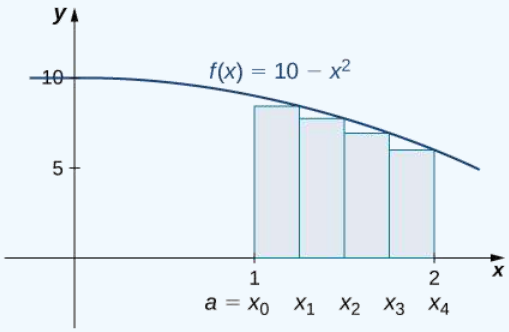


Figure 2.3.13: The graph of $f(x) = 10 - x^2$ is set up for a right-endpoint approximation of the area bounded by the curve and the x-axis on $[1, 2]$, and it shows a lower sum.

The Riemann sum is

$$\begin{aligned}\sum_{k=1}^{4}(10 - x^2)(0.25) &= 0.25[10 - (1.25)^2 + 10 - (1.5)^2 + 10 - (1.75)^2 + 10 - (2)^2] \\ &= 0.25[8.4375 + 7.75 + 6.9375 + 6] \\ &= 7.28 \text{ units}^2.\end{aligned}$$

The area of 7.28 units^2 is a lower sum and an underestimate.

Exercise 2.3.5

a. Find an upper sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$.
b. Sketch the approximation.

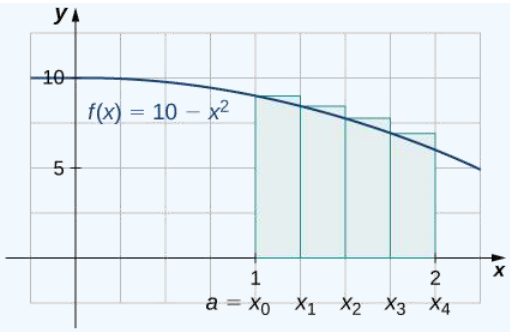
Hint

$f(x)$ is decreasing on $[1, 2]$, so the maximum function values occur at the left endpoints of the subintervals.

Answer

a. Upper sum=8.0313 units^2.

b.



Example 2.3.6: Finding Lower and Upper Sums for $f(x) = \sin x$

Find a lower sum for $f(x) = \sin x$ over the interval $[a, b] = \left[0, \frac{\pi}{2}\right]$; let $n = 6$.

Solution

Let's first look at the graph in Figure 2.3.14 to get a better idea of the area of interest.

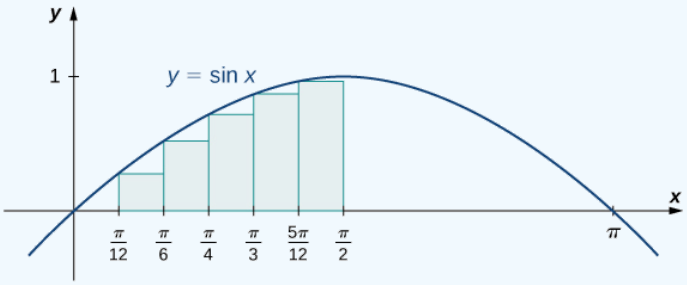


Figure 2.3.14: The graph of $y = \sin x$ is divided into six regions: $\Delta x = \dfrac{\pi/2}{6} = \dfrac{\pi}{12}$.

The intervals are $\left[0, \frac{\pi}{12}\right]$, $\left[\frac{\pi}{12}, \frac{\pi}{6}\right]$, $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$, $\left[\frac{\pi}{4}, \frac{\pi}{3}\right]$, $\left[\frac{\pi}{3}, \frac{5\pi}{12}\right]$, and $\left[\frac{5\pi}{12}, \frac{\pi}{2}\right]$. Note that $f(x) = \sin x$ is increasing on the interval $\left[0, \frac{\pi}{2}\right]$, so a left-endpoint approximation gives us the lower sum. A left-endpoint approximation is the Riemann sum $\sum_{i=0}^{5} \sin x_i \left(\frac{\pi}{12}\right)$. We have

$$A \approx \sin(0)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{12}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{6}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{4}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{\pi}{3}\right)\left(\tfrac{\pi}{12}\right) + \sin\left(\tfrac{5\pi}{12}\right)\left(\tfrac{\pi}{12}\right) \approx 0.863 \text{ units}^2.$$

Exercise 2.3.6

Using the function $f(x) = \sin x$ over the interval $\left[0, \frac{\pi}{2}\right]$, find an upper sum; let $n = 6$.

Hint

Follow the steps from Example 2.3.6

Answer

$A \approx 1.125 \text{ units}^2$

Key Concepts

- The use of sigma (summation) notation of the form $\sum_{i=1}^{n} a_i$ is useful for expressing long sums of values in compact form.
- For a continuous function defined over an interval $[a, b]$, the process of dividing the interval into n equal parts, extending a rectangle to the graph of the function, calculating the areas of the series of rectangles, and then summing the areas yields an approximation of the area of that region.
- When using a regular partition, the width of each rectangle is $\Delta x = \dfrac{b-a}{n}$.

- Riemann sums are expressions of the form $\sum_{i=1}^{n} f(x_i^*)\Delta x$, and can be used to estimate the area under the curve $y = f(x)$. Left- and right-endpoint approximations are special kinds of Riemann sums where the values of x_i^* are chosen to be the left or right endpoints of the subintervals, respectively.
- Riemann sums allow for much flexibility in choosing the set of points x_i^* at which the function is evaluated, often with an eye to obtaining a lower sum or an upper sum.

Key Equations

- **Properties of Sigma Notation**

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i$$

- **Sums and Powers of Integers**

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- **Left-Endpoint Approximation**

$$A \approx L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

- **Right-Endpoint Approximation**

$$A \approx R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x$$

Glossary

left-endpoint approximation

an approximation of the area under a curve computed by using the left endpoint of each subinterval to calculate the height of the vertical sides of each rectangle

lower sum

a sum obtained by using the minimum value of $f(x)$ on each subinterval

partition

a set of points that divides an interval into subintervals

regular partition

a partition in which the subintervals all have the same width

riemann sum

an estimate of the area under the curve of the form $A \approx \sum_{i=1}^{n} f(x_i^*)\Delta x$

right-endpoint approximation

the right-endpoint approximation is an approximation of the area of the rectangles under a curve using the right endpoint of each subinterval to construct the vertical sides of each rectangle

sigma notation

(also, **summation notation**) the Greek letter sigma (Σ) indicates addition of the values; the values of the index above and below the sigma indicate where to begin the summation and where to end it

upper sum

a sum obtained by using the maximum value of $f(x)$ on each subinterval

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2.3E: Approximating Areas (Exercises)

1.) State whether the given sums are equal or unequal.

a. $\sum_{i=1}^{10} i$ and $\sum_{k=1}^{10} k$

b. $\sum_{i=1}^{10} i$ and $\sum_{i=6}^{15} (i-5)$

c. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{j=0}^{9} (j+1)j$

d. $\sum_{i=1}^{10} i(i-1)$ and $\sum_{k=1}^{10} (k^2-k)$

Answer:

a. They are equal; both represent the sum of the first 10 whole numbers.
b. They are equal; both represent the sum of the first 10 whole numbers.
c. They are equal by substituting $j = i-1$.
d. They are equal; the first sum factors the terms of the second.

In exercises 2 - 3, use the rules for sums of powers of integers to compute the sums.

2) $\sum_{i=5}^{10} i$

3) $\sum_{i=5}^{10} i^2$

Answer:

$\sum_{i=5}^{10} i^2 = 385 - 30 = 355$

Suppose that $\sum_{i=1}^{100} a_i = 15$ and $\sum_{i=1}^{100} b_i = -12$.

In exercises 4 - 7, compute the sums.

4) $\sum_{i=1}^{100} (a_i + b_i)$

5) $\sum_{i=1}^{100} (a_i - b_i)$

Answer:

$\sum_{i=1}^{100} (a_i - b_i) = 15 - (-12) = 27$

6) $\sum_{i=1}^{100} (3a_i - 4b_i)$

7) $\sum_{i=1}^{100} (5a_i + 4b_i)$

Answer:

$$\sum_{i=1}^{100}(5a_i+4b_i)=5(15)+4(-12)=27$$

In exercises 8 - 11, use summation properties and formulas to rewrite and evaluate the sums.

8) $\sum_{k=1}^{20}100(k^2-5k+1)$

9) $\sum_{j=1}^{50}(j^2-2j)$

Answer:

$$\sum_{j=1}^{50}j^2-2\sum_{j=1}^{50}j=\frac{(50)(51)(101)}{6}-\frac{2(50)(51)}{2}=40,375$$

10) $\sum_{j=11}^{20}(j^2-10j)$

11) $\sum_{k=1}^{25}[(2k)^2-100k]$

Answer:

$$4\sum_{k=1}^{25}k^2-100\sum_{k=1}^{25}k=\frac{4(25)(26)(51)}{9}-50(25)(26)=-10,400$$

Let L_n denote the left-endpoint sum using n subintervals and let R_n denote the corresponding right-endpoint sum.

In exercises 12 - 19, compute the indicated left and right sums for the given functions on the indicated interval.

12) L_4 for $f(x)=\dfrac{1}{x-1}$ on $[2,3]$

13) R_4 for $g(x)=\cos(\pi x)$ on $[0,1]$

Answer:

$R_4=0.25$

14) L_6 for $f(x)=\dfrac{1}{x(x-1)}$ on $[2,5]$

15) R_6 for $f(x)=\dfrac{1}{x(x-1)}$ on $[2,5]$

Answer:

$R_6=0.372$

16) R_4 for $\dfrac{1}{x^2+1}$ on $[-2,2]$

17) L_4 for $\dfrac{1}{x^2+1}$ on $[-2,2]$

Answer:

$L_4=2.20$

18) R_4 for x^2-2x+1 on $[0,2]$

19) L_8 for x^2-2x+1 on $[0,2]$

Answer:

$L_8 = 0.6875$

20) Compute the left and right Riemann sums— L_4 and R_4, respectively—for $f(x) = (2 - |x|)$ on $[-2, 2]$. Compute their average value and compare it with the area under the graph of f.

21) Compute the left and right Riemann sums— L_6 and R_6, respectively—for $f(x) = (3 - |3 - x|)$ on $[0, 6]$. Compute their average value and compare it with the area under the graph of f.

Answer:

$L_6 = 9.000 = R_6$. The graph of f is a triangle with area 9 units^2.

22) Compute the left and right Riemann sums— L_4 and R_4, respectively—for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ and compare their values.

23) Compute the left and right Riemann sums— L_6 and R_6, respectively—for $f(x) = \sqrt{9 - (x - 3)^2}$ on $[0, 6]$ and compare their values.

Answer:

$L_6 = 13.12899 = R_6$. They are equal.

For exercises 24 - 27, express the following endpoint sums in sigma notation but do not evaluate them.

24) L_{30} for $f(x) = x^2$ on $[1, 2]$

25) L_{10} for $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$

Answer:

$$L_{10} = \frac{4}{10} \sum_{i=1}^{10} \sqrt{4 - (-2 + 4\frac{(i-1)}{10})}$$

26) R_{20} for $f(x) = \sin x$ on $[0, \pi]$

27) R_{100} for $\ln x$ on $[1, e]$

Answer:

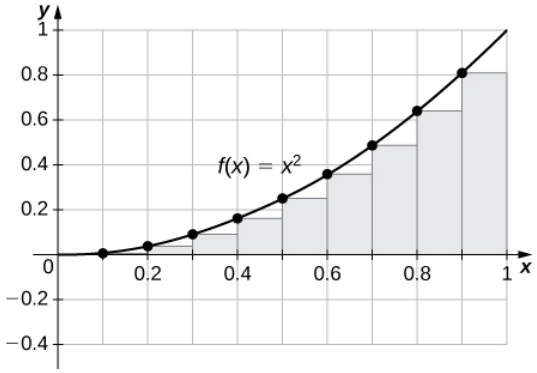
$$R_{100} = \frac{e-1}{100} \sum_{i=1}^{100} ln(1 + (e-1)\frac{i}{100})$$

In exercises 28 - 33, graph the function then use a calculator or a computer program to evaluate the following left and right endpoint sums. Is the area under the curve between the left and right endpoint sums?

28) [T] L_{100} and R_{100} for $y = x^2 - 3x + 1$ on the interval $[-1, 1]$

29) [T] L_{100} and R_{100} for $y = x^2$ on the interval $[0, 1]$

Answer:



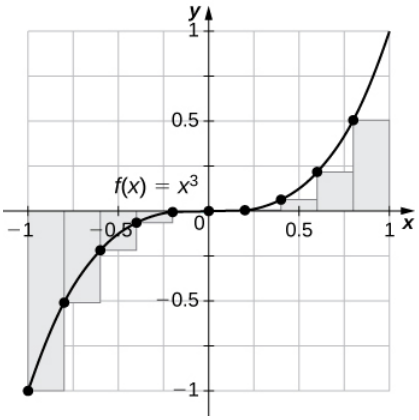
$R_{100} = 0.33835, \quad L_{100} = 0.32835.$

The plot shows that the left Riemann sum is an underestimate because the function is increasing. Similarly, the right Riemann sum is an overestimate. The area lies between the left and right Riemann sums. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

30) [T] L_{50} and R_{50} for $y = \dfrac{x+1}{x^2-1}$ on the interval $[2, 4]$

31) [T] L_{100} and R_{100} for $y = x^3$ on the interval $[-1, 1]$

Answer:



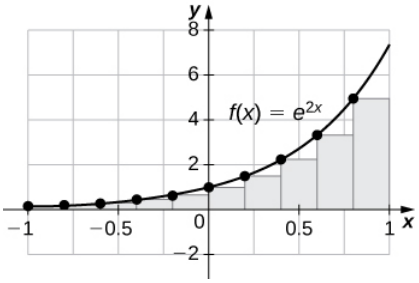
$L_{100} = -0.02, \quad R_{100} = 0.02.$

The left endpoint sum is an underestimate because the function is increasing. Similarly, a right endpoint approximation is an overestimate. The area lies between the left and right endpoint estimates.

32) [T] L_{50} and R_{50} for $y = \tan(x)$ on the interval $[0, \frac{\pi}{4}]$

33) [T] L_{100} and R_{100} for $y = e^{2x}$ on the interval $[-1, 1]$

Answer:



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$L_{100} = 3.555, \quad R_{100} = 3.670$

The plot shows that the left Riemann sum is an underestimate because the function is increasing. Ten rectangles are shown for visual clarity. This behavior persists for more rectangles.

34) Let t_j denote the time that it took Tejay van Garteren to ride the jth stage of the Tour de France in 2014. If there were a total of 21 stages, interpret $\sum_{j=1}^{21} t_j$.

35) Let r_j denote the total rainfall in Portland on the jth day of the year in 2009. Interpret $\sum_{j=1}^{31} r_j$.

Answer:

The sum represents the cumulative rainfall in January 2009.

36) Let d_j denote the hours of daylight and δ_j denote the increase in the hours of daylight from day $j-1$ to day j in Fargo, North Dakota, on the jth day of the year. Interpret $d1 + \sum_{j=2}^{365} \delta_j$.

37) To help get in shape, Joe gets a new pair of running shoes. If Joe runs 1 mi each day in week 1 and adds $\frac{1}{10}$ mi to his daily routine each week, what is the total mileage on Joe's shoes after 25 weeks?

Answer:

The total mileage is $7 \times \sum_{i=1}^{25} (1 + \frac{(i-1)}{10}) = 7 \times 25 + \frac{7}{10} \times 12 \times 25 = 385$ mi.

38) The following table gives approximate values of the average annual atmospheric rate of increase in carbon dioxide (CO_2) each decade since 1960, in parts per million (ppm). Estimate the total increase in atmospheric CO_2 between 1964 and 2013.

Decade	Ppm/y
1964-1973	1.07
1976-1983	1.34
1984-1993	1.40
1994-2003	1.87
2004-2013	2.07

Average Annual Atmospheric CO2 Increase, 1964–2013 Source: http://www.esrl.noaa.gov/gmd/ccgg/trends/.

39) The following table gives the approximate increase in sea level in inches over 20 years starting in the given year. Estimate the net change in mean sea level from 1870 to 2010.

Starting Year	20- Year Change
1870	0.3
1890	1.5
1910	0.2
1930	2.8
1950	0.7
1970	1.1

Starting Year	20- Year Change
1990	1.5

Approximate 20-Year Sea Level Increases, 1870–1990

Source: http://link.springer.com/article/10....712-011-9119-1

Answer:

Add the numbers to get 8.1-in. net increase.

40) The following table gives the approximate increase in dollars in the average price of a gallon of gas per decade since 1950. If the average price of a gallon of gas in 2010 was $2.60, what was the average price of a gallon of gas in 1950?

Starting Year	10- Year Change
1950	0.03
1960	0.05
1970	0.86
1980	−0.03
1990	0.29
2000	1.12

Approximate 10-Year Gas Price Increases, 1950–2000

Source: http://epb.lbl.gov/homepages/Rick_Di...011-trends.pdf.

41) The following table gives the percent growth of the U.S. population beginning in July of the year indicated. If the U.S. population was 281,421,906 in July 2000, estimate the U.S. population in July 2010.

Year	% Change/Year
2000	1.12
2001	0.99
2002	0.93
2003	0.86
2004	0.93
2005	0.93
2006	0.97
2007	0.96
2008	0.95
2009	0.88

Annual Percentage Growth of U.S. Population, 2000–2009

Source: http://www.census.gov/popest/data.

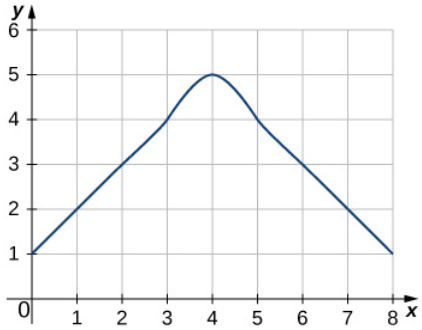
(Hint: To obtain the population in July 2001, multiply the population in July 2000 by 1.0112 to get 284,573,831.)

Answer:

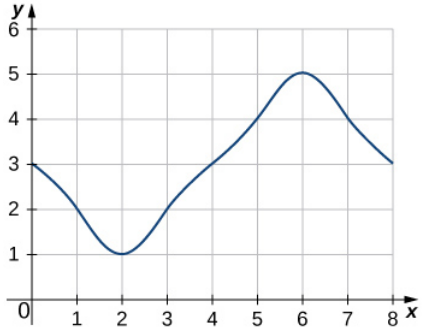
309,389,957

In exercises 42 - 45, estimate the areas under the curves by computing the left Riemann sums, L_8.

42)



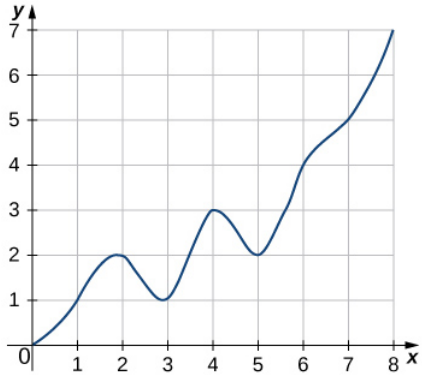
43)



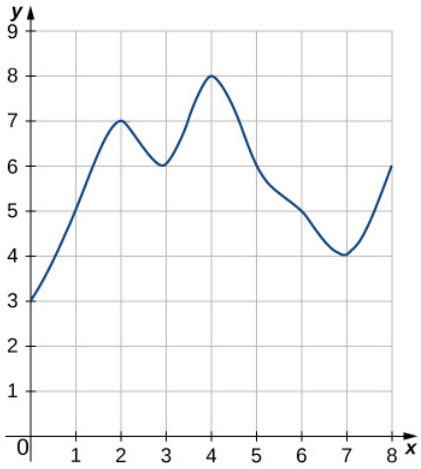
Answer:

$L_8 = 3+2+1+2+3+4+5+4 = 24$

44)



45)



Answer:

$L_8 = 3+5+7+6+8+6+5+4 = 44$

46) [T] Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \sqrt{1-x^2}$ on $[-1, 1]$.

47) [T] Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \dfrac{1}{\sqrt{1+x^2}}$ on $[-1, 1]$.

Answer:

$L_{10} \approx 1.7604, \quad L_{30} \approx 1.7625, \quad L_{50} \approx 1.76265$

48) [T] Use a computer algebra system to compute the Riemann sum, L_N, for $N = 10, 30, 50$ for $f(x) = \sin^2 x$ on $[0, 2\pi]$. Compare these estimates with π.

In exercises 49-50, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$. How do these estimates compare with the exact answers, which you can find via geometry?

49) [T] $y = \cos(\pi x)$ on the interval $[0, 1]$

Answer:

$R_1 = -1, \quad L_1 = 1,$
$R_{10} = -0.1, \quad L_{10} = 0.1,$
$L_{100} = 0.01, \quad R_{100} = -0.01.$
By symmetry of the graph, the exact area is zero.

50) [T] $y = 3x + 2$ on the interval $[3, 5]$

In exercises 51 - 52, use a calculator or a computer program to evaluate the endpoint sums R_N and L_N for $N = 1, 10, 100$.

51) [T] $y = x^4 - 5x^2 + 4$ on the interval $[-2, 2]$, which has an exact area of $\frac{32}{15}$

Answer:

$R_1 = 0, \quad L_1 = 0,$
$R_{10} = 2.4499, \quad L_{10} = 2.4499,$
$R_{100} = 2.1365, \quad L_{100} = 2.1365$

52) [T] $y = \ln x$ on the interval $[1, 2]$, which has an exact area of $2\ln(2) - 1$

53) Explain why, if $f(a) \geq 0$ and f is increasing on $[a, b]$, that the left endpoint estimate is a lower bound for the area below the graph of f on $[a, b]$.

Answer:

If $[c,d]$is a subinterval of $[a,b]$ under one of the left-endpoint sum rectangles, then the area of the rectangle contributing to the left-endpoint estimate is $f(c)(d-c)$. But, $f(c) \le f(x)$ for $c \le x \le d$, so the area under the graph of f between c and d is $f(c)(d-c)$ plus the area below the graph of f but above the horizontal line segment at height $f(c)$, which is positive. As this is true for each left-endpoint sum interval, it follows that the left Riemann sum is less than or equal to the area below the graph of f on $[a,b]$.

54) Explain why, if $f(b) \ge 0$ and f is decreasing on $[a,b]$, that the left endpoint estimate is an upper bound for the area below the graph of f on $[a,b]$.

55) Show that, in general, $R_N - L_N = (b-a) \times \dfrac{f(b)-f(a)}{N}$.

Answer:

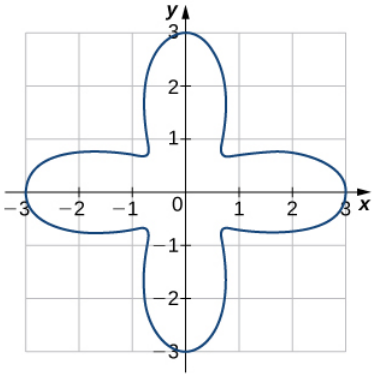
$L_N = \dfrac{b-a}{N} \displaystyle\sum_{i=1}^{N} f(a+(b-a)\frac{i-1}{N}) = \frac{b-a}{N} \sum_{i=0}^{N-1} f(a+(b-a)\frac{i}{N})$ and $R_N = \dfrac{b-a}{N} \displaystyle\sum_{i=1}^{N} f(a+(b-a)\frac{i}{N})$. The left sum has a term corresponding to $i=0$ and the right sum has a term corresponding to $i=N$. In $R_N - L_N$, any term corresponding to $i = 1, 2, \ldots, N-1$ occurs once with a plus sign and once with a minus sign, so each such term cancels and one is left with $R_N - L_N = \dfrac{b-a}{N}(f(a+(b-a))\frac{N}{N}) - (f(a)+(b-a)\frac{0}{N}) = \frac{b-a}{N}(f(b)-f(a))$.

56) Explain why, if f is increasing on $[a,b]$, the error between either L_N or R_N and the area A below the graph of f is at most $(b-a)\dfrac{f(b)-f(a)}{N}$.

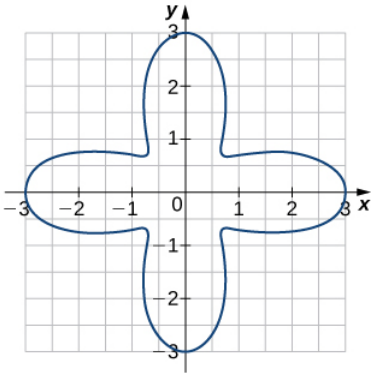
57) For each of the three graphs:

a. Obtain a lower bound $L(A)$ for the area enclosed by the curve by adding the areas of the squares enclosed completely by the curve.

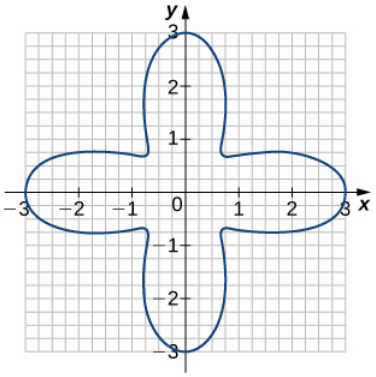
b. Obtain an upper bound $U(A)$ for the area by adding to $L(A)$ the areas $B(A)$ of the squares enclosed partially by the curve.



Graph 1



Graph 2



Graph 3

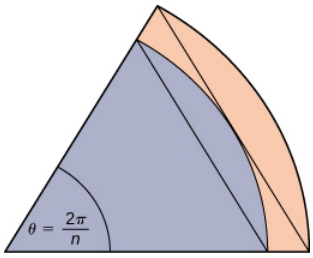
Answer:

Graph 1: a. $L(A) = 0, \quad B(A) = 20; \quad b. U(A) = 20.$
Graph 2: a. $L(A) = 9; \quad b. B(A) = 11, \quad U(A) = 20.$
Graph 3: a. $L(A) = 11.0; \quad b. B(A) = 4.5, \quad U(A) = 15.5.$

58) In the previous exercise, explain why $L(A)$ gets no smaller while $U(A)$ gets no larger as the squares are subdivided into four boxes of equal area.

59) A unit circle is made up of n wedges equivalent to the inner wedge in the figure. The base of the inner triangle is 1 unit and its height is $\sin(\frac{\pi}{n})$. The base of the outer triangle is $B = \cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n})$ and the height is $H = B\sin(\frac{2\pi}{n})$. Use this information to argue that the area of a unit circle is equal to π.

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Answer:

Let A be the area of the unit circle. The circle encloses n congruent triangles each of area $\dfrac{\sin(\frac{2\pi}{n})}{2}$, so $\frac{n}{2}\sin(\frac{2\pi}{n}) \leq A$. Similarly, the circle is contained inside n congruent triangles each of area $\dfrac{BH}{2} = \frac{1}{2}(\cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n}))\sin(\frac{2\pi}{n})$, so $A \leq \frac{n}{2}\sin(\frac{2\pi}{n})(\cos(\frac{\pi}{n})) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n})$. As $n \to \infty, \quad \frac{n}{2}\sin(\frac{2\pi}{n}) = \frac{\pi \sin(\frac{2\pi}{n})}{(\frac{2\pi}{n})} \to \pi$, so we conclude $\pi \leq A$. Also, as $n \to \infty, \quad \cos(\frac{\pi}{n}) + \sin(\frac{\pi}{n})\tan(\frac{\pi}{n}) \to 1$, so we also have $A \leq \pi$. By the squeeze theorem for limits, we conclude that $A = \pi$.

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2.4: The Definite Integral

Learning Objectives

- State the definition of the definite integral.
- Explain the terms integrand, limits of integration, and variable of integration.
- Explain when a function is integrable.
- Describe the relationship between the definite integral and net area.
- Use geometry and the properties of definite integrals to evaluate them.
- Calculate the average value of a function.

In the preceding section we defined the area under a curve in terms of Riemann sums:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

However, this definition came with restrictions. We required $f(x)$ to be continuous and nonnegative. Unfortunately, real-world problems don't always meet these restrictions. In this section, we look at how to apply the concept of the area under the curve to a broader set of functions through the use of the definite integral.

Definition and Notation

The definite integral generalizes the concept of the area under a curve. We lift the requirements that $f(x)$ be continuous and nonnegative, and define the definite integral as follows.

Definition: Definite Integral

If $f(x)$ is a function defined on an interval $[a, b]$, the ***definite integral*** of f from a to b is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

provided the limit exists. If this limit exists, the function $f(x)$ is said to be integrable on $[a, b]$, or is an integrable function.

The integral symbol in the previous definition should look familiar. We have seen similar notation in the chapter on Applications of Derivatives, where we used the indefinite integral symbol (without the a and b above and below) to represent an antiderivative. Although the notation for indefinite integrals may look similar to the notation for a definite integral, they are not the same. A definite integral is a number. An indefinite integral is a family of functions. Later in this chapter we examine how these concepts are related. However, close attention should always be paid to notation so we know whether we're working with a definite integral or an indefinite integral.

Integral notation goes back to the late seventeenth century and is one of the contributions of Gottfried Wilhelm **Leibniz**, who is often considered to be the codiscoverer of calculus, along with Isaac Newton. The integration symbol $\int$ is an elongated S, suggesting sigma or summation. On a definite integral, above and below the summation symbol are the boundaries of the interval, $[a, b]$. The numbers a and b are x-values and are called the **limits of integration**; specifically, a is the lower limit and b is the upper limit. To clarify, we are using the word limit in two different ways in the context of the definite integral. First, we talk about the limit of a sum as $n \to \infty$. Second, the boundaries of the region are called the *limits of integration.*

We call the function $f(x)$ the **integrand**, and the dx indicates that $f(x)$ is a function with respect to x, called the **variable of integration**. Note that, like the index in a sum, the variable of integration is a **dummy variable**, and has no impact on the computation of the integral. We could use any variable we like as the variable of integration:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

Previously, we discussed the fact that if $f(x)$ is continuous on $[a,b]$, then the limit $\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$ exists and is unique. This leads to the following theorem, which we state without proof.

Continuous Functions Are Integrable

If $f(x)$ is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Functions that are not continuous on $[a,b]$ may still be integrable, depending on the nature of the discontinuities. For example, functions with a finite number of jump discontinuities or removable discontinuities on a closed interval are integrable.

It is also worth noting here that we have retained the use of a regular partition in the Riemann sums. This restriction is not strictly necessary. Any partition can be used to form a Riemann sum. However, if a nonregular partition is used to define the definite integral, it is not sufficient to take the limit as the number of subintervals goes to infinity. Instead, we must take the limit as the width of the largest subinterval goes to zero. This introduces a little more complex notation in our limits and makes the calculations more difficult without really gaining much additional insight, so we stick with regular partitions for the Riemann sums.

Example 2.4.1: Evaluating an Integral Using the Definition

Use the definition of the definite integral to evaluate $\int_0^2 x^2\,dx$. Use a right-endpoint approximation to generate the Riemann sum.

Solution

We first want to set up a Riemann sum. Based on the limits of integration, we have $a=0$ and $b=2$. For $i=0,1,2,\ldots,n$, let $P=x_i$ be a regular partition of $[0,2]$. Then

$$\Delta x=\frac{b-a}{n}=\frac{2}{n}.$$

Since we are using a right-endpoint approximation to generate Riemann sums, for each i, we need to calculate the function value at the right endpoint of the interval $[x_{i-1},x_i]$. The right endpoint of the interval is x_i, and since P is a regular partition,

$$x_i=x_0+i\Delta x=0+i\left[\frac{2}{n}\right]=\frac{2i}{n}.$$

Thus, the function value at the right endpoint of the interval is

$$f(x_i)=x_i^2=\left(\frac{2i}{n}\right)^2=\frac{4i^2}{n^2}.$$

Then the Riemann sum takes the form

$$\sum_{i=1}^{n} f(x_i)\Delta x=\sum_{i=1}^{n}\left(\frac{4i^2}{n^2}\right)\frac{2}{n}=\sum_{i=1}^{n}\frac{8i^2}{n^3}=\frac{8}{n^3}\sum_{i=1}^{n} i^2.$$

Using the summation formula for $\sum_{i=1}^{n} i^2$, we have

$$\begin{aligned}\sum_{i=1}^{n} f(x_i)\Delta x &= \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{8}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] \\ &= \frac{8}{n^3}\left[\frac{2n^3+3n^2+n}{6}\right] \\ &= \frac{16n^3+24n^2+n}{6n^3} \\ &= \frac{8}{3}+\frac{4}{n}+\frac{1}{6n^2}.\end{aligned}$$

Now, to calculate the definite integral, we need to take the limit as $n \to \infty$. We get

$$\begin{aligned}\int_0^2 x^2\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \\ &= \lim_{n\to\infty}\left(\frac{8}{3}+\frac{4}{n}+\frac{1}{6n^2}\right) \\ &= \lim_{n\to\infty}\left(\frac{8}{3}\right)+\lim_{n\to\infty}\left(\frac{4}{n}\right)+\lim_{n\to\infty}\left(\frac{1}{6n^2}\right) \\ &= \frac{8}{3}+0+0=\frac{8}{3}.\end{aligned}$$

? Exercise 2.4.1

Use the definition of the definite integral to evaluate $\int_0^3 (2x-1)\,dx$.

Use a right-endpoint approximation to generate the Riemann sum.

Hint

Use the solving strategy from Example 2.4.1

Answer

6

Evaluating Definite Integrals

Evaluating definite integrals this way can be quite tedious because of the complexity of the calculations. Later in this chapter we develop techniques for evaluating definite integrals *without* taking limits of Riemann sums. However, for now, we can rely on the fact that definite integrals represent the area under the curve, and we can evaluate definite integrals by using geometric formulas to calculate that area. We do this to confirm that definite integrals do, indeed, represent areas, so we can then discuss what to do in the case of a curve of a function dropping below the x-axis.

✓ Example 2.4.2: Using Geometric Formulas to Calculate Definite Integrals

Use the formula for the area of a circle to evaluate $\int_3^6 \sqrt{9-(x-3)^2}\,dx$.

Solution

The function describes a semicircle with radius 3. To find

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx$$

we want to find the area under the curve over the interval $[3,6]$. The formula for the area of a circle is $A=\pi r^2$. The area of a semicircle is just one-half the area of a circle, or $A=\frac{1}{2}\pi r^2$. The shaded area in Figure 2.4.1 covers one-half of the semicircle, or $A=\frac{1}{4}\pi r^2$. Thus,

$$\int_3^6 \sqrt{9-(x-3)^2}\,dx=\frac{1}{4}\pi(3)^2=\frac{9}{4}\pi\approx 7.069.$$

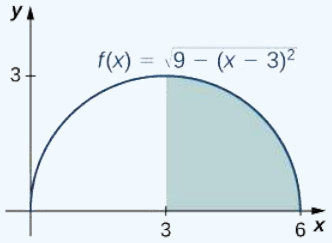


Figure 2.4.1: The value of the integral of the function $f(x)$ over the interval $[3,6]$ is the area of the shaded region.

? Exercise 2.4.2

Use the formula for the area of a trapezoid to evaluate $\int_2^4 (2x+3)\,dx$.

Hint

Graph the function $f(x)$ and calculate the area under the function on the interval $[2,4]$.

Answer

18 square units

Area and the Definite Integral

When we defined the definite integral, we lifted the requirement that $f(x)$ be nonnegative. But how do we interpret "the area under the curve" when $f(x)$ is negative?

Net Signed Area

Let us return to the Riemann sum. Consider, for example, the function $f(x)=2-2x^2$ (shown in Figure 2.4.2) on the interval $[0,2]$. Use $n=8$ and choose $\{x_i^*\}$ as the left endpoint of each interval. Construct a rectangle on each subinterval of height $f(x_i^*)$ and width Δx. When $f(x_i^*)$ is positive, the product $f(x_i^*)\Delta x$ represents the area of the rectangle, as before. When $f(x_i^*)$ is negative, however, the product $f(x_i^*)\Delta x$ represents the negative of the area of the rectangle. The Riemann sum then becomes

$$\sum_{i=1}^{8} f(x_i^*)\Delta x=(\text{Area of rectangles above the } x\text{-axis})-(\text{Area of rectangles below the } x\text{-axis})$$

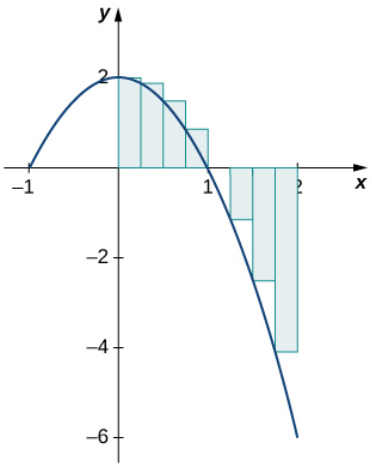


Figure 2.4.2: For a function that is partly negative, the Riemann sum is the area of the rectangles above the x-axis less the area of the rectangles below the x-axis.

Taking the limit as $n \to \infty$, the Riemann sum approaches the area between the curve above the x-axis and the x-axis, less the area between the curve below the x-axis and the x-axis, as shown in Figure 2.4.3. Then,

$$\int_0^2 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = A_1 - A_2.$$

The quantity $A_1 - A_2$ is called the **net signed area.**

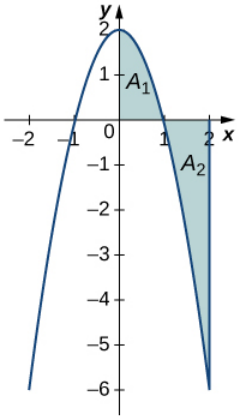


Figure 2.4.3: In the limit, the definite integral equals area A_1 less area A_2, or the net signed area.

Notice that net signed area can be positive, negative, or zero. If the area above the x-axis is larger, the net signed area is positive. If the area below the x-axis is larger, the net signed area is negative. If the areas above and below the x-axis are equal, the net signed area is zero.

Example 2.4.3: Finding the Net Signed Area

Find the net signed area between the curve of the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Solution

The function produces a straight line that forms two triangles: one from $x = -3$ to $x = 0$ and the other from $x = 0$ to $x = 3$ (Figure 2.4.4). Using the geometric formula for the area of a triangle, $A = \frac{1}{2}bh$, the area of triangle A_1, above the axis, is

$$A_1 = \frac{1}{2}3(6) = 9\,,$$

where 3 is the base and $2(3) = 6$ is the height. The area of triangle A_2, below the axis, is

$$A_2 = \frac{1}{2}(3)(6) = 9,$$

where 3 is the base and 6 is the height. Thus, the net area is

$$\int_{-3}^{3} 2x\,dx = A_1 - A_2 = 9 - 9 = 0.$$

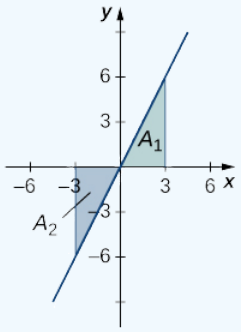


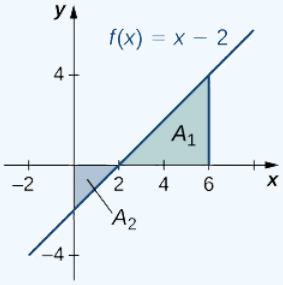
Figure 2.4.4: The area above the curve and below the x-axis equals the area below the curve and above the x-axis.

Analysis

If A_1 is the area above the x-axis and A_2 is the area below the x-axis, then the net area is $A_1 - A_2$. Since the areas of the two triangles are equal, the net area is zero.

? Exercise 2.4.3

Find the net signed area of $f(x) = x - 2$ over the interval $[0, 6]$, illustrated in the following image.



Hint

Use the solving method described in Example 2.4.3

Answer

6

Total Area

One application of the definite integral is finding **displacement** when given a velocity function. If $v(t)$ represents the velocity of an object as a function of time, then the area under the curve tells us how far the object is from its original position. This is a very important application of the definite integral, and we examine it in more detail later in the chapter. For now, we're just going to look at some basics to get a feel for how this works by studying constant velocities.

When velocity is a constant, the area under the curve is just velocity times time. This idea is already very familiar. If a car travels away from its starting position in a straight line at a speed of 70 mph for 2 hours, then it is 140 miles away from its original position (Figure 2.4.5). Using integral notation, we have

$$\int_0^2 70\,dt = 140 \text{ miles}.$$

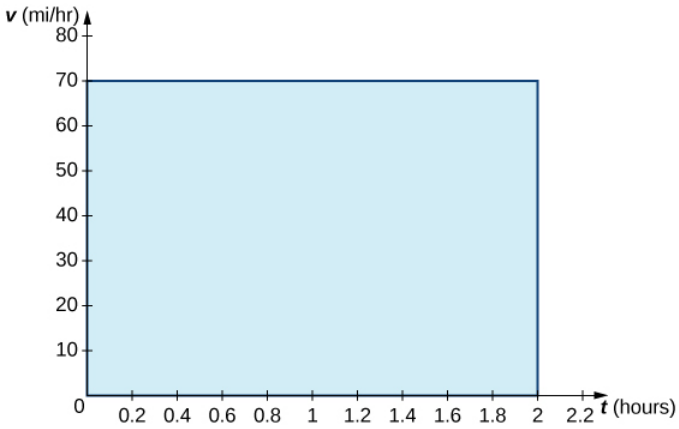


Figure 2.4.5: The area under the curve $v(t) = 70$ tells us how far the car is from its starting point at a given time.

In the context of displacement, net signed area allows us to take direction into account. If a car travels straight north at a speed of 60 mph for 2 hours, it is 120 miles north of its starting position. If the car then turns around and travels south at a speed of 40 mph for 3 hours, it will be back at it starting position (Figure 2.4.6). Again, using integral notation, we have

$$\int_0^2 60\,dt + \int_2^5 -40\,dt = 120 - 120 = 0.$$

In this case the displacement is zero.

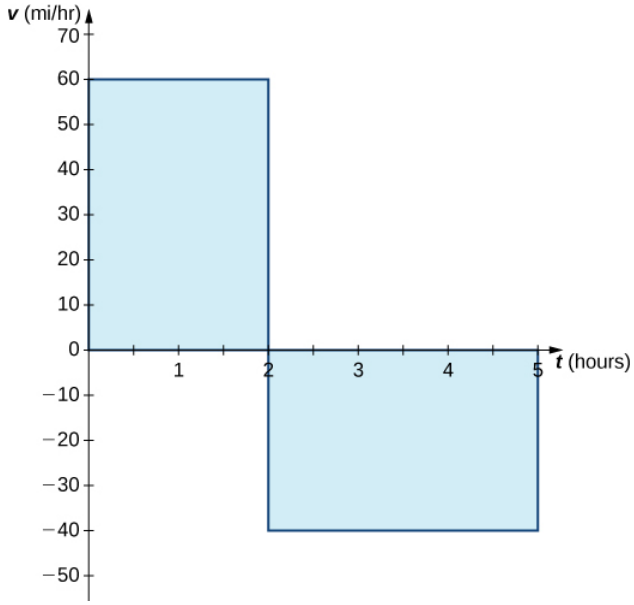


Figure 2.4.6: The area above the axis and the area below the axis are equal, so the net signed area is zero.

Suppose we want to know how far the car travels overall, regardless of direction. In this case, we want to know the area between the curve and the t-axis, regardless of whether that area is above or below the axis. This is called the **total area.**

Graphically, it is easiest to think of calculating total area by adding the areas above the axis and the areas below the axis (rather than subtracting the areas below the axis, as we did with net signed area). To accomplish this mathematically, we use the absolute value function. Thus, the total distance traveled by the car is

$$\int_0^2 |60|\,dt + \int_2^5 |-40|\,dt = \int_0^2 60\,dt + \int_2^5 40\,dt = 120 + 120 = 240.$$

Bringing these ideas together formally, we state the following definitions.

Definition: Net Signed Area

Let $f(x)$ be an integrable function defined on an interval $[a, b]$. Let A_1 represent the area between $f(x)$ and the x-axis that lies above the axis and let A_2 represent the area between $f(x)$ and the x-axis that lies below the axis. Then, the *net signed area* between $f(x)$ and the x-axis is given by

$$\int_a^b f(x)\,dx = A_1 - A_2.$$

The total area between $f(x)$ and the x-axis is given by

$$\int_a^b |f(x)|\,dx = A_1 + A_2.$$

Example 2.4.4: Finding the Total Area

Find the total area between $f(x) = x - 2$ and the x-axis over the interval $[0, 6]$.

Solution

Calculate the x-intercept as $(2, 0)$ (set $y = 0$, solve for x). To find the total area, take the area below the x-axis over the subinterval $[0, 2]$ and add it to the area above the x-axis on the subinterval $[2, 6]$ (Figure 2.4.7).

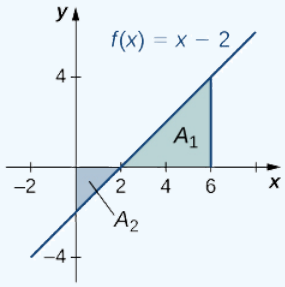


Figure 2.4.7: The total area between the line and the x-axis over $[0, 6]$ is A_2 plus A_1.

We have

$$\int_0^6 |(x-2)|\,dx = A_2 + A_1.$$

Then, using the formula for the area of a triangle, we obtain

$$A_2 = \frac{1}{2}bh = \frac{1}{2}\cdot 2 \cdot 2 = 2$$

$$A_1 = \frac{1}{2}bh = \frac{1}{2}\cdot 4 \cdot 4 = 8.$$

The total area, then, is

$$A_1 + A_2 = 8 + 2 = 10 \text{ units}^2.$$

Exercise 2.4.4

Find the total area between the function $f(x) = 2x$ and the x-axis over the interval $[-3, 3]$.

Hint

Review the solving strategy in Example 2.4.4

Answer

18 units^2

Properties of the Definite Integral

The properties of indefinite integrals apply to definite integrals as well. Definite integrals also have properties that relate to the limits of integration. These properties, along with the rules of integration that we examine later in this chapter, help us manipulate expressions to evaluate definite integrals.

Rule: Properties of the Definite Integral

1.

$$\int_a^a f(x)\,dx = 0$$

If the limits of integration are the same, the integral is just a line and contains no area.

2.

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

If the limits are reversed, then place a negative sign in front of the integral.

3.

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

The integral of a sum is the sum of the integrals.

4.

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

The integral of a difference is the difference of the integrals.

5.

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

for constant c. The integral of the product of a constant and a function is equal to the constant multiplied by the integral of the function.

6.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Although this formula normally applies when c is between a and b, the formula holds for all values of a, b, and c, provided $f(x)$ is integrable on the largest interval.

Example 2.4.5: Using the Properties of the Definite Integral

Use the properties of the definite integral to express the definite integral of $f(x) = -3x^3 + 2x + 2$ over the interval $[-2, 1]$ as the sum of three definite integrals.

Solution

Using integral notation, we have $\int_{-2}^{1} (-3x^3 + 2x + 2)\,dx$. We apply properties 3. and 5. to get

$$\int_{-2}^{1}(-3x^3+2x+2)\,dx=\int_{-2}^{1}-3x^3\,dx+\int_{-2}^{1}2x\,dx+\int_{-2}^{1}2\,dx$$
$$=-3\int_{-2}^{1}x^3\,dx+2\int_{-2}^{1}x\,dx+\int_{-2}^{1}2\,dx.$$

Exercise 2.4.5

Use the properties of the definite integral to express the definite integral of $f(x)=6x^3-4x^2+2x-3$ over the interval $[1,3]$ as the sum of four definite integrals.

Hint

Use the solving strategy from Example 2.4.5and the properties of definite integrals.

Answer

$$6\int_1^3 x^3\,dx-4\int_1^3 x^2\,dx+2\int_1^3 x\,dx-\int_1^3 3\,dx$$

Example 2.4.6: Using the Properties of the Definite Integral

If it is known that $\int_0^8 f(x)\,dx=10$ and $\int_0^5 f(x)\,dx=5$, find the value of $\int_5^8 f(x)\,dx$.

Solution

By property 6,

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx.$$

Thus,

$$\int_0^8 f(x)\,dx=\int_0^5 f(x)\,dx+\int_5^8 f(x)\,dx$$
$$10=5+\int_5^8 f(x)\,dx$$
$$5=\int_5^8 f(x)\,dx.$$

Exercise 2.4.6

If it is known that $\int_1^5 f(x)\,dx=-3$ and $\int_2^5 f(x)\,dx=4$, find the value of $\int_1^2 f(x)\,dx$.

Hint

Use the solving strategy from Example 2.4.6and the rule on properties of definite integrals.

Answer

-7

Comparison Properties of Integrals

A picture can sometimes tell us more about a function than the results of computations. Comparing functions by their graphs as well as by their algebraic expressions can often give new insight into the process of integration. Intuitively, we might say that if a

function $f(x)$ is above another function $g(x)$, then the area between $f(x)$ and the x-axis is greater than the area between $g(x)$ and the x-axis. This is true depending on the interval over which the comparison is made. The properties of definite integrals are valid whether $a < b$, $a = b$, or $a > b$. The following properties, however, concern only the case $a \leq b$, and are used when we want to compare the sizes of integrals.

Comparison Theorem

i. If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq 0.$$

ii. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

iii. If m and M are constants such that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

Example 2.4.7: Comparing Two Functions over a Given Interval

Compare $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ over the interval $[0, 1]$.

Solution

Graphing these functions is necessary to understand how they compare over the interval $[0, 1]$. Initially, when graphed on a graphing calculator, $f(x)$ appears to be above $g(x)$ everywhere. However, on the interval $[0, 1]$, the graphs appear to be on top of each other. We need to zoom in to see that, on the interval $[0, 1]$, $g(x)$ is above $f(x)$. The two functions intersect at $x = 0$ and $x = 1$ (Figure 2.4.8).

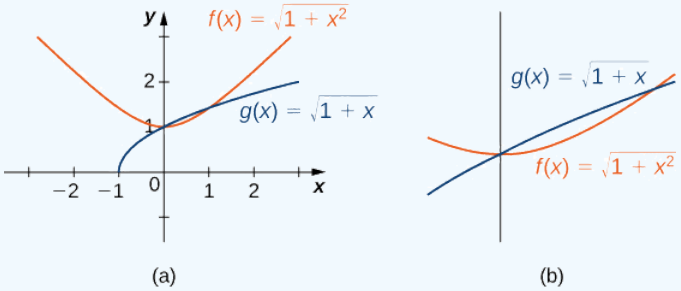


Figure 2.4.8: (a) The function $f(x)$ appears above the function $g(x)$ except over the interval $[0, 1]$ (b) Viewing the same graph with a greater zoom shows this more clearly.

We can see from the graph that over the interval $[0, 1]$, $g(x) \geq f(x)$. Comparing the integrals over the specified interval $[0, 1]$, we also see that $\int_0^1 g(x)\,dx \geq \int_0^1 f(x)\,dx$ (Figure 2.4.9). The thin, red-shaded area shows just how much difference there is between these two integrals over the interval $[0, 1]$.

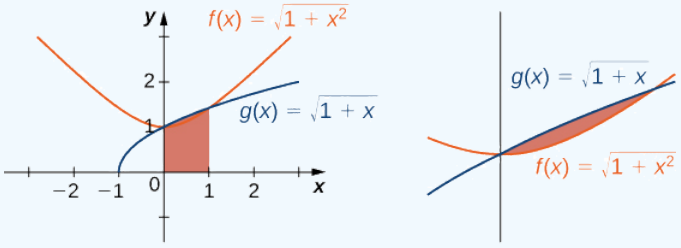


Figure 2.4.9: (a) The graph shows that over the interval $[0,1], g(x) \geq f(x)$, where equality holds only at the endpoints of the interval. (b) Viewing the same graph with a greater zoom shows this more clearly.

Average Value of a Function

We often need to find the average of a set of numbers, such as an average test grade. Suppose you received the following test scores in your algebra class: 89, 90, 56, 78, 100, and 69. Your semester grade is your average of test scores and you want to know what grade to expect. We can find the average by adding all the scores and dividing by the number of scores. In this case, there are six test scores. Thus,

$$\frac{89+90+56+78+100+69}{6} = \frac{482}{6} \approx 80.33.$$

Therefore, your average test grade is approximately 80.33, which translates to a B− at most schools.

Suppose, however, that we have a function $v(t)$ that gives us the speed of an object at any time t, and we want to find the object's average speed. The function $v(t)$ takes on an infinite number of values, so we can't use the process just described. Fortunately, we can use a definite integral to find the average value of a function such as this.

Let $f(x)$ be continuous over the interval $[a,b]$ and let $[a,b]$ be divided into n subintervals of width $\Delta x = (b-a)/n$. Choose a representative x_i^* in each subinterval and calculate $f(x_i^*)$ for $i = 1, 2, \ldots, n$. In other words, consider each $f(x_i^*)$ as a sampling of the function over each subinterval. The average value of the function may then be approximated as

$$f_{ave} \approx \frac{f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)}{n},$$

which is basically the same expression used to calculate the average of discrete values.

But we know $\Delta x = \dfrac{b-a}{n}$, so $n = \dfrac{b-a}{\Delta x}$, and we get

$$f_{ave} \approx \frac{f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)}{n} = \frac{f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)}{\left(\dfrac{b-a}{\Delta x}\right)}.$$

Following through with the algebra, the numerator is a sum that is represented as $\sum_{i=1}^{n} f(x_i^*)$, and we are dividing by a fraction. To divide by a fraction, invert the denominator and multiply. Thus, an approximate value for the average value of the function is given by

$$\frac{\sum_{i=1}^{n} f(x_i^*)}{\left(\dfrac{b-a}{\Delta x}\right)} = \left(\frac{\Delta x}{b-a}\right)\sum_{i=1}^{n} f(x_i^*) = \left(\frac{1}{b-a}\right)\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

This is a Riemann sum. Then, to get the exact average value, take the limit as n goes to infinity. Thus, the average value of a function is given by

$$\frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \frac{1}{b-a}\int_a^b f(x)dx.$$

Definition: Average Value of a Function

Let $f(x)$ be continuous over the interval $[a, b]$. Then, the *average value of the function* $f(x)$ (or f_{ave}) on $[a, b]$ is given by

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{2.4.1}$$

Example 2.4.8: Finding the Average Value of a Linear Function

Find the average value of $f(x) = x + 1$ over the interval $[0, 5]$.

Solution

First, graph the function on the stated interval, as shown in Figure 2.4.10

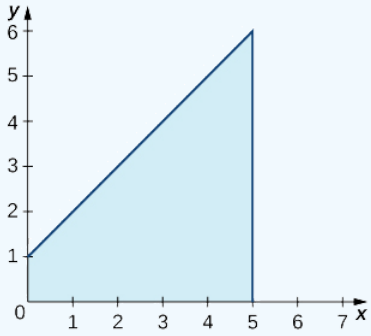


Figure 2.4.10:The graph shows the area under the function $(x) = x + 1$ over $[0, 5]$.

The region is a trapezoid lying on its side, so we can use the area formula for a trapezoid $A = \frac{1}{2}h(a+b)$, where h represents height, and a and b represent the two parallel sides. Then,

$$\int_0^5 x+1\,dx = \frac{1}{2}h(a+b) = \frac{1}{2}\cdot 5\cdot(1+6) = \frac{35}{2} \quad .$$

Thus the average value of the function is

$$\frac{1}{5-0}\int_0^5 x+1\,dx = \frac{1}{5}\cdot\frac{35}{2} = \frac{7}{2}\ .$$

Exercise 2.4.7

Find the average value of $f(x) = 6 - 2x$ over the interval $[0, 3]$.

Hint

Use the average value formula (Equation 2.4.1), and use geometry to evaluate the integral.

Answer

3

Key Concepts

- The definite integral can be used to calculate net signed area, which is the area above the x-axis less the area below the x-axis. Net signed area can be positive, negative, or zero.
- The component parts of the definite integral are the integrand, the variable of integration, and the limits of integration.
- Continuous functions on a closed interval are integrable. Functions that are not continuous may still be integrable, depending on the nature of the discontinuities.
- The properties of definite integrals can be used to evaluate integrals.
- The area under the curve of many functions can be calculated using geometric formulas.

- The average value of a function can be calculated using definite integrals.

Key Equations

- **Definite Integral**

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

- **Properties of the Definite Integral**

$$\int_a^a f(x)\,dx = 0$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx, \text{ for constant } c$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Glossary

average value of a function

(or f_{ave}) the average value of a function on an interval can be found by calculating the definite integral of the function and dividing that value by the length of the interval

definite integral

a primary operation of calculus; the area between the curve and the x-axis over a given interval is a definite integral

integrable function

a function is integrable if the limit defining the integral exists; in other words, if the limit of the Riemann sums as n goes to infinity exists

integrand

the function to the right of the integration symbol; the integrand includes the function being integrated

limits of integration

these values appear near the top and bottom of the integral sign and define the interval over which the function should be integrated

net signed area

the area between a function and the x-axis such that the area below the x-axis is subtracted from the area above the x-axis; the result is the same as the definite integral of the function

total area

total area between a function and the x-axis is calculated by adding the area above the x-axis and the area below the x-axis; the result is the same as the definite integral of the absolute value of the function

variable of integration

indicates which variable you are integrating with respect to; if it is x, then the function in the integrand is followed by dx

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2.5: The Fundamental Theorem of Calculus

Learning Objectives

- Describe the meaning of the Mean Value Theorem for Integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 1.
- Use the Fundamental Theorem of Calculus, Part 1, to evaluate derivatives of integrals.
- State the meaning of the Fundamental Theorem of Calculus, Part 2.
- Use the Fundamental Theorem of Calculus, Part 2, to evaluate definite integrals.
- Explain the relationship between differentiation and integration.

In the previous two sections, we looked at the definite integral and its relationship to the area under the curve of a function. Unfortunately, so far, the only tools we have available to calculate the value of a definite integral are geometric area formulas and limits of Riemann sums, and both approaches are extremely cumbersome. In this section we look at some more powerful and useful techniques for evaluating definite integrals.

These new techniques rely on the relationship between differentiation and integration. This relationship was discovered and explored by both Sir Isaac Newton and Gottfried Wilhelm Leibniz (among others) during the late 1600s and early 1700s, and it is codified in what we now call the **Fundamental Theorem of Calculus**, which has two parts that we examine in this section. Its very name indicates how central this theorem is to the entire development of calculus.

Isaac Newton's contributions to mathematics and physics changed the way we look at the world. The relationships he discovered, codified as Newton's laws and the law of universal gravitation, are still taught as foundational material in physics today, and his calculus has spawned entire fields of mathematics.

Before we get to this crucial theorem, however, let's examine another important theorem, the Mean Value Theorem for Integrals, which is needed to prove the Fundamental Theorem of Calculus.

The Mean Value Theorem for Integrals

The Mean Value Theorem for Integrals states that a continuous function on a closed interval takes on its average value at some point in that interval. The theorem guarantees that if $f(x)$ is continuous, a point c exists in an interval $[a, b]$ such that the value of the function at c is equal to the average value of $f(x)$ over $[a, b]$. We state this theorem mathematically with the help of the formula for the average value of a function that we presented at the end of the preceding section.

Theorem 2.5.1: The Mean Value Theorem for Integrals

If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

This formula can also be stated as

$$\int_a^b f(x)\, dx = f(c)(b-a). \tag{2.5.1}$$

Since $f(x)$ is continuous on $[a, b]$, by the extreme value theorem (see section on Maxima and Minima), it assumes minimum and maximum values—m and M, respectively—on $[a, b]$. Then, for all x in $[a, b]$, we have $m \leq f(x) \leq M$. Therefore, by the comparison theorem (see Section on The Definite Integral), we have

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a).$$

Proof

Since $f(x)$ is continuous on $[a,b]$, by the extreme value theorem (see section on Maxima and Minima), it assumes minimum and maximum values—m and M, respectively—on $[a,b]$. Then, for all x in $[a,b]$, we have $m \le f(x) \le M$. Therefore, by the comparison theorem (see Section on The Definite Integral), we have

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Dividing by $b-a$ gives us

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

Since $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ is a number between m and M, and since $f(x)$ is continuous and assumes the values m and M over $[a,b]$, by the Intermediate Value Theorem, there is a number c over $[a,b]$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

and the proof is complete.

□

Example 2.5.1: Finding the Average Value of a Function

Find the average value of the function $f(x) = 8 - 2x$ over the interval $[0,4]$ and find c such that $f(c)$ equals the average value of the function over $[0,4]$.

Solution

The formula states the mean value of $f(x)$ is given by

$$\frac{1}{4-0}\int_0^4 (8-2x)\,dx.$$

We can see in Figure 2.5.1 that the function represents a straight line and forms a right triangle bounded by the x- and y-axes. The area of the triangle is $A = \frac{1}{2}(base)(height)$. We have

$$A = \frac{1}{2}(4)(8) = 16.$$

The average value is found by multiplying the area by $1/(4-0)$. Thus, the average value of the function is

$$\frac{1}{4}(16) = 4$$

Set the average value equal to $f(c)$ and solve for c.

$$8 - 2c = 4$$
$$c = 2$$

At $c = 2$, $f(2) = 4$.

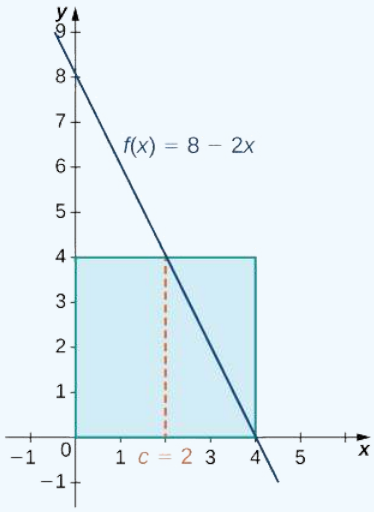


Figure 2.5.1: By the Mean Value Theorem, the continuous function $f(x)$ takes on its average value at c at least once over a closed interval.

Exercise 2.5.1

Find the average value of the function $f(x) = \dfrac{x}{2}$ over the interval $[0, 6]$ and find c such that $f(c)$ equals the average value of the function over $[0, 6]$.

Hint

Use the procedures from Example 2.5.1to solve the problem

Answer

The average value is 1.5 and $c = 3$.

Example 2.5.2: Finding the Point Where a Function Takes on Its Average Value

Given $\int_0^3 x^2\,dx = 9$, find c such that $f(c)$ equals the average value of $f(x) = x^2$ over $[0, 3]$.

Solution

We are looking for the value of c such that

$$f(c) = \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}(9) = 3.$$

Replacing $f(c)$ with c^2, we have

$$c^2 = 3$$
$$c = \pm\sqrt{3}.$$

Since $-\sqrt{3}$ is outside the interval, take only the positive value. Thus, $c = \sqrt{3}$ (Figure 2.5.2).

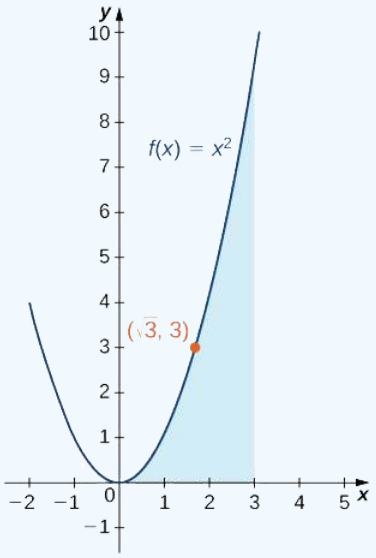


Figure 2.5.2: Over the interval $[0,3]$, the function $f(x)=x^2$ takes on its average value at $c=\sqrt{3}$.

? Exercise 2.5.2

Given $\int_0^3 (2x^2-1)\,dx = 15$, find c such that $f(c)$ equals the average value of $f(x)=2x^2-1$ over $[0,3]$.

Hint

Use the procedures from Example 2.5.2to solve the problem.

Answer

$c=\sqrt{3}$

Fundamental Theorem of Calculus Part 1: Integrals and Antiderivatives

As mentioned earlier, the Fundamental Theorem of Calculus is an extremely powerful theorem that establishes the relationship between differentiation and integration, and gives us a way to evaluate definite integrals without using Riemann sums or calculating areas. The theorem is comprised of two parts, the first of which, the **Fundamental Theorem of Calculus, Part 1,** is stated here. Part 1 establishes the relationship between differentiation and integration.

Theorem 2.5.2: The Fundamental Theorem of Calculus, Part 1

If $f(x)$ is continuous over an interval $[a,b]$, and the function $F(x)$ is defined by

$$F(x)=\int_a^x f(t)\,dt,$$

then $F'(x)=f(x)$ over $[a,b]$.

Before we delve into the proof, a couple of subtleties are worth mentioning here. First, a comment on the notation. Note that we have defined a function, $F(x)$, as the definite integral of another function, $f(t)$, from the point a to the point x. At first glance, this is confusing, because we have said several times that a definite integral is a number, and here it looks like it's a function. The key here is to notice that for any particular value of x, the definite integral is a number. So the function $F(x)$ returns a number (the value of the definite integral) for each value of x.

Second, it is worth commenting on some of the key implications of this theorem. There is a reason it is called the *Fundamental Theorem of Calculus*. Not only does it establish a relationship between integration and differentiation, but also it guarantees that any integrable function has an antiderivative. Specifically, it guarantees that any continuous function has an antiderivative.

Proof: Fundamental Theorem of Calculus, Part 1

Applying the definition of the derivative, we have

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt. \end{aligned}$$

Looking carefully at this last expression, we see $\frac{1}{h}\int_x^{x+h} f(t)\,dt$ is just the average value of the function $f(x)$ over the interval $[x, x+h]$. Therefore, by Equation 2.5.1, there is some number c in $[x, x+h]$ such that

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = f(c).$$

In addition, since c is between x and h, c approaches x as h approaches zero. Also, since $f(x)$ is continuous, we have

$$\lim_{h\to 0} f(c) = \lim_{c\to x} f(c) = f(x)$$

Putting all these pieces together, we have

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c) = f(x),$$

and the proof is complete.

□

Example 2.5.3: Finding a Derivative with the Fundamental Theorem of Calculus

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of

$$g(x) = \int_1^x \frac{1}{t^3+1}\,dt.$$

Solution

According to the Fundamental Theorem of Calculus, the derivative is given by

$$g'(x) = \frac{1}{x^3+1}.$$

Exercise 2.5.3

Use the Fundamental Theorem of Calculus, Part 1 to find the derivative of $g(r) = \int_0^r \sqrt{x^2+4}\,dx$.

Hint

Follow the procedures from Example 2.5.3to solve the problem.

Answer

$g'(r) = \sqrt{r^2+4}$

Example 2.5.4: Using the Fundamental Theorem and the Chain Rule to Calculate Derivatives

Let $F(x) = \int_1^{\sqrt{x}} \sin t \, dt$. Find $F'(x)$.

Solution

Letting $u(x) = \sqrt{x}$, we have $F(x) = \int_1^{u(x)} \sin t \, dt$.

Thus, by the Fundamental Theorem of Calculus and the chain rule,

$$F'(x) = \sin(u(x))\frac{du}{dx} = \sin(u(x)) \cdot \left(\frac{1}{2}x^{-1/2}\right) = \frac{\sin\sqrt{x}}{2\sqrt{x}}.$$

Exercise 2.5.4

Let $F(x) = \int_1^{x^3} \cos t \, dt$. Find $F'(x)$.

Hint

Use the chain rule to solve the problem.

Answer

$F'(x) = 3x^2 \cos x^3$

Example 2.5.5: Using the Fundamental Theorem of Calculus with Two Variable Limits of Integration

Let $F(x) = \int_x^{2x} t^3 \, dt$. Find $F'(x)$.

Solution

We have $F(x) = \int_x^{2x} t^3 \, dt$. Both limits of integration are variable, so we need to split this into two integrals. We get

$$\begin{aligned} F(x) &= \int_x^{2x} t^3 \, dt = \int_x^0 t^3 \, dt + \int_0^{2x} t^3 \, dt \\ &= -\int_0^x t^3 \, dt + \int_0^{2x} t^3 \, dt. \end{aligned}$$

Differentiating the first term, we obtain

$$\frac{d}{dx}\left[-\int_0^x t^3 \, dt\right] = -x^3.$$

Differentiating the second term, we first let $u(x) = 2x$. Then,

$$\begin{aligned} \frac{d}{dx}\left[\int_0^{2x} t^3 \, dt\right] &= \frac{d}{dx}\left[\int_0^{u(x)} t^3 \, dt\right] \\ &= (u(x))^3 \frac{du}{dx} \\ &= (2x)^3 \cdot 2 = 16x^3. \end{aligned}$$

Thus,

$$F'(x) = \frac{d}{dx}\left[-\int_0^x t^3\,dt\right] + \frac{d}{dx}\left[\int_0^{2x} t^3\,dt\right]$$
$$= -x^3 + 16x^3 = 15x^3$$

? Exercise 2.5.5

Let $F(x) = \int_x^{x^2} \cos t\,dt$. Find $F'(x)$.

Hint

Use the procedures from Example 2.5.5to solve the problem

Answer

$F'(x) = 2x\cos x^2 - \cos x$

Fundamental Theorem of Calculus, Part 2: The Evaluation Theorem

The Fundamental Theorem of Calculus, Part 2, is perhaps the most important theorem in calculus. After tireless efforts by mathematicians for approximately 500 years, new techniques emerged that provided scientists with the necessary tools to explain many phenomena. Using calculus, astronomers could finally determine distances in space and map planetary orbits. Everyday financial problems such as calculating marginal costs or predicting total profit could now be handled with simplicity and accuracy. Engineers could calculate the bending strength of materials or the three-dimensional motion of objects. Our view of the world was forever changed with calculus.

After finding approximate areas by adding the areas of n rectangles, the application of this theorem is straightforward by comparison. It almost seems too simple that the area of an entire curved region can be calculated by just evaluating an antiderivative at the first and last endpoints of an interval.

Theorem 2.5.3: The Fundamental Theorem of Calculus, Part 2

If $f(x)$ is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{2.5.2}$$

We often see the notation $F(x)|_a^b$ to denote the expression $F(b) - F(a)$. We use this vertical bar and associated limits a and b to indicate that we should evaluate the function $F(x)$ at the upper limit (in this case, b), and subtract the value of the function $F(x)$ evaluated at the lower limit (in this case, a).

The **Fundamental Theorem of Calculus, Part 2** (also known as the **evaluation theorem**) states that if we can find an antiderivative for the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the endpoints of the interval and subtracting.

Proof

Let $P = x_i, i = 0, 1, \ldots, n$ be a regular partition of $[a, b]$. Then, we can write

$$F(b) - F(a) = F(x_n) - F(x_0)$$
$$= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \ldots + [F(x_1) - F(x_0)]$$
$$= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})].$$

Now, we know F is an antiderivative of f over $[a, b]$, so by the Mean Value Theorem for derivatives (see The Mean Value Theorem) for $i = 0, 1, \ldots, n$ we can find c_i in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1}) = f(c_i)\,\Delta x.$$

Then, substituting into the previous equation, we have

$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\,\Delta x.$$

Taking the limit of both sides as $n \to \infty$, we obtain

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx.$$

□

Example 2.5.6: Evaluating an Integral with the Fundamental Theorem of Calculus

Use Equation 2.5.2 to evaluate

$$\int_{-2}^{2} (t^2 - 4)\,dt.$$

Solution

Recall the power rule for Antiderivatives:

If $y = x^n$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C.$$

Use this rule to find the antiderivative of the function and then apply the theorem. We have

$$\begin{aligned}
\int_{-2}^{2} (t^2 - 4)dt &= \left(\frac{t^3}{3} - 4t\right)\Big|_{-2}^{2} \\
&= \left[\frac{(2)^3}{3} - 4(2)\right] - \left[\frac{(-2)^3}{3} - 4(-2)\right] \\
&= \left[\frac{8}{3} - 8\right] - \left[-\frac{8}{3} + 8\right] \\
&= \frac{8}{3} - 8 + \frac{8}{3} - 8 \\
&= \frac{16}{3} - 16 = -\frac{32}{3}.
\end{aligned}$$

Analysis

Notice that we did not include the "$+C$" term when we wrote the antiderivative. The reason is that, according to the Fundamental Theorem of Calculus, Part 2 (Equation 2.5.2), any antiderivative works. So, for convenience, we chose the antiderivative with $C = 0$. If we had chosen another antiderivative, the constant term would have canceled out. This always happens when evaluating a definite integral.

The region of the area we just calculated is depicted in Figure 2.5.3. Note that the region between the curve and the x-axis is all below the x-axis. Area is always positive, but a definite integral can still produce a negative number (a net signed area). For example, if this were a profit function, a negative number indicates the company is operating at a loss over the given interval.

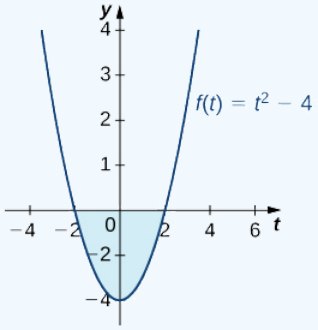


Figure 2.5.3: The evaluation of a definite integral can produce a negative value, even though area is always positive.

Example 2.5.7: Evaluating a Definite Integral Using the Fundamental Theorem of Calculus, Part 2

Evaluate the following integral using the Fundamental Theorem of Calculus, Part 2 (Equation 2.5.2):

$$\int_1^9 \frac{x-1}{\sqrt{x}}\,dx.$$

Solution

First, eliminate the radical by rewriting the integral using rational exponents. Then, separate the numerator terms by writing each one over the denominator:

$$\int_1^9 \frac{x-1}{x^{1/2}}\,dx = \int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right)\,dx.$$

Use the properties of exponents to simplify:

$$\int_1^9 \left(\frac{x}{x^{1/2}} - \frac{1}{x^{1/2}}\right)\,dx = \int_1^9 (x^{1/2} - x^{-1/2})\,dx.$$

Now, integrate using the power rule:

$$\begin{aligned}\int_1^9 (x^{1/2} - x^{-1/2})\,dx &= \left(\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^{1/2}}{\frac{1}{2}}\right)\Bigg|_1^9 \\ &= \left[\frac{(9)^{3/2}}{\frac{3}{2}} - \frac{(9)^{1/2}}{\frac{1}{2}}\right] - \left[\frac{(1)^{3/2}}{\frac{3}{2}} - \frac{(1)^{1/2}}{\frac{1}{2}}\right] \\ &= \left[\frac{2}{3}(27) - 2(3)\right] - \left[\frac{2}{3}(1) - 2(1)\right] \\ &= 18 - 6 - \frac{2}{3} + 2 = \frac{40}{3}.\end{aligned}$$

See Figure 2.5.4.

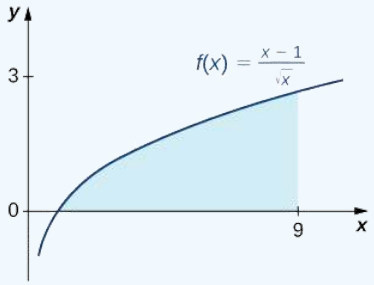


Figure 2.5.4: The area under the curve from $x = 1$ to $x = 9$ can be calculated by evaluating a definite integral.

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Exercise 2.5.6

Use Note to evaluate $\int_1^2 x^{-4}\,dx$.

Hint

Use the power rule.

Answer

$\frac{7}{24}$

Applications of Definite Integrals

Example 2.5.8: A Roller-Skating Race

James and Kathy are racing on roller skates. They race along a long, straight track, and whoever has gone the farthest after 5 sec wins a prize. If James can skate at a velocity of $f(t) = 5 + 2t$ ft/sec and Kathy can skate at a velocity of $g(t) = 10 + \cos\left(\frac{\pi}{2}t\right)$ ft/sec, who is going to win the race?

Solution

We need to integrate both functions over the interval $[0, 5]$ and see which value is bigger. For James, we want to calculate

$$\int_0^5 (5 + 2t)\,dt.$$

Using the power rule, we have

$$\begin{aligned}\int_0^5 (5 + 2t)\,dt &= \left(5t + t^2\right)\Big|_0^5 \\ &= (25 + 25) \\ &= 50.\end{aligned}$$

Thus, James has skated 50 ft after 5 sec. Turning now to Kathy, we want to calculate

$$\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right)\,dt.$$

We know $\sin t$ is an antiderivative of $\cos t$, so it is reasonable to expect that an antiderivative of $\cos\left(\frac{\pi}{2}t\right)$ would involve $\sin\left(\frac{\pi}{2}t\right)$. However, when we differentiate $\sin\left(\pi^2 t\right)$, we get $\pi^2 \cos\left(\pi^2 t\right)$ as a result of the chain rule, so we have to account for this additional coefficient when we integrate. We obtain

$$\begin{aligned}\int_0^5 10 + \cos\left(\frac{\pi}{2}t\right)\,dt &= \left(10t + \frac{2}{\pi}\sin\left(\frac{\pi}{2}t\right)\right)\Big|_0^5 \\ &= \left(50 + \frac{2}{\pi}\right) - \left(0 - \frac{2}{\pi}\sin 0\right) \approx 50.6.\end{aligned}$$

Kathy has skated approximately 50.6 ft after 5 sec. Kathy wins, but not by much!

Exercise 2.5.7

Suppose James and Kathy have a rematch, but this time the official stops the contest after only 3 sec. Does this change the outcome?

Hint

Change the limits of integration from those in Example 2.5.7.

Answer

Kathy still wins, but by a much larger margin: James skates 24 ft in 3 sec, but Kathy skates 29.3634 ft in 3 sec.

A Parachutist in Free Fall

Julie is an avid skydiver with more than 300 jumps under her belt and has mastered the art of making adjustments to her body position in the air to control how fast she falls. If she arches her back and points her belly toward the ground, she reaches a terminal velocity of approximately 120 mph (176 ft/sec). If, instead, she orients her body with her head straight down, she falls faster, reaching a terminal velocity of 150 mph (220 ft/sec).



Figure 2.5.5: Skydivers can adjust the velocity of their dive by changing the position of their body during the free fall. (credit: Jeremy T. Lock)

Since Julie will be moving (falling) in a downward direction, we assume the downward direction is positive to simplify our calculations. Julie executes her jumps from an altitude of 12,500 ft. After she exits the aircraft, she immediately starts falling at a velocity given by $v(t) = 32t$.

She continues to accelerate according to this velocity function until she reaches terminal velocity. After she reaches terminal velocity, her speed remains constant until she pulls her ripcord and slows down to land.

On her first jump of the day, Julie orients herself in the slower "belly down" position (terminal velocity is 176 ft/sec). Using this information, answer the following questions.

1. How long after she exits the aircraft does Julie reach terminal velocity?
2. Based on your answer to question 1, set up an expression involving one or more integrals that represents the distance Julie falls after 30 sec.
3. If Julie pulls her ripcord at an altitude of 3000 ft, how long does she spend in a free fall?
4. Julie pulls her ripcord at 3000 ft. It takes 5 sec for her parachute to open completely and for her to slow down, during which time she falls another 400 ft. After her canopy is fully open, her speed is reduced to 16 ft/sec. Find the total time Julie spends in the air, from the time she leaves the airplane until the time her feet touch the ground. On Julie's second jump of the day, she decides she wants to fall a little faster and orients herself in the "head down" position. Her terminal velocity in this position is 220 ft/sec. Answer these questions based on this velocity:
5. How long does it take Julie to reach terminal velocity in this case?
6. Before pulling her ripcord, Julie reorients her body in the "belly down" position so she is not moving quite as fast when her parachute opens. If she begins this maneuver at an altitude of 4000 ft, how long does she spend in a free fall before beginning the reorientation?

Some jumpers wear "**wingsuits**" (Figure 2.5.6). These suits have fabric panels between the arms and legs and allow the wearer to glide around in a free fall, much like a flying squirrel. (Indeed, the suits are sometimes called "flying squirrel suits.") When wearing these suits, terminal velocity can be reduced to about 30 mph (44 ft/sec), allowing the wearers a much longer time in

the air. Wingsuit flyers still use parachutes to land; although the vertical velocities are within the margin of safety, horizontal velocities can exceed 70 mph, much too fast to land safely.



Figure 2.5.6: The fabric panels on the arms and legs of a wingsuit work to reduce the vertical velocity of a skydiver's fall. (credit: Richard Schneider)

Answer the following question based on the velocity in a wingsuit.

7. If Julie dons a wingsuit before her third jump of the day, and she pulls her ripcord at an altitude of 3000 ft, how long does she get to spend gliding around in the air

Key Concepts

- The Mean Value Theorem for Integrals states that for a continuous function over a closed interval, there is a value c such that $f(c)$ equals the average value of the function.
- The Fundamental Theorem of Calculus, Part 1 shows the relationship between the derivative and the integral.
- The Fundamental Theorem of Calculus, Part 2 is a formula for evaluating a definite integral in terms of an antiderivative of its integrand. The total area under a curve can be found using this formula.

Key Equations

- **Mean Value Theorem for Integrals**

  If $f(x)$ is continuous over an interval $[a, b]$, then there is at least one point $c \in [a, b]$ such that

  $$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

- **Fundamental Theorem of Calculus, Part 1**

  If $f(x)$ is continuous over an interval $[a, b]$, and the function $F(x)$ is defined by

  $$F(x) = \int_a^x f(t)\, dt,$$

  then

  $$F\,'(x) = f(x).$$

- **Fundamental Theorem of Calculus, Part 2**

  If f is continuous over the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then

  $$\int_a^b f(x)\, dx = F(b) - F(a).$$

Glossary

fundamental theorem of calculus

the theorem, central to the entire development of calculus, that establishes the relationship between differentiation and integration

fundamental theorem of calculus, part 1

uses a definite integral to define an antiderivative of a function

fundamental theorem of calculus, part 2

(also, **evaluation theorem**) we can evaluate a definite integral by evaluating the antiderivative of the integrand at the endpoints of the interval and subtracting

mean value theorem for integrals

guarantees that a point c exists such that $f(c)$ is equal to the average value of the function

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2.6: Integration Formulas and the Net Change Theorem

Learning Objectives

- Apply the basic integration formulas.
- Explain the significance of the net change theorem.
- Use the net change theorem to solve applied problems.
- Apply the integrals of odd and even functions.

In this section, we use some basic integration formulas studied previously to solve some key applied problems. It is important to note that these formulas are presented in terms of indefinite integrals. Although definite and indefinite integrals are closely related, there are some key differences to keep in mind. A definite integral is either a number (when the limits of integration are constants) or a single function (when one or both of the limits of integration are variables). An indefinite integral represents a family of functions, all of which differ by a constant. As you become more familiar with integration, you will get a feel for when to use definite integrals and when to use indefinite integrals. You will naturally select the correct approach for a given problem without thinking too much about it. However, until these concepts are cemented in your mind, think carefully about whether you need a definite integral or an indefinite integral and make sure you are using the proper notation based on your choice.

Basic Integration Formulas

Recall the integration formulas given in the section on Antiderivatives and the properties of definite integrals. Let's look at a few examples of how to apply these formulas and properties.

Example 2.6.1: Integrating a Function Using the Power Rule

Use the power rule to integrate the function $\int_1^4 \sqrt{t}(1+t)\,dt$.

Solution

The first step is to rewrite the function and simplify it so we can apply the power rule:

$$\begin{aligned}\int_1^4 \sqrt{t}(1+t)\,dt &= \int_1^4 t^{1/2}(1+t)\,dt \\ &= \int_1^4 (t^{1/2}+t^{3/2})\,dt.\end{aligned}$$

Now apply the power rule:

$$\begin{aligned}\int_1^4 (t^{1/2}+t^{3/2})\,dt &= \left(\frac{2}{3}t^{3/2}+\frac{2}{5}t^{5/2}\right)\Big|_1^4 \\ &= \left[\frac{2}{3}(4)^{3/2}+\frac{2}{5}(4)^{5/2}\right]-\left[\frac{2}{3}(1)^{3/2}+\frac{2}{5}(1)^{5/2}\right] \\ &= \frac{256}{15}.\end{aligned}$$

Exercise 2.6.1

Find the definite integral of $f(x)=x^2-3x$ over the interval $[1,3]$.

Hint

Follow the process from Example 2.6.1to solve the problem.

Answer

$$\int_1^3 \left(x^2 - 3x\right)\, dx = -\frac{10}{3}$$

The Net Change Theorem

The **net change theorem** considers the integral of a **rate of change**. It says that when a quantity changes, the new value equals the initial value plus the integral of the rate of change of that quantity. The formula can be expressed in two ways. The second is more familiar; it is simply the definite integral.

Net Change Theorem

The new value of a changing quantity equals the initial value plus the integral of the rate of change:

$$F(b) = F(a) + \int_a^b F'(x)dx \tag{2.6.1}$$

or

$$\int_a^b F'(x)dx = F(b) - F(a). \tag{2.6.2}$$

Subtracting $F(a)$ from both sides of the Equation 2.6.1 yields Equation 2.6.2. Since they are equivalent formulas, which one we use depends on the application.

The significance of the net change theorem lies in the results. Net change can be applied to area, distance, and volume, to name only a few applications. Net change accounts for negative quantities automatically without having to write more than one integral. To illustrate, let's apply the net change theorem to a **velocity** function in which the result is **displacement.**

We looked at a simple example of this in The Definite Integral section. Suppose a car is moving due north (the positive direction) at 40 mph between 2 p.m. and 4 p.m., then the car moves south at 30 mph between 4 p.m. and 5 p.m. We can graph this motion as shown in Figure 2.6.1.

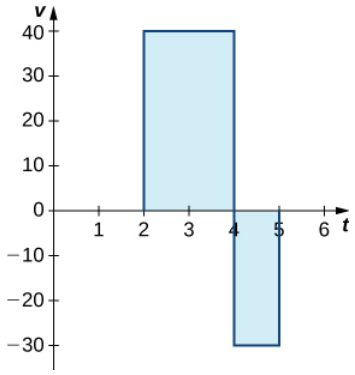


Figure 2.6.1: The graph shows speed versus time for the given motion of a car.

Just as we did before, we can use definite integrals to calculate the net displacement as well as the total distance traveled. The **net displacement** is given by

$$\int_2^5 v(t)\, dt = \int_2^4 40\, dt + \int_4^5 -30\, dt = 80 - 30 = 50.$$

Thus, at 5 p.m. the car is 50 mi north of its starting position. The **total distance** traveled is given by

$$\int_2^5 |v(t)|\, dt = \int_2^4 40\, dt + \int_4^5 30\, dt = 80 + 30 = 110.$$

Therefore, between 2 p.m. and 5 p.m., the car traveled a total of 110 mi.

To summarize, net displacement may include both positive and negative values. In other words, the velocity function accounts for both forward distance and backward distance. To find net displacement, integrate the velocity function over the interval. Total distance traveled, on the other hand, is always positive. To find the total distance traveled by an object, regardless of direction, we need to integrate the absolute value of the velocity function.

Example 2.6.2: Finding Net Displacement

Given a velocity function $v(t) = 3t - 5$ (in meters per second) for a particle in motion from time $t = 0$ to time $t = 3$, find the net displacement of the particle.

Solution

Applying the net change theorem, we have

$$\int_0^3 (3t-5)\,dt = \left(\frac{3t^2}{2} - 5t\right)\Big|_0^3 = \left[\frac{3(3)^2}{2} - 5(3)\right] - 0 = \frac{27}{2} - 15 = \frac{27}{2} - \frac{30}{2} = -\frac{3}{2}.$$

The net displacement is $-\frac{3}{2}$ m (Figure 2.6.2).

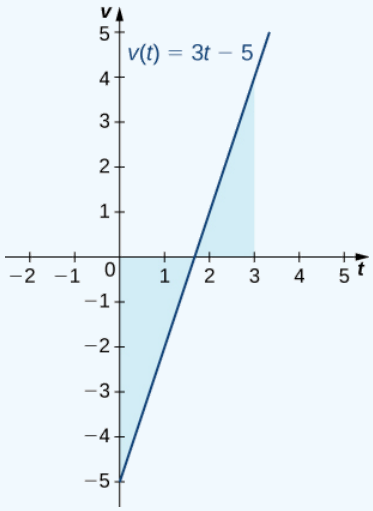


Figure 2.6.2: The graph shows velocity versus time for a particle moving with a linear velocity function.

Example 2.6.3: Finding the Total Distance Traveled

Use Example 2.6.2 to find the total distance traveled by a particle according to the velocity function $v(t) = 3t - 5$ m/sec over a time interval $[0, 3]$.

Solution

The total distance traveled includes both the positive and the negative values. Therefore, we must integrate the absolute value of the velocity function to find the total distance traveled.

To continue with the example, use two integrals to find the total distance. First, find the t-intercept of the function, since that is where the division of the interval occurs. Set the equation equal to zero and solve for t. Thus,

$$\begin{aligned} 3t - 5 &= 0 \\ 3t &= 5 \\ t &= \frac{5}{3}. \end{aligned}$$

The two subintervals are $\left[0, \frac{5}{3}\right]$ and $\left[\frac{5}{3}, 3\right]$. To find the total distance traveled, integrate the absolute value of the function. Since the function is negative over the interval $\left[0, \frac{5}{3}\right]$, we have $|v(t)| = -v(t)$ over that interval. Over $\left[\frac{5}{3}, 3\right]$, the function is positive, so $|v(t)| = v(t)$. Thus, we have

$$\begin{aligned}\int_0^3 |v(t)|\,dt &= \int_0^{5/3} -v(t)\,dt + \int_{5/3}^3 v(t)\,dt \\ &= \int_0^{5/3} 5-3t\,dt + \int_{5/3}^3 3t-5\,dt \\ &= \left(5t-\frac{3t^2}{2}\right)\Big|_0^{5/3} + \left(\frac{3t^2}{2}-5t\right)\Big|_{5/3}^3 \\ &= \left[5(\frac{5}{3})-\frac{3(5/3)^2}{2}\right]-0+\left[\frac{27}{2}-15\right]-\left[\frac{3(5/3)^2}{2}-\frac{25}{3}\right] \\ &= \frac{25}{3}-\frac{25}{6}+\frac{27}{2}-15-\frac{25}{6}+\frac{25}{3} \\ &= \frac{41}{6}\end{aligned}$$

So, the total distance traveled is $\frac{41}{6}$ m.

? Exercise 2.6.2

Find the net displacement and total distance traveled in meters given the velocity function $f(t)=\frac{1}{2}e^t-2$ over the interval $[0,2]$.

Hint

Follow the procedures from Examples 2.6.2and 2.6.3 Note that $f(t)\le 0$ for $t\le \ln 4$ and $f(t)\ge 0$ for $t\ge \ln 4$.

Answer

Net displacement: $\frac{e^2-9}{2}\approx -0.8055$ m; total distance traveled: $4\ln 4-7.5+\frac{e^2}{2}\approx 1.740$ m.

Applying the Net Change Theorem

The net change theorem can be applied to the flow and consumption of fluids, as shown in Example 2.6.4.

✓ Example 2.6.4: How Many Gallons of Gasoline Are Consumed?

If the motor on a motorboat is started at $t=0$ and the boat consumes gasoline at the rate of $5-t^3$ gal/hr, how much gasoline is used in the first 2 hours?

Solution

Express the problem as a definite integral, integrate, and evaluate using the Fundamental Theorem of Calculus. The limits of integration are the endpoints of the interval [0,2]. We have

$$\begin{aligned}\int_0^2 \left(5-t^3\right)\,dt &= \left(5t-\frac{t^4}{4}\right)\Big|_0^2 \\ &= \left[5(2)-\frac{(2)^4}{4}\right]-0 \\ &= 10-\frac{16}{4} \\ &= 6.\end{aligned}$$

Thus, the motorboat uses 6 gal of gas in 2 hours.

Example 2.6.5: Chapter Opener: Iceboats

As we saw at the beginning of the chapter, top **iceboat** racers can attain speeds of up to five times the wind speed. Andrew is an intermediate iceboater, though, so he attains speeds equal to only twice the wind speed.



Figure 2.6.3: (credit: modification of work by Carter Brown, Flickr)

Suppose Andrew takes his iceboat out one morning when a light 5-mph breeze has been blowing all morning. As Andrew gets his iceboat set up, though, the wind begins to pick up. During his first half hour of iceboating, the wind speed increases according to the function $v(t) = 20t + 5$. For the second half hour of Andrew's outing, the wind remains steady at 15 mph. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t+5, & \text{for } 0 \le t \le \frac{1}{2} \\ 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

Recalling that Andrew's iceboat travels at twice the wind speed, and assuming he moves in a straight line away from his starting point, how far is Andrew from his starting point after 1 hour?

Solution

To figure out how far Andrew has traveled, we need to integrate his velocity, which is twice the wind speed. Then

$$\text{Distance} = \int_0^1 2v(t)\,dt.$$

Substituting the expressions we were given for $v(t)$, we get

$$\begin{aligned} \int_0^1 2v(t)\,dt &= \int_0^{1/2} 2v(t)\,dt + \int_{1/2}^1 2v(t)\,dt \\ &= \int_0^{1/2} 2(20t+5)\,dt + \int_{1/3}^1 2(15)\,dt \\ &= \int_0^{1/2} (40t+10)\,dt + \int_{1/2}^1 30\,dt \\ &= \left[20t^2+10t\right]\Big|_0^{1/2} + \left[30t\right]\Big|_{1/2}^1 \\ &= \left(\frac{20}{4}+5\right) - 0 + (30-15) \\ &= 25. \end{aligned}$$

Andrew is 25 mi from his starting point after 1 hour.

Exercise 2.6.3

Suppose that, instead of remaining steady during the second half hour of Andrew's outing, the wind starts to die down according to the function $v(t) = -10t + 15$. In other words, the wind speed is given by

$$v(t) = \begin{cases} 20t + 5, & \text{for } 0 \le t \le \frac{1}{2} \\ -10t + 15, & \text{for } \frac{1}{2} \le t \le 1 \end{cases}.$$

Under these conditions, how far from his starting point is Andrew after 1 hour?

Hint

Don't forget that Andrew's iceboat moves twice as fast as the wind.

Answer

17.5mi

Integrating Even and Odd Functions

We saw in Functions and Graphs that an **even function** is a function in which $f(-x) = f(x)$ for all x in the domain—that is, the graph of the curve is unchanged when x is replaced with $-x$. The graphs of even functions are symmetric about the y-axis. An **odd function** is one in which $f(-x) = -f(x)$ for all x in the domain, and the graph of the function is symmetric about the origin.

Integrals of even functions, when the limits of integration are from $-a$ to a, involve two equal areas, because they are symmetric about the y-axis. Integrals of odd functions, when the limits of integration are similarly $[-a, a]$, evaluate to zero because the areas above and below the x-axis are equal.

Integrals of Even and Odd Functions

For continuous even functions such that $f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

For continuous odd functions such that $f(-x) = -f(x)$,

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Example 2.6.6: Integrating an Even Function

Integrate the even function $\int_{-2}^{2} (3x^8 - 2)\,dx$ and verify that the integration formula for even functions holds.

Solution

The symmetry appears in the graphs in Figure 2.6.4. Graph (a) shows the region below the curve and above the x-axis. We have to zoom in to this graph by a huge amount to see the region. Graph (b) shows the region above the curve and below the x-axis. The signed area of this region is negative. Both views illustrate the symmetry about the y-axis of an even function. We have

$$\begin{aligned}\int_{-2}^{2}(3x^8-2)\,dx &= \left(\frac{x^9}{3}-2x\right)\Big|_{-2}^{2}\\ &= \left[\frac{(2)^9}{3}-2(2)\right]-\left[\frac{(-2)^9}{3}-2(-2)\right]\\ &= \left(\frac{512}{3}-4\right)-\left(-\frac{512}{3}+4\right)\\ &= \frac{1000}{3}.\end{aligned}$$

To verify the integration formula for even functions, we can calculate the integral from 0 to 2 and double it, then check to make sure we get the same answer.

$$\int_{0}^{2}(3x^8-2)\,dx = \left(\frac{x^9}{3}-2x\right)\Big|_{0}^{2} = \frac{512}{3}-4 = \frac{500}{3}$$

Since $2\cdot\frac{500}{3}=\frac{1000}{3}$, we have verified the formula for even functions in this particular example.

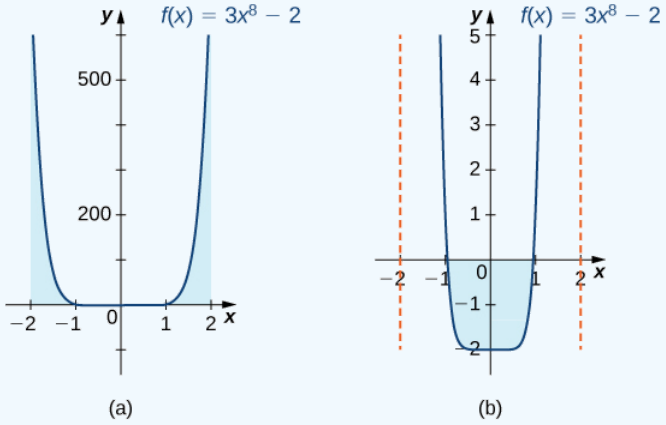


Figure 2.6.4: Graph (a) shows the positive area between the curve and the x-axis, whereas graph (b) shows the negative area between the curve and the x-axis. Both views show the symmetry about the y-axis.

Example 2.6.7: Integrating an Odd Function

Evaluate the definite integral of the odd function $-5\sin x$ over the interval $[-\pi,\pi]$.

Solution

The graph is shown in Figure 2.6.5. We can see the symmetry about the origin by the positive area above the x-axis over $[-\pi,0]$, and the negative area below the x-axis over $[0,\pi]$. we have

$$\begin{aligned}\int_{-\pi}^{\pi}-5\sin x\,dx &= -5(-\cos x)\Big|_{-\pi}^{\pi}\\ &= 5\cos x\Big|_{-\pi}^{\pi}\\ &= [5\cos\pi]-[5\cos(-\pi)]\\ &= -5-(-5)=0.\end{aligned}$$

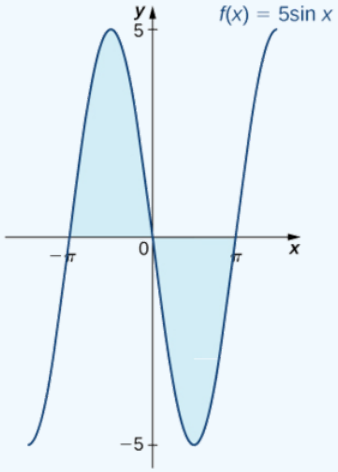


Figure 2.6.5:The graph shows areas between a curve and the x-axis for an odd function.

? Exercise 2.6.4

Integrate the function $\int_{-2}^{2} x^4\,dx$.

Hint

Integrate an even function.

Answer

$\frac{64}{5}$

Key Concepts

- The net change theorem states that when a quantity changes, the final value equals the initial value plus the integral of the rate of change. Net change can be a positive number, a negative number, or zero.
- The area under an even function over a symmetric interval can be calculated by doubling the area over the positive x-axis. For an odd function, the integral over a symmetric interval equals zero, because half the area is negative.

Key Equations

- Net Change Theorem

$$F(b) = F(a) + \int_a^b F'(x)\,dx$$

or

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Glossary

net change theorem

if we know the rate of change of a quantity, the net change theorem says the future quantity is equal to the initial quantity plus the integral of the rate of change of the quantity

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2.7: Substitution

The Fundamental Theorem of Calculus gave us a method to evaluate integrals without using Riemann sums. The drawback of this method, though, is that we must be able to find an antiderivative, and this is not always easy. In this section we examine a technique, called *integration by substitution*, to help us find antiderivatives. Specifically, this method helps us find antiderivatives when the integrand is the result of a chain-rule derivative.

At first, the approach to the substitution procedure may not appear very obvious. However, it is primarily a visual task—that is, the integrand shows you what to do; it is a matter of recognizing the form of the function. So, what are we supposed to see? We are looking for an integrand of the form $f[g(x)]g'(x)dx$. For example, in the integral

$$\int (x^2-3)^3 2x\,dx. \tag{2.7.1}$$

we have

$$f(x)=x^3$$

and

$$g(x)=x^2-3.$$

Then

$$g'(x)=2x.$$

and

$$f[g(x)]g'(x)=(x^2-3)^3(2x),$$

and we see that our integrand is in the correct form. The method is called substitution because we substitute part of the integrand with the variable u and part of the integrand with du. It is also referred to as *change of variables* because we are changing variables to obtain an expression that is easier to work with for applying the integration rules.

Substitution with Indefinite Integrals

Let $u=g(x)$, where $g'(x)$ is continuous over an interval, let $f(x)$ be continuous over the corresponding range of g, and let $F(x)$ be an antiderivative of $f(x)$. Then,

$$\begin{aligned}\int f[g(x)]g'(x)\,dx &= \int f(u)\,du \\ &= F(u)+C \\ &= F(g(x))+C\end{aligned}$$

Proof

Let f, g, u, and F be as specified in the theorem. Then

$$\frac{d}{dx}\big[F(g(x))\big] = F'(g(x))g'(x) = f[g(x)]g'(x). \tag{2.7.2}$$

Integrating both sides with respect to x, we see that

$$\int f[g(x)]g'(x)\,dx = F(g(x))+C. \tag{2.7.3}$$

If we now substitute $u=g(x)$, and $du=g'(x)dx$, we get

$$\int f[g(x)]g'(x)\,dx = \int f(u)\,du = F(u)+C = F(g(x))+C. \tag{2.7.4}$$

□

Returning to the problem we looked at originally, we let $u = x^2 - 3$ and then $du = 2x\,dx$.

Rewrite the integral (Equation 2.7.1) in terms of u:

$$\int (x^2 - 3)^3 (2x\,dx) = \int u^3\,du. \tag{2.7.5}$$

Using the power rule for integrals, we have

$$\int u^3\,du = \frac{u^4}{4} + C. \tag{2.7.6}$$

Substitute the original expression for x back into the solution:

$$\frac{u^4}{4} + C = \frac{(x^2 - 3)^4}{4} + C. \tag{2.7.7}$$

We can generalize the procedure in the following Problem-Solving Strategy.

Problem-Solving Strategy: Integration by Substitution

1. Look carefully at the integrand and select an expression $g(x)$ within the integrand to set equal to u. Let's select $g(x)$ such that $g'(x)$ is also part of the integrand.
2. Substitute $u = g(x)$ and $du = g'(x)dx$. into the integral.
3. We should now be able to evaluate the integral with respect to u. If the integral can't be evaluated we need to go back and select a different expression to use as u.
4. Evaluate the integral in terms of u.
5. Write the result in terms of x and the expression $g(x)$.

Example 2.7.1: Using Substitution to Find an Antiderivative

Use substitution to find the antiderivative of $\int 6x(3x^2 + 4)^4\,dx$.

Solution

The first step is to choose an expression for u. We choose $u = 3x^2 + 4$ because then $du = 6x\,dx$ and we already have du in the integrand. Write the integral in terms of u:

$$\int 6x(3x^2 + 4)^4\,dx = \int u^4\,du.$$

Remember that du is the derivative of the expression chosen for u, regardless of what is inside the integrand. Now we can evaluate the integral with respect to u:

$$\int u^4\,du = \frac{u^5}{5} + C = \frac{(3x^2 + 4)^5}{5} + C.$$

Analysis

We can check our answer by taking the derivative of the result of integration. We should obtain the integrand. Picking a value for C of 1, we let $y = \frac{1}{5}(3x^2 + 4)^5 + 1$. We have

$$y = \frac{1}{5}(3x^2 + 4)^5 + 1,$$

so

$$\begin{aligned} y' &= \left(\frac{1}{5}\right) 5(3x^2 + 4)^4 6x \\ &= 6x(3x^2 + 4)^4. \end{aligned}$$

This is exactly the expression we started with inside the integrand.

Exercise 2.7.1

Use substitution to find the antiderivative of $\int 3x^2(x^3-3)^2\,dx$.

Hint

Let $u = x^3 - 3$.

Answer

$$\int 3x^2(x^3-3)^2\,dx = \frac{1}{3}(x^3-3)^3 + C$$

Sometimes we need to adjust the constants in our integral if they don't match up exactly with the expressions we are substituting.

Example 2.7.2: Using Substitution with Alteration

Use substitution to find the antiderivative of

$$\int z\sqrt{z^2-5}\,dz.$$

Solution

Rewrite the integral as $\int z(z^2-5)^{1/2}\,dz$. Let $u = z^2 - 5$ and $du = 2z\,dz$. Now we have a problem because $du = 2z\,dz$ and the original expression has only $z\,dz$. We have to alter our expression for du or the integral in u will be twice as large as it should be. If we multiply both sides of the du equation by $\frac{1}{2}$. we can solve this problem. Thus,

$$u = z^2 - 5$$

$$du = 2z\,dz$$

$$\frac{1}{2}du = \frac{1}{2}(2z)\,dz = z\,dz.$$

Write the integral in terms of u, but pull the $\frac{1}{2}$ outside the integration symbol:

$$\int z(z^2-5)^{1/2}\,dz = \frac{1}{2}\int u^{1/2}\,du.$$

Integrate the expression in u:

$$\begin{aligned}\frac{1}{2}\int u^{1/2}\,du &= \left(\frac{1}{2}\right)\frac{u^{3/2}}{\frac{3}{2}} + C \\ &= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2} + C \\ &= \frac{1}{3}u^{3/2} + C \\ &= \frac{1}{3}(z^2-5)^{3/2} + C\end{aligned}$$

Exercise 2.7.2

Use substitution to find the antiderivative of $\int x^2(x^3+5)^9\,dx$.

Hint

Multiply the du equation by $\frac{1}{3}$.

Answer

$$\int x^2(x^3+5)^9\,dx = \frac{(x^3+5)^{10}}{30} + C$$

Integration of Trigonometric Functions

The next three examples will help fill in some missing pieces of our antiderivative knowledge. We know the antiderivatives of the sine and cosine functions; what about the other standard functions tangent, cotangent, secant and cosecant? We discover these next.

Example 2.7.3: Integration by substitution: antiderivatives of $\tan x$

Evaluate $\int \tan x\,dx$.

Solution

The previous paragraph established that we did not know the antiderivatives of tangent, hence we must assume that we have learned something in this section that can help us evaluate this indefinite integral.

Rewrite $\tan x$ as $\sin x / \cos x$. While the presence of a composition of functions may not be immediately obvious, recognize that $\cos x$ is "inside" the $1/x$ function. Therefore, we see if setting $u = \cos x$ returns usable results. We have that $du = -\sin x\,dx$, hence $-du = \sin x\,dx$. We can integrate:

$$\begin{aligned}
\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx && (2.7.8)\\
&= \int \underbrace{\frac{1}{\cos x}}_{u}\underbrace{\sin x\,dx}_{-du} && (2.7.9)\\
&= \int \frac{-1}{u}\,du && (2.7.10)\\
&= -\ln|u| + C && (2.7.11)\\
&= -\ln|\cos x| + C. && (2.7.12)
\end{aligned}$$

Some texts prefer to bring the -1 inside the logarithm as a power of $\cos x$, as in:

$$\begin{aligned}
-\ln|\cos x| + C &= \ln|(\cos x)^{-1}| + C && (2.7.13)\\
&= \ln\left|\frac{1}{\cos x}\right| + C && (2.7.14)\\
&= \ln|\sec x| + C. && (2.7.15)
\end{aligned}$$

Thus the result they give is $\int \tan x\,dx = \ln|\sec x| + C$. These two answers are equivalent.

Example 2.7.4: Integrating by substitution: antiderivatives of $\sec x$

Evaluate $\int \sec x\,dx$.

Solution

This example employs a wonderful trick: multiply the integrand by "1" so that we see how to integrate more clearly. In this case, we write "1" as

$$1 = \frac{\sec x + \tan x}{\sec x + \tan x}. \qquad (2.7.16)$$

This may seem like it came out of left field, but it works beautifully. Consider:

$$\int \sec x \, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx \tag{2.7.17}$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx. \tag{2.7.18}$$

Now let $u = \sec x + \tan x$; this means $du = (\sec x \tan x + \sec^2 x) \, dx$, which is our numerator. Thus:

$$= \int \frac{du}{u} \tag{2.7.19}$$

$$= \ln|u| + C \tag{2.7.20}$$

$$= \ln|\sec x + \tan x| + C. \tag{2.7.21}$$

We can use similar techniques to those used in Examples 2.7.3 and 2.7.4 to find antiderivatives of $\cot x$ and $\csc x$ (which the reader can explore in the exercises.) We summarize our results here.

Theorem 2.7.1: Antiderivatives of Trigonometric Functions

1. $\int \sin x \, dx = -\cos x + C$
2. $\int \cos x \, dx = \sin x + C$
3. $\int \tan x \, dx = -\ln|\cos x| + C$
4. $\int \csc x \, dx = -\ln|\csc x + \cot x| + C$
5. $\int \sec x \, dx = \ln|\sec x + \tan x| + C$
6. $\int \cot x \, dx = \ln|\sin x| + C$

Note

Although we have these formulas, at this point it will be best to show supporting work for all but the first two (the antiderivatives of sine and cosine).

Example 2.7.5: Using Substitution with Integrals of Trigonometric Functions

Use substitution to evaluate the integral $\int \frac{\sin t}{\cos^3 t} \, dt$.

Solution

We know the derivative of $\cos t$ is $-\sin t$, so we set $u = \cos t$. Then $du = -\sin t \, dt$.

Substituting into the integral, we have

$$\int \frac{\sin t}{\cos^3 t} \, dt = -\int \frac{du}{u^3}.$$

Evaluating the integral, we get

$$-\int \frac{du}{u^3} = -\int u^{-3} \, du = -\left(-\frac{1}{2}\right) u^{-2} + C.$$

Putting the answer back in terms of t, we get

$$\int \frac{\sin t}{\cos^3 t} \, dt = \frac{1}{2u^2} + C = \frac{1}{2\cos^2 t} + C.$$

Exercise 2.7.3

Use substitution to evaluate the integral $\int \frac{\cos t}{\sin^2 t} \, dt$.

Hint

Use the process from Example 2.7.5to solve the problem.

Answer

$$\int \frac{\cos t}{\sin^2 t}\,dt = -\frac{1}{\sin t} + C$$

Exercise 2.7.4

Use substitution to evaluate the indefinite integral $\int \cos^3 t \sin t\,dt$.

Hint

Use the process from Example 2.7.5to solve the problem.

Answer

$$\int \cos^3 t \sin t\,dt = -\frac{\cos^4 t}{4} + C$$

We explore one more common trigonometric integral.

Example 2.7.6: Integration by substitution: powers of $\cos x$ and $\sin x$

Evaluate $\int \cos^2 x\,dx$.

Solution

We have a composition of functions as $\cos^2 x = \left(\cos x\right)^2$.

However, setting $u = \cos x$ means $du = -\sin x\,dx$, which we do not have in the integral. Another technique is needed.

The process we'll employ is to use a Power Reducing formula for $\cos^2 x$ (perhaps consult the back of this text for this formula), which states

$$\cos^2 x = \frac{1+\cos(2x)}{2}. \tag{2.7.22}$$

The right hand side of this equation is not difficult to integrate. We have:

$$\int \cos^2 x\,dx = \int \frac{1+\cos(2x)}{2}\,dx \tag{2.7.23}$$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\,dx. \tag{2.7.24}$$

Integrating, we obtain:

$$= \frac{1}{2}x + \frac{1}{2}\frac{\sin(2x)}{2} + C \tag{2.7.25}$$

$$= \frac{1}{2}x + \frac{\sin(2x)}{4} + C. \tag{2.7.26}$$

We'll make significant use of this power--reducing technique in future sections.

A u-Substitution with a Twist

Sometimes we need to manipulate an integral in ways that are more complicated than just multiplying or dividing by a constant. We need to eliminate all the expressions within the integrand that are in terms of the original variable. When we are done, u should be the only variable in the integrand. In some cases, this means solving for the original variable in terms of u. This technique should become clear in the next example.

Example 2.7.7: Finding an Antiderivative Using u-Substitution

Use substitution to find the antiderivative of

$$\int \frac{x}{\sqrt{x-1}}\,dx.$$

Solution

If we let $u = x - 1,$ then $du = dx$. But this does not account for the x in the numerator of the integrand. We need to express x in terms of u. If $u = x - 1$, then $x = u + 1$. Now we can rewrite the integral in terms of u:

$$\int \frac{x}{\sqrt{x-1}}\,dx = \int \frac{u+1}{\sqrt{u}}\,du = \int \sqrt{u} + \frac{1}{\sqrt{u}}\,du = \int (u^{1/2} + u^{-1/2})\,du.$$

Then we integrate in the usual way, replace u with the original expression, and factor and simplify the result. Thus,

$$\begin{aligned}\int (u^{1/2} + u^{-1/2})\,du &= \frac{2}{3}u^{3/2} + 2u^{1/2} + C \\ &= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C \\ &= (x-1)^{1/2}\left[\frac{2}{3}(x-1) + 2\right] + C \\ &= (x-1)^{1/2}\left(\frac{2}{3}x - \frac{2}{3} + \frac{6}{3}\right) \\ &= (x-1)^{1/2}\left(\frac{2}{3}x + \frac{4}{3}\right) \\ &= \frac{2}{3}(x-1)^{1/2}(x+2) + C.\end{aligned}$$

Substitution for Definite Integrals

Substitution can be used with definite integrals, too. However, using substitution to evaluate a definite integral requires a change to the limits of integration. If we change variables in the integrand, the limits of integration change as well.

Substitution with Definite Integrals

Let $u = g(x)$ and let g' be continuous over an interval $[a, b]$, and let f be continuous over the range of $u = g(x)$. Then,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du. \tag{2.7.27}$$

Although we will not formally prove this theorem, we justify it with some calculations here. From the substitution rule for indefinite integrals, if $F(x)$ is an antiderivative of $f(x)$, we have

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C. \tag{2.7.28}$$

Then

$$\begin{aligned}\int_a^b f[g(x)]g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= F(u)\Big|_{u=g(a)}^{u=g(b)} \\ &= \int_{g(a)}^{g(b)} f(u)\,du\end{aligned}$$

and we have the desired result.

Example 2.7.8: Using Substitution to Evaluate a Definite Integral

Use substitution to evaluate

$$\int_0^1 x^2(1+2x^3)^5\,dx.$$

Solution

Let $u=1+2x^3$, so $du=6x^2dx$. Since the original function includes one factor of x^2 and $du=6x^2dx$, multiply both sides of the du equation by $1/6$. Then,

$$du=6x^2\,dx$$

$$\text{becomes}\quad \frac{1}{6}du=x^2\,dx.$$

To adjust the limits of integration, note that when $x=0, u=1+2(0)=1,$ and when $x=1, u=1+2(1)=3.$ Then

$$\int_0^1 x^2(1+2x^3)^5dx=\frac{1}{6}\int_1^3 u^5\,du.$$

Evaluating this expression, we get

$$\begin{aligned}\frac{1}{6}\int_1^3 u^5\,du &=(\frac{1}{6})(\frac{u^6}{6})\Big|_1^3\\ &=\frac{1}{36}[(3)^6-(1)^6]\\ &=\frac{182}{9}.\end{aligned}$$

Exercise 2.7.5

Use substitution to evaluate the definite integral $\int_{-1}^0 y(2y^2-3)^5\,dy.$

Hint

Use the steps from Example 2.7.8to solve the problem.

Answer

$$\int_{-1}^0 y(2y^2-3)^5\,dy=\frac{91}{3}$$

Exercise 2.7.6

Use substitution to evaluate $\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx.$

Hint

Use the process from Example 2.7.8to solve the problem.

Answer

$$\int_0^1 x^2\cos\left(\frac{\pi}{2}x^3\right)\,dx=\frac{2}{3\pi}\approx 0.2122$$

Substitution may be only one of the techniques needed to evaluate a definite integral. All of the properties and rules of integration apply independently, and trigonometric functions may need to be rewritten using a trigonometric identity before we can apply substitution. Also, we have the option of replacing the original expression for u after we find the antiderivative, which means that we do not have to change the limits of integration. These two approaches are shown in Example 2.7.9.

Example 2.7.9: Using Substitution to Evaluate a Trigonometric Integral

Use substitution to evaluate

$$\int_0^{\pi/2} \cos^2\theta\, d\theta.$$

Solution

Let us first use a trigonometric identity to rewrite the integral. The trig identity $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ allows us to rewrite the integral as

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta.$$

Then,

$$\begin{aligned}\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta &= \int_0^{\pi/2} \left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right) d\theta \\ &= \frac{1}{2}\int_0^{\pi/2} d\theta + \int_0^{\pi/2} \cos 2\theta\, d\theta.\end{aligned}$$

We can evaluate the first integral as it is, but we need to make a substitution to evaluate the second integral. Let $u = 2\theta$. Then, $du = 2\, d\theta$, or $\dfrac{1}{2}\, du = d\theta$. Also, when $\theta = 0$, $u = 0$, and when $\theta = \pi/2$, $u = \pi$. Expressing the second integral in terms of u, we have

$$\begin{aligned}\frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\int_0^{\pi/2} \cos 2\theta\, d\theta &= \frac{1}{2}\int_0^{\pi/2} d\theta + \frac{1}{2}\left(\frac{1}{2}\right)\int_0^{\pi} \cos u\, du \\ &= \frac{\theta}{2}\Big|_{\theta=0}^{\theta=\pi/2} + \frac{1}{4}\sin u\Big|_{u=0}^{u=\pi} \\ &= \left(\frac{\pi}{4}-0\right)+(0-0) = \frac{\pi}{4}\end{aligned}$$

Key Concepts

- Substitution is a technique that simplifies the integration of functions that are the result of a chain-rule derivative. The term 'substitution' refers to changing variables or substituting the variable u and du for appropriate expressions in the integrand.
- When using substitution for a definite integral, we also have to change the limits of integration.

Key Equations

- **Substitution with Indefinite Integrals**

$$\int f[g(x)]g'(x)\, dx = \int f(u)\, du = F(u)+C = F(g(x))+C$$

- **Substitution with Definite Integrals**

$$\int_a^b f(g(x))g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Glossary

change of variables
the substitution of a variable, such as u, for an expression in the integrand

integration by substitution
a technique for integration that allows integration of functions that are the result of a chain-rule derivative

Contributors

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- Apex Calculus: the subsection on integrating trigonometric functions is mostly from Apex Calculus, Section 6.1.
- Edited by Paul Seeburger (Monroe Community College)

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2.8: 5.5E and 5.6E u-Substitution Exercises

5.5: Substitution

In exercises 1 - 16, find the antiderivative.

1) $\int (x+1)^4\,dx$

Answer:

$$\int (x+1)^4\,dx \quad = \quad \frac{1}{5}(x+1)^5+C$$

2) $\int (x-1)^5\,dx$

3) $\int (2x-3)^{-7}\,dx$

Answer:

$$\int (2x-3)^{-7}\,dx \quad = \quad -\frac{1}{12(2x-3)^6}+C$$

4) $\int (3x-2)^{-11}\,dx$

5) $\int \frac{x}{\sqrt{x^2+1}}\,dx$

Answer:

$$\int \frac{x}{\sqrt{x^2+1}}\,dx \quad = \quad \sqrt{x^2+1}+C$$

6) $\int \frac{x}{\sqrt{1-x^2}}\,dx$

7) $\int (x-1)(x^2-2x)^3\,dx$

Answer:

$$\int (x-1)(x^2-2x)^3\,dx \quad = \quad \frac{1}{8}(x^2-2x)^4+C$$

8) $\int (x^2-2x)(x^3-3x^2)^2\,dx$

9) $\int \cos^3\theta\,d\theta$ (Hint: $\cos^2\theta = 1-\sin^2\theta$)

Answer:

$$\int \cos^3\theta\,d\theta \quad = \quad \sin\theta-\frac{\sin^3\theta}{3}+C$$

10) $\int \sin^3\theta\,d\theta$ (Hint: $\sin^2\theta = 1-\cos^2\theta$)

11) $\int x(1-x)^{99}\,dx$

Answer:

$$\int x(1-x)^{99}\,dx \quad = \quad \frac{(1-x)^{101}}{101} - \frac{(1-x)^{100}}{100} + C$$

12) $\int t(1-t^2)^{10}\,dt$

13) $\int (11x-7)^{-3}\,dx$

Answer:

$$\int (11x-7)^{-3}\,dx \quad = \quad -\frac{1}{22(11x-7)^2} + C$$

14) $\int (7x-11)^4\,dx$

15) $\int \cos^3\theta \sin\theta\,d\theta$

Answer:

$$\int \cos^3\theta \sin\theta\,d\theta \quad = \quad -\frac{\cos^4\theta}{4} + C$$

16) $\int \frac{x^2}{(x^3-3)^2}\,dx$

Answer:

$$\int \frac{x^2}{(x^3-3)^2}\,dx \quad = \quad -\frac{1}{3(x^3-3)} + C$$

u-Substitution with Definite Integrals

In exercises 17 - 22, evaluate the definite integral.

17) $\int_0^1 x\sqrt{1-x^2}\,dx$

18) $\int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx$

Answer:

$$u = 1+x^2, \quad du = 2x\,dx, \quad \int_0^1 \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int_1^2 u^{-1/2}du = \sqrt{2}-1$$

19) $\int_0^2 \frac{t}{\sqrt{5+t^2}}\,dt$

20) $\int_0^1 \frac{t^2}{\sqrt{1+t^3}}\,dt$

Answer:

$$u = 1+t^3, \quad du = 3t^2, \quad \int_0^1 \frac{t^2}{\sqrt{1+t^3}}\,dt = \frac{1}{3}\int_1^2 u^{-1/2}du = \frac{2}{3}(\sqrt{2}-1)$$

21) $\int_0^{\pi/4} \sec^2\theta\tan\theta\,d\theta$

22) $\int_0^{\pi/4} \frac{\sin\theta}{\cos^4\theta}\,d\theta$

Answer:

$$u=\cos\theta,\quad du=-\sin\theta\,d\theta,\quad \int_0^{\pi/4}\frac{\sin\theta}{\cos^4\theta}\,d\theta=-\int_1^{\sqrt{2}/2}u^{-4}\,du=\int_{\sqrt{2}/2}^{1}u^{-4}\,du=\frac{1}{3}(2\sqrt{2}-1)$$

u-Substitution with a Twist

In exercises 23 - 28, find the antiderivative. Then check your answer by showing its derivative can be simplified to the original integrand.

23) $\displaystyle\int x\sqrt{x+1}\,dx$

Answer:

$$\int x\sqrt{x+1}\,dx \quad = \quad \frac{2}{15}(x+1)^{3/2}(3x-2)+C$$

24) $\displaystyle\int \frac{x}{\sqrt{3x+1}}\,dx$

Answer:

$$\int \frac{x}{\sqrt{3x+1}}\,dx \quad = \quad \frac{2}{27}\sqrt{3x+1}\,(3x-2)+C$$

25) $\displaystyle\int \frac{x+2}{(x-1)^{3/2}}\,dx$

Answer:

$$\int \frac{x+2}{(x-1)^{3/2}}\,dx \quad = \quad \frac{2(x-4)}{\sqrt{x-1}}+C$$

Check (using quotient rule):

$$\begin{aligned}\frac{d}{dx}\left(\frac{2(x-4)}{\sqrt{x-1}}+C\right) &= \frac{2\sqrt{x-1}-2(x-4)\cdot\frac{1}{2}(x-1)^{-1/2}}{x-1}\\ &= \frac{(x-1)^{-1/2}\big(2(x-1)-(x-4)\big)}{x-1}\\ &= \frac{2x-2-x+4}{(x-1)^{3/2}}\\ &= \frac{x+2}{(x-1)^{3/2}} \quad \checkmark\end{aligned}$$

26) $\displaystyle\int t^2\sqrt{3-t}\,dt$

Answer:

$$\begin{aligned}\int t^2\sqrt{3-t}\,dt &= -6(3-t)^{3/2}+\frac{12}{5}(3-t)^{5/2}-\frac{2}{7}(3-t)^{7/2}+C\\ &= -\frac{2}{35}(3-t)^{3/2}\left[5t^2+12t+24\right]+C\end{aligned}$$

Check (using product rule):

$$\frac{d}{dx}\left(-\frac{2}{35}(3-t)^{3/2}\left[5t^2+12t+24\right]+C\right) = -\frac{2}{35}\left[(3-t)^{3/2}(10t+12)-\frac{3}{2}(3-t)^{1/2}\left[5t^2+12t+24\right]\right]$$

$$= -\frac{1}{35}(3-t)^{1/2}\left[2(3-t)(10t+12)-3\left[5t^2+12t+24\right]\right]$$

$$= -\frac{1}{35}(3-t)^{1/2}\left[4(3-t)(5t+6)-15t^2-36t-72\right]$$

$$= -\frac{1}{35}(3-t)^{1/2}\left[4(18+9t-5t^2)-15t^2-36t-72\right]$$

$$= -\frac{1}{35}(3-t)^{1/2}\left[72+36t-20t^2-15t^2-36t-72\right]$$

$$= -\frac{1}{35}(3-t)^{1/2}\left[-35t^2\right]$$

$$= t^2\sqrt{3-t} \quad \checkmark$$

27) $\displaystyle\int t^3\sqrt{t^2+5}\,dt$

Answer:

$$\int t^3\sqrt{t^2+5}\,dt = \frac{1}{2}\int(u-5)u^{1/2}\,du$$

$$= \frac{(t^2+5)^{5/2}}{5}-\frac{5}{3}(t^2+5)^{3/2}+C$$

$$= \frac{(t^2+5)^{3/2}}{15}\left[3t^2-10\right]+C$$

Check (using product rule):

$$\frac{d}{dx}\left(\frac{(t^2+5)^{3/2}}{15}\left[3t^2-10\right]+C\right) = \frac{6t(t^2+5)^{3/2}}{15}+(3t^2-10)\cdot\frac{3}{2}\frac{(t^2+5)^{1/2}}{15}\cdot 2t$$

$$= \frac{3t(t^2+5)^{1/2}}{15}\left[2(t^2+5)+(3t^2-10)\right]$$

$$= \frac{t}{5}\sqrt{t^2+5}\left[5t^2\right]$$

$$= t^3\sqrt{t^2+5} \quad \checkmark$$

28) $\displaystyle\int x\sqrt[3]{x-2}\,dx$

Answer:

$$\int x\sqrt[3]{x-2}\,dx = \frac{3}{7}(x-2)^{7/3}+\frac{3}{2}(x-2)^{4/3}+C$$

$$= \frac{3}{14}(x-2)^{4/3}[2x+3]+C$$

Check (using product rule):

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{14}(x-2)^{4/3}[2x+3]+C\right) &= \frac{3}{7}(x-2)^{4/3}+(2x+3)\cdot\frac{3}{14}\cdot\frac{4}{3}(x-2)^{1/3}\\ &= \frac{3}{7}(x-2)^{4/3}+\frac{2}{7}(x-2)^{1/3}(2x+3)\\ &= \frac{1}{7}(x-2)^{1/3}\left[3(x-2)+2(2x+3)\right]\\ &= \frac{1}{7}(x-2)^{1/3}\left[3x-6+4x+6\right]\\ &= \frac{1}{7}(x-2)^{1/3}\left[7x\right]\\ &= x\sqrt[3]{x-2}\quad\checkmark\end{aligned}$$

5.6: Integrals Involving Exponential and Logarithmic Functions

For exercises 1 - 8, compute each indefinite integral.

1) $\int e^{2x}\,dx$

2) $\int e^{-3x}\,dx$

Answer:

$$\int e^{-3x}\,dx \quad = \quad \frac{-1}{3}e^{-3x}+C$$

3) $\int 2^{x}\,dx$

4) $\int 3^{-x}\,dx$

Answer:

$$\int 3^{-x}\,dx \quad = \quad -\frac{3^{-x}}{\ln 3}+C$$

5) $\int \frac{1}{2x}\,dx$

6) $\int \frac{2}{x}\,dx$

Answer:

$$\int \frac{2}{x}\,dx \quad = \quad 2\ln|x|+C \quad = \quad \ln(x^2)+C$$

7) $\int \frac{1}{x^2}\,dx$

8) $\int \frac{1}{\sqrt{x}}\,dx$

Answer:

$$\int \frac{1}{\sqrt{x}}\,dx \quad = \quad 2\sqrt{x}+C$$

In exercises 9 - 16, find each indefinite integral by using appropriate substitutions.

9) $\int \frac{\ln x}{x}\,dx$

10) $\int \frac{dx}{x(\ln x)^2}$

Answer:

$$\int \frac{dx}{x(\ln x)^2} \quad = \quad -\frac{1}{\ln x}+C$$

11) $\int xe^{-x^2}\,dx$

12) $\int x^2e^{-x^3}\,dx$

Answer:

$$\int x^2e^{-x^3}\,dx \quad = \quad \frac{-e^{-x^3}}{3}+C$$

13) $\int e^{\sin x}\cos x\,dx$

14) $\int e^{\tan x}\sec^2 x\,dx$

Answer:

$$\int e^{\tan x}\sec^2 x\,dx \quad = \quad e^{\tan x}+C$$

15) $\int \frac{e^{\ln x}}{x}\,dx$

16) $\int \frac{e^{\ln(1-t)}}{1-t}\,dt$

Answer:

$$\int \frac{e^{\ln(1-t)}}{1-t}\,dt = \int \frac{1-t}{1-t}\,dt = \int 1\,dt \quad = \quad t+C$$

More u-Substitutions with Definite Integrals

In exercises 17 - 21, evaluate the definite integral.

17) $\int_1^2 \frac{1+2x+x^2}{3x+3x^2+x^3}\,dx$

Answer:

$$\int_1^2 \frac{1+2x+x^2}{3x+3x^2+x^3}\,dx \quad = \quad \frac{1}{3}\ln(\frac{26}{7})$$

18) $\int_0^{\pi/4} \tan x\,dx$

19) $\int_0^{\pi/3} \frac{\sin x-\cos x}{\sin x+\cos x}\,dx$

Answer:

$$\int_0^{\pi/3} \frac{\sin x - \cos x}{\sin x + \cos x}\, dx \quad = \quad \ln(\sqrt{3}-1)$$

20) $\int_{\pi/6}^{\pi/2} \csc x\, dx$

21) $\int_{\pi/4}^{\pi/3} \cot x\, dx$

Answer:

$$\int_{\pi/4}^{\pi/3} \cot x\, dx \quad = \quad \frac{1}{2}\ln\frac{3}{2}$$

Some Interesting u-Substitutions

In exercises 22 - 29, integrate using the indicated substitution.

22) $\int \frac{x}{x-100}\, dx; \quad u = x - 100$

23) $\int \frac{y-1}{y+1}\, dy; \quad u = y + 1$

Answer:

$$\int \frac{y-1}{y+1}\, dy \quad = \quad y - 2\ln|y+1| + C$$

24) $\int \frac{1-x^2}{3x-x^3}\, dx; \quad u = 3x - x^3$

25) $\int \frac{\sin x + \cos x}{\sin x - \cos x}\, dx; \quad u = \sin x - \cos x$

Answer:

$$\int \frac{\sin x + \cos x}{\sin x - \cos x}\, dx \quad = \quad \ln|\sin x - \cos x| + C$$

26) $\int e^{2x}\sqrt{1-e^{2x}}\, dx; \quad u = e^{2x}$

27) $\int \ln(x)\frac{\sqrt{1-(\ln x)^2}}{x}\, dx; \quad u = \ln x$

Answer:

$$\int \ln(x)\frac{\sqrt{1-(\ln x)^2}}{x}\, dx \quad = \quad -\frac{1}{3}(1-(\ln x)^2)^{3/2} + C$$

28) $\int \frac{\sqrt{x}}{\sqrt{x}+2}\, dx; \quad u = \sqrt{x} + 2$

Answer:

$$\int \frac{\sqrt{x}}{\sqrt{x}+2}\, dx \quad = \quad (\sqrt{x}+2)^2 - 8(\sqrt{x}+2) + 8\ln(\sqrt{x}+2) + C$$

29) $\int e^x \sec(e^x+1)\tan(e^x+1)\, dx; \quad u = e^x + 1$

Answer:

$$\int e^x \sec(e^x+1)\tan(e^x+1)\, dx \quad = \quad \sec(e^x+1) + C$$

In exercises 29 - 35, use appropriate substitutions to express the trigonometric integrals in terms of compositions with logarithms.

30) $\int \tan(2x)\,dx$

Answer:

$$\int \tan(2x)\,dx \quad = \quad -\frac{1}{2}\ln|\cos(2x)|+C \quad = \quad \frac{1}{2}\ln|\sec(2x)|+C$$

Solution:

$$\int \tan(2x)\,dx = \int \frac{\sin(2x)}{\cos(2x)}\,dx$$

$$= -\frac{1}{2}\int \frac{1}{u}\,du \qquad \text{Letting } u=\cos(2x), \text{ and } du=-2\sin(2x)dx$$

$$= -\frac{1}{2}\ln|u|+C \qquad \text{Integrating in terms of } u$$

$$= -\frac{1}{2}\ln|\cos(2x)|+C \quad = \quad \frac{1}{2}\ln|\sec(2x)|+C \qquad \text{Going back to } x$$

Gives us two equivalent forms of the antiderivative

31) $\int \sec 5x\,dx$

Answer:

$$\int \sec 5x\,dx \quad = \quad \frac{1}{5}\ln|\tan 5x+\sec 5x|+C$$

Solution:

$$\int \sec 5x\,dx = \int \sec 5x\cdot\frac{\sec 5x+\tan 5x}{\sec 5x+\tan 5x}\,dx$$

$$= \frac{1}{5}\int \frac{5(\sec^2 5x+\sec 5x\tan 5x)}{\tan 5x+\sec 5x}\,dx$$

$$= \frac{1}{5}\ln|\tan 5x+\sec 5x|+C$$

32) $\int \frac{x\sin(x^2)}{\cos(x^2)}\,dx$

33) $\int \frac{\sin(3x)-\cos(3x)}{\sin(3x)+\cos(3x)}\,dx$

Answer:

$$\int \frac{\sin(3x)-\cos(3x)}{\sin(3x)+\cos(3x)}\,dx \quad = \quad -\frac{1}{3}\ln|\sin(3x)+\cos(3x)|+C$$

34) $\int x\csc(x^2)\,dx$

Answer:

$$\int x\csc(x^2)\,dx \quad = \quad -\frac{1}{2}\ln|\csc(x^2)+\cot(x^2)|+C$$

35) $\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx$

Answer:

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx \quad = \quad \ln|\, e^x + e^{-x}\,| + C \quad = \quad \ln(e^x + e^{-x}) + C$$

36) $\int \ln(\cos x)\tan x\,dx$

37) $\int \ln(\csc x)\cot x\,dx$

Answer:

$$\int \ln(\csc x)\cot x\,dx \quad = \quad -\frac{1}{2}(\ln(\csc x))^2 + C$$

In exercises 38 - 39, $f(x) \geq 0$ for $a \leq x \leq b$. Find the area under the graph of $f(x)$ between the given values a and b by integrating.

38) $f(x) = 2^{-x}; \quad a = 1, b = 2$

39) $f(x) = 2^{-x}; \quad a = 3, b = 4$

Answer:

$$\frac{1}{\ln(65,536)}$$

40) Find the area under the graph of the function $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

41) Compute the integral of $f(x) = xe^{-x^2}$ and find the smallest value of N such that the area under the graph $f(x) = xe^{-x^2}$ between $x = N$ and $x = N + 10$ is, at most, 0.01.

Answer:

$\int_N^{N+1} xe^{-x^2}\,dx = \frac{1}{2}(e^{-N^2} - e^{-(N+1)^2})$. The quantity is less than $0.01$when $N = 2$.

42) Find the limit, as N tends to infinity, of the area under the graph of $f(x) = xe^{-x^2}$ between $x = 0$ and $x = 5$.

43) Show that $\int_a^b \frac{dt}{t} = \int_{1/b}^{1/a} \frac{dt}{t}$ when $0 < a \leq b$.

Answer:

$$\int_a^b \frac{dx}{x} = \ln(b) - \ln(a) = \ln(\frac{1}{a}) - \ln(\frac{1}{b}) = \int_{1/b}^{1/a} \frac{dx}{x}$$

44) Suppose that $f(x) > 0$ for all x and that f and g are differentiable. Use the identity $f^g = e^{g \ln f}$ and the chain rule to find the derivative of f^g.

45) Use the previous exercise to find the antiderivative of $h(x) = x^x(1 + \ln x)$ and evaluate $\int_2^3 x^x(1 + \ln x)\,dx$.

Answer:

23

46) Show that if $c > 0$, then the integral of $\frac{1}{x}$ from ac to bc (for $0 < a < b$) is the same as the integral of $\frac{1}{x}$ from a to b.

The following exercises are intended to derive the fundamental properties of the natural log starting from the definition $\ln(x) = \int_1^x \frac{dt}{t}$, using properties of the definite integral and making no further assumptions.

47) Use the identity $\ln(x) = \int_1^x \frac{dt}{t}$ to derive the identity $\ln(\frac{1}{x}) = -\ln x$.

Answer:

We may assume that $x > 1$,so $\frac{1}{x} < 1$. Then, $\int_1^{1/x} \frac{dt}{t}$. Now make the substitution $u = \frac{1}{t}$, so $du = -\frac{dt}{t^2}$ and $\frac{du}{u} = -\frac{dt}{t}$, and change endpoints: $\int_1^{1/x} \frac{dt}{t} = -\int_1^{x} \frac{du}{u} = -\ln x$.

48) Use a change of variable in the integral $\int_1^{xy} \frac{1}{t}\,dt$ to show that $\ln xy = \ln x + \ln y$ for $x, y > 0$.

49) Use the identity $\ln x = \int_1^x \frac{dt}{x}$ to show that $\ln(x)$ is an increasing function of x on $[0, \infty)$, and use the previous exercises to show that the range of $\ln(x)$ is $(-\infty, \infty)$. Without any further assumptions, conclude that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$.

50) Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln(x)$, but keep in mind that $\ln(x)$ has an inverse function defined on $(-\infty, \infty)$. Call it E. Use the identity $\ln xy = \ln x + \ln y$ to deduce that $E(a+b) = E(a)E(b)$ for any real numbers a, b.

51) Pretend, for the moment, that we do not know that e^x is the inverse function of $\ln x$, but keep in mind that $\ln x$ has an inverse function defined on $(-\infty, \infty)$. Call it E. Show that $E'(t) = E(t)$.

Answer:

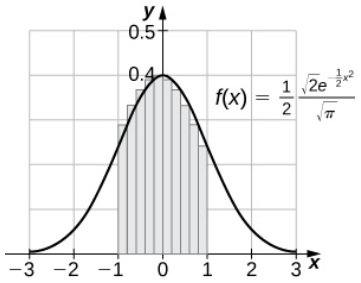
$x = E(\ln(x))$. Then, $1 = \frac{E'(\ln x)}{x}$ or $x = E'(\ln x)$. Since any number t can be written $t = \ln x$ for some x, and for such t we have $x = E(t)$, it follows that for any t, $E'(t) = E(t)$.

52) The sine integral, defined as $S(x) = \int_0^x \frac{\sin t}{t}\,dt$ is an important quantity in engineering. Although it does not have a simple closed formula, it is possible to estimate its behavior for large x. Show that for $k \geq 1$, $|S(2\pi k) - S(2\pi(k+1))| \leq \frac{1}{k(2k+1)\pi}$. (Hint: $\sin(t+\pi) = -\sin t$)

53) [T] The normal distribution in probability is given by $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$, where σ is the standard deviation and μ is the average. The standard normal distribution in probability, p_s, corresponds to $\mu = 0$ and $\sigma = 1$. Compute the left endpoint estimates R_{10} and R_{100} of $\int_{-1}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$.

Answer:

$R_{10} = 0.6811, \quad R_{100} = 0.6827$



54) [T] Compute the right endpoint estimates R_{50} and R_{100} of $\int_{-3}^{5} \frac{1}{2\sqrt{2\pi}} e^{-(x-1)^2/8}$.

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- Paul Seeburger (Monroe Community College) added problems #24 - 28 in 5.5 and #28 - 29, and 31 in 5.6.

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2.9: Integrals Involving Exponential and Logarithmic Functions

Exponential and logarithmic functions are used to model population growth, cell growth, and financial growth, as well as depreciation, radioactive decay, and resource consumption, to name only a few applications. In this section, we explore integration involving exponential and logarithmic functions.

Integrals of Exponential Functions

The exponential function is perhaps the most efficient function in terms of the operations of calculus. The exponential function, $y = e^x$, is its own derivative and its own integral.

Rule: Integrals of Exponential Functions

Exponential functions can be integrated using the following formulas.

$$\int e^x \, dx = e^x + C \tag{2.9.1}$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C \tag{2.9.2}$$

Example 2.9.1: Finding an Antiderivative of an Exponential Function

Find the antiderivative of the exponential function e^{-x}.

Solution

Use substitution, setting $u = -x$, and then $du = -1 \, dx$. Multiply the du equation by -1, so you now have $-du = dx$. Then,

$$\int e^{-x} \, dx = -\int e^u \, du = -e^u + C = -e^{-x} + C.$$

Exercise 2.9.1

Find the antiderivative of the function using substitution: $x^2 e^{-2x^3}$.

Hint

Let u equal the exponent on e.

Answer

$$\int x^2 e^{-2x^3} \, dx = -\frac{1}{6} e^{-2x^3} + C$$

A common mistake when dealing with exponential expressions is treating the exponent on e the same way we treat exponents in polynomial expressions. We cannot use the power rule for the exponent on e. This can be especially confusing when we have both exponentials and polynomials in the same expression, as in the previous checkpoint. In these cases, we should always double-check to make sure we're using the right rules for the functions we're integrating.

Example 2.9.2: Square Root of an Exponential Function

Find the antiderivative of the exponential function $e^x \sqrt{1 + e^x}$.

Solution

First rewrite the problem using a rational exponent:

$$\int e^x \sqrt{1 + e^x} \, dx = \int e^x (1 + e^x)^{1/2} \, dx.$$

Using substitution, choose $u = 1 + e^x$. Then, $du = e^x \, dx$. We have

$$\int e^x (1 + e^x)^{1/2} \, dx = \int u^{1/2} \, du.$$

Then

$$\int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$$

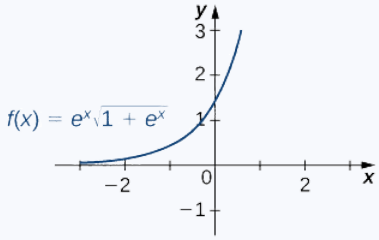


Figure* 2.9.1*: *The graph shows an exponential function times the square root of an exponential function.*

Exercise 2.9.2

Find the antiderivative of $e^x(3e^x-2)^2$.

Hint

Let $u = 3e^x - 2$.

Answer

$$\int e^x(3e^x-2)^2\,dx = \frac{1}{9}(3e^x-2)^3 + C$$

Example 2.9.3: Using Substitution with an Exponential Function

Use substitution to evaluate the indefinite integral $\int 3x^2e^{2x^3}\,dx$.

Solution

Here we choose to let u equal the expression in the exponent on e. Let $u = 2x^3$ and $du = 6x^2\,dx$. Again, du is off by a constant multiplier; the original function contains a factor of $3x^2$, not $6x^2$. Multiply both sides of the equation by $\frac{1}{2}$ so that the integrand in u equals the integrand in x. Thus,

$$\int 3x^2e^{2x^3}\,dx = \frac{1}{2}\int e^u\,du. \tag{2.9.3}$$

Integrate the expression in u and then substitute the original expression in x back into the u-integral:

$$\frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{2x^3} + C. \tag{2.9.4}$$

Exercise 2.9.3

Evaluate the indefinite integral $\int 2x^3e^{x^4}\,dx$.

Hint

Let $u = x^4$.

Answer

$$\int 2x^3e^{x^4}\,dx = \frac{1}{2}e^{x^4} + C$$

As mentioned at the beginning of this section, exponential functions are used in many real-life applications. The number e is often associated with compounded or accelerating growth, as we have seen in earlier sections about the derivative. Although the derivative represents a rate of change or a growth rate, the integral represents the total change or the total growth. Let's look at an example in which integration of an exponential function solves a common business application.

A price–demand function tells us the relationship between the quantity of a product demanded and the price of the product. In general, price decreases as quantity demanded increases. The marginal price–demand function is the derivative of the price–demand function and it tells us how fast the price changes at a given level of production. These functions are used in business to determine the price–elasticity of demand, and to help companies determine whether changing production levels would be profitable.

Example 2.9.4: Finding a Price–Demand Equation

Find the price–demand equation for a particular brand of toothpaste at a supermarket chain when the demand is 50 tubes per week at $2.35 per tube, given that the marginal price—demand function, $p'(x)$, for x number of tubes per week, is given as

$$p'(x) = -0.015e^{-0.01x}. \tag{2.9.5}$$

If the supermarket chain sells 100 tubes per week, what price should it set?

Solution

To find the price–demand equation, integrate the marginal price–demand function. First find the antiderivative, then look at the particulars. Thus,

$$p(x) = \int -0.015e^{-0.01x}\,dx = -0.015\int e^{-0.01x}\,dx. \tag{2.9.6}$$

Using substitution, let $u = -0.01x$ and $du = -0.01\,dx$. Then, divide both sides of the du equation by -0.01. This gives

$$\frac{-0.015}{-0.01}\int e^u\,du = 1.5\int e^u\,du = 1.5e^u + C = 1.5e^{-0.01x} + C. \tag{2.9.7}$$

The next step is to solve for C. We know that when the price is $2.35 per tube, the demand is 50 tubes per week. This means

$$p(50) = 1.5e^{-0.01(50)} + C = 2.35. \tag{2.9.8}$$

Now, just solve for C:

$$C = 2.35 - 1.5e^{-0.5} = 2.35 - 0.91 = 1.44. \tag{2.9.9}$$

Thus,

$$p(x) = 1.5e^{-0.01x} + 1.44. \tag{2.9.10}$$

If the supermarket sells 100 tubes of toothpaste per week, the price would be

$$p(100) = 1.5e^{-0.01(100)} + 1.44 = 1.5e^{-1} + 1.44 \approx 1.99. \tag{2.9.11}$$

The supermarket should charge $1.99 per tube if it is selling 100 tubes per week.

Example 2.9.5: Evaluating a Definite Integral Involving an Exponential Function

Evaluate the definite integral $\int_1^2 e^{1-x}\,dx$.

Solution

Again, substitution is the method to use. Let $u = 1 - x$, so $du = -1\,dx$ or $-du = dx$. Then $\int e^{1-x}\,dx = -\int e^u\,du$.

Next, change the limits of integration. Using the equation $u = 1 - x$, we have:

$$\text{When } x = 1, \quad u = 1 - (1) = 0, \tag{2.9.12}$$

$$\text{and when } x = 2, \quad u = 1 - (2) = -1. \tag{2.9.13}$$

The integral then becomes

$$\int_1^2 e^{1-x}\,dx = -\int_0^{-1} e^u\,du = \int_{-1}^0 e^u\,du = e^u\Big|_{-1}^0 = e^0 - (e^{-1}) = -e^{-1}+1. \tag{2.9.14}$$

See Figure 2.9.2.

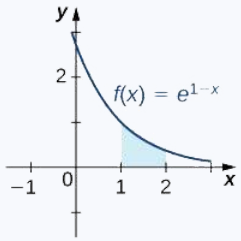


Figure 2.9.2: *The indicated area can be calculated by evaluating a definite integral using substitution.*

Exercise 2.9.4

Evaluate $\int_0^2 e^{2x}\,dx$.

Hint

Let $u = 2x$.

Answer

$\frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}(e^4 - 1)$

Example 2.9.6: Using Substitution with an Exponential Function in a definite integral

Use substitution to evaluate

$$\int_0^1 xe^{4x^2+3}\,dx.$$

Solution

Let $u = 4x^3 + 3$. Then, $du = 8x\,dx$. To adjust the limits of integration, we note that when $x = 0$, $u = 3$, and when $x = 1$, $u = 7$. So our substitution gives

$$\begin{aligned}\int_0^1 xe^{4x^2+3}\,dx &= \frac{1}{8}\int_3^7 e^u\,du \\ &= \frac{1}{8}e^u\Big|_3^7 \\ &= \frac{e^7 - e^3}{8} \\ &\approx 134.568\end{aligned}$$

Example 2.9.7: Growth of Bacteria in a Culture

Suppose the rate of growth of bacteria in a Petri dish is given by $q(t) = 3^t$, where t is given in hours and $q(t)$ is given in thousands of bacteria per hour. If a culture starts with $10,000$ bacteria, find a function $Q(t)$ that gives the number of bacteria in the Petri dish at any time t. How many bacteria are in the dish after 2 hours?

Solution

We have

$$Q(t) = \int 3^t dt = \frac{3^t}{\ln 3} + C. \tag{2.9.15}$$

Then, at $t = 0$ we have $Q(0) = 10 = \frac{1}{\ln 3} + C,$ so $C \approx 9.090$ and we get

$$Q(t) = \frac{3^t}{\ln 3} + 9.090. \tag{2.9.16}$$

At time $t = 2$, we have

$$Q(2) = \frac{3^2}{\ln 3} + 9.090 \tag{2.9.17}$$

$$\approx 17.282. \tag{2.9.18}$$

After 2 hours, there are $17,282$bacteria in the dish.

Exercise 2.9.5

From Example, suppose the bacteria grow at a rate of $q(t) = 2^t$. Assume the culture still starts with $10,000$bacteria. Find $Q(t)$. How many bacteria are in the dish after 3 hours?

Hint

Use the procedure from Example $2.9.7$to solve the problem

Answer

$Q(t) = \frac{2^t}{\ln 2} + 8.557$. $Q(3) \approx 20,099$, so there are $20,099$bacteria in the dish after 3 hours.

Example 2.9.8: Fruit Fly Population Growth

Suppose a population of **fruit flies** increases at a rate of $g(t) = 2e^{0.02t}$, in flies per day. If the initial population of fruit flies is 100 flies, how many flies are in the population after 10 days?

Solution

Let $G(t)$ represent the number of flies in the population at time t. Applying the net change theorem, we have

$$\begin{aligned} G(10) &= G(0) + \textstyle\int_0^{10} 2e^{0.02t}\, dt \\ &= 100 + \left[\frac{2}{0.02} e^{0.02t}\right]\Big|_0^{10} \\ &= 100 + [100e^{0.02t}]\big|_0^{10} \\ &= 100 + 100e^{0.2} - 100 \\ &\approx 122. \end{aligned}$$

There are 122 flies in the population after 10 days.

Exercise 2.9.6

Suppose the rate of growth of the fly population is given by $g(t) = e^{0.01t},$ and the initial fly population is 100 flies. How many flies are in the population after 15 days?

Hint

Use the process from Example $2.9.8$to solve the problem.

Answer

There are 116flies.

Example 2.9.9: Evaluating a Definite Integral Using Substitution

Evaluate the definite integral using substitution:

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx.$$

Solution

This problem requires some rewriting to simplify applying the properties. First, rewrite the exponent on e as a power of x, then bring the x^2 in the denominator up to the numerator using a negative exponent. We have

$$\int_1^2 \frac{e^{1/x}}{x^2}\,dx = \int_1^2 e^{x^{-1}} x^{-2}\,dx.$$

Let $u = x^{-1}$, the exponent on e. Then

$$du = -x^{-2}\,dx$$

$$-du = x^{-2}\,dx.$$

Bringing the negative sign outside the integral sign, the problem now reads

$$-\int e^u\,du.$$

Next, change the limits of integration:

$$u = (1)^{-1} = 1$$

$$u = (2)^{-1} = \frac{1}{2}.$$

Notice that now the limits begin with the larger number, meaning we can multiply by -1 and interchange the limits. Thus,

$$-\int_1^{1/2} e^u\,du = \int_{1/2}^1 e^u\,du = e^u\big|_{1/2}^1 = e - e^{1/2} = e - \sqrt{e}.$$

Exercise 2.9.7

Evaluate the definite integral using substitution:

$$\int_1^2 \frac{1}{x^3} e^{4x^{-2}}\,dx.$$

Hint

Let $u = 4x^{-2}$.

Answer

$$\int_1^2 \frac{1}{x^3} e^{4x^{-2}}\,dx = \frac{1}{8}\left[e^4 - e\right].$$

Integrals Involving Logarithmic Functions

Integrating functions of the form $f(x) = \frac{1}{x}$ or $f(x) = x^{-1}$ result in the absolute value of the natural log function, as shown in the following rule.

Rule: The Basic Integral Resulting in the natural Logarithmic Function

The following formula can be used to evaluate integrals in which the power is -1 and the power rule does not work.

$$\int \frac{1}{x}\,dx = \ln|x| + C \tag{2.9.19}$$

In fact, we can generalize this formula to deal with many rational integrands in which the derivative of the denominator (or its variable part) is present in the numerator. Remember that when we use the chain rule to compute the derivative of $y = \ln[u(x)]$, we obtain:

$$\frac{d}{dx}(\ln[u(x)]) = \frac{1}{u(x)} \cdot u'(x) = \frac{u'(x)}{u(x)} \tag{2.9.20}$$

Rule: General Integrals Resulting in the natural Logarithmic Function

This gives us the more general integration formula,

$$\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C \tag{2.9.21}$$

Example 2.9.10: Finding an Antiderivative Involving $\ln x$

Find the antiderivative of the function

$$\frac{3}{x-10}. \tag{2.9.22}$$

Solution

First factor the 3 outside the integral symbol. Then use the u'/u rule. Thus,

$$\int \frac{3}{x-10}\,dx = 3\int \frac{1}{x-10}\,dx = 3\int \frac{du}{u} = 3\ln|u| + C = 3\ln|x-10| + C, \quad x \neq 10.$$

See Figure 2.9.3.

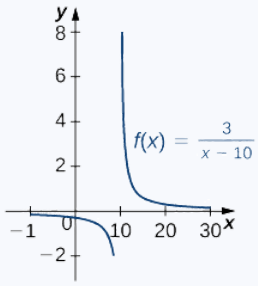


***Figure* 2.9.3:** *The domain of this function is $x \neq 10$.*

Exercise 2.9.8

Find the antiderivative of

$$\frac{1}{x+2}. \tag{2.9.23}$$

Hint

Follow the pattern from Example 2.9.10to solve the problem.

Answer

$$\int \frac{1}{x+2}\,dx = \ln|x+2| + C$$

Example 2.9.11: Finding an Antiderivative of a Rational Function

Find the antiderivative of

$$\frac{2x^3+3x}{x^4+3x^2}.$$

Solution

Use substitution.

Let $u=x^4+3x^2$, then $du=(4x^3+6x)\,dx$. Alter du by factoring out the 2. Thus,

$$du=(4x^3+6x)\,dx=2(2x^3+3x)\,dx$$

$$\frac{1}{2}\,du=(2x^3+3x)\,dx.$$

Rewrite the integrand in u:

$$\int\frac{2x^3+3x}{x^4+3x^2}\,dx=\frac{1}{2}\int\frac{1}{u}\,du.$$

Then we have

$$\frac{1}{2}\int\frac{1}{u}\,du=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln|\,x^4+3x^2\,|+C.$$

Example 2.9.12 is a definite integral of a trigonometric function. With trigonometric functions, we often have to apply a trigonometric property or an identity before we can move forward. Finding the right form of the integrand is usually the key to a smooth integration.

Example 2.9.12: Evaluating a Definite Integral

Find the definite integral of

$$\int_0^{\pi/2}\frac{\sin x}{1+\cos x}\,dx.$$

Solution

We need substitution to evaluate this problem. Let $u=1+\cos x$ so $du=-\sin x\;dx$.

Rewrite the integral in terms of u, changing the limits of integration as well. Thus,

$$u=1+\cos(0)=2 \tag{2.9.24}$$

$$u=1+\cos(\frac{\pi}{2})=1. \tag{2.9.25}$$

Then

$$\int_0^{\pi/2}\frac{\sin x}{1+\cos x}\,dx=-\int_2^1\frac{1}{u}\,du=\int_1^2\frac{1}{u}\,du=\ln|u|\Big|_1^2=[\ln 2-\ln 1]=\ln 2 \tag{2.9.26}$$

Key Concepts

- Exponential and logarithmic functions arise in many real-world applications, especially those involving growth and decay.
- Substitution is often used to evaluate integrals involving exponential functions or logarithms.

Key Equations

- **Integrals of Exponential Functions**

$$\int e^x\,dx=e^x+C$$

$$\int a^x\,dx=\frac{a^x}{\ln a}+C$$

- **Integration Formulas Involving the Natural Logarithmic Function**

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int \frac{u'(x)}{u(x)}\,dx = \ln|u(x)| + C$$

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Edited by Paul Seeburger (Monroe Community College), removing topics requiring integration by parts and adjusting the presentation of integrals resulting in the natural logarithm to a different approach. Also moved Example $2.9.6$ from the previous section where it did not fit as well.

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2.10: Integrals Resulting in Inverse Trigonometric Functions and Related Integration Techniques

In this section we focus on integrals that result in inverse trigonometric functions. We have worked with these functions before. Recall, that trigonometric functions are not one-to-one unless the domains are restricted. When working with inverses of trigonometric functions, we always need to be careful to take these restrictions into account. Also, we previously developed formulas for derivatives of inverse trigonometric functions. The formulas developed there give rise directly to integration formulas involving inverse trigonometric functions.

Integrals that Result in Inverse Trigonometric Functions

Let us begin this last section of the chapter with the three formulas. Along with these formulas, we use substitution to evaluate the integrals. We prove the formula for the inverse sine integral.

Rule: Integration Formulas Resulting in Inverse Trigonometric Functions

The following integration formulas yield inverse trigonometric functions:

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right)+C \tag{2.10.1}$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C \tag{2.10.2}$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C \tag{2.10.3}$$

Proof of the first formula

Let $y = \arcsin\frac{x}{a}$. Then $a\sin y = x$. Now using implicit differentiation, we obtain

$$\frac{d}{dx}(a\sin y) = \frac{d}{dx}(x) \tag{2.10.4}$$

$$a\cos y\frac{dy}{dx} = 1 \tag{2.10.5}$$

$$\frac{dy}{dx} = \frac{1}{a\cos y}. \tag{2.10.6}$$

For $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \cos y \ge 0$. Thus, applying the Pythagorean identity $\sin^2 y + \cos^2 y = 1$, we have $\cos y = \sqrt{1-\sin^2 y}$. This gives

$$\frac{1}{a\cos y} = \frac{1}{a\sqrt{1-\sin^2 y}} \tag{2.10.7}$$

$$= \frac{1}{\sqrt{a^2-a^2\sin^2 y}} \tag{2.10.8}$$

$$= \frac{1}{\sqrt{a^2-x^2}}. \tag{2.10.9}$$

Then for $-a \le x \le a$, we have

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\left(\frac{u}{a}\right)+C. \tag{2.10.10}$$

□

Example 2.10.1: Evaluating a Definite Integral Using Inverse Trigonometric Functions

Evaluate the definite integral

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}.$$

Solution

We can go directly to the formula for the antiderivative in the rule on integration formulas resulting in inverse trigonometric functions, and then evaluate the definite integral. We have

$$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_0^{1/2} = \sin^{-1} \tfrac{1}{2} - \sin^{-1} 0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}.$$

Note that since the integrand is simply the derivative of $\arcsin x$, we are really just using this fact to find the antiderivative here.

Exercise 2.10.1

Find the indefinite integral using an inverse trigonometric function and substitution for $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$.

Hint

Use the formula in the rule on integration formulas resulting in inverse trigonometric functions.

Answer

$$\int \frac{dx}{\sqrt{9-x^2}} \quad = \quad \arcsin\left(\frac{x}{3}\right) + C$$

In many integrals that result in inverse trigonometric functions in the antiderivative, we may need to use substitution to see how to use the integration formulas provided above.

Example 2.10.2: Finding an Antiderivative Involving an Inverse Trigonometric Function using substitution

Evaluate the integral

$$\int \frac{dx}{\sqrt{4-9x^2}}.$$

Solution

Substitute $u = 3x$. Then $du = 3\,dx$ and we have

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}}.$$

Applying the formula with $a = 2$, we obtain

$$\int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int \frac{du}{\sqrt{4-u^2}} = \frac{1}{3}\arcsin\left(\frac{u}{2}\right) + C = \frac{1}{3}\arcsin\left(\frac{3x}{2}\right) + C.$$

Exercise 2.10.2

Find the antiderivative of $\displaystyle\int \frac{dx}{\sqrt{1-16x^2}}$.

Hint

Substitute $u = 4x$.

Answer

$$\int \frac{dx}{\sqrt{1-16x^2}} = \frac{1}{4}\arcsin(4x) + C$$

Example 2.10.3: Evaluating a Definite Integral

Evaluate the definite integral

$$\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}}.$$

Solution

The format of the problem matches the inverse sine formula. Thus,

$$\int_0^{\sqrt{3}/2} \frac{du}{\sqrt{1-u^2}} = \arcsin u \Big|_0^{\sqrt{3}/2} = [\arcsin\left(\frac{\sqrt{3}}{2}\right)] - [\arcsin(0)] = \frac{\pi}{3}.$$

Integrals Resulting in Other Inverse Trigonometric Functions

There are six inverse trigonometric functions. However, only three integration formulas are noted in the rule on integration formulas resulting in inverse trigonometric functions because the remaining three are negative versions of the ones we use. The only difference is whether the integrand is positive or negative. Rather than memorizing three more formulas, if the integrand is negative, simply factor out -1 and evaluate the integral using one of the formulas already provided. To close this section, we examine one more formula: the integral resulting in the inverse tangent function.

Example 2.10.4: Finding an Antiderivative Involving the Inverse Tangent Function

Find the antiderivative of $\int \frac{1}{9+x^2}\,dx$.

Solution

Apply the formula with $a = 3$. Then,

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C.$$

Exercise 2.10.3

Find the antiderivative of $\int \frac{dx}{16+x^2}$.

Hint

Follow the steps in Example 2.10.4

Answer

$$\int \frac{dx}{16+x^2} = \frac{1}{4}\arctan\left(\frac{x}{4}\right) + C$$

Example 2.10.5: Applying the Integration Formulas WITH SUBSTITUTION

Find an antiderivative of $\int \frac{1}{1+4x^2}\,dx$.

Solution

Comparing this problem with the formulas stated in the rule on integration formulas resulting in inverse trigonometric functions, the integrand looks similar to the formula for $\arctan u + C$. So we use substitution, letting $u = 2x$, then $du = 2\,dx$ and $\frac{1}{2}\,du = dx$. Then, we have

$$\frac{1}{2}\int \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan(2x) + C.$$

Exercise 2.10.4

Use substitution to find the antiderivative of $\int \frac{dx}{25+4x^2}$.

Hint

Use the solving strategy from Example 2.10.5 and the rule on integration formulas resulting in inverse trigonometric functions.

Answer

$$\int \frac{dx}{25+4x^2} = \frac{1}{10}\arctan\left(\frac{2x}{5}\right)+C$$

Example 2.10.6: Evaluating a Definite Integral

Evaluate the definite integral $\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2}$.

Solution

Use the formula for the inverse tangent. We have

$$\int_{\sqrt{3}/3}^{\sqrt{3}} \frac{dx}{1+x^2} = \arctan x \Big|_{\sqrt{3}/3}^{\sqrt{3}} = [\arctan(\sqrt{3})] - [\arctan\left(\frac{\sqrt{3}}{3}\right)] = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.$$

Exercise 2.10.5

Evaluate the definite integral $\int_0^2 \frac{dx}{4+x^2}$.

Hint

Follow the procedures from Example 2.10.6to solve the problem.

Answer

$$\int_0^2 \frac{dx}{4+x^2} = \frac{\pi}{8}$$

Simplifying the Integrand using Algebraic Methods

It is common to be reluctant to manipulate the integrand of an integral; at first, our grasp of integration is tenuous and one may think that working with the integrand will improperly change the results. Integration by substitution works using a different logic: as long as *equality* is maintained, the integrand can be manipulated so that its *form* is easier to deal with. The next two examples demonstrate common ways in which using algebra first makes the integration easier to perform.

Example 2.10.7: Integration by substitution: simplifying first using long division

Evaluate $\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\,dx$.

Solution

One may try to start by setting u equal to either the numerator or denominator; in each instance, the result is not workable.

When dealing with rational functions (i.e., quotients made up of polynomial functions), it is an almost universal rule that everything works better when the degree of the numerator is less than the degree of the denominator. Hence we use polynomial long division whenever the integrand is a ratio of two polynomials and the degree of the numerator is not less than the degree of the denominator.

We skip the specifics of the steps, but note that when x^2+2x+1 is divided into x^3+4x^2+8x+5, it goes in $x+2$ times with a remainder of $3x+3$. Thus

$$\frac{x^3+4x^2+8x+5}{x^2+2x+1} = x+2+\frac{3x+3}{x^2+2x+1}. \tag{2.10.11}$$

Integrating $x+2$ is simple. The fraction can be integrated by setting $u=x^2+2x+1$, giving $du=(2x+2)\,dx$. This is very similar to the numerator. Note that $du/2=(x+1)\,dx$ and then consider the following:

$$\begin{aligned}
\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\,dx &= \int\left(x+2+\frac{3x+3}{x^2+2x+1}\right)\,dx && (2.10.12)\\
&= \int (x+2)\,dx + \int \frac{3(x+1)}{x^2+2x+1}\,dx && (2.10.13)\\
&= \frac{1}{2}x^2+2x+C_1+\int \frac{3}{u}\frac{du}{2} && (2.10.14)\\
&= \frac{1}{2}x^2+2x+C_1+\frac{3}{2}\ln|u|+C_2 && (2.10.15)\\
&= \frac{1}{2}x^2+2x+\frac{3}{2}\ln|x^2+2x+1|+C. && (2.10.16)
\end{aligned}$$

In some ways, we "lucked out" in that after dividing, substitution was able to be done. In later sections we'll develop techniques for handling rational functions where substitution is not directly feasible.

Example 2.10.8: Integrals requiring multiple methods

Evaluate $\displaystyle\int \frac{4-x}{\sqrt{16-x^2}}\,dx$.

Solution

This integral requires two different methods to evaluate it. We get to those methods by splitting up the integral:

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = \int \frac{4}{\sqrt{16-x^2}}\,dx - \int \frac{x}{\sqrt{16-x^2}}\,dx. \tag{2.10.17}$$

The first integral is handled using a straightforward application of Theorem 2.10.2; the second integral is handled by substitution, with $u=16-x^2$. We handle each separately.

$\displaystyle\int \frac{4}{\sqrt{16-x^2}}\,dx = 4\arcsin\frac{x}{4}+C.$

$\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$: Set $u=16-x^2$, so $du=-2x\,dx$ and $x\,dx=-du/2$. We have

$$\begin{aligned}
\int \frac{x}{\sqrt{16-x^2}}\,dx &= \int \frac{-du/2}{\sqrt{u}} && (2.10.18)\\
&= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du && (2.10.19)\\
&= -\sqrt{u}+C && (2.10.20)\\
&= -\sqrt{16-x^2}+C. && (2.10.21)
\end{aligned}$$

Combining these together, we have

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = 4\arcsin\frac{x}{4}+\sqrt{16-x^2}+C. \tag{2.10.22}$$

Using Completing the Square in Integration

Sometimes, we will see polynomials in the denominator that are quadratic in form and which we can use the process of completing the square to rewrite them in a form that we will recognize as the derivative of an inverse trigonometric function.

Example 2.10.9: Integrating by substitution: completing the square

Evaluate $\int \frac{1}{x^2-4x+13}\,dx$.

Solution

Initially, this integral seems to have nothing in common with the integrals in Theorem 2.10.2 As it lacks a square root, it almost certainly is not related to arcsine or arcsecant. It is, however, related to the arctangent function.

We see this by *completing the square* in the denominator. We give a brief reminder of the process here.

Start with a quadratic with a leading coefficient of 1. It will have the form of x^2+bx+c. Take 1/2 of b, square it, and add/subtract it back into the expression. I.e.,

$$x^2+bx+c = \underbrace{x^2+bx+\frac{b^2}{4}}_{(x+b/2)^2} - \frac{b^2}{4}+c \tag{2.10.23}$$

$$= \left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4} \tag{2.10.24}$$

In our example, we take half of -4 and square it, getting 4. We add/subtract it into the denominator as follows:

$$\frac{1}{x^2-4x+13} = \frac{1}{\underbrace{x^2-4x+4}_{(x-2)^2}-4+13} \tag{2.10.25}$$

$$= \frac{1}{(x-2)^2+9} \tag{2.10.26}$$

We can now integrate this using the arctangent rule. Technically, we need to substitute first with $u=x-2$, but we can employ Key Idea 10 instead. Thus we have

$$\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x-2)^2+9}\,dx = \frac{1}{3}\arctan\frac{x-2}{3}+C. \tag{2.10.27}$$

Key Concepts

- Formulas for derivatives of inverse trigonometric functions developed in Derivatives of Exponential and Logarithmic Functions lead directly to integration formulas involving inverse trigonometric functions.
- Use the formulas listed in the rule on integration formulas resulting in inverse trigonometric functions to match up the correct format and make alterations as necessary to solve the problem.
- Substitution is often required to put the integrand in the correct form.

Key Equations

- **Integrals That Produce Inverse Trigonometric Functions**

$$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C$$

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- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.

- Paul Seeburger (Monroe Community College) made some added textual clarifications and edits on the OpenStax content in this section, and he also added and edited the last two subsections from Apex Calculus, Section 6.1.
- Apex Calculus Section 6.1 is the source of the material in last two subsections of this section.

2.10: Integrals Resulting in Inverse Trigonometric Functions and Related Integration Techniques is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

2.10E: Exercises for Integrals Resulting in Inverse Trigonometric Functions

In exercises 1 - 6, evaluate each integral in terms of an inverse trigonometric function.

1) $\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}}$

Answer:

$$\int_0^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \arcsin x\Big|_0^{\sqrt{3}/2} = \frac{\pi}{3}$$

2) $\int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}}$

3) $\int_{\sqrt{3}}^{1} \frac{dx}{1+x^2}$

Answer:

$$\int_{\sqrt{3}}^{1} \frac{dx}{1+x^2} = \arctan x\Big|_{\sqrt{3}}^{1} = -\frac{\pi}{12}$$

4) $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2}$

5) $\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}}$

Answer:

$$\int_1^{\sqrt{2}} \frac{dx}{|x|\sqrt{x^2-1}} = \operatorname{arcsec} x\Big|_1^{\sqrt{2}} = \frac{\pi}{4}$$

6) $\int_1^{\frac{2}{\sqrt{3}}} \frac{dx}{|x|\sqrt{x^2-1}}$

In exercises 7 - 12, find each indefinite integral, using appropriate substitutions.

7) $\int \frac{dx}{\sqrt{9-x^2}}$

Answer:

$$\int \frac{dx}{\sqrt{9-x^2}} = \arcsin\left(\frac{x}{3}\right) + C$$

8) $\int \frac{dx}{\sqrt{1-16x^2}}$

Answer:

$$\int \frac{dx}{\sqrt{1-16x^2}} = \frac{1}{4}\int \frac{4\,dx}{\sqrt{1-(4x)^2}} = \frac{1}{4}\arcsin(4x) + C$$

9) $\int \frac{dx}{9+x^2}$

Answer:

$$\int \frac{dx}{9+x^2} = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

10) $\int \frac{dx}{25+16x^2}$

Answer:

$$\int \frac{dx}{25+16x^2} = \int \frac{dx}{5^2+(4x)^2} = \frac{1}{4}\int \frac{4\,dx}{5^2+(4x)^2} = \frac{1}{4}\cdot\frac{1}{5}\arctan\left(\frac{4x}{5}\right)+C \quad = \quad \frac{1}{20}\arctan\left(\frac{4x}{5}\right)+C$$

11) $\int \frac{dx}{x\sqrt{x^2-9}}$

Answer:

$$\int \frac{dx}{x\sqrt{x^2-9}} \quad = \quad \frac{1}{3}\text{arcsec}\left(\frac{|x|}{3}\right)+C$$

12) $\int \frac{dx}{x\sqrt{4x^2-16}}$

Answer:

$$\int \frac{dx}{x\sqrt{4x^2-16}} = \int \frac{2\,dx}{2x\sqrt{(2x)^2-4^2}} = \int \frac{du}{u\sqrt{u^2-4^2}} = \frac{1}{4}\text{arcsec}\left(\frac{|u|}{4}\right)+C \quad = \quad \frac{1}{4}\text{arcsec}\left(\frac{|x|}{2}\right)+C$$

13) Explain the relationship $-\arccos t + C = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C$. Is it true, in general, that $\arccos t = -\arcsin t$?

Answer:

$\cos(\frac{\pi}{2}-\theta) = \sin\theta$. So, $\arcsin t = \frac{\pi}{2} - \arccos t$. They differ by a constant.

14) Explain the relationship $\text{arcsec}\, t + C = \int \frac{dt}{|t|\sqrt{t^2-1}} = -\text{arccsc}\, t + C$. Is it true, in general, that $\text{arcsec}\, t = -\text{arccsc}\, t$?

15) Explain what is wrong with the following integral: $\int_1^2 \frac{dt}{\sqrt{1-t^2}}$.

Answer:

$\sqrt{1-t^2}$ is not defined as a real number when $t > 1$.

16) Explain what is wrong with the following integral: $\int_{-1}^1 \frac{dt}{|t|\sqrt{t^2-1}}$.

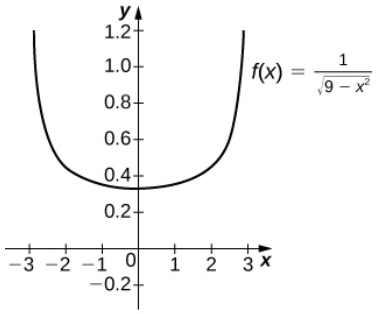
Answer:

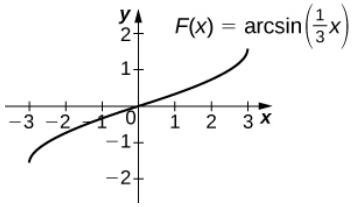
$\sqrt{t^2-1}$ is not defined as a real number when $-1 < t < 1$, and the integrand is undefined when $t = -1$ or $t = 1$.

In exercises 17 - 20, solve for the antiderivative of f with $C = 0$, then use a calculator to graph f and the antiderivative over the given interval $[a, b]$. Identify a value of C such that adding C to the antiderivative recovers the definite integral $F(x) = \int_a^x f(t)\,dt$.

17) [T] $\int \frac{1}{\sqrt{9-x^2}}\,dx$ over $[-3, 3]$

Answer:



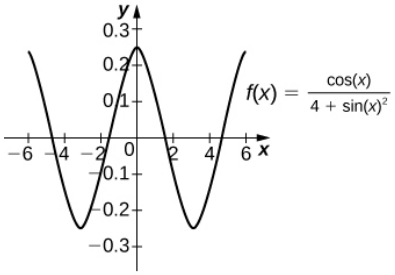


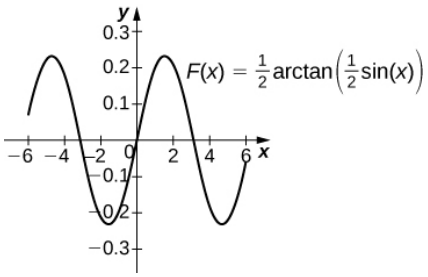
The antiderivative is $\arcsin(\frac{x}{3})+C$. Taking $C=\frac{\pi}{2}$ recovers the definite integral.

18) [T] $\int \frac{9}{9+x^2}\,dx$ over $[-6,6]$

19) [T] $\int \frac{\cos x}{4+\sin^2 x}\,dx$ over $[-6,6]$

Answer:





The antiderivative is $\frac{1}{2}\arctan(\frac{\sin x}{2})+C$. Taking $C=\frac{1}{2}\arctan(\frac{\sin(6)}{2})$ recovers the definite integral.

20) [T] $\int \frac{e^x}{1+e^{2x}}\,dx$ over $[-6,6]$

In exercises 21 - 26, compute the antiderivative using appropriate substitutions.

21) $\int \frac{\arcsin t}{\sqrt{1-t^2}}\,dt$

Answer:

$$\int \frac{\arcsin t\, dt}{\sqrt{1-t^2}} \quad = \quad \tfrac{1}{2}(\arcsin t)^2 + C$$

22) $\displaystyle\int \frac{dt}{\arcsin t\sqrt{1-t^2}}$

Answer:

$$\int \frac{dt}{\arcsin t\sqrt{1-t^2}} \quad = \quad \ln|\arcsin t| + C$$

23) $\displaystyle\int \frac{\arctan(2t)}{1+4t^2}\, dt$

Answer:

$$\int \frac{\arctan(2t)}{1+4t^2}\, dt \quad = \quad \frac{1}{4}(\arctan(2t))^2 + C$$

24) $\displaystyle\int \frac{t\arctan(t^2)}{1+t^4}\, dt$

Answer:

$$\int \frac{t\arctan(t^2)}{1+t^4}\, dt = \frac{1}{2}\int u\, du = \frac{1}{2}\cdot\frac{u^2}{2} + C \quad = \quad \frac{\left(\arctan\left(t^2\right)\right)^2}{4} + C$$

25) $\displaystyle\int \frac{\operatorname{arcsec}\left(\frac{t}{2}\right)}{|t|\sqrt{t^2-4}}\, dt$

Answer:

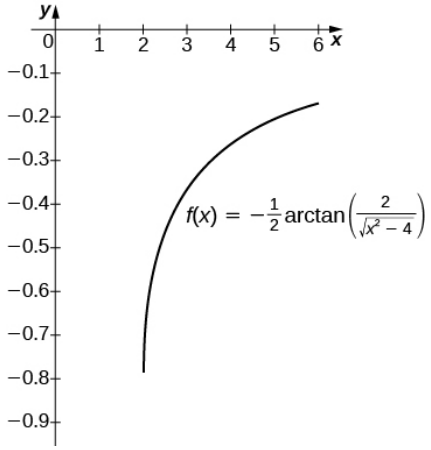
$$\int \frac{\operatorname{arcsec}\left(\frac{t}{2}\right)}{|t|\sqrt{t^2-4}}\, dt \quad = \quad \tfrac{1}{4}\left(\operatorname{arcsec}\left(\tfrac{t}{2}\right)\right)^2 + C$$

26) $\displaystyle\int \frac{t\operatorname{arcsec}(t^2)}{t^2\sqrt{t^4-1}}\, dt$

In exercises 27 - 32, use a calculator to graph the antiderivative of f with $C = 0$ over the given interval $[a, b]$. Approximate a value of C, if possible, such that adding C to the antiderivative gives the same value as the definite integral $F(x) = \displaystyle\int_a^x f(t)\, dt$.

27) [T] $\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\, dx$ over $[2, 6]$

Answer:

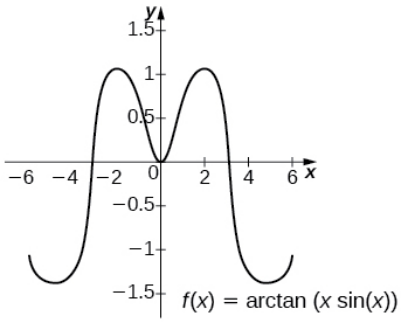


The antiderivative is $\frac{1}{2}\text{arcsec}(\frac{x}{2})+C$. Taking $C=0$ recovers the definite integral over $[2,6]$.

28) [T] $\int \frac{1}{(2x+2)\sqrt{x}}\,dx$ over $[0,6]$

29) [T] $\int \frac{(\sin x + x\cos x)}{1+x^2\sin^2 x\,dx}$ over $[-6,6]$

Answer:

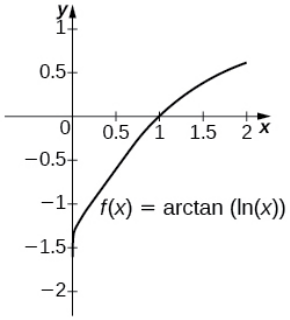


The general antiderivative is $\arctan(x\sin x)+C$. Taking $C=-\arctan(6\sin(6))$ recovers the definite integral.

30) [T] $\int \frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}\,dx$ over $[0,2]$

31) [T] $\int \frac{1}{x+x\ln 2x}$ over $[0,2]$

Answer:



The general antiderivative is $\arctan(\ln x)+C$. Taking $C=\frac{\pi}{2}=\lim_{t\to\infty}\arctan t$ recovers the definite integral.

32) [T] $\int \frac{\arcsin x}{\sqrt{1-x^2}}$ over $[-1,1]$

In exercises 33 - 38, compute each integral using appropriate substitutions.

33) $\int \frac{e^t}{\sqrt{1-e^{2t}}}\,dt$

Answer:

$$\int \frac{e^t}{\sqrt{1-e^{2t}}}\,dt \quad = \quad \arcsin(e^t)+C$$

34) $\int \frac{e^t}{1+e^{2t}}\,dt$

Answer:

$$\int \frac{e^t}{1+e^{2t}}\,dt = \int \frac{e^t}{1+(e^t)^2}\,dt = \int \frac{du}{1+u^2} \quad = \quad \arctan(e^t)+C$$

35) $\displaystyle\int \frac{dt}{t\sqrt{1-\ln^2 t}}$

Answer:

$$\int \frac{dt}{t\sqrt{1-\ln^2 t}} \quad = \quad \arcsin(\ln t)+C$$

36) $\displaystyle\int \frac{dt}{t(1+\ln^2 t)}$

Answer:

$$\int \frac{dt}{t(1+\ln^2 t)} \quad = \quad \arctan(\ln t)+C$$

37) $\displaystyle\int \frac{\arccos(2t)}{\sqrt{1-4t^2}}\,dt$

Answer:

$$\int \frac{\arccos(2t)}{\sqrt{1-4t^2}}\,dt \quad = \quad -\frac{1}{2}(\arccos(2t))^2+C$$

38) $\displaystyle\int \frac{e^t \arccos(e^t)}{\sqrt{1-e^{2t}}}\,dt$

Answer:

$$\int \frac{e^t \arccos(e^t)}{\sqrt{1-e^{2t}}}\,dt \quad = \quad \frac{(\arccos(e^t))^2}{2}+C$$

In exercises 39 - 42, compute each definite integral.

39) $\displaystyle\int_0^{1/2} \frac{\tan(\arcsin t)}{\sqrt{1-t^2}}\,dt$

Answer:

$$\int_0^{1/2} \frac{\tan(\arcsin t)}{\sqrt{1-t^2}}\,dt \quad = \quad \frac{1}{2}\ln\left(\frac{4}{3}\right)$$

40) $\displaystyle\int_{1/4}^{1/2} \frac{\tan(\arccos t)}{\sqrt{1-t^2}}\,dt$

41) $\displaystyle\int_0^{1/2} \frac{\sin(\arctan t)}{1+t^2}\,dt$

Answer:

$$\int_0^{1/2} \frac{\sin(\arctan t)}{1+t^2}\,dt \quad = \quad 1-\frac{2}{\sqrt{5}}$$

42) $\displaystyle\int_0^{1/2} \frac{\cos(\arctan t)}{1+t^2}\,dt$

In exercises 43 - 50, compute each integral using appropriate substitutions and additional techniques.

43) $\displaystyle\int \frac{5}{x^2+10x+34}\,dx$

Answer:

$$\int \frac{5}{x^2+10x+34}\,dx = 5\int \frac{1}{(x^2+10x+25)+9}\,dx = 5\int \frac{1}{(x+5)^2+3^2}\,dx \quad = \quad \frac{5}{3}\arctan\left(\frac{x+5}{3}\right)+C$$

44) $\int \frac{7}{x^2-2x+5}\,dx$

45) $\int \frac{2}{\sqrt{-x^2+8x+3}}\,dx$

Answer:

$$\int \frac{2}{\sqrt{-x^2+8x+3}}\,dx = \int \frac{2}{\sqrt{(\sqrt{19})^2-(x-4)^2}}\,dx \quad = \quad 2\arcsin\left(\frac{x-4}{\sqrt{19}}\right)+C$$

46) $\int \frac{dx}{\sqrt{-x^2-16x}}$

47) $\int \frac{5x}{x^4-16x^2+100}\,dx$

Answer:

$$\int \frac{5x}{x^4-16x^2+100}\,dx = \frac{5}{2}\int \frac{2x}{(x^2-8)^2+36}\,dx \quad = \quad \frac{5}{12}\arctan\left(\frac{x^2-8}{6}\right)+C$$

48) $\int \frac{x}{\sqrt{1-x^4}}\,dx$

49) $\int \frac{8}{x\sqrt{x^4-9}}\,dx$

Answer:

$$\int \frac{8}{x\sqrt{x^4-9}}\,dx = \frac{8}{2}\int \frac{2x}{x^2\sqrt{(x^2)^2-3^2}}\,dx \quad = \quad \frac{4}{3}\operatorname{arcsec}\left(\frac{x^2}{3}\right)+C$$

50) $\int \frac{5}{\sqrt{x^4-16x^2}}\,dx$

Answer:

$$\int \frac{5}{\sqrt{x^4-16x^2}}\,dx = \int \frac{5}{|x|\sqrt{x^2-16}}\,dx \quad = \quad \frac{5}{4}\operatorname{arcsec}\left(\frac{|x|}{4}\right)+C$$

51) $\int \frac{7x^3+5x}{x^4+9}\,dx$

Answer:

$$\int \frac{7x^3+5x}{x^4+9}\,dx = \frac{7}{4}\int \frac{4x^3}{x^4+9}\,dx + \frac{5}{2}\int \frac{2x}{(x^2)^2+3^2}\,dx \quad = \quad \frac{7}{4}\ln|x^4+9| + \frac{5}{6}\arctan\left(\frac{x^2}{3}\right)+C$$

52) $\int \frac{2x+5}{x^2+49}\,dx$

53) $\int \frac{2x-1}{\sqrt{36-25x^2}}\,dx$

Answer:

$$\int \frac{2x-1}{\sqrt{36-25x^2}}\,dx = \frac{2}{-50}\int \frac{-50x}{\sqrt{36-25x^2}}\,dx - \frac{1}{5}\int \frac{5}{\sqrt{6^2-(5x)^2}}\,dx \quad = \quad -\frac{2}{25}\sqrt{36-25x^2} - \frac{1}{5}\arcsin\left(\frac{5x}{6}\right)+C$$

54) $\int \frac{3x^3+4x^2+2x-22}{x^2+3x+5}\,dx$

55) $\int \frac{-x^3+14x^2-46x-7}{x^2-7x+1}\,dx$

Answer:

Use long division first, since the degree of the numerator is not less than the degree of the denominator.

$$\int \frac{-x^3+14x^2-46x-7}{x^2-7x+1}\,dx = \int \left(-x+7+\frac{4x-14}{x^2-7x+1}\right)\,dx \quad = \quad -\frac{x^2}{2}+7x+2\ln|x^2-7x+1|+C$$

56) $\int \frac{x^3-x}{x^2+4x+9}\,dx$

57) $\int \frac{x^2+5x-2}{x^2-10x+32}\,dx$

Answer:

Use long division first, since the degree of the numerator is not less than the degree of the denominator.

$$\int \frac{x^2+5x-2}{x^2-10x+32}\,dx = \int \left(1+\frac{15x-34}{x^2-10x+32}\right)\,dx = \int dx+\frac{15}{2}\int \frac{2x-10}{x^2-10x+32}\,dx+41\int \frac{1}{(x-5)^2+(\sqrt{7})^2}\,dx \quad =$$

$$x+\frac{15}{2}\ln|x^2-10x+32|+\frac{41}{\sqrt{7}}\arctan\left(\frac{x-5}{\sqrt{7}}\right)+C$$

58) $\int \frac{x^3}{x^2+9}\,dx$

Answer:

Use long division first, since the degree of the numerator is not less than the degree of the denominator. Fill in missing terms with zero-terms.

$$\int \frac{x^3}{x^2+9}\,dx \quad = \quad \frac{x^2}{2}-\frac{9}{2}\ln(x^2+9)+C$$

59) $\int \frac{\cos(x)}{\sin^2(x)+1}\,dx$

Answer:

$$\int \frac{\cos(x)}{\sin^2(x)+1}\,dx \quad = \quad \arctan(\sin x)+C$$

60) $\int \frac{\cos(x)}{1-\sin^2(x)}\,dx$

Answer:

$$\int \frac{\cos(x)}{1-\sin^2(x)}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx$$

$$= \int \sec x\,dx = \int \sec x\cdot\frac{\sec x+\tan x}{\sec x+\tan x}\,dx$$

$$= \int \frac{(\sec^2 x+\sec x\tan x)}{\tan x+\sec x}\,dx$$

$$= \ln|\tan x+\sec x|+C$$

61) $\int \frac{3x-3}{\sqrt{x^2-2x-6}}\,dx$

Answer:

$$\int \frac{3x-3}{\sqrt{x^2-2x-6}}\,dx \quad = \quad 3\sqrt{x^2-2x-6}+C$$

62) $\int \frac{x-3}{\sqrt{x^2-6x+8}}\,dx$

63) For $A>0$, compute $I(A)=\int_{-A}^{A}\frac{dt}{1+t^2}$ and evaluate $\lim_{a\to\infty} I(A)$, the area under the graph of $\frac{1}{1+t^2}$ on $[-\infty,\infty]$.

Answer:

$2\arctan(A)\to\pi$ as $A\to\infty$

64) For $1<B<\infty$, compute $I(B)=\int_1^B \frac{dt}{t\sqrt{t^2-1}}$ and evaluate $\lim_{B\to\infty} I(B)$, the area under the graph of $\frac{1}{t\sqrt{t^2-1}}$ over $[1,\infty)$.

65) Use the substitution $u=\sqrt{2}\cot x$ and the identity $1+\cot^2 x=\csc^2 x$ to evaluate $\int \frac{dx}{1+\cos^2 x}$. (Hint: Multiply the top and bottom of the integrand by $\csc^2 x$.)

Answer:

Using the hint, one has $\int \frac{\csc^2 x}{\csc^2 x+\cot^2 x}\,dx=\int \frac{\csc^2 x}{1+2\cot^2 x}\,dx$. Set $u=\sqrt{2}\cot x$. Then, $du=-\sqrt{2}\csc^2 x$ and the integral is $-\frac{1}{\sqrt{2}}\int \frac{du}{1+u^2}=-\frac{\sqrt{2}}{2}\arctan u+C=\frac{\sqrt{2}}{2}\arctan(\sqrt{2}\cot x)+C$. If one uses the identity $\arctan s+\arctan(\frac{1}{s})=\frac{\pi}{2}$, then this can also be written $\frac{\sqrt{2}}{2}\arctan(\frac{\tan x}{\sqrt{2}})+C$.

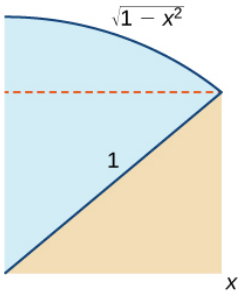
66) [T] Approximate the points at which the graphs of $f(x)=2x^2-1$ and $g(x)=(1+4x^2)^{-3/2}$ intersect, and approximate the area between their graphs accurate to three decimal places.

67) [T] Approximate the points at which the graphs of $f(x)=x^2-1$ and $f(x)=x^2-1$ intersect, and approximate the area between their graphs accurate to three decimal places.

Answer:

$x\approx\pm 1.13525$. The left endpoint estimate with $N=100$ is 2.796 and these decimals persist for $N=500$.

68) Use the following graph to prove that $\int_0^x \sqrt{1-t^2}\,dt=\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\arcsin x$.



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- Paul Seeburger (Monroe Community College) edited this set to use alternate notation for all inverse trig functions and to add solutions for many even problems and to add new problems 43 - 53, except 48 and 50.
- Problems 48, 50, & 54 - 62 are from Apex Calculus.

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2.11: Chapter 5 Review Exercises

In exercises 1 - 4, answer *True or False.* Justify your answer with a proof or a counterexample. Assume all functions f and g are continuous over their domains.

1) If $f(x) > 0,\ f'(x) > 0$ for all x, then the right-hand rule underestimates the integral $\int_a^b f(x)\,dx$. Use a graph to justify your answer.

2) $\int_a^b f(x)^2\,dx = \int_a^b f(x)\,dx$

3) If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

4) All continuous functions have an antiderivative.

In exercises 5 - 8, evaluate the Riemann sums L_4 and R_4 for the given functions over the specified interval. Compare your answer with the exact answer, when possible, or use a calculator to determine the answer.

5) $y = 3x^2 - 2x + 1)$ over $[-1,1]$

6) $y = \ln(x^2+1)$ over $[0,e]$

7) $y = x^2 \sin x$ over $[0,\pi]$

8) $y = \sqrt{x} + \frac{1}{x}$ over $[1,4]$

In exercises 9 - 12, evaluate the integrals.

9) $\int_{-1}^{1} (x^3 - 2x^2 + 4x)\,dx$

10) $\int_0^4 \dfrac{3t}{\sqrt{1+6t^2}}\,dt$

11) $\int_{\pi/3}^{\pi/2} 2\sec(2\theta)\tan(2\theta)\,d\theta$

12) $\int_0^{\pi/4} e^{\cos^2 x} \sin x \cos x\,dx$

In exercises 13 - 16, find the antiderivative.

13) $\int \dfrac{dx}{(x+4)^3}$

14) $\int x \ln(x^2)\,dx$

15) $\int \dfrac{4x^2}{\sqrt{1-x^6}}\,dx$

16) $\int \dfrac{e^{2x}}{1+e^{4x}}\,dx$

In exercises 17 - 20, find the derivative.

17) $\dfrac{d}{dt}\int_0^t \dfrac{\sin x}{\sqrt{1+x^2}}\,dx$

18) $\dfrac{d}{dx}\int_1^{x^3} \sqrt{4-t^2}\,dt$

19) $\dfrac{d}{dx}\int_1^{\ln(x)} (4t + e^t)\,dt$

20) $\frac{d}{dx}\int_0^{\cos x} e^{t^2}\,dt$

In exercises 21 - 23, consider the historic average cost per gigabyte of RAM on a computer.

Year	5-Year Change ($)
1980	0
1985	$-5,468,750$
1990	$-755,495$
1995	$-73,005$
2000	$-29,768$
2005	-918
2010	-177

21) If the average cost per gigabyte of RAM in 2010 is $\$12$, find the average cost per gigabyte of RAM in 1980.

Solution: $6,328,113

22) The average cost per gigabyte of RAM can be approximated by the function $C(t) = 8,500,000(0.65)^t$, where t is measured in years since 1980, and C is cost in US dollars. Find the average cost per gigabyte of RAM for the period from 1980 to 2010.

23) Find the average cost of 1 GB RAM from 2005 to 2010.

24) The velocity of a bullet from a rifle can be approximated by $v(t) = 6400t^2 - 6505t + 2686$, where t is seconds after the shot and v is the velocity measured in feet per second. This equation only models the velocity for the first half-second after the shot: $0 \le t \le 0.5$. What is the total distance the bullet travels in 0.5 sec?

25) What is the average velocity of the bullet for the first half-second?

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CHAPTER OVERVIEW

3: Applications of Integration

In this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.

3.1: Prelude to Applications of Integration
3.2: Areas between Curves
3.2E: Exercises for Section 6.1
3.3: Determining Volumes by Slicing
3.3E: Exercises for Volumes of Common Cross-Section and Disk/Washer Method
3.4: Volumes of Revolution- The Shell Method
3.4b: Volumes of Revolution- Cylindrical Shells OS
3.4E: Exercises for the Shell Method
3.5: Arc Length and Surface Area
3.6: Arc Length of a Curve and Surface Area
3.7: Exercises for Section 6.4
3.8: Using Integration to Determine Work
3.9: More Physical Applications of Integration
3.10: Exercises on Work
3.11: Moments and Centers of Mass
3.11E: Exercises for Section 6.6
3.12: Integrals, Exponential Functions, and Logarithms
3.12E: Exercises for Section 6.7
3.13: Exponential Growth and Decay
3.13E: Exercises for Section 6.8
3.14: Calculus of the Hyperbolic Functions
3.14E: Exercises for Section 6.9
3.15: Chapter 6 Review Exercises

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3.1: Prelude to Applications of Integration

The Hoover Dam is an engineering marvel. When Lake Mead, the reservoir behind the dam, is full, the dam withstands a great deal of force. However, water levels in the lake vary considerably as a result of droughts and varying water demands. Later in this chapter, we use definite integrals to calculate the force exerted on the dam when the reservoir is full and we examine how changing water levels affect that force. Hydrostatic force is only one of the many applications of definite integrals we explore in this chapter. From geometric applications such as surface area and volume, to physical applications such as mass and work, to growth and decay models, definite integrals are a powerful tool to help us understand and model the world around us.



Figure 3.1.1: Hoover Dam is one of the United States' iconic landmarks, and provides irrigation and hydroelectric power for millions of people in the southwest United States. (credit: modification of work by Lynn Betts, Wikimedia).

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3.2: Areas between Curves

Learning Objectives

- Determine the area of a region between two curves by integrating with respect to the independent variable.
- Find the area of a compound region.
- Determine the area of a region between two curves by integrating with respect to the dependent variable.

In Introduction to Integration, we developed the concept of the definite integral to calculate the area below a curve on a given interval. In this section, we expand that idea to calculate the area of more complex regions. We start by finding the area between two curves that are functions of x, beginning with the simple case in which one function value is always greater than the other. We then look at cases when the graphs of the functions cross. Last, we consider how to calculate the area between two curves that are functions of y.

Area of a Region between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$ such that $f(x) \geq g(x)$ on $[a, b]$. We want to find the area between the graphs of the functions, as shown in Figure 3.2.1.

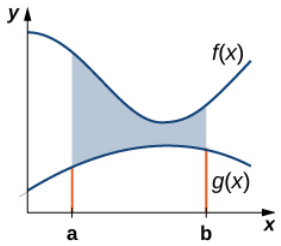


Figure 3.2.1: The area between the graphs of two functions, $f(x)$ and $g(x)$, on the interval $[a, b]$

As we did before, we are going to partition the interval on the **x-axis** and approximate the area between the graphs of the functions with rectangles. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$, and on each interval $[x_{i-1}, x_i]$ construct a rectangle that extends vertically from $g(x_i^*)$ to $f(x_i^*)$. Figure 3.2.2a shows the rectangles when x_i^* is selected to be the left endpoint of the interval and $n = 10$. Figure 3.2.2b shows a representative rectangle in detail.

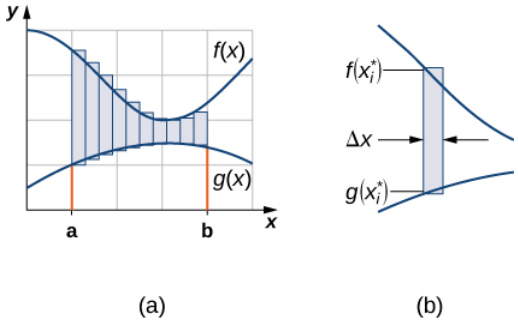


Figure 3.2.2: (a)We can approximate the area between the graphs of two functions, $f(x)$ and $g(x)$, with rectangles. (b) The area of a typical rectangle goes from one curve to the other.

The height of each individual rectangle is $f(x_i^*) - g(x_i^*)$ and the width of each rectangle is Δx. Adding the areas of all the rectangles, we see that the area between the curves is approximated by

$$A \approx \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x.$$

This is a Riemann sum, so we take the limit as $n \to \infty$ and we get

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)] \Delta x = \int_a^b [f(x) - g(x)] dx.$$

These findings are summarized in the following theorem.

Finding the Area between Two Curves

Let $f(x)$ and $g(x)$ be continuous functions such that $f(x) \geq g(x)$ over an interval $[a, b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_a^b [f(x) - g(x)]dx.$$

We apply this theorem in the following example.

Example 3.2.1: Finding the Area of a Region between Two Curves I

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x + 4$ and below by the graph of the function $g(x) = 3 - \frac{x}{2}$ over the interval $[1, 4]$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

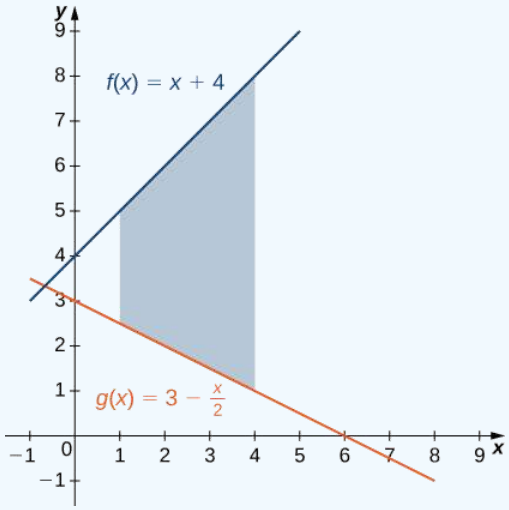


Figure 3.2.3: A region between two curves is shown where one curve is always greater than the other.

We have

$$A = \int_a^b [f(x) - g(x)]\, dx$$

$$= \int_1^4 [(x+4) - (3 - \frac{x}{2})]\, dx = \int_1^4 \left[\frac{3x}{2} + 1\right]\, dx$$

$$= [\frac{3x^2}{4} + x]\Big|_1^4 = (16 - \frac{7}{4}) = \frac{57}{4}.$$

The area of the region is $\frac{57}{4} units^2$.

Exercise 3.2.1

If $\mathbf{R}$ is the region bounded by the graphs of the functions $f(x) = \frac{x}{2} + 5$ and $g(x) = x + \frac{1}{2}$ over the interval $[1, 5]$, find the area of region $\mathbf{R}$.

Hint

Graph the functions to determine which function's graph forms the upper bound and which forms the lower bound, then follow the process used in Example.

Answer

12 units^2

In Example $3.2.1$, we defined the interval of interest as part of the problem statement. Quite often, though, we want to define our interval of interest based on where the graphs of the two functions intersect. This is illustrated in the following example.

Example 3.2.2: Finding the Area of a Region between Two Curves II

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = 9 - (x/2)^2$ and below by the graph of the function $g(x) = 6 - x$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

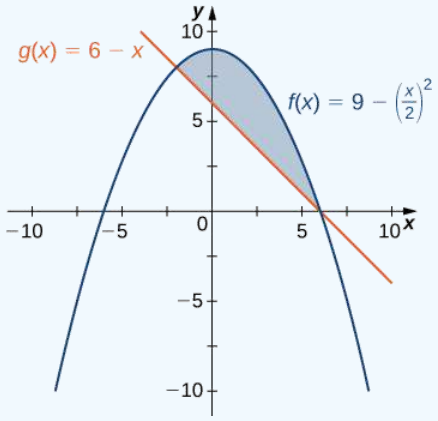


Figure 3.2.4: This graph shows the region below the graph of $f(x)$ and above the graph of $g(x)$.

We first need to compute where the graphs of the functions intersect. Setting $f(x) = g(x)$, we get

$$f(x) = g(x)$$
$$9 - (\frac{x}{2})^2 = 6 - x$$
$$9 - \frac{x^2}{4} = 6 - x$$
$$36 - x^2 = 24 - 4x$$
$$x^2 - 4x - 12 = 0$$
$$(x - 6)(x + 2) = 0.$$

The graphs of the functions intersect when $x = 6$ or $x = -2$, so we want to integrate from -2 to 6. Since $f(x) \geq g(x)$ for $-2 \leq x \leq 6$, we obtain

$$A = \int_a^b [f(x) - g(x)]\, dx$$
$$= \int_{-2}^{6} \left[9 - (\frac{x}{2})^2 - (6 - x)\right]\, dx$$
$$= \int_{-2}^{6} \left[3 - \frac{x^2}{4} + x\right]\, dx$$
$$= \left[3x - \frac{x^3}{12} + \frac{x^2}{2}\right]\Big|_{-2}^{6} = \frac{64}{3}.$$

The area of the region is $64/3$ units^2.

Exercise 3.2.2

If $\mathbf{R}$ is the region bounded above by the graph of the function $f(x) = x$ and below by the graph of the function $g(x) = x^4$, find the area of region $\mathbf{R}$.

Hint

Use the process from Example 3.2.2

Answer

$\frac{3}{10}$ unit^2

Areas of Compound Regions

So far, we have required $f(x) \geq g(x)$ over the entire interval of interest, but what if we want to look at regions bounded by the graphs of functions that cross one another? In that case, we modify the process we just developed by using the absolute value function.

Finding the Area of a Region between Curves That Cross

Let $f(x)$ and $g(x)$ be continuous functions over an interval $[a, b]$. Let $\mathbf{R}$ denote the region between the graphs of $f(x)$ and $g(x)$, and be bounded on the left and right by the lines $x = a$ and $x = b$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_a^b |f(x) - g(x)| dx.$$

In practice, applying this theorem requires us to break up the interval $[a, b]$ and evaluate several integrals, depending on which of the function values is greater over a given part of the interval. We study this process in the following example.

Example 3.2.3: Finding the Area of a Region Bounded by Functions That Cross

If $\mathbf{R}$ is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[0, \pi]$, find the area of region $\mathbf{R}$.

Solution

The region is depicted in the following figure.

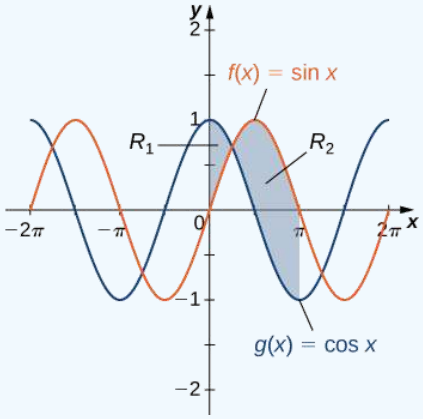


Figure 3.2.5: The region between two curves can be broken into two sub-regions.

The graphs of the functions intersect at $x = \pi/4$. For $x \in [0, \pi/4]$, $\cos x \geq \sin x$, so

$$|f(x) - g(x)| = |\sin x - \cos x| = \cos x - \sin x.$$

On the other hand, for $x \in [\pi/4, \pi]$, $\sin x \geq \cos x$,so

$$|f(x) - g(x)| = |\sin x - \cos x| = \sin x - \cos x.$$

Then

$$A = \int_a^b |f(x) - g(x)|dx$$

$$= \int_0^\pi |\sin x - \cos x|dx = \int_0^{\pi/4} (\cos x - \sin x)dx + \int_{\pi/4}^\pi (\sin x - \cos x)dx$$

$$= [\sin x + \cos x]|_0^{\pi/4} + [-\cos x - \sin x]|_{\pi/4}^\pi$$

$$= (\sqrt{2} - 1) + (1 + \sqrt{2}) = 2\sqrt{2}.$$

The area of the region is $2\sqrt{2}$ units^2.

Exercise 3.2.3

If $\mathbf{R}$ is the region between the graphs of the functions $f(x) = \sin x$ and $g(x) = \cos x$ over the interval $[\pi/2, 2\pi]$ find the area of region $\mathbf{R}$.

Hint

The two curves intersect at $x = (5\pi)/4$.

Answer

$2 + 2\sqrt{2}$ units^2

Example 3.2.4: Finding the Area of a Complex Region

Consider the region depicted in Figure 3.2.6. Find the area of $\mathbf{R}$.

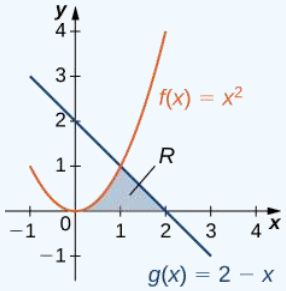


Figure 3.2.6: Two integrals are required to calculate the area of this region.

Solution

As with Example 3.2.3, we need to divide the interval into two pieces. The graphs of the functions intersect at $x = 1$ (set $f(x) = g(x)$ and solve for x), so we evaluate two separate integrals: one over the interval $[0, 1]$ and one over the interval $[1, 2]$.

Over the interval $[0, 1]$, the region is bounded above by $f(x) = x^2$ and below by the x-axis, so we have

$$A_1 = \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}.$$

Over the interval $[1, 2]$, the region is bounded above by $g(x) = 2 - x$ and below by the **x-axis**, so we have

$$A_2 = \int_1^2 (2 - x)dx = [2x - \frac{x^2}{2}] \Big|_1^2 = \frac{1}{2}.$$

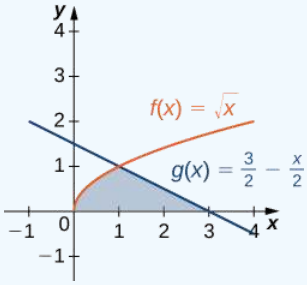
Adding these areas together, we obtain

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The area of the region is $5/6$ units^2.

Exercise 3.2.4

Consider the region depicted in the following figure. Find the area of **R**.



Hint

The two curves intersect at **x=1**

Answer

$\frac{5}{3}$ units^2

Regions Defined with Respect to y

In Example 3.2.4, we had to evaluate two separate integrals to calculate the area of the region. However, there is another approach that requires only one integral. What if we treat the curves as functions of y, instead of as functions of x? Review Figure. Note that the left graph, shown in red, is represented by the function $y = f(x) = x^2$. We could just as easily solve this for x and represent the curve by the function $x = v(y) = \sqrt{y}$. (Note that $x = -\sqrt{y}$ is also a valid representation of the function $y = f(x) = x^2$ as a

function of y. However, based on the graph, it is clear we are interested in the positive square root.) Similarly, the right graph is represented by the function $y = g(x) = 2 - x$, but could just as easily be represented by the function $x = u(y) = 2 - y$. When the graphs are represented as functions of y, we see the region is bounded on the left by the graph of one function and on the right by the graph of the other function. Therefore, if we integrate with respect to y, we need to evaluate one integral only. Let's develop a formula for this type of integration.

Let $u(y)$ and $v(y)$ be continuous functions over an interval $[c, d]$ such that $u(y) \geq v(y)$ for all $y \in [c, d]$. We want to find the area between the graphs of the functions, as shown in Figure 3.2.7.

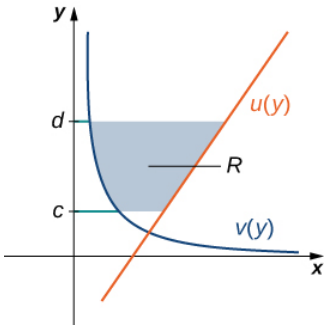


Figure 3.2.7: We can find the area between the graphs of two functions, $u(y)$ and $v(y)$.

This time, we are going to partition the interval on the **y-axis** and use horizontal rectangles to approximate the area between the functions. So, for $i = 0, 1, 2, \ldots, n$, let $Q = y_i$ be a regular partition of $[c, d]$. Then, for $i = 1, 2, \ldots, n$, choose a point $y_i^* \in [y_{i-1}, y_i]$, then over each interval $[y_{i-1}, y_i]$ construct a rectangle that extends horizontally from $v(y^0 *_i)$ to $u(y_i^*)$. Figure 3.2.8$a$shows the rectangles when y_i^* is selected to be the lower endpoint of the interval and $n = 10$. Figure 3.2.8b shows a representative rectangle in detail.

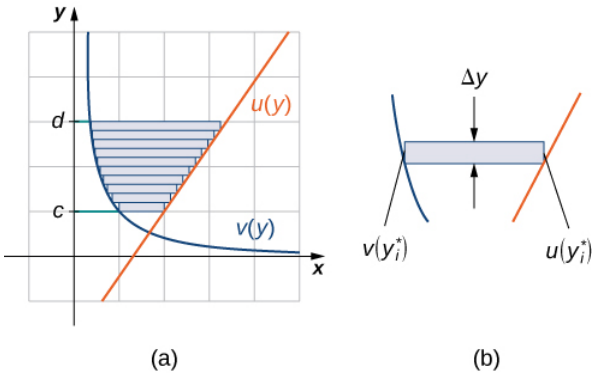


Figure 3.2.8: (a) Approximating the area between the graphs of two functions, $u(y)$ and $v(y)$, with rectangles. (b) The area of a typical rectangle.

The height of each individual rectangle is Δy and the width of each rectangle is $u(y_i^*) - v(y_i^*)$. Therefore, the area between the curves is approximately

$$A \approx \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y.$$

This is a Riemann sum, so we take the limit as $n \to \infty$, obtaining

$$\begin{aligned} A &= \lim_{n \to \infty} \sum_{i=1}^{n} [u(y_i^*) - v(y_i^*)] \Delta y \\ &= \int_c^d [u(y) - v(y)] dy. \end{aligned}$$

These findings are summarized in the following theorem.

Finding the Area between Two Curves, Integrating along the y-axis

Let $u(y)$ and $v(y)$ be continuous functions such that $u(y) \geq v(y)$ for all $y \in [c, d]$. Let $\mathbf{R}$ denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, and above and below by the lines $y = d$ and $y = c$, respectively. Then, the area of $\mathbf{R}$ is given by

$$A = \int_c^d [u(y) - v(y)]dy.$$

Example 3.2.5: Integrating with Respect to y

Let's revisit Example 3.2.4, only this time let's integrate with respect to y. Let $\mathbf{R}$ be the region depicted in Figure 3.2.9. Find the area of $\mathbf{R}$ by integrating with respect to y.

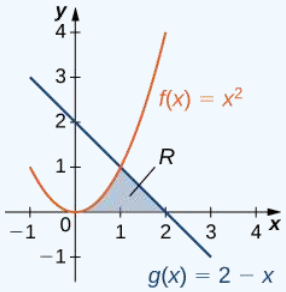


Figure 3.2.9: The area of region $\mathbf{R}$ can be calculated using one integral only when the curves are treated as functions of y.

Solution

We must first express the graphs as functions of y. As we saw at the beginning of this section, the curve on the left can be represented by the function $x = v(y) = \sqrt{y}$, and the curve on the right can be represented by the function $x = u(y) = 2 - y$.

Now we have to determine the limits of integration. The region is bounded below by the x-axis, so the lower limit of integration is $y = 0$. The upper limit of integration is determined by the point where the two graphs intersect, which is the point $(1, 1)$, so the upper limit of integration is $y = 1$. Thus, we have $[c, d] = [0, 1]$.

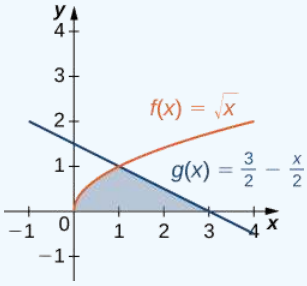
Calculating the area of the region, we get

$$\begin{aligned} A &= \int_c^d [u(y) - v(y)]dy \\ &= \int_0^1 [(2 - y) - \sqrt{y}]dy \\ &= [2y - \frac{y^2}{2} - \frac{2}{3}y^{3/2}]\Big|_0^1 \\ &= \frac{5}{6}. \end{aligned}$$

The area of the region is $5/6$ units^2.

Exercise 3.2.5

Let's revisit the checkpoint associated with Example 3.2.4, only this time, let's integrate with respect to y. Let $\mathbf{R}$ be the region depicted in the following figure. Find the area of $\mathbf{R}$ by integrating with respect to y.



Hint

Follow the process from the previous example.

Answer

$\frac{5}{3}$ units^2

Key Concepts

- Just as definite integrals can be used to find the area under a curve, they can also be used to find the area between two curves.
- To find the area between two curves defined by functions, integrate the difference of the functions.
- If the graphs of the functions cross, or if the region is complex, use the absolute value of the difference of the functions. In this case, it may be necessary to evaluate two or more integrals and add the results to find the area of the region.
- Sometimes it can be easier to integrate with respect to y to find the area. The principles are the same regardless of which variable is used as the variable of integration.

Key Equations

- **Area between two curves, integrating on the x-axis**

$$A = \int_a^b [f(x) - g(x)]dx$$

- **Area between two curves, integrating on the y-axis**

$$A = \int_c^d [u(y) - v(y)]dy$$

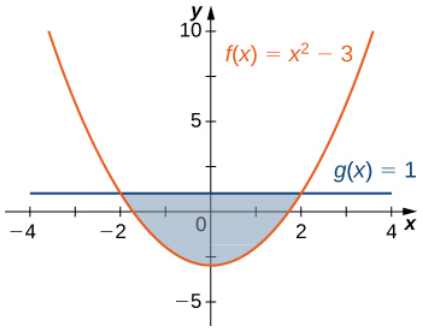
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3.2E: Exercises for Section 6.1

For exercises 1 - 2, determine the area of the region between the two curves in the given figure by integrating over the x-axis.

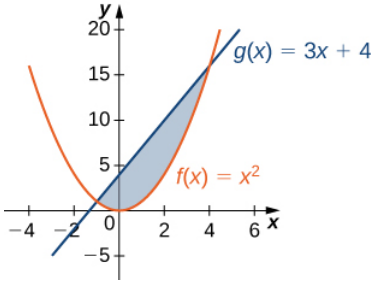
1) $y = x^2 - 3$ and $y = 1$



Answer:

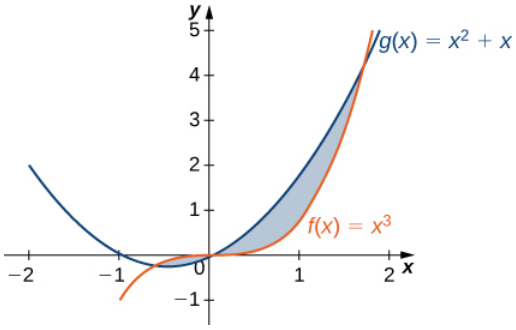
$\frac{32}{3}$ units^2

2) $y = x^2$ and $y = 3x + 4$



For exercises 3 - 4, split the region between the two curves into two smaller regions, then determine the area by integrating over the x-axis. Note that you will have two integrals to solve.

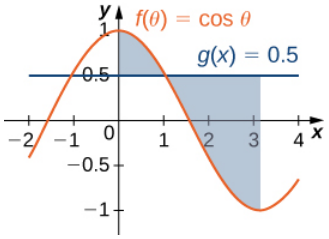
3) $y = x^3$ and $y = x^2 + x$



Answer:

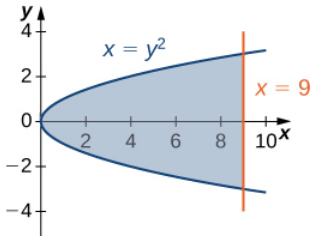
$\frac{13}{12}$ units^2

4) $y = \cos\theta$ and $y = 0.5$, for $0 \le \theta \le \pi$



For exercises 5-6, determine the area of the region between the two curves by integrating over the y-axis.

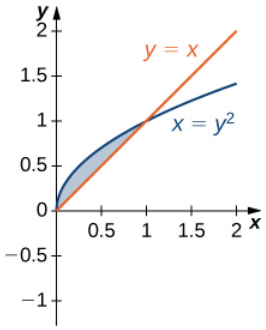
5) $x = y^2$ and $x = 9$



Answer:

36 units^2

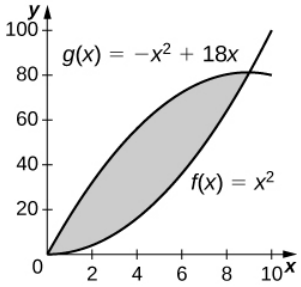
6) $y = x$ and $x = y^2$



For exercises 7 - 13, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis.

7) $y = x^2$ and $y = -x^2 + 18x$

Answer:

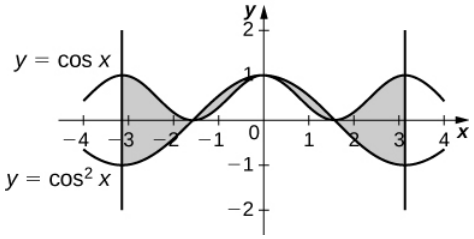


243 square units

8) $y = \dfrac{1}{x}$, $y = \dfrac{1}{x^2}$, and $x = 3$

9) $y = \cos x$ and $y = \cos^2 x$ on $x \in [-\pi, \pi]$

Answer:

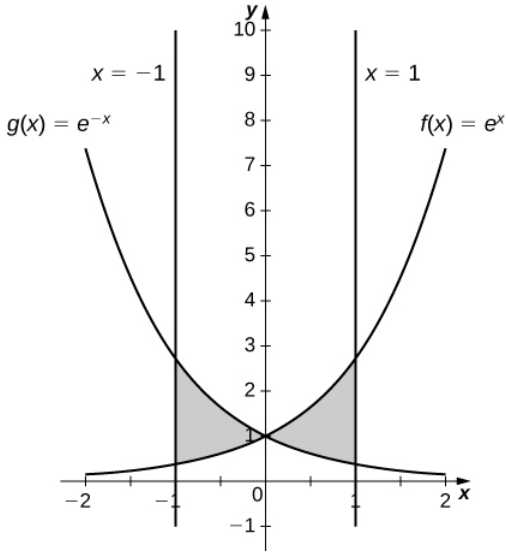


4 square units

10) $y = e^x, \quad y = e^{2x-1}$, and $x = 0$

11) $y = e^x, \quad y = e^{-x}, \quad x = -1$ and $x = 1$

Answer:

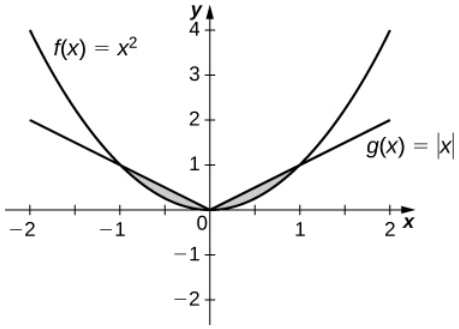


$\frac{2(e-1)^2}{e}$ units^2

12) $y = e, \quad y = e^x$, and $y = e^{-x}$

13) $y = |x|$ and $y = x^2$

Answer:



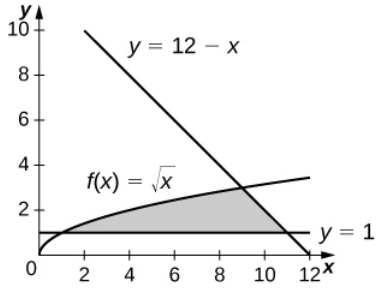
$\frac{1}{3}$ units^2

For exercises 14 - 19, graph the equations and shade the area of the region between the curves. If necessary, break the region into sub-regions to determine its entire area.

14) $y = \sin(\pi x), \quad y = 2x$, and $x > 0$

15) $y = 12 - x, \quad y = \sqrt{x},$ and $y = 1$

Answer:

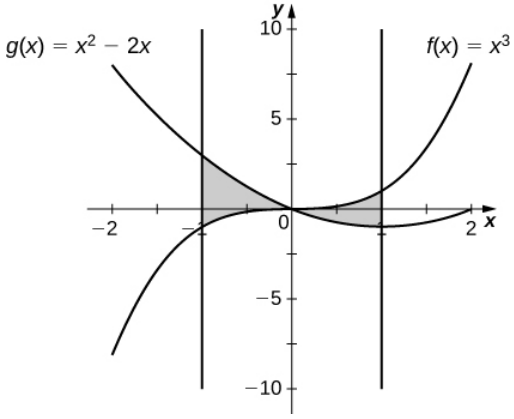


$\frac{34}{3}$ units^2

16) $y = \sin x$ and $y = \cos x$ over $x \in [-\pi, \pi]$

17) $y = x^3$ and $y = x^2 - 2x$ over $x \in [-1, 1]$

Answer:

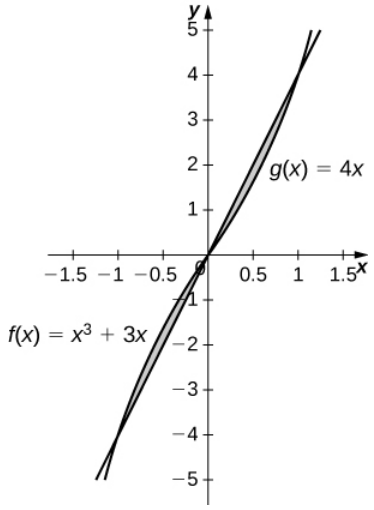


$\frac{5}{2}$ units^2

18) $y = x^2 + 9$ and $y = 10 + 2x$ over $x \in [-1, 3]$

19) $y = x^3 + 3x$ and $y = 4x$

Answer:



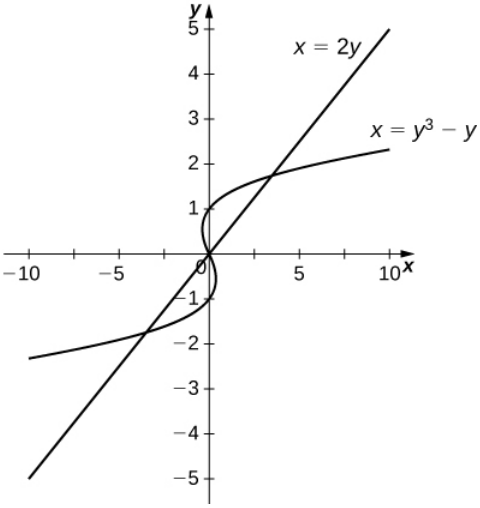
$\frac{1}{2}$ units^2

For exercises 20 -25, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the y-axis.

20) $x = y^3$ and $x = 3y - 2$

21) $x = y$ and $x = y^3 - y$

Answer:



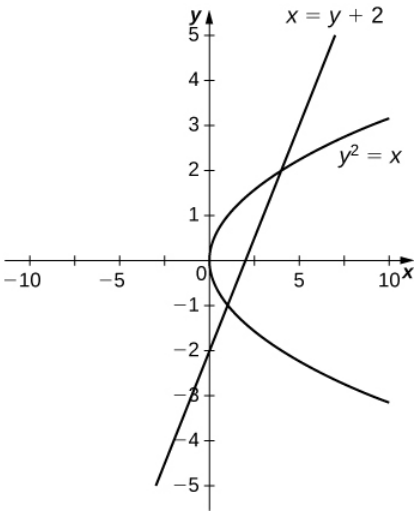
$\frac{9}{2}$ units^2

22) $x = -3 + y^2$ and $x = y - y^2$

23) $y^2 = x$ and $x = y + 2$

Answer:

https://math.libretexts.org/@go/page/127876

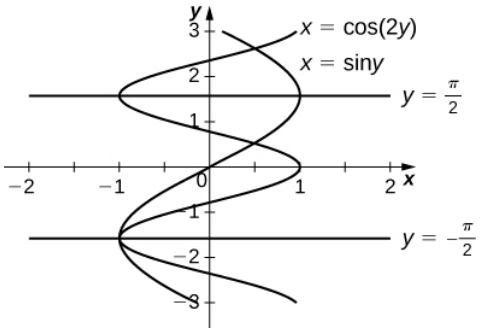


$\frac{9}{2}$ units^2

24) $x = |y|$ and $2x = -y^2 + 2$

25) $x = \sin y, \quad x = \cos(2y), \quad y = \pi/2$, and $y = -\pi/2$

Answer:



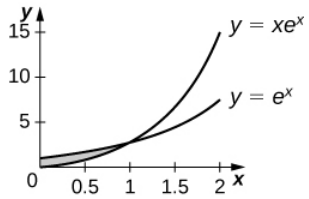
$\frac{3\sqrt{3}}{2}$ units^2

For exercises 26 - 37, graph the equations and shade the area of the region between the curves. Determine its area by integrating over the x-axis or y-axis, whichever seems more convenient.

26) $x = y^4$ and $x = y^5$

27) $y = xe^x, \quad y = e^x, \quad x = 0$, and $x = 1$.

Answer:

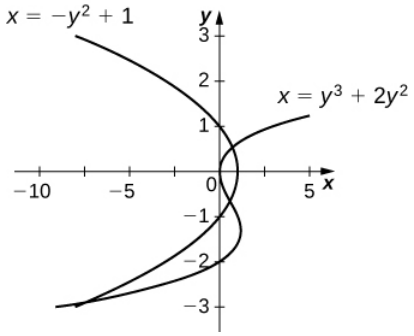


$(e-2)$ units^2

28) $y = x^6$ and $y = x^4$

29) $x = y^3 + 2y^2 + 1$ and $x = -y^2 + 1$

Answer:

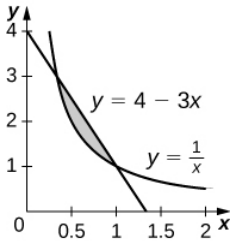


$\frac{27}{4}$ units^2

30) $y = |x|$ and $y = x^2 - 1$

31) $y = 4 - 3x$ and $y = \frac{1}{x}$

Answer:

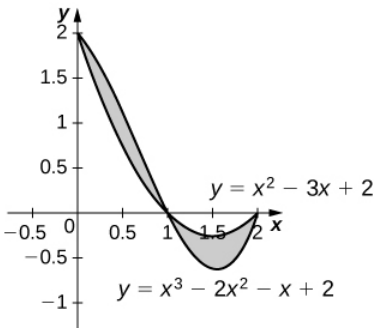


$\left(\frac{4}{3} - \ln(3)\right)$ units^2

32) $y = \sin x, \quad x = -\pi/6, \quad x = \pi/6$, and $y = \cos^3 x$

33) $y = x^2 - 3x + 2$ and $y = x^3 - 2x^2 - x + 2$

Answer:

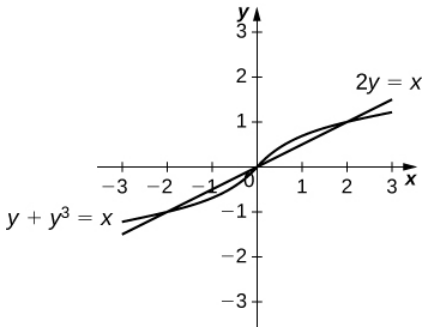


$\frac{1}{2}$ square units

34) $y = 2\cos^3(3x), \quad y = -1, \quad x = \frac{\pi}{4}$, and $x = -\frac{\pi}{4}$

35) $y + y^3 = x$ and $2y = x$

Answer:

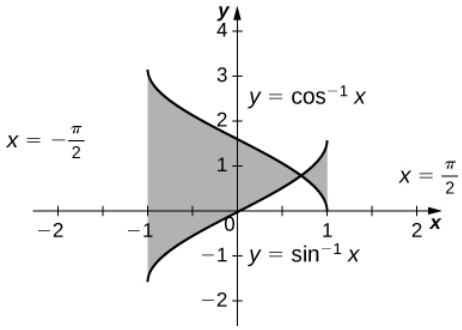


$\frac{1}{2}$ square units

36) $y = \sqrt{1 - x^2}$ and $y = x^2 - 1$

37) $y = \arccos x, \quad y = \arcsin x, \quad x = -1$, and $x = 1$

Answer:



$-2(\sqrt{2} - \pi)$ square units

For exercises 38 - 47, find the exact area of the region bounded by the given equations if possible. If you are unable to determine the intersection points analytically, use a calculator to approximate the intersection points with three decimal places and determine the approximate area of the region.

38) [T] $x = e^y$ and $y = x - 2$

39) [T] $y = x^2$ and $y = \sqrt{1 - x^2}$

Answer:

1.067square units

40) [T] $y = 3x^2 + 8x + 9$ and $3y = x + 24$

41) [T] $x = \sqrt{4 - y^2}$ and $y^2 = 1 + x^2$

Answer:

0.852square units

42) [T] $x^2 = y^3$ and $x = 3y$

43) [T] $y = \sin^3 x + 2, \quad y = \tan x, \quad x = -1.5$, and $x = 1.5$

Answer:

7.523square units

44) [T] $y = \sqrt{1 - x^2}$ and $y^2 = x^2$

45) [T] $y = \sqrt{1 - x^2}$ and $y = x^2 + 2x + 1$

Answer:

$\frac{3\pi - 4}{12}$ square units

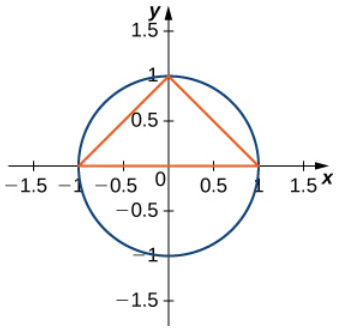
46) [T] $x = 4 - y^2$ and $x = 1 + 3y + y^2$

47) [T] $y = \cos x, \quad y = e^x, \quad x = -\pi, \quad$ and $\quad x = 0$

Answer:

1.429square units

48) The largest triangle with a base on the x-axis that fits inside the upper half of the unit circle $y^2 + x^2 = 1$ is given by $y = 1 + x$ and $y = 1 - x$. See the following figure. What is the area inside the semicircle but outside the triangle?



49) A factory selling cell phones has a marginal cost function $C(x) = 0.01x^2 - 3x + 229$, where x represents the number of cell phones, and a marginal revenue function given by $R(x) = 429 - 2x$. Find the area between the graphs of these curves and $x = 0$. What does this area represent?

Answer:

$33,333.33 total profit for **200** cell phones sold

50) An amusement park has a marginal cost function $C(x) = 1000e - x + 5$, where x represents the number of tickets sold, and a marginal revenue function given by $R(x) = 60 - 0.1x$. Find the total profit generated when selling 550 tickets. Use a calculator to determine intersection points, if necessary, to two decimal places.

51) The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = 1 - \cos((\pi t)/2)$ whereas the speed of the tortoise is $T(t) = (1/2)\arctan(t/4)$, where t is time measured in hours and the speed is measured in miles per hour. Find the area between the curves from time $t = 0$ to the first time after one hour when the tortoise and hare are traveling at the same speed. What does it represent? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

Answer:

3.263mi represents how far ahead the hare is from the tortoise

52) The tortoise versus the hare: The speed of the hare is given by the sinusoidal function $H(t) = (1/2) - (1/2)\cos(2\pi t)$ whereas the speed of the tortoise is $T(t) = \sqrt{t}$, where t is time measured in hours and speed is measured in kilometers per hour. If the race is over in 1 hour, who won the race and by how much? Use a calculator to determine the intersection points, if necessary, accurate to three decimal places.

For exercises 53 - 55, find the area between the curves by integrating with respect to x and then with respect to y. Is one method easier than the other? Do you obtain the same answer?

53) $y = x^2 + 2x + 1$ and $y = -x^2 - 3x + 4$

Answer:

$\frac{343}{24}$ square units

54) $y = x^4$ and $x = y^5$

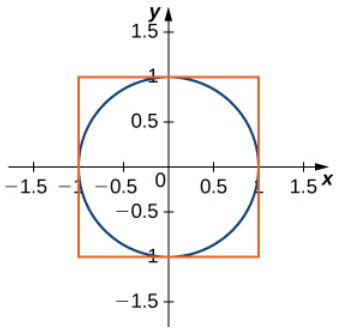
55) $x = y^2 - 2$ and $x = 2y$

Answer:

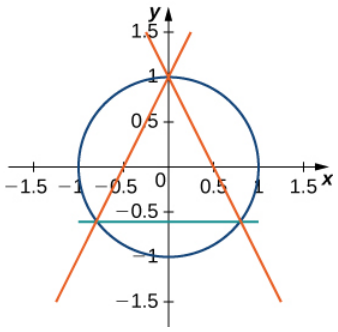
$4\sqrt{3}$ square units

For exercises 56 - 57, solve using calculus, then check your answer with geometry.

56) Determine the equations for the sides of the square that touches the unit circle on all four sides, as seen in the following figure. Find the area between the perimeter of this square and the unit circle. Is there another way to solve this without using calculus?



57) Find the area between the perimeter of the unit circle and the triangle created from $y = 2x + 1$, $y = 1 - 2x$ and $y = -\frac{3}{5}$, as seen in the following figure. Is there a way to solve this without using calculus?



Answer:

$\left(\pi - \frac{32}{25}\right)$ square units

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) edited to use alternate notation for inverse trig functions.

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3.3: Determining Volumes by Slicing

Learning Objectives

- Determine the volume of a solid by integrating a cross-section (the slicing method).
- Find the volume of a solid of revolution using the disk method.
- Find the volume of a solid of revolution with a cavity using the washer method.

In the preceding section, we used definite integrals to find the area between two curves. In this section, we use definite integrals to find volumes of three-dimensional solids. We consider three approaches—slicing, disks, and washers—for finding these volumes, depending on the characteristics of the solid.

Volume and the Slicing Method

Just as area is the numerical measure of a two-dimensional region, volume is the numerical measure of a three-dimensional solid. Most of us have computed volumes of solids by using basic geometric formulas. The volume of a rectangular solid, for example, can be computed by multiplying length, width, and height: $V = lwh$. The formulas for the volumes of:

- a sphere

$$V_{sphere} = \frac{4}{3}\pi r^3,$$

- a cone

$$V_{cone} = \frac{1}{3}\pi r^2 h$$

- and a pyramid

$$V_{pyramid} = \frac{1}{3}Ah$$

have also been introduced. Although some of these formulas were derived using geometry alone, all these formulas can be obtained by using integration.

We can also calculate the volume of a cylinder. Although most of us think of a cylinder as having a circular base, such as a soup can or a metal rod, in mathematics the word cylinder has a more general meaning. To discuss cylinders in this more general context, we first need to define some vocabulary.

We define the *cross-section* of a solid to be the intersection of a plane with the solid. A cylinder is defined as any solid that can be generated by translating a plane region along a line perpendicular to the region, called the axis of the cylinder. Thus, all cross-sections perpendicular to the axis of a cylinder are identical. The solid shown in Figure 3.3.1 is an example of a cylinder with a noncircular base. To calculate the volume of a cylinder, then, we simply multiply the area of the cross-section by the height of the cylinder: $V = A \cdot h$. In the case of a right circular cylinder (soup can), this becomes $V = \pi r^2 h$.

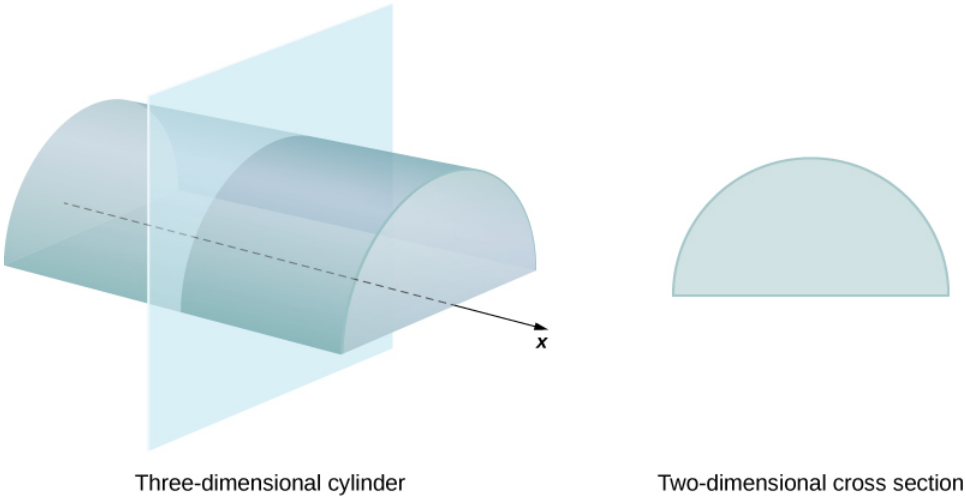


Figure 3.3.1: *Each cross-section of a particular cylinder is identical to the others.*

If a solid does not have a constant cross-section (and it is not one of the other basic solids), we may not have a formula for its volume. In this case, we can use a definite integral to calculate the volume of the solid. We do this by slicing the solid into pieces, estimating the volume of each slice, and then adding those estimated volumes together. The slices should all be parallel to one another, and when we put all the slices together, we should get the whole solid. Consider, for example, the solid S shown in Figure 3.3.2, extending along the x-**axis.**

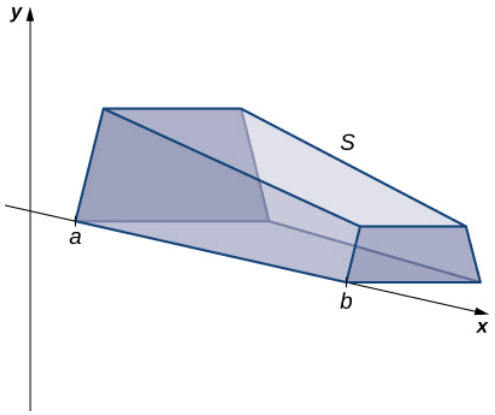


Figure 3.3.2: A solid with a varying cross-section.

We want to divide S into slices perpendicular to the x-**axis**. As we see later in the chapter, there may be times when we want to slice the solid in some other direction—say, with slices perpendicular to the y-axis. The decision of which way to slice the solid is very important. If we make the wrong choice, the computations can get quite messy. Later in the chapter, we examine some of these situations in detail and look at how to decide which way to slice the solid. For the purposes of this section, however, we use slices perpendicular to the x-**axis.**

Because the cross-sectional area is not constant, we let $A(x)$ represent the area of the cross-section at point x. Now let $P = x_0, x_1 \ldots, X_n$ be a regular partition of $[a, b]$, and for $i = 1, 2, \ldots n$, let S_i represent the slice of S stretching from x_{i-1} to x_i. The following figure shows the sliced solid with $n = 3$.

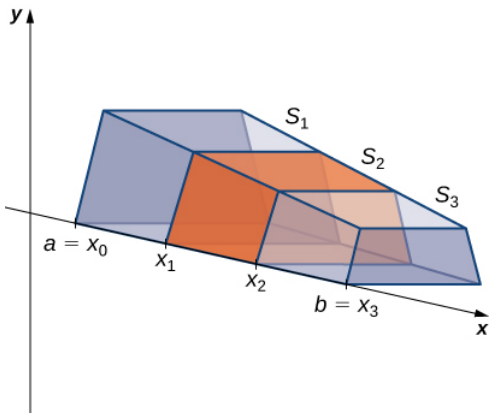


Figure 3.3.3: The solid S has been divided into three slices perpendicular to the x-axis.

Finally, for $i = 1, 2, \ldots n$, let x_i^* be an arbitrary point in $[x_{i-1}, x_i]$. Then the volume of slice S_i can be estimated by $V(S_i) \approx A(x_i^*)\,\Delta x$. Adding these approximations together, we see the volume of the entire solid S can be approximated by

$$V(S) \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x.$$

By now, we can recognize this as a Riemann sum, and our next step is to take the limit as $n \to \infty$. Then we have

$$V(S) = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx.$$

The technique we have just described is called the slicing method. To apply it, we use the following strategy.

Problem-Solving Strategy: Finding Volumes by the Slicing Method

1. Examine the solid and determine the shape of a cross-section of the solid. It is often helpful to draw a picture if one is not provided.
2. Determine a formula for the area of the cross-section.
3. Integrate the area formula over the appropriate interval to get the volume.

Recall that in this section, we assume the slices are perpendicular to the x-**axis**. Therefore, the area formula is in terms of x and the limits of integration lie on the x-**axis.** However, the problem-solving strategy shown here is valid regardless of how we choose to slice the solid.

Example 3.3.1: Deriving the Formula for the Volume of a Pyramid

We know from geometry that the formula for the volume of a pyramid is $V = \frac{1}{3}Ah$. If the pyramid has a square base, this becomes $V = \frac{1}{3}a^2h$, where a denotes the length of one side of the base. We are going to use the slicing method to derive this formula.

Solution

We want to apply the slicing method to a pyramid with a square base. To set up the integral, consider the pyramid shown in Figure 3.3.4, oriented along the x-**axis.**

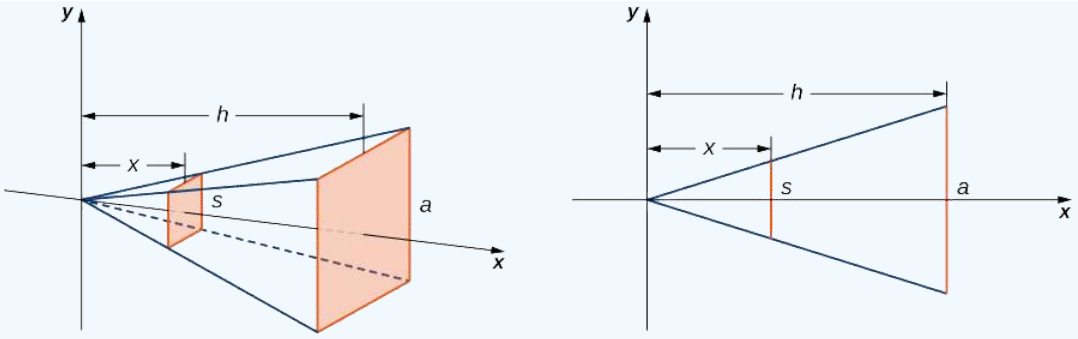


Figure 3.3.4: (a) A pyramid with a square base is oriented along the x-axis. (b) A two-dimensional view of the pyramid is seen from the side.

We first want to determine the shape of a cross-section of the pyramid. We know the base is a square, so the cross-sections are squares as well (step 1). Now we want to determine a formula for the area of one of these cross-sectional squares. Looking at Figure 3.3.4(b), and using a proportion, since these are similar triangles, we have

$$\frac{s}{a} = \frac{x}{h}$$

or

$$s = \frac{ax}{h}.$$

Therefore, the area of one of the cross-sectional squares is

$$A(x) = s^2 = \left(\frac{ax}{h}\right)^2 \qquad \text{(step 2)}$$

Then we find the volume of the pyramid by integrating from 0 to h (step 3):

$$V = \int_0^h A(x)\,dx = \int_0^h \left(\frac{ax}{h}\right)^2 dx = \frac{a^2}{h^2}\int_0^h x^2\,dx = \left[\frac{a^2}{h^2}\left(\frac{1}{3}x^3\right)\right]\Bigg|_0^h = \frac{1}{3}a^2h.$$

This is the formula we were looking for.

? Exercise 3.3.1

Use the slicing method to derive the formula

$$V = \frac{1}{3}\pi r^2 h$$

for the volume of a circular cone.

Hint

Use similar triangles, as in Example 3.3.1

Solids of Revolution

If a region in a plane is revolved around a line in that plane, the resulting solid is called a **solid of revolution**, as shown in the following figure.

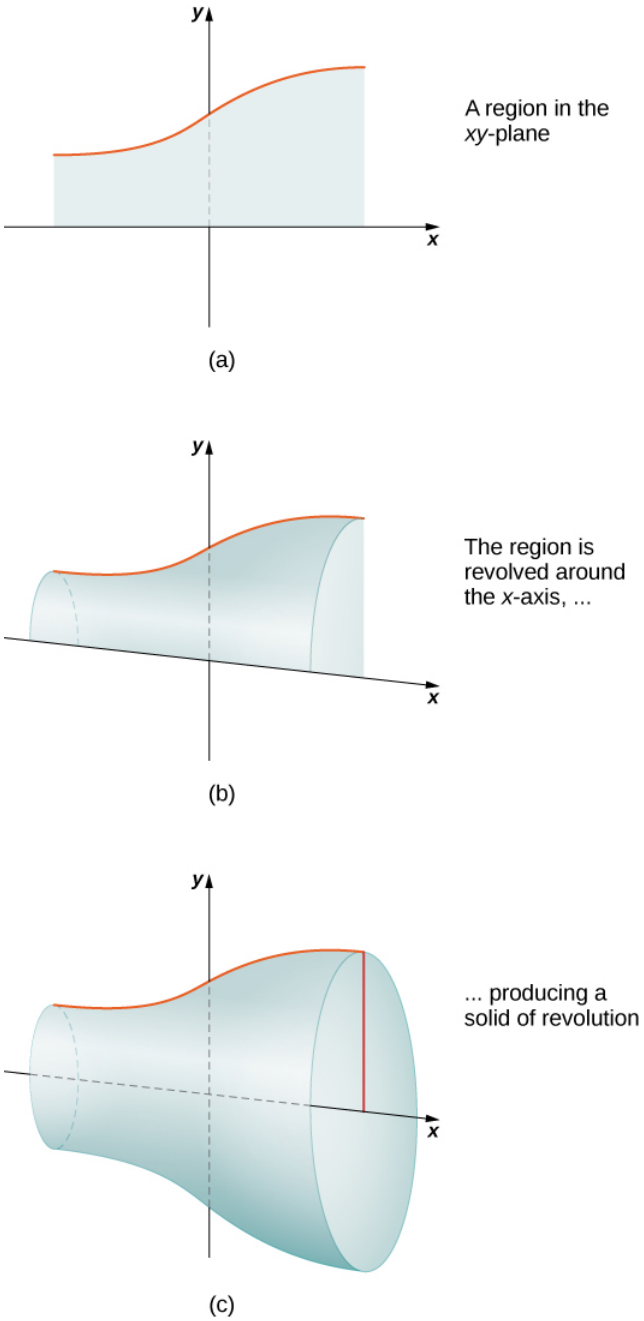


Figure 3.3.5: (a) This is the region that is revolved around the x-axis. (b) As the region begins to revolve around the axis, it sweeps out a solid of revolution. (c) This is the solid that results when the revolution is complete.

Solids of revolution are common in mechanical applications, such as machine parts produced by a lathe. We spend the rest of this section looking at solids of this type. The next example uses the slicing method to calculate the volume of a solid of revolution.

Example 3.3.2: Using the Slicing Method to find the Volume of a Solid of Revolution

Use the slicing method to find the volume of the solid of revolution bounded by the graphs of $f(x) = x^2 - 4x + 5, x = 1$,and $x = 4$, and rotated about the x-axis.

Solution

Using the problem-solving strategy, we first sketch the graph of the quadratic function over the interval $[1, 4]$ as shown in the following figure.

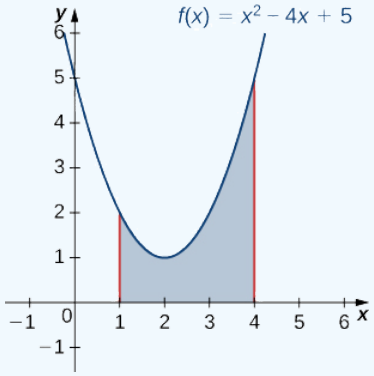
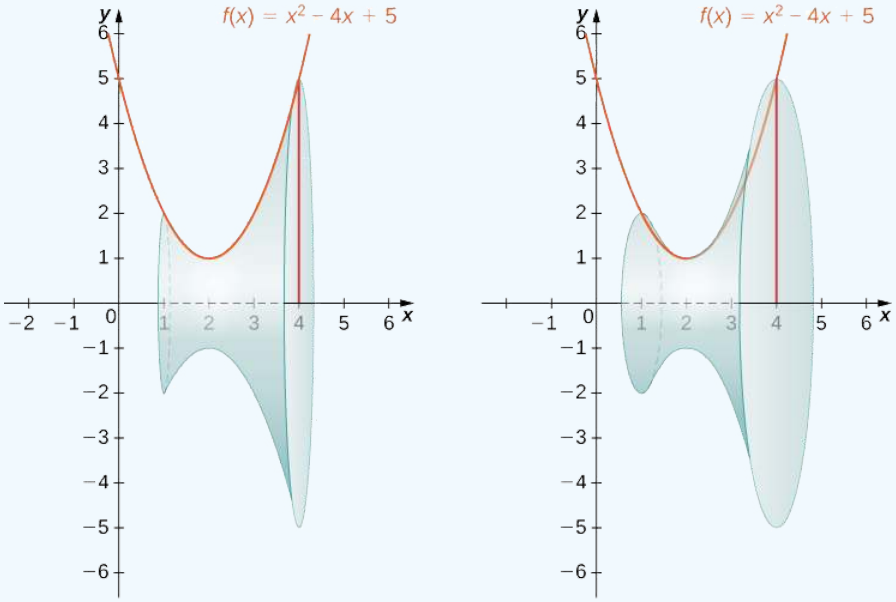


Figure 3.3.6: A region used to produce a solid of revolution.

Next, revolve the region around the x-axis, as shown in the following figure.



(a) (b)

Figure 3.3.7: Two views, (a) and (b), of the solid of revolution produced by revolving the region in Figure 3.3.6 about the x-**axis.**

Since the solid was formed by revolving the region around the x-**axis,** the cross-sections are circles (step 1). The area of the cross-section, then, is the area of a circle, and the radius of the circle is given by $f(x)$. Use the formula for the area of the circle:

$$A(x) = \pi r^2 = \pi[f(x)]^2 = \pi(x^2 - 4x + 5)^2 \qquad \text{(step 2)}.$$

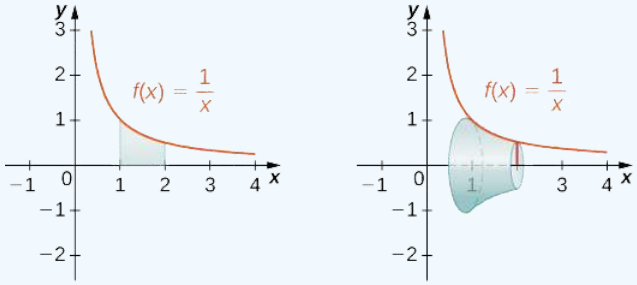
The volume, then, is (step 3)

$$\begin{aligned} V &= \int_a^b A(x)\,dx \\ &= \int_1^4 \pi(x^2 - 4x + 5)^2\,dx \\ &= \pi \int_1^4 (x^4 - 8x^3 + 26x^2 - 40x + 25)\,dx \\ &= \pi \left(\frac{x^5}{5} - 2x^4 + \frac{26x^3}{3} - 20x^2 + 25x \right) \Bigg|_1^4 \\ &= \frac{78}{5}\pi \end{aligned}$$

The volume is $78\pi/5$ units^3.

Exercise 3.3.2

Use the method of slicing to find the volume of the solid of revolution formed by revolving the region between the graph of the function $f(x) = 1/x$ and the x-**axis** over the interval $[1, 2]$ around the x-**axis.** See the following figure.



Hint

Use the problem-solving strategy presented earlier and follow Example 3.3.2to help with step 2.

Answer

$\frac{\pi}{2}$ units^3

The Disk Method

When we use the slicing method with solids of revolution, it is often called the disk method because, for solids of revolution, the slices used to over approximate the volume of the solid are disks. To see this, consider the solid of revolution generated by revolving the region between the graph of the function $f(x) = (x-1)^2 + 1$ and the x-**axis** over the interval $[-1, 3]$ around the x-**axis**. The graph of the function and a representative disk are shown in Figure 3.3.8 (a) and (b). The region of revolution and the resulting solid are shown in Figure 3.3.8 (c) and (d).

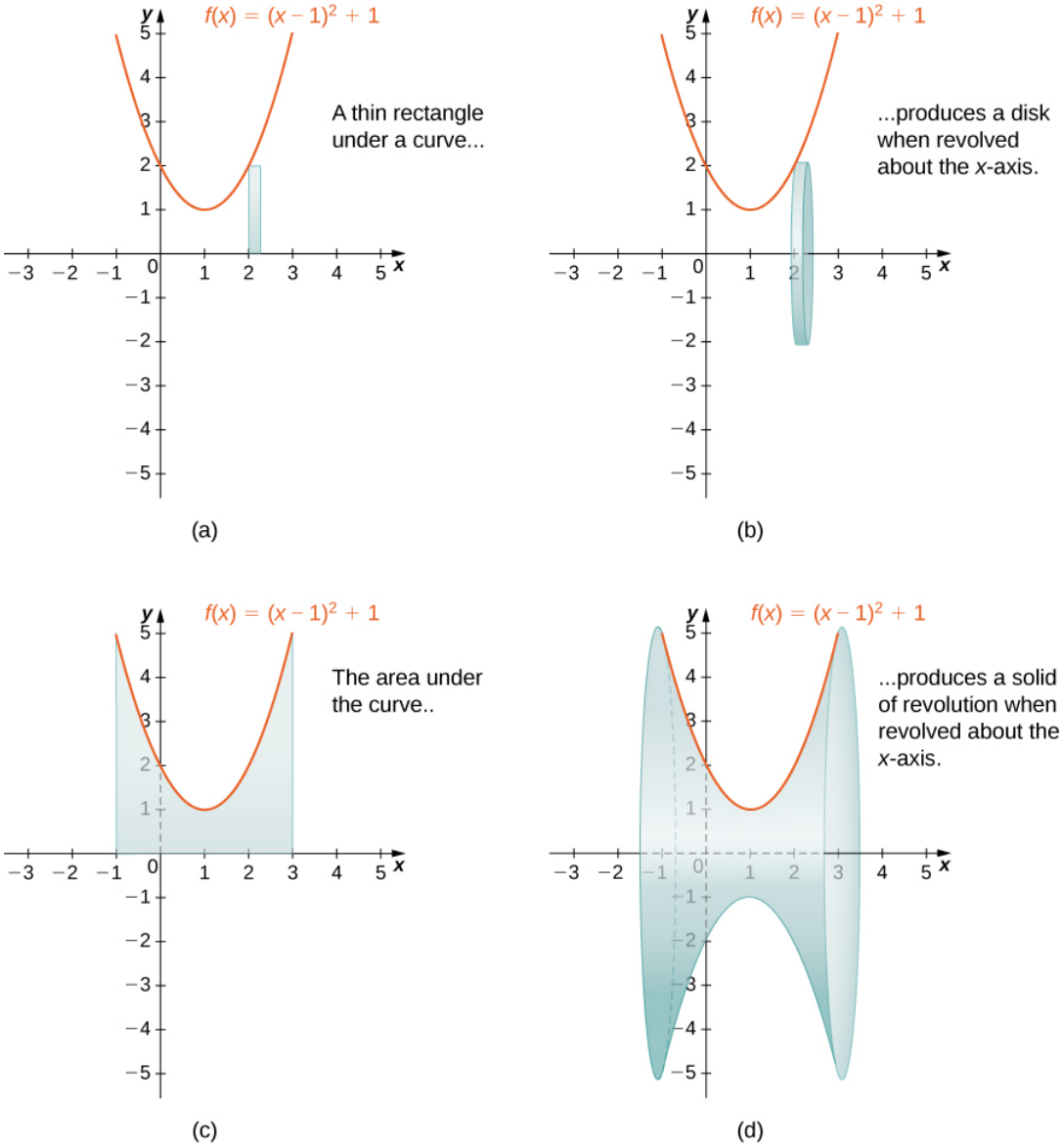


Figure 3.3.8: (a) A thin rectangle for approximating the area under a curve. (b) A representative disk formed by revolving the rectangle about the x**-axis.** (c) The region under the curve is revolved about the x**-axis,** resulting in (d) the solid of revolution.

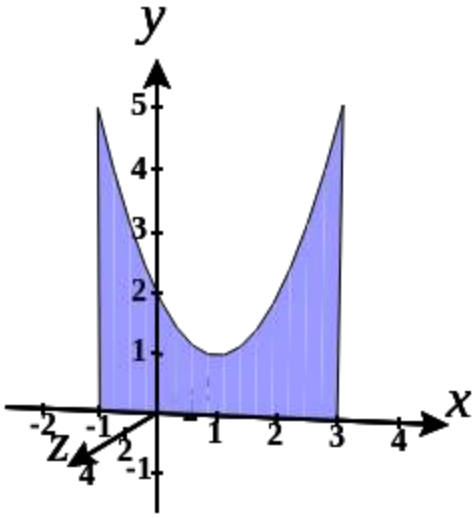


Figure 3.3.8: (e) A dynamic version of this solid of revolution generated using CalcPlot3D.

We already used the formal Riemann sum development of the volume formula when we developed the slicing method. We know that

$$\int_a^b A(x)\,dx.$$

The only difference with the disk method is that we know the formula for the cross-sectional area ahead of time; it is the area of a circle. This gives the following rule.

The Disk Method

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the x-**axis**, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-**axis** is given by

$$V = \int_a^b \pi[f(x)]^2\,dx.$$

The volume of the solid we have been studying (Figure 3.3.8) is given by

$$
\begin{aligned}
V &= \int_a^b \pi[f(x)]^2\,dx \\
&= \int_{-1}^{3} \pi\left[(x-1)^2+1\right]^2\,dx = \pi\int_{-1}^{3}\left[(x-1)^4+2(x-1)^2+1\right]^2\,dx \\
&= \pi\left[\frac{1}{5}(x-1)^5+\frac{2}{3}(x-1)^3+x\right]\Bigg|_{-1}^{3} \\
&= \pi\left[\left(\frac{32}{5}+\frac{16}{3}+3\right)-\left(-\frac{32}{5}-\frac{16}{3}-1\right)\right] \\
&= \frac{412\pi}{15}\ \text{units}^3.
\end{aligned}
$$

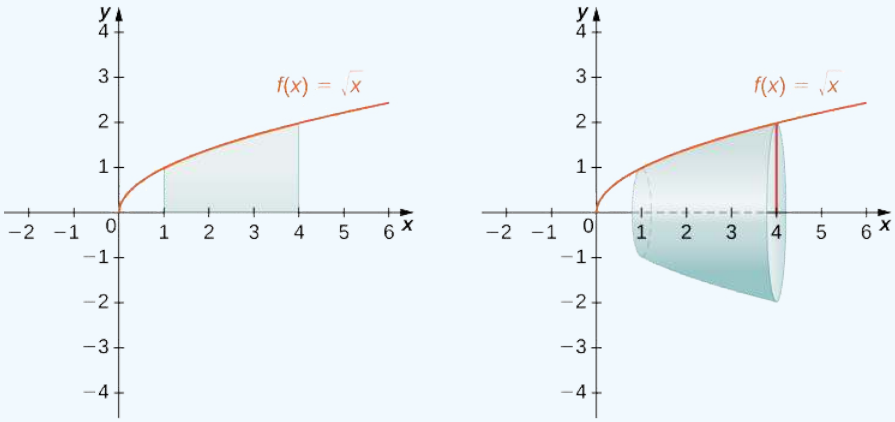
Let's look at some examples.

Example 3.3.3: Using the Disk Method to Find the Volume of a Solid of Revolution 1

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x)=\sqrt{x}$ and the x-**axis** over the interval $[1,4]$ around the x-**axis.**

Solution

The graphs of the function and the solid of revolution are shown in the following figure.



(a) (b)

Figure 3.3.9: (a) The function $f(x)=\sqrt{x}$ over the interval $[1,4]$. (b) The solid of revolution obtained by revolving the region under the graph of $f(x)$ about the x-**axis**.

We have

$$
\begin{aligned}
V &= \int_a^b \pi\left[f(x)\right]^2\,dx \\
&= \int_1^4 \pi[\sqrt{x}]^2\,dx = \pi\int_1^4 x\,dx \\
&= \frac{\pi}{2}x^2\Big|_1^4 = \frac{15\pi}{2}
\end{aligned}
$$

The volume is $(15\pi)/2\ \text{units}^3$.

Exercise 3.3.3

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $f(x)=\sqrt{4-x}$ and the x-**axis** over the interval $[0,4]$ around the x-**axis.**

Hint

Use the procedure from Example 3.3.3

Answer

8π units^3

So far, our examples have all concerned regions revolved around the x-**axis**, but we can generate a solid of revolution by revolving a plane region around any horizontal or vertical line. In the next example, we look at a solid of revolution that has been generated by revolving a region around the y-**axis**. The mechanics of the disk method are nearly the same as when the x-**axis** is the axis of revolution, but we express the function in terms of y and we integrate with respect to y as well. This is summarized in the following rule.

Rule: The Disk Method for Solids of Revolution around the y-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-**axis**, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y-**axis** is given by

$$V = \int_c^d \pi \left[g(y)\right]^2 dy.$$

The next example shows how this rule works in practice.

Example 3.3.4: Using the Disk Method to Find the Volume of a Solid of Revolution 2

Let R be the region bounded by the graph of $g(y) = \sqrt{4-y}$ and the y-**axis** over the y-**axis** interval $[0, 4]$. Use the disk method to find the volume of the solid of revolution generated by rotating R around the y-**axis.**

Solution

Figure 3.3.10 shows the function and a representative disk that can be used to estimate the volume. Notice that since we are revolving the function around the y-**axis,** the disks are horizontal, rather than vertical.

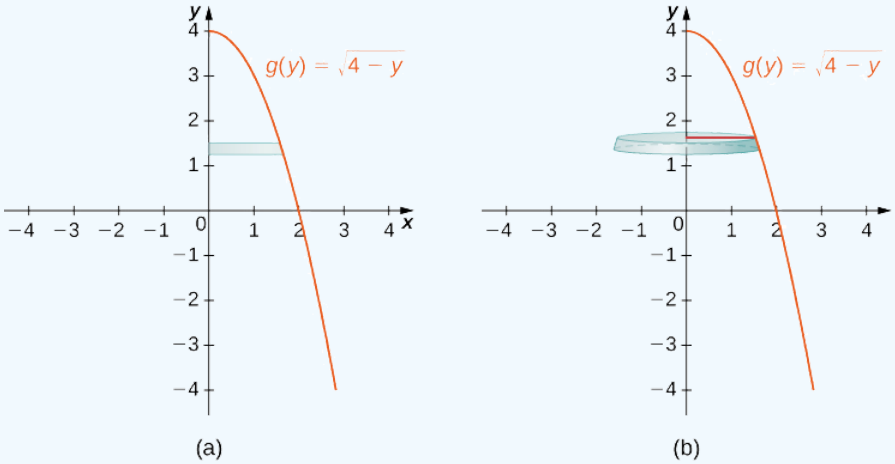


Figure 3.3.10: (a) Shown is a thin rectangle between the curve of the function $g(y) = \sqrt{4-y}$ and the y-**axis.** (b) The rectangle forms a representative disk after revolution around the y-**axis.**

The region to be revolved and the full solid of revolution are depicted in the following figure.

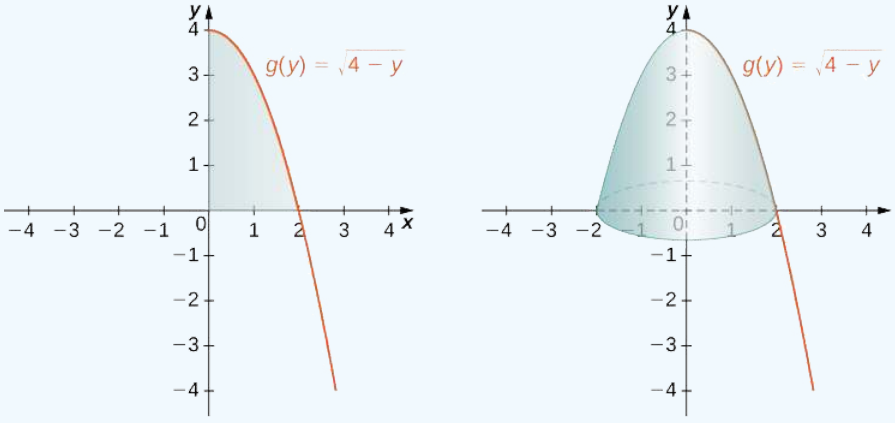


Figure 3.3.11: (a) The region to the left of the function $g(y) = \sqrt{4-y}$ over the y-axis interval $[0,4]$. (b) The solid of revolution formed by revolving the region about the y-axis.

To find the volume, we integrate with respect to y. We obtain

$$V = \int_c^d \pi \left[g(y)\right]^2 dy = \int_0^4 \pi \left[\sqrt{4-y}\right]^2 dy = \pi \int_0^4 (4-y)\, dy = \pi \left[4y - \frac{y^2}{2}\right]\Big|_0^4 = 8\pi.$$

The volume is 8π units^3.

? Exercise 3.3.4

Use the disk method to find the volume of the solid of revolution generated by rotating the region between the graph of $g(y) = y$ and the y-**axis** over the interval $[1,4]$ around the y-**axis.**

Hint

Use the procedure from Example 3.3.4

Answer

21π units^3

The Washer Method

Some solids of revolution have cavities in the middle; they are not solid all the way to the axis of revolution. Sometimes, this is just a result of the way the region of revolution is shaped with respect to the axis of revolution. In other cases, cavities arise when the region of revolution is defined as the region between the graphs of two functions. A third way this can happen is when an axis of revolution other than the x-**axis** or y-**axis** is selected.

When the solid of revolution has a cavity in the middle, the slices used to approximate the volume are not disks, but washers (disks with holes in the center). For example, consider the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the graph of the function $g(x) = 1$ over the interval $[1,4]$. When this region is revolved around the x-**axis**, the result is a solid with a cavity in the middle, and the slices are washers. The graph of the function and a representative washer are shown in Figure 3.3.12 (a) and (b). The region of revolution and the resulting solid are shown in Figure 3.3.12(c) and (d).

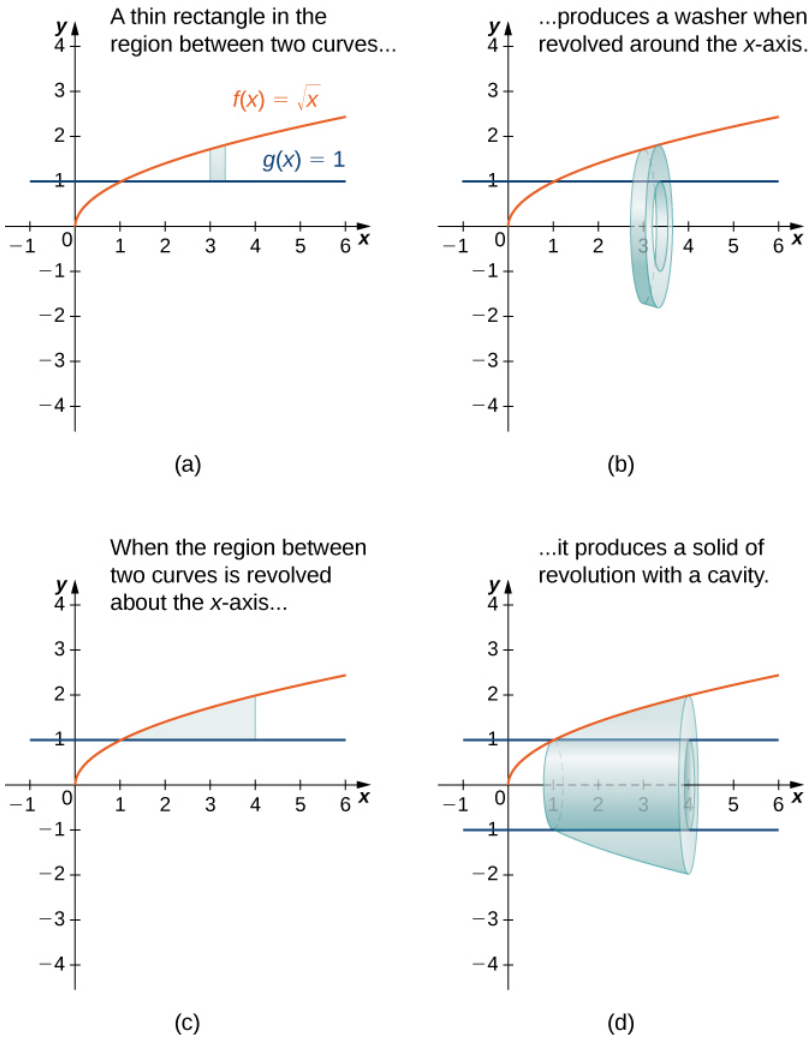


Figure 3.3.12: (a) A thin rectangle in the region between two curves. (b) A representative disk formed by revolving the rectangle about the x-**axis.** (c) The region between the curves over the given interval. (d) The resulting solid of revolution.

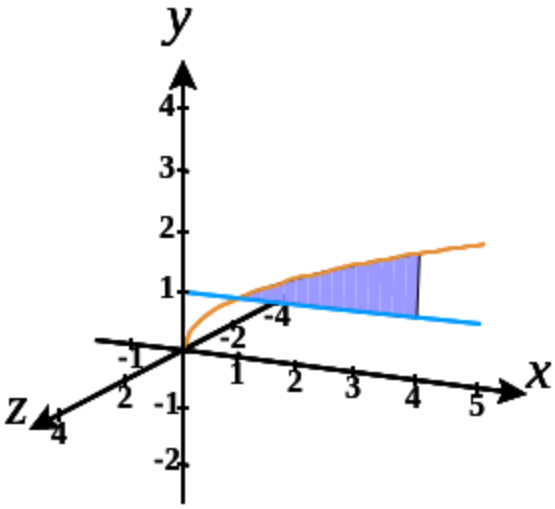


Figure 3.3.12: (e) A dynamic version of this solid of revolution generated using CalcPlot3D.

The cross-sectional area, then, is the area of the outer circle less the area of the inner circle. In this case,

$$A(x) = \pi(\sqrt{x})^2 - \pi(1)^2 = \pi(x-1).$$

Then the volume of the solid is

$$V = \int_a^b A(x)\,dx = \int_1^4 \pi(x-1)\,dx = \pi\left[\frac{x^2}{2} - x\right]\Bigg|_1^4 = \frac{9}{2}\pi \text{ units}^3.$$

Generalizing this process gives the **washer method.**

Rule: The Washer Method

Suppose $f(x)$ and $g(x)$ are continuous, nonnegative functions such that $f(x) \geq g(x)$ over $[a,b]$. Let R denote the region bounded above by the graph of $f(x)$, below by the graph of $g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$. Then, the volume of the solid of revolution formed by revolving R around the x-**axis** is given by

$$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]\,dx.$$

Example 3.3.5: Using the Washer Method

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = 1/x$ over the interval $[1, 4]$ around the x-**axis.**

Solution

The graphs of the functions and the solid of revolution are shown in the following figure.

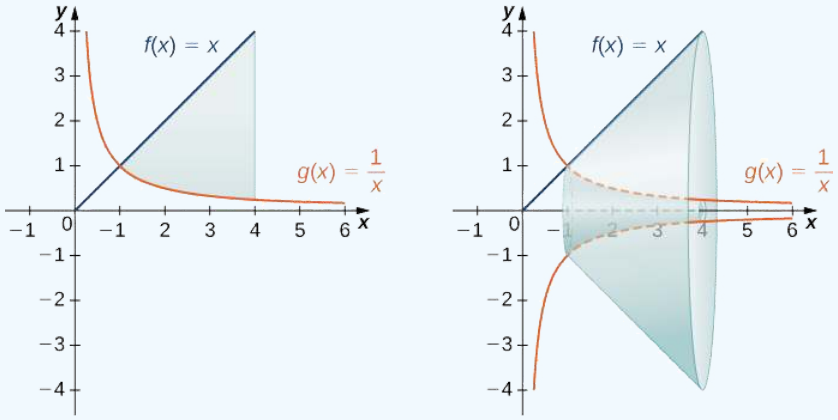


Figure 3.3.13: (a) The region between the graphs of the functions $f(x) = x$ and $g(x) = 1/x$ over the interval $[1, 4]$. (b) Revolving the region about the x-**axis** generates a solid of revolution with a cavity in the middle.

We have

$$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right] dx = \pi \int_1^4 \left[x^2 - \left(\frac{1}{x}\right)^2\right] dx$$
$$= \pi \left[\frac{x^3}{3} + \frac{1}{x}\right]\Bigg|_1^4$$
$$= \frac{81\pi}{4} \text{ units}^3.$$

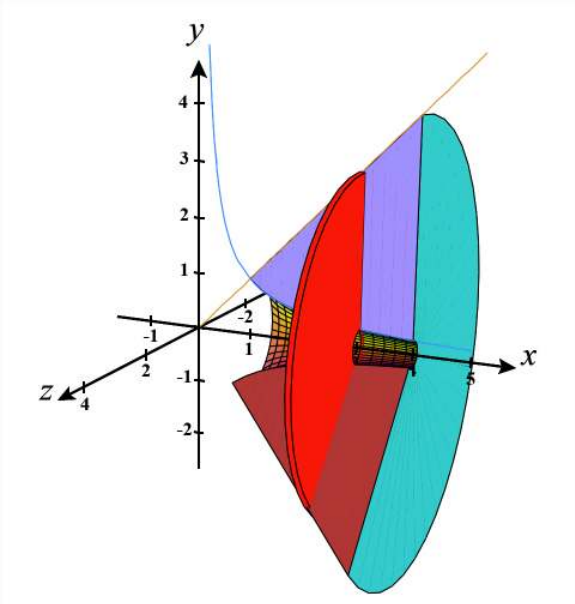


Figure 3.3.13: (c) A dynamic version of this solid of revolution generated using CalcPlot3D.

Exercise 3.3.5

Find the volume of a solid of revolution formed by revolving the region bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = 1/x$ over the interval $[1, 3]$ around the x**-axis.**

Hint

Graph the functions to determine which graph forms the upper bound and which graph forms the lower bound, then use the procedure from Example $3.3.5$

Answer

$\frac{10\pi}{3}$ units^3

As with the disk method, we can also apply the washer method to solids of revolution that result from revolving a region around the y-axis. In this case, the following rule applies.

Rule: The Washer Method for Solids of Revolution around the y-axis

Suppose $u(y)$ and $v(y)$ are continuous, nonnegative functions such that $v(y) \leq u(y)$ for $y \in [c, d]$. Let Q denote the region bounded on the right by the graph of $u(y)$, on the left by the graph of $v(y)$, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the y**-axis** is given by

$$V = \int_c^d \pi \left[(u(y))^2 - (v(y))^2 \right] \, dy.$$

Rather than looking at an example of the washer method with the y**-axis** as the axis of revolution, we now consider an example in which the axis of revolution is a line other than one of the two coordinate axes. The same general method applies, but you may have to visualize just how to describe the cross-sectional area of the volume.

Example 3.3.6:

Find the volume of a solid of revolution formed by revolving the region bounded above by $f(x) = 4 - x$ and below by the x**-axis** over the interval $[0, 4]$ around the line $y = -2$.

Solution

The graph of the region and the solid of revolution are shown in the following figure.

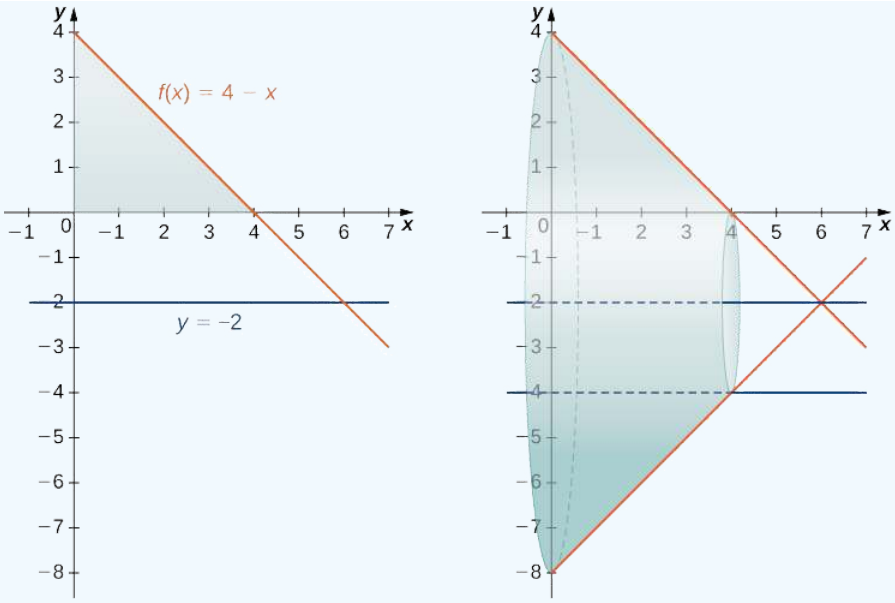


Figure 3.3.14: (a) The region between the graph of the function $f(x) = 4 - x$ and the x-**axis** over the interval $[0, 4]$. (b) Revolving the region about the line $y = -2$ generates a solid of revolution with a cylindrical hole through its middle.

We can't apply the volume formula to this problem directly because the axis of revolution is not one of the coordinate axes. However, we still know that the area of the cross-section is the area of the outer circle less the area of the inner circle. Looking at the graph of the function, we see the radius of the outer circle is given by $f(x) + 2$, which simplifies to

$$f(x) + 2 = (4 - x) + 2 = 6 - x.$$

The radius of the inner circle is $g(x) = 2$. Therefore, we have

$$\begin{aligned} V &= \int_0^4 \pi \left[(6 - x)^2 - (2)^2 \right] \, dx \\ &= \pi \int_0^4 (x^2 - 12x + 32) \, dx = \pi \left[\frac{x^3}{3} - 6x^2 + 32x \right] \Big|_0^4 \\ &= \frac{160\pi}{3} \text{ units}^3. \end{aligned}$$

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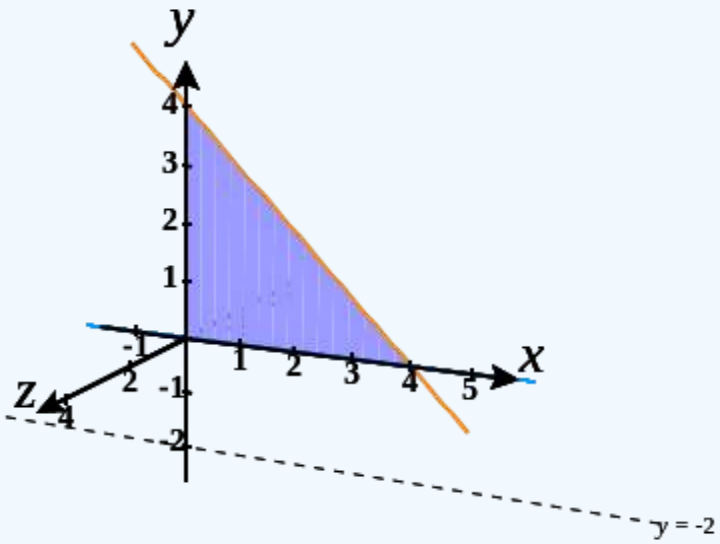


Figure 3.3.14 (c) A dynamic version of this solid of revolution generated using CalcPlot3D.

? Exercise 3.3.6

Find the volume of a solid of revolution formed by revolving the region bounded above by the graph of $f(x) = x + 2$ and below by the x-**axis** over the interval $[0, 3]$ around the line $y = -1$.

Hint

Use the procedure from Example 3.3.6

Answer

60π units^3

Key Concepts

- Definite integrals can be used to find the volumes of solids. Using the slicing method, we can find a volume by integrating the cross-sectional area.
- For solids of revolution, the volume slices are often disks and the cross-sections are circles. The method of disks involves applying the method of slicing in the particular case in which the cross-sections are circles, and using the formula for the area of a circle.
- If a solid of revolution has a cavity in the center, the volume slices are washers. With the method of washers, the area of the inner circle is subtracted from the area of the outer circle before integrating.

Key Equations

- **Disk Method along the x-axis**

$$V = \int_a^b \pi \big[f(x)\big]^2 \, dx$$

- **Disk Method along the y-axis**

$$V = \int_c^d \pi \big[g(y)\big]^2 \, dy$$

- **Washer Method**

$$V = \int_a^b \pi \left[\big(f(x)\big)^2 - \big(g(x)\big)^2 \right] \, dx$$

Glossary

cross-section
the intersection of a plane and a solid object

disk method
a special case of the slicing method used with solids of revolution when the slices are disks

slicing method
a method of calculating the volume of a solid that involves cutting the solid into pieces, estimating the volume of each piece, then adding these estimates to arrive at an estimate of the total volume; as the number of slices goes to infinity, this estimate becomes an integral that gives the exact value of the volume

solid of revolution
a solid generated by revolving a region in a plane around a line in that plane

washer method
a special case of the slicing method used with solids of revolution when the slices are washers

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3.3E: Exercises for Volumes of Common Cross-Section and Disk/Washer Method

Volumes of a Common Cross-Section

1) Derive the formula for the volume of a sphere using the slicing method.

2) Use the slicing method to derive the formula for the volume of a cone.

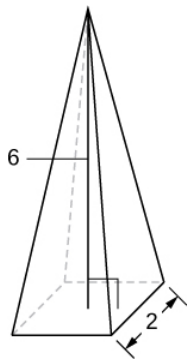
3) Use the slicing method to derive the formula for the volume of a tetrahedron with side length a.

4) Use the disk method to derive the formula for the volume of a trapezoidal cylinder.

5) Explain when you would use the disk method versus the washer method. When are they interchangeable?

For exercises 6 - 10, draw a typical slice and find the volume using the slicing method for the given volume.

6) A pyramid with height 6 units and square base of side 2 units, as pictured here.



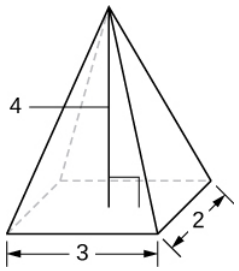
Solution:

Here the cross-sections are squares taken perpendicular to the y-axis.
We use the vertical cross-section of the pyramid through its center to obtain an equation relating x and y.
Here this would be the equation, $y = 6 - 6x$. Since we need the dimensions of the square at each y-level, we solve this equation for x to get, $x = 1 - \frac{y}{6}$.
This is half the distance across the square cross-section at the y-level, so the side length of the square cross-section is, $s = 2\left(1 - \frac{y}{6}\right)$.
Thus, we have the area of a cross-section is,

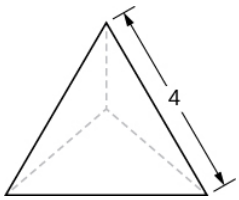
$$A(y) = \left[2\left(1 - \tfrac{y}{6}\right)\right]^2 = 4\left(1 - \tfrac{y}{6}\right)^2.$$

$$\begin{aligned}
\text{Then,}\quad V &= \int_0^6 4\left(1 - \tfrac{y}{6}\right)^2 dy \\
&= -24\int_1^0 u^2\, du, \quad \text{where } u = 1 - \tfrac{y}{6},\ \text{so } du = -\tfrac{1}{6}\, dy, \quad \Longrightarrow \quad -6\, du = dy \\
&= 24\int_0^1 u^2\, du = 24\frac{u^3}{3}\Big|_0^1 \\
&= 8u^3\Big|_0^1 \\
&= 8\left(1^3 - 0^3\right) \quad = \quad 8 \text{ units}^3
\end{aligned}$$

7) A pyramid with height 4 units and a rectangular base with length 2 units and width 3 units, as pictured here.



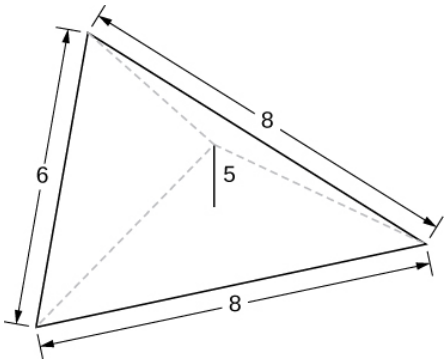
8) A tetrahedron with a base side of 4 units,as seen here.



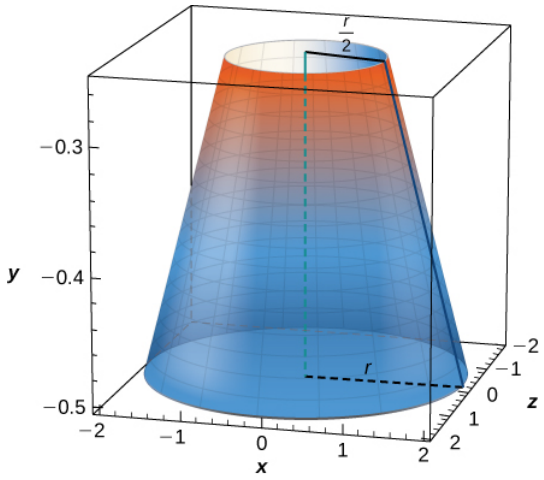
Answer:

$V = \frac{32}{3\sqrt{2}} = \frac{16\sqrt{2}}{3}$ units^3

9) A pyramid with height 5 units, and an isosceles triangular base with lengths of 6 units and 8 units, as seen here.



10) A cone of radius r and height h has a smaller cone of radius $r/2$ and height $h/2$ removed from the top, as seen here. The resulting solid is called a frustum.



Answer:

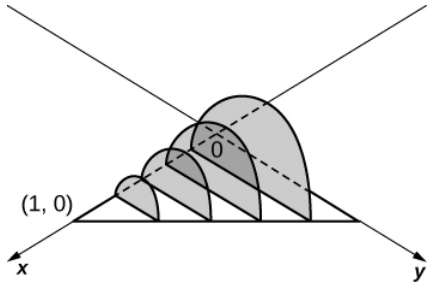
$V = \frac{7\pi}{12}hr^2$ units^3

For exercises 11 - 16, draw an outline of the solid and find the volume using the slicing method.

11) The base is a circle of radius a. The slices perpendicular to the base are squares.

12) The base is a triangle with vertices $(0,0), (1,0), \text{and } (0,1)$. Slices perpendicular to the xy-plane are semicircles.

Answer:

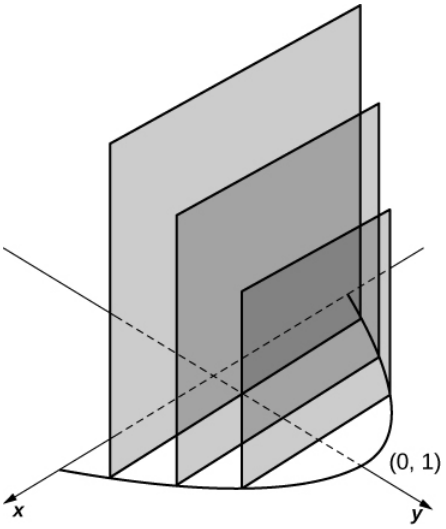


$$V = \int_0^1 \frac{\pi(1-x)^2}{8}\,dx \quad = \quad \frac{\pi}{24} \text{ units}^3$$

13) The base is the region under the parabola $y = 1 - x^2$ in the first quadrant. Slices perpendicular to the xy-plane are squares.

14) The base is the region under the parabola $y = 1 - x^2$ and above the **x-axis**. Slices perpendicular to the **y-axis** are squares.

Answer:

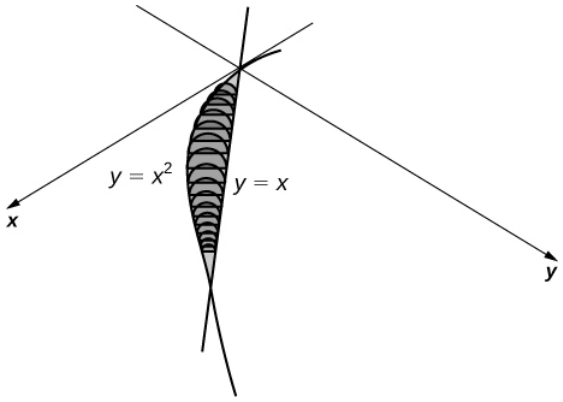


$$V = \int_0^1 4(1-y)\,dy \quad = \quad 2 \text{ units}^3$$

15) The base is the region enclosed by $y = x^2$ and $y = 9$. Slices perpendicular to the x-axis are right isosceles triangles.

16) The base is the area between $y = x$ and $y = x^2$. Slices perpendicular to the x-axis are semicircles.

Answer:



$$V = \int_0^1 \frac{\pi}{8}\left(x - x^2\right)^2 dx \quad = \quad \frac{\pi}{240} \text{ units}^3$$

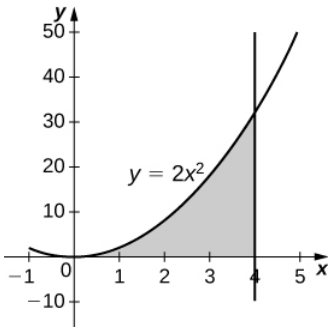
Disk and Washer Method

For exercises 17 - 24, draw the region bounded by the curves. Then, use the disk or washer method to find the volume when the region is rotated around the x-axis.

17) $x + y = 8, \quad x = 0$, and $y = 0$

18) $y = 2x^2, \quad x = 0, \quad x = 4$, and $y = 0$

Answer:

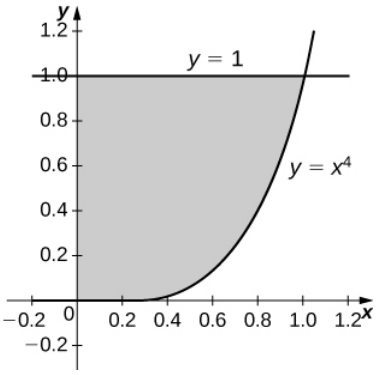


$$V = \int_0^4 4\pi x^4 \, dx \quad = \quad \frac{4096\pi}{5} \text{ units}^3$$

19) $y = e^x + 1, \quad x = 0, \quad x = 1$, and $y = 0$

20) $y = x^4, \quad x = 0$, and $y = 1$

Answer:



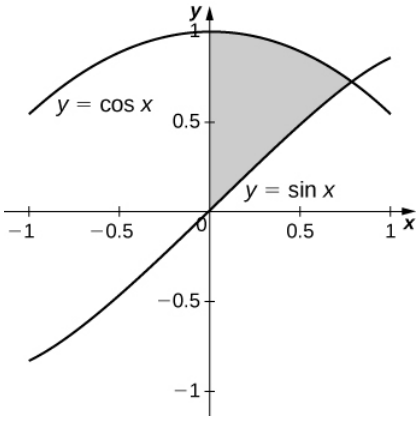
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$$V = \int_0^1 \pi \left(1^2 - \left(x^4\right)^2\right)\, dx = \int_0^1 \pi \left(1 - x^8\right)\, dx \quad = \quad \frac{8\pi}{9} \quad \text{units}^3$$

21) $y = \sqrt{x}, \quad x = 0, \quad x = 4,$ and $y = 0$

22) $y = \sin x, \quad y = \cos x,$ and $x = 0$

Answer:

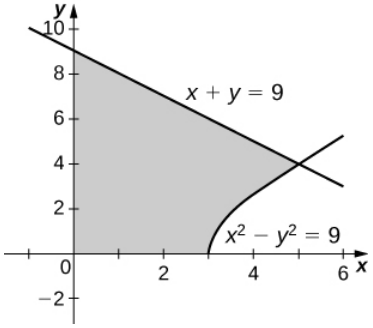


$$V = \int_0^{\pi/4} \pi \left(\cos^2 x - \sin^2 x\right)\, dx = \int_0^{\pi/4} \pi \cos 2x\, dx \quad = \quad \frac{\pi}{2} \quad \text{units}^3$$

23) $y = \dfrac{1}{x}, \quad x = 2$, and $y = 3$

24) $x^2 - y^2 = 9$ and $x + y = 9, \quad y = 0$ and $x = 0$

Answer:



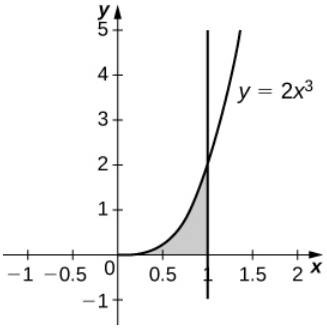
$V = 207\pi$ units^3

For exercises 25 - 32, draw the region bounded by the curves. Then, find the volume when the region is rotated around the y-axis.

25) $y = 4 - \dfrac{1}{2}x, \quad x = 0,$ and $y = 0$

26) $y = 2x^3, \quad x = 0, \quad x = 1,$ and $y = 0$

Answer:

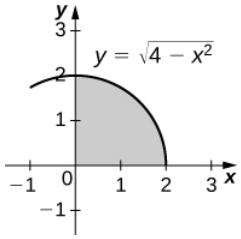


$V = \frac{4\pi}{5}$ units^3

27) $y = 3x^2, \quad x = 0,$ and $y = 3$

28) $y = \sqrt{4 - x^2}, \quad y = 0,$ and $x = 0$

Answer:

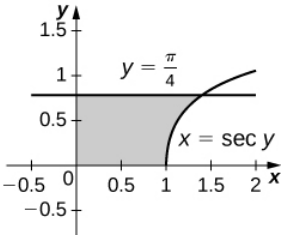


$V = \frac{16\pi}{3}$ units^3

29) $y = \dfrac{1}{\sqrt{x+1}}, \quad x = 0$, and $x = 3$

30) $x = \sec(y)$ and $y = \dfrac{\pi}{4}, \quad y = 0$ and $x = 0$

Answer:

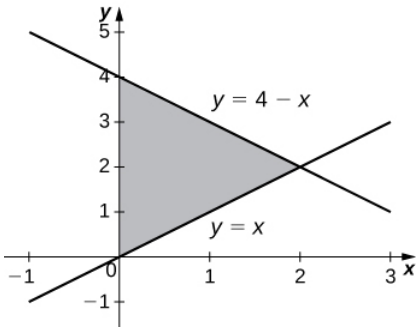


$V = \pi$ units^3

31) $y = \dfrac{1}{x+1}, \quad x = 0,$ and $x = 2$

32) $y = 4 - x, \quad y = x,$ and $x = 0$

Answer:



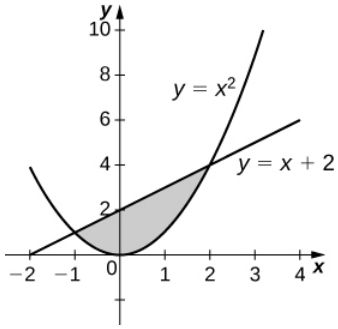
$V = \frac{16\pi}{3}$ units^3

For exercises 33 - 40, draw the region bounded by the curves. Then, find the volume when the region is rotated around the x-axis.

33) $y = x + 2, \quad y = x + 6, \quad x = 0$, and $x = 5$

34) $y = x^2$ and $y = x + 2$

Answer:

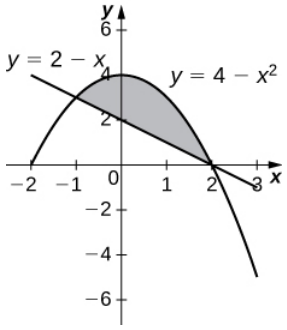


$V = \frac{72\pi}{5}$ units^3

35) $x^2 = y^3$ and $x^3 = y^2$

36) $y = 4 - x^2$ and $y = 2 - x$

Answer:

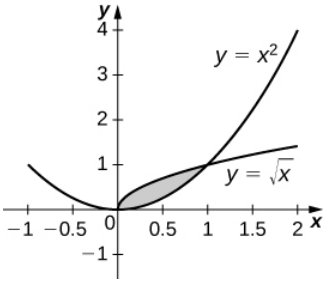


$V = \frac{108\pi}{5}$ units^3

37) [T] $y = \cos x, \quad y = e^{-x}, \quad x = 0$, and $x = 1.2927$

38) $y = \sqrt{x}$ and $y = x^2$

Answer:

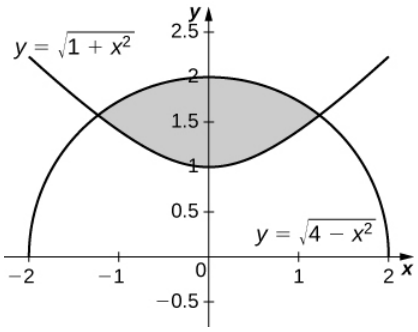


$V = \frac{3\pi}{10}$ units^3

39) $y = \sin x, \quad y = 5\sin x, \quad x = 0$ and $x = \pi$

40) $y = \sqrt{1+x^2}$ and $y = \sqrt{4-x^2}$

Answer:



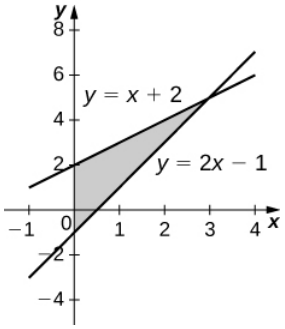
$V = 2\sqrt{6}\pi$ units^3

For exercises 41 - 45, draw the region bounded by the curves. Then, use the washer method to find the volume when the region is revolved around the y-axis.

41) $y = \sqrt{x}, \quad x = 4$, and $y = 0$

42) $y = x + 2, \quad y = 2x - 1$, and $x = 0$

Answer:

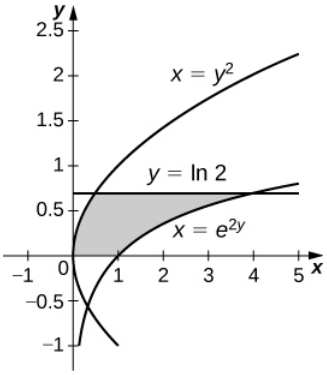


$V = 9\pi$ units^3

43) $y = \dfrac{3}{x}$ and $y = x^3$

44) $x = e^{2y}, \quad x = y^2, \quad y = 0$, and $y = \ln(2)$

Answer:



$$V = \frac{\pi}{20}(75 - 4\ln^5(2)) \text{ units}^3$$

45) $x = \sqrt{9 - y^2}, \quad x = e^{-y}, \quad y = 0$, and $y = 3$

In Exercises 46-51, a region of the Cartesian plane is described. Use the Disk/Washer Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

46) Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:
(a) the x-axis
(b) $y = 1$
(c) the y-axis
(d) $x = 1$

Answer:

(a) $V = \frac{\pi}{2}$ units^3 (c) $V = \frac{4\pi}{5}$ units^3
(b) $V = \frac{5\pi}{6}$ units^3 (d) $V = \frac{8\pi}{15}$ units^3

47) Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:
(a) the x-axis
(b) $y = 4$
(c) $y = -1$
(d) $x = 2$

48) The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.
Rotate about:
(a) the x-axis
(b) $y = 2$
(c) the y-axis
(d) $x = 1$

Answer:

(a) $V = \frac{4\pi}{3}$ units^3 (c) $V = \frac{4\pi}{3}$ units^3
(b) $V = \frac{2\pi}{3}$ units^3 (d) $V = \frac{\pi}{3}$ units^3

49) Region bounded by: $y = y = x^2 - 2x + 2$, and $y = 2x - 1$.
Rotate about:
(a) the x-axis
(b) $y = 1$
(c) $y = 5$

50) Region bounded by: $y = \dfrac{1}{\sqrt{2-x}}$, $x = -2$, $x = 1$ and the x-axis.

Rotate about:
(a) the x-axis
(b) $y = 1$
(c) $y = -1$

Answer:

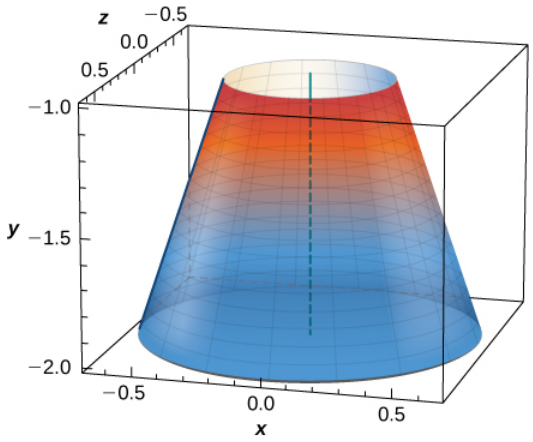
(a) $V = \pi \displaystyle\int_{-2}^{1} \left(\frac{1}{\sqrt{2-x}}\right)^2 dx = (\pi \ln 4)$ units$^3 \approx 4.35517$units^3

(b) $V = \pi \displaystyle\int_{-2}^{1} \left[(1)^2 - \left(1 - \frac{1}{\sqrt{2-x}}\right)^2\right] dx = (4 - \ln 4)\,\pi$ units$^3 \approx 8.21120$units^3

(c) $V = \pi \displaystyle\int_{-2}^{1} \left[\left(\frac{1}{\sqrt{2-x}} + 1\right)^2 - (1)^2\right] dx = (4 + \ln 4)\,\pi$ units$^3 \approx 16.92154$units^3

51) Region bounded by: $y = 2x$, $y = x$ and $x = 2$.
Rotate about:
(a) the x-axis
(b) $y = 4$
(c) the y-axis
(d) $x = 2$

52) Yogurt containers can be shaped like frustums. Rotate the line $y = \left(\frac{1}{m}\right) x$ around the y-axis to find the volume between $y = a$ and $y = b$.

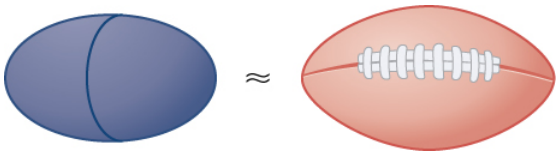




Answer:

$V = \dfrac{m^2\pi}{3}(b^3 - a^3)$ units^3

53) Rotate the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ around the x-axis to approximate the volume of a football, as seen here.

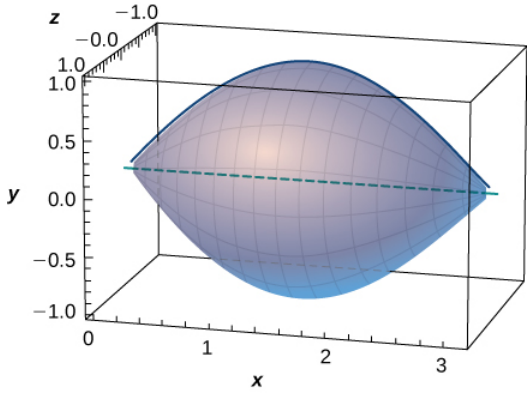


54) Rotate the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ around the y-axis to approximate the volume of a football.

Answer:

$V=\frac{4a^2b\pi}{3}$ units^3

55) A better approximation of the volume of a football is given by the solid that comes from rotating $y=\sin x$ around the x-axis from $x=0$ to $x=\pi$. What is the volume of this football approximation, as seen here?



For exercises 57 - 62, find the volume of the solid described.

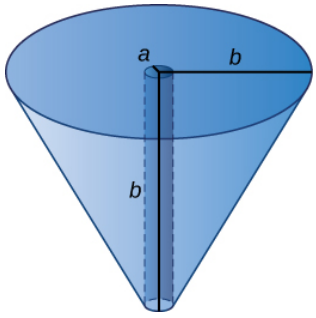
57) The base is the region between $y=x$ and $y=x^2$. Slices perpendicular to the x-axis are semicircles.

58) The base is the region enclosed by the generic ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Slices perpendicular to the x-axis are semicircles.

Answer:

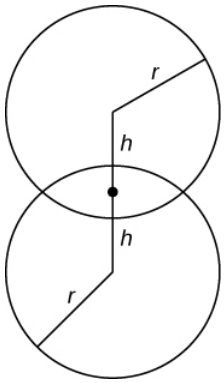
$V=\frac{2ab^2\pi}{3}$ units^3

59) Bore a hole of radius a down the axis of a right cone and through the base of radius b, as seen here.



60) Find the volume common to two spheres of radius r with centers that are $2h$ apart, as shown here.

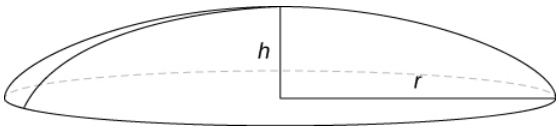
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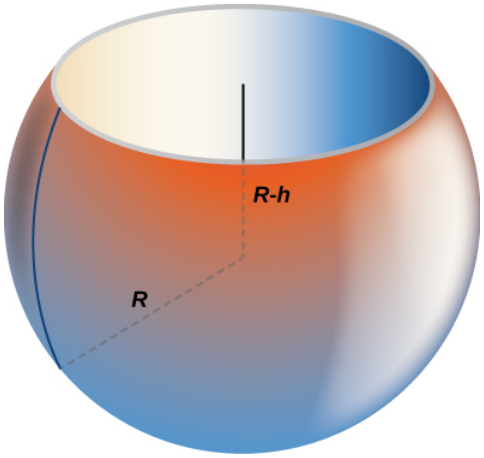
Answer:

$V = \frac{\pi}{12}(r+h)^2(6r-h)$ units^3

61) Find the volume of a spherical cap of height h and radius r where $h < r$, as seen here.



62) Find the volume of a sphere of radius R with a cap of height h removed from the top, as seen here.



Answer:

$V = \frac{\pi}{3}(h+R)(h-2R)^2$ units^3

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) added integrals to solutions for many of these exercises, as well as a full solution to #6, and #50, as a new problem.
- Problems 46 - 49 and 51 are from Apex Calculus Section 7.2

  Gregory Hartman (Virginia Military Institute). Contributions were made by Troy Siemers and Dimplekumar Chalishajar of VMI and Brian Heinold of Mount Saint Mary's University. This content is copyrighted by a Creative Commons Attribution - Noncommercial (BY-NC) License. http://www.apexcalculus.com/

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3.4: Volumes of Revolution- The Shell Method

Often a given problem can be solved in more than one way. A particular method may be chosen out of convenience, personal preference, or perhaps necessity. Ultimately, it is good to have options.

The previous section introduced the Disk and Washer Methods, which computed the volume of solids of revolution by integrating the cross--sectional area of the solid. This section develops another method of computing volume, the **Shell Method**. Instead of slicing the solid perpendicular to the axis of rotation creating cross-sections, we now slice it parallel to the axis of rotation, creating "shells."

Consider Figure 3.4.1, where the region shown in (a) is rotated around the y-axis forming the solid shown in (b). A small slice of the region is drawn in (a), parallel to the axis of rotation. When the region is rotated, this thin slice forms a **cylindrical shell**, as pictured in part (c) of the figure. The previous section approximated a solid with lots of thin disks (or washers); we now approximate a solid with many thin cylindrical shells.

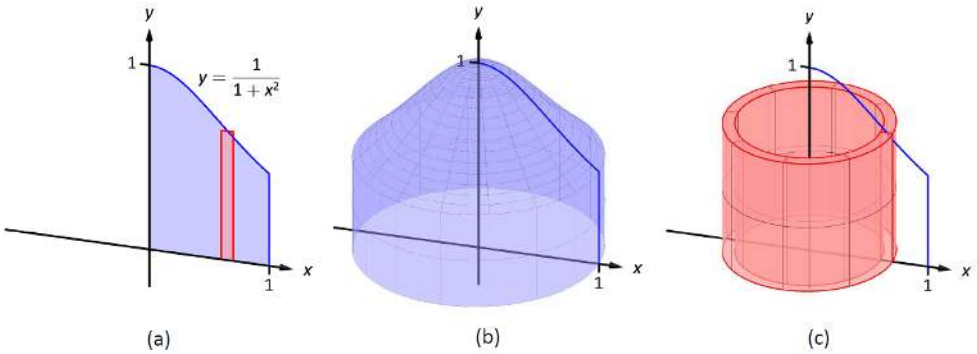


Figure 3.4.1: *Introducing the Shell Method.*

Figure 3.4.1 (d): *A dynamic version of this figure created using CalcPlot3D.*

To compute the volume of one shell, first consider the paper label on a soup can with radius r and height h. What is the area of this label? A simple way of determining this is to cut the label and lay it out flat, forming a rectangle with height h and length $2\pi r$. Thus the area is $A = 2\pi rh$; see Figure 3.4.2a.

Do a similar process with a cylindrical shell, with height h, thickness Δx, and approximate radius r. Cutting the shell and laying it flat forms a rectangular solid with length $2\pi r$, height h and depth dx. Thus the volume is $V \approx 2\pi rh\, dx$; see Figure 3.4.2c. (We say "approximately" since our radius was an approximation.)

By breaking the solid into n cylindrical shells, we can approximate the volume of the solid as

$$V = \sum_{i=1}^{n} 2\pi r_i h_i\, dx_i, \tag{3.4.1}$$

where r_i, h_i and dx_i are the radius, height and thickness of the ith shell, respectively.

This is a Riemann Sum. Taking a limit as the thickness of the shells approaches 0 leads to a definite integral.

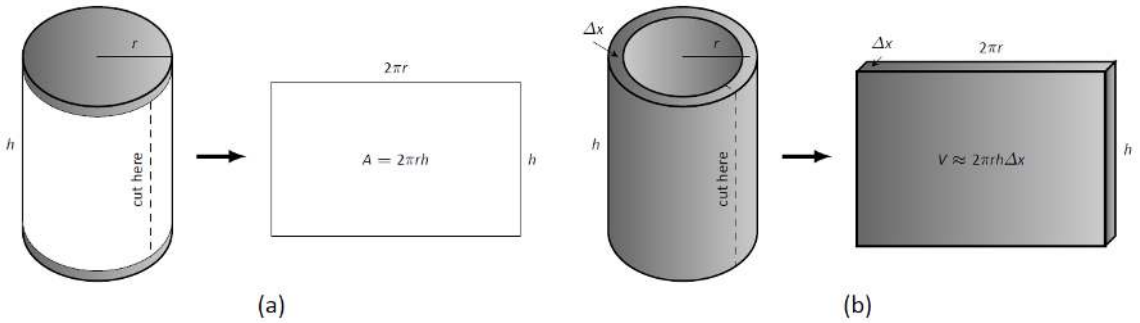


Figure 3.4.2: *Determining the volume of a thin cylindrical shell.}\label{fig:soupcan}*

Key Idea 25: Shell Method

Let a solid be formed by revolving a region R, bounded by $x = a$ and $x = b$, around a vertical axis. Let $r(x)$ represent the distance from the axis of rotation to x (i.e., the radius of a sample shell) and let $h(x)$ represent the height of the solid at x (i.e., the height of the shell). The volume of the solid is

$$V = 2\pi \int_a^b r(x)h(x)\,dx. \tag{3.4.2}$$

Special Cases:

1. When the region R is bounded above by $y = f(x)$ and below by $y = g(x)$, then $h(x) = f(x) - g(x)$.
2. When the axis of rotation is the y-axis (i.e., $x = 0$) then $r(x) = x$.

Let's practice using the Shell Method.

Example 3.4.1: Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region bounded by $y = 0$, $y = 1/(1+x^2)$, $x = 0$ and $x = 1$ about the y-axis.

Solution

This is the region used to introduce the Shell Method in Figure 3.4.1, but is sketched again in Figure 3.4.3 for closer reference. A line is drawn in the region parallel to the axis of rotation representing a shell that will be carved out as the region is rotated about the y-axis. (This is the differential element.)

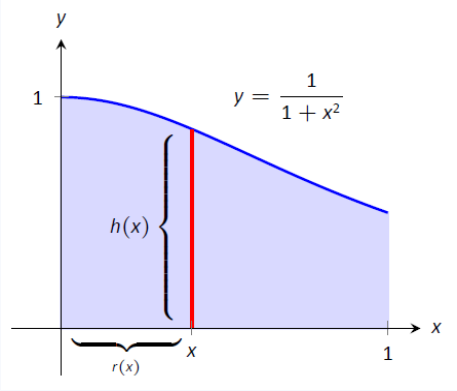


Figure 3.4.3: *Graphing a region in Example 3.4.1.*

The distance this line is from the axis of rotation determines $r(x)$; as the distance from x to the y-axis is x, we have $r(x) = x$. The height of this line determines $h(x)$; the top of the line is at $y = 1/(1+x^2)$, whereas the bottom of the line is at $y = 0$. Thus $h(x) = 1/(1+x^2) - 0 = 1/(1+x^2)$. The region is bounded from $x = 0$ to $x = 1$, so the volume is

$$V = 2\pi \int_0^1 \frac{x}{1+x^2}\,dx. \tag{3.4.3}$$

This requires substitution. Let $u = 1 + x^2$, so $du = 2x\,dx$. We also change the bounds: $u(0) = 1$ and $u(1) = 2$. Thus we have:

$$\begin{aligned} &= \pi \int_1^2 \frac{1}{u}\,du \\ &= \pi \ln u \Big|_1^2 \\ &= \pi \ln 2 - \pi \ln 1 \\ &= \pi \ln 2 \approx 2.178 \text{ units}^3. \end{aligned}$$

Note: in order to find this volume using the Disk Method, two integrals would be needed to account for the regions above and below $y = 1/2$.

With the Shell Method, nothing special needs to be accounted for to compute the volume of a solid that has a hole in the middle, as demonstrated next.

Example 3.4.2: Finding volume using the Shell Method

Find the volume of the solid formed by rotating the triangular region determined by the points $(0,1)$, $(1,1)$ and $(1,3)$ about the line $x=3$.

Solution

The region is sketched in Figure $3.4.4a$ along with the differential element, a line within the region parallel to the axis of rotation. In part (b) of the figure, we see the shell traced out by the differential element, and in part (c) the whole solid is shown.

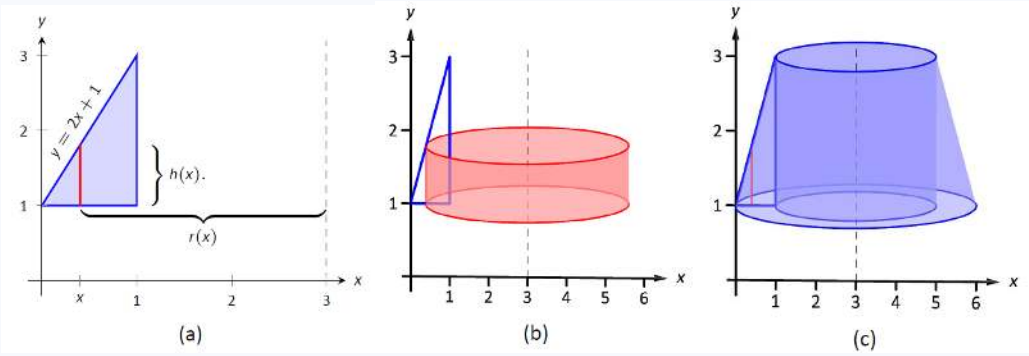


***Figure* 3.4.4:** *Graphing a region in Example* 3.4.2

The height of the differential element is the distance from $y=1$ to $y=2x+1$, the line that connects the points $(0,1)$ and $(1,3)$. Thus $h(x)=2x+1-1=2x$. The radius of the shell formed by the differential element is the distance from x to $x=3$; that is, it is $r(x)=3-x$. The x-bounds of the region are $x=0$ to $x=1$, giving

$$\begin{aligned} V &= 2\pi \int_0^1 (3-x)(2x)\,dx \\ &= 2\pi \int_0^1 \left(6x-2x^2\right)dx \\ &= 2\pi \left(3x^2-\frac{2}{3}x^3\right)\Big|_0^1 \\ &= \frac{14}{3}\pi \approx 14.66 \text{ units}^3. \end{aligned}$$

When revolving a region around a horizontal axis, we must consider the radius and height functions in terms of y, not x.

Example 3.4.3: Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region given in Example $3.4.2$ about the x-axis.

Solution

The region is sketched in Figure $3.4.5a$ with a sample differential element. In part (b) of the figure the shell formed by the differential element is drawn, and the solid is sketched in (c). (Note that the triangular region looks "short and wide" here, whereas in the previous example the same region looked "tall and narrow." This is because the bounds on the graphs are different.)

The height of the differential element is an x-distance, between $x=\frac{1}{2}y-\frac{1}{2}$ and $x=1$. Thus $h(y)=1-(\frac{1}{2}y-\frac{1}{2})=-\frac{1}{2}y+\frac{3}{2}$. The radius is the distance from y to the x-axis, so $r(y)=y$. The y bounds of the region are $y=1$ and $y=3$, leading to the integral

$$
\begin{aligned}
V &= 2\pi \int_1^3 \left[y\left(-\frac{1}{2}y + \frac{3}{2}\right)\right]\, dy \\
&= 2\pi \int_1^3 \left[-\frac{1}{2}y^2 + \frac{3}{2}y\right]\, dy \\
&= 2\pi \left[-\frac{1}{6}y^3 + \frac{3}{4}y^2\right]\Big|_1^3 \\
&= 2\pi \left[\frac{9}{4} - \frac{7}{12}\right] \\
&= \frac{10}{3}\pi \approx 10.472 \text{ units}^3.
\end{aligned}
$$

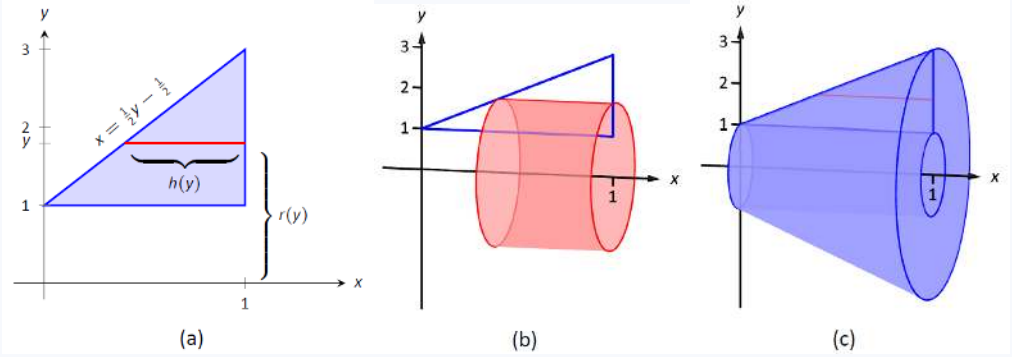


Figure 3.4.5: *Graphing a region in Example* 3.4.3

At the beginning of this section it was stated that "it is good to have options." The next example finds the volume of a solid rather easily with the Shell Method, but using the Washer Method would be quite a chore.

Example 3.4.4: Finding volume using the Shell Method

Find the volume of the solid formed by revolving the region bounded by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ about the y-axis.

Solution

The region and a differential element, the shell formed by this differential element, and the resulting solid are given in Figure 3.4.6.

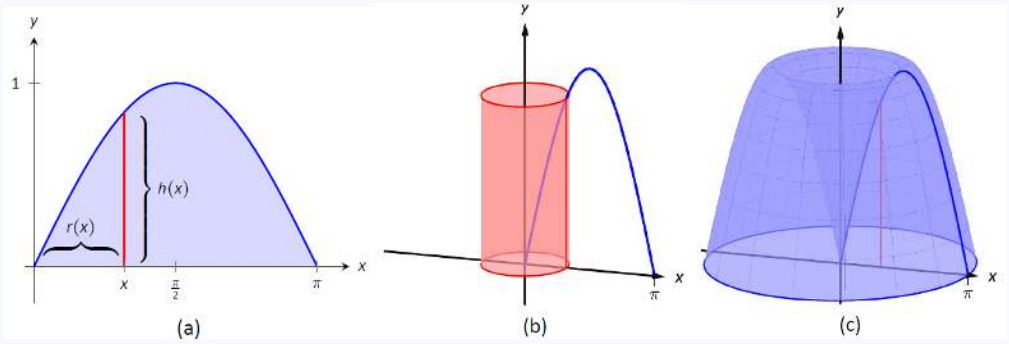


Figure 3.4.6a: *Graphing a region in Example* 3.4.4

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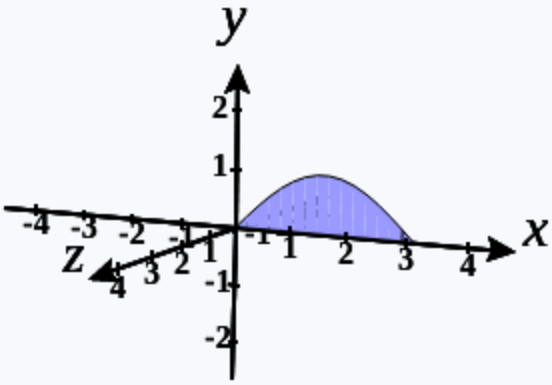


Figure **3.4.6*b*:** Visualizing this figure using CalcPlot3D

The radius of a sample shell is $r(x) = x$; the height of a sample shell is $h(x) = \sin x$, each from $x = 0$ to $x = \pi$. Thus the volume of the solid is

$$V = 2\pi \int_0^{\pi} x \sin x \, dx. \tag{3.4.4}$$

This requires Integration By Parts. Set $u = x$ and $dv = \sin x \, dx$; we leave it to the reader to fill in the rest. We have:

$$\begin{aligned} &= 2\pi \Big[-x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \, dx \Big] \\ &= 2\pi \Big[\pi + \sin x \Big|_0^{\pi} \Big] \\ &= 2\pi \Big[\pi + 0 \Big] \\ &= 2\pi^2 \approx 19.74 \text{ units}^3. \end{aligned}$$

Note that in order to use the Washer Method, we would need to solve $y = \sin x$ for x, requiring the use of the arcsine function. We leave it to the reader to verify that the outside radius function is $R(y) = \pi - \arcsin y$ and the inside radius function is $r(y) = \arcsin y$. Thus the volume can be computed as

$$\pi \int_0^1 \Big[(\pi - \arcsin y)^2 - (\arcsin y)^2 \Big] \, dy. \tag{3.4.5}$$

This integral isn't terrible given that the $\arcsin^2 y$ terms cancel, but it is more onerous than the integral created by the Shell Method.

We end this section with a table summarizing the usage of the Washer and Shell Methods.

Key Idea 26: Summary of the Washer and Shell Methods

Let a region R be given with x-bounds $x = a$ and $x = b$ and y-bounds $y = c$ and $y = d$.

	Washer Method	Shell Method
Horizontal Axis	$\pi \int_a^b \left(R(x)^2 - r(x)^2\right) dx$	$2\pi \int_c^d r(y)h(y)\, dy$
Vertical Axis	$\pi \int_c^d \left(R(y)^2 - r(y)^2\right) dy$	$2\pi \int_a^b r(x)h(x)\, dx$

As in the previous section, the real goal of this section is not to be able to compute volumes of certain solids. Rather, it is to be able to solve a problem by first approximating, then using limits to refine the approximation to give the exact value. In this section, we approximate the volume of a solid by cutting it into thin cylindrical shells. By summing up the volumes of each shell, we get an approximation of the volume. By taking a limit as the number of equally spaced shells goes to infinity, our summation can be evaluated as a definite integral, giving the exact value.

We use this same principle again in the next section, where we find the length of curves in the plane.

Contributors

- Gregory Hartman (Virginia Military Institute). Contributions were made by Troy Siemers and Dimplekumar Chalishajar of VMI and Brian Heinold of Mount Saint Mary's University. This content is copyrighted by a Creative Commons Attribution - Noncommercial (BY-NC) License. http://www.apexcalculus.com/
- Integrated by Justin Marshall.

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3.4b: Volumes of Revolution- Cylindrical Shells OS

Learning Objectives

- Calculate the volume of a solid of revolution by using the method of cylindrical shells.
- Compare the different methods for calculating a volume of revolution.

In this section, we examine the method of cylindrical shells, the final method for finding the volume of a solid of revolution. We can use this method on the same kinds of solids as the disk method or the washer method; however, with the disk and washer methods, we integrate along the coordinate axis parallel to the axis of revolution. With the method of cylindrical shells, we integrate along the coordinate *axis perpendicular* to the axis of revolution. The ability to choose which variable of integration we want to use can be a significant advantage with more complicated functions. Also, the specific geometry of the solid sometimes makes the method of using cylindrical shells more appealing than using the washer method. In the last part of this section, we review all the methods for finding volume that we have studied and lay out some guidelines to help you determine which method to use in a given situation.

The Method of Cylindrical Shells

Again, we are working with a solid of revolution. As before, we define a region R, bounded above by the graph of a function $y = f(x)$, below by the $\boldsymbol{x}$**-axis**, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in Figure $3.4b.1a$. We then revolve this region around the y-axis, as shown in Figure $3.4b.1b$. Note that this is different from what we have done before. Previously, regions defined in terms of functions of x were revolved around the $\boldsymbol{x}$**-axis** or a line parallel to it.

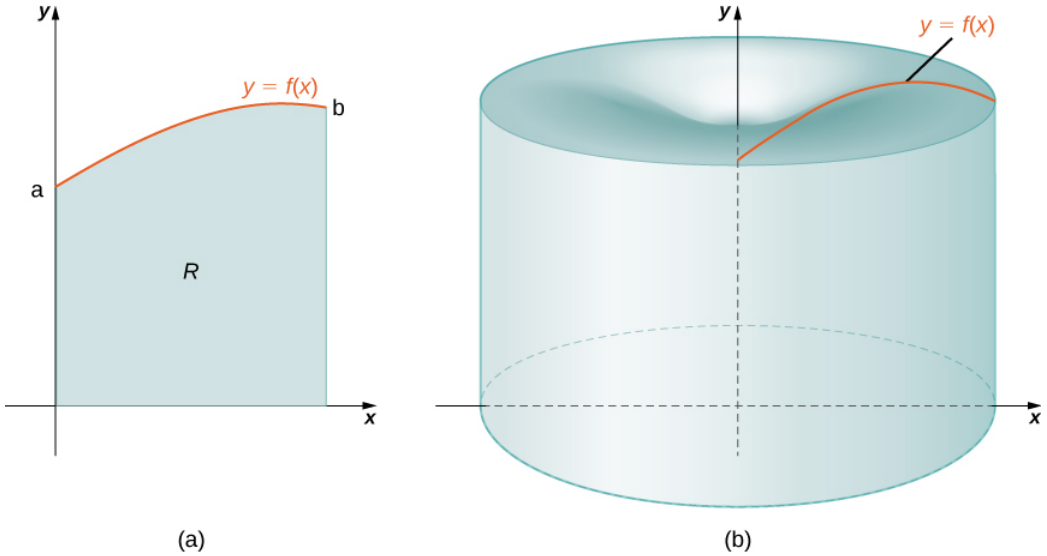


Figure $3.4b.1$: (a) A region bounded by the graph of a function of x. (b) The solid of revolution formed when the region is revolved around the $\boldsymbol{y}$**-axis.**

As we have done many times before, partition the interval $[a, b]$ using a regular partition, $P = x_0, x_1, \ldots, x_n$ and, for $i = 1, 2, \ldots, n$, choose a point $x_i^* \in [x_{i-1}, x_i]$. Then, construct a rectangle over the interval $[x_{i-1}, x_i]$ of height $f(x_i^*)$ and width Δx. A representative rectangle is shown in Figure $3.4b.2a$. When that rectangle is revolved around the y-axis, instead of a disk or a washer, we get a cylindrical shell, as shown in Figure $3.4b.2$.

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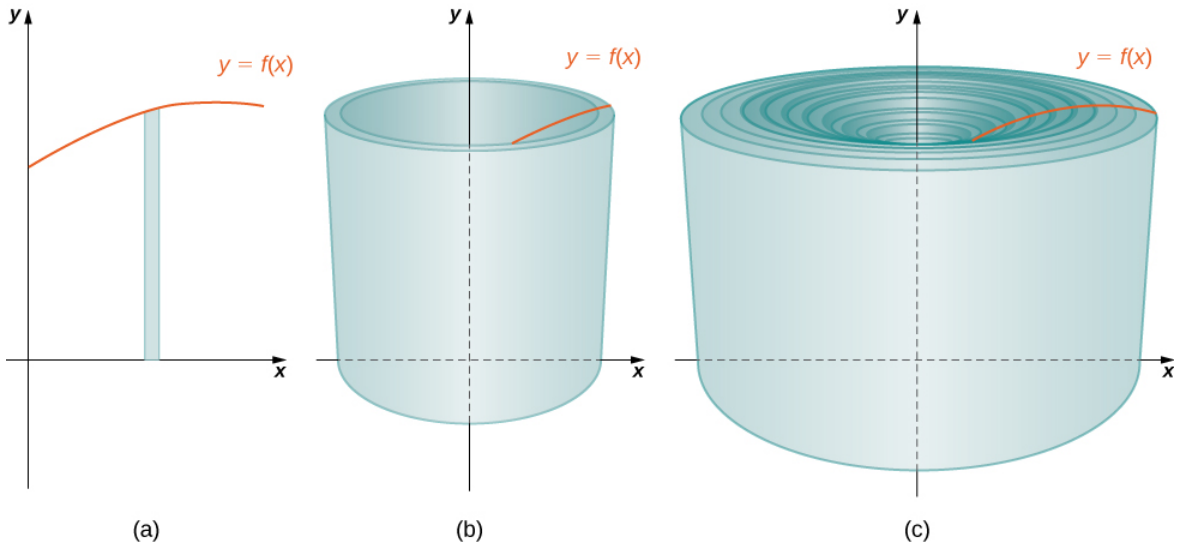


Figure 3.4b.2: (a) A representative rectangle. (b) When this rectangle is revolved around the ***y*-axis**, the result is a cylindrical shell. (c) When we put all the shells together, we get an approximation of the original solid.

To calculate the volume of this shell, consider Figure 3.4b.3.

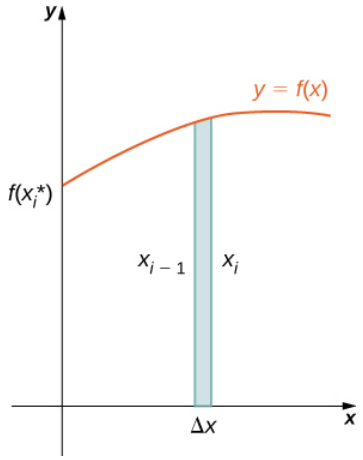


Figure 3.4b.3: Calculating the volume of the shell.

The shell is a cylinder, so its volume is the cross-sectional area multiplied by the height of the cylinder. The cross-sections are annuli (ring-shaped regions—essentially, circles with a hole in the center), with outer radius x_i and inner radius x_{i-1}. Thus, the cross-sectional area is $\pi x_i^2 - \pi x_{i-1}^2$. The height of the cylinder is $f(x_i^*)$. Then the volume of the shell is

$$\begin{aligned} V_{shell} &= f(x_i^*)(\pi\, x_i^2 - \pi\, x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi\, f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

Note that $x_i - x_{i-1} = \Delta x,$ so we have

$$V_{shell} = 2\pi\, f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)\Delta x.$$

Furthermore, $\dfrac{x_i + x_{i-1}}{2}$ is both the midpoint of the interval $[x_{i-1}, x_i]$ and the average radius of the shell, and we can approximate this by x_i^*. We then have

$$V_{shell} \approx 2\pi\, f(x_i^*)x_i^*\, \Delta x.$$

Another way to think of this is to think of making a vertical cut in the shell and then opening it up to form a flat plate (Figure $3.4b.4$).

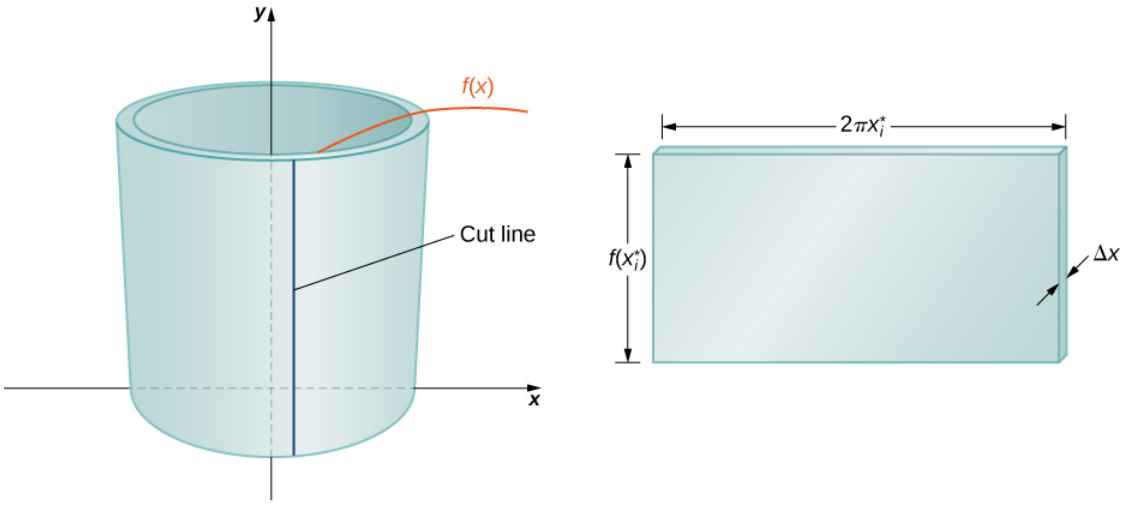


Figure $3.4b.4$: (a) Make a vertical cut in a representative shell. (b) Open the shell up to form a flat plate.

In reality, the outer radius of the shell is greater than the inner radius, and hence the back edge of the plate would be slightly longer than the front edge of the plate. However, we can approximate the flattened shell by a flat plate of height $f(x_i^*)$, width $2\pi x_i^*$, and thickness Δx (Figure). The volume of the shell, then, is approximately the volume of the flat plate. Multiplying the height, width, and depth of the plate, we get

$$V_{shell} \approx f(x_i^*)(2\pi\, x_i^*)\,\Delta x,$$

which is the same formula we had before.

To calculate the volume of the entire solid, we then add the volumes of all the shells and obtain

$$V \approx \sum_{i=1}^{n} (2\pi\, x_i^* f(x_i^*)\,\Delta x).$$

Here we have another Riemann sum, this time for the function $2\pi\, x\, f(x)$.Taking the limit as $n \to \infty$ gives us

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} (2\pi\, x_i^* f(x_i^*)\,\Delta x) = \int_a^b (2\pi\, x\, f(x))\, dx.$$

This leads to the following rule for the *method of cylindrical shells*.

Rule: The Method of Cylindrical Shells

Let $f(x)$ be continuous and nonnegative. Define R as the region bounded above by the graph of $f(x)$, below by the **x-axis,** on the left by the line $x = a$, and on the right by the line $x = b$. Then the volume of the solid of revolution formed by revolving R around the y-axis is given by

$$V = \int_a^b (2\pi\, x\, f(x))\, dx.$$

Now let's consider an example.

Example $3.4b.1$: The Method of Cylindrical Shells I

Define R as the region bounded above by the graph of $f(x) = 1/x$ and below by the **x-axis** over the interval $[1, 3]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Solution

First we must graph the region R and the associated solid of revolution, as shown in Figure $3.4b.5$.

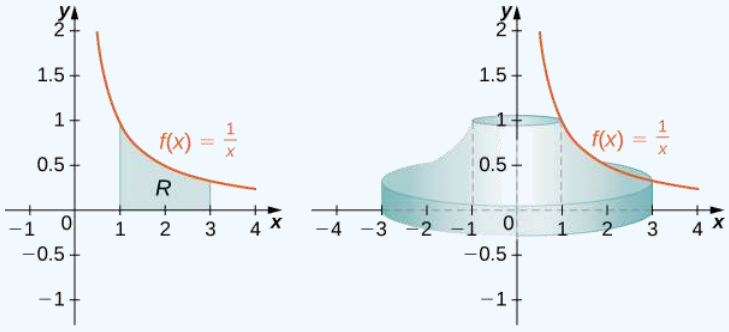


Figure $3.4b.5$: (a) The region R under the graph of $f(x) = 1/x$ over the interval $[1, 3]$. (b) The solid of revolution generated by revolving R about the y-axis.

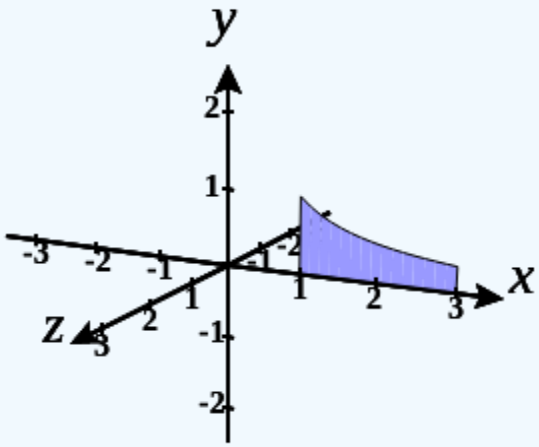


Figure $3.4b.5$ (c) Visualizing the solid of revolution with CalcPlot3D.

Then the volume of the solid is given by

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$$V = \int_a^b (2\pi\, x\, f(x))\, dx$$
$$= \int_1^3 \left(2\pi\, x \left(\frac{1}{x}\right)\right) dx$$
$$= \int_1^3 2\pi\, dx$$
$$= 2\pi\, x\Big|_1^3 = 4\pi \text{ units}^3.$$

? Exercise 3.4b.1

Define R as the region bounded above by the graph of $f(x) = x^2$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint

Use the procedure from Example 3.4b.1.

Answer

$\frac{15\pi}{2}$ units^3

✓ Example 3.4b.2: The Method of Cylindrical Shells II

Define R as the region bounded above by the graph of $f(x) = 2x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the **y-axis.**

Solution

First graph the region R and the associated solid of revolution, as shown in Figure 3.4b.6.

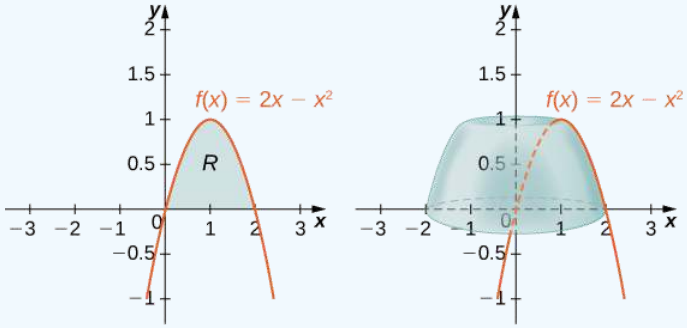


Figure 3.4b.6: (a) The region R *under the graph of* $f(x) = 2x - x^2$ *over the interval* $[0, 2]$. *(b) The volume of revolution obtained by revolving* R *about the* y*-axis.*

Then the volume of the solid is given by

$$
\begin{aligned}
V &= \int_a^b (2\pi\, x\, f(x))\, dx \\
&= \int_0^2 (2\pi\, x(2x - x^2))\, dx \\
&= 2\pi \int_0^2 (2x^2 - x^3)\, dx \\
&= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4} \right] \Big|_0^2 \\
&= \frac{8\pi}{3} \text{ units}^3
\end{aligned}
$$

? Exercise 3.4b. 2

Define R as the region bounded above by the graph of $f(x) = 3x - x^2$ and below by the x-axis over the interval $[0, 2]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint

Use the process from Example 3.4b. 2

Answer

8π units^3

As with the disk method and the washer method, we can use the method of cylindrical shells with solids of revolution, revolved around the x-axis, when we want to integrate with respect to y. The analogous rule for this type of solid is given here.

Rule: The Method of Cylindrical Shells for Solids of Revolution around the x-axis

Let $g(y)$ be continuous and nonnegative. Define Q as the region bounded on the right by the graph of $g(y)$, on the left by the y-axis, below by the line $y = c$, and above by the line $y = d$. Then, the volume of the solid of revolution formed by revolving Q around the x-axis is given by

$$V = \int_c^d (2\pi\, y\, g(y))\, dy.$$

✓ Example 3.4b. 3: The Method of Cylindrical Shells for a Solid Revolved around the x-axis

Define Q as the region bounded on the right by the graph of $g(y) = 2\sqrt{y}$ and on the left by the y-axis for $y \in [0, 4]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Solution

First, we need to graph the region Q and the associated solid of revolution, as shown in Figure 3.4b. 7.

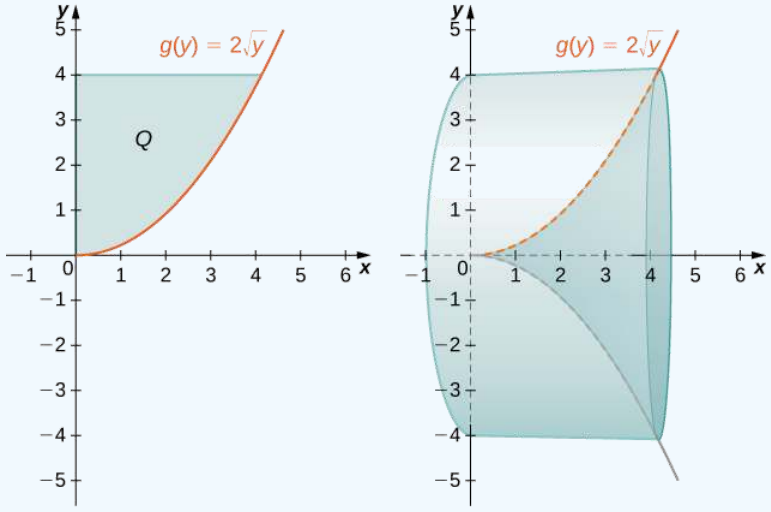


Figure $3.4b.7$: (a) The region Q to the left of the function $g(y)$ over the interval $[0,4]$. (b) The solid of revolution generated by revolving Q around the x-axis.

Label the shaded region Q. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_c^d (2\pi\, y\, g(y))\, dy \\ &= \int_0^4 (2\pi\, y(2\sqrt{y}))\, dy \\ &= 4\pi \int_0^4 y^{3/2}\, dy \\ &= 4\pi \left[\frac{2y^{5/2}}{5}\right]\Bigg|_0^4 \\ &= \frac{256\pi}{5} \text{ units}^3 \end{aligned}$$

? Exercise $3.4b.3$

Define Q as the region bounded on the right by the graph of $g(y) = 3/y$ and on the left by the y-axis for $y \in [1,3]$. Find the volume of the solid of revolution formed by revolving Q around the x-axis.

Hint

Use the process from Example $3.4b.3$

Answer

12π units^3

For the next example, we look at a solid of revolution for which the graph of a function is revolved around a line other than one of the two coordinate axes. To set this up, we need to revisit the development of the method of cylindrical shells. Recall that we found the volume of one of the shells to be given by

$$\begin{aligned} V_{shell} &= f(x_i^*)(\pi\, x_i^2 - \pi\, x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i^2 - x_{i-1}^2) \\ &= \pi\, f(x_i^*)(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= 2\pi\, f(x_i^*)\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}). \end{aligned}$$

This was based on a shell with an outer radius of x_i and an inner radius of x_{i-1}. If, however, we rotate the region around a line other than the y-axis, we have a different outer and inner radius. Suppose, for example, that we rotate the region around the line $x = -k$, where k is some positive constant. Then, the outer radius of the shell is $x_i + k$ and the inner radius of the shell is $x_{i-1} + k$. Substituting these terms into the expression for volume, we see that when a plane region is rotated around the line $x = -k$, the volume of a shell is given by

$$\begin{aligned} V_{shell} &= 2\pi\, f(x_i^*)\left(\frac{(x_i + k) + (x_{i-1} + k)}{2}\right)((x_i + k) - (x_{i-1} + k)) \\ &= 2\pi\, f(x_i^*)\left(\left(\frac{x_i + x_{i-2}}{2}\right) + k\right)\Delta x. \end{aligned}$$

As before, we notice that $\dfrac{x_i + x_{i-1}}{2}$ is the midpoint of the interval $[x_{i-1}, x_i]$ and can be approximated by x_i^*. Then, the approximate volume of the shell is

$$V_{shell} \approx 2\pi(x_i^* + k) f(x_i^*)\Delta x.$$

The remainder of the development proceeds as before, and we see that

$$V = \int_a^b (2\pi(x + k) f(x))dx.$$

We could also rotate the region around other horizontal or vertical lines, such as a vertical line in the right half plane. In each case, the volume formula must be adjusted accordingly. Specifically, the x-term in the integral must be replaced with an expression representing the radius of a shell. To see how this works, consider the following example.

Example $3.4b.4$: A Region of Revolution Revolved around a Line

Define R as the region bounded above by the graph of $f(x) = x$ and below by the x-axis over the interval $[1, 2]$. Find the volume of the solid of revolution formed by revolving R around the line $x = -1$.

Solution

First, graph the region R and the associated solid of revolution, as shown in Figure $3.4b.8$

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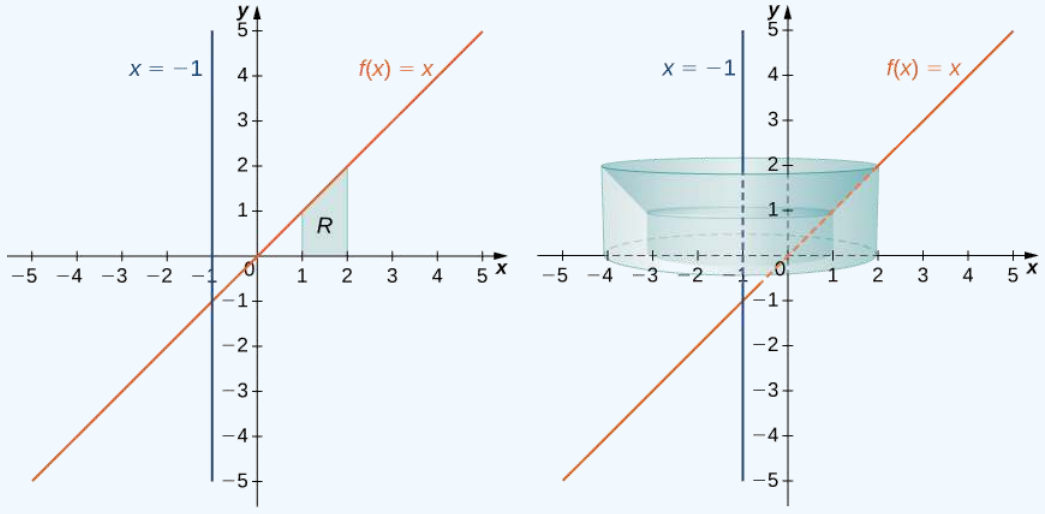


Figure 3.4b.8: (a) The region R between the graph of $f(x)$ and the x-axis over the interval $[1,2]$. (b) The solid of revolution generated by revolving R around the line $x=-1$.

Note that the radius of a shell is given by $x+1$. Then the volume of the solid is given by

$$\begin{aligned} V &= \int_1^2 2\pi(x+1)f(x)\,dx \\ &= \int_1^2 2\pi(x+1)x\,dx = 2\pi\int_1^2 x^2+x\,dx \\ &= 2\pi\left[\frac{x^3}{3}+\frac{x^2}{2}\right]\Big|_1^2 \\ &= \frac{23\pi}{3}\text{ units}^3 \end{aligned}$$

? Exercise 3.4b.4

Define R as the region bounded above by the graph of $f(x)=x^2$ and below by the x-axis over the interval $[0,1]$. Find the volume of the solid of revolution formed by revolving R around the line $x=-2$.

Hint

Use the process from Example 3.4b.4

Answer

$\frac{11\pi}{6}$ units^3

For our final example in this section, let's look at the volume of a solid of revolution for which the region of revolution is bounded by the graphs of two functions.

✓ Example 3.4b.5: A Region of Revolution Bounded by the Graphs of Two Functions

Define R as the region bounded above by the graph of the function $f(x)=\sqrt{x}$ and below by the graph of the function $g(x)=1/x$ over the interval $[1,4]$. Find the volume of the solid of revolution generated by revolving R around the y-axis.

Solution

First, graph the region R and the associated solid of revolution, as shown in Figure 3.4b.9.

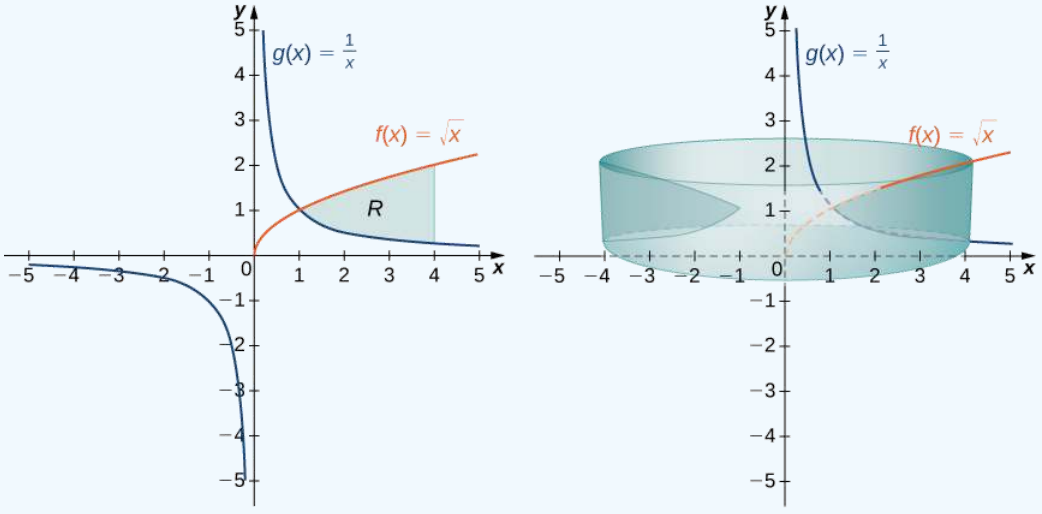


Figure 3.4b.9: (a) The region R between the graph of $f(x)$ and the graph of $g(x)$ over the interval $[1,4]$. (b) The solid of revolution generated by revolving R around the y-axis.

Note that the axis of revolution is the y-axis, so the radius of a shell is given simply by x. We don't need to make any adjustments to the x-term of our integrand. The height of a shell, though, is given by $f(x)-g(x)$, so in this case we need to adjust the $f(x)$ term of the integrand. Then the volume of the solid is given by

$$V = \int_1^4 (2\pi\, x(f(x)-g(x)))\, dx$$

$$= \int_1^4 (2\pi\, x(\sqrt{x} - \frac{1}{x}))\, dx = 2\pi \int_1^4 (x^{3/2} - 1) dx$$

$$= 2\pi \left[\frac{2x^{5/2}}{5} - x\right]\Big|_1^4 = \frac{94\pi}{5} \text{ units}^3.$$

? Exercise 3.4b.5

Define R as the region bounded above by the graph of $f(x) = x$ and below by the graph of $g(x) = x^2$ over the interval $[0,1]$. Find the volume of the solid of revolution formed by revolving R around the y-axis.

Hint

Hint: Use the process from Example 3.4b.5

Answer

$\frac{\pi}{6}$ units^3

Which Method Should We Use?

We have studied several methods for finding the volume of a solid of revolution, but how do we know which method to use? It often comes down to a choice of which integral is easiest to evaluate. Figure 3.4b.10 describes the different approaches for solids of revolution around the x-axis. It's up to you to develop the analogous table for solids of revolution around the y-axis.

Comparing the Methods for Finding the Volume of a Solid Revolution around the x-axis

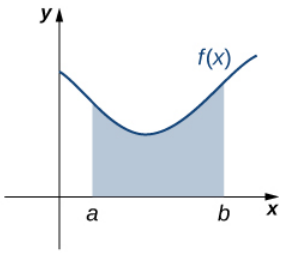
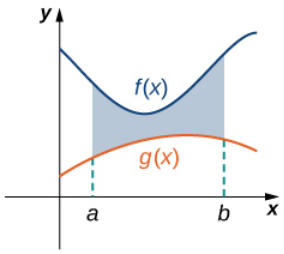
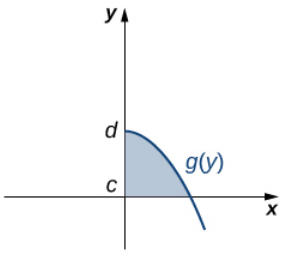
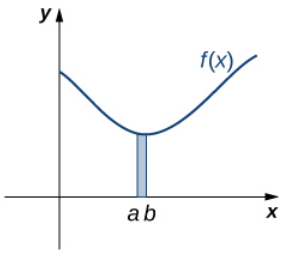
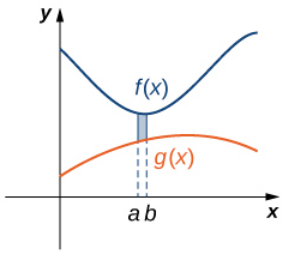
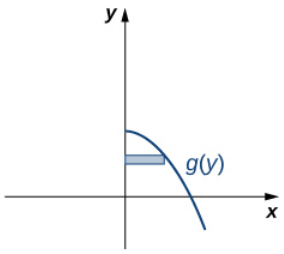
Compare	Disk Method	Washer Method	Shell Method
Volume formula	$V = \int_a^b \pi[f(x)]^2\,dx$	$V = \int_a^b \pi\left[(f(x))^2 - (g(x))^2\right]dx$	$V = \int_c^d 2\pi y\, g(y)\,dy$
Solid	No cavity in the center	Cavity in the center	With or without a cavity in the center
Interval to partition	$[a, b]$ on x-axis	$[a, b]$ on x-axis	$[c, d]$ on y-axis
Rectangle	Vertical	Vertical	Horizontal
Typical region			
Typical element			

Figure 3.4b. 10

Let's take a look at a couple of additional problems and decide on the best approach to take for solving them.

Example 3.4b. 6: Selecting the Best Method

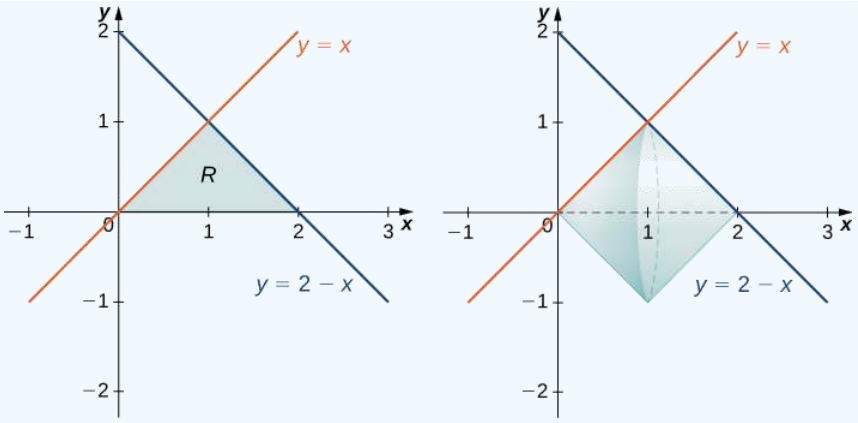
For each of the following problems, select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral).

a. The region bounded by the graphs of $y = x, y = 2 - x,$ and the x-axis.
b. The region bounded by the graphs of $y = 4x - x^2$ and the x-axis.

Solution

a.

First, sketch the region and the solid of revolution as shown.



(a) (b)

Figure 3.4b.11: (a) The region R bounded by two lines and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, if we want to integrate with respect to x, we would have to break the integral into two pieces, because we have different functions bounding the region over $[0, 1]$ and $[1, 2]$. In this case, using the disk method, we would have

$$V = \int_0^1 \pi x^2 \, dx + \int_1^2 \pi (2 - x)^2 \, dx.$$

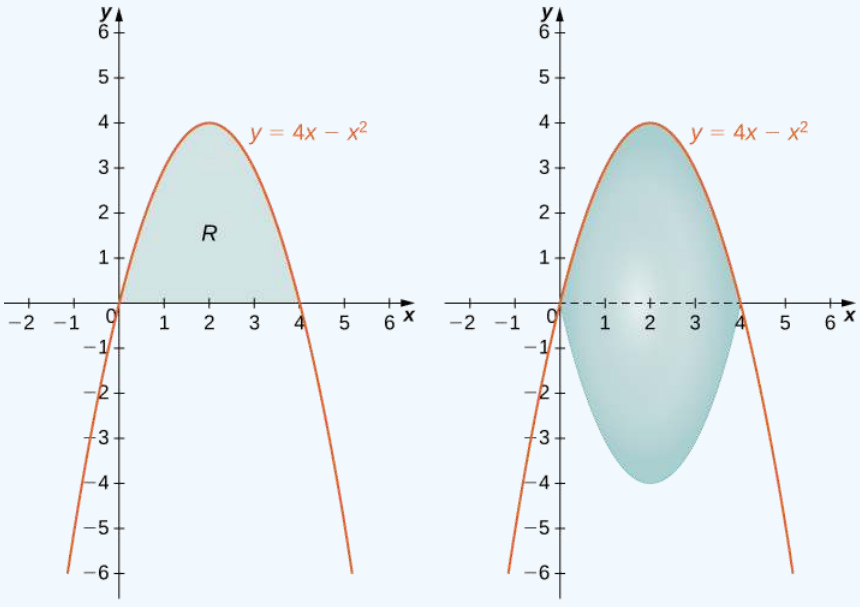
If we used the shell method instead, we would use functions of y to represent the curves, producing

$$V = \int_0^1 2\pi y[(2 - y) - y] \, dy = \int_0^1 2\pi y[2 - 2y] \, dy.$$

Neither of these integrals is particularly onerous, but since the shell method requires only one integral, and the integrand requires less simplification, we should probably go with the shell method in this case.

b.

First, sketch the region and the solid of revolution as shown.



(a) (b)

Figure 3.4b.12: (a) The region R between the curve and the x-axis. (b) The solid of revolution generated by revolving R about the x-axis.

Looking at the region, it would be problematic to define a horizontal rectangle; the region is bounded on the left and right by the same function. Therefore, we can dismiss the method of shells. The solid has no cavity in the middle, so we can use the method of disks. Then

$$V = \int_0^4 \pi \left(4x - x^2\right)^2 \, dx$$

? Exercise $3.4b.6$

Select the best method to find the volume of a solid of revolution generated by revolving the given region around the x-axis, and set up the integral to find the volume (do not evaluate the integral): the region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$.

Hint

Sketch the region and use Figure $3.4b.12$to decide which integral is easiest to evaluate.

Answer

Use the method of washers;

$$V = \int_{-1}^1 \pi \left[\left(2 - x^2\right)^2 - \left(x^2\right)^2 \right] \, dx$$

Key Concepts

- The method of cylindrical shells is another method for using a definite integral to calculate the volume of a solid of revolution. This method is sometimes preferable to either the method of disks or the method of washers because we integrate with respect to the other variable. In some cases, one integral is substantially more complicated than the other.
- The geometry of the functions and the difficulty of the integration are the main factors in deciding which integration method to use.

Key Equations

- **Method of Cylindrical Shells**

$$V = \int_a^b \left(2\pi \, x \, f(x)\right) \, dx$$

Glossary

method of cylindrical shells

a method of calculating the volume of a solid of revolution by dividing the solid into nested cylindrical shells; this method is different from the methods of disks or washers in that we integrate with respect to the opposite variable

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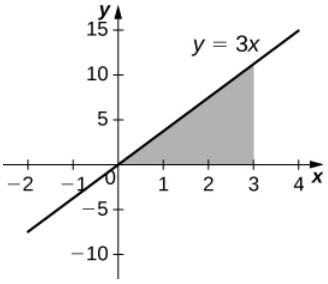
3.4E: Exercises for the Shell Method

For exercises 1 - 6, find the volume generated when the region between the two curves is rotated around the given axis. Use both the shell method and the washer method. Use technology to graph the functions and draw a typical slice by hand.

1) [T] Over the curve of $y = 3x$, $x = 0$, and $y = 3$ rotated around the y-axis.

2) [T] Under the curve of $y = 3x$, $x = 0$, and $x = 3$ rotated around the y-axis.

Answer

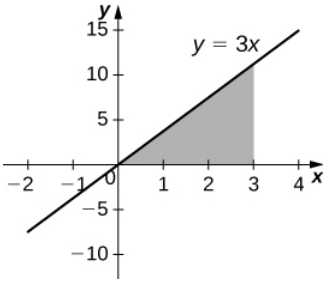


$V = 54\pi$ units^3

3) [T] Over the curve of $y = 3x$, $x = 0$, and $y = 3$ rotated around the x-axis.

4) [T] Under the curve of $y = 3x$, $x = 0$, and $x = 3$ rotated around the x-axis.

Answer

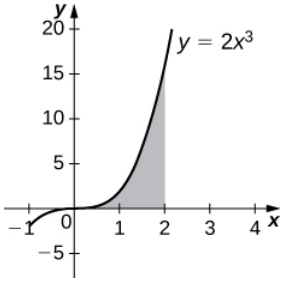


$V = 81\pi$ units^3

5) [T] Under the curve of $y = 2x^3$, $x = 0$, and $x = 2$ rotated around the y-axis.

6) [T] Under the curve of $y = 2x^3$, $x = 0$, and $x = 2$ rotated around the x-axis.

Answer



$V = \frac{512\pi}{7}$ units^3

For exercises 7 - 16, use shells to find the volumes of the given solids. Note that the rotated regions lie between the curve and the x-axis and are rotated around the y-axis.

7) $y = 1 - x^2,\ x = 0$, and $x = 1$

8) $y = 5x^3,\ x = 0$, and $x = 1$

Answer

$V = 2\pi$ units^3

9) $y = \dfrac{1}{x},\ x = 1$, and $x = 100$

10) $y = \sqrt{1 - x^2},\ x = 0$, and $x = 1$

Answer

$V = \frac{2\pi}{3}$ units^3

11) $y = \dfrac{1}{1 + x^2},\ x = 0$,and $x = 3$

12) $y = \sin x^2, x = 0$, and $x = \sqrt{\pi}$

Answer

$V = 2\pi$ units^3

13) $y = \dfrac{1}{\sqrt{1 - x^2}},\ x = 0$, and $x = \frac{1}{2}$

14) $y = \sqrt{x},\ x = 0$, and $x = 1$

Answer

$V = \frac{4\pi}{5}$ units^3

15) $y = (1 + x^2)^3,\ x = 0$, and $x = 1$

16) $y = 5x^3 - 2x^4,\ x = 0$, and $x = 2$

Answer

$V = \frac{64\pi}{3}$ units^3

For exercises 17 - 26, use shells to find the volume generated by rotating the regions between the given curve and $y = 0$ around the x-axis.

17) $y = \sqrt{1 - x^2},\ x = 0$, and $x = 1$

18) $y = x^2,\ x = 0$, and $x = 2$

Answer

$V = \frac{32\pi}{5}$ units^3

19) $y = e^x,\ x = 0$, and $x = 1$

20) $y = \ln(x), x = 1$, and $x = e$

Answer

$V = \pi(e - 2)$ units^3

21) $x = \dfrac{1}{1 + y^2},\ y = 1$, and $y = 4$

22) $x = \dfrac{1 + y^2}{y},\ y = 0$, and $y = 2$

Answer

$V = \frac{28\pi}{3}$ units^3

23) $x = \cos y$, $y = 0$, and $y = \pi$

24) $x = y^3 - 4y^2$, $x = -1$, and $x = 2$

Answer

$V = \frac{84\pi}{5}$ units^3

25) $x = ye^y$, $x = -1$, and $x = 2$

26) $x = e^y \cos y$, $x = 0$, and $x = \pi$

Answer

$V = e^{\pi}\pi^2$ units^3

For exercises 27 - 36, find the volume generated when the region between the curves is rotated around the given axis.

27) $y = 3 - x$, $y = 0$, $x = 0$, and $x = 2$ rotated around the y-axis.

28) $y = x^3$, $y = 0$, $x = 0$, and $y = 8$ rotated around the y-axis.

Answer

$V = \frac{64\pi}{5}$ units^3

29) $y = x^2$, $y = x$, rotated around the y-axis.

30) $y = \sqrt{x}$, $x = 0$, and $x = 1$ rotated around the line $x = 2$.

Answer

$V = \frac{28\pi}{15}$ units^3

31) $y = \dfrac{1}{4 - x}$, $x = 1$, and $x = 2$ rotated around the line $x = 4$.

32) $y = \sqrt{x}$ and $y = x^2$ rotated around the y-axis.

Answer

$V = \frac{3\pi}{10}$ units^3

33) $y = \sqrt{x}$ and $y = x^2$ rotated around the line $x = 2$.

34) $x = y^3$, $y = \dfrac{1}{x}$, $x = 1$, and $y = 2$ rotated around the x-axis.

Answer

$\frac{52\pi}{5}$ units^3

35) $x = y^2$ and $y = x$ rotated around the line $y = 2$.

36) [T] Left of $x = \sin(\pi y)$, right of $y = x$, around the y-axis.

Answer

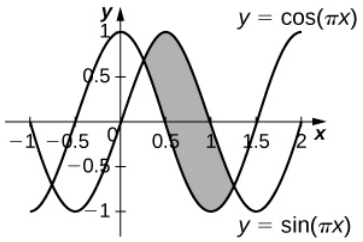
$V \approx 0.9876$ units^3

For exercises 37 - 44, use technology to graph the region. Determine which method you think would be easiest to use to calculate the volume generated when the function is rotated around the specified axis. Then, use your chosen method to find the volume.

37) [T] $y = x^2$ and $y = 4x$ rotated around the y-axis.

38) [T] $y = \cos(\pi x), y = \sin(\pi x), x = \frac{1}{4}$, and $x = \frac{5}{4}$ rotated around the y-axis.

Answer

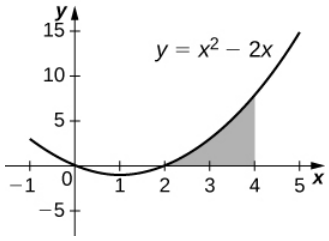


$V = 3\sqrt{2}$ units^3

39) [T] $y = x^2 - 2x,\ x = 2,$ and $x = 4$ rotated around the y-axis.

40) [T] $y = x^2 - 2x,\ x = 2,$ and $x = 4$ rotated around the x-axis.

Answer

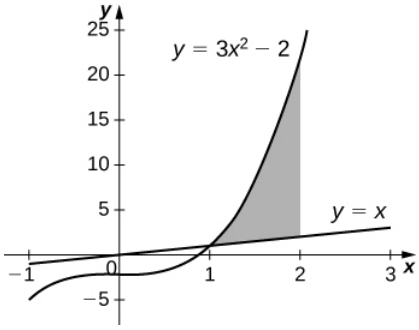


$V = \frac{496\pi}{15}$ units^3

41) [T] $y = 3x^3 - 2,\ y = x$, and $x = 2$ rotated around the x-axis.

42) [T] $y = 3x^3 - 2,\ y = x$, and $x = 2$ rotated around the y-axis.

Answer

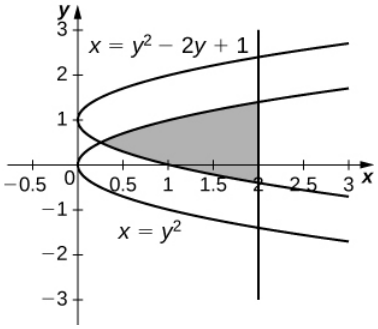


$V = \frac{398\pi}{15}$ units^3

43) [T] $x = \sin(\pi y^2)$ and $x = \sqrt{2}y$ rotated around the x-axis.

44) [T] $x = y^2,\ x = y^2 - 2y + 1$, and $x = 2$ rotated around the y-axis.

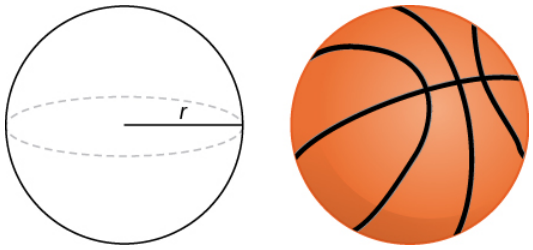
Answer



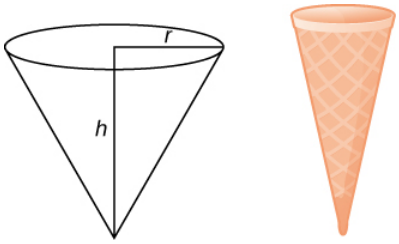
$V = 15.9074 \text{units}^3$

For exercises 45 - 51, use the method of shells to approximate the volumes of some common objects, which are pictured in accompanying figures.

45) Use the method of shells to find the volume of a sphere of radius r.



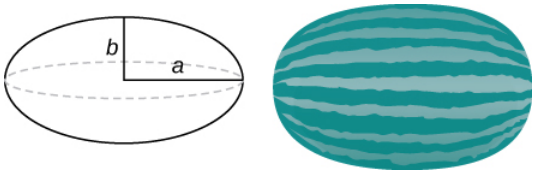
46) Use the method of shells to find the volume of a cone with radius r and height h.



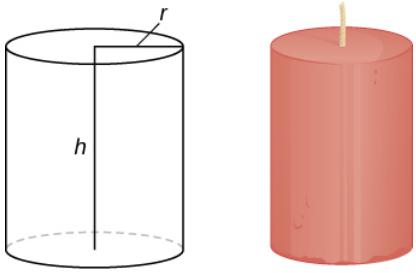
Answer

$V = \frac{1}{3}\pi r^2 h \text{ units}^3$

47) Use the method of shells to find the volume of an ellipse $(x^2/a^2) + (y^2/b^2) = 1$ rotated around the x-axis.



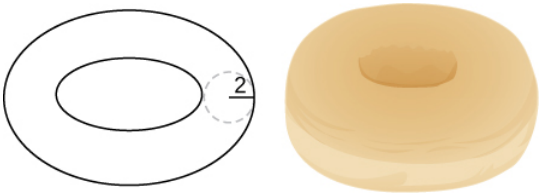
48) Use the method of shells to find the volume of a cylinder with radius r and height h.



Answer

$V = \pi r^2 h$ units^3

49) Use the method of shells to find the volume of the donut created when the circle $x^2 + y^2 = 4$ is rotated around the line $x = 4$.



50) Consider the region enclosed by the graphs of $y = f(x),\ y = 1 + f(x),\ x = 0,\ y = 0,$ and $x = a > 0$. What is the volume of the solid generated when this region is rotated around the y-axis? Assume that the function is defined over the interval $[0, a]$.

Answer

$V = \pi a^2$ units^3

51) Consider the function $y = f(x)$, which decreases from $f(0) = b$ to $f(1) = 0$. Set up the integrals for determining the volume, using both the shell method and the disk method, of the solid generated when this region, with $x = 0$ and $y = 0$, is rotated around the y-axis. Prove that both methods approximate the same volume. Which method is easier to apply? (Hint: Since $f(x)$ is one-to-one, there exists an inverse $f^{-1}(y)$.)

Contributors

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3.5: Arc Length and Surface Area

Arc Length

In previous sections we have used integration to answer the following questions:

1. Given a region, what is its area?
2. Given a solid, what is its volume?

In this section, we address a related question: Given a curve, what is its length? This is often referred to as **arc length**.

Consider the graph of $y = \sin x$ on $[0, \pi]$ given in Figure $3.5.1a$. How long is this curve? That is, if we were to use a piece of string to exactly match the shape of this curve, how long would the string be?

As we have done in the past, we start by approximating; later, we will refine our answer using limits to get an exact solution.

The length of straight--line segments is easy to compute using the Distance Formula. We can approximate the length of the given curve by approximating the curve with straight lines and measuring their lengths.

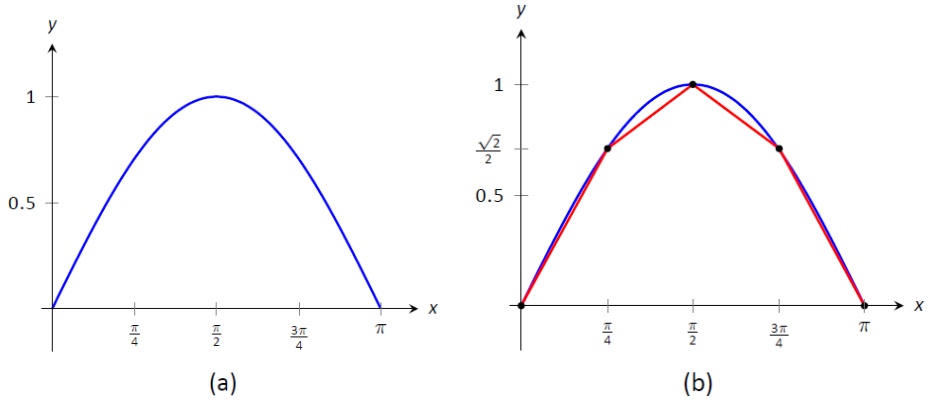


***Figure* 3.5.1:** *Graphing $y = \sin x$ on $[0, \pi]$ and approximating the curve with line segments.*

In Figure $3.5.1b$, the curve $y = \sin x$ has been approximated with 4 line segments (the interval $[0, \pi]$ has been divided into 4 equally--lengthed subintervals). It is clear that these four line segments approximate $y = \sin x$ very well on the first and last subinterval, though not so well in the middle. Regardless, the sum of the lengths of the line segments is 3.79, so we approximate the arc length of $y = \sin x$ on $[0, \pi]$ to be 3.79.

In general, we can approximate the arc length of $y = f(x)$ on $[a, b]$ in the following manner. Let $a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b$ be a partition of $[a, b]$ into n subintervals. Let dx_i represent the length of the ith subinterval $[x_i, x_{i+1}]$.

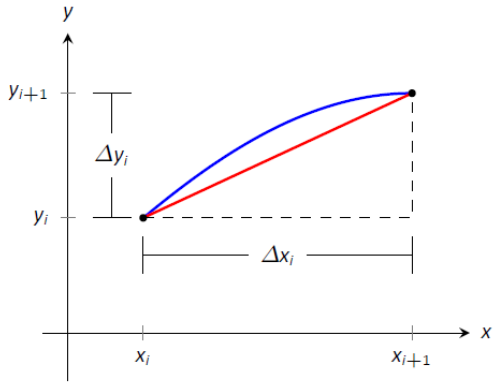


***Figure* 3.5.2:** *Zooming in on the ith subinterval $[x_i, x_{i+1}]$ of a partition of $[a, b]$.*

Figure $3.5.2$ zooms in on the ith subinterval where $y = f(x)$ is approximated by a straight line segment. The dashed lines show that we can view this line segment as they hypotenuse of a right triangle whose sides have length dx_i and dy_i. Using the

Pythagorean Theorem, the length of this line segment is $\sqrt{dx_i^2 + \Delta y_i^2}$. Summing over all subintervals gives an arc length approximation

$$L \approx \sum_{i=1}^{n} \sqrt{dx_i^2 + \Delta y_i^2}. \tag{3.5.1}$$

As shown here, this is *not* a Riemann Sum. While we could conclude that taking a limit as the subinterval length goes to zero gives the exact arc length, we would not be able to compute the answer with a definite integral. We need first to do a little algebra.

In the above expression factor out a dx_i^2 term:

$$\sum_{i=1}^{n} \sqrt{dx_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{dx_i^2 \left(1 + \frac{\Delta y_i^2}{dx_i^2}\right)}. \tag{3.5.2}$$

Now pull the dx_i^2 term out of the square root:

$$= \sum_{i=1}^{n} \sqrt{1 + \frac{\Delta y_i^2}{dx_i^2}}\, dx_i. \tag{3.5.3}$$

This is nearly a Riemann Sum. Consider the $\Delta y_i^2/dx_i^2$ term. The expression $\Delta y_i/dx_i$ measures the "change in y/change in x," that is, the "rise over run" of f on the ith subinterval. The Mean Value Theorem of Differentiation (Theorem 3.2.1) states that there is a c_i in the ith subinterval where $f'(c_i) = \Delta y_i/dx_i$. Thus we can rewrite our above expression as:

$$= \sum_{i=1}^{n} \sqrt{1 + f'(c_i)^2}\, dx_i. \tag{3.5.4}$$

This *is* a Riemann Sum. As long as f' is continuous, we can invoke Theorem 5.3.2 and conclude

$$= \int_a^b \sqrt{1 + f'(x)^2}\, dx. \tag{3.5.5}$$

Key Idea 27: Arc Length

Let f be differentiable on an open interval containing $[a, b]$, where f' is also continuous on $[a, b]$. Then the arc length of f from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx. \tag{3.5.6}$$

As the integrand contains a square root, it is often difficult to use the formula in Key Idea 27 to find the length exactly. When exact answers are difficult to come by, we resort to using numerical methods of approximating definite integrals. The following examples will demonstrate this.

Example 3.5.1: Finding arc length

Find the arc length of $f(x) = x^{3/2}$ from $x = 0$ to $x = 4$.

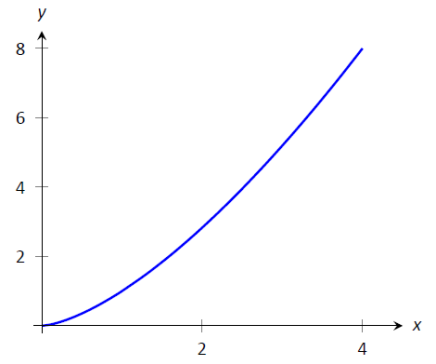


Figure 3.5.3: *A graph of $f(x) = x^{3/2}$ from Example 3.5.1*

Solution

We begin by finding $f'(x) = \frac{3}{2}x^{1/2}$. Using the formula, we find the arc length L as

$$L = \int_0^4 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx \tag{3.5.7}$$

$$= \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx \tag{3.5.8}$$

$$= \int_0^4 \left(1 + \frac{9}{4}x\right)^{1/2} dx \tag{3.5.9}$$

$$= \frac{2}{3}\frac{4}{9}\left(1 + \frac{9}{4}x\right)^{3/2}\Big|_0^4 \tag{3.5.10}$$

$$= \frac{8}{27}\left(10^{3/2} - 1\right) \approx 9.07 \text{units}. \tag{3.5.11}$$

Example 3.5.2: Finding arc length

Find the arc length of $f(x) = \frac{1}{8}x^2 - \ln x$ from $x = 1$ to $x = 2$.

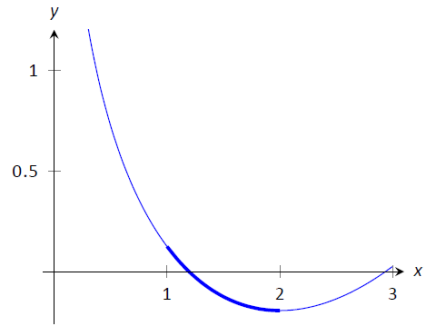


Figure 3.5.4: *A graph of $f(x) = \frac{1}{8}x^2 - \ln x$ from Example 3.5.2.*

Solution

This function was chosen specifically because the resulting integral can be evaluated exactly. We begin by finding $f'(x) = x/4 - 1/x$. The arc length is

$$L = \int_1^2 \sqrt{1 + \left(\frac{x}{4} - \frac{1}{x}\right)^2}\, dx \tag{3.5.12}$$

$$= \int_1^2 \sqrt{1 + \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2}}\, dx \tag{3.5.13}$$

$$= \int_1^2 \sqrt{\frac{x^2}{16} + \frac{1}{2} + \frac{1}{x^2}}\, dx \tag{3.5.14}$$

$$= \int_1^2 \sqrt{\left(\frac{x}{4} + \frac{1}{x}\right)^2}\, dx \qquad = \int_1^2 \left(\frac{x}{4} + \frac{1}{x}\right) dx \tag{3.5.15}$$

$$= \left(\frac{x^2}{8} + \ln x\right)\Big|_1^2 \tag{3.5.16}$$

$$= \frac{3}{8} + \ln 2 \approx 1.07 \text{ units.} \tag{3.5.17}$$

A graph of f is given in Figure 3.5.4; the portion of the curve measured in this problem is in bold.

The previous examples found the arc length exactly through careful choice of the functions. In general, exact answers are much more difficult to come by and numerical approximations are necessary.

Example 3.5.3: Approximating arc length numerically

Find the length of the sine curve from $x = 0$ to $x = \pi$.

Solution

This is somewhat of a mathematical curiosity; in Example 5.4.3 we found the area under one "hump" of the sine curve is 2 square units; now we are measuring its arc length.

The setup is straightforward: $f(x) = \sin x$ and $f'(x) = \cos x$. Thus

$$L = \int_0^\pi \sqrt{1 + \cos^2 x}\, dx. \tag{3.5.18}$$

This integral *cannot* be evaluated in terms of elementary functions so we will approximate it with Simpson's Method with $n = 4$.

x	$\sqrt{1+\cos^2 x}$
0	$\sqrt{2}$
$\pi/4$	$\sqrt{3/2}$
$\pi/2$	1
$3\pi/4$	$\sqrt{3/2}$
π	$\sqrt{2}$

Figure 3.5.5: *A table of values of $y = \sqrt{1+\cos^2 x}$ to evaluate a definite integral in Example 3.5.3.*

x	$\sqrt{1+\cos^2 x}$
0	$\sqrt{2}$
$\pi/4$	$\sqrt{3/2}$
$\pi/2$	1
$3\pi/4$	$\sqrt{3/2}$
π	$\sqrt{2}$

(3.5.19)

Figure ??? gives $\sqrt{1+\cos^2 x}$ evaluated at 5 evenly spaced points in $[0, \pi]$. Simpson's Rule then states that

$$\int_0^{\pi} \sqrt{1+\cos^2 x}\, dx \approx \frac{\pi - 0}{4 \cdot 3}\left(\sqrt{2} + 4\sqrt{3/2} + 2(1) + 4\sqrt{3/2} + \sqrt{2}\right) \tag{3.5.20}$$

$$= 3.82918. \tag{3.5.21}$$

Using a computer with $n = 100$ the approximation is $L \approx 3.8202$; our approximation with $n = 4$ is quite good.

Surface Area of Solids of Revolution

We have already seen how a curve $y = f(x)$ on $[a, b]$ can be revolved around an axis to form a solid. Instead of computing its volume, we now consider its surface area.

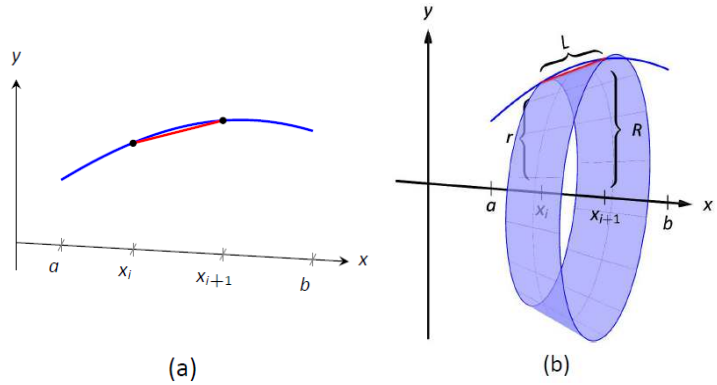


***Figure* 3.5.6:** *Establishing the formula for surface area.*

We begin as we have in the previous sections: we partition the interval $[a, b]$ with n subintervals, where the ith subinterval is $[x_i, x_{i+1}]$. On each subinterval, we can approximate the curve $y = f(x)$ with a straight line that connects $f(x_i)$ and $f(x_{i+1})$ as shown in Figure 3.5.5*a*. Revolving this line segment about the x-axis creates part of a cone (called a *frustum* of a cone) as shown in Figure 3.5.5*b*. The surface area of a frustum of a cone is

$$2\pi \cdot \text{ length } \cdot \text{average of the two radii } R \text{ and } r. \tag{3.5.22}$$

The length is given by L; we use the material just covered by arc length to state that

$$L \approx \sqrt{1 + f'(c_i)}\, dx_i \tag{3.5.23}$$

for some c_i in the ith subinterval. The radii are just the function evaluated at the endpoints of the interval. That is,

$$R = f(x_{i+1}) \quad \text{and} \quad r = f(x_i). \tag{3.5.24}$$

Thus the surface area of this sample frustum of the cone is approximately

$$2\pi \frac{f(x_i) + f(x_{i+1})}{2} \sqrt{1 + f'(c_i)^2}\, dx_i. \tag{3.5.25}$$

Since f is a continuous function, the Intermediate Value Theorem states there is some d_i in $[x_i, x_{i+1}]$ such that $f(d_i) = \frac{f(x_i) + f(x_{i+1})}{2}$; we can use this to rewrite the above equation as

$$2\pi f(d_i) \sqrt{1 + f'(c_i)^2}\, dx_i. \tag{3.5.26}$$

Summing over all the subintervals we get the total surface area to be approximately

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(d_i) \sqrt{1 + f'(c_i)^2}\, dx_i, \tag{3.5.27}$$

which is a Riemann Sum. Taking the limit as the subinterval lengths go to zero gives us the exact surface area, given in the following Key Idea.

Key Idea 28: Surface Area of a Solid of Revolution

Let f be differentiable on an open interval containing $[a, b]$ where f' is also continuous on $[a, b]$.

1. The surface area of the solid formed by revolving the graph of $y = f(x)$, where $f(x) \geq 0$, about the x-axis is

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx. \tag{3.5.28}$$

2. The surface area of the solid formed by revolving the graph of $y = f(x)$ about the y-axis, where $a, b \geq 0$, is

$$\text{Surface Area} = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\, dx. \tag{3.5.29}$$

When revolving $y = f(x)$ about the y-axis, the radii of the resulting frustum are x_i and x_{i+1}; their average value is simply the midpoint of the interval. In the limit, this midpoint is just x. This gives the second part of Key Idea 28.

Example 3.5.4: Finding surface area of a solid of revolution

Find the surface area of the solid formed by revolving $y = \sin x$ on $[0, \pi]$ around the x-axis, as shown in Figure 3.5.6.

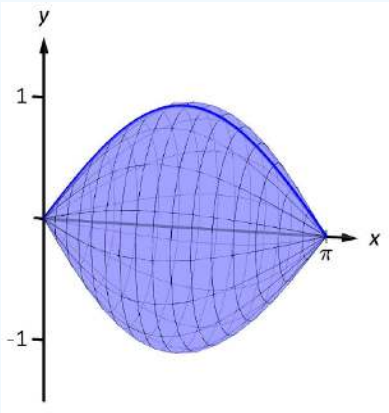


Figure 3.5.7: *Revolving $y = \sin x$ on $[0, \pi]$ about the x-axis.*

Solution

The setup is relatively straightforward. Using Key Idea ???, we have the surface area SA is:

$$\begin{aligned} SA &= 2\pi \int_0^\pi \sin x\sqrt{1 + \cos^2 x}\, dx && (3.5.30)\\ &= -2\pi\frac{1}{2}\left(\sinh^{-1}(\cos x) + \cos x\sqrt{1 + \cos^2 x}\right)\Big|_0^\pi && (3.5.31)\\ &= 2\pi\left(\sqrt{2} + \sinh^{-1} 1\right) && (3.5.32)\\ &\approx 14.42 \text{ units}^2. && (3.5.33) \end{aligned}$$

The integration step above is nontrivial, utilizing an integration method called Trigonometric Substitution.

It is interesting to see that the surface area of a solid, whose shape is defined by a trigonometric function, involves both a square root and an inverse hyperbolic trigonometric function.

Example 3.5.5: Finding surface area of a solid of revolution

Find the surface area of the solid formed by revolving the curve $y = x^2$ on $[0, 1]$ about:

1. the x-axis
2. the y-axis.

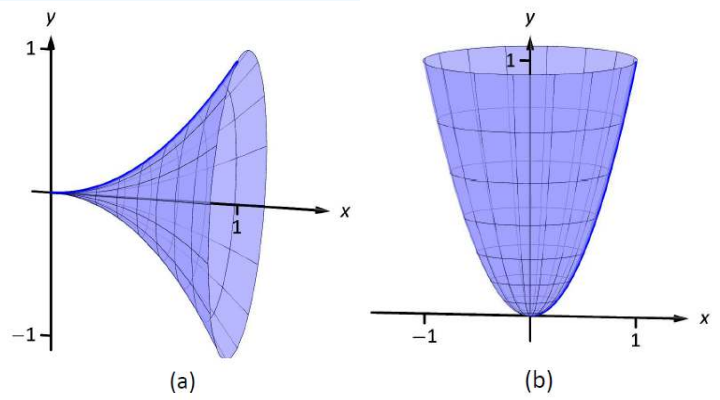


***Figure* 3.5.8:** *The solids used in Example* 3.5.6.

Solution

1. The integral is straightforward to setup:

$$SA = 2\pi \int_0^1 x^2 \sqrt{1+(2x)^2}\,dx. \tag{3.5.34}$$

Like the integral in Example ???, this requires Trigonometric Substitution.

$$= \frac{\pi}{32}\left(2(8x^3+x)\sqrt{1+4x^2} - \sinh^{-1}(2x)\right)\Big|_0^1 \tag{3.5.35}$$
$$= \frac{\pi}{32}\left(18\sqrt{5} - \sinh^{-1} 2\right) \tag{3.5.36}$$
$$\approx 3.81 \text{ units}^2. \tag{3.5.37}$$

The solid formed by revolving $y = x^2$ around the x-axis is graphed in Figure 3.5.7*a*.

2. Since we are revolving around the y-axis, the "radius" of the solid is not $f(x)$ but rather x. Thus the integral to compute the surface area is:

$$SA = 2\pi \int_0^1 x\sqrt{1+(2x)^2}\,dx. \tag{3.5.38}$$

This integral can be solved using substitution. Set $u = 1+4x^2$; the new bounds are $u = 1$ to $u = 5$. We then have

$$= \frac{\pi}{4}\int_1^5 \sqrt{u}\,du \tag{3.5.39}$$
$$= \frac{\pi}{4}\frac{2}{3}u^{3/2}\Big|_1^5 \tag{3.5.40}$$
$$= \frac{\pi}{6}(5\sqrt{5}-1) \tag{3.5.41}$$
$$\approx 5.33 \text{ units}^2. \tag{3.5.42}$$

The solid formed by revolving $y = x^2$ about the y-axis is graphed in Figure 3.5.7*b*

Our final example is a famous mathematical "paradox."

Example 3.5.6: The surface area and volume of Gabriel's Horn

Consider the solid formed by revolving $y = 1/x$ about the x-axis on $[1, \infty)$. Find the volume and surface area of this solid. (This shape, as graphed in Figure 3.5.9, is known as "Gabriel's Horn" since it looks like a very long horn that only a supernatural person, such as an angel, could play.)

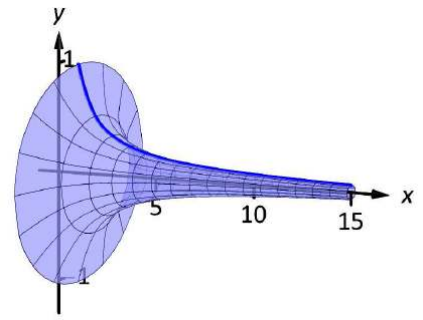


Figure 3.5.9: *A graph of Gabriel's Horn.*

Solution

To compute the volume it is natural to use the Disk Method. We have:

$$V = \pi \int_1^\infty \frac{1}{x^2}\,dx \tag{3.5.43}$$

$$= \lim_{b\to\infty} \pi \int_1^b \frac{1}{x^2}\,dx \tag{3.5.44}$$

$$= \lim_{b\to\infty} \pi \left(\frac{-1}{x}\right)\Big|_1^b \tag{3.5.45}$$

$$= \lim_{b\to\infty} \pi \left(1 - \frac{1}{b}\right) \tag{3.5.46}$$

$$= \pi \text{ units}^3. \tag{3.5.47}$$

Gabriel's Horn has a finite volume of π cubic units. Since we have already seen that regions with infinite length can have a finite area, this is not too difficult to accept.

We now consider its surface area. The integral is straightforward to setup:

$$SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\,dx. \tag{3.5.48}$$

Integrating this expression is not trivial. We can, however, compare it to other improper integrals. Since $1 < \sqrt{1+1/x^4}$ on $[1,\infty)$, we can state that

$$2\pi \int_1^\infty \frac{1}{x}dx < 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}dx. \tag{3.5.49}$$

By Key Idea 21, the improper integral on the left diverges. Since the integral on the right is larger, we conclude it also diverges, meaning Gabriel's Horn has infinite surface area.

Hence the "paradox": we can fill Gabriel's Horn with a finite amount of paint, but since it has infinite surface area, we can never paint it.

Somehow this paradox is striking when we think about it in terms of volume and area. However, we have seen a similar paradox before, as referenced above. We know that the area under the curve $y = 1/x^2$ on $[1,\infty)$ is finite, yet the shape has an infinite perimeter. Strange things can occur when we deal with the infinite.

A standard equation from physics is "Work = force $\times$ distance", when the force applied is constant. In the next section we learn how to compute work when the force applied is variable.

Contributors and Attributions

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3.6: Arc Length of a Curve and Surface Area

Learning Objectives

- Determine the length of a curve, $y = f(x)$, between two points.
- Determine the length of a curve, $x = g(y)$, between two points.
- Find the surface area of a solid of revolution.

In this section, we use definite integrals to find the arc length of a curve. We can think of **arc length** as the distance you would travel if you were walking along the path of the curve. Many real-world applications involve arc length. If a rocket is launched along a parabolic path, we might want to know how far the rocket travels. Or, if a curve on a map represents a road, we might want to know how far we have to drive to reach our destination.

We begin by calculating the arc length of curves defined as functions of x, then we examine the same process for curves defined as functions of y. (The process is identical, with the roles of x and y reversed.) The techniques we use to find arc length can be extended to find the surface area of a surface of revolution, and we close the section with an examination of this concept.

Arc Length of the Curve y = f(x)

In previous applications of integration, we required the function $f(x)$ to be integrable, or at most continuous. However, for calculating arc length we have a more stringent requirement for $f(x)$. Here, we require $f(x)$ to be differentiable, and furthermore we require its derivative, $f'(x)$, to be continuous. Functions like this, which have continuous derivatives, are called **smooth.** (This property comes up again in later chapters.)

Let $f(x)$ be a smooth function defined over $[a, b]$. We want to calculate the length of the curve from the point $(a, f(a))$ to the point $(b, f(b))$. We start by using line segments to approximate the length of the curve. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Although it might seem logical to use either horizontal or vertical line segments, we want our line segments to approximate the curve as closely as possible. Figure 3.6.1 depicts this construct for $n = 5$.

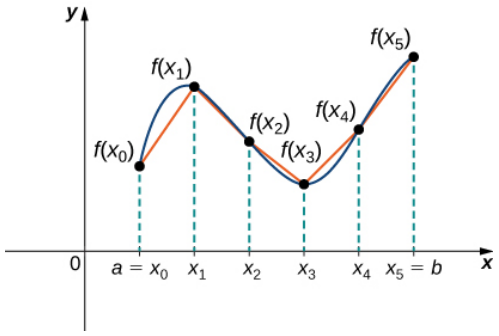


Figure 3.6.1: We can approximate the length of a curve by adding line segments.

To help us find the length of each line segment, we look at the change in vertical distance as well as the change in horizontal distance over each interval. Because we have used a regular partition, the change in horizontal distance over each interval is given by Δx. The change in vertical distance varies from interval to interval, though, so we use $\Delta y_i = f(x_i) - f(x_{i-1})$ to represent the change in vertical distance over the interval $[x_{i-1}, x_i]$, as shown in Figure 3.6.2. Note that some (or all) Δy_i may be negative.

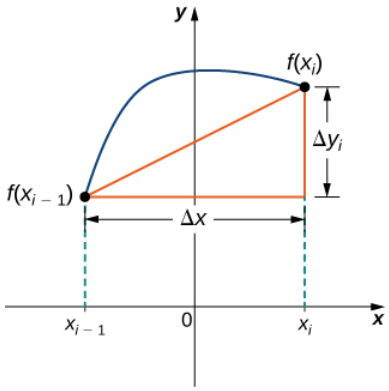


Figure 3.6.2: A representative line segment approximates the curve over the interval $[x_{i-1}, x_i]$.

By the Pythagorean theorem, the length of the line segment is

$$\sqrt{(\Delta x)^2 + (\Delta y_i)^2}.$$

We can also write this as

$$\Delta x \sqrt{1 + ((\Delta y_i)/(\Delta x))^2}.$$

Now, by the Mean Value Theorem, there is a point $x_i^* \in [x_{i-1}, x_i]$ such that $f'(x_i^*) = (\Delta y_i)/(\Delta x)$. Then the length of the line segment is given by

$$\Delta x \sqrt{1 + [f'(x_i^*)]^2}.$$

Adding up the lengths of all the line segments, we get

$$\text{Arc Length} \approx \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x.$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we have

$$\begin{aligned} \text{Arc Length} &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x_i^*)]^2}\, \Delta x \\ &= \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \end{aligned}$$

We summarize these findings in the following theorem.

Arc Length for $y = f(x)$

Let $f(x)$ be a smooth function over the interval $[a, b]$. Then the arc length of the portion of the graph of $f(x)$ from the point $(a, f(a))$ to the point $(b, f(b))$ is given by

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

Note that we are integrating an expression involving $f'(x)$, so we need to be sure $f'(x)$ is integrable. This is why we require $f(x)$ to be smooth. The following example shows how to apply the theorem.

Example 3.6.1: Calculating the Arc Length of a Function of x

Let $f(x) = 2x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Solution

We have $f'(x) = 3x^{1/2}$, so $[f'(x)]^2 = 9x$. Then, the arc length is

$$\begin{aligned}\text{Arc Length} &= \int_a^b \sqrt{1+[f'(x)]^2}dx \\ &= \int_0^1 \sqrt{1+9x}\,dx.\end{aligned}$$

Substitute $u = 1+9x$. Then, $du = 9\,dx$. When $x = 0$, then $u = 1$, and when $x = 1$, then $u = 10$. Thus,

$$\begin{aligned}\text{Arc Length} &= \int_0^1 \sqrt{1+9x}\,dx \\ &= \frac{1}{9}\int_0^1 \sqrt{1+9x}\cdot 9\,dx \\ &= \frac{1}{9}\int_1^{10} \sqrt{u}\,du \\ &= \frac{1}{9}\cdot\frac{2}{3}u^{3/2}\Big|_1^{10} = \frac{2}{27}[10\sqrt{10}-1] \\ &\approx 2.268 \text{ units.}\end{aligned}$$

Exercise 3.6.1

Let $f(x) = (4/3)x^{3/2}$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, 1]$. Round the answer to three decimal places.

Hint

Use the process from the previous example. Don't forget to change the limits of integration.

Answer

$$\frac{1}{6}(5\sqrt{5}-1) \approx 1.697 \text{ units}$$

Although it is nice to have a formula for calculating arc length, this particular theorem can generate expressions that are difficult to integrate. We study some techniques for integration in Introduction to Techniques of Integration. In some cases, we may have to use a computer or calculator to approximate the value of the integral.

Example 3.6.2: Using a Computer or Calculator to Determine the Arc Length of a Function of x

Let $f(x) = x^2$. Calculate the arc length of the graph of $f(x)$ over the interval $[1, 3]$.

Solution

We have $f'(x) = 2x$, so $[f'(x)]^2 = 4x^2$. Then the arc length is given by

$$\begin{aligned}\text{Arc Length} &= \int_a^b \sqrt{1+[f'(x)]^2}\,dx \\ &= \int_1^3 \sqrt{1+4x^2}\,dx.\end{aligned}$$

Using a computer to approximate the value of this integral, we get

$$\int_1^3 \sqrt{1+4x^2}\,dx \approx 8.26815 \text{ units.}$$

Exercise 3.6.2

Let $f(x) = \sin x$. Calculate the arc length of the graph of $f(x)$ over the interval $[0, \pi]$. Use a computer or calculator to approximate the value of the integral.

Hint

Use the process from the previous example.

Answer

$$\text{Arc Length} \approx 3.8202 \text{ units}$$

Arc Length of the Curve x = g(y)

We have just seen how to approximate the length of a curve with line segments. If we want to find the arc length of the graph of a function of y, we can repeat the same process, except we partition the y-axis instead of the x-axis. Figure 3.6.3 shows a representative line segment.

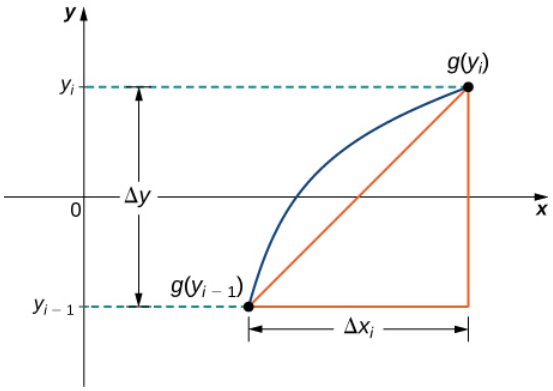


Figure 3.6.3: A representative line segment over the interval $[y_{i-1}, y_i]$.

Then the length of the line segment is

$$\sqrt{(\Delta y)^2 + (\Delta x_i)^2},$$

which can also be written as

$$\Delta y \sqrt{1 + \left(\frac{\Delta x_i}{\Delta y}\right)^2}.$$

If we now follow the same development we did earlier, we get a formula for arc length of a function $x = g(y)$.

Arc Length for $x = g(y)$

Let $g(y)$ be a smooth function over an interval $[c, d]$. Then, the arc length of the graph of $g(y)$ from the point $(c, g(c))$ to the point $(d, g(d))$ is given by

$$\text{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy.$$

Example 3.6.3: Calculating the Arc Length of a Function of y

Let $g(y) = 3y^3$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 2]$.

Solution

We have $g'(y) = 9y^2$, so $[g'(y)]^2 = 81y^4$. Then the arc length is

$$\text{Arc Length} = \int_c^d \sqrt{1+[g'(y)]^2}\, dy$$
$$= \int_1^2 \sqrt{1+81y^4}\, dy.$$

Using a computer to approximate the value of this integral, we obtain

$$\int_1^2 \sqrt{1+81y^4} dy \approx 21.0277 \text{ units}.$$

? Exercise 3.6.3

Let $g(y) = 1/y$. Calculate the arc length of the graph of $g(y)$ over the interval $[1, 4]$. Use a computer or calculator to approximate the value of the integral.

Hint

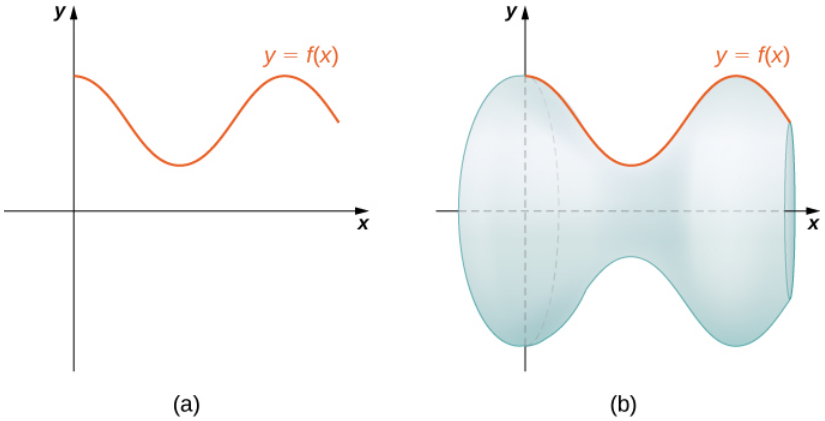
Use the process from the previous example.

Answer

$$\text{Arc Length} = 3.15018 \text{ units}$$

Area of a Surface of Revolution

The concepts we used to find the arc length of a curve can be extended to find the surface area of a surface of revolution. **Surface area** is the total area of the outer layer of an object. For objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces. For curved surfaces, the situation is a little more complex. Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. We wish to find the surface area of the surface of revolution created by revolving the graph of $y = f(x)$ around the x-axis as shown in the following figure.



(a) (b)

Figure 3.6.4: (a) A curve representing the function $f(x)$. (b) The surface of revolution formed by revolving the graph of $f(x)$ around the $x - axis$.

As we have done many times before, we are going to partition the interval $[a, b]$ and approximate the surface area by calculating the surface area of simpler shapes. We start by using line segments to approximate the curve, as we did earlier in this section. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of $[a, b]$. Then, for $i = 1, 2, \ldots, n$, construct a line segment from the point $(x_{i-1}, f(x_{i-1}))$ to the point $(x_i, f(x_i))$. Now, revolve these line segments around the x-axis to generate an approximation of the surface of revolution as shown in the following figure.

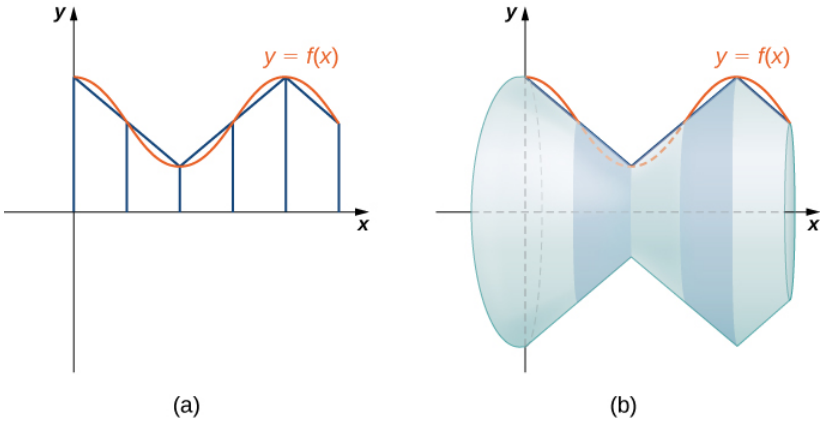


Figure 3.6.5: (a) Approximating $f(x)$ with line segments. (b) The surface of revolution formed by revolving the line segments around the x-axis.

Notice that when each line segment is revolved around the axis, it produces a band. These bands are actually pieces of cones (think of an ice cream cone with the pointy end cut off). A piece of a cone like this is called a **frustum** of a cone.

To find the surface area of the band, we need to find the lateral surface area, S, of the frustum (the area of just the slanted outside surface of the frustum, not including the areas of the top or bottom faces). Let r_1 and r_2 be the radii of the wide end and the narrow end of the frustum, respectively, and let l be the slant height of the frustum as shown in the following figure.

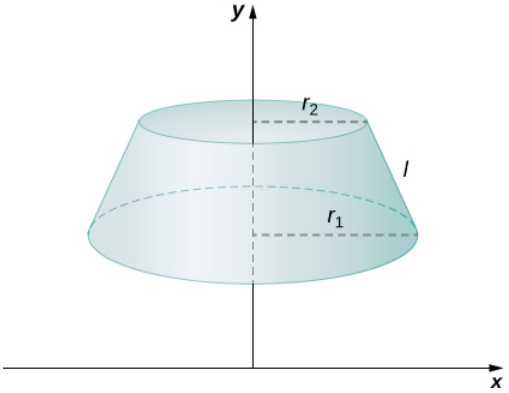


Figure 3.6.6: A frustum of a cone can approximate a small part of surface area.

We know the lateral surface area of a cone is given by

$$\text{Lateral Surface Area } = \pi r s,$$

where r is the radius of the base of the cone and s is the **slant height** (Figure 3.6.7).

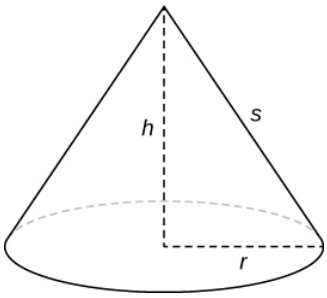


Figure 3.6.7: The lateral surface area of the cone is given by $\pi r s$.

Since a frustum can be thought of as a piece of a cone, the lateral surface area of the frustum is given by the lateral surface area of the whole cone less the lateral surface area of the smaller cone (the pointy tip) that was cut off (Figure 3.6.8).

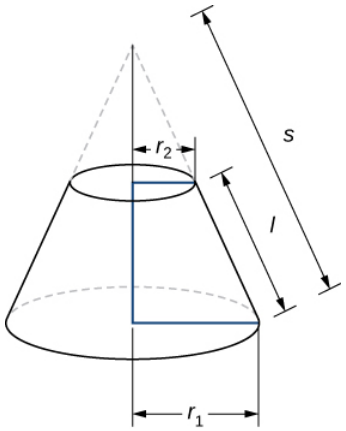


Figure 3.6.8: Calculating the lateral surface area of a frustum of a cone.

The cross-sections of the small cone and the large cone are similar triangles, so we see that

$$\frac{r_2}{r_1} = \frac{s-l}{s}$$

Solving for s, we get

$$\begin{aligned}
\frac{r_2}{r_1} &= \frac{s-l}{s} \\
r_2 s &= r_1(s-l) \\
r_2 s &= r_1 s - r_1 l \\
r_1 l &= r_1 s - r_2 s \\
r_1 l &= (r_1 - r_2)s \\
\frac{r_1 l}{r_1 - r_2} &= s
\end{aligned}$$

Then the lateral surface area (SA) of the frustum is

$$\begin{aligned}
S &= (\text{Lateral SA of large cone}) - (\text{Lateral SA of small cone}) \\
&= \pi r_1 s - \pi r_2 (s-l) \\
&= \pi r_1 \left(\frac{r_1 l}{r_1 - r_2}\right) - \pi r_2 \left(\frac{r_1 l}{r_1 - r_2} - l\right) \\
&= \frac{\pi r_1^2 l}{r^1 - r^2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \pi r_2 l \\
&= \frac{\pi r_1^2 l}{r_1 - r_2} - \frac{\pi r_1 r 2_l}{r_1 - r_2} + \frac{\pi r_2 l (r_1 - r_2)}{r_1 - r_2} \\
&= \frac{\pi r_1^2}{l r_1 - r_2} - \frac{\pi r_1 r_2 l}{r_1 - r_2} + \frac{\pi r_1 r_2 l}{r_1 - r_2} - \frac{\pi r_2^2 l}{r_1 - r_3} \\
&= \frac{\pi(r_1^2 - r_2^2)l}{r_1 - r_2} = \frac{\pi(r_1 - r + 2)(r1 + r2)l}{r_1 - r_2} \\
&= \pi(r_1 + r_2)l.
\end{aligned}$$

Let's now use this formula to calculate the surface area of each of the bands formed by revolving the line segments around the x-axis. A representative band is shown in the following figure.

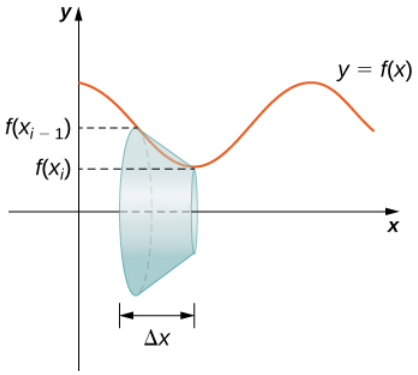


Figure 3.6.9: A representative band used for determining surface area.

Note that the slant height of this frustum is just the length of the line segment used to generate it. So, applying the surface area formula, we have

$$\begin{aligned} S &= \pi(r_1 + r_2)l \\ &= \pi\big(f(x_{i-1}) + f(x_i)\big)\sqrt{\Delta x^2 + (\Delta y_i)^2} \\ &= \pi\big(f(x_{i-1}) + f(x_i)\big)\Delta x\sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2} \end{aligned}$$

Now, as we did in the development of the arc length formula, we apply the Mean Value Theorem to select $x_i^* \in [x_{i-1}, x_i]$ such that $f\,'(x_i^*) = (\Delta y_i)/\Delta x$. This gives us

$$S = \pi\big(f(x_{i-1}) + f(x_i)\big)\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2}$$

Furthermore, since $f(x)$ is continuous, by the *Intermediate Value Theorem*, there is a point $x_i^{**} \in [x_{i-1}, x_i]$ such that $f(x_i^{**}) = \frac{1}{2}\left[f(x_{i-1}) + f(x_i)\right]$,

so we get

$$S = 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2}.$$

Then the approximate surface area of the whole surface of revolution is given by

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2}.$$

This almost looks like a Riemann sum, except we have functions evaluated at two different points, x_i^* and x_i^{**}, over the interval $[x_{i-1}, x_i]$. Although we do not examine the details here, it turns out that because $f(x)$ is smooth, if we let $n \to \infty$, the limit works the same as a Riemann sum even with the two different evaluation points. This makes sense intuitively. Both x_i^* and x_i^{**} are in the interval $[x_{i-1}, x_i]$, so it makes sense that as $n \to \infty$, both x_i^* and x_i^{**} approach x Those of you who are interested in the details should consult an advanced calculus text.

Taking the limit as $n \to \infty$, we get

$$\begin{aligned} \text{Surface Area} &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(x_i^{**})\Delta x\sqrt{1 + \big(f\,'(x_i^*)\big)^2} \\ &= \int_a^b \left(2\pi f(x)\sqrt{1 + \big(f\,'(x)\big)^2}\right) dx \end{aligned}$$

As with arc length, we can conduct a similar development for functions of y to get a formula for the surface area of surfaces of revolution about the y-axis. These findings are summarized in the following theorem.

Surface Area of a Surface of Revolution

Let $f(x)$ be a nonnegative smooth function over the interval $[a, b]$. Then, the surface area of the surface of revolution formed by revolving the graph of $f(x)$ around the x-axis is given by

$$\text{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\right) dx$$

Similarly, let $g(y)$ be a nonnegative smooth function over the interval $[c, d]$. Then, the surface area of the surface of revolution formed by revolving the graph of $g(y)$ around the y-axis is given by

$$\text{Surface Area} = \int_c^d \left(2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\right) dy$$

Example 3.6.4: Calculating the Surface Area of a Surface of Revolution 1.

Let $f(x) = \sqrt{x}$ over the interval $[1, 4]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Solution

The graph of $f(x)$ and the surface of rotation are shown in Figure 3.6.10

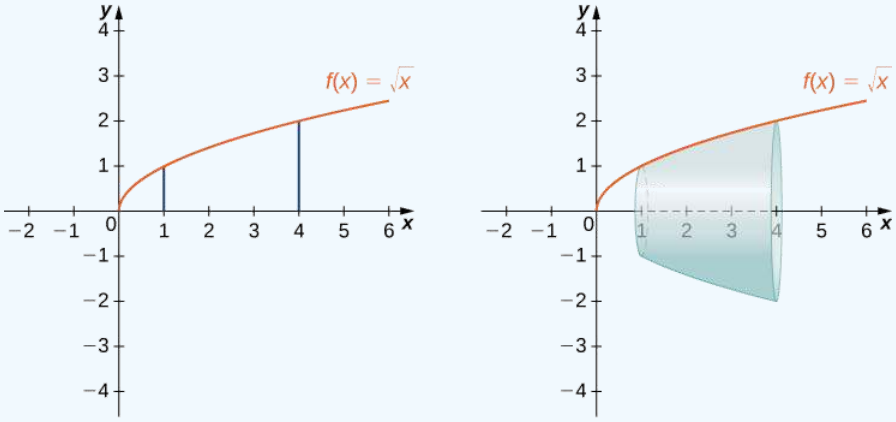


Figure 3.6.10: (a) The graph of $f(x)$. (b) The surface of revolution.

We have $f(x) = \sqrt{x}$. Then, $f'(x) = \frac{1}{2\sqrt{x}}$ and $\left(f'(x)\right)^2 = \frac{1}{4x}$. Then,

$$\begin{aligned}\text{Surface Area} &= \int_a^b \left(2\pi f(x)\sqrt{1+\left(f'(x)\right)^2}\right) dx \\ &= \int_1^4 \left(2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}\right) dx \\ &= \int_1^4 \left(2\pi\sqrt{x+\frac{1}{4}}\right) dx.\end{aligned}$$

Let $u = x + 1/4$. Then, $du = dx$. When $x = 1,\ u = 5/4$, and when $x = 4,\ u = 17/4$. This gives us

$$\begin{aligned}\int_0^1 \left(2\pi\sqrt{x+\frac{1}{4}}\right) dx &= \int_{5/4}^{17/4} 2\pi\sqrt{u}\, du \\ &= 2\pi\left[\frac{2}{3}u^{3/2}\right]\Big|_{5/4}^{17/4} \\ &= \frac{\pi}{6}[17\sqrt{17} - 5\sqrt{5}] \approx 30.846 \text{ units}^2\end{aligned}$$

Exercise 3.6.4

Let $f(x) = \sqrt{1-x}$ over the interval $[0, 1/2]$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the x-axis. Round the answer to three decimal places.

Hint

Use the process from the previous example.

Answer

$$\frac{\pi}{6}(5\sqrt{5} - 3\sqrt{3}) \approx 3.133 \text{ units}^2$$

Example 3.6.5: Calculating the Surface Area of a Surface of Revolution 2

Let $f(x) = y = \sqrt[3]{3x}$. Consider the portion of the curve where $0 \le y \le 2$. Find the surface area of the surface generated by revolving the graph of $f(x)$ around the y-axis.

Solution

Notice that we are revolving the curve around the y-axis, and the interval is in terms of y, so we want to rewrite the function as a function of y. We get $x = g(y) = (1/3)y^3$. The graph of $g(y)$ and the surface of rotation are shown in the following figure.

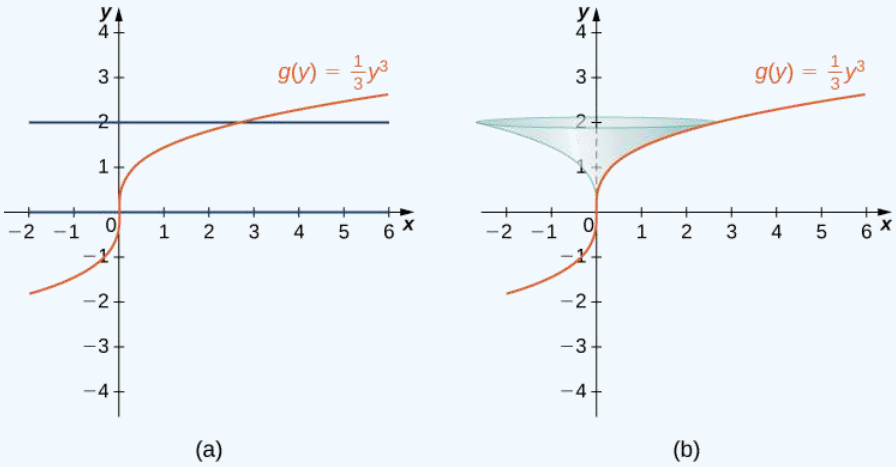


Figure 3.6.11: (a) The graph of $g(y)$. (b) The surface of revolution.

We have $g(y) = (1/3)y^3$, so $g'(y) = y^2$ and $\left(g'(y)\right)^2 = y^4$. Then

$$\begin{aligned} \text{Surface Area} &= \int_c^d \left(2\pi g(y)\sqrt{1+\left(g'(y)\right)^2}\right) dy \\ &= \int_0^2 \left(2\pi\left(\frac{1}{3}y^3\right)\sqrt{1+y^4}\right) dy \\ &= \frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1+y^4}\right) dy. \end{aligned}$$

Let $u = y^4 + 1$. Then $du = 4y^3\, dy$. When $y = 0$, $u = 1$, and when $y = 2$, $u = 17$. Then

$$\begin{aligned} \frac{2\pi}{3}\int_0^2 \left(y^3\sqrt{1+y^4}\right) dy &= \frac{2\pi}{3}\int_1^{17} \frac{1}{4}\sqrt{u}\, du \\ &= \frac{\pi}{6}\left[\frac{2}{3}u^{3/2}\right]\Big|_1^{17} = \frac{\pi}{9}\left[(17)^{3/2} - 1\right] \approx 24.118 \text{ units}^2. \end{aligned}$$

? Exercise 3.6.5

Let $g(y) = \sqrt{9 - y^2}$ over the interval $y \in [0, 2]$. Find the surface area of the surface generated by revolving the graph of $g(y)$ around the y-axis.

Hint

Use the process from the previous example.

Answer

12π units^2

Key Concepts

- The arc length of a curve can be calculated using a definite integral.
- The arc length is first approximated using line segments, which generates a Riemann sum. Taking a limit then gives us the definite integral formula. The same process can be applied to functions of y.
- The concepts used to calculate the arc length can be generalized to find the surface area of a surface of revolution.
- The integrals generated by both the arc length and surface area formulas are often difficult to evaluate. It may be necessary to use a computer or calculator to approximate the values of the integrals.

Key Equations

- **Arc Length of a Function of x**

$$\textbf{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

- **Arc Length of a Function of y**

$$\textbf{Arc Length} = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

- **Surface Area of a Function of x**

$$\textbf{Surface Area} = \int_a^b \left(2\pi f(x)\sqrt{1 + \left(f'(x)\right)^2}\right) dx$$

Glossary

arc length

the arc length of a curve can be thought of as the distance a person would travel along the path of the curve

frustum

a portion of a cone; a frustum is constructed by cutting the cone with a plane parallel to the base

surface area

the surface area of a solid is the total area of the outer layer of the object; for objects such as cubes or bricks, the surface area of the object is the sum of the areas of all of its faces

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3.7: Exercises for Section 6.4

For exercises 1 - 3, find the length of the functions over the given interval.

1) $y = 5x$ from $x = 0$ to $x = 2$

Answer

$s = 2\sqrt{26}$ units

2) $y = -\frac{1}{2}x + 25$ from $x = 1$ to $x = 4$

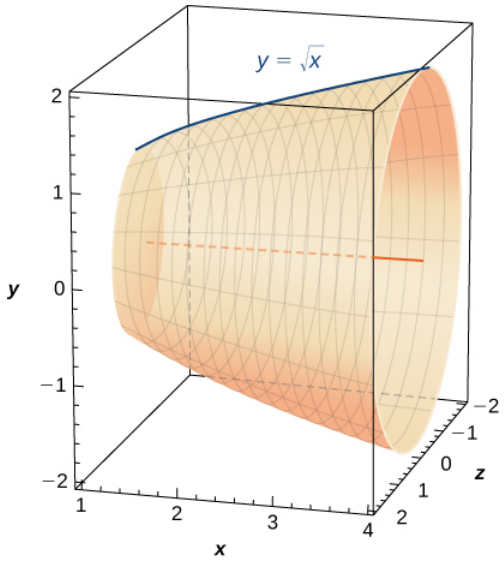
3) $x = 4y$ from $y = -1$ to $y = 1$

Answer

$s = 2\sqrt{17}$ units

4) Pick an arbitrary linear function $x = g(y)$ over any interval of your choice (y_1, y_2). Determine the length of the function and then prove the length is correct by using geometry.

5) Find the surface area of the volume generated when the curve $y = \sqrt{x}$ revolves around the x-axis from $(1, 1)$ to $(4, 2)$, as seen here.



Answer

$A = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$ units^2

6) Find the surface area of the volume generated when the curve $y = x^2$ revolves around the y-axis from $(1, 1)$ to $(3, 9)$.

For exercises 7 - 16, find the lengths of the functions of x over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

7) $y = x^{3/2}$ from $(0, 0)$ to $(1, 1)$

Answer

$s = \frac{13\sqrt{13} - 8}{27}$ units

8) $y = x^{2/3}$ from $(1, 1)$ to $(8, 4)$

9) $y = \frac{1}{3}(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 1$

Answer

$s = \frac{4}{3}$ units

10) $y = \frac{1}{3}(x^2 - 2)^{3/2}$ from $x = 2$ to $x = 4$

11) [T] $y = e^x$ on $x = 0$ to $x = 1$

Answer

$s \approx 2.0035$ units

12) $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ from $x = 1$ to $x = 3$

13) $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ from $x = 1$ to $x = 2$

Answer

$s = \frac{123}{32}$ units

14) $y = \dfrac{2x^{3/2}}{3} - \dfrac{x^{1/2}}{2}$ from $x = 1$ to $x = 4$

15) $y = \frac{1}{27}(9x^2 + 6)^{3/2}$ from $x = 0$ to $x = 2$

Answer

$s = 10$ units

16) [T] $y = \sin x$ on $x = 0$ to $x = \pi$

For exercises 17 - 26, find the lengths of the functions of y over the given interval. If you cannot evaluate the integral exactly, use technology to approximate it.

17) $y = \dfrac{5 - 3x}{4}$ from $y = 0$ to $y = 4$

Answer

$s = \frac{20}{3}$ units

18) $x = \frac{1}{2}(e^y + e^{-y})$ from $y = -1$ to $y = 1$

19) $x = 5y^{3/2}$ from $y = 0$ to $y = 1$

Answer

$s = \frac{1}{675}(229\sqrt{229} - 8)$ units

20) [T] $x = y^2$ from $y = 0$ to $y = 1$

21) $x = \sqrt{y}$ from $y = 0$ to $y = 1$

Answer

$s = \frac{1}{8}(4\sqrt{5} + \ln(9 + 4\sqrt{5}))$ units

22) $x = \frac{2}{3}(y^2 + 1)^{3/2}$ from $y = 1$ to $y = 3$

23) [T] $x = \tan y$ from $y = 0$ to $y = \frac{3}{4}$

Answer

$s \approx 1.201$ units

24) [T] $x = \cos^2 y$ from $y = -\frac{\pi}{2}$ to $y = \frac{\pi}{2}$

25) [T] $x = 4^y$ from $y = 0$ to $y = 2$

Answer

$s \approx 15.2341$units

26) [T] $x = \ln(y)$ on $y = \dfrac{1}{e}$ to $y = e$

For exercises 27 - 34, find the surface area of the volume generated when the following curves revolve around the x-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

27) $y = \sqrt{x}$ from $x = 2$ to $x = 6$

Answer

$A = \frac{49\pi}{3}$ units^2

28) $y = x^3$ from $x = 0$ to $x = 1$

29) $y = 7x$ from $x = -1$ to $x = 1$

Answer

$A = 70\pi\sqrt{2}$ units^2

30) [T] $y = \frac{1}{x^2}$ from $x = 1$ to $x = 3$

31) $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = 2$

Answer

$A = 8\pi$ units^2

32) $y = \sqrt{4 - x^2}$ from $x = -1$ to $x = 1$

33) $y = 5x$ from $x = 1$ to $x = 5$

Answer

$A = 120\pi\sqrt{26}$ units^2

34) [T] $y = \tan x$ from $x = -\frac{\pi}{4}$ to $x = \frac{\pi}{4}$

For exercises 35 - 42, find the surface area of the volume generated when the following curves revolve around the y-axis. If you cannot evaluate the integral exactly, use your calculator to approximate it.

35) $y = x^2$ from $x = 0$ to $x = 2$

Answer

$A = \frac{\pi}{6}(17\sqrt{17} - 1)$ units^2

36) $y = \frac{1}{2}x^2 + \frac{1}{2}$ from $x = 0$ to $x = 1$

37) $y = x + 1$ from $x = 0$ to $x = 3$

Answer

$A = 9\sqrt{2}\pi$ units^2

38) [T] $y = \dfrac{1}{x}$ from $x = \dfrac{1}{2}$ to $x = 1$

39) $y = \sqrt[3]{x}$ from $x = 1$ to $x = 27$

Answer

$A = \frac{10\sqrt{10}\pi}{27}(73\sqrt{73} - 1)$ units^2

40) [T] $y = 3x^4$ from $x = 0$ to $x = 1$

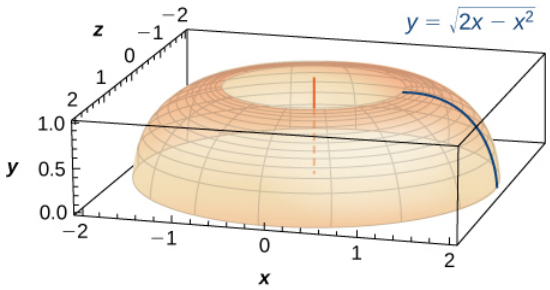
41) [T] $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 3$

Answer

$A \approx 25.645 \text{units}^2$

42) [T] $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$

43) The base of a lamp is constructed by revolving a quarter circle $y = \sqrt{2x - x^2}$ around the y-axis from $x = 1$ to $x = 2$, as seen here. Create an integral for the surface area of this curve and compute it.



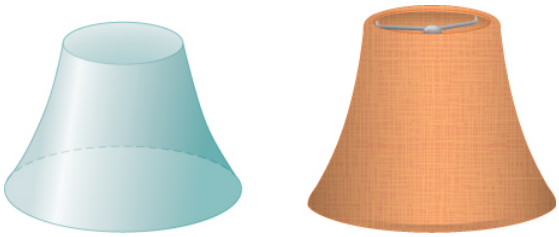
Answer

$A = 2\pi \text{ units}^2$

44) A light bulb is a sphere with radius $1/2$ in. with the bottom sliced off to fit exactly onto a cylinder of radius $1/4$ in. and length $1/3$ in., as seen here. The sphere is cut off at the bottom to fit exactly onto the cylinder, so the radius of the cut is $1/4$ in. Find the surface area (not including the top or bottom of the cylinder).



45) [T] A lampshade is constructed by rotating $y = 1/x$ around the x-axis from $y = 1$ to $y = 2$, as seen here. Determine how much material you would need to construct this lampshade—that is, the surface area—accurate to four decimal places.



Answer

10.5017units^2

46) [T] An anchor drags behind a boat according to the function $y = 24e^{-x/2} - 24$, where y represents the depth beneath the boat and x is the horizontal distance of the anchor from the back of the boat. If the anchor is 23 ft below the boat, how much rope do you have to pull to reach the anchor? Round your answer to three decimal places.

47) [T] You are building a bridge that will span 10 ft. You intend to add decorative rope in the shape of $y = 5|\sin((x\pi)/5)|$, where x is the distance in feet from one end of the bridge. Find out how much rope you need to buy, rounded to the nearest foot.

Answer

23 ft

For exercise 48, find the exact arc length for the following problems over the given interval.

48) $y = \ln(\sin x)$ from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$. (Hint: Recall trigonometric identities.)

49) Draw graphs of $y = x^2, y = x^6$, and $y = x^{10}$. For $y = x^n$, as n increases, formulate a prediction on the arc length from $(0, 0)$ to $(1, 1)$. Now, compute the lengths of these three functions and determine whether your prediction is correct.

Answer

2

50) Compare the lengths of the parabola $x = y^2$ and the line $x = by$ from $(0, 0)$ to (b^2, b) as b increases. What do you notice?

51) Solve for the length of $x = y^2$ from $(0, 0)$ to $(1, 1)$. Show that $x = \dfrac{y^2}{2}$ from $(0, 0)$ to $(2, 2)$ is twice as long. Graph both functions and explain why this is so.

Answer

Answers may vary

52) [T] Which is longer between $(1, 1)$ and $\left(2, \frac{1}{2}\right)$: the hyperbola $y = \dfrac{1}{x}$ or the graph of $x + 2y = 3$?

53) Explain why the surface area is infinite when $y = 1/x$ is rotated around the x-axis for $1 \leq x < \infty,$ but the volume is finite.

Answer

For more information, look up Gabriel's Horn.

Contributors

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3.8: Using Integration to Determine Work

Work is the scientific term used to describe the action of a force which moves an object. When a constant force $\vec{F}$ is applied to move an object a distance d, the amount of work performed is

$$W = \vec{F} \cdot \vec{d}. \tag{3.8.1}$$

The SI unit of force is the Newton, (kg·m/s^2) and the SI unit of distance is a meter (m). The fundamental unit of work is one Newton-meter, or a joule (J). That is, applying a force of one Newton for one meter performs one joule of work. In Imperial units (as used in the United States), force is measured in pounds (lb) and distance is measured in feet (ft), hence work is measured in ft-lb.

Mass vs. Weight

Mass and *weight* are closely related, yet different, concepts. The mass m of an object is a quantitative measure of that object's resistance to acceleration. The weight w of an object is a measurement of the force applied to the object by the acceleration of gravity g.

Since the two measurements are proportional, $w = m \cdot g$, they are often used interchangeably in everyday conversation. When computing work, one must be careful to note which is being referred to. When mass is given, it must be multiplied by the acceleration of gravity to reference the related force.

When force is constant, the measurement of work is straightforward. For instance, lifting a 200 lb object 5 ft performs $200 \cdot 5 = 1000$ ft-lb of work.

What if the force applied is variable? For instance, imagine a climber pulling a 200 ft rope up a vertical face. The rope becomes lighter as more is pulled in, requiring less force and hence the climber performs less work.

In general, let $F(x)$ be a force function on an interval $[a, b]$. We want to measure the amount of work done applying the force F from $x = a$ to $x = b$. We can approximate the amount of work being done by partitioning $[a, b]$ into subintervals $a = x_1 < x_2 < \cdots < x_{n+1} = b$ and assuming that F is constant on each subinterval. Let c_i be a value in the ith subinterval $[x_i, x_{i+1}]$. Then the work done on this interval is approximately

$$W_i \approx F(c_i) \cdot (x_{i+1} - x_i) = F(c_i)\,\Delta x_i,$$

a constant force $\times$ the distance over which it is applied. The total work is

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(c_i)\,\Delta x_i.$$

This, of course, is a Riemann sum. Taking a limit as the subinterval lengths go to zero give an exact value of work which can be evaluated through a definite integral.

Key Idea 29: Work

Let $F(x)$ be a continuous function on $[a, b]$ describing the amount of force being applied to an object in the direction of travel from distance $x = a$ to distance $x = b$. The total work W done on $[a, b]$ is

$$W = \int_a^b F(x)\,dx. \tag{3.8.2}$$

Example 3.8.1: Computing work performed - applying variable force

A 60m climbing rope is hanging over the side of a tall cliff. How much work is performed in pulling the rope up to the top, where the rope has a mass of 66g/m? How much work is performed pulling a 60 m climbing rope up a cliff face, where the rope has a mass of 66 g/m?

Solution

We need to create a force function $F(x)$ on the interval $[0, 60]$. To do so, we must first decide what x is measuring: is it the length of the rope still hanging, or is it the amount of rope pulled in? As long as we are consistent, either approach is fine. We adopt for this example the convention that x is the amount of rope pulled in. This seems to match intuition better; pulling up the first 10 meters of rope involves $x = 0$ to $x = 10$ instead of $x = 60$ to $x = 50$.

As x is the amount of rope pulled in, the amount of rope still hanging is $60 - x$. This length of rope has a mass of 66 g/m, or 0.066 kg/m. The the mass of the rope still hanging is $0.066(60 - x)$ kg; multiplying this mass by the acceleration of gravity, 9.8 m/s^2, gives our variable force function

$$F(x) = (9.8)(0.066)(60 - x) = 0.6468(60 - x).$$

Thus the total work performed in pulling up the rope is

$$W = \int_0^{60} 0.6468(60 - x)\, dx = 1,164.24 \text{ J}.$$

By comparison, consider the work done in lifting the entire rope 60 meters. The rope weights $60 \times 0.066 \times 9.8 = 38.808$ N, so the work applying this force for 60 meters is $60 \times 38.808 = 2,328.48$ J. This is exactly twice the work calculated before (and we leave it to the reader to understand why.)

Example 3.8.2: Computing work performed - applying variable force

Consider again pulling a 60 m rope up a cliff face, where the rope has a mass of 66 g/m. At what point is exactly half the work performed?

Solution

From Example 3.8.1 we know the total work performed is $1,164.24$ J. We want to find a height h such that the work in pulling the rope from a height of $x = 0$ to a height of $x = h$ is 582.12, half the total work. Thus we want to solve the equation

$$\int_0^h 0.6468(60 - x)\, dx = 582.12$$

for h.

$$\begin{aligned} \int_0^h 0.6468(60 - x)\, dx &= 582.12 \\ \left(38.808x - 0.3234x^2\right)\Big|_0^h &= 582.12 \\ 38.808h - 0.3234h^2 &= 582.12 \\ -0.3234h^2 + 38.808h - 582.12 &= 0. \end{aligned}$$

Apply the Quadratic Formula.

$$h = 17.57 \text{ and } 102.43$$

As the rope is only 60 m long, the only sensible answer is $h = 17.57$. Thus about half the work is done pulling up the first 17.5 m the other half of the work is done pulling up the remaining 42.43 m.

Example 3.8.3: Computing work performed: applying variable force

A box of 100 lb of sand is being pulled up at a uniform rate a distance of 50 ft over 1 minute. The sand is leaking from the box at a rate of 1 lb/s. The box itself weighs 5 lb and is pulled by a rope weighing .2 lb/ft.

a. How much work is done lifting just the rope?
b. How much work is done lifting just the box and sand?
c. What is the total amount of work performed?

Solution

a. We start by forming the force function $F_r(x)$ for the rope (where the subscript denotes we are considering the rope). As in the previous example, let x denote the amount of rope, in feet, pulled in. (This is the same as saying x denotes the height of the box.) The weight of the rope with x feet pulled in is

$$F_r(x) = 0.2(50 - x) = 10 - 0.2x.$$

(Note that we do not have to include the acceleration of gravity here, for the *weight* of the rope per foot is given, not its *mass* per meter as before.) The work performed lifting the rope is

$$W_r = \int_0^{50} (10 - 0.2x)\, dx = 250 \text{ ft-lb}.$$

b. The sand is leaving the box at a rate of 1 lb/s. As the vertical trip is to take one minute, we know that 60 lb will have left when the box reaches its final height of 50 ft. Again letting x represent the height of the box, we have two points on the line that describes the weight of the sand: when $x = 0$, the sand weight is 100 lb, producing the point $(0, 100)$; when $x = 50$, the sand in the box weighs 40 lb, producing the point $(50, 40)$. The slope of this line is $\frac{100-40}{0-50} = -1.2$, giving the equation of the weight of the sand at height x as $w(x) = -1.2x + 100$. The box itself weighs a constant 5 lb, so the total force function is $F_b(x) = -1.2x + 105$. Integrating from $x = 0$ to $x = 50$ gives the work performed in lifting box and sand:

$$W_b = \int_0^{50} (-1.2x + 105)\, dx = 3750 \text{ ft-lb}.$$

c. The total work is the sum of W_r and W_b: $250 + 3750 = 4000$ ft-lb. We can also arrive at this via integration:

$$\begin{aligned} W &= \int_0^{50} (F_r(x) + F_b(x))\, dx \\ &= \int_0^{50} (10 - 0.2x - 1.2x + 105)\, dx \\ &= \int_0^{50} (-1.4x + 115)\, dx \\ &= 4000 \text{ ft-lb}. \end{aligned}$$

Hooke's Law and Springs

Hooke's Law states that the force required to compress or stretch a spring x units from its natural length is proportional to x; that is, this force is $F(x) = kx$ for some constant k. For example, if a force of 1 N stretches a given spring 2 cm, then a force of 5 N will stretch the spring 10 cm. Converting the distances to meters, we have that stretching this spring 0.02 m requires a force of $F(0.02) = k(0.02) = 1$ N, hence $k = \frac{1}{0.02} = 50$ N/m.

Example 3.8.4: Computing work performed: stretching a spring

A force of 20 lb stretches a spring from a natural length of 7 inches to a length of 12 inches. How much work was performed in stretching the spring to this length?

Solution

In many ways, we are not at all concerned with the actual length of the spring, only with the amount of its change. Hence, we do not care that 20 lb of force stretches the spring to a length of 12 inches, but rather that a force of 20 lb stretches the spring by 5 in. This is illustrated in Figure 3.8.1; we only measure the change in the spring's length, not the overall length of the spring.

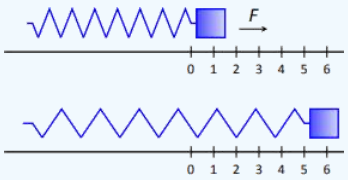


Figure 3.8.1: Illustrating the important aspects of stretching a spring in computing work in Example 3.8.4.

Converting the units of length to feet, we have

$$F\left(\tfrac{5}{12}\right) = \tfrac{5}{12}k = 20 \text{ lb}.$$

Thus $k = 48$ lb/ft and $F(x) = 48x$.

We compute the total work performed by integrating $F(x)$ from $x = 0$ to $x = \frac{5}{12}$:

$$\begin{aligned} W &= \int_0^{5/12} 48x\, dx \\ &= 24x^2\Big|_0^{5/12} \\ &= \frac{25}{6} \approx 4.1667 \text{ ft-lb}. \end{aligned}$$

Pumping Fluids

Another useful example of the application of integration to compute work comes in the pumping of fluids, often illustrated in the context of emptying a storage tank by pumping the fluid out the top. This situation is different than our previous examples for the forces involved are constant. After all, the force required to move one cubic foot of water (about 62.4 lb) is the same regardless of its location in the tank. What is variable is the distance that cubic foot of water has to travel; water closer to the top travels less distance than water at the bottom, producing less work.

Figure 3.8.2: Weight and Mass densities.

Fluid	lb/ft^3	kg/m^3
Concrete	150	2400
Fuel Oil	55.46	890.13
Gasoline	45.93	737.22
Iodine	307	4927
Methanol	49.3	791.3
Mercury	844	1354
Milk	63.6-65.4	1020-1050
Water	62.4	1000

We demonstrate how to compute the total work done in pumping a fluid out of the top of a tank in the next two examples.

Example 3.8.5: Computing work performed: pumping fluids

A cylindrical storage tank with a radius of 10 ft and a height of 30 ft is filled with water, which weighs approximately 62.4 lb/ft^3. Compute the amount of work performed by pumping the water up to a point 5 feet above the top of the tank.

Solution

We will refer often to Figure 3.8.3 which illustrates the salient aspects of this problem.

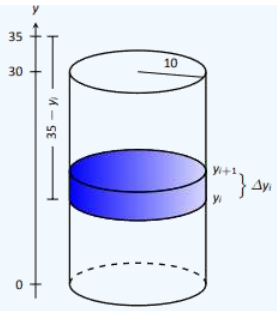


Figure 3.8.3: Illustrating a water tank in order to compute the work required to empty it in Example 7.5.5.

We start as we often do: we partition an interval into subintervals. We orient our tank vertically since this makes intuitive sense with the base of the tank at $y = 0$. Hence the top of the water is at $y = 30$, meaning we are interested in subdividing the y-interval $[0, 30]$ into n subintervals as

$$0 = y_1 < y_2 < \cdots < y_{n+1} = 30.$$

Consider the work W_i of pumping only the water residing in the ith subinterval, illustrated in Figure 3.8.3. The force required to move this water is equal to its weight which we calculate as volume $\times$ density. The volume of water in this subinterval is $V_i = 10^2\pi\Delta y_i$; its density is $62.4\,\text{lb/ft}^3$. Thus the required force is $6240\pi\Delta y_i\,\text{lb}$.

We approximate the distance the force is applied by using any y-value contained in the ith subinterval; for simplicity, we arbitrarily use y_i for now (it will not matter later on). The water will be pumped to a point 5 feet above the top of the tank, that is, to the height of $y = 35$ ft. Thus the distance the water at height y_i travels is $35 - y_i$ ft.

In all, the approximate work W_i performed in moving the water in the ith subinterval to a point 5 feet above the tank is

$$W_i \approx 6240\pi\Delta y_i(35 - y_i).$$

To approximate the total work performed in pumping out all the water from the tank, we sum all the work W_i performed in pumping the water from each of the n subintervals of $[0, 30]$:

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} 6240\pi\Delta y_i(35 - y_i).$$

This is a Riemann sum. Taking the limit as the subinterval length goes to 0 gives

$$\begin{aligned} W &= \int_0^{30} 6240\pi(35 - y)\,dy \\ &= 6240\pi\left(35y - \tfrac{1}{2}y^2\right)\Big|_0^{30} \\ &= 11,762,123\,\text{ft-lb} \\ &\approx 1.176 \times 10^7\,\text{ft-lb}. \end{aligned}$$

We can "streamline" the above process a bit as we may now recognize what the important features of the problem are. Figure 3.8.4 shows the tank from Example 3.8.5 without the ith subinterval identified.

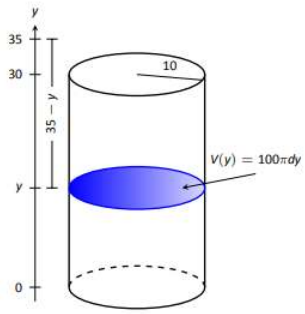


Figure 3.8.4: A simplified illustration for computing work.

Instead, we just draw one differential element. This helps establish the height a small amount of water must travel along with the force required to move it (where the force is volume $\times$ density).

We demonstrate the concepts again in the next examples.

Example 3.8.6: Computing work performed - pumping fluids

A conical water tank has its top at ground level and its base 10 feet below ground. The radius of the cone at ground level is 2 ft. It is filled with water weighing 62.4 lb/ft^3 and is to be emptied by pumping the water to a spigot 3 feet above ground level. Find the total amount of work performed in emptying the tank.

Solution

The conical tank is sketched in Figure 3.8.5. We can orient the tank in a variety of ways; we could let $y = 0$ represent the base of the tank and $y = 10$ represent the top of the tank, but we choose to keep the convention of the wording given in the problem and let $y = 0$ represent ground level and hence $y = -10$ represents the bottom of the tank. The actual "height" of the water does not matter; rather, we are concerned with the distance the water travels.

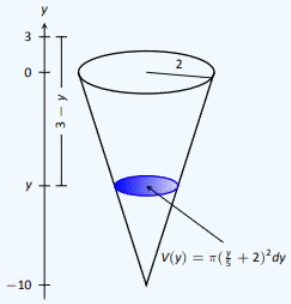


Figure 3.8.5: A graph of the conical water tank in Example 3.8.6.

The figure also sketches a differential element, a cross-sectional circle. The radius of this circle is variable, depending on y. When $y = -10$, the circle has radius 0; when $y = 0$, the circle has radius 2. These two points, $(-10, 0)$ and $(0, 2)$, allow us to find the equation of the line that gives the radius of the cross-sectional circle, which is $r(y) = \frac{1}{5}y + 2$. Hence the volume of water at this height is $V(y) = \pi(\frac{1}{5}y + 2)^2\, dy$, where dy represents a very small height of the differential element. The force required to move the water at height y is $F(y) = 62.4 \times V(y)$.

The distance the water at height y travels is given by $h(y) = 3 - y$. Thus the total work done in pumping the water from the tank is

$$\begin{aligned} W &= \int_{-10}^{0} 62.4\pi(\tfrac{1}{5}y + 2)^2(3 - y)\, dy \\ &= 62.4\pi \int_{-10}^{0} \left(-\tfrac{1}{25}y^3 - \tfrac{17}{25}y^2 - \tfrac{8}{5}y + 12\right)\, dy \\ &= 62.2\pi \cdot \frac{220}{3} \approx 14,376 \text{ ft-lb.} \end{aligned}$$

Example 3.8.7: Computing work performed - pumping fluids

A rectangular swimming pool is 20 ft wide and has a 3 ft "shallow end" and a 6 ft "deep end." It is to have its water pumped out to a point 2 ft above the current top of the water.

The cross-sectional dimensions of the water in the pool are given in Figure 3.8.6; note that the dimensions are for the water, not the pool itself. Compute the amount of work performed in draining the pool.

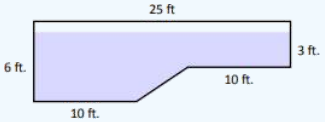


Figure 3.8.6: The cross-section of a swimming pool filled with water in Example 3.8.7.

Solution

For the purposes of this problem we choose to set $y = 0$ to represent the bottom of the pool, meaning the top of the water is at $y = 6$.

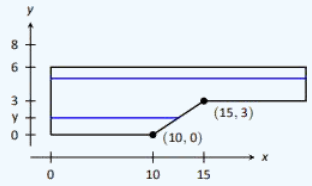


Figure 3.8.7: Orienting the pool and showing differential elements for Example 3.8.7.

Figure 3.8.7 shows the pool oriented with this y-axis, along with 2 differential elements as the pool must be split into two different regions.

The top region lies in the y-interval of $[3, 6]$, where the length of the differential element is 25 ft as shown. As the pool is 20 ft wide, this differential element represents a this slice of water with volume $V(y) = 20 \cdot 25 \cdot dy$. The water is to be pumped to a height of $y = 8$, so the height function is $h(y) = 8 - y$. The work done in pumping this top region of water is

$$W_t = 62.4 \int_3^6 500(8 - y)\, dy = 327,600 \text{ ft-lb.}$$

The bottom region lies in the y-interval of $[0, 3]$; we need to compute the length of the differential element in this interval.

One end of the differential element is at $x = 0$ and the other is along the line segment joining the points $(10, 0)$ and $(15, 3)$. The equation of this line is $y = \frac{3}{5}(x - 10)$; as we will be integrating with respect to y, we rewrite this equation as $x = \frac{5}{3}y + 10$. So the length of the differential element is a difference of x-values: $x = 0$ and $x = \frac{5}{3}y + 10$, giving a length of $x = \frac{5}{3}y + 10$.

Again, as the pool is 20 ft wide, this differential element represents a thin slice of water with volume $V(y) = 20 \cdot (\frac{5}{3}y + 10) \cdot dy$; the height function is the same as before at $h(y) = 8 - y$. The work performed in emptying this part of the pool is

$$W_b = 62.4 \int_0^3 20(\tfrac{5}{3}y + 10)(8 - y)\, dy = 299,520 \text{ ft-lb.}$$

The total work in emptying the pool is

$$W = W_b + W_t = 327,600 + 299,520 = 627,120 \text{ ft-lb.}$$

Notice how the emptying of the bottom of the pool performs almost as much work as emptying the top. The top portion travels a shorter distance but has more water. In the end, this extra water produces more work.

The next section introduces one final application of the definite integral, the calculation of fluid force on a plate.

Contributors and Attributions

- Gregory Hartman (Virginia Military Institute). Contributions were made by Troy Siemers and Dimplekumar Chalishajar of VMI and Brian Heinold of Mount Saint Mary's University. This content is copyrighted by a Creative Commons Attribution - Noncommercial (BY-NC) License. http://www.apexcalculus.com/

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3.9: More Physical Applications of Integration

Learning Objectives

- Determine the mass of a one-dimensional object from its linear density function.
- Determine the mass of a two-dimensional circular object from its radial density function.
- Calculate the work done by a variable force acting along a line.
- Calculate the work done in pumping a liquid from one height to another.
- Find the hydrostatic force against a submerged vertical plate.

In this section, we examine some physical applications of integration. Let's begin with a look at calculating mass from a density function. We then turn our attention to work, and close the section with a study of hydrostatic force.

Mass and Density

We can use integration to develop a formula for calculating mass based on a density function. First we consider a thin rod or wire. Orient the rod so it aligns with the x-axis, with the left end of the rod at $x = a$ and the right end of the rod at $x = b$ (Figure 3.9.1). Note that although we depict the rod with some thickness in the figures, for mathematical purposes we assume the rod is thin enough to be treated as a one-dimensional object.

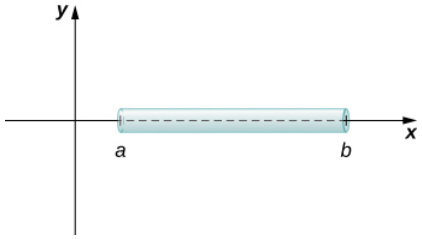


Figure 3.9.1: We can calculate the mass of a thin rod oriented along the x-axis by integrating its density function.

If the rod has constant density ρ, given in terms of mass per unit length, then the mass of the rod is just the product of the density and the length of the rod: $(b-a)\rho$. If the density of the rod is not constant, however, the problem becomes a little more challenging. When the density of the rod varies from point to point, we use a linear density function, $\rho(x)$, to denote the density of the rod at any point, x. Let $\rho(x)$ be an integrable linear density function. Now, for $i = 0, 1, 2, \ldots, n$ let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$ choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure 3.9.2 shows a representative segment of the rod.

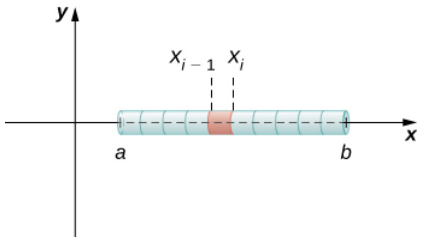


Figure 3.9.2: A representative segment of the rod.

The mass m_i of the segment of the rod from x_{i-1} to x_i is approximated by

$$\begin{aligned} m_i &\approx \rho(x_i^*)(x_i - x_{i-1}) \\ &= \rho(x_i^*)\Delta x. \end{aligned}$$

Adding the masses of all the segments gives us an approximation for the mass of the entire rod:

$$\begin{aligned} m &= \sum_{i=1}^{n} m_i \\ &\approx \sum_{i=1}^{n} \rho(x_i^*)\Delta x. \end{aligned}$$

This is a Riemann sum. Taking the limit as $n \to \infty$, we get an expression for the exact mass of the rod:

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho(x_i^*)\Delta x \\ &= \int_a^b \rho(x)dx. \end{aligned}$$

We state this result in the following theorem.

Mass–Density Formula of a One-Dimensional Object

Given a thin rod oriented along the x-axis over the interval $[a, b]$, let $\rho(x)$ denote a linear density function giving the density of the rod at a point x in the interval. Then the mass of the rod is given by

$$m = \int_a^b \rho(x)dx. \tag{3.9.1}$$

We apply this theorem in the next example.

Example 3.9.1: Calculating Mass from Linear Density

Consider a thin rod oriented on the x-axis over the interval $[\pi/2, \pi]$ If the density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

Solution

Applying Equation 3.9.1 directly, we have

$$\begin{aligned} m &= \int_a^b \rho(x)dx \\ &= \int_{\pi/2}^{\pi} \sin x\, dx \\ &= -\cos x\Big|_{\pi/2}^{\pi} \\ &= 1. \end{aligned}$$

Exercise 3.9.1

Consider a thin rod oriented on the x-axis over the interval $[1, 3]$. If the density of the rod is given by $\rho(x) = 2x^2 + 3,$ what is the mass of the rod?

Hint

Use the process from the previous example.

Solution

$70/3$

We now extend this concept to find the mass of a two-dimensional disk of radius r. As with the rod we looked at in the one-dimensional case, here we assume the disk is thin enough that, for mathematical purposes, we can treat it as a two-dimensional object. We assume the density is given in terms of mass per unit area (called **area density**), and further assume the density varies only along the disk's radius (called **radial density**). We orient the disk in the $xy-plane$, with the center at the origin. Then, the density of the disk can be treated as a function of x, denoted $\rho(x)$. We assume $\rho(x)$ is integrable. Because density is a function of x, we partition the interval from $[0, r]$ along the x-axis. For $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[0, r]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Now, use the partition to break up the disk into thin (two-dimensional) washers. A disk and a representative washer are depicted in the following figure.

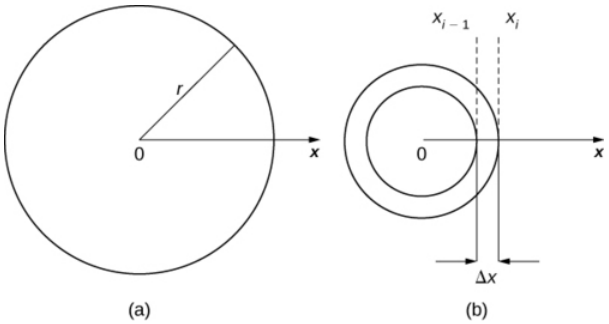


Figure 3.9.3: (a) A thin disk in the *xy*-plane. (b) A representative washer.

We now approximate the density and area of the washer to calculate an approximate mass, m_i. Note that the area of the washer is given by

$$\begin{aligned} A_i &= \pi(x_i)^2 - \pi(x_{i-1})^2 \\ &= \pi[x_i^2 - x_{i-1}^2] \\ &= \pi(x_i + x_{i-1})(x_i - x_{i-1}) \\ &= \pi(x_i + x_{i-1})\Delta x. \end{aligned}$$

You may recall that we had an expression similar to this when we were computing volumes by shells. As we did there, we use $x_i^* \approx (x_i + x_{i-1})/2$ to approximate the average radius of the washer. We obtain

$$A_i = \pi(x_i + x_{i-1})\Delta x \approx 2\pi x_i^* \Delta x.$$

Using $\rho(x_i^*)$ to approximate the density of the washer, we approximate the mass of the washer by

$$m_i \approx 2\pi x_i^* \rho(x_i^*)\Delta x.$$

Adding up the masses of the washers, we see the mass m of the entire disk is approximated by

$$m = \sum_{i=1}^{n} m_i \approx \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x.$$

We again recognize this as a Riemann sum, and take the limit as $n \to \infty$. This gives us

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\pi x_i^* \rho(x_i^*)\Delta x \\ &= \int_0^r 2\pi x \rho(x) dx. \end{aligned}$$

We summarize these findings in the following theorem.

Mass–Density Formula of a Circular Object

Let $\rho(x)$ be an integrable function representing the radial density of a disk of radius r. Then the mass of the disk is given by

$$m = \int_0^r 2\pi x \rho(x) dx. \tag{3.9.2}$$

Example 3.9.2: Calculating Mass from Radial Density

Let $\rho(x) = \sqrt{x}$ represent the radial density of a disk. Calculate the mass of a disk of radius 4.

Solution

Applying Equation 3.9.2, we find

$$m = \int_0^r 2\pi x \rho(x)dx$$
$$= \int_0^4 2\pi x\sqrt{x}dx = 2\pi \int_0^4 x^{3/2}dx$$
$$= 2\pi \frac{2}{5}x^{5/2}\,|_0^4 = \frac{4\pi}{5}[32]$$
$$= \frac{128\pi}{5}.$$

Exercise 3.9.2

Let $\rho(x) = 3x + 2$ represent the radial density of a disk. Calculate the mass of a disk of radius 2.

Hint

Use the process from the previous example.

Solution

24π

Work Done by a Force

We now consider work. In physics, work is related to force, which is often intuitively defined as a push or pull on an object. When a force moves an object, we say the force does work on the object. In other words, work can be thought of as the amount of energy it takes to move an object. According to physics, when we have a constant force, work can be expressed as the product of force and distance.

In the English system, the unit of force is the pound and the unit of distance is the foot, so work is given in foot-pounds. In the metric system, kilograms and meters are used. One newton is the force needed to accelerate 1 kilogram of mass at the rate of 1 m/sec^2. Thus, the most common unit of work is the newton-meter. This same unit is also called the joule. Both are defined as kilograms times meters squared over seconds squared $(kg \cdot m^2/s^2)$.

When we have a constant force, things are pretty easy. It is rare, however, for a force to be constant. The work done to compress (or elongate) a spring, for example, varies depending on how far the spring has already been compressed (or stretched). We look at springs in more detail later in this section.

Suppose we have a variable force $F(x)$ that moves an object in a positive direction along the x-axis from point a to point b. To calculate the work done, we partition the interval $[a, b]$ and estimate the work done over each subinterval. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. To calculate the work done to move an object from point x_{i-1} to point x_i, we assume the force is roughly constant over the interval, and use $F(x_i^*)$ to approximate the force. The work done over the interval $[x_{i-1}, x_i]$, then, is given by

$$W_i \approx F(x_i^*)(x_i - x_{i-1}) = F(x_i^*)\Delta x.$$

Therefore, the work done over the interval $[a, b]$ is approximately

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(x_i^*)\Delta x.$$

Taking the limit of this expression as $n \to \infty$ gives us the exact value for work:

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} F(x_i^*)\Delta x$$
$$= \int_a^b F(x)dx.$$

Thus, we can define work as follows.

Definition: Work

If a variable force $F(x)$ moves an object in a positive direction along the x-axis from point a to point b, then the work done on the object is

$$W = \int_a^b F(x)dx. \tag{3.9.3}$$

Note that if F is constant, the integral evaluates to $F \cdot (b-a) = F \cdot d$, which is the formula we stated at the beginning of this section.

Now let's look at the specific example of the work done to compress or elongate a spring. Consider a block attached to a horizontal spring. The block moves back and forth as the spring stretches and compresses. Although in the real world we would have to account for the force of friction between the block and the surface on which it is resting, we ignore friction here and assume the block is resting on a frictionless surface. When the spring is at its natural length (at rest), the system is said to be at equilibrium. In this state, the spring is neither elongated nor compressed, and in this equilibrium position the block does not move until some force is introduced. We orient the system such that $x = 0$ corresponds to the equilibrium position (Figure 3.9.4).

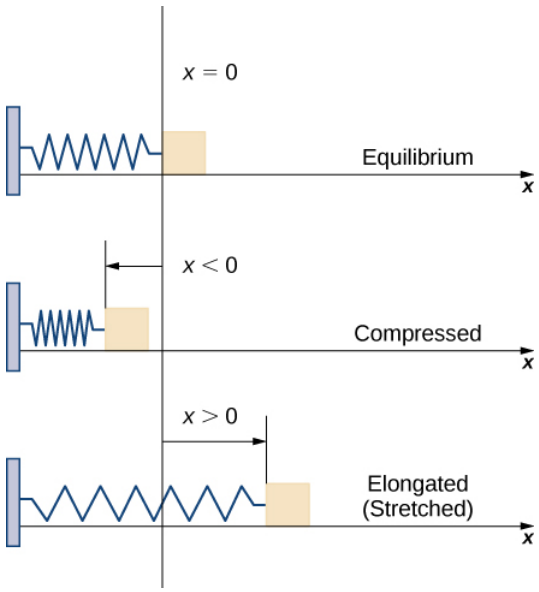


Figure 3.9.4: A block attached to a horizontal spring at equilibrium, compressed, and elongated.

According to Hooke's law, the force required to compress or stretch a spring from an equilibrium position is given by $F(x) = kx$, for some constant k. The value of k depends on the physical characteristics of the spring. The constant k is called the spring constant and is always positive. We can use this information to calculate the work done to compress or elongate a spring, as shown in the following example.

Example 3.9.3: The Work Required to Stretch or Compress a Spring

Suppose it takes a force of 10 N (in the negative direction) to compress a spring 0.2 m from the equilibrium position. How much work is done to stretch the spring 0.5 m from the equilibrium position?

Solution

First find the spring constant, k. When $x = -0.2$, we know $F(x) = -10$, so

$$\begin{aligned} F(x) &= kx \\ -10 &= k(-0.2) \\ k &= 50 \end{aligned}$$

and $F(x) = 50x$. Then, to calculate work, we integrate the force function, obtaining

$$\begin{aligned} W &= \int_a^b F(x)dx \\ &= \int_0^{0.5} 50x\, dx \\ &= 25x^2\Big|_0^{0.5} \\ &= 6.25. \end{aligned}$$

The work done to stretch the spring is 6.25 J.

? Exercise 3.9.3

Suppose it takes a force of 8 lb to stretch a spring 6 in. from the equilibrium position. How much work is done to stretch the spring 1 ft from the equilibrium position?

Hint

Use the process from the previous example. Be careful with units.

Solution

8 ft-lb

Work Done in Pumping

Consider the work done to pump water (or some other liquid) out of a tank. Pumping problems are a little more complicated than spring problems because many of the calculations depend on the shape and size of the tank. In addition, instead of being concerned about the work done to move a single mass, we are looking at the work done to move a volume of water, and it takes more work to move the water from the bottom of the tank than it does to move the water from the top of the tank.

We examine the process in the context of a cylindrical tank, then look at a couple of examples using tanks of different shapes. Assume a cylindrical tank of radius 4 m and height 10 m is filled to a depth of 8 m. How much work does it take to pump all the water over the top edge of the tank?

The first thing we need to do is define a frame of reference. We let x represent the vertical distance below the top of the tank. That is, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (Figure $3.9.5$).

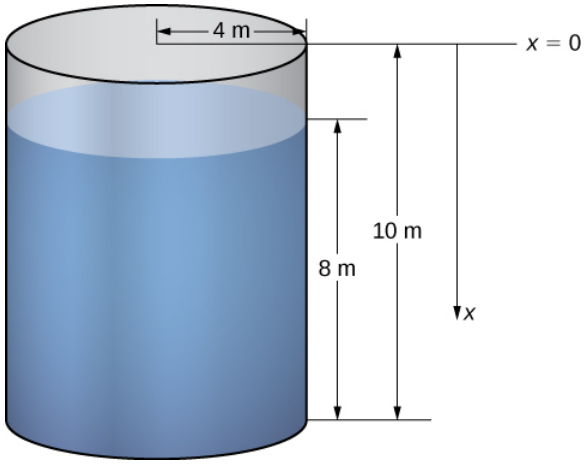


Figure $3.9.5$: How much work is needed to empty a tank partially filled with water?

Using this coordinate system, the water extends from $x = 2$ to $x = 10$. Therefore, we partition the interval $[2, 10]$ and look at the work required to lift each individual "layer" of water. So, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[2, 10]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. Figure $3.9.6$ shows a representative layer.

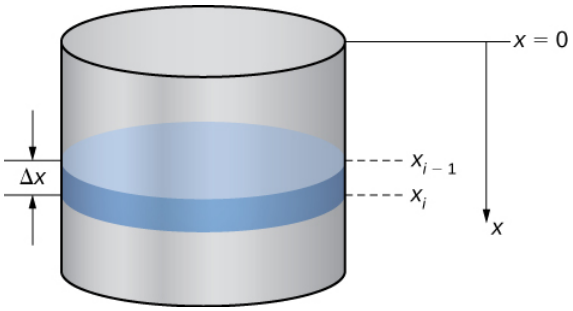


Figure 3.9.6: A representative layer of water.

In pumping problems, the force required to lift the water to the top of the tank is the force required to overcome gravity, so it is equal to the weight of the water. Given that the weight-density of water is $9800\,\text{N/m}^3$, or $62.4\,\text{lb/ft}^3$, calculating the volume of each layer gives us the weight. In this case, we have

$$V = \pi(4)^2 \Delta x = 16\pi\Delta x.$$

Then, the force needed to lift each layer is

$$F = 9800 \cdot 16\pi\Delta x = 156,800\pi\Delta x.$$

Note that this step becomes a little more difficult if we have a noncylindrical tank. We look at a noncylindrical tank in the next example.

We also need to know the distance the water must be lifted. Based on our choice of coordinate systems, we can use x_i^* as an approximation of the distance the layer must be lifted. Then the work to lift the ith layer of water W_i is approximately

$$W_i \approx 156,800\pi x_i^* \Delta x.$$

Adding the work for each layer, we see the approximate work to empty the tank is given by

$$W = \sum_{i=1}^{n} W_i$$

$$\approx \sum_{i=1}^{n} 156,800\pi x_i^* \Delta x.$$

This is a Riemann sum, so taking the limit as $n \to \infty$, we get

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 156,800\pi x_i^* \Delta x$$

$$= 156,800\pi \int_2^{10} x\,dx$$

$$= 156,800\pi \left(\frac{x^2}{2}\right)\Big|_2^{10} = 7,526,400\pi \approx 23,644,883.$$

The work required to empty the tank is approximately 23,650,000 J.

For pumping problems, the calculations vary depending on the shape of the tank or container. The following problem-solving strategy lays out a step-by-step process for solving pumping problems.

Problem-Solving Strategy: Solving Pumping Problems

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.
4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to get an estimate of the work needed to lift the layer of water.

6. Sum the work required to lift all the layers. This expression is an estimate of the work required to pump out the desired amount of water, and it is in the form of a Riemann sum.
7. Take the limit as $n \to \infty$ and evaluate the resulting integral to get the exact work required to pump out the desired amount of water.

We now apply this problem-solving strategy in an example with a noncylindrical tank.

Example 3.9.4: A Pumping Problem with a Noncylindrical Tank

Assume a tank in the shape of an inverted cone, with height 12 ft and base radius 4 ft. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4 ft. How much work is required to pump out that amount of water?

Solution

The tank is depicted in Figure 3.9.7. As we did in the example with the cylindrical tank, we orient the x-axis vertically, with the origin at the top of the tank and the downward direction being positive (step 1).

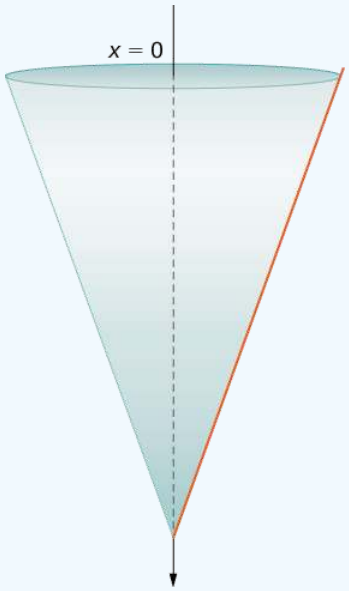


Figure 3.9.7: A water tank in the shape of an inverted cone.

The tank starts out full and ends with 4 ft of water left, so, based on our chosen frame of reference, we need to partition the interval $[0, 8]$. Then, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[0, 8]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. We can approximate the volume of a layer by using a disk, then use similar triangles to find the radius of the disk (Figure 3.9.8).

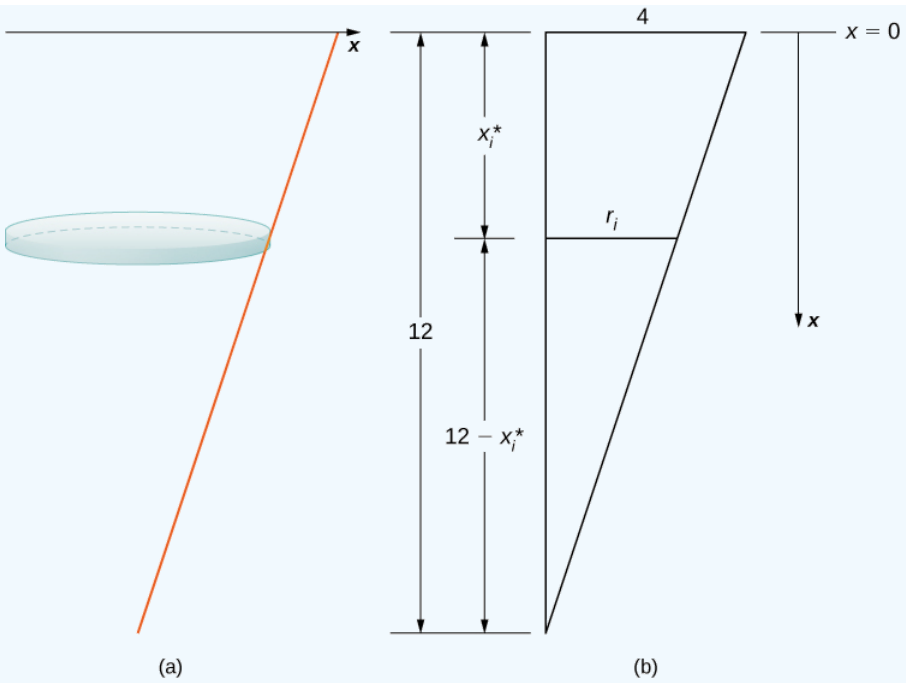


Figure 3.9.8: Using similar triangles to express the radius of a disk of water.

From properties of similar triangles, we have

$$\begin{aligned} \frac{r_i}{12-x_i^*} &= \frac{4}{12} = \frac{1}{3} && \text{(step 1)} \\ 3r_i &= 12 - x_i^* \\ r_i &= \frac{12-x_i^*}{3} \\ &= 4 - \frac{x_i^*}{3}. \end{aligned}$$

Then the volume of the disk is

$$V_i = \pi \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 2)}$$

The weight-density of water is 62.4lb/ft^3, so the force needed to lift each layer is approximately

$$F_i \approx 62.4\pi \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x \qquad \text{(step 3)}$$

Based on the diagram, the distance the water must be lifted is approximately x_i^* feet (step 4), so the approximate work needed to lift the layer is

$$W_i \approx 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 5)}$$

Summing the work required to lift all the layers, we get an approximate value of the total work:

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} 62.4\pi x_i^* \left(4 - \frac{x_i^*}{3}\right)^2 \Delta x. \qquad \text{(step 6)}$$

Taking the limit as $n \to \infty$, we obtain

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} 62.4\pi x_i^*(4 - \frac{x_i^*}{3})^2 \Delta x$$

$$= \int_0^8 62.4\pi x \left(4 - \frac{x}{3}\right)^2 dx$$

$$= 62.4\pi \int_0^8 x\left(16 - \frac{8x}{3} + \frac{x^2}{9}\right) dx = 62.4\pi \int_0^8 \left(16x - \frac{8x^2}{3} + \frac{x^3}{9}\right) dx$$

$$= 62.4\pi \left[8x^2 - \frac{8x^3}{9} + \frac{x^4}{36}\right]\Big|_0^8 = 10,649.6\pi \approx 33,456.7.$$

It takes approximately $33,450$ ft-lb of work to empty the tank to the desired level.

Exercise 3.9.4

A tank is in the shape of an inverted cone, with height 10 ft and base radius 6 ft. The tank is filled to a depth of 8 ft to start with, and water is pumped over the upper edge of the tank until 3 ft of water remain in the tank. How much work is required to pump out that amount of water?

Hint

Use the process from the previous example.

Solution

Approximately $43,255.2$ ft-lb

Hydrostatic Force and Pressure

In this last section, we look at the force and pressure exerted on an object submerged in a liquid. In the English system, force is measured in pounds. In the metric system, it is measured in newtons. Pressure is force per unit area, so in the English system we have pounds per square foot (or, perhaps more commonly, pounds per square inch, denoted psi). In the metric system we have newtons per square meter, also called **pascals.**

Let's begin with the simple case of a plate of area A submerged horizontally in water at a depth s (Figure 3.9.9). Then, the force exerted on the plate is simply the weight of the water above it, which is given by $F = \rho As$, where ρ is the weight density of water (weight per unit volume). To find the **hydrostatic pressure**—that is, the pressure exerted by water on a submerged object—we divide the force by the area. So the pressure is $p = F/A = \rho s$.

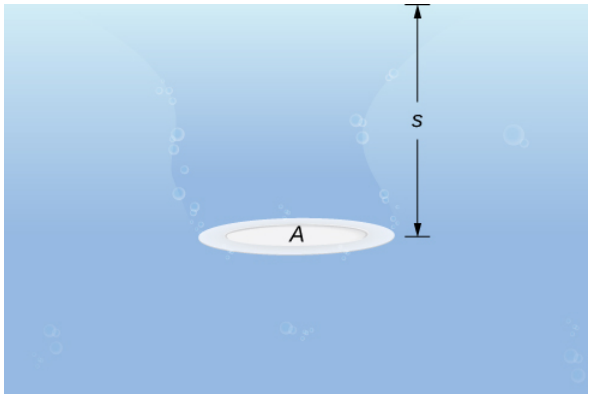


Figure 3.9.9: A plate submerged horizontally in water.

By Pascal's principle, the pressure at a given depth is the same in all directions, so it does not matter if the plate is submerged horizontally or vertically. So, as long as we know the depth, we know the pressure. We can apply Pascal's principle to find the force exerted on surfaces, such as dams, that are oriented vertically. We cannot apply the formula $F = \rho As$ directly, because the depth varies from point to point on a vertically oriented surface. So, as we have done many times before, we form a partition, a Riemann sum, and, ultimately, a definite integral to calculate the force.

Suppose a thin plate is submerged in water. We choose our frame of reference such that the x-axis is oriented vertically, with the downward direction being positive, and point $x = 0$ corresponding to a logical reference point. Let $s(x)$ denote the depth at point x. Note we often let $x = 0$ correspond to the surface of the water. In this case, depth at any point is simply given by $s(x) = x$. However, in some cases we may want to select a different reference point for $x = 0$, so we proceed with the development in the more general case. Last, let $w(x)$ denote the width of the plate at the point x.

Assume the top edge of the plate is at point $x = a$ and the bottom edge of the plate is at point $x = b$. Then, for $i = 0, 1, 2, \ldots, n$, let $P = x_i$ be a regular partition of the interval $[a, b]$, and for $i = 1, 2, \ldots, n$, choose an arbitrary point $x_i^* \in [x_{i-1}, x_i]$. The partition divides the plate into several thin, rectangular strips (Figure 3.9.10).

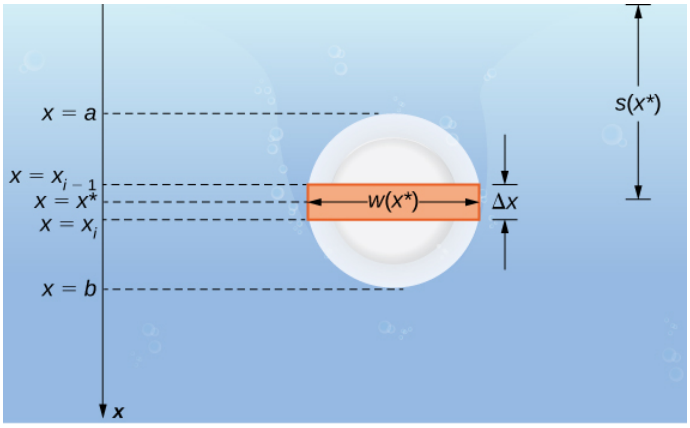


Figure 3.9.10: A thin plate submerged vertically in water.

Let's now estimate the force on a representative strip. If the strip is thin enough, we can treat it as if it is at a constant depth, $s(x_i^*)$. We then have

$$F_i = \rho A s = \rho[w(x_i^*)\Delta x]s(x_i^*).$$

Adding the forces, we get an estimate for the force on the plate:

$$F \approx \sum_{i=1}^{n} F_i = \sum_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*).$$

This is a Riemann sum, so taking the limit gives us the exact force. We obtain

$$F = \lim_{n\to\infty} \sum_{i=1}^{n} \rho[w(x_i^*)\Delta x]s(x_i^*) = \int_a^b \rho w(x)s(x)dx. \tag{3.9.4}$$

Evaluating this integral gives us the force on the plate. We summarize this in the following problem-solving strategy.

Problem-Solving Strategy: Finding Hydrostatic Force

1. Sketch a picture and select an appropriate frame of reference. (Note that if we select a frame of reference other than the one used earlier, we may have to adjust Equation 3.9.4 accordingly.)
2. Determine the depth and width functions, $s(x)$ and $w(x)$.
3. Determine the weight-density of whatever liquid with which you are working. The weight-density of water is $62.4\,\text{lb/ft}^3$, or $9800\,\text{N/m}^3$.
4. Use the equation to calculate the total force.

Example 3.9.5: Finding Hydrostatic Force

A water trough 15 ft long has ends shaped like inverted isosceles triangles, with base 8 ft and height 3 ft. Find the force on one end of the trough if the trough is full of water.

Solution

Figure 3.9.11 shows the trough and a more detailed view of one end.

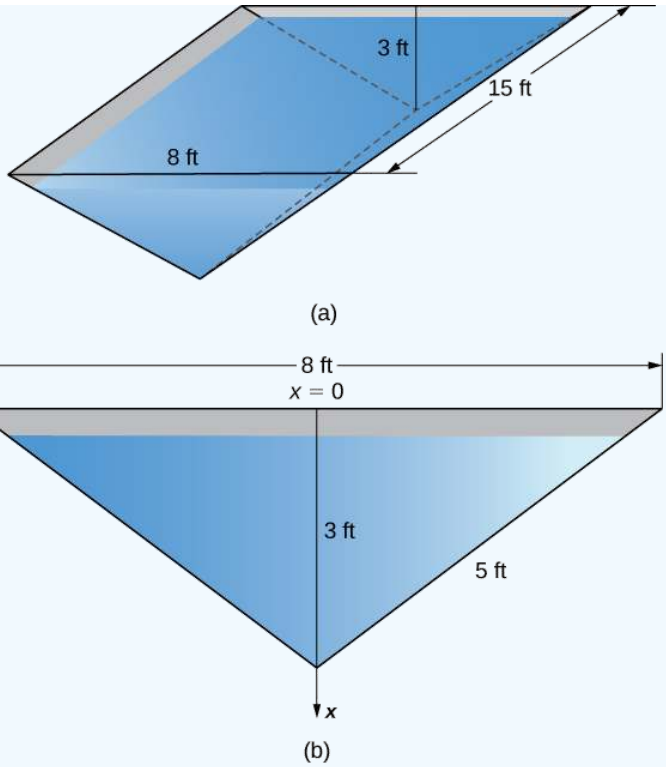


Figure 3.9.11: (a) A water trough with a triangular cross-section. (b) Dimensions of one end of the water trough.

Select a frame of reference with the x-axis oriented vertically and the downward direction being positive. Select the top of the trough as the point corresponding to $x = 0$ (step 1). The depth function, then, is $s(x) = x$. Using similar triangles, we see that $w(x) = 8 - (8/3)x$ (step 2). Now, the weight density of water is $62.4\,\text{lb/ft}^3$ (step 3), so applying Equation 3.9.4, we obtain

$$F = \int_a^b \rho w(x) s(x) dx$$

$$= \int_0^3 62.4 \left(8 - \frac{8}{3}x\right) x\, dx = 62.4 \int_0^3 \left(8x - \frac{8}{3}x^2\right) dx$$

$$= 62.4 \left[4x^2 - \frac{8}{9}x^3\right]\Big|_0^3 = 748.8.$$

The water exerts a force of 748.8 lb on the end of the trough (step 4).

Exercise 3.9.5

A water trough 12 m long has ends shaped like inverted isosceles triangles, with base 6 m and height 4 m. Find the force on one end of the trough if the trough is full of water.

Hint

Follow the problem-solving strategy and the process from the previous example.

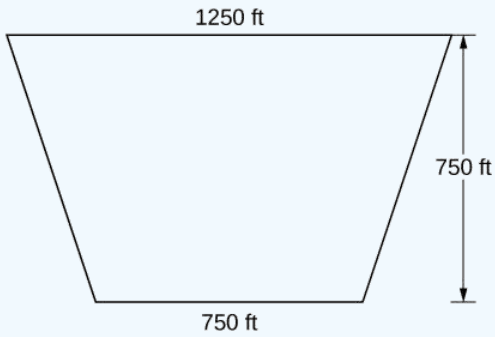
Solution

$156,800\text{N}$

Example 3.9.6: Finding Hydrostatic Force

We now return our attention to the Hoover Dam, mentioned at the beginning of this chapter. The actual dam is arched, rather than flat, but we are going to make some simplifying assumptions to help us with the calculations. Assume the face of the Hoover Dam is shaped like an isosceles trapezoid with lower base 750 ft, upper base 1250 ft, and height 750 ft (see the following figure).





When the reservoir is full, Lake Mead's maximum depth is about 530 ft, and the surface of the lake is about 10 ft below the top of the dam (see the following figure).

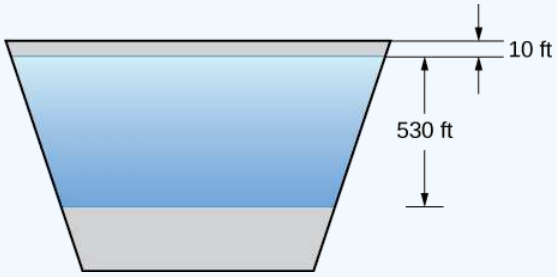


Figure 3.9.12: A simplified model of the Hoover Dam with assumed dimensions.

a. Find the force on the face of the dam when the reservoir is full.
b. The southwest United States has been experiencing a drought, and the surface of Lake Mead is about 125 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

Solution:

a.

We begin by establishing a frame of reference. As usual, we choose to orient the x-axis vertically, with the downward direction being positive. This time, however, we are going to let $x = 0$ represent the top of the dam, rather than the surface of the water. When the reservoir is full, the surface of the water is 10 ft below the top of the dam, so $s(x) = x - 10$ (see the following figure).

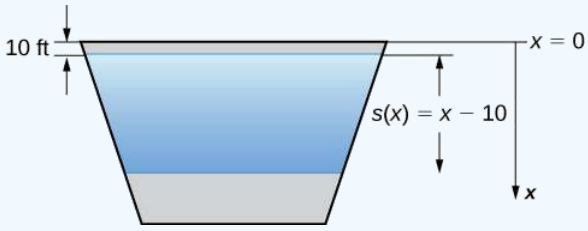


Figure 3.9.13: We first choose a frame of reference.

To find the width function, we again turn to similar triangles as shown in the figure below.

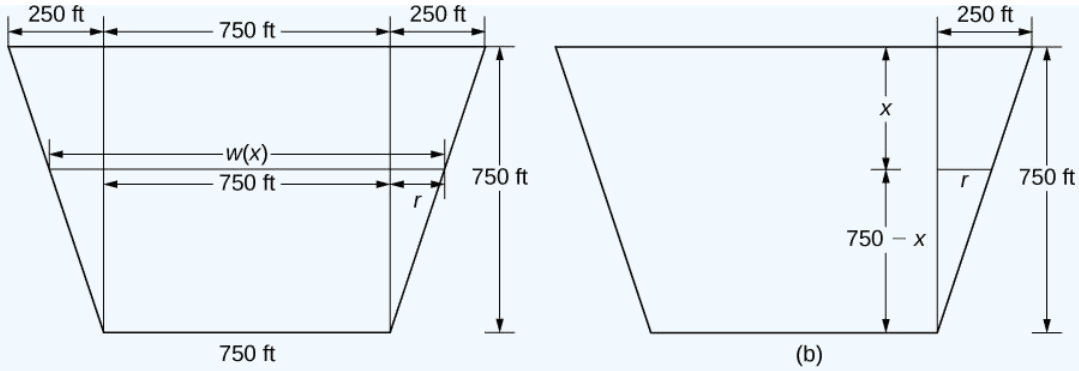


Figure 3.9.14: We use similar triangles to determine a function for the width of the dam. (a) Assumed dimensions of the dam; (b) highlighting the similar triangles.

From the figure, we see that $w(x) = 750 + 2r$. Using properties of similar triangles, we get $r = 250 - (1/3)x$. Thus,

$$w(x) = 1250 - \frac{2}{3}x \qquad \text{(step 2)}$$

Using a weight-density of 62.4lb/ft^3 (step 3) and applying Equation 3.9.4, we get

$$\begin{aligned} F &= \int_a^b \rho w(x) s(x)\, dx \\ &= \int_{10}^{540} 62.4 \left(1250 - \frac{2}{3}x\right)(x - 10)\, dx \\ &= 62.4 \int_{10}^{540} -\frac{2}{3}[x^2 - 1885x + 18750]\, dx \\ &= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - \frac{1885x^2}{2} + 18750x\right]\Big|_{10}^{540} \approx 8,832,245,000\,\text{lb} = 4,416,122.5\,\text{t}. \end{aligned}$$

Note the change from pounds to tons (2000lb = 1 ton) (step 4). This changes our depth function, $s(x)$, and our limits of integration. We have $s(x) = x - 135$. The lower limit of integration is 135. The upper limit remains 540. Evaluating the integral, we get

$$\begin{aligned} F &= \int_a^b \rho w(x) s(x)\, dx \\ &= \int_{135}^{540} 62.4 \left(1250 - \frac{2}{3}x\right)(x - 135)\, dx \\ &= -62.4\left(\frac{2}{3}\right)\int_{135}^{540} (x - 1875)(x - 135)\, dx = -62.4\left(\frac{2}{3}\right)\int_{135}^{540} (x^2 - 2010x + 253125)\, dx \\ &= -62.4\left(\frac{2}{3}\right)\left[\frac{x^3}{3} - 1005x^2 + 253125x\right]\Big|_{135}^{540} \approx 5,015,230,000\,\text{lb} = 2,507,615\,\text{t}. \end{aligned}$$

Exercise 3.9.6

When the reservoir is at its average level, the surface of the water is about 50 ft below where it would be if the reservoir were full. What is the force on the face of the dam under these circumstances?

Hint

Change the depth function, $s(x)$, and the limits of integration.

Solution

Approximately 7,164,520,000 lb or 3,582,260 t

Key Concepts

- Several physical applications of the definite integral are common in engineering and physics.
- Definite integrals can be used to determine the mass of an object if its density function is known.
- Work can also be calculated from integrating a force function, or when counteracting the force of gravity, as in a pumping problem.
- Definite integrals can also be used to calculate the force exerted on an object submerged in a liquid.

Key Equations

- **Mass of a one-dimensional object**

$$m = \int_a^b \rho(x)dx$$

- **Mass of a circular object**

$$m = \int_0^r 2\pi x\rho(x)dx$$

- **Work done on an object**

$$W = \int_a^b F(x)dx$$

- **Hydrostatic force on a plate**

$$F = \int_a^b \rho w(x)s(x)dx$$

Glossary

density function
: a density function describes how mass is distributed throughout an object; it can be a linear density, expressed in terms of mass per unit length; an area density, expressed in terms of mass per unit area; or a volume density, expressed in terms of mass per unit volume; weight-density is also used to describe weight (rather than mass) per unit volume

Hooke's law
: this law states that the force required to compress (or elongate) a spring is proportional to the distance the spring has been compressed (or stretched) from equilibrium; in other words, $F = kx$, where k is a constant

hydrostatic pressure
: the pressure exerted by water on a submerged object

work
: the amount of energy it takes to move an object; in physics, when a force is constant, work is expressed as the product of force and distance

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3.10: Exercises on Work

Basic Work Problems

For exercises 1 - 6, find the work done.

1) Find the work done when a constant force $F = 12$ lb moves a chair from $x = 0.9$ to $x = 1.1$ ft.

2) How much work is done when a person lifts a 50 lb box of comics onto a truck that is 3 ft off the ground?

Answer:

$W = 150$ ft-lb

3) What is the work done lifting a 20 kg child from the floor to a height of 2 m? (Note that 1 kg equates to 9.8 N)

4) Find the work done when you push a box along the floor 2 m, when you apply a constant force of $F = 100$ N.

Answer:

$W = 200$ J

5) Compute the work done for a force $F = \dfrac{12}{x^2}$ N from $x = 1$ to $x = 2$ m.

6) What is the work done moving a particle from $x = 0$ to $x = 1$ m if the force acting on it is $F = 3x^2$ N?

Answer:

$W = 1$ J

Spring Work Problems

7) A 12-in. spring is stretched to 15 in. by a force of 75 lb. What is the spring constant?

8) A spring has a natural length of 10 cm. It takes 2 J to stretch the spring to 15 cm. How much work would it take to stretch the spring from 15 cm to 20 cm?

Answer:

$W = 6$ J

9) A 1-m spring requires 10 J to stretch the spring to 1.1 m. How much work would it take to stretch the spring from 1 m to 1.2 m?

10) A spring requires 5 J to stretch the spring from 8 cm to 12 cm, and an additional 4 J to stretch the spring from 12 cm to 14 cm. What is the natural length of the spring?

Answer:

The natural length is5 cm.

11) A shock absorber is compressed 1 in. by a weight of 1 ton. What is the spring constant?

12) A force of $F = \left(20x - x^3\right)$ N stretches a nonlinear spring by x meters. What work is required to stretch the spring from $x = 0$ to $x = 2$ m?

Answer:

$W = 36$ J

Cable and Chain Work Problems

13) Find the work done by winding up a hanging cable of length 100 ft and weight-density 5 lb/ft.

14) For the cable in the preceding exercise, how much work is done to lift the cable 50 ft?

Answer:

$W = 18,750$ ft-lb

15) For the cable in the preceding exercise, how much additional work is done by hanging a 200 lb weight at the end of the cable?

Pyramid & Satellite/Rocket Work Problems

16) [T] A pyramid of height 500 ft has a square base 800 ft by 800 ft. Find the area A at height h. If the rock used to build the pyramid weighs approximately $w = 100\,\text{lb}/\text{ft}^3$, how much work did it take to lift all the rock?

Answer:

$W = \frac{32}{3} \times 10^9$ ft-lb

17) [T] For the pyramid in the preceding exercise, assume there were 1000 workers each working 10 hours a day, 5 days a week, 50 weeks a year. If each of the workers, on average, lifted ten 100-lb rocks 2 ft/hr, how long did it take to build the pyramid?

18) [T] The force of gravity on a mass m is $F = -((GMm)/x^2)$ newtons. For a rocket of mass $m = 1000$ kg, compute the work to lift the rocket from $x = 6400$ to $x = 6500$ km. (Note: $G = 6 \times 10^{-17}\ \text{N m}^2/\text{kg}^2$ and $M = 6 \times 10^{24}$ kg.)

Answer:

$W = 8.65 \times 10^5$ J

19) [T] For the rocket in the preceding exercise, find the work to lift the rocket from $x = 6400$ to $x = \infty$.

Pumping Work Problems

20) [T] Find the work required to pump all the water out of a cylinder that has a circular base of radius 5ft and height 200 ft. Use the fact that the density of water is $62\ \text{lb}/\text{ft}^3$.

21) [T] Find the work required to pump all the water out of the cylinder in the preceding exercise if the cylinder is only half full.

Answer:

$W = 23.25\pi$ million ft-lb

22) [T] How much work is required to pump out a swimming pool if the area of the base is $800\ \text{ft}^2$, the water is 4 ft deep, and the top is 1 ft above the water level? Assume that the density of water is $62\ \text{lb}/\text{ft}^3$.

23) A cylinder of depth H and cross-sectional area A stands full of water at density ρ. Compute the work to pump all the water to the top.

Answer:

$$W = \frac{A\rho H^2}{2}$$

24) For the cylinder in the preceding exercise, compute the work to pump all the water to the top if the cylinder is only half full.

25) A cone-shaped tank has a cross-sectional area that increases with its depth: $A = \frac{\pi r^2 h^2}{H^3}$. Show that the work to empty it is half the work for a cylinder with the same height and base.

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3.11: Moments and Centers of Mass

Learning Objectives

- Find the center of mass of objects distributed along a line.
- Locate the center of mass of a thin plate.
- Use symmetry to help locate the centroid of a thin plate.
- Apply the theorem of Pappus for volume.

In this section, we consider centers of mass (also called *centroids*, under certain conditions) and moments. The basic idea of the center of mass is the notion of a balancing point. Many of us have seen performers who spin plates on the ends of sticks. The performers try to keep several of them spinning without allowing any of them to drop. If we look at a single plate (without spinning it), there is a sweet spot on the plate where it balances perfectly on the stick. If we put the stick anywhere other than that sweet spot, the plate does not balance and it falls to the ground. (That is why performers spin the plates; the spin helps keep the plates from falling even if the stick is not exactly in the right place.) Mathematically, that sweet spot is called the *center of mass of the plate.*

In this section, we first examine these concepts in a one-dimensional context, then expand our development to consider centers of mass of two-dimensional regions and symmetry. Last, we use centroids to find the volume of certain solids by applying the theorem of Pappus.

Center of Mass and Moments

Let's begin by looking at the center of mass in a one-dimensional context. Consider a long, thin wire or rod of negligible mass resting on a fulcrum, as shown in Figure $3.11.1a$. Now suppose we place objects having masses m_1 and m_2 at distances d_1 and d_2 from the fulcrum, respectively, as shown in Figure $3.11.1b$

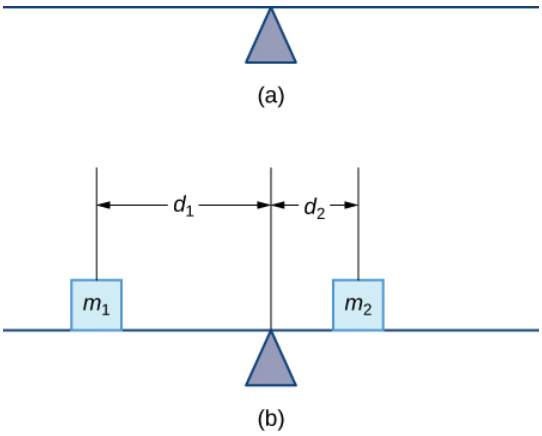


Figure 3.11.1: (a) A thin rod rests on a fulcrum. (b) Masses are placed on the rod.

The most common real-life example of a system like this is a playground seesaw, or teeter-totter, with children of different weights sitting at different distances from the center. On a seesaw, if one child sits at each end, the heavier child sinks down and the lighter child is lifted into the air. If the heavier child slides in toward the center, though, the seesaw balances. Applying this concept to the masses on the rod, we note that the masses balance each other if and only if

$$m_1 d_1 = m_2 d_2.$$

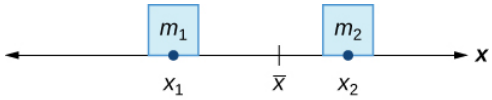


Figure 3.11.2: The center of mass $\bar{x}$ is the balance point of the system.

In the seesaw example, we balanced the system by moving the masses (children) with respect to the fulcrum. However, we are really interested in systems in which the masses are not allowed to move, and instead we balance the system by moving the

fulcrum. Suppose we have two point masses, m_1 and m_2, located on a number line at points x_1 and x_2, respectively (Figure 3.11.2). The center of mass, $\bar{x}$, is the point where the fulcrum should be placed to make the system balance.

Thus, we have

$$\begin{aligned} m_1|x_1-\bar{x}| &= m_2|x_2-\bar{x}| \\ m_1(\bar{x}-x_1) &= m_2(x_2-\bar{x}) \\ m_1\bar{x}-m_1x_1 &= m_2x_2-m_2\bar{x} \\ \bar{x}(m_1+m_2) &= m_1x_1+m_2x_2 \end{aligned}$$

or

$$\bar{x}=\frac{m_1x_1+m_2x_2}{m_1+m_2} \tag{3.11.1}$$

The expression in the numerator of Equation 3.11.1, $m_1x_1+m_2x_2$, is called the *first moment of the system* with respect to the origin. If the context is clear, we often drop the word first and just refer to this expression as the **moment** of the system. The expression in the denominator, m_1+m_2 , is the total mass of the system. Thus, the **center of mass** of the system is the point at which the total mass of the system could be concentrated without changing the moment.

This idea is not limited just to two point masses. In general, if n masses, $m_1, m_2, \ldots, m_n$, are placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, then the center of mass of the system is given by

$$\bar{x}=\frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i}$$

Center of Mass of Objects on a Line

Let $m_1, m_2, \ldots, m_n$ be point masses placed on a number line at points $x_1, x_2, \ldots, x_n$, respectively, and let $m=\sum_{i=1}^{n} m_i$ denote the total mass of the system. Then, the *moment of the system* with respect to the origin is given by

$$M=\sum_{i=1}^{n} m_ix_i \tag{3.11.2}$$

and the *center of mass of the system* is given by

$$\bar{x}=\frac{M}{m}. \tag{3.11.3}$$

We apply this theorem in the following example.

Example 3.11.1: Finding the Center of Mass of Objects along a Line

Suppose four point masses are placed on a number line as follows:

- $m_1=30\,kg,$ placed at $x_1=-2m$
- $m_2=5\,kg,$ placed at $x_2=3m$
- $m_3=10\,kg,$ placed at $x_3=6m$
- $m_4=15\,kg,$ placed at $x_4=-3m.$

Solution

Find the moment of the system with respect to the origin and find the center of mass of the system.

First, we need to calculate the moment of the system (Equation 3.11.2):

$$\begin{aligned} M &= \sum_{i=1}^{4} m_i x_i \\ &= -60 + 15 + 60 - 45 \\ &= -30. \end{aligned}$$

Now, to find the center of mass, we need the total mass of the system:

$$\begin{aligned} m &= \sum_{i=1}^{4} m_i \\ &= 30 + 5 + 10 + 15 \\ &= 60\ kg \end{aligned}$$

Then we have (from Equation 3.11.3)

$$\bar{x} - = \frac{M}{m} = -\frac{30}{60} = -\frac{1}{2}\ .$$

The center of mass is located 1/2 m to the left of the origin.

? Exercise 3.11.1

Suppose four point masses are placed on a number line as follows:

- $m_1 = 12\ kg$ placed at $x_1 = -4m$
- $m_2 = 12\ kg$ placed at $x_2 = 4m$
- $m_3 = 30\ kg$ placed at $x_3 = 2m$
- $m_4 = 6\ kg,$ placed at $x_4 = -6m.$

Find the moment of the system with respect to the origin and find the center of mass of the system.

Hint

Use the process from the previous example.

Answer

$M = 24, \bar{x} = \frac{2}{5}m$

We can generalize this concept to find the center of mass of a system of point masses in a plane. Let m_1 be a point mass located at point (x_1, y_1) in the plane. Then the moment M_x of the mass with respect to the x-axis is given by $M_x = m_1 y_1$. Similarly, the moment M_y with respect to the y-axis is given by

$$M_y = m_1 x_1.$$

Notice that the x-coordinate of the point is used to calculate the moment with respect to the y-axis, and vice versa. The reason is that the x-coordinate gives the distance from the point mass to the y-axis, and the y-coordinate gives the distance to the x-axis (see the following figure).

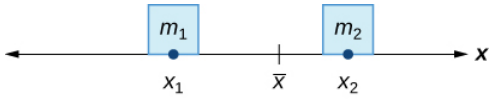


Figure 3.11.3: Point mass m_1 is located at point (x_1, y_1) in the plane.

If we have several point masses in the xy-plane, we can use the moments with respect to the x- and y-axes to calculate the x- and y-coordinates of the center of mass of the system.

Center of Mass of Objects in a Plane

Let m_1, m_2, ..., m_n be point masses located in the xy-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, respectively, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then the moments M_x and M_y of the system with respect to the x- and y-axes, respectively, are given by

$$M_x = \sum_{i=1}^{n} m_i y_i \tag{3.11.4}$$

and

$$M_y = \sum_{i=1}^{n} m_i x_i. \tag{3.11.5}$$

Also, the coordinates of the center of mass $(\bar{x}, \bar{y})$ of the system are

$$\bar{x} = \frac{M_y}{m} \tag{3.11.6}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{3.11.7}$$

The next example demonstrates how to the center of mass formulas (Equations 3.11.4- 3.11.7) may be applied.

Example 3.11.2: Finding the Center of Mass of Objects in a Plane

Suppose three point masses are placed in the xy-plane as follows (assume coordinates are given in meters):

- $m_1 = 2\,kg$ placed at $(-1, 3)$,
- $m_2 = 6\,kg$ placed at $(1, 1)$,
- $m_3 = 4\,kg$ placed at $(2, -2)$.

Find the center of mass of the system.

Solution

First we calculate the total mass of the system:

$$m = \sum_{i=1}^{3} m_i = 2 + 6 + 4 = 12\,kg.$$

Next we find the moments with respect to the x- and y-axes:

$$M_y = \sum_{i=1}^{3} m_i x_i = -2 + 6 + 8 = 12,$$

$$M_x = \sum_{i=1}^{3} m_i y_i = 6 + 6 - 8 = 4.$$

Then we have

$$\bar{x} = \frac{M_y}{m} = \frac{12}{12} = 1$$

and

$$\bar{y} = \frac{M_x}{m} = \frac{4}{12} = \frac{1}{3}.$$

The center of mass of the system is $(1, 1/3)$, in meters.

Exercise 3.11.2

Suppose three point masses are placed on a number line as follows (assume coordinates are given in meters):

- $m_1 = 5\,kg$, placed at $(-2, -3)$,
- $m_2 = 3\,kg$, placed at $(2, 3)$,
- $m_3 = 2\,kg$, placed at $(-3, -2)$.

Find the center of mass of the system.

Hint

Use the process from the previous example.

Answer

$(-1, -1)$ m

Center of Mass of Thin Plates

So far we have looked at systems of point masses on a line and in a plane. Now, instead of having the mass of a system concentrated at discrete points, we want to look at systems in which the mass of the system is distributed continuously across a thin sheet of material. For our purposes, we assume the sheet is thin enough that it can be treated as if it is two-dimensional. Such a sheet is called a **lamina**. Next we develop techniques to find the center of mass of a lamina. In this section, we also assume the density of the lamina is constant.

Laminas are often represented by a two-dimensional region in a plane. The geometric center of such a region is called its **centroid**. Since we have assumed the density of the lamina is constant, the center of mass of the lamina depends only on the shape of the corresponding region in the plane; it does not depend on the density. In this case, the center of mass of the lamina corresponds to the centroid of the delineated region in the plane. As with systems of point masses, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes.

We first consider a lamina in the shape of a rectangle. Recall that the center of mass of a lamina is the point where the lamina balances. For a rectangle, that point is both the horizontal and vertical center of the rectangle. Based on this understanding, it is clear that the center of mass of a rectangular lamina is the point where the diagonals intersect, which is a result of the **symmetry principle**, and it is stated here without proof.

The Symmetry Principle

If a region R is symmetric about a line l, then the centroid of R lies on l.

Let's turn to more general laminas. Suppose we have a lamina bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively, as shown in the following figure.

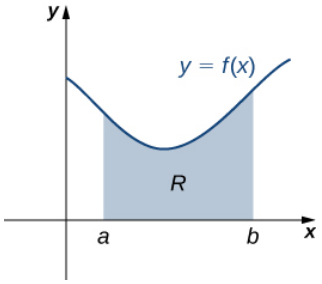


Figure 3.11.4: A region in the plane representing a lamina.

As with systems of point masses, to find the center of mass of the lamina, we need to find the total mass of the lamina, as well as the moments of the lamina with respect to the x- and y-axes. As we have done many times before, we approximate these quantities

by partitioning the interval $[a, b]$ and constructing rectangles.

For $i = 0, 1, 2, \ldots, n$,let $P = x_i$ be a regular partition of $[a, b]$. Recall that we can choose any point within the interval $[x_{i-1}, x_i]$ as our x_i^*. In this case, we want x_i^* to be the x-coordinate of the centroid of our rectangles. Thus, for $i = 1, 2, \ldots, n$, we select $x_i^* \in [x_{i-1}, x_i]$ such that x_i^* is the midpoint of the interval. That is, $x_i^* = (x_{i-1} + x_i)/2$. Now, for $i = 1, 2, \ldots, n$, construct a rectangle of height $f(x_i^*)$ on $[x_{i-1}, x_i]$. The center of mass of this rectangle is $(x_i^*, (f(x_i^*))/2)$, as shown in the following figure.

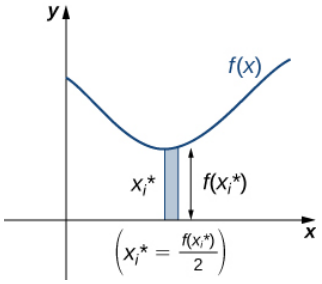


Figure 3.11.5: A representative rectangle of the lamina.

Next, we need to find the total mass of the rectangle. Let ρ represent the density of the lamina (note that ρ is a constant). In this case, ρ is expressed in terms of mass per unit area. Thus, to find the total mass of the rectangle, we multiply the area of the rectangle by ρ. Then, the mass of the rectangle is given by $\rho f(x_i^*)\Delta x$.

To get the approximate mass of the lamina, we add the masses of all the rectangles to get

$$m \approx \sum_{i=1}^{n} \rho f(x_i^*)\Delta x. \tag{3.11.8}$$

Equation 3.11.8is a Riemann sum. Taking the limit as $n \to \infty$ gives the exact mass of the lamina:

$$\begin{aligned} m &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho f(x_i^*)\Delta x \\ &= \rho \int_a^b f(x)dx. \end{aligned}$$

Next, we calculate the moment of the lamina with respect to the x-axis. Returning to the representative rectangle, recall its center of mass is $(x_i^*, (f(x_i^*))/2)$. Recall also that treating the rectangle as if it is a point mass located at the center of mass does not change the moment. Thus, the moment of the rectangle with respect to the x-axis is given by the mass of the rectangle, $\rho f(x_i^*)\Delta x$, multiplied by the distance from the center of mass to the x-axis: $(f(x_i^*))/2$. Therefore, the moment with respect to the x-axis of the rectangle is $\rho([f(x_i^*)]^2/2)\Delta x$. Adding the moments of the rectangles and taking the limit of the resulting Riemann sum, we see that the moment of the lamina with respect to the x-axis is

$$\begin{aligned} M_x &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho \frac{[f(x_i^*)]^2}{2}\Delta x \\ &= \rho \int_a^b \frac{[f(x)]^2}{2}dx. \end{aligned}$$

We derive the moment with respect to the y-axis similarly, noting that the distance from the center of mass of the rectangle to the y-axis is x_i^*. Then the moment of the lamina with respect to the y-axis is given by

$$\begin{aligned} M_y &= \lim_{n\to\infty} \sum_{i=1}^{n} \rho x_i^* f(x^*)i)\Delta x \\ &= \rho \int_a^b x f(x)dx. \end{aligned}$$

We find the coordinates of the center of mass by dividing the moments by the total mass to give $\bar{x} = M_y/m$ and $\bar{y} = M_x/m$. If we look closely at the expressions for M_x, M_y, and m, we notice that the constant ρ cancels out when $\bar{x}$ and $\bar{y}$ are calculated.

We summarize these findings in the following theorem.

Center of Mass of a Thin Plate in the xy-Plane

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b f(x)dx. \tag{3.11.9}$$

ii. The moments M_x and M_y of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2} dx \tag{3.11.10}$$

and

$$M_y = \rho \int_a^b x f(x)dx. \tag{3.11.11}$$

iii. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m} \tag{3.11.12}$$

and

$$\bar{y} = \frac{M_x}{m}. \tag{3.11.13}$$

In the next example, we use this theorem to find the center of mass of a lamina.

Example 3.11.3: Finding the Center of Mass of a Lamina

Let R be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the x-axis over the interval $[0, 4]$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

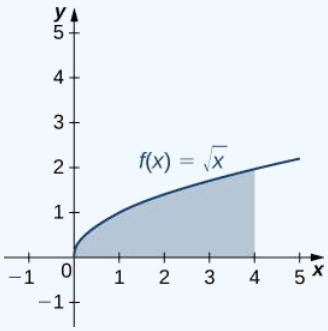


Figure 3.11.6: Finding the center of mass of a lamina.

Since we are only asked for the centroid of the region, rather than the mass or moments of the associated lamina, we know the density constant ρ cancels out of the calculations eventually. Therefore, for the sake of convenience, let's assume $\rho = 1$.

First, we need to calculate the total mass (Equation 3.11.9):

$$
\begin{aligned}
m &= \rho \int_a^b f(x)dx \\
&= \int_0^4 \sqrt{x}dx \\
&= \frac{2}{3}x^{3/2}\Big|_0^4 \\
&= \frac{2}{3}[8-0] \\
&= \frac{16}{3}.
\end{aligned}
$$

Next, we compute the moments (Equation 3.11.12):

$$
\begin{aligned}
M_x &= \rho \int_a^b \frac{[f(x)]^2}{2}dx \\
&= \int_0^4 \frac{x}{2}dx \\
&= \frac{1}{4}x^2\Big|_0^4 \\
&= 4
\end{aligned}
$$

and (Equation 3.11.11):

$$
\begin{aligned}
M_y &= \rho \int_a^b xf(x)dx \\
&= \int_0^4 x\sqrt{x}dx \\
&= \int_0^4 x^{3/2}dx \\
&= \frac{2}{5}x^{5/2}\Big|_0^4 \\
&= \frac{2}{5}[32-0] \\
&= \frac{64}{5}.
\end{aligned}
$$

Thus, we have (Equation 3.11.12):

$$
\begin{aligned}
\bar{x} &= \frac{M_y}{m} \\
&= \frac{64/5}{16/3} \\
&= \frac{64}{5}\cdot\frac{3}{16} \\
&= \frac{12}{5}
\end{aligned}
$$

and (Equation 3.11.13):

$$\begin{aligned}\bar{y} &= \frac{M_x}{y} \\ &= \frac{4}{16/3} \\ &= 4 \cdot \frac{3}{16} \\ &= \frac{3}{4}.\end{aligned}$$

The centroid of the region is $(12/5, 3/4)$.

Exercise 3.11.3

Let R be the region bounded above by the graph of the function $f(x) = x^2$ and below by the x-axis over the interval $[0, 2]$. Find the centroid of the region.

Hint

Use the process from the previous example.

Answer

The centroid of the region is $(3/2, 6/5)$.

We can adapt this approach to find centroids of more complex regions as well. Suppose our region is bounded above by the graph of a continuous function $f(x)$, as before, but now, instead of having the lower bound for the region be the x-axis, suppose the region is bounded below by the graph of a second continuous function, $g(x)$, as shown in Figure 3.11.7.

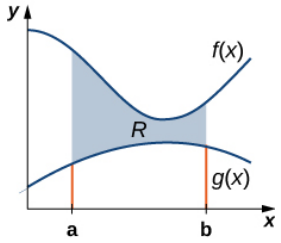


Figure 3.11.7: A region between two functions.

Again, we partition the interval $[a, b]$ and construct rectangles. A representative rectangle is shown in Figure 3.11.8

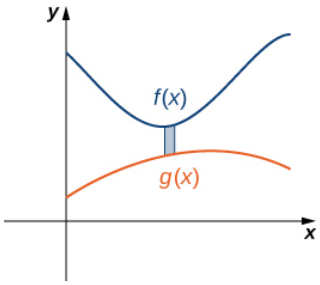


Figure 3.11.8: A representative rectangle of the region between two functions.

Note that the centroid of this rectangle is $(x_i^*, (f(x_i^*) + g(x_i^*))/2)$. We won't go through all the details of the Riemann sum development, but let's look at some of the key steps. In the development of the formulas for the mass of the lamina and the moment with respect to the y-axis, the height of each rectangle is given by $f(x_i^*) - g(x_i^*)$, which leads to the expression $f(x) - g(x)$ in the integrands.

In the development of the formula for the moment with respect to the x-axis, the moment of each rectangle is found by multiplying the area of the rectangle, $\rho[f(x_i^*) - g(x_i^*)]\Delta x$, by the distance of the centroid from the x-axis, $(f(x_i^*) + g(x_i^*))/2$, which gives $\rho(1/2)[f(x_i^*)]^2 - [g(x_i^*)]^2 \Delta x$. Summarizing these findings, we arrive at the following theorem.

Center of Mass of a Lamina Bounded by Two Functions

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the graph of the continuous function $g(x)$, and on the left and right by the lines $x = a$ and $x = b$, respectively. Let ρ denote the density of the associated lamina. Then we can make the following statements:

i. The mass of the lamina is

$$m = \rho \int_a^b [f(x) - g(x)]dx.$$

ii. The moments M_x and M_y of the lamina with respect to the x- and y-axes, respectively, are

$$M_x = \rho \int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)dx$$

and

$$M_y = \rho \int_a^b x[f(x) - g(x)]dx.$$

iii. The coordinates of the center of mass $\bar{x}, \bar{y})$ are

$$\bar{x} = \frac{M_y}{m}$$

and

$$\bar{y} = \frac{M_x}{m}$$

We illustrate this theorem in the following example.

Example 3.11.4: Finding the Centroid of a Region Bounded by Two Functions

Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by the graph of the function $g(x) = x - 1$. Find the centroid of the region.

Solution

The region is depicted in the following figure.

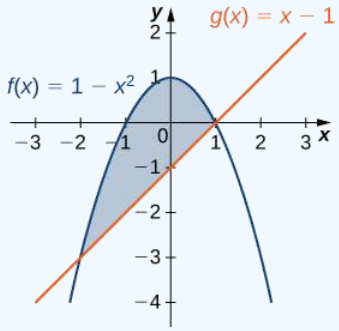


Figure 3.11.9: Finding the centroid of a region between two curves.

The graphs of the functions intersect at $(-2, -3)$ and $(1, 0)$, so we integrate from -2 to 1. Once again, for the sake of convenience, assume $\rho = 1$.

First, we need to calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b [f(x) - g(x)]dx \\ &= \int_{-2}^1 [1 - x^2 - (x-1)]dx \\ &= \int_{-2}^1 (2 - x^2 - x)dx \\ &= \left[2x - \frac{1}{3}x^3 - \frac{1}{2}x^2\right]\Big|_{-2}^1 \\ &= \left[2 - \frac{1}{3} - \frac{1}{2}\right] - \left[-4 + \frac{8}{3} - 2\right] \\ &= \frac{9}{2}. \end{aligned}$$

Next, we compute the moments:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{1}{2}([f(x)]^2 - [g(x)]^2)dx \\ &= \frac{1}{2}\int_{-2}^1 ((1-x^2)^2 - (x-1)^2)dx \\ &= \frac{1}{2}\int_{-2}^1 (x^4 - 3x^2 + 2x)dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - x^3 + x^2\right]\Big|_{-2}^1 \\ &= -\frac{27}{10} \end{aligned}$$

and

$$\begin{aligned} M_y &= \rho \int_a^b x[f(x) - g(x)]dx \\ &= \int_{-2}^1 x[(1-x^2) - (x-1)]dx \\ &= \int_{-2}^1 x[2 - x^2 - x]dx \\ &= \int_{-2}^1 (2x - x^3 - x^2)dx \\ &= \left[x^2 - \frac{x^4}{4} - \frac{x^3}{3}\right]\Big|_{-2}^1 \\ &= -\frac{9}{4}. \end{aligned}$$

Therefore, we have

$$\begin{aligned}\bar{x} &= \frac{M_y}{m} \\ &= -\frac{9}{4} \cdot \frac{2}{9} \\ &= -\frac{1}{2}\end{aligned}$$

and

$$\begin{aligned}\bar{y} &= \frac{M_x}{y} \\ &= -\frac{27}{10} \cdot \frac{2}{9} \\ &= -\frac{3}{5}.\end{aligned}$$

The centroid of the region is $(-(1/2), -(3/5))$.

Exercise 3.11.4

Let R be the region bounded above by the graph of the function $f(x) = 6 - x^2$ and below by the graph of the function $g(x) = 3 - 2x$. Find the centroid of the region.

Hint

Use the process from the previous example.

Answer

The centroid of the region is $(1, 13/5)$.

The Symmetry Principle

We stated the symmetry principle earlier, when we were looking at the centroid of a rectangle. The symmetry principle can be a great help when finding centroids of regions that are symmetric. Consider the following example.

Example 3.11.5: Finding the Centroid of a Symmetric Region

Let R be the region bounded above by the graph of the function $f(x) = 4 - x^2$ and below by the x-axis. Find the centroid of the region.

Solution

The region is depicted in the following figure

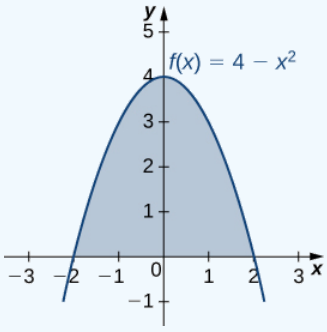


Figure 3.11.10: We can use the symmetry principle to help find the centroid of a symmetric region.

The region is symmetric with respect to the y-axis. Therefore, the x-coordinate of the centroid is zero. We need only calculate $\bar{y}$. Once again, for the sake of convenience, assume $\rho = 1$.

First, we calculate the total mass:

$$\begin{aligned} m &= \rho \int_a^b f(x)dx \\ &= \int_{-2}^{2} (4-x^2)dx \\ &= \left[4x - \frac{x^3}{3}\right]\Big|_{-2}^{2} \\ &= \frac{32}{3}. \end{aligned}$$

Next, we calculate the moments. We only need M_x:

$$\begin{aligned} M_x &= \rho \int_a^b \frac{[f(x)]^2}{2} dx \\ &= \frac{1}{2}\int_{-2}^{2} \left[4-x^2\right]^2 dx = \frac{1}{2}\int_{-2}^{2} (16 - 8x^2 + x^4)dx \\ &= \frac{1}{2}\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right] |_{-2}^{2} = \frac{256}{15} \end{aligned}$$

Then we have

$$\bar{y} = \frac{M_x}{y} = \frac{256}{15} \cdot \frac{3}{32} = \frac{8}{5}.$$

The centroid of the region is $(0, 8/5)$.

Exercise 3.11.5

Let R be the region bounded above by the graph of the function $f(x) = 1 - x^2$ and below by x-axis. Find the centroid of the region.

Hint

Use the process from the previous example.

Answer

The centroid of the region is $(0, 2/5)$.

The Grand Canyon Skywalk

The Grand Canyon Skywalk opened to the public on March 28, 2007. This engineering marvel is a horseshoe-shaped observation platform suspended 4000 ft above the Colorado River on the West Rim of the Grand Canyon. Its crystal-clear glass floor allows stunning views of the canyon below (see the following figure).



Figure 3.11.11: The Grand Canyon Skywalk offers magnificent views of the canyon. (credit: 10da_ralta, Wikimedia Commons)

The Skywalk is a cantilever design, meaning that the observation platform extends over the rim of the canyon, with no visible means of support below it. Despite the lack of visible support posts or struts, cantilever structures are engineered to be very stable and the Skywalk is no exception. The observation platform is attached firmly to support posts that extend 46 ft down into bedrock. The structure was built to withstand 100-mph winds and an 8.0-magnitude earthquake within 50 mi, and is capable of supporting more than 70,000,000 lb.

One factor affecting the stability of the Skywalk is the center of gravity of the structure. We are going to calculate the center of gravity of the Skywalk, and examine how the center of gravity changes when tourists walk out onto the observation platform.

The observation platform is U-shaped. The legs of the U are 10 ft wide and begin on land, under the visitors' center, 48 ft from the edge of the canyon. The platform extends 70 ft over the edge of the canyon.

To calculate the center of mass of the structure, we treat it as a lamina and use a two-dimensional region in the xy-plane to represent the platform. We begin by dividing the region into three subregions so we can consider each subregion separately. The first region, denoted R_1, consists of the curved part of the U. We model R_1 as a semicircular annulus, with inner radius 25 ft and outer radius 35 ft, centered at the origin (Figure 3.11.12).

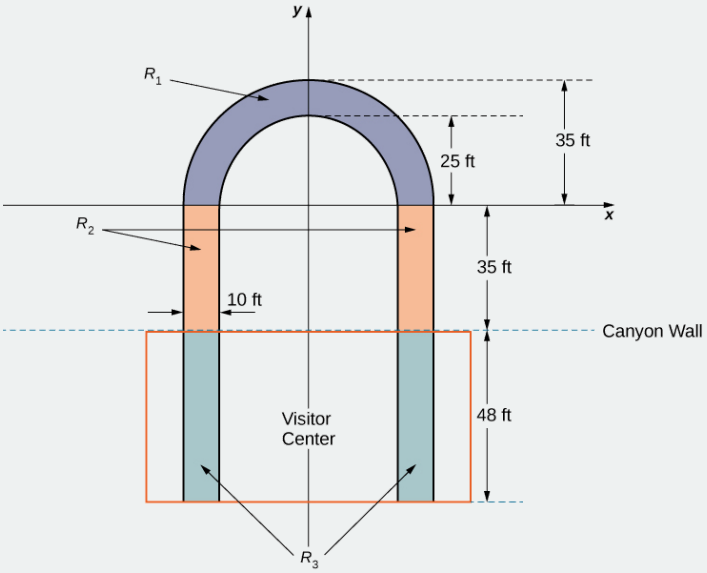


Figure 3.11.12: We model the Skywalk with three sub-regions.

The legs of the platform, extending 35 ft between R_1 and the canyon wall, comprise the second sub-region, R_2. Last, the ends of the legs, which extend 48 ft under the visitor center, comprise the third sub-region, R_3. Assume the density of the lamina is constant and assume the total weight of the platform is 1,200,000 lb (not including the weight of the visitor center; we will consider that later). Use $g = 32\ ft/sec^2$.

1. Compute the area of each of the three sub-regions. Note that the areas of regions R_2 and R_3 should include the areas of the legs only, not the open space between them. Round answers to the nearest square foot.
2. Determine the mass associated with each of the three sub-regions.
3. Calculate the center of mass of each of the three sub-regions.
4. Now, treat each of the three sub-regions as a point mass located at the center of mass of the corresponding sub-region. Using this representation, calculate the center of mass of the entire platform.
5. Assume the visitor center weighs 2,200,000 lb, with a center of mass corresponding to the center of mass of R_3.Treating the visitor center as a point mass, recalculate the center of mass of the system. How does the center of mass change?
6. Although the Skywalk was built to limit the number of people on the observation platform to 120, the platform is capable of supporting up to 800 people weighing 200 lb each. If all 800 people were allowed on the platform, and all of them went to the farthest end of the platform, how would the center of gravity of the system be affected? (Include the visitor center in the calculations and represent the people by a point mass located at the farthest edge of the platform, 70 ft from the canyon wall.)

Theorem of Pappus

This section ends with a discussion of the *theorem of Pappus for volume*, which allows us to find the volume of particular kinds of solids by using the centroid. (There is also a theorem of Pappus for surface area, but it is much less useful than the theorem for volume.)

Theorem of Pappus for Volume

Let R be a region in the plane and let l be a line in the plane that does not intersect R. Then the volume of the solid of revolution formed by revolving R around l is equal to the area of R multiplied by the distance d traveled by the centroid of R.

Proof

We can prove the case when the region is bounded above by the graph of a function $f(x)$ and below by the graph of a function $g(x)$ over an interval $[a,b]$, and for which the axis of revolution is the y-axis. In this case, the area of the region is $A=\int_a^b [f(x)-g(x)]\,dx$. Since the axis of rotation is the y-axis, the distance traveled by the centroid of the region depends only on the x-coordinate of the centroid, $\bar{x}$, which is

$$x=\frac{M_y}{m},$$

where

$$m=\rho\int_a^b [f(x)-g(x)]dx$$

and

$$M_y=\rho\int_a^b x[f(x)-g(x)]dx.$$

Then,

$$d=2\pi\frac{\rho\int_a^b x[f(x)-g(x)]dx}{\rho\int_a^b [f(x)-g(x)]dx}$$

and thus

$$d\cdot A=2\pi\int_a^b x[f(x)-g(x)]dx.$$

However, using the method of cylindrical shells, we have

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx.$$

So,

$$V = d \cdot A$$

and the proof is complete.

□

Example 3.11.6: Using the Theorem of Pappus for Volume

Let R be a circle of radius 2 centered at $(4, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

Solution

The region and torus are depicted in the following figure.

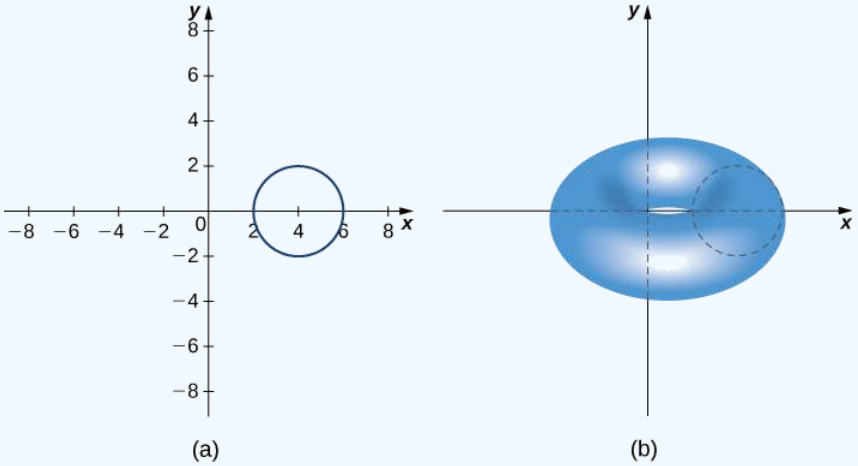


Figure 3.11.13: Determining the volume of a torus by using the theorem of Pappus. (a) A circular region R in the plane; (b) the torus generated by revolving R about the y-axis.

The region R is a circle of radius 2, so the area of R is $A = 4\pi$ units^2. By the symmetry principle, the centroid of R is the center of the circle. The centroid travels around the y-axis in a circular path of radius 4, so the centroid travels $d = 8\pi$ units. Then, the volume of the torus is $A \cdot d = 32\pi^2$ units^3.

Exercise 3.11.6

Let R be a circle of radius 1 centered at $(3, 0)$. Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

Hint

Use the process from the previous example.

Answer

$6\pi^2$ units^3

Key Concepts

- Mathematically, the center of mass of a system is the point at which the total mass of the system could be concentrated without changing the moment. Loosely speaking, the center of mass can be thought of as the balancing point of the system.

- For point masses distributed along a number line, the moment of the system with respect to the origin is $M = \sum_{i=1}^{n} m_i x_i$. For point masses distributed in a plane, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \sum_{i=1}^{n} m_i y_i$ and $M_y = \sum_{i=}^{n} m_i x_i$, respectively.
- For a lamina bounded above by a function $f(x)$, the moments of the system with respect to the x- and y-axes, respectively, are $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\, dx$ and $M_y = \rho \int_a^b x f(x)\, dx$.
- The x- and y-coordinates of the center of mass can be found by dividing the moments around the y-axis and around the x-axis, respectively, by the total mass. The symmetry principle says that if a region is symmetric with respect to a line, then the centroid of the region lies on the line.
- The theorem of Pappus for volume says that if a region is revolved around an external axis, the volume of the resulting solid is equal to the area of the region multiplied by the distance traveled by the centroid of the region.

Key Equations

- **Mass of a lamina**

$$m = \rho \int_a^b f(x) dx$$

- **Moments of a lamina**

$$M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\, dx \text{ and } M_y = \rho \int_a^b x f(x)\, dx$$

- **Center of mass of a lamina**

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m}$$

Glossary

center of mass

the point at which the total mass of the system could be concentrated without changing the moment

centroid

the centroid of a region is the geometric center of the region; laminas are often represented by regions in the plane; if the lamina has a constant density, the center of mass of the lamina depends only on the shape of the corresponding planar region; in this case, the center of mass of the lamina corresponds to the centroid of the representative region

lamina

a thin sheet of material; laminas are thin enough that, for mathematical purposes, they can be treated as if they are two-dimensional

moment

if n masses are arranged on a number line, the moment of the system with respect to the origin is given by $M = \sum_{i=1}^{n} m_i x_i$; if, instead, we consider a region in the plane, bounded above by a function $f(x)$ over an interval $[a, b]$, then the moments of the region with respect to the x- and y-axes are given by $M_x = \rho \int_a^b \frac{[f(x)]^2}{2}\, dx$ and $M_y = \rho \int_a^b x f(x)\, dx$, respectively

symmetry principle

the symmetry principle states that if a region R is symmetric about a line I, then the centroid of R lies on I

theorem of Pappus for volume

this theorem states that the volume of a solid of revolution formed by revolving a region around an external axis is equal to the area of the region multiplied by the distance traveled by the centroid of the region

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3.11E: Exercises for Section 6.6

In exercises 1 - 6, calculate the center of mass for the collection of masses given.

1) $m_1 = 2$ at $x_1 = 1$ and $m_2 = 4$ at $x_2 = 2$

2) $m_1 = 1$ at $x_1 = -1$ and $m_2 = 3$ at $x_2 = 2$

Answer

$x = \frac{5}{4}$

3) $m = 3$ at $x = 0, 1, 2, 6$

4) Unit masses at $(x, y) = (1, 0), (0, 1), (1, 1)$

Answer

$\left(\frac{2}{3}, \frac{2}{3}\right)$

5) $m_1 = 1$ at $(1, 0)$ and $m_2 = 4$ at $(0, 1)$

6) $m_1 = 1$ at $(1, 0)$ and $m_2 = 3$ at $(2, 2)$

Answer

$\left(\frac{7}{4}, \frac{3}{2}\right)$

In exercises 7 - 16, compute the center of mass $\bar{x}$.

7) $\rho = 1$ for $x \in (-1, 3)$

8) $\rho = x^2$ for $x \in (0, L)$

Answer

$\frac{3L}{4}$

9) $\rho = 1$ for $x \in (0, 1)$ and $\rho = 2$ for $x \in (1, 2)$

10) $\rho = \sin x$ for $x \in (0, \pi)$

Answer

$\frac{\pi}{2}$

11) $\rho = \cos x$ for $x \in \left(0, \frac{\pi}{2}\right)$

12) $\rho = e^x$ for $x \in (0, 2)$

Answer

$\frac{e^2 + 1}{e^2 - 1}$

13) $\rho = x^3 + xe^{-x}$ for $x \in (0, 1)$

14) $\rho = x \sin x$ for $x \in (0, \pi)$

Answer

$\frac{\pi^2 - 4}{\pi}$

15) $\rho = \sqrt{x}$ for $x \in (1, 4)$

16) $\rho = \ln x$ for $x \in (1, e)$

Answer

$\frac{1}{4}(1+e^2)$

In exercises 17 - 19, compute the center of mass $(\bar{x},\bar{y})$. Use symmetry to help locate the center of mass whenever possible.

17) $\rho=7$ in the square $0\le x\le 1,\ 0\le y\le 1$

18) $\rho=3$ in the triangle with vertices $(0,0),\ (a,0)$, and $(0,b)$

Answer

$\left(\frac{a}{3},\ \frac{b}{3}\right)$

19) $\rho=2$ for the region bounded by $y=\cos(x),\ y=-\cos(x),\ x=-\frac{\pi}{2}$, and $x=\frac{\pi}{2}$

In exercises 20 - 26, use a calculator to draw the region, then compute the center of mass $(\bar{x},\bar{y})$. Use symmetry to help locate the center of mass whenever possible.

20) [T] The region bounded by $y=\cos(2x),\ x=-\frac{\pi}{4}$, and $x=\frac{\pi}{4}$

Answer

$\left(0,\frac{\pi}{8}\right)$

21) [T] The region between $y=2x^2,\ y=0,\ x=0$, and $x=1$

22) [T] The region between $y=\frac{5}{4}x^2$ and $y=5$

Answer

$(0,3)$

23) [T] Region between $y=\sqrt{x},\ y=\ln x,\ x=1$, and $x=4$

24) [T] The region bounded by $y=0$ and $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$

Answer

$\left(0,\frac{4}{\pi}\right)$

25) [T] The region bounded by $y=0,\ x=0$, and $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$

26) [T] The region bounded by $y=x^2$ and $y=x^4$ in the first quadrant

Answer

$\left(\frac{5}{8},\ \frac{1}{3}\right)$

In exercises 27 - 31, use the theorem of Pappus to determine the volume of the shape.

27) Rotating $y=mx$ around the x-axis between $x=0$ and $x=1$

28) Rotating $y=mx$ around the y-axis between $x=0$ and $x=1$

Answer

$V=\frac{m\pi}{3}$ units^3

29) A general cone created by rotating a triangle with vertices $(0,0),\ (a,0)$, and $(0,b)$ around the y-axis. Does your answer agree with the volume of a cone?

30) A general cylinder created by rotating a rectangle with vertices $(0,0),\ (a,0),\ (0,b)$, and (a,b) around the y-axis. Does your answer agree with the volume of a cylinder?

Answer

$V = \pi a^2 b$ units³

31) A sphere created by rotating a semicircle with radius a around the y-axis. Does your answer agree with the volume of a sphere?

In exercises 32 - 36, use a calculator to draw the region enclosed by the curve. Find the area M and the centroid $(\bar{x}, \bar{y})$ for the given shapes. Use symmetry to help locate the center of mass whenever possible.

32) [T] Quarter-circle: $y = \sqrt{1 - x^2}$, $y = 0$, and $x = 0$

Answer

$\left(\frac{4}{3\pi}, \frac{4}{3\pi}\right)$

33) [T] Triangle: $y = x$, $y = 2 - x$, and $y = 0$

34) [T] Lens: $y = x^2$ and $y = x$

Answer

$\left(\frac{1}{2}, \frac{2}{5}\right)$

35) [T] Ring: $y^2 + x^2 = 1$ and $y^2 + x^2 = 4$

36) [T] Half-ring: $y^2 + x^2 = 1$, $y^2 + x^2 = 4$, and $y = 0$

Answer

$\left(0, \frac{28}{9\pi}\right)$

37) Find the generalized center of mass in the sliver between $y = x^a$ and $y = x^b$ with $a > b$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y-axis.

38) Find the generalized center of mass between $y = a^2 - x^2$, $x = 0$, and $y = 0$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y-axis.

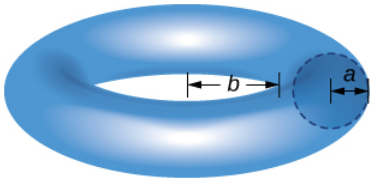
Answer

Center of mass: $\left(\frac{a}{6}, \frac{4a^2}{5}\right)$,

Volume: $\frac{2\pi a^4}{9}$ units³

39) Find the generalized center of mass between $y = b\sin(ax)$, $x = 0$, and $x = \frac{\pi}{a}$. Then, use the Pappus theorem to find the volume of the solid generated when revolving around the y-axis.

40) Use the theorem of Pappus to find the volume of a torus (pictured here). Assume that a disk of radius a is positioned with the left end of the circle at $x = b$, $b > 0$, and is rotated around the y-axis.



Answer

Volume: $V = 2\pi^2 a^2 (b + a)$

41) Find the center of mass $(\bar{x}, \bar{y})$ for a thin wire along the semicircle $y = \sqrt{1 - x^2}$ with unit mass. (Hint: Use the theorem of Pappus.)

Contributors

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3.12: Integrals, Exponential Functions, and Logarithms

Learning Objectives

- Write the definition of the natural logarithm as an integral.
- Recognize the derivative of the natural logarithm.
- Integrate functions involving the natural logarithmic function.
- Define the number e through an integral.
- Recognize the derivative and integral of the exponential function.
- Prove properties of logarithms and exponential functions using integrals.
- Express general logarithmic and exponential functions in terms of natural logarithms and exponentials.

We already examined exponential functions and logarithms in earlier chapters. However, we glossed over some key details in the previous discussions. For example, we did not study how to treat exponential functions with exponents that are irrational. The definition of the number e is another area where the previous development was somewhat incomplete. We now have the tools to deal with these concepts in a more mathematically rigorous way, and we do so in this section.

For purposes of this section, assume we have not yet defined the natural logarithm, the number e, or any of the integration and differentiation formulas associated with these functions. By the end of the section, we will have studied these concepts in a mathematically rigorous way (and we will see they are consistent with the concepts we learned earlier). We begin the section by defining the natural logarithm in terms of an integral. This definition forms the foundation for the section. From this definition, we derive differentiation formulas, define the number e, and expand these concepts to logarithms and exponential functions of any base.

The Natural Logarithm as an Integral

Recall the power rule for integrals:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1.$$

Clearly, this does not work when $n = -1,$ as it would force us to divide by zero. So, what do we do with $\int \frac{1}{x}\,dx$? Recall from the Fundamental Theorem of Calculus that $\int_1^x \frac{1}{t}dt$ is an antiderivative of $\frac{1}{x}$. Therefore, we can make the following definition.

Definition: The Natural Logarithm

For $x > 0$, define the natural logarithm function by

$$\ln x = \int_1^x \frac{1}{t}\,dt.$$

For $x > 1$, this is just the area under the curve $y = \frac{1}{t}$ from 1 to x. For $x < 1$, we have

$$\int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt,$$

so in this case it is the negative of the area under the curve from x to 1 (see the following figure).

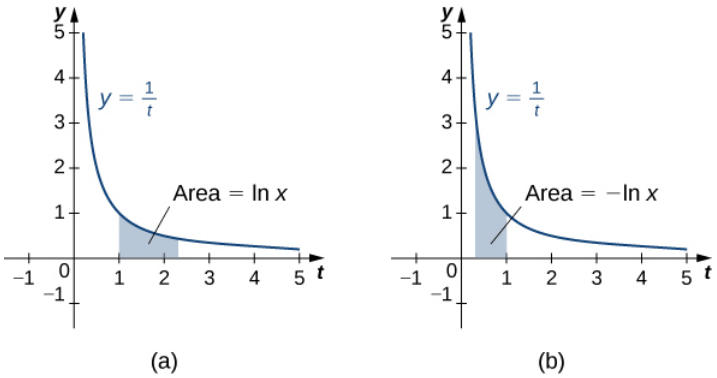


Figure 3.12.1: (a) When $x > 1$, the natural logarithm is the area under the curve $y = 1/t$ from 1 to x. (b) When $x < 1$, the natural logarithm is the negative of the area under the curve from x to 1.

Notice that $\ln 1 = 0$. Furthermore, the function $y = \dfrac{1}{t} > 0$ for $x > 0$. Therefore, by the properties of integrals, it is clear that $\ln x$ is increasing for $x > 0$.

Properties of the Natural Logarithm

Because of the way we defined the natural logarithm, the following differentiation formula falls out immediately as a result of to the Fundamental Theorem of Calculus.

Definition: Derivative of the Natural Logarithm

For $x > 0$, the derivative of the natural logarithm is given by

$$\frac{d}{dx}\Big(\ln x\Big) = \frac{1}{x}.$$

Corollary to the Derivative of the Natural Logarithm

The function $\ln x$ is differentiable; therefore, it is continuous.

A graph of $\ln x$ is shown in Figure. Notice that it is continuous throughout its domain of $(0, \infty)$.

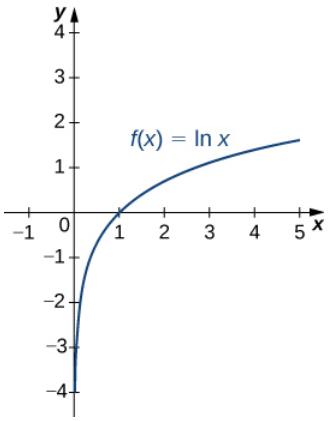


Figure 3.12.2: The graph of $f(x) = \ln x$ shows that it is a continuous function.

Example 3.12.1: Calculating Derivatives of Natural Logarithms

Calculate the following derivatives:

a. $\dfrac{d}{dx}\Big(\ln(5x^3 - 2)\Big)$

b. $\dfrac{d}{dx}\Big((\ln(3x))^2\Big)$

Solution

We need to apply the chain rule in both cases.

a. $\dfrac{d}{dx}\Big(\ln(5x^3-2)\Big) = \dfrac{15x^2}{5x^3-2}$

b. $\dfrac{d}{dx}\Big((\ln(3x))^2\Big) = \dfrac{2(\ln(3x))\cdot 3}{3x} = \dfrac{2(\ln(3x))}{x}$

Exercise 3.12.1

Calculate the following derivatives:

a. $\dfrac{d}{dx}\Big(\ln(2x^2+x)\Big)$

b. $\dfrac{d}{dx}\Big((\ln(x^3))^2\Big)$

Hint

Apply the differentiation formula just provided and use the chain rule as necessary.

Answer

a. $\dfrac{d}{dx}\Big(\ln(2x^2+x)\Big) = \dfrac{4x+1}{2x^2+x}$

b. $\dfrac{d}{dx}\Big((\ln(x^3))^2\Big) = \dfrac{6\ln(x^3)}{x}$

Note that if we use the absolute value function and create a new function $\ln|x|$, we can extend the domain of the natural logarithm to include $x<0$. Then $\dfrac{d}{dx}\Big(\ln x\Big) = \dfrac{1}{x}$. This gives rise to the familiar integration formula.

Integral of $\frac{1}{u}\,du$

The natural logarithm is the antiderivative of the function $f(u) = \dfrac{1}{u}$:

$$\int \frac{1}{u}\,du = \ln|u| + C.$$

Example 3.12.2: Calculating Integrals Involving Natural Logarithms

Calculate the integral $\displaystyle\int \frac{x}{x^2+4}\,dx$.

Solution

Using u-substitution, let $u = x^2+4$. Then $du = 2x\,dx$ and we have

$$\int \frac{x}{x^2+4}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+4| + C = \frac{1}{2}\ln(x^2+4) + C.$$

Exercise 3.12.2

Calculate the integral $\displaystyle\int \frac{x^2}{x^3+6}\,dx$.

Hint

Apply the integration formula provided earlier and use u-substitution as necessary.

Answer

$$\int \frac{x^2}{x^3+6}\,dx = \frac{1}{3}\ln\,|\,x^3+6\,|\,+C$$

Although we have called our function a "logarithm," we have not actually proved that any of the properties of logarithms hold for this function. We do so here.

Properties of the Natural Logarithm

If $a,\ b>0$ and r is a rational number, then

i. $\ln 1 = 0$
ii. $\ln(ab) = \ln a + \ln b$
iii. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
iv. $\ln(a^r) = r\ln a$

Proof

i. By definition, $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0.$

ii. We have

$$\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt.$$

Use $u-substitution$ on the last integral in this expression. Let $u = t/a$. Then $du = (1/a)dt$. Furthermore, when $t=a,\ u=1$, and when $t=ab,\ u=b$. So we get

$$\ln(ab) = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_1^{ab} \frac{a}{t}\cdot\frac{1}{a}\,dt = \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{u}\,du = \ln a + \ln b.$$

iii. Note that

$$\frac{d}{dx}\Big(\ln(x^r)\Big) = \frac{rx^{r-1}}{x^r} = \frac{r}{x}\ .$$

Furthermore,

$$\frac{d}{dx}\Big((r\ln x)\Big) = \frac{r}{x}.$$

Since the derivatives of these two functions are the same, by the Fundamental Theorem of Calculus, they must differ by a constant. So we have

$$\ln(x^r) = r\ln x + C$$

for some constant C. Taking $x=1$, we get

$$\ln(1^r) = r\ln(1) + C$$
$$0 = r(0) + C$$
$$C = 0.$$

Thus $\ln(x^r) = r\ln x$ and the proof is complete. Note that we can extend this property to irrational values of r later in this section.

Part iii. follows from parts ii. and iv. and the proof is left to you.

□

Example 3.12.3: Using Properties of Logarithms

Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 9 - 2\ln 3 + \ln\left(\tfrac{1}{3}\right).$$

Solution

We have

$$\ln 9 - 2\ln 3 + \ln\left(\tfrac{1}{3}\right) = \ln(3^2) - 2\ln 3 + \ln(3^{-1}) = 2\ln 3 - 2\ln 3 - \ln 3 = -\ln 3.$$

Exercise 3.12.3

Use properties of logarithms to simplify the following expression into a single logarithm:

$$\ln 8 - \ln 2 - \ln\left(\tfrac{1}{4}\right)$$

Hint

Apply the properties of logarithms.

Answer

$4\ln 2$

Defining the Number e

Now that we have the natural logarithm defined, we can use that function to define the number e.

Definition: e

The number e is defined to be the real number such that

$$\ln e = 1$$

To put it another way, the area under the curve $y = 1/t$ between $t = 1$ and $t = e$ is 1 (Figure). The proof that such a number exists and is unique is left to you. (Hint: Use the Intermediate Value Theorem to prove existence and the fact that $\ln x$ is increasing to prove uniqueness.)

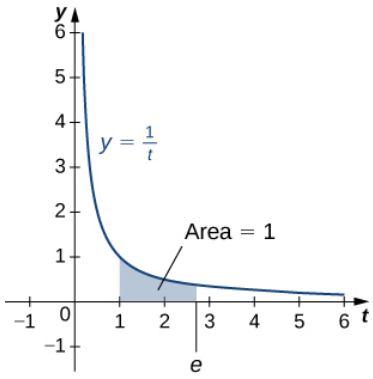


Figure 3.12.3:The area under the curve from 1 to e is equal to one.

The number e can be shown to be *irrational*, although we won't do so here (see the Student Project in Taylor and Maclaurin Series). Its approximate value is given by

$$e \approx 2.71828182846.$$

The Exponential Function

We now turn our attention to the function e^x. Note that the natural logarithm is one-to-one and therefore has an inverse function. For now, we denote this inverse function by $\exp x$. Then,

$$\exp(\ln x) = x$$

for $x > 0$ and

$$\ln(\exp x) = x$$

for all x.

The following figure shows the graphs of $\exp x$ and $\ln x$.

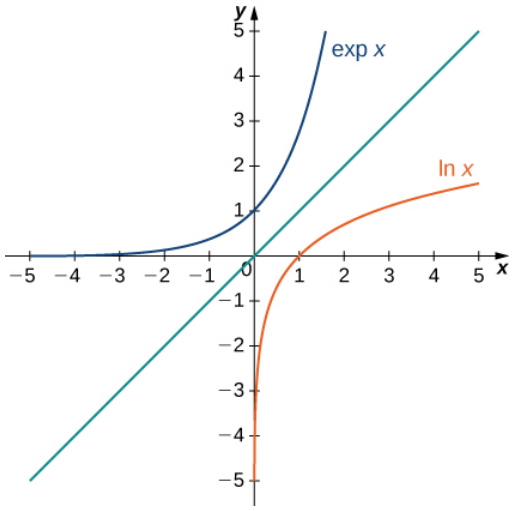


Figure 3.12.4: The graphs of $\ln x$ and $\exp x$.

We hypothesize that $\exp x = e^x$. For rational values of x, this is easy to show. If x is rational, then we have $\ln(e^x) = x \ln e = x$. Thus, when x is rational, $e^x = \exp x$. For irrational values of x, we simply define e^x as the inverse function of $\ln x$.

Definition

For any real number x, define $y = e^x$ to be the number for which

$$\ln y = \ln(e^x) = x.$$

Then we have $e^x = \exp x$ for all x, and thus

$$e^{\ln x} = x \text{ for } x > 0 \text{ and } \ln(e^x) = x$$

for all x.

Properties of the Exponential Function

Since the exponential function was defined in terms of an inverse function, and not in terms of a power of e we must verify that the usual laws of exponents hold for the function e^x.

Properties of the Exponential Function

If p and q are any real numbers and r is a rational number, then

i. $e^p e^q = e^{p+q}$
ii. $\dfrac{e^p}{e^q} = e^{p-q}$
iii. $(e^p)^r = e^{pr}$

Proof

Note that if p and q are rational, the properties hold. However, if p or q are irrational, we must apply the inverse function definition of e^x and verify the properties. Only the first property is verified here; the other two are left to you. We have

$$\ln(e^p e^q) = \ln(e^p) + \ln(e^q) = p + q = \ln(e^{p+q}).$$

Since $\ln x$ is one-to-one, then

$$e^p e^q = e^{p+q}.$$

□

As with part iv. of the logarithm properties, we can extend property iii. to irrational values of r, and we do so by the end of the section.

We also want to verify the differentiation formula for the function $y = e^x$. To do this, we need to use implicit differentiation. Let $y = e^x$. Then

$$\begin{aligned} \ln y &= x \\ \frac{d}{dx}\left(\ln y\right) &= \frac{d}{dx}\left(x\right) \\ \frac{1}{y}\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= y. \end{aligned}$$

Thus, we see

$$\frac{d}{dx}\left(e^x\right) = e^x$$

as desired, which leads immediately to the integration formula

$$\int e^x \, dx = e^x + C.$$

We apply these formulas in the following examples.

Example 3.12.4: Using Properties of Exponential Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(e^{3t}e^{t^2}\right)$

b. $\frac{d}{dx}\left(e^{3x^2}\right)$

Solution

We apply the chain rule as necessary.

a. $\frac{d}{dt}\left(e^{3t}e^{t^2}\right) = \frac{d}{dt}\left(e^{3t+t^2}\right) = e^{3t+t^2}(3+2t)$

b. $\frac{d}{dx}\left(e^{3x^2}\right) = e^{3x^2}6x$

Exercise 3.12.4

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\frac{e^{x^2}}{e^{5x}}\right)$

b. $\dfrac{d}{dt}\left((e^{2t})^3\right)$

Hint

Use the properties of exponential functions and the chain rule as necessary.

Answer

a. $\dfrac{d}{dx}\left(\dfrac{e^{x^2}}{e^{5x}}\right) = e^{x^{2-5x}}(2x-5)$

b. $\dfrac{d}{dt}\left((e^{2t})^3\right) = 6e^{6t}$

Example 3.12.5: Using Properties of Exponential Functions

Evaluate the following integral: $\displaystyle\int 2xe^{-x^2}\,dx$.

Solution

Using u-substitution, let $u = -x^2$. Then $du = -2x\,dx$, and we have

$$\int 2xe^{-x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{-x^2} + C.$$

Exercise 3.12.5

Evaluate the following integral: $\displaystyle\int \frac{4}{e^{3x}}\,dx$.

Hint

Use the properties of exponential functions and $u-substitution$ as necessary.

Answer

$$\int \frac{4}{e^{3x}}\,dx = -\frac{4}{3}e^{-3x} + C$$

General Logarithmic and Exponential Functions

We close this section by looking at exponential functions and logarithms with bases other than e. Exponential functions are functions of the form $f(x) = a^x$. Note that unless $a = e$, we still do not have a mathematically rigorous definition of these functions for irrational exponents. Let's rectify that here by defining the function $f(x) = a^x$ in terms of the exponential function e^x. We then examine logarithms with bases other than e as inverse functions of exponential functions.

Definition: Exponential Function

For any $a > 0$, and for any real number x, define $y = a^x$ as follows:

$$y = a^x = e^{x\ln a}.$$

Now a^x is defined rigorously for all values of x. This definition also allows us to generalize property iv. of logarithms and property iii. of exponential functions to apply to both rational and irrational values of r. It is straightforward to show that properties of exponents hold for general exponential functions defined in this way.

Let's now apply this definition to calculate a differentiation formula for a^x. We have

$$\frac{d}{dx}\left(a^x\right) = \frac{d}{dx}\left(e^{x\ln a}\right) = e^{x\ln a}\ln a = a^x \ln a.$$

The corresponding integration formula follows immediately.

Derivatives and Integrals Involving General Exponential Functions

Let $a > 0$. Then,

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a$$

and

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C.$$

If $a \neq 1$, then the function a^x is one-to-one and has a well-defined inverse. Its inverse is denoted by $\log_a x$. Then,

$$y = \log_a x \text{ if and only if } x = a^y.$$

Note that general logarithm functions can be written in terms of the natural logarithm. Let $y = \log_a x$. Then, $x = a^y$. Taking the natural logarithm of both sides of this second equation, we get

$$\begin{aligned} \ln x &= \ln(a^y) \\ \ln x &= y \ln a \\ y &= \frac{\ln x}{\ln a} \\ \log_a x &= \frac{\ln x}{\ln a}. \end{aligned}$$

Thus, we see that all logarithmic functions are constant multiples of one another. Next, we use this formula to find a differentiation formula for a logarithm with base a. Again, let $y = \log_a x$. Then,

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\log_a x\right) \\ &= \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) \\ &= (\frac{1}{\ln a})\frac{d}{dx}\left(\ln x\right) \\ &= \frac{1}{\ln a} \cdot \frac{1}{x} = \frac{1}{x \ln a} \end{aligned}$$

Derivatives of General Logarithm Functions

Let $a > 0$. Then,

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x \ln a}.$$

Example 3.12.6: Calculating Derivatives of General Exponential and Logarithm Functions

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right)$

b. $\frac{d}{dx}\left(\log_8(7x^2+4)\right)$

Solution: We need to apply the chain rule as necessary.

a. $\frac{d}{dt}\left(4^t \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t} \cdot 2^{t^2}\right) = \frac{d}{dt}\left(2^{2t+t^2}\right) = 2^{2t+t^2} \ln(2)(2+2t)$

b. $\frac{d}{dx}\left(\log_8(7x^2+4)\right) = \frac{1}{(7x^2+4)(\ln 8)}(14x)$

? Exercise 3.12.6

Evaluate the following derivatives:

a. $\frac{d}{dt}\left(4^{t^4}\right)$

b. $\frac{d}{dx}\left(\log_3(\sqrt{x^2+1})\right)$

Hint

Use the formulas and apply the chain rule as necessary.

Answer

a. $\frac{d}{dt}\left(4^{t^4}\right) = 4^{t^4}(\ln 4)(4t^3)$

b. $\frac{d}{dx}\left(\log_3(\sqrt{x^2+1})\right) = \frac{x}{(\ln 3)(x^2+1)}$

✓ Example 3.12.7: Integrating General Exponential Functions

Evaluate the following integral: $\int \frac{3}{2^{3x}}\,dx$.

Solution

Use $u-substitution$ and let $u=-3x$. Then $du=-3\,dx$ and we have

$$\int \frac{3}{2^{3x}}\,dx = \int 3\cdot 2^{-3x}\,dx = -\int 2^u\,du = -\frac{1}{\ln 2}2^u + C = -\frac{1}{\ln 2}2^{-3x} + C.$$

? Exercise 3.12.7

Evaluate the following integral: $\int x^2 2^{x^3}\,dx$.

Hint

Use the properties of exponential functions and u-substitution

Answer

$$\int x^2 2^{x^3}\,dx = \frac{1}{3\ln 2}2^{x^3} + C$$

Key Concepts

- The earlier treatment of logarithms and exponential functions did not define the functions precisely and formally. This section develops the concepts in a mathematically rigorous way.
- The cornerstone of the development is the definition of the natural logarithm in terms of an integral.
- The function e^x is then defined as the inverse of the natural logarithm. General exponential functions are defined in terms of e^x, and the corresponding inverse functions are general logarithms.
- Familiar properties of logarithms and exponents still hold in this more rigorous context.

Key Equations

- **Natural logarithm function**
- $\ln x = \int_1^x \frac{1}{t}\, dt$
- **Exponential function** $y = e^x$
- $\ln y = \ln(e^x) = x$

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3.12E: Exercises for Section 6.7

In exercises 1 - 3, find the derivative $\frac{dy}{dx}$.

1) $y = \ln(2x)$

Answer

$$\frac{dy}{dx} = \frac{1}{x}$$

2) $y = \ln(2x+1)$

3) $y = \dfrac{1}{\ln x}$

Answer

$$\frac{dy}{dx} = -\frac{1}{x(\ln x)^2}$$

In exercises 4 - 5, find the indefinite integral.

4) $\displaystyle\int \frac{dt}{3t}$

5) $\displaystyle\int \frac{dx}{1+x}$

Answer

$$\int \frac{dx}{1+x} = \ln|x+1| + C$$

In exercises 6 - 15, find the derivative $\frac{dy}{dx}$. (You can use a calculator to plot the function and the derivative to confirm that it is correct.)

6) [T] $y = \dfrac{\ln x}{x}$

7) [T] $y = x \ln x$

Answer

$$\frac{dy}{dx} = \ln(x) + 1$$

8) [T] $y = \log_{10} x$

9) [T] $y = \ln(\sin x)$

Answer

$$\frac{dy}{dx} = \cot x$$

10) [T] $y = \ln(\ln x)$

11) [T] $y = 7\ln(4x)$

Answer

$$\frac{dy}{dx} = \frac{7}{x}$$

12) [T] $y = \ln\left((4x)^7\right)$

13) [T] $y = \ln(\tan x)$

Answer

$$\frac{dy}{dx} = \csc x \sec x$$

14) [T] $y = \ln(\tan 3x)$

15) [T] $y = \ln(\cos^2 x)$

Answer

$$\frac{dy}{dx} = -2 \tan x$$

In exercises 16 - 25, find the definite or indefinite integral.

16) $\int_0^1 \frac{dx}{3+x}$

17) $\int_0^1 \frac{dt}{3+2t}$

Answer

$$\int_0^1 \frac{dt}{3+2t} = \tfrac{1}{2}\ln\left(\tfrac{5}{3}\right)$$

18) $\int_0^2 \frac{x}{x^2+1}\,dx$

19) $\int_0^2 \frac{x^3}{x^2+1}\,dx$

Answer

$$\int_0^2 \frac{x^3}{x^2+1}\,dx = 2 - \tfrac{1}{2}\ln(5)$$

20) $\int_2^e \frac{dx}{x \ln x}$

21) $\int_2^e \frac{dx}{(x \ln x)^2}$

Answer

$$\int_2^e \frac{dx}{(x \ln x)^2} = \frac{1}{\ln(2)} - 1$$

22) $\int \frac{\cos x}{\sin x}\,dx$

23) $\int_0^{\pi/4} \tan x\,dx$

Answer

$$\int_0^{\pi/4} \tan x\,dx = \tfrac{1}{2}\ln(2)$$

24) $\int \cot(3x)\,dx$

25) $\int \frac{(\ln x)^2}{x}\,dx$

Answer

$$\int \frac{(\ln x)^2}{x}\,dx = \tfrac{1}{3}(\ln x)^3$$

In exercises 26 - 35, compute $\frac{dy}{dx}$ by differentiating $\ln y$.

26) $y = \sqrt{x^2+1}$

27) $y = \sqrt{x^2+1}\sqrt{x^2-1}$

Answer

$$\frac{dy}{dx} = \frac{2x^3}{\sqrt{x^2+1}\sqrt{x^2-1}}$$

28) $y = e^{\sin x}$

29) $y = x^{-1/x}$

Answer

$$\frac{dy}{dx} = x^{-2-(1/x)}(\ln x - 1)$$

30) $y = e^{ex}$

31) $y = x^e$

Answer

$$\frac{dy}{dx} = ex^{e-1}$$

32) $y = x^{(ex)}$

33) $y = \sqrt{x}\sqrt[3]{x}\sqrt[6]{x}$

Answer

$$\frac{dy}{dx} = 1$$

34) $y = x^{-1/\ln x}$

35) $y = e^{-\ln x}$

Answer

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

In exercises 36 - 40, evaluate by any method.

36) $\int_5^{10} \frac{dt}{t} - \int_{5x}^{10x} \frac{dt}{t}$

37) $\int_1^{e^\pi} \frac{dx}{x} + \int_{-2}^{-1} \frac{dx}{x}$

Answer

$\pi - \ln(2)$

38) $\dfrac{d}{dx}\left[\displaystyle\int_x^1 \dfrac{dt}{t}\right]$

39) $\dfrac{d}{dx}\left[\displaystyle\int_x^{x^2} \dfrac{dt}{t}\right]$

Answer

$\dfrac{1}{x}$

40) $\dfrac{d}{dx}\left[\ln(\sec x + \tan x)\right]$

In exercises 41 - 44, use the function $\ln x$. If you are unable to find intersection points analytically, use a calculator.

41) Find the area of the region enclosed by $x = 1$ and $y = 5$ above $y = \ln x$.

Answer

$(e^5 - 6)$ units^2

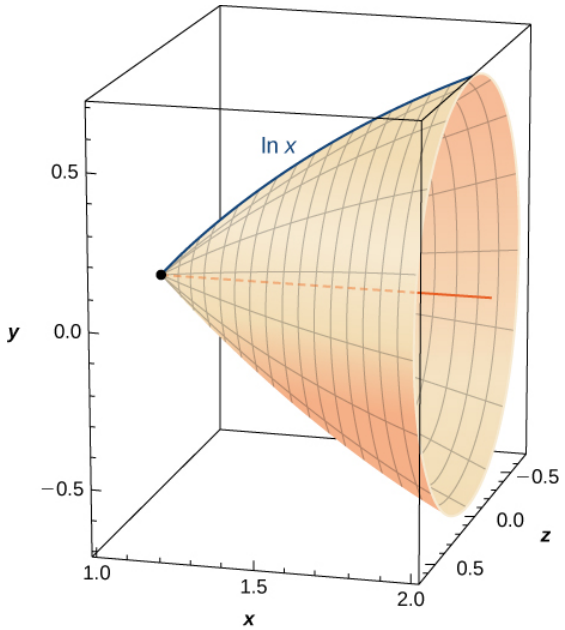
42) [T] Find the arc length of $\ln x$ from $x = 1$ to $x = 2$.

43) Find the area between $\ln x$ and the x-axis from $x = 1$ to $x = 2$.

Answer

$\ln(4) - 1)$ units^2

44) Find the volume of the shape created when rotating this curve from $x = 1$ to $x = 2$ around the x-axis, as pictured here.



45) [T] Find the surface area of the shape created when rotating the curve in the previous exercise from $x = 1$ to $x = 2$ around the x-axis.

Answer

2.8656 units^2

If you are unable to find intersection points analytically in the following exercises, use a calculator.

46) Find the area of the hyperbolic quarter-circle enclosed by $x = 2$ and $y = 2$ above $y = 1/x$.

47) [T] Find the arc length of $y = 1/x$ from $x = 1$ to $x = 4$.

Answer

$s = 3.1502$ units

48) Find the area under $y = 1/x$ and above the x-axis from $x = 1$ to $x = 4$.

In exercises 49 - 53, verify the derivatives and antiderivatives.

49) $\frac{d}{dx}\left[\ln(x+\sqrt{x^2+1})\right] = \frac{1}{\sqrt{1+x^2}}$

50) $\frac{d}{dx}\left[\ln\left(\frac{x-a}{x+a}\right)\right] = \frac{2a}{(x^2-a^2)}$

51) $\frac{d}{dx}\left[\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)\right] = -\frac{1}{x\sqrt{1-x^2}}$

52) $\frac{d}{dx}\left[\ln(x+\sqrt{x^2-a^2})\right] = \frac{1}{\sqrt{x^2-a^2}}$

53) $\int \frac{dx}{x\ln(x)\ln(\ln x)} = \ln|\ln(\ln x)| + C$

Contributors

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3.13: Exponential Growth and Decay

Learning Objectives

- Use the exponential growth model in applications, including population growth and compound interest.
- Explain the concept of doubling time.
- Use the exponential decay model in applications, including radioactive decay and Newton's law of cooling.
- Explain the concept of half-life.

One of the most prevalent applications of exponential functions involves growth and decay models. Exponential growth and decay show up in a host of natural applications. From population growth and continuously compounded interest to radioactive decay and Newton's law of cooling, exponential functions are ubiquitous in nature. In this section, we examine exponential growth and decay in the context of some of these applications.

Exponential Growth Model

Many systems exhibit exponential growth. These systems follow a model of the form $y = y_0 e^{kt}$, where y_0 represents the initial state of the system and k is a positive constant, called the growth constant. Notice that in an exponential growth model, we have

$$y' = ky_0 e^{kt} = ky. \tag{3.13.1}$$

That is, the rate of growth is proportional to the current function value. This is a key feature of exponential growth. Equation 3.13.1 involves derivatives and is called *a differential equation*.

Exponential Growth

Systems that exhibit *exponential growth* increase according to the mathematical model

$$y = y_0 e^{kt}$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the growth constant.

Population growth is a common example of exponential growth. Consider a population of bacteria, for instance. It seems plausible that the rate of population growth would be proportional to the size of the population. After all, the more bacteria there are to reproduce, the faster the population grows. Figure 3.13.1 and Table 3.13.1 represent the growth of a population of bacteria with an initial population of 200 bacteria and a growth constant of 0.02. Notice that after only 2 hours (120 minutes), the population is 10 times its original size!

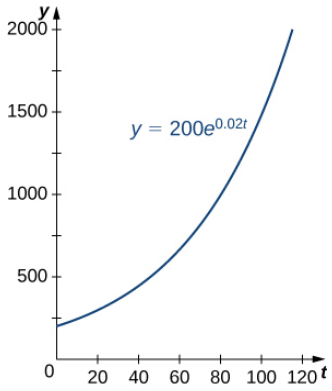


Figure 3.13.1: An example of exponential growth for bacteria.

Table 3.13.1: Exponential Growth of a Bacterial Population

Time(min)	Population Size (no. of bacteria)
10	244
20	298

Time(min)	Population Size (no. of bacteria)
30	364
40	445
50	544
60	664
70	811
80	991
90	1210
100	1478
110	1805
120	2205

Note that we are using a continuous function to model what is inherently discrete behavior. At any given time, the real-world population contains a whole number of bacteria, although the model takes on noninteger values. When using exponential growth models, we must always be careful to interpret the function values in the context of the phenomenon we are modeling.

Example 3.13.1: Population Growth

Consider the population of bacteria described earlier. This population grows according to the function $f(t) = 200e^{0.02t}$, where t is measured in minutes. How many bacteria are present in the population after 5 hours (300 minutes)? When does the population reach $100,000$ bacteria?

Solution

We have $f(t) = 200e^{0.02t}$. Then

$$f(300) = 200e^{0.02(300)} \approx 80,686.$$

There are $80,686$ bacteria in the population after 5 hours.

To find when the population reaches $100,000$ bacteria, we solve the equation

$$\begin{aligned} 100,000 &= 200e^{0.02t} \\ 500 &= e^{0.02t} \\ \ln 500 &= 0.02t \\ t &= \frac{\ln 500}{0.02} \approx 310.73. \end{aligned}$$

The population reaches $100,000$ bacteria after 310.73 minutes.

Exercise 3.13.1

Consider a population of bacteria that grows according to the function $f(t) = 500e^{0.05t}$, where t is measured in minutes. How many bacteria are present in the population after 4 hours? When does the population reach 100 million bacteria?

Answer

Use the process from the previous example.

Answer

There are $81,377,396$ bacteria in the population after 4 hours. The population reaches 100 million bacteria after 244.12 minutes.

Let's now turn our attention to a financial application: **compound interest**. Interest that is not compounded is called simple interest. ***Simple interest*** is paid once, at the end of the specified time period (usually 1 year). So, if we put $\$1000$ in a savings account earning 2 simple interest per year, then at the end of the year we have

$$1000(1+0.02)=\$1020.$$

Compound interest is paid multiple times per year, depending on the compounding period. Therefore, if the bank compounds the interest every 6 months, it credits half of the year's interest to the account after 6 months. During the second half of the year, the account earns interest not only on the initial $\$1000$, but also on the interest earned during the first half of the year. Mathematically speaking, at the end of the year, we have

$$1000\left(1+\frac{0.02}{2}\right)^2=\$1020.10.$$

Similarly, if the interest is compounded every 4 months, we have

$$1000\left(1+\frac{0.02}{3}\right)^3=\$1020.13,$$

and if the interest is compounded daily (365 times per year), we have $\$1020.20$ If we extend this concept, so that the interest is compounded continuously, after t years we have

$$1000\lim_{n\to\infty}\left(1+\frac{0.02}{n}\right)^{nt}.$$

Now let's manipulate this expression so that we have an exponential growth function. Recall that the number e can be expressed as a limit:

$$e=\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m.$$

Based on this, we want the expression inside the parentheses to have the form $(1+1/m)$. Let $n=0.02m$. Note that as $n\to\infty, m\to\infty$ as well. Then we get

$$1000\lim_{n\to\infty}\left(1+\frac{0.02}{n}\right)^{nt}=1000\lim_{m\to\infty}\left(1+\frac{0.02}{0.02m}\right)^{0.02mt}=1000\left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^{0.02t}.$$

We recognize the limit inside the brackets as the number e. So, the balance in our bank account after t years is given by $1000e^{0.02t}$. Generalizing this concept, we see that if a bank account with an initial balance of $\$P$ earns interest at a rate of r, compounded continuously, then the balance of the account after t years is

$$\text{Balance }=Pe^{rt}.$$

Example 3.13.2: Compound Interest

A 25-year-old student is offered an opportunity to invest some money in a retirement account that pays 5 annual interest compounded continuously. How much does the student need to invest today to have $\$1$ million when she retires at age 65? What if she could earn 6 annual interest compounded continuously instead?

Solution

We have

$$1,000,000=Pe^{0.05(40)}$$

$$P=135,335.28.$$

She must invest $\$135,335.28$at 5 interest.

If, instead, she is able to earn 6 then the equation becomes

$$1,000,000 = Pe^{0.06(40)}$$

$$P = 90,717.95.$$

In this case, she needs to invest only $\$90,717.95$.This is roughly two-thirds the amount she needs to invest at 5. The fact that the interest is compounded continuously greatly magnifies the effect of the 1 increase in interest rate.

Exercise 3.13.2

Suppose instead of investing at age $25\sqrt{b^2-4ac}$, the student waits until age 35. How much would she have to invest at 5? At 6?

Hint

Use the process from the previous example.

Answer

At 5 interest, she must invest $\$223,130.16$At 6 interest, she must invest $\$165,298.89$.

If a quantity grows exponentially, the time it takes for the quantity to double remains constant. In other words, it takes the same amount of time for a population of bacteria to grow from 100 to 200 bacteria as it does to grow from $10,000$ to $20,000$ bacteria. This time is called the doubling time. To calculate the doubling time, we want to know when the quantity reaches twice its original size. So we have

$$\begin{aligned} 2y_0 &= y_0 e^{kt} \\ 2 &= e^{kt} \\ \ln 2 &= kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Definition: Doubling Time

If a quantity grows exponentially, the **doubling time** is the amount of time it takes the quantity to double. It is given by

$$\text{Doubling time} = \frac{\ln 2}{k}.$$

Example 3.13.3: Using the Doubling Time

Assume a population of fish grows exponentially. A pond is stocked initially with 500 fish. After 6 months, there are 1000 fish in the pond. The owner will allow his friends and neighbors to fish on his pond after the fish population reaches $10,000$. When will the owner's friends be allowed to fish?

Solution

We know it takes the population of fish 6 months to double in size. So, if t represents time in months, by the doubling-time formula, we have $6 = (\ln 2)/k$. Then, $k = (\ln 2)/6$. Thus, the population is given by $y = 500e^{((\ln 2)/6)t}$. To figure out when the population reaches $10,000$ fish, we must solve the following equation:

$$\begin{aligned} 10,000 &= 500e^{(\ln 2/6)t} \\ 20 &= e^{(\ln 2/6)t} \\ \ln 20 &= \left(\frac{\ln 2}{6}\right) t \\ t &= \frac{6(\ln 20)}{\ln 2} \\ &\approx 25.93. \end{aligned}$$

The owner's friends have to wait 25.93 months (a little more than 2 years) to fish in the pond.

Exercise 3.13.3

Suppose it takes 9 months for the fish population in Example $3.13.3$ to reach 1000 fish. Under these circumstances, how long do the owner's friends have to wait?

Hint

Use the process from the previous example.

Answer

38.90 months

Exponential Decay Model

Exponential functions can also be used to model populations that shrink (from disease, for example), or chemical compounds that break down over time. We say that such systems exhibit exponential decay, rather than exponential growth. The model is nearly the same, except there is a negative sign in the exponent. Thus, for some positive constant k, we have

$$y = y_0 e^{-kt}.$$

As with exponential growth, there is a differential equation associated with exponential decay. We have

$$y' = -ky_0 e^{-kt} = -ky.$$

Exponential Decay

Systems that exhibit exponential decay behave according to the model

$$y = y_0 e^{-kt},$$

where y_0 represents the initial state of the system and $k > 0$ is a constant, called the decay constant.

Figure $3.13.2$ shows a graph of a representative exponential decay function.

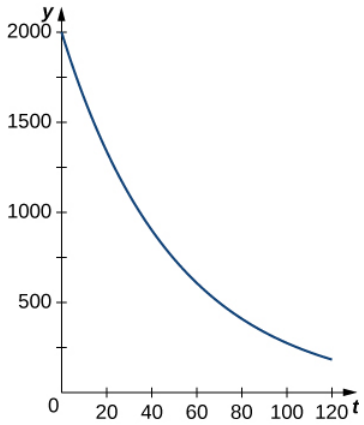


Figure 3.13.2: An example of exponential decay.

Let's look at a physical application of exponential decay. *Newton's law of cooling* says that an object cools at a rate proportional to the difference between the temperature of the object and the temperature of the surroundings. In other words, if T represents the temperature of the object and T_a represents the ambient temperature in a room, then

$$T' = -k(T - T_a).$$

Note that this is not quite the right model for exponential decay. We want the derivative to be proportional to the function, and this expression has the additional T_a term. Fortunately, we can make a change of variables that resolves this issue. Let $y(t) = T(t) - T_a$. Then $y'(t) = T'(t) - 0 = T'(t)$, and our equation becomes

$$y' = -ky.$$

From our previous work, we know this relationship between y and its derivative leads to exponential decay. Thus,

$$y = y_0 e^{-kt},$$

and we see that

$$T - T_a = (T_0 - T_a)e^{-kt}$$

$$T = (T_0 - T_a)e^{-kt} + T_a$$

where T_0 represents the initial temperature. Let's apply this formula in the following example.

Example 3.13.4: Newton's Law of Cooling

According to experienced baristas, the optimal temperature to serve coffee is between $155^\circ F$ and $175^\circ F$. Suppose coffee is poured at a temperature of $200^\circ F$, and after 2 minutes in a $70^\circ F$ room it has cooled to $180^\circ F$. When is the coffee first cool enough to serve? When is the coffee too cold to serve? Round answers to the nearest half minute.

Solution

We have

$$\begin{aligned} T &= (T_0 - T_a)e^{-kt} + T_a \\ 180 &= (200-70)e^{-k(2)} + 70 \\ 110 &= 130e^{-2k} \\ \frac{11}{13} &= e^{-2k} \\ \ln\frac{11}{13} &= -2k \\ \ln 11 - \ln 13 &= -2k \\ k &= \frac{\ln 13 - \ln 11}{2} \end{aligned}$$

Then, the model is

$$T = 130e^{(\ln 11 - \ln 13/2)t} + 70.$$

The coffee reaches $175^\circ F$ when

$$\begin{aligned} 175 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\ 105 &= 130e^{(\ln 11 - \ln 13/2)t} \\ \frac{21}{26} &= e^{(\ln 11 - \ln 13/2)t} \\ \ln\frac{21}{26} &= \frac{\ln 11 - \ln 13}{2}t \\ \ln 21 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\ t &= \frac{2(\ln 21 - \ln 26)}{\ln 11 - \ln 13} \\ &\approx 2.56. \end{aligned}$$

The coffee can be served about 2.5 minutes after it is poured. The coffee reaches $155^\circ F$ at

$$\begin{aligned} 155 &= 130e^{(\ln 11 - \ln 13/2)t} + 70 \\ 85 &= 130e^{(\ln 11 - \ln 13)t} \\ \frac{17}{26} &= e^{(\ln 11 - \ln 13)t} \\ \ln 17 - \ln 26 &= \left(\frac{\ln 11 - \ln 13}{2}\right)t \\ t &= \frac{2(\ln 17 - \ln 26)}{\ln 11 - \ln 13} \\ &\approx 5.09. \end{aligned}$$

The coffee is too cold to be served about 5 minutes after it is poured.

? Exercise 3.13.4

Suppose the room is warmer $(75^\circ F)$ and, after 2 minutes, the coffee has cooled only to $185^\circ F$. When is the coffee first cool enough to serve? When is the coffee be too cold to serve? Round answers to the nearest half minute.

Hint

Use the process from the previous example.

Answer

The coffee is first cool enough to serve about 3.5 minutes after it is poured. The coffee is too cold to serve about 7 minutes after it is poured.

Just as systems exhibiting exponential growth have a constant doubling time, systems exhibiting exponential decay have a constant half-life. To calculate the half-life, we want to know when the quantity reaches half its original size. Therefore, we have

$$\begin{aligned} \frac{y_0}{2} &= y_0 e^{-kt} \\ \frac{1}{2} &= e^{-kt} \\ -\ln 2 &= -kt \\ t &= \frac{\ln 2}{k}. \end{aligned}$$

Note: This is the same expression we came up with for doubling time.

Definition: Half-Life

If a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by

$$\text{Half-life} = \frac{\ln 2}{k}.$$

Example 3.13.5: Radiocarbon Dating

One of the most common applications of an exponential decay model is carbon dating. Carbon-14 decays (emits a radioactive particle) at a regular and consistent exponential rate. Therefore, if we know how much carbon-14 was originally present in an object and how much carbon-14 remains, we can determine the age of the object. The half-life of carbon-14 is approximately 5730 years—meaning, after that many years, half the material has converted from the original carbon-14 to the new nonradioactive nitrogen-14. If we have 100 g carbon-14 today, how much is left in 50 years? If an artifact that originally contained 100 g of carbon-14 now contains 10 g of carbon-14, how old is it? Round the answer to the nearest hundred years.

Solution

We have

$$\begin{aligned} 5730 &= \frac{\ln 2}{k} \\ k &= \frac{\ln 2}{5730}. \end{aligned}$$

So, the model says

$$y = 100e^{-(\ln 2/5730)t}.$$

In 50 years, we have

$$y = 100e^{-(\ln 2/5730)(50)} \approx 99.40$$

Therefore, in 50 years, 99.40 g of carbon-14 remains.

To determine the age of the artifact, we must solve

$$\begin{aligned} 10 &= 100e^{-(\ln 2/5730)t} \\ \frac{1}{10} &= e^{-(\ln 2/5730)t} \\ t &\approx 19035. \end{aligned}$$

The artifact is about $19,000$ years old.

Exercise 3.13.5: Carbon-14 Decay

If we have 100 g of carbon-14 , how much is left after 500 years? If an artifact that originally contained 100 g of carbon-14 now contains 20 g of carbon-14, how old is it? Round the answer to the nearest hundred years.

Hint

Use the process from the previous example.

Answer

A total of 94.13 g of carbon-14 remains after 500 years. The artifact is approximately 13,300 years old.

Key Concepts

- Exponential growth and exponential decay are two of the most common applications of exponential functions.
- Systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$.
- In exponential growth, the rate of growth is proportional to the quantity present. In other words, $y' = ky$.
- Systems that exhibit exponential growth have a constant doubling time, which is given by $(\ln 2)/k$.
- Systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$.
- Systems that exhibit exponential decay have a constant half-life, which is given by $(\ln 2)/k$.

Glossary

doubling time

if a quantity grows exponentially, the doubling time is the amount of time it takes the quantity to double, and is given by $(\ln 2)/k$

exponential decay

systems that exhibit exponential decay follow a model of the form $y = y_0 e^{-kt}$

exponential growth

systems that exhibit exponential growth follow a model of the form $y = y_0 e^{kt}$

half-life

if a quantity decays exponentially, the half-life is the amount of time it takes the quantity to be reduced by half. It is given by $(\ln 2)/k$

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3.13E: Exercises for Section 6.8

In exercises 1 - 2, answer *True or False*? If true, prove it. If false, find the true answer.

1) The doubling time for $y = e^{ct}$ is $\dfrac{\ln 2}{\ln c}$.

2) If you invest $\$500$, an annual rate of interest of 3% yields more money in the first year than a 2.5% continuous rate of interest.

Answer

True

3) If you leave a $100^\circ C$ pot of tea at room temperature ($25^\circ C$) and an identical pot in the refrigerator ($5^\circ C$), with $k = 0.02$, the tea in the refrigerator reaches a drinkable temperature ($70^\circ C$) more than 5 minutes before the tea at room temperature.

4) If given a half-life of t years, the constant k for $y = e^{kt}$ is calculated by $k = \dfrac{\ln\left(\frac{1}{2}\right)}{t}$.

Answer

False; $k = \dfrac{\ln 2}{t}$

In exercises 5 - 18, use $y = y_0 e^{kt}$.

5) If a culture of bacteria doubles in 3 hours, how many hours does it take to multiply by 10?

6) If bacteria increase by a factor of 10 in 10 hours, how many hours does it take to increase by 100?

Answer

20 hours

7) How old is a skull that contains one-fifth as much radiocarbon as a modern skull? Note that the half-life of radiocarbon is 5730 years.

8) If a relic contains 90% as much radiocarbon as new material, can it have come from the time of Christ (approximately 2000 years ago)? Note that the half-life of radiocarbon is 5730 years.

Answer

No. The relic is approximately 871 years old.

9) The population of Cairo grew from 5 million to 10 million in 20 years. Use an exponential model to find when the population was 8 million.

10) The populations of New York and Los Angeles are growing at 1% and 1.4% a year, respectively. Starting from 8 million (New York) and 6 million (Los Angeles), when are the populations equal?

Answer

71.92 years

11) Suppose the value of $\$1$ in Japanese yen decreases at 2% per year. Starting from $\$1 = ¥250$, when will $\$1 = ¥1$?

12) The effect of advertising decays exponentially. If 40% of the population remembers a new product after 3 days, how long will 20% remember it?

Answer

5 days 6 hours 27 minutes

13) If $y = 1000$ at $t = 3$ and $y = 3000$ at $t = 4$, what was y_0 at $t = 0$?

14) If $y = 100$ at $t = 4$ and $y = 10$ at $t = 8$, when does $y = 1$?

Answer

At $t = 12$

15) If a bank offers annual interest of 7.5% or continuous interest of 7.25%, which has a better annual yield?

16) What continuous interest rate has the same yield as an annual rate of 9%?

Answer

8.618%

17) If you deposit $\$5000$ at 8% annual interest, how many years can you withdraw $\$500$ (starting after the first year) without running out of money?

18) You are trying to save $\$50,000$ in 20 years for college tuition for your child. If interest is a continuous 10%, how much do you need to invest initially?

Answer

$6766.76

19) You are cooling a turkey that was taken out of the oven with an internal temperature of $165^\circ F$. After 10 minutes of resting the turkey in a $70^\circ F$ apartment, the temperature has reached $155^\circ F$. What is the temperature of the turkey 20 minutes after taking it out of the oven?

20) You are trying to thaw some vegetables that are at a temperature of $1^\circ F$. To thaw vegetables safely, you must put them in the refrigerator, which has an ambient temperature of $44^\circ F$. You check on your vegetables 2 hours after putting them in the refrigerator to find that they are now $12^\circ F$. Plot the resulting temperature curve and use it to determine when the vegetables reach 33°.

Answer

9hours 13minutes

21) You are an archeologist and are given a bone that is claimed to be from a Tyrannosaurus Rex. You know these dinosaurs lived during the Cretaceous Era (146 million years to 65 million years ago), and you find by radiocarbon dating that there is 0.000001% the amount of radiocarbon. Is this bone from the Cretaceous?

22) The spent fuel of a nuclear reactor contains plutonium-239, which has a half-life of $24,000$ years. If 1 barrel containing 10 kg of plutonium-239 is sealed, how many years must pass until only 10 g of plutonium-239 is left?

Answer

$239,179$years

For exercises 23 - 26, use the following table, which features the world population by decade.

Years since 1950	Population (millions)
0	2,556
10	3,039
20	3,706
30	4,453
40	5,279
50	6,083
60	6,849

Source: http:/www.factmonster.com/ipka/A0762181.html.

23) [T] The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 2686e^{0.01604t}$. Use a graphing calculator to graph the data and the exponential curve together.

24) [T] Find and graph the derivative y' of your equation. Where is it increasing and what is the meaning of this increase?

Answer

$P'(t) = 43e^{0.01604t}$. The population is always increasing.

25) [T] Find and graph the second derivative of your equation. Where is it increasing and what is the meaning of this increase?

26) [T] Find the predicted date when the population reaches 10 billion. Using your previous answers about the first and second derivatives, explain why exponential growth is unsuccessful in predicting the future.

Answer

The population reaches 10 billion people in 2027.

For exercises 27 - 29, use the following table, which shows the population of San Francisco during the 19th century.

Years since 1850	Population (thousands)
0	21.00
10	56.80
20	149.5
30	234.0

Source: http:/www.sfgenealogy.com/sf/history/hgpop.htm.

27) [T] The best-fit exponential curve to the data of the form $P(t) = ae^{bt}$ is given by $P(t) = 35.26e^{0.06407t}$. Use a graphing calculator to graph the data and the exponential curve together.

28) [T] Find and graph the derivative y' of your equation. Where is it increasing? What is the meaning of this increase? Is there a value where the increase is maximal?

Answer

$P'(t) = 2.259e^{0.06407t}$. The population is always increasing.

29) [T] Find and graph the second derivative of your equation. Where is it increasing? What is the meaning of this increase?

Contributors

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3.14: Calculus of the Hyperbolic Functions

Learning Objectives

- Apply the formulas for derivatives and integrals of the hyperbolic functions.
- Apply the formulas for the derivatives of the inverse hyperbolic functions and their associated integrals.
- Describe the common applied conditions of a catenary curve.

We were introduced to hyperbolic functions previously, along with some of their basic properties. In this section, we look at differentiation and integration formulas for the hyperbolic functions and their inverses.

Derivatives and Integrals of the Hyperbolic Functions

Recall that the hyperbolic sine and hyperbolic cosine are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

and

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

The other hyperbolic functions are then defined in terms of $\sinh x$ and $\cosh x$. The graphs of the hyperbolic functions are shown in Figure 3.14.1.

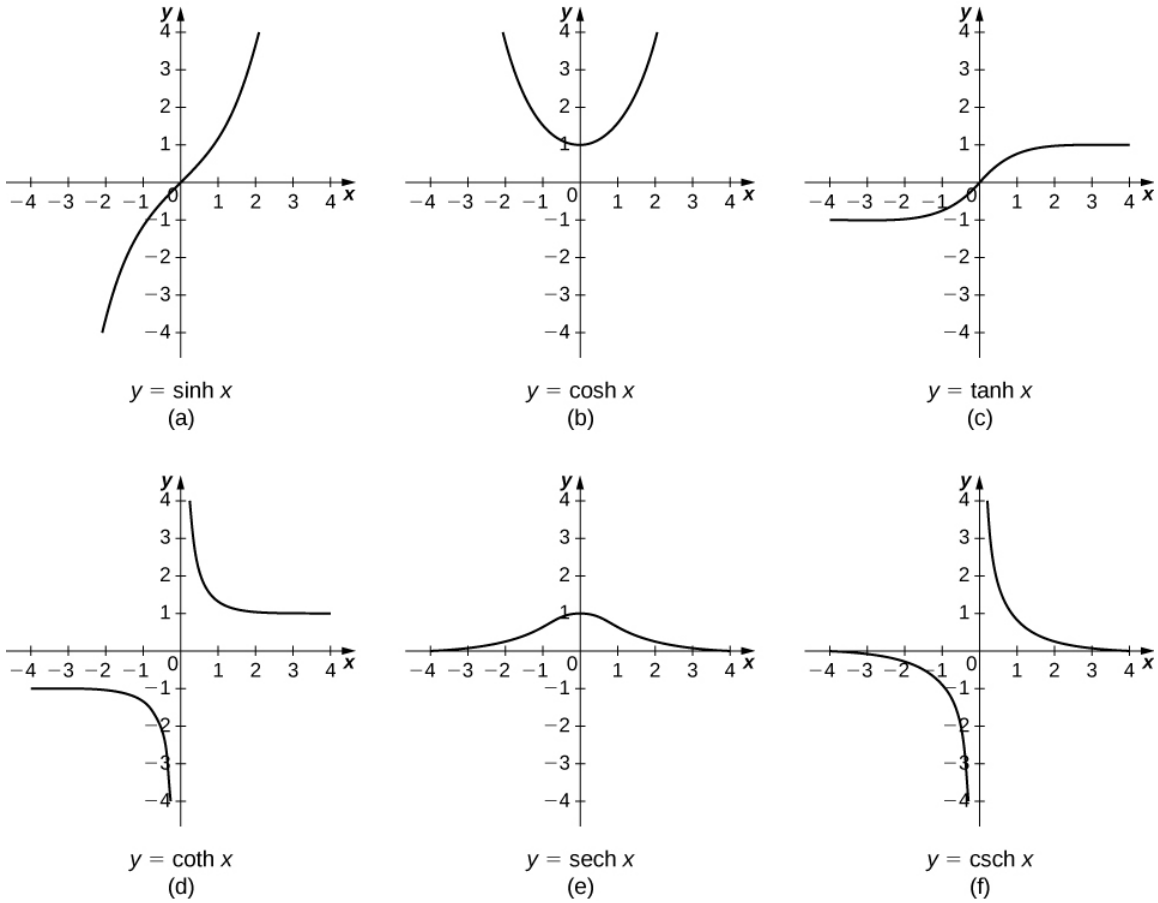


Figure 3.14.1: Graphs of the hyperbolic functions.

It is easy to develop differentiation formulas for the hyperbolic functions. For example, looking at $\sinh x$ we have

$$\begin{aligned}\frac{d}{dx}(\sinh x) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) \\ &= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right] \\ &= \frac{1}{2}[e^x + e^{-x}] \\ &= \cosh x.\end{aligned}$$

Similarly,

$$\frac{d}{dx}\cosh x = \sinh x.$$

We summarize the differentiation formulas for the hyperbolic functions in Table 3.14.1.

Table 3.14.1: Derivatives of the Hyperbolic Functions

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{csch}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$

Let's take a moment to compare the derivatives of the hyperbolic functions with the derivatives of the standard trigonometric functions. There are a lot of similarities, but differences as well. For example, the derivatives of the sine functions match:

$$\frac{d}{dx}\sin x = \cos x$$

and

$$\frac{d}{dx}\sinh x = \cosh x.$$

The derivatives of the cosine functions, however, differ in sign:

$$\frac{d}{dx}\cos x = -\sin x,$$

but

$$\frac{d}{dx}\cosh x = \sinh x.$$

As we continue our examination of the hyperbolic functions, we must be mindful of their similarities and differences to the standard trigonometric functions. These differentiation formulas for the hyperbolic functions lead directly to the following integral formulas.

$$\int \sinh u\, du = \cosh u + C \tag{3.14.1}$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C \tag{3.14.2}$$

$$\int \cosh u\, du = \sinh u + C \tag{3.14.3}$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech}\ u + C - \operatorname{csch} u + C \tag{3.14.4}$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C \tag{3.14.5}$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C \tag{3.14.6}$$

Example 3.14.1: Differentiating Hyperbolic Functions

Evaluate the following derivatives:

a. $\dfrac{d}{dx}(\sinh(x^2))$

b. $\dfrac{d}{dx}(\cosh x)^2$

Solution:

Using the formulas in Table 3.14.1 and the chain rule, we get

a. $\dfrac{d}{dx}(\sinh(x^2)) = \cosh(x^2) \cdot 2x$

b. $\dfrac{d}{dx}(\cosh x)^2 = 2\cosh x \sinh x$

Exercise 3.14.1

Evaluate the following derivatives:

a. $\dfrac{d}{dx}(\tanh(x^2+3x))$

b. $\dfrac{d}{dx}\left(\dfrac{1}{(\sinh x)^2}\right)$

Hint

Use the formulas in Table 3.14.1 and apply the chain rule as necessary.

Answer a

$$\frac{d}{dx}(\tanh(x^2+3x)) = (\operatorname{sech}^2(x^2+3x))(2x+3)$$

Answer b

$$\frac{d}{dx}\left(\frac{1}{(\sinh x)^2}\right) = \frac{d}{dx}(\sinh x)^{-2} = -2(\sinh x)^{-3}\cosh x$$

Example 3.14.2: Integrals Involving Hyperbolic Functions

Evaluate the following integrals:

a. $\displaystyle\int x\cosh(x^2)dx$

b. $\int \tanh x \, dx$

Solution

We can use u-substitution in both cases.

a. Let $u = x^2$. Then, $du = 2x \, dx$ and

$$\begin{aligned} \int x \cosh(x^2) dx &= \int \frac{1}{2} \cosh u \, du \\ &= \frac{1}{2} \sinh u + C \\ &= \frac{1}{2} \sinh(x^2) + C. \end{aligned}$$

b. Let $u = \cosh x$. Then, $du = \sinh x \, dx$ and

$$\begin{aligned} \int \tanh x \, dx &= \int \frac{\sinh x}{\cosh x} \, dx \\ &= \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |\cosh x| + C. \end{aligned}$$

Note that $\cosh x > 0$ for all x, so we can eliminate the absolute value signs and obtain

$$\int \tanh x \, dx = \ln(\cosh x) + C.$$

? Exercise 3.14.2

Evaluate the following integrals:

a. $\int \sinh^3 x \cosh x \, dx$

b. $\int \text{sech}^2(3x) \, dx$

Hint

Use the formulas above and apply u-substitution as necessary.

Answer a

$$\int \sinh^3 x \cosh x \, dx = \frac{\sinh^4 x}{4} + C$$

Answer b

$$\int \text{sech}^2(3x) \, dx = \frac{\tanh(3x)}{3} + C$$

Calculus of Inverse Hyperbolic Functions

Looking at the graphs of the hyperbolic functions, we see that with appropriate range restrictions, they all have inverses. Most of the necessary range restrictions can be discerned by close examination of the graphs. The domains and ranges of the inverse hyperbolic functions are summarized in Table 3.14.2

Table 3.14.2: Domains and Ranges of the Inverse Hyperbolic Functions

Function	Domain	Range

Function	Domain	Range
$\sinh^{-1} x$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\cosh^{-1} x$	$(1,\infty)$	$[0,\infty)$
$\tanh^{-1} x$	$(-1,1)$	$(-\infty,\infty)$
$\coth^{-1} x$	$(-\infty,1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$
$\operatorname{sech}^{-1} x$	$(0,1)$	$[0,\infty)$
$\operatorname{csch}^{-1} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$

The graphs of the inverse hyperbolic functions are shown in the following figure.

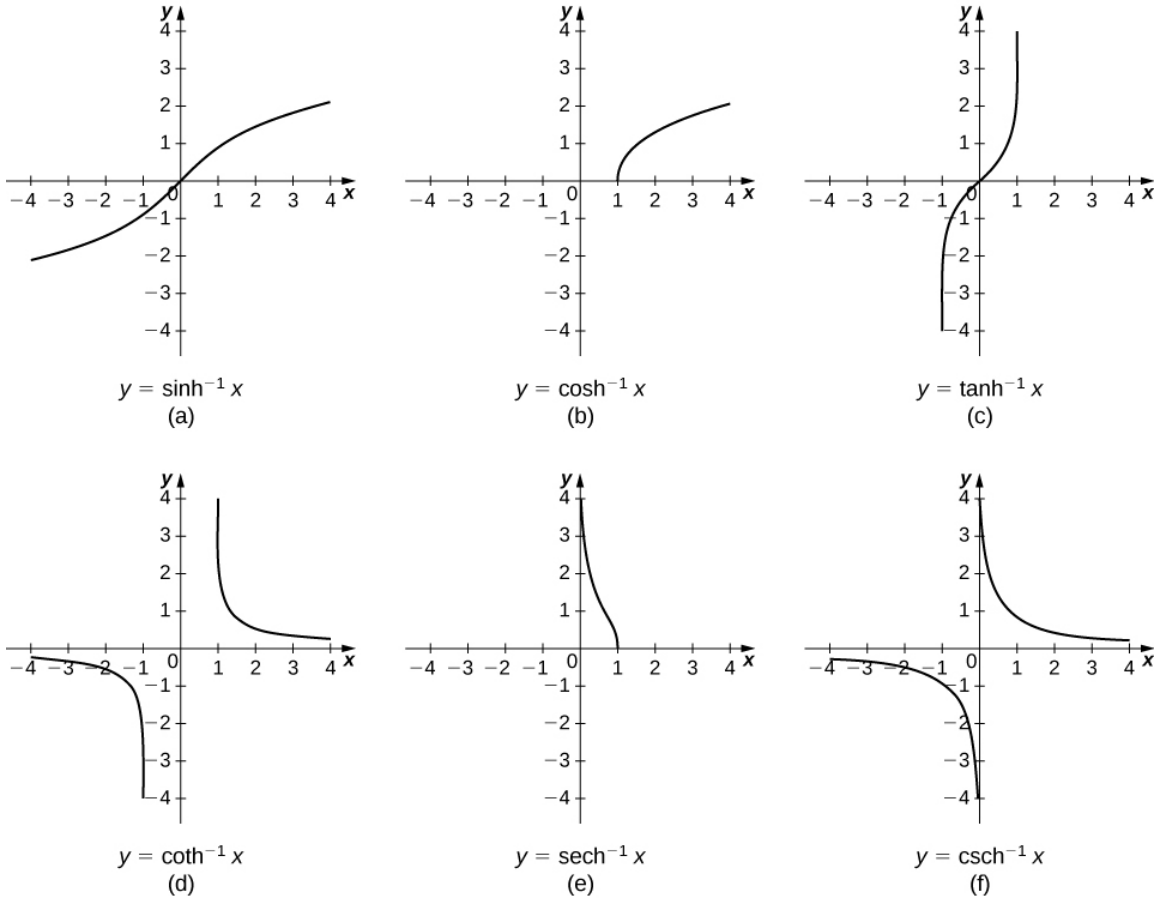


Figure 3.14.3: Graphs of the inverse hyperbolic functions.

To find the derivatives of the inverse functions, we use implicit differentiation. We have

$$y = \sinh^{-1} x \tag{3.14.7}$$

$$\sinh y = x \tag{3.14.8}$$

$$\frac{d}{dx}\sinh y = \frac{d}{dx}x \tag{3.14.9}$$

$$\cosh y \frac{dy}{dx} = 1. \tag{3.14.10}$$

Recall that $\cosh^2 y - \sinh^2 y = 1$, so $\cosh y = \sqrt{1 + \sinh^2 y}$. Then,

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

We can derive differentiation formulas for the other inverse hyperbolic functions in a similar fashion. These differentiation formulas are summarized in Table 3.14.3

Table 3.14.3: Derivatives of the Inverse Hyperbolic Functions

$f(x)$	$\frac{d}{dx}f(x)$
$\sinh^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$
$\coth^{-1} x$	$\frac{1}{1-x^2}$
$\text{sech}^{-1} x$	$\frac{-1}{x\sqrt{1-x^2}}$
$\text{csch}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Note that the derivatives of $\tanh^{-1} x$ and $\coth^{-1} x$ are the same. Thus, when we integrate $1/(1-x^2)$, we need to select the proper antiderivative based on the domain of the functions and the values of x. Integration formulas involving the inverse hyperbolic functions are summarized as follows.

$$\int \frac{1}{\sqrt{1+u^2}}du = \sinh^{-1} u + C$$

$$\int \frac{1}{u\sqrt{1-u^2}}du = -\text{sech}^{-1}|u| + C$$

$$\int \frac{1}{\sqrt{u^2-1}}du = \cosh^{-1} u + C$$

$$\int \frac{1}{u\sqrt{1+u^2}}du = -\text{csch}^{-1}|u| + C$$

$$\int \frac{1}{1-u^2}du = \begin{cases} \tanh^{-1} u + C & \text{if } |u| < 1 \\ \coth^{-1} u + C & \text{if } |u| > 1 \end{cases}$$

Example 3.14.3: Differentiating Inverse Hyperbolic Functions

Evaluate the following derivatives:

a. $\frac{d}{dx}\left(\sinh^{-1}\left(\frac{x}{3}\right)\right)$

b. $\frac{d}{dx}\left(\tanh^{-1} x\right)^2$

Solution

Using the formulas in Table 3.14.3 and the chain rule, we obtain the following results:

a. $\frac{d}{dx}(\sinh^{-1}(\frac{x}{3})) = \frac{1}{3\sqrt{1+\frac{x^2}{9}}} = \frac{1}{\sqrt{9+x^2}}$

b. $\dfrac{d}{dx}(\tanh^{-1} x)^2 = \dfrac{2(\tanh^{-1} x)}{1-x^2}$

Exercise 3.14.3

Evaluate the following derivatives:

a. $\dfrac{d}{dx}(\cosh^{-1}(3x))$

b. $\dfrac{d}{dx}(\coth^{-1} x)^3$

Hint

Use the formulas in Table 3.14.3and apply the chain rule as necessary.

Answer a

$$\frac{d}{dx}(\cosh^{-1}(3x)) = \frac{3}{\sqrt{9x^2-1}}$$

Answer b

$$\frac{d}{dx}(\coth^{-1} x)^3 = \frac{3(\coth^{-1} x)^2}{1-x^2}$$

Example 3.14.4: Integrals Involving Inverse Hyperbolic Functions

Evaluate the following integrals:

a. $\displaystyle\int \frac{1}{\sqrt{4x^2-1}}\,dx$

b. $\displaystyle\int \frac{1}{2x\sqrt{1-9x^2}}\,dx$

Solution

We can use ***u*-substitution** in both cases.

Let $u = 2x$. Then, $du = 2\,dx$ and we have

$$\begin{aligned}\int \frac{1}{\sqrt{4x^2-1}}\,dx &= \int \frac{1}{2\sqrt{u^2-1}}\,du \\ &= \frac{1}{2}\cosh^{-1} u + C \\ &= \frac{1}{2}\cosh^{-1}(2x) + C.\end{aligned}$$

Let $u = 3x$. Then, $du = 3\,dx$ and we obtain

$$\begin{aligned}\int \frac{1}{2x\sqrt{1-9x^2}}\,dx &= \frac{1}{2}\int \frac{1}{u\sqrt{1-u^2}}\,du \\ &= -\frac{1}{2}\operatorname{sech}^{-1}|u| + C \\ &= -\frac{1}{2}\operatorname{sech}^{-1}|3x| + C\end{aligned}$$

Exercise 3.14.4

Evaluate the following integrals:

a. $\int \frac{1}{\sqrt{x^2-4}}dx, x>2$

b. $\int \frac{1}{\sqrt{1-e^{2x}}}dx$

Hint

Use the formulas above and apply u-substitution as necessary.

Answer a

$$\int \frac{1}{\sqrt{x^2-4}}dx = \cosh^{-1}(\frac{x}{2})+C$$

Answer b

$$\int \frac{1}{\sqrt{1-e^{2x}}}dx = -\text{sech}^{-1}(e^x)+C$$

Applications

One physical application of hyperbolic functions involves hanging cables. If a cable of uniform density is suspended between two supports without any load other than its own weight, the cable forms a curve called a **catenary**. High-voltage power lines, chains hanging between two posts, and strands of a spider's web all form catenaries. The following figure shows chains hanging from a row of posts.



Figure 3.14.3: Chains between these posts take the shape of a catenary. (credit: modification of work by OKFoundryCompany, Flickr)

Hyperbolic functions can be used to model catenaries. Specifically, functions of the form $y = a \cdot \cosh(x/a)$ are catenaries. Figure 3.14.4shows the graph of $y = 2\cosh(x/2)$.

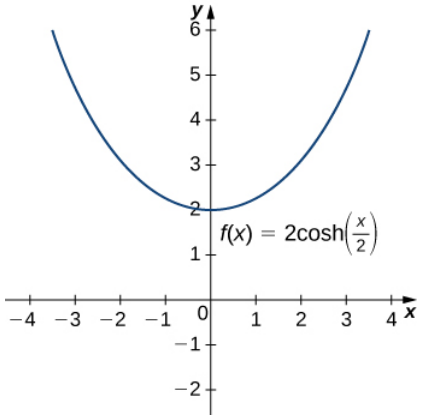


Figure 3.14.4: A hyperbolic cosine function forms the shape of a catenary.

Example 3.14.5: Using a Catenary to Find the Length of a Cable

Assume a hanging cable has the shape $10\cosh(x/10)$ for $-15 \le x \le 15$, where x is measured in feet. Determine the length of the cable (in feet).

Solution

Recall from Section 6.4 that the formula for arc length is

$$\underbrace{\int_a^b \sqrt{1+[f'(x)]^2}\,dx.}_{\text{Arc Length}}$$

We have $f(x) = 10\cosh(x/10)$, so $f'(x) = \sinh(x/10)$. Then the arc length is

$$\int_a^b \sqrt{1+[f'(x)]^2}\,dx = \int_{-15}^{15} \sqrt{1+\sinh^2\left(\frac{x}{10}\right)}\,dx.$$

Now recall that

$$1+\sinh^2 x = \cosh^2 x,$$

so we have

$$\begin{aligned}
\text{Arc Length} &= \int_{-15}^{15} \sqrt{1+\sinh^2\left(\frac{x}{10}\right)}\,dx \\
&= \int_{-15}^{15} \cosh\left(\frac{x}{10}\right)dx \\
&= 10\sinh\left(\frac{x}{10}\right)\Big|_{-15}^{15} \\
&= 10\left[\sinh\left(\frac{3}{2}\right) - \sinh\left(-\frac{3}{2}\right)\right] \\
&= 20\sinh\left(\frac{3}{2}\right) \\
&\approx 42.586 \text{ ft.}
\end{aligned}$$

Exercise 3.14.5:

Assume a hanging cable has the shape $15\cosh(x/15)$ for $-20 \le x \le 20$. Determine the length of the cable (in feet).

Answer

52.95ft

Key Concepts

- Hyperbolic functions are defined in terms of exponential functions.
- Term-by-term differentiation yields differentiation formulas for the hyperbolic functions. These differentiation formulas give rise, in turn, to integration formulas.
- With appropriate range restrictions, the hyperbolic functions all have inverses.
- Implicit differentiation yields differentiation formulas for the inverse hyperbolic functions, which in turn give rise to integration formulas.
- The most common physical applications of hyperbolic functions are calculations involving catenaries.

Glossary

catenary

a curve in the shape of the function $y = a \cdot \cosh(x/a)$ is a catenary; a cable of uniform density suspended between two supports assumes the shape of a catenary

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3.14E: Exercises for Section 6.9

1) [T] Find expressions for $\cosh x + \sinh x$ and $\cosh x - \sinh x$. Use a calculator to graph these functions and ensure your expression is correct.

Answer

e^x and e^{-x}

2) From the definitions of $\cosh(x)$ and $\sinh(x)$, find their antiderivatives.

3) Show that $\cosh(x)$ and $\sinh(x)$ satisfy $y'' = y$.

Answer

Answers may vary

4) Use the quotient rule to verify that $\frac{d}{dx}\left(\tanh(x)\right) = \operatorname{sech}^2(x)$.

5) Derive $\cosh^2(x) + \sinh^2(x) = \cosh(2x)$ from the definition.

Answer

Answers may vary

6) Take the derivative of the previous expression to find an expression for $\sinh(2x)$.

7) Prove $\sinh(x+y) = \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$ by changing the expression to exponentials.

Answer

Answers may vary

8) Take the derivative of the previous expression to find an expression for $\cosh(x+y)$.

In exercises 9 - 18, find the derivatives of the given functions and graph along with the function to ensure your answer is correct.

9) [T] $\cosh(3x+1)$

Answer

$3\sinh(3x+1)$

10) [T] $\sinh(x^2)$

11) [T] $\dfrac{1}{\cosh(x)}$

Answer

$-\tanh(x)\operatorname{sech}(x)$

12) [T] $\sinh(\ln(x))$

13) [T] $\cosh^2(x) + \sinh^2(x)$

Answer

$4\cosh(x)\sinh(x)$

14) [T] $\cosh^2(x) - \sinh^2(x)$

15) [T] $\tanh(\sqrt{x^2+1})$

Answer

$$\frac{x\operatorname{sech}^2(\sqrt{x^2+1})}{\sqrt{x^2+1}}$$

16) [T] $\dfrac{1+\tanh(x)}{1-\tanh(x)}$

17) [T] $\sinh^6(x)$

Answer

$6\sinh^5(x)\cosh(x)$

18) [T] $\ln(\operatorname{sech}(x)+\tanh(x))$

In exercises 19 - 28, find the antiderivatives for the given functions.

19) $\cosh(2x+1)$

Answer

$\frac{1}{2}\sinh(2x+1)+C$

20) $\tanh(3x+2)$

21) $x\cosh(x^2)$

Answer

$\frac{1}{2}\sinh^2(x^2)+C$

22) $3x^3\tanh(x^4)$

23) $\cosh^2(x)\sinh(x)$

Answer

$\frac{1}{3}\cosh^3(x)+C$

24) $\tanh^2(x)\operatorname{sech}^2(x)$

25) $\dfrac{\sinh(x)}{1+\cosh(x)}$

Answer

$\ln(1+\cosh(x))+C$

26) $\coth(x)$

27) $\cosh(x)+\sinh(x)$

Answer

$\cosh(x)+\sinh(x)+C$

28) $(\cosh(x)+\sinh(x))^n$

In exercises 29 - 35, find the derivatives for the functions.

29) $\tanh^{-1}(4x)$

Answer

$\dfrac{4}{1-16x^2}$

30) $\sinh^{-1}(x^2)$

31) $\sinh^{-1}(\cosh(x))$

Answer

$$\frac{\sinh(x)}{\sqrt{\cosh^2(x)+1}}$$

32) $\cosh^{-1}(x^3)$

33) $\tanh^{-1}(\cos(x))$

Answer

$$-\csc(x)$$

34) $e^{\sinh^{-1}(x)}$

35) $\ln(\tanh^{-1}(x))$

Answer

$$-\frac{1}{(x^2-1)\tanh^{-1}(x)}$$

In exercises 36 - 42, find the antiderivatives for the functions.

36) $\int \frac{dx}{4-x^2}$

37) $\int \frac{dx}{a^2-x^2}$

Answer

$$\frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C$$

38) $\int \frac{dx}{\sqrt{x^2+1}}$

39) $\int \frac{x\,dx}{\sqrt{x^2+1}}$

Answer

$$\sqrt{x^2+1}+C$$

40) $\int -\frac{dx}{x\sqrt{1-x^2}}$

41) $\int \frac{e^x}{\sqrt{e^{2x}-1}}$

Answer

$$\cosh^{-1}(e^x)+C$$

42) $\int -\frac{2x}{x^4-1}$

In exercises 43 - 45, use the fact that a falling body with friction equal to velocity squared obeys the equation $\frac{dv}{dt}=g-v^2$.

43) Show that $v(t)=\sqrt{g}\tanh(\sqrt{g}t)$ satisfies this equation.

Answer

Answers may vary

44) Derive the previous expression for $v(t)$ by integrating $\dfrac{dv}{g-v^2}=dt$.

45) [T] Estimate how far a body has fallen in 12seconds by finding the area underneath the curve of $v(t)$.

Answer

37.30

In exercises 46 - 48, use this scenario: A cable hanging under its own weight has a slope $S=\dfrac{dy}{dx}$ that satisfies $\dfrac{dS}{dx}=c\sqrt{1+S^2}$. The constant c is the ratio of cable density to tension.

46) Show that $S=\sinh(cx)$ satisfies this equation.

47) Integrate $\dfrac{dy}{dx}=\sinh(cx)$ to find the cable height $y(x)$ if $y(0)=1/c$.

Answer

$y=\frac{1}{c}\cosh(cx)$

48) Sketch the cable and determine how far down it sags at $x=0$.

In exercises 49 - 52, solve each problem.

49) [T] A chain hangs from two posts 2m apart to form a catenary described by the equation $y=2\cosh(x/2)-1$. Find the slope of the catenary at the left fence post.

Answer

-0.521095

50) [T] A chain hangs from two posts four meters apart to form a catenary described by the equation $y=4\cosh(x/4)-3$. Find the total length of the catenary (arc length).

51) [T] A high-voltage power line is a catenary described by $y=10\cosh(x/10)$. Find the ratio of the area under the catenary to its arc length. What do you notice?

Answer

10

52) A telephone line is a catenary described by $y=a\cosh(x/a)$. Find the ratio of the area under the catenary to its arc length. Does this confirm your answer for the previous question?

53) Prove the formula for the derivative of $y=\sinh^{-1}(x)$ by differentiating $x=\sinh(y)$.

(Hint: Use hyperbolic trigonometric identities.)

54) Prove the formula for the derivative of $y=\cosh^{-1}(x)$ by differentiating $x=\cosh(y)$.

(Hint: Use hyperbolic trigonometric identities.)

55) Prove the formula for the derivative of $y=\operatorname{sech}^{-1}(x)$ by differentiating $x=\operatorname{sech}(y)$.

(Hint: Use hyperbolic trigonometric identities.)

56) Prove that $\cosh(x)+\sinh(x))^n=\cosh(nx)+\sinh(nx)$.

57) Prove the expression for $\sinh^{-1}(x)$. Multiply $x=\sinh(y)=\dfrac{e^y-e^{-y}}{2}$ by $2e^y$ and solve for y. Does your expression match the textbook?

58) Prove the expression for $\cosh^{-1}(x)$. Multiply $x = \cosh(y) = \dfrac{e^y + e^{-y}}{2}$ by $2e^y$ and solve for y. Does your expression match the textbook?

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3.15: Chapter 6 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1) The amount of work to pump the water out of a half-full cylinder is half the amount of work to pump the water out of the full cylinder.

Answer

False

2) If the force is constant, the amount of work to move an object from $x = a$ to $x = b$ is $F(b - a)$.

3) The disk method can be used in any situation in which the washer method is successful at finding the volume of a solid of revolution.

Answer

False

4) If the half-life of $seaborgium - 266$ is 360 ms, then $k = \dfrac{\ln 2}{360}$.

For exercises 5 - 8, use the requested method to determine the volume of the solid.

5) The volume that has a base of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ and cross-sections of an equilateral triangle perpendicular to the y-axis. Use the method of slicing.

Answer

$V = 32\sqrt{3}\,\text{units}^3$

6) $y = x^2 - x$, from $x = 1$ to $x = 4$, rotated around the y-axis using the washer method

7) $x = y^2$ and $x = 3y$ rotated around the y-axis using the washer method

Answer

$V = \frac{162\pi}{5}\ \text{units}^3$

8) $x = 2y^2 - y^3,\ x = 0$,and $y = 0$ rotated around the x-axis using cylindrical shells

For exercises 9 - 14, find

a. the area of the region,

b.the volume of the solid when rotated around the x-axis, and

c. the volume of the solid when rotated around the y-axis. Use whichever method seems most appropriate to you.

9) $y = x^3, x = 0, y = 0$, and $x = 2$

Answer

a. $A = 4\ \text{units}^2$
b. $V = \frac{128\pi}{7}\ \text{units}^3$
c. $V = \frac{64\pi}{5}\ \text{units}^3$

10) $y = x^2 - x$ and $x = 0$

11) [T] $y = \ln(x) + 2$ and $y = x$

Answer

a. $A \approx 1.949\,\text{units}^2$
b. $V \approx 21.952\,\text{units}^3$

c. $V = \approx 17.099$ units^3

12) $y = x^2$ and $y = \sqrt{x}$

13) $y = 5 + x, y = x^2, x = 0$, and $x = 1$

Answer

a. $A = \frac{31}{6}$ units^2
b. $V = \frac{452\pi}{15}$ units^3
c. $V = \frac{31\pi}{6}$ units^3

14) Below $x^2 + y^2 = 1$ and above $y = 1 - x$

15) Find the mass of $\rho = e^{-x}$ on a disk centered at the origin with radius 4.

Answer

$m \approx 245.282$

16) Find the center of mass for $\rho = \tan^2 x$ on $x \in (-\frac{\pi}{4}, \frac{\pi}{4})$.

17) Find the mass and the center of mass of $\rho = 1$ on the region bounded by $y = x^5$ and $y = \sqrt{x}$.

Answer

Mass: $\frac{1}{2}$,
Center of mass: $(\frac{18}{35}, \frac{9}{11})$

For exercises 18 - 19, find the requested arc lengths.

18) The length of x for $y = \cosh(x)$ from $x = 0$ to $x = 2$.

19) The length of y for $x = 3 - \sqrt{y}$ from $y = 0$ to $y = 4$

Answer

$s = \left[\sqrt{17} + \frac{1}{8}\ln(33 + 8\sqrt{17})\right]$ units

For exercises 20 - 21, find the surface area and volume when the given curves are revolved around the specified axis.

20) The shape created by revolving the region between $y = 4 + x$, $y = 3 - x$, $x = 0$, and $x = 2$ rotated around the y-axis.

21) The loudspeaker created by revolving $y = \dfrac{1}{x}$ from $x = 1$ to $x = 4$ around the x-axis.

Answer

Volume: $V = \frac{3\pi}{4}$ units^3
Surface area: $A = \pi\left(\sqrt{2} - \sinh^{-1}(1) + \sinh^{-1}(16) - \frac{\sqrt{257}}{16}\right)$ units^2

For exercise 22, consider the Karun-3 dam in Iran. Its shape can be approximated as an isosceles triangle with height 205 m and width 388 m. Assume the current depth of the water is 180 m. The density of water is 1000 kg/m^3.

22) Find the total force on the wall of the dam.

23) You are a crime scene investigator attempting to determine the time of death of a victim. It is noon and 45 °F outside and the temperature of the body is 78 °F. You know the cooling constant is $k = 0.00824$ °F/min. When did the victim die, assuming that a human's temperature is 98 °F?

Answer

11:02 a.m.

For the following exercise, consider the stock market crash in 1929 in the United States. The table lists the Dow Jones industrial average per year leading up to the crash.

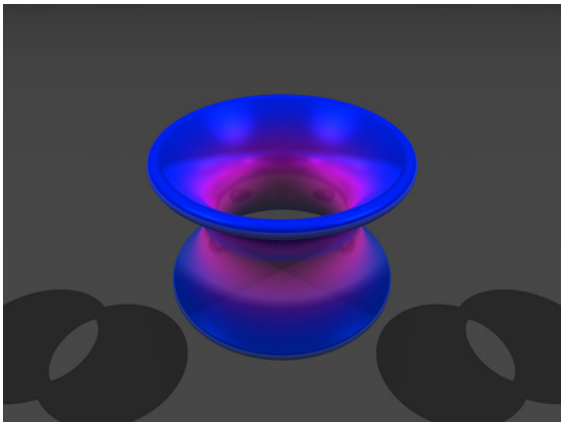
Year after 1920	Value ($)
1	63.90
3	100
5	110
7	160
9	381.17

Source: http:/stockcharts.com/freecharts/hi...a19201940.html

24) [T] The best-fit exponential curve to these data is given by $y = 40.71 + 1.224^x$. Why do you think the gains of the market were unsustainable? Use first and second derivatives to help justify your answer. What would this model predict the Dow Jones industrial average to be in **2014** ?

For exercises 25 - 26, consider the catenoid, the only solid of revolution that has a minimal surface, or zero mean curvature. A catenoid in nature can be found when stretching soap between two rings.

25) Find the volume of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-**axis,** as shown here.



Answer

$V = \pi\big(1 + \sinh(1)\cosh(1)\big)$ units^3

26) Find surface area of the catenoid $y = \cosh(x)$ from $x = -1$ to $x = 1$ that is created by rotating this curve around the x-**axis.**

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CHAPTER OVERVIEW

4: Techniques of Integration

We saw in the previous chapter how important integration can be for all kinds of different topics—from calculations of volumes to flow rates, and from using a velocity function to determine a position to locating centers of mass. It is no surprise, then, that techniques for finding antiderivatives (or indefinite integrals) are important to know for everyone who uses them. We have already discussed some basic integration formulas and the method of integration by substitution. In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

4.1: Prelude to Techniques of Integration
4.2: Integration by Parts
4.2E: Exercises for Integration by Parts
4.3: Trigonometric Integrals
4.3E: Exercises for Trigonometric Integrals
4.4: Trigonometric Substitution
4.4E: Exercises for Trigonometric Substitution
4.5: Partial Fractions
4.5E: Exercises for Integration by Partial Fractions
4.6: Other Strategies for Integration
4.6E: Exercises for Section 7.5
4.7: Numerical Integration
4.7E: Exercises for Numerical Integration
4.8: L'Hôpital's Rule
4.8E: Exercises for L'Hôpital's Rule
4.9: Improper Integrals
4.9E: Exercises for Improper Integrals
4.10: Chapter 7 Review Exercises

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4.1: Prelude to Techniques of Integration

In a large city, accidents occurred at an average rate of one every three months at a particularly busy intersection. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the eight-month interval without an accident a result of chance? We explore this question later in this chapter and see that integration is an essential part of determining the answer.



Figure 4.1.1: Careful planning of traffic signals can prevent or reduce the number of accidents at busy intersections. (credit: modification of work by David McKelvey, Flickr)

We saw in the previous chapter how important integration can be for all kinds of different topics—from calculations of volumes to flow rates, and from using a velocity function to determine a position to locating centers of mass. It is no surprise, then, that techniques for finding antiderivatives (or indefinite integrals) are important to know for everyone who uses them. We have already discussed some basic integration formulas and the method of integration by substitution. In this chapter, we study some additional techniques, including some ways of approximating definite integrals when normal techniques do not work.

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4.2: Integration by Parts

Learning Objectives

- Recognize when to use integration by parts.
- Use the integration-by-parts formula to solve integration problems.
- Use the integration-by-parts formula for definite integrals.

By now we have a fairly thorough procedure for how to evaluate many basic integrals. However, although we can integrate $\int x\sin(x^2)\,dx$ by using the substitution, $u=x^2$, something as simple looking as $\int x\sin x\;dx$ defies us. Many students want to know whether there is a product rule for integration. There is not, but there is a technique based on the product rule for differentiation that allows us to exchange one integral for another. We call this technique **integration by parts.**

The Integration-by-Parts Formula

If, $h(x)=f(x)g(x)$, then by using the product rule, we obtain

$$h'(x)=f'(x)g(x)+g'(x)f(x). \tag{4.2.1}$$

Although at first it may seem counterproductive, let's now integrate both sides of Equation 4.2.1:

$$\int h'(x)\;dx=\int (g(x)f'(x)+f(x)g'(x))\;dx.$$

This gives us

$$h(x)=f(x)g(x)=\int g(x)f'(x)\,dx+\int f(x)g'(x)\;dx.$$

Now we solve for $\int f(x)g'(x)\;dx$:

$$\int f(x)g'(x)\,dx=f(x)g(x)-\int g(x)f'(x)\;dx.$$

By making the substitutions $u=f(x)$ and $v=g(x)$, which in turn make $du=f'(x)\,dx$ and $dv=g'(x)\,dx$, we have the more compact form

$$\int u\,dv=uv-\int v\,du.$$

Integration by Parts

Let $u=f(x)$ and $v=g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int u\,dv=uv-\int v\,du. \tag{4.2.2}$$

The advantage of using the integration-by-parts formula is that we can use it to exchange one integral for another, possibly easier, integral. The following example illustrates its use.

Example 4.2.1: Using Integration by Parts

Use integration by parts with $u=x$ and $dv=\sin x\;dx$ to evaluate

$$\int x\sin x\;dx.$$

Solution

By choosing $u=x$, we have $du=1\;dx$. Since $dv=\sin x\;dx$, we get

$$v = \int \sin x \; dx = -\cos x.$$

It is handy to keep track of these values as follows:

- $u = x$
- $dv = \sin x \; dx$
- $du = 1 \, dx$
- $v = \int \sin x \; dx = -\cos x.$

Applying the integration-by-parts formula (Equation 4.2.2) results in

$$\begin{aligned} \int x \sin x \; dx &= (x)(-\cos x) - \int (-\cos x)(1 \; dx) && \text{(Substitute)} \\ &= -x \cos x + \int \cos x \; dx && \text{(Simplify)} \end{aligned}$$

Then use

$$\int \cos x \; dx = \sin x + C.$$

to obtain

$$\int x \sin x \; dx = -x \cos x + \sin x + C.$$

Analysis

At this point, there are probably a few items that need clarification. First of all, you may be curious about what would have happened if we had chosen $u = \sin x$ and $dv = x \, dx$. If we had done so, then we would have $du = \cos x$ and $v = \frac{1}{2}x^2$. Thus, after applying integration by parts (Equation 4.2.2), we have

$$\int x \sin x \; dx = \frac{1}{2}x^2 \sin x - \int \frac{1}{2}x^2 \cos x \; dx.$$

Unfortunately, with the new integral, we are in no better position than before. It is important to keep in mind that when we apply integration by parts, we may need to try several choices for u and dv before finding a choice that works.

Second, you may wonder why, when we find $v = \int \sin x \; dx = -\cos x$, we do not use $v = -\cos x + K$. To see that it makes no difference, we can rework the problem using $v = -\cos x + K$:

$$\begin{aligned} \int x \sin x \; dx &= (x)(-\cos x + K) - \int (-\cos x + K)(1 \; dx) \\ &= -x \cos x + Kx + \int \cos x \; dx - \int K \; dx \\ &= -x \cos x + Kx + \sin x - Kx + C \\ &= -x \cos x + \sin x + C. \end{aligned}$$

As you can see, it makes no difference in the final solution.

Last, we can check to make sure that our antiderivative is correct by differentiating $-x \cos x + \sin x + C$:

$$\begin{aligned} \frac{d}{dx}(-x \cos x + \sin x + C) &= \cancel{(-1)\cos x} + (-x)(-\sin x) + \cancel{\cos x} \\ &= x \sin x \end{aligned}$$

Therefore, the antiderivative checks out.

? Exercise 4.2.1

Evaluate $\int xe^{2x}\,dx$ using the integration-by-parts formula (Equation 4.2.2) with $u = x$ and $dv = e^{2x}\,dx$.

Hint

Find du and v, and use the previous example as a guide.

Answer

$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$$

The natural question to ask at this point is: How do we know how to choose u and dv? Sometimes it is a matter of trial and error; however, the acronym **LIATE** can often help to take some of the guesswork out of our choices. This acronym stands for **L**ogarithmic Functions, **I**nverse Trigonometric Functions, **A**lgebraic Functions, **T**rigonometric Functions, and **E**xponential Functions. This mnemonic serves as an aid in determining an appropriate choice for u. The type of function in the integral that appears first in the list should be our first choice of u.

For example, if an integral contains a **logarithmic function** and an **algebraic function,** we should choose u to be the logarithmic function, because L comes before A in LIATE. The integral in Example $4.2.1$ has a trigonometric function ($\sin x$) and an algebraic function (x). Because A comes before T in LIATE, we chose u to be the algebraic function. When we have chosen u, dv is selected to be the remaining part of the function to be integrated, together with dx.

Why does this mnemonic work? Remember that whatever we pick to be dv must be something we can integrate. Since we do not have integration formulas that allow us to integrate simple logarithmic functions and inverse trigonometric functions, it makes sense that they should not be chosen as values for dv. Consequently, they should be at the head of the list as choices for u. Thus, we put LI at the beginning of the mnemonic. (We could just as easily have started with IL, since these two types of functions won't appear together in an integration-by-parts problem.) The exponential and trigonometric functions are at the end of our list because they are fairly easy to integrate and make good choices for dv. Thus, we have TE at the end of our mnemonic. (We could just as easily have used ET at the end, since when these types of functions appear together it usually doesn't really matter which one is u and which one is dv.) Algebraic functions are generally easy both to integrate and to differentiate, and they come in the middle of the mnemonic.

✓ Example 4.2.2: Using Integration by Parts

Evaluate

$$\int \frac{\ln x}{x^3}\,dx.$$

Solution

Begin by rewriting the integral:

$$\int \frac{\ln x}{x^3}\,dx = \int x^{-3}\ln x\,dx.$$

Since this integral contains the algebraic function x^{-3} and the logarithmic function $\ln x$, choose $u = \ln x$, since L comes before A in LIATE. After we have chosen $u = \ln x$, we must choose $dv = x^{-3}\,dx$.

Next, since $u = \ln x$, we have $du = \frac{1}{x}\,dx$. Also, $v = \int x^{-3}\,dx = -\frac{1}{2}x^{-2}$. Summarizing,

- $u = \ln x$
- $du = \frac{1}{x}\,dx$
- $dv = x^{-3}\,dx$
- $v = \int x^{-3}\,dx = -\frac{1}{2}x^{-2}$.

Substituting into the integration-by-parts formula (Equation 4.2.2) gives

$$\begin{aligned}\int \frac{\ln x}{x^3}\,dx &= \int x^{-3}\ln x\,dx = (\ln x)(-\frac{1}{2}x^{-2}) - \int (-\frac{1}{2}x^{-2})(\frac{1}{x}\,dx)\\ &= -\frac{1}{2}x^{-2}\ln x + \int \frac{1}{2}x^{-3}\,dx\\ &= -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C\\ &= -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C\end{aligned}$$

? Exercise 4.2.2

Evaluate

$$\int x\ln x\,\,dx.$$

Hint

Use $u = \ln x$ and $dv = x\,\,dx$.

Answer

$$\int x\ln x\,\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

In some cases, as in the next two examples, it may be necessary to apply integration by parts more than once.

✓ Example 4.2.3A: Applying Integration by Parts More Than Once

Evaluate

$$\int x^2 e^{3x}\,dx.$$

Solution

Using LIATE, choose $u = x^2$ and $dv = e^{3x}\,dx$. Thus, $du = 2x\,dx$ and $v = \int e^{3x}\,dx = \left(\frac{1}{3}\right)e^{3x}$. Therefore,

- $u = x^2$
- $du = 2x\,dx$
- $dv = e^{3x}\,dx$
- $v = \int e^{3x}\,dx = \frac{1}{3}e^{3x}$.

Substituting into Equation 4.2.2 produces

$$\int x^2 e^{3x}\,dx = \frac{1}{3}x^2 e^{3x} - \int \frac{2}{3}xe^{3x}\,dx. \tag{4.2.3}$$

We still cannot integrate $\int \frac{2}{3}xe^{3x}\,dx$ directly, but the integral now has a lower power on x. We can evaluate this new integral by using integration by parts again. To do this, choose

$$u = x$$

and

$$dv = \frac{2}{3}e^{3x}\,dx.$$

Thus,

$$du = dx$$

and

$$v = \int \left(\frac{2}{3}\right) e^{3x}\, dx = \left(\frac{2}{9}\right) e^{3x}.$$

Now we have

- $u = x$
- $du = dx$
- $dv = \frac{2}{3} e^{3x}\, dx$
- $v = \int \frac{2}{3} e^{3x}\, dx = \frac{2}{9} e^{3x}$.

Substituting back into Equation 4.2.3 yields

$$\int x^2 e^{3x}\, dx = \frac{1}{3} x^2 e^{3x} - \left(\frac{2}{9} x e^{3x} - \int \frac{2}{9} e^{3x}\, dx\right).$$

After evaluating the last integral and simplifying, we obtain

$$\int x^2 e^{3x}\, dx = \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C.$$

Example 4.2.3B: Applying Integration by Parts When LIATE Does not Quite Work

Evaluate

$$\int t^3 e^{t^2}\, dt.$$

Solution

If we use a strict interpretation of the mnemonic LIATE to make our choice of u, we end up with $u = t^3$ and $dv = e^{t^2}\, dt$. Unfortunately, this choice won't work because we are unable to evaluate $\int e^{t^2}\, dt$. However, since we can evaluate $\int t e^{t^2}\, dx$, we can try choosing $u = t^2$ and $dv = t e^{t^2}\, dt$. With these choices we have

- $u = t^2$
- $du = 2t dt$
- $dv = t e^{t^2}\, dt$
- $v = \int t e^{t^2}\, dt = \frac{1}{2} e^{t^2}$.

Thus, we obtain

$$\begin{aligned}\int t^3 e^{t^2}\, dt &= \frac{1}{2} t^2 e^{t^2} - \int \frac{1}{2} e^{t^2} 2t\, dt \\ &= \frac{1}{2} t^2 e^{t^2} - \frac{1}{2} e^{t^2} + C.\end{aligned}$$

Example 4.2.3C: Applying Integration by Parts More Than Once

Evaluate

$$\int \sin(\ln x)\, dx.$$

Solution

This integral appears to have only one function—namely, $\sin(\ln x)$—however, we can always use the constant function 1 as the other function. In this example, let's choose $u = \sin(\ln x)$ and $dv = 1\, dx$. (The decision to use $u = \sin(\ln x)$ is easy. We

can't choose $dv = \sin(\ln x)\,dx$ because if we could integrate it, we wouldn't be using integration by parts in the first place!) Consequently, $du = (1/x)\cos(\ln x)\,dx$ and $v = \int 1\,dx = x$. After applying integration by parts to the integral and simplifying, we have

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx.$$

Unfortunately, this process leaves us with a new integral that is very similar to the original. However, let's see what happens when we apply integration by parts again. This time let's choose $u = \cos(\ln x)$ and $dv = 1\,dx$, making $du = -(1/x)\sin(\ln x)\,dx$ and $v = \int 1\,dx = x$.

Substituting, we have

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - (x\cos(\ln x) - \int -\sin(\ln x)\,dx).$$

After simplifying, we obtain

$$\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx.$$

The last integral is now the same as the original. It may seem that we have simply gone in a circle, but now we can actually evaluate the integral. To see how to do this more clearly, substitute $I = \int \sin(\ln x)\,dx$. Thus, the equation becomes

$$I = x\sin(\ln x) - x\cos(\ln x) - I.$$

First, add I to both sides of the equation to obtain

$$2I = x\sin(\ln x) - x\cos(\ln x).$$

Next, divide by 2:

$$I = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

Substituting $I = \int \sin(\ln x)\,dx$ again, we have

$$\int \sin(\ln x)\,dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x).$$

From this we see that $(1/2)x\sin(\ln x) - (1/2)x\cos(\ln x)$ is an antiderivative of $\sin(\ln x)\,dx$. For the most general antiderivative, add $+C$:

$$\int \sin(\ln x)\,dx = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C.$$

Analysis

If this method feels a little strange at first, we can check the answer by differentiation:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x)\right) &= \frac{1}{2}(\sin(\ln x)) + \cos(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x - \left(\frac{1}{2}\cos(\ln x) - \sin(\ln x)\cdot\frac{1}{x}\cdot\frac{1}{2}x\right)\\ &= \sin(\ln x).\end{aligned}$$

? Exercise 4.2.3

Evaluate

$$\int x^2 \sin x\,dx.$$

Hint

This is similar to Examples $4.2.3A$- $4.2.3C$.

Answer

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

Integration by Parts for Definite Integrals

Now that we have used integration by parts successfully to evaluate *indefinite integrals*, we turn our attention to definite integrals. The integration technique is really the same, only we add a step to evaluate the integral at the upper and lower limits of integration.

Integration by Parts for Definite Integrals

Let $u = f(x)$ and $v = g(x)$ be functions with continuous derivatives on $[a, b]$. Then

$$\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$$

Example $4.2.4A$: Finding the Area of a Region

Find the area of the region bounded above by the graph of $y = \tan^{-1} x$ and below by the x-axis over the interval $[0, 1]$.

Solution

This region is shown in Figure $4.2.1$. To find the area, we must evaluate

$$\int_0^1 \tan^{-1} x \; dx.$$

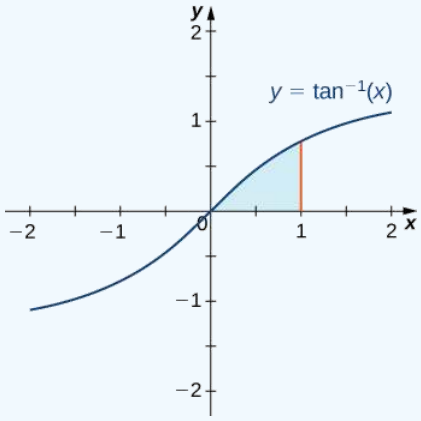


Figure 4.2.1: To find the area of the shaded region, we have to use integration by parts.

For this integral, let's choose $u = tan^{-1} x$ and $dv = dx$, thereby making $du = \dfrac{1}{x^2+1} dx$ and $v = x$. After applying the integration-by-parts formula (Equation 4.2.2) we obtain

$$\text{Area} = x \tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{x^2+1} \, dx.$$

Use u-substitution to obtain

$$\int_0^1 \frac{x}{x^2+1} \, dx = \frac{1}{2} \ln\left(x^2+1\right)\Big|_0^1.$$

Thus,

$$\text{Area} = x\tan^{-1}x\Big|_0^1 - \frac{1}{2}\ln(x^2+1)\Big|_0^1 = \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) \text{ units}^2.$$

At this point it might not be a bad idea to do a "reality check" on the reasonableness of our solution. Since $\frac{\pi}{4} - \frac{1}{2}\ln 2 \approx 0.4388 \text{ units}^2$, and from Figure 4.2.1 we expect our area to be slightly less than 0.5 units^2, this solution appears to be reasonable.

Example 4.2.4B: Finding a Volume of Revolution

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = e^{-x}$, the x-axis, the y-axis, and the line $x = 1$ about the y-axis.

Solution

The best option to solving this problem is to use the shell method. Begin by sketching the region to be revolved, along with a typical rectangle (Figure 4.2.2).

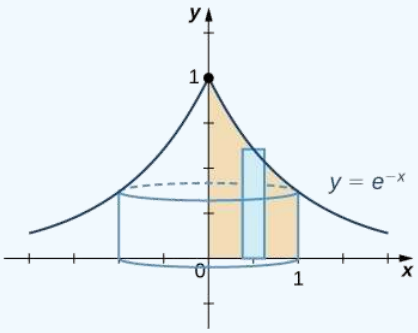


Figure 4.2.2: We can use the shell method to find a volume of revolution.

To find the volume using shells, we must evaluate

$$2\pi \int_0^1 xe^{-x}\,dx. \tag{4.2.4}$$

To do this, let $u = x$ and $dv = e^{-x}$. These choices lead to $du = dx$ and $v = \int e^{-x}\,dx = -e^{-x}$. Using the Shell Method formula, we obtain

$$\begin{aligned}
\text{Volume} &= 2\pi \int_0^1 xe^{-x}\,dx \\
&= 2\pi\left(-xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx\right) && \text{(Use integration by parts)} \\
&= 2\pi\left(-e^{-1} + 0 - e^{-x}\Big|_0^1\right) \\
&= 2\pi\left(-e^{-1} - e^{-1} + 1\right) \\
&= 2\pi\left(1 - \frac{2}{e}\right) \text{ units}^3. && \text{(Evaluate and simplify)}
\end{aligned}$$

Analysis

Again, it is a good idea to check the reasonableness of our solution. We observe that the solid has a volume slightly less than that of a cylinder of radius 1 and height of $1/e$ added to the volume of a cone of base radius 1 and height of $1 - \frac{1}{e}$. Consequently, the solid should have a volume a bit less than

$$\pi(1)^2\frac{1}{e} + \left(\frac{\pi}{3}\right)(1)^2\left(1 - \frac{1}{e}\right) = \frac{2\pi}{3e} + \frac{\pi}{3} \approx 1.8177 \text{ units}^3.$$

Since $2\pi - \frac{4\pi}{e} \approx 1.6603$, we see that our calculated volume is reasonable.

? Exercise 4.2.4

Evaluate

$$\int_0^{\pi/2} x \cos x \, dx.$$

Hint

Use Equation 4.2.2with $u = x$ and $dv = \cos x \, dx$.

Answer

$$\int_0^{\pi/2} x \cos x \, dx = \frac{\pi}{2} - 1$$

Key Concepts

- The integration-by-parts formula (Equation 4.2.2) allows the exchange of one integral for another, possibly easier, integral.
- Integration by parts applies to both definite and indefinite integrals.

Key Equations

- **Integration by parts formula**

  $$\int u \, dv = uv - \int v \, du$$

- **Integration by parts for definite integrals**

  $$\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$$

Glossary

integration by parts

a technique of integration that allows the exchange of one integral for another using the formula $\int u \, dv = uv - \int v \, du$

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4.2E: Exercises for Integration by Parts

In using the technique of integration by parts, you must carefully choose which expression is u. For each of the following problems, use the guidelines in this section to choose u. Do not evaluate the integrals.

1) $\int x^3 e^{2x}\,dx$

Answer

$u = x^3$

2) $\int x^3 \ln(x)\,dx$

3) $\int y^3 \cos y\,dy$

Answer

$u = y^3$

4) $\int x^2 \arctan x\,dx$

5) $\int e^{3x} \sin(2x)\,dx$

Answer

$u = \sin(2x)$

In exercises 6 - 37, find the integral by using the simplest method. Not all problems require integration by parts.

6) $\int v \sin v\,dv$

7) $\int \ln x\,dx$ (Hint: $\int \ln x\,dx$ is equivalent to $\int 1 \cdot \ln(x)\,dx$.)

Answer

$$\int \ln x\,dx \quad = \quad -x + x\ln x + C$$

8) $\int x \cos x\,dx$

9) $\int \tan^{-1} x\,dx$

Answer

$$\int \tan^{-1} x\,dx \quad = \quad x\tan^{-1} x - \tfrac{1}{2}\ln(1+x^2) + C$$

10) $\int x^2 e^x\,dx$

11) $\int x \sin(2x)\,dx$

Answer

$$\int x \sin(2x)\,dx \quad = \quad -\tfrac{1}{2}x\cos(2x) + \tfrac{1}{4}\sin(2x) + C$$

12) $\int xe^{4x}\,dx$

13) $\int xe^{-x}\,dx$

Answer

$$\int xe^{-x}\,dx \quad = \quad e^{-x}(-1-x)+C$$

14) $\int x\cos 3x\,dx$

15) $\int x^2\cos x\,dx$

Answer

$$\int x^2\cos x\,dx \quad = \quad 2x\cos x+(-2+x^2)\sin x+C$$

16) $\int x\ln x\,dx$

17) $\int \ln(2x+1)\,dx$

Answer

$$\int \ln(2x+1)\,dx \quad = \quad \tfrac{1}{2}(1+2x)(-1+\ln(1+2x))+C$$

18) $\int x^2e^{4x}\,dx$

19) $\int e^x\sin x\,dx$

Answer

$$\int e^x\sin x\,dx \quad = \quad \tfrac{1}{2}e^x(-\cos x+\sin x)+C$$

20) $\int e^x\cos x\,dx$

21) $\int xe^{-x^2}\,dx$

Answer

$$\int xe^{-x^2}\,dx \quad = \quad -\frac{e^{-x^2}}{2}+C$$

22) $\int x^2e^{-x}\,dx$

23) $\int \sin(\ln(2x))\,dx$

Answer

$$\int \sin(\ln(2x))\,dx \quad = \quad -\tfrac{1}{2}x\cos[\ln(2x)]+\tfrac{1}{2}x\sin[\ln(2x)]+C$$

24) $\int \cos(\ln x)\,dx$

25) $\int (\ln x)^2\,dx$

Answer

$$\int (\ln x)^2\,dx \quad = \quad 2x - 2x\ln x + x(\ln x)^2 + C$$

26) $\int \ln(x^2)\,dx$

27) $\int x^2 \ln x\,dx$

Answer

$$\int x^2 \ln x\,dx \quad = \quad -\frac{x^3}{9} + \tfrac{1}{3}x^3 \ln x + C$$

28) $\int \sin^{-1} x\,dx$

29) $\int \cos^{-1}(2x)\,dx$

Answer

$$\int \cos^{-1}(2x)\,dx \quad = \quad -\tfrac{1}{2}\sqrt{1-4x^2} + x\cos^{-1}(2x) + C$$

30) $\int x \arctan x\,dx$

31) $\int x^2 \sin x\,dx$

Answer

$$\int x^2 \sin x\,dx \quad = \quad -(-2+x^2)\cos x + 2x\sin x + C$$

32) $\int x^3 \cos x\,dx$

33) $\int x^3 \sin x\,dx$

Answer

$$\int x^3 \sin x\,dx \quad = \quad -x(-6+x^2)\cos x + 3(-2+x^2)\sin x + C$$

34) $\int x^3 e^x\,dx$

35) $\int x \sec^{-1} x\,dx$

Answer

$$\int x \sec^{-1} x\,dx \quad = \quad \tfrac{1}{2}x\left(-\sqrt{1-\frac{1}{x^2}} + x\cdot\sec^{-1} x\right) + C$$

36) $\int x \sec^2 x\,dx$

37) $\int x \cosh x\,dx$

Answer

$$\int x\cosh x\,dx \quad = \quad -\cosh x + x\sinh x + C$$

In exercises 38 - 46, compute the definite integrals. Use a graphing utility to confirm your answers.

38) $\int_{1/e}^{1} \ln x\,dx$

39) $\int_{0}^{1} xe^{-2x}\,dx$ (Express the answer in exact form.)

Answer

$$\int_{0}^{1} xe^{-2x}\,dx \quad = \quad \frac{1}{4} - \frac{3}{4e^2}$$

40) $\int_{0}^{1} e^{\sqrt{x}}\,dx \quad (\text{let } u = \sqrt{x})$

41) $\int_{1}^{e} \ln(x^2)\,dx$

Answer

$$\int_{1}^{e} \ln(x^2)\,dx \quad = \quad 2$$

42) $\int_{0}^{\pi} x\cos x\,dx$

43) $\int_{-\pi}^{\pi} x\sin x\,dx$ (Express the answer in exact form.)

Answer

$$\int_{-\pi}^{\pi} x\sin x\,dx \quad = \quad 2\pi$$

44) $\int_{0}^{3} \ln(x^2+1)\,dx$ (Express the answer in exact form.)

45) $\int_{0}^{\pi/2} x^2\sin x\,dx$ (Express the answer in exact form.)

Answer

$$\int_{0}^{\pi/2} x^2\sin x\,dx \quad = \quad -2+\pi$$

46) $\int_{0}^{1} x5^x\,dx$ (Express the answer using five significant digits.)

47) Evaluate $\int \cos x\ln(\sin x)\,dx$

Answer

$$\int \cos x\ln(\sin x)\,dx \quad = \quad -\sin(x) + \ln[\sin(x)]\sin x + C$$

In exercises 48 - 50, derive the following formulas using the technique of integration by parts. Assume that n is a positive integer. These formulas are called *reduction formulas* because the exponent in the x term has been reduced by one in each case. The second integral is simpler than the original integral.

48) $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

49) $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$

Answer

Answers vary

50) $\int x^n \sin x\,dx =$_____

51) Integrate $\int 2x\sqrt{2x-3}\,dx$ using two methods:

a. Using parts, letting $dv = \sqrt{2x-3}\,dx$

b. Substitution, letting $u = 2x-3$

Answer

a. $\int 2x\sqrt{2x-3}\,dx = \frac{2}{5}(1+x)(-3+2x)^{3/2}+C$

b. $\int 2x\sqrt{2x-3}\,dx = \frac{2}{5}(1+x)(-3+2x)^{3/2}+C$

In exercises 52 - 57, state whether you would use integration by parts to evaluate the integral. If so, identify u and dv. If not, describe the technique used to perform the integration without actually doing the problem.

52) $\int x\ln x\,dx$

53) $\int \frac{\ln^2 x}{x}\,dx$

Answer

Do not use integration by parts. Choose u to be $\ln x$, and the integral is of the form $\int u^2\,du$.

54) $\int xe^x\,dx$

55) $\int xe^{x^2-3}\,dx$

Answer

Do not use integration by parts. Let $u = x^2-3$, and the integral can be put into the form $\int e^u\,du$.

56) $\int x^2\sin x\,dx$

57) $\int x^2\sin(3x^3+2)\,dx$

Answer

Do not use integration by parts. Choose u to be $u = 3x^3+2$ and the integral can be put into the form $\int \sin(u)\,du$.

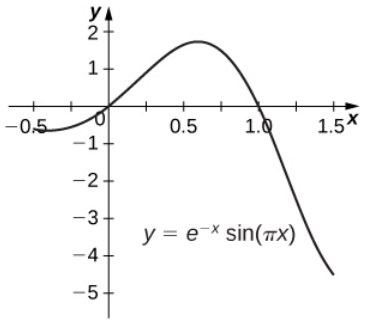
In exercises 58-59, sketch the region bounded above by the curve, the x-axis, and $x = 1$, and find the area of the region. Provide the exact form or round answers to the number of places indicated.

58) $y = 2xe^{-x}$ (Approximate answer to four decimal places.)

59) $y = e^{-x}\sin(\pi x)$ (Approximate answer to five decimal places.)

Answer

The area under graph is 0.39535 units^2.



In exercises 60 - 61, find the volume generated by rotating the region bounded by the given curves about the specified line. Express the answers in exact form or approximate to the number of decimal places indicated.

60) $y = \sin x,\ y = 0,\ x = 2\pi,\ x = 3\pi$; about the y-axis (Express the answer in exact form.)

61) $y = e^{-x},\ y = 0,\ x = -1,\ x = 0;$ about $x = 1$ (Express the answer in exact form.)

Answer

$V = 2\pi e \text{ units}^3$

62) A particle moving along a straight line has a velocity of $v(t) = t^2 e^{-t}$ after t sec. How far does it travel in the first 2 sec? (Assume the units are in feet and express the answer in exact form.)

63) Find the area under the graph of $y = \sec^3 x$ from $x = 0$ to $x = 1$. (Round the answer to two significant digits.)

Answer

$A = 2.05 \text{ units}^2$

64) Find the area between $y = (x - 2)e^x$ and the x-axis from $x = 2$ to $x = 5$. (Express the answer in exact form.)

65) Find the area of the region enclosed by the curve $y = x \cos x$ and the x-axis for $\frac{11\pi}{2} \leq x \leq \frac{13\pi}{2}$. (Express the answer in exact form.)

Answer

$A = 12\pi \text{ units}^2$

66) Find the volume of the solid generated by revolving the region bounded by the curve $y = \ln x$, the x-axis, and the vertical line $x = e^2$ about the x-axis. (Express the answer in exact form.)

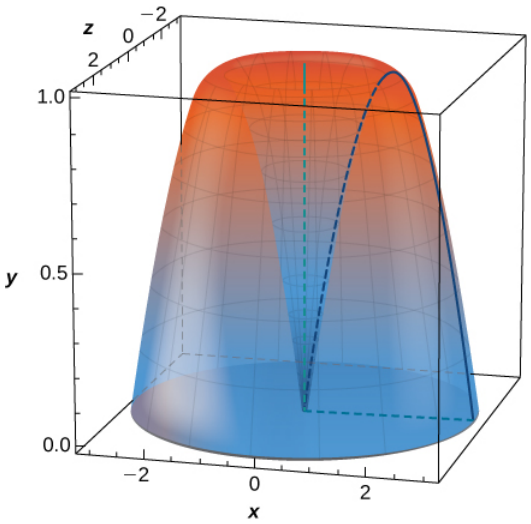
67) Find the volume of the solid generated by revolving the region bounded by the curve $y = 4 \cos x$ and the x-axis, $\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$, about the x-axis. (Express the answer in exact form.)

Answer

$V = 8\pi^2 \text{ units}^3$

68) Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = e^x$ and the x-axis, from $x = 0$ to $x = \ln(7)$, about the y-axis. (Express the answer in exact form.)

69) What is the volume of the Bundt cake that comes from rotating $y = \sin x$ around the y-axis from $x = 0$ to $x = \pi$?





Answer

$V = 2\pi^2$ units^3

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4.3: Trigonometric Integrals

Learning Objectives

- Solve integration problems involving products and powers of $\sin x$ and $\cos x$.
- Solve integration problems involving products and powers of $\tan x$ and $\sec x$.
- Use reduction formulas to solve trigonometric integrals.

In this section we look at how to integrate a variety of products of trigonometric functions. These integrals are called **trigonometric integrals**. They are an important part of the integration technique called *trigonometric substitution*, which is featured in Trigonometric Substitution. This technique allows us to convert algebraic expressions that we may not be able to integrate into expressions involving trigonometric functions, which we may be able to integrate using the techniques described in this section. In addition, these types of integrals appear frequently when we study polar, cylindrical, and spherical coordinate systems later. Let's begin our study with products of $\sin x$ and $\cos x$.

Integrating Products and Powers of sin x and cos x

A key idea behind the strategy used to integrate combinations of products and powers of $\sin x$ and $\cos x$ involves rewriting these expressions as sums and differences of integrals of the form $\int \sin^j x \cos x \, dx$ or $\int \cos^j x \sin x \, dx$. After rewriting these integrals, we evaluate them using u-substitution. Before describing the general process in detail, let's take a look at the following examples.

Example 4.3.1: Integrating $\int \cos^j x \sin x \, dx$

Evaluate $\int \cos^3 x \sin x \, dx$.

Solution

Use u-substitution and let $u = \cos x$. In this case, $du = -\sin x \, dx$.

Thus,

$$\int \cos^3 x \sin x \, dx = -\int u^3 \, du = -\frac{1}{4}u^4 + C = -\frac{1}{4}\cos^4 x + C.$$

Exercise 4.3.1

Evaluate $\int \sin^4 x \cos x \, dx$.

Hint

Let $u = \sin x$.

Answer

$$\int \sin^4 x \cos x \, dx = \frac{1}{5}\sin^5 x + C$$

Example 4.3.2: A Preliminary Example: Integrating $\int \cos^j x \sin^k x \, dx$ where k is Odd

Evaluate $\int \cos^2 x \sin^3 x \, dx$.

Solution

To convert this integral to integrals of the form $\int \cos^j x \sin x \, dx$, rewrite $\sin^3 x = \sin^2 x \sin x$ and make the substitution $\sin^2 x = 1 - \cos^2 x$.

Thus,

$$\int \cos^2 x \sin^3 x\, dx = \int \cos^2 x (1-\cos^2 x)\sin x\, dx \qquad \text{Let } u = \cos x;\ \text{then } du = -\sin x\, dx.$$

$$= -\int u^2(1-u^2)\, du$$

$$= \int (u^4 - u^2)\, du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

$$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C.$$

? Exercise 4.3.2

Evaluate $\int \cos^3 x \sin^2 x\, dx$.

Hint

Write $\cos^3 x = \cos^2 x \cos x = (1-\sin^2 x)\cos x$ and let $u = \sin x$.

Answer

$$\int \cos^3 x \sin^2 x\, dx = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C$$

In the next example, we see the strategy that must be applied when there are only even powers of $\sin x$ and $\cos x$. For integrals of this type, the identities

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1-\cos(2x)}{2}$$

and

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x) = \frac{1+\cos(2x)}{2}$$

are invaluable. These identities are sometimes known as **power-reducing identities** and they may be derived from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$ and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

✓ Example 4.3.3: Integrating an Even Power of $\sin x$

Evaluate $\int \sin^2 x\, dx$.

Solution

To evaluate this integral, let's use the trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$. Thus,

$$\int \sin^2 x\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right) dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C.$$

? Exercise 4.3.3

Evaluate $\int \cos^2 x\, dx$.

Hint

$\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

Answer

$$\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$$

The general process for integrating products of powers of $\sin x$ and $\cos x$ is summarized in the following set of guidelines.

Problem-Solving Strategy: Integrating Products and Powers of $\sin x$ and $cosx$

To integrate $\int \cos^j x \sin^k x\, dx$ use the following strategies:

1. If k is odd, rewrite $\sin^k x = \sin^{k-1} x \sin x$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to rewrite $\sin^{k-1} x$ in terms of $\cos x$. Integrate using the substitution $u = \cos x$. This substitution makes $du = -\sin x\, dx$.
2. If j is odd, rewrite $\cos^j x = \cos^{j-1} x \cos x$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to rewrite $\cos^{j-1} x$ in terms of $\sin x$. Integrate using the substitution $u = \sin x$. This substitution makes $du = \cos x\, dx$. (Note: If both j and k are odd, either strategy 1 or strategy 2 may be used.)
3. If both j and k are even, use $\sin^2 x = \dfrac{1 - \cos(2x)}{2}$ and $\cos^2 x = \dfrac{1 + \cos(2x)}{2}$. After applying these formulas, simplify and reapply strategies 1 through 3 as appropriate.

Example 4.3.4: Integrating $\int \cos^j x \sin^k x\, dx$ where k is Odd

Evaluate $\int \cos^8 x \sin^5 x\, dx$.

Solution

Since the power on $\sin x$ is odd, use strategy 1. Thus,

$$\begin{aligned}
\int \cos^8 x \sin^5 x\, dx &= \int \cos^8 x \sin^4 x \sin x\, dx && \text{Break off } \sin x. \\
&= \int \cos^8 x (\sin^2 x)^2 \sin x\, dx && \text{Rewrite } \sin^4 x = (\sin^2 x)^2. \\
&= \int \cos^8 x (1 - \cos^2 x)^2 \sin x\, dx && \text{Substitute } \sin^2 x = 1 - \cos^2 x. \\
&= \int u^8 (1 - u^2)^2 (-du) && \text{Let } u = \cos x \text{ and } du = -\sin x\, dx. \\
&= \int (-u^8 + 2u^{10} - u^{12}) du && \text{Expand.} \\
&= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C && \text{Evaluate the integral.} \\
&= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C && \text{Substitute } u = \cos x.
\end{aligned}$$

Example 4.3.5: Integrating $\int \cos^j x \sin^k x\, dx$ where k and j are Even

Evaluate $\int \sin^4 x\, dx$.

Solution: Since the power on $\sin x$ is even $(k = 4)$ and the power on $\cos x$ is even $(j = 0)$, we must use strategy 3. Thus,

$$\begin{aligned}
\int \sin^4 x\,dx &= \int \left(\sin^2 x\right)^2 dx && \text{Rewrite } \sin^4 x = \left(\sin^2 x\right)^2.\\
&= \int \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2 dx && \text{Substitute } \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x).\\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\cos^2(2x)\right) dx && \text{Expand } \left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)^2.\\
&= \int \left(\frac{1}{4} - \frac{1}{2}\cos(2x) + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos(4x)\right)\right) dx && \text{Use Power Reduction Formula on } \cos^2(2x).\\
&= \int \left(\frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)\right) dx && \text{Simplify.}\\
&= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C && \text{Evaluate the integral.}
\end{aligned}$$

? Exercise 4.3.4

Evaluate $\int \cos^3 x\,dx$.

Hint

Use strategy 2. Write $\cos^3 x = \cos^2 x \cos x$ and substitute $\cos^2 x = 1 - \sin^2 x$.

Answer

$$\int \cos^3 x\,dx = \sin x - \frac{1}{3}\sin^3 x + C$$

? Exercise 4.3.5

Evaluate $\int \cos^2(3x)\,dx$.

Hint

Use strategy 3. Substitute $\cos^2(3x) = \frac{1}{2} + \frac{1}{2}\cos(6x)$

Answer

$$\int \cos^2(3x)\,dx = \frac{1}{2}x + \frac{1}{12}\sin(6x) + C$$

Product to Sum Formulas

In some areas of physics, such as quantum mechanics, signal processing, and the computation of Fourier series, it is often necessary to integrate products that include $sin(ax), sin(bx), cos(ax)$, and $cos(bx)$. These integrals are evaluated by applying trigonometric identities, as outlined in the following rule.

Rule: Integrating Products of Sines and Cosines of Different Angles

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$, and $\cos(bx)$, use the substitutions

$$\sin(ax)\sin(bx) = \frac{1}{2}\cos((a-b)x) - \frac{1}{2}\cos((a+b)x)$$

$$\sin(ax)\cos(bx) = \frac{1}{2}\sin((a-b)x) + \frac{1}{2}\sin((a+b)x)$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\cos((a-b)x) + \frac{1}{2}\cos((a+b)x)$$

These formulas may be derived from the sum-of-angle formulas for sine and cosine.

Example 4.3.6: Evaluating $\int \sin(ax)\cos(bx)\,dx$

Evaluate $\int \sin(5x)\cos(3x)\,dx$.

Solution: Apply the identity $\sin(5x)\cos(3x) = \frac{1}{2}\sin(2x) + \frac{1}{2}\sin(8x)$. Thus,

$$\int \sin(5x)\cos(3x)\,dx = \int \frac{1}{2}\sin(2x) + \frac{1}{2}\sin(8x)\,dx = -\frac{1}{4}\cos(2x) - \frac{1}{16}\cos(8x) + C.$$

Exercise 4.3.6

Evaluate $\int \cos(6x)\cos(5x)\,dx$.

Hint

Substitute $\cos(6x)\cos(5x) = \frac{1}{2}\cos x + \frac{1}{2}\cos(11x)$.

Answer

$$\int \cos(6x)\cos(5x)\,dx = \frac{1}{2}\sin x + \frac{1}{22}\sin(11x) + C$$

Integrating Products and Powers of tan x and sec x

Before discussing the integration of products and powers of $\tan x$ and $\sec x$, it is useful to recall the integrals involving $\tan x$ and $\sec x$ we have already learned:

1. $\int \sec^2 x\,dx = \tan x + C$
2. $\int \sec x \tan x\,dx = \sec x + C$
3. $\int \tan x\,dx = \ln|\sec x| + C$
4. $\int \sec x\,dx = \ln|\sec x + \tan x| + C.$

For most integrals of products and powers of $\tan x$ and $\sec x$, we rewrite the expression we wish to integrate as the sum or difference of integrals of the form $\int \tan^j x \sec^2 x\,dx$ or $\int \sec^j x \tan x\,dx$. As we see in the following example, we can evaluate these new integrals by using u-substitution.

Example 4.3.7: Evaluating $\int \sec^j x \tan x\,dx$

Evaluate $\int \sec^5 x \tan x\,dx$.

Solution: Start by rewriting $\sec^5 x \tan x$ as $\sec^4 x \sec x \tan x$.

$$\begin{aligned}\int \sec^5 x \tan x\, dx &= \int \sec^4 x \sec x \tan x\, dx \\ &= \int u^4\, du && \text{Let } u = \sec x;\ \text{then, } du = \sec x \tan x\, dx. \\ &= \tfrac{1}{5}u^5 + C && \text{Evaluate the integral.} \\ &= \tfrac{1}{5}\sec^5 x + C && \text{Substitute } \sec x = u.\end{aligned}$$

You can read some interesting information at this website to learn about a common integral involving the secant.

Exercise 4.3.7

Evaluate $\int \tan^5 x \sec^2 x\, dx$.

Hint

Let $u = \tan x$ and $du = \sec^2 x$.

Answer

$$\int \tan^5 x \sec^2 x\, dx = \tfrac{1}{6}\tan^6 x + C$$

We now take a look at the various strategies for integrating products and powers of $\sec x$ and $\tan x$.

Problem-Solving Strategy: Integrating $\int \tan^k x \sec^j x\, dx$

To integrate $\int \tan^k x \sec^j x\, dx$, use the following strategies:

1. If j is even and $j \geq 2$, rewrite $\sec^j x = \sec^{j-2} x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite $\sec^{j-2} x$ in terms of $\tan x$. Let $u = \tan x$ and $du = \sec^2 x$.

2. If k is odd and $j \geq 1$, rewrite $\tan^k x \sec^j x = \tan^{k-1} x \sec^{j-1} x \sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to rewrite $\tan^{k-1} x$ in terms of $\sec x$. Let $u = \sec x$ and $du = \sec x \tan x\, dx$. (Note: If j is even and k is odd, then either strategy 1 or strategy 2 may be used.)

3. If k is odd where $k \geq 3$ and $j = 0$, rewrite $\tan^k x = \tan^{k-2} x \tan^2 x = \tan^{k-2} x(\sec^2 x - 1) = \tan^{k-2} x \sec^2 x - \tan^{k-2} x$. It may be necessary to repeat this process on the $\tan^{k-2} x$ term.

4. If k is even and j is odd, then use $\tan^2 x = \sec^2 x - 1$ to express $\tan^k x$ in terms of $\sec x$. Use integration by parts to integrate odd powers of $\sec x$.

Example 4.3.8: Integrating $\int \tan^k x \sec^j x\, dx$ when j is Even

Evaluate $\int \tan^6 x \sec^4 x\, dx$.

Solution

Since the power on $\sec x$ is even, rewrite $\sec^4 x = \sec^2 x \sec^2 x$ and use $\sec^2 x = \tan^2 x + 1$ to rewrite the first $\sec^2 x$ in terms of $\tan x$. Thus,

$$\begin{aligned}
\int \tan^6 x \sec^4 x\, dx &= \int \tan^6 x(\tan^2 x+1)\sec^2 x\, dx \\
&= \int u^6(u^2+1)\, du && \text{Let } u=\tan x \text{ and } du=\sec^2 x. \\
&= \int (u^8+u^6)\, du && \text{Expand.} \\
&= \frac{1}{9}u^9+\frac{1}{7}u^7+C && \text{Evaluate the integral.} \\
&= \frac{1}{9}\tan^9 x+\frac{1}{7}\tan^7 x+C. && \text{Substitute } \tan x=u.
\end{aligned}$$

Example 4.3.9: Integrating $\int \tan^k x \sec^j x\, dx$ when k is Odd

Evaluate $\int \tan^5 x \sec^3 x\, dx$.

Solution

Since the power on $\tan x$ is odd, begin by rewriting $\tan^5 x \sec^3 x = \tan^4 x \sec^2 x \sec x \tan x$. Thus,

$$\begin{aligned}
\int \tan^5 x \sec^3 x\, dx &= \int \tan^4 x \sec^2 x \sec x \tan x\, dx. \\
&= \int (\tan^2 x)^2 \sec^2 x \sec x \tan x\, dx && \text{Write } \tan^4 x=(\tan^2 x)^2. \\
&= \int (\sec^2 x-1)^2 \sec^2 x \sec x \tan x\, dx && \text{Use } \tan^2 x=\sec^2 x-1. \\
&= \int (u^2-1)^2 u^2\, du && \text{Let } u=\sec x \text{ and } du=\sec x \tan x\, dx \\
&= \int (u^6-2u^4+u^2)du && \text{Expand.} \\
&= \frac{1}{7}u^7-\frac{2}{5}u^5+\frac{1}{3}u^3+C && \text{Integrate.} \\
&= \frac{1}{7}\sec^7 x-\frac{2}{5}\sec^5 x+\frac{1}{3}\sec^3 x+C && \text{Substitute } \sec x=u.
\end{aligned}$$

Example 4.3.10: Integrating $\int \tan^k x\, dx$ where k is Odd and $k \geq 3$

Evaluate $\int \tan^3 x\, dx$.

Solution

Begin by rewriting $\tan^3 x = \tan x \tan^2 x = \tan x(\sec^2 x-1) = \tan x \sec^2 x - \tan x$. Thus,

$$\begin{aligned}
\int \tan^3 x\, dx &= \int (\tan x \sec^2 x - \tan x)\, dx \\
&= \int \tan x \sec^2 x\, dx - \int \tan x\, dx \\
&= \frac{1}{2}\tan^2 x - \ln|\sec x| + C.
\end{aligned}$$

For the first integral, use the substitution $u=\tan x$. For the second integral, use the formula.

Example 4.3.11: Integrating $\int \sec^3 x\,dx$

Integrate $\int \sec^3 x\,dx$.

Solution

This integral requires integration by parts. To begin, let $u = \sec x$ and $dv = \sec^2 x$. These choices make $du = \sec x \tan x$ and $v = \tan x$. Thus,

$$\begin{aligned}
\int \sec^3 x\,dx &= \sec x \tan x - \int \tan x \sec x \tan x\,dx \\
&= \sec x \tan x - \int \tan^2 x \sec x\,dx && \text{Simplify.} \\
&= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx && \text{Substitute } \tan^2 x = \sec^2 x - 1. \\
&= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx && \text{Rewrite.} \\
&= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx. && \text{Evaluate } \int \sec x\,dx.
\end{aligned}$$

We now have

$$\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x\,dx.$$

Since the integral $\int \sec^3 x\,dx$ has reappeared on the right-hand side, we can solve for $\int \sec^3 x\,dx$ by adding it to both sides. In doing so, we obtain

$$2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x|.$$

Dividing by 2, we arrive at

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C$$

Exercise 4.3.8

Evaluate $\int \tan^3 x \sec^7 x\,dx$.

Hint

Use Example 4.3.9as a guide.

Answer

$$\int \tan^3 x \sec^7 x\,dx = \frac{1}{9}\sec^9 x - \frac{1}{7}\sec^7 x + C$$

Reduction Formulas

Evaluating $\int \sec^n x\,dx$ for values of n where n is odd requires integration by parts. In addition, we must also know the value of $\int \sec^{n-2} x\,dx$ to evaluate $\int \sec^n x\,dx$. The evaluation of $\int \tan^n x\,dx$ also requires being able to integrate $\int \tan^{n-2} x\,dx$. To

make the process easier, we can derive and apply the following power reduction formulas. These rules allow us to replace the integral of a power of $\sec x$ or $\tan x$ with the integral of a lower power of $\sec x$ or $\tan x$.

Rule: Reduction Formulas for $\int \sec^n x\,dx$ and $\int \tan^n x\,dx$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

The first power reduction rule may be verified by applying integration by parts. The second may be verified by following the strategy outlined for integrating odd powers of $\tan x$.

Example 4.3.12: Revisiting $\int \sec^3 x\,dx$

Apply a reduction formula to evaluate $\int \sec^3 x\,dx$.

Solution: By applying the first reduction formula, we obtain

$$\begin{aligned}\int \sec^3 x\,dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\int \sec x\,dx \\ &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

Example 4.3.13: Using a Reduction Formula

Evaluate $\int \tan^4 x\,dx$.

Solution: Applying the reduction formula for $\int \tan^4 x\,dx$ we have

$$\begin{aligned}\int \tan^4 x\,dx &= \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx \\ &= \frac{1}{3}\tan^3 x - (\tan x - \int \tan^0 x\,dx) && \text{Apply the reduction formula to } \int \tan^2 x\,dx. \\ &= \frac{1}{3}\tan^3 x - \tan x + \int 1\,dx && \text{Simplify.} \\ &= \frac{1}{3}\tan^3 x - \tan x + x + C && \text{Evaluate } \int 1\,dx\end{aligned}$$

Exercise 4.3.9

Apply the reduction formula to $\int \sec^5 x\,dx$.

Hint

Use reduction formula 1 and let $n = 5$.

Answer

$$\int \sec^5 x\,dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x$$

Key Concepts

Integrals of trigonometric functions can be evaluated by the use of various strategies. These strategies include

1. Applying trigonometric identities to rewrite the integral so that it may be evaluated by u-substitution
2. Using integration by parts
3. Applying trigonometric identities to rewrite products of sines and cosines with different arguments as the sum of individual sine and cosine functions
4. Applying reduction formulas

Key Equations

To integrate products involving $\sin(ax)$, $\sin(bx)$, $\cos(ax)$,and $\cos(bx)$, use the substitutions.

- **Sine Products**

  $$\sin(ax)\sin(bx) = \tfrac{1}{2}\cos((a-b)x) - \tfrac{1}{2}\cos((a+b)x)$$

- **Sine and Cosine Products**

  $$\sin(ax)\cos(bx) = \tfrac{1}{2}\sin((a-b)x) + \tfrac{1}{2}\sin((a+b)x)$$

- **Cosine Products**

  $$\cos(ax)\cos(bx) = \tfrac{1}{2}\cos((a-b)x) + \tfrac{1}{2}\cos((a+b)x)$$

- **Power Reduction Formula**

  $$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

- **Power Reduction Formula**

  $$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

Glossary

power reduction formula
a rule that allows an integral of a power of a trigonometric function to be exchanged for an integral involving a lower power

trigonometric integral
an integral involving powers and products of trigonometric functions

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4.3E: Exercises for Trigonometric Integrals

Fill in the blank to make a true statement.

1) $\sin^2 x + \_\_\_\_\_ = 1$

Answer

$\cos^2 x$

2) $\sec^2 x - 1 = \_\_\_\_\_$

Answer

$\tan^2 x$

Use an identity to reduce the power of the trigonometric function to a trigonometric function raised to the first power.

3) $\sin^2 x = \_\_\_\_\_$

Answer

$$\frac{1-\cos(2x)}{2}$$

4) $\cos^2 x = \_\_\_\_\_$

Answer

$$\frac{1+\cos(2x)}{2}$$

Evaluate each of the following integrals by u-substitution.

5) $\int \sin^3 x \cos x \, dx$

Answer

$$\int \sin^3 x \cos x \, dx = \frac{\sin^4 x}{4} + C$$

6) $\int \sqrt{\cos x} \sin x \, dx$

7) $\int \tan^5(2x) \sec^2(2x) \, dx$

Answer

$$\int \tan^5(2x) \sec^2(2x) \, dx = \tfrac{1}{12}\tan^6(2x) + C$$

8) $\int \sin^7(2x) \cos(2x) \, dx$

9) $\int \tan(\frac{x}{2}) \sec^2(\frac{x}{2}) \, dx$

Answer

$$\int \tan(\frac{x}{2}) \sec^2(\frac{x}{2}) \, dx = \tan^2(\frac{x}{2}) + C$$

10) $\int \tan^2 x \sec^2 x \, dx$

Compute the following integrals using the guidelines for integrating powers of trigonometric functions. Use a CAS to check the solutions. (*Note*: Some of the problems may be done using techniques of integration learned previously.)

11) $\int \sin^3 x\, dx$

Answer

$$\int \sin^3 x\, dx \quad = \quad -\frac{3\cos x}{4} + \tfrac{1}{12}\cos(3x) + C = -\cos x + \frac{\cos^3 x}{3} + C$$

12) $\int \cos^3 x\, dx$

13) $\int \sin x \cos x\, dx$

Answer

$$\int \sin x \cos x\, dx \quad = \quad -\tfrac{1}{2}\cos^2 x + C$$

14) $\int \cos^5 x\, dx$

15) $\int \sin^5 x \cos^2 x\, dx$

Answer

$$\int \sin^5 x \cos^2 x\, dx \quad = \quad -\frac{5\cos x}{64} - \tfrac{1}{192}\cos(3x) + \tfrac{3}{320}\cos(5x) - \tfrac{1}{448}\cos(7x) + C$$

16) $\int \sin^3 x \cos^3 x\, dx$

17) $\int \sqrt{\sin x}\cos x\, dx$

Answer

$$\int \sqrt{\sin x}\cos x\, dx \quad = \quad \tfrac{2}{3}(\sin x)^{3/2} + C$$

18) $\int \sqrt{\sin x}\cos^3 x\, dx$

19) $\int \sec x \tan x\, dx$

Answer

$$\int \sec x \tan x\, dx \quad = \quad \sec x + C$$

20) $\int \tan(5x)\, dx$

21) $\int \tan^2 x \sec x\, dx$

Answer

$$\int \tan^2 x \sec x\, dx \quad = \quad \tfrac{1}{2}\sec x \tan x - \tfrac{1}{2}\ln(\sec x + \tan x) + C$$

22) $\int \tan x \sec^3 x\, dx$

23) $\int \sec^4 x\, dx$

Answer

$$\int \sec^4 x\, dx \quad = \quad \frac{2\tan x}{3} + \tfrac{1}{3}\sec^2 x \tan x = \tan x + \frac{\tan^3 x}{3} + C$$

24) $\int \cot x\, dx$

25) $\int \csc x\, dx$

Answer

$$\int \csc x\, dx \quad = \quad -\ln|\cot x + \csc x| + C$$

26) $\int \frac{\tan^3 x}{\sqrt{\sec x}}\, dx$

For exercises 27 - 28, find a general formula for the integrals.

27) $\int \sin^2 ax \cos ax\, dx$

Answer

$$\int \sin^2 ax \cos ax\, dx \quad = \quad \frac{\sin^3(ax)}{3a} + C$$

28) $\int \sin ax \cos ax\, dx.$

Use the double-angle formulas to evaluate the integrals in exercises 29 - 34.

29) $\int_0^{\pi} \sin^2 x\, dx$

Answer

$$\int_0^{\pi} \sin^2 x\, dx \quad = \quad \frac{\pi}{2}$$

30) $\int_0^{\pi} \sin^4 x\, dx$

31) $\int \cos^2 3x\, dx$

Answer

$$\int \cos^2 3x\, dx \quad = \quad \frac{x}{2} + \tfrac{1}{12}\sin(6x) + C$$

32) $\int \sin^2 x \cos^2 x\, dx$

33) $\int \sin^2 x\, dx + \int \cos^2 x\, dx$

Answer

$$\int \sin^2 x\, dx + \int \cos^2 x\, dx \quad = \quad x + C$$

34) $\int \sin^2 x \cos^2(2x)\, dx$

For exercises 35 - 43, evaluate the definite integrals. Express answers in exact form whenever possible.

35) $\int_0^{2\pi} \cos x \sin 2x\, dx$

Answer

$$\int_0^{2\pi} \cos x \sin 2x\, dx \quad = \quad 0$$

36) $\int_0^{\pi} \sin 3x \sin 5x\, dx$

37) $\int_0^{\pi} \cos(99x) \sin(101x)\, dx$

Answer

$$\int_0^{\pi} \cos(99x) \sin(101x)\, dx \quad = \quad 0$$

38) $\int_{-\pi}^{\pi} \cos^2(3x)\, dx$

39) $\int_0^{2\pi} \sin x \sin(2x) \sin(3x)\, dx$

Answer

$$\int_0^{2\pi} \sin x \sin(2x) \sin(3x)\, dx \quad = \quad 0$$

40) $\int_0^{4\pi} \cos(x/2) \sin(x/2)\, dx$

41) $\int_{\pi/6}^{\pi/3} \dfrac{\cos^3 x}{\sqrt{\sin x}}\, dx$ (Round this answer to three decimal places.)

Answer

$$\int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\sqrt{\sin x}}\, dx \quad \approx \quad 0.239$$

42) $\int_{-\pi/3}^{\pi/3} \sqrt{\sec^2 x - 1}\, dx$

43) $\int_0^{\pi/2} \sqrt{1 - \cos(2x)}\, dx$

Answer

$$\int_0^{\pi/2} \sqrt{1 - \cos(2x)}\, dx \quad = \quad \sqrt{2}$$

44) Find the area of the region bounded by the graphs of the equations $y = \sin x$, $y = \sin^3 x$, $x = 0$, and $x = \frac{\pi}{2}$.

45) Find the area of the region bounded by the graphs of the equations $y = \cos^2 x$, $y = \sin^2 x$, $x = -\frac{\pi}{4}$, and $x = \frac{\pi}{4}$.

Answer

$A = 1 \text{ unit}^2$

46) A particle moves in a straight line with the velocity function $v(t) = \sin(\omega t)\cos^2(\omega t)$. Find its position function $x = f(t)$ if $f(0) = 0$.

47) Find the average value of the function $f(x) = \sin^2 x \cos^3 x$ over the interval $[-\pi, \pi]$.

Answer

0

For exercises 48 - 49, solve the differential equations.

48) $\frac{dy}{dx} = \sin^2 x$. The curve passes through point $(0, 0)$.

49) $\frac{dy}{d\theta} = \sin^4(\pi\theta)$

Answer

$$f(x) = \frac{3\theta}{8} - \frac{1}{4\pi}\sin(2\pi\theta) + \frac{1}{32\pi}\sin(4\pi\theta) + C$$

50) Find the length of the curve $y = \ln(\csc x)$, for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$.

51) Find the length of the curve $y = \ln(\sin x)$, for $\frac{\pi}{3} \le x \le \frac{\pi}{2}$.

Answer

$s = \ln(\sqrt{3})$

52) Find the volume generated by revolving the curve $y = \cos(3x)$ about the x-axis, for $0 \le x \le \frac{\pi}{36}$.

For exercises 53 - 54, use this information: The inner product of two functions f and g over $[a, b]$ is defined by $f(x) \cdot g(x) = \langle f, g\rangle = \int_a^b f \cdot g\, dx$. Two distinct functions f and g are said to be orthogonal if $\langle f, g\rangle = 0$.

53) Show that $\sin(2x)$, $\cos(3x)$ are orthogonal over the interval $[-\pi, \pi]$.

Answer

$$\int_{-\pi}^{\pi} \sin(2x)\cos(3x)\, dx = 0$$

54) Evaluate $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\, dx$.

55) Integrate $y' = \sqrt{\tan x}\sec^4 x$.

Answer

$$y = \int \sqrt{\tan x}\sec^4 x\, dx \quad = \quad \frac{2}{3}(\tan x)^{3/2} + \frac{2}{7}(\tan x)^{7/2} + C = \frac{2}{21}(\tan x)^{3/2}\left[7 + 3\tan^2 x\right] + C$$

For each pair of integrals in exercises 56 - 57, determine which one is more difficult to evaluate. Explain your reasoning.

56) $\int \sin^{456} x \cos x\, dx$ or $\int \sin^2 x \cos^2 x\, dx$

57) $\int \tan^{350} x \sec^2 x\, dx$ or $\int \tan^{350} x \sec x\, dx$

Answer

The second integral is more difficult because the first integral is simply a u-substitution type.

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4.4: Trigonometric Substitution

Learning Objectives

- Solve integration problems involving the square root of a sum or difference of two squares.

In this section, we explore integrals containing expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, and $\sqrt{x^2-a^2}$, where the values of a are positive. We have already encountered and evaluated integrals containing some expressions of this type, but many still remain inaccessible. The technique of *trigonometric substitution* comes in very handy when evaluating these integrals. This technique uses substitution to rewrite these integrals as trigonometric integrals.

Integrals Involving $\sqrt{a^2-x^2}$

Before developing a general strategy for integrals containing $\sqrt{a^2-x^2}$, consider the integral $\int \sqrt{9-x^2}\,dx$. This integral cannot be evaluated using any of the techniques we have discussed so far. However, if we make the substitution $x=3\sin\theta$, we have $dx=3\cos\theta\,d\theta$. After substituting into the integral, we have

$$\int \sqrt{9-x^2}\,dx=\int \sqrt{9-(3\sin\theta)^2}\cdot 3\cos\theta\,d\theta.$$

After simplifying, we have

$$\int \sqrt{9-x^2}\,dx=\int 9\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta.$$

Letting $1-\sin^2\theta=\cos^2\theta$, we now have

$$\int \sqrt{9-x^2}\,dx=\int 9\sqrt{\cos^2\theta}\cos\theta\,d\theta.$$

Assuming that $\cos\theta\geq 0$, we have

$$\int \sqrt{9-x^2}\,dx=\int 9\cos^2\theta\,d\theta.$$

At this point, we can evaluate the integral using the techniques developed for integrating powers and products of trigonometric functions. Before completing this example, let's take a look at the general theory behind this idea.

To evaluate integrals involving $\sqrt{a^2-x^2}$, we make the substitution $x=a\sin\theta$ and $dx=a\cos\theta$. To see that this actually makes sense, consider the following argument: The domain of $\sqrt{a^2-x^2}$ is $[-a,a]$. Thus,

$$-a\leq x\leq a.$$

Consequently,

$$-1\leq\frac{x}{a}\leq 1.$$

Since the range of $\sin x$ over $[-(\pi/2),\pi/2]$is $[-1,1]$, there is a unique angle θ satisfying $-(\pi/2)\leq\theta\leq\pi/2$ so that $\sin\theta=x/a$, or equivalently, so that $x=a\sin\theta$. If we substitute $x=a\sin\theta$ into $\sqrt{a^2-x^2}$, we get

$$\begin{aligned}
\sqrt{a^2-x^2} &= \sqrt{a^2-(a\sin\theta)^2} && \text{Let } x = a\sin\theta \text{ where } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.\\
& && \text{Simplify.}\\
&= \sqrt{a^2-a^2\sin^2\theta} && \text{Factor out } a^2.\\
&= \sqrt{a^2(1-\sin^2\theta)} && \text{Substitute } 1-\sin^2 x = \cos^2 x.\\
&= \sqrt{a^2\cos^2\theta} && \text{Take the square root.}\\
&= |a\cos\theta|\\
&= a\cos\theta
\end{aligned}$$

Since $\cos x \ge 0$ on $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $a > 0$, $|a\cos\theta| = a\cos\theta$. We can see, from this discussion, that by making the substitution $x = a\sin\theta$, we are able to convert an integral involving a radical into an integral involving trigonometric functions. After we evaluate the integral, we can convert the solution back to an expression involving x. To see how to do this, let's begin by assuming that $0 < x < a$. In this case, $0 < \theta < \frac{\pi}{2}$. Since $\sin\theta = \frac{x}{a}$, we can draw the reference triangle in Figure 4.4.1 to assist in expressing the values of $\cos\theta$, $\tan\theta$, and the remaining trigonometric functions in terms of x. It can be shown that this triangle actually produces the correct values of the trigonometric functions evaluated at θ for all θ satisfying $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. It is useful to observe that the expression $\sqrt{a^2-x^2}$ actually appears as the length of one side of the triangle. Last, should θ appear by itself, we use $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

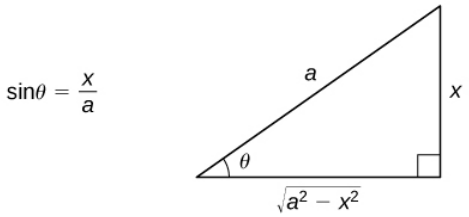


Figure 4.4.1: A reference triangle can help express the trigonometric functions evaluated at θ in terms of x.

The essential part of this discussion is summarized in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2-x^2}$

1. It is a good idea to make sure the integral cannot be evaluated easily in another way. For example, although this method can be applied to integrals of the form $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$, $\int \frac{x}{\sqrt{a^2-x^2}}\,dx$, and $\int x\sqrt{a^2-x^2}\,dx$, they can each be integrated directly either by formula or by a simple u-substitution.
2. Make the substitution $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$. Note: This substitution yields $\sqrt{a^2-x^2} = a\cos\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 1 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \sin^{-1}\left(\frac{x}{a}\right)$.

The following example demonstrates the application of this problem-solving strategy.

Example 4.4.1: Integrating an Expression Involving $\sqrt{a^2-x^2}$

Evaluate

$$\int \sqrt{9-x^2}\,dx.$$

Solution

Begin by making the substitutions $x = 3\sin\theta$ and $dx = 3\cos\theta\, d\theta$. Since $\sin\theta = \dfrac{x}{3}$, we can construct the reference triangle shown in Figure 2.

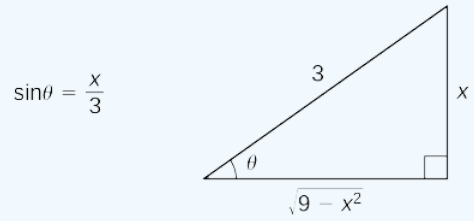


Figure 4.4.2: A reference triangle can be constructed for Example 4.4.1.

Thus,

$$\begin{aligned}
\int \sqrt{9-x^2}\, dx &= \int \sqrt{9-(3\sin\theta)^2}\, 3\cos\theta\, d\theta \quad \text{Substitute } x = 3\sin\theta \text{ and } dx = 3\cos\theta\, d\theta. \\
&= \int \sqrt{9(1-\sin^2\theta)} \cdot 3\cos\theta\, d\theta \quad \text{Simplify.} \\
&= \int \sqrt{9\cos^2\theta} \cdot 3\cos\theta\, d\theta \quad \text{Substitute } \cos^2\theta = 1-\sin^2\theta\,. \\
&= \int 3|\cos\theta| 3\cos\theta\, d\theta \quad \text{Take the square root.} \\
&= \int 9\cos^2\theta\, d\theta \quad \text{Simplify. Since } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \cos\theta \ge 0 \text{ and } |\cos\theta| = \cos\theta. \\
&= \int 9\left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta \quad \text{Use the strategy for integrating an even power of } \cos\theta. \\
&= \frac{9}{2}\theta + \frac{9}{4}\sin(2\theta) + C \quad \text{Evaluate the integral.} \\
&= \frac{9}{2}\theta + \frac{9}{4}(2\sin\theta\cos\theta) + C \quad \text{Substitute } \sin(2\theta) = 2\sin\theta\cos\theta\,. \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3} + C \quad \text{Substitute } \sin^{-1}\left(\frac{x}{3}\right) = \theta \text{ and } \sin\theta = \tfrac{x}{3}\,. \\
&\qquad \text{Use the reference triangle to see that } \cos\theta = \frac{\sqrt{9-x^2}}{3} \text{ and make this substitution. Simplify.} \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C. \quad \text{Simplify.}
\end{aligned}$$

Example 4.4.2: Integrating an Expression Involving $\sqrt{a^2 - x^2}$

Evaluate

$$\int \frac{\sqrt{4-x^2}}{x}\, dx.$$

Solution

First make the substitutions $x = 2\sin\theta$ and $dx = 2\cos\theta\, d\theta$. Since $\sin\theta = \dfrac{x}{2}$, we can construct the reference triangle shown in Figure 4.4.3.

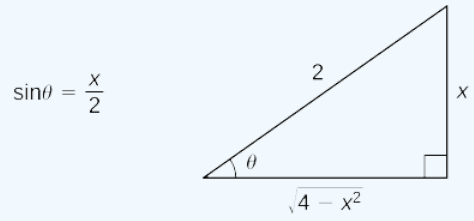


Figure 4.4.3: A reference triangle can be constructed for Example 4.4.2.

Thus,

$$\int \frac{\sqrt{4-x^2}}{x}dx = \int \frac{\sqrt{4-(2\sin\theta)^2}}{2\sin\theta}2\cos\theta\,d\theta \text{ Substitute } x=2\sin\theta \text{ and } dx=2\cos\theta\,d\theta.$$

$$= \int \frac{2\cos^2\theta}{\sin\theta}\,d\theta \text{ Substitute } \cos^2\theta = 1-\sin^2\theta \text{ and simplify.}$$

$$= \int \frac{2(1-\sin^2\theta)}{\sin\theta}\,d\theta \text{ Substitute } \cos^2\theta = 1-\sin^2\theta\,.$$

$$= \int (2\csc\theta - 2\sin\theta)\,d\theta \text{ Separate the numerator, simplify, and use } \csc\theta = \frac{1}{\sin\theta}\,.$$

$$= 2\ln|\csc\theta - \cot\theta| + 2\cos\theta + C \text{ Evaluate the integral.}$$

$$= 2\ln\left|\frac{2}{x} - \frac{\sqrt{4-x^2}}{x}\right| + \sqrt{4-x^2} + C. \text{ Use the reference triangle to rewrite the expression in terms of } x \text{ and simplify.}$$

In the next example, we see that we sometimes have a choice of methods.

Example 4.4.3: Integrating an Expression Involving $\sqrt{a^2-x^2}$ Two Ways

Evaluate $\int x^3\sqrt{1-x^2}\,dx$ two ways: first by using the substitution $u=1-x^2$ and then by using a trigonometric substitution.

Method 1

Let $u=1-x^2$ and hence $x^2=1-u$. Thus, $du=-2x\,dx$. In this case, the integral becomes

$$\int x^3\sqrt{1-x^2}\,dx = -\frac{1}{2}\int x^2\sqrt{1-x^2}\,(-2x\,dx) \text{ Make the substitution.}$$

$$= -\frac{1}{2}\int (1-u)\sqrt{u}\,du \text{ Expand the expression.}$$

$$= -\frac{1}{2}\int (u^{1/2}-u^{3/2})\,du \text{ Evaluate the integral.}$$

$$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C \text{ Rewrite in terms of x.}$$

$$= -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C.$$

Method 2

Let $x=\sin\theta$. In this case, $dx=\cos\theta\,d\theta$. Using this substitution, we have

$$\int x^3\sqrt{1-x^2}\,dx = \int \sin^3\theta\cos^2\theta\,d\theta$$

$$= \int (1-\cos^2\theta)\cos^2\theta\sin\theta\,d\theta \text{ Let } u=\cos\theta. \text{ Thus,} du=-\sin\theta\,d\theta.$$

$$= \int (u^4-u^2)\,du$$

$$= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \text{ Substitute } \cos\theta = u.$$

$$= \frac{1}{5}\cos^5\theta - \frac{1}{3}\cos^3\theta + C \text{ Use a reference triangle to see that } \cos\theta = \sqrt{1-x^2}\,.$$

$$= \frac{1}{5}(1-x^2)^{5/2} - \frac{1}{3}(1-x^2)^{3/2} + C.$$

Exercise 4.4.1

Rewrite the integral $\displaystyle\int \frac{x^3}{\sqrt{25-x^2}}\,dx$ using the appropriate trigonometric substitution (do not evaluate the integral).

Hint

Substitute $x = 5\sin\theta$ and $dx = 5\cos\theta\, d\theta$.

Answer

$$\int 125\sin^3\theta\, d\theta$$

Integrating Expressions Involving $\sqrt{a^2 + x^2}$

For integrals containing $\sqrt{a^2+x^2}$,let's first consider the domain of this expression. Since $\sqrt{a^2+x^2}$ is defined for all real values of x, we restrict our choice to those trigonometric functions that have a range of all real numbers. Thus, our choice is restricted to selecting either $x = a\tan\theta$ or $x = a\cot\theta$. Either of these substitutions would actually work, but the standard substitution is $x = a\tan\theta$ or, equivalently, $\tan\theta = x/a$. With this substitution, we make the assumption that $-(\pi/2) < \theta < \pi/2$, so that we also have $\theta = \tan^{-1}(x/a)$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrating Expressions Involving $\sqrt{a^2 + x^2}$

1. Check to see whether the integral can be evaluated easily by using another method. In some cases, it is more convenient to use an alternative method.
2. Substitute $x = a\tan\theta$ and $dx = a\sec^2\theta\, d\theta$. This substitution yields $\sqrt{a^2+x^2} = \sqrt{a^2+(a\tan\theta)^2} = \sqrt{a^2(1+\tan^2\theta)} = \sqrt{a^2 sec^2\theta} = |a\sec\theta| = a\sec\theta$. (Since $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ and $\sec\theta > 0$ over this interval, $|a\sec\theta| = a\sec\theta$.)
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangle from Figure 4.4.4 to rewrite the result in terms of x. You may also need to use some trigonometric identities and the relationship $\theta = \tan^{-1}\left(\dfrac{x}{a}\right)$. (Note: The reference triangle is based on the assumption that $x > 0$; however, the trigonometric ratios produced from the reference triangle are the same as the ratios for which $x \le 0$.)

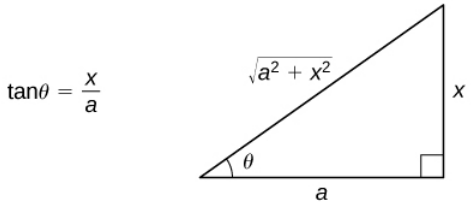


Figure 4.4.4: A reference triangle can be constructed to express the trigonometric functions evaluated at θ in terms of x.

Example 4.4.4: Integrating an Expression Involving $\sqrt{a^2 + x^2}$

Evaluate $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}$ and check the solution by differentiating.

Solution

Begin with the substitution $x = \tan\theta$ and $dx = sec^2\theta\, d\theta$. Since $\tan\theta = x$, draw the reference triangle in Figure 4.4.5.

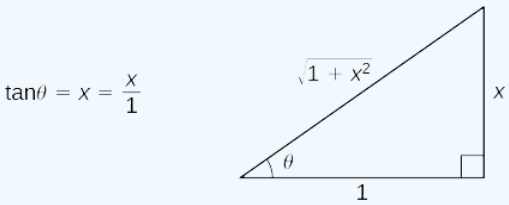


Figure 4.4.5: The reference triangle for Example 4.4.4.

Thus,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta \qquad \text{Substitute } x=\tan\theta \text{ and } dx=\sec^2\theta\,d\theta.$$

$$\text{This substitution makes } \sqrt{1+x^2}=\sec\theta. \text{ Simplify.}$$

$$= \int \sec\theta\,d\theta \qquad \text{Evaluate the integral.}$$

$$= \ln|\sec\theta+\tan\theta|+C \qquad \text{Use the reference triangle to express the result in terms of } x.$$

$$= \ln|\sqrt{1+x^2}+x|+C$$

To check the solution, differentiate:

$$\frac{d}{dx}\left(\ln|\sqrt{1+x^2}+x|\right) = \frac{1}{\sqrt{1+x^2}+x}\cdot\left(\frac{x}{\sqrt{1+x^2}}+1\right) = \frac{1}{\sqrt{1+x^2}+x}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}}.$$

Since $\sqrt{1+x^2}+x>0$ for all values of x, we could rewrite $\ln|\sqrt{1+x^2}+x|+C=\ln(\sqrt{1+x^2}+x)+C$, if desired.

Example 4.4.5: Evaluating $\int \frac{dx}{\sqrt{1+x^2}}$ Using a Different Substitution

Use the substitution $x=\sinh\theta$ to evaluate $\int \frac{dx}{\sqrt{1+x^2}}$.

Solution

Because $\sinh\theta$ has a range of all real numbers, and $1+\sinh^2\theta=\cosh^2\theta$, we may also use the substitution $x=\sinh\theta$ to evaluate this integral. In this case, $dx=\cosh\theta\,d\theta$. Consequently,

$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh\theta}{\sqrt{1+\sinh^2\theta}}\,d\theta \qquad \text{Substitute } x=\sinh\theta \text{ and } dx=\cosh\theta\,d\theta.$$

$$\text{Substitute } 1+\sinh^2\theta=\cosh^2\theta.$$

$$= \int \frac{\cosh\theta}{\sqrt{\cosh^2\theta}}\,d\theta \qquad \text{Since } \sqrt{\cosh^2\theta}=|\cosh\theta|$$

$$= \int \frac{\cosh\theta}{|\cosh\theta|}\,d\theta \qquad |\cosh\theta|=\cosh\theta \text{ since } \cosh\theta>0 \text{ for all } \theta.$$

$$= \int \frac{\cosh\theta}{\cosh\theta}\,d\theta \qquad \text{Simplify.}$$

$$= \int 1\,d\theta \qquad \text{Evaluate the integral.}$$

$$= \theta+C \qquad \text{Since } x=\sinh\theta, \text{ we know } \theta=\sinh^{-1}x.$$

$$= \sinh^{-1}x+C.$$

Analysis

This answer looks quite different from the answer obtained using the substitution $x=\tan\theta$. To see that the solutions are the same, set $y=\sinh^{-1}x$. Thus, $\sinh y=x$. From this equation we obtain:

$$\frac{e^y-e^{-y}}{2}=x.$$

After multiplying both sides by $2e^y$ and rewriting, this equation becomes:

$$e^{2y}-2xe^y-1=0.$$

Use the quadratic equation to solve for e^y:

$$e^y=\frac{2x\pm\sqrt{4x^2+4}}{2}.$$

Simplifying, we have:

$$e^y = x \pm \sqrt{x^2+1}.$$

Since $x - \sqrt{x^2+1} < 0$, it must be the case that $e^y = x + \sqrt{x^2+1}$. Thus,

$$y = \ln(x + \sqrt{x^2+1}).$$

Last, we obtain

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}).$$

After we make the final observation that, since $x + \sqrt{x^2+1} > 0$,

$$\ln(x + \sqrt{x^2+1}) = \ln | \sqrt{1+x^2} + x |,$$

we see that the two different methods produced equivalent solutions.

Example 4.4.6: Finding an Arc Length

Find the length of the curve $y = x^2$ over the interval $[0, \frac{1}{2}]$.

Solution

Because $\frac{dy}{dx} = 2x$, the arc length is given by

$$\int_0^{1/2} \sqrt{1+(2x)^2}\,dx = \int_0^{1/2} \sqrt{1+4x^2}\,dx.$$

To evaluate this integral, use the substitution $x = \frac{1}{2}\tan\theta$ and $dx = \frac{1}{2}\sec^2\theta\,d\theta$. We also need to change the limits of integration. If $x = 0$, then $\theta = 0$ and if $x = \frac{1}{2}$, then $\theta = \frac{\pi}{4}$. Thus,

$\int_0^{1/2} \sqrt{1+4x^2}\,dx = \int_0^{\pi/4} \sqrt{1+\tan^2\theta} \cdot \frac{1}{2}\sec^2\theta\,d\theta$ After substitution, $\sqrt{1+4x^2} = \sec\theta$. (Substitute $1+\tan^2\theta = \sec^2\theta$ and simplify.)

$= \frac{1}{2}\int_0^{\pi/4} \sec^3\theta\,d\theta$ We derived this integral in the previous section.

$= \frac{1}{2}\left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4}$ Evaluate and simplify.

$= \frac{1}{4}(\sqrt{2} + \ln(\sqrt{2}+1)).$

Exercise 4.4.2

Rewrite $\int x^3\sqrt{x^2+4}\,dx$ by using a substitution involving $\tan\theta$.

Hint

Use $x = 2\tan\theta$ and $dx = 2\sec^2\theta\,d\theta$.

Answer

$$\int 32\tan^3\theta\sec^3\theta\,d\theta$$

Integrating Expressions Involving $\sqrt{x^2 - a^2}$

The domain of the expression $\sqrt{x^2 - a^2}$ is $(-\infty, -a] \cup [a, +\infty)$. Thus, either $x \leq -a$ or $x \geq a$. Hence, $\frac{x}{a} \leq -1$ or $\frac{x}{a} \geq 1$. Since these intervals correspond to the range of $\sec\theta$ on the set $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, it makes sense to use the substitution $\sec\theta = \frac{x}{a}$ or, equivalently, $x = a\sec\theta$, where $0 \leq \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \leq \pi$. The corresponding substitution for dx is $dx = a\sec\theta\tan\theta\, d\theta$. The procedure for using this substitution is outlined in the following problem-solving strategy.

Problem-Solving Strategy: Integrals Involving $\sqrt{x^2 - a^2}$

1. Check to see whether the integral cannot be evaluated using another method. If so, we may wish to consider applying an alternative technique.
2. Substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\, d\theta$. This substitution yields

$$\sqrt{x^2 - a^2} = \sqrt{(a\sec\theta)^2 - a^2} = \sqrt{a^2(\sec^2\theta - 1)} = \sqrt{a^2\tan^2\theta} = |a\tan\theta|.$$

   For $x \geq a$, $|a\tan\theta| = a\tan\theta$ and for $x \leq -a$, $|a\tan\theta| = -a\tan\theta$.
3. Simplify the expression.
4. Evaluate the integral using techniques from the section on trigonometric integrals.
5. Use the reference triangles from Figure 4.4.6 to rewrite the result in terms of x.
6. You may also need to use some trigonometric identities and the relationship $\theta = \sec^{-1}\left(\frac{x}{a}\right)$. (Note: We need both reference triangles, since the values of some of the trigonometric ratios are different depending on whether $x > a$ or $x < -a$.)

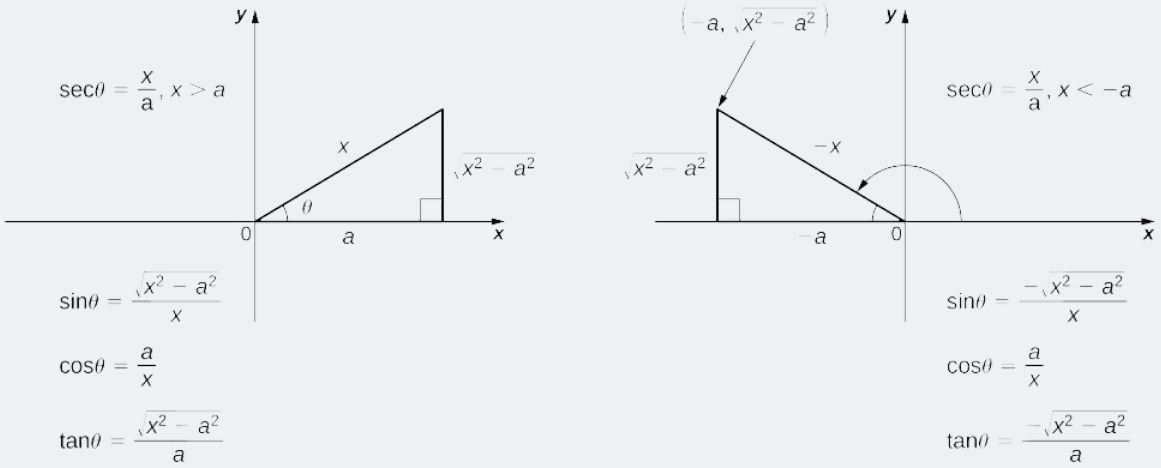


Figure 4.4.6: Use the appropriate reference triangle to express the trigonometric functions evaluated at θ in terms of x.

Example 4.4.7: Finding the Area of a Region

Find the area of the region between the graph of $f(x) = \sqrt{x^2 - 9}$ and the x-axis over the interval $[3, 5]$.

Solution

First, sketch a rough graph of the region described in the problem, as shown in the following figure.

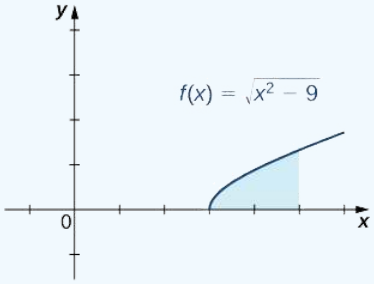


Figure 4.4.7: Calculating the area of the shaded region requires evaluating an integral with a trigonometric substitution.

We can see that the area is $A = \int_3^5 \sqrt{x^2-9}\,dx$. To evaluate this definite integral, substitute $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta\,d\theta$. We must also change the limits of integration. If $x = 3$, then $3 = 3\sec\theta$ and hence $\theta = 0$. If $x = 5$, then $\theta = \sec^{-1}(\frac{5}{3})$. After making these substitutions and simplifying, we have

$$\begin{aligned}
\textbf{Area} &= \int_3^5 \sqrt{x^2-9}\,dx \\
&= \int_0^{\sec^{-1}(5/3)} 9\tan^2\theta\sec\theta\,d\theta \quad \text{Use } \tan^2\theta = \sec^2\theta - 1. \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^2\theta - 1)\sec\theta\,d\theta \quad \text{Expand.} \\
&= \int_0^{\sec^{-1}(5/3)} 9(\sec^3\theta - \sec\theta)\,d\theta \quad \text{Evaluate the integral.} \\
&= (\frac{9}{2}\ln|\sec\theta+\tan\theta| + \frac{9}{2}\sec\theta\tan\theta) - 9\ln|\sec\theta+\tan\theta|\Big|_0^{\sec^{-1}(5/3)} \quad \text{Simplify.} \\
&= \frac{9}{2}\sec\theta\tan\theta - \frac{9}{2}\ln|\sec\theta+\tan\theta|\Big|_0^{\sec^{-1}(5/3)} \quad \text{Evaluate. Use } \sec(\sec^{-1}\frac{5}{3}) = \frac{5}{3} \text{ and } \tan(\sec^{-1}\frac{5}{3}) = \frac{4}{3}. \\
&= \frac{9}{2}\cdot\frac{5}{3}\cdot\frac{4}{3} - \frac{9}{2}\ln|\frac{5}{3}+\frac{4}{3}| - (\frac{9}{2}\cdot 1\cdot 0 - \frac{9}{2}\ln|1+0|) \\
&= 10 - \frac{9}{2}\ln 3
\end{aligned}$$

Exercise 4.4.3

Evaluate

$$\int \frac{dx}{\sqrt{x^2-4}}.$$

Assume that $x > 2$.

Hint

Substitute $x = 2\sec\theta$ and $dx = 2\sec\theta\tan\theta\,d\theta$.

Answer

$$\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C$$

Key Concepts

- For integrals involving $\sqrt{a^2-x^2}$, use the substitution $x = a\sin\theta$ and $dx = a\cos\theta\,d\theta$.
- For integrals involving $\sqrt{a^2+x^2}$, use the substitution $x = a\tan\theta$ and $dx = a\sec^2\theta\,d\theta$.
- For integrals involving $\sqrt{x^2-a^2}$, substitute $x = a\sec\theta$ and $dx = a\sec\theta\tan\theta\,d\theta$.

Glossary

trigonometric substitution

an integration technique that converts an algebraic integral containing expressions of the form $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ into a trigonometric integral

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4.4E: Exercises for Trigonometric Substitution

Simplify the expressions in exercises 1 - 5 by writing each one using a single trigonometric function.

1) $4-4\sin^2\theta$

2) $9\sec^2\theta-9$

Answer

$9\sec^2\theta-9 \quad = \quad 9\tan^2\theta$

3) $a^2+a^2\tan^2\theta$

4) $a^2+a^2\sinh^2\theta$

Answer

$a^2+a^2\sinh^2\theta \quad = \quad a^2\cosh^2\theta$

5) $16\cosh^2\theta-16$

Use the technique of completing the square to express each trinomial in exercises 6 - 8 as the square of a binomial.

6) $4x^2-4x+1$

Answer

$4(x-\frac{1}{2})^2$

7) $2x^2-8x+3$

8) $-x^2-2x+4$

Answer

$-(x+1)^2+5$

In exercises 9 - 28, integrate using the method of trigonometric substitution. Express the final answer in terms of the original variable.

9) $\int \frac{dx}{\sqrt{4-x^2}}$

10) $\int \frac{dx}{\sqrt{x^2-a^2}}$

Answer

$\int \frac{dx}{\sqrt{x^2-a^2}} \quad = \quad \ln|\,x+\sqrt{-a^2+x^2}\,|+C$

11) $\int \sqrt{4-x^2}\,dx$

12) $\int \frac{dx}{\sqrt{1+9x^2}}$

Answer

$\int \frac{dx}{\sqrt{1+9x^2}} \quad = \quad \frac{1}{3}\ln|\,\sqrt{9x^2+1}+3x\,|+C$

13) $\int \frac{x^2\,dx}{\sqrt{1-x^2}}$

14) $\int \frac{dx}{x^2\sqrt{1-x^2}}$

Answer

$$\int \frac{dx}{x^2\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x}+C$$

15) $\int \frac{dx}{(1+x^2)^2}$

16) $\int \sqrt{x^2+9}\,dx$

Answer

$$\int \sqrt{x^2+9}\,dx = 9\left[\frac{x\sqrt{x^2+9}}{18}+\tfrac{1}{2}\ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right|\right]+C$$

17) $\int \frac{\sqrt{x^2-25}}{x}\,dx$

18) $\int \frac{\theta^3}{\sqrt{9-\theta^2}}\,d\theta$

Answer

$$\int \frac{\theta^3 d\theta}{\sqrt{9-\theta^2}}\,d\theta = -\tfrac{1}{3}\sqrt{9-\theta^2}\,(18+\theta^2)+C$$

19) $\int \frac{dx}{\sqrt{x^6-x^2}}$

20) $\int \sqrt{x^6-x^8}\,dx$

Answer

$$\int \sqrt{x^6-x^8}\,dx = \frac{(-1+x^2)(2+3x^2)\sqrt{x^6-x^8}}{15x^3}+C$$

21) $\int \frac{dx}{(1+x^2)^{3/2}}$

22) $\int \frac{dx}{(x^2-9)^{3/2}}$

Answer

$$\int \frac{dx}{(x^2-9)^{3/2}} = -\frac{x}{9\sqrt{x^2-9}}+C$$

23) $\int \frac{\sqrt{1+x^2}}{x}\,dx$

24) $\int \frac{x^2}{\sqrt{x^2-1}}\,dx$

Answer

$$\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \tfrac{1}{2}(\ln|\,x+\sqrt{x^2-1}\,|+x\sqrt{x^2-1}\,)+C$$

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25) $\int \frac{x^2}{x^2+4}\,dx$

26) $\int \frac{dx}{x^2\sqrt{x^2+1}}$

Answer

$$\int \frac{dx}{x^2\sqrt{x^2+1}} \quad = \quad -\frac{\sqrt{1+x^2}}{x}+C$$

27) $\int \frac{x^2}{\sqrt{1+x^2}}\,dx$

28) $\int_{-1}^{1}(1-x^2)^{3/2}\,dx$

Answer

$$\int_{-1}^{1}(1-x^2)^{3/2}\,dx \quad = \quad \tfrac{1}{8}\left(x(5-2x^2)\sqrt{1-x^2}+3\arcsin x\right)+C$$

In exercises 29 - 34, use the substitutions $x=\sinh\theta$, $\cosh\theta$, or $\tanh\theta$. Express the final answers in terms of the variable x.

29) $\int \frac{dx}{\sqrt{x^2-1}}$

30) $\int \frac{dx}{x\sqrt{1-x^2}}$

Answer

$$\int \frac{dx}{x\sqrt{1-x^2}} \quad = \quad \ln x - \ln\left|\,1+\sqrt{1-x^2}\,\right|+C$$

31) $\int \sqrt{x^2-1}\,dx$

32) $\int \frac{\sqrt{x^2-1}}{x^2}\,dx$

Answer

$$\int \frac{\sqrt{x^2-1}}{x^2}\,dx \quad = \quad -\frac{\sqrt{-1+x^2}}{x}+\ln\left|x+\sqrt{-1+x^2}\,\right|+C$$

33) $\int \frac{dx}{1-x^2}$

34) $\int \frac{\sqrt{1+x^2}}{x^2}\,dx$

Answer

$$\int \frac{\sqrt{1+x^2}}{x^2}\,dx \quad = \quad -\frac{\sqrt{1+x^2}}{x}+\operatorname{arcsinh} x+C$$

Use the technique of completing the square to evaluate the integrals in exercises 35 - 39.

35) $\int \frac{1}{x^2-6x}\,dx$

36) $\int \frac{1}{x^2+2x+1}\,dx$

Answer

$$\int \frac{1}{x^2+2x+1}\,dx \quad = \quad -\frac{1}{1+x}+C$$

37) $\int \frac{1}{\sqrt{-x^2+2x+8}}\,dx$

38) $\int \frac{1}{\sqrt{-x^2+10x}}\,dx$

Answer

$$\int \frac{1}{\sqrt{-x^2+10x}}\,dx \quad = \quad \arcsin\left(\frac{x-5}{5}\right)+C$$

39) $\int \frac{1}{\sqrt{x^2+4x-12}}\,dx$

40) Evaluate the integral without using calculus: $\int_{-3}^{3} \sqrt{9-x^2}\,dx$.

Answer

$\int_{-3}^{3} \sqrt{9-x^2}\,dx \quad = \quad \frac{9\pi}{2}$; area of a semicircle with radius 3

41) Find the area enclosed by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

42) Evaluate the integral $\int \frac{dx}{\sqrt{1-x^2}}$ using two different substitutions. First, let $x=\cos\theta$ and evaluate using trigonometric substitution. Second, let $x=\sin\theta$ and use trigonometric substitution. Are the answers the same?

Answer

$\int \frac{dx}{\sqrt{1-x^2}} \quad = \quad \arcsin(x)+C$ is the common answer.

43) Evaluate the integral $\int \frac{dx}{x\sqrt{x^2-1}}$ using the substitution $x=\sec\theta$. Next, evaluate the same integral using the substitution $x=\csc\theta$. Show that the results are equivalent.

44) Evaluate the integral $\int \frac{x}{x^2+1}\,dx$ using the form $\int \frac{1}{u}\,du$. Next, evaluate the same integral using $x=\tan\theta$. Are the results the same?

Answer

$\int \frac{x}{x^2+1}\,dx \quad = \quad \frac{1}{2}\ln(1+x^2)+C$ is the result using either method.

45) State the method of integration you would use to evaluate the integral $\int x\sqrt{x^2+1}\,dx$. Why did you choose this method?

46) State the method of integration you would use to evaluate the integral $\int x^2\sqrt{x^2-1}\,dx$. Why did you choose this method?

Answer

Use trigonometric substitution. Let $x=\sec(\theta)$.

47) Evaluate $\int_{-1}^{1} \frac{x}{x^2+1}\,dx$

48) Find the length of the arc of the curve over the specified interval: $y = \ln x, \quad [1, 5]$. Round the answer to three decimal places.

Answer

$s = 4.367$ units

49) Find the surface area of the solid generated by revolving the region bounded by the graphs of $y = x^2$, $y = 0$, $x = 0$, and $x = \sqrt{2}$ about the x-axis. (Round the answer to three decimal places).

50) The region bounded by the graph of $f(x) = \dfrac{1}{1+x^2}$ and the x-axis between $x = 0$ and $x = 1$ is revolved about the x-axis. Find the volume of the solid that is generated.

Answer

$V = \left(\frac{\pi^2}{8} + \frac{\pi}{4}\right)$ units^3

In exercises 51 - 52, solve the initial-value problem for y as a function of x.

51) $(x^2 + 36)\dfrac{dy}{dx} = 1, \quad y(6) = 0$

52) $(64 - x^2)\dfrac{dy}{dx} = 1, \quad y(0) = 3$

Answer

$y = \frac{1}{16}\ln\left|\dfrac{x+8}{x-8}\right| + 3$

53) Find the area bounded by $y = \dfrac{2}{\sqrt{64 - 4x^2}}$, $x = 0$, $y = 0$, and $x = 2$.

54) An oil storage tank can be described as the volume generated by revolving the area bounded by $y = \dfrac{16}{\sqrt{64 + x^2}}$, $x = 0$, $y = 0$, $x = 2$ about the x-axis. Find the volume of the tank (in cubic meters).

Answer

$V = 24.6$ m^3

55) During each cycle, the velocity v (in feet per second) of a robotic welding device is given by $v = 2t - \dfrac{14}{4 + t^2}$, where t is time in seconds. Find the expression for the displacement s (in feet) as a function of t if $s = 0$ when $t = 0$.

56) Find the length of the curve $y = \sqrt{16 - x^2}$ between $x = 0$ and $x = 2$.

Answer

$s = \frac{2\pi}{3}$ units

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4.5: Partial Fractions

Learning Objectives

- Integrate a rational function using the method of partial fractions.
- Recognize simple linear factors in a rational function.
- Recognize repeated linear factors in a rational function.
- Recognize quadratic factors in a rational function.

We have seen some techniques that allow us to integrate specific rational functions. For example, we know that

$$\int \frac{du}{u} = \ln|u| + C$$

and

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C.$$

However, we do not yet have a technique that allows us to tackle arbitrary quotients of this type. Thus, it is not immediately obvious how to go about evaluating

$$\int \frac{3x}{x^2 - x - 2}\, dx.$$

However, we know from material previously developed that

$$\int \left(\frac{1}{x+1} + \frac{2}{x-2}\right) dx = \ln|x+1| + 2\ln|x-2| + C.$$

In fact, by getting a common denominator, we see that

$$\frac{1}{x+1} + \frac{2}{x-2} = \frac{3x}{x^2 - x - 2}.$$

Consequently,

$$\int \frac{3x}{x^2 - x - 2}\, dx = \int \left(\frac{1}{x+1} + \frac{2}{x-2}\right) dx.$$

In this section, we examine the method of **partial fraction decomposition**, which allows us to decompose **rational functions** into sums of simpler, more easily integrated rational functions. Using this method, we can rewrite an expression such as:

$$\frac{3x}{x^2 - x - 2}$$

as an expression such as

$$\frac{1}{x+1} + \frac{2}{x-2}.$$

The key to the method of partial fraction decomposition is being able to anticipate the form that the decomposition of a rational function will take. As we shall see, this form is both predictable and highly dependent on the factorization of the denominator of the rational function. It is also extremely important to keep in mind that partial fraction decomposition can be applied to a rational function $\frac{P(x)}{Q(x)}$ only if $deg(P(x)) < deg(Q(x))$. In the case when $deg(P(x)) \geq deg(Q(x))$, we must first perform long division to rewrite the quotient $\frac{P(x)}{Q(x)}$ in the form $A(x) + \frac{R(x)}{Q(x)}$, where $deg(R(x)) < deg(Q(x))$. We then do a partial fraction

decomposition on $\frac{R(x)}{Q(x)}$. The following example, although not requiring partial fraction decomposition, illustrates our approach to integrals of rational functions of the form $\int \frac{P(x)}{Q(x)}\,dx$, where $deg(P(x)) \geq deg(Q(x))$.

Example 4.5.1: Integrating $\int \frac{P(x)}{Q(x)}\,dx$, where $deg(P(x)) \geq deg(Q(x))$

Evaluate

$$\int \frac{x^2+3x+5}{x+1}\,dx.$$

Solution

Since $deg(x^2+3x+5) \geq deg(x+1)$, we perform long division to obtain

$$\frac{x^2+3x+5}{x+1} = x+2+\frac{3}{x+1}.$$

Thus,

$$\int \frac{x^2+3x+5}{x+1}\,dx = \int \left(x+2+\frac{3}{x+1}\right)\,dx = \frac{1}{2}x^2+2x+3\ln|x+1|+C.$$

Visit this website for a review of long division of polynomials.

Exercise 4.5.1

Evaluate

$$\int \frac{x-3}{x+2}\,dx.$$

Hint

Use long division to obtain $\frac{x-3}{x+2} = 1-\frac{5}{x+2}$.

Answer

$$x-5\ln|x+2|+C$$

To integrate $\int \frac{P(x)}{Q(x)}\,dx$, where $deg(P(x)) < deg(Q(x))$, we must begin by factoring $Q(x)$.

Nonrepeated Linear Factors

If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_n}{a_nx+b_n}. \tag{4.5.1}$$

The proof that such constants exist is beyond the scope of this course.

In this next example, we see how to use partial fractions to integrate a rational function of this type.

Example 4.5.2: Partial Fractions with Nonrepeated Linear Factors

Evaluate $\int \frac{3x+2}{x^3-x^2-2x}\,dx$.

Solution

Since $deg(3x+2) < deg(x^3-x^2-2x)$, we begin by factoring the denominator of $\frac{3x+2}{x^3-x^2-2x}$. We can see that $x^3-x^2-2x = x(x-2)(x+1)$. Thus, there are constants A, B, and C satisfying Equation 4.5.1 such that

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+1}.$$

We must now find these constants. To do so, we begin by getting a common denominator on the right. Thus,

$$\frac{3x+2}{x(x-2)(x+1)} = \frac{A(x-2)(x+1)+Bx(x+1)+Cx(x-2)}{x(x-2)(x+1)}.$$

Now, we set the numerators equal to each other, obtaining

$$3x+2 = A(x-2)(x+1)+Bx(x+1)+Cx(x-2). \tag{4.5.2}$$

There are two different strategies for finding the coefficients A, B, and C. We refer to these as the *method of equating coefficients* and the *method of strategic substitution*.

Strategy one: Method of Equating Coefficients

Rewrite Equation 4.5.2 in the form

$$3x+2 = (A+B+C)x^2+(-A+B-2C)x+(-2A).$$

Equating coefficients produces the system of equations

$$\begin{aligned} A+B+C &= 0 \\ -A+B-2C &= 3 \\ -2A &= 2. \end{aligned}$$

To solve this system, we first observe that $-2A = 2 \Rightarrow A = -1$. Substituting this value into the first two equations gives us the system

$$\begin{aligned} B+C &= 1 \\ B-2C &= 2\,. \end{aligned}$$

Multiplying the second equation by -1 and adding the resulting equation to the first produces

$$-3C = 1,$$

which in turn implies that $C = -\frac{1}{3}$. Substituting this value into the equation $B+C=1$ yields $B = \frac{4}{3}$. Thus, solving these equations yields $A = -1, B = \frac{4}{3}$, and $C = -\frac{1}{3}$.

It is important to note that the system produced by this method is consistent if and only if we have set up the decomposition correctly. If the system is inconsistent, there is an error in our decomposition.

Strategy two: Method of Strategic Substitution

The method of strategic substitution is based on the assumption that we have set up the decomposition correctly. If the decomposition is set up correctly, then there must be values of A, B, and C that satisfy Equation 4.5.2 for all values of x. That is, this equation must be true for any value of x we care to substitute into it. Therefore, by choosing values of x carefully and substituting them into the equation, we may find A, B, and C easily. For example, if we substitute $x = 0$, the equation reduces to $2 = A(-2)(1)$. Solving for A yields $A = -1$. Next, by substituting $x = 2$, the equation reduces to $8 = B(2)(3)$, or

equivalently $B=4/3$. Last, we substitute $x=-1$ into the equation and obtain $-1=C(-1)(-3)$. Solving, we have $C=-\dfrac{1}{3}$.

It is important to keep in mind that if we attempt to use this method with a decomposition that has not been set up correctly, we are still able to find values for the constants, but these constants are meaningless. If we do opt to use the method of strategic substitution, then it is a good idea to check the result by recombining the terms algebraically.

Now that we have the values of A, B, and C, we rewrite the original integral:

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx=\int\left(-\frac{1}{x}+\frac{4}{3}\cdot\frac{1}{x-2}-\frac{1}{3}\cdot\frac{1}{x+1}\right)dx.$$

Evaluating the integral gives us

$$\int \frac{3x+2}{x^3-x^2-2x}\,dx=-\ln|x|+\frac{4}{3}\ln|x-2|-\frac{1}{3}\ln|x+1|+C.$$

In the next example, we integrate a rational function in which the degree of the numerator is not less than the degree of the denominator.

Example 4.5.3: Dividing before Applying Partial Fractions

Evaluate $\displaystyle\int \frac{x^2+3x+1}{x^2-4}\,dx$.

Solution

Since $deg(x^2+3x+1)\geq deg(x^2-4)$, we must perform long division of polynomials. This results in

$$\frac{x^2+3x+1}{x^2-4}=1+\frac{3x+5}{x^2-4}$$

Next, we perform partial fraction decomposition on $\dfrac{3x+5}{x^2-4}=\dfrac{3x+5}{(x+2)(x-2)}$. We have

$$\frac{3x+5}{(x-2)(x+2)}=\frac{A}{x-2}+\frac{B}{x+2}.$$

Thus,

$$3x+5=A(x+2)+B(x-2).$$

Solving for A and B using either method, we obtain $A=11/4$ and $B=1/4$.

Rewriting the original integral, we have

$$\int \frac{x^2+3x+1}{x^2-4}\,dx=\int\left(1+\frac{11}{4}\cdot\frac{1}{x-2}+\frac{1}{4}\cdot\frac{1}{x+2}\right)dx.$$

Evaluating the integral produces

$$\int \frac{x^2+3x+1}{x^2-4}\,dx=x+\frac{11}{4}\ln|x-2|+\frac{1}{4}\ln|x+2|+C.$$

As we see in the next example, it may be possible to apply the technique of partial fraction decomposition to a nonrational function. The trick is to convert the nonrational function to a rational function through a substitution.

Example 4.5.4: Applying Partial Fractions after a Substitution

Evaluate $\displaystyle\int \frac{\cos x}{\sin^2 x-\sin x}\,dx$.

Solution

Let's begin by letting $u = \sin x$. Consequently, $du = \cos x\, dx$. After making these substitutions, we have

$$\int \frac{\cos x}{\sin^2 x - \sin x}\, dx = \int \frac{du}{u^2 - u} = \int \frac{du}{u(u-1)}.$$

Applying partial fraction decomposition to $\frac{1}{u(u-1)}$ gives $\frac{1}{u(u-1)} = -\frac{1}{u} + \frac{1}{u-1}$.

Thus,

$$\int \frac{\cos x}{\sin^2 x - \sin x}\, dx = -\ln|u| + \ln|u-1| + C = -\ln|\sin x| + \ln|\sin x - 1| + C.$$

? Exercise 4.5.2

Evaluate $\int \frac{x+1}{(x+3)(x-2)}\, dx$.

Hint

$$\frac{x+1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}$$

Answer

$$\frac{2}{5}\ln|x+3| + \frac{3}{5}\ln|x-2| + C$$

Repeated Linear Factors

For some applications, we need to integrate rational expressions that have denominators with repeated linear factors—that is, rational functions with at least one factor of the form $(ax+b)^n$, where n is a positive integer greater than or equal to 2. If the denominator contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}. \tag{4.5.3}$$

As we see in our next example, the basic technique used for solving for the coefficients is the same, but it requires more algebra to determine the numerators of the partial fractions.

✓ Example 4.5.5: Partial Fractions with Repeated Linear Factors

Evaluate $\int \frac{x-2}{(2x-1)^2(x-1)}\, dx$.

Solution

We have $deg(x-2) < deg((2x-1)^2(x-1))$, so we can proceed with the decomposition. Since $(2x-1)^2$ is a repeated linear factor, include

$$\frac{A}{2x-1} + \frac{B}{(2x-1)^2}$$

in the decomposition in Equation 4.5.3. Thus,

$$\frac{x-2}{(2x-1)^2(x-1)} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} + \frac{C}{x-1}.$$

After getting a common denominator and equating the numerators, we have

$$x - 2 = A(2x-1)(x-1) + B(x-1) + C(2x-1)^2. \tag{4.5.4}$$

We then use the method of equating coefficients to find the values of A, B, and C.

$$x-2=(2A+4C)x^2+(-3A+B-4C)x+(A-B+C).$$

Equating coefficients yields $2A+4C=0$, $-3A+B-4C=1$, and $A-B+C=-2$. Solving this system yields $A=2, B=3,$ and $C=-1$.

Alternatively, we can use the method of strategic substitution. In this case, substituting $x=1$ and $x=1/2$ into Equation 4.5.4 easily produces the values $B=3$ and $C=-1$. At this point, it may seem that we have run out of good choices for x, however, since we already have values for B and C, we can substitute in these values and choose any value for x not previously used. The value $x=0$ is a good option. In this case, we obtain the equation $-2=A(-1)(-1)+3(-1)+(-1)(-1)^2$ or, equivalently, $A=2$.

Now that we have the values for A, B, and C, we rewrite the original integral and evaluate it:

$$\begin{aligned}\int \frac{x-2}{(2x-1)^2(x-1)}\,dx &= \int\left(\frac{2}{2x-1}+\frac{3}{(2x-1)^2}-\frac{1}{x-1}\right)dx\\ &= \ln|2x-1|-\frac{3}{2(2x-1)}-\ln|x-1|+C.\end{aligned}$$

? Exercise 4.5.3

Set up the partial fraction decomposition for

$$\int \frac{x+2}{(x+3)^3(x-4)^2}\,dx.$$

(Do not solve for the coefficients or complete the integration.)

Hint

Use the problem-solving method of Example 4.5.5for guidance.

Answer

$$\frac{x+2}{(x+3)^3(x-4)^2}=\frac{A}{x+3}+\frac{B}{(x+3)^2}+\frac{C}{(x+3)^3}+\frac{D}{(x-4)}+\frac{E}{(x-4)^2}$$

The General Method

Now that we are beginning to get the idea of how the technique of partial fraction decomposition works, let's outline the basic method in the following problem-solving strategy.

Problem-Solving Strategy: Partial Fraction Decomposition

To decompose the rational function $P(x)/Q(x)$, use the following steps:

1. Make sure that $deg(P(x))<deg(Q(x))$. If not, perform long division of polynomials.
2. Factor $Q(x)$ into the product of linear and irreducible quadratic factors. An irreducible quadratic is a quadratic that has no real zeros.
3. Assuming that $deg(P(x))<deg(Q(x)$, the factors of $Q(x)$ determine the form of the decomposition of $P(x)/Q(x)$.
   a. If $Q(x)$ can be factored as $(a_1x+b_1)(a_2x+b_2)\ldots(a_nx+b_n)$, where each linear factor is distinct, then it is possible to find constants $A_1, A_2, \ldots A_n$ satisfying

$$\frac{P(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\cdots+\frac{A_n}{a_nx+b_n}.$$

   b. If $Q(x)$ contains the repeated linear factor $(ax+b)^n$, then the decomposition must contain

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_n}{(ax+b)^n}.$$

c. For each irreducible quadratic factor ax^2+bx+c that $Q(x)$ contains, the decomposition must include

$$\frac{Ax+B}{ax^2+bx+c}.$$

d. For each repeated irreducible quadratic factor $(ax^2+bx+c)^n$, the decomposition must include

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}.$$

e. After the appropriate decomposition is determined, solve for the constants.

f. Last, rewrite the integral in its decomposed form and evaluate it using previously developed techniques or integration formulas.

Simple Quadratic Factors

Now let's look at integrating a rational expression in which the denominator contains an irreducible quadratic factor. Recall that the quadratic ax^2+bx+c is irreducible if $ax^2+bx+c=0$ has no real zeros—that is, if $b^2-4ac<0$.

Example 4.5.6: Rational Expressions with an Irreducible Quadratic Factor

Evaluate

$$\int\frac{2x-3}{x^3+x}\,dx.$$

Solution

Since $deg(2x-3)<deg(x^3+x)$, factor the denominator and proceed with partial fraction decomposition. Since $x^3+x=x(x^2+1)$ contains the irreducible quadratic factor x^2+1, include $\frac{Ax+B}{x^2+1}$ as part of the decomposition, along with $\frac{C}{x}$ for the linear term x. Thus, the decomposition has the form

$$\frac{2x-3}{x(x^2+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x}.$$

After getting a common denominator and equating the numerators, we obtain the equation

$$2x-3=(Ax+B)x+C(x^2+1).$$

Solving for A, B, and C, we get $A=3$, $B=2$, and $C=-3$.

Thus,

$$\frac{2x-3}{x^3+x}=\frac{3x+2}{x^2+1}-\frac{3}{x}.$$

Substituting back into the integral, we obtain

$$\begin{aligned}\int\frac{2x-3}{x^3+x}\,dx &= \int\left(\frac{3x+2}{x^2+1}-\frac{3}{x}\right)dx \\ &= 3\int\frac{x}{x^2+1}\,dx+2\int\frac{1}{x^2+1}\,dx-3\int\frac{1}{x}\,dx && \text{Split up the integral} \\ &= \frac{3}{2}\ln|x^2+1|+2\tan^{-1}x-3\ln|x|+C. && \text{Evaluate each integral}\end{aligned}$$

Note: We may rewrite $\ln|x^2+1|=\ln(x^2+1)$, if we wish to do so, since $x^2+1>0$.

Example 4.5.7: Partial Fractions with an Irreducible Quadratic Factor

Evaluate $\int \frac{dx}{x^3-8}$.

Solution: We can start by factoring $x^3-8=(x-2)(x^2+2x+4)$. We see that the quadratic factor x^2+2x+4 is irreducible since $2^2-4(1)(4)=-12<0$. Using the decomposition described in the problem-solving strategy, we get

$$\frac{1}{(x-2)(x^2+2x+4)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+4}.$$

After obtaining a common denominator and equating the numerators, this becomes

$$1=A(x^2+2x+4)+(Bx+C)(x-2).$$

Applying either method, we get $A=\frac{1}{12}, B=-\frac{1}{12},$ and $C=-\frac{1}{3}$.

Rewriting $\int \frac{dx}{x^3-8}$, we have

$$\int \frac{dx}{x^3-8}=\frac{1}{12}\int \frac{1}{x-2}\,dx-\frac{1}{12}\int \frac{x+4}{x^2+2x+4}\,dx.$$

We can see that

$$\int \frac{1}{x-2}\,dx=\ln|x-2|+C,$$

but

$$\int \frac{x+4}{x^2+2x+4}\,dx$$

requires a bit more effort. Let's begin by **completing the square** on x^2+2x+4 to obtain

$$x^2+2x+4=(x+1)^2+3.$$

By letting $u=x+1$ and consequently $du=dx$, we see that

$$\begin{aligned}
\int \frac{x+4}{x^2+2x+4}\,dx &= \int \frac{x+4}{(x+1)^2+3}\,dx && \text{Complete the square on the denominator}\\
&= \int \frac{u+3}{u^2+3}\,du && \text{Substitute } u=x+1,\ x=u-1, \text{ and } du=dx\\
&= \int \frac{u}{u^2+3}du+\int \frac{3}{u^2+3}du && \text{Split the numerator apart}\\
&= \frac{1}{2}\ln|u^2+3|+\frac{3}{\sqrt{3}}\tan^{-1}\frac{u}{\sqrt{3}}+C && \text{Evaluate each integral}\\
&= \frac{1}{2}\ln|x^2+2x+4|+\sqrt{3}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C && \text{Rewrite in terms of } x \text{ and simplify}
\end{aligned}$$

Substituting back into the original integral and simplifying gives

$$\int \frac{dx}{x^3-8}=\frac{1}{12}\ln|x-2|-\frac{1}{24}\ln|x^2+2x+4|-\frac{\sqrt{3}}{12}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right)+C.$$

Here again, we can drop the absolute value if we wish to do so, since $x^2+2x+4>0$ for all x.

Example 4.5.8: Finding a Volume

Find the volume of the solid of revolution obtained by revolving the region enclosed by the graph of $f(x)=\dfrac{x^2}{(x^2+1)^2}$ and the x-axis over the interval $[0,1]$ about the y-axis.

Solution

Let's begin by sketching the region to be revolved (see Figure 4.5.1). From the sketch, we see that the shell method is a good choice for solving this problem.

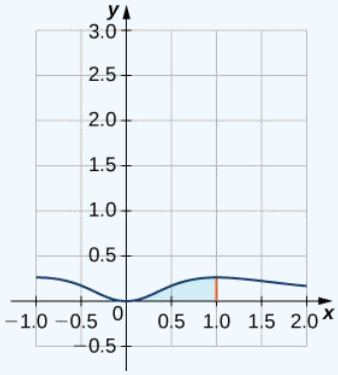


Figure 4.5.1: We can use the shell method to find the volume of revolution obtained by revolving the region shown about the y-axis.

The volume is given by

$$V=2\pi\int_0^1 x\cdot\frac{x^2}{(x^2+1)^2}\,dx=2\pi\int_0^1\frac{x^3}{(x^2+1)^2}\,dx.$$

Since $deg((x^2+1)^2)=4>3=deg(x^3),$ we can proceed with partial fraction decomposition. Note that $(x^2+1)^2$ is a repeated irreducible quadratic. Using the decomposition described in the problem-solving strategy, we get

$$\frac{x^3}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{(x^2+1)^2}.$$

Finding a common denominator and equating the numerators gives

$$x^3=(Ax+B)(x^2+1)+Cx+D.$$

Solving, we obtain $A=1, B=0, C=-1,$ and $D=0$. Substituting back into the integral, we have

$$V=2\pi\int_0^1\frac{x^3}{(x^2+1)^2}\,dx=2\pi\int_0^1\left(\frac{x}{x^2+1}-\frac{x}{(x^2+1)^2}\right)dx=2\pi\left(\frac{1}{2}\ln(x^2+1)+\frac{1}{2}\cdot\frac{1}{x^2+1}\right)\Big|_0^1=$$
$$\pi\left(\ln 2-\tfrac{1}{2}\right).$$

Exercise 4.5.4

Set up the partial fraction decomposition for

$$\int\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2}\,dx.$$

Hint

Use the problem-solving strategy.

Answer

$$\frac{x^2+3x+1}{(x+2)(x-3)^2(x^2+4)^2}=\frac{A}{x+2}+\frac{B}{x-3}+\frac{C}{(x-3)^2}+\frac{Dx+E}{x^2+4}+\frac{Fx+G}{(x^2+4)^2}$$

Key Concepts

- Partial fraction decomposition is a technique used to break down a rational function into a sum of simple rational functions that can be integrated using previously learned techniques.
- When applying partial fraction decomposition, we must make sure that the degree of the numerator is less than the degree of the denominator. If not, we need to perform long division before attempting partial fraction decomposition.
- The form the decomposition takes depends on the type of factors in the denominator. The types of factors include nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors, and repeated irreducible quadratic factors.

Glossary

partial fraction decomposition

a technique used to break down a rational function into the sum of simple rational functions

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4.5E: Exercises for Integration by Partial Fractions

Use partial fraction decomposition (or a simpler technique) to express the rational function as a sum or difference of two or more simpler rational expressions.

1) $\dfrac{1}{(x-3)(x-2)}$

2) $\dfrac{x^2+1}{x(x+1)(x+2)}$

Answer

$$\frac{x^2+1}{x(x+1)(x+2)} = -\frac{2}{x+1}+\frac{5}{2(x+2)}+\frac{1}{2x}$$

3) $\dfrac{1}{x^3-x}$

4) $\dfrac{3x+1}{x^2}$

Answer

$$\frac{3x+1}{x^2} = \frac{1}{x^2}+\frac{3}{x}$$

5) $\dfrac{3x^2}{x^2+1}$ (Hint: Use long division first.)

6) $\dfrac{2x^4}{x^2-2x}$

Answer

$$\frac{2x^4}{x^2-2x} = 2x^2+4x+8+\frac{16}{x-2}$$

7) $\dfrac{1}{(x-1)(x^2+1)}$

8) $\dfrac{1}{x^2(x-1)}$

Answer

$$\frac{1}{x^2(x-1)} = -\frac{1}{x^2}-\frac{1}{x}+\frac{1}{x-1}$$

9) $\dfrac{x}{x^2-4}$

10) $\dfrac{1}{x(x-1)(x-2)(x-3)}$

Answer

$$\frac{1}{x(x-1)(x-2)(x-3)} = -\frac{1}{2(x-2)}+\frac{1}{2(x-1)}-\frac{1}{6x}+\frac{1}{6(x-3)}$$

11) $\dfrac{1}{x^4-1} = \dfrac{1}{(x+1)(x-1)(x^2+1)}$

12) $\dfrac{3x^2}{x^3-1} = \dfrac{3x^2}{(x-1)(x^2+x+1)}$

Answer

$$\frac{3x^2}{x^3-1} = \frac{1}{x-1}+\frac{2x+1}{x^2+x+1}$$

13) $\dfrac{2x}{(x+2)^2}$

14) $\dfrac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2}$

Answer

$$\frac{3x^4+x^3+20x^2+3x+31}{(x+1)(x^2+4)^2} = \frac{2}{x+1}+\frac{x}{x^2+4}-\frac{1}{(x^2+4)^2}$$

In exercises 15 - 25, use the method of partial fractions to evaluate each of the following integrals.

15) $\int \frac{dx}{(x-3)(x-2)}$

16) $\int \frac{3x}{x^2+2x-8}\,dx$

Answer

$$\int \frac{3x}{x^2+2x-8}\,dx = 2\ln|x+4|+\ln|x-2|+C = \ln\left|(x+4)^2(x-2)\right|+C$$

17) $\int \frac{dx}{x^3-x}$

18) $\int \frac{x}{x^2-4}\,dx$

Answer

Note that you don't need Partial Fractions here. We use a simple u-substitution.

$$\int \frac{x}{x^2-4}\,dx = \tfrac{1}{2}\ln|4-x^2|+C$$

19) $\int \frac{dx}{x(x-1)(x-2)(x-3)}$

20) $\int \frac{2x^2+4x+22}{x^2+2x+10}\,dx$

Answer

Note that since the degree of the numerator is equal to the degree of the denominator, we need to start with long division.
Then note that we will need to use completing the square to continue since we cannot factor the trinomial in the denominator.

$$\int \frac{2x^2+4x+22}{x^2+2x+10}\,dx = 2\left(x+\tfrac{1}{3}\arctan\left(\frac{1+x}{3}\right)\right)+C$$

21) $\int \frac{dx}{x^2-5x+6}$

22) $\int \frac{2-x}{x^2+x}\,dx$

Answer

$$\int \frac{2-x}{x^2+x}\,dx = 2\ln|x|-3\ln|1+x|+C = \ln\left|\frac{x^2}{(1+x)^3}\right|+C$$

23) $\int \frac{2}{x^2-x-6}\,dx$

24) $\int \frac{dx}{x^3-2x^2-4x+8}$

Answer

$$\int \frac{dx}{x^3-2x^2-4x+8} = \tfrac{1}{16}\left(-\frac{4}{-2+x}-\ln|-2+x|+\ln|2+x|\right)+C = \tfrac{1}{16}\left(-\frac{4}{-2+x}+\ln\left|\frac{x+2}{x-2}\right|\right)+C$$

25) $\int \frac{dx}{x^4-10x^2+9}$

In exercises 26 - 29, evaluate the integrals with irreducible quadratic factors in the denominators.

26) $\int \frac{2}{(x-4)(x^2+2x+6)}\,dx$

Answer

$$\int \frac{2}{(x-4)(x^2+2x+6)}\,dx = \tfrac{1}{30}(-2\sqrt{5}\arctan\left[\frac{1+x}{\sqrt{5}}\right]+2\ln|-4+x|-\ln|6+2x+x^2|)+C$$

27) $\int \frac{x^2}{x^3-x^2+4x-4}\,dx$

28) $\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx$

Answer

Note that we need to use long division first, since the degree of the numerator is greater than the degree of the denominator.

$$\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}\,dx \quad = \quad -\frac{3}{x}+4\ln|x+2|+x+C$$

29) $\int \frac{x}{(x-1)(x^2+2x+2)^2}\,dx$

In exercises 30 - 32, use the method of partial fractions to evaluate the integrals.

30) $\int \frac{3x+4}{(x^2+4)(3-x)}\,dx$

Answer

$$\int \frac{3x+4}{(x^2+4)(3-x)}\,dx \quad = \quad -\ln|3-x|+\tfrac{1}{2}\ln|x^2+4|+C$$

31) $\int \frac{2}{(x+2)^2(2-x)}\,dx$

32) $\int \frac{3x+4}{x^3-2x-4}\,dx$ (Hint: Use the rational root theorem.)

Answer

$$\int \frac{3x+4}{x^3-2x-4}\,dx \quad = \quad \ln|x-2|-\tfrac{1}{2}\ln|x^2+2x+2|+C$$

In exercises 33 - 46, use substitution to convert the integrals to integrals of rational functions. Then use partial fractions to evaluate the integrals.

33) $\int_0^1 \frac{e^x}{36-e^{2x}}\,dx$ (Give the exact answer and the decimal equivalent. Round to five decimal places.)

34) $\int \frac{e^x}{e^{2x}-e^x}\,dx$

Answer

$$\int \frac{e^x}{e^{2x}-e^x}\,dx \quad = \quad -x+\ln|1-e^x|+C$$

35) $\int \frac{\sin x\,dx}{1-\cos^2 x}$

36) $\int \frac{\sin x}{\cos^2 x+\cos x-6}\,dx$

Answer

$$\int \frac{\sin x}{\cos^2 x+\cos x-6}\,dx \quad = \quad \tfrac{1}{5}\ln\left|\frac{\cos x+3}{\cos x-2}\right|+C$$

37) $\int \frac{1-\sqrt{x}}{1+\sqrt{x}}\,dx$

38) $\int \frac{dt}{(e^t-e^{-t})^2}$

Answer

$$\int \frac{dt}{(e^t-e^{-t})^2} \quad = \quad \frac{1}{2-2e^{2t}}+C$$

39) $\int \frac{1+e^x}{1-e^x}\,dx$

40) $\int \frac{dx}{1+\sqrt{x+1}}$

Answer

$$\int \frac{dx}{1+\sqrt{x+1}} \quad = \quad 2\sqrt{1+x}-2\ln|1+\sqrt{1+x}|+C$$

41) $\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}}$

42) $\int \frac{\cos x}{\sin x(1-\sin x)}\,dx$

Answer

$$\int \frac{\cos x}{\sin x(1-\sin x)}\,dx \quad = \quad \ln\left|\frac{\sin x}{1-\sin x}\right|+C$$

43) $\int \frac{e^x}{(e^{2x}-4)^2}\,dx$

44) $\int_1^2 \frac{1}{x^2\sqrt{4-x^2}}\,dx$

Answer

$$\int_1^2 \frac{1}{x^2\sqrt{4-x^2}}\,dx \quad = \quad \frac{\sqrt{3}}{4}$$

45) $\int \frac{1}{2+e^{-x}}\,dx$

46) $\int \frac{1}{1+e^x}\,dx$

Answer

$$\int \frac{1}{1+e^x}\,dx \quad = \quad x-\ln(1+e^x)+C$$

In exercises 47 - 48, use the given substitution to convert the integral to an integral of a rational function, then evaluate.

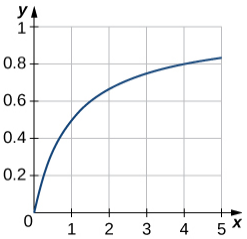
47) $\int \frac{1}{t-\sqrt[3]{t}}\,dt; \quad t=x^3$

48) $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx; \quad x=u^6$

Answer

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx \quad = \quad 6x^{1/6}-3x^{1/3}+2\sqrt{x}-6\ln(1+x^{1/6})+C$$

49) Graph the curve $y=\frac{x}{1+x}$ over the interval $[0,5]$. Then, find the area of the region bounded by the curve, the x-axis, and the line $x=4$.



50) Find the volume of the solid generated when the region bounded by $y=\frac{1}{\sqrt{x(3-x)}}$, $y=0$, $x=1$, and $x=2$ is revolved about the x-axis.

Answer

$V=\frac{4}{3}\pi\operatorname{arctanh}\left[\frac{1}{3}\right]=\frac{1}{3}\pi\ln 4$ units^3

51) The velocity of a particle moving along a line is a function of time given by $v(t)=\frac{88t^2}{t^2+1}$. Find the distance that the particle has traveled after $t=5$ sec.

In exercises 52 - 54, solve the initial-value problem for x as a function of t.

52) $(t^2-7t+12)\frac{dx}{dt}=1, \quad t>4,\ x(5)=0$

Answer

$$x=-\ln|t-3|+\ln|t-4|+\ln 2=\ln\left|\frac{2(t-4)}{t-3}\right|$$

53) $(t+5)\dfrac{dx}{dt} = x^2+1, \quad t > -5,\ x(1) = \tan 1$

54) $(2t^3 - 2t^2 + t - 1)\dfrac{dx}{dt} = 3, \quad x(2) = 0$

Answer

$x = \ln|t-1| - \sqrt{2}\arctan(\sqrt{2}t) - \frac{1}{2}\ln(t^2+\frac{1}{2}) + \sqrt{2}\arctan(2\sqrt{2}) + \frac{1}{2}\ln 4.5$

55) Find the x-coordinate of the centroid of the area bounded by $y(x^2-9) = 1,\ y = 0,\ x = 4,$ and $x = 5$. (Round the answer to two decimal places.)

56) Find the volume generated by revolving the area bounded by $y = \dfrac{1}{x^3+7x^2+6x}$, $x = 1,\ x = 7$, and $y = 0$ about the y-axis.

Answer

$V = \frac{2}{5}\pi \ln\frac{28}{13}$ units^3

57) Find the area bounded by $y = \dfrac{x-12}{x^2-8x-20}$, $y = 0,\ x = 2,$ and $x = 4$. (Round the answer to the nearest hundredth.)

58) Evaluate the integral $\displaystyle\int \frac{dx}{x^3+1}$.

Answer

$$\int \frac{dx}{x^3+1} = \frac{\arctan[\frac{-1+2x}{\sqrt{3}}]}{\sqrt{3}} + \frac{1}{3}\ln|1+x| - \frac{1}{6}\ln|\,1-x+x^2\,| + C$$

For problems 59 - 62, use the substitutions $\tan(\frac{x}{2}) = t$, $dx = \dfrac{2}{1+t^2}\,dt$, $\sin x = \dfrac{2t}{1+t^2}$, **and** $\cos x = \dfrac{1-t^2}{1+t^2}$.

59) $\displaystyle\int \frac{dx}{3-5\sin x}$

60) Find the area under the curve $y = \dfrac{1}{1+\sin x}$ between $x = 0$ and $x = \pi$. (Assume the dimensions are in inches.)

Answer

2.0 in.^{2}

61) Given $\tan\left(\frac{x}{2}\right) = t,$ derive the formulas $dx = \dfrac{2}{1+t^2}dt,$ $\sin x = \dfrac{2t}{1+t^2}$, and $\cos x = \dfrac{1-t^2}{1+t^2}$.

62) Evaluate $\displaystyle\int \frac{\sqrt[3]{x-8}}{x}\,dx$.

Answer

$$\int \frac{\sqrt[3]{x-8}}{x}\,dx = 3(-8+x)^{1/3} - 2\sqrt{3}\arctan\left[\frac{-1+(-8+x)^{1/3}}{\sqrt{3}}\right] - 2\ln\left[2+(-8+x)^{1/3}\right] + \ln\left[4-2(-8+x)^{1/3}+(-8+x)^{2/3}\right] + C$$

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4.6: Other Strategies for Integration

Learning Objectives

- Use a table of integrals to solve integration problems.
- Use a computer algebra system (CAS) to solve integration problems.

In addition to the techniques of integration we have already seen, several other tools are widely available to assist with the process of integration. Among these tools are **integration tables**, which are readily available in many books, including the appendices to this one. Also widely available are **computer algebra systems (CAS)**, which are found on calculators and in many campus computer labs, and are free online.

Tables of Integrals

Integration tables, if used in the right manner, can be a handy way either to evaluate or check an integral quickly. Keep in mind that when using a table to check an answer, it is possible for two completely correct solutions to look very different. For example, in Trigonometric Substitution, we found that, by using the substitution $x = \tan\theta$, we can arrive at

$$\int \frac{dx}{\sqrt{1+x^2}} = \ln\left|x+\sqrt{x^2+1}\right| + C.$$

However, using $x = \sinh\theta$, we obtained a different solution—namely,

$$\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C.$$

We later showed algebraically that the two solutions are equivalent. That is, we showed that $\sinh^{-1} x = \ln\left|x+\sqrt{x^2+1}\right|$. In this case, the two antiderivatives that we found were actually equal. This need not be the case. However, as long as the difference in the two antiderivatives is a constant, they are equivalent.

Example 4.6.1: Using a Formula from a Table to Evaluate an Integral

Use the table formula

$$\int \frac{\sqrt{a^2-u^2}}{u^2}du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C$$

to evaluate $\displaystyle\int \frac{\sqrt{16-e^{2x}}}{e^x}dx$.

Solution

If we look at integration tables, we see that several formulas contain expressions of the form $\sqrt{a^2-u^2}$. This expression is actually similar to $\sqrt{16-e^{2x}}$, where $a=4$ and $u=e^x$. Keep in mind that we must also have $du=e^x$. Multiplying the numerator and the denominator of the given integral by e^x should help to put this integral in a useful form. Thus, we now have

$$\int \frac{\sqrt{16-e^{2x}}}{e^x}dx = \int \frac{\sqrt{16-e^{2x}}}{e^{2x}}e^x dx.$$

Substituting $u=e^x$ and $du=e^x\,dx$ produces $\displaystyle\int \frac{\sqrt{a^2-u^2}}{u^2}du$. From the integration table (#88 in Appendix A),

$$\int \frac{\sqrt{a^2-u^2}}{u^2}du = -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\frac{u}{a} + C.$$

Thus,

$$\begin{aligned}\int \frac{\sqrt{16-e^{2x}}}{e^x}dx &= \int \frac{\sqrt{16-e^{2x}}}{e^{2x}}e^x dx \text{ Substitute } u=e^x \text{ and } du=e^x dx.\\ &= \int \frac{\sqrt{4^2-u^2}}{u^2}du \text{ Apply the formula using } a=4.\end{aligned}$$

$$= -\frac{\sqrt{4^2-u^2}}{u} - \sin^{-1}\frac{u}{4} + C \quad \text{Substitute } u = e^x.$$
$$= -\frac{\sqrt{16-e^{2x}}}{e^x} - \sin^{-1}\left(\frac{e^x}{4}\right) + C$$

Computer Algebra Systems

If available, a CAS is a faster alternative to a table for solving an integration problem. Many such systems are widely available and are, in general, quite easy to use.

Example 4.6.2: Using a Computer Algebra System to Evaluate an Integral

Use a computer algebra system to evaluate $\int \frac{dx}{\sqrt{x^2-4}}$. Compare this result with $\ln\left|\frac{\sqrt{x^2-4}}{2} + \frac{x}{2}\right| + C$, a result we might have obtained if we had used trigonometric substitution.

Solution

Using Wolfram Alpha, we obtain

$$\int \frac{dx}{\sqrt{x^2-4}} = \ln\left|\sqrt{x^2-4} + x\right| + C.$$

Notice that

$$\ln\left|\frac{\sqrt{x^2-4}}{2} + \frac{x}{2}\right| + C = \ln\left|\frac{\sqrt{x^2-4}+x}{2}\right| + C = \ln\left|\sqrt{x^2-4}+x\right| - \ln 2 + C.$$

Since these two antiderivatives differ by only a constant, the solutions are equivalent. We could have also demonstrated that each of these antiderivatives is correct by differentiating them.

You can access an integral calculator for more examples.

Example 4.6.3: Using a CAS to Evaluate an Integral

Evaluate $\int \sin^3 x\, dx$ using a CAS. Compare the result to $\frac{1}{3}\cos^3 x - \cos x + C$, the result we might have obtained using the technique for integrating odd powers of $\sin x$ discussed earlier in this chapter.

Solution

Using Wolfram Alpha, we obtain

$$\int \sin^3 x\, dx = \frac{1}{12}(\cos(3x) - 9\cos x) + C.$$

This looks quite different from $\frac{1}{3}\cos^3 x - \cos x + C$. To see that these antiderivatives are equivalent, we can make use of a few trigonometric identities:

$$\begin{aligned}
\frac{1}{12}(\cos(3x) - 9\cos x) &= \frac{1}{12}(\cos(x+2x) - 9\cos x) \\
&= \frac{1}{12}(\cos(x)\cos(2x) - \sin(x)\sin(2x) - 9\cos x) \\
&= \frac{1}{12}(\cos x(2\cos^2 x - 1) - \sin x(2\sin x\cos x) - 9\cos x) \\
&= \frac{1}{12}(2\cos^3 x - \cos x - 2\cos x(1-\cos^2 x) - 9\cos x) \\
&= \frac{1}{12}(4\cos^3 x - 12\cos x)
\end{aligned}$$

$$= \frac{1}{3}\cos^3 x - \cos x.$$

Thus, the two antiderivatives are identical.

We may also use a CAS to compare the graphs of the two functions, as shown in the following figure.

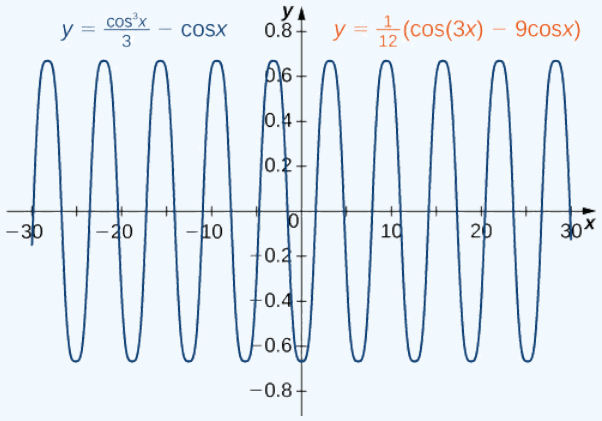


Figure 4.6.1: The graphs of $y = \frac{1}{3}\cos^3 x - \cos x$ and $y = \frac{1}{12}(\cos(3x) - 9\cos x)$ are identical.

Exercise 4.6.1

Use a CAS to evaluate $\int \frac{dx}{\sqrt{x^2+4}}$.

Hint

Answers may vary.

Answer

Possible solutions include $\sinh^{-1}\left(\frac{x}{2}\right) + C$ and $\ln\left|\sqrt{x^2+4} + x\right| + C$.

Key Concepts

- An integration table may be used to evaluate indefinite integrals.
- A CAS (or computer algebra system) may be used to evaluate indefinite integrals.
- It may require some effort to reconcile equivalent solutions obtained using different methods.

Glossary

computer algebra system (CAS)

technology used to perform many mathematical tasks, including integration

integration table

a table that lists integration formulas

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4.6E: Exercises for Section 7.5

Use a table of integrals to evaluate the following integrals.

1) $\int_0^4 \frac{x}{\sqrt{1+2x}}\,dx$

2) $\int \frac{x+3}{x^2+2x+2}\,dx$

Answer

$$\int \frac{x+3}{x^2+2x+2}\,dx = \tfrac{1}{2}\ln|x^2+2x+2| + 2\arctan(x+1) + C$$

3) $\int x^3\sqrt{1+2x^2}\,dx$

4) $\int \frac{1}{\sqrt{x^2+6x}}\,dx$

Answer

$$\int \frac{1}{\sqrt{x^2+6x}}\,dx = \cosh^{-1}\left(\frac{x+3}{3}\right) + C$$

5) $\int \frac{x}{x+1}\,dx$

6) $\int x\cdot 2^{x^2}\,dx$

Answer

$$\int x\cdot 2^{x^2}\,dx = \frac{2^{x^2-1}}{\ln 2} + C$$

7) $\int \frac{1}{4x^2+25}\,dx$

8) $\int \frac{dy}{\sqrt{4-y^2}}$

Answer

$$\int \frac{dy}{\sqrt{4-y^2}} = \arcsin\left(\frac{y}{2}\right) + C$$

9) $\int \sin^3(2x)\cos(2x)\,dx$

10) $\int \csc(2w)\cot(2w)\,dw$

Answer

$$\int \csc(2w)\cot(2w)\,dw = -\tfrac{1}{2}\csc(2w) + C$$

11) $\int 2^y\,dy$

12) $\int_0^1 \frac{3x}{\sqrt{x^2+8}}\,dx$

Answer

$$\int_0^1 \frac{3x}{\sqrt{x^2+8}}\,dx = 9-6\sqrt{2}$$

13) $\int_{-1/4}^{1/4} \sec^2(\pi x)\tan(\pi x)\,dx$

14) $\int_0^{\pi/2} \tan^2\left(\frac{x}{2}\right)dx$

Answer

$$\int_0^{\pi/2} \tan^2\left(\frac{x}{2}\right)dx = 2-\frac{\pi}{2}$$

15) $\int \cos^3 x\,dx$

16) $\int \tan^5(3x)\,dx$

Answer

$$\int \tan^5(3x)\,dx = \tfrac{1}{12}\tan^4(3x) - \tfrac{1}{6}\tan^2(3x) + \tfrac{1}{3}\ln|\sec 3x| + C$$

17) $\int \sin^2 y\cos^3 y\,dy$

Use a CAS to evaluate the following integrals. Tables can also be used to verify the answers.

18) [T] $\int \frac{dw}{1+\sec\left(\frac{w}{2}\right)}$

Answer

$$\int \frac{dw}{1+\sec\left(\frac{w}{2}\right)} = 2\cot\left(\tfrac{w}{2}\right) - 2\csc\left(\tfrac{w}{2}\right) + w + C$$

19) [T] $\int \frac{dw}{1-\cos(7w)}$

20) [T] $\int_0^t \frac{dt}{4\cos t + 3\sin t}$

Answer

$$\int_0^t \frac{dt}{4\cos t+3\sin t} = \tfrac{1}{5}\ln\left|\frac{2(5+4\sin t-3\cos t)}{4\cos t+3\sin t}\right|$$

21) [T] $\int \frac{\sqrt{x^2-9}}{3x}\,dx$

22) [T] $\int \frac{dx}{x^{1/2}+x^{1/3}}$

Answer

$$\int \frac{dx}{x^{1/2}+x^{1/3}} = 6x^{1/6} - 3x^{1/3} + 2\sqrt{x} - 6\ln[1+x^{1/6}] + C$$

23) [T] $\int \frac{dx}{x\sqrt{x-1}}$

24) [T] $\int x^3 \sin x \, dx$

Answer

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C$$

25) [T] $\int x\sqrt{x^4 - 9} \, dx$

26) [T] $\int \frac{x}{1 + e^{-x^2}} \, dx$

Answer

$$\int \frac{x}{1 + e^{-x^2}} \, dx = \frac{1}{2}\left(x^2 + \ln|1 + e^{-x^2}|\right) + C$$

27) [T] $\int \frac{\sqrt{3 - 5x}}{2x} \, dx$

28) [T] $\int \frac{dx}{x\sqrt{x - 1}}$

Answer

$$\int \frac{dx}{x\sqrt{x - 1}} = 2 \arctan\left(\sqrt{x - 1}\right) + C$$

29) [T] $\int e^x \cos^{-1}(e^x) \, dx$

Use a calculator or CAS to evaluate the following integrals.

30) [T] $\int_0^{\pi/4} \cos 2x \, dx$

Answer

$$\int_0^{\pi/4} \cos 2x \, dx = 0.5 = \frac{1}{2}$$

31) [T] $\int_0^1 x \cdot e^{-x^2} \, dx$

32) [T] $\int_0^8 \frac{2x}{\sqrt{x^2 + 36}} \, dx$

Answer

$$\int_0^8 \frac{2x}{\sqrt{x^2 + 36}} \, dx = 8.0$$

33) [T] $\int_0^{2/\sqrt{3}} \frac{1}{4 + 9x^2} \, dx$

34) [T] $\int \frac{dx}{x^2 + 4x + 13}$

Answer

$$\int \frac{dx}{x^2 + 4x + 13} = \frac{1}{3}\arctan\left(\frac{1}{3}(x + 2)\right) + C$$

35) [T] $\int \frac{dx}{1+\sin x}$

Use tables to evaluate the integrals. You may need to complete the square or change variables to put the integral into a form given in the table.

36) $\int \frac{dx}{x^2+2x+10}$

Answer

$$\int \frac{dx}{x^2+2x+10} = \tfrac{1}{3}\arctan\left(\frac{x+1}{3}\right)+C$$

37) $\int \frac{dx}{\sqrt{x^2-6x}}$

38) $\int \frac{e^x}{\sqrt{e^{2x}-4}}\,dx$

Answer

$$\int \frac{e^x}{\sqrt{e^{2x}-4}}\,dx = \ln\left(e^x+\sqrt{4+e^{2x}}\right)+C$$

39) $\int \frac{\cos x}{\sin^2 x+2\sin x}\,dx$

40) $\int \frac{\arctan(x^3)}{x^4}\,dx$

Answer

$$\int \frac{\arctan(x^3)}{x^4}\,dx = \ln x - \tfrac{1}{6}\ln(x^6+1) - \frac{\arctan(x^3)}{3x^3}+C$$

41) $\int \frac{\ln|x|\arcsin(\ln|x|)}{x}\,dx$

Use tables to perform the integration.

42) $\int \frac{dx}{\sqrt{x^2+16}}$

Answer

$$\int \frac{dx}{\sqrt{x^2+16}} = \ln|x|+\sqrt{16+x^2}\ |+C$$

43) $\int \frac{3x}{2x+7}\,dx$

44) $\int \frac{dx}{1-\cos 4x}$

Answer

$$\int \frac{dx}{1-\cos 4x} = -\frac{1}{4}\cot 2x+C$$

45) $\int \frac{dx}{\sqrt{4x+1}}$

46) Find the area bounded by $y(4+25x^2)=5,\ x=0,\ y=0,$ and $x=4$. Use a table of integrals or a CAS.

Answer

$\frac{1}{2}\arctan 10 \text{ units}^2$

47) The region bounded between the curve $y = \dfrac{1}{\sqrt{1+\cos x}}$, $0.3 \le x \le 1.1$, and the x-axis is revolved about the x-axis to generate a solid. Use a table of integrals to find the volume of the solid generated. (Round the answer to two decimal places.)

48) Use substitution and a table of integrals to find the area of the surface generated by revolving the curve $y = e^x$, $0 \le x \le 3$, about the x-axis. (Round the answer to two decimal places.)

Answer

1276.14units^2

49) [T] Use an integral table and a calculator to find the area of the surface generated by revolving the curve $y = \dfrac{x^2}{2}$, $0 \le x \le 1$, about the x-axis. (Round the answer to two decimal places.)

50) [T] Use a CAS or tables to find the area of the surface generated by revolving the curve $y = \cos x$, $0 \le x \le \frac{\pi}{2}$, about the x-axis. (Round the answer to two decimal places.)

Answer

7.21units^2

51) Find the length of the curve $y = \dfrac{x^2}{4}$ over $[0, 8]$.

52) Find the length of the curve $y = e^x$ over $[0, \ln(2)]$.

Answer

$\left(\sqrt{5} - \sqrt{2} + \ln\left|\frac{2+2\sqrt{2}}{1+\sqrt{5}}\right|\right)$ units

53) Find the area of the surface formed by revolving the graph of $y = 2\sqrt{x}$ over the interval $[0, 9]$ about the x-axis.

54) Find the average value of the function $f(x) = \dfrac{1}{x^2+1}$ over the interval $[-3, 3]$.

Answer

$\frac{1}{3}\arctan(3) \approx 0.416$

55) Approximate the arc length of the curve $y = \tan \pi x$ over the interval $\left[0, \frac{1}{4}\right]$. (Round the answer to three decimal places.)

Contributors

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4.7: Numerical Integration

Learning Objectives

- Approximate the value of a definite integral by using the midpoint and trapezoidal rules.
- Determine the absolute and relative error in using a numerical integration technique.
- Estimate the absolute and relative error using an error-bound formula.
- Recognize when the midpoint and trapezoidal rules over- or underestimate the true value of an integral.
- Use Simpson's rule to approximate the value of a definite integral to a given accuracy.

The antiderivatives of many functions either cannot be expressed or cannot be expressed easily in closed form (that is, in terms of known functions). Consequently, rather than evaluate definite integrals of these functions directly, we resort to various techniques of **numerical integration** to approximate their values. In this section, we explore several of these techniques. In addition, we examine the process of estimating the error in using these techniques.

The Midpoint Rule

Earlier in this text we defined the definite integral of a function over an interval as the limit of **Riemann sums**. In general, any Riemann sum of a function $f(x)$ over an interval $[a, b]$ may be viewed as an estimate of $\int_a^b f(x)\,dx$. Recall that a Riemann sum of a function $f(x)$ over an interval $[a, b]$ is obtained by selecting a partition

$$P = \{x_0, x_1, x_2, \ldots, x_n\}$$

where $a = x_0 < x_1 < x_2 < \cdots < x_n = b$

and a set

$$S = \{x_1^*, x_2^*, \ldots, x_n^*\}$$

where $x_{i-1} \le x_i^* \le x_i \quad$ for all i.

The Riemann sum corresponding to the partition P and the set S is given by $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$, the length of the ith subinterval.

The midpoint rule for estimating a definite integral uses a Riemann sum with subintervals of equal width and the midpoints, m_i, of each subinterval in place of x_i^*. Formally, we state a theorem regarding the convergence of the midpoint rule as follows.

The Midpoint Rule

Assume that $f(x)$ is continuous on $[a, b]$. Let n be a positive integer and $\Delta x = \dfrac{b-a}{n}$. If $[a, b]$ is divided into n subintervals, each of length Δx, and m_i is the midpoint of the ith subinterval, set

$$M_n = \sum_{i=1}^{n} f(m_i)\Delta x.$$

Then $\lim_{n\to\infty} M_n = \int_a^b f(x)\,dx$.

As we can see in Figure 4.7.1, if $f(x) \ge 0$ over $[a, b]$, then $\sum_{i=1}^{n} f(m_i)\Delta x$ corresponds to the sum of the areas of rectangles approximating the area between the graph of $f(x)$ and the x-axis over $[a, b]$. The graph shows the rectangles corresponding to M_4 for a nonnegative function over a closed interval $[a, b]$.

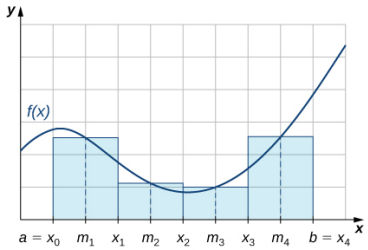


Figure 4.7.1: The midpoint rule approximates the area between the graph of $f(x)$ and the x-axis by summing the areas of rectangles with midpoints that are points on $f(x)$.

Example 4.7.1: Using the Midpoint Rule with M_4

Use the midpoint rule to estimate $\int_0^1 x^2\,dx$ using four subintervals. Compare the result with the actual value of this integral.

Solution: Each subinterval has length $\Delta x = \dfrac{1-0}{4} = \dfrac{1}{4}$. Therefore, the subintervals consist of

$$\left[0, \tfrac{1}{4}\right], \left[\tfrac{1}{4}, \tfrac{1}{2}\right], \left[\tfrac{1}{2}, \tfrac{3}{4}\right], \text{ and } \left[\tfrac{3}{4}, 1\right].$$

The midpoints of these subintervals are $\left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$. Thus,

$$\begin{aligned} M_4 &= \frac{1}{4}\cdot f\left(\frac{1}{8}\right)+\frac{1}{4}\cdot f\left(\frac{3}{8}\right)+\frac{1}{4}\cdot f\left(\frac{5}{8}\right)+\frac{1}{4}\cdot f\left(\frac{7}{8}\right) \\ &= \frac{1}{4}\cdot\frac{1}{64}+\frac{1}{4}\cdot\frac{9}{64}+\frac{1}{4}\cdot\frac{25}{64}+\frac{1}{4}\cdot\frac{49}{64} \\ &= \frac{21}{64}=0.328125. \end{aligned}$$

Since

$$\int_0^1 x^2\,dx=\frac{1}{3},$$

the absolute error in this approximation is:

$$\left|\frac{1}{3}-\frac{21}{64}\right|=\frac{1}{192}\approx 0.0052,$$

and we see that the midpoint rule produces an estimate that is somewhat close to the actual value of the definite integral.

Example 4.7.2: Using the Midpoint Rule with M_6

Use M_6 to estimate the length of the curve $y=\frac{1}{2}x^2$ on $[1,4]$.

Solution: The length of $y=\frac{1}{2}x^2$ on $[1,4]$ is

$$s=\int_1^4\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Since $\dfrac{dy}{dx}=x$, this integral becomes $\displaystyle\int_1^4\sqrt{1+x^2}\,dx$.

If $[1,4]$ is divided into six subintervals, then each subinterval has length $\Delta x=\dfrac{4-1}{6}=\dfrac{1}{2}$ and the midpoints of the subintervals are $\left\{\frac{5}{4},\frac{7}{4},\frac{9}{4},\frac{11}{4},\frac{13}{4},\frac{15}{4}\right\}$. If we set $f(x)=\sqrt{1+x^2}$,

$$\begin{aligned} M_6 &= \tfrac{1}{2}\cdot f\left(\frac{5}{4}\right)+\tfrac{1}{2}\cdot f\left(\frac{7}{4}\right)+\frac{1}{2}\cdot f\left(\frac{9}{4}\right)+\frac{1}{2}\cdot f\left(\frac{11}{4}\right)+\frac{1}{2}\cdot f\left(\frac{13}{4}\right)+\frac{1}{2}\cdot f\left(\frac{15}{4}\right) \\ &\approx \frac{1}{2}(1.6008+2.0156+2.4622+2.9262+3.4004+3.8810)=8.1431 \text{ units.} \end{aligned}$$

Exercise 4.7.1

Use the midpoint rule with $n=2$ to estimate $\displaystyle\int_1^2\frac{1}{x}\,dx$.

Hint

$\Delta x=\frac{1}{2}$, $m_1=\frac{5}{4}$, and $m_2=\frac{7}{4}$.

Answer

$\dfrac{24}{35}\approx 0.685714$

The Trapezoidal Rule

We can also approximate the value of a definite integral by using trapezoids rather than rectangles. In Figure 4.7.2, the area beneath the curve is approximated by trapezoids rather than by rectangles.

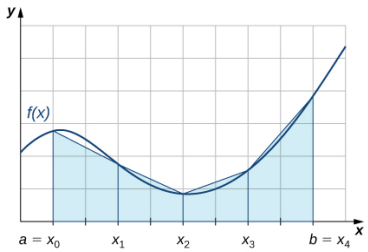


Figure 4.7.2: Trapezoids may be used to approximate the area under a curve, hence approximating the definite integral.

The **trapezoidal rule** for estimating definite integrals uses trapezoids rather than rectangles to approximate the area under a curve. To gain insight into the final form of the rule, consider the trapezoids shown in Figure 4.7.2. We assume that the length of each subinterval is given by Δx. First, recall that the area of a trapezoid with a height of h and bases of length b_1 and b_2 is given by $\text{Area}=\frac{1}{2}h(b_1+b_2)$. We see that the first trapezoid has a height Δx and parallel bases of length $f(x_0)$ and $f(x_1)$. Thus, the area of the first trapezoid in Figure 4.7.2 is

$$\frac{1}{2}\Delta x\Big(f(x_0)+f(x_1)\Big).$$

The areas of the remaining three trapezoids are

$$\frac{1}{2}\Delta x\Big(f(x_1)+f(x_2)\Big),\ \frac{1}{2}\Delta x\Big(f(x_2)+f(x_3)\Big),\text{ and }\frac{1}{2}\Delta x\Big(f(x_3)+f(x_4)\Big).$$

Consequently,

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\Delta x\Big(f(x_0)+f(x_1)\Big)+\frac{1}{2}\Delta x\Big(f(x_1)+f(x_2)\Big)+\frac{1}{2}\Delta x\Big(f(x_2)+f(x_3)\Big)+\frac{1}{2}\Delta x\Big(f(x_3)+f(x_4)\Big).$$

After taking out a common factor of $\frac{1}{2}\Delta x$ and combining like terms, we have

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\Big[f(x_0)+2\,f(x_1)+2\,f(x_2)+2\,f(x_3)+f(x_4)\Big].$$

Generalizing, we formally state the following rule.

The Trapezoidal Rule

Assume that $f(x)$ is continuous over $[a,b]$. Let n be a positive integer and $\Delta x = \dfrac{b-a}{n}$. Let $[a,b]$ be divided into n subintervals, each of length Δx, with endpoints at $P=\{x_0,x_1,x_2\ldots,x_n\}$.

Set

$$T_n = \frac{\Delta x}{2}\Big[f(x_0)+2\,f(x_1)+2\,f(x_2)+\cdots+2\,f(x_{n-1})+f(x_n)\Big].$$

Then, $\lim\limits_{n\to+\infty} T_n = \int_a^b f(x)\,dx$.

Before continuing, let's make a few observations about the trapezoidal rule. First of all, it is useful to note that

$$T_n = \frac{1}{2}(L_n+R_n) \text{ where } L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x \text{ and } R_n = \sum_{i=1}^{n} f(x_i)\Delta x.$$

That is, L_n and R_n approximate the integral using the left-hand and right-hand endpoints of each subinterval, respectively. In addition, a careful examination of Figure 4.7.3 leads us to make the following observations about using the trapezoidal rules and midpoint rules to estimate the definite integral of a nonnegative function. The trapezoidal rule tends to overestimate the value of a definite integral systematically over intervals where the function is concave up and to underestimate the value of a definite integral systematically over intervals where the function is concave down. On the other hand, the midpoint rule tends to average out these errors somewhat by partially overestimating and partially underestimating the value of the definite integral over these same types of intervals. This leads us to hypothesize that, in general, the midpoint rule tends to be more accurate than the trapezoidal rule.

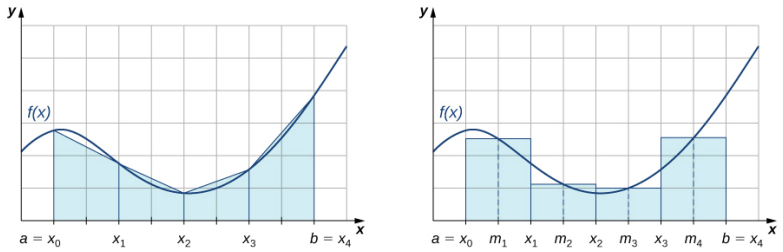


Figure 4.7.3:The trapezoidal rule tends to be less accurate than the midpoint rule.

Example 4.7.3: Using the Trapezoidal Rule

Use the trapezoidal rule to estimate $\int_0^1 x^2\,dx$ using four subintervals.

Solution

The endpoints of the subintervals consist of elements of the set $P=\left\{0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right\}$ and $\Delta x = \frac{1-0}{4}=\frac{1}{4}$. Thus,

$$\begin{aligned}\int_0^1 x^2dx &\approx \frac{1}{2}\cdot\frac{1}{4}\Big[f(0)+2\,f\left(\tfrac{1}{4}\right)+2\,f\left(\tfrac{1}{2}\right)+2\,f\left(\tfrac{3}{4}\right)+f(1)\Big]\\ &= \tfrac{1}{8}\left(0+2\cdot\tfrac{1}{16}+2\cdot\tfrac{1}{4}+2\cdot\tfrac{9}{16}+1\right)\\ &= \frac{11}{32}=0.34375\end{aligned}$$

Exercise 4.7.2

Use the trapezoidal rule with $n=2$ to estimate $\int_1^2 \frac{1}{x}\,dx$.

Hint

Set $\Delta x = \dfrac{1}{2}$. The endpoints of the subintervals are the elements of the set $P=\left\{1,\frac{3}{2},2\right\}$.

Answer

$\dfrac{17}{24}\approx 0.708333$

Absolute and Relative Error

An important aspect of using these numerical approximation rules consists of calculating the error in using them for estimating the value of a definite integral. We first need to define absolute error and relative error.

Definition: absolute and relative error

If B is our estimate of some quantity having an actual value of A, then the **absolute error** is given by $|A-B|$.

The **relative error** is the error as a percentage of the actual value and is given by

$$\left|\frac{A-B}{A}\right| \cdot 100\%.$$

Example 4.7.4: Calculating Error in the Midpoint Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the midpoint rule, found in Example 4.7.1.

Solution: The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $M_4 = \frac{21}{64}$. Thus, the absolute error is given by $\left|\frac{1}{3} - \frac{21}{64}\right| = \frac{1}{192} \approx 0.0052$.

The relative error is

$$\frac{1/192}{1/3} = \frac{1}{64} \approx 0.015625 \approx 1.6\%.$$

Example 4.7.5: Calculating Error in the Trapezoidal Rule

Calculate the absolute and relative error in the estimate of $\int_0^1 x^2\,dx$ using the trapezoidal rule, found in Example 4.7.3.

Solution: The calculated value is $\int_0^1 x^2\,dx = \frac{1}{3}$ and our estimate from the example is $T_4 = \frac{11}{32}$. Thus, the absolute error is given by $\left|\frac{1}{3} - \frac{11}{32}\right| = \frac{1}{96} \approx 0.0104$.

The relative error is given by

$$\frac{1/96}{1/3} = 0.03125 \approx 3.1\%.$$

Exercise 4.7.3

In an earlier checkpoint, we estimated $\int_1^2 \frac{1}{x}\,dx$ to be $\frac{24}{35}$ using M_2. The actual value of this integral is $\ln 2$. Using $\frac{24}{35} \approx 0.6857$ and $\ln 2 \approx 0.6931$, calculate the absolute error and the relative error.

Hint

Use the previous examples as a guide.

Answer

absolute error ≈ 0.0074,and relative error $\approx 1.1\%$

Error Bounds on the Midpoint and Trapezoidal Rules

In the two previous examples, we were able to compare our estimate of an integral with the actual value of the integral; however, we do not typically have this luxury. In general, if we are approximating an integral, we are doing so because we cannot compute the exact value of the integral itself easily. Therefore, it is often helpful to be able to determine an upper bound for the error in an approximation of an integral. The following theorem provides error bounds for the midpoint and trapezoidal rules. The theorem is stated without proof.

Error Bounds for the Midpoint and Trapezoidal Rules

Let $f(x)$ be a continuous function over $[a,b]$, having a second derivative $f''(x)$ over this interval. If M is the maximum value of $|f''(x)|$ over $[a,b]$, then the upper bounds for the error in using M_n and T_n to estimate $\int_a^b f(x)\,dx$ are

$$\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2} \tag{4.7.1}$$

and

$$\text{Error in } T_n \le \frac{M(b-a)^3}{12n^2}$$

.

We can use these bounds to determine the value of n necessary to guarantee that the error in an estimate is less than a specified value.

Example 4.7.6: Determining the Number of Intervals to Use

What value of n should be used to guarantee that an estimate of $\int_0^1 e^{x^2}\,dx$ is accurate to within 0.01 if we use the midpoint rule?

Solution

We begin by determining the value of M, the maximum value of $|f''(x)|$ over $[0,1]$ for $f(x) = e^{x^2}$. Since $f'(x) = 2xe^{x^2}$, we have

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2}.$$

Thus,

$$|f''(x)| = 2e^{x^2}(1+2x^2) \le 2 \cdot e \cdot 3 = 6e.$$

From the error-bound Equation 4.7.1, we have

$$\text{Error in } M_n \le \frac{M(b-a)^3}{24n^2} \le \frac{6e(1-0)^3}{24n^2} = \frac{6e}{24n^2}.$$

Now we solve the following inequality for n:

$$\frac{6e}{24n^2} \le 0.01.$$

Thus, $n \ge \sqrt{\frac{600e}{24}} \approx 8.24$. Since n must be an integer satisfying this inequality, a choice of $n = 9$ would guarantee that

$$\left| \int_0^1 e^{x^2}\,dx - M_n \right| < 0.01.$$

Analysis

We might have been tempted to round 8.24 down and choose $n = 8$, but this would be incorrect because we must have an integer greater than or equal to 8.24. We need to keep in mind that the error estimates provide an upper bound only for the error. The actual estimate may, in fact, be a much better approximation than is indicated by the error bound.

? Exercise 4.7.4

Use Equation 4.7.1 to find an upper bound for the error in using M_4 to estimate $\int_0^1 x^2\,dx$.

Hint

$f''(x) = 2,$ so $M = 2.$

Answer

$\frac{1}{192}$

Simpson's Rule

With the midpoint rule, we estimated areas of regions under curves by using rectangles. In a sense, we approximated the curve with piecewise constant functions. With the trapezoidal rule, we approximated the curve by using piecewise linear functions. What if we were, instead, to approximate a curve using piecewise quadratic functions? With **Simpson's rule,** we do just this. We partition the interval into an even number of subintervals, each of equal width. Over the first pair of subintervals we approximate $\int_{x_0}^{x_2} f(x)\,dx$ with $\int_{x_0}^{x_2} p(x)\,dx$, where $p(x) = Ax^2 + Bx + C$ is the quadratic function passing through $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$ (Figure 4.7.4). Over the next pair of subintervals we approximate $\int_{x_2}^{x_4} f(x)\,dx$ with the integral of another quadratic function passing through $(x_2, f(x_2))$, $(x_3, f(x_3))$, and $(x_4, f(x_4))$. This process is continued with each successive pair of subintervals.

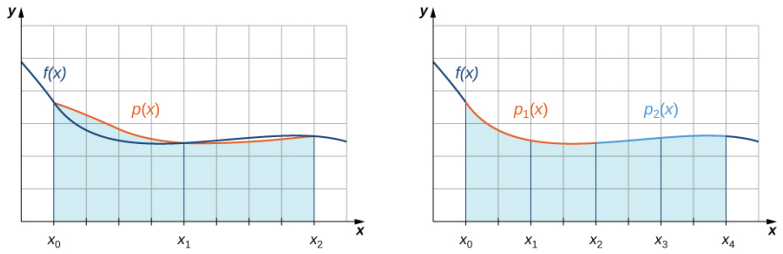


Figure 4.7.4: With Simpson's rule, we approximate a definite integral by integrating a piecewise quadratic function.

To understand the formula that we obtain for Simpson's rule, we begin by deriving a formula for this approximation over the first two subintervals. As we go through the derivation, we need to keep in mind the following relationships:

$$f(x_0) = p(x_0) = Ax_0^2 + Bx_0 + C$$
$$f(x_1) = p(x_1) = Ax_1^2 + Bx_1 + C$$
$$f(x_2) = p(x_2) = Ax_2^2 + Bx_2 + C$$

$x_2 - x_0 = 2\Delta x$, where Δx is the length of a subinterval.

$$x_2 + x_0 = 2x_1, \text{ since } x_1 = \frac{(x_2 + x_0)}{2}.$$

Thus,

$$\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} p(x)\,dx$$

$$= \int_{x_0}^{x_2} (Ax^2+Bx+C)\,dx$$

$$= \left(\frac{A}{3}x^3+\frac{B}{2}x^2+Cx\right)\Big|_{x_0}^{x_2} \qquad \text{Find the antiderivative.}$$

$$= \frac{A}{3}(x_2^3-x_0^3)+\frac{B}{2}(x_2^2-x_0^2)+C(x_2-x_0) \qquad \text{Evaluate the antiderivative.}$$

$$= \frac{A}{3}(x_2-x_0)(x_2^2+x_2x_0+x_0^2)+\frac{B}{2}(x_2-x_0)(x_2+x_0)+C(x_2-x_0)$$

$$= \frac{x_2-x_0}{6}\left(2A(x_2^2+x_2x_0+x_0^2)+3B(x_2+x_0)+6C\right) \qquad \text{Factor out } \frac{x_2-x_0}{6}.$$

$$= \frac{\Delta x}{3}\left((Ax_2^2+Bx_2+C)+(Ax_0^2+Bx_0+C)+A(x_2^2+2x_2x_0+x_0^2)+2B(x_2+x_0)+4C\right) \qquad \text{Rearrange the terms. Note: } \Delta x = \frac{x_2-x_0}{2}$$

$$= \frac{\Delta x}{3}\left(f(x_2)+f(x_0)+A(x_2+x_0)^2+2B(x_2+x_0)+4C\right) \qquad \text{Factor and substitute:}$$

$$f(x_2) = Ax_2^2+Bx_2+C \text{ and } f(x_0) = Ax_0^2+Bx_0+C.$$

$$= \frac{\Delta x}{3}\left(f(x_2)+f(x_0)+A(2x_1)^2+2B(2x_1)+4C\right) \qquad \text{Substitute } x_2+x_0 = 2x_1.$$

$$\text{Note: } x_1 = \frac{x_2+x_0}{2} \text{ is the midpoint.}$$

$$= \frac{\Delta x}{3}\left(f(x_2)+4f(x_1)+f(x_0)\right). \qquad \text{Expand and substitute } f(x_1) = Ax_1^2+Bx_1+C.$$

If we approximate $\int_{x_2}^{x_4} f(x)\,dx$ using the same method, we see that we have

$$\int_{x_2}^{x_4} f(x)\,dx \approx \frac{\Delta x}{3}(f(x_4)+4\,f(x_3)+f(x_2)).$$

Combining these two approximations, we get

$$\int_{x_0}^{x_4} f(x)\,dx \approx \frac{\Delta x}{3}(f(x_0)+4\,f(x_1)+2\,f(x_2)+4\,f(x_3)+f(x_4)).$$

The pattern continues as we add pairs of subintervals to our approximation. The general rule may be stated as follows.

Simpson's Rule

Assume that $f(x)$ is continuous over $[a,b]$. Let n be a positive even integer and $\Delta x = \frac{b-a}{n}$. Let $[a,b]$ be divided into n subintervals, each of length Δx, with endpoints at $P = \{x_0, x_1, x_2, \ldots, x_n\}$. Set

$$S_n = \frac{\Delta x}{3}\Big[f(x_0)+4\,f(x_1)+2\,f(x_2)+4\,f(x_3)+2\,f(x_4)+\cdots+2\,f(x_{n-2})+4\,f(x_{n-1})+f(x_n)\Big].$$

Then,

$$\lim_{n\to+\infty} S_n = \int_a^b f(x)\,dx.$$

Just as the trapezoidal rule is the average of the left-hand and right-hand rules for estimating definite integrals, Simpson's rule may be obtained from the midpoint and trapezoidal rules by using a weighted average. It can be shown that $S_{2n} = \left(\frac{2}{3}\right) M_n + \left(\frac{1}{3}\right) T_n$.

It is also possible to put a bound on the error when using Simpson's rule to approximate a definite integral. The bound in the error is given by the following rule:

Rule: Error Bound for Simpson's Rule

Let $f(x)$ be a continuous function over $[a,b]$ having a fourth derivative, $f^{(4)}(x)$, over this interval. If M is the maximum value of $\left|f^{(4)}(x)\right|$ over $[a,b]$, then the upper bound for the error in using S_n to estimate $\int_a^b f(x)\,dx$ is given by

$$\text{Error in } S_n \le \frac{M(b-a)^5}{180n^4}.$$

Example 4.7.7: Applying Simpson's Rule 1

Use S_2 to approximate $\int_0^1 x^3\,dx$. Estimate a bound for the error in S_2.

Solution

Since $[0,1]$ is divided into two intervals, each subinterval has length $\Delta x = \frac{1-0}{2} = \frac{1}{2}$. The endpoints of these subintervals are $\left\{0, \frac{1}{2}, 1\right\}$. If we set $f(x) = x^3$, then

$$S_2 = \frac{1}{3}\cdot\frac{1}{2}(f(0)+4\,f(\tfrac{1}{2})+f(1)) = \frac{1}{6}(0+4\cdot\frac{1}{8}+1) = \frac{1}{4}.$$

Since $f^{(4)}(x) = 0$ and consequently $M = 0$, we see that

$$\text{Error in } S_2 \le \frac{0(1)^5}{180 \cdot 2^4} = 0.$$

This bound indicates that the value obtained through Simpson's rule is exact. A quick check will verify that, in fact, $\int_0^1 x^3\,dx = \frac{1}{4}$.

Example 4.7.8: Applying Simpson's Rule 2

Use S_6 to estimate the length of the curve $y = \frac{1}{2}x^2$ over $[1, 4]$.

Solution

The length of $y = \frac{1}{2}x^2$ over $[1, 4]$ is $\int_1^4 \sqrt{1+x^2}\,dx$. If we divide $[1, 4]$ into six subintervals, then each subinterval has length $\Delta x = \frac{4-1}{6} = \frac{1}{2}$, and the endpoints of the subintervals are $\{1, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4\}$. Setting $f(x) = \sqrt{1+x^2}$,

$$S_6 = \frac{1}{3} \cdot \frac{1}{2}(f(1) + 4f(\frac{3}{2}) + 2f(2) + 4f(\frac{5}{2}) + 2f(3) + 4f(\frac{7}{2}) + f(4)).$$

After substituting, we have

$$S_6 = \frac{1}{6}(1.4142 + 4 \cdot 1.80278 + 2 \cdot 2.23607 + 4 \cdot 2.69258 + 2 \cdot 3.16228 + 4 \cdot 3.64005 + 4.12311) \approx 8.14594 \text{ units}.$$

Exercise 4.7.5

Use S_2 to estimate $\int_1^2 \frac{1}{x}\,dx$.

Hint

$$S_2 = \frac{1}{3}\Delta x\,(f(x_0) + 4f(x_1) + f(x_2))$$

Answer

$\frac{25}{36} \approx 0.694444$

Key Concepts

- We can use numerical integration to estimate the values of definite integrals when a closed form of the integral is difficult to find or when an approximate value only of the definite integral is needed.
- The most commonly used techniques for numerical integration are the midpoint rule, trapezoidal rule, and Simpson's rule.
- The midpoint rule approximates the definite integral using rectangular regions whereas the trapezoidal rule approximates the definite integral using trapezoidal approximations.
- Simpson's rule approximates the definite integral by first approximating the original function using piecewise quadratic functions.

Key Equations

- **Midpoint rule**

  $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$

- **Trapezoidal rule**

  $T_n = \frac{\Delta x}{2}\left[f(x_0) + 2\,f(x_1) + 2\,f(x_2) + \cdots + 2\,f(x_{n-1}) + f(x_n)\right]$

- **Simpson's rule**

  $S_n = \frac{\Delta x}{3}\left[f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + 2\,f(x_4) + 4\,f(x_5) + \cdots + 2\,f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)\right]$

- **Error bound for midpoint rule**

  Error in $M_n \le \frac{M(b-a)^3}{24n^2}$, where M is the maximum value of $|f''(x)|$ over $[a, b]$.

- **Error bound for trapezoidal rule**

  Error in $T_n \le \frac{M(b-a)^3}{12n^2}$, where M is the maximum value of $|f''(x)|$ over $[a, b]$.

- **Error bound for Simpson's rule**

  Error in $S_n \le \frac{M(b-a)^5}{180n^4}$, where M is the maximum value of $\left|f^{(4)}(x)\right|$ over $[a, b]$.

Glossary

absolute error

if B is an estimate of some quantity having an actual value of A, then the absolute error is given by $|A - B|$

midpoint rule

a rule that uses a Riemann sum of the form $M_n = \sum_{i=1}^{n} f(m_i)\Delta x$, where m_i is the midpoint of the ith subinterval to approximate $\int_a^b f(x)\,dx$

numerical integration

the variety of numerical methods used to estimate the value of a definite integral, including the midpoint rule, trapezoidal rule, and Simpson's rule

relative error

error as a percentage of the actual value, given by

$$\text{relative error} = \left| \frac{A - B}{A} \right| \cdot 100\%$$

Simpson's rule

a rule that approximates $\int_a^b f(x)\,dx$ using the area under a piecewise quadratic function.

The approximation S_n to $\int_a^b f(x)\,dx$ is given by

$$S_n = \frac{\Delta x}{3}\big(f(x_0) + 4\,f(x_1) + 2\,f(x_2) + 4\,f(x_3) + 2\,f(x_4) + \cdots + 2\,f(x_{n-2}) + 4\,f(x_{n-1}) + f(x_n)\big).$$

trapezoidal rule

a rule that approximates $\int_a^b f(x)\,dx$ using the area of trapezoids.

The approximation T_n to $\int_a^b f(x)\,dx$ is given by

$$T_n = \frac{\Delta x}{2}\big(f(x_0) + 2\,f(x_1) + 2\,f(x_2) + \cdots + 2\,f(x_{n-1}) + f(x_n)\big).$$

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- Edited by Paul Seeburger (Monroe Community College). Notes added to development of area under a parabola and typos fixed in original text.

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4.7E: Exercises for Numerical Integration

In exercises 1 - 5, approximate the following integrals using either the midpoint rule, trapezoidal rule, or Simpson's rule as indicated. (Round answers to three decimal places.)

1) $\int_1^2 \frac{dx}{x}$; trapezoidal rule; $n = 5$

Answer

0.696

2) $\int_0^3 \sqrt{4+x^3}\ dx$; trapezoidal rule; $n = 6$

3) $\int_0^3 \sqrt{4+x^3}\ dx$; Simpson's rule; $n = 6$

Answer

9.279

4) $\int_0^{12} x^2\ dx$; midpoint rule; $n = 6$

5) $\int_0^1 \sin^2(\pi x)\ dx$; midpoint rule; $n = 3$

Answer

0.500

6) Use the midpoint rule with eight subdivisions to estimate $\int_2^4 x^2\ dx$.

7) Use the trapezoidal rule with four subdivisions to estimate $\int_2^4 x^2\ dx$.

Answer

$T_4 = 18.75$

8) Find the exact value of $\int_2^4 x^2\ dx$. Find the error of approximation between the exact value and the value calculated using the trapezoidal rule with four subdivisions. Draw a graph to illustrate.

Approximate the integral to four decimal places using the indicated rule.

9) $\int_0^1 \sin^2(\pi x)\ dx$; trapezoidal rule; $n = 6$

Answer

0.5000

10) $\int_0^3 \frac{1}{1+x^3}\ dx$; trapezoidal rule; $n = 6$

11) $\int_0^3 \frac{1}{1+x^3}\ dx$; Simpson's rule; $n = 6$

Answer

1.1614

12) $\int_0^{0.8} e^{-x^2}\ dx$; trapezoidal rule; $n = 4$

13) $\int_0^{0.8} e^{-x^2}\ dx$; Simpson's rule; $n = 4$

Answer

0.6577

14) $\int_0^{0.4} \sin(x^2)\ dx$; trapezoidal rule; $n = 4$

15) $\int_0^{0.4} \sin(x^2)\ dx$; Simpson's rule; $n = 4$

Answer

0.0213

16) $\int_{0.1}^{0.5} \frac{\cos x}{x}\ dx$; trapezoidal rule; $n = 4$

17) $\int_{0.1}^{0.5} \frac{\cos x}{x}\ dx$; Simpson's rule; $n = 4$

Answer

1.5629

18) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ exactly and show that the result is $\pi/4$. Then, find the approximate value of the integral using the trapezoidal rule with $n = 4$ subdivisions. Use the result to approximate the value of π.

19) Approximate $\int_2^4 \frac{1}{\ln x}\ dx$ using the midpoint rule with four subdivisions to four decimal places.

Answer

1.9133

20) Approximate $\int_2^4 \frac{1}{\ln x}\ dx$ using the trapezoidal rule with eight subdivisions to four decimal places.

21) Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3\ dx$ to four decimal places.

Answer

$T(4) = 0.1088$

22) Use the trapezoidal rule with four subdivisions to estimate $\int_0^{0.8} x^3\ dx$. Compare this value with the exact value and find the error estimate.

23) Using Simpson's rule with four subdivisions, find $\int_0^{\pi/2} \cos(x)\ dx$.

Answer

$\int_0^{\pi/2} \cos(x)\ dx \approx \quad 1.0$

24) Show that the exact value of $\int_0^1 xe^{-x}\,dx = 1 - \frac{2}{e}$. Find the absolute error if you approximate the integral using the midpoint rule with 16 subdivisions.

25) Given $\int_0^1 xe^{-x}\,dx = 1 - \frac{2}{e}$, use the trapezoidal rule with 16 subdivisions to approximate the integral and find the absolute error.

Answer

Approximate error is 0.000325.

26) Find an upper bound for the error in estimating $\int_0^3 (5x+4)\,dx$ using the trapezoidal rule with six steps.

27) Find an upper bound for the error in estimating $\int_4^5 \frac{1}{(x-1)^2}\,dx$ using the trapezoidal rule with seven subdivisions.

Answer

$\frac{1}{7938}$

28) Find an upper bound for the error in estimating $\int_0^3 (6x^2-1)\,dx$ using Simpson's rule with $n = 10$ steps.

29) Find an upper bound for the error in estimating $\int_2^5 \frac{1}{x-1}\,dx$ using Simpson's rule with $n = 10$ steps.

Answer

$\frac{81}{25{,}000}$

30) Find an upper bound for the error in estimating $\int_0^\pi 2x\cos(x)\,dx$ using Simpson's rule with four steps.

31) Estimate the minimum number of subintervals needed to approximate the integral $\int_1^4 (5x^2+8)\,dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

Answer

475

32) Determine a value of n such that the trapezoidal rule will approximate $\int_0^1 \sqrt{1+x^2}\,dx$ with an error of no more than 0.01.

33) Estimate the minimum number of subintervals needed to approximate the integral $\int_2^3 (2x^3+4x)\,dx$ with an error of magnitude less than 0.0001 using the trapezoidal rule.

Answer

174

34) Estimate the minimum number of subintervals needed to approximate the integral $\int_3^4 \frac{1}{(x-1)^2}\,dx$ with an error magnitude of less than 0.0001 using the trapezoidal rule.

35) Use Simpson's rule with four subdivisions to approximate the area under the probability density function $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ from $x = 0$ to $x = 0.4$.

Answer

0.1554

36) Use Simpson's rule with $n = 14$ to approximate (to three decimal places) the area of the region bounded by the graphs of $y = 0, x = 0$, and $x = \pi/2$.

37) The length of one arch of the curve $y = 3\sin(2x)$ is given by $L = \int_0^{\pi/2} \sqrt{1 + 36\cos^2(2x)}\, dx$. Estimate L using the trapezoidal rule with $n = 6$.

Answer

6.2807

38) The length of the ellipse $x = a\cos(t), y = b\sin(t), 0 \le t \le 2\pi$ is given by $L = 4a\int_0^{\pi/2} \sqrt{1 - e^2\cos^2(t)}\, dt$, where e is the eccentricity of the ellipse. Use Simpson's rule with $n = 6$ subdivisions to estimate the length of the ellipse when $a = 2$ and $e = 1/3$.

39) Estimate the area of the surface generated by revolving the curve $y = \cos(2x), 0 \le x \le \frac{\pi}{4}$ about the x-axis. Use the trapezoidal rule with six subdivisions.

Answer

4.606

40) Estimate the area of the surface generated by revolving the curve $y = 2x^2, 0 \le x \le 3$ about the x-axis. Use Simpson's rule with $n = 6$.

41) The growth rate of a certain tree (in feet) is given by $y = \dfrac{2}{t+1} + e^{-t^2/2}$, where t is time in years. Estimate the growth of the tree through the end of the second year by using Simpson's rule, using two subintervals. (Round the answer to the nearest hundredth.)

Answer

3.41ft

42) [T] Use a calculator to approximate $\int_0^1 \sin(\pi x)\, dx$ using the midpoint rule with 25 subdivisions. Compute the relative error of approximation.

43) [T] Given $\int_1^5 (3x^2 - 2x)\, dx = 100$, approximate the value of this integral using the midpoint rule with 16 subdivisions and determine the absolute error.

Answer

$T_{16} = 100.125$;absolute error = 0.125

44) Given that we know the Fundamental Theorem of Calculus, why would we want to develop numerical methods for definite integrals?

45) The table represents the coordinates (x, y) that give the boundary of a lot. The units of measurement are meters. Use the trapezoidal rule to estimate the number of square meters of land that is in this lot.

x	y	x	y
0	125	600	95
100	125	700	88
200	120	800	75
300	112	900	35
400	90	1000	0

500	90	

Answer

about 89,250 m^2

46) Choose the correct answer. When Simpson's rule is used to approximate the definite integral, it is necessary that the number of partitions be____

a. an even number

b. odd number

c. either an even or an odd number

d. a multiple of 4

47) The "Simpson" sum is based on the area under a ____.

Answer

parabola

48) The error formula for Simpson's rule depends on___.

a. $f(x)$

b. $f'(x)$

c. $f^{(4)}(x)$

d. the number of steps

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4.8: L'Hôpital's Rule

Learning Objectives

- Recognize when to apply L'Hôpital's rule.
- Identify indeterminate forms produced by quotients, products, subtractions, and powers, and apply L'Hôpital's rule in each case.
- Describe the relative growth rates of functions.

In this section, we examine a powerful tool for evaluating limits. This tool, known as **L'Hôpital's rule,** uses derivatives to calculate limits. With this rule, we will be able to evaluate many limits we have not yet been able to determine. Instead of relying on numerical evidence to conjecture that a limit exists, we will be able to show definitively that a limit exists and to determine its exact value.

Applying L'Hôpital's Rule

L'Hôpital's rule can be used to evaluate limits involving the quotient of two functions. Consider

$$\lim_{x\to a}\frac{f(x)}{g(x)}.$$

If $\lim_{x\to a} f(x)=L_1$ and $\lim_{x\to a} g(x)=L_2\neq 0,$ then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{L_1}{L_2}.$$

However, what happens if $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$? We call this one of the **indeterminate forms,** of type $\frac{0}{0}$. This is considered an indeterminate form because we cannot determine the exact behavior of $\frac{f(x)}{g(x)}$ as $x\to a$ without further analysis. We have seen examples of this earlier in the text. For example, consider

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

and

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

For the first of these examples, we can evaluate the limit by factoring the numerator and writing

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x-2}=\lim_{x\to 2}(x+2)=2+2=4.$$

For $\lim_{x\to 0}\frac{\sin x}{x}$ we were able to show, using a geometric argument, that

$$\lim_{x\to 0}\frac{\sin x}{x}=1.$$

Here we use a different technique for evaluating limits such as these. Not only does this technique provide an easier way to evaluate these limits, but also, and more importantly, it provides us with a way to evaluate many other limits that we could not calculate previously.

The idea behind L'Hôpital's rule can be explained using local linear approximations. Consider two differentiable functions f and g such that $\lim_{x\to a} f(x)=0=\lim_{x\to a} g(x)$ and such that $g'(a)\neq 0$ For x near a,we can write

$$f(x)\approx f(a)+f'(a)(x-a)$$

and

$$g(x) \approx g(a) + g'(a)(x-a).$$

Therefore,

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)}.$$

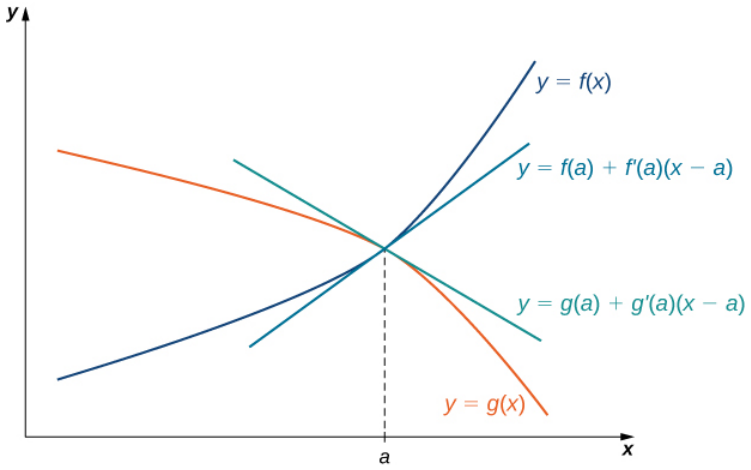


Figure 4.8.1: If $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, then the ratio $f(x)/g(x)$ is approximately equal to the ratio of their linear approximations near a.

Since f is differentiable at a, then f is continuous at a, and therefore $f(a) = \lim_{x \to a} f(x) = 0$. Similarly, $g(a) = \lim_{x \to a} g(x) = 0$. If we also assume that f' and g' are continuous at $x = a$, then $f'(a) = \lim_{x \to a} f'(x)$ and $g'(a) = \lim_{x \to a} g'(x)$. Using these ideas, we conclude that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)(x-a)}{g'(x)(x-a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Note that the assumption that f' and g' are continuous at a and $g'(a) \neq 0$ can be loosened. We state L'Hôpital's rule formally for the indeterminate form $\frac{0}{0}$. Also note that the notation $\frac{0}{0}$ does not mean we are actually dividing zero by zero. Rather, we are using the notation $\frac{0}{0}$ to represent a quotient of limits, each of which is zero.

L'Hôpital's Rule (0/0 Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if we are considering one-sided limits, or if $a = \infty$ or $a = -\infty$.

Proof

We provide a proof of this theorem in the special case when f, g, f', and g' are all continuous over an open interval containing a. In that case, since $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and f and g are continuous at a, it follows that $f(a) = 0 = g(a)$. Therefore,

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)}{g(x)} &= \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} && \text{Since } f(a)=0=g(a)\\
&= \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} && \text{Multiply numerator and denominator by } \frac{1}{x-a}\\
&= \frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}} && \text{The limit of a quotient is the quotient of the limits.}\\
&= \frac{f'(a)}{g'(a)} && \text{By the definition of the derivative}\\
&= \frac{\lim\limits_{x\to a}f'(x)}{\lim\limits_{x\to a}g'(x)} && \text{By the continuity of } f' \text{ and } g'\\
&= \lim_{x\to a}\frac{f'(x)}{g'(x)}. && \text{The limit of a quotient}
\end{aligned}$$

Note that L'Hôpital's rule states we can calculate the limit of a quotient $\frac{f}{g}$ by considering the limit of the quotient of the derivatives $\frac{f'}{g'}$. It is important to realize that we are not calculating the derivative of the quotient $\frac{f}{g}$.

□

Example 4.8.1: Applying L'Hôpital's Rule (0/0 Case)

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}$

b. $\lim\limits_{x\to 1}\dfrac{\sin(\pi x)}{\ln x}$

c. $\lim\limits_{x\to\infty}\dfrac{e^{1/x}-1}{1/x}$

d. $\lim\limits_{x\to 0}\dfrac{\sin x - x}{x^2}$

Solution

a. Since the numerator $1-\cos x\to 0$ and the denominator $x\to 0$, we can apply L'Hôpital's rule to evaluate this limit. We have

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{\frac{d}{dx}(1-\cos x)}{\frac{d}{dx}(x)}=\lim_{x\to 0}\frac{\sin x}{1}=\frac{\lim\limits_{x\to 0}\sin x}{\lim\limits_{x\to 0}1}=\frac{0}{1}=0.$$

b. As $x\to 1$, the numerator $\sin(\pi x)\to 0$ and the denominator $\ln(x)\to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\begin{aligned}
\lim_{x\to 1}\frac{\sin(\pi x)}{\ln x} &= \lim_{x\to 1}\frac{\pi\cos(\pi x)}{1/x}\\
&= \lim_{x\to 1}(\pi x)\cos(\pi x)\\
&= (\pi\cdot 1)(-1)=-\pi.
\end{aligned}$$

c. As $x \to \infty$, the numerator $e^{1/x} - 1 \to 0$ and the denominator $\frac{1}{x} \to 0$. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to\infty} \frac{e^{1/x}-1}{\frac{1}{x}} = \lim_{x\to\infty} \frac{e^{1/x}\left(\frac{-1}{x^2}\right)}{\left(\frac{-1}{x^2}\right)} = \lim_{x\to\infty} e^{1/x} = e^0 = 1.$$

d. As $x \to 0$, both the numerator and denominator approach zero. Therefore, we can apply L'Hôpital's rule. We obtain

$$\lim_{x\to 0} \frac{\sin x - x}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{2x}.$$

Since the numerator and denominator of this new quotient both approach zero as $x \to 0$, we apply L'Hôpital's rule again. In doing so, we see that

$$\lim_{x\to 0} \frac{\cos x - 1}{2x} = \lim_{x\to 0} \frac{-\sin x}{2} = 0.$$

Therefore, we conclude that

$$\lim_{x\to 0} \frac{\sin x - x}{x^2} = 0.$$

Exercise 4.8.1

Evaluate $\lim_{x\to 0} \frac{x}{\tan x}$.

Hint

$\frac{d}{dx}\left(\tan x\right) = \sec^2 x$

Answer

1

We can also use L'Hôpital's rule to evaluate limits of quotients $\frac{f(x)}{g(x)}$ in which $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$. Limits of this form are classified as indeterminate forms of type ∞/∞. Again, note that we are not actually dividing ∞ by ∞. Since ∞ is not a real number, that is impossible; rather, ∞/∞ is used to represent a quotient of limits, each of which is ∞ or $-\infty$.

L'Hôpital's Rule (∞/∞ Case)

Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose $\lim_{x\to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x\to a} g(x) = \infty$ (or $-\infty$). Then,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f\,'(x)}{g'(x)}$$

assuming the limit on the right exists or is ∞ or $-\infty$. This result also holds if the limit is infinite, if $a = \infty$ or $-\infty$, or the limit is one-sided.

Example 4.8.2: Applying L'Hôpital's Rule (∞/∞) Case

Evaluate each of the following limits by applying L'Hôpital's rule.

a. $\lim_{x\to\infty} \frac{3x+5}{2x+1}$

b. $\lim_{x\to 0^+} \frac{\ln x}{\cot x}$

Solution

a. Since $3x+5$ and $2x+1$ are first-degree polynomials with positive leading coefficients, $\lim_{x\to\infty}(3x+5)=\infty$ and $\lim_{x\to\infty}(2x+1)=\infty$. Therefore, we apply L'Hôpital's rule and obtain

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3}{2}=\frac{3}{2}.$$

Note that this limit can also be calculated without invoking L'Hôpital's rule. Earlier in the chapter we showed how to evaluate such a limit by dividing the numerator and denominator by the highest power of x in the denominator. In doing so, we saw that

$$\lim_{x\to\infty}\frac{3x+5}{2x+1}=\lim_{x\to\infty}\frac{3+5/x}{2+1/x}=\frac{3}{2}.$$

L'Hôpital's rule provides us with an alternative means of evaluating this type of limit.

b. Here, $\lim_{x\to0^+}\ln x=-\infty$ and $\lim_{x\to0^+}\cot x=\infty$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to0^+}\frac{\ln x}{\cot x}=\lim_{x\to0^+}\frac{1/x}{-\csc^2 x}=\lim_{x\to0^+}\frac{1}{-x\csc^2 x}.$$

Now as $x\to0^+$, $\csc^2 x\to\infty$. Therefore, the first term in the denominator is approaching zero and the second term is getting really large. In such a case, anything can happen with the product. Therefore, we cannot make any conclusion yet. To evaluate the limit, we use the definition of $\csc x$ to write

$$\lim_{x\to0^+}\frac{1}{-x\csc^2 x}=\lim_{x\to0^+}\frac{\sin^2 x}{-x}.$$

Now $\lim_{x\to0^+}\sin^2 x=0$ and $\lim_{x\to0^+}-x=0$, so we apply L'Hôpital's rule again. We find

$$\lim_{x\to0^+}\frac{\sin^2 x}{-x}=\lim_{x\to0^+}\frac{2\sin x\cos x}{-1}=\frac{0}{-1}=0.$$

We conclude that

$$\lim_{x\to0^+}\frac{\ln x}{\cot x}=0.$$

? Exercise 4.8.2

Evaluate $\lim_{x\to\infty}\frac{\ln x}{5x}$.

Hint

$\frac{d}{dx}(\ln x)=\frac{1}{x}$

Answer

0

As mentioned, L'Hôpital's rule is an extremely useful tool for evaluating limits. It is important to remember, however, that to apply L'Hôpital's rule to a quotient $\frac{f(x)}{g(x)}$, it is essential that the limit of $\frac{f(x)}{g(x)}$ be of the form $\frac{0}{0}$ or ∞/∞. Consider the following example.

Example 4.8.3: When L'Hôpital's Rule Does Not Apply

Consider $\lim\limits_{x\to 1}\dfrac{x^2+5}{3x+4}$.

Show that the limit cannot be evaluated by applying L'Hôpital's rule.

Solution

Because the limits of the numerator and denominator are not both zero and are not both infinite, we cannot apply L'Hôpital's rule. If we try to do so, we get

$$\frac{d}{dx}(x^2+5)=2x$$

and

$$\frac{d}{dx}(3x+4)=3.$$

At which point we would conclude erroneously that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\lim_{x\to 1}\frac{2x}{3}=\frac{2}{3}.$$

However, since $\lim\limits_{x\to 1}(x^2+5)=6$ and $\lim\limits_{x\to 1}(3x+4)=7$, we actually have

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}=\frac{6}{7}.$$

We can conclude that

$$\lim_{x\to 1}\frac{x^2+5}{3x+4}\neq\lim_{x\to 1}\frac{\dfrac{d}{dx}(x^2+5)}{\dfrac{d}{dx}(3x+4)}.$$

Exercise 4.8.3

Explain why we cannot apply L'Hôpital's rule to evaluate $\lim\limits_{x\to 0^+}\dfrac{\cos x}{x}$. Evaluate $\lim\limits_{x\to 0^+}\dfrac{\cos x}{x}$ by other means.

Hint

Determine the limits of the numerator and denominator separately.

Answer

$\lim\limits_{x\to 0^+}\cos x=1$. Therefore, we cannot apply L'Hôpital's rule. The limit of the quotient is ∞.

Other Indeterminate Forms

L'Hôpital's rule is very useful for evaluating limits involving the indeterminate forms $\dfrac{0}{0}$ and ∞/∞. However, we can also use L'Hôpital's rule to help evaluate limits involving other indeterminate forms that arise when evaluating limits. The expressions $0\cdot\infty, \infty-\infty, 1^\infty, \infty^0$, and 0^0 are all considered indeterminate forms. These expressions are not real numbers. Rather, they represent forms that arise when trying to evaluate certain limits. Next we realize why these are indeterminate forms and then understand how to use L'Hôpital's rule in these cases. The key idea is that we must rewrite the indeterminate forms in such a way that we arrive at the indeterminate form $\dfrac{0}{0}$ or ∞/∞.

Indeterminate Form of Type $0\cdot\infty$

Suppose we want to evaluate $\lim_{x\to a}(f(x)\cdot g(x))$, where $f(x)\to 0$ and $g(x)\to\infty$ (or $-\infty$) as $x\to a$. Since one term in the product is approaching zero but the other term is becoming arbitrarily large (in magnitude), anything can happen to the product. We use the notation $0\cdot\infty$ to denote the form that arises in this situation. The expression $0\cdot\infty$ is considered indeterminate because we cannot determine without further analysis the exact behavior of the product $f(x)g(x)$ as $x\to\infty$. For example, let n be a positive integer and consider

$$f(x)=\frac{1}{(x^n+1)} \text{ and } g(x)=3x^2.$$

As $x\to\infty$, $f(x)\to 0$ and $g(x)\to\infty$. However, the limit as $x\to\infty$ of $f(x)g(x)=\dfrac{3x^2}{(x^n+1)}$ varies, depending on n. If $n=2$, then $\lim_{x\to\infty} f(x)g(x)=3$. If $n=1$, then $\lim_{x\to\infty} f(x)g(x)=\infty$. If $n=3$, then $\lim_{x\to\infty} f(x)g(x)=0$. Here we consider another limit involving the indeterminate form $0\cdot\infty$ and show how to rewrite the function as a quotient to use L'Hôpital's rule.

Example 4.8.4: Indeterminate Form of Type $0\cdot\infty$

Evaluate $\lim_{x\to 0^+} x\ln x$.

Solution

First, rewrite the function $x\ln x$ as a quotient to apply L'Hôpital's rule. If we write

$$x\ln x=\frac{\ln x}{1/x}$$

we see that $\ln x\to-\infty$ as $x\to 0^+$ and $\dfrac{1}{x}\to\infty$ as $x\to 0^+$. Therefore, we can apply L'Hôpital's rule and obtain

$$\lim_{x\to 0^+}\frac{\ln x}{1/x}=\lim_{x\to 0^+}\frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(1/x)}=\lim_{x\to 0^+}\frac{1/x}{-1/x^2}=\lim_{x\to 0^+}(-x)=0.$$

We conclude that

$$\lim_{x\to 0^+} x\ln x=0.$$

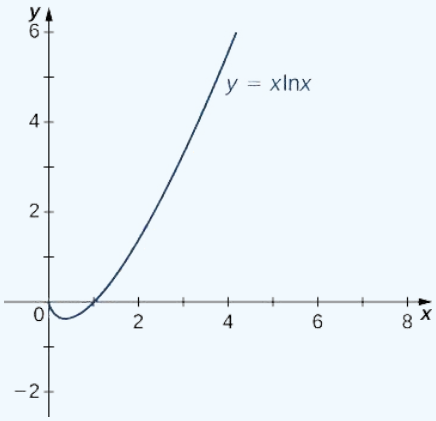


Figure 4.8.2: Finding the limit at $x=0$ of the function $f(x)=x\ln x$.

Exercise 4.8.4

Evaluate $\lim_{x\to 0} x\cot x$.

Hint

Write $x \cot x = \dfrac{x \cos x}{\sin x}$

Answer

1

Indeterminate Form of Type ∞-∞

Another type of indeterminate form is $\infty - \infty$. Consider the following example. Let n be a positive integer and let $f(x) = 3x^n$ and $g(x) = 3x^2 + 5$. As $x \to \infty$, $f(x) \to \infty$ and $g(x) \to \infty$. We are interested in $\lim\limits_{x \to \infty} (f(x) - g(x))$. Depending on whether $f(x)$ grows faster, $g(x)$ grows faster, or they grow at the same rate, as we see next, anything can happen in this limit. Since $f(x) \to \infty$ and $g(x) \to \infty$, we write $\infty - \infty$ to denote the form of this limit. As with our other indeterminate forms, $\infty - \infty$ has no meaning on its own and we must do more analysis to determine the value of the limit. For example, suppose the exponent n in the function $f(x) = 3x^n$ is $n = 3$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^3 - 3x^2 - 5) = \infty.$$

On the other hand, if $n = 2$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x^2 - 3x^2 - 5) = -5.$$

However, if $n = 1$, then

$$\lim_{x \to \infty} (f(x) - g(x)) = \lim_{x \to \infty} (3x - 3x^2 - 5) = -\infty.$$

Therefore, the limit cannot be determined by considering only $\infty - \infty$. Next we see how to rewrite an expression involving the indeterminate form $\infty - \infty$ as a fraction to apply L'Hôpital's rule.

Example 4.8.5: Indeterminate Form of Type $\infty - \infty$

Evaluate $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x^2} - \dfrac{1}{\tan x} \right)$.

Solution

By combining the fractions, we can write the function as a quotient. Since the least common denominator is $x^2 \tan x$, we have

$$\frac{1}{x^2} - \frac{1}{\tan x} = \frac{(\tan x) - x^2}{x^2 \tan x}.$$

As $x \to 0^+$, the numerator $\tan x - x^2 \to 0$ and the denominator $x^2 \tan x \to 0$. Therefore, we can apply L'Hôpital's rule. Taking the derivatives of the numerator and the denominator, we have

$$\lim_{x \to 0^+} \frac{(\tan x) - x^2}{x^2 \tan x} = \lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x}.$$

As $x \to 0^+$, $(\sec^2 x) - 2x \to 1$ and $x^2 \sec^2 x + 2x \tan x \to 0$. Since the denominator is positive as x approaches zero from the right, we conclude that

$$\lim_{x \to 0^+} \frac{(\sec^2 x) - 2x}{x^2 \sec^2 x + 2x \tan x} = \infty.$$

Therefore,

$$\lim_{x \to 0^+} \left(\frac{1}{x^2} - \frac{1}{\tan x} \right) = \infty.$$

Exercise 4.8.5

Evaluate $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

Hint

Rewrite the difference of fractions as a single fraction.

Answer

0

Indeterminate Forms Involving Exponents

Another type of indeterminate form that arises when evaluating limits involves exponents. The expressions 0^0, ∞^0, and 1^∞ are all indeterminate forms. On their own, these expressions are meaningless because we cannot actually evaluate these expressions as we would evaluate an expression involving real numbers. Rather, these expressions represent forms that arise when finding limits. Now we examine how L'Hôpital's rule can be used to evaluate limits involving these indeterminate forms.

Since L'Hôpital's rule applies to quotients, we use the natural logarithm function and its properties to reduce a problem evaluating a limit involving exponents to a related problem involving a limit of a quotient. For example, suppose we want to evaluate $\lim_{x\to a} f(x)^{g(x)}$ and we arrive at the indeterminate form ∞^0. (The indeterminate forms 0^0 and 1^∞ can be handled similarly.) We proceed as follows. Let

$$y = f(x)^{g(x)}.$$

Then,

$$\ln y = \ln(f(x)^{g(x)}) = g(x)\ln(f(x)).$$

Therefore,

$$\lim_{x\to a}[\ln(y)] = \lim_{x\to a}[g(x)\ln(f(x))].$$

Since $\lim_{x\to a} f(x) = \infty$, we know that $\lim_{x\to a}\ln(f(x)) = \infty$. Therefore, $\lim_{x\to a} g(x)\ln(f(x))$ is of the indeterminate form $0\cdot\infty$, and we can use the techniques discussed earlier to rewrite the expression $g(x)\ln(f(x))$ in a form so that we can apply L'Hôpital's rule. Suppose $\lim_{x\to a} g(x)\ln(f(x)) = L$, where L may be ∞ or $-\infty$. Then

$$\lim_{x\to a}[\ln(y)] = L.$$

Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to a} y\right) = L,$$

which gives us

$$\lim_{x\to a} y = \lim_{x\to a} f(x)^{g(x)} = e^L.$$

Example 4.8.6: Indeterminate Form of Type ∞^0

Evaluate $\lim_{x\to\infty} x^{1/x}$.

Solution

Let $y = x^{1/x}$.Then,

$$\ln(x^{1/x}) = \frac{1}{x}\ln x = \frac{\ln x}{x}.$$

We need to evaluate $\lim_{x\to\infty}\dfrac{\ln x}{x}$. Applying L'Hôpital's rule, we obtain

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{1/x}{1}=0.$$

Therefore, $\lim_{x\to\infty}\ln y=0$. Since the natural logarithm function is continuous, we conclude that

$$\ln\left(\lim_{x\to\infty}y\right)=0,$$

which leads to

$$\lim_{x\to\infty}x^{1/x}=\lim_{x\to\infty}y=e^{\ln\left(\lim_{x\to\infty}y\right)}=e^0=1.$$

Hence,

$$\lim_{x\to\infty}x^{1/x}=1.$$

Exercise 4.8.6

Evaluate $\lim_{x\to\infty}x^{1/\ln(x)}$.

Hint

Let $y=x^{1/\ln(x)}$ and apply the natural logarithm to both sides of the equation.

Answer

e

Example 4.8.7: Indeterminate Form of Type 0^0

Evaluate $\lim_{x\to0^+}x^{\sin x}$.

Solution

Let

$$y=x^{\sin x}.$$

Therefore,

$$\ln y=\ln(x^{\sin x})=\sin x\ln x.$$

We now evaluate $\lim_{x\to0^+}\sin x\ln x$. Since $\lim_{x\to0^+}\sin x=0$ and $\lim_{x\to0^+}\ln x=-\infty$, we have the indeterminate form $0\cdot\infty$. To apply L'Hôpital's rule, we need to rewrite $\sin x\ln x$ as a fraction. We could write

$$\sin x\ln x=\frac{\sin x}{1/\ln x}$$

or

$$\sin x\ln x=\frac{\ln x}{1/\sin x}=\frac{\ln x}{\csc x}.$$

Let's consider the first option. In this case, applying L'Hôpital's rule, we would obtain

$$\lim_{x\to0^+}\sin x\ln x=\lim_{x\to0^+}\frac{\sin x}{1/\ln x}=\lim_{x\to0^+}\frac{\cos x}{-1/(x(\ln x)^2)}=\lim_{x\to0^+}(-x(\ln x)^2\cos x).$$

Unfortunately, we not only have another expression involving the indeterminate form $0 \cdot \infty$, but the new limit is even more complicated to evaluate than the one with which we started. Instead, we try the second option. By writing

$$\sin x \ln x = \frac{\ln x}{1/\sin x} = \frac{\ln x}{\csc x},$$

and applying L'Hôpital's rule, we obtain

$$\lim_{x \to 0^+} \sin x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\csc x} = \lim_{x \to 0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x \to 0^+} \frac{-1}{x \csc x \cot x}.$$

Using the fact that $\csc x = \dfrac{1}{\sin x}$ and $\cot x = \dfrac{\cos x}{\sin x}$, we can rewrite the expression on the right-hand side as

$$\lim_{x \to 0^+} \frac{-\sin^2 x}{x \cos x} = \lim_{x \to 0^+} \left[\frac{\sin x}{x} \cdot (-\tan x) \right] = \left(\lim_{x \to 0^+} \frac{\sin x}{x} \right) \cdot \left(\lim_{x \to 0^+} (-\tan x) \right) = 1 \cdot 0 = 0.$$

We conclude that $\lim\limits_{x \to 0^+} \ln y = 0$. Therefore, $\ln \left(\lim\limits_{x \to 0^+} y \right) = 0$ and we have

$$\lim_{x \to 0^+} y = \lim_{x \to 0^+} x^{\sin x} = e^0 = 1.$$

Hence,

$$\lim_{x \to 0^+} x^{\sin x} = 1.$$

? Exercise 4.8.7

Evaluate $\lim\limits_{x \to 0^+} x^x$.

Hint

Let $y = x^x$ and take the natural logarithm of both sides of the equation.

Answer

1

Growth Rates of Functions

Suppose the functions f and g both approach infinity as $x \to \infty$. Although the values of both functions become arbitrarily large as the values of x become sufficiently large, sometimes one function is growing more quickly than the other. For example, $f(x) = x^2$ and $g(x) = x^3$ both approach infinity as $x \to \infty$. However, as Table 4.8.1 shows, the values of x^3 are growing much faster than the values of x^2.

Table 4.8.1: Comparing the Growth Rates of x^2 and x^3

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = x^3$	1000	1,000,000	1,000,000,000	1,000,000,000,000

In fact,

$$\lim_{x \to \infty} \frac{x^3}{x^2} = \lim_{x \to \infty} x = \infty.$$

or, equivalently

$$\lim_{x \to \infty} \frac{x^2}{x^3} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

As a result, we say x^3 is growing more rapidly than x^2 as $x \to \infty$. On the other hand, for $f(x) = x^2$ and $g(x) = 3x^2 + 4x + 1$, although the values of $g(x)$ are always greater than the values of $f(x)$ for $x > 0$, each value of $g(x)$ is roughly three times the corresponding value of $f(x)$ as $x \to \infty$, as shown in Table 4.8.2. In fact,

$$\lim_{x \to \infty} \frac{x^2}{3x^2 + 4x + 1} = \frac{1}{3}.$$

Table 4.8.2: Comparing the Growth Rates of x^2 and $3x^2 + 4x + 1$

x	10	100	1000	10,000
$f(x) = x^2$	100	10,000	1,000,000	100,000,000
$g(x) = 3x^2 + 4x + 1$	341	30,401	3,004,001	300,040,001

In this case, we say that x^2 and $3x^2 + 4x + 1$ are growing at the same rate as $x \to \infty$.

More generally, suppose f and g are two functions that approach infinity as $x \to \infty$. We say g grows more rapidly than f as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = \infty \quad \text{or, equivalently,} \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

On the other hand, if there exists a constant $M \neq 0$ such that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = M,$$

we say f and g grow at the same rate as $x \to \infty$.

Next we see how to use L'Hôpital's rule to compare the growth rates of power, exponential, and logarithmic functions.

Example 4.8.8: Comparing the Growth Rates of $\ln(x)$, x^2, and e^x

For each of the following pairs of functions, use L'Hôpital's rule to evaluate

$$\lim_{x \to \infty} \frac{f(x)}{g(x)}.$$

a. $f(x) = x^2$ and $g(x) = e^x$
b. $f(x) = \ln(x)$ and $g(x) = x^2$

Solution

a. Since $\lim_{x \to \infty} x^2 = \infty$ and $\lim_{x \to \infty} e^x = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x \to \infty} \left[\frac{x^2}{e^x}\right]$. We obtain

$$\lim_{x \to \infty} \frac{x^2}{e^x} = \lim_{x \to \infty} \frac{2x}{e^x}.$$

Since $\lim_{x \to \infty} 2x = \infty$ and $\lim_{x \to \infty} e^x = \infty$, we can apply L'Hôpital's rule again. Since

$$\lim_{x \to \infty} \frac{2x}{e^x} = \lim_{x \to \infty} \frac{2}{e^x} = 0,$$

we conclude that

$$\lim_{x \to \infty} \frac{x^2}{e^x} = 0.$$

Therefore, e^x grows more rapidly than x^2 as $x \to \infty$ (See Figure 4.8.3 and Table 4.8.3)

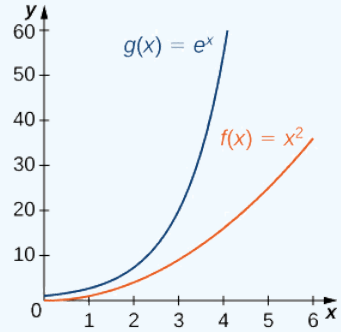


Figure 4.8.3: An exponential function grows at a faster rate than a power function.

Table 4.8.3: Growth rates of a power function and an exponential function.

x	5	10	15	20
x^2	25	100	225	400
e^x	148	22,026	3,269,017	485,165,195

b. Since $\lim_{x\to\infty} \ln x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, we can use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \dfrac{\ln x}{x^2}$. We obtain

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{1/x}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0.$$

Thus, x^2 grows more rapidly than $\ln x$ as $x \to \infty$ (see Figure 4.8.4 and Table 4.8.4).

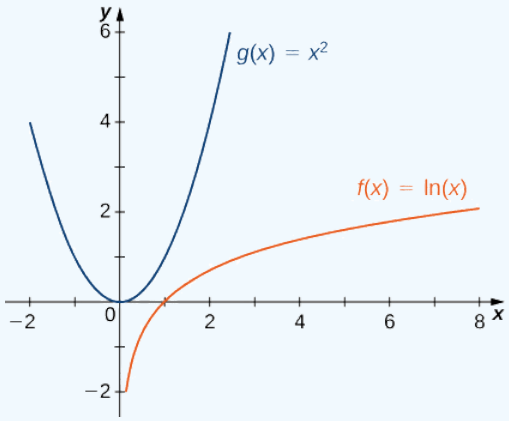


Figure 4.8.4: A power function grows at a faster rate than a logarithmic function.

Table 4.8.4: Growth rates of a power function and a logarithmic function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
x^2	100	10,000	1,000,000	100,000,000

? Exercise 4.8.8

Compare the growth rates of x^{100} and 2^x.

Hint

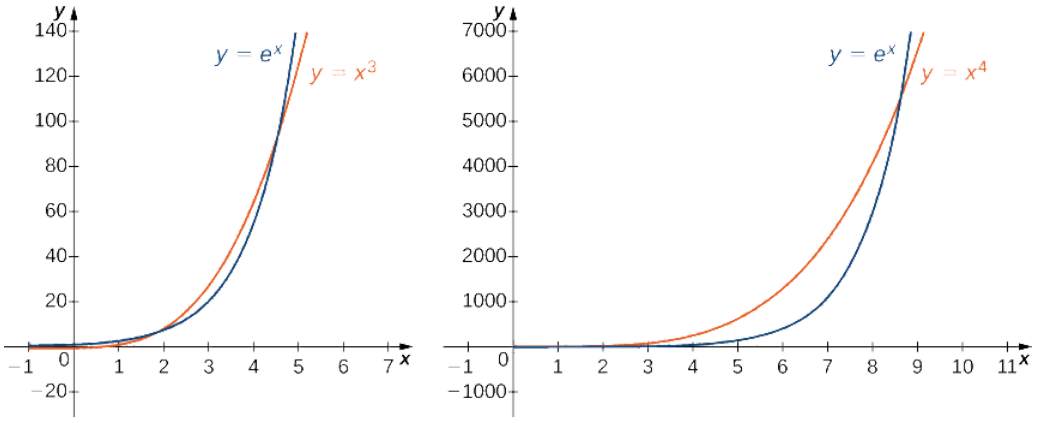
Apply L'Hôpital's rule to $x^{100}/2^x$.

Answer

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The function 2^x grows faster than x^{100}.

Using the same ideas as in Example 4.8.8*a*, it is not difficult to show that e^x grows more rapidly than x^p for any $p > 0$. In Figure 4.8.5 and Table 4.8.5, we compare e^x with x^3 and x^4 as $x \to \infty$.



(a) (b)

Figure 4.8.5: The exponential function e^x grows faster than x^p for any $p > 0$. (a) A comparison of e^x with x^3. (b) A comparison of e^x with x^4.

Table 4.8.5: An exponential function grows at a faster rate than any power function

x	5	10	15	20
x^3	125	1000	3375	8000
x^4	625	10,000	50,625	160,000
e^x	148	22,026	3,269,017	485,165,195

Similarly, it is not difficult to show that x^p grows more rapidly than $\ln x$ for any $p > 0$. In Figure 4.8.6 and Table 4.8.6, we compare $\ln x$ with $\sqrt[3]{x}$ and $\sqrt{x}$.

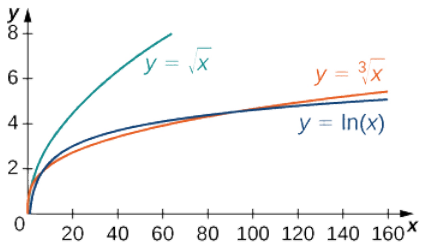


Figure 4.8.6: The function $y = \ln(x)$ grows more slowly than x^p for any $p > 0$ as $x \to \infty$.

Table 4.8.6: A logarithmic function grows at a slower rate than any root function

x	10	100	1000	10,000
$\ln(x)$	2.303	4.605	6.908	9.210
$\sqrt[3]{x}$	2.154	4.642	10	21.544
$\sqrt{x}$	3.162	10	31.623	100

Key Concepts

- L'Hôpital's rule can be used to evaluate the limit of a quotient when the indeterminate form $\frac{0}{0}$ or ∞/∞ arises.
- L'Hôpital's rule can also be applied to other indeterminate forms if they can be rewritten in terms of a limit involving a quotient that has the indeterminate form $\frac{0}{0}$ or ∞/∞.

- The exponential function e^x grows faster than any power function $x^p, p > 0$.
- The logarithmic function $\ln x$ grows more slowly than any power function $x^p, p > 0$.

Glossary

indeterminate forms

When evaluating a limit, the forms $\frac{0}{0}, \infty/\infty, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0$, and 1^∞ are considered indeterminate because further analysis is required to determine whether the limit exists and, if so, what its value is.

L’Hôpital’s rule

If f and g are differentiable functions over an interval a, except possibly at a, and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ or $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are infinite, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f\,'(x)}{g'(x)}$, assuming the limit on the right exists or is ∞ or $-\infty$.

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4.8E: Exercises for L'Hôpital's Rule

Terms and Concepts

1. List the different indeterminate forms described in this section.

Answer:

The forms $0 \cdot \infty, \infty - \infty, 1^{\infty}, \infty^{0}$, and 0^{0} are all considered indeterminate.

2. List similar looking forms that are not indeterminate.

Answer:

Among others, the forms $\infty \cdot \infty, \infty + \infty, -\infty - \infty, 0^{\infty}, 1^{0}$, and ∞^{∞} are not considered indeterminate, as these limits can be determined clearly.

3. T/F: l'Hôpital's Rule states that $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)}{g'(x)}$.

Answer:

False. L'Hôpital's Rule is a method for taking limits of rational functions in certain cases. It does not replace the Quotient Rule when taking the derivative of these rational functions.

4. Explain what the indeterminate form "1^{∞}" means. Why is it indeterminate?

Answer:

When a limit has the form "1^{∞}", this means that the function in the base of the exponent is approaching 1, while the function in the exponent is approaching ∞. It's indeterminate, since if the base function is approaching 1, but always is less than 1, then the limit could be 0, while if the base function were approaching 1, but always is greater than 1, the limit could be ∞. But since this uncertainty exists, the limit could, in fact, be anything.

5. Explain why limits of the form $\infty - \infty$ are indeterminate.

Answer:

Limits with this form depend on the relative speed with which the two terms are approaching ∞. If the first term approaches ∞ faster than the second term, the limit would be ∞. If the second term approaches ∞ faster than the first term, the limit would be $-\infty$. But if they both appraoch ∞ at about the same rates, the limit could be anything!

6. Fill in the blanks" The Quotient Rule is applied to $\frac{f(x)}{g(x)}$ when taking its _____; l'Hôpital's Rule is applied when taking _____ of $\frac{f(x)}{g(x)}$ when the form is _____ or _____.

Answer:

derivative; limits; $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

7. Create (but do not evaluate) a limit that initially has the form "∞^{0}".

8. Create a function $f(x)$ such that $\lim_{x \to 1} f(x)$ initially has the form "0^{0}".

Problems

For exercises 1 - 6, evaluate the limit.

1) Evaluate the limit $\lim_{x \to \infty} \frac{e^x}{x}$.

2) Evaluate the limit $\lim_{x \to \infty} \frac{e^x}{x^k}$.

Answer:

$$\lim_{x\to\infty}\frac{e^x}{x^k} \quad = \quad \infty$$

3) Evaluate the limit $\lim_{x\to\infty}\frac{\ln x}{x^k}$.

4) Evaluate the limit $\lim_{x\to a}\frac{x-a}{x^2-a^2}$.

Answer:

$$\lim_{x\to a}\frac{x-a}{x^2-a^2} \quad = \quad \frac{1}{2a}$$

5. Evaluate the limit $\lim_{x\to a}\frac{x-a}{x^3-a^3}$.

6. Evaluate the limit $\lim_{x\to a}\frac{x-a}{x^n-a^n}$.

Answer:

$$\lim_{x\to a}\frac{x-a}{x^n-a^n} \quad = \quad \frac{1}{na^{n-1}}$$

For exercises 7 - 11, determine whether you can apply L'Hôpital's rule directly. Explain why or why not. Then, indicate if there is some way you can alter the limit so you can apply L'Hôpital's rule.

7) $\lim_{x\to 0^+} x^2 \ln x$

8) $\lim_{x\to\infty} x^{1/x}$

Answer:

Cannot apply directly; use logarithms

9) $\lim_{x\to 0} x^{2/x}$

10) $\lim_{x\to 0}\frac{x^2}{1/x}$

Answer:

Cannot apply directly; rewrite as $\lim_{x\to 0} x^3$

11) $\lim_{x\to\infty}\frac{e^x}{x}$

For exercises 12 - 44, evaluate the limits with either L'Hôpital's rule or previously learned methods.

12) $\lim_{x\to 3}\frac{x^2-9}{x-3}$

Answer:

$$\lim_{x\to 3}\frac{x^2-9}{x-3} \quad = \quad 6$$

13) $\lim_{x\to 3}\frac{x^2-9}{x+3}$

14) $\lim_{x\to 0}\frac{(1+x)^{-2}-1}{x}$

Answer:

$$\lim_{x\to 0}\frac{(1+x)^{-2}-1}{x} = -2$$

15) $\lim_{x\to \pi/2}\frac{\cos x}{\frac{\pi}{2}-x}$

16) $\lim_{x\to \pi}\frac{x-\pi}{\sin x}$

Answer:

$$\lim_{x\to \pi}\frac{x-\pi}{\sin x} = -1$$

17) $\lim_{x\to 1}\frac{x-1}{\sin x}$

18) $\lim_{x\to 0}\frac{(1+x)^n-1}{x}$

Answer:

$$\lim_{x\to 0}\frac{(1+x)^n-1}{x} = n$$

19) $\lim_{x\to 0}\frac{(1+x)^n-1-nx}{x^2}$

20) $\lim_{x\to 0}\frac{\sin x-\tan x}{x^3}$

Answer:

$$\lim_{x\to 0}\frac{\sin x-\tan x}{x^3} = -\frac{1}{2}$$

21) $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$

22) $\lim_{x\to 0}\frac{e^x-x-1}{x^2}$

Answer:

$$\lim_{x\to 0}\frac{e^x-x-1}{x^2} = \frac{1}{2}$$

23) $\lim_{x\to 0}\frac{\tan x}{\sqrt{x}}$

24) $\lim_{x\to 1}\frac{x-1}{\ln x}$

Answer:

$$\lim_{x\to 1}\frac{x-1}{\ln x} = 1$$

25) $\lim_{x\to 0}(x+1)^{1/x}$

26) $\lim_{x\to 1}\frac{\sqrt{x}-\sqrt[3]{x}}{x-1}$

Answer:

$$\lim_{x\to 1}\frac{\sqrt{x}-\sqrt[3]{x}}{x-1} = \frac{1}{6}$$

27) $\lim_{x\to 0^+} x^{2x}$

28) $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right)$

Answer:

$$\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) \quad = \quad 1$$

29) $\lim_{x\to 0} \dfrac{\sin x - x}{x^2}$

30) $\lim_{x\to 0^+} x\ln\left(x^4\right)$

Answer:

$$\lim_{x\to 0^+} x\ln\left(x^4\right) \quad = \quad 0$$

31) $\lim_{x\to\infty} \left(x - e^x\right)$

32) $\lim_{x\to\infty} x^2 e^{-x}$

Answer:

$$\lim_{x\to\infty} x^2 e^{-x} \quad = \quad 0$$

33) $\lim_{x\to 1^+} \left[\dfrac{1}{\ln x} - \dfrac{1}{1-x}\right]$

34) $\lim_{x\to 3^+} \left[\dfrac{5}{x^2-9} - \dfrac{x}{x-3}\right]$

Answer:

$$\lim_{x\to 3^+} \left[\frac{5}{x^2-9} - \frac{x}{x-3}\right] = \lim_{x\to 3^+} \frac{5-x^2-3x}{x^2-9} \quad = \quad -\infty$$

35) $\lim_{x\to\infty} \dfrac{\sqrt{2x^2-3}}{x+2}$

Note:

L'Hôpital's rule fails to help us find this limit, although the form seems appropriate. But you can evaluate this limit using techniques you learned earlier in calculus.

36) $\lim_{x\to\infty} \left(\dfrac{x+7}{x+3}\right)^x$

Answer:

$$\lim_{x\to\infty} \left(\frac{x+7}{x+3}\right)^x \quad = \quad e^4$$

37) $\lim_{x\to 0} \dfrac{3^x - 2^x}{x}$

38) $\lim_{x\to 0} \dfrac{1+1/x}{1-1/x}$

Answer:

$$\lim_{x\to 0} \frac{1+1/x}{1-1/x} \quad = \quad -1$$

39) $\lim_{x\to\pi/4}(1-\tan x)\cot x$

40) $\lim_{x\to\infty} xe^{1/x}$

Answer:

$$\lim_{x\to\infty} xe^{1/x} = \infty$$

41) $\lim_{x\to 0} x^{1/\cos x}$

42) $\lim_{x\to 0^+} x^{1/x}$

Answer:

$$\lim_{x\to 0^+} x^{1/x} = 0$$

43) $\lim_{x\to 0}\left(1-\frac{1}{x}\right)^x$

44) $\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x$

Answer:

$$\lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x = \frac{1}{e}$$

For exercises 45 - 54, use a calculator to graph the function and estimate the value of the limit, then use L'Hôpital's rule to find the limit directly.

45) [T] $\lim_{x\to 0}\frac{e^x-1}{x}$

46) [T] $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$

Answer:

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$$

47) [T] $\lim_{x\to 1}\frac{x-1}{1-\cos(\pi x)}$

48) [T] $\lim_{x\to 1}\frac{e^{x-1}-1}{x-1}$

Answer:

$$\lim_{x\to 1}\frac{e^{x-1}-1}{x-1} = 1$$

49) [T] $\lim_{x\to 1}\frac{(x-1)^2}{\ln x}$

50) [T] $\lim_{x\to\pi}\frac{1+\cos x}{\sin x}$

Answer:

$$\lim_{x\to\pi}\frac{1+\cos x}{\sin x} = 0$$

51) [T] $\lim_{x\to 0}\left(\csc x-\frac{1}{x}\right)$

52) [T] $\lim_{x \to 0^+} \tan(x^x)$

Answer:

$$\lim_{x \to 0^+} \tan(x^x) = \tan 1$$

53) [T] $\lim_{x \to 0^+} \dfrac{\ln x}{\sin x}$

54) [T] $\lim_{x \to 0} \dfrac{e^x - e^{-x}}{x}$

Answer:

$$\lim_{x \to 0} \frac{e^x - e^{-x}}{x} = 2$$

Contributors

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4.9: Improper Integrals

Learning Objectives

- Evaluate an integral over an infinite interval.
- Evaluate an integral over a closed interval with an infinite discontinuity within the interval.
- Use the comparison theorem to determine whether a definite integral is convergent.

Is the area between the graph of $f(x) = \dfrac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ finite or infinite? If this same region is revolved about the x-axis, is the volume finite or infinite? Surprisingly, the area of the region described is infinite, but the volume of the solid obtained by revolving this region about the x-axis is finite.

In this section, we define integrals over an infinite interval as well as integrals of functions containing a discontinuity on the interval. Integrals of these types are called improper integrals. We examine several techniques for evaluating improper integrals, all of which involve taking limits.

Integrating over an Infinite Interval

How should we go about defining an integral of the type $\displaystyle\int_a^{+\infty} f(x)\,dx$? We can integrate $\displaystyle\int_a^t f(x)\,dx$ for any value of t, so it is reasonable to look at the behavior of this integral as we substitute larger values of t. Figure 4.9.1 shows that $\displaystyle\int_a^t f(x)\,dx$ may be interpreted as area for various values of t. In other words, we may define an improper integral as a limit, taken as one of the limits of integration increases or decreases without bound.

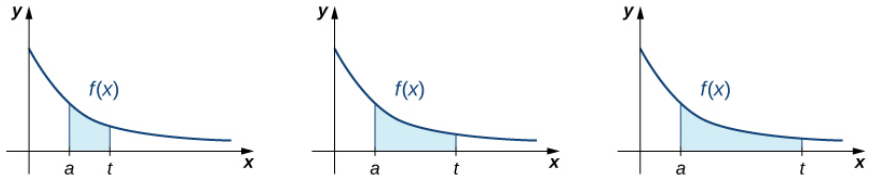


Figure 4.9.1: To integrate a function over an infinite interval, we consider the limit of the integral as the upper limit increases without bound.

Definition: Improper Integral

1. Let $f(x)$ be continuous over an interval of the form $[a, +\infty)$. Then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx, \tag{4.9.1}$$

   provided this limit exists.

2. Let $f(x)$ be continuous over an interval of the form $(-\infty, b]$. Then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx, \tag{4.9.2}$$

   provided this limit exists.

In each case, if the limit exists, then the *improper integral* is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. Let $f(x)$ be continuous over $(-\infty, +\infty)$. Then

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^{+\infty} f(x)\,dx, \tag{4.9.3}$$

   provided that $\displaystyle\int_{-\infty}^0 f(x)\,dx$ and $\displaystyle\int_0^{+\infty} f(x)\,dx$ both converge.

   If either of these two integrals diverge, then $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx$ diverges.

(It can be shown that, in fact, $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^{+\infty} f(x)\,dx$ for any value of a.).

In our first example, we return to the question we posed at the start of this section: Is the area between the graph of $f(x) = \dfrac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ finite or infinite?

Example 4.9.1: Finding an Area

Determine whether the area between the graph of $f(x) = \dfrac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ is finite or infinite.

Solution

We first do a quick sketch of the region in question, as shown in Figure 4.9.2.

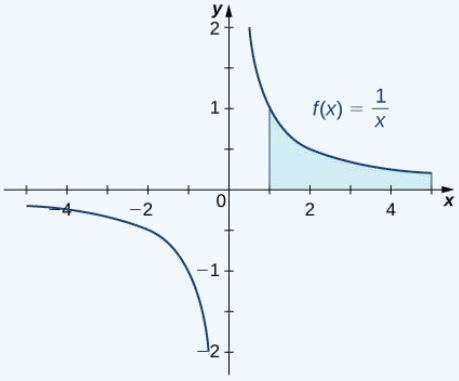


Figure 4.9.2: We can find the area between the curve $f(x) = 1/x$ and the x-axis on an infinite interval.

We can see that the area of this region is given by

$$A = \int_1^{\infty} \frac{1}{x}\, dx.$$

which can be evaluated using Equation 4.9.1:

$$\begin{aligned} A &= \int_1^{\infty} \frac{1}{x}\, dx \\ &= \lim_{t \to +\infty} \int_1^{t} \frac{1}{x}\, dx && \text{Rewrite the improper integral as a limit} \\ &= \lim_{t \to +\infty} \ln|x| \Big|_1^{t} && \text{Find the antiderivative} \\ &= \lim_{t \to +\infty} (\ln|t| - \ln 1) && \text{Evaluate the antiderivative} \\ &= +\infty. && \text{Evaluate the limit.} \end{aligned}$$

Since the improper integral diverges to $+\infty$, the area of the region is infinite.

Example 4.9.2: Finding a Volume

Find the volume of the solid obtained by revolving the region bounded by the graph of $f(x) = \dfrac{1}{x}$ and the x-axis over the interval $[1, +\infty)$ about the x-axis.

Solution

The solid is shown in Figure 4.9.3. Using the disk method, we see that the volume V is

$$V = \pi \int_1^{+\infty} \frac{1}{x^2}\, dx.$$

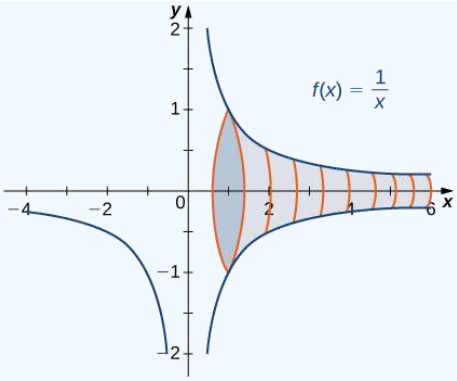


Figure 4.9.3: The solid of revolution can be generated by rotating an infinite area about the x-axis.

Then we have

$$
\begin{aligned}
V &= \pi \int_1^{+\infty} \frac{1}{x^2}\,dx \\
&= \pi \lim_{t\to+\infty} \int_1^t \frac{1}{x^2}\,dx && \text{Rewrite as a limit.} \\
&= \pi \lim_{t\to+\infty} -\frac{1}{x}\Big|_1^t && \text{Find the antiderivative.} \\
&= \pi \lim_{t\to+\infty} \left(-\frac{1}{t}+1\right) && \text{Evaluate the antiderivative.} \\
&= \pi \lim_{t\to+\infty} \left(-\frac{1}{\cancel{t}^{\,0}}+1\right) && \text{Evaluate the antiderivative.} \\
&= \pi
\end{aligned}
$$

The improper integral converges to π. Therefore, the volume of the solid of revolution is π.

In conclusion, although the area of the region between the x-axis and the graph of $f(x) = 1/x$ over the interval $[1, +\infty)$ is infinite, the volume of the solid generated by revolving this region about the x-axis is finite. The solid generated is known as Gabriel's Horn.

Note: Gabriel's horn (also called Torricelli's trumpet) is a geometric figure which has *infinite* surface area, but *finite* volume. The name refers to the tradition identifying the Archangel Gabriel as the angel who blows the horn to announce Judgment Day, associating the divine, or infinite, with the finite. The properties of this figure were first studied by Italian physicist and mathematician Evangelista Torricelli in the 17th century.

Example 4.9.3: Traffic Accidents in a City

Suppose that at a busy intersection, traffic accidents occur at an average rate of one every three months. After residents complained, changes were made to the traffic lights at the intersection. It has now been eight months since the changes were made and there have been no accidents. Were the changes effective or is the 8-month interval without an accident a result of chance?



Figure 4.9.4: Modification of work by David McKelvey, Flickr.

Probability theory tells us that if the average time between events is k, the *probability* that X, the time between events, is between a and b is given by

$$(P(a \le x \le b) = \int_a^b f(x)\,dx$$

where

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ ke^{-kx}, & \text{if } x \geq 0 \end{cases}.$$

Thus, if accidents are occurring at a rate of one every 3 months, then the probability that X, the time between accidents, is between a and b is given by

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx$$

where

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 3e^{-3x}, & \text{if } x \geq 0 \end{cases}.$$

To answer the question, we must compute $P(X \geq 8) = \int_8^{+\infty} 3e^{-3x}\,dx$ and decide whether it is likely that 8 months could have passed without an accident if there had been no improvement in the traffic situation.

Solution

We need to calculate the probability as an improper integral:

$$\begin{aligned} P(X \geq 8) &= \int_8^{+\infty} 3e^{-3x}\,dx \\ &= \lim_{t \to +\infty} \int_8^t 3e^{-3x}\,dx \\ &= \lim_{t \to +\infty} -e^{-3x}\Big|_8^t \\ &= \lim_{t \to +\infty} (-e^{-3t} + e^{-24}) \\ &\approx 3.8 \times 10^{-11}. \end{aligned}$$

The value 3.8×10^{-11} represents the probability of no accidents in 8 months under the initial conditions. Since this value is very, very small, it is reasonable to conclude the changes were effective.

Example 4.9.4: Evaluating an Improper Integral over an Infinite Interval

Evaluate $\int_{-\infty}^0 \frac{1}{x^2+4}\,dx$. State whether the improper integral converges or diverges.

Solution

Begin by rewriting $\int_{-\infty}^0 \frac{1}{x^2+4}\,dx$ as a limit using Equation 4.9.2 from the definition. Thus,

$$\begin{aligned} \int_{-\infty}^0 \frac{1}{x^2+4}\,dx &= \lim_{t \to -\infty} \int_t^0 \frac{1}{x^2+4}\,dx && \text{Rewrite as a limit.} \\ &= \lim_{t \to -\infty} \frac{1}{2}\tan^{-1}\frac{x}{2}\Big|_t^0 && \text{Find the antiderivative.} \\ &= \lim_{t \to -\infty} \left(\frac{1}{2}\tan^{-1} 0 - \frac{1}{2}\tan^{-1}\frac{t}{2}\right) && \text{Evaluate the antiderivative.} \\ &= \frac{\pi}{4}. && \text{Evaluate the limit and simplify.} \end{aligned}$$

The improper integral converges to $\frac{\pi}{4}$.

Example 4.9.5: Evaluating an Improper Integral on $(-\infty, +\infty)$

Evaluate $\int_{-\infty}^{+\infty} xe^x\,dx$. State whether the improper integral converges or diverges.

Solution

Start by splitting up the integral:

$$\int_{-\infty}^{+\infty} xe^x\,dx = \int_{-\infty}^{0} xe^x\,dx + \int_{0}^{+\infty} xe^x\,dx.$$

If either $\int_{-\infty}^{0} xe^x\,dx$ or $\int_{0}^{+\infty} xe^x\,dx$ diverges, then $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges.

Compute each integral separately.

For the first integral,

$$\begin{aligned}
\int_{-\infty}^{0} xe^x\,dx &= \lim_{t\to-\infty}\int_{t}^{0} xe^x\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to-\infty}(xe^x - e^x)\Big|_{t}^{0} && \text{Use integration by parts to find the antiderivative. (Here } u = x \text{ and } dv = e^x\,dx.\text{)}\\
&= \lim_{t\to-\infty}(-1 - te^t + e^t) && \text{Evaluate the antiderivative.}\\
&= -1.
\end{aligned}$$

Evaluate the limit. Note: $\lim_{t\to-\infty} te^t$ is indeterminate of the form $0\cdot\infty$.Thus, $\lim_{t\to-\infty} te^t = \lim_{t\to-\infty}\frac{t}{e^{-t}} = \lim_{t\to-\infty}\frac{-1}{e^{-t}} = \lim_{t\to-\infty} -e^t = 0$ by L'Hôpital's Rule.

The first improper integral converges. For the second integral,

$$\begin{aligned}
\int_{0}^{+\infty} xe^x\,dx &= \lim_{t\to+\infty}\int_{0}^{t} xe^x\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to+\infty}(xe^x - e^x)\Big|_{0}^{t} && \text{Find the antiderivative.}\\
&= \lim_{t\to+\infty}(te^t - e^t + 1) && \text{Evaluate the antiderivative.}\\
&= \lim_{t\to+\infty}((t-1)e^t + 1) && \text{Rewrite. } (te^t - e^t \text{ is indeterminate.})\\
&= +\infty. && \text{Evaluate the limit.}
\end{aligned}$$

Thus, $\int_{0}^{+\infty} xe^x\,dx$ diverges. Since this integral diverges, $\int_{-\infty}^{+\infty} xe^x\,dx$ diverges as well.

? Exercise 4.9.1

Evaluate $\int_{-3}^{+\infty} e^{-x}\,dx$. State whether the improper integral converges or diverges.

Hint

$$\int_{-3}^{+\infty} e^{-x}\,dx = \lim_{t\to+\infty}\int_{-3}^{t} e^{-x}\,dx$$

Answer

It converges to e^3.

Integrating a Discontinuous Integrand

Now let's examine integrals of functions containing an infinite discontinuity in the interval over which the integration occurs. Consider an integral of the form $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over $[a,b)$ and discontinuous at b. Since the function $f(x)$ is continuous over $[a,t]$ for all values of t satisfying $a \le t < b$, the integral $\int_a^t f(x)\,dx$ is defined for all such values of t. Thus, it makes sense to consider the values of $\int_a^t f(x)\,dx$ as t approaches b for $a \le t < b$. That is, we define $\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$, provided this limit exists. Figure 4.9.5 illustrates $\int_a^t f(x)\,dx$ as areas of regions for values of t approaching b.

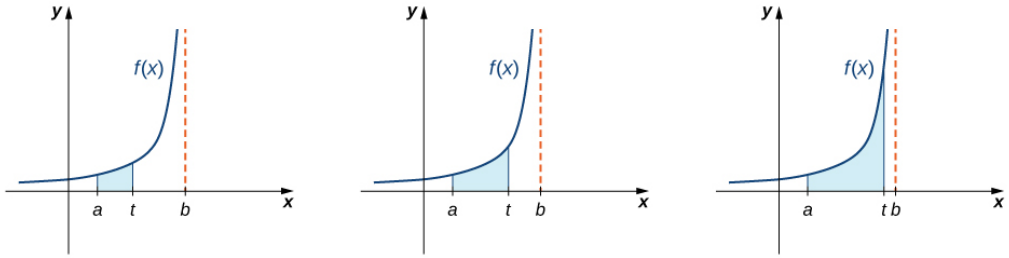


Figure 4.9.5: As t approaches b from the left, the value of the area from a to t approaches the area from a to b.

We use a similar approach to define $\int_a^b f(x)\,dx$, where $f(x)$ is continuous over $(a, b]$ and discontinuous at a. We now proceed with a formal definition.

Definition: Converging and Diverging Improper Integral

1. Let $f(x)$ be continuous over $[a, b)$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx. \tag{4.9.4}$$

2. Let $f(x)$ be continuous over $(a, b]$. Then,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx. \tag{4.9.5}$$

In each case, if the limit exists, then the improper integral is said to converge. If the limit does not exist, then the improper integral is said to diverge.

3. If $f(x)$ is continuous over $[a, b]$ except at a point c in (a, b), then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \tag{4.9.6}$$

provided both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ converge. If either of these integrals diverges, then $\int_a^b f(x)\,dx$ diverges.

The following examples demonstrate the application of this definition.

Example 4.9.6: Integrating a Discontinuous Integrand

Evaluate $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$, if possible. State whether the integral converges or diverges.

Solution

The function $f(x) = \frac{1}{\sqrt{4-x}}$ is continuous over $[0, 4)$ and discontinuous at 4. Using Equation 4.9.4 from the definition, rewrite $\int_0^4 \frac{1}{\sqrt{4-x}}\,dx$ as a limit:

$$\begin{aligned}
\int_0^4 \frac{1}{\sqrt{4-x}}\,dx &= \lim_{t\to 4^-} \int_0^t \frac{1}{\sqrt{4-x}}\,dx && \text{Rewrite as a limit.}\\
&= \lim_{t\to 4^-} \left(-2\sqrt{4-x}\right)\Big|_0^t && \text{Find the antiderivative.}\\
&= \lim_{t\to 4^-} \left(-2\sqrt{4-t}+4\right) && \text{Evaluate the antiderivative.}\\
&= 4. && \text{Evaluate the limit.}
\end{aligned}$$

The improper integral converges.

Example 4.9.7: Integrating a Discontinuous Integrand

Evaluate $\int_0^2 x \ln x \, dx$. State whether the integral converges or diverges.

Solution

Since $f(x) = x \ln x$ is continuous over $(0, 2]$ and is discontinuous at zero, we can rewrite the integral in limit form using Equation 4.9.5:

$$\int_0^2 x \ln x \, dx = \lim_{t \to 0^+} \int_t^2 x \ln x \, dx \qquad \text{Rewrite as a limit.}$$

$$= \lim_{t \to 0^+} \left(\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2 \right) \Big|_t^2 \qquad \text{Evaluate } \int x \ln x \, dx \text{ using integration by parts with } u = \ln x \text{ and } dv = x.$$

$$= \lim_{t \to 0^+} \left(2 \ln 2 - 1 - \frac{1}{2} t^2 \ln t + \frac{1}{4} t^2 \right). \qquad \text{Evaluate the antiderivative.}$$

$$= 2 \ln 2 - 1. \qquad \text{Evaluate the limit. Note that } \lim_{t \to 0^+} t^2 \ln t \text{ is indeterminate.}$$

To evaluate it, rewrite it as a quotient and apply L'Hôpital's rule.

The improper integral converges.

Example 4.9.8: Integrating a Discontinuous Integrand

Evaluate $\int_{-1}^1 \frac{1}{x^3} \, dx$. State whether the improper integral converges or diverges.

Solution

Since $f(x) = 1/x^3$ is discontinuous at zero, using Equation 4.9.6, we can write

$$\int_{-1}^1 \frac{1}{x^3} \, dx = \int_{-1}^0 \frac{1}{x^3} \, dx \quad + \quad \int_0^1 \frac{1}{x^3} \, dx.$$

If either of the two integrals diverges, then the original integral diverges. Begin with $\int_{-1}^0 \frac{1}{x^3} \, dx$:

$$\int_{-1}^0 \frac{1}{x^3} \, dx = \lim_{t \to 0^-} \int_{-1}^t \frac{1}{x^3} \, dx \qquad \text{Rewrite as a limit.}$$

$$= \lim_{t \to 0^-} \left(-\frac{1}{2x^2} \right) \Big|_{-1}^t \qquad \text{Find the antiderivative.}$$

$$= \lim_{t \to 0^-} \left(-\frac{1}{2t^2} + \frac{1}{2} \right) \qquad \text{Evaluate the antiderivative.}$$

$$= +\infty. \qquad \text{Evaluate the limit.}$$

Therefore, $\int_{-1}^0 \frac{1}{x^3} \, dx$ diverges.

Since $\int_{-1}^0 \frac{1}{x^3} \, dx$ diverges, $\int_{-1}^1 \frac{1}{x^3} \, dx$ also diverges.

Exercise 4.9.2

Evaluate $\int_0^2 \frac{1}{x} \, dx$. State whether the integral converges or diverges.

Hint

Write $\int_0^2 \frac{1}{x} \, dx$ in limit form using Equation 4.9.5

Answer

$+\infty$, It diverges.

A Comparison Theorem

It is not always easy or even possible to evaluate an improper integral directly; however, by comparing it with another carefully chosen integral, it may be possible to determine its convergence or divergence. To see this, consider two continuous functions $f(x)$ and $g(x)$ satisfying $0 \le f(x) \le g(x)$ for $x \ge a$ (Figure 4.9.6). In this case, we may view integrals of these functions over intervals of the form $[a, t]$ as areas, so we have the relationship

$$0 \le \int_a^t f(x)\,dx \le \int_a^t g(x)\,dx$$

for $t \ge a$.

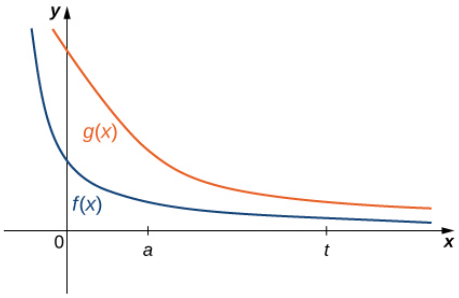


Figure 4.9.6: If $0 \le f(x) \le g(x)$ for $x \ge a$, then for $t \ge a$, $\int_a^t f(x)\,dx \le \int_a^t g(x)\,dx$.

Thus, if

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = +\infty,$$

then

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = +\infty$$

as well. That is, if the area of the region between the graph of $f(x)$ and the x-axis over $[a, +\infty)$ is infinite, then the area of the region between the graph of $g(x)$ and the x-axis over $[a, +\infty)$ is infinite too.

On the other hand, if

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = L$$

for some real number L, then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

must converge to some value less than or equal to L, since $\int_a^t f(x)\,dx$ increases as t increases and $\int_a^t f(x)\,dx \le L$ for all $t \ge a$.

If the area of the region between the graph of $g(x)$ and the x-axis over $[a, +\infty)$ is finite, then the area of the region between the graph of $f(x)$ and the x-axis over $[a, +\infty)$ is also finite.

These conclusions are summarized in the following theorem.

A Comparison Theorem

Let $f(x)$ and $g(x)$ be continuous over $[a, +\infty)$. Assume that $0 \le f(x) \le g(x)$ for $x \ge a$.

i. If

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx = +\infty,$$

then

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty} \int_a^t g(x)\,dx = +\infty.$$

ii. If

$$\int_a^{+\infty} g(x)\,dx = \lim_{t\to+\infty}\int_a^t g(x)\,dx = L,$$

where L is a real number, then

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty}\int_a^t f(x)\,dx = M$$

for some real number $M \le L$.

Example 4.9.9: Applying the Comparison Theorem

Use a comparison to show that $\int_1^{+\infty} \frac{1}{xe^x}\,dx$.

converges.

Solution

We can see that $0 \le \frac{1}{xe^x} \le \frac{1}{e^x} = e^{-x}$,

so if $\int_1^{+\infty} e^{-x}\,dx$ converges, then so does $\int_1^{+\infty} \frac{1}{xe^x}\,dx$.

To evaluate $\int_1^{+\infty} e^{-x}\,dx$, first rewrite it as a limit:

$$\begin{aligned}\int_1^{+\infty} e^{-x}\,dx &= \lim_{t\to+\infty}\int_1^t e^{-x}\,dx \\ &= \lim_{t\to+\infty}\left(-e^{-x}\right)\Big|_1^t \\ &= \lim_{t\to+\infty}\left(-e^{-t}+e^{-1}\right) \\ &= e^{-1}.\end{aligned}$$

Since $\int_1^{+\infty} e^{-x}\,dx$ converges, so does $\int_1^{+\infty} \frac{1}{xe^x}\,dx$.

Example 4.9.10: Applying the Comparison Theorem

Use the comparison theorem to show that $\int_1^{+\infty} \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Solution

For $p < 1$, $\frac{1}{x} \le \frac{1}{x^p}$ over $[1, +\infty)$.

In Example 4.9.1, we showed that $\int_1^{+\infty} \frac{1}{x}\,dx = +\infty$.

Therefore, $\int_1^{+\infty} \frac{1}{x^p}\,dx$ diverges for all $p < 1$.

Exercise 4.9.3

Use a comparison to show that $\int_e^{+\infty} \frac{\ln x}{x}\,dx$ diverges.

Hint

$\frac{1}{x} \le \frac{\ln x}{x}$ on $[e, +\infty)$

Answer

Since $\int_e^{+\infty} \frac{1}{x}\,dx = +\infty$, $\int_e^{+\infty} \frac{\ln x}{x}\,dx$ diverges.

Laplace Transforms

Laplace Transforms

In the last few chapters, we have looked at several ways to use integration for solving real-world problems. For this next project, we are going to explore a more advanced application of integration: integral transforms. Specifically, we describe the Laplace transform and some of its properties. The **Laplace transform** is used in engineering and physics to simplify the computations needed to solve some problems. It takes functions expressed in terms of time and *transforms* them to functions expressed in terms of frequency. It turns out that, in many cases, the computations needed to solve problems in the frequency domain are much simpler than those required in the time domain.

The Laplace transform is defined in terms of an integral as

$$\mathscr{L}\big(f(t)\big) = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Note that the input to a Laplace transform is a function of time, $f(t)$, and the output is a function of frequency, $F(s)$. Although many real-world examples require the use of complex numbers (involving the imaginary number $i = \sqrt{-1}$), in this project we limit ourselves to functions of real numbers.

Let's start with a simple example. Here we calculate the Laplace transform of $f(t) = t$. We have

$$\mathscr{L}(t) = \int_0^\infty te^{-st}\, dt.$$

This is an improper integral, so we express it in terms of a limit, which gives

$$\mathscr{L}(t) = \int_0^\infty te^{-st}\, dt = \lim_{z\to\infty} \int_0^z te^{-st}\, dt.$$

Now we use integration by parts to evaluate the integral. Note that we are integrating with respect to t, so we treat the variable s as a constant. We have

$$\begin{aligned} u &= t & dv &= e^{-st}\, dt \\ du &= dt & v &= -\tfrac{1}{s}e^{-st} \end{aligned}$$

Then we obtain

$$\begin{aligned} \lim_{z\to\infty} \int_0^z te^{-st}\, dt &= \lim_{z\to\infty} \left[-\frac{t}{s}e^{-st}\Big|_0^z + \frac{1}{s}\int_0^z e^{-st}\, dt \right] \\ &= \lim_{z\to\infty} \left[\left[-\frac{z}{s}e^{-sz} + \frac{0}{s}e^{-0s} \right] + \frac{1}{s}\int_0^z e^{-st}\, dt \right] \\ &= \lim_{z\to\infty} \left[\left[-\frac{z}{s}e^{-sz} + 0 \right] - \frac{1}{s}\left[\frac{e^{-st}}{s} \right] \Big|_0^z \right] \\ &= \lim_{z\to\infty} \left[-\frac{z}{s}e^{-sz} - \frac{1}{s^2}\left[e^{-sz} - 1 \right] \right] \\ &= \lim_{z\to\infty} \left[-\frac{z}{se^{sz}} \right] - \lim_{z\to\infty} \frac{1}{s^2e^{sz}} + \lim_{z\to\infty} \frac{1}{s^2} \\ &= 0 - 0 + \frac{1}{s^2} \qquad \text{Note that the first limit requires L'Hôpital's rule. See work below.} \\ &= \frac{1}{s^2}. \end{aligned}$$

As noted above, $\lim\limits_{z\to\infty} \left[-\dfrac{z}{se^{sz}} \right]$ is indeterminate of the form $-\dfrac{\infty}{\infty}$ and requires L'Hôpital's rule to evaluate. Using L'Hôpital's rule,

$$\lim_{z\to\infty} \left[-\frac{z}{se^{sz}} \right] = \lim_{z\to\infty} \left[-\frac{1}{s^2e^{sz}} \right] = 0.$$

1. Calculate the Laplace transform of $f(t) = 1$.
2. Calculate the Laplace transform of $f(t) = e^{-3t}$.
3. Calculate the Laplace transform of $f(t) = t^2$. (Note, you will have to integrate by parts twice.)

Laplace transforms are often used to solve differential equations. Differential equations are not covered in detail until later in this book; but, for now, let's look at the relationship between the Laplace transform of a function and the Laplace transform of its derivative.

Let's start with the definition of the Laplace transform. We have

$$\mathscr{L}\big(f(t)\big) = \int_0^{\infty} e^{-st} f(t)\, dt = \lim_{z\to\infty} \int_0^{z} e^{-st} f(t)\, dt.$$

Use integration by parts to evaluate $\lim_{z\to\infty} \int_0^{z} e^{-st} f(t)\, dt$. (Let $u = f(t)$ and $dv = e^{-st}\, dt$.)

After integrating by parts and evaluating the limit, you should see that

$$\mathscr{L}\big(f(t)\big) = \frac{f(0)}{s} + \frac{1}{s}\big[\mathscr{L}\big(f'(t)\big)\big].$$

Then,

$$\mathscr{L}\big(f'(t)\big) = s\mathscr{L}\big(f(t)\big) - f(0).$$

Thus, differentiation in the time domain simplifies to multiplication by s in the frequency domain.

The final thing we look at in this project is how the Laplace transforms of $f(t)$ and its antiderivative are related. Let $g(t) = \int_0^{t} f(u)\, du$.

Then,

$$\mathscr{L}\big(g(t)\big) = \int_0^{\infty} e^{-st} g(t)\, dt = \lim_{z\to\infty} \int_0^{z} e^{-st} g(t)\, dt.$$

Use integration by parts to evaluate $\lim_{z\to\infty} \int_0^{z} e^{-st} g(t)\, dt$. (Let $u = g(t)$ and $dv = e^{-st}\, dt$. Note, by the way that we have defined $g(t)$, $du = f(t)\, dt$.)

As you might expect, you should see that

$$\mathscr{L}\big(g(t)\big) = \frac{1}{s} \cdot \mathscr{L}\big(f(t)\big).$$

Integration in the time domain simplifies to division by s in the frequency domain.

Key Concepts

- Integrals of functions over infinite intervals are defined in terms of limits.
- Integrals of functions over an interval for which the function has a discontinuity at an endpoint may be defined in terms of limits.
- The convergence or divergence of an improper integral may be determined by comparing it with the value of an improper integral for which the convergence or divergence is known.

Key Equations

- **Improper integrals**

$$\int_a^{+\infty} f(x)\, dx = \lim_{t\to+\infty} \int_a^{t} f(x)\, dx$$

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{t\to-\infty} \int_t^{b} f(x)\, dx$$

$$\int_{-\infty}^{+\infty} f(x)\, dx = \int_{-\infty}^{0} f(x)\, dx + \int_0^{+\infty} f(x)\, dx$$

Glossary

improper integral

an integral over an infinite interval or an integral of a function containing an infinite discontinuity on the interval; an improper integral is defined in terms of a limit. The improper integral converges if this limit is a finite real number; otherwise, the improper integral diverges

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4.9E: Exercises for Improper Integrals

Terms and Concepts

1. The definite integral was defined with what two stipulations?

2. If $\lim_{b\to\infty}\int_0^b f(x)\,dx$ exists, then the integral $\int_0^\infty f(x)\,dx$ is said to _____.

3. If $\int_1^\infty f(x)\,dx = 10,$ and $0 \le g(x) \le f(x)$ for all x, then we know that $\int_1^\infty g(x)\,dx$ _____.

Problems

In exercises 1 - 8, evaluate the following integrals. If the integral is not convergent, answer "It diverges."

1) $\int_2^4 \dfrac{dx}{(x-3)^2}$

Answer:

It diverges.

2) $\int_0^\infty \dfrac{1}{4+x^2}\,dx$

3) $\int_0^2 \dfrac{1}{\sqrt{4-x^2}}\,dx$

Answer:

Converges to $\frac{\pi}{2}$

4) $\int_1^\infty \dfrac{1}{x\ln x}\,dx$

5) $\int_1^\infty xe^{-x}\,dx$

Answer:

Converges to $\frac{2}{e}$

6) $\int_{-\infty}^\infty \dfrac{x}{x^2+1}\,dx$

7) Without integrating, determine whether the integral $\int_1^\infty \dfrac{1}{\sqrt{x^3+1}}\,dx$ converges or diverges by comparing the function $f(x) = \dfrac{1}{\sqrt{x^3+1}}$ with $g(x) = \dfrac{1}{\sqrt{x^3}}$.

Answer:

It converges.

8) Without integrating, determine whether the integral $\int_1^\infty \dfrac{1}{\sqrt{x+1}}\,dx$ converges or diverges.

In exercises 9 - 25, determine whether the improper integrals converge or diverge. If possible, determine the value of the integrals that converge.

9) $\int_0^\infty e^{-x}\cos x\,dx$

Answer:

Converges to $\frac{1}{2}$.

10) $\int_1^{\infty} \frac{\ln x}{x}\, dx$

11) $\int_0^1 \frac{\ln x}{\sqrt{x}}\, dx$

Answer:

Converges to -4.

12) $\int_0^1 \ln x\, dx$

13) $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx$

Answer:

Converges to π.

14) $\int_1^5 \frac{dx}{\sqrt{x-1}}$

15) $\int_{-2}^2 \frac{dx}{(1+x)^2}$

Answer:

It diverges.

16) $\int_0^{\infty} e^{-x}\, dx$

17) $\int_0^{\infty} \sin x\, dx$

Answer:

It diverges.

18) $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}\, dx$

19) $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

Answer:

Converges to 1.5.

20) $\int_0^2 \frac{dx}{x^3}$

21) $\int_{-1}^2 \frac{dx}{x^3}$

Answer:

It diverges.

22) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

23) $\int_0^3 \frac{1}{x-1}\,dx$

Answer:

It diverges.

24) $\int_1^\infty \frac{5}{x^3}\,dx$

25) $\int_3^5 \frac{5}{(x-4)^2}\,dx$

Answer:

It diverges.

In exercises 26 and 27, determine the convergence of each of the following integrals by comparison with the given integral. If the integral converges, find the number to which it converges.

26) $\int_1^\infty \frac{dx}{x^2+4x}$; compare with $\int_1^\infty \frac{dx}{x^2}$.

27) $\int_1^\infty \frac{dx}{\sqrt{x}+1}$; compare with $\int_1^\infty \frac{dx}{2\sqrt{x}}$.

Answer:

Both integrals diverge.

In exercises 28 - 38, evaluate the integrals. If the integral diverges, answer "It diverges."

28) $\int_1^\infty \frac{dx}{x^e}$

29) $\int_0^1 \frac{dx}{x^\pi}$

Answer:

It diverges.

30) $\int_0^1 \frac{dx}{\sqrt{1-x}}$

31) $\int_0^1 \frac{dx}{1-x}$

Answer:

It diverges.

32) $\int_{-\infty}^0 \frac{dx}{x^2+1}$

33) $\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}}$

Answer:

Converges to π.

34) $\int_0^1 \frac{\ln x}{x}\,dx$

35) $\int_0^e \ln(x)\,dx$

Answer:

Converges to 0.

36) $\int_0^\infty xe^{-x}\,dx$

37) $\int_{-\infty}^\infty \frac{x}{(x^2+1)^2}\,dx$

Answer:

Converges to 0.

38) $\int_0^\infty e^{-x}\,dx$

In exercises 39 - 44, evaluate the improper integrals. Each of these integrals has an infinite discontinuity either at an endpoint or at an interior point of the interval.

39) $\int_0^9 \frac{dx}{\sqrt{9-x}}$

Answer:

Converges to 6.

40) $\int_{-27}^1 \frac{dx}{x^{2/3}}$

41) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

Answer:

Converges to $\frac{\pi}{2}$.

42) $\int_6^{24} \frac{dt}{t\sqrt{t^2-36}}$

43) $\int_0^4 x\ln(4x)\,dx$

Answer:

Converges to $8\ln(16)-4$.

44) $\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx$

45) Evaluate $\int_{.5}^t \frac{dx}{\sqrt{1-x^2}}$. (Be careful!) (Express your answer using three decimal places.)

Answer:

Converges to about 1.047.

46) Evaluate $\int_1^4 \frac{dx}{\sqrt{x^2-1}}$. (Express the answer in exact form.)

47) Evaluate $\int_2^\infty \frac{dx}{(x^2-1)^{3/2}}$.

Answer:

Converges to $-1+\frac{2}{\sqrt{3}}$.

48) Find the area of the region in the first quadrant between the curve $y=e^{-6x}$ and the x-axis.

49) Find the area of the region bounded by the curve $y=\frac{7}{x^2}$, the x-axis, and on the left by $x=1$.

Answer:

$A=7.0$ units.^{2}

50) Find the area under the curve $y=\frac{1}{(x+1)^{3/2}}$, bounded on the left by $x=3$.

51) Find the area under $y=\frac{5}{1+x^2}$ in the first quadrant.

Answer:

$A=\frac{5\pi}{2}$ units.^{2}

52) Find the volume of the solid generated by revolving about the x-axis the region under the curve $y=\frac{3}{x}$ from $x=1$ to $x=\infty$.

53) Find the volume of the solid generated by revolving about the y-axis the region under the curve $y=6e^{-2x}$ in the first quadrant.

Answer:

$V=3\pi$ units^3

54) Find the volume of the solid generated by revolving about the x-axis the area under the curve $y=3e^{-x}$ in the first quadrant.

The **Laplace transform** of a continuous function over the interval $[0,\infty)$ is defined by $F(s)=\int_0^\infty e^{-sx}f(x)\,dx$ (see the Student Project). This definition is used to solve some important initial-value problems in differential equations, as discussed later. The domain of F is the set of all real numbers s such that the improper integral converges. Find the Laplace transform F of each of the following functions and give the domain of F.

55) $f(x)=1$

Answer:

$\frac{1}{s}, \quad s>0$

56) $f(x)=x$

57) $f(x)=\cos(2x)$

Answer:

$\frac{s}{s^2+4}, \quad s>0$

58) $f(x)=e^{ax}$

59) Use the formula for arc length to show that the circumference of the circle $x^2+y^2=1$ is 2π.

Answer:

Answers will vary.

A function is a **probability density function** if it satisfies the following definition: $\int_{-\infty}^{\infty} f(t)\,dt = 1$. The probability that a random variable x lies between a and b is given by $P(a \leq x \leq b) = \int_{a}^{b} f(t)\,dt$.

60) Show that $f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 7e^{-7x}, & \text{if } x \geq 0 \end{cases}$ is a probability density function.

61) Find the probability that x is between 0 and 0.3. (Use the function defined in the preceding problem.) Use four-place decimal accuracy.

Answer:

0.8775

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- Terms and Concepts questions from Apex Calculus

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4.10: Chapter 7 Review Exercises

In exercises 1 - 4, determine whether the statement is true or false. Justify your answer with a proof or a counterexample.

1) $\int e^x \sin(x)\,dx$ cannot be integrated by parts.

2) $\int \frac{1}{x^4+1}\,dx$ cannot be integrated using partial fractions.

Answer

False

3) In numerical integration, increasing the number of points decreases the error.

4) Integration by parts can always yield the integral.

Answer

False

In exercises 5 - 10, evaluate the integral using the specified method.

5) $\int x^2 \sin(4x)\,dx$, using integration by parts

6) $\int \frac{1}{x^2\sqrt{x^2+16}}\,dx$, using trigonometric substitution

Answer

$$\int \frac{1}{x^2\sqrt{x^2+16}}\,dx = -\frac{\sqrt{x^2+16}}{16x} + C$$

7) $\int \sqrt{x}\ln x\,dx$, using integration by parts

8) $\int \frac{3x}{x^3+2x^2-5x-6}\,dx$, using partial fractions

Answer

$$\int \frac{3x}{x^3+2x^2-5x-6}\,dx = \frac{1}{10}\left(4\ln|2-x| + 5\ln|x+1| - 9\ln|x+3|\right) + C$$

9) $\int \frac{x^5}{(4x^2+4)^{5/2}}\,dx$, using trigonometric substitution

10) $\int \frac{\sqrt{4-\sin^2(x)}}{\sin^2(x)}\cos(x)\,dx$, using a table of integrals or a CAS

Answer

$$\int \frac{\sqrt{4-\sin^2(x)}}{\sin^2(x)}\cos(x)\,dx = -\frac{\sqrt{4-\sin^2(x)}}{\sin(x)} - \frac{x}{2} + C$$

In exercises 11 - 15, integrate using whatever method you choose.

11) $\int \sin^2 x \cos^2 x\,dx$

12) $\int x^3\sqrt{x^2+2}\,dx$

Answer

$$\int x^3\sqrt{x^2+2}\,dx = \frac{1}{15}(x^2+2)^{3/2}(3x^2-4)+C$$

13) $\displaystyle\int \frac{3x^2+1}{x^4-2x^3-x^2+2x}\,dx$

14) $\displaystyle\int \frac{1}{x^4+4}\,dx$

Answer

$$\int \frac{1}{x^4+4}\,dx = \frac{1}{16}\ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right) - \frac{1}{8}\tan^{-1}(1-x) + \frac{1}{8}\tan^{-1}(x+1)+C$$

15) $\displaystyle\int \frac{\sqrt{3+16x^4}}{x^4}\,dx$

In exercises 16 - 18, approximate the integrals using the midpoint rule, trapezoidal rule, and Simpson's rule using four subintervals, rounding to three decimals.

16) [T] $\displaystyle\int_1^2 \sqrt{x^5+2}\,dx$

Answer

$M_4 = 3.312,$
$T_4 = 3.354,$
$S_4 = 3.326$

17) [T] $\displaystyle\int_0^{\sqrt{\pi}} e^{-\sin(x^2)}\,dx$

18) [T] $\displaystyle\int_1^4 \frac{\ln(1/x)}{x}\,dx$

Answer

$M_4 = -0.982,$
$T_4 = -0.917,$
$S_4 = -0.952$

In exercises 19 - 20, evaluate the integrals, if possible.

19) $\displaystyle\int_1^\infty \frac{1}{x^n}\,dx$, for what values of n does this integral converge or diverge?

20) $\displaystyle\int_1^\infty \frac{e^{-x}}{x}\,dx$

Answer

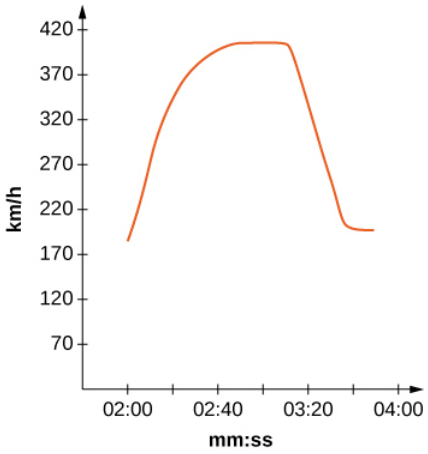
approximately 0.2194

In exercises 21 - 22, consider the gamma function given by $\Gamma(a) = \displaystyle\int_0^\infty e^{-y}y^{a-1}\,dy.$

21) Show that $\Gamma(a) = (a-1)\Gamma(a-1)$.

22) Extend to show that $\Gamma(a) = (a-1)!,$ assuming a is a positive integer.

The fastest car in the world, the Bugati Veyron, can reach a top speed of 408 km/h. The graph represents its velocity.



23) [T] Use the graph to estimate the velocity every 20 sec and fit to a graph of the form $v(t) = ae^{bx}\sin(cx) + d$. (Hint: Consider the time units.)

24) [T] Using your function from the previous problem, find exactly how far the Bugati Veyron traveled in the 1 min 40 sec included in the graph.

Answer

Answers may vary. Ex: 9.405km

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5: Introduction to Differential Equations

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5.1: Prelude to Differential Equations

Many real-world phenomena can be modeled mathematically by using differential equations. Population growth, radioactive decay, predator-prey models, and spring-mass systems are four examples of such phenomena. In this chapter we study some of these applications. Suppose we wish to study a population of deer over time and determine the total number of animals in a given area. We can first observe the population over a period of time, estimate the total number of deer, and then use various assumptions to derive a mathematical model for different scenarios. Some factors that are often considered are environmental impact, threshold population values, and predators. In this chapter we see how differential equations can be used to predict populations over time.



Figure 5.1.1: The white-tailed deer (Odocoileus virginianus) of the eastern United States. Differential equations can be used to study animal populations. (credit: modification of work by Rachel Kramer, Flickr)

Another goal of this chapter is to develop solution techniques for different types of differential equations. As the equations become more complicated, the solution techniques also become more complicated, and in fact an entire course could be dedicated to the study of these equations. In this chapter we study several types of differential equations and their corresponding methods of solution.

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5.2: Basics of Differential Equations

Learning Objectives

- Identify the order of a differential equation.
- Explain what is meant by a solution to a differential equation.
- Distinguish between the general solution and a particular solution of a differential equation.
- Identify an initial-value problem.
- Identify whether a given function is a solution to a differential equation or an initial-value problem.

Calculus is the mathematics of change, and rates of change are expressed by derivatives. Thus, one of the most common ways to use calculus is to set up an equation containing an unknown function $y = f(x)$ and its derivative, known as a differential equation. Solving such equations often provides information about how quantities change and frequently provides insight into how and why the changes occur.

Techniques for solving differential equations can take many different forms, including direct solution, use of graphs, or computer calculations. We introduce the main ideas in this chapter and describe them in a little more detail later in the course. In this section we study what differential equations are, how to verify their solutions, some methods that are used for solving them, and some examples of common and useful equations.

General Differential Equations

Consider the equation $y' = 3x^2$, which is an example of a differential equation because it includes a derivative. There is a relationship between the variables x and y : y is an unknown function of x. Furthermore, the left-hand side of the equation is the derivative of y. Therefore we can interpret this equation as follows: Start with some function $y = f(x)$ and take its derivative. The answer must be equal to $3x^2$. What function has a derivative that is equal to $3x^2$? One such function is $y = x^3$, so this function is considered a *solution* to a differential equation.

Definition: differential equation

A *differential equation* is an equation involving an unknown function $y = f(x)$ and one or more of its derivatives. A solution to a differential equation is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.

Go to this website to explore more on this topic.

Some examples of differential equations and their solutions appear in Table 5.2.1.

Table 5.2.1: Examples of Differential Equations and Their Solutions

Equation	Solution
$y' = 2x$	$y = x^2$
$y' + 3y = 6x + 11$	$y = e^{-3x} + 2x + 3$
$y'' - 3y' + 2y = 24e^{-2x}$	$y = 3e^x - 4e^{2x} + 2e^{-2x}$

Note that a solution to a differential equation is not necessarily unique, primarily because the derivative of a constant is zero. For example, $y = x^2 + 4$ is also a solution to the first differential equation in Table 5.2.1. We will return to this idea a little bit later in this section. For now, let's focus on what it means for a function to be a solution to a differential equation.

Example 5.2.1: Verifying Solutions of Differential Equations

Verify that the function $y = e^{-3x} + 2x + 3$ is a solution to the differential equation $y' + 3y = 6x + 11$.

Solution

To verify the solution, we first calculate y' using the chain rule for derivatives. This gives $y' = -3e^{-3x} + 2$. Next we substitute y and y' into the left-hand side of the differential equation:

$$(-3e^{-2x} + 2) + 3(e^{-2x} + 2x + 3).$$

The resulting expression can be simplified by first distributing to eliminate the parentheses, giving

$$-3e^{-2x} + 2 + 3e^{-2x} + 6x + 9.$$

Combining like terms leads to the expression $6x + 11$, which is equal to the right-hand side of the differential equation. This result verifies that $y = e^{-3x} + 2x + 3$ is a solution of the differential equation.

Exercise 5.2.1

Verify that $y = 2e^{3x} - 2x - 2$ is a solution to the differential equation $y' - 3y = 6x + 4$.

Hint

First calculate y' then substitute both y' and y into the left-hand side.

It is convenient to define characteristics of differential equations that make it easier to talk about them and categorize them. The most basic characteristic of a differential equation is its order.

Definition: order of a differential equation

The *order of a differential equation* is the highest order of any derivative of the unknown function that appears in the equation.

Example 5.2.2: Identifying the Order of a Differential Equation

The highest derivative in the equation is y',

What is the order of each of the following differential equations?

a. $y' - 4y = x^2 - 3x + 4$
b. $x^2y''' - 3xy'' + xy' - 3y = \sin x$
c. $\frac{4}{x}y^{(4)} - \frac{6}{x^2}y'' + \frac{12}{x^4}y = x^3 - 3x^2 + 4x - 12$

Solution

a. The highest derivative in the equation is y',so the order is 1.
b. The highest derivative in the equation is y''', so the order is 3.
c. The highest derivative in the equation is $y^{(4)}$, so the order is 4.

Exercise 5.2.2

What is the order of the following differential equation?

$$(x^4 - 3x)y^{(5)} - (3x^2 + 1)y' + 3y = \sin x \cos x$$

Hint

What is the highest derivative in the equation?

Answer

5

General and Particular Solutions

We already noted that the differential equation $y' = 2x$ has at least two solutions: $y = x^2$ and $y = x^2 + 4$. The only difference between these two solutions is the last term, which is a constant. What if the last term is a different constant? Will this expression

still be a solution to the differential equation? In fact, any function of the form $y = x^2 + C$, where C represents any constant, is a solution as well. The reason is that the derivative of $x^2 + C$ is $2x$, regardless of the value of C. It can be shown that any solution of this differential equation must be of the form $y = x^2 + C$. This is an example of a **general solution** to a differential equation. A graph of some of these solutions is given in Figure 5.2.1. (Note: in this graph we used even integer values for C ranging between -4 and 4. In fact, there is no restriction on the value of C; it can be an integer or not.)

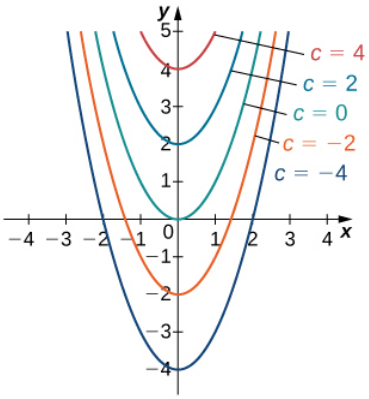


Figure 5.2.1: Family of solutions to the differential equation $y' = 2x$.

In this example, we are free to choose any solution we wish; for example, $y = x^2 - 3$ is a member of the family of solutions to this differential equation. This is called a particular solution to the differential equation. A particular solution can often be uniquely identified if we are given additional information about the problem.

Example 5.2.3: Finding a Particular Solution

Find the particular solution to the differential equation $y' = 2x$ passing through the point $(2, 7)$.

Solution

Any function of the form $y = x^2 + C$ is a solution to this differential equation. To determine the value of C, we substitute the values $x = 2$ and $y = 7$ into this equation and solve for C:

$$\begin{aligned} y &= x^2 + C \\ 7 &= 2^2 + C \\ &= 4 + C \\ C &= 3. \end{aligned}$$

Therefore the particular solution passing through the point $(2, 7)$ is $y = x^2 + 3$.

Exercise 5.2.3

Find the particular solution to the differential equation

$$y' = 4x + 3$$

passing through the point $(1, 7)$, given that $y = 2x^2 + 3x + C$ is a general solution to the differential equation.

Hint

First substitute $x = 1$ and $y = 7$ into the equation, then solve for C.

Answer

$$y = 2x^2 + 3x + 2$$

Initial-Value Problems

Usually a given differential equation has an infinite number of solutions, so it is natural to ask which one we want to use. To choose one solution, more information is needed. Some specific information that can be useful is an *initial value*, which is an ordered pair that is used to find a particular solution.

A differential equation together with one or more initial values is called an *initial-value problem*. The general rule is that the number of initial values needed for an initial-value problem is equal to the order of the differential equation. For example, if we have the differential equation $y' = 2x$, then $y(3) = 7$ is an initial value, and when taken together, these equations form an initial-value problem. The differential equation $y'' - 3y' + 2y = 4e^x$ is second order, so we need two initial values. With initial-value problems of order greater than one, the same value should be used for the independent variable. An example of initial values for this second-order equation would be $y(0) = 2$ and $y'(0) = -1$. These two initial values together with the differential equation form an initial-value problem. These problems are so named because often the independent variable in the unknown function is t, which represents time. Thus, a value of $t = 0$ represents the beginning of the problem.

Example 5.2.4: Verifying a Solution to an Initial-Value Problem

Verify that the function $y = 2e^{-2t} + e^t$ is a solution to the initial-value problem

$$y' - y = -6e^{-2t}, \quad y(0) = 3.$$

Solution

For a function to satisfy an initial-value problem, it must satisfy both the differential equation and the initial condition. To show that y satisfies the differential equation, we start by calculating y'. This gives $y' = -4e^{-2t} + e^t$. Next we substitute both y and y' into the left-hand side of the differential equation and simplify:

$$\begin{aligned} y' - y &= (-4e^{-2t} + e^t) - (2e^{-2t} + e^t) \\ &= -4e^{-2t} + e^t - 2e^{-2t} - e^t = -6e^{-2t}. \end{aligned}$$

This is equal to the right-hand side of the differential equation, so $y = 2e^{-2t} + e^t$ solves the differential equation. Next we calculate $y(0)$:

$$y(0) = 2e^{-2(0)} + e^0 = 2 + 1 = 3.$$

This result verifies the initial value. Therefore the given function satisfies the initial-value problem.

Exercise 5.2.4

Verify that $y = 3e^{2t} + 4\sin t$ is a solution to the initial-value problem

$$y' - 2y = 4\cos t - 8\sin t, y(0) = 3.$$

Hint

First verify that y solves the differential equation. Then check the initial value.

In Example 5.2.4, the initial-value problem consisted of two parts. The first part was the differential equation $y' - y = -6e^{-2t}$, and the second part was the initial value $y(0) = 3$. These two equations together formed the initial-value problem.

The same is true in general. An initial-value problem will consists of two parts: the differential equation and the initial condition. The differential equation has a family of solutions, and the initial condition determines the value of C. The family of solutions to the differential equation in Example 5.2.4 is given by $y = 2e^{-2t} + Ce^t$. This family of solutions is shown in Figure 5.2.2, with the particular solution $y = 2e^{-2t} + e^t$ labeled.

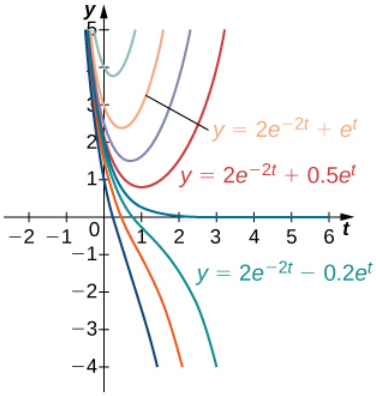


Figure 5.2.2: A family of solutions to the differential equation $y'-y=-6e^{-2t}$. The particular solution $y=2e^{-2t}+e^{t}$ is labeled.

Example 5.2.5: Solving an Initial-value Problem

Solve the following initial-value problem:

$$y'=3e^{x}+x^{2}-4, y(0)=5.$$

Solution

The first step in solving this initial-value problem is to find a general family of solutions. To do this, we find an antiderivative of both sides of the differential equation

$$\int y'\,dx=\int(3e^{x}+x^{2}-4)\,dx,$$

namely,

$$y+C_1=3e^{x}+\tfrac{1}{3}x^{3}-4x+C_2 \quad .$$

We are able to integrate both sides because the y term appears by itself. Notice that there are two integration constants: C_1 and C_2. Solving this equation for y gives

$$y=3e^{x}+\tfrac{1}{3}x^{3}-4x+C_2-C_1.$$

Because C_1 and C_2 are both constants, C_2-C_1 is also a constant. We can therefore define $C=C_2-C_1$, which leads to the equation

$$y=3e^{x}+\tfrac{1}{3}x^{3}-4x+C.$$

Next we determine the value of C. To do this, we substitute $x=0$ and $y=5$ into this equation and solve for C:

$$\begin{aligned}5 &= 3e^{0}+\frac{1}{3}0^{3}-4(0)+C\\ 5 &= 3+C\\ C &= 2\end{aligned}$$

Now we substitute the value $C=2$ into the general equation. The solution to the initial-value problem is $y=3e^{x}+\tfrac{1}{3}x^{3}-4x+2$.

Analysis

The difference between a general solution and a particular solution is that a general solution involves a family of functions, either explicitly or implicitly defined, of the independent variable. The initial value or values determine which particular solution in the family of solutions satisfies the desired conditions.

Exercise 5.2.5

Solve the initial-value problem

$$y' = x^2 - 4x + 3 - 6e^x, y(0) = 8.$$

Hint

First take the antiderivative of both sides of the differential equation. Then substitute $x = 0$ and $y = 8$ into the resulting equation and solve for C.

Answer

$y = \frac{1}{3}x^3 - 2x^2 + 3x - 6e^x + 14$

Applications

In physics and engineering applications, we often consider the forces acting upon an object, and use this information to understand the resulting motion that may occur. For example, if we start with an object at Earth's surface, the primary force acting upon that object is gravity. Physicists and engineers can use this information, along with Newton's **second law of motion** (in equation form $F = ma$, where F represents force, m represents mass, and a represents acceleration), to derive an equation that can be solved.

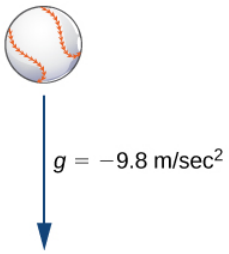


Figure 5.2.3: For a baseball falling in air, the only force acting on it is gravity (neglecting air resistance).

In Figure 5.2.3 we assume that the only force acting on a baseball is the force of gravity. This assumption ignores air resistance. (The force due to air resistance is considered in a later discussion.) The acceleration due to gravity at Earth's surface, g, is approximately $9.8\ \text{m/s}^2$. We introduce a frame of reference, where Earth's surface is at a height of 0 meters. Let $v(t)$ represent the velocity of the object in meters per second. If $v(t) > 0$, the ball is rising, and if $v(t) < 0$, the ball is falling (Figure).



Figure 5.2.4: Possible velocities for the rising/falling baseball.

Our goal is to solve for the velocity $v(t)$ at any time t. To do this, we set up an initial-value problem. Suppose the mass of the ball is m, where m is measured in kilograms. We use Newton's second law, which states that the force acting on an object is equal to its mass times its acceleration $(F = ma)$. Acceleration is the derivative of velocity, so $a(t) = v'(t)$. Therefore the force acting on the baseball is given by $F = mv'(t)$. However, this force must be equal to the force of gravity acting on the object, which (again using Newton's second law) is given by $F_g = -mg$, since this force acts in a downward direction. Therefore we obtain the equation $F = F_g$, which becomes $mv'(t) = -mg$. Dividing both sides of the equation by m gives the equation

$$v'(t) = -g.$$

Notice that this differential equation remains the same regardless of the mass of the object.

We now need an initial value. Because we are solving for velocity, it makes sense in the context of the problem to assume that we know the **initial velocity**, or the velocity at time $t = 0$. This is denoted by $v(0) = v_0$.

Example 5.2.6: Velocity of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of 10 m/s, and the only force acting on it is gravity. The ball has a mass of 0.15 kg at Earth's surface.

a. Find the velocity $v(t)$ of the basevall at time t.
b. What is its velocity after 2 seconds?

Solution

a. From the preceding discussion, the differential equation that applies in this situation is

$$v'(t) = -g,$$

where $g = 9.8\,\text{m/s}^2$. The initial condition is $v(0) = v_0$, where $v_0 = 10$ m/s. Therefore the initial-value problem is $v'(t) = -9.8\,\text{m/s}^2$, $v(0) = 10$ m/s.

The first step in solving this initial-value problem is to take the antiderivative of both sides of the differential equation. This gives

$$\int v'(t)\,dt = \int -9.8\,dt$$

$$v(t) = -9.8t + C.$$

The next step is to solve for C. To do this, substitute $t = 0$ and $v(0) = 10$:

$$\begin{aligned} v(t) &= -9.8t + C \\ v(0) &= -9.8(0) + C \\ 10 &= C. \end{aligned}$$

Therefore $C = 10$ and the velocity function is given by $v(t) = -9.8t + 10$.

b. To find the velocity after 2 seconds, substitute $t = 2$ into $v(t)$.

$$\begin{aligned} v(t) &= -9.8t + 10 \\ v(2) &= -9.8(2) + 10 \\ v(2) &= -9.6 \end{aligned}$$

The units of velocity are meters per second. Since the answer is negative, the object is falling at a speed of 9.6 m/s.

Exercise 5.2.6

Suppose a rock falls from rest from a height of 100 meters and the only force acting on it is gravity. Find an equation for the velocity $v(t)$ as a function of time, measured in meters per second.

Hint

What is the initial velocity of the rock? Use this with the differential equation in Example 5.2.6 to form an initial-value problem, then solve for $v(t)$.

Answer

$v(t) = -9.8t$

A natural question to ask after solving this type of problem is how high the object will be above Earth's surface at a given point in time. Let $s(t)$ denote the height above Earth's surface of the object, measured in meters. Because velocity is the derivative of position (in this case height), this assumption gives the equation $s'(t) = v(t)$. An initial value is necessary; in this case the initial height of the object works well. Let the initial height be given by the equation $s(0) = s_0$. Together these assumptions give the initial-value problem

$$s'(t) = v(t), s(0) = s_0.$$

If the velocity function is known, then it is possible to solve for the position function as well.

Example 5.2.7: Height of a Moving Baseball

A baseball is thrown upward from a height of 3 meters above Earth's surface with an initial velocity of $10 m/s$, and the only force acting on it is gravity. The ball has a mass of 0.15 kilogram at Earth's surface.

a. Find the position $s(t)$ of the baseball at time t.
b. What is its height after 2 seconds?

Solution

We already know the velocity function for this problem is $v(t) = -9.8t + 10$. The initial height of the baseball is 3 meters, so $s_0 = 3$. Therefore the initial-value problem for this example is

To solve the initial-value problem, we first find the antiderivatives:

$$\int s'(t)\,dt = \int (-9.8t + 10)\,dt$$

$$s(t) = -4.9t^2 + 10t + C.$$

Next we substitute $t = 0$ and solve for C:

$$s(t) = -4.9t^2 + 10t + C$$

$$s(0) = -4.9(0)^2 + 10(0) + C$$

$$3 = C.$$

Therefore the position function is $s(t) = -4.9t^2 + 10t + 3$.

b. The height of the baseball after 2 sec is given by $s(2)$:

$$s(2) = -4.9(2)^2 + 10(2) + 3 = -4.9(4) + 23 = 3.4.$$

Therefore the baseball is 3.4 meters above Earth's surface after 2 seconds. It is worth noting that the mass of the ball cancelled out completely in the process of solving the problem.

Key Concepts

- A differential equation is an equation involving a function $y = f(x)$ and one or more of its derivatives. A solution is a function $y = f(x)$ that satisfies the differential equation when f and its derivatives are substituted into the equation.
- The order of a differential equation is the highest order of any derivative of the unknown function that appears in the equation.
- A differential equation coupled with an initial value is called an initial-value problem. To solve an initial-value problem, first find the general solution to the differential equation, then determine the value of the constant. Initial-value problems have many applications in science and engineering.

Glossary

differential equation

an equation involving a function $y = y(x)$ and one or more of its derivatives

general solution (or family of solutions)

the entire set of solutions to a given differential equation

initial value(s)
a value or set of values that a solution of a differential equation satisfies for a fixed value of the independent variable

initial velocity
the velocity at time $t = 0$

initial-value problem
a differential equation together with an initial value or values

order of a differential equation
the highest order of any derivative of the unknown function that appears in the equation

particular solution
member of a family of solutions to a differential equation that satisfies a particular initial condition

solution to a differential equation
a function $y = f(x)$ that satisfies a given differential equation

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5.2E: Exercises for Basics of Differential Equations

In exercises 1 - 7, determine the order of each differential equation.

1) $y' + y = 3y^2$

Answer

1st-order

2) $(y')^2 = y' + 2y$

3) $y''' + y''y' = 3x^2$

Answer

3rd-order

4) $y' = y'' + 3t^2$

5) $\dfrac{dy}{dt} = t$

Answer

1st-order

6) $\dfrac{dy}{dx} + \dfrac{d^2y}{dx^2} = 3x^4$

7) $\left(\dfrac{dy}{dt}\right)^2 + 8\dfrac{dy}{dt} + 3y = 4t$

Answer

1st-order

In exercises 8 - 17, verify that the given function is a solution to the given differential equation.

8) $y = \dfrac{x^3}{3}$ solves $y' = x^2$

9) $y = 2e^{-x} + x - 1$ solves $y' = x - y$

10) $y = e^{3x} - \dfrac{e^x}{2}$ solves $y' = 3y + e^x$

11) $y = \dfrac{1}{1-x}$ solves $y' = y^2$

12) $y = \dfrac{e^{x^2}}{2}$ solves $y' = 2xy$

13) $y = 4 + \ln x$ solves $xy' = 1$

14) $y = 3 - x + x\ln x$ solves $y' = \ln x$

15) $y = 2e^x - x - 1$ solves $y' = y + x$

16) $y = e^x + \dfrac{\sin x}{2} - \dfrac{\cos x}{2}$ solves $y' = \cos x + y$

17) $y = \pi e^{-\cos x}$ solves $y' = y\sin x$

In exercises 18 - 27, verify the given general solution and find the particular solution.

18) Find the particular solution to the differential equation $y' = 4x^2$ that passes through $(-3, -30)$, given that $y = C + \dfrac{4x^3}{3}$ is a general solution.

19) Find the particular solution to the differential equation $y' = 3x^3$ that passes through $(1, 4.75)$, given that $y = C + \dfrac{3x^4}{4}$ is a general solution.

Answer

$y = 4 + \dfrac{3x^4}{4}$

20) Find the particular solution to the differential equation $y' = 3x^2 y$ that passes through $(0, 12)$, given that $y = Ce^{x^3}$ is a general solution.

21) Find the particular solution to the differential equation $y' = 2xy$ that passes through $\left(0, \frac{1}{2}\right)$, given that $y = Ce^{x^2}$ is a general solution.

Answer

$y = \frac{1}{2}e^{x^2}$

22) Find the particular solution to the differential equation $y' = (2xy)^2$ that passes through $\left(1, -\frac{1}{2}\right)$, given that $y = -\dfrac{3}{C + 4x^3}$ is a general solution.

23) Find the particular solution to the differential equation $y'x^2 = y$ that passes through $\left(1, \frac{2}{e}\right)$, given that $y = Ce^{-1/x}$ is a general solution.

Answer

$y = 2e^{-1/x}$

24) Find the particular solution to the differential equation $8\dfrac{dx}{dt} = -2\cos(2t) - \cos(4t)$ that passes through (π, π), given that $x = C - \frac{1}{8}\sin(2t) - \frac{1}{32}\sin(4t)$ is a general solution.

25) Find the particular solution to the differential equation $\dfrac{du}{dt} = \tan u$ that passes through $\left(1, \frac{\pi}{2}\right)$, given that $u = \sin^{-1}\left(e^{C+t}\right)$ is a general solution.

Answer

$u = \sin^{-1}\left(e^{-1+t}\right)$

26) Find the particular solution to the differential equation $\dfrac{dy}{dt} = e^{t+y}$ that passes through $(1, 0)$, given that $y = -\ln(C - e^t)$ is a general solution.

27) Find the particular solution to the differential equation $y'(1 - x^2) = 1 + y$ that passes through $(0, -2)$, given that $y = C\dfrac{\sqrt{x+1}}{\sqrt{1-x}} - 1$ is a general solution.

Answer

$y = -\dfrac{\sqrt{x+1}}{\sqrt{1-x}} - 1$

In exercises 28 - 37, find the general solution to the differential equation.

28) $y' = 3x + e^x$

29) $y' = \ln x + \tan x$

Answer

$y = C - x + x\ln x - \ln(\cos x)$

30) $y' = \sin x e^{\cos x}$

31) $y' = 4^x$

Answer

$$y = C + \frac{4^x}{\ln 4}$$

32) $y' = \sin^{-1}(2x)$

33) $y' = 2t\sqrt{t^2+16}$

Answer

$$y = \frac{2}{3}\sqrt{t^2+16}\left(t^2+16\right) + C$$

34) $x' = \coth t + \ln t + 3t^2$

35) $x' = t\sqrt{4+t}$

Answer

$$x = \frac{2}{15}\sqrt{4+t}\left(3t^2+4t-32\right) + C$$

36) $y' = y$

37) $y' = \dfrac{y}{x}$

Answer

$$y = Cx$$

In exercises 38 - 42, solve the initial-value problems starting from $y(t=0)=1$ and $y(t=0)=-1$. Draw both solutions on the same graph.

38) $\dfrac{dy}{dt} = 2t$

39) $\dfrac{dy}{dt} = -t$

Answer

$$y = 1 - \frac{t^2}{2}, \text{ and } y = -\frac{t^2}{2} - 1$$

40) $\dfrac{dy}{dt} = 2y$

41) $\dfrac{dy}{dt} = -y$

Answer

$$y = e^{-t} \text{ and } y = -e^{-t}$$

42) $\dfrac{dy}{dt} = 2$

In exercises 43 - 47, solve the initial-value problems starting from $y_0 = 10$. At what time does y increase to 100 or drop to 1?

43) $\dfrac{dy}{dt} = 4t$

Answer

$$y = 2(t^2+5), \text{ When } t = 3\sqrt{5},\ y \text{ will increase to } 100.$$

44) $\frac{dy}{dt} = 4y$

45) $\frac{dy}{dt} = -2y$

Answer

$y = 10e^{-2t}$, When $t = -\frac{1}{2}\ln\left(\frac{1}{10}\right)$, y will decrease to 1.

46) $\frac{dy}{dt} = e^{4t}$

47) $\frac{dy}{dt} = e^{-4t}$

Answer

$y = \frac{1}{4}(41 - e^{-4t})$, Neither condition will ever happen.

Recall that a family of solutions includes solutions to a differential equation that differ by a constant. For exercises 48 - 52, use your calculator to graph a family of solutions to the given differential equation. Use initial conditions from $y(t = 0) = -10$ to $y(t = 0) = 10$ increasing by 2. Is there some critical point where the behavior of the solution begins to change?

48) [T] $y' = y(x)$

49) [T] $xy' = y$

Answer

Solution changes from increasing to decreasing at $y(0) = 0$.

50) [T] $y' = t^3$

51) [T] $y' = x + y$ (Hint: $y = Ce^x - x - 1$ is the general solution)

Answer

Solution changes from increasing to decreasing at $y(0) = 0$.

52) [T] $y' = x\ln x + \sin x$

53) Find the general solution to describe the velocity of a ball of mass 1 lb that is thrown upward at a rate of a ft/sec.

Answer

$v(t) = -32t + a$

54) In the preceding problem, if the initial velocity of the ball thrown into the air is $a = 25$ ft/s, write the particular solution to the velocity of the ball. Solve to find the time when the ball hits the ground.

55) You throw two objects with differing masses m_1 and m_2 upward into the air with the same initial velocity of a ft/s. What is the difference in their velocity after 1 second?

Answer

0 ft/s

56) [T] You throw a ball of mass 1 kilogram upward with a velocity of $a = 25$ m/s on Mars, where the force of gravity is $g = -3.711$ m/s^2. Use your calculator to approximate how much longer the ball is in the air on Mars.

57) [T] For the previous problem, use your calculator to approximate how much higher the ball went on Mars.

Answer

4.86meters

58) [T] A car on the freeway accelerates according to $a = 15\cos(\pi t)$, where t is measured in hours. Set up and solve the differential equation to determine the velocity of the car if it has an initial speed of 51 mph. After 40 minutes of driving, what is the driver's velocity?

59) [T] For the car in the preceding problem, find the expression for the distance the car has traveled in time t, assuming an initial distance of 0. How long does it take the car to travel 100 miles? Round your answer to hours and minutes.

Answer

$x = 50t - \frac{15}{\pi^2}\cos(\pi t) + \frac{3}{\pi^2}, 2$ hours 1 minute

60) [T] For the previous problem, find the total distance traveled in the first hour.

61) Substitute $y = Be^{3t}$ into $y' - y = 8e^{3t}$ to find a particular solution.

Answer

$y = 4e^{3t}$

62) Substitute $y = a\cos(2t) + b\sin(2t)$ into $y' + y = 4\sin(2t)$ to find a particular solution.

63) Substitute $y = a + bt + ct^2$ into $y' + y = 1 + t^2$ to find a particular solution.

Answer

$y = 1 - 2t + t^2$

64) Substitute $y = ae^t\cos t + be^t\sin t$ into $y' = 2e^t\cos t$ to find a particular solution.

65) Solve $y' = e^{kt}$ with the initial condition $y(0) = 0$ and solve $y' = 1$ with the same initial condition. As k approaches 0, what do you notice?

Answer

$y = \frac{1}{k}(e^{kt} - 1)$ and $y = t$

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5.3: Direction Fields and Numerical Methods

Learning Objectives

- Draw the direction field for a given first-order differential equation.
- Use a direction field to draw a solution curve of a first-order differential equation.
- Use Euler's Method to approximate the solution to a first-order differential equation.

For the rest of this chapter we will focus on various methods for solving differential equations and analyzing the behavior of the solutions. In some cases it is possible to predict properties of a solution to a differential equation without knowing the actual solution. We will also study numerical methods for solving differential equations, which can be programmed by using various computer languages or even by using a spreadsheet program, such as Microsoft Excel.

Creating Direction Fields

Direction fields (also called slope fields) are useful for investigating first-order differential equations. In particular, we consider a first-order differential equation of the form

$$y' = f(x, y).$$

An applied example of this type of differential equation appears in Newton's law of cooling, which we will solve explicitly later in this chapter. First, though, let us create a direction field for the differential equation

$$T'(t) = -0.4(T - 72).$$

Here $T(t)$ represents the temperature (in degrees Fahrenheit) of an object at time t, and the ambient temperature is $72^\circ F$. Figure $5.3.1$ shows the direction field for this equation.

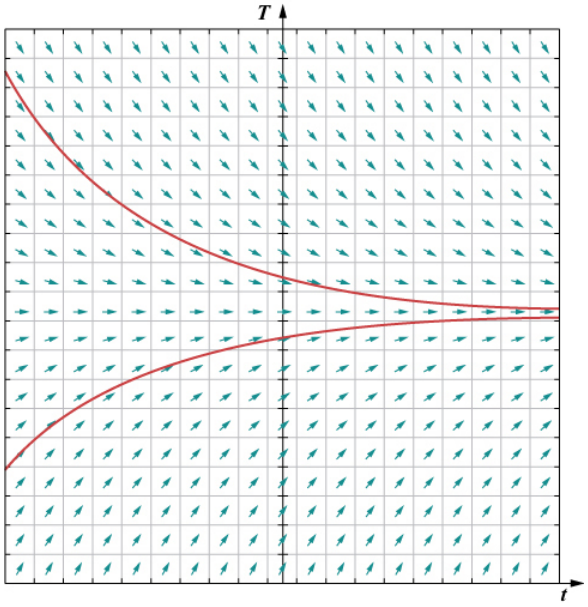


Figure $5.3.1$: Direction field for the differential equation $T'(t) = -0.4(T - 72)$. Two solutions are plotted: one with initial temperature less than $72^\circ F$ and the other with initial temperature greater than $72^\circ F$.

The idea behind a direction field is the fact that the derivative of a function evaluated at a given point is the slope of the tangent line to the graph of that function at the same point. Other examples of differential equations for which we can create a direction field include

$$y' = 3x + 2y - 4$$

$$y' = x^2 - y^2$$

$$y' = \frac{2x+4}{y-2}.$$

To create a direction field, we start with the first equation: $y' = 3x + 2y - 4$. We let (x_0, y_0) be any ordered pair, and we substitute these numbers into the right-hand side of the differential equation. For example, if we choose $x = 1$ and $y = 2$, substituting into the right-hand side of the differential equation yields

$$y' = 3x + 2y - 4 = 3(1) + 2(2) - 4 = 3.$$

This tells us that if a solution to the differential equation $y' = 3x + 2y - 4$ passes through the point $(1, 2)$, then the slope of the solution at that point must equal 3. To start creating the direction field, we put a short line segment at the point $(1, 2)$ having slope 3. We can do this for any point in the domain of the function $f(x, y) = 3x + 2y - 4$, which consists of all ordered pairs (x, y) in R^2. Therefore any point in the Cartesian plane has a slope associated with it, assuming that a solution to the differential equation passes through that point. The direction field for the differential equation $y' = 3x + 2y - 4$ is shown in Figure 5.3.2.

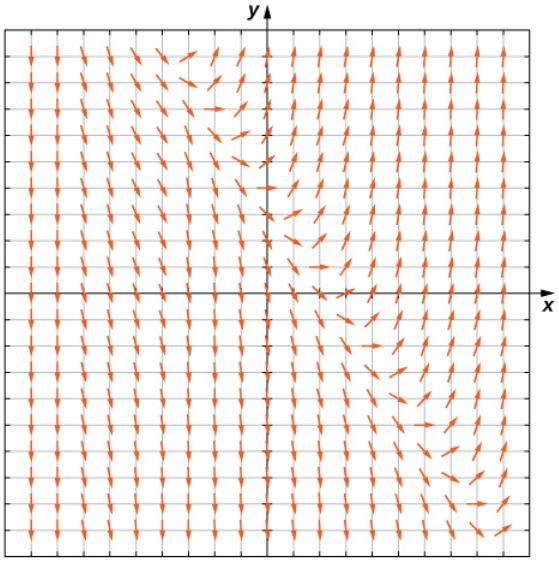


Figure 5.3.2: Direction field for the differential equation $y' = 3x + 2y - 4$.

We can generate a direction field of this type for any differential equation of the form $y' = f(x, y)$.

Definition: Direction Field (Slope Field)

A direction field (slope field) is a mathematical object used to graphically represent solutions to a first-order differential equation. At each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point.

Using Direction Fields

We can use a direction field to predict the behavior of solutions to a differential equation without knowing the actual solution. For example, the direction field in Figure 5.3.3 serves as a guide to the behavior of solutions to the differential equation $y' = 3x + 2y - 4$.

To use a direction field, we start by choosing any point in the field. The line segment at that point serves as a signpost telling us what direction to go from there. For example, if a solution to the differential equation passes through the point $(0, 1)$, then the slope of the solution passing through that point is given by $y' = 3(0) + 2(1) - 4 = -2$. Now let x increase slightly, say to $x = 0.1$. Using the method of linear approximations gives a formula for the approximate value of y for $x = 0.1$. In particular,

$$L(x) = y_0 + f'(x_0)(x - x_0) = 1 - 2(x_0 - 0) = 1 - 2x_0.$$

Substituting $x_0 = 0.1$ into $L(x)$ gives an approximate y value of 0.8.

At this point the slope of the solution changes (again according to the differential equation). We can keep progressing, recalculating the slope of the solution as we take small steps to the right, and watching the behavior of the solution. Figure 5.3.3 shows a graph

of the solution passing through the point $(0, 1)$.

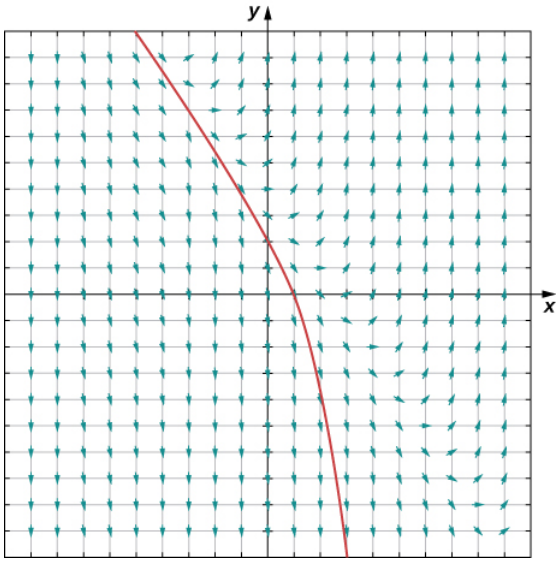


Figure 5.3.3: Direction field for the differential equation $y' = 3x + 2y - 4$ with the solution passing through the point $(0, 1)$.

The curve is the graph of the solution to the initial-value problem

$$y' = 3x + 2y - 4,\ y(0) = 1.$$

This curve is called a solution curve passing through the point $(0, 1)$. The exact solution to this initial-value problem is

$$y = -\frac{3}{2}x + \frac{5}{4} - \frac{1}{4}e^{2x},$$

and the graph of this solution is identical to the curve in Figure 5.3.3.

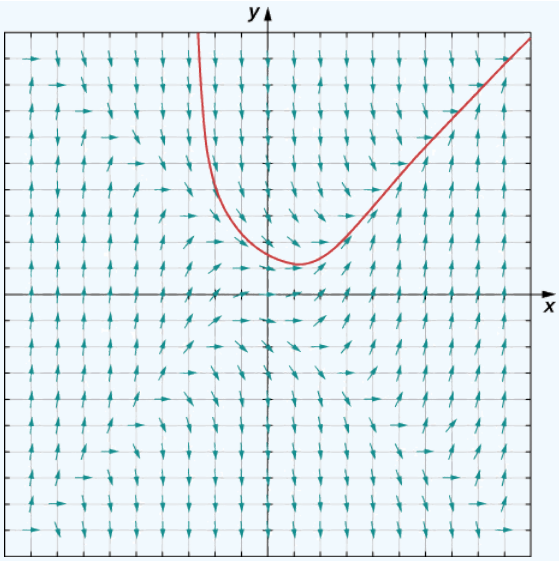
? Exercise 5.3.1

Create a direction field for the differential equation $y' = x^2 - y^2$ and sketch a solution curve passing through the point $(-1, 2)$.

Hint

Use x and y values ranging from -5 to 5. For each coordinate pair, calculate y' using the right-hand side of the differential equation.

Answer



Now consider the direction field for the differential equation $y' = (x-3)(y^2-4)$, shown in Figure 5.3.4. This direction field has several interesting properties. First of all, at $y = -2$ and $y = 2$, horizontal dashes appear all the way across the graph. This means that if $y = -2$, then $y' = 0$. Substituting this expression into the right-hand side of the differential equation gives

$$\begin{aligned}(x-3)(y^2-4) &= (x-3)((-2)^2-4)\\ &= (x-3)(0)\\ &= 0\\ &= y'.\end{aligned}$$

Therefore $y = -2$ is a solution to the differential equation. Similarly, $y = 2$ is a solution to the differential equation. These are the only constant-valued solutions to the differential equation, as we can see from the following argument. Suppose $y = k$ is a constant solution to the differential equation. Then $y' = 0$. Substituting this expression into the differential equation yields $0 = (x-3)(k^2-4)$. This equation must be true for all values of x, so the second factor must equal zero. This result yields the equation $k^2 - 4 = 0$. The solutions to this equation are $k = -2$ and $k = 2$, which are the constant solutions already mentioned. These are called the equilibrium solutions to the differential equation.

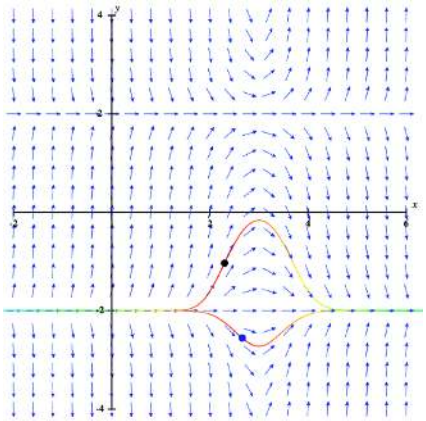


Figure 5.3.4: Direction field for the differential equation $y' = (x-3)(y^2-4)$ showing two solutions. These solutions are very close together, but one is barely above the equilibrium solution $y = -2$ and the other is barely below the same equilibrium solution.

Definition: Equilibrium Solutions

Consider the differential equation $y' = f(x, y)$. An *equilibrium solution* is any solution to the differential equation of the form $y = c$, where c is a constant.

To determine the equilibrium solutions to the differential equation $y' = f(x, y)$, set the right-hand side equal to zero. An equilibrium solution of the differential equation is any function of the form $y = k$ such that $f(x, k) = 0$ for all values of x in the domain of f.

An important characteristic of equilibrium solutions concerns whether or not they approach the line $y = k$ as an asymptote for large values of x.

Definition: asymptotically Stable, Unstable and Semi-Stable Solutions

Consider the differential equation $y' = f(x, y)$, and assume that all solutions to this differential equation are defined for $x \geq x_0$. Let $y = k$ be an equilibrium solution to the differential equation.

1. $y = k$ is an *asymptotically stable solution* to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon,\ k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y),\ y(x_0) = c$ approaches k as x approaches infinity.
2. $y = k$ is an *asymptotically unstable solution* to the differential equation if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon,\ k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y),\ y(x_0) = c$ never approaches k as x approaches infinity.
3. $y = k$ is an *asymptotically semi-stable solution* to the differential equation if it is neither asymptotically stable nor asymptotically unstable.

Now we return to the differential equation $y' = (x - 3)(y^2 - 4)$, with the initial condition $y(0) = 0.5$. The direction field for this initial-value problem, along with the corresponding solution, is shown in Figure 5.3.5.

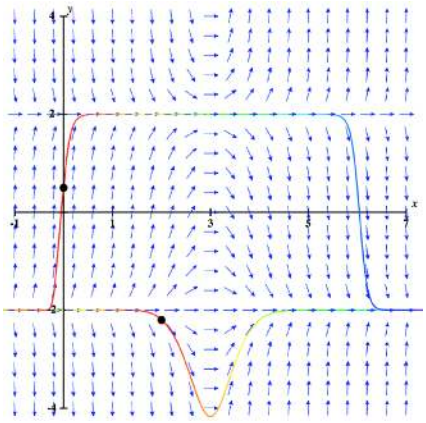


Figure 5.3.5: Direction field for the initial-value problem $y' = (x - 3)(y^2 - 4),\ y(0) = 0.5$. A second solution curve is graphed with initial condition $(2, -2.2)$.

The values of the solution to this initial-value problem stay between $y = -2$ and $y = 2$, which are the equilibrium solutions to the differential equation. Furthermore, as x approaches infinity, although they initially appear to approach the line y = 2, the y-coordinates clearly approach -2. The behavior of solutions is similar if the initial value is below -2, for example, $y(2) = -2.2$. In this case, the solutions increase and approach $y = -2$ as x approaches infinity. Therefore $y = -2$ is an asymptotically stable solution to this differential equation.

What happens when the initial value is above $y = 2$? This scenario is illustrated in Figure 5.3.6, with the initial value $y(0) = 3$.

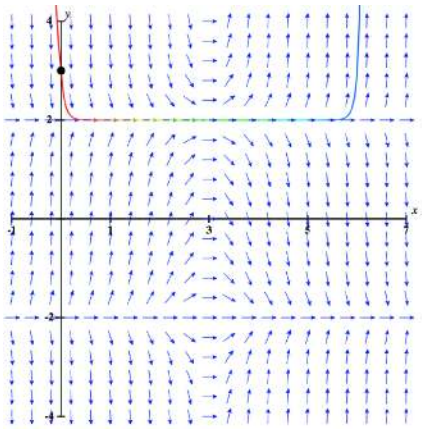


Figure 5.3.6: Direction field for the initial-value problem $y' = (x-3)(y^2-4),\ y(0) = 3$.

The solution increases rapidly toward positive infinity as x approaches infinity. Furthermore, if the initial value is slightly below 2, then the solution approaches -2, which is the other equilibrium solution. Therefore in neither case does the solution approach $y = 2$, so $y = 2$ is called an asymptotically unstable, or unstable, equilibrium solution.

Example 5.3.1: Stability of an Equilibrium Solution

Create a direction field for the differential equation $y' = (y-3)^2(y^2+y-2)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

Solution

The direction field is shown in Figure 5.3.7.

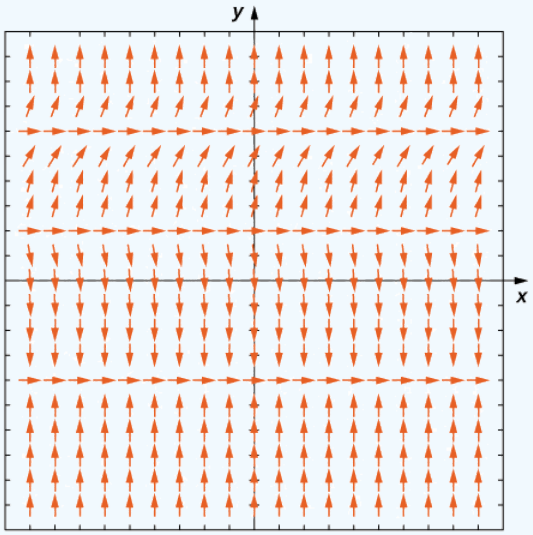


Figure 5.3.7: Direction field for the differential equation $y' = (y-3)^2(y^2+y-2)$.

The equilibrium solutions are $y = -2,\ y = 1,$ and $y = 3$. To classify each of the solutions, look at an arrow directly above or below each of these values. For example, at $y = -2$ the arrows directly below this solution point up, and the arrows directly above the solution point down. Therefore all initial conditions close to $y = -2$ approach $y = -2$, and the solution is stable. For the solution $y = 1$, all initial conditions above and below $y = 1$ are repelled (pushed away) from $y = 1$, so this solution is unstable. The solution $y = 3$ is semi-stable, because for initial conditions slightly greater than 3, the solution approaches infinity, and for initial conditions slightly less than 3, the solution approaches $y = 1$.

Analysis

It is possible to find the equilibrium solutions to the differential equation by setting the right-hand side equal to zero and solving for y. This approach gives the same equilibrium solutions as those we saw in the direction field.

Exercise 5.3.2

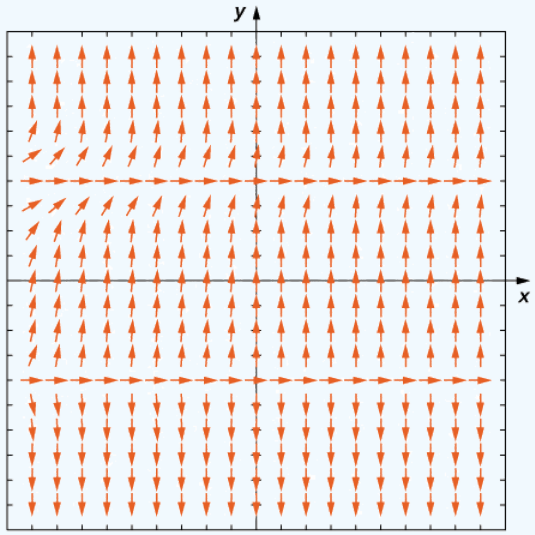
Create a direction field for the differential equation $y' = (x+5)(y+2)(y^2-4y+4)$ and identify any equilibrium solutions. Classify each of the equilibrium solutions as stable, unstable, or semi-stable.

Hint

First create the direction field and look for horizontal dashes that go all the way across. Then examine the slope lines directly above and below the equilibrium solutions.

Answer

The equilibrium solutions are $y = -2$ and $y = 2$. For this equation, $y = -2$ is an unstable equilibrium solution, and $y = 2$ is a semi-stable equilibrium solution.



Euler's Method

Consider the initial-value problem

$$y' = 2x - 3,\ y(0) = 3.$$

Integrating both sides of the differential equation gives $y = x^2 - 3x + C,$ and solving for C yields the particular solution $y = x^2 - 3x + 3.$ The solution for this initial-value problem appears as the parabola in Figure 5.3.8.

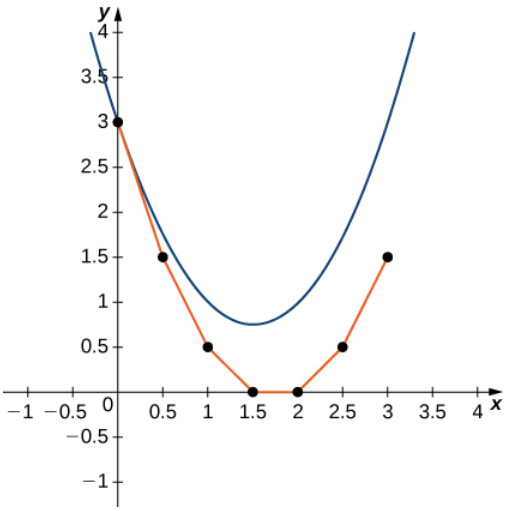


Figure 5.3.8: Euler's Method for the initial-value problem $y' = 2x - 3, y(0) = 3$.

The red graph consists of line segments that approximate the solution to the initial-value problem. The graph starts at the same initial value of $(0, 3)$. Then the slope of the solution at any point is determined by the right-hand side of the differential equation, and the length of the line segment is determined by increasing the x value by 0.5 each time (the step size). This approach is the basis of Euler's Method.

Before we state Euler's Method as a theorem, let's consider another initial-value problem:

$$y' = x^2 - y^2,\ y(-1) = 2.$$

The idea behind direction fields can also be applied to this problem to study the behavior of its solution. For example, at the point $(-1, 2)$, the slope of the solution is given by $y' = (-1)^2 - 2^2 = -3$, so the slope of the tangent line to the solution at that point is also equal to -3. Now we define $x_0 = -1$ and $y_0 = 2$. Since the slope of the solution at this point is equal to -3, we can use the method of linear approximation to approximate y near $(-1, 2)$.

$$L(x) = y_0 + f'(x_0)(x - x_0).$$

Here $x_0 = -1, y_0 = 2,$ and $f'(x_0) = -3,$ so the linear approximation becomes

$$L(x) = 2 - 3(x - (-1)) = 2 - 3x - 3 = -3x - 1.$$

Now we choose a **step size**. The step size is a small value, typically 0.1 or less, that serves as an increment for x; it is represented by the variable h. In our example, let $h = 0.1$. Incrementing x_0 by h gives our next x value:

$$x_1 = x_0 + h = -1 + 0.1 = -0.9.$$

We can substitute $x_1 = -0.9$ into the linear approximation to calculate y_1.

$$y_1 = L(x_1) = -3(-0.9) - 1 = 1.7.$$

Therefore the approximate y value for the solution when $x = -0.9$ is $y = 1.7$. We can then repeat the process, using $x_1 = -0.9$ and $y_1 = 1.7$ to calculate x_2 and y_2. The new slope is given by $y' = (-0.9)^2 - (1.7)^2 = -2.08$. First, $x_2 = x_1 + h = -0.9 + 0.1 = -0.8$. Using linear approximation gives

$$\begin{aligned} L(x) &= y_1 + f'(x_1)(x - x_1) \\ &= 1.7 - 2.08(x - (-0.9)) \\ &= 1.7 - 2.08x - 1.872 \\ &= -2.08x - 0.172. \end{aligned}$$

Finally, we substitute $x_2 = -0.8$ into the linear approximation to calculate y_2.

$$\begin{aligned} y_2 &= L(x_2) \\ &= -2.08x_2 - 0.172 \\ &= -2.08(-0.8) - 0.172 \\ &= 1.492. \end{aligned}$$

Therefore the approximate value of the solution to the differential equation is $y = 1.492$ when $x = -0.8$.

What we have just shown is the idea behind **Euler's Method**. Repeating these steps gives a list of values for the solution. These values are shown in the table, rounded off to four decimal places.

Table 5.3.1: Using Euler's Method to Approximate Solutions to a Differential Equation

n	0	1	2	3	4	5
x_n	−1	−0.9	−0.8	−0.7	−0.6	−0.5
y_n	2	1.7	1.492	1.3334	1.2046	1.0955
n	6	7	8	9	10	
x_n	−0.4	−0.3	−0.2	−0.1	0	
y_n	1.0004	1.9164	1.8414	1.7746	1.7156	

Euler's method

Consider the initial-value problem

$$y' = f(x, y),\ y(x_0) = y_0.$$

To approximate a solution to this problem using Euler's method, define

$$x_n = x_0 + nh$$
$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})\ .$$

Here $h > 0$ represents the step size and n is an integer, starting with 1. The number of steps taken is counted by the variable n.

Typically h is a small value, say 0.1 or 0.05. The smaller the value of h, the more calculations are needed. The higher the value of h, the fewer calculations are needed. However, the tradeoff results in a lower degree of accuracy for larger step size, as illustrated in Figure $5.3.9$.

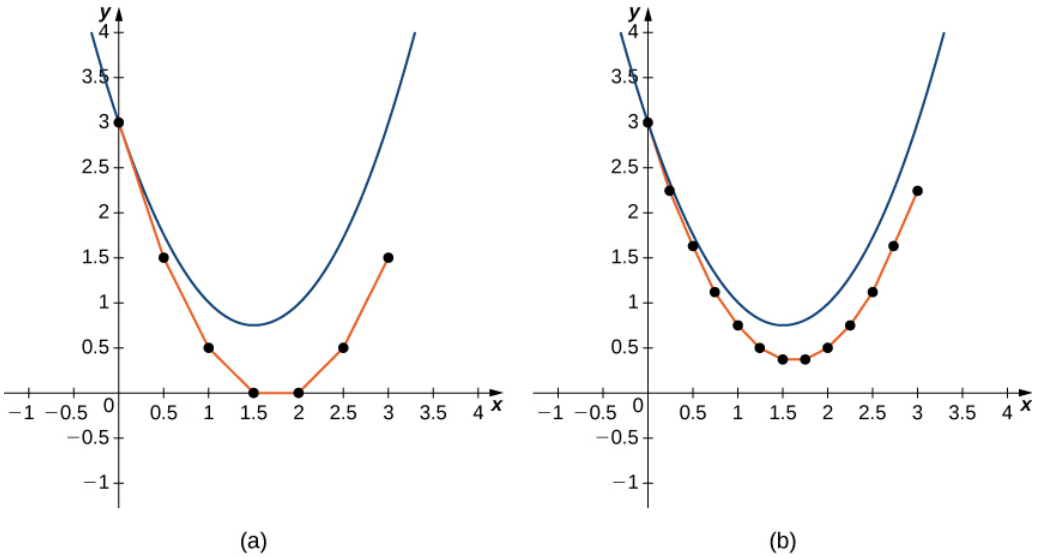


Figure 5.3.9: Euler's method for the initial-value problem $y' = 2x - 3,\ y(0) = 3$ with (a) a step size of $h = 0.5$; and (b) a step size of $h = 0.25$.

Example 5.3.2: Using Euler's Method

Consider the initial-value problem

$$y' = 3x^2 - y^2 + 1,\ y(0) = 2.$$

Use Euler's method with a step size of 0.1 to generate a table of values for the solution for values of x between 0 and 1.

Solution

We are given $h = 0.1$ and $f(x, y) = 3x^2 - y^2 + 1$. Furthermore, the initial condition $y(0) = 2$ gives $x_0 = 0$ and $y_0 = 2$. Using Equation with $n = 0$, we can generate this table.

Table 5.3.2: Using Euler's Method to Approximate Solutions to a Differential Equation

n	x_n	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
0	0	2
1	0.1	$y_1 = y_0 + hf(x_0, y_0) = 1.7$
2	0.2	$y_2 = y_1 + hf(x_1, y_1) = 1.514$
3	0.3	$y_3 = y_2 + hf(x_2, y_2) = 1.3968$

n	x_n	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
4	0.4	$y_4 = y_3 + hf(x_3, y_3) = 1.3287$
5	0.5	$y_5 = y_4 + hf(x_4, y_4) = 1.3001$
6	0.6	$y_6 = y_5 + hf(x_5, y_5) = 1.3061$
7	0.7	$y_7 = y_6 + hf(x_6, y_6) = 1.3435$
8	0.8	$y_8 = y_7 + hf(x_7, y_7) = 1.4100$
9	0.9	$y_9 = y_8 + hf(x_8, y_8) = 1.5032$
10	1.0	$y_{10} = y_9 + hf(x_9, y_9) = 1.6202$

With ten calculations, we are able to approximate the values of the solution to the initial-value problem for values of x between 0 and 1.

Go to this website for a chance to visually explore Euler's method.

Exercise 5.3.3

Consider the initial-value problem

$$y' = x^3 + y^2,\ y(1) = -2.$$

Using a step size of 0.1, generate a table with approximate values for the solution to the initial-value problem for values of x between 1 and 2.

Hint

Start by identifying the value of h, then figure out what $f(x, y)$ is. Then use the formula for Euler's Method to calculate y_1, y_2, and so on.

Answer

Table 5.3.3: Using Euler's Method to approximate solutions to the differential equation in Exercise 5.3.3.

n	\(x_n)	$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$
0	1	-2
1	1.1	$y_1 = y_0 + hf(x_0, y_0) = -1.5$
2	1.2	$y_2 = y_1 + hf(x_1, y_1) = -1.1419$
3	1.3	$y_3 = y_2 + hf(x_2, y_2) = -0.8387$
4	1.4	$y_4 = y_3 + hf(x_3, y_3) = -0.5487$
5	1.5	$y_5 = y_4 + hf(x_4, y_4) = -0.2442$
6	1.6	$y_6 = y_5 + hf(x_5, y_5) = 0.0993$
7	1.7	$y_7 = y_6 + hf(x_6, y_6) = 0.5099$
8	1.8	$y_8 = y_7 + hf(x_7, y_7) = 1.0272$
9	1.9	$y_9 = y_8 + hf(x_8, y_8) = 1.7159$
10	2	$y_{10} = y_9 + hf(x_9, y_9) = 2.6962$

Key Concepts

- A direction field is a mathematical object used to graphically represent solutions to a first-order differential equation.
- Euler's Method is a numerical technique that can be used to approximate solutions to a differential equation.

Key Equations

- **Euler's Method**

  $x_n = x_0 + nh$

  $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$, where h is the step size

Glossary

asymptotically semi-stable solution

$y = k$ if it is neither asymptotically stable nor asymptotically unstable

asymptotically stable solution

$y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon,\ k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y),\ y(x_0) = c$ approaches k as x approaches infinity

asymptotically unstable solution

$y = k$ if there exists $\varepsilon > 0$ such that for any value $c \in (k - \varepsilon,\ k + \varepsilon)$ the solution to the initial-value problem $y' = f(x, y),\ y(x_0) = c$ never approaches k as $x$approaches infinity

direction field (slope field)

a mathematical object used to graphically represent solutions to a first-order differential equation; at each point in a direction field, a line segment appears whose slope is equal to the slope of a solution to the differential equation passing through that point

equilibrium solution

any solution to the differential equation of the form $y = c$, where c is a constant

Euler's Method

a numerical technique used to approximate solutions to an initial-value problem

solution curve

a curve graphed in a direction field that corresponds to the solution to the initial-value problem passing through a given point in the direction field

step size

the increment hh that is added to the xx value at each step in Euler's Method

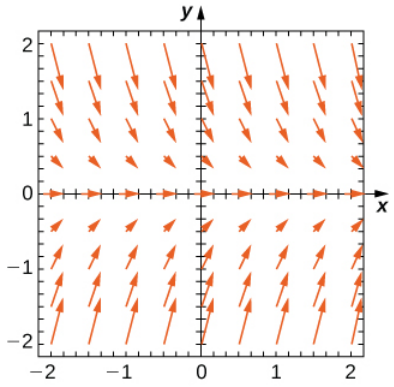
- Gilbert Strang (MIT) and Edwin "Jed" Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Paul Seeburger (Monroe Community College) edited this section to adjust the explanation of equilibrium points in the example shown in Figures 5.3.4 - 5.3.6. He also created these figures.

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5.3E: Exercises for Direction Fields and Numerical Methods

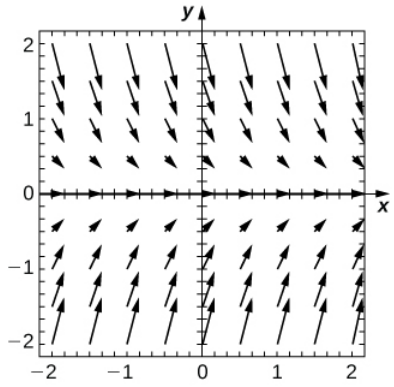
For exercises 1 - 3, use the direction field below from the differential equation $y' = -2y$. Sketch the graph of the solution for the given initial conditions.



1) $y(0) = 1$

2) $y(0) = 0$

Answer



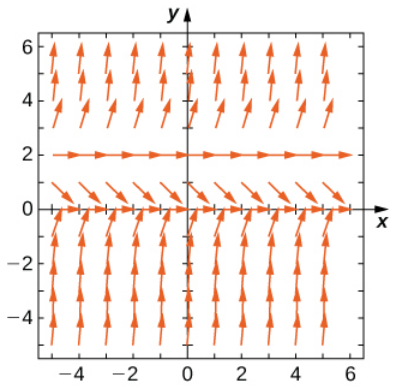
3) $y(0) = -1$

4) Are there any equilibria among the solutions of the differential equation from exercises 1 - 3? List any equilibria along with their stabilities.

Answer

$y = 0$ is a stable equilibrium

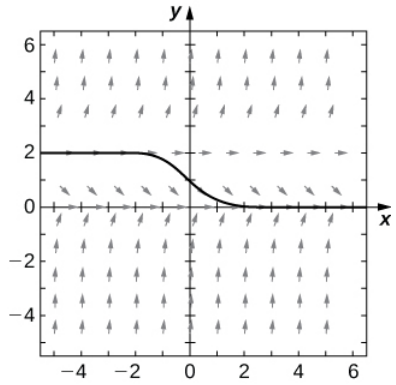
For exercises 5 - 7, use the direction field below from the differential equation $y' = y^2 - 2y$. Sketch the graph of the solution for the given initial conditions.



5) $y(0) = 3$

6) $y(0) = 1$

Answer



7) $y(0) = -1$

8) Are there any equilibria among the solutions of the differential equation from exercises 5 - 7? List any equilibria along with their stabilities.

Answer

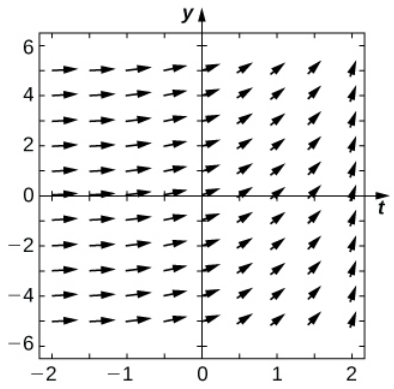
$y = 0$ is a stable equilibrium and $y = 2$ is unstable

In exercises 9 - 13, draw the direction field for the following differential equations, then solve the differential equation. Draw your solution on top of the direction field. Does your solution follow along the arrows on your direction field?

9) $y' = t^3$

10) $y' = e^t$

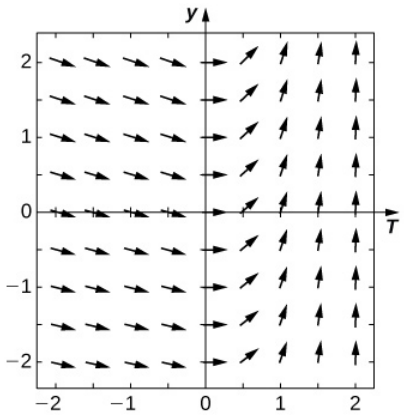
Answer



11) $\frac{dy}{dx} = x^2 \cos x$

12) $\frac{dy}{dt} = te^t$

Answer



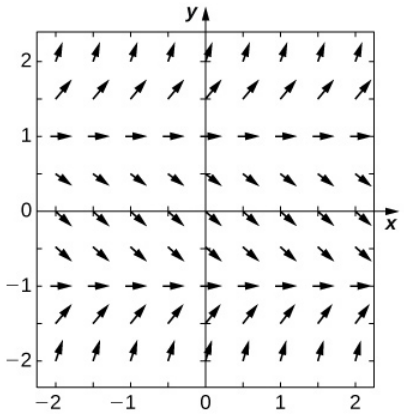
13) $\frac{dx}{dt} = \cosh(t)$

In exercises 14 - 18, draw the directional field for the following differential equations. What can you say about the behavior of the solution? Are there equilibria? What stability do these equilibria have?

14) $y' = y^2 - 1$

Answer

There appear to be equlibria at $y = -1$ (stable) and $y = 1$ (unstable).

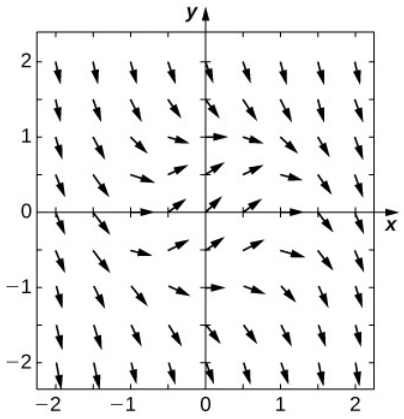


15) $y' = y - x$

16) $y' = 1 - y^2 - x^2$

Answer

There do not appear to be any equilibria.

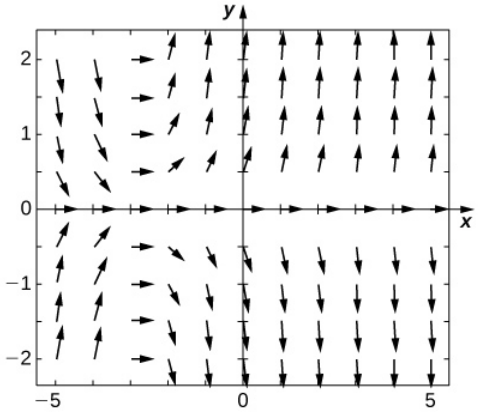


17) $y' = t^2 \sin y$

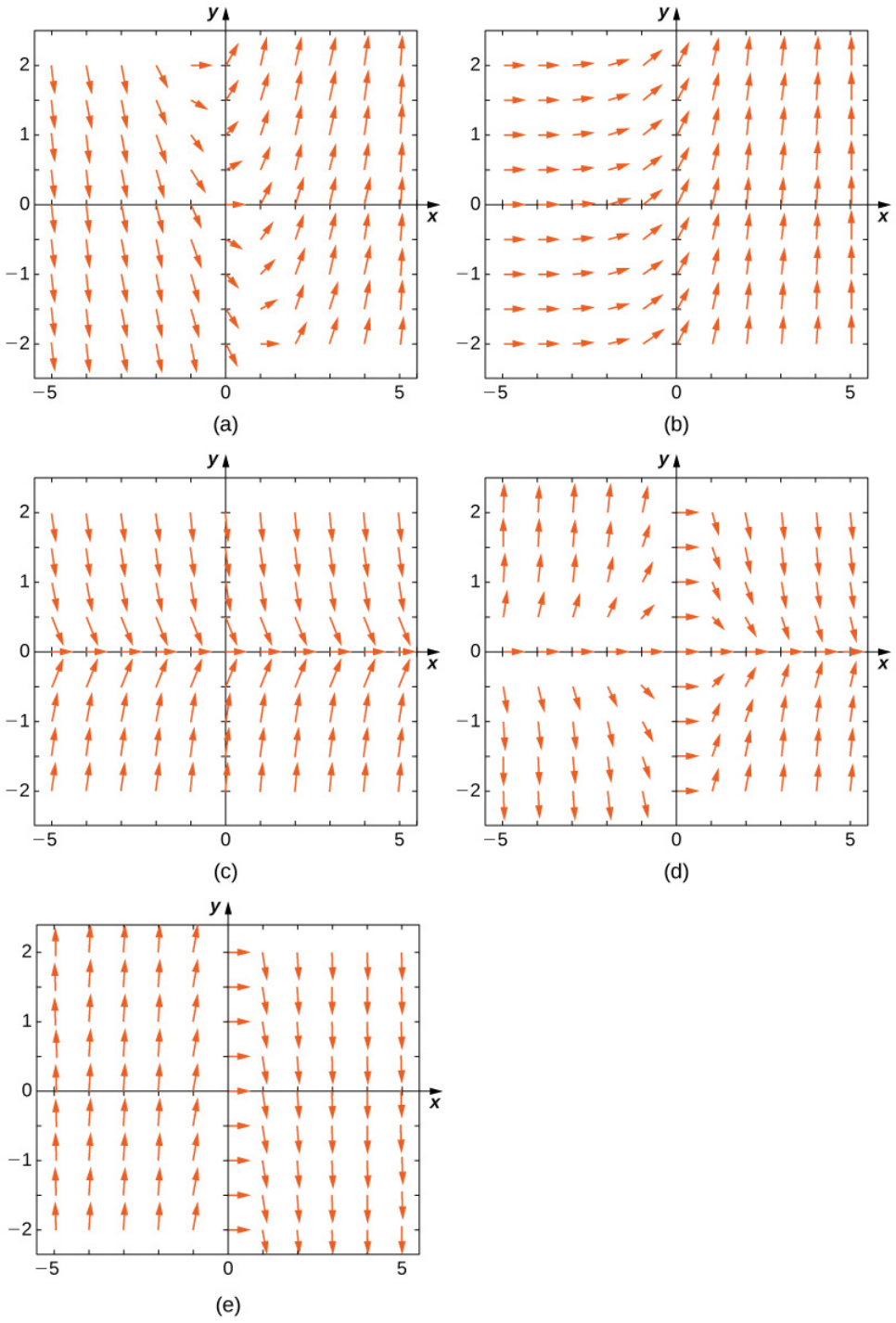
18) $y' = 3y + xy$

Answer

There appears to be an unstable equilibrium at $y = 0$.



Match the direction field with the given differential equations. Explain your selections.



19) $y' = -3y$

20) $y' = -3t$

Answer

E

21) $y' = e^t$

22) $y' = \frac{1}{2}y + t$

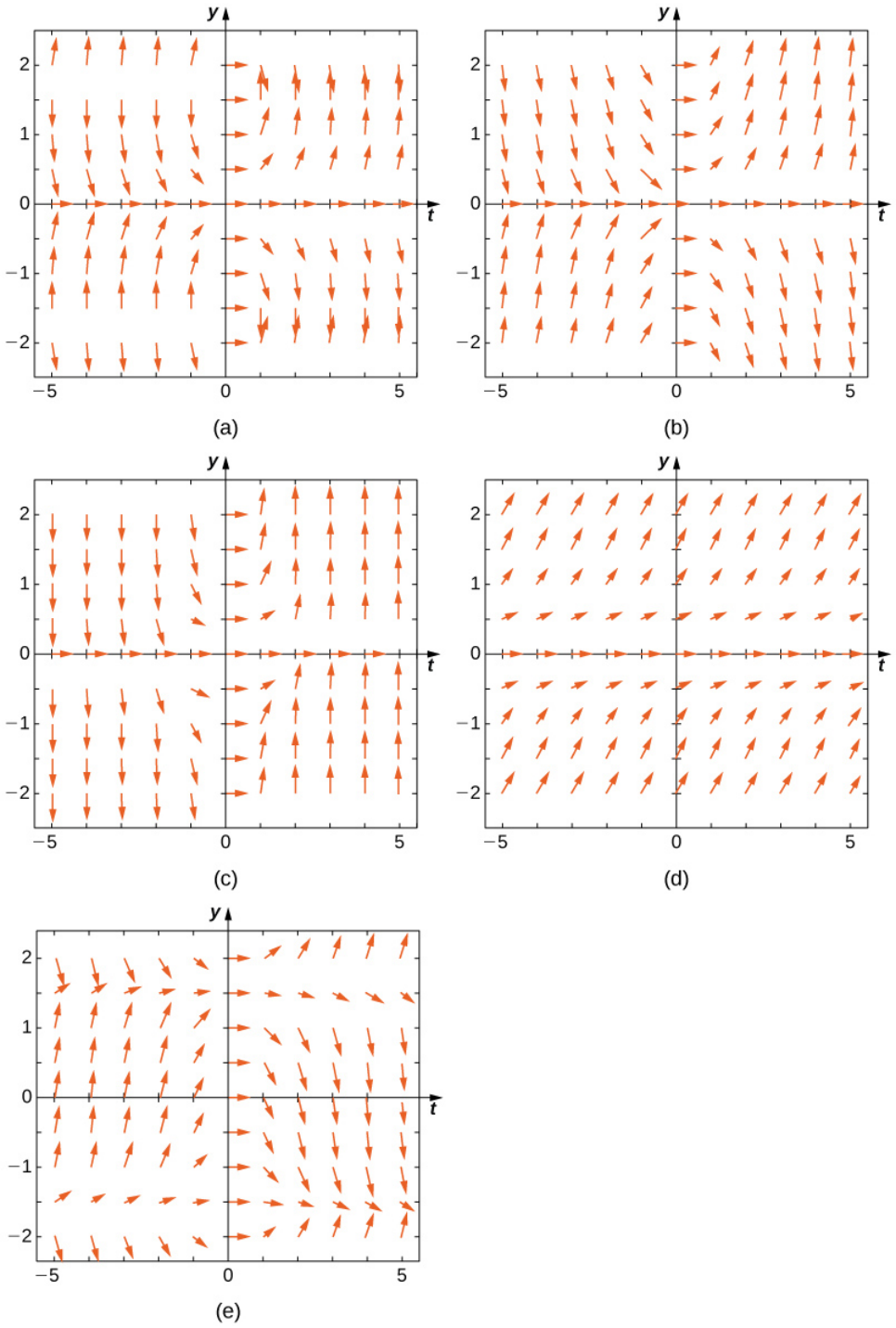
Answer

https://math.libretexts.org/@go/page/127921

A

23) $y' = -ty$

Match the direction field with the given differential equations. Explain your selections.



24) $y' = t \sin y$

Answer

B

25) $y' = -t \cos y$

26) $y' = t \tan y$

Answer

A

27) $y' = \sin^2 y$

28) $y' = y^2 t^3$

Answer

C

Estimate the following solutions using Euler's method with $n = 5$ steps over the interval $t = [0, 1]$. If you are able to solve the initial-value problem exactly, compare your solution with the exact solution. If you are unable to solve the initial-value problem, the exact solution will be provided for you to compare with Euler's method. How accurate is Euler's method?

29) $y' = -3y, \quad y(0) = 1$

30) $y' = t^2, \quad y(0) = 1$

Answer

1.24,exact: $\frac{4}{3} \approx 1.33$

Solution:

31) $y' = 3t - y, \quad y(0) = 1$. Exact solution is $y = 3t + 4e^{-t} - 3$

32) $y' = y + t^2, \quad y(0) = 3$. Exact solution is $y = 5e^t - 2 - t^2 - 2t$

Answer

7.739364,exact: $5(e-1)$

33) $y' = 2t, \quad y(0) = 0$

34) [T] $y' = e^{x+y}, y(0) = -1$. Exact solution is $y = -\ln(e + 1 - e^x)$

Answer

-0.2535,exact: 0

35) $y' = y^2 \ln(x+1), \quad y(0) = 1$. Exact solution is $y = -\dfrac{1}{(x+1)(\ln(x+1) - 1)}$

36) $y' = 2^x, \quad y(0) = 0$. Exact solution is $y = \dfrac{2^x - 1}{\ln 2}$

Answer

1.345,exact: $\frac{1}{\ln(2)}$

37) $y' = y, \quad y(0) = -1$. Exact solution is $y = -e^x$.

38) $y' = -5t, \quad y(0) = -2$. Exact solution is $y = -\frac{5}{2}t^2 - 2$

Answer

-4, exact: $-9/2 = -4.5$

Differential equations can be used to model **disease epidemics.** In the next set of problems, we examine the change of size of two sub-populations of people living in a city: individuals who are infected and individuals who are susceptible to infection. S represents the size of the susceptible population, and I represents the size of the infected population. We assume that if a susceptible person interacts with an infected person, there is a probability c that the susceptible person will become infected. Each infected person recovers from the infection at a rate r and becomes susceptible again. We consider the case of influenza, where we

assume that no one dies from the disease, so we assume that the total population size of the two sub-populations is a constant number, N. The differential equations that model these population sizes are

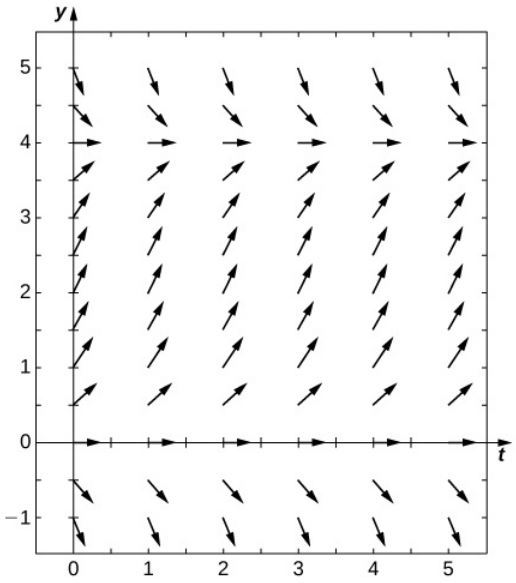
$$S' = rI - cSI \text{ and } I' = cSI - rI.$$

Here c represents the contact rate and r is the recovery rate.

39) Show that, by our assumption that the total population size is constant $(S + I = N)$, you can reduce the system to a single differential equation in I : $I' = c(N - I)I - rI$.

40) Assuming the parameters are $c = 0.5, N = 5,$ and $r = 0.5$, draw the resulting directional field.

Answer



41) [T] Use computational software or a calculator to compute the solution to the initial-value problem $y' = ty, y(0) = 2$ using Euler's Method with the given step size h. Find the solution at $t = 1$. For a hint, here is "pseudo-code" for how to write a computer program to perform Euler's Method for $y' = f(t, y), y(0) = 2$:

Create function $f(t, y)$

Define parameters $y(1) = y_0, t(0) = 0,$ step size h, and total number of steps, N

Write a for-loop:

for $k = 1$ to N

$fn = f(t(k), y(k))$

$y(k + 1) = y(k) + h * fn$

$t(k + 1) = t(k) + h$

42) Solve the initial-value problem for the exact solution.

Answer

$y' = 2e^{t^2/2}$

43) Draw the directional field

44) $h = 1$

Answer

2

45) [T] $h = 10$

46) [T] $h = 100$

Answer

3.2756

47) [T] $h = 1000$

48) [T] Evaluate the exact solution at $t = 1$. Make a table of errors for the relative error between the Euler's method solution and the exact solution. How much does the error change? Can you explain?

Answer

Exact solution: y =$2\sqrt{e}$.

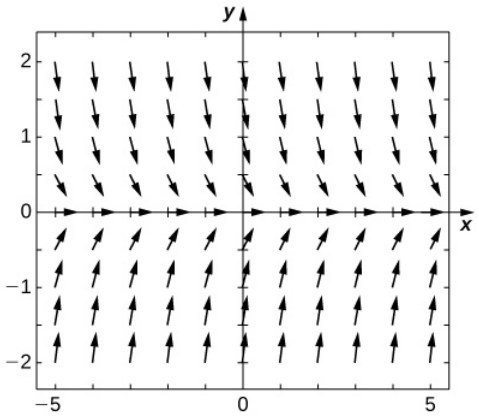
Step Size	Error
$h = 1$	0.3935
$h = 10$	0.06163
$h = 100$	0.006612
$h = 10000$	0.0006661

For exercises 49 - 53, consider the initial-value problem $y' = -2y$, with $y(0) = 2$.

49) Show that $y = 2e^{-2x}$ solves this initial-value problem.

50) Draw the directional field of this differential equation.

Answer



51) [T] By hand or by calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 5$.

52) [T] By calculator or computer, approximate the solution using Euler's Method at $t = 10$ using $h = 100$.

Answer

$4.0741e^{-10}$

53) [T] Plot exact answer and each Euler approximation (for $h = 5$ and $h = 100$) at each h on the directional field. What do you notice?

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5.4: Separable Differential Equations

Learning Objectives

- Use separation of variables to solve a differential equation.
- Solve applications using separation of variables.

We now examine a solution technique for finding exact solutions to a class of differential equations known as separable differential equations. These equations are common in a wide variety of disciplines, including physics, chemistry, and engineering. We illustrate a few applications at the end of the section.

Separation of Variables

We start with a definition and some examples.

Definition: Separable Differential Equations

A *separable differential equation* is any equation that can be written in the form

$$y' = f(x)g(y). \tag{5.4.1}$$

The term 'separable' refers to the fact that the right-hand side of Equation 5.4.1 can be separated into a function of x times a function of y. Examples of separable differential equations include

$$y' = (x^2-4)(3y+2) \tag{5.4.2}$$

$$y' = 6x^2+4x \tag{5.4.3}$$

$$y' = \sec y + \tan y \tag{5.4.4}$$

$$y' = xy+3x-2y-6. \tag{5.4.5}$$

Equation 5.4.3 is separable with $f(x) = 6x^2+4x$ and $g(y) = 1$, Equation 5.4.4 is separable with $f(x) = 1$ and $g(y) = \sec y + \tan y$, and the right-hand side of Equation 5.4.5 can be factored as $(x+3)(y-2)$, so it is separable as well. Equation 5.4.4 is also called an *autonomous differential equation* because the right-hand side of the equation is a function of y alone. If a differential equation is separable, then it is possible to solve the equation using the method of *separation of variables*.

Problem-Solving Strategy: Separation of Variables

1. Check for any values of y that make $g(y) = 0$. These correspond to constant solutions.
2. Rewrite the differential equation in the form
$$\frac{dy}{g(y)} = f(x)dx.$$
3. Integrate both sides of the equation.
4. Solve the resulting equation for y if possible.
5. If an initial condition exists, substitute the appropriate values for x and y into the equation and solve for the constant.

Note that Step 4 states "Solve the resulting equation for y if possible." It is not always possible to obtain y as an explicit function of x. Quite often we have to be satisfied with finding y as an implicit function of x.

Example 5.4.1: Using Separation of Variables

Find a general solution to the differential equation $y' = (x^2-4)(3y+2)$ using the method of separation of variables.

Solution

Follow the five-step method of separation of variables.

1. In this example, $f(x) = x^2 - 4$ and $g(y) = 3y + 2$. Setting $g(y) = 0$ gives $y = -\frac{2}{3}$ as a constant solution.

2. Rewrite the differential equation in the form

$$\frac{dy}{3y+2} = (x^2 - 4)\,dx.$$

3. Integrate both sides of the equation:

$$\int \frac{dy}{3y+2} = \int (x^2 - 4)\,dx.$$

Let $u = 3y + 2$. Then $du = 3\frac{dy}{dx}\,dx$, so the equation becomes

$$\frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}x^3 - 4x + C$$

$$\frac{1}{3}\ln|u| = \frac{1}{3}x^3 - 4x + C$$

$$\frac{1}{3}\ln|3y+2| = \frac{1}{3}x^3 - 4x + C.$$

4. To solve this equation for y, first multiply both sides of the equation by 3.

$$\ln|3y+2| = x^3 - 12x + 3C$$

Now we use some logic in dealing with the constant C. Since C represents an arbitrary constant, $3C$ also represents an arbitrary constant. If we call the second arbitrary constant C_1, where $C_1 = 3C$, the equation becomes

$$\ln|3y+2| = x^3 - 12x + C_1.$$

Now exponentiate both sides of the equation (i.e., make each side of the equation the exponent for the base e).

$$e^{\ln|3y+2|} = e^{x^3 - 12x + C_1}$$

$$|3y+2| = e^{C_1}e^{x^3 - 12x}$$

Again define a new constant $C_2 = e^{C_1}$ (note that $C_2 > 0$):

$$|3y+2| = C_2 e^{x^3 - 12x}.$$

Because of the absolute value on the left side of the equation, this corresponds to two separate equations:

$$3y+2 = C_2 e^{x^3 - 12x}$$

and

$$3y+2 = -C_2 e^{x^3 - 12x}.$$

The solution to either equation can be written in the form

$$y = \frac{-2 \pm C_2 e^{x^3 - 12x}}{3}.$$

Since $C_2 > 0$, it does not matter whether we use plus or minus, so the constant can actually have either sign. Furthermore, the subscript on the constant C is entirely arbitrary, and can be dropped. Therefore the solution can be written as

$$y = \frac{-2 + Ce^{x^3 - 12x}}{3}, \text{ where } C = \pm C_2 \text{ or } C = 0.$$

Note that in writing a single general solution in this way, we are also allowing C to equal 0. This gives us the singular solution, $y = -\frac{2}{3}$, for the given differential equation. Check that this is indeed a solution of this differential equation!

5. No initial condition is imposed, so we are finished.

Exercise 5.4.1

Use the method of separation of variables to find a general solution to the differential equation

$$y' = 2xy + 3y - 4x - 6.$$

Hint

First factor the right-hand side of the equation by grouping, then use the five-step strategy of separation of variables.

Answer

$$y = 2 + Ce^{x^2+3x}$$

Example 5.4.2: Solving an Initial-Value Problem

Using the method of separation of variables, solve the initial-value problem

$$y' = (2x+3)(y^2-4), \quad y(0) = -1.$$

Solution

Follow the five-step method of separation of variables.

1. In this example, $f(x) = 2x+3$ and $g(y) = y^2 - 4$. Setting $g(y) = 0$ gives $y = \pm 2$ as constant solutions.

2. Divide both sides of the equation by $y^2 - 4$ and multiply by dx. This gives the equation

$$\frac{dy}{y^2-4} = (2x+3)\,dx.$$

3. Next integrate both sides:

$$\int \frac{1}{y^2-4}\,dy = \int (2x+3)\,dx. \tag{5.4.6}$$

To evaluate the left-hand side, use the method of partial fraction decomposition. This leads to the identity

$$\frac{1}{y^2-4} = \frac{1}{4}\left(\frac{1}{y-2} - \frac{1}{y+2}\right).$$

Then Equation 5.4.6 becomes

$$\frac{1}{4}\int \left(\frac{1}{y-2} - \frac{1}{y+2}\right) dy = \int (2x+3)\,dx$$

$$\frac{1}{4}(\ln|y-2| - \ln|y+2|) = x^2 + 3x + C.$$

Multiplying both sides of this equation by 4 and replacing $4C$ with C_1 gives

$$\ln|y-2| - \ln|y+2| = 4x^2 + 12x + C_1$$

$$\ln\left|\frac{y-2}{y+2}\right| = 4x^2 + 12x + C_1.$$

4. It is possible to solve this equation for y. First exponentiate both sides of the equation and define $C_2 = e^{C_1}$:

$$\left|\frac{y-2}{y+2}\right| = C_2 e^{4x^2+12x}.$$

Next we can remove the absolute value and let a new constant C_3 be positive, negative, or zero, i.e., $C_3 = \pm C_2$ or $C_3 = 0$.

Then multiply both sides by $y+2$.

$$y-2=C_3(y+2)e^{4x^2+12x}$$

$$y-2=C_3ye^{4x^2+12x}+2C_3e^{4x^2+12x}.$$

Now collect all terms involving y on one side of the equation, and solve for y:

$$y-C_3ye^{4x^2+12x}=2+2C_3e^{4x^2+12x}$$

$$y\left(1-C_3e^{4x^2+12x}\right)=2+2C_3e^{4x^2+12x}$$

$$y=\frac{2+2C_3e^{4x^2+12x}}{1-C_3e^{4x^2+12x}}.$$

5. To determine the value of C_3, substitute $x=0$ and $y=-1$ into the general solution. Alternatively, we can put the same values into an earlier equation, namely the equation $\frac{y-2}{y+2}=C_3e^{4x^2+12}$. This is much easier to solve for C_3:

$$\frac{y-2}{y+2}=C_3e^{4x^2+12x}$$

$$\frac{-1-2}{-1+2}=C_3e^{4(0)^2+12(0)}$$

$$C_3=-3.$$

Therefore the solution to the initial-value problem is

$$y=\frac{2-6e^{4x^2+12x}}{1+3e^{4x^2+12x}}.$$

A graph of this solution appears in Figure 5.4.1.

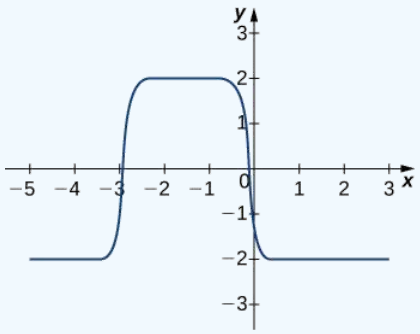


Figure 5.4.1: Graph of the solution to the initial-value problem $y'=(2x+3)(y^2-4), y(0)=-1$.

Exercise 5.4.2

Find the solution to the initial-value problem

$$6y'=(2x+1)(y^2-2y-8)$$

with $y(0)=-3$ using the method of separation of variables.

Hint

Follow the steps for separation of variables to solve the initial-value problem.

Answer

$$y=\frac{4+14e^{x^2+x}}{1-7e^{x^2+x}}$$

Applications of Separation of Variables

Many interesting problems can be described by separable equations. We illustrate two types of problems: solution concentrations and Newton's law of cooling.

Solution concentrations

Consider a tank being filled with a salt solution. We would like to determine the amount of salt present in the tank as a function of time. We can apply the process of separation of variables to solve this problem and similar problems involving *solution concentrations*.

Example 5.4.3: Determining Salt Concentration over Time

A tank containing 100 L of a brine solution initially has 4 kg of salt dissolved in the solution. At time $t = 0$, another brine solution flows into the tank at a rate of 2 L/min. This brine solution contains a concentration of 0.5 kg/L of salt. At the same time, a stopcock is opened at the bottom of the tank, allowing the combined solution to flow out at a rate of 2 L/min, so that the level of liquid in the tank remains constant (Figure 5.4.2). Find the amount of salt in the tank as a function of time (measured in minutes), and find the limiting amount of salt in the tank, assuming that the solution in the tank is well mixed at all times.

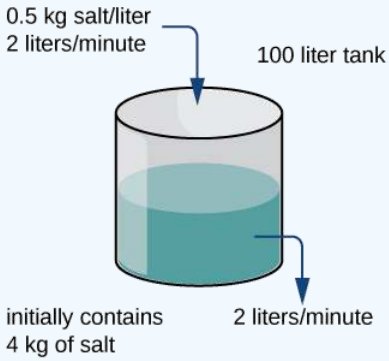


Figure 5.4.2: A brine tank with an initial amount of salt solution accepts an input flow and delivers an output flow. How does the amount of salt change with time?

Solution

First we define a function $u(t)$ that represents the amount of salt in kilograms in the tank as a function of time. Then $\frac{du}{dt}$ represents the rate at which the amount of salt in the tank changes as a function of time. Also, $u(0)$ represents the amount of salt in the tank at time $t = 0$, which is 4 kilograms.

The general setup for the differential equation we will solve is of the form

$$\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}.$$

INFLOW RATE represents the rate at which salt enters the tank, and OUTFLOW RATE represents the rate at which salt leaves the tank. Because solution enters the tank at a rate of 2 L/min, and each liter of solution contains 0.5 kilogram of salt, every minute $2(0.5) = 1$ kilogram of salt enters the tank. Therefore INFLOW RATE = 1.

To calculate the rate at which salt leaves the tank, we need the concentration of salt in the tank at any point in time. Since the actual amount of salt varies over time, so does the concentration of salt. However, the volume of the solution remains fixed at 100 liters. The number of kilograms of salt in the tank at time t is equal to $u(t)$. Thus, the concentration of salt is $\frac{u(t)}{100}$ kg/L, and the solution leaves the tank at a rate of 2 L/min. Therefore salt leaves the tank at a rate of $\frac{u(t)}{100} \cdot 2 = \frac{u(t)}{50}$ kg/min, and OUTFLOW RATE is equal to $\frac{u(t)}{50}$. Therefore the differential equation becomes $\frac{du}{dt} = 1 - \frac{u}{50}$, and the initial condition is $u(0) = 4$. The initial-value problem to be solved is

$$\frac{du}{dt} = 1 - \frac{u}{50}, u(0) = 4.$$

The differential equation is a separable equation, so we can apply the five-step strategy for solution.

Step 1. Setting $1-\frac{u}{50}=0$ gives $u=50$ as a constant solution. Since the initial amount of salt in the tank is 4 kilograms, this solution does not apply.

Step 2. Rewrite the equation as

$$\frac{du}{dt}=\frac{50-u}{50}.$$

Then multiply both sides by dt and divide both sides by $50-u$:

$$\frac{du}{50-u}=\frac{dt}{50}.$$

Step 3. Integrate both sides:

$$\begin{aligned}\int\frac{du}{50-u}&=\int\frac{dt}{50}\\ -\ln|50-u|&=\frac{t}{50}+C.\end{aligned}$$

Step 4. Solve for $u(t)$:

$$\ln|50-u|=-\frac{t}{50}-C$$

$$e^{\ln|50-u|}=e^{-(t/50)-C}$$

$$|50-u|=C_1e^{-t/50},\text{ where }C_1=e^{-C}.$$

Eliminate the absolute value by allowing the constant to be positive, negative, or zero, i.e., $C_1=\pm e^{-C}$ or $C_1=0$:

$$50-u=C_1e^{-t/50}.$$

Finally, solve for $u(t)$:

$$u(t)=50-C_1e^{-t/50}.$$

Step 5. Solve for C_1:

$$\begin{aligned}u(0)&=50-C_1e^{-0/50}\\ 4&=50-C_1\\ C_1&=46.\end{aligned}$$

The solution to the initial value problem is $u(t)=50-46e^{-t/50}$. To find the limiting amount of salt in the tank, take the limit as t approaches infinity:

$$\begin{aligned}\lim_{t\to\infty}u(t)&=50-46e^{-t/50}\\ &=50-46(0)=50.\end{aligned}$$

Note that this was the constant solution to the differential equation. If the initial amount of salt in the tank is 50 kilograms, then it remains constant. If it starts at less than 50 kilograms, then it approaches 50 kilograms over time.

Exercise 5.4.3

A tank contains 3 kilograms of salt dissolved in 75 liters of water. A salt solution of 0.4 kg salt/L is pumped into the tank at a rate of 6 L/min and is drained at the same rate. Solve for the salt concentration at time t. Assume the tank is well mixed at all times.

Hint

Follow the steps in Example 5.4.3 and determine an expression for INFLOW and OUTFLOW. Formulate an initial-value problem, and then solve it.

Initial value problem:

$$\frac{du}{dt} = 2.4 - \frac{2u}{25},\ u(0) = 3$$

Answer

$$u(t) = 30 - 27e^{-t/50}$$

Newton's Law of Cooling

Newton's law of cooling states that the rate of change of an object's temperature is proportional to the difference between its own temperature and the ambient temperature (i.e., the temperature of its surroundings). If we let $T(t)$ represent the temperature of an object as a function of time, then $\frac{dT}{dt}$ represents the rate at which that temperature changes. The temperature of the object's surroundings can be represented by T_s. Then Newton's law of cooling can be written in the form

$$\frac{dT}{dt} = k(T(t) - T_s)$$

or simply

$$\frac{dT}{dt} = k(T - T_s).$$

The temperature of the object at the beginning of any experiment is the initial value for the initial-value problem. We call this temperature T_0. Therefore the initial-value problem that needs to be solved takes the form

$$\frac{dT}{dt} = k(T - T_s) \tag{5.4.7}$$

with $T(0) = T_0$, where k is a constant that needs to be either given or determined in the context of the problem. We use these equations in Example 5.4.4.

Example 5.4.4: Waiting for a Pizza to Cool

A pizza is removed from the oven after baking thoroughly, and the temperature of the oven is $350^\circ F$. The temperature of the kitchen is $75^\circ F$, and after 5 minutes the temperature of the pizza is $340^\circ F$. We would like to wait until the temperature of the pizza reaches $300^\circ F$ before cutting and serving it (Figure 5.4.3). How much longer will we have to wait?

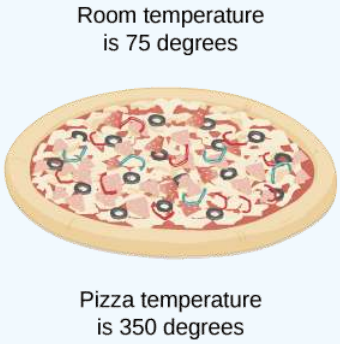


Figure 5.4.3: From Newton's law of cooling, if the pizza cools $10^\circ F$ in 5 minutes, how long before it cools to $300^\circ F$?

Solution

The ambient temperature (surrounding temperature) is $75^\circ F$, so $T_s = 75$. The temperature of the pizza when it comes out of the oven is $350^\circ F$, which is the initial temperature (i.e., initial value), so $T_0 = 350$. Therefore Equation 5.4.7 becomes

$$\frac{dT}{dt} = k(T - 75)$$

with $T(0) = 350$.

To solve the differential equation, we use the five-step technique for solving separable equations.

1. Setting the right-hand side equal to zero gives $T = 75$ as a constant solution. Since the pizza starts at $350^\circ F$, this is not the solution we are seeking.

2. Rewrite the differential equation by multiplying both sides by dt and dividing both sides by $T - 75$:

$$\frac{dT}{T-75} = k\,dt.$$

3. Integrate both sides:

$$\begin{aligned}\int \frac{dT}{T-75} &= \int k\,dt \\ \ln|T-75| &= kt + C.\end{aligned}$$

4. Solve for T by first exponentiating both sides:

$$\begin{aligned} e^{\ln|T-75|} &= e^{kt+C} \\ |T-75| &= C_1 e^{kt}, && \text{where } C_1 = e^C. \\ T-75 &= \pm C_1 e^{kt} \\ T-75 &= Ce^{kt}, && \text{where } C = \pm C_1 \text{ or } C = 0. \\ T(t) &= 75 + Ce^{kt}. \end{aligned}$$

5. Solve for C by using the initial condition $T(0) = 350$:

$$\begin{aligned} T(t) &= 75 + Ce^{kt} \\ T(0) &= 75 + Ce^{k(0)} \\ 350 &= 75 + C \\ C &= 275. \end{aligned}$$

Therefore the solution to the initial-value problem is

$$T(t) = 75 + 275e^{kt}.$$

To determine the value of k, we need to use the fact that after 5 minutes the temperature of the pizza is $340^\circ F$. Therefore $T(5) = 340$. Substituting this information into the solution to the initial-value problem, we have

$$T(t) = 75 + 275e^{kt}$$

$$T(5) = 340 = 75 + 275e^{5k}$$

$$265 = 275e^{5k}$$

$$e^{5k} = \frac{53}{55}$$

$$\ln e^{5k} = \ln\left(\frac{53}{55}\right)$$

$$5k = \ln\left(\frac{53}{55}\right)$$

$$k = \frac{1}{5}\ln\left(\frac{53}{55}\right) \approx -0.007408.$$

So now we have $T(t) = 75 + 275e^{-0.007048t}$. When is the temperature $300^\circ F$? Solving for t, we find

$$T(t) = 75 + 275e^{-0.007048t}$$

$$300 = 75 + 275e^{-0.007048t}$$

$$225 = 275e^{-0.007048t}$$

$$e^{-0.007048t} = \frac{9}{11}$$

$$\ln e^{-0.007048t} = \ln \frac{9}{11}$$

$$-0.007048t = \ln \frac{9}{11}$$

$$t = -\frac{1}{0.007048} \ln \frac{9}{11} \approx 28.5.$$

Therefore we need to wait an additional 23.5 minutes (after the temperature of the pizza reached $340^\circ F$). That should be just enough time to finish this calculation.

Exercise 5.4.4

A cake is removed from the oven after baking thoroughly, and the temperature of the oven is $450^\circ F$. The temperature of the kitchen is $70^\circ F$, and after 10 minutes the temperature of the cake is $430^\circ F$.

a. Write the appropriate initial-value problem to describe this situation.
b. Solve the initial-value problem for $T(t)$.
c. How long will it take until the temperature of the cake is within $5^\circ F$ of room temperature?

Hint

Determine the values of T_s and T_0 then use Equation 5.4.7.

Answer a

Initial-value problem

$$\frac{dT}{dt} = k(T - 70), \quad T(0) = 450$$

Answer b

$$T(t) = 70 + 380e^{kt}$$

Answer c

Approximately 114 minutes.

Key Concepts

- A separable differential equation is any equation that can be written in the form $y' = f(x)g(y)$.
- The method of separation of variables is used to find the general solution to a separable differential equation.

Key Equations

- **Separable differential equation**

  $y' = f(x)g(y)$

- **Solution concentration**

  $\frac{du}{dt} = \text{INFLOW RATE} - \text{OUTFLOW RATE}$

- **Newton's law of cooling**

  $\frac{dT}{dt} = k(T - T_s)$

Glossary

autonomous differential equation

an equation in which the right-hand side is a function of y alone

separable differential equation

any equation that can be written in the form $y' = f(x)g(y)$

separation of variables

a method used to solve a separable differential equation

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5.4E: Exercises for Separable Differential Equations

In exercises 1 - 4, solve the following initial-value problems with the initial condition $y_0 = 0$ and graph the solution.

1) $\frac{dy}{dt} = y + 1$

Answer

$y = e^t - 1$

2) $\frac{dy}{dt} = y - 1$

3) $\frac{dy}{dt} = -y + 1$

Answer

$y = 1 - e^{-t}$

4) $\frac{dy}{dt} = -y - 1$

In exercises 5 - 14, find the general solution to the differential equation.

5) $x^2 y' = (x+1)y$

Answer

$y = Cxe^{-1/x}$

6) $y' = \tan(y)x$

7) $y' = 2xy^2$

Answer

$y = \frac{1}{C - x^2}$

8) $\frac{dy}{dt} = y\cos(3t+2)$

9) $2x\frac{dy}{dx} = y^2$

Answer

$y = -\frac{2}{C + \ln|x|}$

10) $y' = e^y x^2$

11) $(1+x)y' = (x+2)(y-1)$

Answer

$y = Ce^x(x+1) + 1$

12) $\frac{dx}{dt} = 3t^2(x^2+4)$

13) $t\frac{dy}{dt} = \sqrt{1-y^2}$

Answer

$y = \sin(\ln|t| + C)$

14) $y' = e^x e^y$

In exercises 15 - 24, find the solution to the initial-value problem.

15) $y' = e^{y-x}, \quad y(0) = 0$

Answer

$y = -\ln(e^{-x})$ which simplifies to $y = x$

16) $y' = y^2(x+1), \quad y(0) = 2$

17) $\dfrac{dy}{dx} = y^3 x e^{x^2}, \quad y(0) = 1$

Answer

$$y = \frac{1}{\sqrt{2 - e^{x^2}}}$$

18) $\dfrac{dy}{dt} = y^2 e^x \sin(3x), \quad y(0) = 1$

19) $y' = \dfrac{x}{\operatorname{sech}^2 y}, \quad y(0) = 0$

Answer

$$y = \tanh^{-1}\left(\frac{x^2}{2}\right)$$

20) $y' = 2xy(1+2y), \quad y(0) = -1$

21) $\dfrac{dx}{dt} = \ln(t)\sqrt{1-x^2}, \quad x(1) = 0$

Answer

$x = \sin(1 - t + t\ln t)$

22) $y' = 3x^2(y^2+4), \quad y(0) = 0$

23) $y' = e^y 5^x, \quad y(0) = \ln(\ln(5))$

Answer

$y = \ln(\ln(5)) - \ln(2 - 5^x)$

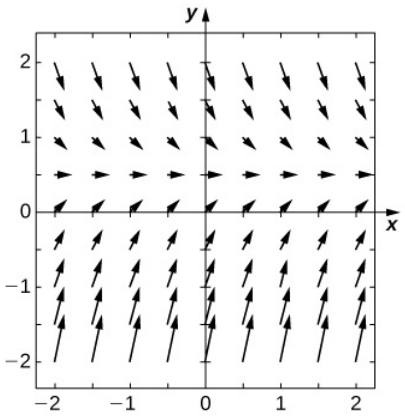
24) $y' = -2x\tan(y), \quad y(0) = \dfrac{\pi}{2}$

For problems 25 - 29, use a software program or your calculator to generate the directional fields. Solve explicitly and draw solution curves for several initial conditions. Are there some critical initial conditions that change the behavior of the solution?

25) [T] $y' = 1 - 2y$

Answer

$$y = Ce^{-2}x + \frac{1}{2}$$

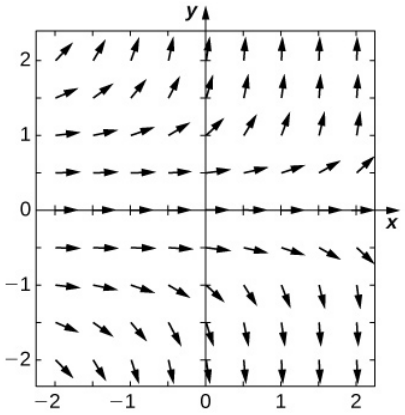


26) [T] $y' = y^2 x^3$

27) [T] $y' = y^3 e^x$

Answer

$$y = \frac{1}{\sqrt{2}\sqrt{C - e^x}}$$

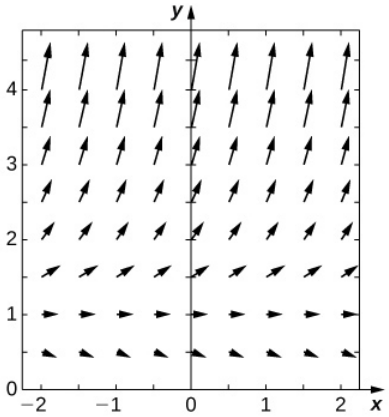


28) [T] $y' = e^y$

29) [T] $y' = y \ln(x)$

Answer

$y = Ce^{-x} x^x$



30) Most **drugs in the bloodstream** decay according to the equation $y' = cy$, where y is the concentration of the drug in the bloodstream. If the half-life of a drug is 2 hours, what fraction of the initial dose remains after 6 hours?

31) A drug is administered intravenously to a patient at a rate r mg/h and is cleared from the body at a rate proportional to the amount of drug still present in the body, d. Set up and solve the differential equation, assuming there is no drug initially present in the body.

Answer

$y = \frac{r}{d}(1 - e^{-dt})$

32) [T] How often should a drug be taken if its dose is 3 mg, it is cleared at a rate $c = 0.1$ mg/h, and 1 mg is required to be in the bloodstream at all times?

33) A tank contains 1 kilogram of salt dissolved in 100 liters of water. A salt solution of 0.1 kg salt/L is pumped into the tank at a rate of 2 L/min and is drained at the same rate. Solve for the salt concentration at time t. Assume the tank is well mixed.

Answer

Since the amount of salt at time t is given by the equation $A(t) = 10 - 9e^{-t/50}$, and the tank volume stays constant at 100 liters, the concentration at time t would be $c(t) = \frac{1}{100}\left(10 - 9e^{-t/50}\right)$.

34) A tank containing 10 kilograms of salt dissolved in 1000 liters of water has two salt solutions pumped in. The first solution of 0.2 kg salt/L is pumped in at a rate of 20 L/min and the second solution of 0.05 kg salt/L is pumped in at a rate of 5 L/min. The tank drains at 25 L/min. Assume the tank is well mixed. Solve for the salt concentration at time t.

35) [T] For the preceding problem, find how much salt is in the tank 1 hour after the process begins.

Answer

134.3 kilograms

36) Torricelli's law states that for a water tank with a hole in the bottom that has a cross-section of A and with a height of water h above the bottom of the tank, the rate of change of volume of water flowing from the tank is proportional to the square root of the height of water, according to $\frac{dV}{dt} = -A\sqrt{2gh}$, where g is the acceleration due to gravity. Note that $\frac{dV}{dt} = A\frac{dh}{dt}$. Solve the resulting initial-value problem for the height of the water, assuming a tank with a hole of radius 2 ft. The initial height of water is 100 ft.

37) For the preceding problem, determine how long it takes the tank to drain.

Answer

720 seconds

For problems 38 - 44, use Newton's law of cooling.

38) The liquid base of an ice cream has an initial temperature of $200^\circ F$ before it is placed in a freezer with a constant temperature of $0^\circ F$. After 1 hour, the temperature of the ice-cream base has decreased to $140^\circ F$. Formulate and solve the initial-value problem to determine the temperature of the ice cream.

39) [T] The liquid base of an ice cream has an initial temperature of $210^\circ F$ before it is placed in a freezer with a constant temperature of $20^\circ F$. After 2 hours, the temperature of the ice-cream base has decreased to $170^\circ F$. At what time will the ice cream be ready to eat? (Assume $30^\circ F$ is the optimal eating temperature.)

Answer

24 hours 55 minutes

40) [T] You are organizing an ice cream social. The outside temperature is $80^\circ F$ and the ice cream is at $10^\circ F$. After 10 minutes, the ice cream temperature has risen by $10^\circ F$. How much longer can you wait before the ice cream melts at $40^\circ F$?

41) You have a cup of coffee at temperature $70^\circ C$ and the ambient temperature in the room is $20^\circ C$. Assuming a cooling rate k of 0.125, write and solve the differential equation to describe the temperature of the coffee with respect to time.

Answer

$T(t) = 20 + 50e^{-0.125t}$

42) [T] You have a cup of coffee at temperature $70^\circ C$ that you put outside, where the ambient temperature is $0^\circ C$. After 5 minutes, how much colder is the coffee?

43) You have a cup of coffee at temperature $70^\circ C$ and you immediately pour in 1 part milk to 5 parts coffee. The milk is initially at temperature $1^\circ C$. Write and solve the differential equation that governs the temperature of this coffee.

Answer

$T(t) = 20 + 38.5e^{-0.125t}$

44) You have a cup of coffee at temperature $70^\circ C$, which you let cool 10 minutes before you pour in the same amount of milk at $1^\circ C$ as in the preceding problem. How does the temperature compare to the previous cup after 10 minutes?

45) Solve the generic problem $y' = ay + b$ with initial condition $y(0) = c$.

Answer

$y = (c + ba)e^{ax} - \frac{b}{a}$

46) Prove the basic continual compounded interest equation. Assuming an initial deposit of P_0 and an interest rate of r, set up and solve an equation for continually compounded interest.

47) Assume an initial nutrient amount of I kilograms in a tank with L liters. Assume a concentration of c kg/L being pumped in at a rate of r L/min. The tank is well mixed and is drained at a rate of r L/min. Find the equation describing the amount of nutrient in the tank.

Answer

$y(t) = cL + (I - cL)e^{-rt/L}$

48) Leaves accumulate on the forest floor at a rate of 2 g/cm^2/yr and also decompose at a rate of 90% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor, assuming at time 0 there is no leaf litter on the ground. Does this amount approach a steady value? What is that value?

49) Leaves accumulate on the forest floor at a rate of 4 g/cm^2/yr. These leaves decompose at a rate of 10% per year. Write a differential equation governing the number of grams of leaf litter per square centimeter of forest floor. Does this amount approach a steady value? What is that value?

Answer

Differential Equation: $\frac{dy}{dt} = 4 - 0.1y$
Solution, the model for this situation: $y = 40(1 - e^{-0.1t})$,
Amount approaches a steady value of 40 g/cm^2

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5.5: The Logistic Equation

Learning Objectives

- Describe the concept of environmental carrying capacity in the logistic model of population growth.
- Draw a direction field for a logistic equation and interpret the solution curves.
- Solve a logistic equation and interpret the results.

Differential equations can be used to represent the size of a population as it varies over time. We saw this in an earlier chapter in the section on exponential growth and decay, which is the simplest model. A more realistic model includes other factors that affect the growth of the population. In this section, we study the logistic differential equation and see how it applies to the study of population dynamics in the context of biology.

Population Growth and Carrying Capacity

To model population growth using a differential equation, we first need to introduce some variables and relevant terms. The variable t. will represent time. The units of time can be hours, days, weeks, months, or even years. Any given problem must specify the units used in that particular problem. The variable P will represent population. Since the population varies over time, it is understood to be a function of time. Therefore we use the notation $P(t)$ for the population as a function of time. If $P(t)$ is a differentiable function, then the first derivative $\frac{dP}{dt}$ represents the instantaneous rate of change of the population as a function of time.

In Exponential Growth and Decay, we studied the exponential growth and decay of populations and radioactive substances. An example of an exponential growth function is $P(t) = P_0 e^{rt}$. In this function, $P(t)$ represents the population at time t, P_0 represents the **initial population** (population at time $t = 0$), and the constant $r > 0$ is called the **growth rate**. Figure $5.5.1$ shows a graph of $P(t) = 100e^{0.03t}$. Here $P_0 = 100$ and $r = 0.03$.

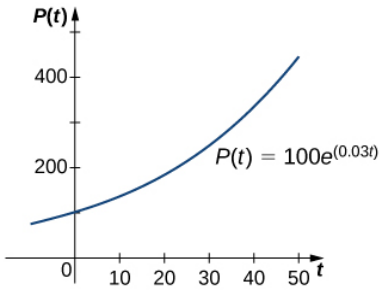


Figure 5.5.1: An exponential growth model of population.

We can verify that the function $P(t) = P_0 e^{rt}$ satisfies the initial-value problem

$$\frac{dP}{dt} = rP$$

with $P(0) = P_0$.

This differential equation has an interesting interpretation. The left-hand side represents the rate at which the population increases (or decreases). The right-hand side is equal to a positive constant multiplied by the current population. Therefore the differential equation states that the rate at which the population increases is proportional to the population at that point in time. Furthermore, it states that the constant of proportionality never changes.

One problem with this function is its prediction that as time goes on, the population grows without bound. This is unrealistic in a real-world setting. Various factors limit the rate of growth of a particular population, including birth rate, death rate, food supply, predators, and so on. The growth constant r usually takes into consideration the birth and death rates but none of the other factors, and it can be interpreted as a net (birth minus death) percent growth rate per unit time. A natural question to ask is whether the population growth rate stays constant, or whether it changes over time. Biologists have found that in many biological systems, the population grows until a certain steady-state population is reached. This possibility is not taken into account with exponential growth. However, the concept of carrying capacity allows for the possibility that in a given area, only a certain number of a given organism or animal can thrive without running into resource issues.

Definition: Carrying Capacity

The *carrying capacity* of an organism in a given environment is defined to be the maximum population of that organism that the environment can sustain indefinitely.

We use the variable K to denote the carrying capacity. The growth rate is represented by the variable r. Using these variables, we can define the logistic differential equation.

Definition: Logistic Differential Equation

Let K represent the carrying capacity for a particular organism in a given environment, and let r be a real number that represents the growth rate. The function $P(t)$ represents the population of this organism as a function of time t, and the constant P_0 represents the initial population (population of the organism at time $t = 0$). Then the *logistic differential equation* is

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{5.5.1}$$

The logistic equation was first published by Pierre Verhulst in 1845. This differential equation can be coupled with the initial condition $P(0) = P_0$ to form an initial-value problem for $P(t)$.

Suppose that the initial population is small relative to the carrying capacity. Then $\frac{P}{K}$ is small, possibly close to zero. Thus, the quantity in parentheses on the right-hand side of Equation 5.5.1 is close to 1, and the right-hand side of this equation is close to rP. If $r > 0$, then the population grows rapidly, resembling exponential growth.

However, as the population grows, the ratio $\frac{P}{K}$ also grows, because K is constant. If the population remains below the carrying capacity, then $\frac{P}{K}$ is less than 1, so $1 - \frac{P}{K} > 0$. Therefore the right-hand side of Equation 5.5.1 is still positive, but the quantity in parentheses gets smaller, and the growth rate decreases as a result. If $P = K$ then the right-hand side is equal to zero, and the population does not change.

Now suppose that the population starts at a value higher than the carrying capacity. Then $\frac{P}{K} > 1$, and $1 - \frac{P}{K} < 0$. Then the right-hand side of Equation 5.5.1 is negative, and the population decreases. As long as $P > K$, the population decreases. It never actually reaches K because $\frac{dP}{dt}$ will get smaller and smaller, but the population approaches the carrying capacity as t approaches infinity. This analysis can be represented visually by way of a phase line. A **phase line** describes the general behavior of a solution to an autonomous differential equation, depending on the initial condition. For the case of a carrying capacity in the logistic equation, the phase line is as shown in Figure 5.5.2.



Figure 5.5.2: A phase line for the differential equation $\frac{dP}{dt} = rP(1 - \frac{P}{K})$.

This phase line shows that when P is less than zero or greater than K, the population decreases over time. When P is between 0 and K, the population increases over time.

Example 5.5.1: Examining the Carrying Capacity of a Deer Population

Let's consider the population of white-tailed deer (*Odocoileus virginianus*) in the state of Kentucky. The Kentucky Department of Fish and Wildlife Resources (KDFWR) sets guidelines for hunting and fishing in the state. Before the hunting season of 2004, it estimated a population of 900,000 deer. Johnson notes: "A deer population that has plenty to eat and is not hunted by humans or other predators will double every three years." (George Johnson, "The Problem of Exploding Deer Populations Has No Attractive Solutions," January 12,2001, accessed April 9, 2015)



Figure 5.5.3: (credit: modification of work by Rachel Kramer, Flickr)

This observation corresponds to a rate of increase $r = \dfrac{\ln(2)}{3} = 0.2311$, so the approximate growth rate is 23.11% per year. (This assumes that the population grows exponentially, which is reasonable—at least in the short term—with plentiful food supply and no predators.) The KDFWR also reports deer population densities for 32 counties in Kentucky, the average of which is approximately 27 deer per square mile. Suppose this is the deer density for the whole state (39,732 square miles). The carrying capacity K is 39,732 square miles times 27 deer per square mile, or 1,072,764 deer.

a. For this application, we have $P_0 = 900,000$, $K = 1,072,764$, and $r = 0.2311$. Substitute these values into Equation 5.5.1 and form the initial-value problem.
b. Solve the initial-value problem from part a.
c. According to this model, what will be the population in 3 years? Recall that the doubling time predicted by Johnson for the deer population was 3 years. How do these values compare?

Suppose the population managed to reach **1,200,000** What does the logistic equation predict will happen to the population in this scenario?

Solution

a. The initial value problem is

$$\frac{dP}{dt} = 0.2311P\left(1 - \frac{P}{1,072,764}\right), \quad P(0) = 900,000.$$

b. The logistic equation is an autonomous differential equation, so we can use the method of separation of variables.

Step 1: Setting the right-hand side equal to zero gives $P = 0$ and $P = 1,072,764$. This means that if the population starts at zero it will never change, and if it starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation and multiply both sides by:

$$\frac{dP}{dt} = 0.2311P\left(\frac{1,072,764 - P}{1,072,764}\right)$$

$$dP = 0.2311P\left(\frac{1,072,764 - P}{1,072,764}\right)dt$$

$$\frac{dP}{P(1,072,764 - P)} = \frac{0.2311}{1,072,764}dt.$$

Step 3: Integrate both sides of the equation using partial fraction decomposition:

$$\int \frac{dP}{P(1,072,764-P)} = \int \frac{0.2311}{1,072,764} dt$$

$$\frac{1}{1,072,764} \int \left(\frac{1}{P} + \frac{1}{1,072,764-P} \right) dP = \frac{0.2311t}{1,072,764} + C$$

$$\frac{1}{1,072,764} (\ln|P| - \ln|1,072,764-P|) = \frac{0.2311t}{1,072,764} + C.$$

Step 4: Multiply both sides by 1,072,764 and use the quotient rule for logarithms:

$$\ln\left|\frac{P}{1,072,764-P}\right| = 0.2311t + C_1.$$

Here $C_1 = 1,072,764C$. Next exponentiate both sides and eliminate the absolute value:

$$e^{\ln\left|\frac{P}{1,072,764-P}\right|} = e^{0.2311t+C_1}$$

$$\left|\frac{P}{1,072,764-P}\right| = C_2 e^{0.2311t}$$

$$\frac{P}{1,072,764-P} = C_2 e^{0.2311t}.$$

Here $C_2 = e^{C_1}$ but after eliminating the absolute value, it can be negative as well. Now solve for:

$$P = C_2 e^{0.2311t}(1,072,764-P)$$

$$P = 1,072,764 C_2 e^{0.2311t} - C_2 P e^{0.2311t}$$

$$P + C_2 P e^{0.2311t} = 1,072,764 C_2 e^{0.2311t}$$

$$P(1 + C_2 e^{0.2311t} = 1,072,764 C_2 e^{0.2311t}$$

$$P(t) = \frac{1,072,764 C_2 e^{0.2311t}}{1 + C_2 e^{0.2311t}}.$$

Step 5: To determine the value of C_2, it is actually easier to go back a couple of steps to where C_2 was defined. In particular, use the equation

$$\frac{P}{1,072,764-P} = C_2 e^{0.2311t}.$$

The initial condition is $P(0) = 900,000$. Replace P with $900,000$ and t with zero:

$$\frac{P}{1,072,764-P} = C_2 e^{0.2311t}$$

$$\frac{900,000}{1,072,764-900,000} = C_2 e^{0.2311(0)}$$

$$\frac{900,000}{172,764} = C_2$$

$$C_2 = \frac{25,000}{4,799}$$

$$\approx 5.209.$$

Therefore

$$P(t) = \frac{1,072,764\left(\frac{25000}{4799}\right)e^{0.2311t}}{1+(250004799)e^{0.2311t}}$$
$$= \frac{1,072,764(25000)e^{0.2311t}}{4799+25000e^{0.2311t}}.$$

Dividing the numerator and denominator by **25,000** gives

$$P(t) = \frac{1,072,764e^{0.2311t}}{0.19196+e^{0.2311t}}.$$

Figure is a graph of this equation.

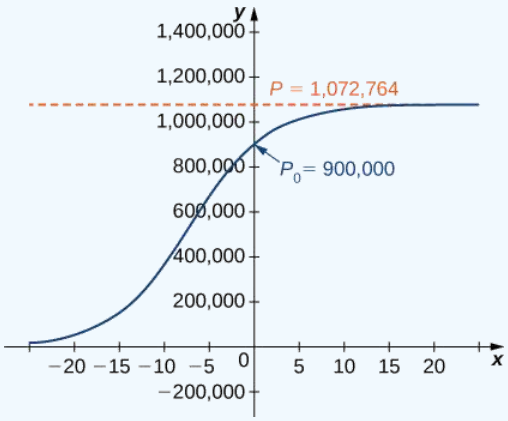


Figure 5.5.4: Logistic curve for the deer population with an initial population of **900,000** deer.

c. Using this model we can predict the population in **3** years.

$$P(3) = \frac{1,072,764e^{0.2311(3)}}{0.19196+e^{0.2311(3)}} \approx 978,830\, deer$$

This is far short of twice the initial population of $900,000$. Remember that the doubling time is based on the assumption that the growth rate never changes, but the logistic model takes this possibility into account.

d. If the population reached **1,200,000** deer, then the new initial-value problem would be

$$\frac{dP}{dt} = 0.2311P\left(1-\frac{P}{1,072,764}\right),\ P(0) = 1,200,000.$$

The general solution to the differential equation would remain the same.

$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1+C_2e^{0.2311t}}$$

To determine the value of the constant, return to the equation

$$\frac{P}{1,072,764-P} = C_2e^{0.2311t}.$$

Substituting the values $t = 0$ and $P = 1,200,000$, you get

$$C_2e^{0.2311(0)} = \frac{1,200,000}{1,072,764-1,200,000}$$
$$C_2 = -\frac{100,000}{10,603} \approx -9.431.$$

Therefore

$$P(t) = \frac{1,072,764C_2e^{0.2311t}}{1+C_2e^{0.2311t}}$$

$$= \frac{1,072,764\left(-\frac{100,000}{10,603}\right)e^{0.2311t}}{1+\left(-\frac{100,000}{10,603}\right)e^{0.2311t}}$$

$$= -\frac{107,276,400,000e^{0.2311t}}{100,000e^{0.2311t}-10,603}$$

$$\approx \frac{10,117,551e^{0.2311t}}{9.43129e^{0.2311t}-1}$$

This equation is graphed in Figure 5.5.5.

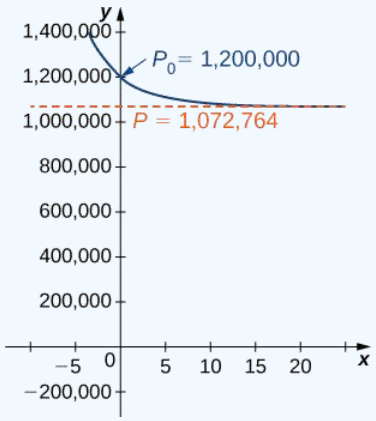


Figure 5.5.5: Logistic curve for the deer population with an initial population of **1,200,000** deer.

Solving the Logistic Differential Equation

The logistic differential equation is an autonomous differential equation, so we can use separation of variables to find the general solution, as we just did in Example 5.5.1.

Step 1: Setting the right-hand side equal to zero leads to $P = 0$ and $P = K$ as constant solutions. The first solution indicates that when there are no organisms present, the population will never grow. The second solution indicates that when the population starts at the carrying capacity, it will never change.

Step 2: Rewrite the differential equation in the form

$$\frac{dP}{dt} = \frac{rP(K-P)}{K}.$$

Then multiply both sides by dt and divide both sides by $P(K-P)$. This leads to

$$\frac{dP}{P(K-P)} = \frac{r}{K}dt.$$

Multiply both sides of the equation by K and integrate:

$$\int \frac{K}{P(K-P)}dP = \int rdt. \tag{5.5.2}$$

The left-hand side of this equation can be integrated using partial fraction decomposition. We leave it to you to verify that

$$\frac{K}{P(K-P)} = \frac{1}{P} + \frac{1}{K-P}.$$

Then the Equation 5.5.2 becomes

$$\int \frac{1}{P} + \frac{1}{K-P} dP = \int r dt$$

$$\ln|P| - \ln|K-P| = rt + C$$

$$\ln\left|\frac{P}{K-P}\right| = rt + C.$$

Now exponentiate both sides of the equation to eliminate the natural logarithm:

$$e^{\ln\left|\frac{P}{K-P}\right|} = e^{rt+C}$$

$$\left|\frac{P}{K-P}\right| = e^{C} e^{rt}.$$

We define $C_1 = e^c$ so that the equation becomes

$$\frac{P}{K-P} = C_1 e^{rt}. \tag{5.5.3}$$

To solve this equation for $P(t)$, first multiply both sides by $K-P$ and collect the terms containing P on the left-hand side of the equation:

$$P = C_1 e^{rt}(K-P)$$

$$= C_1 K e^{rt} - C_1 P e^{rt}$$

$$P + C_1 P e^{rt} = C_1 K e^{rt}.$$

Next, factor P from the left-hand side and divide both sides by the other factor:

$$P(1 + C_1 e^{rt}) = C_1 K e^{rt}$$

$$P(t) = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}}.$$

The last step is to determine the value of C_1. The easiest way to do this is to substitute $t = 0$ and P_0 in place of P in Equation and solve for C_1:

$$\frac{P}{K-P} = C_1 e^{rt}$$

$$\frac{P_0}{K-P_0} = C_1 e^{r(0)}$$

$$C_1 = \frac{P_0}{K-P_0}.$$

Finally, substitute the expression for C_1 into Equation 5.5.3:

$$P(t) = \frac{C_1 K e^{rt}}{1 + C_1 e^{rt}} = \frac{\frac{P_0}{K-P_0} K e^{rt}}{1 + \frac{P_0}{K-P_0} e^{rt}}$$

Now multiply the numerator and denominator of the right-hand side by $(K-P_0)$ and simplify:

$$P(t) = \frac{\frac{P_0}{K-P_0}Ke^{rt}}{1+\frac{P_0}{K-P_0}e^{rt}}$$

$$= \frac{\frac{P_0}{K-P_0}Ke^{rt}}{1+\frac{P_0}{K-P_0}e^{rt}} \cdot \frac{K-P_0}{K-P_0} = \frac{P_0Ke^{rt}}{(K-P_0)+P_0e^{rt}}.$$

We state this result as a theorem.

Solution of the Logistic Differential Equation

Consider the logistic differential equation subject to an initial population of P_0 with carrying capacity K and growth rate r. The solution to the corresponding initial-value problem is given by

$$P(t) = \frac{P_0Ke^{rt}}{(K-P_0)+P_0e^{rt}}$$

.

Now that we have the solution to the initial-value problem, we can choose values for P_0, r, and K and study the solution curve. For example, in Example we used the values $r = 0.2311, K = 1,072,764,$ and an initial population of $900,000$ deer. This leads to the solution

$$P(t) = \frac{P_0Ke^{rt}}{(K-P_0)+P_0e^{rt}}$$

$$= \frac{900,000(1,072,764)e^{0.2311t}}{(1,072,764-900,000)+900,000e^{0.2311t}}$$

$$= \frac{900,000(1,072,764)e^{0.2311t}}{172,764+900,000e^{0.2311t}}.$$

Dividing top and bottom by $900,000$ gives

$$P(t) = \frac{1,072,764e^{0.2311t}}{0.19196+e^{0.2311t}}.$$

This is the same as the original solution. The graph of this solution is shown again in blue in Figure $5.5.6$, superimposed over the graph of the exponential growth model with initial population $900,000$ and growth rate 0.2311 (appearing in green). The red dashed line represents the carrying capacity, and is a horizontal asymptote for the solution to the logistic equation.

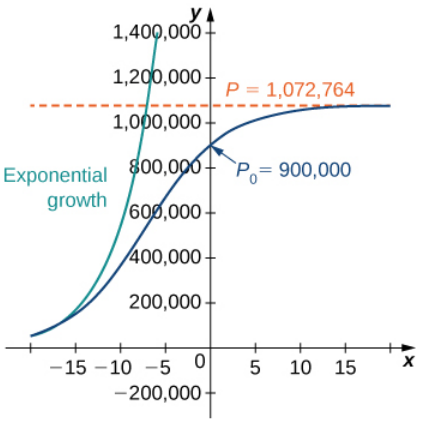


Figure $5.5.6$: A comparison of exponential versus logistic growth for the same initial population of $900,000$ organisms and growth rate of 23.11

Working under the assumption that the population grows according to the logistic differential equation, this graph predicts that approximately 20 years earlier (1984), the growth of the population was very close to exponential. The net growth rate at that time would have been around 23.1 per year. As time goes on, the two graphs separate. This happens because the population increases, and the logistic differential equation states that the growth rate decreases as the population increases. At the time the population was measured (2004), it was close to carrying capacity, and the population was starting to level off.

The solution to the logistic differential equation has a point of inflection. To find this point, set the second derivative equal to zero:

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$

$$P'(t) = \frac{rP_0 K(K - P0)e^{rt}}{((K - P_0) + P_0 e^{rt})^2}$$

$$\begin{aligned} P''(t) &= \frac{r^2 P_0 K(K - P_0)^2 e^{rt} - r^2 P_0^2 K(K - P_0) e^{2rt}}{((K - P_0) + P_0 e^{rt})^3} \\ &= \frac{r^2 P_0 K(K - P_0) e^{rt}((K - P_0) - P_0 e^{rt})}{((K - P_0) + P_0 e^{rt})^3}. \end{aligned}$$

Setting the numerator equal to zero,

$$r^2 P_0 K(K - P_0) e^{rt}((K - P_0) - P_0 e^{rt}) = 0.$$

As long as $P_0 \neq K$, the entire quantity before and including $e^{rt}$is nonzero, so we can divide it out:

$$(K - P_0) - P_0 e^{rt} = 0.$$

Solving for t,

$$P_0 e^{rt} = K - P_0$$

$$e^{rt} = \frac{K - P_0}{P_0}$$

$$\ln e^{rt} = \ln \frac{K - P_0}{P_0}$$

$$rt = \ln \frac{K - P_0}{P_0}$$

$$t = \frac{1}{r} \ln \frac{K - P_0}{P_0}.$$

Notice that if $P_0 > K$, then this quantity is undefined, and the graph does not have a point of inflection. In the logistic graph, the point of inflection can be seen as the point where the graph changes from concave up to concave down. This is where the "leveling off" starts to occur, because the net growth rate becomes slower as the population starts to approach the carrying capacity.

? Exercise 5.5.1

A population of rabbits in a meadow is observed to be 200 rabbits at time $t = 0$. After a month, the rabbit population is observed to have increased by 4. Using an initial population of 200 and a growth rate of 0.04, with a carrying capacity of 750 rabbits,

a. Write the logistic differential equation and initial condition for this model.
b. Draw a slope field for this logistic differential equation, and sketch the solution corresponding to an initial population of 200 rabbits.
c. Solve the initial-value problem for $P(t)$.
d. Use the solution to predict the population after 1 year.

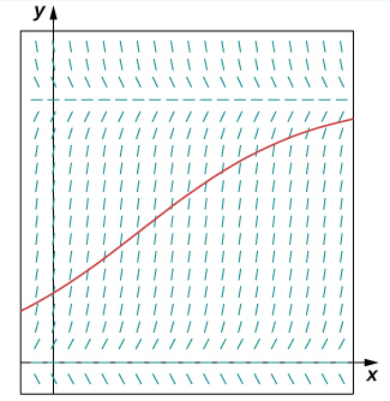
Hint

First determine the values of $r, K,$ and P_0. Then create the initial-value problem, draw the direction field, and solve the problem.

Answer

a. $\frac{dP}{dt} = 0.04(1 - \frac{P}{750}), P(0) = 200$

b.



c. $P(t) = \frac{3000e^{.04t}}{11 + 4e^{.04t}}$

d. After 12 months, the population will be $P(12) \approx 278$ rabbits.

Student Project: Logistic Equation with a Threshold Population

An improvement to the logistic model includes a **threshold population.** The threshold population is defined to be the minimum population that is necessary for the species to survive. We use the variable T to represent the threshold population. A differential equation that incorporates both the threshold population T and carrying capacity K is

$$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$$

where r represents the growth rate, as before.

1. The threshold population is useful to biologists and can be utilized to determine whether a given species should be placed on the endangered list. A group of Australian researchers say they have determined the threshold population for any species to survive: 5000 adults. (Catherine Clabby, “A Magic Number,” American Scientist 98(1): 24, doi:10.1511/2010.82.24. accessed April 9, 2015, www.americanscientist.org/iss...a-magic-number). Therefore we use $T = 5000$ as the threshold population in this project. Suppose that the environmental carrying capacity in Montana for elk is $25,000$. Set up Equation using the carrying capacity of $25,000$ and threshold population of 5000. Assume an annual net growth rate of 18%.
2. Draw the direction field for the differential equation from step 1, along with several solutions for different initial populations. What are the constant solutions of the differential equation? What do these solutions correspond to in the original population model (i.e., in a biological context)?
3. What is the limiting population for each initial population you chose in step 2? (Hint: use the slope field to see what happens for various initial populations, i.e., look for the horizontal asymptotes of your solutions.)
4. This equation can be solved using the method of separation of variables. However, it is very difficult to get the solution as an explicit function of t. Using an initial population of $18,000$ elk, solve the initial-value problem and express the solution as an implicit function of t, or solve the general initial-value problem, finding a solution in terms of $r, K, T,$ and P_0.

Key Concepts

- When studying population functions, different assumptions—such as exponential growth, logistic growth, or threshold population—lead to different rates of growth.

- The logistic differential equation incorporates the concept of a carrying capacity. This value is a limiting value on the population for any given environment.
- The logistic differential equation can be solved for any positive growth rate, initial population, and carrying capacity.

Key Equations

- **Logistic differential equation and initial-value problem**

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right), \quad P(0) = P_0$$

- **Solution to the logistic differential equation/initial-value problem**

$$P(t) = \frac{P_0 K e^{rt}}{(K - P_0) + P_0 e^{rt}}$$

- **Threshold population model**

$$\frac{dP}{dt} = -rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{T}\right)$$

Glossary

carrying capacity
the maximum population of an organism that the environment can sustain indefinitely

growth rate
the constant $r > 0$ in the exponential growth function $P(t) = P_0 e^{rt}$

initial population
the population at time $t = 0$

logistic differential equation
a differential equation that incorporates the carrying capacity K and growth rate rr into a population model

phase line
a visual representation of the behavior of solutions to an autonomous differential equation subject to various initial conditions

threshold population
the minimum population that is necessary for a species to survive

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5.5E: Exercises for the Logistic Equation

Basic Logistic Model

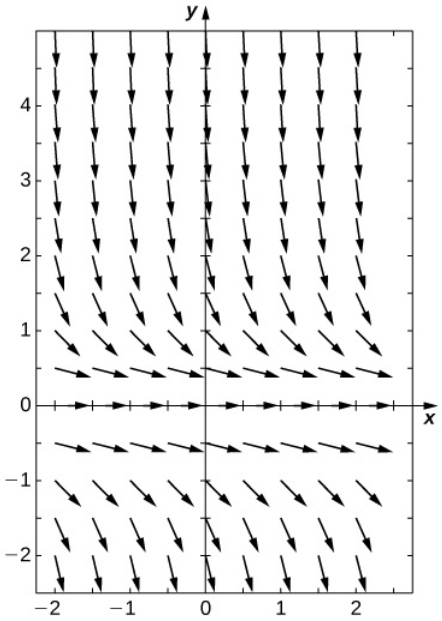
For problems 1 - 11, consider the logistic equation in the form $P' = CP - P^2$. Draw the directional field and find the stability of the equilibria.

1) $C = 3$

2) $C = 0$

Answer:

$P = 0$ semi-stable



3) $C = -3$

4) Solve the logistic equation for $C = 10$ and an initial condition of $P(0) = 2$.

Answer:

$$P = \frac{10e^{10x}}{e^{10x} + 4}$$

5) Solve the logistic equation for $C = -10$ and an initial condition of $P(0) = 2$.

6) A population of deer inside a park has a carrying capacity of 200 and a growth rate of 2. If the initial population is 50 deer, what is the population of deer at any given time?

Answer:

$$P(t) = \frac{10000e^{0.02t}}{150 + 50e^{0.02t}}$$

7) A population of frogs in a pond has a growth rate of 5 If the initial population is 1000 frogs and the carrying capacity is 6000, what is the population of frogs at any given time?

8) [T] Bacteria grow at a rate of 20 per hour in a petri dish. If there is initially one bacterium and a carrying capacity of 1 million cells, how long does it take to reach $500,000$ cells?

Answer:

69 hours 5 minutes

9) [T] Rabbits in a park have an initial population of 10 and grow at a rate of 4 per year. If the carrying capacity is 500, at what time does the population reach 100 rabbits?

10) [T] Two monkeys are placed on an island. After 5 years, there are 8 monkeys, and the estimated carrying capacity is 25 monkeys. When does the population of monkeys reach 16 monkeys?

Answer:

8 years 11 months

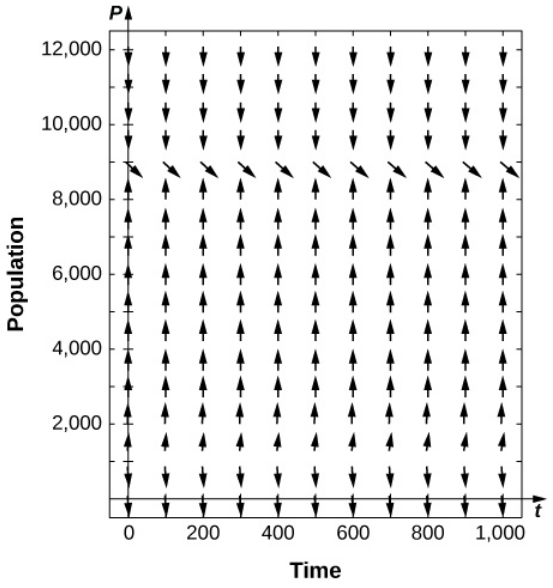
11) [T] A butterfly sanctuary is built that can hold 2000 butterflies, and 400 butterflies are initially moved in. If after 2 months there are now 800 butterflies, when does the population get to 1500 butterflies?

Logistic Population Model with Depletion

The following problems consider the logistic equation with an added term for depletion, either through death or emigration.

12) [T] The population of trout in a pond is given by $P' = 0.4P\left(1 - \frac{P}{10000}\right) - 400$, where 400 trout are caught per year. Use your calculator or computer software to draw a directional field and draw a few sample solutions. What do you expect for the behavior?

Answer:

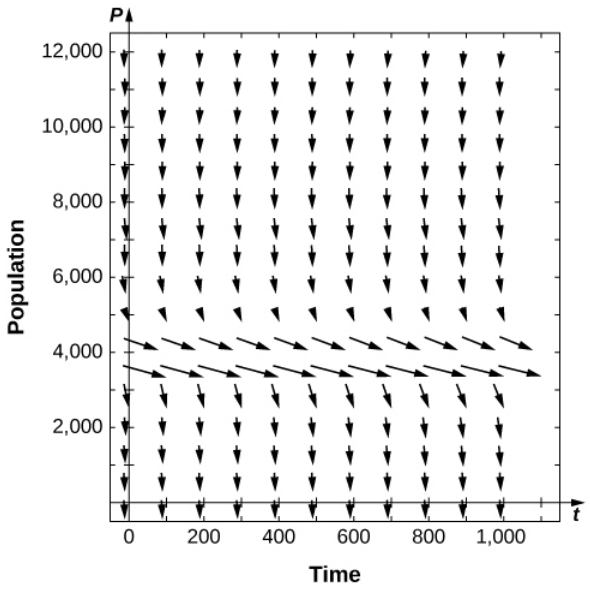


13) In the preceding problem, what are the stabilities of the equilibria $0 < P_1 < P_2$?

14) [T] For the preceding problem, use software to generate a directional field for the value $f = 400$. What are the stabilities of the equilibria?

Answer:

P_1 semi-stable

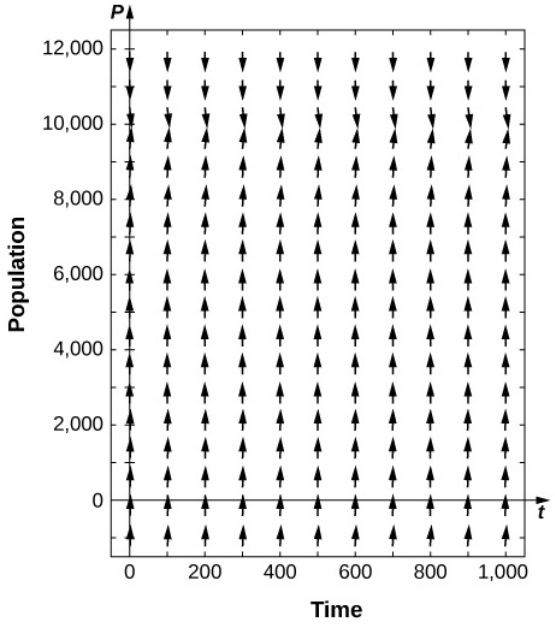


15) [T] For the preceding problems, use software to generate a directional field for the value $f = 600$. What are the stabilities of the equilibria?

16) [T] For the preceding problems, consider the case where a certain number of fish are added to the pond, or $f = -200$. What are the nonnegative equilibria and their stabilities?

Answer:

$P_2 > 0$ stable



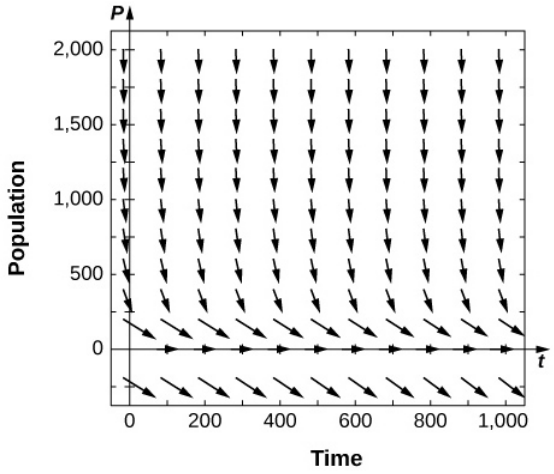
It is more likely that the amount of fishing is governed by the current number of fish present, so instead of a constant number of fish being caught, the rate is proportional to the current number of fish present, with proportionality constant k, as $P' = 0.4P\left(1 - \dfrac{P}{10000}\right) - kP$.

17) [T] For the previous fishing problem, draw a directional field assuming $k = 0.1$. Draw some solutions that exhibit this behavior. What are the equilibria and what are their stabilities?

18) [T] Use software or a calculator to draw directional fields for $k = 0.4$. What are the nonnegative equilibria and their stabilities?

Answer:

$P_1 = 0$ is semi-stable



19) [T] Use software or a calculator to draw directional fields for $k = 0.6$. What are the equilibria and their stabilities?

20) Solve this equation, assuming a value of $k = 0.05$ and an initial condition of 2000 fish.

Answer:

$$y = \frac{-20}{4 \times 10^{-6} - 0.002e^{0.01t}}$$

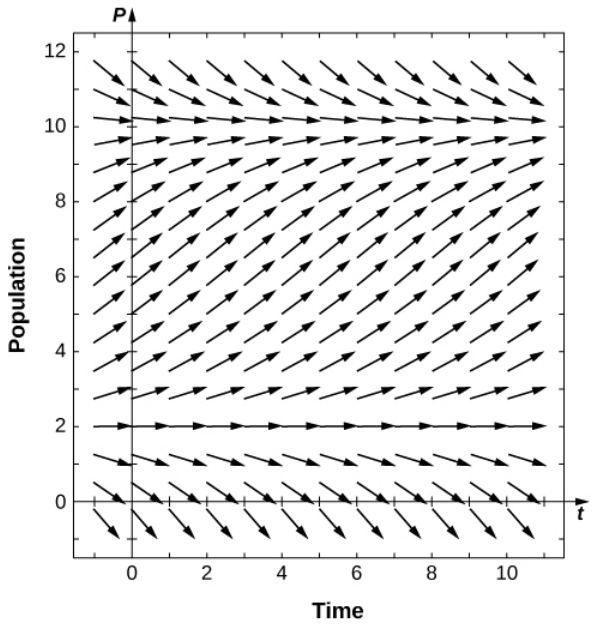
21) Solve this equation, assuming a value of $k = 0.05$ and an initial condition of 5000 fish.

Minimal Sustainable Population Thresholds

The following problems add in a minimal threshold value for the species to survive, T, which changes the differential equation to $P'(t) = rP\left(1 - \frac{P}{K}\right)\left(1 - \frac{T}{P}\right)$.

22) Draw the directional field of the threshold logistic equation, assuming $K = 10, r = 0.1, T = 2$. When does the population survive? When does it go extinct?

Answer:



23) For the preceding problem, solve the logistic threshold equation, assuming the initial condition $P(0) = P_0$.

24) Bengal tigers in a conservation park have a carrying capacity of 100 and need a minimum of 10 to survive. If they grow in population at a rate of 1 per year, with an initial population of 15 tigers, solve for the number of tigers present.

Answer:

$$P(t) = \frac{850 + 500e^{0.009t}}{85 + 5e^{0.009t}}$$

25) A forest containing ring-tailed lemurs in Madagascar has the potential to support 5000 individuals, and the lemur population grows at a rate of 5 per year. A minimum of 500 individuals is needed for the lemurs to survive. Given an initial population of 600 lemurs, solve for the population of lemurs.

26) The population of mountain lions in Northern Arizona has an estimated carrying capacity of 250 and grows at a rate of 0.25 per year and there must be 25 for the population to survive. With an initial population of 30 mountain lions, how many years will it take to get the mountain lions off the endangered species list (at least 100)?

Answer:

13 years months

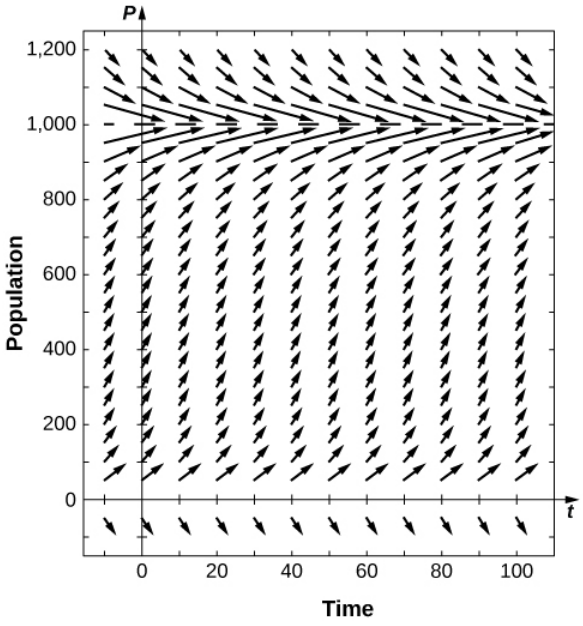
The Gompertz Equation

The following questions consider the Gompertz equation, a modification for logistic growth, which is often used for modeling cancer growth, specifically the number of tumor cells.

27) The Gompertz equation is given by $P(t)' = \alpha \ln\left(\frac{K}{P(t)}\right) P(t)$.Draw the directional fields for this equation assuming all parameters are positive, and given that $K = 1$.

28) Assume that for a population, $K = 1000$ and $\alpha = 0.05$. Draw the directional field associated with this differential equation and draw a few solutions. What is the behavior of the population?

Answer:



29) Solve the Gompertz equation for generic α and K and $P(0) = P_0$.

30) [T] The Gompertz equation has been used to model tumor growth in the human body. Starting from one tumor cell on day 1 and assuming $\alpha = 0.1$ and a carrying capacity of 10 million cells, how long does it take to reach "detection" stage at 5 million cells?

Answer:

31.465days

31) [T] It is estimated that the world human population reached 3 billion people in 1959 and 6 billion in 1999. Assuming a carrying capacity of 16 billion humans, write and solve the differential equation for logistic growth, and determine what year the population reached 7 billion.

32) [T] It is estimated that the world human population reached 3 billion people in 1959 and 6 billion in 1999. Assuming a carrying capacity of 16 billion humans, write and solve the differential equation for Gompertz growth, and determine what year the population reached 7 billion. Was logistic growth or Gompertz growth more accurate, considering world population reached 7 billion on October $31, 2011$?

Answer:

September 2008

33) Show that the population grows fastest when it reaches half the carrying capacity for the logistic equation $P' = rP\left(1 - \dfrac{P}{K}\right)$.

34) When does population increase the fastest in the threshold logistic equation $P'(t) = rP\left(1 - \dfrac{P}{K}\right)\left(1 - \dfrac{T}{P}\right)$?

Answer:

$\dfrac{K+T}{2}$

35) When does population increase the fastest for the Gompertz equation $P(t)' = \alpha \ln\left(\frac{K}{P(t)}\right)P(t)$?

Below is a table of the populations of whooping cranes in the wild from 1940 to 2000. The population rebounded from near extinction after conservation efforts began. The following problems consider applying population models to fit the data. Assume a

carrying capacity of $10,000$ cranes. Fit the data assuming years since 1940 (so your initial population at time 0 would be 22 cranes).

Year (years since conservation began)	Whooping Crane Population
1940(0)	22
1950(10)	31
1960(20)	36
1970(30)	57
1980(40)	91
1990(50)	159
2000(60)	256

Source: https://www.savingcranes.org/images/...wc_numbers.pdf

36) Find the equation and parameter r that best fit the data for the logistic equation.

Answer:

$r = 0.0405$

37) Find the equation and parameters r and T that best fit the data for the threshold logistic equation.

38) Find the equation and parameter α that best fit the data for the Gompertz equation.

Answer:

$\alpha = 0.0081$

39) Graph all three solutions and the data on the same graph. Which model appears to be most accurate?

40) Using the three equations found in the previous problems, estimate the population in 2010 (year 70 after conservation). The real population measured at that time was 437. Which model is most accurate?

Answer:

Logistic: 361, Threshold: 436, Gompertz: 309.

Contributors

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5.6: First-order Linear Equations

Learning Objectives

- Write a first-order linear differential equation in standard form.
- Find an integrating factor and use it to solve a first-order linear differential equation.
- Solve applied problems involving first-order linear differential equations.

Earlier, we studied an application of a first-order differential equation that involved solving for the velocity of an object. In particular, if a ball is thrown upward with an initial velocity of v_0 ft/s, then an initial-value problem that describes the velocity of the ball after t seconds is given by

$$\frac{dv}{dt} = -32$$

with $v(0) = v_0$.

This model assumes that the only force acting on the ball is gravity. Now we add to the problem by allowing for the possibility of **air resistance** acting on the ball.

Air resistance always acts in the direction opposite to motion. Therefore if an object is rising, air resistance acts in a downward direction. If the object is falling, air resistance acts in an upward direction (Figure 5.6.1). There is no exact relationship between the velocity of an object and the air resistance acting on it. For very small objects, air resistance is proportional to velocity; that is, the force due to air resistance is numerically equal to some constant k times v. For larger (e.g., baseball-sized) objects, depending on the shape, air resistance can be approximately proportional to the square of the velocity. In fact, air resistance may be proportional to $v^{1.5}$, or $v^{0.9}$, or some other power of v.

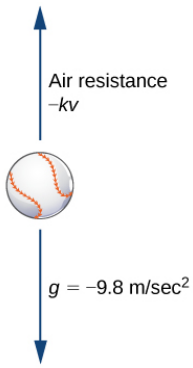


Figure 5.6.1: Forces acting on a moving baseball: gravity acts in a downward direction and air resistance acts in a direction opposite to the direction of motion.

We will work with the linear approximation for air resistance. If we assume $k > 0$, then the expression for the force F_A due to air resistance is given by $F_A = -kv$. Therefore the sum of the forces acting on the object is equal to the sum of the gravitational force and the force due to air resistance. This, in turn, is equal to the mass of the object multiplied by its acceleration at time t(Newton's second law). This gives us the differential equation

$$m\frac{dv}{dt} = -kv - mg.$$

Finally, we impose an initial condition $v(0) = v_0$, where v_0 is the initial velocity measured in meters per second. This makes $g = 9.8m/s^2$. The initial-value problem becomes

$$m\frac{dv}{dt} = -kv - mg$$

with $v(0) = v_0$.

The differential equation in this initial-value problem is an example of a first-order linear differential equation. (Recall that a differential equation is first-order if the highest-order derivative that appears in the equation is 1.) In this section, we study first-

order linear equations and examine a method for finding a general solution to these types of equations, as well as solving initial-value problems involving them.

Definition: Linear first-order differential equation

A first-order differential equation is *linear* if it can be written in the form

$$a(x)y'+b(x)y=c(x), \tag{5.6.1}$$

where $a(x), b(x),$ and $c(x)$ are arbitrary functions of x.

Remember that the unknown function y depends on the variable x; that is, x is the independent variable and y is the dependent variable. Some examples of first-order linear differential equations are

$$\begin{aligned}(3x^2-4)y'+(x-3)y &= \sin x\\ (\sin x)y'-(\cos x)y &= \cot x\\ 4xy'+(3\ln x)y &= x^3-4x.\end{aligned}$$

Examples of first-order nonlinear differential equations include

$$\begin{aligned}(y')^4-(y')^3 &= (3x-2)(y+4)\\ 4y'+3y^3 &= 4x-5\\ (y')^2 &= \sin y+\cos x.\end{aligned}$$

These equations are nonlinear because of terms like $(y')^4, y^3$, etc. Due to these terms, it is impossible to put these equations into the same form as Equation 5.6.1.

Standard Form

Consider the differential equation

$$(3x^2-4)y'+(x-3)y=\sin x.$$

Our main goal in this section is to derive a solution method for equations of this form. It is useful to have the coefficient of y' be equal to 1. To make this happen, we divide both sides by $3x^2-4$.

$$y'+\left(\frac{x-3}{3x^2-4}\right)y=\frac{\sin x}{3x^2-4}$$

This is called the **standard form** of the differential equation. We will use it later when finding the solution to a general first-order linear differential equation. Returning to Equation 5.6.1, we can divide both sides of the equation by $a(x)$. This leads to the equation

$$y'+\frac{b(x)}{a(x)}y=\frac{c(x)}{a(x)}. \tag{5.6.2}$$

Now define

$$p(x)=\frac{b(x)}{a(x)}$$

and

$$q(x)=\frac{c(x)}{a(x)}$$

Then Equation 5.6.2 becomes

$$y'+p(x)y=q(x).$$

We can write any first-order linear differential equation in this form, and this is referred to as the standard form for a first-order linear differential equation.

Example 5.6.1: Writing First-Order Linear Equations in Standard Form

Put each of the following first-order linear differential equations into standard form. Identify $p(x)$ and $q(x)$ for each equation.

a. $y' = 3x - 4y$

b. $\dfrac{3xy'}{4y-3} = 2$ (here $x > 0$)

c. $y = 3y' - 4x^2 + 5$

Solution

a. Add $4y$ to both sides:

$$y' + 4y = 3x.$$

In this equation, $p(x) = 4$ and $q(x) = 3x$.

b. Multiply both sides by $4y - 3$, then subtract $8y$ from each side:

$$\begin{aligned} \frac{3xy'}{4y-3} &= 2 \\ 3xy' &= 2(4y-3) \\ 3xy' &= 8y - 6 \\ 3xy' - 8y &= -6. \end{aligned}$$

Finally, divide both sides by $3x$ to make the coefficient of y' equal to 1:

$$y' - \frac{8}{3x}y = -\frac{2}{3x}.$$

This is allowable because in the original statement of this problem we assumed that $x > 0$. (If $x = 0$ then the original equation becomes $0 = 2$, which is clearly a false statement.)

In this equation, $p(x) = -\dfrac{8}{3x}$ and $q(x) = -\dfrac{2}{3x}$.

c. Subtract y from each side and add $4x^2 - 5$:

$$3y' - y = 4x^2 - 5.$$

Next divide both sides by 3:

$$y' - \frac{1}{3}y = \frac{4}{3}x^2 - \frac{5}{3}.$$

In this equation, $p(x) = -\dfrac{1}{3}$ and $q(x) = \dfrac{4}{3}x^2 - \dfrac{5}{3}$.

Exercise 5.6.1

Put the equation $\dfrac{(x+3)y'}{2x - 3y - 4} = 5$ into standard form and identify $p(x)$ and $q(x)$.

Hint

Multiply both sides by the common denominator, then collect all terms involving y on one side.

Answer

$$y' + \frac{15}{x+3}y = \frac{10x - 20}{x+3}$$

$$p(x) = \frac{15}{x+3}$$

and

$$q(x) = \frac{10x-20}{x+3}$$

Integrating Factors

We now develop a solution technique for any first-order linear differential equation. We start with the standard form of a first-order linear differential equation:

$$y' + p(x)y = q(x). \tag{5.6.3}$$

The first term on the left-hand side of Equation 5.6.3 is the derivative of the unknown function, and the second term is the product of a known function with the unknown function. This is somewhat reminiscent of the power rule. If we multiply Equation 5.6.3 by a yet-to-be-determined function $\mu(x)$, then the equation becomes

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{5.6.4}$$

The left-hand side Equation 5.6.4 can be matched perfectly to the product rule:

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

Matching term by term gives $y = f(x), g(x) = \mu(x)$, and $g'(x) = \mu(x)p(x)$. Taking the derivative of $g(x) = \mu(x)$ and setting it equal to the right-hand side of $g'(x) = \mu(x)p(x)$ leads to

$$\mu'(x) = \mu(x)p(x).$$

This is a first-order, separable differential equation for $\mu(x)$. We know $p(x)$ because it appears in the differential equation we are solving. Separating variables and integrating yields

$$\begin{aligned}
\frac{\mu'(x)}{\mu(x)} &= p(x) \\
\int \frac{\mu'(x)}{\mu(x)}dx &= \int p(x)dx \\
\ln|\mu(x)| &= \int p(x)dx + C \\
e^{\ln|\mu(x)|} &= e^{\int p(x)dx + C} \\
|\mu(x)| &= C_1 e^{\int p(x)dx} \\
\mu(x) &= C_2 e^{\int p(x)dx}.
\end{aligned}$$

Here C_2 can be an arbitrary (positive or negative) constant. This leads to a general method for solving a first-order linear differential equation. We first multiply both sides of Equation 5.6.3 by this integrating factor by **integrating factor** $\mu(x)$. This gives

$$\mu(x)y' + \mu(x)p(x)y = \mu(x)q(x). \tag{5.6.5}$$

Note that, since the non-zero constant C_2 can be divided out of this equation (as it will appear in every term on both sides), we do not need to use it in this process. That is, we can simply use

$$\mu(x) = e^{\int p(x)dx}.$$

The left-hand side of Equation 5.6.5 can be rewritten as $\frac{d}{dx}(\mu(x)y)$.

$$\frac{d}{dx}(\mu(x)y) = \mu(x)q(x). \tag{5.6.6}$$

Next integrate both sides of Equation 5.6.6 with respect to x.

$$\int \frac{d}{dx}(\mu(x)y)dx = \int \mu(x)q(x)dx$$

$$\mu(x)y = \int \mu(x)q(x)dx$$

Divide both sides of Equation 5.6.6 by $\mu(x)$:

$$y = \frac{1}{\mu(x)}\left[\int \mu(x)q(x)dx + C\right]. \tag{5.6.7}$$

Since $\mu(x)$ was previously calculated, we are now finished. An important note about the integrating constant C: It may seem that we are inconsistent in the usage of the integrating constant. However, the integral involving $p(x)$ is necessary in order to find an integrating factor for Equation 5.6.3. Only one integrating factor is needed in order to solve the equation; therefore, it is safe to assign a value for C for this integral. We chose $C = 0$. When calculating the integral inside the brackets in Equation 5.6.7, it is necessary to keep our options open for the value of the integrating constant, because our goal is to find a general family of solutions to Equation 5.6.3. This integrating factor guarantees just that.

Problem-Solving Strategy: Solving a First-order Linear Differential Equation

1. Put the equation into standard form and identify $p(x)$ and $q(x)$.
2. Calculate the integrating factor

$$\mu(x) = e^{\int p(x)dx}. \tag{5.6.8}$$

3. Multiply both sides of the differential equation by $\mu(x)$.
4. Integrate both sides of the equation obtained in step 3, and divide both sides by $\mu(x)$.
5. If there is an initial condition, determine the value of C.

Example 5.6.2: Solving a First-order Linear Equation

Find a general solution for the differential equation $xy' + 3y = 4x^2 - 3x$. Assume $x > 0$.

Solution

1. To put this differential equation into standard form, divide both sides by x:

$$y' + \frac{3}{x}y = 4x - 3.$$

Therefore $p(x) = \dfrac{3}{x}$ and $q(x) = 4x - 3$.

2. The integrating factor is $\mu(x) = e^{\int (3/x)dx} = e^{3\ln x} = x^3$.

3. Multiplying both sides of the differential equation by $\mu(x)$ gives us

$$\begin{aligned} x^3y' + x^3\left(\frac{3}{x}\right)y &= x^3(4x-3) \\ x^3y' + 3x^2y &= 4x^4 - 3x^3 \\ \frac{d}{dx}(x^3y) &= 4x^4 - 3x^3. \end{aligned}$$

4. Integrate both sides of the equation.

$$\int \frac{d}{dx}(x^3 y)dx = \int 4x^4 - 3x^3 dx$$

$$x^3 y = \frac{4x^5}{5} - \frac{3x^4}{4} + C$$

$$y = \frac{4x^2}{5} - \frac{3x}{4} + Cx^{-3}.$$

5. There is no initial value, so the problem is complete.

Analysis

You may have noticed the condition that was imposed on the differential equation; namely, $x > 0$. For any nonzero value of C, the general solution is not defined at $x = 0$. Furthermore, when $x < 0$, the integrating factor changes. The integrating factor is given by Equation 5.6.8 as $\mu(x) = e^{\int p(x)dx}$. For this $p(x)$ we get

$$e^{\int p(x)dx} = e^{\int (3/x)dx}$$

$$= e^{3\ln|x|}$$

$$= |x|^3$$

since $x < 0$. The behavior of the general solution changes at $x = 0$ largely due to the fact that $p(x)$ is not defined there.

Exercise 5.6.2

Find the general solution to the differential equation $(x-2)y' + y = 3x^2 + 2x$. Assume $x > 2$.

Hint

Use the method outlined in the problem-solving strategy for first-order linear differential equations.

Answer

$$y = \frac{x^3 + x^2 + C}{x - 2}$$

Now we use the same strategy to find the solution to an initial-value problem.

Example 5.6.3: A First-order Linear Initial-Value Problem

Solve the initial-value problem

$$y' + 3y = 2x - 1, y(0) = 3.$$

Solution

1. This differential equation is already in standard form with $p(x) = 3$ and $q(x) = 2x - 1$.

2. The integrating factor is $\mu(x) = e^{\int 3dx} = e^{3x}$.

3. Multiplying both sides of the differential equation by $\mu(x)$ gives

$$e^{3x}y' + 3e^{3x}y = (2x-1)e^{3x}$$

$$\frac{d}{dx}[ye^{3x}] = (2x-1)e^{3x}.$$

Integrate both sides of the equation:

$$\int \frac{d}{dx}[ye^{3x}]dx = \int (2x-1)e^{3x}dx$$

$$ye^{3x} = \frac{e^{3x}}{3}(2x-1) - \int \frac{2}{3}e^{3x}dx$$

$$ye^{3x} = \frac{e^{3x}(2x-1)}{3} - \frac{2e^{3x}}{9} + C$$

$$y = \frac{2x-1}{3} - \frac{2}{9} + Ce^{-3x}$$

$$y = \frac{2x}{3} - \frac{5}{9} + Ce^{-3x}.$$

4. Now substitute $x = 0$ and $y = 3$ into the general solution and solve for C:

$$y = \frac{2}{3}x - \frac{5}{9} + Ce^{-3x}$$

$$3 = \frac{2}{3}(0) - \frac{5}{9} + Ce^{-3(0)}$$

$$3 = -\frac{5}{9} + C$$

$$C = \frac{32}{9}.$$

Therefore the solution to the initial-value problem is

$$y = \frac{2}{3}x - \frac{5}{9} + \frac{32}{9}e^{-3x}.$$

Example 5.6.4:

Solve the initial-value problem

$$y' - 2y = 4x + 3 y(0) = -2.$$

Solution

$$y = -2x - 4 + 2e^{2x}$$

Applications of First-order Linear Differential Equations

We look at two different applications of first-order linear differential equations. The first involves air resistance as it relates to objects that are rising or falling; the second involves an electrical circuit. Other applications are numerous, but most are solved in a similar fashion.

Free fall with air resistance

We discussed air resistance at the beginning of this section. The next example shows how to apply this concept for a ball in vertical motion. Other factors can affect the force of air resistance, such as the size and shape of the object, but we ignore them here.

Example 5.6.5: A Ball with Air Resistance

A racquetball is hit straight upward with an initial velocity of $2\ \text{m/s}$. The mass of a racquetball is approximately $0.0427\ \text{kg}$. Air resistance acts on the ball with a force numerically equal to $0.5v$, where v represents the velocity of the ball at time t.

a. Find the velocity of the ball as a function of time.
b. How long does it take for the ball to reach its maximum height?
c. If the ball is hit from an initial height of 1 meter, how high will it reach?

Solution

a. The mass $m = 0.0427\ kg, k = 0.5,$ and $g = 9.8\ m/s^2$. The initial velocity is $v_0 = 2\ m/s$. Therefore the initial-value problem is

$$0.0427\frac{dv}{dt} = -0.5v - 0.0427(9.8), v_0 = 2.$$

Dividing the differential equation by 0.0427 gives

$$\frac{dv}{dt} = -11.7096v - 9.8, v_0 = 2.$$

The differential equation is linear. Using the problem-solving strategy for linear differential equations:

Step 1. Rewrite the differential equation as $\frac{dv}{dt} + 11.7096v = -9.8$. This gives $p(t) = 11.7096$ and $q(t) = -9.8$

Step 2. The integrating factor is $\mu(t) = e^{\int 11.7096dt} = e^{11.7096t}$.

Step 3. Multiply the differential equation by $\mu(t)$:

$$e^{11.7096t}\frac{dv}{dt} + 11.7096ve^{11.7096t} = -9.8e^{11.7096t}$$

$$\frac{d}{dt}[ve^{11.7096t}] = -9.8e^{11.7096t}.$$

Step 4. Integrate both sides:

$$\int \frac{d}{dt}[ve^{11.7096t}]dt = \int -9.8e^{11.7096t}dt$$

$$ve^{11.7096t} = \frac{-9.8}{11.7096}e^{11.7096t} + C$$

$$v(t) = -0.8369 + Ce^{-11.7096t}.$$

Step 5. Solve for C using the initial condition $v_0 = v(0) = 2$:

$$v(t) = -0.8369 + Ce^{-11.7096t}$$

$$v(0) = -0.8369 + Ce^{-11.7096(0)}$$

$$2 = -0.8369 + C$$

$$C = 2.8369.$$

Therefore the solution to the initial-value problem is

$$v(t) = 2.8369e^{-11.7096t} - 0.8369.$$

b. The ball reaches its maximum height when the velocity is equal to zero. The reason is that when the velocity is positive, it is rising, and when it is negative, it is falling. Therefore when it is zero, it is neither rising nor falling, and is at its maximum height:

$$2.8369e^{-11.7096t} - 0.8369 = 0$$

$$2.8369e^{-11.7096t} = 0.8369$$

$$e^{-11.7096t} = \frac{0.8369}{2.8369} \approx 0.295$$

$$\ln e^{-11.7096t} = \ln 0.295 \approx -1.221$$

$$-11.7096t = -1.221$$

$$t \approx 0.104.$$

Therefore it takes approximately 0.104 second to reach maximum height.

c. To find the height of the ball as a function of time, use the fact that the derivative of position is velocity, i.e., if $h(t)$ represents the height at time t, then $h'(t) = v(t)$. Because we know $v(t)$ and the initial height, we can form an initial-value problem:

$$h'(t) = 2.8369e^{-11.7096t} - 0.8369,\ h(0) = 1.$$

Integrating both sides of the differential equation with respect to t gives

$$\begin{aligned}\int h'(t)dt &= \int 2.8369e^{-11.7096t} - 0.8369\ dt \\ h(t) &= -\frac{2.8369}{11.7096}e^{-11.7096t} - 0.8369t + C \\ h(t) &= -0.2423e^{-11.7096t} - 0.8369t + C.\end{aligned}$$

Solve for C by using the initial condition:

$$\begin{aligned}h(t) &= -0.2423e^{-11.7096t} - 0.8369t + C \\ h(0) &= -0.2423e^{-11.7096(0)} - 0.8369(0) + C \\ 1 &= -0.2423 + C \\ C &= 1.2423.\end{aligned}$$

Therefore

$$h(t) = -0.2423e^{-11.7096t} - 0.8369t + 1.2423.$$

After 0.104 second, the height is given by

$$\begin{aligned}h(0.2) &= -0.2423e^{-11.7096t} - 0.8369t + 1.2423 \\ &\approx 1.0836 \text{ meter.}\end{aligned}$$

? Exercise 5.6.3

The weight of a penny is 2.5 grams (United States Mint, "Coin Specifications," accessed April 9, 2015, www.usmint.gov/about_the_mint...specifications), and the upper observation deck of the Empire State Building is 369 meters above the street. Since the penny is a small and relatively smooth object, air resistance acting on the penny is actually quite small. We assume the air resistance is numerically equal to $0.0025u$ Furthermore, the penny is dropped with no initial velocity imparted to it.

a. Set up an initial-value problem that represents the falling penny.
b. Solve the problem for $v(t)$.
c. What is the terminal velocity of the penny (i.e., calculate the limit of the velocity as t approaches infinity)?

Hint

Set up the differential equation the same way as Example ? 5.6.5 Remember to convert from grams to kilograms.

Answer

a. $\dfrac{dv}{dt} = -v - 9.8\ \ v(0) = 0$

b. $v(t) = 9.8(e^{-t} - 1)$

c. $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty}(9.8(e^{-t} - 1)) = -9.8 \text{ m/s} \approx -21.922 \text{ mph}$

Electrical Circuits

A source of electromotive force (e.g., a battery or generator) produces a flow of current in a closed circuit, and this current produces a voltage drop across each resistor, inductor, and capacitor in the circuit. Kirchhoff's Loop Rule states that the sum of the voltage drops across resistors, inductors, and capacitors is equal to the total electromotive force in a closed circuit. We have the following three results:

1. The voltage drop across a resistor is given by

$$E_R = Ri,$$

where R is a constant of proportionality called the resistance, and i is the current.

2. The voltage drop across an inductor is given by

$$E_L = Li',$$

where L is a constant of proportionality called the inductance, and i again denotes the current.

3. The voltage drop across a capacitor is given by

$$E_C = \frac{1}{C}q,$$

where C is a constant of proportionality called the capacitance, and q is the instantaneous charge on the capacitor. The relationship between i and q is $i = q'$.

We use units of volts (V) to measure voltage E, amperes (A) to measure current i, coulombs (C) to measure charge q, ohms (Ω) to measure resistance R, henrys (H) to measure inductance L, and farads (F) to measure capacitance C. Consider the circuit in Figure 5.6.2.

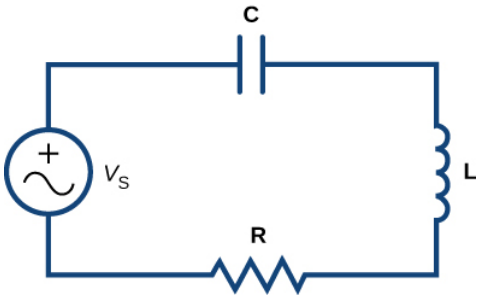


Figure 5.6.2: A typical electric circuit, containing a voltage generator (V_S), capacitor (C), inductor (L), and resistor (R).

Applying Kirchhoff's Loop Rule to this circuit, we let E denote the electromotive force supplied by the voltage generator. Then

$$E_L + E_R + E_C = E \ .$$

Substituting the expressions for E_L, E_R, and E_C into this equation, we obtain

$$Li' + Ri + \frac{1}{C}q = E. \tag{5.6.9}$$

If there is no capacitor in the circuit, then the equation becomes

$$Li' + Ri = E. \tag{5.6.10}$$

This is a first-order differential equation in i. The circuit is referred to as an **LR circuit**.

Next, suppose there is no inductor in the circuit, but there is a capacitor and a resistor, so $L = 0,\ R \neq 0,$ and $C \neq 0$. Then, since $q' = i$, Equation 5.6.9 can be rewritten as

$$Rq' + \frac{1}{C}q = E, \tag{5.6.11}$$

which is a first-order linear differential equation in q. This is referred to as an **RC circuit**. In either case, we can set up and solve an initial-value problem.

Electric Circuit

A circuit has in series an electromotive force given by $E = 50\sin 20tV$, a resistor of 5Ω, and an inductor of $0.4H$. If the initial current is 0, find the current at time $t > 0$.

Solution

We have a resistor and an inductor in the circuit, so we have an LR circuit and can use Equation 5.6.10. The voltage drop across the resistor is given by $E_R = Ri = 5i$. The voltage drop across the inductor is given by $E_L = Li' = 0.4i'$. The electromotive force becomes the right-hand side of Equation 5.6.10. Therefore Equation 5.6.10 becomes

$$0.4i' + 5i = 50\sin 20t.$$

Dividing both sides by 0.4 gives the equation

$$i' + 12.5i = 125\sin 20t.$$

Since the initial current is 0, this result gives an initial condition of $i(0) = 0$. We can solve this initial-value problem using the five-step strategy for solving first-order differential equations.

Step 1. Rewrite the differential equation as $i' + 12.5i = 125\sin 20t$. This gives $p(t) = 12.5$ and $q(t) = 125\sin 20t$.

Step 2. The integrating factor is $\mu(t) = e^{\int 12.5dt} = e^{12.5t}$.

Step 3. Multiply the differential equation by $\mu(t)$:

$$\begin{aligned} e^{12.5t}i' + 12.5e^{12.5t}i &= 125e^{12.5t}\sin 20t \\ \frac{d}{dt}[ie^{12.5t}] &= 125e^{12.5t}\sin 20t. \end{aligned}$$

Step 4. Integrate both sides:

$$\begin{aligned} \int \frac{d}{dt}[ie^{12.5t}]dt &= \int 125e^{12.5t}\sin 20t\, dt \\ ie^{12.5t} &= \left(\frac{250\sin 20t - 400\cos 20t}{89}\right)e^{12.5t} + C \\ i(t) &= \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t}. \end{aligned}$$

Step 5. Solve for C using the initial condition $i(0) = 0$:

$$\begin{aligned} i(t) &= \frac{250\sin 20t - 400\cos 20t}{89} + Ce^{-12.5t} \\ i(0) &= \frac{250\sin 20(0) - 400\cos 20(0)}{89} + Ce^{-12.5(0)} \\ 0 &= -\frac{400}{89} + C \\ C &= \frac{400}{89}. \end{aligned}$$

Therefore the solution to the initial-value problem is

$$i(t) = \frac{250\sin 20t - 400\cos 20t + 400e^{-12.5t}}{89} = \frac{250\sin 20t - 400\cos 20t}{89} + \frac{400e^{-12.5t}}{89}.$$

The first term can be rewritten as a single cosine function. First, multiply and divide by $\sqrt{250^2 + 400^2} = 50\sqrt{89}$:

$$\frac{250\sin 20t - 400\cos 20t}{89} = \frac{50\sqrt{89}}{89}\left(\frac{250\sin 20t - 400\cos 20t}{50\sqrt{89}}\right) = -\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}} - \frac{5\sin 20t}{\sqrt{89}}\right).$$

Next, define φ to be an acute angle such that $\cos\varphi = \frac{8}{\sqrt{89}}$. Then $\sin\varphi = \frac{5}{\sqrt{89}}$ and

$$-\frac{50\sqrt{89}}{89}\left(\frac{8\cos 20t}{\sqrt{89}}-\frac{5\sin 20t}{\sqrt{89}}\right)=-\frac{50\sqrt{89}}{89}(\cos\varphi\cos 20t-\sin\varphi\sin 20t)=-\frac{50\sqrt{89}}{89}\cos(20t+\varphi).$$

Therefore the solution can be written as

$$i(t)=-\frac{50\sqrt{89}}{89}cos(20t+\varphi)+\frac{400e^{-12.5t}}{89}\ .$$

The second term is called the attenuation term, because it disappears rapidly as t grows larger. The phase shift is given by φ, and the amplitude of the steady-state current is given by $\frac{50\sqrt{89}}{89}$. The graph of this solution appears in Figure 5.6.3:

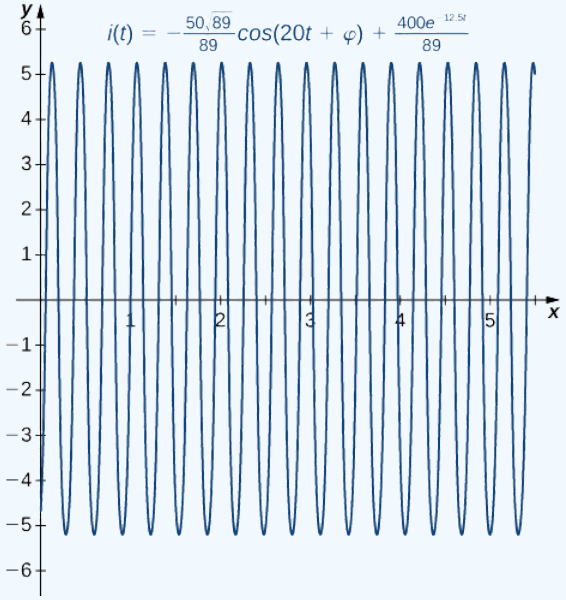


Figure 5.6.3.

? Exercise 5.6.4

A circuit has in series an electromotive force given by $E=20\sin 5t$ V, a capacitor with capacitance $0.02F$, and a resistor of 8Ω. If the initial charge is $4C$, find the charge at time $t>0$.

Hint

Use Equation 5.6.11 for an RC circuit to set up an initial-value problem.

Answer

Initial-value problem:

$$8q'+\frac{1}{0.02}q=20\sin 5t, q(0)=4$$

$$q(t)=\frac{10\sin 5t-8\cos 5t+172e^{-6.25t}}{41}$$

Key Concepts

- Any first-order linear differential equation can be written in the form $y'+p(x)y=q(x)$.
- We can use a five-step problem-solving strategy for solving a first-order linear differential equation that may or may not include an initial value.
- Applications of first-order linear differential equations include determining motion of a rising or falling object with air resistance and finding current in an electrical circuit.

Key Equations

- **standard form**

  $$y' + p(x)y = q(x)$$

- **integrating factor**

  $$\mu(x) = e^{\int p(x)dx}$$

Glossary

integrating factor
any function $f(x)$ that is multiplied on both sides of a differential equation to make the side involving the unknown function equal to the derivative of a product of two functions

linear
description of a first-order differential equation that can be written in the form $a(x)y' + b(x)y = c(x)$

standard form
the form of a first-order linear differential equation obtained by writing the differential equation in the form $y' + p(x)y = q(x)$

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5.6E: Exercises for Section 8.5

In exercises 1 - 5, state if each of the following differential equations is linear? Explain your reasoning.

1) $\frac{dy}{dx} = x^2 y + \sin x$

2) $\frac{dy}{dt} = ty$

Answer

Yes

3) $\frac{dy}{dt} + y^2 = x$

4) $y' = x^3 + e^x$

Answer

Yes

5) $y' = y + e^y$

In exercises 6 - 10, write the following first-order differential equations in standard form.

6) $y' = x^3 y + \sin x$

Answer

$y' - x^3 y = \sin x$

7) $y' + 3y - \ln x = 0$

8) $-xy' = (3x + 2)y + xe^x$

Answer

$y' + \frac{(3x+2)}{x} y = -e^x$

9) $\frac{dy}{dt} = 4y + ty + \tan t$

10) $\frac{dy}{dt} = yx(x + 1)$

Answer

$\frac{dy}{dt} - yx(x + 1) = 0$

In exercises 11 - 15, state the integrating factors for each of the following differential equations.

11) $y' = xy + 3$

12) $y' + e^x y = \sin x$

Answer

e^x

13) $y' = x \ln(x) y + 3x$

14) $\frac{dy}{dx} = \tanh(x) y + 1$

Answer

$-\ln(\cosh x)$

15) $\frac{dy}{dt}+3ty=e^t y$

In exercises 16 - 25, solve each differential equation by using integrating factors.

16) $y'=3y+2$

Answer

$y=Ce^{3x}-\frac{2}{3}$

17) $y'=2y-x^2$

18) $xy'=3y-6x^2$

Answer

$y=Cx^3+6x^2$

19) $(x+2)y'=3x+y$

20) $y'=3x+xy$

Answer

$y=Ce^{x^2/2}-3$

21) $xy'=x+y$

22) $\sin(x)y'=y+2x$

Answer

$y=C\tan\left(\frac{x}{2}\right)-2x+4\tan\left(\frac{x}{2}\right)\ln\left(\sin(\frac{x}{2})\right)$

23) $y'=y+e^x$

24) $xy'=3y+x^2$

Answer

$y=Cx^3-x^2$

25) $y'+\ln x=\frac{y}{x}$

In exercises 26 - 33, solve the given differential equation. Use your calculator to draw a family of solutions. Are there certain initial conditions that change the behavior of the solution?

26) [T] $(x+2)y'=2y-1$

Answer

$y=C(x+2)^2+\frac{1}{2}$

27) [T] $y'=3e^{t/3}-2y$

28) [T] $xy'+\frac{y}{2}=\sin(3t)$

Answer

$y=\frac{C}{\sqrt{x}}+2\sin(3t)$

29) [T] $xy'=2\frac{\cos x}{x}-3y$

30) [T] $(x+1)y'=3y+x^2+2x+1$

Answer

$y = C(x+1)^3 - x^2 - 2x - 1$

31) [T] $\sin(x)y' + \cos(x)y = 2x$

32) [T] $\sqrt{x^2+1}\,y' = y + 2$

Answer

$y = Ce^{\sinh^{-1} x} - 2$

33) [T] $x^3y' + 2x^2y = x + 1$

In exercises 34 - 43, solve each initial-value problem by using integrating factors.

34) $y' + y = x, \quad y(0) = 3$

Answer

$y = x + 4e^x - 1$

35) $y' = y + 2x^2, \quad y(0) = 0$

36) $xy' = y - 3x^3, \quad y(1) = 0$

Answer

$y = -\dfrac{3x}{2}(x^2 - 1)$

37) $x^2y' = xy - \ln x, \quad y(1) = 1$

38) $(1+x^2)y' = y - 1, \quad y(0) = 0$

Answer

$y = 1 - e^{\tan^{-1} x}$

39) $xy' = y + 2x\ln x, \quad y(1) = 5$

40) $(2+x)y' = y + 2 + x, \quad y(0) = 0$

Answer

$y = (x+2)\ln\left(\dfrac{x+2}{2}\right)$

41) $y' = xy + 2xe^x, \quad y(0) = 2$

42) $\sqrt{x}y' = y + 2x, \quad y(0) = 1$

Answer

$y = 2e^{2\sqrt{x}} - 2x - 2\sqrt{x} - 1$

43) $y' = 2y + xe^x, \quad y(0) = -1$

44) A falling object of mass m can reach terminal velocity when the drag force is proportional to its velocity, with proportionality constant k. Set up the differential equation and solve for the velocity given an initial velocity of 0.

Answer

$v(t) = \dfrac{gm}{k}\left(1 - e^{-kt/m}\right)$

45) Using your expression from the preceding problem, what is the terminal velocity? (Hint: Examine the limiting behavior; does the velocity approach a value?)

46) [T] Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is $9.8\,\text{m/s}^2$ and the proportionality constant is 4?

Answer

40.451seconds

47) A more accurate way to describe terminal velocity is that the drag force is proportional to the square of velocity, with a proportionality constant k. Set up the differential equation and solve for the velocity.

48) Using your expression from the preceding problem, what is the terminal velocity? (Hint: Examine the limiting behavior: Does the velocity approach a value?)

Answer

$\sqrt{\frac{gm}{k}}$

49) [T] Using your equation for terminal velocity, solve for the distance fallen. How long does it take to fall 5000 meters if the mass is 100 kilograms, the acceleration due to gravity is $9.8\,\text{m/s}^2$ and the proportionality constant is 4? Does it take more or less time than your initial estimate?

In exercises 50 - 54, determine how parameter a affects the solution.

50) Solve the generic equation $y' = ax + y$. How does varying a change the behavior?

Answer

$y = Ce^x - a(x+1)$

51) Solve the generic equation $y' = ax + y$. How does varying a change the behavior?

52) Solve the generic equation $y' = ax + xy$. How does varying a change the behavior?

Answer

$y = Ce^{x^2/2} - a$

53) Solve the generic equation $y' = x + axy$. How does varying a change the behavior?

54) Solve $y' - y = e^{kt}$ with the initial condition $y(0) = 0$. As k approaches 1, what happens to your formula?

Answer

$y = \frac{e^{kt} - e^t}{k-1}$

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5.7: Chapter 8 Review Exercises

True or False? Justify your answer with a proof or a counterexample.

1) The differential equation $y' = 3x^2y - \cos(x)y''$ is linear.

2) The differential equation $y' = x - y$ is separable.

Answer

F

3) You can explicitly solve all first-order differential equations by separation or by the method of integrating factors.

4) You can determine the behavior of all first-order differential equations using directional fields or Euler's method.

Answer

T

For the following problems, find the general solution to the differential equations.

5) $y' = x^2 + 3e^x - 2x$

6) $y' = 2^x + \cos^{-1} x$

Answer

$y(x) = \frac{2^x}{ln(2)} + xcos^{-1}x - \sqrt{1-x^2} + C$

7) $y' = y(x^2 + 1)$

8) $y' = e^{-y} \sin x$

Answer

$y(x) = \ln(C - \cos x)$

9) $y' = 3x - 2y$

10) $y' = y \ln y$

Answer

$y(x) = e^{e^{C+x}}$

For the following problems, find the solution to the initial value problem.

11) $y' = 8x - \ln x - 3x^4, \quad y(1) = 5$

12) $y' = 3x - \cos x + 2, \quad y(0) = 4$

Answer

$y(x) = 4 + \frac{3}{2}x^2 + 2x - \sin x$

13) $xy' = y(x - 2), \quad y(1) = 3$

14) $y' = 3y^2(x + \cos x), \quad y(0) = -2$

Answer

$$y(x) = -\frac{2}{1 + 3(x^2 + 2\sin x)}$$

15) $(x - 1)y' = y - 2, \quad y(0) = 0$

16) $y' = 3y - x + 6x^2, \quad y(0) = -1$

Answer

$y(x) = -2x^2 - 2x - \frac{1}{3} - \frac{2}{3}e^{3x}$

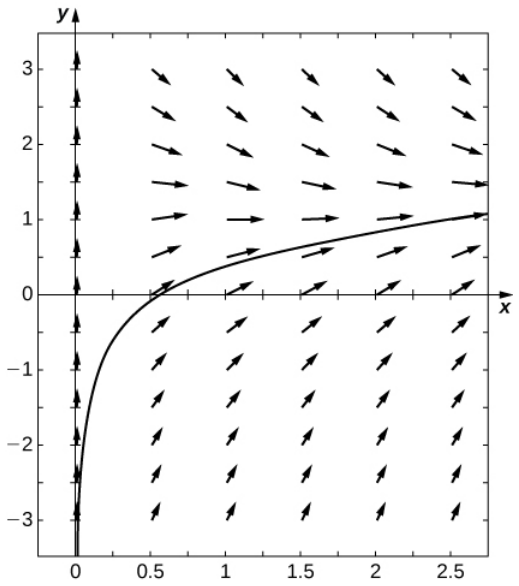
For the following problems, draw the directional field associated with the differential equation, then solve the differential equation. Draw a sample solution on the directional field.

17) $y' = 2y - y^2$

18) $y' = \dfrac{1}{x} + \ln x - y, \quad \text{for } x > 0$

Answer

$y(x) = Ce^{-x} + \ln x$



For the following problems, use Euler's Method with $n = 5$ steps over the interval $t = [0, 1]$. Then solve the initial-value problem exactly. How close is your Euler's Method estimate?

19) $y' = -4yx, \quad y(0) = 1$

20) $y' = 3^x - 2y, \quad y(0) = 0$

Answer

Euler: 0.6939

Exact solution: $y(x) = \dfrac{3^x - e^{-2x}}{2 + \ln(3)}$

For the following problems, set up and solve the differential equations.

21) A car drives along a freeway, accelerating according to $a = 5\sin(\pi t)$, where t represents time in minutes. Find the velocity at any time t, assuming the car starts with an initial speed of 60 mph.

22) You throw a ball of mass 2 kilograms into the air with an upward velocity of 8 m/s. Find exactly the time the ball will remain in the air, assuming that gravity is given by $g = 9.8\,\text{m/s}^2$.

Answer

$\frac{40}{49}$ second

23) You drop a ball with a mass of 5 kilograms out an airplane window at a height of 5000 m. How long does it take for the ball to reach the ground?

24) You drop the same ball of mass 5 kilograms out of the same airplane window at the same height, except this time you assume a drag force proportional to the ball's velocity, using a proportionality constant of 3 and the ball reaches terminal velocity. Solve for the distance fallen as a function of time. How long does it take the ball to reach the ground?

Answer

$x(t) = 5000 + \frac{245}{9} - \frac{49}{3}t - \frac{245}{9}e^{-5/3t}, \quad t = 307.8$ seconds

25) A drug is administered to a patient every 24 hours and is cleared at a rate proportional to the amount of drug left in the body, with proportionality constant 0.2. If the patient needs a baseline level of 5 mg to be in the bloodstream at all times, how large should the dose be?

26) A 1000-liter tank contains pure water and a solution of 0.2 kg salt/L is pumped into the tank at a rate of 1 L/min and is drained at the same rate. Solve for total amount of salt in the tank at time t.

Answer

$T(t) = 200\left(1 - e^{-t/1000}\right)$

27) You boil water to make tea. When you pour the water into your teapot, the temperature is $100^\circ C$. After 5 minutes in your $15^\circ C$ room, the temperature of the tea is $85^\circ C$. Solve the equation to determine the temperatures of the tea at time t. How long must you wait until the tea is at a drinkable temperature ($72^\circ C$)?

28) The human population (in thousands) of Nevada in 1950 was roughly 160. If the carrying capacity is estimated at 10 million individuals, and assuming a growth rate of 2% per year, develop a logistic growth model and solve for the population in Nevada at any time (use 1950 as time = 0). What population does your model predict for 2000? How close is your prediction to the true value of $1,998,257$?

Answer

$P(t) = \dfrac{1600000e^{0.02t}}{9840 + 160e^{0.02t}}$

29) Repeat the previous problem but use Gompertz growth model. Which is more accurate?

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6: Sequences and Series

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6.1: Prelude to Sequence and Series

The Koch snowflake is constructed from an infinite number of nonoverlapping equilateral triangles. Consequently, we can express its area as a sum of infinitely many terms. How do we add an infinite number of terms? Can a sum of an infinite number of terms be finite? To answer these questions, we need to introduce the concept of an infinite series, a sum with infinitely many terms. Having defined the necessary tools, we will be able to calculate the area of the Koch snowflake.

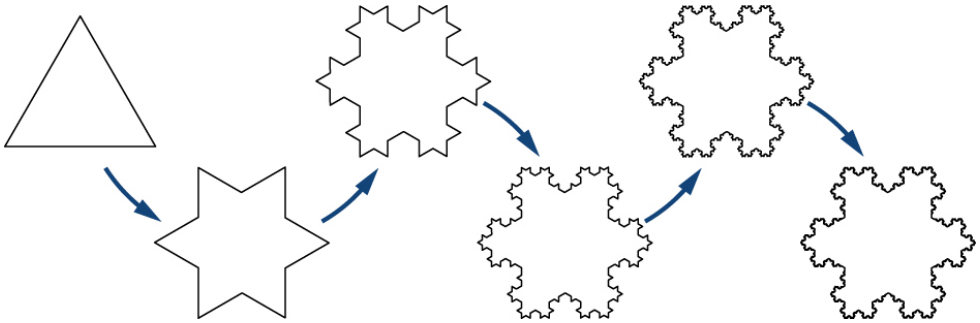


Figure 6.1.1: The Koch snowflake is constructed by using an iterative process. Starting with an equilateral triangle, at each step of the process the middle third of each line segment is removed and replaced with an equilateral triangle pointing outward.

The topic of infinite series may seem unrelated to differential and integral calculus. In fact, an infinite series whose terms involve powers of a variable is a powerful tool that we can use to express functions as “infinite polynomials.” We can use infinite series to evaluate complicated functions, approximate definite integrals, and create new functions. In addition, infinite series are used to solve differential equations that model physical behavior, from tiny electronic circuits to Earth-orbiting satellites.

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6.2: Sequences

Learning Objectives

- Find the formula for the general term of a sequence.
- Calculate the limit of a sequence if it exists.
- Determine the convergence or divergence of a given sequence.

In this section, we introduce sequences and define what it means for a sequence to converge or diverge. We show how to find limits of sequences that converge, often by using the properties of limits for functions discussed earlier. We close this section with the Monotone Convergence Theorem, a tool we can use to prove that certain types of sequences converge.

Terminology of Sequences

To work with this new topic, we need some new terms and definitions. First, an infinite sequence is an ordered list of numbers of the form

$$a_1, a_2, a_3, \ldots, a_n, \ldots.$$

Each of the numbers in the sequence is called a term. The symbol n is called the index variable for the sequence. We use the notation

$$\{a_n\}_{n=1}^{\infty},$$

or simply $\{a_n\}$, to denote this sequence. A similar notation is used for sets, but a sequence is an ordered list, whereas a set is not ordered. Because a particular number a_n exists for each positive integer n, we can also define a sequence as a function whose domain is the set of positive integers.

Let's consider the infinite, ordered list

$$2, 4, 8, 16, 32, \ldots.$$

This is a sequence in which the first, second, and third terms are given by $a_1 = 2, a_2 = 4,$ and $a_3 = 8.$ You can probably see that the terms in this sequence have the following pattern:

$$a_1 = 2^1,\ a_2 = 2^2,\ a_3 = 2^3,\ a_4 = 2^4 \text{ and } a_5 = 2^5.$$

Assuming this pattern continues, we can write the nth term in the sequence by the explicit formula $a_n = 2^n.$ Using this notation, we can write this sequence as

$$\{2^n\}_{n=1}^{\infty}$$

or

$$\{2^n\}.$$

Alternatively, we can describe this sequence in a different way. Since each term is twice the previous term, this sequence can be defined recursively by expressing the nth term a_n in terms of the previous term a_{n-1}. In particular, we can define this sequence as the sequence $\{a_n\}$ where $a_1 = 2$ and for all $n \geq 2$, each term an is defined by the **recurrence relation**

$$a_n = 2a_{n-1}.$$

Definition: Infinite Sequence

An **infinite sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$a_1,\ a_2,\ \ldots,\ a_n,\ \ldots.$$

The subscript n is called the **index variable** of the sequence. Each number a_n is a term of the sequence. Sometimes sequences are defined by **explicit formulas,** in which case $a_n = f(n)$ for some function $f(n)$ defined over the positive integers. In other cases, sequences are defined by using a **recurrence relation**. In a recurrence relation, one term (or more) of the sequence is given explicitly, and subsequent terms are defined in terms of earlier terms in the sequence.

Note that the index does not have to start at $n = 1$ but could start with other integers. For example, a sequence given by the explicit formula $a_n = f(n)$ could start at $n = 0$, in which case the sequence would be

$$a_0,\ a_1,\ a_2, \ldots .$$

Similarly, for a sequence defined by a recurrence relation, the term a_0 may be given explicitly, and the terms a_n for $n \geq 1$ may be defined in terms of a_{n-1}. Since a sequence $\{a_n\}$ has exactly one value for each positive integer n, it can be described as a function whose domain is the set of positive integers. As a result, it makes sense to discuss the graph of a sequence. The graph of a sequence $\{a_n\}$ consists of all points (n, a_n) for all positive integers n. Figure 6.2.1 shows the graph of 2^n.

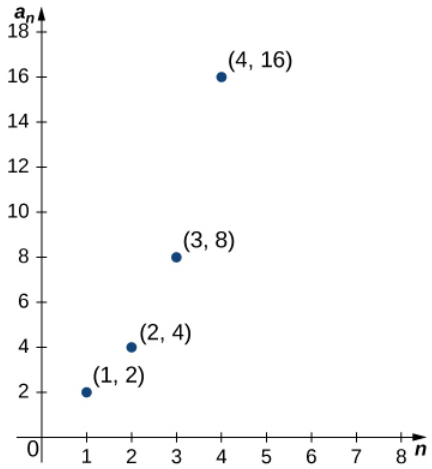


Figure 6.2.1: The plotted points are a graph of the sequence $\{2^n\}$.

Two types of sequences occur often and are given special names: arithmetic sequences and geometric sequences. In an arithmetic sequence, the difference between every pair of consecutive terms is the same. For example, consider the sequence

$$3,\ 7,\ 11,\ 15, 1\,9, \ldots$$

You can see that the difference between every consecutive pair of terms is 4. Assuming that this pattern continues, this sequence is an arithmetic sequence. It can be described by using the recurrence relation

$$\begin{cases} a_1 = 3 \\ a_n = a_{n-1} + 4, \text{ for } n \geq 2 \end{cases}.$$

Note that

$$a_2 = 3 + 4$$

$$a_3 = 3 + 4 + 4 = 3 + 2 \cdot 4$$

$$a_4 = 3 + 4 + 4 + 4 = 3 + 3 \cdot 4.$$

Thus the sequence can also be described using the explicit formula

$$a_n = 3 + 4(n-1) = 4n - 1.$$

In general, an arithmetic sequence is any sequence of the form $a_n = cn + b$.

In a geometric sequence, the ratio of every pair of consecutive terms is the same. For example, consider the sequence

$$2,\ -\frac{2}{3},\ \frac{2}{9},\ -\frac{2}{27},\ \frac{2}{81}, \ldots .$$

We see that the ratio of any term to the preceding term is $-\frac{1}{3}$. Assuming this pattern continues, this sequence is a geometric sequence. It can be defined recursively as

$$a_1 = 2$$

$$a_n = -\frac{1}{3} \cdot a_{n-1}, \text{ for } n \geq 2.$$

Alternatively, since

$$\begin{aligned} a_2 &= -\frac{1}{3} \cdot 2 \\ a_3 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^2 \cdot 2 \\ a_4 &= \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)(2) = \left(-\frac{1}{3}\right)^3 \cdot 2, \end{aligned}$$

we see that the sequence can be described by using the explicit formula

$$a_n = 2\left(-\frac{1}{3}\right)^{n-1}.$$

The sequence $\{2^n\}$ that we discussed earlier is a geometric sequence, where the ratio of any term to the previous term is 2. In general, a geometric sequence is any sequence of the form $a_n = cr^n$.

Example 6.2.1: Finding Explicit Formulas

For each of the following sequences, find an explicit formula for the nth term of the sequence.

a. $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \ldots$
b. $\frac{3}{4}, \frac{9}{7}, \frac{27}{10}, \frac{81}{13}, \frac{243}{16}, \ldots$.

Solution

a. First, note that the sequence is alternating from negative to positive. The odd terms in the sequence are negative, and the even terms are positive. Therefore, the nth term includes a factor of $(-1)^n$. Next, consider the sequence of numerators $1, 2, 3, \ldots$ and the sequence of denominators $2, 3, 4, \ldots$. We can see that both of these sequences are arithmetic sequences. The nth term in the sequence of numerators is n, and the nth term in the sequence of denominators is $n+1$. Therefore, the sequence can be described by the explicit formula

$$a_n = \frac{(-1)^n n}{n+1}.$$

b. The sequence of numerators $3, 9, 27, 81, 243, \ldots$ is a geometric sequence. The numerator of the nth term is 3^n The sequence of denominators $4, 7, 10, 13, 16, \ldots$ is an arithmetic sequence. The denominator of the nth term is $4 + 3(n-1) = 3n+1$. Therefore, we can describe the sequence by the explicit formula $a_n = \frac{3^n}{3n+1.}$

Exercise 6.2.1

Find an explicit formula for the nth term of the sequence $\left\{\frac{1}{5}, -\frac{1}{7}, \frac{1}{9}, -\frac{1}{11}, \ldots\right\}$.

Hint

The denominators form an arithmetic sequence.

Answer

$$a_n = \frac{(-1)^{n+1}}{3+2n}$$

Example 6.2.2: Defined by Recurrence Relations

For each of the following recursively defined sequences, find an explicit formula for the sequence.

a. $a_1 = 2,\ a_n = -3a_{n-1}$ for $n \geq 2$

b. $a_1 = \left(\frac{1}{2}\right),\ a_n = a_{n-1} + \left(\frac{1}{2}\right)^n$ for $n \geq 2$

Solution

a. Writing out the first few terms, we have

$$\begin{aligned} a_1 &= 2 \\ a_2 &= -3a_1 = -3(2) \\ a_3 &= -3a_2 = (-3)^2 2 \\ a_4 &= -3a_3 = (-3)^3 2. \end{aligned}$$

In general,

$$a_n = 2(-3)^{n-1}.$$

b. Write out the first few terms:

$$a_1 = \frac{1}{2}$$

$$a_2 = a_1 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$a_3 = a_2 + \left(\frac{1}{2}\right)^3 = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$$

$$a_4 = a_3 + \left(\frac{1}{2}\right)^4 = \frac{7}{8} + \frac{1}{16} = \frac{15}{16} \quad .$$

From this pattern, we derive the explicit formula

$$a_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n} \ .$$

Exercise 6.2.2

Find an explicit formula for the sequence defined recursively such that $a_1 = -4$ and $a_n = a_{n-1} + 6$.

Hint

This is an arithmetic sequence.

Answer

$a_n = 6n - 10$

Limit of a Sequence

A fundamental question that arises regarding infinite sequences is the behavior of the terms as n gets larger. Since a sequence is a function defined on the positive integers, it makes sense to discuss the limit of the terms as $n \to \infty$. For example, consider the following four sequences and their different behaviors as $n \to \infty$ (Figure 6.2.2):

a. $\{1 + 3n\} = \{4, 7, 10, 13, \ldots\}$.The terms $1 + 3n$ become arbitrarily large as $n \to \infty$. In this case, we say that $1 + 3n \to \infty$ as $n \to \infty$.

b. $\left\{1 - \left(\frac{1}{2}\right)^n\right\} = \left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16} \ldots\right\}$. The terms $1 - \left(\frac{1}{2}\right)^n \to 1$ as $n \to \infty$.

c. $\{(-1)^n\} = \{-1, 1, -1, 1, \ldots\}$.The terms alternate but do not approach one single value as $n \to \infty$.

d. $\left\{\dfrac{(-1)^n}{n}\right\} = \left\{-1, \dfrac{1}{2}, -\dfrac{1}{3}, \dfrac{1}{4}, \ldots\right\}$. The terms alternate for this sequence as well, but $\dfrac{(-1)^n}{n} \to 0$ as $n \to \infty$.

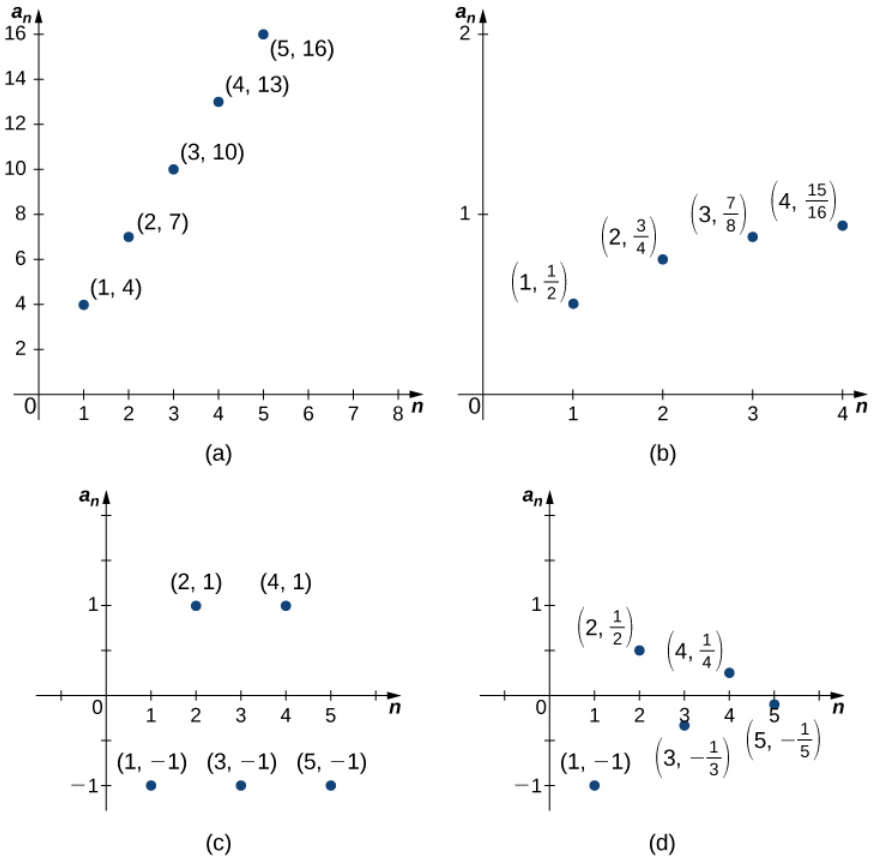


Figure 6.2.2: (a) The terms in the sequence become arbitrarily large as $n \to \infty$. (b) The terms in the sequence approach 1 as $n \to \infty$. (c) The terms in the sequence alternate between 1 and -1 as $n \to \infty$. (d) The terms in the sequence alternate between positive and negative values but approach 0 as $n \to \infty$.

From these examples, we see several possibilities for the behavior of the terms of a sequence as $n \to \infty$. In two of the sequences, the terms approach a finite number as $n \to \infty$. In the other two sequences, the terms do not. If the terms of a sequence approach a finite number L as $n \to \infty$, we say that the sequence is a convergent sequence and the real number L is the limit of the sequence. We can give an informal definition here.

Definition: Convergent and Divergent Sequences

Given a sequence $\{a_n\}$, if the terms a_n become arbitrarily close to a finite number L as n becomes sufficiently large, we say $\{a_n\}$ is a **convergent sequence** and L is the **limit of the sequence.** In this case, we write

$$\lim_{n\to\infty} a_n = L.$$

If a sequence $\{a_n\}$ is not convergent, we say it is a **divergent sequence.**

From Figure $6.2.2b$, we see that the terms in the sequence $\left\{1 - \left(\dfrac{1}{2}\right)^n\right\}$ are becoming arbitrarily close to 1 as n becomes very large. We conclude that $\left\{1 - \left(\dfrac{1}{2}\right)^n\right\}$ is a convergent sequence and its limit is 1. In contrast, from Figure $6.2.2a$, we see that the terms in the sequence $\{1 + 3n\}$ are not approaching a finite number as n becomes larger. We say that $\{1 + 3n\}$ is a divergent sequence.

In the informal definition for the limit of a sequence, we used the terms “arbitrarily close” and “sufficiently large.” Although these phrases help illustrate the meaning of a converging sequence, they are somewhat vague. To be more precise, we now present the

more formal definition of limit for a sequence and show these ideas graphically in Figure 6.2.3.

Definition: Convergence

A sequence $\{a_n\}$ converges to a real number L if for all $\varepsilon > 0$, there exists an integer N such that for all $n \geq N$ $|a_n - L| < \varepsilon$. The number L is the limit of the sequence and we write

$$\lim_{n\to\infty} a_n = L \text{ or } a_n \to L.$$

In this case, we say the sequence $\{a_n\}$ is a convergent sequence. If a sequence does not converge, it is a divergent sequence, and we say the limit does not exist.

We remark that the convergence or divergence of a sequence $\{a_n\}$ depends only on what happens to the terms a_n as $n \to \infty$. Therefore, if a finite number of terms $b_1, b_2, \ldots, b_N$ are placed before a_1 to create a new sequence

$$b_1,\ b_2,\ \ldots,\ b_N,\ a_1,\ a_2,\ \ldots,$$

this new sequence will converge if $\{a_n\}$ converges and diverge if $\{a_n\}$ diverges. Further, if the sequence $\{a_n\}$ converges to L, this new sequence will also converge to L.

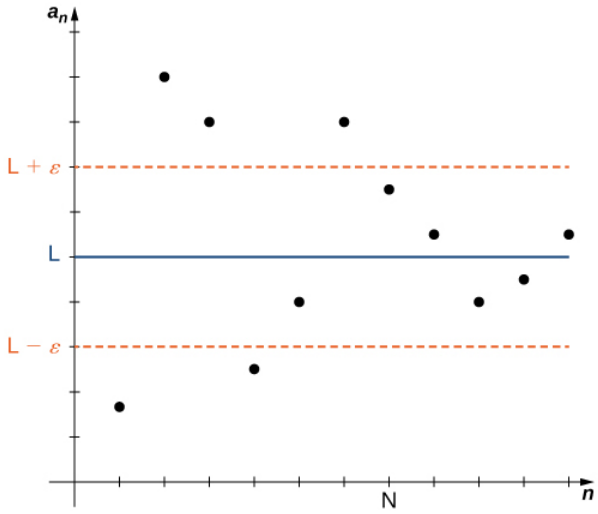


Figure 6.2.3: As n increases, the terms a_n become closer to L. For values of $n \geq N$, the distance between each point (n, a_n) and the line $y = L$ is less than ε.

As defined above, if a sequence does not converge, it is said to be a divergent sequence. For example, the sequences $\{1+3n\}$ and $\{(-1)^n\}$ shown in Figure 6.2.2 diverge. However, different sequences can diverge in different ways. The sequence $\{(-1)^n\}$ diverges because the terms alternate between 1 and -1, but do not approach one value as $n \to \infty$. On the other hand, the sequence $\{1+3n\}$ diverges because the terms $1+3n \to \infty$ as $n \to \infty$. We say the sequence $\{1+3n\}$ diverges to infinity and write $\lim_{n\to\infty}(1+3n) = \infty$. It is important to recognize that this notation does not imply the limit of the sequence $\{1+3n\}$ exists. The sequence is, in fact, divergent. Writing that the limit is infinity is intended only to provide more information about why the sequence is divergent. A sequence can also diverge to negative infinity. For example, the sequence $\{-5n+2\}$ diverges to negative infinity because $-5n+2 \to -\infty$ as $n \to -\infty$. We write this as $\lim_{n\to\infty}(-5n+2) =\to -\infty$.

Because a sequence is a function whose domain is the set of positive integers, we can use properties of limits of functions to determine whether a sequence converges. For example, consider a sequence $\{a_n\}$ and a related function f defined on all positive real numbers such that $f(n) = a_n$ for all integers $n \geq 1$. Since the domain of the sequence is a subset of the domain of f, if $\lim_{x\to\infty} f(x)$ exists, then the sequence converges and has the same limit. For example, consider the sequence $\left\{\frac{1}{n}\right\}$ and the related function $f(x) = \frac{1}{x}$. Since the function f defined on all real numbers $x > 0$ satisfies $f(x) = \frac{1}{x} \to 0$ as $x \to \infty$, the sequence $\left\{\frac{1}{n}\right\}$ must satisfy $\frac{1}{n} \to 0$ as $n \to \infty$.

Limit of a Sequence Defined by a Function

Consider a sequence $\{a_n\}$ such that $a_n = f(n)$ for all $n \geq 1$. If there exists a real number L such that

$$\lim_{x\to\infty} f(x) = L,$$

then $\{a_n\}$ converges and

$$\lim_{n\to\infty} a_n = L.$$

We can use this theorem to evaluate $\lim_{n\to\infty} r^n$ for $0 \leq r \leq 1$. For example, consider the sequence $\{(1/2)^n\}$ and the related exponential function $f(x) = (1/2)^x$. Since $\lim_{x\to\infty} (1/2)^x = 0$, we conclude that the sequence $\{(1/2)^n\}$ converges and its limit is 0. Similarly, for any real number r such that $0 \leq r < 1$, $\lim_{x\to\infty} r^x = 0$, and therefore the sequence $\{r^n\}$ converges. On the other hand, if $r = 1$, then $\lim_{x\to\infty} r^x = 1$, and therefore the limit of the sequence $\{1^n\}$ is 1. If $r > 1$, $\lim_{x\to\infty} r^x = \infty$, and therefore we cannot apply this theorem. However, in this case, just as the function r^x grows without bound as $n \to \infty$, the terms r^n in the sequence become arbitrarily large as $n \to \infty$, and we conclude that the sequence $\{r^n\}$ diverges to infinity if $r > 1$.

We summarize these results regarding the geometric sequence r^n:

$$r^n \to 0 \text{ if } 0 < r < 1$$

$$r^n \to 1 \text{ if } r = 1$$

$$r^n \to \infty \text{ if } r > 1.$$

Later in this section we consider the case when $r < 0$.

We now consider slightly more complicated sequences. For example, consider the sequence $\{(2/3)^n + (1/4)^n\}$. The terms in this sequence are more complicated than other sequences we have discussed, but luckily the limit of this sequence is determined by the limits of the two sequences $\{(2/3)^n\}$ and $\{(1/4)^n\}$. As we describe in the following algebraic limit laws, since $\{(2/3)^n\}$ and $\{1/4)^n\}$ both converge to 0, the sequence $\{(2/3)^n + (1/4)^n\}$ converges to $0 + 0 = 0$. Just as we were able to evaluate a limit involving an algebraic combination of functions f and g by looking at the limits of f and g (see Introduction to Limits), we are able to evaluate the limit of a sequence whose terms are algebraic combinations of a_n and b_n by evaluating the limits of $\{a_n\}$ and $\{b_n\}$.

Algebraic Limit Laws

Given sequences $\{a_n\}$ and $\{b_n\}$ and any real number c, if there exist constants A and B such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, then

i. $\lim_{n\to\infty} c = c$

ii. $\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = cA$

iii. $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = A \pm B$

iv. $\lim_{n\to\infty} (a_n \cdot b_n) = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right) = A \cdot B$

v. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, provided $B \neq 0$ and each $b_n \neq 0$.

Proof

We prove part iii.

Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = A$, there exists a constant positive integer N_1 such that for all $n \geq N_1$. Since $\lim_{n\to\infty} b_n = B$, there exists a constant N_2 such that $|b_n - B| < \varepsilon/2$ for all $n \geq N_2$. Let N be the largest of N_1 and N_2. Therefore, for all $n \geq N$, $|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

□

The algebraic limit laws allow us to evaluate limits for many sequences. For example, consider the sequence $a_n = \frac{1}{n^2}$. As shown earlier, $\lim_{n\to\infty} \frac{1}{n} = 0$. Similarly, for any positive integer k, we can conclude that

$$\lim_{n\to\infty} \frac{1}{n^k} = 0.$$

In the next example, we make use of this fact along with the limit laws to evaluate limits for other sequences.

Example 6.2.3: Determining Convergence and Finding Limits

For each of the following sequences, determine whether or not the sequence converges. If it converges, find its limit.

a. $\left\{5 - \frac{3}{n^2}\right\}$

b. $\left\{\frac{3n^4 - 7n^2 + 5}{6 - 4n^4}\right\}$

c. $\left\{\frac{2^n}{n^2}\right\}$

d. $\left\{\left(1 + \frac{4}{n}\right)^n\right\}$

Solution

a. We know that $\lim_{n\to\infty} \frac{1}{n} = 0$. Using this fact, we conclude that

$$\lim_{n\to\infty} \frac{1}{n^2} = \lim_{n\to\infty} \frac{1}{n} . \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore,

$$\lim_{n\to\infty} \left(5 - \frac{3}{n^2}\right) = \lim_{n\to\infty} 5 - 3 \lim_{n\to\infty} \frac{1}{n^2} = 5 - 3.0 = 5.$$

The sequence converges and its limit is 5.

b. By factoring n^4 out of the numerator and denominator and using the limit laws above, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{3n^4 - 7n^2 + 5}{6 - 4n^4} &= \lim_{n\to\infty} \frac{3 - \frac{7}{n^2} + \frac{5}{n^4}}{\frac{6}{n^4} - 4} \\
&= \frac{\lim_{n\to\infty} (3 - \frac{7}{n^2} + \frac{5}{n^4})}{\lim_{n\to\infty} (\frac{6}{n^4} - 4)} \\
&= \frac{\lim_{n\to\infty} (3) - \lim_{n\to\infty} \frac{7}{n^2} + \lim_{n\to\infty} \frac{5}{n^4}}{\lim_{n\to\infty} \frac{6}{n^4} - \lim_{n\to\infty} (4)} \\
&= \frac{\lim_{n\to\infty} (3) - 7 \cdot \lim_{n\to\infty} \frac{1}{n^2} + 5 \cdot \lim_{n\to\infty} \frac{1}{n^4}}{6 \cdot \lim_{n\to\infty} \frac{1}{n^4} - \lim_{n\to\infty} (4)} \\
&= \frac{3 - 7 \cdot 0 + 5 \cdot 0}{6 \cdot 0 - 4} = -\frac{3}{4}.
\end{aligned}$$

The sequence converges and its limit is $-3/4$.

c. Consider the related function $f(x) = 2^x/x^2$ defined on all real numbers $x > 0$. Since $2^x \to \infty$ and $x^2 \to \infty$ as $x \to \infty$, apply L'Hôpital's rule and write

$$\begin{aligned} \lim_{x\to\infty} \frac{2^x}{x^2} &= \lim_{x\to\infty} \frac{2^x \ln 2}{2x} && \text{Take the derivatives of the numerator and denominator.} \\ &= \lim_{x\to\infty} \frac{2^x (\ln 2)^2}{2} && \text{Take the derivatives again.} \\ &= \infty. \end{aligned}$$

We conclude that the sequence diverges.

d. Consider the function $f(x) = \left(1 + \frac{4}{x}\right)^x$ defined on all real numbers $x > 0$. This function has the indeterminate form 1^∞ as $x \to \infty$. Let

$$y = \lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x.$$

Now taking the natural logarithm of both sides of the equation, we obtain

$$\ln(y) = \ln\left[\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x\right].$$

Since the function $f(x) = \ln(x)$ is continuous on its domain, we can interchange the limit and the natural logarithm. Therefore,

$$\ln(y) = \lim_{x\to\infty} \left[\ln\left(1 + \frac{4}{x}\right)^x\right].$$

Using properties of logarithms, we write

$$\lim_{x\to\infty} \left[\ln\left(1 + \frac{4}{x}\right)^x\right] = \lim_{x\to\infty} x \ln\left(1 + \frac{4}{x}\right).$$

Since the right-hand side of this equation has the indeterminate form $\infty \cdot 0$, rewrite it as a fraction to apply L'Hôpital's rule. Write

$$\lim_{x\to\infty} x \ln\left(1 + \frac{4}{x}\right) = \lim_{x\to\infty} \frac{\ln(1 + 4/x)}{1/x}.$$

Since the right-hand side is now in the indeterminate form 0/0, we are able to apply L'Hôpital's rule. We conclude that

$$\lim_{x\to\infty} \frac{\ln(1 + 4/x)}{1/x} = \lim_{x\to\infty} \frac{4}{1 + 4/x} = 4.$$

Therefore, $\ln(y) = 4$ and $y = e^4$. Therefore, since $\lim_{x\to\infty} \left(1 + \frac{4}{x}\right)^x = e^4$, we can conclude that the sequence $\left\{\left(1 + \frac{4}{n}\right)^n\right\}$ converges to e^4.

? Exercise 6.2.3

Consider the sequence $\left\{(5n^2 + 1)/e^n\right\}$. Determine whether or not the sequence converges. If it converges, find its limit.

Hint

Use L'Hôpital's rule.

Answer

The sequence converges, and its limit is 0

Recall that if f is a continuous function at a value L, then $f(x) \to f(L)$ as $x \to L$. This idea applies to sequences as well. Suppose a sequence $a_n \to L$, and a function f is continuous at L. Then $f(a_n) \to f(L)$. This property often enables us to find limits for complicated sequences. For example, consider the sequence $\sqrt{5 - \frac{3}{n^2}}$. From Example a. we know the sequence $5 - \frac{3}{n^2} \to 5$. Since $\sqrt{x}$ is a continuous function at $x = 5$,

$$\lim_{n\to\infty} \sqrt{5 - \frac{3}{n^2}} = \sqrt{\lim_{n\to\infty} (5 - \frac{3}{n^2})} = \sqrt{5}.$$

Continuous Functions Defined on Convergent Sequences

Consider a sequence $\{a_n\}$ and suppose there exists a real number L such that the sequence $\{a_n\}$ converges to L. Suppose f is a continuous function at L. Then there exists an integer N such that f is defined at all values an for $n \geq N$, and the sequence $\{f(a_n)\}$ converges to $f(L)$ (Figure 6.2.4).

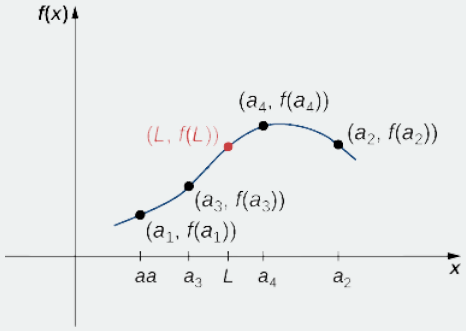


Figure 6.2.4: Because f is a continuous function as the inputs $a_1, a_2, a_3, \ldots$ approach L, the outputs $f(a_1), f(a_2), f(a_3), \ldots$ approach $f(L)$.

Proof

Let $\epsilon > 0$. Since f is continuous at L, there exists $\delta > 0$ such that $|f(x) - f(L)| < \varepsilon$ if $|x - L| < \delta$. Since the sequence $\{a_n\}$ converges to L, there exists N such that $|a_n - L| < \delta$ for all $n \geq N$. Therefore, for all $n \geq N$, $|a_n - L| < \delta$, which implies $|f(a_n) - f(L)| < \varepsilon$. We conclude that the sequence $\{f(a_n)\}$ converges to $f(L)$.

□

Example 6.2.4: Limits Involving Continuous Functions Defined on Convergent Sequences

Determine whether the sequence $\{\cos(3/n^2)\}$ converges. If it converges, find its limit.

Solution:

Since the sequence $\{3/n^2\}$ converges to 0 and $\cos x$ is continuous at $x = 0$, we can conclude that the sequence $\{\cos(3/n^2)\}$ converges and

$$\lim_{n\to\infty} \cos\left(\frac{3}{n^2}\right) = \cos 0 = 1.$$

Exercise 6.2.4

Determine if the sequence $\left\{\sqrt{\frac{2n+1}{3n+5}}\right\}$ converges. If it converges, find its limit.

Hint

Consider the sequence $\left\{\frac{2n+1}{3n+5}\right\}$.

Answer

The sequence converges, and its limit is $\sqrt{2/3}$.

Another theorem involving limits of sequences is an extension of the Squeeze Theorem for limits discussed in Introduction to Limits.

Squeeze Theorem for Sequences

Consider sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$. Suppose there exists an integer N such that

$$a_n \leq b_n \leq c_n \text{ for all } n \geq N.$$

If there exists a real number L such that

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$ (Figure 6.2.5).

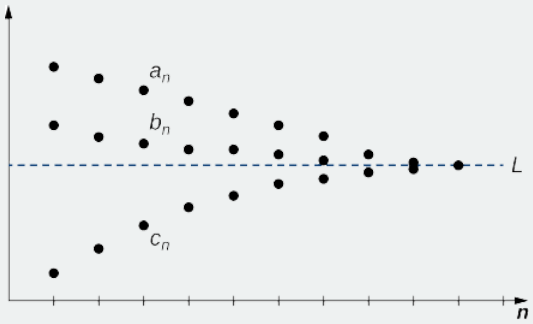


Figure 6.2.5: Each term bn satisfies $a_n \leq b_n \leq c_n$ and the sequences $\{a_n\}$ and $\{c_n\}$ converge to the same limit, so the sequence $\{b_n\}$ must converge to the same limit as well.

Proof

Let $\varepsilon > 0$. Since the sequence $\{a_n\}$ converges to L, there exists an integer N_1 such that $|a_n - L| < \varepsilon$ for all $n \geq N_1$. Similarly, since $\{c_n\}$ converges to L, there exists an integer N_2 such that $|c_n - L| < \varepsilon$ for all $n \geq N_2$. By assumption, there exists an integer N such that $a_n \leq b_n \leq c_n$ for all $n \geq N$. Let M be the largest of N_1, N_2, and N. We must show that $|b_n - L| < \varepsilon$ for all $n \geq M$. For all $n \geq M$,

$$-\varepsilon < -|a_n - L| \leq a_n - L \leq b_n - L \leq c_n - L \leq |c_n - L| < \varepsilon$$

Therefore, $-\varepsilon < b_n - L < \varepsilon$, and we conclude that $|b_n - L| < \varepsilon$ for all $n \geq M$, and we conclude that the sequence b_n converges to L.

□

Example 6.2.5: Using the Squeeze Theorem

Use the Squeeze Theorem to find the limit of each of the following sequences.

a. $\left\{\dfrac{\cos n}{n^2}\right\}$

b. $\left\{\left(-\dfrac{1}{2}\right)^n\right\}$

Solution

a. Since $-1 \leq \cos n \leq 1$ for all integers n, we have

$$-\frac{1}{n^2} \leq \frac{\cos n}{n^2} \leq \frac{1}{n^2}.$$

Since $-1/n^2 \to 0$ and $1/n^2 \to 0$, we conclude that $\cos n/n^2 \to 0$ as well.

b. Since

$$-\frac{1}{2^n} \leq \left(-\frac{1}{2}\right)^n \leq \frac{1}{2^n}$$

for all positive integers n, $-1/2^n \to 0$ and $1/2^n \to 0$, we can conclude that $(-1/2)^n \to 0$.

Exercise 6.2.5

Find $\lim\limits_{n\to\infty} \dfrac{2n-\sin n}{n}$.

Hint

Use the fact that $-1 \leq \sin n \leq 1$.

Answer

2

Using the idea from Example 6.2.5 b we conclude that $r^n \to 0$ for any real number r such that $-1 < r < 0$. If $r < -1$, the sequence r^n diverges because the terms oscillate and become arbitrarily large in magnitude. If $r = -1$, the sequence $r^n = (-1)^n$ diverges, as discussed earlier. Here is a summary of the properties for geometric sequences.

$$r^n \to 0 \text{ if } |r| < 1$$

$$r^n \to 1 \text{ if } r = 1$$

$$r^n \to \infty \text{ if } r > 1$$

$$\{r^n\} \text{ diverges if } r \leq -1$$

Bounded Sequences

We now turn our attention to one of the most important theorems involving sequences: the Monotone Convergence Theorem. Before stating the theorem, we need to introduce some terminology and motivation. We begin by defining what it means for a sequence to be bounded.

Definition: Bound Sequences

A sequence $\{a_n\}$ is **bounded above** if there exists a real number M such that

$$a_n \leq M$$

for all positive integers n.

A sequence $\{a_n\}$ is **bounded below** if there exists a real number m such that

$$m \leq a_n$$

for all positive integers n.

A sequence $\{a_n\}$ is a **bounded sequence** if it is bounded above and bounded below.

If a sequence is not bounded, it is an **unbounded sequence.**

For example, the sequence $\{1/n\}$ is bounded above because $1/n \leq 1$ for all positive integers n. It is also bounded below because $1/n \geq 0$ for all positive integers n. Therefore, $\{1/n\}$ is a bounded sequence. On the other hand, consider the sequence $\{2^n\}$. Because $2^n \geq 2$ for all $n \geq 1$, the sequence is bounded below. However, the sequence is not bounded above. Therefore, $\{2^n\}$ is an unbounded sequence.

We now discuss the relationship between boundedness and convergence. Suppose a sequence $\{a_n\}$ is unbounded. Then it is not bounded above, or not bounded below, or both. In either case, there are terms an that are arbitrarily large in magnitude as n gets

larger. As a result, the sequence $\{a_n\}$ cannot converge. Therefore, being bounded is a necessary condition for a sequence to converge.

Convergent Sequences Are Bounded

If a sequence $\{a_n\}$ converges, then it is bounded.

Note that a sequence being bounded is not a sufficient condition for a sequence to converge. For example, the sequence $\{(-1)^n\}$ is bounded, but the sequence diverges because the sequence oscillates between 1 and -1 and never approaches a finite number. We now discuss a sufficient (but not necessary) condition for a bounded sequence to converge.

Consider a bounded sequence $\{a_n\}$. Suppose the sequence $\{a_n\}$ is increasing. That is, $a_1 \leq a_2 \leq a_3 \ldots$. Since the sequence is increasing, the terms are not oscillating. Therefore, there are two possibilities. The sequence could diverge to infinity, or it could converge. However, since the sequence is bounded, it is bounded above and the sequence cannot diverge to infinity. We conclude that $\{a_n\}$ converges. For example, consider the sequence

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}.$$

Since this sequence is increasing and bounded above, it converges. Next, consider the sequence

$$\left\{2, 0, 3, 0, 4, 0, 1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right\}.$$

Even though the sequence is not increasing for all values of n, we see that $-1/2 < -1/3 < -1/4 < \cdots$. Therefore, starting with the eighth term, $a_8 = -1/2$, the sequence is increasing. In this case, we say the sequence is eventually increasing. Since the sequence is bounded above, it converges. It is also true that if a sequence is decreasing (or eventually decreasing) and bounded below, it also converges.

Definition

A sequence $\{a_n\}$ is increasing for all $n \geq n_0$ if

$$a_n \leq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is decreasing for all $n \geq n_0$ if

$$a_n \geq a_{n+1} \text{ for all } n \geq n_0.$$

A sequence $\{a_n\}$ is a *monotone sequence* for all $n \geq n_0$ if it is increasing for all $n \geq n_0$ or decreasing for all $n \geq n_0$.

We now have the necessary definitions to state the Monotone Convergence Theorem, which gives a sufficient condition for convergence of a sequence.

Definition: Monotone Convergence Theorem

If $\{a_n\}$ is a bounded sequence and there exists a positive integer n_0 such that $\{a_n\}$ is monotone for all $n \geq n_0$, then $\{a_n\}$ converges.

The proof of this theorem is beyond the scope of this text. Instead, we provide a graph to show intuitively why this theorem makes sense (Figure 6.2.6).

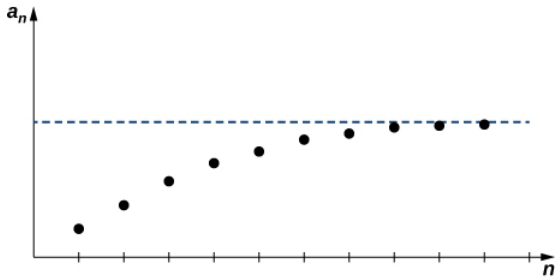


Figure 6.2.6: Since the sequence $\{a_n\}$ is increasing and bounded above, it must converge.

In the following example, we show how the Monotone Convergence Theorem can be used to prove convergence of a sequence.

Example 6.2.6: Using the Monotone Convergence Theorem

For each of the following sequences, use the Monotone Convergence Theorem to show the sequence converges and find its limit.

a. $\left\{\dfrac{4^n}{n!}\right\}$

b. $\{a_n\}$ defined recursively such that

$$a_1 = 2 \text{ and } a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \text{ for all } n \geq 2.$$

Solution

a. Writing out the first few terms, we see that

$$\left\{\frac{4^n}{n!}\right\} = \left\{4,\ 8,\ \frac{32}{3},\ \frac{32}{3},\ \frac{128}{15},\ \ldots\right\}.$$

At first, the terms increase. However, after the third term, the terms decrease. In fact, the terms decrease for all $n \geq 3$. We can show this as follows.

$$a_{n+1} = \frac{4^{n+1}}{(n+1)!} = \frac{4}{n+1} \cdot \frac{4^n}{n!} = \frac{4}{n+1} \cdot a_n \leq a_n \quad \text{if } n \geq 3.$$

Therefore, the sequence is decreasing for all $n \geq 3$. Further, the sequence is bounded below by 0 because $4n/n! \geq 0$ for all positive integers n. Therefore, by the Monotone Convergence Theorem, the sequence converges.

To find the limit, we use the fact that the sequence converges and let $L = \lim\limits_{n\to\infty} a_n$. Now note this important observation. Consider $\lim\limits_{n\to\infty} a_{n+1}$. Since

$$\{a_{n+1}\} = \{a_2,\ a_3,\ a_4,\ \ldots\},$$

the only difference between the sequences $\{a_{n+1}\}$ and $\{a_n\}$ is that $\{a_{n+1}\}$ omits the first term. Since a finite number of terms does not affect the convergence of a sequence,

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = L.$$

Combining this fact with the equation

$$a_{n+1} = \frac{4}{n+1} a_n$$

and taking the limit of both sides of the equation

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{4}{n+1} a_n,$$

we can conclude that

$$L = 0 \cdot L = 0.$$

b. Writing out the first several terms,

$$\left\{2, \frac{5}{4}, \frac{41}{40}, \frac{3281}{3280}, \ldots\right\}.$$

we can conjecture that the sequence is decreasing and bounded below by 1. To show that the sequence is bounded below by 1, we can show that

$$\frac{a_n}{2} + \frac{1}{2a_n} \geq 1.$$

To show this, first rewrite

$$\frac{a_n}{2} + \frac{1}{2a_n} = \frac{a_n^2 + 1}{2a_n} \ .$$

Since $a_1 > 0$ and a_2 is defined as a sum of positive terms, $a_2 > 0$. Similarly, all terms $a_n > 0$. Therefore,

$$\frac{a_n^2 + 1}{2a_n} \geq 1$$

if and only if

$$a_n^2 + 1 \geq 2a_n \ .$$

Rewriting the inequality $a_n^2 + 1 \geq 2a_n$ as $a_n^2 - 2a_n + 1 \geq 0$, and using the fact that

$$a_n^2 - 2a_n + 1 = (a_n - 1)^2 \geq 0$$

because the square of any real number is nonnegative, we can conclude that

$$\frac{a^n}{2} + \frac{1}{2a_n} \geq 1.$$

To show that the sequence is decreasing, we must show that $a_{n+1} \leq a_n$ for all $n \geq 1$. Since $1 \leq a_n^2$, it follows that

$$a_n^2 + 1 \leq 2a_n^2 \ .$$

Dividing both sides by $2a_n$, we obtain

$$\frac{a_n}{2} + \frac{1}{2a_n} \leq a_n.$$

Using the definition of a_{n+1}, we conclude that

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \leq a_n \ .$$

Since $\{a_n\}$ is bounded below and decreasing, by the Monotone Convergence Theorem, it converges.

To find the limit, let $L = \lim_{n\to\infty} a_n$. Then using the recurrence relation and the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$, we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{a_n}{2} + \frac{1}{2a_n}\right),$$

and therefore

$$L = \frac{L}{2} + \frac{1}{2L} \ .$$

Multiplying both sides of this equation by $2L$, we arrive at the equation

$$2L^2 = L^2 + 1 \, .$$

Solving this equation for L, we conclude that $L^2 = 1$, which implies $L = \pm 1$. Since all the terms are positive, the limit $L = 1$.

Exercise 6.2.6

Consider the sequence $\{a_n\}$ defined recursively such that $a_1 = 1$, $a_n = \dfrac{a_{n-1}}{2}$. Use the Monotone Convergence Theorem to show that this sequence converges and find its limit.

Hint

Show the sequence is decreasing and bounded below.

Answer

0.

Definition: Fibonacci Numbers

The *Fibonacci numbers* are defined recursively by the sequence $\{F_n\}$ where $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}.$$

Here we look at properties of the Fibonacci numbers.

1. Write out the first twenty Fibonacci numbers.

2. Find a closed formula for the Fibonacci sequence by using the following steps.

   a. Consider the recursively defined sequence x_n where $x_0 = c$ and $x_{n+1} = ax_n$. Show that this sequence can be described by the closed formula $x_n = ca^n$ for all $n \geq 0$.

   b. Using the result from part a. as motivation, look for a solution of the equation

   $F_n = F_{n-1} + F_{n-2}$

   of the form $F_n = c\lambda^n$. Determine what two values for λ will allow F_n to satisfy this equation.

   c. Consider the two solutions from part b.: λ_1 and λ_2. Let $F_n = c_1\lambda_1^n + c_2\lambda_2^n$. Use the initial conditions F_0 and F_1 to determine the values for the constants c_1 and c_2 and write the closed formula F_n.

3. Use the answer in 2 c. to show that

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}.$$

The number $\phi = (1+\sqrt{5})/2$ is known as the **golden ratio** (Figure 6.2.7 and Figure 6.2.8).



Figure 6.2.7: The seeds in a sunflower exhibit spiral patterns curving to the left and to the right. The number of spirals in each direction is always a Fibonacci number—always. (credit: modification of work by Esdras Calderan, Wikimedia Commons)

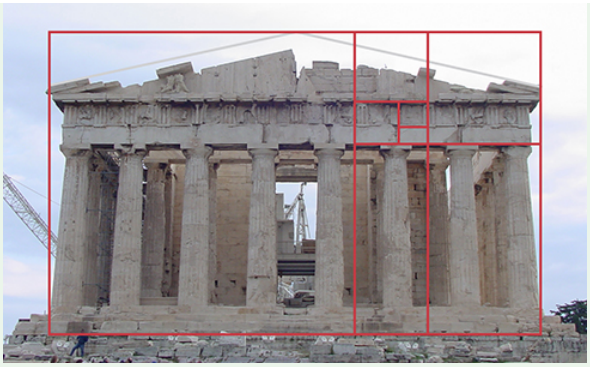


Figure 6.2.8: The proportion of the golden ratio appears in many famous examples of art and architecture. The ancient Greek temple known as the Parthenon was designed with these proportions, and the ratio appears again in many of the smaller details. (credit: modification of work by TravelingOtter, Flickr).

Key Concepts

- To determine the convergence of a sequence given by an explicit formula $a_n = f(n)$, we use the properties of limits for functions.
- If $\{a_n\}$ and $\{b_n\}$ are convergent sequences that converge to A and B, respectively, and c is any real number, then the sequence $\{ca_n\}$converges to $c \cdot A$, the sequences $\{a_n \pm b_n\}$ converge to $A \pm B$, the sequence $\{a_n \cdot b_n\}$ converges to $A \cdot B$, and the sequence $\{a_n / b_n\}$ converges to A/B, provided $B \neq 0$.
- If a sequence is bounded and monotone, then it converges, but not all convergent sequences are monotone.
- If a sequence is unbounded, it diverges, but not all divergent sequences are unbounded.
- The geometric sequence $\{r^n\}$ converges if and only if $|r| < 1$ or $r = 1$.

Glossary

arithmetic sequence

a sequence in which the difference between every pair of consecutive terms is the same is called an arithmetic sequence

bounded above

a sequence $\{a_n\}$ is bounded above if there exists a constant M such that $a_n \leq M$ for all positive integers n

bounded below

a sequence $\{a_n\}$ is bounded below if there exists a constant M such that $M \leq a_n$ for all positive integers n

bounded sequence

a sequence $\{a_n\}$ is bounded if there exists a constant M such that $|a_n| \leq M$ for all positive integers n

convergent sequence

a convergent sequence is a sequence $\{a_n\}$ for which there exists a real number L such that a_n is arbitrarily close to L as long as n is sufficiently large

divergent sequence

a sequence that is not convergent is divergent

explicit formula

a sequence may be defined by an explicit formula such that $a_n = f(n)$

geometric sequence

a sequence $\{a_n\}$ in which the ratio a_{n+1}/a_n is the same for all positive integers n is called a geometric sequence

index variable

the subscript used to define the terms in a sequence is called the index

limit of a sequence

the real number L to which a sequence converges is called the limit of the sequence

monotone sequence

an increasing or decreasing sequence

recurrence relation

a recurrence relation is a relationship in which a term a_n in a sequence is defined in terms of earlier terms in the sequence

sequence

an ordered list of numbers of the form $a_1, a_2, a_3, \ldots$ is a sequence

term

the number a_n in the sequence $\{a_n\}$ is called the nth term of the sequence

unbounded sequence

a sequence that is not bounded is called unbounded

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6.2E: Exercises for Sequences

In exercises 1 - 4, find the first six terms of each sequence, starting with $n=1$.

1) $a_n = 1+(-1)^n$ for $n \geq 1$

Answer

$a_n = 0$ if n is odd and $a_n = 2$ if n is even

2) $a_n = n^2 - 1$ for $n \geq 1$

3) $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$

Answer

$a_n = 1, 3, 6, 10, 15, 21, \ldots$

4) $a_1 = 1, a_2 = 1$ and $a_n + 2 = a_n + a_{n+1}$ for $n \geq 1$

5) Find an explicit formula for a_n where $a_1 = 1$ and $a_n = a_{n-1} + n$ for $n \geq 2$.

Answer

$$a_n = \frac{n(n+1)}{2}$$

6) Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = 1$ such that $a_{n+1} - a_n = 17$ for $n \geq 1$.

7) Find a formula a_n for the nth term of the arithmetic sequence whose first term is $a_1 = -3$ such that $a_{n+1} - a_n = 4$ for $n \geq 1$.

Answer

$a_n = 4n - 7$

8) Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 1$ such that $\frac{a_{n+1}}{a_n} = 10$ for $n \geq 1$.

9) Find a formula a_n for the nth term of the geometric sequence whose first term is $a_1 = 3$ such that $\frac{a_{n+1}}{a_n} = 1/10$ for $n \geq 1$.

Answer

$a_n = 3.10^{1-n} = 30.10^{-n}$

10) Find an explicit formula for the nth term of the sequence whose first several terms are $0, 3, 8, 15, 24, 35, 48, 63, 80, 99, \ldots$. (Hint: First add one to each term.)

11) Find an explicit formula for the nth term of the sequence satisfying $a_1 = 0$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

Answer

$a_n = 2^{n-1} - 1$

In exercises 12 and 13, find a formula for the general term a_n of each of the following sequences.

12) $1, 0, -1, 0, 1, 0, -1, 0, \ldots$ (Hint: Find where $\sin x$ takes these values)

13) $1, -1/3, 1/5, -1/7, \ldots$

Answer

$$a_n = \frac{(-1)^{n-1}}{2n-1}$$

In exercises 14-18, find a function $f(n)$ that identifies the nth term a_n of the following recursively defined sequences, as $a_n = f(n)$.

14) $a_1 = 1$ and $a_{n+1} = -a_n$ for $n \geq 1$

15) $a_1 = 2$ and $a_{n+1} = 2a_n$ for $n \geq 1$

Answer

$f(n) = 2^n$

16) $a_1 = 1$ and $a_{n+1} = (n+1)a_n$ for $n \geq 1$

17) $a_1 = 2$ and $a_{n+1} = (n+1)a_n/2$ for $n \geq 1$

Answer

$$f(n) = \frac{n!}{2^{n-2}}$$

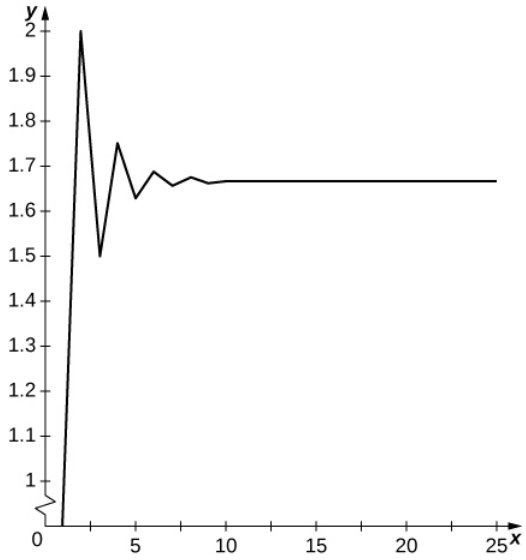
18) $a_1 = 1$ and $a_{n+1} = a_n/2^n$ for $n \geq 1$

In exercises 19 - 22, plot the first N terms of the given sequence. State whether the graphical evidence suggests that the sequence converges or diverges.

19) [T] $a_1 = 1, a_2 = 2$, and for $n \geq 2, a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$; $N = 30$

Answer

Terms oscillate above and below $5/3$ and appear to converge to $5/3$.

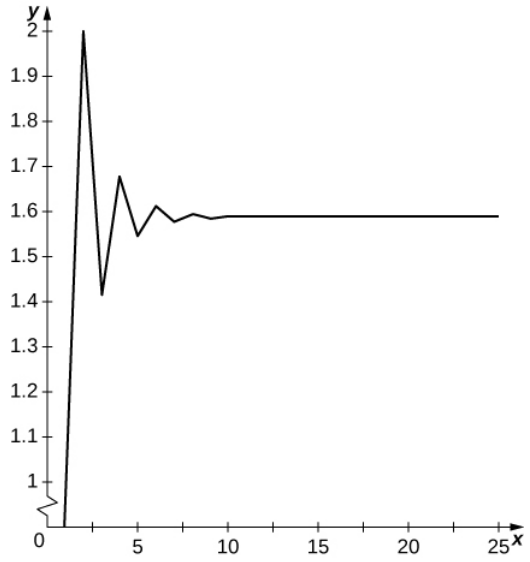


20) [T] $a_1 = 1, a_2 = 2, a_3 = 3$ and for $n \geq 4, a_n = \frac{1}{3}(a_{n-1} + a_{n-2} + a_{n-3}), N = 30$

21) [T] $a_1 = 1, a_2 = 2$, and for $n \geq 3, a_n = \sqrt{a_{n-1}a_{n-2}}; N = 30$

Answer

Terms oscillate above and below $y \approx 1.57..$ and appear to converge to a limit.



22) [T] $a_1 = 1, a_2 = 2, a_3 = 3$, and for $n \geq 4, a_n = \sqrt{a_{n-1}a_{n-2}a_{n-3}}; N = 30$

In exercises 23 - 16, suppose that $\lim_{n\to\infty} a_n = 1$, $\lim_{n\to\infty} b_n = -1$, and $0 < -b_n < a_n$ for all n.

Using this information, evaluate each of the following limits, state that the limit does not exist, or state that there is not enough information to determine whether the limit exists.

23) $\lim_{n\to\infty} 3a_n - 4b_n$

Answer

$$\lim_{n\to\infty} 3a_n - 4b_n = 7$$

24) $\lim_{n\to\infty} \frac{1}{2}b_n - \frac{1}{2}a_n$

25) $\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n}$

Answer

$$\lim_{n\to\infty} \frac{a_n + b_n}{a_n - b_n} = 0$$

26) $\lim_{n\to\infty} \frac{a_n - b_n}{a_n + b_n}$

In exercises 27 - 30, find the limit of each of the following sequences, using L'Hôpital's rule when appropriate.

27) $\frac{n^2}{2^n}$

Answer

$$\lim_{n\to\infty} \frac{n^2}{2^n} = 0$$

28) $\frac{(n-1)^2}{(n+1)^2}$

29) $\frac{\sqrt{n}}{\sqrt{n+1}}$

Answer

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}} \quad = \quad 1$$

30) $n^{1/n}$ (Hint: $n^{1/n}=e^{\frac{1}{n}\ln n}$)

In exercises 31 - 37, state whether each sequence is bounded and whether it is eventually monotone, increasing, or decreasing.

31) $n/2^n, n\geq 2$

Answer

bounded, decreasing for $n\geq 1$

32) $\ln\left(1+\frac{1}{n}\right)$

33) $\sin n$

Answer

bounded, not monotone

34) $\cos(n^2)$

35) $n^{1/n}, \quad n\geq 3$

Answer

bounded, decreasing

36) $n^{-1/n}, \quad n\geq 3$

37) $\tan n$

Answer

not monotone, not bounded

In exercises 38 - 39, determine whether the given sequence has a limit. If it does, find the limit.

38) $a_1=\sqrt{2}, a_2=\sqrt{2\sqrt{2}}. a_3=\sqrt{2\sqrt{2\sqrt{2}}}$ etc.

39) $a_1=3, a_n=\sqrt{2a_{n-1}}, n=2,3,\ldots.$

Answer

a_n is decreasing and bounded below by 2. The limit a must satisfy $a=\sqrt{2a}$ so $a=2$, independent of the initial value.

Use the Squeeze Theorem to find the limit of each sequence in exercises 40 - 43.

40) $n\sin(1/n)$

41) $\frac{\cos(1/n)-1}{1/n}$

Answer

0

42) $a_n=\frac{n!}{n^n}$

43) $a_n=\sin n\sin(1/n)$

Answer

0 since $|\sin x| \leq |x|$ and $|\sin x| \leq 1$ so $-\frac{1}{n} \leq a_n \leq \frac{1}{n}$) .

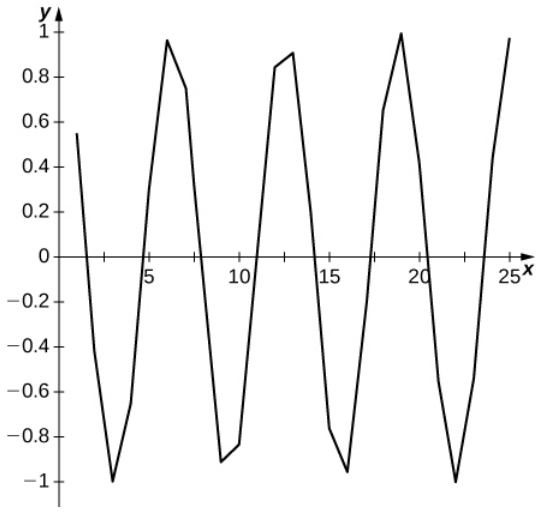
For the sequences in exercises 44 and 45, plot the first 25 terms of the sequence and state whether the graphical evidence suggests that the sequence converges or diverges.

44) [T] $a_n = \sin n$

45) [T] $a_n = \cos n$

Answer

Graph oscillates and suggests no limit.



In exercises 46 - 52, determine the limit of the sequence or show that the sequence diverges. If it converges, find its limit.

46) $a_n = \tan^{-1}(n^2)$

47) $a_n = (2n)^{1/n} - n^{1/n}$

Answer

$n^{1/n} \to 1$ and $2^{1/n} \to 1$, so $a_n \to 0$

48) $a_n = \dfrac{\ln(n^2)}{\ln(2n)}$

49) $a_n = \left(1 - \frac{2}{n}\right)^n$

Answer

Since $(1+1/n)^n \to e$, one has $(1-2/n)^n \approx (1+k)^{-2k} \to e^{-2}$ as $k \to \infty$.

50) $a_n = \ln\left(\dfrac{n+2}{n^2-3}\right)$

51) $a_n = \dfrac{2^n + 3^n}{4^n}$

Answer

$2^n + 3^n \leq 2 \cdot 3^n$ and $3^n/4^n \to 0$ as $n \to \infty$, so $a_n \to 0$ as $n \to \infty$.

52) $a_n = \dfrac{(1000)^n}{n!}$

53) $a_n = \dfrac{(n!)^2}{(2n)!}$

Answer

$\dfrac{a_{n+1}}{a_n} = n!/(n+1)(n+2)\cdots(2n) = \dfrac{1\cdot 2\cdot 3\cdots n}{(n+1)(n+2)\cdots(2n)} < 1/2^n$. In particular, $a_{n+1}/a_n \le 1/2$, so $a_n \to 0$ as $n \to \infty$.

Newton's method seeks to approximate a solution $f(x) = 0$ that starts with an initial approximation x_0 and successively defines a sequence $x_{n+1} = x_n - \dfrac{f(x_n)}{f\,'(x_n)}$. For the given choice of f and x_0, write out the formula for x_{n+1}. If the sequence appears to converge, give an exact formula for the solution x, then identify the limit x accurate to four decimal places and the smallest n such that x_n agrees with x up to four decimal places.

54) [T] $f(x) = x^2 - 2, \quad x_0 = 1$

55) [T] $f(x) = (x-1)^2 - 2, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - ((x_n - 1)^2 - 2)/2(x_n - 1);\ x = 1 + \sqrt{2},\ x \approx 2.4142,\ n = 5$

56) [T] $f(x) = e^x - 2, \quad x_0 = 1$

57) [T] $f(x) = \ln x - 1, \quad x_0 = 2$

Answer

$x_{n+1} = x_n - x_n(\ln(x_n) - 1);\ x = e,\ x \approx 2.7183,\ n = 5$

58) [T] Suppose you start with one liter of vinegar and repeatedly remove 0.1 L, replace with water, mix, and repeat.

a. Find a formula for the concentration after n steps.

b. After how many steps does the mixture contain less than 10% vinegar?

59) [T] A lake initially contains 2000 fish. Suppose that in the absence of predators or other causes of removal, the fish population increases by 6% each month. However, factoring in all causes, 150 fish are lost each month.

a. Explain why the fish population after n months is modeled by $P_n = 1.06P_{n-1} - 150$ with $P_0 = 2000$.

b. How many fish will be in the pond after one year?

Answer

a. Without losses, the population would obey $P_n = 1.06P_{n-1}$. The subtraction of 150 accounts for fish losses.
b. After 12 months, we have $P_{12} \approx 1494$.

60) [T] A bank account earns 5% interest compounded monthly. Suppose that $\$1000$ is initially deposited into the account, but that $\$10$ is withdrawn each month.

a. Show that the amount in the account after n months is $A_n = (1 + .05/12)A_{n-1} - 10;\ A_0 = 1000$.

b. How much money will be in the account after 1 year?

c. Is the amount increasing or decreasing?

d. Suppose that instead of $\$10$, a fixed amount d dollars is withdrawn each month. Find a value of d such that the amount in the account after each month remains $\$1000$.

e. What happens if d is greater than this amount?

61) [T] A student takes out a college loan of $\$10,000$ at an annual percentage rate of 6%, compounded monthly.

a. If the student makes payments of $\$100$ per month, how much does the student owe after 12 months?

b. After how many months will the loan be paid off?

Answer

a. The student owes $9383 after 12 months.

b. The loan will be paid in full after 139 months or eleven and a half years.

62) [T] Consider a series combining geometric growth and arithmetic decrease. Let $a_1 = 1$. Fix $a > 1$ and $0 < b < a$. Set $a_{n+1} = a.\, a_n - b$. Find a formula for a_{n+1} in terms of a_n, a, and b and a relationship between a and b such that a_n converges.

63) [T] The binary representation $x = 0.b_1 b_2 b_3 \ldots$ of a number x between 0 and 1 can be defined as follows. Let $b_1 = 0$ if $x < 1/2$ and $b_1 = 1$ if $1/2 \le x < 1$. Let $x_1 = 2x - b_1$. Let $b_2 = 0$ if $x_1 < 1/2$ and $b_2 = 1$ if $1/2 \le x < 1$. Let $x_2 = 2x_1 - b_2$ and in general, $x_n = 2x_{n-1} - b_n$ and $b_{n-}1 = 0$ if $x_n < 1/2$ and $b_{n-1} = 1$ if $1/2 \le x_n < 1$. Find the binary expansion of $1/3$.

Answer

$b_1 = 0, x_1 = 2/3, b_2 = 1, x_2 = 4/3 - 1 = 1/3,$ so the pattern repeats, and $1/3 = 0.010101\ldots$.

64) [T] To find an approximation for π, set $a_0 = \sqrt{2+1}, a_1 = \sqrt{2+a_0}$, and, in general, $a_{n+1} = \sqrt{2+a_n}$. Finally, set $p_n = 3.2^n\sqrt{2-a_n}$. Find the first ten terms of p_n and compare the values to π.

For the following two exercises, assume that you have access to a computer program or Internet source that can generate a list of zeros and ones of any desired length. Pseudo-random number generators (PRNGs) play an important role in simulating random noise in physical systems by creating sequences of zeros and ones that appear like the result of flipping a coin repeatedly. One of the simplest types of PRNGs recursively defines a random-looking sequence of N integers $a_1, a_2, \ldots, a_N$ by fixing two special integers $(K$ and M and letting a_{n+1} be the remainder after dividing $K.\, a_n$ into M, then creates a bit sequence of zeros and ones whose n^{th} term b_n is equal to one if a_n is odd and equal to zero if a_n is even. If the bits b_n are pseudo-random, then the behavior of their average $(b_1 + b_2 + \cdots + b_N)/N$ should be similar to behavior of averages of truly randomly generated bits.

65) [T] Starting with $K = 16,807$ and $M = 2,147,483,647$, using ten different starting values of a_1, compute sequences of bits b_n up to $n = 1000$, and compare their averages to ten such sequences generated by a random bit generator.

Answer

For the starting values $a_1 = 1, a_2 = 2, \ldots, a_1 = 10,$ the corresponding bit averages calculated by the method indicated are $0.5220, 0.5000, 0.4960, 0.4870, 0.4860, 0.4680, 0.5130, 0.5210, 0.5040,$ and 0.4840. Here is an example of ten corresponding averages of strings of 1000 bits generated by a random number generator:
$0.4880, 0.4870, 0.5150, 0.5490, 0.5130, 0.5180, 0.4860, 0.5030, 0.5050,$ and 0.4980. There is no real pattern in either type of average. The random-number-generated averages range between 0.4860 and 0.5490, a range of 0.0630, whereas the calculated PRNG bit averages range between 0.4680 and 0.5220, a range of 0.0540.

66) [T] Find the first 1000 digits of π using either a computer program or Internet resource. Create a bit sequence b_n by letting $b_n = 1$ if the nth digit of π is odd and $b_n = 0$ if the nth digit of π is even. Compute the average value of b_n and the average value of $d_n = |b_{n+1} - b_n|, n = 1, \ldots, 999$. Does the sequence b_n appear random? Do the differences between successive elements of b_n appear random?

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6.3: Infinite Series

Learning Objectives

- Explain the meaning of the sum of an infinite series.
- Calculate the sum of a geometric series.
- Evaluate a telescoping series.

We have seen that a sequence is an ordered set of terms. If you add these terms together, you get a series. In this section we define an infinite series and show how series are related to sequences. We also define what it means for a series to converge or diverge. We introduce one of the most important types of series: the geometric series. We will use geometric series in the next chapter to write certain functions as polynomials with an infinite number of terms. This process is important because it allows us to evaluate, differentiate, and integrate complicated functions by using polynomials that are easier to handle. We also discuss the harmonic series, arguably the most interesting divergent series because it just fails to converge.

Sums and Series

An infinite series is a sum of infinitely many terms and is written in the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

But what does this mean? We cannot add an infinite number of terms in the same way we can add a finite number of terms. Instead, the value of an infinite series is defined in terms of the limit of partial sums. A partial sum of an infinite series is a finite sum of the form

$$\sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k.$$

To see how we use partial sums to evaluate infinite series, consider the following example. Suppose oil is seeping into a lake such that 1000 gallons enters the lake the first week. During the second week, an additional 500 gallons of oil enters the lake. The third week, 250 more gallons enters the lake. Assume this pattern continues such that each week half as much oil enters the lake as did the previous week. If this continues forever, what can we say about the amount of oil in the lake? Will the amount of oil continue to get arbitrarily large, or is it possible that it approaches some finite amount? To answer this question, we look at the amount of oil in the lake after k weeks. Letting S_k denote the amount of oil in the lake (measured in thousands of gallons) after k weeks, we see that

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + 0.5 = 1 + \tfrac{1}{2} \\
S_3 &= 1 + 0.5 + 0.25 = 1 + \tfrac{1}{2} + \tfrac{1}{4} \\
S_4 &= 1 + 0.5 + 0.25 + 0.125 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} \\
S_5 &= 1 + 0.5 + 0.25 + 0.125 + 0.0625 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16}.
\end{aligned}$$

Looking at this pattern, we see that the amount of oil in the lake (in thousands of gallons) after k weeks is

$$S_k = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{k-1}} = \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1}.$$

We are interested in what happens as $k \to \infty$. Symbolically, the amount of oil in the lake as $k \to \infty$ is given by the infinite series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots.$$

At the same time, as $k \to \infty$, the amount of oil in the lake can be calculated by evaluating $\lim_{k\to\infty} S_k$. Therefore, the behavior of the infinite series can be determined by looking at the behavior of the sequence of partial sums S_k. If the sequence of partial sums S_k

converges, we say that the infinite series converges, and its sum is given by $\lim_{k\to\infty} S_k$. If the sequence S_k diverges, we say the infinite series diverges. We now turn our attention to determining the limit of this sequence S_k.

First, simplifying some of these partial sums, we see that

$$S_1 = 1$$
$$S_2 = 1 + \tfrac{1}{2} = \tfrac{3}{2}$$
$$S_3 = 1 + \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{7}{4}$$
$$S_4 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{15}{8}$$
$$S_5 = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{31}{16}.$$

Plotting some of these values in Figure 6.3.1, it appears that the sequence S_k could be approaching 2.

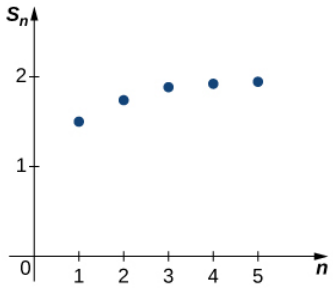


Figure 6.3.1: The graph shows the sequence of partial sums S_k. It appears that the sequence is approaching the value 2.

Let's look for more convincing evidence. In the following table, we list the values of S_k for several values of k.

k	5	10	15	20
S_k	1.9375	1.998	1.999939	1.999998

These data supply more evidence suggesting that the sequence S_k converges to 2. Later we will provide an analytic argument that can be used to prove that $\lim_{k\to\infty} S_k = 2$. For now, we rely on the numerical and graphical data to convince ourselves that the sequence of partial sums does actually converge to 2. Since this sequence of partial sums converges to 2, we say the infinite series converges to 2 and write

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 2.$$

Returning to the question about the oil in the lake, since this infinite series converges to 2, we conclude that the amount of oil in the lake will get arbitrarily close to 2000 gallons as the amount of time gets sufficiently large.

This series is an example of a geometric series. We discuss geometric series in more detail later in this section. First, we summarize what it means for an infinite series to converge.

Definition

An **infinite series** is an expression of the form

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots.$$

For each positive integer k, the sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

is called the kth **partial sum** of the infinite series. The partial sums form a sequence S_k. If the sequence of partial sums converges to a real number S, the infinite series converges. If we can describe the **convergence of a series** to S, we call S the sum of the series, and we write

$$\sum_{n=1}^{\infty} a_n = S.$$

If the sequence of partial sums diverges, we have the **divergence of a series.**

Note that the index for a series need not begin with $n = 1$ but can begin with any value. For example, the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1}$$

can also be written as

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \text{ or } \sum_{n=5}^{\infty} \left(\frac{1}{2}\right)^{n-5}.$$

Often it is convenient for the index to begin at 1, so if for some reason it begins at a different value, we can re-index by making a change of variables. For example, consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

By introducing the variable $m = n - 1$, so that $n = m + 1$, we can rewrite the series as

$$\sum_{m=1}^{\infty} \frac{1}{(m+1)^2}.$$

Example 6.3.1: Evaluating Limits of Sequences of Partial Sums

For each of the following series, use the sequence of partial sums to determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+1}$

b. $\displaystyle\sum_{n=1}^{\infty} (-1)^n$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution

a. The sequence of partial sums S_k satisfies

$$S_1 = \frac{1}{2}$$
$$S_2 = \frac{1}{2} + \frac{2}{3}$$
$$S_3 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4}$$
$$S_4 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} \quad .$$

Notice that each term added is greater than $1/2$. As a result, we see that

$$S_1 = \frac{1}{2}$$

$$S_2 = \frac{1}{2} + \frac{2}{3} > \frac{1}{2} + \frac{1}{2} = 2\left(\frac{1}{2}\right)$$
$$S_3 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 3\left(\frac{1}{2}\right)$$
$$S_4 = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 4\left(\frac{1}{2}\right).$$

From this pattern we can see that $S_k > k\left(\frac{1}{2}\right)$ for every integer k. Therefore, S_k is unbounded and consequently, diverges. Therefore, the infinite series $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

b. The sequence of partial sums S_k satisfies

$$S_1 = -1$$
$$S_2 = -1 + 1 = 0$$
$$S_3 = -1 + 1 - 1 = -1$$
$$S_4 = -1 + 1 - 1 + 1 = 0.$$

From this pattern we can see the sequence of partial sums is

$$S_k = -1, 0, -1, 0, \ldots.$$

Since this sequence diverges, the infinite series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

c. The sequence of partial sums S_k satisfies

$$S_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$$
$$S_2 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$
$$S_3 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$$
$$S_4 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} = \frac{4}{5}$$
$$S_5 = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} = \frac{5}{6}.$$

From this pattern, we can see that the kth partial sum is given by the explicit formula

$$S_k = \frac{k}{k+1}$$

.

Since $k/(k+1) \to 1,$ we conclude that the sequence of partial sums converges, and therefore the infinite series converges to 1. We have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Exercise 6.3.1

Determine whether the series $\sum_{n=1}^{\infty} \frac{n+1}{n}$ converges or diverges.

Hint

Look at the sequence of partial sums.

Answer

The series diverges because the kth partial sum $S_k > k$.

The Harmonic Series

A useful series to know about is the **harmonic series.** The harmonic series is defined as

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots.$$

This series is interesting because it diverges, but it diverges very slowly. By this we mean that the terms in the sequence of partial sums S_k approach infinity, but do so very slowly. We will show that the series diverges, but first we illustrate the slow growth of the terms in the sequence S_k in the following table.

k	10	100	1000	10,00	100,000	1,000,000
S_k	2.92897	5.18738	7.48547	9.78761	12.09015	14.39273

Even after $1,000,000$terms, the partial sum is still relatively small. From this table, it is not clear that this series actually diverges. However, we can show analytically that the sequence of partial sums diverges, and therefore the series diverges.

To show that the sequence of partial sums diverges, we show that the sequence of partial sums is unbounded. We begin by writing the first several partial sums:

$$\begin{aligned} S_1 &= 1 \\ S_2 &= 1 + \frac{1}{2} \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{3} \\ S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \quad . \end{aligned}$$

Notice that for the last two terms in S_4,

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4}$$

Therefore, we conclude that

$$S_4 > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{1}{2} + \frac{1}{2} = 1 + 2\left(\frac{1}{2}\right).$$

Using the same idea for S_8, we see that

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + 3\left(\frac{1}{2}\right).$$

From this pattern, we see that $S_1 = 1, S_2 = 1 + 1/2, S_4 > 1 + 2(1/2),$ and $S_8 > 1 + 3(1/2)$. More generally, it can be shown that $S_{2^j} > 1 + j(1/2)$ for all $j > 1$. Since $1 + j(1/2) \to \infty,$ we conclude that the sequence S_k is unbounded and therefore diverges. In the previous section, we stated that convergent sequences are bounded. Consequently, since S_k is unbounded, it diverges. Thus, the harmonic series diverges.

Algebraic Properties of Convergent Series

Since the sum of a convergent infinite series is defined as a limit of a sequence, the algebraic properties for series listed below follow directly from the algebraic properties for sequences.

Note 6.3.1: Algebraic Properties of Convergent Series

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be convergent series. Then the following algebraic properties hold.

i. The series $\sum_{n=1}^{\infty}(a_n+b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty} a_n+\sum_{n=1}^{\infty} b_n$. (Sum Rule)

ii. The series $\sum_{n=1}^{\infty}(a_n-b_n)$ converges, and $\sum_{n=1}^{\infty}(a_n-b_n)=\sum_{n=1}^{\infty} a_n-\sum_{n=1}^{\infty} b_n$. (Difference Rule)

iii. For any real number c, the series $\sum_{n=1}^{\infty} ca_n$ converges, and $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$. (Constant Multiple Rule)

Example 6.3.2: Using Algebraic Properties of Convergent Series

Evaluate $\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right]$.

Solution

We showed earlier that

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=1$$

and

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=2.$$

Since both of those series converge, we can apply the properties of Note 6.3.1 to evaluate

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right].$$

Using the sum rule, write

$$\sum_{n=1}^{\infty}\left[\frac{3}{n(n+1)}+\left(\frac{1}{2}\right)^{n-2}\right]=\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2}.$$

Then, using the constant multiple rule and the sums above, we can conclude that

$$\sum_{n=1}^{\infty}\frac{3}{n(n+1)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-2}=3\sum_{n=1}^{\infty}\frac{1}{n(n+1)}+\left(\frac{1}{2}\right)^{-1}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=3(1)+\left(\frac{1}{2}\right)^{-1}(2)=3+2(2)=7.$$

Exercise 6.3.2

Evaluate $\sum_{n=1}^{\infty}\frac{5}{2^{n-1}}$.

Hint

Rewrite as $\sum_{n=1}^{\infty}5\left(\frac{1}{2}\right)^{n-1}$.

Answer

10

Geometric Series

A **geometric series** is any series that we can write in the form

$$a+ar+ar^2+ar^3+\cdots=\sum_{n=1}^{\infty}ar^{n-1}.$$

Because the ratio of each term in this series to the previous term is r, the number r is called the ratio. We refer to a as the initial term because it is the first term in the series. For example, the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

is a geometric series with initial term $a=1$ and ratio $r=1/2$.

In general, when does a geometric series converge? Consider the geometric series

$$\sum_{n=1}^{\infty}ar^{n-1}$$

when $a>0$. Its sequence of partial sums S_k is given by

$$S_k=\sum_{n=1}^{k}ar^{n-1}=a+ar+ar^2+\cdots+ar^{k-1}.$$

Consider the case when $r=1$. In that case,

$$S_k=a+a(1)+a(1)^2+\cdots+a(1)^{k-1}=ak.$$

Since $a>0$, we know $ak\to\infty$ as $k\to\infty$. Therefore, the sequence of partial sums is unbounded and thus diverges. Consequently, the infinite series diverges for $r=1$. For $r\neq 1$, to find the limit of S_k, multiply Equation by $1-r$. Doing so, we see that

$$\begin{aligned}(1-r)S_k=a(1-r)(1+r+r^2+r^3+\cdots+r^{k-1})&=a[(1+r+r^2+r^3+\cdots+r^{k-1})-(r+r^2+r^3+\cdots+r^k)]\\&=a(1-r^k).\end{aligned}$$

All the other terms cancel out.

Therefore,

$$S_k=\frac{a(1-r^k)}{1-r}\text{ for }r\neq 1.$$

From our discussion in the previous section, we know that the geometric sequence $r^k\to 0$ if $|r|<1$ and that r^k diverges if $|r|>1$ or $r=\pm 1$. Therefore, for $|r|<1$, $S_k\to\dfrac{a}{1-r}$ and we have

$$\sum_{n=1}^{\infty}ar^{n-1}=\frac{a}{1-r}\text{ if }|r|<1.$$

If $|r|\geq 1$, S_k diverges, and therefore

$$\sum_{n=1}^{\infty}ar^{n-1}\text{ diverges if }|r|\geq 1.$$

Definitions: Diverging and Converging Series

A geometric series is a series of the form

$$\sum_{n=1}^{\infty}ar^{n-1}=a+ar+ar^2+ar^3+\cdots. \tag{6.3.1}$$

If $|r| < 1$, the series **converges**, and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \text{ for } |r| < 1.$$

If $|r| \geq 1$, the series **diverges**.

Geometric series sometimes appear in slightly different forms. For example, sometimes the index begins at a value other than $n = 1$ or the exponent involves a linear expression for n other than $n - 1$. As long as we can rewrite the series in the form given by Equation 6.3.1, it is a geometric series. For example, consider the series

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2}.$$

To see that this is a geometric series, we write out the first several terms:

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+2} = \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots = \frac{4}{9} + \frac{4}{9} \cdot \left(\frac{2}{3}\right) + \frac{4}{9} \cdot \left(\frac{2}{3}\right)^2 + \cdots.$$

We see that the initial term is $a = 4/9$ and the ratio is $r = 2/3$. Therefore, the series can be written as

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1}.$$

Since $r = 2/3 < 1$, this series converges, and its sum is given by

$$\sum_{n=1}^{\infty} \frac{4}{9} \cdot \left(\frac{2}{3}\right)^{n-1} = \frac{4/9}{1 - 2/3} = \frac{4}{3}.$$

Example 6.3.3: Determining Convergence or Divergence of a Geometric Series

Determine whether each of the following geometric series converges or diverges, and if it converges, find its sum.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}}$

b. $\displaystyle\sum_{n=1}^{\infty} e^{2n}$

Solution

a. Writing out the first several terms in the series, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{4^{n-1}} &= \frac{(-3)^2}{4^0} + \frac{(-3)^3}{4} + \frac{(-3)^4}{4^2} + \cdots \\ &= (-3)^2 + (-3)^2 \cdot \left(\frac{-3}{4}\right) + (-3)^2 \cdot \left(\frac{-3}{4}\right)^2 + \cdots \\ &= 9 + 9 \cdot \left(\frac{-3}{4}\right) + 9 \cdot \left(\frac{-3}{4}\right)^2 + \cdots.\end{aligned}$$

The initial term $a = -3$ and the ratio $r = -3/4$. Since $|r| = 3/4 < 1$, the series converges to

$$\frac{9}{1 - (-3/4)} = \frac{9}{7/4} = \frac{36}{7}.$$

b. Writing this series as

$$e^2 \sum_{n=1}^{\infty} (e^2)^{n-1}$$

we can see that this is a geometric series where $r = e^2 > 1$. Therefore, the series diverges.

Exercise 6.3.3

Determine whether the series $\sum_{n=1}^{\infty} \left(\frac{-2}{5}\right)^{n-1}$ converges or diverges. If it converges, find its sum.

Hint

$r = -2/5$

Answer

$5/7$

We now turn our attention to a nice application of geometric series. We show how they can be used to write repeating decimals as fractions of integers.

Example 6.3.4: Writing Repeating Decimals as Fractions of Integers

Use a geometric series to write $3.2\bar{6}$ as a fraction of integers.

Solution

Since $3.2\bar{6} = 3.262626\ldots$, first we write

$$\begin{aligned} 3.262626\ldots &= 3 + \frac{26}{100} + \frac{26}{10,000} + \frac{26}{100,000} + \cdots \\ &= 3 + \frac{26}{10^2} + \frac{26}{10^4} + \frac{26}{10^6} + \cdots. \end{aligned}$$

Ignoring the term 3, the rest of this expression is a geometric series with initial term $a = 26/10^2$ and ratio $r = 1/10^2$. Therefore, the sum of this series is

$$\frac{26/10^2}{1-(1/10^2)} = \frac{26/10^2}{99/10^2} = \frac{26}{99}.$$

Thus,

$$3.262626\ldots = 3 + \tfrac{26}{99} = \tfrac{323}{99}.$$

Exercise 6.3.4

Write $5.2\bar{7}$ as a fraction of integers.

Hint

By expressing this number as a series, find a geometric series with initial term $a = 7/100$ and ratio $r = 1/10$.

Answer

$475/90$

Example 6.3.5: Finding the Area of the Koch Snowflake

Define a sequence of figures $\{F_n\}$ recursively as follows (Figure 6.3.2). Let F_0 be an equilateral triangle with sides of length 1. For $n \geq 1$, let F_n be the curve created by removing the middle third of each side of F_{n-1} and replacing it with an equilateral triangle pointing outward. The limiting figure as $n \to \infty$ is known as **Koch's snowflake.**

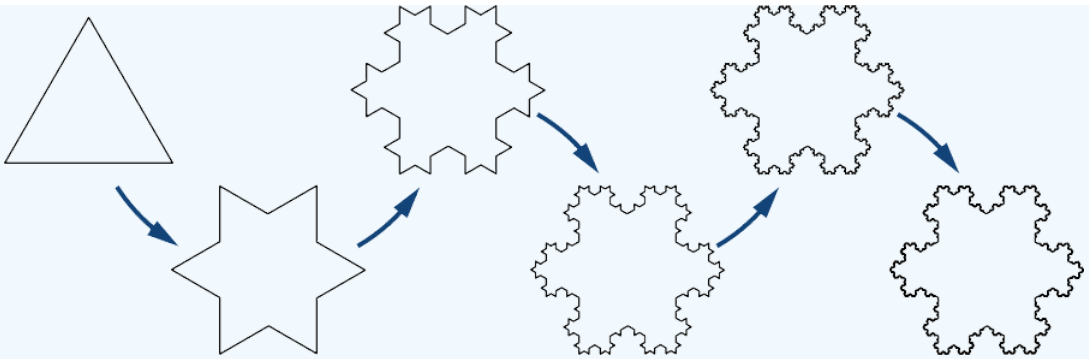


Figure 6.3.2: The first four figures, F_0, F_1, F_2, and F_3, in the construction of the Koch snowflake.

a. Find the length L_n of the perimeter of F_n. Evaluate $\lim_{n\to\infty} L_n$ to find the length of the perimeter of Koch's snowflake.

b. Find the area A_n of figure F_n. Evaluate $\lim_{n\to\infty} A_n$ to find the area of Koch's snowflake.

Solution

a. Let N_n denote the number of sides of figure F_n. Since F_0 is a triangle, $N_0 = 3$. Let l_n denote the length of each side of F_n. Since F_0 is an equilateral triangle with sides of length $l_0 = 1$, we now need to determine N_1 and l_1. Since F_1 is created by removing the middle third of each side and replacing that line segment with two line segments, for each side of F_0, we get four sides in F_1. Therefore, the number of sides for F_1 is

$$N_1 = 4 \cdot 3\,.$$

Since the length of each of these new line segments is $1/3$ the length of the line segments in F_0, the length of the line segments for F_1 is given by

$$l_1 = \tfrac{1}{3} \cdot 1 = \tfrac{1}{3}\ .$$

Similarly, for F_2, since the middle third of each side of F_1 is removed and replaced with two line segments, the number of sides in F_2 is given by

$$N_2 = 4N_1 = 4(4 \cdot 3) = 4^2 \cdot 3.$$

Since the length of each of these sides is $1/3$ the length of the sides of F_1, the length of each side of figure F_2 is given by

$$l_2 = \tfrac{1}{3} \cdot l_1 = \tfrac{1}{3} \cdot \tfrac{1}{3} = \left(\tfrac{1}{3}\right)^2\ .$$

More generally, since F_n is created by removing the middle third of each side of F_{n-1} and replacing that line segment with two line segments of length $\frac{1}{3}l_{n-1}$ in the shape of an equilateral triangle, we know that $N_n = 4N_{n-1}$ and $l_n = \dfrac{l_{n-1}}{3}$. Therefore, the number of sides of figure F_n is

$$N_n = 4^n \cdot 3$$

and the length of each side is

$$l_n = \left(\frac{1}{3}\right)^n.$$

Therefore, to calculate the perimeter of F_n, we multiply the number of sides N_n and the length of each side l_n. We conclude that the perimeter of F_n is given by

$$L_n = N_n \cdot l_n = 3 \cdot \left(\frac{4}{3}\right)^n$$

Therefore, the length of the perimeter of Koch's snowflake is

$$L = \lim_{n\to\infty} L_n = \infty.$$

b. Let T_n denote the area of each new triangle created when forming F_n. For $n = 0, T_0$ is the area of the original equilateral triangle. Therefore, $T_0 = A_0 = \sqrt{3}/4$. For $n \geq 1$, since the lengths of the sides of the new triangle are $1/3$ the length of the

sides of F_{n-1}, we have

$$T_n = \left(\frac{1}{3}\right)^2 \cdot T_{n-1} = \frac{1}{9} \cdot T_{n-1}.$$

Therefore, $T_n = \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4}$. Since a new triangle is formed on each side of F_{n-1},

$$A_n = A_{n-1} + N_{n-1} \cdot T_n = A_{n-1} + (3 \cdot 4_{n-1}) \cdot \left(\frac{1}{9}\right)^n \cdot \frac{\sqrt{3}}{4} = A_{n-1} + \frac{3}{4} \cdot \left(\frac{4}{9}\right)^n \cdot \frac{\sqrt{3}}{4}.$$

Writing out the first few terms A_0, A_1, A_2, we see that

$$A_0 = \tfrac{\sqrt{3}}{4}$$

$$A_1 = A_0 + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right) \cdot \tfrac{\sqrt{3}}{4} = \tfrac{\sqrt{3}}{4} + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right) \cdot \tfrac{\sqrt{3}}{4} = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right)\right]$$

$$A_2 = A_1 + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right)^2 \cdot \tfrac{\sqrt{3}}{4} = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right)\right] + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right)^2 \cdot \tfrac{\sqrt{3}}{4} = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right) + \tfrac{3}{4} \cdot \left(\tfrac{4}{9}\right)^2\right] \quad .$$

More generally,

$$A_n = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{3}{4}\left(\tfrac{4}{9} + \left(\tfrac{4}{9}\right)^2 + \cdots + \left(\tfrac{4}{9}\right)^n\right)\right] \quad .$$

Factoring $4/9$ out of each term inside the inner parentheses, we rewrite our expression as

$$A_n = \tfrac{\sqrt{3}}{4}\left[1 + \tfrac{1}{3}\left(1 + \tfrac{4}{9} + \left(\tfrac{4}{9}\right)^2 + \cdots + \left(\tfrac{4}{9}\right)^{n-1}\right)\right] \quad .$$

The expression $1 + \left(\frac{4}{9}\right) + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}$ is a geometric sum. As shown earlier, this sum satisfies

$$1 + \tfrac{4}{9} + \left(\tfrac{4}{9}\right)^2 + \cdots + \left(\tfrac{4}{9}\right)^{n-1} = \frac{1 - (4/9)^n}{1 - (4/9)}.$$

Substituting this expression into the expression above and simplifying, we conclude that

$$A_n = \frac{\sqrt{3}}{4}\left[1 + \frac{1}{3}\left(\frac{1 - (4/9)^n}{1 - (4/9)}\right)\right] = \frac{\sqrt{3}}{4}\left[\frac{8}{5} - \frac{3}{5}\left(\frac{4}{9}\right)^n\right].$$

Therefore, the area of Koch's snowflake is

$$A = \lim_{n \to \infty} A_n = \frac{2\sqrt{3}}{5}.$$

Analysis

The Koch snowflake is interesting because it has finite area, yet infinite perimeter. Although at first this may seem impossible, recall that you have seen similar examples earlier in the text. For example, consider the region bounded by the curve $y = 1/x^2$ and the x-axis on the interval $[1, \infty)$. Since the improper integral

$$\int_1^\infty \frac{1}{x^2}\, dx$$

converges, the area of this region is finite, even though the perimeter is infinite.

Telescoping Series

Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. We discussed this series in Example 6.3.1c, showing that the series converges by writing out the first several partial sums $S_1, S_2, \ldots, S_6$ and noticing that they are all of the form $S_k = \dfrac{k}{k+1}$. Here we use a different technique to show that this series converges. By using partial fractions, we can write

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Therefore, the series can be written as

$$\sum_{n=1}^{\infty}\left[\frac{1}{n}-\frac{1}{n+1}\right]=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots.$$

Writing out the first several terms in the sequence of partial sums S_k, we see that

$$\begin{aligned}S_1&=1-\tfrac{1}{2}\\S_2&=\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)=1-\tfrac{1}{3}\\S_3&=\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)=1-\tfrac{1}{4}\quad.\end{aligned}$$

In general,

$$S_k=\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{2}-\tfrac{1}{3}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\cdots+\left(\tfrac{1}{k}-\tfrac{1}{k+1}\right)=1-\frac{1}{k+1}\quad.$$

We notice that the middle terms cancel each other out, leaving only the first and last terms. In a sense, the series collapses like a spyglass with tubes that disappear into each other to shorten the telescope. For this reason, we call a series that has this property a telescoping series. For this series, since $S_k=1-1/(k+1)$ and $1/(k+1)\to 0$ as $k\to\infty$, the sequence of partial sums converges to 1, and therefore the series converges to 1.

Definition

A **telescoping series** is a series in which most of the terms cancel in each of the partial sums, leaving only some of the first terms and some of the last terms.

For example, any series of the form

$$\sum_{n=1}^{\infty}[b_n-b_{n+1}]=(b_1-b_2)+(b_2-b_3)+(b_3-b_4)+\cdots$$

is a telescoping series. We can see this by writing out some of the partial sums. In particular, we see that

$$\begin{aligned}S_1&=b_1-b_2\\S_2&=(b_1-b_2)+(b_2-b_3)=b_1-b_3\\S_3&=(b_1-b_2)+(b_2-b_3)+(b_3-b_4)=b_1-b_4.\end{aligned}$$

In general, the kth partial sum of this series is

$$S_k=b_1-b_{k+1}\ .$$

Since the kth partial sum can be simplified to the difference of these two terms, the sequence of partial sums S_k will converge if and only if the sequence b_{k+1} converges. Moreover, if the sequence b_{k+1} converges to some finite number B, then the sequence of partial sums converges to b_1-B, and therefore

$$\sum_{n=1}^{\infty}[b_n-b_{n+1}]=b_1-B.$$

In the next example, we show how to use these ideas to analyze a telescoping series of this form.

Example 6.3.6: Evaluating a Telescoping Series

Determine whether the telescoping series

$$\sum_{n=1}^{\infty}\left[\cos\left(\frac{1}{n}\right)-\cos\left(\frac{1}{n+1}\right)\right]$$

converges or diverges. If it converges, find its sum.

Solution

By writing out terms in the sequence of partial sums, we can see that

$$S_1=\cos(1)-\cos(\tfrac{1}{2})$$

$$S_2 = (\cos(1) - \cos(\tfrac{1}{2})) + (\cos(\tfrac{1}{2}) - \cos(\tfrac{1}{3})) = \cos(1) - \cos(\tfrac{1}{3})$$

$$S_3 = (\cos(1) - \cos(\tfrac{1}{2})) + (\cos(\tfrac{1}{2}) - \cos(\tfrac{1}{3})) + (\cos(\tfrac{1}{3}) - \cos(\tfrac{1}{4}))$$

$$= \cos(1) - \cos(\tfrac{1}{4}).$$

In general,

$$S_k = \cos(1) - \cos\left(\tfrac{1}{k+1}\right).$$

Since $1/(k+1) \to 0$ as $k \to \infty$ and $\cos x$ is a continuous function, $\cos(1/(k+1)) \to \cos(0) = 1$. Therefore, we conclude that $S_k \to \cos(1) - 1$. The telescoping series converges and the sum is given by

$$\sum_{n=1}^{\infty} \left[\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right)\right] = \cos(1) - 1.$$

Exercise 6.3.5

Determine whether $\sum_{n=1}^{\infty} [e^{1/n} - e^{1/(n+1)}]$ converges or diverges. If it converges, find its sum.

Hint

Write out the sequence of partial sums to see which terms cancel.

Answer

$e - 1$

Euler's Constant

We have shown that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Here we investigate the behavior of the partial sums S_k as $k \to \infty$. In particular, we show that they behave like the natural logarithm function by showing that there exists a constant γ such that

$$\sum_{n=1}^{k} \left(\frac{1}{n} - \ln k\right) \to \gamma \text{ as } k \to \infty.$$

This constant γ is known as **Euler's constant.**

1. Let $T_k = \sum_{n=1}^{k} \left(\frac{1}{n} - \ln k\right)$. Evaluate T_k for various values of k.

2. For T_k as defined in part 1. show that the sequence T_k converges by using the following steps.

   a. Show that the sequence T_k is monotone decreasing. (Hint: Show that $\ln(1 + 1/k > 1/(k+1))$

   b. Show that the sequence T_k is bounded below by zero. (Hint: Express $\ln k$ as a definite integral.)

   c. Use the Monotone Convergence Theorem to conclude that the sequence T_k converges. The limit γ is Euler's constant.

3. Now estimate how far T_k is from γ for a given integer k. Prove that for $k \geq 1, 0 < T_k - \gamma \leq 1/k$ by using the following steps.

   a. Show that $\ln(k+1) - \ln k < 1/k$.

   b. Use the result from part a. to show that for any integer k,

   $$T_k - T_{k+1} < \frac{1}{k} - \frac{1}{k+1}.$$

   c. For any integers k and j such that $j > k$, express $T_k - T_j$ as a telescoping sum by writing

$$T_k - T_j = (T_k - T_{k+1}) + (T_{k+1} - T_{k+2}) + (T_{k+2} - T_{k+3}) + \cdots + (T_{j-1} - T_j).$$

Use the result from part b. combined with this telescoping sum to conclude that

$$T_k - T_j < \frac{1}{k} - \frac{1}{j}.$$

a. Apply the limit to both sides of the inequality in part c. to conclude that

$$T_k - \gamma \leq \frac{1}{k}.$$

e. Estimate γ to an accuracy of within **0.001**.

Key Concepts

- Given the infinite series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

and the corresponding sequence of partial sums S_k where

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k \quad ,$$

the series converges if and only if the sequence S_k converges.

- The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \geq 1$. For $|r| < 1$,

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \, .$$

- The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

diverges.

- A series of the form $\sum_{n=1}^{\infty} [b_n - b_{n+1}] = [b_1 - b_2] + [b_2 - b_3] + [b_3 - b_4] + \cdots + [b_n - b_{n+1}] + \cdots$ is a telescoping series. The kth partial sum of this series is given by $S_k = b_1 - b_{k+1}$. The series will converge if and only if $\lim_{k \to \infty} b_{k+1}$ exists. In that case,

$$\sum_{n=1}^{\infty} [b_n - b_{n+1}] = b_1 - \lim_{k \to \infty} (b_{k+1}) \, .$$

Key Equations

- **Harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

- **Sum of a geometric series**

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad \text{for } |r| < 1$$

Glossary

convergence of a series

a series converges if the sequence of partial sums for that series converges

divergence of a series

a series diverges if the sequence of partial sums for that series diverges

geometric series

a geometric series is a series that can be written in the form

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$$

harmonic series

the harmonic series takes the form

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

infinite series

an infinite series is an expression of the form

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n$$

partial sum

the kth partial sum of the infinite series $\sum_{n=1}^{\infty} a_n$ is the finite sum

$$S_k = \sum_{n=1}^{k} a_n = a_1 + a_2 + a_3 + \cdots + a_k$$

telescoping series

a telescoping series is one in which most of the terms cancel in each of the partial sums

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6.3E: Exercises for Infinite Series

In exercises 1 - 4, use sigma notation to write each expressions as an infinite series.

1) $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

Answer

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

2) $1-1+1-1+\cdots$

3) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$

Answer

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

4) $\sin 1+\sin\frac{1}{2}+\sin\frac{1}{3}+\sin\frac{1}{4}+\cdots$

In exercises 5 - 8, compute the first four partial sums $S_1,\ldots,S_4$ for the series having n^{th} term a_n starting with $n=1$ as follows.

5) $a_n=n$

Answer

$1,3,6,10$

6) $a_n=1/n$

7) $a_n=\sin\frac{n\pi}{2}$

Answer

$1,1,0,0$

8) $a_n=(-1)^n$

In exercises 9 - 12, compute the general term a_n of the series with the given partial sum S_n. If the sequence of partial sums converges, find its limit S.

9) $S_n=1-\frac{1}{n},\quad n\geq 2$

Answer

$a_n=S_n-S_{n-1}=\dfrac{1}{n-1}-\dfrac{1}{n}.$ Since $S=\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\left(1-\dfrac{1}{n}\right)=1,$ the series converges to $S=1.$

10) $S_n=\dfrac{n(n+1)}{2},\quad n\geq 1$

11) $S_n=\sqrt{n},\quad n\geq 2$

Answer

$a_n=S_n-S_{n-1}=\sqrt{n}-\sqrt{n-1}=\dfrac{1}{\sqrt{n-1}+\sqrt{n}}.$

The series diverges because the partial sums are unbounded.

That is, $\lim\limits_{n\to\infty}S_n=\lim\limits_{n\to\infty}\sqrt{n}=\infty.$

12) $S_n=2-\dfrac{n+2}{2^n},\quad n\geq 1$

For each series in exercises 13 - 16, use the sequence of partial sums to determine whether the series converges or diverges.

13) $\sum_{n=1}^{\infty}\dfrac{n}{n+2}$

Answer

$S_1=1/3,$

$S_2=1/3+2/4>1/3+1/3=2/3,$

$S_3 = 1/3 + 2/4 + 3/5 > 3 \cdot (1/3) = 1$.
In general $S_k > k/3$, so the series diverges.
Note that the nth Term Test for Divergence could also be used to prove that this series diverges.

14) $\sum_{n=1}^{\infty}(1-(-1)^n))$

15) $\sum_{n=1}^{\infty}\dfrac{1}{(n+1)(n+2)}$ (Hint: Use a partial fraction decomposition like that for $\sum_{n=1}^{\infty}\dfrac{1}{n(n+1)}$.)

Answer

$S_1 = 1/(2\cdot 3) = 1/6 = 2/3 - 1/2,$
$S_2 = 1/(2\cdot 3) + 1/(3\cdot 4) = 2/12 + 1/12 = 1/4 = 3/4 - 1/2,$
$S_3 = 1/(2\cdot 3) + 1/(3\cdot 4) + 1/(4\cdot 5) = 10/60 + 5/60 + 3/60 = 3/10 = 4/5 - 1/2,$
$S_4 = 1/(2\cdot 3) + 1/(3\cdot 4) + 1/(4\cdot 5) + 1/(5\cdot 6) = 10/60 + 5/60 + 3/60 + 2/60 = 1/3 = 5/6 - 1/2.$

The pattern is $S_k = \dfrac{k+1}{k+2} - \dfrac{1}{2}$.

Then $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\dfrac{k+1}{k+2} - \dfrac{1}{2}\right) = \dfrac{1}{2}$, so the series converges to $1/2$.

16) $\sum_{n=1}^{\infty}\dfrac{1}{2n+1}$ (Hint: Follow the reasoning for $\sum_{n=1}^{\infty}\dfrac{1}{n}$.)

Suppose that $\sum_{n=1}^{\infty} a_n = 1$, that $\sum_{n=1}^{\infty} b_n = -1$, that $a_1 = 2$, and $b_1 = -3$. Use this information to find the sum of the indicated series in exercises 17 - 20.

17) $\sum_{n=1}^{\infty}(a_n + b_n)$

Answer

$$\sum_{n=1}^{\infty}(a_n+b_n) \quad = \quad \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \quad = \quad 1 + (-1) \quad = \quad 0$$

18) $\sum_{n=1}^{\infty}(a_n - 2b_n)$

19) $\sum_{n=2}^{\infty}(a_n - b_n)$

Answer

$$\sum_{n=2}^{\infty}(a_n-b_n) \quad = \quad \sum_{n=2}^{\infty} a_n - \sum_{n=2}^{\infty} b_n \quad = \quad \left(\sum_{n=1}^{\infty} a_n - a_1\right) - \left(\sum_{n=1}^{\infty} b_n - b_1\right) \quad = \quad (1-2) - (-1-(-3)) = -1-2 \quad = \quad -3$$

20) $\sum_{n=1}^{\infty}(3a_{n+1} - 4b_{n+1})$

In exercises 21 - 26, state whether the given series converges or diverges and explain why.

21) $\sum_{n=1}^{\infty}\dfrac{1}{n+1000}$ (Hint: Rewrite using a change of index.)

Answer

The series diverges, $\sum_{n=1001}^{\infty}\dfrac{1}{n}$

22) $\sum_{n=1}^{\infty}\dfrac{1}{n+10^{80}}$ (Hint: Rewrite using a change of index.)

23) $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$

Answer

This is a convergent geometric series, since $r = \frac{1}{10} < 1$

24) $1 + \frac{e}{\pi} + \frac{e^2}{\pi^2} + \frac{e^3}{\pi^3} + \cdots$

25) $1 + \frac{\pi}{e^2} + \frac{\pi^2}{e^4} + \frac{\pi^3}{e^6} + \frac{\pi^4}{e^8} + \cdots$

Answer

This is a convergent geometric series, since $r = \pi/e^2 < 1$

26) $1 - \sqrt{\frac{\pi}{3}} + \sqrt{\frac{\pi^2}{9}} - \sqrt{\frac{\pi^3}{27}} + \cdots$

For each a_n in exercises 27 - 30, write its sum as a geometric series of the form $\sum_{n=1}^{\infty} ar^n$. State whether the series converges and if it does, find the exact value of its sum.

27) $a_1 = -1$ and $\dfrac{a_n}{a_{n+1}} = -5$ for $n \geq 1$.

Answer

$\sum_{n=1}^{\infty} 5 \cdot (-1/5)^n$, converges to $-5/6$

28) $a_1 = 2$ and $\dfrac{a_n}{a_{n+1}} = 1/2$ for $n \geq 1$.

29) $a_1 = 10$ and $\dfrac{a_n}{a_{n+1}} = 10$ for $n \geq 1$.

Answer

$\sum_{n=1}^{\infty} 100 \cdot (1/10)^n$,converges to $\frac{100}{9}$

30) $a_1 = \frac{1}{10}$ and $a_n/a_{n+1} = -10$ for $n \geq 1$.

In exercises 31 - 34, use the identity $\dfrac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ (which is true for $|y| < 1$) to express each function as a geometric series in the indicated term.

31) $\dfrac{x}{1+x}$ in x

Answer

$x \sum_{n=0}^{\infty} (-x)^n = \sum_{n=1}^{\infty} (-1)^{n-1} x^n$

32) $\dfrac{\sqrt{x}}{1 - x^{3/2}}$ in $\sqrt{x}$

33) $\dfrac{1}{1 + \sin^2 x}$ in $\sin x$

Answer

$\sum_{n=0}^{\infty} (-1)^n \sin^{2n}(x)$

34) $\sec^2 x$ in $\sin x$

In exercises 35 - 38, evaluate the telescoping series or state whether the series diverges.

35) $\sum_{n=1}^{\infty} 2^{1/n} - 2^{1/(n+1)}$

Answer

$S_k = 2 - 2^{1/(k+1)} \to 1$ as $k \to \infty$.

36) $\sum_{n=1}^{\infty} \frac{1}{n^{13}} - \frac{1}{(n+1)^{13}}$

37) $\sum_{n=1}^{\infty} (\sqrt{n} - \sqrt{n+1})$

Answer

$S_k = 1 - \sqrt{k+1}$ diverges

38) $\sum_{n=1}^{\infty} (\sin n - \sin(n+1))$

Express each series in exercises 39 - 42 as a telescoping sum and evaluate its nth partial sum.

39) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

Answer

$\sum_{n=1}^{\infty} [\ln n - \ln(n+1)],$
$S_k = -\ln(k+1)$

40) $\sum_{n=1}^{\infty} \frac{2n+1}{(n^2+n)^2}$ (Hint: Factor denominator and use partial fractions.)

41) $\sum_{n=2}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{(\ln n)\ln(n+1)}$

Answer

$a_n = \frac{1}{\ln n} - \frac{1}{\ln(n+1)}$ and $S_k = \frac{1}{\ln(2)} - \frac{1}{\ln(k+1)} \to \frac{1}{\ln(2)}$

42) $\sum_{n=1}^{\infty} \frac{(n+2)}{n(n+1)2^{n+1}}$ (Hint: Look at $1/(n2^n)$.

A general telescoping series is one in which all but the first few terms cancel out after summing a given number of successive terms.

43) Let $a_n = f(n) - 2f(n+1) + f(n+2)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

Answer

$\sum_{n=1}^{\infty} a_n = f(1) - f(2)$

44) $a_n = f(n) - f(n+1) - f(n+2) + f(n+3)$, in which $f(n) \to 0$ as $n \to \infty$. Find $\sum_{n=1}^{\infty} a_n$.

45) Suppose that $a_n = c_0 f(n) + c_1 f(n+1) + c_2 f(n+2) + c_3 f(n+3) + c_4 f(n+4)$, where $f(n) \to 0$ as $n \to \infty$. Find a condition on the coefficients $c_0, \ldots, c_4$ that make this a general telescoping series.

Answer

$c_0 + c_1 + c_2 + c_3 + c_4 = 0$

46) Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ (Hint: $\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$)

47) Evaluate $\sum_{n=2}^{\infty} \frac{2}{n^3 - n}$.

Answer

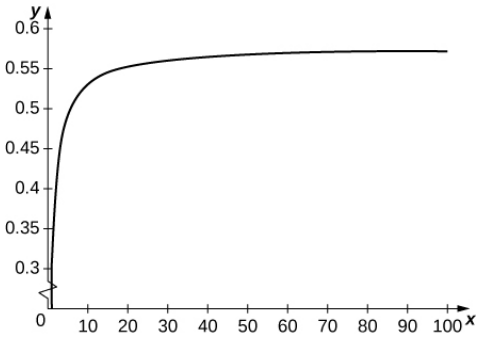
$\frac{2}{n^3-1} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1},$
$S_n = (1 - 1 + 1/3) + (1/2 - 2/3 + 1/4) + (1/3 - 2/4 + 1/5) + (1/4 - 2/5 + 1/6) + \cdots = 1/2$

48) Find a formula for $\sum_{n=1}^{\infty} \frac{1}{n(n+N)}$ where N is a positive integer.

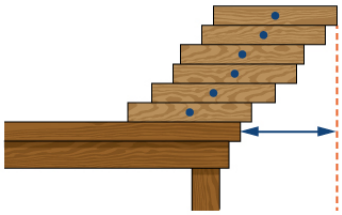
49) [T] Define a sequence $t_k = \sum_{n=1}^{k-1} (1/k) - \ln k$. Use the graph of $1/x$ to verify that t_k is increasing. Plot t_k for $k = 1 \ldots 100$ and state whether it appears that the sequence converges.

Answer

t_k converges to $0.57721\ldots$ $t_k$is a sum of rectangles of height $1/k$over the interval $[k, k+1]$ which lie above the graph of $1/x$.



50) [T] Suppose that N equal uniform rectangular blocks are stacked one on top of the other, allowing for some overhang. Archimedes' law of the lever implies that the stack of N blocks is stable as long as the center of mass of the top $(N-1)$ blocks lies at the edge of the bottom block. Let x denote the position of the edge of the bottom block, and think of its position as relative to the center of the next-to-bottom block. This implies that $(N-1)x = \left(\frac{1}{2} - x\right)$ or $x = 1/(2N)$. Use this expression to compute the maximum overhang (the position of the edge of the top block over the edge of the bottom block.) See the following figure.



Each of the following infinite series converges to the given multiple of π or $1/\pi$.

In each case, find the minimum value of N such that the Nth partial sum of the series accurately approximates the left-hand side to the given number of decimal places, and give the desired approximate value. Up to 15 decimals place, $\pi = 3.141592653589793\ldots$.

51) [T] $\pi = -3 + \sum_{n=1}^{\infty} \frac{n 2^n n!^2}{(2n)!}$, error < 0.0001

Answer

$N = 22$,
$S_N = 6.1415$

52) [T] $\frac{\pi}{2} = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty} \frac{2^k k!^2}{(2k+1)!}$, error $< 10^{-4}$

53) [T] $\frac{9801}{2\pi} = \frac{4}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}$, error $< 10^{-12}$

Answer

$N = 3$,
$S_N = 1.559877597243667\ldots$

54) [T] $\frac{1}{12\pi} = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$, error $< 10^{-15}$

55) [T] A fair coin is one that has probability $1/2$ of coming up heads when flipped.

a. What is the probability that a fair coin will come up tails n times in a row?

b. Find the probability that a coin comes up heads for the first time after an even number of coin flips.

Answer

a. The probability of any given ordered sequence of outcomes for n coin flips is $1/2^n$.

b. The probability of coming up heads for the first time on the n th flip is the probability of the sequence $TT\ldots TH$ which is $1/2^n$. The probability of coming up heads for the first time on an even flip is $\sum_{n=1}^{\infty} 1/2^{2n}$ or $1/3$.

56) [T] Find the probability that a fair coin is flipped a multiple of three times before coming up heads.

57) [T] Find the probability that a fair coin will come up heads for the second time after an even number of flips.

Answer

$5/9$

58) [T] Find a series that expresses the probability that a fair coin will come up heads for the second time on a multiple of three flips.

59) [T] The expected number of times that a fair coin will come up heads is defined as the sum over $n = 1, 2, \ldots$ of n times the probability that the coin will come up heads exactly n times in a row, or $\frac{n}{2^{n+1}}$. Compute the expected number of consecutive times that a fair coin will come up heads.

Answer

$E = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1,$ as can be shown using summation by parts

60) [T] A person deposits $\$10$ at the beginning of each quarter into a bank account that earns 4 annual interest compounded quarterly (four times a year).

a. Show that the interest accumulated after n quarters is $\$10(\frac{1.01^{n+1}-1}{0.01} - n)$.

b. Find the first eight terms of the sequence.

c. How much interest has accumulated after 2 years?

61) [T] Suppose that the amount of a drug in a patient's system diminishes by a multiplicative factor $r < 1$ each hour. Suppose that a new dose is administered every N hours. Find an expression that gives the amount $A(n)$ in the patient's system after n hours for each n in terms of the dosage d and the ratio r. (Hint: Write $n = mN + k$, where $0 \le k < N$, and sum over values from the different doses administered.)

Answer

The part of the first dose after n hours is dr^n, the part of the second dose is dr^{n-N}, and, in general, the part remaining of the mth dose is dr^{n-mN}, so $A(n) = \sum_{l=0}^{m} dr^{n-lN} = \sum_{l=0}^{m} dr^{k+(m-l)N} = \sum_{q=0}^{m} dr^{k+qN} = dr^k \sum_{q=0}^{m} r^{Nq} = dr^k \frac{1-r^{(m+1)N}}{1-r^N},$ where $n = k + mN$.

62) [T] A certain drug is effective for an average patient only if there is at least 1 mg per kg in the patient's system, while it is safe only if there is at most 2 mg per kg in an average patient's system. Suppose that the amount in a patient's system diminishes by a multiplicative factor of 0.9 each hour after a dose is administered. Find the maximum interval N of hours between doses, and corresponding dose range d (in mg/kg) for this N that will enable use of the drug to be both safe and effective in the long term.

63) Suppose that $a_n \ge 0$ is a sequence of numbers. Explain why the sequence of partial sums of a_n is increasing.

Answer

$S_{N+1} = a_{N+1} + S_N \ge S_N$

64) [T] Suppose that a_n is a sequence of positive numbers and the sequence S_n of partial sums of a_n is bounded above. Explain why $\sum_{n=1}^{\infty} a_n$ converges. Does the conclusion remain true if we remove the hypothesis $a_n \ge 0$?

65) [T] Suppose that $a_1 = S_1 = 1$ and that, for given numbers $S > 1$ and $0 < k < 1$, one defines $a_{n+1} = k(S - S_n)$ and $S_{n+1} = a_{n+1} + S_n$. Does S_n converge? If so, to what? (Hint: First argue that $S_n < S$ for all n and S_n is increasing.)

Answer

Since $S > 1, a_2 > 0,$ and since $k < 1, S_2 = 1 + a_2 < 1 + (S - 1) = S$. If $S_n > S$ for some n, then there is a smallest n. For this $n, S > S_{n-1}$, so $S_n = S_{n-1} + k(S - S_{n-1}) = kS + (1 - k)S_{n-1} < S$, a contradiction. Thus $S_n < S$ and $a_{n+1} > 0$ for all n, so S_n is

increasing and bounded by S. Let $S_* = \lim S_n$. If $S_* < S$, then $\delta = k(S - S_*) > 0$, but we can find n such that $S_* - S_n < \delta/2$, which implies that $S_{n+1} = S_n + k(S - S_n) > S_* + \delta/2$, contradicting that Sn is increasing to S_*. Thus $S_n \to S$.

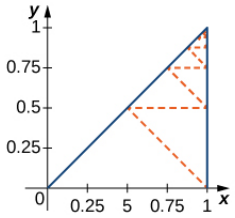
66) [T] A version of **von Bertalanffy growth** can be used to estimate the age of an individual in a homogeneous species from its length if the annual increase in year $n+1$ satisfies $a_{n+1} = k(S - S_n)$, with S_n as the length at year n, S as a limiting length, and k as a relative growth constant. If $S_1 = 3, S = 9,$ and $k = 1/2$, numerically estimate the smallest value of n such that $S_n \geq 8$. Note that $S_{n+1} = S_n + a_{n+1}$. Find the corresponding n when $k = 1/4$.

67) [T] Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms. Explain why $\lim_{N\to\infty} \sum_{n=N+1}^{\infty} a_n = 0$.

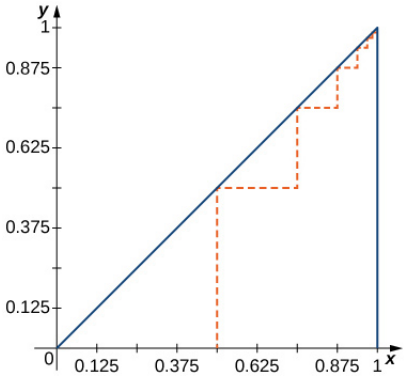
Answer

Let $S_k = \sum_{n=1}^{k} a_n$ and $S_k \to L$. Then S_k eventually becomes arbitrarily close to L, which means that $L - S_N = \sum_{n=N+1}^{\infty} a_n$ becomes arbitrarily small as $N \to \infty$.

68) [T] Find the length of the dashed zig-zag path in the following figure.



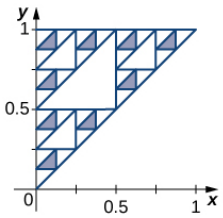
69) [T] Find the total length of the dashed path in the following figure.



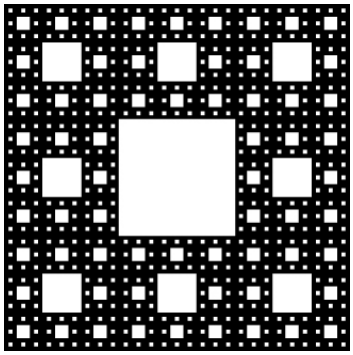
Answer

$L = \left(1 + \frac{1}{2}\right) \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{3}{2}$.

70) [T] The **Sierpinski triangle** is obtained from a triangle by deleting the middle fourth as indicated in the first step, by deleting the middle fourths of the remaining three congruent triangles in the second step, and in general deleting the middle fourths of the remaining triangles in each successive step. Assuming that the original triangle is shown in the figure, find the areas of the remaining parts of the original triangle after N steps and find the total length of all of the boundary triangles after N steps.



71) [T] The Sierpinski gasket is obtained by dividing the unit square into nine equal sub-squares, removing the middle square, then doing the same at each stage to the remaining sub-squares. The figure shows the remaining set after four iterations. Compute the total area removed after N stages, and compute the length the total perimeter of the remaining set after N stages.



Answer

At stage one a square of area $1/9$ is removed, at stage 2 one removes 8 squares of area $1/9^2$, at stage three one removes 8^2 squares of area $1/9^3$, and so on. The total removed area after N stages is $\sum_{n=0}^{N-1} \frac{8^N}{9^{N+1}} = \frac{1}{8} \cdot \frac{1-(8/9)^N}{1-8/9} \to 1$ as $N \to \infty$. The total perimeter is $4 + 4\sum_{n=0}^{\infty} \frac{8^N}{3^{N+1}} \to \infty$.

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6.4: The Divergence and Integral Tests

Learning Objectives

- Use the nth Term Test for Divergence to determine if a series diverges.
- Use the Integral Test to determine the convergence or divergence of a series.
- Estimate the value of a series by finding bounds on its remainder term.

In the previous section, we determined the convergence or divergence of several series by explicitly calculating the limit of the sequence of partial sums S_k. In practice, explicitly calculating this limit can be difficult or impossible. Luckily, several tests exist that allow us to determine convergence or divergence for many types of series. In this section, we discuss two of these tests: the divergence test and the integral test. We will examine several other tests in the rest of this chapter and then summarize how and when to use them.

The nth Term Test for Divergence

For a series $\sum_{n=1}^{\infty} a_n$ to converge, the nth term a_n must satisfy $a_n \to 0$ as $n \to \infty$. Therefore, from the algebraic limit properties of sequences,

$$\begin{aligned}\lim_{k\to\infty} a_k &= \lim_{k\to\infty} (S_k - S_{k-1}) \\ &= \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} \\ &= S - S = 0.\end{aligned}$$

Therefore, if $\sum_{n=1}^{\infty} a_n$ converges, the nth term $a_n \to 0$ as $n \to \infty$. An important consequence of this fact is the following statement:

$$\text{If } a_n \nrightarrow 0 \text{ as } n \to \infty, \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

This test is known as the divergence test because it provides a way of proving that a series diverges.

Theorem: The nth Term Test for Divergence

If $\lim_{n\to\infty} a_n = c \neq 0$ or $\lim_{n\to\infty} a_n$ does not exist, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

It is important to note that the inverse statement of this theorem is not true. That is, if $\lim_{n\to\infty} a_n = 0$, we cannot make any conclusion about the convergence of $\sum_{n=1}^{\infty} a_n$.

For example, $\lim_{n\to 0} \frac{1}{n} = 0$, but the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. In this section and the remaining sections of this chapter, we show many more examples of such series. Consequently, although we can use the divergence test to show that a series diverges, we cannot use it to prove that a series converges. Specifically, if $a_n \to 0$, the divergence test is inconclusive.

Example 6.4.1: Using the nth Term Test for Divergence

For each of the following series, apply the ✔ n^{th} Term Test for Divergence. If this test proves that the series diverges, state so. Otherwise, indicate that the ✔ n^{th} Term Test for Divergence is inconclusive.

a. $\sum_{n=1}^{\infty} \frac{n}{3n-1}$

b. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

c. $\sum_{n=1}^{\infty} e^{1/n^2}$

Solution

a. Since $\lim_{n\to\infty} \frac{n}{3n-1} = \frac{1}{3} \neq 0$, by the ✔ n^{th} Term Test for Divergence, we can conclude that $\sum_{n=1}^{\infty} \frac{n}{3n-1}$ diverges.

b. Since $\lim_{n\to\infty} \frac{1}{n^3} = 0$, the divergence test is inconclusive.

c. Since $\lim_{n\to\infty} e^{1/n^2} = 1 \neq 0$, by the ✔ n^{th} Term Test for Divergence, the series $\sum_{n=1}^{\infty} e^{1/n^2}$ diverges.

? Exercise 6.4.1

What does the ? n^{th} Term Test for Divergence tell us about the series $\sum_{n=1}^{\infty} \cos(1/n^2)$?

Hint

Look at $\lim_{n\to\infty} \cos(1/n^2)$.

Answer

The series diverges.

Integral Test

In the previous section, we proved that the harmonic series diverges by looking at the sequence of partial sums S_k and showing that $S_{2^k} > 1 + k/2$ for all positive integers k. In this section we use a different technique to prove the divergence of the harmonic series. This technique is important because it is used to prove the divergence or convergence of many other series. This test, called the **Integral Test,** compares an infinite sum to an improper integral. It is important to note that this test can only be applied when we are considering a series whose terms are all positive.

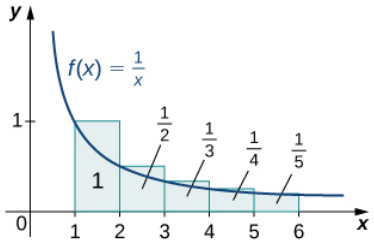


Figure 6.4.1: *The sum of the areas of the rectangles is greater than the area between the curve $f(x) = 1/x$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is infinite (as calculated by an improper integral), the sum of the areas of the rectangles is also infinite.*

To illustrate how the integral test works, use the harmonic series as an example. In Figure 6.4.1, we depict the harmonic series by sketching a sequence of rectangles with areas $1, 1/2, 1/3, 1/4, \ldots$ along with the function $f(x) = 1/x$. From the graph, we see that

$$\sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} > \int_1^{k+1} \frac{1}{x}\, dx.$$

Therefore, for each k, the kth partial sum S_k satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n} > \int_1^{k+1} \frac{1}{x}\,dx = \ln x\Big|_1^{k+1}$$
$$= \ln(k+1) - \ln(1)$$
$$= \ln(k+1).$$

Since $\lim_{k\to\infty} \ln(k+1) = \infty$, we see that the sequence of partial sums S_k is unbounded. Therefore, S_k diverges, and, consequently, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ also diverges.

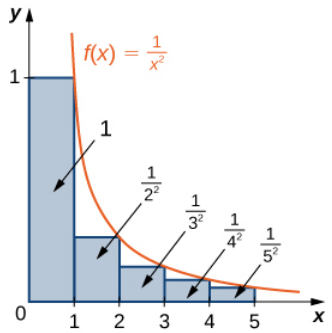


Figure 6.4.2: The sum of the areas of the rectangles is less than the sum of the area of the first rectangle and the area between the curve $f(x) = 1/x^2$ and the x-axis for $x \geq 1$. Since the area bounded by the curve is finite, the sum of the areas of the rectangles is also finite.

Now consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We show how an integral can be used to prove that this series converges. In Figure 6.4.2, we sketch a sequence of rectangles with areas $1, 1/2^2, 1/3^2, \ldots$ along with the function $f(x) = \frac{1}{x^2}$. From the graph we see that

$$\sum_{n=1}^{k} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} < 1 + \int_1^k \frac{1}{x^2}\,dx.$$

Therefore, for each k, the kth partial sum S_k satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2} < 1 + \int_1^k \frac{1}{x^2}\,dx = 1 - \frac{1}{x}\Big|_1^k$$
$$= 1 - \frac{1}{k} + 1$$
$$= 2 - \frac{1}{k} < 2.$$

We conclude that the sequence of partial sums S_k is bounded. We also see that S_k is an increasing sequence:

$$S_k = S_{k-1} + \frac{1}{k^2}$$

for $k \geq 2$.

Since S_k is increasing and bounded, by the **Monotone Convergence Theorem**, it converges. Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

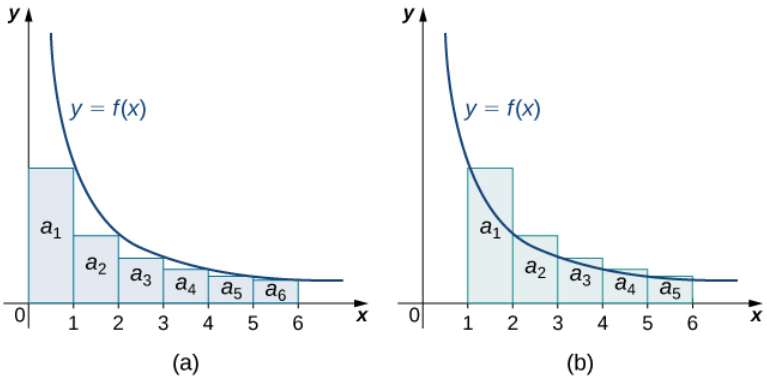


Figure 6.4.3: (a) If we can inscribe rectangles inside a region bounded by a curve $y = f(x)$ and the x-axis, and the area bounded by those curves for $x \geq 1$ is finite, then the sum of the areas of the rectangles is also finite. (b) If a set of rectangles circumscribes the region bounded by $y = f(x)$ and the x axis for $x \geq 1$ and the region has infinite area, then the sum of the areas of the rectangles is also infinite.

We can extend this idea to prove convergence or divergence for many different series. Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n such that there exists a continuous, positive, decreasing function f where $f(n) = a_n$ for all positive integers. Then, as in Figure 6.4.3*a*, for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k < a_1 + \int_1^k f(x)\,dx < 1 + \int_1^{\infty} f(x)\,dx.$$

Therefore, if $\int_1^{\infty} f(x)\,dx$ converges, then the sequence of partial sums S_k is bounded. Since S_k is an increasing sequence, if it is also a bounded sequence, then by the Monotone Convergence Theorem, it converges. We conclude that if $\int_1^{\infty} f(x)\,dx$ converges, then the series $\sum_{n=1}^{\infty} a_n$ also converges. On the other hand, from Figure 6.4.3*b*, for any integer k, the kth partial sum S_k satisfies

$$S_k = a_1 + a_2 + a_3 + \cdots + a_k > \int_1^{k+1} f(x)\,dx.$$

If

$$\lim_{k\to\infty} \int_1^{k+1} f(x)\,dx = \infty,$$

then S_k is an unbounded sequence and therefore diverges. As a result, the series $\sum_{n=1}^{\infty} a_n$ also diverges. Since f is a positive function, if $\int_1^{\infty} f(x)\,dx$ diverges, then

$$\lim_{k\to\infty} \int_1^{k+1} f(x)\,dx = \infty.$$

We conclude that if $\int_1^{\infty} f(x)\,dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Definition: The Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n. Suppose there exists a function f and a positive integer N such that the following three conditions are satisfied:

i. f is continuous,
ii. f is decreasing, and

iii. $f(n) = a_n$ for all integers $n \geq N$.

Then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_N^{\infty} f(x)\,dx$$

both converge or both diverge (Figure 6.4.3).

Although convergence of $\int_N^{\infty} f(x)\,dx$ implies convergence of the related series $\sum_{n=1}^{\infty} a_n$, it does not imply that the value of the integral and the series are the same. They may be different, and often are. For example,

$$\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n = \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \left(\frac{1}{e}\right)^3 + \cdots$$

is a geometric series with initial term $a = 1/e$ and ratio $r = 1/e$, which converges to

$$\frac{1/e}{1-(1/e)} = \frac{1/e}{(e-1)/e} = \frac{1}{e-1}.$$

However, the related integral $\int_1^{\infty} (1/e)^x\,dx$ satisfies

$$\int_1^{\infty} \left(\frac{1}{e}\right)^x dx = \int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty} \left[-e^{-b} + e^{-1}\right] = \frac{1}{e}.$$

Example 6.4.2: Using the Integral Test

For each of the following series, use the integral test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$

Solution

a. Compare

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \text{ and } \int_1^{\infty} \frac{1}{x^3}\,dx.$$

We have

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\Big|_1^b\right] = \lim_{b\to\infty} \left[-\frac{1}{2b^2} + \frac{1}{2}\right] = \frac{1}{2}.$$

Thus the integral $\int_1^{\infty} \frac{1}{x^3}\,dx$ converges, and therefore so does the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

b. Compare

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}} \text{ and } \int_1^{\infty} \frac{1}{\sqrt{2x-1}}\,dx.$$

Since

$$\int_1^{\infty} \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{\sqrt{2x-1}}\,dx = \lim_{b\to\infty} \sqrt{2x-1}\Big|_1^{b} = \lim_{b\to\infty} \left[\sqrt{2b-1}-1\right] = \infty,$$

the integral $\int_1^{\infty} \frac{1}{\sqrt{2x-1}}\,dx$ diverges, and therefore

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}}$$

diverges.

? Exercise 6.4.2

Use the integral test to determine whether the series $\sum_{n=1}^{\infty} \frac{n}{3n^2+1}$ converges or diverges.

Hint

Compare to the integral $\int_1^{\infty} \frac{x}{3x^2+1}\,dx$.

Answer

The series diverges.

The p-Series

The harmonic series $\sum_{n=1}^{\infty} 1/n$ and the series $\sum_{n=1}^{\infty} 1/n^2$ are both examples of a type of series called a p-series.

Definition: p-series

For any real number p, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a *p-series*.

We know the p-series converges if $p=2$ and diverges if $p=1$. What about other values of p? In general, it is difficult, if not impossible, to compute the exact value of most p-series. However, we can use the tests presented thus far to prove whether a p-series converges or diverges.

If $p<0$, then $1/n^p \to \infty$, and if $p=0$, then $1/n^p \to 1$. Therefore, by the divergence test,

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

diverges if $p \le 0$.

If $p>0$, then $f(x)=1/x^p$ is a positive, continuous, decreasing function. Therefore, for $p>0$, we use the integral test, comparing

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

and

$$\int_1^\infty \frac{1}{x^p}\,dx.$$

We have already considered the case when $p=1$. Here we consider the case when $p>0, p\neq 1$. For this case,

$$\int_1^\infty \frac{1}{x^p}\,dx=\lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx=\lim_{b\to\infty}\frac{1}{1-p}x^{1-p}\,|_1^b=\lim_{b\to\infty}\frac{1}{1-p}[b^{1-p}-1].$$

Because

$$b^{1-p}\to 0 \text{ if } p>1 \text{ and } b^{1-p}\to\infty \text{ if } p<1,$$

we conclude that

$$\int_1^\infty \frac{1}{x^p}\,dx=\begin{cases}\dfrac{1}{p-1}, \text{if } p>1\\ \infty, \text{if } p<1.\end{cases}$$

Therefore, $\displaystyle\sum_{n=1}^\infty 1/n^p$ converges if $p>1$ and diverges if $0<p<1$.

In summary,

$$\sum_{n=1}^\infty \frac{1}{n^p} \quad \begin{cases}\text{converges} & \text{if } p>1\\ \text{diverges} & \text{if } p\leq 1\end{cases}$$

.

Example 6.4.3: Testing for Convergence of p-series

For each of the following series, determine whether it converges or diverges.

a. $\displaystyle\sum_{n=1}^\infty \frac{1}{n^4}$

b. $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{2/3}}$

Solution

a. This is a p-series with $p=4>1$,so the series converges.

b. Since $p=2/3<1$, the series diverges.

Exercise 6.4.3

Does the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{5/4}}$ converge or diverge?

Hint

$p=5/4$

Answer

The series converges.

Estimating the Value of a Series

Suppose we know that a series $\displaystyle\sum_{n=1}^\infty a_n$ converges and we want to estimate the sum of that series. Certainly we can approximate that sum using any finite sum $\displaystyle\sum_{n=1}^N a_n$ where N is any positive integer. The question we address here is, for a convergent series $\displaystyle\sum_{n=1}^\infty a_n$,

how good is the approximation $\sum_{n=1}^{N} a_n$?

More specifically, if we let

$$R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$$

be the remainder when the sum of an infinite series is approximated by the Nth partial sum, how large is R_N? For some types of series, we are able to use the ideas from the integral test to estimate R_N.

Note 6.4.1: Remainder Estimate from the Integral Test

Suppose $\sum_{n=1}^{\infty} a_n$ is a convergent series with positive terms. Suppose there exists a function f satisfying the following three conditions:

i. f is continuous,
ii. f is decreasing, and
iii. $f(n) = a_n$ for all integers $n \geq 1$.

Let S_N be the Nth partial sum of $\sum_{n=1}^{\infty} a_n$. For all positive integers N,

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)\,dx.$$

In other words, the remainder $R_N = \sum_{n=1}^{\infty} a_n - S_N = \sum_{n=N+1}^{\infty} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx.$$

This is known as the **remainder estimate.**

We illustrate Note 6.4.1 in Figure 6.4.4. In particular, by representing the remainder $R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots$ as the sum of areas of rectangles, we see that the area of those rectangles is bounded above by $\int_{N}^{\infty} f(x)\,dx$ and bounded below by $\int_{N+1}^{\infty} f(x)\,dx$. In other words,

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^{\infty} f(x)\,dx$$

and

$$R_N = a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_{N}^{\infty} f(x)\,dx.$$

We conclude that

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx.$$

Since

$$\sum_{n=1}^{\infty} a_n = S_N + R_N,$$

where S_N is the Nth partial sum, we conclude that

$$S_N + \int_{N+1}^{\infty} f(x)\,dx < \sum_{n=1}^{\infty} a_n < S_N + \int_{N}^{\infty} f(x)\,dx.$$

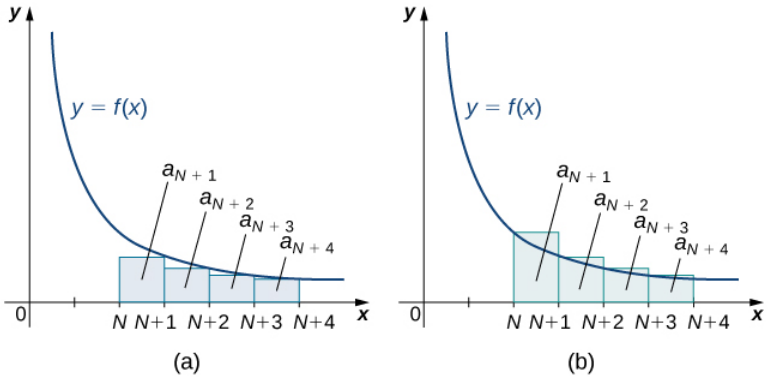


Figure 6.4.4: Given a continuous, positive, decreasing function f and a sequence of positive terms a_n such that $a_n = f(n)$ for all positive integers n, (a) the areas $a_{N+1} + a_{N+2} + a_{N+3} + \cdots < \int_N^{\infty} f(x)\,dx$, *or (b) the areas* $a_{N+1} + a_{N+2} + a_{N+3} + \cdots > \int_{N+1}^{\infty} f(x)\,dx$ *. Therefore, the integral is either an overestimate or an underestimate of the error.*

Example 6.4.4: Estimating the Value of a Series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

a. Calculate $S_{10} = \sum_{n=1}^{10} \frac{1}{n^3}$ and estimate the error.

b. Determine the least value of N necessary such that S_N will estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ to within 0.001.

Solution

a. Using a calculating utility, we have

$$S_{10} = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots + \frac{1}{10^3} \approx 1.19753.$$

By the remainder estimate, we know

$$R_N < \int_N^{\infty} \frac{1}{x^3}\,dx.$$

We have

$$\int_{10}^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_{10}^{b} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_N^b = \lim_{b\to\infty} \left[-\frac{1}{2b^2} + \frac{1}{2N^2}\right] = \frac{1}{2N^2}.$$

Therefore, the error is $R_{10} < 1/2(10)^2 = 0.005$.

b. Find N such that $R_N < 0.001$. In part a. we showed that $R_N < 1/2N^2$. Therefore, the remainder $R_N < 0.001$ as long as $1/2N^2 < 0.001$. That is, we need $2N^2 > 1000$. Solving this inequality for N, we see that we need $N > 22.36$. To ensure that the remainder is within the desired amount, we need to round up to the nearest integer. Therefore, the minimum necessary value is $N = 23$.

Exercise 6.4.4

For $\sum_{n=1}^{\infty} \frac{1}{n^4}$, calculate S_5 and estimate the error R_5.

Hint

Use the remainder estimate $R_N < \int_N^{\infty} \frac{1}{x^4}\, dx$.

Answer

$S_5 \approx 1.09035, R_5 < 0.00267$

Key Concepts

- If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\lim_{n\to\infty} a_n = 0$, the series $\sum_{n=1}^{\infty} a_n$ may converge or diverge.
- If $\sum_{n=1}^{\infty} a_n$ is a series with positive terms a_n and f is a continuous, decreasing function such that $f(n) = a_n$ for all positive integers n, then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_1^{\infty} f(x)\, dx$$

either both converge or both diverge. Furthermore, if $\sum_{n=1}^{\infty} a_n$ converges, then the Nth partial sum approximation S_N is accurate up to an error R_N where $\int_{N+1}^{\infty} f(x)\, dx < R_N < \int_N^{\infty} f(x)\, dx$.

- The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Key Equations

- **Divergence test**

  If $a_n \nrightarrow 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} a_n$ diverges.

- **p-series**

  $$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges, if } p > 1 \\ \text{diverges, if } p \leq 1 \end{cases}$$

- **Remainder estimate from the integral test**

  $$\int_{N+1}^{\infty} f(x)\, dx < R_N < \int_N^{\infty} f(x)\, dx$$

Glossary

divergence test

if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges

integral test

for a series $\sum_{n=1}^{\infty} a_n$ with positive terms a_n, if there exists a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, then

$$\sum_{n=1}^{\infty} a_n$$

and

$$\int_1^{\infty} f(x)\,dx$$

either both converge or both diverge

***p*-series**

a series of the form $\sum_{n=1}^{\infty} 1/n^p$

remainder estimate

for a series $\sum_{n=}^{\infty} 1a_n$ with positive terms a_n and a continuous, decreasing function f such that $f(n) = a_n$ for all positive integers n, the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ satisfies the following estimate:

$$\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_N^{\infty} f(x)\,dx$$

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6.4E: Exercises for Divergence and Integral Tests

Divergence Test Problems

Consider the sequence for each series in exercises 1 - 14, if the divergence test applies, either state that $\lim_{n\to\infty} a_n$ does not exist or find $\lim_{n\to\infty} a_n$. If the divergence test does not apply, state why.

1) $\sum_{n=1}^{\infty} \frac{n}{n+2}$

2) $\sum_{n=1}^{\infty} \frac{n}{5n^2-3}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

3) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{3n^2+2n+1}}$

4) $\sum_{n=1}^{\infty} \frac{(2n+1)(n-1)}{(n+1)^2}$

Answer

$\lim_{n\to\infty} a_n = 2$. So the series diverges by the nth-Term Test for Divergence.

5) $\sum_{n=1}^{\infty} \frac{(2n+1)^{2n}}{(3n^2+1)^n}$

6) $\sum_{n=1}^{\infty} \frac{2^n}{3^{n/2}}$

Answer

$\lim_{n\to\infty} a_n = \infty$ (does not exist). So the series diverges by the nth-Term Test for Divergence.

7) $\sum_{n=1}^{\infty} \frac{2^n+3^n}{10^{n/2}}$

8) $\sum_{n=1}^{\infty} e^{-2/n}$

Answer

$\lim_{n\to\infty} a_n = 1$. So the series diverges by the nth-Term Test for Divergence.

9) $\sum_{n=1}^{\infty} \cos n$

10) $\sum_{n=1}^{\infty} \tan n$

Answer

$\lim_{n\to\infty} a_n$ does not exist. So the series diverges by the nth-Term Test for Divergence.

11) $\sum_{n=1}^{\infty} \frac{1-\cos^2(1/n)}{\sin^2(2/n)}$

12) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{2n}$

Answer

$\lim_{n\to\infty} a_n = 1/e^2$. So the series diverges by the nth-Term Test for Divergence.

13) $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

14) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{\sqrt{n}}$

Answer

$\lim_{n\to\infty} a_n = 0$. The Divergence Test does not apply.

p-Series Problems & Integral Test Problems

In exercises 15 - 20, state whether the given p-series converges.

15) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

16) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

Answer

The series converges, since $p = 3/2 > 1$.

17) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$

18) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$

Answer

The series converges, since $p = 4/3 > 1$.

19) $\sum_{n=1}^{\infty} \frac{n^e}{n^\pi}$

20) $\sum_{n=1}^{\infty} \frac{n^\pi}{n^{2e}}$

Answer

The series converges, since $p = 2e - \pi > 1$.

In exercises 21 - 27, use the integral test to determine whether the following sums converge.

21) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$

22) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n+5}}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{dx}{(x+5)^{1/3}}$ can be shown to diverge.

23) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

24) $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

Answer

The series diverges by the Integral Test since $\int_1^{\infty} \frac{x}{1+x^2}\, dx$ can be shown to diverge.

25) $\sum_{n=1}^{\infty} \frac{e^n}{1+e^{2n}}$

26) $\sum_{n=1}^{\infty} \frac{2n}{1+n^4}$

Answer

The series converges by the Integral Test since $\int_1^{\infty} \frac{2x}{1+x^4}\, dx$ converges.

27) $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

Express the sums in exercises 28 - 31 as p-series and determine whether each converges.

28) $\sum_{n=1}^{\infty} 2^{-\ln n}$ $\left(\text{Hint: } 2^{-\ln n} = \frac{1}{n^{\ln 2}}.\right)$

Answer

$2^{-\ln n} = 1/n^{\ln 2}$. Since $p = \ln 2 < 1$, this series diverges by the p-series test.

29) $\sum_{n=1}^{\infty} 3^{-\ln n}$ $\left(\text{Hint: } 3^{-\ln n} = \frac{1}{n^{\ln 3}}.\right)$

30) $\sum_{n=1}^{n} 2^{-2 \ln n}$

Answer

$2^{-2 \ln n} = 1/n^{2 \ln 2}$. Since $p = 2 \ln 2 - 1 < 1$, this series diverges by the p-series test.

31) $\sum_{n=1}^{\infty} n 3^{-2 \ln n}$

In exercises 32 - 35, use the estimate $R_N \leq \int_N^{\infty} f(t)\, dt$ to find a bound for the remainder $R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n$ where $a_n = f(n)$.

32) $\sum_{n=1}^{1000} \frac{1}{n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{t^2} = \lim_{b\to\infty} -\frac{1}{t}\Big|_{1000}^{b} = \lim_{b\to\infty}\left(-\frac{1}{b}+\frac{1}{1000}\right) = 0.001$$

33) $\displaystyle\sum_{n=1}^{1000} \frac{1}{n^3}$

34) $\displaystyle\sum_{n=1}^{1000} \frac{1}{1+n^2}$

Answer

$$R_{1000} \le \int_{1000}^{\infty} \frac{dt}{1+t^2} = \lim_{b\to\infty}\left(\tan^{-1} b - \tan^{-1}(1000)\right) = \pi/2 - \tan^{-1}(1000) \approx 0.000999$$

35) $\displaystyle\sum_{n=1}^{100} \frac{n}{2^n}$

[T] In exercises 36 - 40, find the minimum value of N such that the remainder estimate $\displaystyle\int_{N+1}^{\infty} f(x)\,dx < R_N < \int_{N}^{\infty} f(x)\,dx$ guarantees that $\displaystyle\sum_{n=1}^{N} a_n$ estimates $\displaystyle\sum_{n=1}^{\infty} a_n$, accurate to within the given error.

36) $a_n = \dfrac{1}{n^2}$, error $< 10^{-4}$

Answer

$$R_N < \int_{N}^{\infty} \frac{dx}{x^2} = 1/N, \text{ for } N > 10^4$$

37) $a_n = \dfrac{1}{n^{1.1}}$, error $< 10^{-4}$

38) $a_n = \dfrac{1}{n^{1.01}}$, error $< 10^{-4}$

Answer

$$R_N < \int_{N}^{\infty} \frac{dx}{x^{1.01}} = 100N^{-0.01}, \text{ for } N > 10^{600}$$

39) $a_n = \dfrac{1}{n\ln^2 n}$, error $< 10^{-3}$

40) $a_n = \dfrac{1}{1+n^2}$, error $< 10^{-3}$

Answer

$$R_N < \int_{N}^{\infty} \frac{dx}{1+x^2} = \pi/2 - \tan^{-1}(N), \text{ for } N > \tan(\pi/2 - 10^{-3}) \approx 1000$$

In exercises 41 - 45, find a value of N such that R_N is smaller than the desired error. Compute the corresponding sum $\displaystyle\sum_{n=1}^{N} a_n$ and compare it to the given estimate of the infinite series.

41) $a_n = \dfrac{1}{n^{11}}$, error $< 10^{-4}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{11}} = 1.000494\ldots$

42) $a_n = \dfrac{1}{e^n}$, error $< 10^{-5}$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e-1} = 0.581976\ldots$

Answer

$R_N < \int_N^\infty \frac{dx}{e^x} = e^{-N}$, for $N > 5\ln(10)$, okay if $N = 12$; $\sum_{n=1}^{12} e^{-n} = 0.581973\ldots$Estimate agrees with $1/(e-1)$ to five decimal places.

43) $a_n = \frac{1}{e^{n^2}}$, error $< 10^{-5}$, $\sum_{n=1}^{\infty} \frac{1}{e^{n^2}} = 0.40488139857\ldots$

44) $a_n = \frac{1}{n^4}$, error $< 10^{-4}$, $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} = 1.08232\ldots$

Answer

$R_N < \int_N^\infty \frac{dx}{x^4} = 4/N^3$, for $N > (4.10^4)^{1/3}$, okay if $N = 35$; $\sum_{n=1}^{35} \frac{1}{n^4} = 1.08231\ldots$.Estimate agrees with the sum to four decimal places.

45) $a_n = \frac{1}{n^6}$, error $< 10^{-6}$, $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} = 1.01734306\ldots$,

46) Find the limit as $n \to \infty$ of $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$. $\left(\text{Hint: Compare to } \int_n^{2n} \frac{1}{t}\, dt.\right)$

Answer

$\ln(2)$

47) Find the limit as $n \to \infty$ of $\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{3n}$

The next few exercises are intended to give a sense of applications in which partial sums of the harmonic series arise.

48) In certain applications of probability, such as the so-called Watterson estimator for predicting mutation rates in population genetics, it is important to have an accurate estimate of the number $H_k = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k})$. Recall that $T_k = H_k - \ln k$ is decreasing. Compute $T = \lim_{k\to\infty} T_k$ to four decimal places.

$\left(\text{Hint: } \frac{1}{k+1} < \int_k^{k+1} \frac{1}{x}\, dx.\right)$

Answer

$T = 0.5772\ldots$

49) [T] Complete sampling with replacement, sometimes called the **coupon collector's problem**, is phrased as follows: Suppose you have N unique items in a bin. At each step, an item is chosen at random, identified, and put back in the bin. The problem asks what is the expected number of steps $E(N)$ that it takes to draw each unique item at least once. It turns out that $E(N) = N.H_N = N\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N}\right)$. Find $E(N)$ for $N = 10, 20$, and 50.

50) [T] The simplest way to shuffle cards is to take the top card and insert it at a random place in the deck, called top random insertion, and then repeat. We will consider a deck to be randomly shuffled once enough top random insertions have been made that the card originally at the bottom has reached the top and then been randomly inserted. If the deck has n cards, then the probability that the insertion will be below the card initially at the bottom (call this card B) is $1/n$. Thus the expected number of top random insertions before B is no longer at the bottom is n. Once one card is below B, there are two places below B and the probability that a randomly inserted card will fall below B is $2/n$. The expected number of top random insertions before this happens is $n/2$. The two cards below B are now in random order. Continuing this way, find a formula for the expected number of top random insertions needed to consider the deck to be randomly shuffled.

Answer

The expected number of random insertions to get B to the top is $n+n/2+n/3+\cdots+n/(n-1)$. Then one more insertion puts B back in at random. Thus, the expected number of shuffles to randomize the deck is $n(1+1/2+\cdots+1/n)$.

51) Suppose a scooter can travel 100 km on a full tank of fuel. Assuming that fuel can be transferred from one scooter to another but can only be carried in the tank, present a procedure that will enable one of the scooters to travel $100H_N$ km, where $H_N=1+1/2+\cdots+1/N$.

52) Show that for the remainder estimate to apply on $[N,\infty)$ it is sufficient that $f(x)$ be decreasing on $[N,\infty)$, but f need not be decreasing on $[1,\infty)$.

Answer

Set $b_n=a_{n+N}$ and $g(t)=f(t+N)$ such that f is decreasing on $[t,\infty)$.

53) [T] Use the remainder estimate and integration by parts to approximate $\displaystyle\sum_{n=1}^{\infty}\frac{n}{e^n}$ within an error smaller than 0.0001.

54) Does $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$ converge if p is large enough? If so, for which p?

Answer

The series converges for $p>1$ by integral test using change of variable.

55) [T] Suppose a computer can sum one million terms per second of the divergent series $\displaystyle\sum_{n=1}^{N}\frac{1}{n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

56) [T] A fast computer can sum one million terms per second of the divergent series $\displaystyle\sum_{n=2}^{N}\frac{1}{n\ln n}$. Use the integral test to approximate how many seconds it will take to add up enough terms for the partial sum to exceed 100.

Answer

$N=e^{e^{100}}\approx e^{10^{43}}$ terms are needed.

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6.5: Comparison Tests

Learning Objectives

- Use the comparison test to test a series for convergence.
- Use the limit comparison test to determine convergence of a series.

We have seen that the integral test allows us to determine the convergence or divergence of a series by comparing it to a related improper integral. In this section, we show how to use comparison tests to determine the convergence or divergence of a series by comparing it to a series whose convergence or divergence is known. Typically these tests are used to determine convergence of series that are similar to geometric series or p-series.

Comparison Test

In the preceding two sections, we discussed two large classes of series: geometric series and p-series. We know exactly when these series converge and when they diverge. Here we show how to use the convergence or divergence of these series to prove convergence or divergence for other series, using a method called the **comparison test.**

For example, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

This series looks similar to the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since the terms in each of the series are positive, the sequence of partial sums for each series is monotone increasing. Furthermore, since

$$0 < \frac{1}{n^2+1} < \frac{1}{n^2}$$

for all positive integers n, the kth partial sum S_k of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+1} < \sum_{n=1}^{k} \frac{1}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

(See Figure $6.5.1a$ and Table $6.5.1$.) Since the series on the right converges, the sequence S_k is bounded above. We conclude that S_k is a monotone increasing sequence that is bounded above. Therefore, by the Monotone Convergence Theorem, S_k converges, and thus

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

converges.

Similarly, consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}.$$

This series looks similar to the divergent series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

The sequence of partial sums for each series is monotone increasing and

$$\frac{1}{n-1/2} > \frac{1}{n} > 0$$

for every positive integer n. Therefore, the kth partial sum S_k of

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}$$

satisfies

$$S_k = \sum_{n=1}^{k} \frac{1}{n-1/2} > \sum_{n=1}^{k} \frac{1}{n}.$$

(See Figure $6.5.1n$and Table $6.5.1$). Since the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to infinity, the sequence of partial sums $\sum_{n=1}^{k} \frac{1}{n}$ is unbounded. Consequently, S_k is an unbounded sequence, and therefore diverges. We conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n-1/2}$$

diverges.

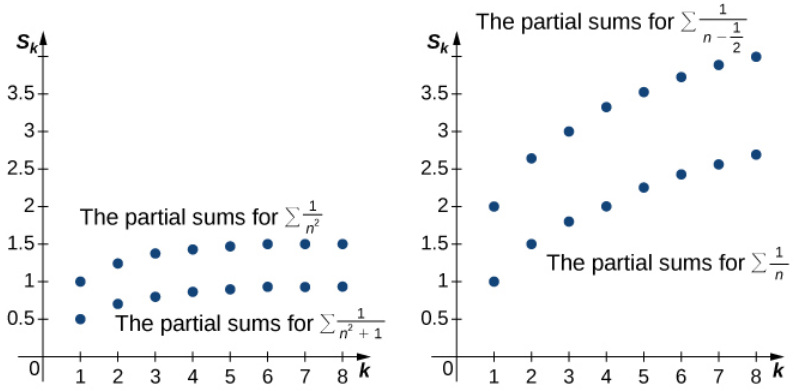


Figure 6.5.1: (a) Each of the partial sums for the given series is less than the corresponding partial sum for the converging $p-series$. (b) Each of the partial sums for the given series is greater than the corresponding partial sum for the diverging harmonic series.

Table 6.5.1: Comparing a series with a p-series ($p = 2$)

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n^2+1}$	0.5	0.7	0.8	0.8588	0.8973	0.9243	0.9443	0.9597
$\sum_{n=1}^{k} \frac{1}{n^2}$	1	1.25	1.3611	1.4236	1.4636	1.4914	1.5118	1.5274

Table 6.5.2: Comparing a series with the harmonic series

k	1	2	3	4	5	6	7	8
$\sum_{n=1}^{k} \frac{1}{n-1/2}$	2	2.6667	3.0667	3.3524	3.5746	3.7564	3.9103	4.0436

$\sum_{n=1}^{k} \frac{1}{n}$	1	1.5	1.8333	2.0933	2.2833	2.45	2.5929	2.7179

Comparison Test

i. Suppose there exists an integer N such that $0 \le a_n \le b_n$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. Suppose there exists an integer N such that $a_n \ge b_n \ge 0$ for all $n \ge N$. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

We prove part i. The proof of part ii. is the contrapositive of part i. Let S_k be the sequence of partial sums associated with $\sum_{n=1}^{\infty} a_n$, and let $L = \sum_{n=1}^{\infty} b_n$. Since the terms $a_n \ge 0$,

$$S_k = a_1 + a_2 + \cdots + a_k \le a_1 + a_2 + \cdots + a_k + a_{k+1} = S_{k+1}.$$

Therefore, the sequence of partial sums is increasing. Further, since $a_n \le b_n$ for all $n \ge N$, then

$$\sum_{n=N}^{k} a_n \le \sum_{n=N}^{k} b_n \le \sum_{n=1}^{\infty} b_n = L.$$

Therefore, for all $k \ge 1$,

$$S_k = (a_1 + a_2 + \cdots + a_{N-1}) + \sum_{n=N}^{k} a_n \le (a_1 + a_2 + \cdots + a_{N-1}) + L.$$

Since $a_1 + a_2 + \cdots + a_{N-1}$ is a finite number, we conclude that the sequence S_k is bounded above. Therefore, S_k is an increasing sequence that is bounded above. By the Monotone Convergence Theorem, we conclude that S_k converges, and therefore the series $\sum_{n=1}^{\infty} a_n$ converges.

□

To use the comparison test to determine the convergence or divergence of a series $\sum_{n=1}^{\infty} a_n$, it is necessary to find a suitable series with which to compare it. Since we know the convergence properties of geometric series and p-series, these series are often used. If there exists an integer N such that for all $n \ge N$, each term an is less than each corresponding term of a known convergent series, then $\sum_{n=1}^{\infty} a_n$ converges. Similarly, if there exists an integer N such that for all $n \ge N$, each term an is greater than each corresponding term of a known divergent series, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 6.5.1: Using the Comparison Test

For each of the following series, use the comparison test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{1}{n^3 + 3n + 1}$

b. $\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$

c. $\sum_{n=2}^{\infty} \frac{1}{\ln n}$

Solution

a. Compare to $\sum_{n=1}^{\infty} \frac{1}{n^3}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p = 3$, it converges. Further,

$$\frac{1}{n^3+3n+1} < \frac{1}{n^3}$$

for every positive integer n. Therefore, we can conclude that $\sum_{n=1}^{\infty} \frac{1}{n^3+3n+1}$ converges.

b. Compare to $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. Since $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a geometric series with $r = \frac{1}{2}$ and $\left|\frac{1}{2}\right| < 1$, it converges. Also,

$$\frac{1}{2^n+1} < \frac{1}{2^n}$$

for every positive integer n. Therefore, we see that $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

c. Compare to $\sum_{n=2}^{\infty} \frac{1}{n}$. Since

$$\frac{1}{\ln n} > \frac{1}{n}$$

for every integer $n \geq 2$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges, we have that $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

? Exercise 6.5.1

Use the comparison test to determine if the series $\sum_{n=1}^{\infty} \frac{n}{n^3+n+1}$ converges or diverges.

Hint

Find a value p such that $\frac{n}{n^3+n+1} \leq \frac{1}{n^p}$.

Answer

The series converges.

Limit Comparison Test

The comparison test works nicely if we can find a comparable series satisfying the hypothesis of the test. However, sometimes finding an appropriate series can be difficult. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}.$$

It is natural to compare this series with the convergent series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

However, this series does not satisfy the hypothesis necessary to use the comparison test because

$$\frac{1}{n^2-1} > \frac{1}{n^2}$$

for all integers $n \geq 2$. Although we could look for a different series with which to compare $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$, instead we show how we can use the **limit comparison test** to compare

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

and

$$\sum_{n=2}^{\infty} \frac{1}{n^2}.$$

Let us examine the idea behind the limit comparison test. Consider two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$. with positive terms a_n and b_n and evaluate

$$\lim_{n\to\infty} \frac{a_n}{b_n}.$$

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \neq 0,$$

then, for n sufficiently large, $a_n \approx Lb_n$. Therefore, either both series converge or both series diverge. For the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$, we see that

$$\lim_{n\to\infty} \frac{1/(n^2-1)}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = 1.$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges, we conclude that

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

converges.

The limit comparison test can be used in two other cases. Suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

In this case, a_n/b_n is a bounded sequence. As a result, there exists a constant M such that $a_n \leq Mb_n$. Therefore, if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. On the other hand, suppose

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty.$$

In this case, a_n/b_n is an unbounded sequence. Therefore, for every constant M there exists an integer N such that $a_n \geq Mb_n$ for all $n \geq N$. Therefore, if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges as well.

Limit Comparison Test

Let $a_n, b_n \geq 0$ for all $n \geq 1$.

i. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L \neq 0,$ then $\sum\limits_{n=1}^{\infty} a_n$ and $\sum\limits_{n=1}^{\infty} b_n$ both converge or both diverge.

ii. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum\limits_{n=1}^{\infty} b_n$ converges, then $\sum\limits_{n=1}^{\infty} a_n$ converges.

iii. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\sum\limits_{n=1}^{\infty} b_n$ diverges, then $\sum\limits_{n=1}^{\infty} a_n$ diverges.

Note that if $\dfrac{a_n}{b_n} \to 0$ and $\sum\limits_{n=1}^{\infty} b_n$ diverges, the limit comparison test gives no information. Similarly, if $\dfrac{a_n}{b_n} \to \infty$ and $\sum\limits_{n=1}^{\infty} b_n$ converges, the test also provides no information. For example, consider the two series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ and $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$. These series are both p-series with $p = \frac{1}{2}$ and $p = 2$, respectively. Since $p = \frac{1}{2} < 1,$ the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ diverges. On the other hand, since $p = 2 > 1,$ the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ converges. However, suppose we attempted to apply the limit comparison test, using the convergent $\boldsymbol{p}$**−series** $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$ as our comparison series. First, we see that

$$\frac{1/\sqrt{n}}{1/n^3} = \frac{n^3}{\sqrt{n}} = n^{5/2} \to \infty \text{ as } n \to \infty.$$

Similarly, we see that

$$\frac{1/n^2}{1/n^3} = n \to \infty \text{ as } n \to \infty.$$

Therefore, if $\dfrac{a_n}{b_n} \to \infty$ when $\sum\limits_{n=1}^{\infty} b_n$ converges, we do not gain any information on the convergence or divergence of $\sum\limits_{n=1}^{\infty} a_n$.

Example 6.5.2: Using the Limit Comparison Test

For each of the following series, use the limit comparison test to determine whether the series converges or diverges. If the test does not apply, say so.

a. $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}+1}$

b. $\sum\limits_{n=1}^{\infty} \dfrac{2^n+1}{3^n}$

c. $\sum\limits_{n=1}^{\infty} \dfrac{\ln(n)}{n^2}$

Solution

a. Compare this series to $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$. Calculate

$$\lim_{n\to\infty} \frac{1/(\sqrt{n}+1)}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1/\sqrt{n}}{1+1/\sqrt{n}} = 1.$$

By the limit comparison test, since $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ diverges, then $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}+1}$ diverges.

b. Compare this series to $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$. We see that

$$\lim_{n\to\infty}\frac{(2^n+1)/3^n}{2^n/3^n}=\lim_{n\to\infty}\frac{2^n+1}{3^n}\cdot\frac{3^n}{2^n}=\lim_{n\to\infty}\frac{2^n+1}{2^n}=\lim_{n\to\infty}\left[1+\left(\tfrac{1}{2}\right)^n\right]=1.$$

Therefore,

$$\lim_{n\to\infty}\frac{(2^n+1)/3^n}{2^n/3^n}=1.$$

Since $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ converges, we conclude that $\sum_{n=1}^{\infty}\frac{2^n+1}{3^n}$ converges.

c. Since $\ln n<n,$ compare with $\sum_{n=1}^{\infty}\frac{1}{n}$. We see that

$$\lim_{n\to\infty}\frac{\ln n/n^2}{1/n}=\lim_{n\to\infty}\frac{\ln n}{n^2}\cdot\frac{n}{1}=\lim_{n\to\infty}\frac{\ln n}{n}.$$

In order to evaluate $\lim_{n\to\infty}\ln n/n$, evaluate the limit as $x\to\infty$ of the real-valued function $\ln(x)/x$. These two limits are equal, and making this change allows us to use L'Hôpital's rule. We obtain

$$\lim_{x\to\infty}\frac{lnx}{x}=\lim_{x\to\infty}\frac{1}{x}=0.$$

Therefore, $\lim_{n\to\infty}\frac{\ln n}{n}=0$, and, consequently,

$$\lim_{n\to\infty}\frac{(\ln n)/n^2}{1/n}=0.$$

Since the limit is 0 but $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, the limit comparison test does not provide any information.

Compare with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ instead. In this case,

$$\lim_{n\to\infty}\frac{(\ln n)/n^2}{1/n^2}=\lim_{n\to\infty}\frac{\ln n}{n^2}\cdot\frac{n^2}{1}=\lim_{n\to\infty}\ln n=\infty.$$

Since the limit is ∞ but $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, the test still does not provide any information.

So now we try a series between the two we already tried. Choosing the series $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$, we see that

$$\lim_{n\to\infty}\frac{(\ln n)/n^2}{1/n^{3/2}}=\lim_{n\to\infty}\frac{\ln n}{n^2}\cdot\frac{n^{3/2}}{1}=\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}\ .$$

As above, in order to evaluate $\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}$, evaluate the limit as $x\to\infty$ of the real-valued function $\frac{\ln n}{\sqrt{n}}$. Using L'Hôpital's rule,

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{2\sqrt{x}}{x}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0\ .$$

Since the limit is 0 and $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges, we can conclude that $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ converges.

Exercise 6.5.2

Use the limit comparison test to determine whether the series $\sum_{n=1}^{\infty} \frac{5^n}{3^n+2}$ converges or diverges.

Hint

Compare with a geometric series.

Answer

The series diverges.

Key Concepts

- The comparison tests are used to determine convergence or divergence of series with positive terms.
- When using the comparison tests, a series $\sum_{n=1}^{\infty} a_n$ is often compared to a geometric or p-series.

Glossary

comparison test

If $0 \le a_n \le b_n$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges; if $a_n \ge b_n \ge 0$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

limit comparison test

Suppose $a_n, b_n \ge 0$ for all $n \ge 1$. If $\lim_{n\to\infty} a_n/b_n \to L \ne 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge; if $\lim_{n\to\infty} a_n/b_n \to 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. If $\lim_{n\to\infty} a_n/b_n \to \infty$, and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

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6.5E: Exercises for Comparison Test

Use the Comparison Test to determine whether each series in exercises 1 - 13 converges or diverges.

1) $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{2}{n(n+1)}$

2) $\sum_{n=1}^{\infty} a_n$ where $a_n = \dfrac{1}{n(n+1/2)}$

Answer

Converges by comparison with $\dfrac{1}{n^2}$.

3) $\sum_{n=1}^{\infty} \dfrac{1}{2(n+1)}$

4) $\sum_{n=1}^{\infty} \dfrac{1}{2n-1}$

Answer

Since $2n-1 < 2n$, we have $\dfrac{1}{2n} < \dfrac{1}{2n-1} \rightarrow \sum_{n=1}^{\infty} \dfrac{1}{2n} < \sum_{n=1}^{\infty} \dfrac{1}{2n-1}$.

But the series on the left is just half of the harmonic series, since $\sum_{n=1}^{\infty} \dfrac{1}{2n} = \frac{1}{2} \sum_{n=1}^{\infty} \dfrac{1}{n}$. This clearly diverges, so the given series also diverges, by the Comparison Test.

5) $\sum_{n=2}^{\infty} \dfrac{1}{(n \ln n)^2}$

6) $\sum_{n=1}^{\infty} \dfrac{n!}{(n+2)!}$

Answer

$a_n = 1/(n+1)(n+2) < 1/n^2$. Converges by comparison with p-series, $p = 2 > 1$.

7) $\sum_{n=1}^{\infty} \dfrac{1}{n!}$

8) $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{n}$

Answer

$\sin(1/n) \leq 1/n$, so converges by comparison with p-series, $p = 2 > 1$.

9) $\sum_{n=1}^{\infty} \dfrac{\sin^2 n}{n^2}$

10) $\sum_{n=1}^{\infty} \dfrac{\sin(1/n)}{\sqrt{n}}$

Answer

$\sin(1/n) \leq 1$, so converges by comparison with p-series, $p = 3/2 > 1$.

11) $\sum_{n=1}^{\infty} \frac{n^{1.2}-1}{n^{2.3}+1}$

12) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{n}$

Answer

Since $\sqrt{n+1}-\sqrt{n}=1/(\sqrt{n+1}+\sqrt{n}) \leq 2/\sqrt{n}$, series converges by comparison with p-series for $p=1.5>1$.

13) $\sum_{n=1}^{\infty} \frac{\sqrt[4]{n}}{\sqrt[3]{n^4+n^2}}$

Use the Limit Comparison Test to determine whether each series in exercises 14 - 28 converges or diverges.

14) $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^2$

Answer

Converges by limit comparison with p-series for $p>1$.

15) $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n^{0.6}}\right)^2$

16) $\sum_{n=1}^{\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n}$

Answer

Converges by limit comparison with p-series, $p=2>1$.

17) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

18) $\sum_{n=1}^{\infty} \frac{1}{4^n-3^n}$

Answer

Converges by limit comparison with 4^{-n}.

19) $\sum_{n=1}^{\infty} \frac{1}{n^2-n\sin n}$

20) $\sum_{n=1}^{\infty} \frac{1}{e^{(1.1)n}-3^n}$

Answer

Converges by limit comparison with $1/e^{1.1n}$.

21) $\sum_{n=1}^{\infty} \frac{1}{e^{(1.01)n}-3^n}$

22) $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Answer

Diverges by limit comparison with harmonic series.

23) $\sum_{n=1}^{\infty} \frac{1}{2^{1+1/n}} n^{1+1/n}$

24) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \sin\left(\frac{1}{n}\right)\right)$

Answer

Converges by limit comparison with p-series, $p = 3 > 1$.

25) $\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$

26) $\sum_{n=1}^{\infty} \frac{1}{n}\left(\tan^{-1} n - \frac{\pi}{2}\right)$

Answer

Converges by limit comparison with p-series, $p = 3 > 1$.

27) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^{n.n}$ (Hint: $\left(1 - \frac{1}{n}\right)^{n} \to 1/e$.)

28) $\sum_{n=1}^{\infty} \left(1 - e^{-1/n}\right)$ (Hint: $1/e \approx (1 - 1/n)^n$, so $1 - e^{-1/n} \approx 1/n$.)

Answer

Diverges by limit comparison with $1/n$.

29) Does $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^p}$ converge if p is large enough? If so, for which p?

30) Does $\sum_{n=1}^{\infty} \left(\frac{\ln n}{n}\right)^p$ converge if p is large enough? If so, for which p?

Answer

Converges for $p > 1$ by comparison with a p series for slightly smaller p.

31) For which p does the series $\sum_{n=1}^{\infty} 2^{pn}/3^n$ converge?

32) For which $p > 0$ does the series $\sum_{n=1}^{\infty} \frac{n^p}{2^n}$ converge?

Answer

Converges for all $p > 0$.

33) For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{r^{n^2}}{2^n}$ converge?

34) For which $r > 0$ does the series $\sum_{n=1}^{\infty} \frac{2^n}{r^{n^2}}$ converge?

Answer

Converges for all $r > 1$. If $r > 1$ then $r^n > 4$, say, once $n > \ln(2)/\ln(r)$ and then the series converges by limit comparison with a geometric series with ratio $1/2$

35) Find all values of p and q such that $\sum_{n=1}^{\infty} \frac{n^p}{(n!)^q}$ converges.

36) Does $\sum_{n=1}^{\infty} \frac{\sin^2(nr/2)}{n}$ converge or diverge? Explain.

Answer

The numerator is equal to 1 when n is odd and 0 when n is even, so the series can be rewritten $\sum_{n=1}^{\infty} \frac{1}{2n+1}$, which diverges by limit comparison with the harmonic series.

37) Explain why, for each n, at least one of $|\sin n|, |\sin(n+1)|, \ldots, |\sin(n+6)|$ is larger than $1/2$. Use this relation to test convergence of $\sum_{n=1}^{\infty} \frac{|\sin n|}{\sqrt{n}}$.

38) Suppose that $a_n \geq 0$ and $b_n \geq 0$ and that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges and $\sum_{n=1}^{\infty} a_n b_n \leq \frac{1}{2}\left(\sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right)$.

Answer

$(a-b)^2 = a^2 - 2ab + b^2$ or $a^2 + b^2 \geq 2ab$, so convergence follows from comparison of $2a_nb_n$ with $a_n^2 + b_n^2$. Since the partial sums on the left are bounded by those on the right, the inequality holds for the infinite series.

39) Does $\sum_{n=1}^{\infty} 2^{-\ln \ln n}$ converge? (Hint: Write $2^{\ln \ln n}$ as a power of $\ln n$.)

40) Does $\sum_{n=1}^{\infty} (\ln n)^{-\ln n}$ converge? (Hint: Use $t = e^{\ln(t)}$ to compare to a $p-$series.)

Answer

$(\ln n)^{-\ln n} = e^{-\ln(n)\ln\ln(n)}$. If n is sufficiently large, then $\ln \ln n > 2$, so $(\ln n)^{-\ln n} < 1/n^2$, and the series converges by comparison to a $p-$series.

41) Does $\sum_{n=2}^{\infty} (\ln n)^{-\ln \ln n}$ converge? (Hint: Compare a_n to $1/n$.)

42) Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges. If $\sum_{n=1}^{\infty} a_n^2$ converges, does $\sum_{n=1}^{\infty} a_n$ necessarily converge?

Answer

$a_n \to 0$, so $a_n^2 \leq |a_n|$ for large n. Convergence follows from limit comparison. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not, so the fact that $\sum_{n=1}^{\infty} a_n^2$ converges does not imply that $\sum_{n=1}^{\infty} a_n$ converges.

43) Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ converges. Suppose that b_n is an arbitrary sequence of zeros and ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

44) Suppose that $a_n > 0$ for all n and that $\sum_{n=1}^{\infty} a_n$ diverges. Suppose that b_n is an arbitrary sequence of zeros and ones with infinitely many terms equal to one. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily diverge?

Answer

No. $\sum_{n=1}^{\infty} 1/n$ diverges. Let $b_k = 0$ unless $k = n^2$ for some n. Then $\sum_k b_k/k = \sum 1/k^2$ converges.

45) Complete the details of the following argument: If $\sum_{n=1}^{\infty} \frac{1}{n}$ converges to a finite sum s, then $\frac{1}{2}s = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots$ and $s - \frac{1}{2}s = 1 + \frac{1}{3} + \frac{1}{5} + \cdots$. Why does this lead to a contradiction?

46) Show that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} \sin^2(a_n)$ converges.

Answer

$|\sin t| \leq |t|$, so the result follows from the comparison test.

47) Suppose that $a_n/b_n \to 0$ in the comparison test, where $a_n \geq 0$ and $b_n \geq 0$. Prove that if $\sum b_n$ converges, then $\sum a_n$ converges.

48) Let b_n be an infinite sequence of zeros and ones. What is the largest possible value of $x = \sum_{n=1}^{\infty} b_n/2^n$?

Answer

By the comparison test, $x = \sum_{n=1}^{\infty} b_n/2^n \leq \sum_{n=1}^{\infty} 1/2^n = 1$.

49) Let d_n be an infinite sequence of digits, meaning d_n takes values in $\{0, 1, \ldots, 9\}$. What is the largest possible value of $x = \sum_{n=1}^{\infty} d_n/10^n$ that converges?

50) Explain why, if $x > 1/2$, then x cannot be written $x = \sum_{n=2}^{\infty} \frac{b_n}{2^n}$ $(b_n = 0 \text{ or } 1,\ b_1 = 0)$.

Answer

If $b_1 = 0$, then, by comparison, $x \leq \sum_{n=2}^{\infty} 1/2^n = 1/2$.

51) [T] Evelyn has a perfect balancing scale, an unlimited number of 1-kg weights, and one each of $1/2$-kg, $1/4$-kg, $1/8$-kg, and so on weights. She wishes to weigh a meteorite of unspecified origin to arbitrary precision. Assuming the scale is big enough, can she do it? What does this have to do with infinite series?

52) [T] Robert wants to know his body mass to arbitrary precision. He has a big balancing scale that works perfectly, an unlimited collection of 1-kg weights, and nine each of 0.1-kg, 0.01-kg, 0.001-kg, and so on weights. Assuming the scale is big enough, can he do this? What does this have to do with infinite series?

Answer

Yes. Keep adding 1-kg weights until the balance tips to the side with the weights. If it balances perfectly, with Robert standing on the other side, stop. Otherwise, remove one of the 1-kg weights, and add 0.1-kg weights one at a time. If it balances after adding some of these, stop. Otherwise if it tips to the weights, remove the last 0.1-kg weight. Start adding 0.01-kg weights. If it balances, stop. If it tips to the side with the weights, remove the last 0.01-kg weight that was added.

Continue in this way for the 0.001-kg weights, and so on. After a finite number of steps, one has a finite series of the form $A+\sum_{n=1}^{N} s_n/10^n$ where A is the number of full kg weights and d_n is the number of $1/10^n$-kg weights that were added. If at some state this series is Robert's exact weight, the process will stop. Otherwise it represents the Nth partial sum of an infinite series that gives Robert's exact weight, and the error of this sum is at most $1/10^N$.

53) The series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is half the harmonic series and hence diverges. It is obtained from the harmonic series by deleting all terms in which n is odd. Let $m>1$ be fixed. Show, more generally, that deleting all terms $1/n$ where $n=mk$ for some integer k also results in a divergent series.

54) In view of the previous exercise, it may be surprising that a subseries of the harmonic series in which about one in every five terms is deleted might converge. A depleted harmonic series is a series obtained from $\sum_{n=1}^{\infty} \frac{1}{n}$ by removing any term $1/n$ if a given digit, say 9, appears in the decimal expansion of n. Argue that this depleted harmonic series converges by answering the following questions.

a. How many whole numbers n have d digits?

b. How many d-digit whole numbers $h(d)$. do not contain 9 as one or more of their digits?

c. What is the smallest d-digit number $m(d)$?

d. Explain why the deleted harmonic series is bounded by $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$.

e. Show that $\sum_{d=1}^{\infty} \frac{h(d)}{m(d)}$ converges.

Answer

a. $10^d-10^{d-1}<10^d$

b. $h(d)<9^d$

c. $m(d)=10^{d-1}+1$

d. Group the terms in the deleted harmonic series together by number of digits. $h(d)$ bounds the number of terms, and each term is at most $\frac{1}{m(d)}$.

Then $\sum_{d=1}^{\infty} h(d)/m(d) \le \sum_{d=1}^{\infty} 9^d/(10)^{d-1} \le 90$. One can actually use comparison to estimate the value to smaller than 80. The actual value is smaller than 23.

55) Suppose that a sequence of numbers $a_n>0$ has the property that $a_1=1$ and $a_{n+1}=\frac{1}{n+1}S_n$, where $S_n=a_1+\cdots+a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: S_n is monotone.)

56) Suppose that a sequence of numbers $a_n>0$ has the property that $a_1=1$ and $a_{n+1}=\frac{1}{(n+1)^2}S_n$, where $S_n=a_1+\cdots+a_n$. Can you determine whether $\sum_{n=1}^{\infty} a_n$ converges? (Hint: $S_2=a_2+a_1=a_2+S_1=a_2+1=1+1/4=(1+1/4)S_1, S_3=\frac{1}{3^2}S_2+S_2=(1+1/9)S_2$ $=(1+1/9)(1+1/4)S_1$, etc. Look at $\ln(S_n)$, and use $\ln(1+t)\le t, t>0$.)

Answer

Continuing the hint gives $S_N = (1+1/N^2)(1+1/(N-1)^2 \dots (1+1/4))$. Then $\ln(S_N) = \ln(1+1/N^2) + \ln(1+1/(N-1)^2) + \cdots + \ln(1+1/4)$. Since $\ln(1+t)$ is bounded by a constant times t, when $0 < t < 1$ one has $\ln(S_N) \leq C \sum_{n=1}^{N} \frac{1}{n^2}$, which converges by comparison to the p-series for $p = 2$.

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6.6: Alternating Series

Learning Objectives

- Use the alternating series test to test an alternating series for convergence.
- Estimate the sum of an alternating series.
- Explain the meaning of absolute convergence and conditional convergence.

So far in this chapter, we have primarily discussed series with positive terms. In this section we introduce alternating series—those series whose terms alternate in sign. We will show in a later chapter that these series often arise when studying power series. After defining alternating series, we introduce the alternating series test to determine whether such a series converges.

The Alternating Series Test

A series whose terms alternate between positive and negative values is an **alternating series.** For example, the series

$$\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n = -\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\ldots \tag{6.6.1}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots \tag{6.6.2}$$

are both alternating series.

Definition: Alternating Series

Any series whose terms alternate between positive and negative values is called an *alternating series*. An alternating series can be written in the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1-b_2+b_3-b_4+\ldots \tag{6.6.3}$$

or

$$\sum_{n-1}^{\infty}(-1)^{n}b_n = -b_1+b_2-b_3+b_4-\ldots \tag{6.6.4}$$

Where $b_n \geq 0$ for all positive integers n.

Series (1), shown in Equation 6.6.1, is a *geometric series*. Since $|r| = |-1/2| < 1,$ the series converges. Series (2), shown in Equation 6.6.2, is called the *alternating harmonic series*. We will show that whereas the harmonic series diverges, the alternating harmonic series converges. To prove this, we look at the sequence of partial sums $\{S_k\}$ (Figure 6.6.1).

Proof

Consider the odd terms S_{2k+1} for $k \geq 0$. Since $1/(2k+1) < 1/2k$,

$$S_{2k+1} = S_{2k-1} - \frac{1}{2k}+\frac{1}{2k+1} < S_{2k-1}.$$

Therefore, $\{S_{2k+1}\}$ is a decreasing sequence. Also,

$$S_{2k+1} = \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{2k-1}-\frac{1}{2k}\right)+\frac{1}{2k+1} > 0.$$

Therefore, $\{S_{2k+1}\}$ is bounded below. Since $\{S_{2k+1}\}$ is a decreasing sequence that is bounded below, by the Monotone Convergence Theorem, $\{S_{2k+1}\}$ converges. Similarly, the even terms $\{S_{2k}\}$ form an increasing sequence that is bounded

above because

$$S_{2k} = S_{2k-2} + \frac{1}{2k-1} - \frac{1}{2k} > S_{2k-2}$$

and

$$S_{2k} = 1 + \left(-\frac{1}{2} + \frac{1}{3}\right) + \ldots + \left(-\frac{1}{2k-2} + \frac{1}{2k-1}\right) - \frac{1}{2k} < 1.$$

Therefore, by the Monotone Convergence Theorem, the sequence $\{S_{2k}\}$ also converges. Since

$$S_{2k+1} = S_{2k} + \frac{1}{2k+1},$$

we know that

$$\lim_{k\to\infty} S_{2k+1} = \lim_{k\to\infty} S_{2k} + \lim_{k\to\infty} \frac{1}{2k+1}.$$

Letting $S = \lim_{k\to\infty} S_{2k+1}$ and using the fact that $1/(2k+1) \to 0$, we conclude that $\lim_{k\to\infty} S_{2k} = S$. Since the odd terms and the even terms in the sequence of partial sums converge to the same limit S, it can be shown that the sequence of partial sums converges to S, and therefore the alternating harmonic series converges to S.

It can also be shown that $S = \ln 2$, and we can write

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + a \ldots = \ln(2).$$

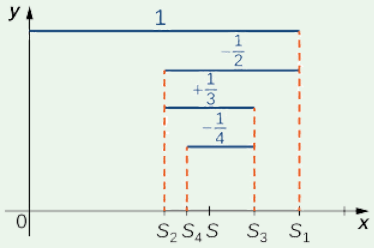


Figure 6.6.1: For the alternating harmonic series, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

□

More generally, any alternating series of form (3) (Equation 6.6.3) or (4) (Equation 6.6.4) converges as long as $b_1 \geq b_2 \geq b_3 \geq \cdots$ and $b_n \to 0$ (Figure 6.6.2). The proof is similar to the proof for the alternating harmonic series.

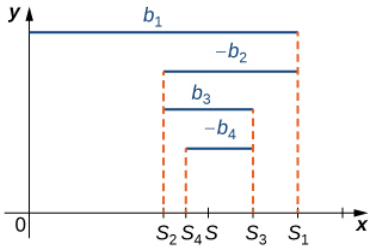


Figure 6.6.2: For an alternating series $b_1 - b_2 + b_3 - \cdots$ in which $b_1 > b_2 > b_3 > \cdots$, the odd terms S_{2k+1} in the sequence of partial sums are decreasing and bounded below. The even terms S_{2k} are increasing and bounded above.

Alternating Series Test

An alternating series of the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} b_n$$

or

$$\sum_{n=1}^{\infty}(-1)^n b_n$$

converges if

i. $0 \le b_{n+1} \le b_n$ for all $n \ge 1$ and
ii. $\lim_{n\to\infty} b_n = 0$.

This is known as the *alternating series test*.

We remark that this theorem is true more generally as long as there exists some integer N such that $0 \le b_{n+1} \le b_n$ for all $n \ge N$.

Example 6.6.1: Convergence of Alternating Series

For each of the following alternating series, determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

b. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n+1}$

Solution

a. Since $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ and $\frac{1}{n^2} \to 0$, the series converges.

b. Since $n/(n+1) \nrightarrow 0$ as $n \to \infty$, we cannot apply the alternating series test. Instead, we use the nth term test for divergence. Since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \ne 0$, the series diverges.

Exercise 6.6.1

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2^n}$ converges or diverges.

Hint

Is $\left\{\frac{n}{2^n}\right\}$ decreasing? What is $\lim_{n\to\infty}\frac{n}{2^n}$?

Answer

The series converges.

Remainder of an Alternating Series

It is difficult to explicitly calculate the sum of most alternating series, so typically the sum is approximated by using a partial sum. When doing so, we are interested in the amount of error in our approximation. Consider an alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n$$

satisfying the hypotheses of the alternating series test. Let S denote the sum of this series and S_k be the corresponding sequence of partial sums. From Figure 6.6.2, we see that for any integer $N \ge 1$, the remainder R_N satisfies

$$|R_N| = |S - S_N| \le |S_{N+1} - S_N| = b_{n+1}.$$

Remainders in Alternating Series

Consider an alternating series of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n$$

or

$$\sum_{n=1}^{\infty}(-1)^{n}b_n$$

that satisfies the hypotheses of the alternating series test. Let S denote the sum of the series and S_N denote the Nth partial sum. For any integer $N \geq 1$, the remainder $R_N = S - S_N$ satisfies

$$|R_N| \leq b_{N+1}.$$

In other words, if the conditions of the alternating series test apply, then the error in approximating the infinite series by the Nth partial sum S_N is in magnitude at most the size of the next term b_{N+1}.

Example 6.6.2: Estimating the Remainder of an Alternating Series

Consider the alternating series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}.$$

Use the remainder estimate to determine a bound on the error R_{10} if we approximate the sum of the series by the partial sum S_{10}.

Solution

From the theorem stated above,

$$|R_{10}| \leq b_{11} = \frac{1}{11^2} \approx 0.008265.$$

Exercise 6.6.2

Find a bound for R_{20} when approximating $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ by S_{20}.

Hint

$|R_{20}| \leq b_{21}$

Answer

0.04762

Absolute and Conditional Convergence

Consider a series $\sum_{n=1}^{\infty}a_n$ and the related series $\sum_{n=1}^{\infty}|a_n|$. Here we discuss possibilities for the relationship between the convergence of these two series. For example, consider the alternating harmonic series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. The series whose terms are the absolute

value of these terms is the harmonic series, since $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n}\right|=\sum_{n=1}^{\infty}\frac{1}{n}$. Since the alternating harmonic series converges, but the harmonic series diverges, we say the alternating harmonic series exhibits conditional convergence.

By comparison, consider the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$. The series whose terms are the absolute values of the terms of this series is the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Since both of these series converge, we say the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ exhibits absolute convergence.

Definition: Absolute Convergence

A series $\sum_{n=1}^{\infty}a_n$ exhibits *absolute convergence* if $\sum_{n=1}^{\infty}|a_n|$ converges. A series $\sum_{n=1}^{\infty}a_n$ exhibits *conditional convergence* if $\sum_{n=1}^{\infty}a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ diverges.

As shown by the alternating harmonic series, a series $\sum_{n=1}^{\infty}a_n$ may converge, but $\sum_{n=1}^{\infty}|a_n|$ may diverge. In the following theorem, however, we show that if $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Absolute Convergence Implies Convergence

If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Proof

Suppose that $\sum_{n=1}^{\infty}|a_n|$ converges. We show this by using the fact that $a_n=|a_n|$ or $a_n=-|a_n|$ and therefore $|a_n|+a_n=2|a_n|$ or $|a_n|+a_n=0$. Therefore, $0\le|a_n|+a_n\le 2|a_n|$. Consequently, by the comparison test, since $2\sum_{n=1}^{\infty}|a_n|$ converges, the series

$$\sum_{n=1}^{\infty}(|a_n|+a_n)$$

converges. By using the algebraic properties for convergent series, we conclude that

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(|a_n|+a_n)-\sum_{n=1}^{\infty}|a_n|$$

converges.

□

Example 6.6.3: Absolute versus Conditional Convergence

For each of the following series, determine whether the series converges absolutely, converges conditionally, or diverges.

a. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n+1}$

b. $\sum_{n=1}^{\infty}\frac{\cos(n)}{n^2}$

Solution

a. We can see that

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{3n+1}\right|=\sum_{n=1}^{\infty}\frac{1}{3n+1}$$

diverges by using the limit comparison test with the harmonic series. In fact,

$$\lim_{n\to\infty}\frac{1/(3n+1)}{1/n}=\frac{1}{3}.$$

Therefore, the series does not converge absolutely. However, since

$$\frac{1}{3(n+1)+1}<\frac{1}{3n+1}\text{ and }\frac{1}{3n+1}\to 0,$$

the series converges. We can conclude that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3n+1}$ converges conditionally.

b. Noting that $|\cos n|\le 1$, to determine whether the series converges absolutely, compare

$$\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|$$

with the series $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, by the comparison test, $\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right|$ converges, and therefore $\sum_{n=1}^{\infty}\frac{\cos n}{n^2}$ converges absolutely.

Exercise 6.6.3

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{2n^3+1}$ converges absolutely, converges conditionally, or diverges.

Hint

Check for absolute convergence first.

Answer

The series converges absolutely.

To see the difference between absolute and conditional convergence, look at what happens when we rearrange the terms of the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$. We show that we can rearrange the terms so that the new series diverges. Certainly if we rearrange the terms of a finite sum, the sum does not change. When we work with an infinite sum, however, interesting things can happen.

Begin by adding enough of the positive terms to produce a sum that is larger than some real number $M=10$ For example, let $M=10$, and find an integer k such that

$$1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{2k-1}>10$$

(We can do this because the series $\sum_{n=1}^{\infty}\frac{1}{2n-1}$ diverges to infinity.) Then subtract $1/2$. Then add more positive terms until the sum reaches 100. That is, find another integer $j>k$ such that

$$(1+\frac{1}{3}+\cdots+\frac{1}{2k-1}-\frac{1}{2}+\frac{1}{2k+1}+\ldots+\frac{1}{2j+1}>100.$$

Then subtract $1/4$. Continuing in this way, we have found a way of rearranging the terms in the alternating harmonic series so that the sequence of partial sums for the rearranged series is unbounded and therefore diverges.

The terms in the alternating harmonic series can also be rearranged so that the new series converges to a different value. In Example, we show how to rearrange the terms to create a new series that converges to $3\ln(2)/2$. We point out that the alternating harmonic series can be rearranged to create a series that converges to any real number r; however, the proof of that fact is beyond the scope of this text.

In general, any series $\sum_{n=1}^{\infty} a_n$ that converges conditionally can be rearranged so that the new series diverges or converges to a different real number. A series that converges absolutely does not have this property. For any series $\sum_{n=1}^{\infty} a_n$ that converges absolutely, the value of $\sum_{n=1}^{\infty} a_n$ is the same for any rearrangement of the terms. This result is known as the **Riemann Rearrangement Theorem**, which is beyond the scope of this book.

Example 6.6.4: Rearranging Series

Use the fact that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\ln 2$$

to rearrange the terms in the alternating harmonic series so the sum of the rearranged series is $3\ln(2)/2$.

Solution

Let

$$\sum_{n=1}^{\infty} a_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots.$$

Since $\sum_{n=1}^{\infty} a_n = \ln(2)$, by the algebraic properties of convergent series,

$$\sum_{n=1}^{\infty} \frac{1}{2}a_n = \frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\cdots=\frac{1}{2}\sum_{n=1}^{\infty} a_n=\frac{\ln 2}{2}.$$

Now introduce the series $\sum_{n=1}^{\infty} b_n$ such that for all $n\geq 1, b_{2n-1}=0$ and $b_{2n}=a_n/2$. Then

$$\sum_{n=1}^{\infty} b_n = 0+\frac{1}{2}+0-\frac{1}{4}+0+\frac{1}{6}+0-\frac{1}{8}+\cdots=\frac{\ln 2}{2}.$$

Then using the algebraic limit properties of convergent series, since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, the series $\sum_{n=1}^{\infty}(a_n+b_n)$ converges and

$$\sum_{n=1}^{\infty}(a_n+b_n)=\sum_{n=1}^{\infty} a_n+\sum_{n=1}^{\infty} b_n=\ln 2+\frac{\ln 2}{2}=\frac{3\ln 2}{2}.$$

Now adding the corresponding terms, a_n and b_n, we see that

$$\sum_{n=1}^{\infty}(a_n+b_n)=(1+0)+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{3}+0\right)+\left(-\frac{1}{4}-14\right)+\left(\frac{1}{5}+0\right)+\left(-\frac{1}{6}+\frac{1}{6}\right)+\left(\frac{1}{7}+0\right)$$
$$+\left(\frac{1}{8}-\frac{1}{8}\right)+\cdots=1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots.$$

We notice that the series on the right side of the equal sign is a rearrangement of the alternating harmonic series. Since $\sum_{n=1}^{\infty}(a_n+b_n)=3\ln(2)/2,$ we conclude that

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\cdots=\frac{3\ln(2)}{2}.$$

Therefore, we have found a rearrangement of the alternating harmonic series having the desired property.

Key Concepts

- For an alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$, if $b_{k+1}\leq b_k$ for all k and $b_k\to 0$ as $k\to\infty$, the alternating series converges.
- If $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ converges.

Key Equations

- **Alternating series**

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n=b_1-b_2+b_3-b_4+\cdots \quad \text{or}$$

$$\sum_{n=1}^{\infty}(-1)^{n}b_n=-b_1+b_2-b_3+b_4-\cdots$$

Glossary

absolute convergence

if the series $\sum_{n=1}^{\infty}|a_n|$ converges, the series $\sum_{n=1}^{\infty}a_n$ is said to converge absolutely

alternating series

a series of the form $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^{n}b_n$, where $b_n\geq 0$, is called an alternating series

alternating series test

for an alternating series of either form, if $b_{n+1}\leq b_n$ for all integers $n\geq 1$ and $b_n\to 0$, then an alternating series converges

conditional convergence

if the series $\sum_{n=1}^{\infty}a_n$ converges, but the series $\sum_{n=1}^{\infty}|a_n|$ diverges, the series $\sum_{n=1}^{\infty}a_n$ is said to converge conditionally

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6.6E: Exercises for Alternating Series

In exercises 1 - 30, state whether each of the following series converges absolutely, conditionally, or not at all.

1) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n}{n+3}$

2) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n}+1}{\sqrt{n}+3}$

Answer

This series diverges by the divergence test. Terms do not tend to zero.

3) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{\sqrt{n+3}}$

4) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{\sqrt{n+3}}{n}$

Answer

Converges conditionally by alternating series test, since $\sqrt{n+3}\,/n$ is decreasing and its limit is 0. Does not converge absolutely by comparison with p-series, $p=1/2$.

5) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{1}{n!}$

6) $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{3^n}{n!}$

Answer

Converges absolutely by limit comparison to $3^n/4^n$, for example.

7) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n-1}{n}\right)^n$

8) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\dfrac{n+1}{n}\right)^n$

Answer

Diverges by divergence test since $\lim_{n\to\infty}|a_n|=e$ and not 0.

9) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2 n$

10) $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2 n$

Answer

Diverges by the divergence test, since its terms do not tend to zero. The limit of the sequence of its terms does not exist.

11) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin^2(1/n)$

12) $\sum_{n=1}^{\infty}(-1)^{n+1}\cos^2(1/n)$

Answer

$\lim_{n\to\infty} \cos^2(1/n) = 1$. Diverges by divergence test.

13) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(1/n)$

14) $\sum_{n=1}^{\infty}(-1)^{n+1}\ln(1+\frac{1}{n})$

Answer

Converges by alternating series test.

15) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{1+n^4}$

16) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^e}{1+n^\pi}$

Answer

Converges conditionally by alternating series test. Does not converge absolutely by limit comparison with p-series, $p = \pi - e$

Solution:

17) $\sum_{n=1}^{\infty}(-1)^{n+1}2^{1/n}$

18) $\sum_{n=1}^{\infty}(-1)^{n+1}n^{1/n}$

Answer

Diverges; terms do not tend to zero.

19) $\sum_{n=1}^{\infty}(-1)^{n}(1-n^{1/n})$ (Hint: $n^{1/n} \approx 1+\ln(n)/n$ for large n.)

20) $\sum_{n=1}^{\infty}(-1)^{n+1}n\left(1-\cos\left(\frac{1}{n}\right)\right)$ (Hint: $\cos(1/n) \approx 1-1/n^2$ for large n.)

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

21) $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt{n+1}-\sqrt{n})$ (Hint: Rationalize the numerator.)

22) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\right)$ (Hint: Cross-multiply then rationalize numerator.)

Answer

Converges absolutely by limit comparison with p-series, $p = 3/2$, after applying the hint.

23) $\sum_{n=1}^{\infty}(-1)^{n+1}(\ln(n+1)-\ln n)$

24) $\sum_{n=1}^{\infty}(-1)^{n+1}n(\tan^{-1}(n+1)-\tan^{-1} n)$ (Hint: Use Mean Value Theorem.)

Answer

Converges by alternating series test since $n(\tan^{-1}(n+1)-\tan^{-1} n)$ is decreasing to zero for large n.Does not converge absolutely by limit comparison with harmonic series after applying hint.

25) $\sum_{n=1}^{\infty}(-1)^{n+1}((n+1)^2-n^2)$

26) $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

Answer

Converges absolutely, since $a_n=\frac{1}{n}-\frac{1}{n+1}$ are terms of a telescoping series.

27) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

28) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n^{1/n}}$

Answer

Terms do not tend to zero. Series diverges by divergence test.

29) $\sum_{n=1}^{\infty}\frac{1}{n}\sin(\frac{n\pi}{2})$

30) $\sum_{n=1}^{\infty}\sin(n\pi/2)\sin(1/n)$

Answer

Converges by alternating series test. Does not converge absolutely by limit comparison with harmonic series.

In exercises 31 - 36, use the estimate $|R_N|\le b_{N+1}$ to find a value of N that guarantees that the sum of the first N terms of the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ differs from the infinite sum by at most the given error. Calculate the partial sum S_N for this N.

31) [T] $b_n=1/n$, error $<10^{-5}$

32) [T] $b_n=1/\ln(n), n\ge 2,$ error $<10^{-1}$

Answer

$\ln(N+1)>10, N+1>e^{10}, N\ge 22026; S_{22026}=0.0257\ldots$

33) [T] $b_n=1/\sqrt{n}$, error $<10^{-3}$

34) [T] $b_n=1/2^n$, error $<10^{-6}$

Answer

$2^{N+1}>10^6$ or $N+1>6\ln(10)/\ln(2)=19.93.$ or $N\ge 19; S_{19}=0.333333969\ldots$

35) [T] $b_n=ln(1+\frac{1}{n}),$ error $<10^{-3}$

36) [T] $b_n=1/n^2$, error $<10^{-6}$

Answer

$(N+1)^2>10^6$ or $N>999; S_{1000}\approx 0.822466.$

For exercises 37 - 45, indicate whether each of the following statements is true or false. If the statement is false, provide an example in which it is false.

37) If $b_n \geq 0$ is decreasing and $\lim_{n\to\infty} b_n = 0$, then $\sum_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.

38) If $b_n \geq 0$ is decreasing, then $\sum_{n=1}^{\infty}(b_{2n-1} - b_{2n})$ converges absolutely.

Answer

True. b_n need not tend to zero since if $c_n = b_n - \lim b_n$, then $c_{2n-1} - c_{2n} = b_{2n-1} - b_{2n}$.

39) If $b_n \geq 0$ and $\lim_{n\to\infty} b_n = 0$ then $\sum_{n=1}^{\infty}(\frac{1}{2}(b_{3n-2} + b_{3n-1}) - b_{3n})$ converges.

40) If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(b_{3n-2} + b_{3n-1} - b_{3n})$ converges then $\sum_{n=1}^{\infty} b_{3n-2}$ converges.

Answer

True. $b_{3n-1} - b_{3n} \geq 0,$ so convergence of $\sum b_{3n-2}$ follows from the comparison test.

41) If $b_n \geq 0$ is decreasing and $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ converges conditionally but not absolutely, then b_n does not tend to zero.

42) Let $a_n^+ = a_n$ if $a_n \geq 0$ and $a_n^- = -a_n$ if $a_n < 0$. (Also, $a_n^+ = 0$ if $a_n < 0$ and $a_n^- = 0$ if $a_n \geq 0$.) If $\sum_{n=1}^{\infty} a_n$ converges conditionally but not absolutely, then neither $\sum_{n=1}^{\infty} a_n^+$ nor $\sum_{n=1}^{\infty} a_n^-$ converge.

Answer

True. If one converges, then so must the other, implying absolute convergence.

43) Suppose that a_n is a sequence of positive real numbers and that $\sum_{n=1}^{\infty} a_n$ converges.

44) Suppose that b_n is an arbitrary sequence of ones and minus ones. Does $\sum_{n=1}^{\infty} a_n b_n$ necessarily converge?

45) Suppose that a_n is a sequence such that $\sum_{n=1}^{\infty} a_n b_n$ converges for every possible sequence b_n of zeros and ones. Does $\sum_{n=1}^{\infty} a_n$ converge absolutely?

Answer

Yes. Take $b_n = 1$ if $a_n \geq 0$ and $b_n = 0$ if $a_n < 0$. Then $\sum_{n=1}^{\infty} a_n b_n = \sum_{n:a_n \geq 0} a_n$ converges. Similarly, one can show $\sum_{n:a_n<0} a_n$ converges. Since both series converge, the series must converge absolutely.

In exercises 46 - 49, the series do not satisfy the hypotheses of the alternating series test as stated. In each case, state which hypothesis is not satisfied. State whether the series converges absolutely.

46) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin^2 n}{n}$

47) $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos^2 n}{n}$

Answer

Not decreasing. Does not converge absolutely.

48) $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\cdots$

49) $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\cdots$

Answer

Not alternating. Can be expressed as $\sum_{n=1}^{\infty}\left(\frac{1}{3n-2}+\frac{1}{3n-1}-\frac{1}{3n}\right)$, which diverges by comparison with $\sum_{n=1}^{\infty}\frac{1}{3n-2}$.

50) Show that the alternating series $1-\frac{1}{2}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}+\frac{1}{4}-\frac{1}{8}+\cdots$ does not converge. What hypothesis of the alternating series test is not met?

51) Suppose that $\sum a_n$ converges absolutely. Show that the series consisting of the positive terms a_n also converges.

Answer

Let $a_n^+=a_n$ if $a_n\geq 0$ and $a_n^+=0$ if $a_n<0$. Then $a_n^+\leq |a_n|$ for all n so the sequence of partial sums of a_n^+ is increasing and bounded above by the sequence of partial sums of $|a_n|$, which converges; hence, $\sum_{n=1}^{\infty}a_n^+$ converges.

52) Show that the alternating series $\frac{2}{3}-\frac{3}{5}+\frac{4}{7}-\frac{5}{9}+\cdots$ does not converge. What hypothesis of the alternating series test is not met?

53) The formula $\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots$ will be derived in the next chapter. Use the remainder $|R_N|\leq b_{N+1}$ to find a bound for the error in estimating $\cos\theta$ by the fifth partial sum $1-\theta^2/2!+\theta^4/4!-\theta^6/6!+\theta^8/8!$ for $\theta=1,\theta=\pi/6,$ and $\theta=\pi$.

Answer

For $N=5$ one has $\left|R_N\right| b_6=\theta^{10}/10!$. When $\theta=1, R_5\leq 1/10!\approx 2.75\times 10^{-7}$. When $\theta=\pi/6$, $R_5\leq(\pi/6)^{10}/10!\approx 4.26\times 10^{-10}$. When $\theta=\pi, R_5\leq\pi^{10}/10!=0.0258$.

54) The formula $\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots$ will be derived in the next chapter. Use the remainder $|R_N|\leq b_{N+1}$ to find a bound for the error in estimating $\sin\theta$ by the fifth partial sum $\theta-\theta^3/3!+\theta^5/5!-\theta^7/7!+\theta^9/9!$ for $\theta=1,\theta=\pi/6,$ and $\theta=\pi$.

55) How many terms in $\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\cdots$ are needed to approximate $\cos 1$ accurate to an error of at most 0.00001?

Answer

Let $b_n=1/(2n-2)!$. Then $R_N\leq 1/(2N)!<0.00001$ when $(2N)!>10^5$ or $N=5$ and $1-\frac{1}{2!}+\frac{1}{4!}-\frac{1}{6!}+\frac{1}{8!}=0.540325\ldots$, whereas $\cos 1=0.5403023\ldots$

56) How many terms in $\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots$ are needed to approximate $\sin 1$ accurate to an error of at most 0.00001?

57) Sometimes the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges to a certain fraction of an absolutely convergent series $\sum_{n=1}^{\infty}b_n$ at a faster rate. Given that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, find $S=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots$. Which of the series $6\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $S\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$ gives a better estimation of π^2 using 1000 terms?

Answer

Let $T = \sum \frac{1}{n^2}$. Then $T - S = \frac{1}{2}T$, so $S = T/2$.

$\sqrt{6 \times \sum_{n=1}^{1000} 1/n^2} = 3.140638\ldots$; $\sqrt{\frac{1}{2} \times \sum_{n=1}^{1000} (-1)^{n-1}/n^2} = 3.141591\ldots$; $\pi = 3.141592\ldots$The alternating series is more accurate for 1000terms.

The alternating series in exercises 58 & 59 converge to given multiples of π. Find the value of N predicted by the remainder estimate such that the Nth partial sum of the series accurately approximates the left-hand side to within the given error. Find the minimum N for which the error bound holds, and give the desired approximate value in each case. Up to 15 decimals places, $\pi = 3.141592653589793\ldots$.

58) [T] $\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$, error < 0.0001

59) [T] $\frac{\pi}{\sqrt{12}} = \sum_{k=0}^{\infty} \frac{(-3)^{-k}}{2k+1}$, error < 0.0001

Answer

$N = 6, S_N = 0.9068$

60) [T] The series $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ plays an important role in signal processing. Show that $\sum_{n=0}^{\infty} \frac{\sin(x+\pi n)}{x+\pi n}$ converges whenever $0 < x < \pi$. (Hint: Use the formula for the sine of a sum of angles.)

61) [T] If $\sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n} \to ln2$, what is $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12} + \cdots$?

Answer

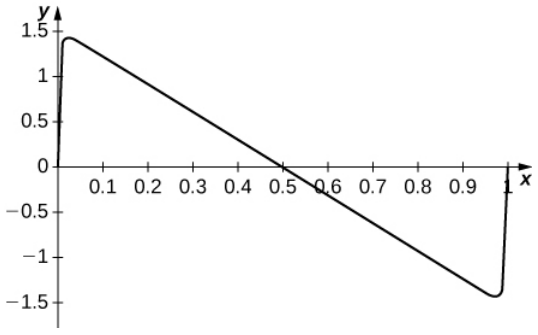
$\ln(2)$. The nth partial sum is the same as that for the alternating harmonic series.

62) [T] Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ for $0 \le x < 1$. Explain why $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n}$ diverges when $x = 0, 1$. How does the series behave for other x?

63) [T] Plot the series $\sum_{n=1}^{100} \frac{\sin(2\pi nx)}{n}$ for $0 \le x < 1$ and comment on its behavior

Answer

The series jumps rapidly near the endpoints. For x away from the endpoints, the graph looks like $\pi(1/2 - x)$.



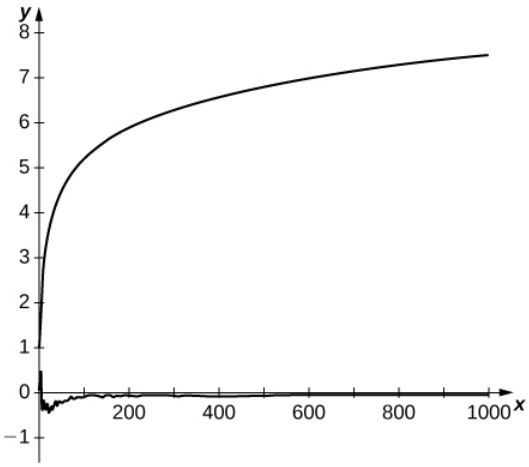
64) [T] Plot the series $\sum_{n=1}^{100} \frac{\cos(2\pi nx)}{n^2}$ for $0 \le x < 1$ and describe its graph.

https://math.libretexts.org/@go/page/127940

65) [T] The alternating harmonic series converges because of cancelation among its terms. Its sum is known because the cancelation can be described explicitly. A random harmonic series is one of the form $\sum_{n=1}^{\infty} \frac{S_n}{n}$, where s_n is a randomly generated sequence of $\pm 1's$ in which the values ± 1 are equally likely to occur. Use a random number generator to produce 1000 random $\pm 1's$ and plot the partial sums $S_N = \sum_{n=1}^{N} \frac{s_n}{n}$ of your random harmonic sequence for $N = 1$ to 1000. Compare to a plot of the first 1000 partial sums of the harmonic series.

Answer

Here is a typical result. The top curve consists of partial sums of the harmonic series. The bottom curve plots partial sums of a random harmonic series.



66) [T] Estimates of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ can be accelerated by writing its partial sums as $\sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=1}^{N} \frac{1}{n(n+1)} + \sum_{n=1}^{N} \frac{1}{n^2(n+1)}$ and recalling that $\sum_{n=1}^{N} \frac{1}{n(n+1)} = 1 - \frac{1}{N+1}$ converges to one as $N \to \infty$. Compare the estimate of $\pi^2/6$ using the sums $\sum_{n=1}^{1000} \frac{1}{n^2}$ with the estimate using $1 + \sum_{n=1}^{1000} \frac{1}{n^2(n+1)}$.

67) [T] The **Euler transform** rewrites $S = \sum_{n=0}^{\infty} (-1)^n b_n$ as $S = \sum_{n=0}^{\infty} (-1)^n 2^{-n-1} \sum_{m=0}^{n} \binom{n}{m} b_{n-m}$. For the alternating harmonic series, it takes the form $\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{1}{n2^n}$. Compute partial sums of $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ until they approximate $\ln(2)$ accurate to within 0.0001. How many terms are needed? Compare this answer to the number of terms of the alternating harmonic series are needed to estimate $\ln(2)$.

Answer

By the alternating series test, $|S_n - S| \leq b_{n+1},$ so one needs 10^4 terms of the alternating harmonic series to estimate $\ln(2)$ to within 0.0001 The first 10 partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ are (up to four decimals)

$0.5000, 0.6250, 0.6667, 0.6823, 0.6885, 0.6911, 0.6923, 0.6928, 0.6930, 0.6931$, and the tenth partial sum is within 0.0001 of $\ln(2) = 0.6931\ldots$.

68) [T] In the text it was stated that a conditionally convergent series can be rearranged to converge to any number. Here is a slightly simpler, but similar, fact. If $a_n \geq 0$ is such that $a_n \to 0$ as $n \to \infty$ but $\sum_{n=1}^{\infty} a_n$ diverges, then, given any number A there is

a sequence s_n of $\pm 1's$ such that $\sum_{n=1}^{\infty} a_n s_n \to A$. Show this for $A > 0$ as follows.

a. Recursively define s_n by $s_n = 1$ if $S_{n-1} = \sum_{k=1}^{n-1} a_k s_k < A$ and $s_n = -1$ otherwise.

b. Explain why eventually $S_n \geq A$, and for any m larger than this n, $A - a_m \leq S_m \leq A + a_m$.

c. Explain why this implies that $S_n \to A$ as $n \to \infty$.

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6.7: Ratio and Root Tests

Learning Objectives

- Use the ratio test to determine absolute convergence of a series.
- Use the root test to determine absolute convergence of a series.
- Describe a strategy for testing the convergence of a given series.

In this section, we prove the last two series convergence tests: the ratio test and the root test. These tests are particularly nice because they do not require us to find a comparable series. The ratio test will be especially useful in the discussion of power series in the next chapter. Throughout this chapter, we have seen that no single convergence test works for all series. Therefore, at the end of this section we discuss a strategy for choosing which convergence test to use for a given series.

Ratio Test

Consider a series $\sum_{n=1}^{\infty} a_n$. From our earlier discussion and examples, we know that $\lim_{n\to\infty} a_n = 0$ is not a sufficient condition for the series to converge. Not only do we need $a_n \to 0$, but we need $a_n \to 0$ quickly enough. For example, consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. We know that $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$. However, only the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because the terms in the sequence $\left\{\frac{1}{n}\right\}$ do not approach zero fast enough as $n \to \infty$. Here we introduce the **ratio test,** which provides a way of measuring how fast the terms of a series approach zero.

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms. Let

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

Proof

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms.

We begin with the proof of part i. In this case, $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. Since $0 \le \rho < 1$, there exists R such that $0 \le \rho < R < 1$. Let $\varepsilon = R - \rho > 0$. By the definition of limit of a sequence, there exists some integer N such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \varepsilon, \text{ for all } n \ge N.$$

Therefore,

$$\left|\frac{a_{n+1}}{a_n}\right| < \rho + \varepsilon = R, \text{ for all } n \ge N$$

and, thus,

$$
\begin{aligned}
&|a_{N+1}| < R|a_N| \\
&|a_{N+2}| < R|a_{N+1}| < R^2|a_N| \\
&|a_{N+3}| < R|a_{N+2}| < R^2|a_{N+1}| < R^3|a_N| \\
&|a_{N+4}| < R|a_{N+3}| < R^2|a_{N+2}| < R^3|a_{N+1}| < R^4|a_N| \\
&\vdots
\end{aligned}
$$

Since $R < 1$, the geometric series

$$R|a_N| + R^2|a_N| + R^3|a_N| + \cdots$$

converges. Given the inequalities above, we can apply the comparison test and conclude that the series

$$|a_{N+1}| + |a_{N+2}| + |a_{N+3}| + |a_{N+4}| + \cdots$$

converges. Therefore, since

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|$$

where $\sum_{n=1}^{N} |a_n|$ is a finite sum and $\sum_{n=N+1}^{\infty} |a_n|$ converges, we conclude that $\sum_{n=1}^{\infty} |a_n|$ converges.

For part ii.

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1.$$

Since $\rho > 1$, there exists R such that $\rho > R > 1$. Let $\varepsilon = \rho - R > 0$. By the definition of the limit of a sequence, there exists an integer N such that

$$\left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \varepsilon, \text{ for all } n \geq N.$$

Therefore,

$$R = \rho - \varepsilon < \left|\frac{a_{n+1}}{a_n}\right|, \text{ for all } n \geq N,$$

and, thus,

$$
\begin{aligned}
&|a_{N+1}| > R|a_N| \\
&|a_{N+2}| > R|a_{N+1}| > R^2|a_N| \\
&|a_{N+3}| > R|a_{N+2}| > R^2|a_{N+1}| > R^3|a_N| \\
&|a_{N+4}| > R|a_{N+3}| > R^2|a_{N+2}| > R^3|a_{N+1}| > R^4|a_N|.
\end{aligned}
$$

Since $R > 1$, the geometric series

$$R|a_N| + R^2|a_N| + R^3|a_N| + \cdots$$

diverges. Applying the comparison test, we conclude that the series

$$|a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

diverges, and therefore the series $\sum_{n=1}^{\infty} |a_n|$ diverges.

For part iii. we show that the test does not provide any information if $\rho = 1$ by considering the $p-series$ $\sum_{n=1}^{\infty} \frac{1}{n^p}$. For any real number p,

$$\rho = \lim_{n\to\infty} \frac{1/(n+1)^p}{1/n^p} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = 1.$$

However, we know that if $p \le 1$, the **p−series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges, whereas $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

□

The ratio test is particularly useful for series whose terms contain factorials or exponential, where the ratio of terms simplifies the expression. The ratio test is convenient because it does not require us to find a comparative series. The drawback is that the test sometimes does not provide any information regarding convergence.

Example 6.7.1: Using the Ratio Test

For each of the following series, use the ratio test to determine whether the series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

b. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

c. $\sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2}{(2n)!}$

Solution

a. From the ratio test, we can see that

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}.$$

Since $(n+1)! = (n+1)\cdot n!$,

$$\rho = \lim_{n\to\infty} \frac{2}{n+1} = 0.$$

Since $\rho < 1$, the series converges.

b. We can see that

$$\rho = \lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e.$$

Since $\rho > 1$, the series diverges.

c. Since

$$\left| \frac{(-1)^{n+1}((n+1)!)^2/(2(n+1))!}{(-1)^n (n!)^2/(2n)!} \right| = \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!n!} = \frac{(n+1)(n+1)}{(2n+2)(2n+1)}$$

we see that

$$\rho = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{1}{4}.$$

Since $\rho < 1$, the series converges.

Exercise 6.7.1

Use the ratio test to determine whether the series $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ converges or diverges.

Hint

Evaluate $\lim_{n\to\infty} \dfrac{(n+1)^3}{3^{n+1}} \cdot \dfrac{3^n}{n^3}$.

Answer

The series converges.

Root Test

The approach of the **root test** is similar to that of the ratio test. Consider a series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho$ for some real number ρ. Then for N sufficiently large, $|a_N| \approx \rho^N$. Therefore, we can approximate $\sum_{n=N}^{\infty} |a_n|$ by writing

$$|a_N| + |a_{N+1}| + |a_{N+2}| + \cdots \approx \rho^N + \rho^{N+1} + \rho^{N+2} + \cdots.$$

The expression on the right-hand side is a geometric series. As in the ratio test, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely if $0 \le \rho < 1$ and the series diverges if $\rho \ge 1$. If $\rho = 1$, the test does not provide any information. For example, for any p-series, $\sum_{n=1}^{\infty} \dfrac{1}{n^p}$, we see that

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^p}\right|} = \lim_{n\to\infty} \frac{1}{n^{p/n}}$$

.

To evaluate this limit, we use the natural logarithm function. Doing so, we see that

$$\ln \rho = \ln\left(\lim_{n\to\infty} \frac{1}{n^{p/n}}\right) = \lim_{n\to\infty} \ln\left(\frac{1}{n}\right)^{p/n} = \lim_{n\to\infty} \frac{p}{n} \cdot \ln\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{p\ln(1/n)}{n}.$$

Using L'Hôpital's rule, it follows that $\ln \rho = 0$, and therefore $\rho = 1$ for all p. However, we know that the p-series only converges if $p > 1$ and diverges if $p < 1$.

Root Test

Consider the series $\sum_{n=1}^{\infty} a_n$. Let

$$\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$$

.

i. If $0 \le \rho < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

ii. If $\rho > 1$ or $\rho = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

iii. If $\rho = 1$, the test does not provide any information.

The root test is useful for series whose terms involve exponentials. In particular, for a series whose terms a_n satisfy $|a_n| = (b_n)^n$, then $\sqrt[n]{|a_n|} = b_n$ and we need only evaluate $\lim_{n\to\infty} b_n$.

Example 6.7.2: Using the Root Test

For each of the following series, use the root test to determine whether the series converges or diverges.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{(n^2+3n)^n}{(4n^2+5)^n}$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{(\ln(n))^n}$

Solution

a. To apply the root test, we compute

$$\rho=\lim_{n\to\infty}\sqrt[n]{\frac{(n^2+3n)^n}{(4n^2+5)^n}}=\lim_{n\to\infty}\frac{n^2+3n}{4n^2+5}=\frac{1}{4}.$$

Since $\rho<1$, the series converges absolutely.

b. We have

$$\rho=\lim_{n\to\infty}\sqrt[n]{\frac{n^n}{(\ln(n))^n}}=\lim_{n\to\infty}\frac{n}{\ln n}=\infty \quad \text{by L'Hôpital's rule.}$$

Since $\rho=\infty$, the series diverges.

Exercise 6.7.2

Use the root test to determine whether the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^n}$ converges or diverges.

Hint

Evaluate $\displaystyle\lim_{n\to\infty}\sqrt[n]{\frac{1}{n^n}}$.

Answer

The series converges.

Choosing a Convergence Test

At this point, we have a long list of convergence tests. However, not all tests can be used for all series. When given a series, we must determine which test is the best to use. Here is a strategy for finding the best test to apply.

Problem-Solving Strategy: Choosing a Convergence Test for a Series

Consider a series $\displaystyle\sum_{n=1}^{\infty}a_n$. In the steps below, we outline a strategy for determining whether the series converges.

1. Is $\displaystyle\sum_{n=1}^{\infty}a_n$ a familiar series? For example, is it the harmonic series (which diverges) or the alternating harmonic series (which converges)? Is it a **p−series** or geometric series? If so, check the power p or the ratio r to determine if the series converges.
2. Is it an alternating series? Are we interested in absolute convergence or just convergence? If we are just interested in whether the series converges, apply the alternating series test. If we are interested in absolute convergence, proceed to step 3, considering the series of absolute values $\displaystyle\sum_{n=1}^{\infty}|a_n|$.
3. Is the series similar to a **p−series** or geometric series? If so, try the comparison test or limit comparison test.

4. Do the terms in the series contain a factorial or power? If the terms are powers such that $a_n = (b_n)^n$, try the root test first. Otherwise, try the ratio test first.
5. Use the divergence test. If this test does not provide any information, try the integral test.

Example 6.7.3: Using Convergence Tests

For each of the following series, determine which convergence test is the best to use and explain why. Then determine if the series converges or diverges. If the series is an alternating series, determine whether it converges absolutely, converges conditionally, or diverges.

a. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2n}{n^3+3n^2+1}$

b. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(3n+1)}{n!}$

c. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^3}$

d. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{(n+1)^n}$

Solution

a. Step 1. The series is not a **p–series** or geometric series.

Step 2. The series is not alternating.

Step 3. For large values of n, we approximate the series by the expression

$$\frac{n^2+2n}{n^3+3n^2+1} \approx \frac{n^2}{n^3} = \frac{1}{n}.$$

Therefore, it seems reasonable to apply the comparison test or limit comparison test using the series $\displaystyle\sum_{n=1}^{\infty} 1/n$.

Using the limit comparison test, we see that

$$\lim_{n\to\infty} \frac{(n^2+2n)/(n^3+3n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^3+2n^2}{n^3+3n^2+1} = 1.$$

Since the series $\displaystyle\sum_{n=1}^{\infty} 1/n$

diverges, this series diverges as well.

b. Step 1.The series is not a familiar series.

Step 2. The series is alternating. Since we are interested in absolute convergence, consider the series

$$\sum_{n=1}^{\infty} \frac{3n}{(n+1)!}.$$

Step 3. The series is not similar to a p-series or geometric series.

Step 4. Since each term contains a factorial, apply the ratio test. We see that

$$\lim_{n\to\infty} \frac{(3(n+1))/(n+1)!}{(3n+1)/n!} = \lim_{n\to\infty} \frac{3n+3}{(n+1)!} \cdot \frac{n!}{3n+1} = \lim_{n\to\infty} \frac{3n+3}{(n+1)(3n+1)} = 0.$$

Therefore, this series converges, and we conclude that the original series converges absolutely, and thus converges.

c. Step 1. The series is not a familiar series.

Step 2. It is not an alternating series.

Step 3. There is no obvious series with which to compare this series.

Step 4. There is no factorial. There is a power, but it is not an ideal situation for the root test.

Step 5. To apply the divergence test, we calculate that

$$\lim_{n\to\infty}\frac{e^n}{n^3}=\infty.$$

Therefore, by the divergence test, the series diverges.

d. Step 1. This series is not a familiar series.

Step 2. It is not an alternating series.

Step 3. There is no obvious series with which to compare this series.

Step 4. Since each term is a power of n,we can apply the root test. Since

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{3}{n+1}\right)^n}=\lim_{n\to\infty}\frac{3}{n+1}=0,$$

by the root test, we conclude that the series converges.

? Exercise 6.7.3

For the series $\sum_{n=1}^{\infty}\frac{2^n}{3^n+n}$, determine which convergence test is the best to use and explain why.

Hint

The series is similar to the geometric series $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$.

Answer

The comparison test because $\frac{2^n}{3^n+n}<\frac{2^n}{3^n}$ for all positive integers n. The limit comparison test could also be used.

In Table, we summarize the convergence tests and when each can be applied. Note that while the comparison test, limit comparison test, and integral test require the series $\sum_{n=1}^{\infty}a_n$ to have nonnegative terms, if $\sum_{n=1}^{\infty}a_n$ has negative terms, these tests can be applied to $\sum_{n=1}^{\infty}|a_n|$ to test for absolute convergence.

Summary of Convergence Tests

Series or Test	Conclusions	Comments
Divergence Test For any series $\sum_{n=1}^{\infty}a_n$, evaluate $\lim_{n\to\infty}a_n$.	If $\lim_{n\to\infty}a_n=0$, the test is inconclusive.	This test cannot prove convergence of a series.
	If $\lim_{n\to\infty}a_n\neq 0$, the series diverges.	
Geometric Series $\sum_{n=1}^{\infty}ar^{n-1}$	If $\lvert r\rvert<1$, the series converges to $a/(1-r)$.	Any geometric series can be reindexed to be written in the form $a+ar+ar^2+\cdots$, where a is the initial term and r is the ratio.
	If $\lvert r\rvert\geq 1$, the series diverges.	
p-Series $\sum_{n=1}^{\infty}\frac{1}{n^p}$	If $p>1$, the series converges.	For $p=1$, we have the harmonic series $\sum_{n=1}^{\infty}1/n$.
	If $p\leq 1$, the series diverges.	

Series or Test	Conclusions	Comments
Comparison Test For $\sum_{n=1}^{\infty} a_n$ with nonnegative terms, compare with a known series $\sum_{n=1}^{\infty} b_n$.	If $a_n \le b_n$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	Typically used for a series similar to a geometric or p-series. It can sometimes be difficult to find an appropriate series.
	If $a_n \ge b_n$ for all $n \ge N$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Limit Comparison Test For $\sum_{n=1}^{\infty} a_n$ with positive terms, compare with a series $\sum_{n=1}^{\infty} b_n$ by evaluating $L = \lim_{n\to\infty} \frac{a_n}{b_n}$.	If L is a real number and $L \ne 0$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or both diverge.	Typically used for a series similar to a geometric or p-series. Often easier to apply than the comparison test.
	If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.	
	If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.	
Integral Test If there exists a positive, continuous, decreasing function f such that $a_n = f(n)$ for all $n \ge N$, evaluate $\int_N^{\infty} f(x)dx$.	$\int_N^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ both converge or both diverge.	Limited to those series for which the corresponding function can be easily integrated.
Alternating Series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ or $\sum_{n=1}^{\infty}(-1)^{n}b_n$	If $b_{n+1} \le b_n$ for all $n \ge 1$ and $b_n \to 0$, then the series converges.	Only applies to alternating series.
Ratio Test For any series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$	If $0 \le \rho < 1$, the series converges absolutely.	Often used for series involving factorials or exponentials.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	
Root Test For any series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert}$.	If $0 \le \rho < 1$, the series converges absolutely.	Often used for series where $\lvert a_n \rvert = (b_n)^n$.
	If $\rho > 1$ or $\rho = \infty$, the series diverges.	
	If $\rho = 1$, the test is inconclusive.	

Series Converging to π and $1/\pi$

Dozens of series exist that converge to π or an algebraic expression containing π. Here we look at several examples and compare their rates of convergence. By rate of convergence, we mean the number of terms necessary for a partial sum to be within a certain amount of the actual value. The series representations of π in the first two examples can be explained using Maclaurin series, which are discussed in the next chapter. The third example relies on material beyond the scope of this text.

1. The series

https://math.libretexts.org/@go/page/127941

$$\pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

was discovered by Gregory and Leibniz in the late 1600s. This result follows from the Maclaurin series for $f(x) = \tan^{-1} x$. We will discuss this series in the next chapter.

a. Prove that this series converges.

b. Evaluate the partial sums S_n for $n = 10, 20, 50, 100$.

c. Use the remainder estimate for alternating series to get a bound on the error R_n.

d. What is the smallest value of N that guarantees $|R_N| < 0.01$? Evaluate S_N.

2. The series

$$\pi = 6\sum_{n=0}^{\infty}\frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)} = 6\left(\frac{1}{2} + \frac{1}{2\cdot 3}\left(\frac{1}{2}\right)^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}\cdot\left(\frac{1}{2}\right)^5 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}\left(\frac{1}{2}\right)^7 + \cdots\right)$$

has been attributed to Newton in the late 1600s. The proof of this result uses the Maclaurin series for $f(x) = \sin^{-1} x$.

a. Prove that the series converges.

b. Evaluate the partial sums S_n for $n = 5, 10, 20$.

c. Compare S_n to π for $n = 5, 10, 20$ and discuss the number of correct decimal places.

3. The series

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

was discovered by **Ramanujan** in the early 1900s. William Gosper, Jr., used this series to calculate π to an accuracy of more than 17 million digits in the mid-1980s. At the time, that was a world record. Since that time, this series and others by Ramanujan have led mathematicians to find many other series representations for π and $1/\pi$.

a. Prove that this series converges.

b. Evaluate the first term in this series. Compare this number with the value of π from a calculating utility. To how many decimal places do these two numbers agree? What if we add the first two terms in the series?

c. Investigate the life of Srinivasa Ramanujan (1887–1920) and write a brief summary. Ramanujan is one of the most fascinating stories in the history of mathematics. He was basically self-taught, with no formal training in mathematics, yet he contributed in highly original ways to many advanced areas of mathematics.

Key Concepts

- For the ratio test, we consider

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

If $\rho < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, the series diverges. If $\rho = 1$, the test does not provide any information. This test is useful for series whose terms involve factorials.

- For the root test, we consider

$$\rho = \lim_{n\to\infty}\sqrt[n]{|a_n|}.$$

If $\rho < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges absolutely. If $\rho > 1$, the series diverges. If $\rho = 1$, the test does not provide any information. The root test is useful for series whose terms involve powers.

- For a series that is similar to a geometric series or **p−series,** consider one of the comparison tests.

Glossary

ratio test

for a series $\sum_{n=1}^{\infty} a_n$ with nonzero terms, let $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

root test

for a series $\sum_{n=1}^{\infty} a_n$, let $\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}$; if $0 \le \rho < 1$, the series converges absolutely; if $\rho > 1$, the series diverges; if $\rho = 1$, the test is inconclusive

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6.7E: Exercises for Ratio and Root Tests

In exercises 1 - 11, use the ratio test to determine whether each series $\sum_{n=1}^{\infty} a_n$ converges or diverges. State if the ratio test is inconclusive.

1) $\sum_{n=1}^{\infty} \frac{1}{n!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$. Converges by the Ratio Test.

2) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

3) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2} < 1.$ Converges by the Ratio Test.

4) $\sum_{n=1}^{\infty} \frac{n^{10}}{2^n}$

5) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{27} < 1.$ Converges by the Ratio Test.

6) $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!}$

7) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{4}{e^2} < 1.$ Converges by the Ratio Test.

8) $\sum_{n=1}^{\infty} \frac{(2n)!}{(2n)^n}$

9) $\sum_{n=1}^{\infty} \frac{n!}{(n/e)^n}$

Answer

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$ Ratio test is inconclusive.

10) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n/e)^{2n}}$

11) $\sum_{n=1}^{\infty} \frac{(2^n n!)^2}{(2n)^{2n}}$

Answer

$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{1}{e^2} < 1$. Converges by the Ratio Test.

In exercises 12 - 21, use the root test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, where a_n is as follows.

12) $a_k = \left(\frac{k-1}{2k+3}\right)^k$

13) $a_k = \left(\frac{2k^2-1}{k^2+3}\right)^k$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 2 > 1$. Diverges by the Root Test.

14) $a_n = \frac{(\ln n)^{2n}}{n^n}$

15) $a_n = n/2^n$

Answer

$\lim_{n\to\infty} (a_n)^{1/n} = 1/2 < 1$. Converges by the Root Test.

16) $a_n = n/e^n$

17) $a_k = \frac{k^e}{e^k}$

Answer

$\lim_{k\to\infty} (a_k)^{1/k} = 1/e < 1$. Converges by the Root Test.

18) $a_k = \frac{\pi^k}{k^\pi}$

19) $a_n = \left(\frac{1}{e} + \frac{1}{n}\right)^n$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{1}{e} + \frac{1}{n} = \frac{1}{e} < 1$. Converges by the Root Test.

20) $a_k = \frac{1}{(1+\ln k)^k}$

21) $a_n = \frac{(\ln(1+\ln n))^n}{(\ln n)^n}$

Answer

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{(\ln(1+\ln n))}{(\ln n)} = 0$ by L'Hôpital's rule. Converges by the Root Test.

In exercises 22 - 28, use either the ratio test or the root test as appropriate to determine whether the series $\sum_{k=1}^{\infty} a_k$ with given terms a_k converges, or state if the test is inconclusive.

22) $a_k = \dfrac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$

23) $a_k = \dfrac{2 \cdot 4 \cdot 6 \cdots 2k}{(2k)!}$

Answer

$\lim_{k\to\infty} \dfrac{a_{k+1}}{a_k} = \lim_{k\to\infty} \dfrac{1}{2k+1} = 0$. Converges by the Ratio Test.

24) $a_k = \dfrac{1 \cdot 4 \cdot 7 \cdots (3k-2)}{3^k k!}$

25) $a_n = \left(1 - \dfrac{1}{n}\right)^{n^2}$

Answer

$\lim_{n\to\infty} (a_n)^{1/n} = 1/e$. Converges by the Root Test.

26) $a_k = \left(\dfrac{1}{k+1} + \dfrac{1}{k+2} + \cdots + \dfrac{1}{2k}\right)^k$ $\left(\text{Hint: Compare } a_k^{1/k} \text{ to } \int_k^{2k} \dfrac{dt}{t}.\right)$

27) $a_k = \left(\dfrac{1}{k+1} + \dfrac{1}{k+2} + \cdots + \dfrac{1}{3k}\right)^k$

Answer

$\lim_{k\to\infty} a_k^{1/k} = \ln(3) > 1$. Diverges by the Root Test.

28) $a_n = \left(n^{1/n} - 1\right)^n$

In exercises 29 - 30, use the ratio test to determine whether $\sum_{n=1}^{\infty} a_n$ converges, or state if the ratio test is inconclusive.

29) $\sum_{n=1}^{\infty} \dfrac{3^{n^2}}{2^{n^3}}$

Answer

$\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \dfrac{3^{2n+1}}{2^{3n^2+3n+1}} = 0$. Converges by the Ratio Test.

30) $\sum_{n=1}^{\infty} \dfrac{2^{n^2}}{n^n n!}$

In exercise 31, use the root and limit comparison tests to determine whether $\sum_{n=1}^{\infty} a_n$ converges.

31) $\sum_{n=1}^{\infty} \dfrac{1}{x_n^n}$ where $x_{n+1} = \frac{1}{2}x_n + \dfrac{1}{x_n}, x_1 = 1$ (Hint: Find limit of x_n.)

Answer

Converges by the Root Test and Limit Comparison Test since $\lim_{n\to\infty} x_n = \sqrt{2}$.

In exercises 32 - 43, use an appropriate test to determine whether the series converges.

32) $\sum_{n=1}^{\infty} \dfrac{n+1}{n^3+n^2+n+1}$

33) $\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}(n+1)}{n^3+3n^2+3n+1}$

Answer

Converges absolutely by limit comparison with p**−series,** $p=2$.

34) $\sum_{n=1}^{\infty} \dfrac{(n+1)^2}{n^3+(1.1)^n}$

35) $\sum_{n=1}^{\infty} \dfrac{(n-1)^n}{(n+1)^n}$

Answer

$\lim_{n\to\infty} a_n = 1/e^2 \neq 0$. Series diverges by the Divergence Test.

36) $a_n = \left(1+\dfrac{1}{n^2}\right)^n$ $\left(\text{Hint: } \left(1+\dfrac{1}{n^2}\right)^{n^2} \approx e.\right)$

37) $a_k = 1/2^{\sin^2 k}$

Answer

Terms do not tend to zero: $a_k \geq 1/2$, since $\sin^2 x \leq 1$.

38) $a_k = 2^{-\sin(1/k)}$

39) $a_n = 1/\binom{n+2}{n}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$a_n = \dfrac{2}{(n+1)(n+2)}$, which converges by comparison with p**−series** for $p=2$.

40) $a_k = 1/\binom{2k}{k}$

41) $a_k = 2^k/\binom{3k}{k}$

Answer

$a_k = \dfrac{2^k 1\cdot 2\cdots k}{(2k+1)(2k+2)\cdots 3k} \leq (2/3)^k$ converges by comparison with geometric series.

42) $a_k = \left(\dfrac{k}{k+\ln k}\right)^k$ $\left(\text{Hint: } a_k = \left(1+\dfrac{\ln k}{k}\right)^{-(k/\ln k)\ln k} \approx e^{-\ln k}.\right)$

43) $a_k = \left(\dfrac{k}{k+\ln k}\right)^{2k}$ $\left(\text{Hint: } a_k = \left(1+\dfrac{\ln k}{k}\right)^{-(k/\ln k)\ln k^2}.\right)$

Answer

$a_k \approx e^{-\ln k^2} = 1/k^2$. Series converges by limit comparison with p**−series**, $p=2$.

The series in exercises 44 - 47 converge by the ratio test. Use summation by parts, $\sum_{k=1}^{n} a_k(b_{k+1}-b_k) = [a_{n+1}b_{n+1}-a_1b_1] - \sum_{k=1}^{n} b_{k+1}(a_{k+1}-a_k)$ **, to find the sum of the given series.**

44) $\sum_{k=1}^{\infty} \frac{k}{2^k}$ (Hint: Take $a_k = k$ and $b_k = 2^{1-k}$.)

45) $\sum_{k=1}^{\infty} \frac{k}{c^k}$, where $c > 1$ (Hint: Take $a_k = k$ and $b_k = c^{1-k}/(c-1)$.)

Answer

If $b_k = c^{1-k}/(c-1)$ and $a_k = k$, then $b_{k+1} - b_k = -c^{-k}$ and $\sum_{n=1}^{\infty} \frac{k}{c^k} = a_1 b_1 + \frac{1}{c-1} \sum_{k=1}^{\infty} c^{-k} = \frac{c}{(c-1)^2}$.

46) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

47) $\sum_{n=1}^{\infty} \frac{(n+1)^2}{2^n}$

Answer

$6+4+1=11$

The kth term of each of the following series has a factor x^k. Find the range of x for which the ratio test implies that the series converges.

48) $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$

49) $\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2}$

Answer

$|x| \le 1$

50) $\sum_{k=1}^{\infty} \frac{x^{2k}}{3^k}$

51) $\sum_{k=1}^{\infty} \frac{x^k}{k!}$

Answer

$|x| < \infty$

52) Does there exist a number p such that $\sum_{n=1}^{\infty} \frac{2^n}{n^p}$ converges?

53) Let $0 < r < 1$. For which real numbers p does $\sum_{n=1}^{\infty} n^p r^n$ converge?

Answer

All real numbers p by the Ratio Test.

54) Suppose that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = p$. For which values of p must $\sum_{n=1}^{\infty} 2^n a_n$ converge?

55) Suppose that $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = p$. For which values of $r > 0$ is $\sum_{n=1}^{\infty} r^n a_n$ guaranteed to converge?

Answer

$r < 1/p$

56) Suppose that $\left|\frac{a_{n+1}}{a_n}\right| \le (n+1)^p$ for all $n = 1, 2, \ldots$ where p is a fixed real number. For which values of p is $\sum_{n=1}^{\infty} n! a_n$ guaranteed to converge?

57) For which values of $r > 0$, if any, does $\sum_{n=1}^{\infty} r^{\sqrt{n}}$ converge? $\left(\text{Hint: } sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} \sum_{n=k^2}^{(k+1)^2-1} a_n.\right)$

Answer

$0 < r < 1$. Note that the ratio and root tests are inconclusive. Using the hint, there are $2k$ terms $r^{\sqrt{n}}$ for $k^2 \le n < (k+1)^2$, and for $r < 1$ each term is at least r^k. Thus, $\sum_{n=1}^{\infty} r^{\sqrt{n}} = \sum_{k=1}^{\infty} \sum_{n=k^2}^{(k+1)^2-1} r^{\sqrt{n}} \ge \sum_{k=1}^{\infty} 2k r^k$, which converges by the ratio test for $r < 1$. For $r \ge 1$ the series diverges by the divergence test.

58) Suppose that $\left|\frac{a_{n+2}}{a_n}\right| \le r < 1$ for all n. Can you conclude that $\sum_{n=1}^{\infty} a_n$ converges?

59) Let $a_n = 2^{-[n/2]}$ where $[x]$ is the greatest integer less than or equal to x. Determine whether $\sum_{n=1}^{\infty} a_n$ converges and justify your answer.

Answer

One has $a_1 = 1, a_2 = a_3 = 1/2, \ldots a_{2n} = a_{2n+1} = 1/2^n$. The ratio test does not apply because $a_{n+1}/a_n = 1$ if n is even. However, $a_{n+2}/a_n = 1/2$, so the series converges according to the previous exercise. Of course, the series is just a duplicated geometric series.

The following *advanced exercises* use a generalized ratio test to determine convergence of some series that arise in particular applications when tests in this chapter, including the ratio and root test, are not powerful enough to determine their convergence. The test states that if $\lim_{n\to\infty} \frac{a_{2n}}{a_n} < 1/2$, then $\sum a_n$ converges, while if $\lim_{n\to\infty} \frac{a_{2n+1}}{a_n} > 1/2$, then $\sum a_n$ diverges.

60) Let $a_n = \frac{1}{4}\frac{3}{6}\frac{5}{8}\cdots\frac{2n-1}{2n+2} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}$. Explain why the ratio test cannot determine convergence of $\sum_{n=1}^{\infty} a_n$. Use the fact that $1 - 1/(4k)$ is increasing k to estimate $\lim_{n\to\infty} \frac{a_{2n}}{a_n}$.

61) Let $a_n = \frac{1}{1+x}\frac{2}{2+x}\cdots\frac{n}{n+x}\frac{1}{n} = \frac{(n-1)!}{(1+x)(2+x)\cdots(n+x).}$ Show that $a_{2n}/a_n \le e^{-x/2}/2$. For which $x > 0$ does the generalized ratio test imply convergence of $\sum_{n=1}^{\infty} a_n$? (Hint: Write $2a_{2n}/a_n$ as a product of n factors each smaller than $1/(1 + x/(2n)).$)

Answer

$a_{2n}/a_n = \frac{1}{2}\cdot\frac{n+1}{n+1+x}\frac{n+2}{n+2+x}\cdots\frac{2n}{2n+x}$. The inverse of the kth factor is $(n+k+x)/(n+k) > 1 + x/(2n)$ so the product is less than $(1 + x/(2n))^{-n} \approx e^{-x/2}$. Thus for $x > 0$, $\frac{a_{2n}}{a_n} \le \frac{1}{2}e^{-x/2}$. The series converges for $x > 0$.

62) Let $a_n = \frac{n^{\ln n}}{(\ln n)^n}$. Show that $\frac{a_{2n}}{a_n} \to 0$ as $n \to \infty$.

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6.8: Chapter 9 Review Exercises

***True or False?* Justify your answer with a proof or a counterexample.**

1) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Answer

false

2) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

3) If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Answer

true

4) If $\sum_{n=1}^{\infty} 2^n a_n$ converges, then $\sum_{n=1}^{\infty} (-2)^n a_n$ converges.

Is the sequence bounded, monotone, and convergent or divergent? If it is convergent, find the limit.

5) $a_n = \dfrac{3+n^2}{1-n}$

Answer

unbounded, not monotone, divergent

6) $a_n = \ln\left(\frac{1}{n}\right)$

7) $a_n = \dfrac{\ln(n+1)}{\sqrt{n+1}}$

Answer

bounded, monotone, convergent, 0

8) $a_n = \dfrac{2^{n+1}}{5^n}$

9) $a_n = \dfrac{\ln(\cos n)}{n}$

Answer

unbounded, not monotone, divergent

Is the series convergent or divergent?

10) $\sum_{n=1}^{\infty} \dfrac{1}{n^2+5n+4}$

11) $\sum_{n=1}^{\infty} \ln\left(\dfrac{n+1}{n}\right)$

Answer

diverges

12) $\sum_{n=1}^{\infty} \frac{2^n}{n^4}$

13) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$

Answer

converges

14) $\sum_{n=1}^{\infty} n^{-(n+1/n)}$

Is the series convergent or divergent? If convergent, is it absolutely convergent?

15) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Answer

converges, but not absolutely

16) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{3^n}$

17) $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n}$

Answer

converges absolutely

18) $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$

19) $\sum_{n=1}^{\infty} \cos(\pi n) e^{-n}$

Answer

converges absolutely

Evaluate.

20) $\sum_{n=1}^{\infty} \frac{2^{n+4}}{7^n}$

21) $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

Answer

$\frac{1}{2}$

22) A legend from India tells that a mathematician invented chess for a king. The king enjoyed the game so much he allowed the mathematician to demand any payment. The mathematician asked for one grain of rice for the first square on the chessboard, two grains of rice for the second square on the chessboard, and so on. Find an exact expression for the total payment (in grains of rice) requested by the mathematician. Assuming there are $30,000$ grains of rice in 1 pound, and 2000 pounds in 1 ton, how many tons of rice did the mathematician attempt to receive?

The following problems consider a simple population model of the housefly, which can be exhibited by the recursive formula $x_{n+1} = bx_n$, where x_n is the population of houseflies at generation n, and b is the average number of offspring per housefly who

survive to the next generation. Assume a starting population x_0.

23) Find $\lim_{n\to\infty} x_n$ if $b > 1,\ b < 1$, and $b = 1$.

Answer

$\infty,\ 0,\ x_0$

24) Find an expression for $S_n = \sum_{i=0}^{n} x_i$ in terms of b and x_0. What does it physically represent?

25) If $b = \frac{3}{4}$ and $x_0 = 100$, find S_{10} and $\lim_{n\to\infty} S_n$

Answer

$S_{10} \approx 383, \quad \lim_{n\to\infty} S_n = 400$

26) For what values of b will the series converge and diverge? What does the series converge to?

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CHAPTER OVERVIEW

7: Power Series

A power series (in one variable) is an infinite series. Any polynomial can be easily expressed as a power series around any center c, although most of the coefficients will be zero since a power series has infinitely many terms by definition. One can view power series as being like "polynomials of infinite degree," although power series are not polynomials. The content in this Textmap's chapter is complemented by Guichard's Calculus Textmap.

7.1: Prelude to Power Series

7.2: Power Series and Functions

7.2E: Exercises for Section 10.1

7.3: Properties of Power Series

7.3E: Exercises for Section 10.2

7.4: Taylor and Maclaurin Series

7.4E: Exercises for Taylor Polynomials and Taylor Series

7.5: Working with Taylor Series

7.5E: Exercises for Section 10.4

7.6: Chapter 10 Review Exercises

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7.1: Prelude to Power Series

When winning a lottery, sometimes an individual has an option of receiving winnings in one lump-sum payment or receiving smaller payments over fixed time intervals. For example, you might have the option of receiving 20 million dollars today or receiving 1.5 million dollars each year for the next 20 years. Which is the better deal? Certainly 1.5 million dollars over 20 years is equivalent to 30 million dollars. However, receiving the 20 million dollars today would allow you to invest the money.



Figure 7.1.1: If you win a lottery, do you get more money by taking a lump-sum payment or by accepting fixed payments over time? (credit: modification of work by Robert Huffstutter, Flickr)

Alternatively, what if you were guaranteed to receive 1 million dollars every year indefinitely (extending to your heirs) or receive 20 million dollars today. Which would be the better deal? To answer these questions, you need to know how to use infinite series to calculate the value of periodic payments over time in terms of today's dollars.

An infinite series of the form

$$\sum_{n=0}^{\infty} c_n x^n$$

is known as a **power series**. Since the terms contain the variable x, power series can be used to define functions. They can be used to represent given functions, but they are also important because they allow us to write functions that cannot be expressed any other way than as "infinite polynomials." In addition, power series can be easily differentiated and integrated, thus being useful in solving differential equations and integrating complicated functions. An infinite series can also be truncated, resulting in a finite polynomial that we can use to approximate functional values. Power series have applications in a variety of fields, including physics, chemistry, biology, and economics. As we will see in this chapter, representing functions using power series allows us to solve mathematical problems that cannot be solved with other techniques.

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7.2: Power Series and Functions

Learning Objectives

- Identify a power series and provide examples of them.
- Determine the radius of convergence and interval of convergence of a power series.
- Use a power series to represent a function.

A power series is a type of series with terms involving a variable. More specifically, if the variable is x, then all the terms of the series involve powers of x. As a result, a power series can be thought of as an infinite polynomial. Power series are used to represent common functions and also to define new functions. In this section we define power series and show how to determine when a power series converges and when it diverges. We also show how to represent certain functions using power series.

Form of a Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots,$$

where x is a variable and the coefficients c_n are constants, is known as a **power series.** The series

$$1 + x + x^2 + \ldots = \sum_{n=0}^{\infty} x^n$$

is an example of a power series. Since this series is a geometric series with ratio $r = |x|$, we know that it converges if $|x| < 1$ and diverges if $|x| \geq 1$.

Definition 7.2.1: Power Series

A series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

is a power series centered at $x = 0$. A series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

is a *power series* centered at $x = a$.

To make this definition precise, we stipulate that $x^0 = 1$ and $(x-a)^0 = 1$ even when $x = 0$ and $x = a$, respectively.

The series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

and

$$\sum_{n=0}^{\infty} n! x^n = 1 + x + 2! x^2 + 3! x^3 + \ldots$$

are both power series centered at $x = 0$. The series

$$\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n} = 1 + \frac{x-2}{2 \cdot 3} + \frac{(x-2)^2}{3 \cdot 3^2} + \frac{(x-2)^3}{4 \cdot 3^3} + \ldots$$

is a power series centered at $x = 2$.

Convergence of a Power Series

Since the terms in a power series involve a variable x, the series may converge for certain values of x and diverge for other values of x. For a power series centered at $x = a$, the value of the series at $x = a$ is given by c_0. Therefore, a power series always converges at its center. Some power series converge only at that value of x. Most power series, however, converge for more than one value of x. In that case, the power series either converges for all real numbers x or converges for all x in a finite interval. For example, the geometric series $\sum_{n=0}^{\infty} x^n$ converges for all x in the interval $(-1, 1)$, but diverges for all x outside that interval. We now summarize these three possibilities for a general power series.

Note 7.2.1: Convergence of a Power Series

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The series satisfies exactly one of the following properties:

i. The series converges at $x = a$ and diverges for all $x \neq a$.
ii. The series converges for all real numbers x.
iii. There exists a real number $R > 0$ such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$. At the values x where $|x-a| = R$, the series may converge or diverge.

Proof

Suppose that the power series is centered at $a = 0$. (For a series centered at a value of a other than zero, the result follows by letting $y = x - a$ and considering the series

$$\sum_{n=1}^{\infty} c_n y^n.$$

We must first prove the following fact:

If there exists a real number $d \neq 0$ such that $\sum_{n=0}^{\infty} c_n d^n$ converges, then the series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for all x such that $|x| < |d|$.

Since $\sum_{n=0}^{\infty} c_n d^n$ converges, the nth term $c_n d^n \to 0$ as $n \to \infty$. Therefore, there exists an integer N such that $|c_n d^n| \leq 1$ for all $n \geq N$. Writing

$$|c_n x^n| = |c_n d^n| \left|\frac{x}{d}\right|^n,$$

we conclude that, for all n≥N,

$$|c_n x^n| \leq \left|\frac{x}{d}\right|^n.$$

The series

$$\sum_{n=N}^{\infty} \left|\frac{x}{d}\right|^n$$

is a geometric series that converges if $|\frac{x}{d}| < 1$. Therefore, by the comparison test, we conclude that $\sum_{n=N}^{\infty} c_n x^n$ also converges for $|x| < |d|$. Since we can add a finite number of terms to a convergent series, we conclude that $\sum_{n=0}^{\infty} c_n x^n$ converges for $|x| < |d|$.

With this result, we can now prove the theorem. Consider the series

$$\sum_{n=0}^{\infty} a_n x^n$$

and let S be the set of real numbers for which the series converges. Suppose that the set $S = 0$. Then the series falls under case i.

Suppose that the set S is the set of all real numbers. Then the series falls under case ii. Suppose that $S \neq 0$ and S is not the set of real numbers. Then there exists a real number $x* \neq 0$ such that the series does not converge. Thus, the series cannot converge for any x such that $|x| > |x*|$. Therefore, the set S must be a bounded set, which means that it must have a smallest upper bound. (This fact follows from the **Least Upper Bound Property** for the real numbers, which is beyond the scope of this text and is covered in real analysis courses.) Call that smallest upper bound R. Since $S \neq 0$, the number $R > 0$. Therefore, the series converges for all x such that $|x| < R$, and the series falls into case iii.

□

If a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ falls into case iii. of Note 7.2.1, then the series converges for all x such that $|x-a| < R$ for some $R > 0$, and diverges for all x such that $|x-a| > R$. The series may converge or diverge at the values x where $|x-a| = R$. The set of values x for which the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges is known as the interval of convergence. Since the series diverges for all values x where $|x-a| > R$, the length of the interval is $2R$, and therefore, the radius of the interval is R. The value R is called the radius of convergence. For example, since the series $\sum_{n=0}^{\infty} x^n$ converges for all values x in the interval $(-1, 1)$ and diverges for all values x such that $|x| \geq 1$, the interval of convergence of this series is $(-1, 1)$. Since the length of the interval is 2, the radius of convergence is 1.

Definition: Radius of Convergence

Consider the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$. The set of real numbers x where the series converges is the interval of convergence. If there exists a real number $R > 0$ such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence. If the series converges only at $x = a$, we say the radius of convergence is $R = 0$. If the series converges for all real numbers x, we say the radius of convergence is $R = \infty$ (Figure 7.2.1).

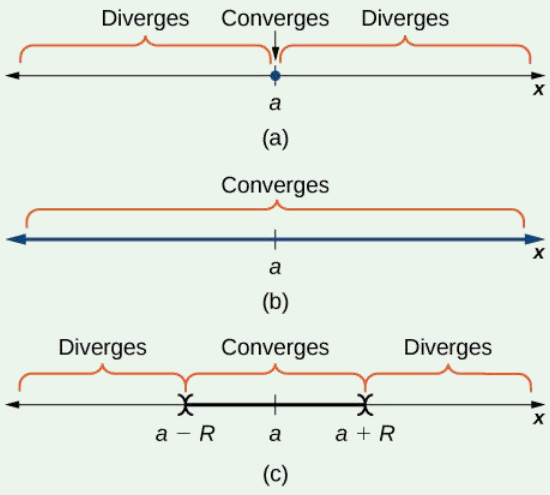


Figure 7.2.1: For a series $\sum_{n=0}^{\infty} c_n(x-a)^n$ graph (a) shows a radius of convergence at $R = 0$, graph (b) shows a radius of convergence at $R = \infty$, and graph (c) shows a radius of convergence at R. For graph (c) we note that the series may or may not converge at the endpoints $x = a + R$ and $x = a - R$.

To determine the interval of convergence for a power series, we typically apply the ratio test. In Example 7.2.1, we show the three different possibilities illustrated in Figure 7.2.1.

Example 7.2.1: Finding the Interval and Radius of Convergence

For each of the following series, find the interval and radius of convergence.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

b. $\sum_{n=0}^{\infty} n! x^n$

c. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{(n+1)3^n}$

Solution

a. To check for convergence, apply the ratio test. We have

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)\cdot n!} \cdot \frac{n!}{x^n} \right| \\
&= \lim_{n\to\infty} \left| \frac{x}{n+1} \right| \\
&= |x| \lim_{n\to\infty} \frac{1}{n+1} \\
&= 0 < 1
\end{aligned}$$

for all values of x. Therefore, the series converges for all real numbers x. The interval of convergence is $(-\infty, \infty)$ and the radius of convergence is $R = \infty$.

b. Apply the ratio test. For $x \neq 0$, we see that

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \left| \frac{(n+1)! x^{n+1}}{n! x^n} \right| \\
&= \lim_{n\to\infty} |(n+1)x| \\
&= |x| \lim_{n\to\infty} (n+1) \\
&= \infty.
\end{aligned}$$

Therefore, the series diverges for all $x \neq 0$. Since the series is centered at $x = 0$, it must converge there, so the series converges only for $x \neq 0$. The interval of convergence is the single value $x = 0$ and the radius of convergence is $R = 0$.

c. In order to apply the ratio test, consider

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+2)3^{n+1}}}{\frac{(x-2)^n}{(n+1)3^n}} \right|$$
$$= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{(x-2)^n} \right|$$
$$= \lim_{n\to\infty} \left| \frac{(x-2)(n+1)}{3(n+2)} \right|$$
$$= \frac{|x-2|}{3}.$$

The ratio $\rho < 1$ if $|x-2| < 3$. Since $|x-2| < 3$ implies that $-3 < x-2 < 3$, the series converges absolutely if $-1 < x < 5$. The ratio $\rho > 1$ if $|x-2| > 3$. Therefore, the series diverges if $x < -1$ or $x > 5$. The ratio test is inconclusive if $\rho = 1$. The ratio $\rho = 1$ if and only if $x = -1$ or $x = 5$. We need to test these values of x separately. For $x = -1$, the series is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

Since this is the alternating harmonic series, it converges. Thus, the series converges at $x = -1$. For $x = 5$, the series is given by

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.$$

This is the harmonic series, which is divergent. Therefore, the power series diverges at $x = 5$. We conclude that the interval of convergence is $[-1, 5)$ and the radius of convergence is $R = 3$.

? Exercise 7.2.1

Find the interval and radius of convergence for the series

$$\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}.$$

Hint

Apply the ratio test to check for absolute convergence.

Answer

The interval of convergence is $[-1, 1)$. The radius of convergence is $R = 1$.

Representing Functions as Power Series

Being able to represent a function by an "infinite polynomial" is a powerful tool. Polynomial functions are the easiest functions to analyze, since they only involve the basic arithmetic operations of addition, subtraction, multiplication, and division. If we can represent a complicated function by an infinite polynomial, we can use the polynomial representation to differentiate or integrate it. In addition, we can use a truncated version of the polynomial expression to approximate values of the function. So, the question is, when can we represent a function by a power series?

Consider again the geometric series

$$1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n.$$

Recall that the geometric series

$$a + ar + ar^2 + ar^3 + \ldots$$

converges if and only if $|r| < 1$. In that case, it converges to $\dfrac{a}{1-r}$. Therefore, if $|x| < 1$, the series in Example 7.2.1 converges to $\dfrac{1}{1-x}$ and we write

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x} \text{ for } |x| < 1.$$

As a result, we are able to represent the function $f(x) = \dfrac{1}{1-x}$ by the power series

$$1 + x + x^2 + x^3 + \ldots \text{ when } |x| < 1.$$

We now show graphically how this series provides a representation for the function $f(x) = \dfrac{1}{1-x}$ by comparing the graph of f with the graphs of several of the partial sums of this infinite series.

Example 7.2.2: Graphing a Function and Partial Sums of its Power Series

Sketch a graph of $f(x) = \dfrac{1}{1-x}$ and the graphs of the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^n$ for $N = 2, 4, 6$ on the interval $(-1, 1)$. Comment on the approximation S_N as N increases.

Solution

From the graph in Figure 7.2.2, you see that as N increases, S_N becomes a better approximation for $f(x) = \dfrac{1}{1-x}$ for x in the interval $(-1, 1)$.

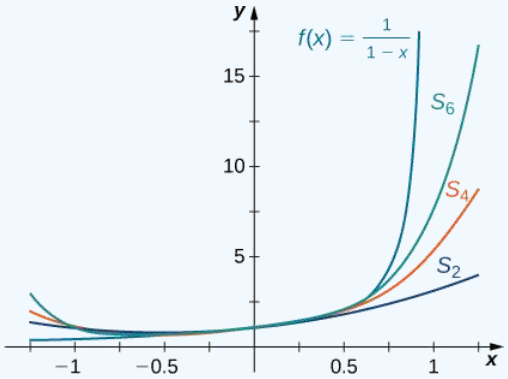


Figure 7.2.2: The graph shows a function and three approximations of it by partial sums of a power series.

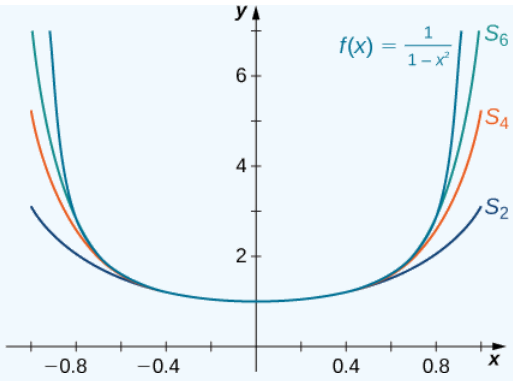
Exercise 7.2.2

Sketch a graph of $f(x) = \dfrac{1}{1-x^2}$ and the corresponding partial sums $S_N(x) = \sum_{n=0}^{N} x^{2n}$ for $N = 2, 4, 6$ on the interval $(-1, 1)$.

Hint

$$S_N(x) = 1 + x^2 + \ldots + x^{2N} = \frac{1 - x^{2(N+1)}}{1 - x^2}$$

Answer



Next we consider functions involving an expression similar to the sum of a geometric series and show how to represent these functions using power series.

Example 7.2.3: Representing a Function with a Power Series

Use a power series to represent each of the following functions f. Find the interval of convergence.

a. $f(x) = \dfrac{1}{1+x^3}$

b. $f(x) = \dfrac{x^2}{4-x^2}$

Solution

a. You should recognize this function f as the sum of a geometric series, because

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)}.$$

Using the fact that, for $|r|<1$, $\dfrac{a}{1-r}$ is the sum of the geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots,$$

we see that, for $|-x^3|<1$,

$$\begin{aligned}\frac{1}{1+x^3} &= \frac{1}{1-(-x^3)} \\ &= \sum_{n=0}^{\infty}(-x^3)^n \\ &= 1 - x^3 + x^6 - x^9 + \ldots.\end{aligned}$$

Since this series converges if and only if $|-x^3|<1$, the interval of convergence is $(-1,1)$, and we have

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \ldots \text{ for } |x|<1.$$

b. This function is not in the exact form of a sum of a geometric series. However, with a little algebraic manipulation, we can relate f to a geometric series. By factoring 4 out of the two terms in the denominator, we obtain

$$\frac{x^2}{4-x^2} = \frac{x^2}{4\left(\frac{1-x^2}{4}\right)}$$
$$= \frac{x^2}{4\left(1-\left(\frac{x}{2}\right)^2\right)}.$$

Therefore, we have

$$\frac{x^2}{4-x^2} = \frac{x^2}{4\left(1-\left(\frac{x}{2}\right)^2\right)}$$
$$= \frac{\frac{x^2}{4}}{1-\left(\frac{x}{2}\right)^2}$$
$$= \sum_{n=0}^{\infty} \frac{x^2}{4}\left(\frac{x}{2}\right)^{2n}.$$

The series converges as long as $\left|\left(\frac{x}{2}\right)^2\right| < 1$ (note that when $\left|\left(\frac{x}{2}\right)^2\right| = 1$ the series does not converge). Solving this inequality, we conclude that the interval of convergence is $(-2, 2)$ and

$$\frac{x^2}{4-x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{4^{n+1}}$$
$$= \frac{x^2}{4} + \frac{x^4}{4^2} + \frac{x^6}{4^3} + \ldots$$

for $|x| < 2$.

? Exercise 7.2.3

Represent the function $f(x) = \frac{x^3}{2-x}$ using a power series and find the interval of convergence.

Hint

Rewrite f in the form $f(x) = \frac{g(x)}{1-h(x)}$ for some functions g and h.

Answer

$\sum_{n=0}^{\infty} \frac{x^{n+3}}{2^{n+1}}$ with interval of convergence $(-2, 2)$

In the remaining sections of this chapter, we will show ways of deriving power series representations for many other functions, and how we can make use of these representations to evaluate, differentiate, and integrate various functions.

Key Concepts

- For a power series centered at $x = a$, one of the following three properties hold:
  - i. The power series converges only at $x = a$. In this case, we say that the radius of convergence is $R = 0$.
  - ii. The power series converges for all real numbers x. In this case, we say that the radius of convergence is $R = \infty$.
  - iii. There is a real number R such that the series converges for $|x-a| < R$ and diverges for $|x-a| > R$. In this case, the radius of convergence is R.

- If a power series converges on a finite interval, the series may or may not converge at the endpoints.
- The ratio test may often be used to determine the radius of convergence.
- The geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$ allows us to represent certain functions using geometric series.

Key Equations

- **Power series centered at $x = 0$**

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots n$$

- **Power series centered at $x = a$**

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots$$

Glossary

interval of convergence

the set of real numbers x for which a power series converges

power series

a series of the form $\sum_{n=0}^{\infty} c_n x^n$ is a power series centered at $x = 0$; a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series centered at $x = a$

radius of convergence

if there exists a real number $R > 0$ such that a power series centered at $x = a$ converges for $|x-a| < R$ and diverges for $|x-a| > R$, then R is the radius of convergence; if the power series only converges at $x = a$, the radius of convergence is $R = 0$; if the power series converges for all real numbers x, the radius of convergence is $R = \infty$

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7.2E: Exercises for Section 10.1

In exercises 1 - 4, state whether each statement is true, or give an example to show that it is false.

1) If $\sum_{n=1}^{\infty} a_n x^n$ converges, then $a_n x^n \to 0$ as $n \to \infty$.

Answer

True. If a series converges then its terms tend to zero.

2) $\sum_{n=1}^{\infty} a_n x^n$ converges at $x = 0$ for any real numbers a_n.

3) Given any sequence a_n, there is always some $R > 0$, possibly very small, such that $\sum_{n=1}^{\infty} a_n x^n$ converges on $(-R, R)$.

Answer

False. It would imply that $a_n x^n \to 0$ for $|x| < R$. If $a_n = n^n$, then $a_n x^n = (nx)^n$ does not tend to zero for any $x \neq 0$.

4) If $\sum_{n=1}^{\infty} a_n x^n$ has radius of convergence $R > 0$ and if $|b_n| \leq |a_n|$ for all n, then the radius of convergence of $\sum_{n=1}^{\infty} b_n x^n$ is greater than or equal to R.

5) Suppose that $\sum_{n=0}^{\infty} a_n (x-3)^n$ converges at $x = 6$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 1$

b. $x = 2$

c. $x = 3$

d. $x = 0$

e. $x = 5.99$

f. $x = 0.000001$

Answer

It must converge on $(0, 6]$ and hence at: a. $x = 1$; b. $x = 2$; c. $x = 3$; d. $x = 0$; e. $x = 5.99$; and f. $x = 0.000001$

6) Suppose that $\sum_{n=0}^{\infty} a_n (x+1)^n$ converges at $x = -2$. At which of the following points must the series also converge? Use the fact that if $\sum a_n (x-c)^n$ converges at x, then it converges at any point closer to c than x.

a. $x = 2$

b. $x = -1$

c. $x = -3$

d. $x = 0$

e. $x = 0.99$

f. $x = 0.000001$

In the following exercises, suppose that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$. Find the radius of convergence for each series.

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7) $\sum_{n=0}^{\infty} a_n 2^n x^n$

Answer

$$\left|\frac{a_{n+1}2^{n+1}x^{n+1}}{a_n 2^n x^n}\right| = 2|x|\left|\frac{a_{n+1}}{a_n}\right| \to 2|x| \text{ so } R = \tfrac{1}{2}$$

8) $\sum_{n=0}^{\infty} \frac{a_n x^n}{2^n}$

9) $\sum_{n=0}^{\infty} \frac{a_n \pi^n x^n}{e^n}$

Answer

$$\left|\frac{a_{n+1}(\frac{\pi}{e})^{n+1}x^{n+1}}{a_n(\frac{\pi}{e})^n x^n}\right| = \frac{\pi|x|}{e}\left|\frac{a_{n+1}}{a_n}\right| \to \frac{\pi|x|}{e} \text{ so } R = \tfrac{e}{\pi}$$

10) $\sum_{n=0}^{\infty} \frac{a_n(-1)^n x^n}{10^n}$

11) $\sum_{n=0}^{\infty} a_n(-1)^n x^{2n}$

Answer

$$\left|\frac{a_{n+1}(-1)^{n+1}x^{2n+2}}{a_n(-1)^n x^{2n}}\right| = |x^2|\left|\frac{a_{n+1}}{a_n}\right| \to |x^2| \text{ so } R = 1$$

12) $\sum_{n=0}^{\infty} a_n(-4)^n x^{2n}$

In exercises 13 - 22, find the radius of convergence R and interval of convergence for $\sum a_n x^n$ with the given coefficients a_n.

13) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$

Answer

$a_n = \frac{2^n}{n}$ so $\frac{a_{n+1}x}{a_n} \to 2x$. so $R = \frac{1}{2}$. When $x = \frac{1}{2}$ the series is harmonic and diverges. When $x = -\frac{1}{2}$ the series is alternating harmonic and converges. The interval of convergence is $I = \left[-\frac{1}{2}, \frac{1}{2}\right)$.

14) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}$

15) $\sum_{n=1}^{\infty} \frac{nx^n}{2^n}$

Answer

$a_n = \frac{n}{2^n}$ so $\frac{a_{n+1}x}{a_n} \to \frac{x}{2}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

16) $\sum_{n=1}^{\infty} \frac{nx^n}{e^n}$

17) $\sum_{n=1}^{\infty} \frac{n^2 x^n}{2^n}$

Answer

$a_n = \frac{n^2}{2^n}$ so $R = 2$. When $x = \pm 2$ the series diverges by the divergence test. The interval of convergence is $I = (-2, 2)$.

18) $\sum_{k=1}^{\infty} \frac{k^e x^k}{e^k}$

19) $\sum_{k=1}^{\infty} \frac{\pi^k x^k}{k^\pi}$

Answer

$a_k = \frac{\pi^k}{k^\pi}$ so $R = \frac{1}{\pi}$. When $x = \pm \frac{1}{\pi}$ the series is an absolutely convergent p-series. The interval of convergence is $I = \left[-\frac{1}{\pi}, \frac{1}{\pi}\right]$.

20) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

21) $\sum_{n=1}^{\infty} \frac{10^n x^n}{n!}$

Answer

$a_n = \frac{10^n}{n!}, \frac{a_{n+1} x}{a_n} = \frac{10x}{n+1} \to 0 < 1$ so the series converges for all x by the ratio test and $I = (-\infty, \infty)$.

22) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\ln(2n)}$

In exercises 23 - 28, find the radius of convergence of each series.

23) $\sum_{k=1}^{\infty} \frac{(k!)^2 x^k}{(2k)!}$

Answer

$a_k = \frac{(k!)^2}{(2k)!}$ so $\frac{a_{k+1}}{a_k} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4}$ so $R = 4$

24) $\sum_{n=1}^{\infty} \frac{(2n)! x^n}{n^{2n}}$

25) $\sum_{k=1}^{\infty} \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)} x^k$

Answer

$a_k = \frac{k!}{1 \cdot 3 \cdot 5 \cdots (2k-1)}$ so $\frac{a_{k+1}}{a_k} = \frac{k+1}{2k+1} \to \frac{1}{2}$ so $R = 2$

26) $\sum_{k=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdots 2k}{(2k)!} x^k$

27) $\sum_{n=1}^{\infty} \frac{x^n}{\binom{2n}{n}}$ where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Answer

$$a_n = \frac{1}{\binom{2n}{n}} \text{ so } \frac{a_{n+1}}{a_n} = \frac{\left((n+1)!\right)^2}{(2n+2)!}\frac{2n!}{(n!)^2} = \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4} \text{ so } R = 4$$

28) $\sum_{n=1}^{\infty} \sin^2 n x^n$

In exercises 29 - 32, use the ratio test to determine the radius of convergence of each series.

29) $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!} x^n$

Answer

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27} \text{ so } R = 27$$

30) $\sum_{n=1}^{\infty} \frac{2^{3n}(n!)^3}{(3n)!} x^n$

31) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

Answer

$$a_n = \frac{n!}{n^n} \text{ so } \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n \to \frac{1}{e} \text{ so } R = e$$

32) $\sum_{n=1}^{\infty} \frac{(2n)!}{n^{2n}} x^n$

In the following exercises, given that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ with convergence in $(-1, 1)$, find the power series for each function with the given center a, and identify its interval of convergence.

33) $f(x) = \frac{1}{x}; a = 1$ (Hint: $\frac{1}{x} = \frac{1}{1-(1-x)}$)

Answer

$$f(x) = \sum_{n=0}^{\infty} (1-x)^n \text{ on } I = (0, 2)$$

34) $f(x) = \frac{1}{1-x^2}; a = 0$

35) $f(x) = \frac{x}{1-x^2}; a = 0$

Answer

$$\sum_{n=0}^{\infty} x^{2n+1} \text{ on } I = (-1, 1)$$

36) $f(x) = \frac{1}{1+x^2}; a = 0$

37) $f(x) = \frac{x^2}{1+x^2}; a = 0$

Answer

$$\sum_{n=0}^{\infty}(-1)^n x^{2n+2} \text{ on } I=(-1,1)$$

38) $f(x)=\dfrac{1}{2-x}; a=1$

39) $f(x)=\dfrac{1}{1-2x}; a=0.$

Answer

$$\sum_{n=0}^{\infty}2^n x^n \text{ on } \left(-\tfrac{1}{2},\tfrac{1}{2}\right)$$

40) $f(x)=\dfrac{1}{1-4x^2}; a=0$

41) $f(x)=\dfrac{x^2}{1-4x^2}; a=0$

Answer

$$\sum_{n=0}^{\infty}4^n x^{2n+2} \text{ on } \left(-\tfrac{1}{2},\tfrac{1}{2}\right)$$

42) $f(x)=\dfrac{x^2}{5-4x+x^2}; a=2$

Use the result of exercise 43 to find the radius of convergence of the given series in the subsequent exercises (44 - 47).

43) Explain why, if $|a_n|^{1/n}\to r>0,$ then $|a_n x^n|^{1/n}\to |x|r<1$ whenever $|x|<\frac{1}{r}$ and, therefore, the radius of convergence of $\sum_{n=1}^{\infty}a_n x^n$ is $R=\frac{1}{r}$.

Answer

$|a_n x^n|^{1/n}=|a_n|^{1/n}|x|\to |x|r$ as $n\to\infty$ and $|x|r<1$ when $|x|<\frac{1}{r}$. Therefore, $\sum_{n=1}^{\infty}a_n x^n$ converges when $|x|<\frac{1}{r}$ by the nth root test.

44) $\sum_{n=1}^{\infty}\dfrac{x^n}{n^n}$

45) $\sum_{k=1}^{\infty}\left(\dfrac{k-1}{2k+3}\right)^k x^k$

Answer

$a_k=\left(\dfrac{k-1}{2k+3}\right)^k$ so $(a_k)^{1/k}\to\frac{1}{2}<1$ so $R=2$

46) $\sum_{k=1}^{\infty}(\dfrac{2k^2-1}{k^2+3})^k x^k$

47) $\sum_{n=1}^{\infty}a_n=(n^{1/n}-1)^n x^n$

Answer

$a_n=(n^{1/n}-1)^n$ so $(a_n)^{1/n}\to 0$ so $R=\infty$

48) Suppose that $p(x)=\sum_{n=0}^{\infty} a_n x^n$ such that $a_n=0$ if n is even. Explain why $p(x)=p(-x)$.

49) Suppose that $p(x)=\sum_{n=0}^{\infty} a_n x^n$ such that $a_n=0$ if n is odd. Explain why $p(x)=-p(-x)$.

Answer

We can rewrite $p(x)=\sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$ and $p(x)=p(-x)$ since $x^{2n+1}=-(-x)^{2n+1}$.

50) Suppose that $p(x)=\sum_{n=0}^{\infty} a_n x^n$ converges on $(-1,1]$. Find the interval of convergence of $p(Ax)$.

51) Suppose that $p(x)=\sum_{n=0}^{\infty} a_n x^n$ converges on $(-1,1]$. Find the interval of convergence of $p(2x-1)$.

Answer

If $x\in[0,1]$, then $y=2x-1\in[-1,1]$ so $p(2x-1)=p(y)=\sum_{n=0}^{\infty} a_n y^n$ converges.

In the following exercises, suppose that $p(x)=\sum_{n=0}^{\infty} a_n x^n$ satisfies $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1$ where $a_n\geq 0$ for each n. State whether each series converges on the full interval $(-1,1)$, or if there is not enough information to draw a conclusion. Use the comparison test when appropriate.

52) $\sum_{n=0}^{\infty} a_n x^{2n}$

53) $\sum_{n=0}^{\infty} a_{2n} x^{2n}$

Answer

Converges on $(-1,1)$ by the ratio test

54) $\sum_{n=0}^{\infty} a_{2n} x^n$ (Hint:$x=\pm\sqrt{x^2}$)

55) $\sum_{n=0}^{\infty} a_{n^2} x^{n^2}$ (Hint: Let $b_k=a_k$ if $k=n^2$ for some n, otherwise $b_k=0$.)

Answer

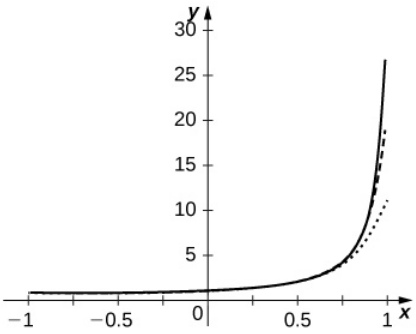
Consider the series $\sum b_k x^k$ where $b_k=a_k$ if $k=n^2$ and $b_k=0$ otherwise. Then $b_k\leq a_k$ and so the series converges on $(-1,1)$ by the comparison test.

56) Suppose that $p(x)$ is a polynomial of degree N. Find the radius and interval of convergence of $\sum_{n=1}^{\infty} p(n)x^n$.

57) [T] Plot the graphs of $\frac{1}{1-x}$ and of the partial sums $S_N=\sum_{n=0}^{N} x^n$ for $n=10,20,30$ on the interval $[-0.99,0.99]$ Comment on the approximation of $\frac{1}{1-x}$ by S_N near $x=-1$ and near $x=1$ as N increases.

Answer

The approximation is more accurate near $x = -1$. The partial sums follow $\dfrac{1}{1-x}$ more closely as N increases but are never accurate near $x = 1$ since the series diverges there.

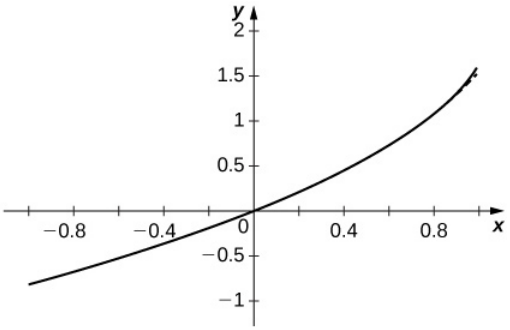


58) [T] Plot the graphs of $-\ln(1-x)$ and of the partial sums $S_N = \displaystyle\sum_{n=1}^{N} \frac{x^n}{n}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$ Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

59) [T] Plot the graphs of the partial sums $S_n = \displaystyle\sum_{n=1}^{N} \frac{x^n}{n^2}$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$ Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

Answer

The approximation appears to stabilize quickly near both $x = \pm 1$.

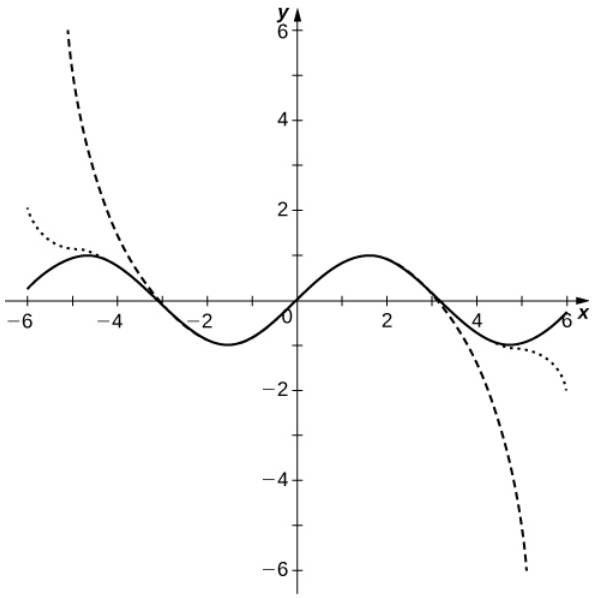


60) [T] Plot the graphs of the partial sums $S_N = \displaystyle\sum_{n=1}^{N} (\sin n) x^n$ for $n = 10, 50, 100$ on the interval $[-0.99, 0.99]$ Comment on the behavior of the sums near $x = -1$ and near $x = 1$ as N increases.

61) [T] Plot the graphs of the partial sums $S_N = \displaystyle\sum_{n=0}^{N} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$ Comment on how these plots approximate $\sin x$ as N increases.

Answer

The polynomial curves have roots close to those of $\sin x$ up to their degree and then the polynomials diverge from $\sin x$.



62) [T] Plot the graphs of the partial sums $S_N = \sum_{n=0}^{N} (-1)^n \frac{x^{2n}}{(2n)!}$ for $n = 3, 5, 10$ on the interval $[-2\pi, 2\pi]$. Comment on how these plots approximate $\cos x$ as N increases.

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7.3: Properties of Power Series

Learning Objectives

- Combine power series by addition or subtraction.
- Create a new power series by multiplication by a power of the variable or a constant, or by substitution.
- Multiply two power series together.
- Differentiate and integrate power series term-by-term.

In the preceding section on power series and functions we showed how to represent certain functions using power series. In this section we discuss how power series can be combined, differentiated, or integrated to create new power series. This capability is particularly useful for a couple of reasons. First, it allows us to find power series representations for certain elementary functions, by writing those functions in terms of functions with known power series. For example, given the power series representation for $f(x) = \dfrac{1}{1-x}$, we can find a power series representation for $f'(x) = \dfrac{1}{(1-x)^2}$. Second, being able to create power series allows us to define new functions that cannot be written in terms of elementary functions. This capability is particularly useful for solving differential equations for which there is no solution in terms of elementary functions.

Combining Power Series

If we have two power series with the same interval of convergence, we can add or subtract the two series to create a new power series, also with the same interval of convergence. Similarly, we can multiply a power series by a power of x or evaluate a power series at x^m for a positive integer m to create a new power series. Being able to do this allows us to find power series representations for certain functions by using power series representations of other functions. For example, since we know the power series representation for $f(x) = \frac{1}{1-x}$, we can find power series representations for related functions, such as

$$y = \frac{3x}{1-x^2}$$

and

$$y = \frac{1}{(x-1)(x-3)}.$$

In Theorem 7.3.1, we state results regarding addition or subtraction of power series, composition of a power series, and multiplication of a power series by a power of the variable. For simplicity, we state the theorem for power series centered at $x = 0$. Similar results hold for power series centered at $x = a$.

Theorem 7.3.1: Combining Power Series

Suppose that the two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on a common interval I.

i. The power series $\sum_{n=0}^{\infty} (c_n x^n \pm d_n x^n)$ converges to $f \pm g$ on I.

ii. For any integer $m \geq 0$ and any real number b, the power series $\sum_{n=0}^{\infty} b x^m x^n$ converges to $b x^m f(x)$ on I.

iii. For any integer $m \geq 0$ and any real number b, the series $\sum_{n=0}^{\infty} c_n (b x^m)^n$ converges to $f(b x^m)$ for all x such that $b x^m$ is in I.

Proof

We prove $i.$ in the case of the series $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to the functions f and g, respectively, on the interval I. Let x be a point in I and let $S_N(x)$ and $T_N(x)$ denote the Nth partial sums of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$, respectively. Then the sequence $S_N(x)$ converges to $f(x)$ and the sequence $T_N(x)$ converges to $g(x)$. Furthermore, the Nth partial sum of $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$ is

$$\begin{aligned}\sum_{n=0}^{N}(c_n x^n + d_n x^n) &= \sum_{n=0}^{N} c_n x^n + \sum_{n=0}^{N} d_n x^n \\ &= S_N(x) + T_N(x).\end{aligned}$$

Because

$$\begin{aligned}\lim_{N\to\infty}(S_N(x) + T_N(x)) &= \lim_{N\to\infty} S_N(x) + \lim_{N\to\infty} T_N(x) \\ &= f(x) + g(x),\end{aligned}$$

we conclude that the series $\sum_{n=0}^{\infty}(c_n x^n + d_n x^n)$ converges to $f(x) + g(x)$.

□

We examine products of power series in a later theorem. First, we show several applications of Theorem 7.3.1 and how to find the interval of convergence of a power series given the interval of convergence of a related power series.

Example 7.3.1: Combining Power Series

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ is a power series whose interval of convergence is $(-1, 1)$, and suppose that $\sum_{n=0}^{\infty} b_n x^n$ is a power series whose interval of convergence is $(-2, 2)$.

a. Find the interval of convergence of the series $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$.

b. Find the interval of convergence of the series $\sum_{n=0}^{\infty} a_n 3^n x^n$.

Solution

a. Since the interval $(-1, 1)$ is a common interval of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$, the interval of convergence of the series $\sum_{n=0}^{\infty}(a_n x^n + b_n x^n)$ is $(-1, 1)$.

b. Since $\sum_{n=0}^{\infty} a_n x^n$ is a power series centered at zero with radius of convergence 1, it converges for all x in the interval $(-1, 1)$. By Theorem 7.3.1, the series

$$\sum_{n=0}^{\infty} a_n 3^n x^n = \sum_{n=0}^{\infty} a_n (3x)^n$$

converges if $3x$ is in the interval $(-1, 1)$. Therefore, the series converges for all x in the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

Exercise 7.3.1

Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has an interval of convergence of $(-1,1)$. Find the interval of convergence of $\sum_{n=0}^{\infty} a_n (\frac{x}{2})^n$.

Hint

Find the values of x such that $\frac{x}{2}$ is in the interval $(-1,1)$.

Answer

Interval of convergence is $(-2,2)$.

In the next example, we show how to use Theorem 7.3.1 and the power series for a function f to construct power series for functions related to f. Specifically, we consider functions related to the function $f(x)=\frac{1}{1-x}$ and we use the fact that

$$\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\ldots$$

for $|x|<1$.

Example 7.3.2: Constructing Power Series from Known Power Series

Use the power series representation for $f(x)=\frac{1}{1-x}$ combined with Theorem 7.3.1 to construct a power series for each of the following functions. Find the interval of convergence of the power series.

a. $f(x)=\frac{3x}{1+x^2}$

b. $f(x)=\frac{1}{(x-1)(x-3)}$

Solution

a. First write $f(x)$ as

$$f(x)=3x\left(\frac{1}{1-(-x^2)}\right).$$

Using the power series representation for $f(x)=\frac{1}{1-x}$ and parts ii. and iii. of Theorem 7.3.1, we find that a power series representation for f is given by

$$\sum_{n=0}^{\infty} 3x(-x^2)^n = \sum_{n=0}^{\infty} 3(-1)^n x^{2n+1}.$$

Since the interval of convergence of the series for $\frac{1}{1-x}$ is $(-1,1)$, the interval of convergence for this new series is the set of real numbers x such that $|x^2|<1$. Therefore, the interval of convergence is $(-1,1)$.

b. To find the power series representation, use partial fractions to write $f(x)=\frac{1}{(x-1)(x-3)}$ as the sum of two fractions. We have

$$\frac{1}{(x-1)(x-3)}=\frac{-1/2}{x-1}+\frac{1/2}{x-3}=\frac{1/2}{1-x}-\frac{1/2}{3-x}=\frac{1/2}{1-x}-\frac{1/6}{1-\frac{x}{3}}.$$

First, using part ii. of Theorem 7.3.1, we obtain

$$\frac{1/2}{1-x} = \sum_{n=0}^{\infty} \frac{1}{2} x^n \quad \text{for } |x| < 1.$$

Then, using parts ii. and iii. of Theorem 7.3.1, we have

$$\frac{1/6}{1-x/3} = \sum_{n=0}^{\infty} \frac{1}{6} \left(\frac{x}{3}\right)^n \quad \text{for } |x| < 3.$$

Since we are combining these two power series, the interval of convergence of the difference must be the smaller of these two intervals. Using this fact and part i. of Theorem 7.3.1, we have

$$\frac{1}{(x-1)(x-3)} = \sum_{n=0}^{\infty} \left(\frac{1}{2} - \frac{1}{6 \cdot 3^n}\right) x^n$$

where the interval of convergence is $(-1, 1)$.

Exercise 7.3.2

Use the series for $f(x) = \dfrac{1}{1-x}$ on $|x| < 1$ to construct a series for $\dfrac{1}{(1-x)(x-2)}$. Determine the interval of convergence.

Hint

Use partial fractions to rewrite $\dfrac{1}{(1-x)(x-2)}$ as the difference of two fractions.

Answer

$\displaystyle\sum_{n=0}^{\infty} \left(-1 + \frac{1}{2^{n+1}}\right) x^n$. The interval of convergence is $(-1, 1)$.

In Example 7.3.2, we showed how to find power series for certain functions. In Example 7.3.3 we show how to do the opposite: given a power series, determine which function it represents.

Example 7.3.3: Finding the Function Represented by a Given Power Series

Consider the power series $\displaystyle\sum_{n=0}^{\infty} 2^n x^n$. Find the function f represented by this series. Determine the interval of convergence of the series.

Solution

Writing the given series as

$$\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n,$$

we can recognize this series as the power series for

$$f(x) = \frac{1}{1-2x}.$$

Since this is a geometric series, the series converges if and only if $|2x| < 1$. Therefore, the interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Exercise 7.3.3

Find the function represented by the power series $\sum_{n=0}^{\infty} \frac{1}{3^n} x^n$.

Determine its interval of convergence.

Hint

Write $\frac{1}{3^n} x^n = \left(\frac{x}{3}\right)^n$.

Answer

$f(x) = \frac{3}{3-x}$. The interval of convergence is $(-3, 3)$.

Recall the questions posed in the chapter opener about which is the better way of receiving payouts from lottery winnings. We now revisit those questions and show how to use series to compare values of payments over time with a lump sum payment today. We will compute how much future payments are worth in terms of today's dollars, assuming we have the ability to invest winnings and earn interest. The value of future payments in terms of today's dollars is known as the *present value* of those payments.

Example 7.3.4: Present Value of Future Winnings

Suppose you win the lottery and are given the following three options:

- Receive 20 million dollars today;
- Receive 1.5 million dollars per year over the next 20 years; or
- Receive 1 million dollars per year indefinitely (being passed on to your heirs).

Which is the best deal, assuming that the annual interest rate is 5%? We answer this by working through the following sequence of questions.

a. How much is the 1.5 million dollars received annually over the course of 20 years worth in terms of today's dollars, assuming an annual interest rate of 5%?
b. Use the answer to part a. to find a general formula for the **present value** of payments of C dollars received each year over the next n years, assuming an average annual interest rate r.
c. Find a formula for the present value if annual payments of C dollars continue indefinitely, assuming an average annual interest rate r.
d. Use the answer to part c. to determine the present value of 1 million dollars paid annually indefinitely.
e. Use your answers to parts a. and d. to determine which of the three options is best.



Figure 7.3.1: (credit: modification of work by Robert Huffstutter, Flickr)

Solution

a. Consider the payment of 1.5 million dollars made at the end of the first year. If you were able to receive that payment today instead of one year from now, you could invest that money and earn 5% interest. Therefore, the present value of that money P_1 satisfies $P_1(1+0.05) = 1.5$ **million dollars.** We conclude that

$$P_1 = \frac{1.5}{1.05} = 1.429 \text{ million dollars.}$$

Similarly, consider the payment of 1.5 million dollars made at the end of the second year. If you were able to receive that payment today, you could invest that money for two years, earning 5% interest, compounded annually. Therefore, the present value of that money P_2 satisfies $P_2(1+0.05)^2 = 1.5$ **million dollars**. We conclude that

$$P_2 = 1.5(1.05)^2 = 1.361 \text{ million dollars.}$$

The value of the future payments today is the sum of the present values $P_1, P_2, \ldots, P_{20}$ of each of those annual payments. The present value P_k satisfies

$$P_k = \frac{1.5}{(1.05)^k}.$$

Therefore,

$$P = \frac{1.5}{1.05} + \frac{1.5}{(1.05)^2} + \ldots + \frac{1.5}{(1.05)^{20}} = 18.693 \quad \text{million dollars.}$$

b. Using the result from part a. we see that the present value P of C dollars paid annually over the course of n years, assuming an annual interest rate r, is given by

$$P = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^n} \quad \text{dollars.}$$

c. Using the result from part b. we see that the present value of an annuity that continues indefinitely is given by the infinite series

$$P = \sum_{n=0}^{\infty} \frac{C}{(1+r)^{n+1}}.$$

We can view the present value as a power series in r, which converges as long as $\left|\frac{1}{1+r}\right| < 1$. Since $r > 0$, this series converges. Rewriting the series as

$$P = \frac{C}{(1+r)} \sum_{n=0}^{\infty} \left(\frac{1}{1+r}\right)^n,$$

we recognize this series as the power series for

$$f(r) = \frac{1}{1-\left(\frac{1}{1+r}\right)} = \frac{1}{\left(\frac{r}{1+r}\right)} = \frac{1+r}{r}.$$

We conclude that the present value of this annuity is

$$P = \frac{C}{1+r} \cdot \frac{1+r}{r} = \frac{C}{r}.$$

d. From the result to part c. we conclude that the present value P of $C = 1$ **million dollars** paid out every year indefinitely, assuming an annual interest rate $r = 0.05$, is given by

$$P = \frac{1}{0.05} = 20 \text{ million dollars.}$$

e. From part a. we see that receiving \$1.5 million dollars over the course of 20 years is worth \$18.693 million dollars in today's dollars. From part d. we see that receiving \$1 million dollars per year indefinitely is worth \$20 million dollars in today's dollars. Therefore, either receiving a lump-sum payment of \$20 million dollars today or receiving \$1 million dollars indefinitely have the same present value.

Multiplication of Power Series

We can also create new power series by multiplying power series. Being able to multiply two power series provides another way of finding power series representations for functions. The way we multiply them is similar to how we multiply polynomials. For example, suppose we want to multiply

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

and

$$\sum_{n=0}^{\infty} d_n x^n = d_0 + d_1 x + d_2 x^2 + \ldots.$$

It appears that the product should satisfy

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=-0}^{\infty} d_n x^n\right) = (c_0 + c_1 x + c_2 x^2 + \ldots)\cdot(d_0 + d_1 x + d_2 x^2 + \ldots) = c_0 d_0 + (c_1 d_0 + c_0 d_1)x$$
$$+ (c_2 d_0 + c_1 d_1 + c_0 d_2)x^2 + \ldots.$$

In Theorem 7.3.2, we state the main result regarding multiplying power series, showing that if $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge on a common interval I, then we can multiply the series in this way, and the resulting series also converges on the interval I.

Theorem 7.3.2: Multiplying Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ converge to f and g, respectively, on a common interval I. Let

$$e_n = c_0 d_n + c_1 d_{n-1} + c_2 d_{n-2} + \ldots + c_{n-1} d_1 + c_n d_0 = \sum_{k=0}^{n} c_k d_{n-k}.$$

Then

$$\left(\sum_{n=0}^{\infty} c_n x^n\right)\left(\sum_{n=0}^{\infty} d_n x^n\right) = \sum_{n=0}^{\infty} e_n x^n$$

and

$$\sum_{n=0}^{\infty} e_n x^n \text{ converges to } f(x)\cdot g(x) \text{ on } I.$$

The series $\sum_{n=0}^{\infty} e_n x^n$ is known as the *Cauchy product* of the series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$.

We omit the proof of this theorem, as it is beyond the level of this text and is typically covered in a more advanced course. We now provide an example of this theorem by finding the power series representation for

$$f(x) = \frac{1}{(1-x)(1-x^2)}$$

using the power series representations for

$$y = \frac{1}{1-x} \text{ and } y = \frac{1}{1-x^2}$$

.

Example 7.3.5: Multiplying Power Series

Multiply the power series representation

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

for $|x|<1$ with the power series representation

$$\frac{1}{1-x^2}=\sum_{n=0}^{\infty}\left(x^2\right)^n=1+x^2+x^4+x^6+\ldots$$

for $|x|<1$ to construct a power series for $f(x)=\dfrac{1}{(1-x)(1-x^2)}$ on the interval $(-1,1)$.

Solution

We need to multiply

$$(1+x+x^2+x^3+\ldots)(1+x^2+x^4+x^6+\ldots).$$

Writing out the first several terms, we see that the product is given by

$$\begin{aligned}&(1+x^2+x^4+x^6+\ldots)+(x+x^3+x^5+x^7+\ldots)+(x^2+x^4+x^6+x^8+\ldots)+(x^3+x^5+x^7+x^9+\ldots)\\&=1+x+(1+1)x^2+(1+1)x^3+(1+1+1)x^4+(1+1+1)x^5+\ldots=1+x+2x^2+2x^3+3x^4+3x^5+\ldots.\end{aligned}$$

Since the series for $y=\dfrac{1}{1-x}$ and $y=\dfrac{1}{1-x^2}$ both converge on the interval $(-1,1)$, the series for the product also converges on the interval $(-1,1)$.

? Exercise 7.3.4

Multiply the series $\dfrac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ by itself to construct a series for $\dfrac{1}{(1-x)(1-x)}$.

Hint

Multiply the first few terms of $(1+x+x^2+x^3+\ldots)(1+x+x^2+x^3+\ldots)$

Answer

$1+2x+3x^2+4x^3+\ldots$

Differentiating and Integrating Power Series

Consider a power series $\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+\ldots$ that converges on some interval I, and let f be the function defined by this series. Here we address two questions about f.

- Is f differentiable, and if so, how do we determine the derivative $f\,'$?
- How do we evaluate the indefinite integral $\int f(x)\,dx$?

We know that, for a polynomial with a finite number of terms, we can evaluate the derivative by differentiating each term separately. Similarly, we can evaluate the indefinite integral by integrating each term separately. Here we show that we can do the same thing for convergent power series. That is, if

$$f(x)=c_nx^n=c_0+c_1x+c_2x^2+\ldots$$

converges on some interval I, then

$$f\,'(x)=c_1+2c_2x+3c_3x^2+\ldots$$

and

$$\int f(x)\,dx=C+c_0x+c_1\frac{x^2}{2}+c_2\frac{x^3}{3}+\ldots.$$

Evaluating the derivative and indefinite integral in this way is called **term-by-term differentiation of a power series** and **term-by-term integration of a power series**, respectively. The ability to differentiate and integrate power series term-by-term also allows us to use known power series representations to find power series representations for other functions. For example, given the power series for $f(x)=\dfrac{1}{1-x}$, we can differentiate term-by-term to find the power series for $f\,'(x)=\dfrac{1}{(1-x)^2}$. Similarly, using the power series for $g(x)=\dfrac{1}{1+x}$, we can integrate term-by-term to find the power series for $G(x)=\ln(1+x)$, an antiderivative of g. We show how to do this in Example 7.3.6 and Example 7.3.7. First, we state Theorem 7.3.3, which provides the main result regarding differentiation and integration of power series.

Theorem 7.3.3: Term-by-Term Differentiation and Integration for Power Series

Suppose that the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges on the interval $(a-R,a+R)$ for some $R>0$. Let f be the function defined by the series

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+\ldots$$

for $|x-a|<R$. Then f is differentiable on the interval $(a-R,a+R)$ and we can find $f\,'$ by differentiating the series term-by-term:

$$f\,'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}=c_1+2c_2(x-a)+3c_3(x-a)^2+\ldots$$

for $|x-a|<R$. Also, to find $\int f(x)\,dx$, we can integrate the series term-by-term. The resulting series converges on $(a-R,a+R)$, and we have

$$\int f(x)\,dx=C+\sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}=C+c_0(x-a)+c_1\frac{(x-a)^2}{2}+c_2\frac{(x-a)^3}{3}+\ldots$$

for $|x-a|<R$.

The proof of this result is beyond the scope of the text and is omitted. Note that although Theorem 7.3.3 guarantees the same radius of convergence when a power series is differentiated or integrated term-by-term, it says nothing about what happens at the endpoints. It is possible that the differentiated and integrated power series have different behavior at the endpoints than does the original series. We see this behavior in the next examples.

Example 7.3.6: Differentiating Power Series

a. Use the power series representation

$$f(x)=\frac{1}{1-x}=\sum_{n=0}^{\infty} x^n=1+x+x^2+x^3+\ldots$$

for $|x|<1$ to find a power series representation for

$$g(x)=\frac{1}{(1-x)^2}$$

on the interval $(-1,1)$. Determine whether the resulting series converges at the endpoints.

b. Use the result of part a. to evaluate the sum of the series $\sum_{n=0}^{\infty}\dfrac{n+1}{4^n}$.

Solution

a. Since $g(x)=\dfrac{1}{(1-x)^2}$ is the derivative of $f(x)=\dfrac{1}{1-x}$, we can find a power series representation for g by differentiating the power series for f term-by-term. The result is

$$g(x)=\frac{1}{(1-x)^2}=\frac{d}{dx}\left(\frac{1}{1-x}\right)=\sum_{n=0}^{\infty}\frac{d}{dx}(x^n)=\frac{d}{dx}(1+x+x^2+x^3+\ldots)=0+1+2x+3x^2+4x^3+\ldots$$

$$=\sum_{n=0}^{\infty}(n+1)x^n$$

for $|x|<1$.

Theorem 7.3.1 does not guarantee anything about the behavior of this series at the endpoints. Testing the endpoints by using the divergence test, we find that the series diverges at both endpoints $x=\pm 1$. Note that this is the same result found in Example 7.3.5.

b. From part a. we know that

$$\sum_{n=0}^{\infty}(n+1)x^n=\frac{1}{(1-x)^2}.$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{n+1}{4^n}&=\sum_{n=0}^{\infty}(n+1)\left(\frac{1}{4}\right)^n\\&=\frac{1}{\left(1-\frac{1}{4}\right)^2}\\&=\frac{1}{\left(\frac{3}{4}\right)^2}\\&=\frac{16}{9}\end{aligned}$$

Exercise 7.3.5

Differentiate the series $\dfrac{1}{(1-x)^2}=\sum_{n=0}^{\infty}(n+1)x^n$ term-by-term to find a power series representation for $\dfrac{2}{(1-x)^3}$ on the interval $(-1,1)$.

Hint

Write out the first several terms and apply the power rule.

Answer

$$\sum_{n=0}^{\infty}(n+2)(n+1)x^n$$

Example 7.3.7: Integrating Power Series

For each of the following functions f, find a power series representation for f by integrating the power series for f' and find its interval of convergence.

a. $f(x)=\ln(1+x)$
b. $f(x)=\tan^{-1}x$

Solution:

a. For $f(x)=\ln(1+x)$,the derivative is $f\,'(x)=\dfrac{1}{1+x}$. We know that

$$\frac{1}{1+x}=\frac{1}{1-(-x)}=\sum_{n=0}^{\infty}(-x)^n=1-x+x^2-x^3+\ldots$$

for $|x|<1$. To find a power series for $f(x)=\ln(1+x)$, we integrate the series term-by-term.

$$\int f\,'(x)\,dx=\int(1-x+x^2-x^3+\ldots)\,dx=C+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots$$

Since $f(x)=\ln(1+x)$ is an antiderivative of $\dfrac{1}{1+x}$, it remains to solve for the constant C. Since $\ln(1+0)=0$, we have $C=0$. Therefore, a power series representation for $f(x)=\ln(1+x)$ is

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}\text{ for }|x|<1.$$

Theorem 7.3.3 does not guarantee anything about the behavior of this power series at the endpoints. However, checking the endpoints, we find that at $x=1$ the series is the alternating harmonic series, which converges. Also, at $x=-1$, the series is the harmonic series, which diverges. It is important to note that, even though this series converges at $x=1$, Theorem 7.3.3 does not guarantee that the series actually converges to $\ln(2)$. In fact, the series does converge to $\ln(2)$, but showing this fact requires more advanced techniques. (Abel's theorem, covered in more advanced texts, deals with this more technical point.) The interval of convergence is $(-1,1]$.

b. The derivative of $f(x)=\tan^{-1}x$ is $f\,'(x)=\dfrac{1}{1+x^2}$. We know that

$$\frac{1}{1+x^2}=\frac{1}{1-(-x^2)}=\sum_{n=0}^{\infty}(-x^2)^n=1-x^2+x^4-x^6+\ldots$$

for $|x|<1$. To find a power series for $f(x)=\tan^{-1}x$, we integrate this series term-by-term.

$$\int f\,'(x)\,dx=\int(1-x^2+x^4-x^6+\ldots)\,dx=C+x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$$

Since $\tan^{-1}(0)=0$, we have $C=0$. Therefore, a power series representation for $f(x)=\tan^{-1}x$ is

$$\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

for $|x|<1$. Again, Theorem 7.3.3 does not guarantee anything about the convergence of this series at the endpoints. However, checking the endpoints and using the alternating series test, we find that the series converges at $x=1$ and $x=-1$. As discussed in part a., using Abel's theorem, it can be shown that the series actually converges to $\tan^{-1}(1)$ and $\tan^{-1}(-1)$ at $x=1$ and $x=-1$, respectively. Thus, the interval of convergence is $[-1,1]$.

? Exercise 7.3.6

Integrate the power series $\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{x^n}{n}$ term-by-term to evaluate $\int\ln(1+x)\,dx$.

Hint

Use the fact that $\dfrac{x^{n+1}}{(n+1)n}$ is an antiderivative of $\dfrac{x^n}{n}$.

Answer

$$\sum_{n=2}^{\infty} \frac{(-1)^n x^n}{n(n-1)}$$

Up to this point, we have shown several techniques for finding power series representations for functions. However, how do we know that these power series are unique? That is, given a function f and a power series for f at a, is it possible that there is a different power series for f at a that we could have found if we had used a different technique? The answer to this question is no. This fact should not seem surprising if we think of power series as polynomials with an infinite number of terms. Intuitively, if

$$c_0 + c_1 x + c_2 x^2 + \ldots = d_0 + d_1 x + d_2 x^2 + \ldots$$

for all values x in some open interval I about zero, then the coefficients c_n should equal d_n for $n \geq 0$. We now state this result formally.

Theorem 7.3.4: Uniqueness of Power Series

Let $\sum_{n=0}^{\infty} c_n (x-a)^n$ and $\sum_{n=0}^{\infty} d_n (x-a)^n$ be two convergent power series such that

$$\sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} d_n (x-a)^n$$

for all x in an open interval containing a. Then $c_n = d_n$ for all $n \geq 0$.

Proof

Let

$$\begin{aligned} f(x) &= c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \\ &= d_0 + d_1(x-a) + d_2(x-a)^2 + d_3(x-a)^3 + \ldots . \end{aligned}$$

Then $f(a) = c_0 = d_0$. By Theorem 7.3.3, we can differentiate both series term-by-term. Therefore,

$$\begin{aligned} f'(x) &= c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots \\ &= d_1 + 2d_2(x-a) + 3d_3(x-a)^2 + \ldots , \end{aligned}$$

and thus, $f'(a) = c_1 = d_1$. Similarly,

$$\begin{aligned} f''(x) &= 2c_2 + 3 \cdot 2c_3(x-a) + \ldots \\ &= 2d_2 + 3 \cdot 2d_3(x-a) + \ldots \end{aligned}$$

implies that $f''(a) = 2c_2 = 2d_2$, and therefore, $c_2 = d_2$. More generally, for any integer $n \geq 0$, $f^{(n)}(a) = n!c_n = n!d_n$, and consequently, $c_n = d_n$ for all $n \geq 0$.

□

In this section we have shown how to find power series representations for certain functions using various algebraic operations, differentiation, or integration. At this point, however, we are still limited as to the functions for which we can find power series representations. Next, we show how to find power series representations for many more functions by introducing Taylor series.

Key Concepts

- Given two power series $\sum_{n=0}^{\infty} c_n x^n$ and $\sum_{n=0}^{\infty} d_n x^n$ that converge to functions f and g on a common interval I, the sum and difference of the two series converge to $f \pm g$, respectively, on I. In addition, for any real number b and integer $m \geq 0$, the series $\sum_{n=0}^{\infty} b x^m c_n x^n$ converges to $b x^m f(x)$ and the series $\sum_{n=0}^{\infty} c_n (b x^m)^n$ converges to $f(b x^m)$ whenever $b x^m$ is in the interval I.

- Given two power series that converge on an interval $(-R, R)$, the Cauchy product of the two power series converges on the interval $(-R, R)$.
- Given a power series that converges to a function f on an interval $(-R, R)$, the series can be differentiated term-by-term and the resulting series converges to f' on $(-R, R)$. The series can also be integrated term-by-term and the resulting series converges to $\int f(x)\,dx$ on $(-R, R)$.

Glossary

term-by-term differentiation of a power series

a technique for evaluating the derivative of a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ by evaluating the derivative of each term separately to create the new power series $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

term-by-term integration of a power series

a technique for integrating a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ by integrating each term separately to create the new power series $C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

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7.3E: Exercises for Section 10.2

1) If *[Math Processing Error]* and *[Math Processing Error]*, find the power series of *[Math Processing Error]* and of *[Math Processing Error]*.

Answer

[Math Processing Error] and *[Math Processing Error]*.

2) If *[Math Processing Error]* and *[Math Processing Error]*, find the power series of *[Math Processing Error]* and of *[Math Processing Error]*.

In exercises 3 - 6, use partial fractions to find the power series of each function.

3) *[Math Processing Error]*

Answer

[Math Processing Error]

4) *[Math Processing Error]*

5) *[Math Processing Error]*

Answer

[Math Processing Error]

6) *[Math Processing Error]*

In exercises 7 - 10, express each series as a rational function.

7) *[Math Processing Error]*

Answer

[Math Processing Error]

8) *[Math Processing Error]*

9) *[Math Processing Error]*

Answer

[Math Processing Error]

10) *[Math Processing Error]*

Exercises 11 - 16 explore applications of *annuities*.

11) Calculate the present values *[Math Processing Error]* of an annuity in which $10,000 is to be paid out annually for a period of 20 years, assuming interest rates of *[Math Processing Error]*, and *[Math Processing Error]*.

Answer

[Math Processing Error] where *[Math Processing Error]*. Then *[Math Processing Error]*. When *[Math Processing Error]* When *[Math Processing Error]* When *[Math Processing Error]*.

12) Calculate the present values *[Math Processing Error]* of annuities in which $9,000 is to be paid out annually perpetually, assuming interest rates of *[Math Processing Error]* and *[Math Processing Error]*.

13) Calculate the annual payouts *[Math Processing Error]* to be given for 20 years on annuities having present value $100,000 assuming respective interest rates of *[Math Processing Error]* and *[Math Processing Error]*

Answer

In general, *[Math Processing Error]* for *[Math Processing Error]* years of payouts, or *[Math Processing Error]*. For *[Math Processing Error]* and *[Math Processing Error]*, one has *[Math Processing Error]* when *[Math Processing Error]* when *[Math Processing Error]*; and *[Math Processing Error]* when *[Math Processing Error]*.

14) Calculate the annual payouts *[Math Processing Error]* to be given perpetually on annuities having present value $100,000 assuming respective interest rates of *[Math Processing Error]* and *[Math Processing Error]*.

15) Suppose that an annuity has a present value *[Math Processing Error]* **million dollars.** What interest rate *[Math Processing Error]* would allow for perpetual annual payouts of $50,000?

Answer

In general, *[Math Processing Error]* Thus, *[Math Processing Error]*

16) Suppose that an annuity has a present value *[Math Processing Error]* **million dollars.** What interest rate *[Math Processing Error]* would allow for perpetual annual payouts of $100,000?

In exercises 17 - 20, express the sum of each power series in terms of geometric series, and then express the sum as a rational function.

17) *[Math Processing Error]* (Hint: Group powers *[Math Processing Error]* and *[Math Processing Error]*.)

Answer

[Math Processing Error]

18) *[Math Processing Error]* (Hint: Group powers *[Math Processing Error]* etc.)

19) *[Math Processing Error]* (Hint: Group powers *[Math Processing Error]*, and *[Math Processing Error]*.)

Answer

[Math Processing Error]

20) *[Math Processing Error]* (Hint: Group powers *[Math Processing Error]* and *[Math Processing Error]*.)

In exercises 21 - 24, find the power series of *[Math Processing Error]* **given** *[Math Processing Error]* **and** *[Math Processing Error]* **as defined.**

21) *[Math Processing Error]*

Answer

[Math Processing Error] so *[Math Processing Error]* and *[Math Processing Error]*

22) *[Math Processing Error]*. Express the coefficients of *[Math Processing Error]* in terms of *[Math Processing Error]*.

23) *[Math Processing Error]*

Answer

[Math Processing Error] so *[Math Processing Error]* and *[Math Processing Error]*

24) *[Math Processing Error]*

In exercises 25 - 26, differentiate the given series expansion of *[Math Processing Error]* **term-by-term to obtain the corresponding series expansion for the derivative of** *[Math Processing Error]***.**

25) *[Math Processing Error]*

Answer

The derivative of *[Math Processing Error]* is *[Math Processing Error]*.

26) *[Math Processing Error]*

In exercises 27 - 28, integrate the given series expansion of *[Math Processing Error]* **term-by-term from zero to** *[Math Processing Error]* **to obtain the corresponding series expansion for the indefinite integral of** *[Math Processing Error]***.**

27) *[Math Processing Error]*

Answer

The indefinite integral of *[Math Processing Error]* is *[Math Processing Error]*.

28) *[Math Processing Error]*

In exercises 29 - 32, evaluate each infinite series by identifying it as the value of a derivative or integral of geometric series.

29) Evaluate *[Math Processing Error]* as *[Math Processing Error]* where *[Math Processing Error]*.

Answer

[Math Processing Error] so *[Math Processing Error]*

30) Evaluate *[Math Processing Error]* as *[Math Processing Error]* where *[Math Processing Error]*.

31) Evaluate *[Math Processing Error]* as *[Math Processing Error]* where *[Math Processing Error]*.

Answer

[Math Processing Error] so *[Math Processing Error]*

32) Evaluate *[Math Processing Error]* as *[Math Processing Error]* where *[Math Processing Error]*.

In exercises 33 - 39, given that *[Math Processing Error]***, use term-by-term differentiation or integration to find power series for each function centered at the given point.**

33) *[Math Processing Error]* centered at *[Math Processing Error]* (Hint: *[Math Processing Error]*)

Answer

[Math Processing Error]

34) *[Math Processing Error]* at *[Math Processing Error]*

35) *[Math Processing Error]* at *[Math Processing Error]*

Answer

[Math Processing Error]

36) *[Math Processing Error]* at *[Math Processing Error]*

37) *[Math Processing Error]* at *[Math Processing Error]*

Answer

[Math Processing Error]

38) *[Math Processing Error]* at *[Math Processing Error]*

39) *[Math Processing Error]* where *[Math Processing Error]*

Answer

Term-by-term integration gives *[Math Processing Error]*

40) [T] Evaluate the power series expansion *[Math Processing Error]* at *[Math Processing Error]* to show that *[Math Processing Error]* is the sum of the alternating harmonic series. Use the alternating series test to determine how many terms of the sum are needed to estimate *[Math Processing Error]* accurate to within *[Math Processing Error]* and find such an approximation.

41) [T] Subtract the infinite series of *[Math Processing Error]* from *[Math Processing Error]* to get a power series for *[Math Processing Error]*. Evaluate at *[Math Processing Error]*. What is the smallest *[Math Processing Error]* such that the *[Math*

Processing Error] partial sum of this series approximates *[Math Processing Error]* with an error less than *[Math Processing Error]*?

Answer

We have *[Math Processing Error]* so *[Math Processing Error]*. Thus, *[Math Processing Error]*. When *[Math Processing Error]* we obtain *[Math Processing Error]*. We have *[Math Processing Error]*, while *[Math Processing Error]* and *[Math Processing Error]* therefore, *[Math Processing Error]*.

In exercises 42 - 45, using a substitution if indicated, express each series in terms of elementary functions and find the radius of convergence of the sum.

42) *[Math Processing Error]*

43) *[Math Processing Error]*

Answer

[Math Processing Error] so *[Math Processing Error]*. The radius of convergence is equal to *[Math Processing Error]* by the ratio test.

44) *[Math Processing Error]* using *[Math Processing Error]*

45) *[Math Processing Error]* using *[Math Processing Error]*

Answer

If *[Math Processing Error]*, then *[Math Processing Error]*. If *[Math Processing Error]*, then *[Math Processing Error]* when *[Math Processing Error]*. So the series converges for all *[Math Processing Error]*.

46) Show that, up to powers *[Math Processing Error]* and *[Math Processing Error]*, *[Math Processing Error]* satisfies *[Math Processing Error]*.

47) Differentiate the series *[Math Processing Error]* term-by-term to show that *[Math Processing Error]* is equal to its derivative.

Answer

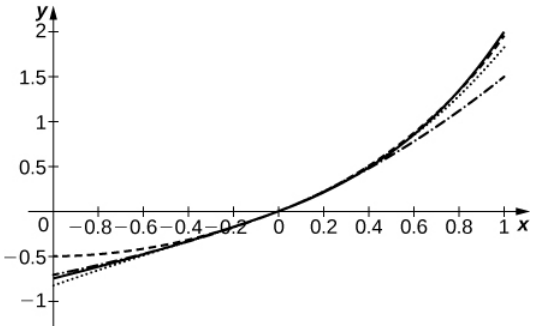
Answers will vary.

48) Show that if *[Math Processing Error]* is a sum of even powers, that is, *[Math Processing Error]* if *[Math Processing Error]* is odd, then *[Math Processing Error]* is a sum of odd powers, while if *[Math Processing Error]* is a sum of odd powers, then *[Math Processing Error]* is a sum of even powers.

49) [T] Suppose that the coefficients an of the series *[Math Processing Error]* are defined by the recurrence relation *[Math Processing Error]*. For *[Math Processing Error]* and *[Math Processing Error]*, compute and plot the sums *[Math Processing Error]* for *[Math Processing Error]* on *[Math Processing Error]*

Answer

The solid curve is *[Math Processing Error]*. The dashed curve is *[Math Processing Error]*, dotted is *[Math Processing Error]*, and dash-dotted is *[Math Processing Error]*



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50) [T] Suppose that the coefficients an of the series *[Math Processing Error]* are defined by the recurrence relation *[Math Processing Error]*. For *[Math Processing Error]* and *[Math Processing Error]*, compute and plot the sums *[Math Processing Error]* for *[Math Processing Error]* on *[Math Processing Error]*.

51) [T] Given the power series expansion *[Math Processing Error]*, determine how many terms *[Math Processing Error]* of the sum evaluated at *[Math Processing Error]* are needed to approximate *[Math Processing Error]* accurate to within *[Math Processing Error]* Evaluate the corresponding partial sum *[Math Processing Error]*.

Answer

When *[Math Processing Error]*. Since *[Math Processing Error]* one has *[Math Processing Error]* whereas *[Math Processing Error]* therefore, *[Math Processing Error]*

52) [T] Given the power series expansion *[Math Processing Error]*, use the alternating series test to determine how many terms *[Math Processing Error]* of the sum evaluated at *[Math Processing Error]* are needed to approximate *[Math Processing Error]* accurate to within *[Math Processing Error]* Evaluate the corresponding partial sum *[Math Processing Error]*.

53) [T] Recall that *[Math Processing Error]* Assuming an exact value of *[Math Processing Error]*, estimate *[Math Processing Error]* by evaluating partial sums *[Math Processing Error]* of the power series expansion *[Math Processing Error]* at *[Math Processing Error]*. What is the smallest number *[Math Processing Error]* such that *[Math Processing Error]* approximates *[Math Processing Error]* accurately to within *[Math Processing Error]*? How many terms are needed for accuracy to within *[Math Processing Error]*?

Answer

[Math Processing Error] One has *[Math Processing Error]* and *[Math Processing Error]* so *[Math Processing Error]* is the smallest partial sum with accuracy to within *[Math Processing Error]* Also, *[Math Processing Error]* while *[Math Processing Error]* so *[Math Processing Error]* is the smallest *[Math Processing Error]* to give accuracy to within *[Math Processing Error]*

Contributors

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7.4: Taylor and Maclaurin Series

Learning Objectives

- Describe the procedure for finding a Taylor polynomial of a given order for a function.
- Explain the meaning and significance of Taylor's theorem with remainder.
- Estimate the remainder for a Taylor series approximation of a given function.

In the previous two sections we discussed how to find power series representations for certain types of functions—specifically, functions related to geometric series. Here we discuss power series representations for other types of functions. In particular, we address the following questions: Which functions can be represented by power series and how do we find such representations? If we can find a power series representation for a particular function f and the series converges on some interval, how do we prove that the series actually converges to f?

Overview of Taylor/Maclaurin Series

Consider a function f that has a power series representation at $x=a$. Then the series has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots. \tag{7.4.1}$$

What should the coefficients be? For now, we ignore issues of convergence, but instead focus on what the series should be, if one exists. We return to discuss convergence later in this section. If the series Equation 7.4.1 is a representation for f at $x=a$, we certainly want the series to equal $f(a)$ at $x=a$. Evaluating the series at $x=a$, we see that

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(a-a) + c_2(a-a)^2 + \cdots = c_0. \tag{7.4.2}$$

Thus, the series equals $f(a)$ if the coefficient $c_0 = f(a)$. In addition, we would like the first derivative of the power series to equal $f'(a)$ at $x=a$. Differentiating Equation 7.4.2 term-by-term, we see that

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots. \tag{7.4.3}$$

Therefore, at $x=a$, the derivative is

$$f'(a) = c_1 + 2c_2(a-a) + 3c_3(a-a)^2 + \cdots = c_1. \tag{7.4.4}$$

Therefore, the derivative of the series equals $f'(a)$ if the coefficient $c_1 = f'(a)$. Continuing in this way, we look for coefficients c_n such that all the derivatives of the power series Equation 7.4.4 will agree with all the corresponding derivatives of f at $x=a$. The second and third derivatives of Equation 7.4.3 are given by

$$\frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 2c_2 + 3\cdot 2c_3(x-a) + 4\cdot 3c_4(x-a)^2 + \ldots \tag{7.4.5}$$

and

$$\frac{d^3}{dx^3}\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + 5\cdot 4\cdot 3c_5(x-a)^2 + \cdots. \tag{7.4.6}$$

Therefore, at $x=a$, the second and third derivatives

$$f''(a) = 2c_2 + 3\cdot 2c_3(a-a) + 4\cdot 3c_4(a-a)^2 + \cdots = 2c_2 \tag{7.4.7}$$

and

$$f'''(a) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(a-a) + 5\cdot 4\cdot 3c_5(a-a)^2 + \cdots = 3\cdot 2c_3 \tag{7.4.8}$$

equal $f''(a)$ and $f'''(a)$, respectively, if $c_2 = \dfrac{f''(a)}{2}$ and $c_3 = \dfrac{f'''(a)}{3 \cdot 2}$. More generally, we see that if f has a power series representation at $x = a$, then the coefficients should be given by $c_n = \dfrac{f^{(n)}(a)}{n!}$. That is, the series should be

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

This power series for f is known as the Taylor series for f at a. If $a = 0$, then this series is known as the Maclaurin series for f.

Definition 7.4.1: Maclaurin and Taylor series

If f has derivatives of all orders at $x = a$, then the *Taylor series* for the function f at a is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

The Taylor series for f at 0 is known as the *Maclaurin series* for f.

Later in this section, we will show examples of finding Taylor series and discuss conditions under which the Taylor series for a function will converge to that function. Here, we state an important result. Recall that power series representations are unique. Therefore, if a function f has a power series at a, then it must be the Taylor series for f at a.

Theorem 7.4.1: Uniqueness of Taylor Series

If a function f has a power series at a that converges to f on some open interval containing a, then that power series is the Taylor series for f at a.

The proof follows directly from that discussed previously.

To determine if a Taylor series converges, we need to look at its sequence of partial sums. These partial sums are finite polynomials, known as **Taylor polynomials.**

Taylor Polynomials

The nth partial sum of the Taylor series for a function f at a is known as the nth-degree Taylor polynomial. For example, the 0th, 1st, 2nd, and 3rd partial sums of the Taylor series are given by

$$\begin{aligned} p_0(x) &= f(a) \\ p_1(x) &= f(a) + f'(a)(x-a) \\ p_2(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \\ p_3(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \end{aligned}$$

respectively. These partial sums are known as the 0th, 1st, 2nd, and 3rd degree Taylor polynomials of f at a, respectively. If $a = 0$, then these polynomials are known as *Maclaurin polynomials* for f. We now provide a formal definition of Taylor and Maclaurin polynomials for a function f.

Definition 7.4.2: Maclaurin polynomial

If f has n derivatives at $x = a$, then the nth-degree Taylor polynomial of f at a is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

The nth-degree Taylor polynomial for f at 0 is known as the nth-degree Maclaurin polynomial for f.

We now show how to use this definition to find several Taylor polynomials for $f(x) = \ln x$ at $x = 1$.

Example 7.4.1: Finding Taylor Polynomials

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \ln x$ at $x = 1$. Use a graphing utility to compare the graph of f with the graphs of p_0, p_1, p_2 and p_3.

Solution

To find these Taylor polynomials, we need to evaluate f and its first three derivatives at $x = 1$.

$$f(x) = \ln x \qquad f(1) = 0$$
$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$
$$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$$
$$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$$

Therefore,

$$\begin{aligned} p_0(x) &= f(1) = 0, \\ p_1(x) &= f(1) + f'(1)(x-1) = x - 1, \\ p_2(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2 \\ p_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \end{aligned}$$

The graphs of $y = f(x)$ and the first three Taylor polynomials are shown in Figure 7.4.1.

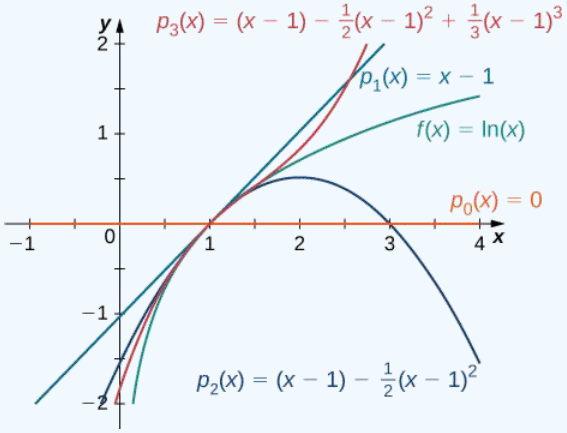


Figure 7.4.1: The function $y = \ln x$ and the Taylor polynomials p_0, p_1, p_2 and p_3 at $x = 1$ are plotted on this graph.

Exercise 7.4.1

Find the Taylor polynomials p_0, p_1, p_2 and p_3 for $f(x) = \dfrac{1}{x^2}$ at $x = 1$.

Hint

Find the first three derivatives of f and evaluate them at $x = 1$.

Answer

$$
\begin{aligned}
p_0(x) &= 1 \\
p_1(x) &= 1-2(x-1) \\
p_2(x) &= 1-2(x-1)+3(x-1)^2 \\
p_3(x) &= 1-2(x-1)+3(x-1)^2-4(x-1)^3
\end{aligned}
$$

We now show how to find Maclaurin polynomials for $e^x, \sin x$, and $\cos x$. As stated above, Maclaurin polynomials are Taylor polynomials centered at zero.

Example 7.4.2: Finding Maclaurin Polynomials

For each of the following functions, find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3. Find a formula for the nth-degree Maclaurin polynomial and write it using sigma notation. Use a graphing utility to compare the graphs of p_0, p_1, p_2 and p_3 with f.

a. $f(x) = e^x$
b. $f(x) = \sin x$
c. $f(x) = \cos x$

Solution

Since $f(x) = e^x$,we know that $f(x) = f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x$ for all positive integers n. Therefore,

$$f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 1$$

for all positive integers n. Therefore, we have

$$
\begin{aligned}
p_0(x) &= f(0) = 1, \\
p_1(x) &= f(0) + f'(0)x = 1 + x, \\
p_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{1}{2}x^2, \\
p_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3,
\end{aligned}
$$

$$
\begin{aligned}
p_n(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \\
&= \sum_{k=0}^{n} \frac{x^k}{k!}
\end{aligned}
$$

The function and the first three Maclaurin polynomials are shown in Figure 7.4.2.

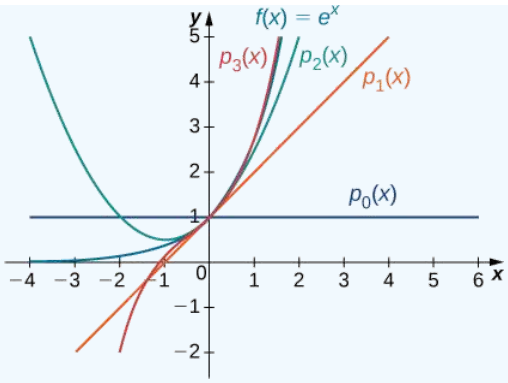


Figure 7.4.2: *The graph shows the function $y = e^x$ and the Maclaurin polynomials p_0, p_1, p_2 and p_3.*

b. For $f(x) = \sin x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x & f^{(4)}(0) &= 0. \end{aligned}$$

Since the fourth derivative is $\sin x$, the pattern repeats. That is, $f^{(2m)}(0) = 0$ and $f^{(2m+1)}(0) = (-1)^m$ for $m \geq 0$. Thus, we have

$$\begin{aligned} p_0(x) &= 0, \\ p_1(x) &= 0 + x = x, \\ p_2(x) &= 0 + x + 0 = x, \\ p_3(x) &= 0 + x + 0 - \frac{1}{3!}x^3 = x - \frac{x^3}{3!}, \\ p_4(x) &= 0 + x + 0 - \frac{1}{3!}x^3 + 0 = x - \frac{x^3}{3!}, \\ p_5(x) &= 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 = x - \frac{x^3}{3!} + \frac{x^5}{5!}, \end{aligned}$$

and for $m \geq 0$,

$$\begin{aligned} p_{2m+1}(x) = p_{2m+2}(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} \\ &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{(2k+1)!}. \end{aligned}$$

Graphs of the function and its Maclaurin polynomials are shown in Figure 7.4.3.

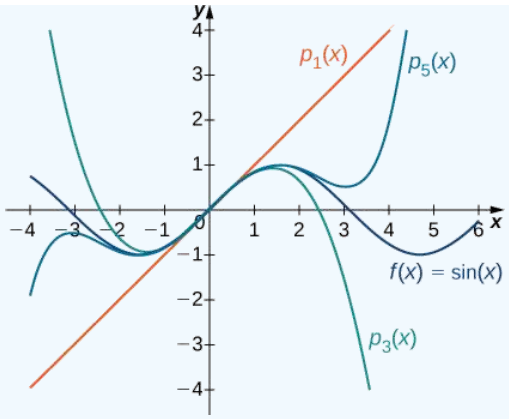


Figure 7.4.3: *The graph shows the function* $y = \sin x$ *and the Maclaurin polynomials* p_1, p_3 *and* p_5.

c. For $f(x) = \cos x$, the values of the function and its first four derivatives at $x = 0$ are given as follows:

$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1. \end{aligned}$$

Since the fourth derivative is $\sin x$, the pattern repeats. In other words, $f^{(2m)}(0) = (-1)^m$ and $f^{(2m+1)} = 0$ for $m \geq 0$. Therefore,

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= 1 + 0 = 1, \\ p_2(x) &= 1 + 0 - \frac{1}{2!}x^2 = 1 - \frac{x^2}{2!}, \\ p_3(x) &= 1 + 0 - \frac{1}{2!}x^2 + 0 = 1 - \frac{x^2}{2!}, \\ p_4(x) &= 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}, \\ p_5(x) &= 1 + 0 - \frac{1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + 0 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}, \end{aligned}$$

and for $n \geq 0$,

$$\begin{aligned} p_{2m}(x) &= p_{2m+1}(x) \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^m \frac{x^{2m}}{(2m)!} \\ &= \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!}. \end{aligned}$$

Graphs of the function and the Maclaurin polynomials appear in Figure 7.4.4.

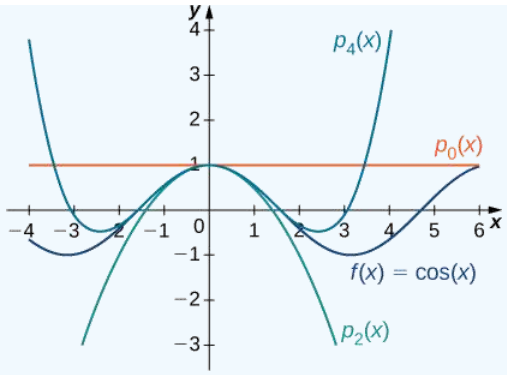


Figure 7.4.4: *The function $y = \cos x$ and the Maclaurin polynomials p_0, p_2 and p_4 are plotted on this graph.*

? Exercise 7.4.2

Find formulas for the Maclaurin polynomials p_0, p_1, p_2 and p_3 for $f(x) = \dfrac{1}{1+x}$.

Find a formula for the nth-degree Maclaurin polynomial. Write your answer using sigma notation.

Hint

Evaluate the first four derivatives of f and look for a pattern.

Answer

$p_0(x) = 1, \quad p_1(x) = 1 - x, \quad p_2(x) = 1 - x + x^2, \quad p_3(x) = 1 - x + x^2 - x^3,$

$p_n(x) = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n \quad = \sum_{k=0}^{n} (-1)^k x^k$

Taylor's Theorem with Remainder

Recall that the nth-degree Taylor polynomial for a function f at a is the nth partial sum of the Taylor series for f at a. Therefore, to determine if the Taylor series converges, we need to determine whether the sequence of Taylor polynomials p_n converges. However, not only do we want to know if the sequence of Taylor polynomials converges, we want to know if it converges to f. To answer this question, we define the remainder $R_n(x)$ as

$$R_n(x) = f(x) - p_n(x).$$

For the sequence of Taylor polynomials to converge to f, we need the remainder R_n to converge to zero. To determine if R_n converges to zero, we introduce **Taylor's theorem with remainder**. Not only is this theorem useful in proving that a Taylor series converges to its related function, but it will also allow us to quantify how well the nth-degree Taylor polynomial approximates the function.

Here we look for a bound on $|R_n|$. Consider the simplest case: $n = 0$. Let p_0 be the 0th Taylor polynomial at a for a function f. The remainder R_0 satisfies

$$R_0(x) = f(x) - p_0(x) = f(x) - f(a).$$

If f is differentiable on an interval I containing a and x, then by the Mean Value Theorem there exists a real number c between a and x such that $f(x) - f(a) = f'(c)(x - a)$. Therefore,

$$R_0(x) = f'(c)(x - a).$$

Using the Mean Value Theorem in a similar argument, we can show that if f is n times differentiable on an interval I containing a and x, then the nth remainder R_n satisfies

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - a)^{n+1}$$

for some real number c between a and x. It is important to note that the value c in the numerator above is not the center a, but rather an unknown value c between a and x. This formula allows us to get a bound on the remainder R_n. If we happen to know that $\left|f^{(n+1)}(x)\right|$ is bounded by some real number M on this interval I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x in the interval I.

We now state Taylor's theorem, which provides the formal relationship between a function f and its nth-degree Taylor polynomial $p_n(x)$. This theorem allows us to bound the error when using a Taylor polynomial to approximate a function value, and will be important in proving that a Taylor series for f converges to f.

Theorem 7.4.2: Taylor's Theorem with Remainder

Let f be a function that can be differentiated $n+1$ times on an interval I containing the real number a. Let p_n be the nth-degree Taylor polynomial of f at a and let

$$R_n(x) = f(x) - p_n(x)$$

be the nth remainder. Then for each x in the interval I, there exists a real number c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

.

If there exists a real number M such that $\left|f^{(n+1)}(x)\right| \leq M$ for all $x \in I$, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

for all x in I.

Proof

Fix a point $x \in I$ and introduce the function g such that

$$g(t) = f(x) - f(t) - f'(t)(x-t) - \frac{f''(t)}{2!}(x-t)^2 - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n - R_n(x)\frac{(x-t)^{n+1}}{(x-a)^{n+1}}.$$

We claim that g satisfies the criteria of Rolle's theorem. Since g is a polynomial function (in t), it is a differentiable function. Also, g is zero at $t=a$ and $t=x$ because

$$\begin{aligned} g(a) &= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n - R_n(x) \\ &= f(x) - p_n(x) - R_n(x) \\ &= 0, \\ g(x) &= f(x) - f(x) - 0 - \cdots - 0 \\ &= 0. \end{aligned}$$

Therefore, g satisfies Rolle's theorem, and consequently, there exists c between a and x such that $g'(c) = 0$. We now calculate g'. Using the product rule, we note that

$$\frac{d}{dt}\left[\frac{f^{(n)}(t)}{n!}(x-t)^n\right] = -\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} + \frac{f^{(n+1)}(t)}{n!}(x-t)^n.$$

Consequently,

$$g'(t) = -f'(t) + [f'(t) - f''(t)(x-t)] + \left[f''(t)(x-t) - \frac{f'''(t)}{2!}(x-t)^2\right] + \cdots$$
$$+ \left[\frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} - \frac{f^{(n+1)}(t)}{n!}(x-t)^n\right] + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}$$

.

Notice that there is a telescoping effect. Therefore,

$$g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n + (n+1)R_n(x)\frac{(x-t)^n}{(x-a)^{n+1}}$$

.

By Rolle's theorem, we conclude that there exists a number c between a and x such that $g'(c) = 0$. Since

$$g'(c) = -\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}}$$

we conclude that

$$-\frac{f^{(n+1)}(c)}{n!}(x-c)^n + (n+1)R_n(x)\frac{(x-c)^n}{(x-a)^{n+1}} = 0.$$

Adding the first term on the left-hand side to both sides of the equation and dividing both sides of the equation by $n+1$, we conclude that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

as desired. From this fact, it follows that if there exists M such that $\left|f^{(n+1)}(x)\right| \leq M$ for all x in I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

.

□

Not only does Taylor's theorem allow us to prove that a Taylor series converges to a function, but it also allows us to estimate the accuracy of Taylor polynomials in approximating function values. We begin by looking at linear and quadratic approximations of $f(x) = \sqrt[3]{x}$ at $x = 8$ and determine how accurate these approximations are at estimating $\sqrt[3]{11}$.

Example 7.4.3: Using Linear and Quadratic Approximations to Estimate Function Values

Consider the function $f(x) = \sqrt[3]{x}$.

a. Find the first and second Taylor polynomials for f at $x = 8$. Use a graphing utility to compare these polynomials with f near $x = 8$.
b. Use these two polynomials to estimate $\sqrt[3]{11}$.
c. Use Taylor's theorem to bound the error.

Solution:

a. For $f(x) = \sqrt[3]{x}$, the values of the function and its first two derivatives at $x = 8$ are as follows:

$$f(x) = \sqrt[3]{x}, \qquad f(8) = 2$$
$$f'(x) = \frac{1}{3x^{2/3}}, \qquad f'(8) = \frac{1}{12}$$
$$f''(x) = \frac{-2}{9x^{5/3}}, \qquad f''(8) = -\frac{1}{144.}$$

Thus, the first and second Taylor polynomials at $x = 8$ are given by

$$\begin{aligned} p_1(x) &= f(8) + f'(8)(x-8) \\ &= 2 + \frac{1}{12}(x-8) \\ p_2(x) &= f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2 \\ &= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2. \end{aligned}$$

The function and the Taylor polynomials are shown in Figure 7.4.5.

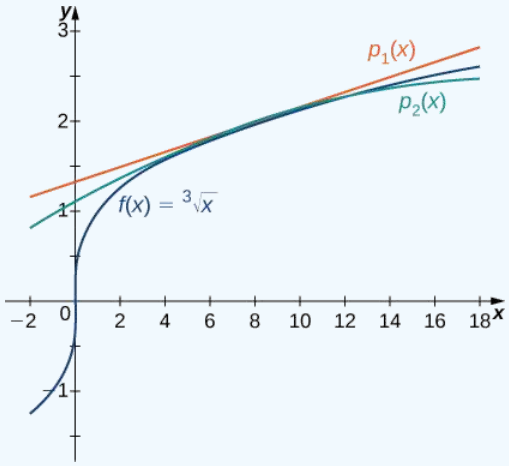


Figure 7.4.5: *The graphs of $f(x) = \sqrt[3]{x}$ and the linear and quadratic approximations $p_1(x)$ and $p_2(x)$*

b. Using the first Taylor polynomial at $x = 8$, we can estimate

$$\sqrt[3]{11} \approx p_1(11) = 2 + \frac{1}{12}(11-8) = 2.25.$$

Using the second Taylor polynomial at $x = 8$, we obtain

$$\sqrt[3]{11} \approx p_2(11) = 2 + \frac{1}{12}(11-8) - \frac{1}{288}(11-8)^2 = 2.21875.$$

c. By Theorem 7.4.2, there exists a c in the interval $(8, 11)$ such that the remainder when approximating $\sqrt[3]{11}$ by the first Taylor polynomial satisfies

$$R_1(11) = \frac{f''(c)}{2!}(11-8)^2.$$

We do not know the exact value of c, so we find an upper bound on $R_1(11)$ by determining the maximum value of f'' on the interval $(8, 11)$. Since $f''(x) = -\dfrac{2}{9x^{5/3}}$, the largest value for $|f''(x)|$ on that interval occurs at $x = 8$. Using the fact that $f''(8) = -\dfrac{1}{144}$, we obtain

$$|R_1(11)| \le \frac{1}{144 \cdot 2!}(11-8)^2 = 0.03125.$$

Similarly, to estimate $R_2(11)$, we use the fact that

$$R_2(11) = \frac{f'''(c)}{3!}(11-8)^3.$$

Since $f'''(x) = \dfrac{10}{27x^{8/3}}$, the maximum value of f''' on the interval $(8, 11)$ is $f'''(8) \approx 0.0014468$. Therefore, we have

$$|R_2(11)| \le \frac{0.0011468}{3!}(11-8)^3 \approx 0.0065104.$$

Exercise 7.4.3:

Find the first and second Taylor polynomials for $f(x)=\sqrt{x}$ at $x=4$. Use these polynomials to estimate $\sqrt{6}$. Use Taylor's theorem to bound the error.

Hint

Evaluate $f(4)$, $f'(4)$,and $f''(4)$.

Answer

$p_1(x)=2+\frac{1}{4}(x-4); p_2(x)=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2; p_1(6)=2.5; p_2(6)=2.4375;$

$|R_1(6)|\le 0.0625; |R_2(6)|\le 0.015625$

Example 7.4.4: Approximating $\sin x$ Using Maclaurin Polynomials

From Example 7.4.2*b*, the Maclaurin polynomials for $\sin x$ are given by

$$p_{2m+1}(x)=p_{2m+2}(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+(-1)^m\frac{x^{2m+1}}{(2m+1)!}$$

for $m=0,1,2,\ldots$.

a. Use the fifth Maclaurin polynomial for $\sin x$ to approximate $\sin\left(\frac{\pi}{18}\right)$ and bound the error.

b. For what values of x does the fifth Maclaurin polynomial approximate $\sin x$ to within 0.0001?

Solution

a.

The fifth Maclaurin polynomial is

$$p_5(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}$$

.

Using this polynomial, we can estimate as follows:

$$\sin\left(\frac{\pi}{18}\right)\approx p_5\left(\frac{\pi}{18}\right)=\frac{\pi}{18}-\frac{1}{3!}\left(\frac{\pi}{18}\right)^3+\frac{1}{5!}\left(\frac{\pi}{18}\right)^5\approx 0.173648.$$

To estimate the error, use the fact that the sixth Maclaurin polynomial is $p_6(x)=p_5(x)$ and calculate a bound on $R_6(\frac{\pi}{18})$. By Theorem 7.4.2, the remainder is

$$R_6\left(\frac{\pi}{18}\right)=\frac{f^{(7)}(c)}{7!}\left(\frac{\pi}{18}\right)^7$$

for some c between 0 and $\frac{\pi}{18}$. Using the fact that $\left|f^{(7)}(x)\right|\le 1$ for all x, we find that the magnitude of the error is at most

$$\frac{1}{7!}\cdot\left(\frac{\pi}{18}\right)^7\le 9.8\times 10^{-10}.$$

b.

We need to find the values of x such that

$$\frac{1}{7!}|x|^7\le 0.0001.$$

Solving this inequality for x, we have that the fifth Maclaurin polynomial gives an estimate to within 0.0001 as long as $|x|<0.907$.

Exercise 7.4.4

Use the fourth Maclaurin polynomial for $\cos x$ to approximate $\cos\left(\frac{\pi}{12}\right)$.

Hint

The fourth Maclaurin polynomial is $p_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$.

Answer

0.96593

Now that we are able to bound the remainder $R_n(x)$, we can use this bound to prove that a Taylor series for f at a converges to f.

Representing Functions with Taylor and Maclaurin Series

We now discuss issues of convergence for Taylor series. We begin by showing how to find a Taylor series for a function, and how to find its interval of convergence.

Example 7.4.5: Finding a Taylor Series

Find the Taylor series for $f(x) = \frac{1}{x}$ at $x = 1$. Determine the interval of convergence.

Solution

For $f(x) = \frac{1}{x}$, the values of the function and its first four derivatives at $x = 1$ are

$$f(x) = \frac{1}{x} \qquad f(1) = 1$$

$$f'(x) = -\frac{1}{x^2} \qquad f'(1) = -1$$

$$f''(x) = \frac{2}{x^3} \qquad f''(1) = 2!$$

$$f'''(x) = -\frac{3 \cdot 2}{x^4} \qquad f'''(1) = -3!$$

$$f^{(4)}(x) = \frac{4 \cdot 3 \cdot 2}{x^5} \qquad f^{(4)}(1) = 4!.$$

That is, we have $f^{(n)}(1) = (-1)^n n!$ for all $n \geq 0$. Therefore, the Taylor series for f at $x = 1$ is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^n \ .$$

To find the interval of convergence, we use the ratio test. We find that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|(-1)^{n+1}(x-1)n^{+1}|}{|(-1)^n (x-1)^n|} = |x-1| \ .$$

Thus, the series converges if $|x-1| < 1$. That is, the series converges for $0 < x < 2$. Next, we need to check the endpoints. At $x = 2$, we see that

$$\sum_{n=0}^{\infty} (-1)^n (2-1)^n = \sum_{n=0}^{\infty} (-1)^n$$

diverges by the divergence test. Similarly, at $x = 0$,

$$\sum_{n=0}^{\infty} (-1)^n (0-1)^n = \sum_{n=0}^{\infty} (-1)^{2n} = \sum_{n=0}^{\infty} 1$$

diverges. Therefore, the interval of convergence is $(0, 2)$.

Exercise 7.4.5

Find the Taylor series for $f(x) = \dfrac{1}{2x}$ at $x = 2$ and determine its interval of convergence.

Hint

$f^{(n)}(2) = \dfrac{(-1)^n n!}{2^{n+2}}$

Answer

$\dfrac{1}{4}\displaystyle\sum_{n=0}^{\infty}\left(\frac{2-x}{2}\right)^n$. The interval of convergence is $(0, 4)$.

We know that the Taylor series found in this example converges on the interval $(0, 2)$, but how do we know it actually converges to f? We consider this question in more generality in a moment, but for this example, we can answer this question by writing

$$f(x) = \frac{1}{x} = \frac{1}{1-(1-x)}.$$

That is, f can be represented by the geometric series $\displaystyle\sum_{n=0}^{\infty}(1-x)^n$. Since this is a geometric series, it converges to $\dfrac{1}{x}$ as long as $|1-x| < 1$. Therefore, the Taylor series found in Example 7.4.5 does converge to $f(x) = \dfrac{1}{x}$ on $(0, 2)$.

We now consider the more general question: if a Taylor series for a function f converges on some interval, how can we determine if it actually converges to f? To answer this question, recall that a series converges to a particular value if and only if its sequence of partial sums converges to that value. Given a Taylor series for f at a, the nth partial sum is given by the nth-degree Taylor polynomial p_n. Therefore, to determine if the Taylor series converges to f, we need to determine whether

$$\lim_{n\to\infty} p_n(x) = f(x).$$

Since the remainder $R_n(x) = f(x) - p_n(x)$, the Taylor series converges to f if and only if

$$\lim_{n\to\infty} R_n(x) = 0.$$

We now state this theorem formally.

Theorem 7.4.3: Convergence of Taylor Series

Suppose that f has derivatives of all orders on an interval I containing a. Then the Taylor series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$$

converges to $f(x)$ for all x in I if and only if

$$\lim_{n\to\infty} R_n(x) = 0$$

for all x in I.

With this theorem, we can prove that a Taylor series for f at a converges to f if we can prove that the remainder $R_n(x) \to 0$. To prove that $R_n(x) \to 0$, we typically use the bound

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

from Taylor's theorem with remainder.

In the next example, we find the Maclaurin series for e^x and $\sin x$ and show that these series converge to the corresponding functions for all real numbers by proving that the remainders $R_n(x) \to 0$ for all real numbers x.

Example 7.4.6: Finding Maclaurin Series

For each of the following functions, find the Maclaurin series and its interval of convergence. Use Theorem 7.4.2 to prove that the Maclaurin series for f converges to f on that interval.

a. e^x
b. $\sin x$

Solution

a. Using the nth-degree Maclaurin polynomial for e^x found in Example 7.4.2*a*, we find that the Maclaurin series for e^x is given by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

To determine the interval of convergence, we use the ratio test. Since

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{n+1}}{(n+1)!} \cdot \frac{n!}{|x|^n} = \frac{|x|}{n+1},$$

we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0$$

for all x. Therefore, the series converges absolutely for all x, and thus, the interval of convergence is $(-\infty, \infty)$. To show that the series converges to e^x for all x, we use the fact that $f^{(n)}(x) = e^x$ for all $n \geq 0$ and e^x is an increasing function on $(-\infty, \infty)$. Therefore, for any real number b, the maximum value of e^x for all $|x| \leq b$ is e^b. Thus,

$$|R_n(x)| \leq \frac{e^b}{(n+1)!}|x|^{n+1}.$$

Since we just showed that

$$\sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

converges for all x, by the divergence test, we know that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$$

for any real number x. By combining this fact with the squeeze theorem, the result is $\lim_{n\to\infty} R_n(x) = 0$.

b. Using the nth-degree Maclaurin polynomial for $\sin x$ found in Example 7.4.2*b*, we find that the Maclaurin series for $\sin x$ is given by

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

In order to apply the ratio test, consider

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{|x|^{2n+1}} = \frac{|x|^2}{(2n+3)(2n+2)}.$$

Since

$$\lim_{n\to\infty}\frac{|x|^2}{(2n+3)(2n+2)}=0$$

for all x, we obtain the interval of convergence as $(-\infty,\infty)$. To show that the Maclaurin series converges to $\sin x$, look at $R_n(x)$. For each x there exists a real number c between 0 and x such that

$$R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Since $\left|f^{(n+1)}(c)\right|\leq 1$ for all integers n and all real numbers c, we have

$$|R_n(x)|\leq\frac{|x|^{n+1}}{(n+1)!}$$

for all real numbers x. Using the same idea as in part a., the result is $\lim_{n\to\infty}R_n(x)=0$ for all x, and therefore, the Maclaurin series for $\sin x$ converges to $\sin x$ for all real x.

Exercise 7.4.6

Find the Maclaurin series for $f(x)=\cos x$. Use the ratio test to show that the interval of convergence is $(-\infty,\infty)$. Show that the Maclaurin series converges to $\cos x$ for all real numbers x.

Hint

Use the Maclaurin polynomials for $\cos x$.

Answer

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

By the ratio test, the interval of convergence is $(-\infty,\infty)$. Since $|R_n(x)|\leq\frac{|x|^{n+1}}{(n+1)!}$, the series converges to $\cos x$ for all real x.

Project: Proving that e is Irrational

In this project, we use the Maclaurin polynomials for e^x to prove that e is irrational. The proof relies on supposing that e is rational and arriving at a contradiction. Therefore, in the following steps, we suppose $e=r/s$ for some integers r and s where $s\neq 0$.

1. Write the Maclaurin polynomials $p_0(x),p_1(x),p_2(x),p_3(x),p_4(x)$for e^x. Evaluate $p_0(1),p_1(1),p_2(1),p_3(1),p_4(1)$to estimate e.
2. Let $R_n(x)$ denote the remainder when using $p_n(x)$ to estimate e^x. Therefore, $R_n(x)=e^x-p_n(x)$, and $R_n(1)=e-p_n(1)$. Assuming that $e=\frac{r}{s}$ for integers r and s, evaluate $R_0(1),R_1(1),R_2(1),R_3(1),R_4(1)$.
3. Using the results from part 2, show that for each remainder $R_0(1),R_1(1),R_2(1),R_3(1),R_4(1)$, we can find an integer k such that $kR_n(1)$ is an integer for $n=0,1,2,3,4$.
4. Write down the formula for the nth-degree Maclaurin polynomial $p_n(x)$ for e^x and the corresponding remainder $R_n(x)$. Show that $sn!R_n(1)$ is an integer.
5. Use Taylor's theorem to write down an explicit formula for $R_n(1)$. Conclude that $R_n(1)\neq 0$, and therefore, $sn!R_n(1)\neq 0$.
6. Use Taylor's theorem to find an estimate on $R_n(1)$. Use this estimate combined with the result from part 5 to show that $|sn!R_n(1)|<\frac{se}{n+1}$. Conclude that if n is large enough, then $|sn!R_n(1)|<1$. Therefore, $sn!R_n(1)$ is an integer with magnitude less than 1. Thus, $sn!R_n(1)=0$. But from part 5, we know that $sn!R_n(1)\neq 0$. We have arrived at a contradiction, and consequently, the original supposition that e is rational must be false.

Key Concepts

- Taylor polynomials are used to approximate functions near a value $x = a$. Maclaurin polynomials are Taylor polynomials at $x = 0$.
- The nth-degree Taylor polynomials for a function f are the partial sums of the Taylor series for f.
- If a function f has a power series representation at $x = a$, then it is given by its Taylor series at $x = a$.
- A Taylor series for f converges to f if and only if $\lim_{n\to\infty} R_n(x) = 0$ where $R_n(x) = f(x) - p_n(x)$.
- The Taylor series for e^x, $\sin x$, and $\cos x$ converge to the respective functions for all real x.

Key Equations

- **Taylor series for the function f at the point $x = a$**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Glossary

Maclaurin polynomial

a Taylor polynomial centered at 0; the nth-degree Taylor polynomial for f at 0 is the nth-degree Maclaurin polynomial for f

Maclaurin series

a Taylor series for a function f at $x = 0$ is known as a Maclaurin series for f

Taylor polynomials

the nth-degree Taylor polynomial for f at $x = a$ is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor series

a power series at a that converges to a function f on some open interval containing a.

Taylor's theorem with remainder

for a function f and the nth-degree Taylor polynomial for f at $x = a$, the remainder $R_n(x) = f(x) - p_n(x)$ satisfies $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$

for some$c$ between x and a; if there exists an interval I containing a and a real number M such that $\left|f^{(n+1)}(x)\right| \leq M$ for all x in I, then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$

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7.4E: Exercises for Taylor Polynomials and Taylor Series

Taylor Polynomials

In exercises 1 - 8, find the Taylor polynomials of degree two approximating the given function centered at the given point.

1) *[Math Processing Error]* at *[Math Processing Error]*

2) *[Math Processing Error]* at *[Math Processing Error]*

Answer:

[Math Processing Error]

3) *[Math Processing Error]* at *[Math Processing Error]*

4) *[Math Processing Error]* at *[Math Processing Error]*

Answer:

[Math Processing Error]

5) *[Math Processing Error]* at *[Math Processing Error]*

6) *[Math Processing Error]* at *[Math Processing Error]*

Answer:

[Math Processing Error]

7) *[Math Processing Error]* at *[Math Processing Error]*

8) *[Math Processing Error]* at *[Math Processing Error]*

Answer:

[Math Processing Error]

Taylor Remainder Theorem

In exercises 9 - 14, verify that the given choice of *[Math Processing Error]* in the remainder estimate *[Math Processing Error]*, where *[Math Processing Error]* is the maximum value of *[Math Processing Error]* on the interval between *[Math Processing Error]* and the indicated point, yields *[Math Processing Error]*. Find the value of the Taylor polynomial *[Math Processing Error]* of *[Math Processing Error]* at the indicated point.

9) [T] *[Math Processing Error]*

10) [T] *[Math Processing Error]*

Answer:

[Math Processing Error] when *[Math Processing Error]* so the remainder estimate applies to the linear approximation *[Math Processing Error]*, which gives *[Math Processing Error]*, while *[Math Processing Error]*

11) [T] *[Math Processing Error]*

12) [T] *[Math Processing Error]*

Answer:

Using the estimate *[Math Processing Error]* we can use the Taylor expansion of order 9 to estimate *[Math Processing Error]* at *[Math Processing Error]*. as *[Math Processing Error]*… whereas *[Math Processing Error]*

13) [T] *[Math Processing Error]*

14) [T] *[Math Processing Error]*

Answer:

Since *[Math Processing Error]*. One has *[Math Processing Error]* whereas *[Math Processing Error]*

Approximating Definite Integrals Using Taylor Series

15) Integrate the approximation *[Math Processing Error]* evaluated at *[Math Processing Error]*t to approximate *[Math Processing Error]*.

16) Integrate the approximation *[Math Processing Error]* evaluated at *[Math Processing Error]* to approximate *[Math Processing Error]*

Answer:

[Math Processing Error] whereas *[Math Processing Error]*

More Taylor Remainder Theorem Problems

In exercises 17 - 20, find the smallest value of *[Math Processing Error]* such that the remainder estimate *[Math Processing Error]*, where *[Math Processing Error]* is the maximum value of *[Math Processing Error]* on the interval between *[Math Processing Error]* and the indicated point, yields *[Math Processing Error]* on the indicated interval.

17) *[Math Processing Error]* on *[Math Processing Error]*

18) *[Math Processing Error]* on *[Math Processing Error]*

Answer:

Since *[Math Processing Error]* is *[Math Processing Error]* or *[Math Processing Error]*, we have *[Math Processing Error]*. Since *[Math Processing Error]*, we seek the smallest *[Math Processing Error]* such that *[Math Processing Error]*. The smallest such value is *[Math Processing Error]*. The remainder estimate is *[Math Processing Error]*

19) *[Math Processing Error]* on *[Math Processing Error]*

20) *[Math Processing Error]* on *[Math Processing Error]*

Answer:

Since *[Math Processing Error]* one has *[Math Processing Error]*. Since *[Math Processing Error]*, one seeks the smallest *[Math Processing Error]* such that *[Math Processing Error]*. The smallest such value is *[Math Processing Error]*. The remainder estimate is *[Math Processing Error]*

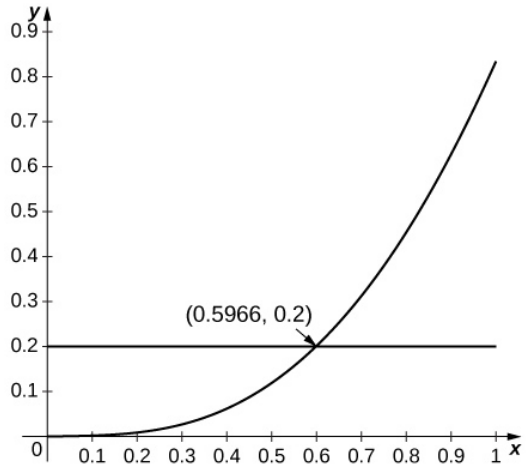
In exercises 21 - 24, the maximum of the right-hand side of the remainder estimate *[Math Processing Error]* on *[Math Processing Error]* occurs at *[Math Processing Error]* or *[Math Processing Error]*. Estimate the maximum value of *[Math Processing Error]* such that *[Math Processing Error]* on *[Math Processing Error]* by plotting this maximum as a function of *[Math Processing Error]*.

21) [T] *[Math Processing Error]* approximated by *[Math Processing Error]*

22) [T] *[Math Processing Error]* approximated by *[Math Processing Error]*

Answer:

Since *[Math Processing Error]* is increasing for small *[Math Processing Error]* and since *[Math Processing Error]*, the estimate applies whenever *[Math Processing Error]*, which applies up to *[Math Processing Error]*

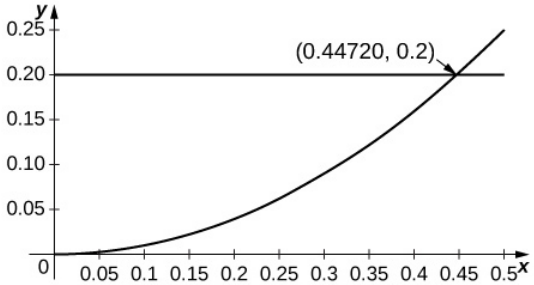


23) [T] *[Math Processing Error]* approximated by *[Math Processing Error]*

24) [T] *[Math Processing Error]* approximated by *[Math Processing Error]*

Answer:

Since the second derivative of *[Math Processing Error]* is *[Math Processing Error]* and since *[Math Processing Error]* is decreasing away from *[Math Processing Error]*, the estimate applies when *[Math Processing Error]* or *[Math Processing Error]*.



Taylor Series

In exercises 25 - 35, find the Taylor series of the given function centered at the indicated point.

25) *[Math Processing Error]* at *[Math Processing Error]*

26) *[Math Processing Error]* at *[Math Processing Error]*

Answer:

[Math Processing Error]

27) *[Math Processing Error]* at *[Math Processing Error]*

28) *[Math Processing Error]* at *[Math Processing Error]*

Answer:

Values of derivatives are the same as for *[Math Processing Error]* so *[Math Processing Error]*

29) *[Math Processing Error]* at *[Math Processing Error]*

30) *[Math Processing Error]* at *[Math Processing Error]*

Answer:

[Math Processing Error] so *[Math Processing Error]*, which is also *[Math Processing Error]*.

31) *[Math Processing Error]* at *[Math Processing Error]*

32) *[Math Processing Error]* at *[Math Processing Error]*

Answer:

The derivatives are *[Math Processing Error]* so *[Math Processing Error]*

33) *[Math Processing Error]* at *[Math Processing Error]* (Hint: Differentiate the Taylor Series for*[Math Processing Error]*.)

34) *[Math Processing Error]* at *[Math Processing Error]*

Answer:

[Math Processing Error]

35) *[Math Processing Error]* at a=0 (Note: *[Math Processing Error]* is the Taylor series of *[Math Processing Error]*

In exercises 36 - 44, compute the Taylor series of each function around *[Math Processing Error]*.

36) *[Math Processing Error]*

Answer:

[Math Processing Error]

37) *[Math Processing Error]*

38) *[Math Processing Error]*

Answer:

[Math Processing Error]

39) *[Math Processing Error]*

40) *[Math Processing Error]*

Answer:

[Math Processing Error]

41) *[Math Processing Error]*

42) *[Math Processing Error]*

Answer:

[Math Processing Error]

43) *[Math Processing Error]*

44) *[Math Processing Error]*

Answer:

[Math Processing Error]

Maclaurin Series

[T] In exercises 45 - 48, identify the value of *[Math Processing Error]* such that the given series *[Math Processing Error]* is the value of the Maclaurin series of *[Math Processing Error]* at *[Math Processing Error]*. Approximate the value of *[Math Processing Error]* using *[Math Processing Error]*.

45) *[Math Processing Error]*

46) *[Math Processing Error]*

Answer:

[Math Processing Error]

47) *[Math Processing Error]*

48) *[Math Processing Error]*

Answer:

[Math Processing Error]

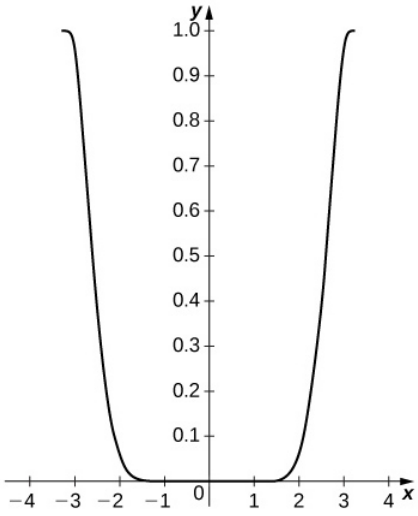
In exercises 49 - 52 use the functions *[Math Processing Error]* **and** *[Math Processing Error]* **on** *[Math Processing Error]***.**

49) [T] Plot *[Math Processing Error]* on *[Math Processing Error]*. Compare the maximum difference with the square of the Taylor remainder estimate for *[Math Processing Error]*

50) [T] Plot *[Math Processing Error]* on *[Math Processing Error]*. Compare the maximum difference with the square of the Taylor remainder estimate for *[Math Processing Error]*.

Answer:

The difference is small on the interior of the interval but approaches *[Math Processing Error]* near the endpoints. The remainder estimate is *[Math Processing Error]*

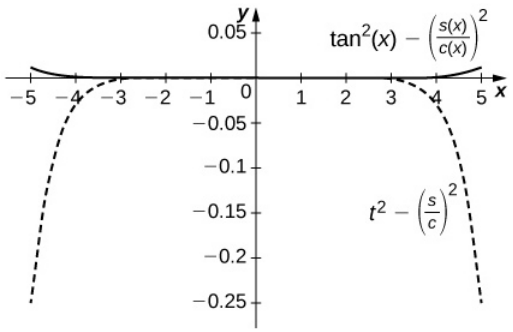


51) [T] Plot *[Math Processing Error]* on *[Math Processing Error]*.

52) [T] Compare *[Math Processing Error]* on *[Math Processing Error]* to *[Math Processing Error]*. Compare this with the Taylor remainder estimate for the approximation of *[Math Processing Error]* by *[Math Processing Error]*.

Answer:

The difference is on the order of *[Math Processing Error]* on *[Math Processing Error]* while the Taylor approximation error is around *[Math Processing Error]* near *[Math Processing Error]*. The top curve is a plot of *[Math Processing Error]* and the lower dashed plot shows *[Math Processing Error]*.



53) [T] Plot *[Math Processing Error]* where *[Math Processing Error]* on *[Math Processing Error]*. Compare the maximum error with the Taylor remainder estimate.

54) (Taylor approximations and root finding.) Recall that Newton's method *[Math Processing Error]* approximates solutions of *[Math Processing Error]* near the input *[Math Processing Error]*.

a. If *[Math Processing Error]* and *[Math Processing Error]* are inverse functions, explain why a solution of *[Math Processing Error]* is the value *[Math Processing Error]* of *[Math Processing Error]*.

b. Let *[Math Processing Error]* be the *[Math Processing Error]* degree Maclaurin polynomial of *[Math Processing Error]*. Use Newton's method to approximate solutions of *[Math Processing Error]* for *[Math Processing Error]*

c. Explain why the approximate roots of *[Math Processing Error]* are approximate values of *[Math Processing Error]*

Answer:

a. Answers will vary.
b. The following are the *[Math Processing Error]* values after *[Math Processing Error]* iterations of Newton's method to approximation a root of *[Math Processing Error]*: for *[Math Processing Error]* for *[Math Processing Error]* for *[Math Processing Error]* (Note: *[Math Processing Error]*)
c. Answers will vary.

Evaluating Limits using Taylor Series

In exercises 55 - 58, use the fact that if *[Math Processing Error]* converges in an interval containing *[Math Processing Error]*, then *[Math Processing Error]* to evaluate each limit using Taylor series.

55) *[Math Processing Error]*

56) *[Math Processing Error]*

Answer:

[Math Processing Error]

57) *[Math Processing Error]*

58) *[Math Processing Error]*

Answer:

[Math Processing Error]

Contributors

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7.5: Working with Taylor Series

Learning Objectives

- Write the terms of the binomial series.
- Recognize the Taylor series expansions of common functions.
- Recognize and apply techniques to find the Taylor series for a function.
- Use Taylor series to solve differential equations.
- Use Taylor series to evaluate non-elementary integrals.

In the preceding section, we defined Taylor series and showed how to find the Taylor series for several common functions by explicitly calculating the coefficients of the Taylor polynomials. In this section we show how to use those Taylor series to derive Taylor series for other functions. We then present two common applications of power series. First, we show how power series can be used to solve differential equations. Second, we show how power series can be used to evaluate integrals when the antiderivative of the integrand cannot be expressed in terms of elementary functions. In one example, we consider $\int e^{-x^2}\,dx$, an integral that arises frequently in probability theory.

The Binomial Series

Our first goal in this section is to determine the Maclaurin series for the function $f(x)=(1+x)^r$ for all real numbers r. The Maclaurin series for this function is known as the **binomial series.** We begin by considering the simplest case: r is a nonnegative integer. We recall that, for $r=0,\ 1,\ 2,\ 3,\ 4,\ f(x)=(1+x)^r$ can be written as

$$\begin{aligned} f(x) &= (1+x)^0 = 1, \\ f(x) &= (1+x)^1 = 1+x, \\ f(x) &= (1+x)^2 = 1+2x+x^2, \\ f(x) &= (1+x)^3 = 1+3x+3x^2+x^3 \\ f(x) &= (1+x)^4 = 1+4x+6x^2+4x^3+x^4. \end{aligned}$$

The expressions on the right-hand side are known as binomial expansions and the coefficients are known as binomial coefficients. More generally, for any nonnegative integer r, the binomial coefficient of x^n in the binomial expansion of $(1+x)^r$ is given by

$$\binom{r}{n}=\frac{r!}{n!(r-n)!} \tag{7.5.1}$$

and

$$\begin{aligned} f(x) &= (1+x)^r \\ &= \binom{r}{0}+\binom{r}{1}x+\binom{r}{2}x^2+\binom{r}{3}x^3+\cdots+\binom{r}{r-1}x^{r-1}+\binom{r}{r}x^r \\ &= \sum_{n=0}^{r}\binom{r}{n}x^n. \end{aligned} \tag{7.5.2}$$

For example, using this formula for $r=5$, we see that

$$\begin{aligned} f(x) &= (1+x)^5 \\ &= \binom{5}{0}1+\binom{5}{1}x+\binom{5}{2}x^2+\binom{5}{3}x^3+\binom{5}{4}x^4+\binom{5}{5}x^5 \\ &= \frac{5!}{0!5!}1+\frac{5!}{1!4!}x+\frac{5!}{2!3!}x^2+\frac{5!}{3!2!}x^3+\frac{5!}{4!1!}x^4+\frac{5!}{5!0!}x^5 \\ &= 1+5x+10x^2+10x^3+5x^4+x^5. \end{aligned}$$

We now consider the case when the exponent r is any real number, not necessarily a nonnegative integer. If r is not a nonnegative integer, then $f(x)=(1+x)^r$ cannot be written as a finite polynomial. However, we can find a power series for f. Specifically, we look for the Maclaurin series for f. To do this, we find the derivatives of f and evaluate them at $x=0$.

$$\begin{aligned} f(x) &= (1+x)^r & f(0) &= 1 \\ f'(x) &= r(1+x)^{r-1} & f'(0) &= r \\ f''(x) &= r(r-1)(1+x)^{r-2} & f''(0) &= r(r-1) \\ f'''(x) &= r(r-1)(r-2)(1+x)^{r-3} & f'''(0) &= r(r-1)(r-2) \\ f^{(n)}(x) &= r(r-1)(r-2)\cdots(r-n+1)(1+x)^{r-n} & f^{(n)}(0) &= r(r-1)(r-2)\cdots(r-n+1) \end{aligned}$$

We conclude that the coefficients in the binomial series are given by

$$\frac{f^{(n)}(0)}{n!}=\frac{r(r-1)(r-2)\cdots(r-n+1)}{n!}. \tag{7.5.3}$$

We note that if r is a nonnegative integer, then the $(r+1)^{\text{st}}$ derivative $f^{(r+1)}$ is the zero function, and the series terminates. In addition, if r is a nonnegative integer, then Equation 7.5.3 for the coefficients agrees with Equation 7.5.1 for the coefficients, and the formula for the binomial series agrees with Equation 7.5.2 for the finite binomial expansion. More generally, to denote the binomial coefficients for any real number r, we define

$$\binom{r}{n}=\frac{(r-1)(r-2)\cdots(r-n+1)}{n!}.$$

With this notation, we can write the binomial series for $(1+x)^r$ as

$$\sum_{n=0}^{\infty}\binom{r}{n}x^n=1+rx+\frac{r(r-1)}{2!}x^2+\cdots+\frac{r(r-1)\cdots(r-n+1)}{n!}x^n+\cdots. \tag{7.5.4}$$

We now need to determine the interval of convergence for the binomial series Equation 7.5.4. We apply the ratio test. Consequently, we consider

$$\begin{aligned} \frac{|a_{n+1}|}{|a_n|} &= \frac{|r(r-1)(r-2)\cdots(r-n)||x||^{n+1}}{(n+1)!}\cdot\frac{n!}{|r(r-1)(r-2)\cdots(r-n+1)||x|^n} \\ &= \frac{|r-n||x|}{|n+1|} \end{aligned}$$

.

Since

$$\lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}=|x|<1$$

if and only if $|x|<1$, we conclude that the interval of convergence for the binomial series is $(-1,1)$. The behavior at the endpoints depends on r. It can be shown that for $r\geq 0$ the series converges at both endpoints; for $-1<r<0$, the series converges at $x=1$ and diverges at $x=-1$; and for $r<-1$, the series diverges at both endpoints. The binomial series does converge to $(1+x)^r$ in $(-1,1)$ for all real numbers r, but proving this fact by showing that the remainder $R_n(x)\to 0$ is difficult.

Definition: Binomial Series

For any real number r, the Maclaurin series for $f(x)=(1+x)^r$ is the binomial series. It converges to f for $|x|<1$, and we write

$$(1+x)^r=\sum_{n=0}^{\infty}\binom{r}{n}x^n=1+rx+\frac{r(r-1)}{2!}x^2+\cdots+r\frac{(r-1)\cdots(r-n+1)}{n!}x^n+\cdots$$

for $|x|<1$.

We can use this definition to find the binomial series for $f(x)=\sqrt{1+x}$ and use the series to approximate $\sqrt{1.5}$.

Example 7.5.1: Finding Binomial Series

a. Find the binomial series for $f(x)=\sqrt{1+x}$.
b. Use the third-order Maclaurin polynomial $p_3(x)$ to estimate $\sqrt{1.5}$. Use Taylor's theorem to bound the error. Use a graphing utility to compare the graphs of f and p_3.

Solution

a. Here $r=\dfrac{1}{2}$. Using the definition for the binomial series, we obtain

$$\begin{aligned}\sqrt{1+x} &= 1+\frac{1}{2}x+\frac{(1/2)(-1/2)}{2!}x^2+\frac{(1/2)(-1/2)(-3/2)}{3!}x^3+\cdots \\ &= 1+\frac{1}{2}x-\frac{1}{2!}\frac{1}{2^2}x^2+\frac{1}{3!}\frac{1\cdot 3}{2^3}x^3-\cdots+\frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n+\cdots \\ &= 1+\frac{1}{2}x+\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n}x^n.\end{aligned}$$

b. From the result in part a. the third-order Maclaurin polynomial is

$$p_3(x)=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3 \quad .$$

Therefore,

$$\sqrt{1.5}=\sqrt{1+0.5}\approx 1+\frac{1}{2}(0.5)-\frac{1}{8}(0.5)^2+\frac{1}{16}(0.5)^3\approx 1.2266.$$

From Taylor's theorem, the error satisfies

$$R_3(0.5)=\frac{f^{(4)}(c)}{4!}(0.5)^4$$

for some c between 0 and 0.5. Since $f^{(4)}(x)=-\dfrac{15}{2^4(1+x)^{7/2}}$, and the maximum value of $\left|f^{(4)}(x)\right|$ on the interval $(0,0.5)$ occurs at $x=0$, we have

$$|R_3(0.5)|\le\frac{15}{4!2^4}(0.5)^4\approx 0.00244.$$

The function and the Maclaurin polynomial p_3 are graphed in Figure 7.5.1.

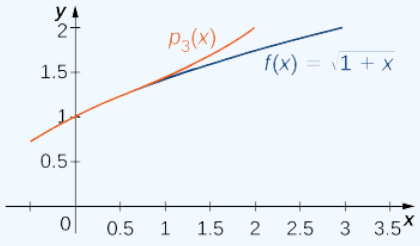


Figure 7.5.1: The third-order Maclaurin polynomial $p_3(x)$ provides a good approximation for $f(x)=\sqrt{1+x}$ for x near zero.

Exercise 7.5.1

Find the binomial series for $f(x)=\dfrac{1}{(1+x)^2}$.

Hint

Use the definition of binomial series for $r=-2$.

Answer

$$\sum_{n=0}^{\infty}(-1)^n(n+1)x^n$$

Common Functions Expressed as Taylor Series

At this point, we have derived Maclaurin series for exponential, trigonometric, and logarithmic functions, as well as functions of the form $f(x)=(1+x)^r$. In Table 7.5.1, we summarize the results of these series. We remark that the convergence of the Maclaurin series for $f(x)=\ln(1+x)$ at the endpoint $x=1$ and the Maclaurin series for $f(x)=\tan^{-1}x$ at the endpoints $x=1$ and $x=-1$ relies on a more advanced theorem than we present here. (Refer to Abel's theorem for a discussion of this more technical point.)

Table 7.5.1: Maclaurin Series for Common Functions

Function	Maclaurin Series	Interval of Convergence
$f(x)=\dfrac{1}{1-x}$	$\displaystyle\sum_{n=0}^{\infty}x^n$	$-1<x<1$
$f(x)=e^x$	$\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$	$-\infty<x<\infty$
$f(x)=\sin x$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$	$-\infty<x<\infty$
$f(x)=\cos x$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$	$-\infty<x<\infty$
$f(x)=\ln(1+x)$	$\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n}$	$-1<x\le 1$
$f(x)=\tan^{-1}x$	$\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$	$-1\le x\le 1$
$f(x)=(1+x)^r$	$\displaystyle\sum_{n=0}^{\infty}\binom{r}{n}x^n$	$-1<x<1$ Convergence at the endpoints depends on the value of r

Earlier in the chapter, we showed how you could combine power series to create new power series. Here we use these properties, combined with the Maclaurin series in Table 7.5.1, to create Maclaurin series for other functions.

Example 7.5.2: Deriving Maclaurin Series from Known Series

Find the Maclaurin series of each of the following functions by using one of the series listed in Table 7.5.1.

a. $f(x)=\cos\sqrt{x}$
b. $f(x)=\sinh x$

Solution

a. Using the Maclaurin series for $\cos x$ we find that the Maclaurin series for $\cos\sqrt{x}$ is given by

$$\sum_{n=0}^{\infty}\frac{(-1)^n(\sqrt{x})^{2n}}{(2n)!}=\sum_{n=0}^{\infty}\frac{(-1)^nx^n}{(2n)!}=1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\frac{x^4}{8!}-\cdots.$$

This series converges to $\cos\sqrt{x}$ for all x in the domain of $\cos\sqrt{x}$; that is, for all $x\ge 0$.

b. To find the Maclaurin series for $\sinh x$, we use the fact that

$$\sinh x=\frac{e^x-e^{-x}}{2}.$$

Using the Maclaurin series for e^x, we see that the nth term in the Maclaurin series for $\sinh x$ is given by

$$\frac{x^n}{n!} - \frac{(-x)^n}{n!}.$$

For n even, this term is zero. For n odd, this term is $\frac{2x^n}{n!}$. Therefore, the Maclaurin series for $\sinh x$ has only odd-order terms and is given by

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots.$$

? Exercise 7.5.2

Find the Maclaurin series for $\sin(x^2)$.

Hint

Use the Maclaurin series for $\sin x$.

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

We also showed previously in this chapter how power series can be differentiated term by term to create a new power series. In Example 7.5.3, we differentiate the binomial series for $\sqrt{1+x}$ term by term to find the binomial series for $\frac{1}{\sqrt{1+x}}$. Note that we could construct the binomial series for $\frac{1}{\sqrt{1+x}}$ directly from the definition, but differentiating the binomial series for $\sqrt{1+x}$ is an easier calculation.

✓ Example 7.5.3: Differentiating a Series to Find a New Series

Use the binomial series for $\sqrt{1+x}$ to find the binomial series for $\frac{1}{\sqrt{1+x}}$.

Solution

The two functions are related by

$$\frac{d}{dx}\sqrt{1+x} = \frac{1}{2\sqrt{1+x}},$$

so the binomial series for $\frac{1}{\sqrt{1+x}}$ is given by

$$\frac{1}{\sqrt{1+x}} = 2\frac{d}{dx}\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^n.$$

? Exercise 7.5.3

Find the binomial series for $f(x) = \frac{1}{(1+x)^{3/2}}$

Hint

Differentiate the series for $\frac{1}{\sqrt{1+x}}$

Answer

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2^n} x^n$$

In this example, we differentiated a known Taylor series to construct a Taylor series for another function. The ability to differentiate power series term by term makes them a powerful tool for solving differential equations. We now show how this is accomplished.

Solving Differential Equations with Power Series

Consider the differential equation

$$y'(x) = y.$$

Recall that this is a first-order separable equation and its solution is $y = Ce^x$. This equation is easily solved using techniques discussed earlier in the text. For most differential equations, however, we do not yet have analytical tools to solve them. Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form $y = \sum_{n=0}^{\infty} c_n x^n$ and determine what the coefficients would need to be. In the next example, we consider an initial-value problem involving $y' = y$ to illustrate the technique.

Example 7.5.4: Power Series Solution of a Differential Equation

Use power series to solve the initial-value problem $y' = y, \quad y(0) = 3.$

Solution

Suppose that there exists a power series solution

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots.$$

Differentiating this series term by term, we obtain

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots.$$

If y satisfies the differential equation, then

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = c_1 + 2c_2 x + 3c_3 x^2 + 4c_3 x^3 + \cdots.$$

Using the uniqueness of power series representations, we know that these series can only be equal if their coefficients are equal. Therefore,

$$\begin{aligned} c_0 &= c_1, \\ c_1 &= 2c_2, \\ c_2 &= 3c_3, \\ c_3 &= 4c_4, \\ &\vdots \end{aligned}$$

Using the initial condition $y(0) = 3$ combined with the power series representation

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \ ,$$

we find that $c_0 = 3$. We are now ready to solve for the rest of the coefficients. Using the fact that $c_0 = 3$, we have

$$
\begin{aligned}
c_1 &= c_0 = 3 = \frac{3}{1!},\\
c_2 &= \frac{c_1}{2} = \frac{3}{2} = \frac{3}{2!},\\
c_3 &= \frac{c_2}{3} = \frac{3}{3\cdot 2} = \frac{3}{3!},\\
c_4 &= \frac{c_3}{4} = \frac{3}{4\cdot 3\cdot 2} = \frac{3}{4!}.
\end{aligned}
$$

Therefore,

$$
y = 3\left[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3\frac{1}{4!}x^4 + \cdots\right] = 3\sum_{n=0}^{\infty}\frac{x^n}{n!}.
$$

You might recognize

$$
\sum_{n=0}^{\infty}\frac{x^n}{n!}
$$

as the Taylor series for e^x. Therefore, the solution is $y = 3e^x$.

? Exercise 7.5.4

Use power series to solve $y' = 2y, \quad y(0) = 5.$

Hint

The equations for the first several coefficients c_n will satisfy $c_0 = 2c_1$, $c_1 = 2\cdot 2c_2$, $c_2 = 2\cdot 3c_3$, ... In general, for all $n \geq 0$, $c_n = 2(n+1)C_{n+1}$.

Answer

$y = 5e^{2x}$

We now consider an example involving a differential equation that we cannot solve using previously discussed methods. This differential equation

$$
y'' - xy = 0
$$

is known as **Airy's equation**. It has many applications in mathematical physics, such as modeling the diffraction of light. Here we show how to solve it using power series.

✓ Example 7.5.5: Power Series Solution of Airy's Equation

Use power series to solve $y'' - xy = 0$ with the initial conditions $y(0) = a$ and $y'(0) = b$.

Solution

We look for a solution of the form

$$
y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots
$$

Differentiating this function term by term, we obtain

$$
\begin{aligned}
y' &= c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots,\\
y'' &= 2\cdot 1c_2 + 3\cdot 2c_3 x + 4\cdot 3c_4 x^2 + \cdots.
\end{aligned}
$$

If y satisfies the equation $y'' = xy$, then

$$2\cdot 1c_2+3\cdot 2c_3x+4\cdot 3c_4x^2+\cdots=x(c_0+c_1x+c_2x^2+c_3x^3+\cdots).$$

Using the Uniqueness of Power Series Theorem (from an earlier Section) on the uniqueness of power series representations, we know that coefficients of the same degree must be equal. Therefore,

$$\begin{aligned}2\cdot 1c_2&=0,\\3\cdot 2c_3&=c_0,\\4\cdot 3c_4&=c_1,\\5\cdot 4c_5&=c_2,\\&\vdots\end{aligned}$$

More generally, for $n\geq 3$, we have $n\cdot(n-1)c_n=c_{n-3}$. In fact, all coefficients can be written in terms of c_0 and c_1. To see this, first note that $c_2=0$. Then

$$\begin{aligned}c_3&=\frac{c_0}{3\cdot 2},\\c_4&=\frac{c_1}{4\cdot 3}.\end{aligned}$$

For c_5, c_6, c_7, we see that

$$\begin{aligned}c_5&=\frac{c_2}{5\cdot 4}=0,\\c_6&=\frac{c_3}{6\cdot 5}=\frac{c_0}{6\cdot 5\cdot 3\cdot 2},\\c_7&=\frac{c_4}{7\cdot 6}=\frac{c_1}{7\cdot 6\cdot 4\cdot 3}.\end{aligned}$$

Therefore, the series solution of the differential equation is given by

$$y=c_0+c_1x+0\cdot x^2+\frac{c_0}{3\cdot 2}x^3+\frac{c_1}{4\cdot 3}x^4+0\cdot x^5+\frac{c_0}{6\cdot 5\cdot 3\cdot 2}x^6+\frac{c_1}{7\cdot 6\cdot 4\cdot 3}x^7+\cdots.$$

The initial condition $y(0)=a$ implies $c_0=a$. Differentiating this series term by term and using the fact that $y'(0)=b$, we conclude that $c_1=b$.

Therefore, the solution of this initial-value problem is

$$y=a\left(1+\frac{x^3}{3\cdot 2}+\frac{x^6}{6\cdot 5\cdot 3\cdot 2}+\cdots\right)+b\left(x+\frac{x^4}{4\cdot 3}+\frac{x^7}{7\cdot 6\cdot 4\cdot 3}+\cdots\right).$$

? Exercise 7.5.5

Use power series to solve $y''+x^2y=0$ with the initial condition $y(0)=a$ and $y'(0)=b$.

Hint

The coefficients satisfy $c_0=a$, $c_1=b$, $c_2=0$, $c_3=0$, and for $n\geq 4$, $n(n-1)c_n=-c_{n-4}$.

Answer

$$y=a\left(1-\frac{x^4}{3\cdot 4}+\frac{x^8}{3\cdot 4\cdot 7\cdot 8}-\cdots\right)+b\left(x-\frac{x^5}{4\cdot 5}+\frac{x^9}{4\cdot 5\cdot 8\cdot 9}-\cdots\right)$$

Evaluating Non-elementary Integrals

Solving differential equations is one common application of power series. We now turn to a second application. We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions.

One integral that arises often in applications in probability theory is $\int e^{-x^2}\,dx$. Unfortunately, the antiderivative of the integrand e^{-x^2} is not an elementary function. By elementary function, we mean a function that can be written using a finite number of algebraic combinations or compositions of exponential, logarithmic, trigonometric, or power functions. We remark that the term "elementary function" is not synonymous with noncomplicated function. For example, the function $f(x)=\sqrt{x^2-3x}+e^{x^3}-\sin(5x+4)$ is an elementary function, although not a particularly simple-looking function. Any integral of the form $\int f(x)\,dx$ where the antiderivative of f cannot be written as an elementary function is considered a **non-elementary integral.**

Non-elementary integrals cannot be evaluated using the basic integration techniques discussed earlier. One way to evaluate such integrals is by expressing the integrand as a power series and integrating term by term. We demonstrate this technique by considering $\int e^{-x^2}\,dx$.

Example 7.5.6: Using Taylor Series to Evaluate a Definite Integral

a. Express $\int e^{-x^2}\,dx$ as an infinite series.

b. Evaluate $\int_0^1 e^{-x^2}\,dx$ to within an error of 0.01.

Solution

a. The Maclaurin series for e^{-x^2} is given by

$$\begin{aligned} e^{-x^2} &= \sum_{n=0}^{\infty}\frac{(-x^2)^n}{n!} \\ &= 1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots+(-1)^n\frac{x^{2n}}{n!}+\cdots \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{n!}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int e^{-x^2}\,dx &= \int\left(1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots+(-1)^n\frac{x^{2n}}{n!}+\cdots\right)dx \\ &= C+x-\frac{x^3}{3}+\frac{x^5}{5.2!}-\frac{x^7}{7.3!}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+1)n!}+\cdots. \end{aligned}$$

b. Using the result from part a. we have

$$\int_0^1 e^{-x^2}\,dx = 1-\frac{1}{3}+\frac{1}{10}-\frac{1}{42}+\frac{1}{216}-\cdots.$$

The sum of the first four terms is approximately 0.74. By the alternating series test, this estimate is accurate to within an error of less than $\frac{1}{216}\approx 0.0046296<0.01$.

Exercise 7.5.6

Express $\int \cos\sqrt{x}\,dx$ as an infinite series. Evaluate $\int_0^1 \cos\sqrt{x}\,dx$ to within an error of 0.01.

Hint

Use the series found in Example $7.5.2$

Answer

$C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n(2n-2)!}$ The definite integral is approximately 0.764 to within an error of 0.01.

As mentioned above, the integral $\int e^{-x^2}\,dx$ arises often in probability theory. Specifically, it is used when studying data sets that are normally distributed, meaning the data values lie under a bell-shaped curve. For example, if a set of data values is normally distributed with mean μ and standard deviation σ, then the probability that a randomly chosen value lies between $x = a$ and $x = b$ is given by

$$\frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{-(x-\mu)^2/(2\sigma^2)}\,dx. \tag{7.5.5}$$

(See Figure 7.5.2.)

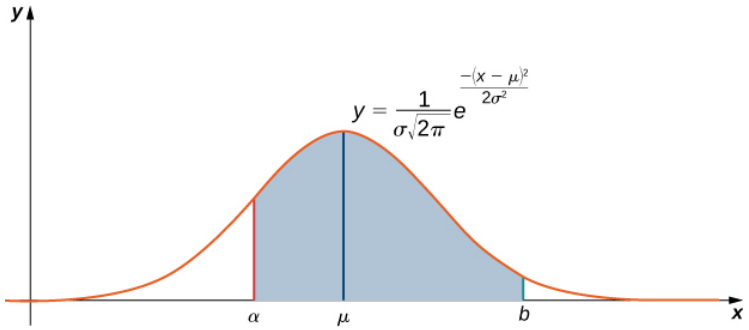


Figure 7.5.2: If data values are normally distributed with mean μ and standard deviation σ, the probability that a randomly selected data value is between a and b is the area under the curve $y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$ between $x = a$ and $x = b$.

To simplify this integral, we typically let $z = \frac{x-\mu}{\sigma}$. This quantity z is known as the z score of a data value. With this simplification, integral Equation 7.5.5 becomes

$$\frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} e^{-z^2/2}\,dz.$$

In Example 7.5.7, we show how we can use this integral in calculating probabilities.

Example 7.5.7: Using Maclaurin Series to Approximate a Probability

Suppose a set of standardized test scores are normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 50$. Use Equation 7.5.5 and the first six terms in the Maclaurin series for $e^{-x^2/2}$ to approximate the probability that a randomly selected test score is between $x = 100$ and $x = 200$. Use the alternating series test to determine how accurate your approximation is.

Solution

Since $\mu = 100, \sigma = 50$, and we are trying to determine the area under the curve from $a = 100$ to $b = 200$, integral Equation 7.5.5 becomes

$$\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2}\,dz. \tag{7.5.6}$$

The Maclaurin series for $e^{-x^2/2}$ is given by

$$
\begin{aligned}
e^{-x^2/2} &= \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^n}{n!} \\
&= 1 - \frac{x^2}{2^1 \cdot 1!} + \frac{x^4}{2^2 \cdot 2!} - \frac{x^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n}}{2^n \cdot n}! + \cdots \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2^n \cdot n!}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int e^{-z^2/2}\, dz &= \frac{1}{\sqrt{2\pi}} \int \left(1 - \frac{z^2}{2^1 \cdot 1!} + \frac{z^4}{2^2 \cdot 2!} - \frac{z^6}{2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n}}{2^n \cdot n!} + \cdots\right) dz \\
&= \frac{1}{\sqrt{2\pi}} \left(C + z - \frac{z^3}{3 \cdot 2^1 \cdot 1!} + \frac{z^5}{5 \cdot 2^2 \cdot 2!} - \frac{z^7}{7 \cdot 2^3 \cdot 3!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1) 2^n \cdot n!} + \cdots\right)
\end{aligned}
$$

We now use this result to evaluate the definite integral from Equation 7.5.6:

$$
\frac{1}{\sqrt{2\pi}} \int_0^2 e^{-z^2/2}\, dz = \frac{1}{\sqrt{2\pi}} \left(2 - \frac{8}{6} + \frac{32}{40} - \frac{128}{336} + \frac{512}{3456} - \frac{2^{11}}{11 \cdot 2^5 \cdot 5!} + \cdots\right)
$$

Using the first five terms, we estimate that the probability is approximately **0.4729.** By the alternating series test, we see that this estimate is accurate to within

$$
\frac{1}{\sqrt{2\pi}} \frac{2^{13}}{13 \cdot 2^6 \cdot 6!} \approx 0.00546.
$$

Analysis

If you are familiar with probability theory, you may know that the probability that a data value is within two standard deviations of the mean is approximately 95%. Here we calculated the probability that a data value is between the mean and two standard deviations above the mean, so the estimate should be around 47.5%. The estimate, combined with the bound on the accuracy, falls within this range.

Alternative method. If term-by-term integration is done instead, the result of evaluating the definite integral is

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \int_0^2 z^{2n}\, dz &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) 2^n n!} \left[2^{2n+1} - 0^{2n+1}\right] = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{n!(2n+1)} \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{2}{1 \cdot 0!} - \frac{4}{3 \cdot 1!} + \frac{8}{5 \cdot 2!} - \frac{16}{7 \cdot 3!} + \frac{32}{9 \cdot 4!} - \frac{64}{11 \cdot 5!} + \cdots\right) \\
&= \frac{1}{\sqrt{2\pi}} \left(2 - \frac{4}{3} + \frac{4}{5} - \frac{8}{21} + \frac{4}{27} - \frac{8}{165} + \cdots\right)
\end{aligned}
$$

? Exercise 7.5.7

Given a set of normally distributed standardized test scores with mean $\mu = 100$ and standard deviation $\sigma = 50$, use the first five terms of the Maclaurin series for $e^{-x^2/2}$ to estimate the probability that a randomly selected test score is between 100 and 150. Use the alternating series test to determine the accuracy of this estimate.

Hint

Evaluate $\int_0^1 e^{-z^2/2}\, dz$ using the first five terms of the Maclaurin series for $e^{-z^2/2}$.

Answer

The estimate is approximately 0.3414.This estimate is accurate to within 0.0000094.

Another application in which a non-elementary integral arises involves the period of a pendulum. The integral is

$$\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

.

An integral of this form is known as an **elliptic integral** of the first kind. Elliptic integrals originally arose when trying to calculate the arc length of an ellipse. We now show how to use power series to approximate this integral.

Example 7.5.8: Period of a Pendulum

The period of a pendulum is the time it takes for a pendulum to make one complete back-and-forth swing. For a pendulum with length L that makes a maximum angle θ_{max} with the vertical, its period T is given by

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

where g is the acceleration due to gravity and $k = \sin\left(\frac{\theta_{max}}{2}\right)$ (see Figure 7.5.3). (We note that this formula for the period arises from a non-linearized model of a pendulum. In some cases, for simplification, a linearized model is used and $\sin\theta$ is approximated by θ.)

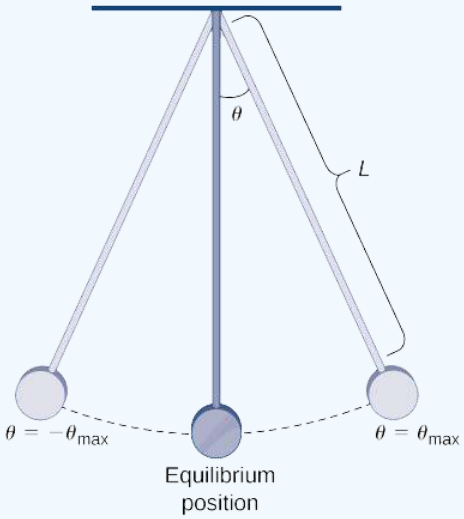


Figure 7.5.3: This pendulum has length L and makes a maximum angle θ_{max} with the vertical.

Use the binomial series

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}x^n$$

to estimate the period of this pendulum. Specifically, approximate the period of the pendulum if

a. you use only the first term in the binomial series, and
b. you use the first two terms in the binomial series.

Solution

We use the binomial series, replacing x with $-k^2\sin^2\theta$. Then we can write the period as

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta+\frac{1\cdot 3}{2!2^2}k^4\sin^4\theta+\cdots\right)d\theta.$$

a. Using just the first term in the integrand, the first-order estimate is

$$T \approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2} d\theta = 2\pi\sqrt{\frac{L}{g}}.$$

If θ_{max} is small, then $k = \sin\left(\frac{\theta_{max}}{2}\right)$ is small. We claim that when k is small, this is a good estimate. To justify this claim, consider

$$\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta+\frac{1\cdot 3}{2!2^2}k^4\sin^4\theta+\cdots\right)d\theta.$$

Since $|\sin x| \leq 1$, this integral is bounded by

$$\int_0^{\pi/2}\left(\frac{1}{2}k^2+\frac{1.3}{2!2^2}k^4+\cdots\right)d\theta \;<\; \frac{\pi}{2}\left(\frac{1}{2}k^2+\frac{1\cdot 3}{2!2^2}k^4+\cdots\right).$$

Furthermore, it can be shown that each coefficient on the right-hand side is less than 1 and, therefore, that this expression is bounded by

$$\frac{\pi k^2}{2}(1+k^2+k^4+\cdots) = \frac{\pi k^2}{2}\cdot\frac{1}{1-k^2} \quad ,$$

which is small for k small.

b. For larger values of θ_{max}, we can approximate T by using more terms in the integrand. By using the first two terms in the integral, we arrive at the estimate

$$T \approx 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1+\frac{1}{2}k^2\sin^2\theta\right)d\theta = 2\pi\sqrt{\frac{L}{g}}\left(1+\frac{k^2}{4}\right).$$

The applications of Taylor series in this section are intended to highlight their importance. In general, Taylor series are useful because they allow us to represent known functions using polynomials, thus providing us a tool for approximating function values and estimating complicated integrals. In addition, they allow us to define new functions as power series, thus providing us with a powerful tool for solving differential equations.

Key Concepts

- The binomial series is the Maclaurin series for $f(x) = (1+x)^r$. It converges for $|x| < 1$.
- Taylor series for functions can often be derived by algebraic operations with a known Taylor series or by differentiating or integrating a known Taylor series.
- Power series can be used to solve differential equations.
- Taylor series can be used to help approximate integrals that cannot be evaluated by other means.

Glossary

binomial series

the Maclaurin series for $f(x) = (1+x)^r$; it is given by

$$(1+x)^r = \sum_{n=0}^{\infty}\binom{r}{n}x^n = 1+rx+\frac{r(r-1)}{2!}x^2+\cdots+\frac{r(r-1)\cdots(r-n+1)}{n!}x^n+\cdots \qquad \text{for } |x|<1$$

non-elementary integral

an integral for which the antiderivative of the integrand cannot be expressed as an elementary function

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7.5E: Exercises for Section 10.4

In exercises 1 - 4, use appropriate substitutions to write down the Maclaurin series for the given binomial.

1) $(1-x)^{1/3}$

2) $(1+x^2)^{-1/3}$

Answer

$$(1+x^2)^{-1/3}=\sum_{n=0}^{\infty}\left(n^{-\frac{1}{3}}\right)x^{2n}$$

3) $(1-x)^{1.01}$

4) $(1-2x)^{2/3}$

Answer

$$(1-2x)^{2/3}=\sum_{n=0}^{\infty}(-1)^n 2^n\left(n^{\frac{2}{3}}\right)x^n$$

In exercises 5 - 12, use the substitution $(b+x)^r=(b+a)^r\left(1+\dfrac{x-a}{b+a}\right)^r$ **in the binomial expansion to find the Taylor series of each function with the given center.**

5) $\sqrt{x+2}$ at $a=0$

6) $\sqrt{x^2+2}$ at $a=0$

Answer

$$\sqrt{2+x^2}=\sum_{n=0}^{\infty}2^{(1/2)-n}\left(n^{\frac{1}{2}}\right)x^{2n};(|x^2|<2)$$

7) $\sqrt{x+2}$ at $a=1$

8) $\sqrt{2x-x^2}$ at $a=1$ (Hint: $2x-x^2=1-(x-1)^2$)

Answer

$$\sqrt{2x-x^2}=\sqrt{1-(x-1)^2}\text{ so }\sqrt{2x-x^2}=\sum_{n=0}^{\infty}(-1)^n\left(n^{\frac{1}{2}}\right)(x-1)^{2n}$$

9) $(x-8)^{1/3}$ at $a=9$

10) $\sqrt{x}$ at $a=4$

Answer

$$\sqrt{x}=2\sqrt{1+\tfrac{x-4}{4}}\text{ so }\sqrt{x}=\sum_{n=0}^{\infty}2^{1-2n}\left(n^{\frac{1}{2}}\right)(x-4)^n$$

11) $x^{1/3}$ at $a=27$

12) $\sqrt{x}$ at $x=9$

Answer

$$\sqrt{x}=\sum_{n=0}^{\infty}3^{1-3n}\left(n^{\frac{1}{2}}\right)(x-9)^n$$

In exercises 13 - 14, use the binomial theorem to estimate each number, computing enough terms to obtain an estimate accurate to an error of at most 1/1000.

13) [T] $(15)^{1/4}$ using $(16-x)^{1/4}$

14) [T] $(1001)^{1/3}$ using $(1000+x)^{1/3}$

Answer

$10(1+\dfrac{x}{1000})^{1/3}=\sum_{n=0}^{\infty}10^{1-3n}(\frac{1}{n})x^n$. Using, for example, a fourth-degree estimate at $x=1$ gives
$(1001)^{1/3}\approx 10\left(1+\left(1^{\frac{1}{3}}\right)10^{-3}+\left(2^{\frac{1}{3}}\right)10^{-6}+\left(3^{\frac{1}{3}}\right)10^{-9}+\left(3^{\frac{1}{3}}\right)10^{-12}\right)=10\left(1+\frac{1}{3.10^3}-\frac{1}{9.10^6}+\frac{5}{81.10^9}-\frac{10}{243.10^{12}}\right)=10.00333222\ldots$
whereas $(1001)^{1/3}=10.00332222839093$. Two terms would suffice for three-digit accuracy.

In exercises 15 - 18, use the binomial approximation $\sqrt{1-x} \approx 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256}$ for $|x| < 1$ to approximate each number. Compare this value to the value given by a scientific calculator.

15) [T] $\frac{1}{\sqrt{2}}$ using $x = \frac{1}{2}$ in $(1-x)^{1/2}$

16) [T] $\sqrt{5} = 5 \times \frac{1}{\sqrt{5}}$ using $x = \frac{4}{5}$ in $(1-x)^{1/2}$

Answer

The approximation is 2.3152 the CAS value is $2.23\ldots$.

17) [T] $\sqrt{3} = \frac{3}{\sqrt{3}}$ using $x = \frac{2}{3}$ in $(1-x)^{1/2}$

18) [T] $\sqrt{6}$ using $x = \frac{5}{6}$ in $(1-x)^{1/2}$

Answer

The approximation is $2.583\ldots$; the CAS value is $2.449\ldots$.

19) Integrate the binomial approximation of $\sqrt{1-x}$ to find an approximation of $\int_0^x \sqrt{1-t}\, dt$.

20) [T] Recall that the graph of $\sqrt{1-x^2}$ is an upper semicircle of radius 1. Integrate the binomial approximation of $\sqrt{1-x^2}$ up to order 8 from $x = -1$ to $x = 1$ to estimate $\frac{\pi}{2}$.

Answer

$\sqrt{1-x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128} + \cdots$. Thus

$\int_{-1}^{1} \sqrt{1-x^2}\, dx = \left[x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{7 \cdot 16} - \frac{5x^9}{9 \cdot 128} + \cdots\right]\Big|_{-1}^{1} \approx 2 - \frac{1}{3} - \frac{1}{20} - \frac{1}{56} - \frac{10}{9 \cdot 128} + error = 1.590\ldots$ whereas $\frac{\pi}{2} = 1.570\ldots$

In exercises 21 - 24, use the expansion $(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots$ to write the first five terms (not necessarily a quartic polynomial) of each expression.

21) $(1+4x)^{1/3};\ a = 0$

22) $(1+4x)^{4/3};\ a = 0$

Answer

$(1+x)^{4/3} = (1+x)(1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots) = 1 + \frac{4x}{3} + \frac{2x^2}{9} - \frac{4x^3}{81} + \frac{5x^4}{243} + \cdots$

23) $(3+2x)^{1/3};\ a = -1$

24) $(x^2+6x+10)^{1/3};\ a = -3$

Answer

$(1+(x+3)^2)^{1/3} = 1 + \frac{1}{3}(x+3)^2 - \frac{1}{9}(x+3)^4 + \frac{5}{81}(x+3)^6 - \frac{10}{243}(x+3)^8 + \cdots$

25) Use $(1+x)^{1/3} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + \cdots$ with $x = 1$ to approximate $2^{1/3}$.

26) Use the approximation $(1-x)^{2/3} = 1 - \frac{2x}{3} - \frac{x^2}{9} - \frac{4x^3}{81} - \frac{7x^4}{243} - \frac{14x^5}{729} + \cdots$ for $|x| < 1$ to approximate $2^{1/3} = 2.2^{-2/3}$.

Answer

Twice the approximation is $1.260\ldots$whereas $2^{1/3} = 1.2599\ldots$.

27) Find the 25th derivative of $f(x) = (1+x^2)^{13}$ at $x = 0$.

28) Find the 99th derivative of $f(x) = (1+x^4)^{25}$.

Answer

$f^{(99)}(0) = 0$

In exercises 29 - 36, find the Maclaurin series of each function.

29) $f(x) = xe^{2x}$

30) $f(x) = 2^x$

Answer

$\sum_{n=0}^{\infty} \frac{(\ln(2)x)^n}{n!}$

31) $f(x) = \dfrac{\sin x}{x}$

32) $f(x) = \dfrac{\sin(\sqrt{x})}{\sqrt{x}}, (x > 0),$

Answer

For $x > 0$, $\sin(\sqrt{x}) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)/2}}{\sqrt{x}(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n+1)!}$.

33) $f(x) = \sin(x^2)$

34) $f(x) = e^{x^3}$

Answer

$$e^{x^3} = \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

35) $f(x) = \cos^2 x$ using the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$

36) $f(x) = \sin^2 x$ using the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

Answer

$$\sin^2 x = -\sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} x^{2k}}{(2k)!}$$

In exercises 37 - 44, find the Maclaurin series of $F(x) = \displaystyle\int_0^x f(t)\,dt$ by integrating the Maclaurin series of f term by term. If f is not strictly defined at zero, you may substitute the value of the Maclaurin series at zero.

37) $F(x) = \displaystyle\int_0^x e^{-t^2}\,dt;\ f(t) = e^{-t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!}$

38) $F(x) = \tan^{-1} x;\ f(t) = \dfrac{1}{1+t^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n t^{2n}$

Answer

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

39) $F(x) = \tanh^{-1} x;\ f(t) = \dfrac{1}{1-t^2} = \displaystyle\sum_{n=0}^{\infty} t^{2n}$

40) $F(x) = \sin^{-1} x;\ f(t) = \dfrac{1}{\sqrt{1-t^2}} = \displaystyle\sum_{k=0}^{\infty} \left(k^{\frac{1}{2}}\right) \frac{t^{2k}}{k!}$

Answer

$$\sin^{-1} x = \sum_{n=0}^{\infty} \left(n^{\frac{1}{2}}\right) \frac{x^{2n+1}}{(2n+1)n!}$$

41) $F(x) = \displaystyle\int_0^x \frac{\sin t}{t}\,dt;\ f(t) = \frac{\sin t}{t} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}$

42) $F(x) = \displaystyle\int_0^x \cos(\sqrt{t})\,dt;\ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$

Answer

$$F(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)(2n)!}$$

43) $F(x) = \displaystyle\int_0^x \frac{1-\cos t}{t^2}\,dt;\ f(t) = \frac{1-\cos t}{t^2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+2)!}$

44) $F(x) = \displaystyle\int_0^x \frac{\ln(1+t)}{t}\,dt;\ f(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{n+1}$

Answer

$$F(x)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n^2}$$

In exercises 45 - 52, compute at least the first three nonzero terms (not necessarily a quadratic polynomial) of the Maclaurin series of f.

45) $f(x)=\sin\left(x+\frac{\pi}{4}\right)=\sin x\cos\left(\frac{\pi}{4}\right)+\cos x\sin\left(\frac{\pi}{4}\right)$

46) $f(x)=\tan x$

Answer

$$x+\frac{x^3}{3}+\frac{2x^5}{15}+\cdots$$

47) $f(x)=\ln(\cos x)$

48) $f(x)=e^x\cos x$

Answer

$$1+x-\frac{x^3}{3}-\frac{x^4}{6}+\cdots$$

49) $f(x)=e^{\sin x}$

50) $f(x)=\sec^2 x$

Answer

$$1+x^2+\frac{2x^4}{3}+\frac{17x^6}{45}+\cdots$$

51) $f(x)=\tanh x$

52) $f(x)=\dfrac{\tan\sqrt{x}}{\sqrt{x}}$ (see expansion for $\tan x$)

Answer

Using the expansion for $\tan x$ gives $1+\dfrac{x}{3}+\dfrac{2x^2}{15}$.

In exercises 53 - 56, find the radius of convergence of the Maclaurin series of each function.

53) $\ln(1+x)$

54) $\dfrac{1}{1+x^2}$

Answer

$\dfrac{1}{1+x^2}=\displaystyle\sum_{n=0}^{\infty}(-1)^n x^{2n}$ so $R=1$ by the ratio test.

55) $\tan^{-1}x$

56) $\ln(1+x^2)$

Answer

$\ln(1+x^2)=\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^{2n}$ so $R=1$ by the ratio test.

57) Find the Maclaurin series of $\sinh x=\dfrac{e^x-e^{-x}}{2}$.

58) Find the Maclaurin series of $\cosh x=\dfrac{e^x+e^{-x}}{2}$.

Answer

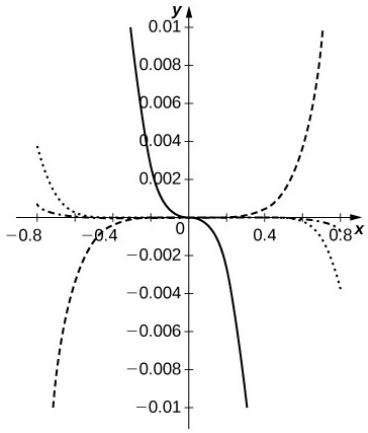
Add series of e^x and e^{-x} term by term. Odd terms cancel and $\cosh x=\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$.

59) Differentiate term by term the Maclaurin series of $\sinh x$ and compare the result with the Maclaurin series of $\cosh x$.

60) [T] Let $S_n(x)=\sum_{k=0}^{n}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$ and $C_n(x)=\sum_{n=0}^{n}(-1)^k\frac{x^{2k}}{(2k)!}$ denote the respective Maclaurin polynomials of degree $2n+1$ of $\sin x$ and degree $2n$ of $\cos x$. Plot the errors $\frac{S_n(x)}{C_n(x)}-\tan x$ for $n=1,\ldots,5$ and compare them to $x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}-\tan x$ on $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$.

Answer

The ratio $\frac{S_n(x)}{C_n(x)}$ approximates $\tan x$ better than does $p_7(x)=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}$ for $N\geq 3$. The dashed curves are $\frac{S_n}{C_n}-\tan x$ for $n=1,\ 2$. The dotted curve corresponds to $n=3$, and the dash-dotted curve corresponds to $n=4$. The solid curve is $p_7-\tan x$.



61) Use the identity $2\sin x\cos x=\sin(2x)$ to find the power series expansion of $\sin^2 x$ at $x=0$. (Hint: Integrate the Maclaurin series of $\sin(2x)$ term by term.)

62) If $y=\sum_{n=0}^{\infty}a_nx^n$, find the power series expansions of xy' and x^2y''.

Answer

By the term-by-term differentiation theorem, $y'=\sum_{n=1}^{\infty}na_nx^{n-1}$ so $y'=\sum_{n=1}^{\infty}na_nx^{n-1}xy'=\sum_{n=1}^{\infty}na_nx^n$, whereas $y'=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}$ so $xy''=\sum_{n=2}^{\infty}n(n-1)a_nx^n$.

63) [T] Suppose that $y=\sum_{k=0}^{\infty}a^kx^k$ satisfies $y'=-2xy$ and $y(0)=0$. Show that $a_{2k+1}=0$ for all k and that $a_{2k+2}=\frac{-a_{2k}}{k+1}$. Plot the partial sum S_{20} of y on the interval $[-4,4]$.

64) [T] Suppose that a set of standardized test scores is normally distributed with mean $\mu=100$ and standard deviation $\sigma=10$. Set up an integral that represents the probability that a test score will be between 90 and 110 and use the integral of the degree 10 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ to estimate this probability.

Answer

The probability is $p=\frac{1}{\sqrt{2\pi}}\int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma}e^{-x^2/2}\,dx$ where $a=90$ and $b=100$, that is,

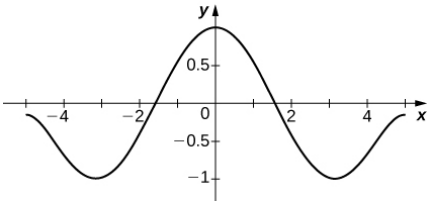
$$p=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}e^{-x^2/2}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\sum_{n=0}^{5}(-1)^n\frac{x^{2n}}{2^n n!}\,dx=\frac{2}{\sqrt{2\pi}}\sum_{n=0}^{5}(-1)^n\frac{1}{(2n+1)2^n n!}\approx 0.6827.$$

65) [T] Suppose that a set of standardized test scores is normally distributed with mean $\mu=100$ and standard deviation $\sigma=10$. Set up an integral that represents the probability that a test score will be between 70 and 130 and use the integral of the degree 50 Maclaurin polynomial of $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ to estimate this probability.

66) [T] Suppose that $\sum_{n=0}^{\infty}a_nx^n$ converges to a function $f(x)$ such that $f(0)=1,\ f'(0)=0$, and $f''(x)=-f(x)$. Find a formula for a_n and plot the partial sum S_N for $N=20$ on $[-5,5]$.

Answer

As in the previous problem one obtains $a_n = 0$ if n is odd and $a_n = -(n+2)(n+1)a_{n+2}$ if n is even, so $a_0 = 1$ leads to $a_{2n} = \dfrac{(-1)^n}{(2n)!}$.



67) [T] Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function $f(x)$ such that $f(0) = 0,\ f'(0) = 1$, and $f''(x) = -f(x)$. Find a formula for an and plot the partial sum S_N for $N = 10$ on $[-5, 5]$.

68) Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y' + y = 0$ where $y(0) = 1$ and $y'(0) = 0$. Find a formula that relates $a_{n+2},\ a_{n+1}$, and an and compute $a_0, \ldots, a_5$.

Answer

$y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$ and $y' = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ so $y'' - y' + y = 0$ implies that $(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} + a_n = 0$ or $a_n = \dfrac{a_{n-1}}{n} - \dfrac{a_{n-2}}{n(n-1)}$ for all $n \cdot y(0) = a_0 = 1$ and $y'(0) = a_1 = 0$, so $a_2 = \frac{1}{2},\ a_3 = \frac{1}{6}$, $a_4 = 0$, and $a_5 = -\frac{1}{120}$.

69) Suppose that $\sum_{n=0}^{\infty} a_n x^n$ converges to a function y such that $y'' - y' + y = 0$ where $y(0) = 0$ and $y'(0) = 1$. Find a formula that relates $a_{n+2},\ a_{n+1}$, and an and compute $a_1, \ldots, a_5$.

The error in approximating the integral $\int_a^b f(t)\,dt$ by that of a Taylor approximation $\int_a^b Pn(t)\,dt$ is at most $\int_a^b R_n(t)\,dt$. In exercises 70 - 71, the Taylor remainder estimate $R_n \le \frac{M}{(n+1)!}|x-a|^{n+1}$ guarantees that the integral of the Taylor polynomial of the given order approximates the integral of f with an error less than $\frac{1}{10}$.

a. Evaluate the integral of the appropriate Taylor polynomial and verify that it approximates the CAS value with an error less than $\frac{1}{100}$.

b. Compare the accuracy of the polynomial integral estimate with the remainder estimate.

70) [T] $\int_0^{\pi} \dfrac{\sin t}{t}\,dt;\quad P_s = 1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \dfrac{x^8}{9!}$ (You may assume that the absolute value of the ninth derivative of $\frac{\sin t}{t}$ is bounded by 0.1.)

Answer

a. (Proof)

b. We have $R_s \le \frac{0.1}{(9)!}\pi^9 \approx 0.0082 < 0.01$. We have $\int_0^{\pi}\left(1 - \dfrac{x^2}{3!} + \dfrac{x^4}{5!} - \dfrac{x^6}{7!} + \dfrac{x^8}{9!}\right)dx = \pi - \dfrac{\pi^3}{3\cdot 3!} + \dfrac{\pi^5}{5\cdot 5!} - \dfrac{\pi^7}{7\cdot 7!} + \dfrac{\pi^9}{9\cdot 9!} = 1.852\ldots$, whereas $\int_0^{\pi} \dfrac{\sin t}{t}\,dt = 1.85194..$, so the actual error is approximately 0.00006.

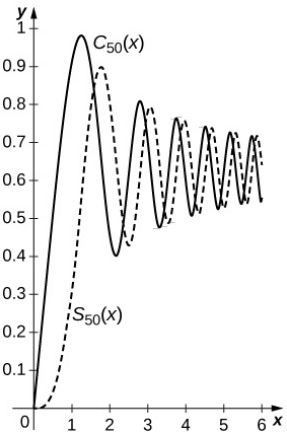
71) [T] $\int_0^2 e^{-x^2}\,dx;\ p_{11} = 1 - x^2 + \dfrac{x^4}{2} - \dfrac{x^6}{3!} + \cdots - \dfrac{x^{22}}{11!}$ (You may assume that the absolute value of the 23rd derivative of e^{-x^2} is less than 2×10^{14}.)

The following exercises (72-73) deal with *Fresnel integrals*.

72) The Fresnel integrals are defined by $C(x) = \int_0^x \cos(t^2)\,dt$ and $S(x) = \int_0^x \sin(t^2)\,dt$. Compute the power series of $C(x)$ and $S(x)$ and plot the sums $C_N(x)$ and $S_N(x)$ of the first $N = 50$ nonzero terms on $[0, 2\pi]$.

Answer

Since $\cos(t^2) = \sum_{n=0}^{\infty}(-1)^n \dfrac{t^{4n}}{(2n)!}$ and $\sin(t^2) = \sum_{n=0}^{\infty}(-1)^n \dfrac{t^{4n+2}}{(2n+1)!}$, one has $S(x) = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{4n+3}}{(4n+3)(2n+1)!}$ and $C(x) = \sum_{n=0}^{\infty}(-1)^n \dfrac{x^{4n+1}}{(4n+1)(2n)!}$. The sums of the first 50 nonzero terms are plotted below with $C_{50}(x)$ the solid curve and $S_{50}(x)$ the dashed curve.



73) [T] The Fresnel integrals are used in design applications for roadways and railways and other applications because of the curvature properties of the curve with coordinates $(C(t), S(t))$. Plot the curve (C_{50}, S_{50}) for $0 \le t \le 2\pi$, the coordinates of which were computed in the previous exercise.

74) Estimate $\int_0^{1/4} \sqrt{x - x^2}\, dx$ by approximating $\sqrt{1-x}$ using the binomial approximation $1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{2128} - \frac{7x^5}{256}$.

Answer

$\int_0^{1/4} \sqrt{x}\left(1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} - \frac{7x^5}{256}\right) dx = \frac{2}{3}2^{-3} - \frac{1}{2}\frac{2}{5}2^{-5} - \frac{1}{8}\frac{2}{7}2^{-7} - \frac{1}{16}\frac{2}{9}2^{-9} - \frac{5}{128}\frac{2}{11}2^{-11} - \frac{7}{256}\frac{2}{13}2^{-13} = 0.0767732\ldots$
whereas $\int_0^{1/4} \sqrt{x - x^2}\, dx = 0.076773.$

75) [T] Use Newton's approximation of the binomial $\sqrt{1-x^2}$ to approximate π as follows. The circle centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$ has upper semicircle $y = \sqrt{x}\sqrt{1-x}$. The sector of this circle bounded by the x-axis between $x = 0$ and $x = \frac{1}{2}$ and by the line joining $(\frac{1}{4}, \frac{\sqrt{3}}{4})$ corresponds to $\frac{1}{6}$ of the circle and has area $\frac{\pi}{24}$. This sector is the union of a right triangle with height $\frac{\sqrt{3}}{4}$ and base $\frac{1}{4}$ and the region below the graph between $x = 0$ and $x = \frac{1}{4}$. To find the area of this region you can write $y = \sqrt{x}\sqrt{1-x} = \sqrt{x} \times (\text{binomial expansion of} \sqrt{1-x})$ and integrate term by term. Use this approach with the binomial approximation from the previous exercise to estimate π.

76) Use the approximation $T \approx 2\pi\sqrt{\frac{L}{g}}(1 + \frac{k^2}{4})$ to approximate the period of a pendulum having length 10 meters and maximum angle $\theta_{max} = \frac{\pi}{6}$ where $k = \sin\left(\frac{\theta_{max}}{2}\right)$. Compare this with the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$.

Answer

$T \approx 2\pi\sqrt{\frac{10}{9.8}}\left(1 + \frac{\sin^2(\theta/12)}{4}\right) \approx 6.453$ seconds. The small angle estimate is $T \approx 2\pi\sqrt{\frac{10}{9.8}} \approx 6.347$. The relative error is around 2 percent.

77) Suppose that a pendulum is to have a period of 2 seconds and a maximum angle of $\theta_{max} = \frac{\pi}{6}$. Use $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4}\right)$ to approximate the desired length of the pendulum. What length is predicted by the small angle estimate $T \approx 2\pi\sqrt{\frac{L}{g}}$?

78) Evaluate $\int_0^{\pi/2} \sin^4\theta\, d\theta$ in the approximation $T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left(1 + \frac{1}{2}k^2\sin^2\theta + \frac{3}{8}k^4\sin^4\theta + \cdots\right) d\theta$ to obtain an improved estimate for T.

Answer

$\int_0^{\pi/2} \sin^4\theta\, d\theta = \frac{3\pi}{16}$. Hence $T \approx 2\pi\sqrt{\frac{L}{g}}\left(1 + \frac{k^2}{4} + \frac{9}{256}k^4\right)$.

79) [T] An equivalent formula for the period of a pendulum with amplitude θ_{max} is $T(\theta_{max}) = 2\sqrt{2}\sqrt{\frac{L}{g}}\int_0^{\theta_{max}} \frac{d\theta}{\sqrt{\cos\theta - \cos(\theta_{max})}}$ where L is the pendulum length and g is the gravitational acceleration constant. When $\theta_{max} = \frac{\pi}{3}$ we get $\frac{1}{\sqrt{\cos t - 1/2}} \approx \sqrt{2}\left(1 + \frac{t^2}{2} + \frac{t^4}{3} + \frac{181t^6}{720}\right)$. Integrate this approximation to estimate $T(\frac{\pi}{3})$ in terms of L and g. Assuming $g = 9.806$ meters per second squared, find an approximate length L such that $T(\frac{\pi}{3}) = 2$ seconds.

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7.6: Chapter 10 Review Exercises

True or False? In exercises 1 - 4, justify your answer with a proof or a counterexample.

1) If the radius of convergence for a power series *[Math Processing Error]* is *[Math Processing Error]*, then the radius of convergence for the series *[Math Processing Error]* is also *[Math Processing Error]*.

2) Power series can be used to show that the derivative of *[Math Processing Error]* is *[Math Processing Error]*. (Hint: Recall that *[Math Processing Error]*)

3) For small values of *[Math Processing Error]* *[Math Processing Error]*

4) The radius of convergence for the Maclaurin series of *[Math Processing Error]* is *[Math Processing Error]*.

In exercises 5 - 8, find the radius of convergence and the interval of convergence for the given series.

5) *[Math Processing Error]*

6) *[Math Processing Error]*

7) *[Math Processing Error]*

8) *[Math Processing Error]*

In exercises 9 - 10, find the power series representation for the given function. Determine the radius of convergence and the interval of convergence for that series.

9) *[Math Processing Error]*

10) *[Math Processing Error]*

In exercises 11 - 12, find the power series for the given function using term-by-term differentiation or integration.

11) *[Math Processing Error]*

12) *[Math Processing Error]*

In exercises 13 - 14, evaluate the Taylor series expansion of degree four for the given function at the specified point. What is the error in the approximation?

13) *[Math Processing Error]*

14) *[Math Processing Error]*

In exercises 15 - 16, find the Maclaurin series for the given function.

15) *[Math Processing Error]*

16) *[Math Processing Error]*

In exercises 17 - 18, find the Taylor series at the given value.

17) *[Math Processing Error]*

18) *[Math Processing Error]*

In exercises 19 - 20, find the Maclaurin series for the given function.

19) *[Math Processing Error]*

20) *[Math Processing Error]*

In exercises 21 - 23, find the Maclaurin series for *[Math Processing Error]* **by integrating the Maclaurin series of** *[Math Processing Error]* **term by term.**

21) *[Math Processing Error]*

22) *[Math Processing Error]*

23) Use power series to prove **Euler's formula:** *[Math Processing Error]*

Exercises 24 - 26 consider problems of *annuity payments*.

24) For annuities with a present value of *[Math Processing Error]* million, calculate the annual payouts given over *[Math Processing Error]* years assuming interest rates of *[Math Processing Error]*, and *[Math Processing Error]*

25) A lottery winner has an annuity that has a present value of *[Math Processing Error]* million. What interest rate would they need to live on perpetual annual payments of *[Math Processing Error]*?

26) Calculate the necessary present value of an annuity in order to support annual payouts of *[Math Processing Error]* given over *[Math Processing Error]* years assuming interest rates of *[Math Processing Error]*,and *[Math Processing Error]*

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CHAPTER OVERVIEW

8: Appendices

8.1: Trigonometric Identities
8.2: Review of Derivative Rules
8.3: Table of Derivatives
8.4: Table of Integrals

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8.1: Trigonometric Identities

Pythagorean Identities

$\cos^2 x + \sin^2 x = 1$

$\sec^2 x - \tan^2 x = 1$

Double-Angle Identities

$\sin 2x = 2 \sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$

Half-Angle Identities

$\cos^2 x = \dfrac{1 + \cos 2x}{2}$

$\sin^2 x = \dfrac{1 - \cos 2x}{2}$

Angle Sum and Difference Identities

$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Angle Reflections and Shifts

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\sin\left(x \pm \frac{\pi}{2}\right) = \pm \cos x$

$\cos\left(x \pm \frac{\pi}{2}\right) = \mp \sin x$

Angle Supplement Identities

$\sin(\pi - x) = \sin x$

$\cos(\pi - x) = -\cos x$

$\tan(\pi - x) = -\tan x$

Periodicity Identities

$\sin(x + 2\pi) = \sin x$

$\cos(x + 2\pi) = \cos x$

$\tan(x + \pi) = \tan x$

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8.2: Review of Derivative Rules

Here is a quick review of the basic derivative rules in words. See the Table of Derivatives for a list of Derivative Formulas.

- The derivative of a constant function is zero.
- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative.
- The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.
- The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g.
- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function.
- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.

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8.3: Table of Derivatives

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{for real numbers } n$
5. $\frac{d}{dx}(cf(x)) = cf'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$
13. $\frac{d}{dx}(\cot x) = -\csc^2 x$
14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$
17. $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$
18. $\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$
19. $\frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2}$
20. $\frac{d}{dx}(\operatorname{arccsc} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$
22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
23. $\frac{d}{dx}(b^x) = b^x \ln b$
24. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$
26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
28. $\frac{d}{dx}(\cosh x) = \sinh x$
29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}}$
32. $\frac{d}{dx}(\operatorname{arctanh} x) = \frac{1}{1-x^2} \quad (|x| < 1)$
33. $\frac{d}{dx}(\operatorname{arcsech} x) = \frac{-1}{x\sqrt{1-x^2}} \quad (0 < x < 1)$
34. $\frac{d}{dx}(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
35. $\frac{d}{dx}(\operatorname{arccoth} x) = \frac{1}{1-x^2} \quad (|x| > 1)$
36. $\frac{d}{dx}(\operatorname{arccsch} x) = \frac{-1}{|x|\sqrt{1+x^2}} \quad (x \neq 0)$

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- Modified to change inverse trig notation by Paul Seeburger (Monroe Community College)

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8.4: Table of Integrals

For this course, all work must be shown to obtain most of these integral forms. **Of the integration formulas listed below, the only ones that can be applied without further work are #1 - 10, 15 - 17, and 49 and 50.** And even these will require work to be shown if a substitution is involved.

All others may be helpful for checking your final answers, but cannot be used to skip the necessary work to show you understand how to use the integration techniques taught in this course.

As you look through these formulas, you should be able to recognize which integration technique was needed to obtain the general formula. It may be very useful for you to try to obtain the general formula yourself using the techniques we learn in this course.

Basic Integrals

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$
7. $\int \sec^2 u\,du = \tan u + C$
8. $\int \csc^2 u\,du = -\cot u + C$
9. $\int \sec u \tan u\,du = \sec u + C$
10. $\int \csc u \cot u\,du = -\csc u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \cot u\,du = \ln|\sin u| + C$
13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C$
16. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$
17. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

Trigonometric Integrals

18. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$

19. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

20. $\int \tan^2 u\,du = \tan u - u + C$

21. $\int \cot^2 u\,du = -\cot u - u + C$

22. $\int \sin^3 u\,du = -\frac{1}{3}(2+\sin^2 u)\cos u + C$

23. $\int \cos^3 u\,du = \frac{1}{3}(2+\cos^2 u)\sin u + C$

24. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$

25. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$

26. $\int \sec^3 u\,du = \frac{1}{2}\sec u\tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$

27. $\int \csc^3 u\,du = -\frac{1}{2}\csc u\cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\,du = \frac{-1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$

29. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$

30. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$

31. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$

32. $\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$

33. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$

34. $\int \sin au\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

37. $\int u\sin u\,du = \sin u - u\cos u + C$

38. $\int u\cos u\,du = \cos u + u\sin u + C$

39. $\int u^n \sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$

40. $\int u^n \cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$

41. $\int \sin^n u \cos^m u \, du =$ Use the methods for powers of sine and cosine

Exponential and Logarithmic Integrals

42. $\int ue^{au} \, du = \frac{1}{a^2}(au-1)e^{au} + C$

43. $\int u^n e^{au} \, du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au} \, du$

44. $\int e^{au} \sin bu \, du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$

45. $\int e^{au} \cos bu \, du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$

46. $\int \ln u \, du = u\ln u - u + C$

47. $\int u^n \ln u \, du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$

48. $\int \frac{1}{u\ln u} \, du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u \, du = \cosh u + C$

50. $\int \cosh u \, du = \sinh u + C$

51. $\int \tanh u \, du = \ln\cosh u + C$

52. $\int \coth u \, du = \ln|\sinh u| + C$

53. $\int \operatorname{sech} u \, du = \arctan|\sinh u| + C$

54. $\int \operatorname{csch} u \, du = \ln|\tanh\frac{1}{2}u| + C$

55. $\int \operatorname{sech}^2 u \, du = \tanh u + C$

56. $\int \operatorname{csch}^2 u \, du = -\coth u + C$

57. $\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$

58. $\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$

Inverse Trigonometric Integrals

59. $\int \arcsin u \, du = u\arcsin u + \sqrt{1-u^2} + C$

60. $\int \arccos u \, du = u\arccos u - \sqrt{1-u^2} + C$

61. $\int \arctan u \, du = u\arctan u - \frac{1}{2}\ln(1+u^2) + C$

62. $\int u \arcsin u \, du = \frac{2u^2-1}{4}\arcsin u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u \arccos u \, du = \frac{2u^2-1}{4}\arccos u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u \arctan u \, du = \frac{u^2+1}{2}\arctan u - \frac{u}{2} + C$

65. $\int u^n \arcsin u \, du = \frac{1}{n+1}\left[u^{n+1}\arcsin u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\int u^n \arccos u \, du = \frac{1}{n+1}\left[u^{n+1}\arccos u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

67. $\int u^n \arctan u \, du = \frac{1}{n+1}\left[u^{n+1}\arctan u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2+u^2}\, du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

69. $\int u^2\sqrt{a^2+u^2}\, du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

70. $\int \frac{\sqrt{a^2+u^2}}{u}\, du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2+u^2}}{u^2}\, du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

73. $\int \frac{u^2}{\sqrt{a^2+u^2}}\, du = \frac{u}{2}\left(\sqrt{a^2+u^2}\right) - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{-1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

75. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

76. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2-a^2}\, du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

78. $\int u^2\sqrt{u^2-a^2}\, du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

79. $\int \frac{\sqrt{u^2-a^2}}{u}\, du = \sqrt{u^2-a^2} - a\arccos\frac{a}{|u|} + C$

80. $\int \frac{\sqrt{u^2-a^2}}{u^2}\, du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

81. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

82. $$\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$$

83. $$\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2 u} + C$$

84. $$\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\frac{u}{a} + C$$

86. $$\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\arcsin\frac{u}{a} + C$$

87. $$\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

88. $$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \frac{-1}{u}\sqrt{a^2-u^2} - \arcsin\frac{u}{a} + C$$

89. $$\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{1}{2}\left(-u\sqrt{a^2-u^2} + a^2\arcsin\frac{u}{a}\right) + C$$

90. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$$

91. $$\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2 u}\sqrt{a^2-u^2} + C$$

92. $$\int \left(a^2-u^2\right)^{3/2}\,du = -\frac{u}{8}\left(2u^2-5a^2\right)\sqrt{a^2-u^2} + \frac{3a^4}{8}\arcsin\frac{u}{a} + C$$

93. $$\int \frac{du}{(a^2-u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2-u^2}} + C$$

Integrals Involving $2au - u^2$, $a > 0$

94. $$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\arccos\left(\frac{a-u}{a}\right) + C$$

95. $$\int \frac{du}{\sqrt{2au-u^2}} = \arccos\left(\frac{a-u}{a}\right) + C$$

96. $$\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\arccos\left(\frac{a-u}{a}\right) + C$$

97. $$\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$$

Integrals Involving $a + bu$, $a \neq 0$

98. $$\int \frac{u}{a+bu}\,du = \frac{1}{b^2}(a+bu-a\ln|a+bu|) + C$$

99. $$\int \frac{u^2}{a+bu}\,du = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$$

100. $$\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$$

101. $$\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

102. $$\int \frac{u}{(a+bu)^2}\,du = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$$

103. $$\int \frac{u}{u(a+bu)^2}\,du = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$$

104. $$\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$$

105. $$\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$$

106. $$\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$$

107. $$\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$$

108. $$\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, & \text{if } a > 0 \\ \frac{\sqrt{2}}{\sqrt{-a}}\arctan\sqrt{\frac{a+bu}{-a}} + C, & \text{if } a < 0 \end{cases}$$

109. $$\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$$

110. $$\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$$

111. $$\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$$

112. $$\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du$$

113. $$\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

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Index

A

absolute convergence
6.6: Alternating Series
absolute error
4.7: Numerical Integration
air resistance
5.6: First-order Linear Equations
Airy's equation
7.5: Working with Taylor Series
Alternating Harmonic Series
6.6: Alternating Series
alternating series
6.6: Alternating Series
Alternating Series Test
6.6: Alternating Series
arc length
3.6: Arc Length of a Curve and Surface Area
AREA BETWEEN TWO CURVES
3.2: Areas between Curves
Areas of Compound Regions
3.2: Areas between Curves
arithmetic sequence
6.2: Sequences
asymptotically stable solution
5.3: Direction Fields and Numerical Methods
asymptotically unstable solution
5.3: Direction Fields and Numerical Methods
autonomous differential equation
5.4: Separable Differential Equations
average value of a function
2.4: The Definite Integral

B

binomial series
7.5: Working with Taylor Series

C

carrying capacity
5.5: The Logistic Equation
catenary
3.14: Calculus of the Hyperbolic Functions
change of variables
2.7: Substitution
comparison test
6.5: Comparison Tests
compound interest
3.13: Exponential Growth and Decay
computer algebra system
4.6: Other Strategies for Integration
conditional convergence
6.6: Alternating Series
convergence of a series
6.3: Infinite Series
convergent sequence
6.2: Sequences

D

definite integral
2.4: The Definite Integral
DENSITY FUNCTION
3.9: More Physical Applications of Integration
Derivative of exponential functions
1.1: Derivatives of Exponential and Logarithmic Functions
Derivative of Hyperbolic Functions
3.14: Calculus of the Hyperbolic Functions
Derivative of logarithmic functions
1.1: Derivatives of Exponential and Logarithmic Functions
Derivative of the general exponential function
1.1: Derivatives of Exponential and Logarithmic Functions
Derivative of the general logarithmic function
1.1: Derivatives of Exponential and Logarithmic Functions
Derivative of the natural exponential function
1.1: Derivatives of Exponential and Logarithmic Functions
Derivative of the natural logarithmic function
1.1: Derivatives of Exponential and Logarithmic Functions
differential equation
5.2: Basics of Differential Equations
disk method
3.3: Determining Volumes by Slicing
divergence of a series
6.3: Infinite Series
Divergence Test
6.4: The Divergence and Integral Tests
divergent sequence
6.2: Sequences
doubling time
3.13: Exponential Growth and Decay

E

equilibrium solution
5.3: Direction Fields and Numerical Methods
Euler's Method
5.3: Direction Fields and Numerical Methods
exponential function
3.12: Integrals, Exponential Functions, and Logarithms

F

family of solutions
5.2: Basics of Differential Equations
Fibonacci Numbers
6.2: Sequences
first moment
3.11: Moments and Centers of Mass
frustum
3.6: Arc Length of a Curve and Surface Area
fundamental theorem of calculus
2.5: The Fundamental Theorem of Calculus
fundamental theorem of calculus, part 1
2.5: The Fundamental Theorem of Calculus
fundamental theorem of calculus, part 2
2.5: The Fundamental Theorem of Calculus

G

general solution
5.2: Basics of Differential Equations
Geometric Sequences
6.2: Sequences
geometric series
6.3: Infinite Series
6.6: Alternating Series
growth rate
5.5: The Logistic Equation

H

harmonic series
6.3: Infinite Series
Hooke's law
3.9: More Physical Applications of Integration
hydrostatic pressure
3.9: More Physical Applications of Integration

I

improper integral
4.9: Improper Integrals
index variable
6.2: Sequences
infinite series
6.3: Infinite Series
initial population
5.5: The Logistic Equation
initial value
5.2: Basics of Differential Equations
initial velocity
5.2: Basics of Differential Equations
integrable function
2.4: The Definite Integral
Integral Test
6.4: The Divergence and Integral Tests
Integrals of Exponential Functions
2.9: Integrals Involving Exponential and Logarithmic Functions
Integrals That Produce Inverse Trigonometric Functions
2.10: Integrals Resulting in Inverse Trigonometric Functions and Related Integration Techniques
integrand
2.4: The Definite Integral
integrating factor
5.6: First-order Linear Equations
integration by parts
4.2: Integration by Parts
integration by substitution
2.7: Substitution
Integration Formulas Involving Logarithmic Functions
2.9: Integrals Involving Exponential and Logarithmic Functions
integration table
4.6: Other Strategies for Integration
interval of convergence
7.2: Power Series and Functions

L

Limit Comparison Test
6.5: Comparison Tests
limit of a sequence
6.2: Sequences
linear
5.6: First-order Linear Equations
logistic differential equation
5.5: The Logistic Equation

lower sum
2.3: Approximating Areas

M

Maclaurin polynomial
7.4: Taylor and Maclaurin Series
Maclaurin series
7.4: Taylor and Maclaurin Series
Mean Value Theorem for Integrals
2.5: The Fundamental Theorem of Calculus
method of cylindrical shells
3.4b: Volumes of Revolution- Cylindrical Shells OS
method of equating coefficients
4.5: Partial Fractions
method of exhaustion
2.3: Approximating Areas
method of strategic substitution
4.5: Partial Fractions
Midpoint Rule
4.7: Numerical Integration
Moments of Inertia
3.11: Moments and Centers of Mass
Monotone Convergence Theorem
6.2: Sequences
monotone sequence
6.2: Sequences

N

Natural logarithm function
3.12: Integrals, Exponential Functions, and Logarithms
net change theorem
2.6: Integration Formulas and the Net Change Theorem
net signed area
2.4: The Definite Integral
Newton’s Law of Cooling
3.13: Exponential Growth and Decay
5.4: Separable Differential Equations
nonelementary integral
7.5: Working with Taylor Series
numerical integration
4.7: Numerical Integration

O

order of a differential equation
5.2: Basics of Differential Equations

P

partial fraction decomposition
4.5: Partial Fractions
partial sum
6.3: Infinite Series
particular solution
5.2: Basics of Differential Equations
partition
2.3: Approximating Areas
phase line
5.5: The Logistic Equation
population growth
3.13: Exponential Growth and Decay
Power Reduction Formula
4.3: Trigonometric Integrals
power series
7: Power Series
7.2: Power Series and Functions

R

radial density
3.9: More Physical Applications of Integration
radius of convergence
7.2: Power Series and Functions
ratio test
6.7: Ratio and Root Tests
recurrence relation
6.2: Sequences
regular partition
2.3: Approximating Areas
relative error
4.7: Numerical Integration
Remainder Estimate
6.4: The Divergence and Integral Tests
repeated linear factors
4.5: Partial Fractions
Riemann Rearrangement Theorem
6.6: Alternating Series
Riemann sum
2.3: Approximating Areas
root test
6.7: Ratio and Root Tests

S

separation of variables
5.4: Separable Differential Equations
sequence
6.2: Sequences
sequences
6.2: Sequences
shell method
3.4: Volumes of Revolution- The Shell Method
sigma notation
2.3: Approximating Areas
slicing method
3.3: Determining Volumes by Slicing
slope field
5.3: Direction Fields and Numerical Methods
solid of revolution
3.3: Determining Volumes by Slicing
3.4b: Volumes of Revolution- Cylindrical Shells OS
solution curve
5.3: Direction Fields and Numerical Methods
solution to a differential equation
5.2: Basics of Differential Equations
Squeeze Theorem for Sequences
6.2: Sequences
step size
5.3: Direction Fields and Numerical Methods
summation notation
2.3: Approximating Areas
surface area
3.6: Arc Length of a Curve and Surface Area
surface of revolution
3.6: Arc Length of a Curve and Surface Area

T

Taylor polynomials
7.4: Taylor and Maclaurin Series
Taylor series
7.4: Taylor and Maclaurin Series
Taylor’s theorem with remainder
7.4: Taylor and Maclaurin Series
telescoping series
6.3: Infinite Series
term
6.2: Sequences
The Logistic Equation
5.5: The Logistic Equation
Theorem of Pappus
3.11: Moments and Centers of Mass
threshold population
5.5: The Logistic Equation
total area
2.4: The Definite Integral
trigonometric integral
4.3: Trigonometric Integrals
trigonometric substitution
4.4: Trigonometric Substitution

U

upper sum
2.3: Approximating Areas

V

variable of integration
2.4: The Definite Integral
Volume by Shells
3.4: Volumes of Revolution- The Shell Method
3.4b: Volumes of Revolution- Cylindrical Shells OS

W

washer method
3.3: Determining Volumes by Slicing
work
3.9: More Physical Applications of Integration
work done by a force
3.8: Using Integration to Determine Work

Glossary

Sample Word 1 | Sample Definition 1

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      - 2.3E: Approximating Areas (Exercises) - *CC BY-NC-SA 4.0*
    - 2.4: The Definite Integral - *CC BY-NC-SA 4.0*
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      - 2.10E: Exercises for Integrals Resulting in Inverse Trigonometric Functions - *CC BY-NC-SA 4.0*
    - 2.11: Chapter 5 Review Exercises - *CC BY-NC-SA 4.0*
  - 3: Applications of Integration - *CC BY-NC-SA 4.0*
    - 3.1: Prelude to Applications of Integration - *CC BY-NC-SA 4.0*
    - 3.2: Areas between Curves - *CC BY-NC-SA 4.0*
      - 3.2E: Exercises for Section 6.1 - *CC BY-NC-SA 4.0*
    - 3.3: Determining Volumes by Slicing - *CC BY-NC-SA 4.0*
      - 3.3E: Exercises for Volumes of Common Cross-Section and Disk/Washer Method - *CC BY-NC-SA 4.0*
    - 3.4: Volumes of Revolution- The Shell Method - *CC BY-NC-SA 3.0*
      - 3.4b: Volumes of Revolution- Cylindrical Shells OS - *CC BY-NC-SA 4.0*
      - 3.4E: Exercises for the Shell Method - *CC BY-NC-SA 4.0*
    - 3.5: Arc Length and Surface Area - *CC BY-NC-SA 4.0*
    - 3.6: Arc Length of a Curve and Surface Area - *CC BY-NC-SA 4.0*
    - 3.7: Exercises for Section 6.4 - *CC BY-NC-SA 4.0*
    - 3.8: Using Integration to Determine Work - *CC BY-NC-SA 4.0*
    - 3.9: More Physical Applications of Integration - *CC BY-NC-SA 4.0*
    - 3.10: Exercises on Work - *CC BY-NC-SA 4.0*
    - 3.11: Moments and Centers of Mass - *CC BY-NC-SA 4.0*
      - 3.11E: Exercises for Section 6.6 - *CC BY-NC-SA 4.0*
    - 3.12: Integrals, Exponential Functions, and Logarithms - *CC BY-NC-SA 4.0*

- 3.12E: Exercises for Section 6.7 - *CC BY-NC-SA 4.0*
- 3.13: Exponential Growth and Decay - *CC BY-NC-SA 4.0*
  - 3.13E: Exercises for Section 6.8 - *CC BY-NC-SA 4.0*
- 3.14: Calculus of the Hyperbolic Functions - *CC BY-NC-SA 4.0*
  - 3.14E: Exercises for Section 6.9 - *CC BY-NC-SA 4.0*
- 3.15: Chapter 6 Review Exercises - *CC BY-NC-SA 4.0*

4: Techniques of Integration - *CC BY-NC-SA 4.0*
- 4.1: Prelude to Techniques of Integration - *CC BY-NC-SA 4.0*
- 4.2: Integration by Parts - *CC BY-NC-SA 4.0*
  - 4.2E: Exercises for Integration by Parts - *CC BY-NC-SA 4.0*
- 4.3: Trigonometric Integrals - *CC BY-NC-SA 4.0*
  - 4.3E: Exercises for Trigonometric Integrals - *CC BY-NC-SA 4.0*
- 4.4: Trigonometric Substitution - *CC BY-NC-SA 4.0*
  - 4.4E: Exercises for Trigonometric Substitution - *CC BY-NC-SA 4.0*
- 4.5: Partial Fractions - *CC BY-NC-SA 4.0*
  - 4.5E: Exercises for Integration by Partial Fractions - *CC BY-NC-SA 4.0*
- 4.6: Other Strategies for Integration - *CC BY-NC-SA 4.0*
  - 4.6E: Exercises for Section 7.5 - *CC BY-NC-SA 4.0*
- 4.7: Numerical Integration - *CC BY-NC-SA 4.0*
  - 4.7E: Exercises for Numerical Integration - *CC BY-NC-SA 4.0*
- 4.8: L'Hôpital's Rule - *CC BY-NC-SA 4.0*
  - 4.8E: Exercises for L'Hôpital's Rule - *CC BY-NC-SA 4.0*
- 4.9: Improper Integrals - *CC BY-NC-SA 4.0*
  - 4.9E: Exercises for Improper Integrals - *CC BY-NC-SA 4.0*
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- 5.1: Prelude to Differential Equations - *CC BY-NC-SA 4.0*
- 5.2: Basics of Differential Equations - *CC BY-NC-SA 4.0*
  - 5.2E: Exercises for Basics of Differential Equations - *CC BY-NC-SA 4.0*
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  - 5.3E: Exercises for Direction Fields and Numerical Methods - *CC BY-NC-SA 4.0*
- 5.4: Separable Differential Equations - *CC BY-NC-SA 4.0*
  - 5.4E: Exercises for Separable Differential Equations - *CC BY-NC-SA 4.0*
- 5.5: The Logistic Equation - *CC BY-NC-SA 4.0*
  - 5.5E: Exercises for the Logistic Equation - *CC BY-NC-SA 4.0*
- 5.6: First-order Linear Equations - *CC BY-NC-SA 4.0*
  - 5.6E: Exercises for Section 8.5 - *CC BY-NC-SA 4.0*
- 5.7: Chapter 8 Review Exercises - *CC BY-NC-SA 4.0*

6: Sequences and Series - *CC BY-NC-SA 4.0*
- 6.1: Prelude to Sequence and Series - *CC BY-NC-SA 4.0*
- 6.2: Sequences - *CC BY-NC-SA 4.0*
  - 6.2E: Exercises for Sequences - *CC BY-NC-SA 4.0*
- 6.3: Infinite Series - *CC BY-NC-SA 4.0*
  - 6.3E: Exercises for Infinite Series - *CC BY-NC-SA 4.0*
- 6.4: The Divergence and Integral Tests - *CC BY-NC-SA 4.0*
  - 6.4E: Exercises for Divergence and Integral Tests - *CC BY-NC-SA 4.0*
- 6.5: Comparison Tests - *CC BY-NC-SA 4.0*
  - 6.5E: Exercises for Comparison Test - *CC BY-NC-SA 4.0*
- 6.6: Alternating Series - *CC BY-NC-SA 4.0*
  - 6.6E: Exercises for Alternating Series - *CC BY-NC-SA 4.0*
- 6.7: Ratio and Root Tests - *CC BY-NC-SA 4.0*
  - 6.7E: Exercises for Ratio and Root Tests - *CC BY-NC-SA 4.0*
- 6.8: Chapter 9 Review Exercises - *CC BY-NC-SA 4.0*

7: Power Series - *CC BY-NC-SA 4.0*
- 7.1: Prelude to Power Series - *CC BY-NC-SA 4.0*
- 7.2: Power Series and Functions - *CC BY-NC-SA 4.0*
  - 7.2E: Exercises for Section 10.1 - *CC BY-NC-SA 4.0*
- 7.3: Properties of Power Series - *CC BY-NC-SA 4.0*
  - 7.3E: Exercises for Section 10.2 - *CC BY-NC-SA 4.0*

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  - 7.4E: Exercises for Taylor Polynomials and Taylor Series - *CC BY-NC-SA 4.0*
- 7.5: Working with Taylor Series - *CC BY-NC-SA 4.0*
  - 7.5E: Exercises for Section 10.4 - *CC BY-NC-SA 4.0*
- 7.6: Chapter 10 Review Exercises - *CC BY-NC-SA 4.0*

8: Appendices - *CC BY-NC-SA 4.0*

- 8.1: Trigonometric Identities - *CC BY-NC-SA 4.0*
- 8.2: Review of Derivative Rules - *CC BY-NC-SA 4.0*
- 8.3: Table of Derivatives - *CC BY-NC-SA 4.0*
- 8.4: Table of Integrals - *CC BY-NC-SA 4.0*

Back Matter - *Undeclared*

- Index - *Undeclared*
- Glossary - *Undeclared*
- Detailed Licensing - *Undeclared*
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- Math C185: Calculus II (Everett) - *CC BY-NC-SA 4.0*
  - Front Matter - *Undeclared*
    - TitlePage - *Undeclared*
    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
    - Licensing - *Undeclared*
  - 1: Calculus I Review - *CC BY-NC-SA 4.0*
    - 1.1: Derivatives of Exponential and Logarithmic Functions - *CC BY-NC-SA 4.0*
    - 1.2: Exercises on Derivatives of Logarithms and Exponential Functions - *CC BY-NC-SA 4.0*
    - 1.3: Derivatives of the Inverse Trigonometric Functions - *CC BY-NC-SA 4.0*
  - 2: Integration - *CC BY-NC-SA 4.0*
    - 2.1: Antiderivatives - *CC BY-NC-SA 4.0*
      - 2.1E: Antiderivative and Indefinite Integral Exercises - *CC BY-NC-SA 4.0*
    - 2.2: Prelude to Integration - *CC BY-NC-SA 4.0*
    - 2.3: Approximating Areas - *CC BY-NC-SA 4.0*
      - 2.3E: Approximating Areas (Exercises) - *CC BY-NC-SA 4.0*
    - 2.4: The Definite Integral - *CC BY-NC-SA 4.0*
    - 2.5: The Fundamental Theorem of Calculus - *CC BY-NC-SA 4.0*
    - 2.6: Integration Formulas and the Net Change Theorem - *CC BY-NC-SA 4.0*
    - 2.7: Substitution - *CC BY-NC-SA 4.0*
    - 2.8: 5.5E and 5.6E u-Substitution Exercises - *CC BY-NC-SA 4.0*
    - 2.9: Integrals Involving Exponential and Logarithmic Functions - *CC BY-NC-SA 4.0*
    - 2.10: Integrals Resulting in Inverse Trigonometric Functions and Related Integration Techniques - *CC BY-NC-SA 4.0*
      - 2.10E: Exercises for Integrals Resulting in Inverse Trigonometric Functions - *CC BY-NC-SA 4.0*
    - 2.11: Chapter 5 Review Exercises - *CC BY-NC-SA 4.0*
  - 3: Applications of Integration - *CC BY-NC-SA 4.0*
    - 3.1: Prelude to Applications of Integration - *CC BY-NC-SA 4.0*
    - 3.2: Areas between Curves - *CC BY-NC-SA 4.0*
      - 3.2E: Exercises for Section 6.1 - *CC BY-NC-SA 4.0*
    - 3.3: Determining Volumes by Slicing - *CC BY-NC-SA 4.0*
      - 3.3E: Exercises for Volumes of Common Cross-Section and Disk/Washer Method - *CC BY-NC-SA 4.0*
    - 3.4: Volumes of Revolution- The Shell Method - *CC BY-NC-SA 3.0*
      - 3.4b: Volumes of Revolution- Cylindrical Shells OS - *CC BY-NC-SA 4.0*
      - 3.4E: Exercises for the Shell Method - *CC BY-NC-SA 4.0*
    - 3.5: Arc Length and Surface Area - *CC BY-NC-SA 4.0*
    - 3.6: Arc Length of a Curve and Surface Area - *CC BY-NC-SA 4.0*
    - 3.7: Exercises for Section 6.4 - *CC BY-NC-SA 4.0*
    - 3.8: Using Integration to Determine Work - *CC BY-NC-SA 4.0*
    - 3.9: More Physical Applications of Integration - *CC BY-NC-SA 4.0*
    - 3.10: Exercises on Work - *CC BY-NC-SA 4.0*
    - 3.11: Moments and Centers of Mass - *CC BY-NC-SA 4.0*
      - 3.11E: Exercises for Section 6.6 - *CC BY-NC-SA 4.0*
    - 3.12: Integrals, Exponential Functions, and Logarithms - *CC BY-NC-SA 4.0*

- 3.12E: Exercises for Section 6.7 - *CC BY-NC-SA 4.0*
- 3.13: Exponential Growth and Decay - *CC BY-NC-SA 4.0*
  - 3.13E: Exercises for Section 6.8 - *CC BY-NC-SA 4.0*
- 3.14: Calculus of the Hyperbolic Functions - *CC BY-NC-SA 4.0*
  - 3.14E: Exercises for Section 6.9 - *CC BY-NC-SA 4.0*
- 3.15: Chapter 6 Review Exercises - *CC BY-NC-SA 4.0*

4: Techniques of Integration - *CC BY-NC-SA 4.0*
- 4.1: Prelude to Techniques of Integration - *CC BY-NC-SA 4.0*
- 4.2: Integration by Parts - *CC BY-NC-SA 4.0*
  - 4.2E: Exercises for Integration by Parts - *CC BY-NC-SA 4.0*
- 4.3: Trigonometric Integrals - *CC BY-NC-SA 4.0*
  - 4.3E: Exercises for Trigonometric Integrals - *CC BY-NC-SA 4.0*
- 4.4: Trigonometric Substitution - *CC BY-NC-SA 4.0*
  - 4.4E: Exercises for Trigonometric Substitution - *CC BY-NC-SA 4.0*
- 4.5: Partial Fractions - *CC BY-NC-SA 4.0*
  - 4.5E: Exercises for Integration by Partial Fractions - *CC BY-NC-SA 4.0*
- 4.6: Other Strategies for Integration - *CC BY-NC-SA 4.0*
  - 4.6E: Exercises for Section 7.5 - *CC BY-NC-SA 4.0*
- 4.7: Numerical Integration - *CC BY-NC-SA 4.0*
  - 4.7E: Exercises for Numerical Integration - *CC BY-NC-SA 4.0*
- 4.8: L'Hôpital's Rule - *CC BY-NC-SA 4.0*
  - 4.8E: Exercises for L'Hôpital's Rule - *CC BY-NC-SA 4.0*
- 4.9: Improper Integrals - *CC BY-NC-SA 4.0*
  - 4.9E: Exercises for Improper Integrals - *CC BY-NC-SA 4.0*
- 4.10: Chapter 7 Review Exercises - *CC BY-NC-SA 4.0*

5: Introduction to Differential Equations - *CC BY-NC-SA 4.0*
- 5.1: Prelude to Differential Equations - *CC BY-NC-SA 4.0*
- 5.2: Basics of Differential Equations - *CC BY-NC-SA 4.0*
  - 5.2E: Exercises for Basics of Differential Equations - *CC BY-NC-SA 4.0*
- 5.3: Direction Fields and Numerical Methods - *CC BY-NC-SA 4.0*
  - 5.3E: Exercises for Direction Fields and Numerical Methods - *CC BY-NC-SA 4.0*
- 5.4: Separable Differential Equations - *CC BY-NC-SA 4.0*
  - 5.4E: Exercises for Separable Differential Equations - *CC BY-NC-SA 4.0*
- 5.5: The Logistic Equation - *CC BY-NC-SA 4.0*
  - 5.5E: Exercises for the Logistic Equation - *CC BY-NC-SA 4.0*
- 5.6: First-order Linear Equations - *CC BY-NC-SA 4.0*
  - 5.6E: Exercises for Section 8.5 - *CC BY-NC-SA 4.0*
- 5.7: Chapter 8 Review Exercises - *CC BY-NC-SA 4.0*

6: Sequences and Series - *CC BY-NC-SA 4.0*
- 6.1: Prelude to Sequence and Series - *CC BY-NC-SA 4.0*
- 6.2: Sequences - *CC BY-NC-SA 4.0*
  - 6.2E: Exercises for Sequences - *CC BY-NC-SA 4.0*
- 6.3: Infinite Series - *CC BY-NC-SA 4.0*
  - 6.3E: Exercises for Infinite Series - *CC BY-NC-SA 4.0*
- 6.4: The Divergence and Integral Tests - *CC BY-NC-SA 4.0*
  - 6.4E: Exercises for Divergence and Integral Tests - *CC BY-NC-SA 4.0*
- 6.5: Comparison Tests - *CC BY-NC-SA 4.0*
  - 6.5E: Exercises for Comparison Test - *CC BY-NC-SA 4.0*
- 6.6: Alternating Series - *CC BY-NC-SA 4.0*
  - 6.6E: Exercises for Alternating Series - *CC BY-NC-SA 4.0*
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  - 6.7E: Exercises for Ratio and Root Tests - *CC BY-NC-SA 4.0*
- 6.8: Chapter 9 Review Exercises - *CC BY-NC-SA 4.0*

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- 7.1: Prelude to Power Series - *CC BY-NC-SA 4.0*
- 7.2: Power Series and Functions - *CC BY-NC-SA 4.0*
  - 7.2E: Exercises for Section 10.1 - *CC BY-NC-SA 4.0*
- 7.3: Properties of Power Series - *CC BY-NC-SA 4.0*
  - 7.3E: Exercises for Section 10.2 - *CC BY-NC-SA 4.0*

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  - 7.5E: Exercises for Section 10.4 - *CC BY-NC-SA 4.0*
- 7.6: Chapter 10 Review Exercises - *CC BY-NC-SA 4.0*

8: Appendices - *CC BY-NC-SA 4.0*

- 8.1: Trigonometric Identities - *CC BY-NC-SA 4.0*
- 8.2: Review of Derivative Rules - *CC BY-NC-SA 4.0*
- 8.3: Table of Derivatives - *CC BY-NC-SA 4.0*
- 8.4: Table of Integrals - *CC BY-NC-SA 4.0*

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- Index - *Undeclared*
- Glossary - *Undeclared*
- Detailed Licensing - *Undeclared*
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