COLLEGE ALGEBRA (NWTC)



Northeast Wisconsin Technical College

College Algebra (NWTC)

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College Algebra (NWTC)

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Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Functions

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1.1E: Functions and Function Notation

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1.1: Functions and Function Notation

What is a Function?

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Definition: Function

A **function** is a rule for a relationship between an input, or independent, quantity and an output, or dependent, quantity in which each input value uniquely determines one output value. We say "the output is a function of the input."

Example 1.1.1

In the height and age example above, is height a function of age? Is age a function of height?

Solution

In the height and age example above, it would be correct to say that height is a function of age, since each age uniquely determines a height. For example, on my 18th birthday, I had exactly one height of 69 inches.

However, age is not a function of height, since one height input might correspond with more than one output age. For example, for an input height of 70 inches, there is more than one output of age since I was 70 inches at the age of 20 and 21.

Example 1.1.2

At a coffee shop, the menu consists of items and their prices. Is price a function of the item? Is the item a function of the price?

Solution

We could say that price is a function of the item, since each input of an item has one output of a price corresponding to it. We could not say that item is a function of price, since two items might have the same price.

Example 1.1.3

In many classes the overall percentage you earn in the course corresponds to a decimal grade point. Is decimal grade a function of percentage? Is percentage a function of decimal grade?

Solution

For any percentage earned, there would be a decimal grade associated, so we could say that the decimal grade is a function of percentage. That is, if you input the percentage, your output would be a decimal grade. Percentage may or may not be a function of decimal grade, depending upon the teacher's grading scheme. With some grading systems, there are a range of percentages that correspond to the same decimal grade.

Definition: One-to-One Function

Sometimes in a relationship each input corresponds to exactly one output, and every output corresponds to exactly one input. We call this kind of relationship a **one-to-one function**.

From Example 1.1.3, if each unique percentage corresponds to one unique decimal grade point and each unique decimal grade point corresponds to one unique percentage then it is a one-to-one function.

Exercise 1.1.1

Let's consider bank account information.

a. Is your balance a function of your bank account number? *(if you input a bank account number does it make sense that the output is your balance?)*
b. Is your bank account number a function of your balance?*(if you input a balance does it make sense that the output is your bank account number?)*

Answer a

Yes: for each bank account, there would be one balance associated

Answer b

No: there could be several bank accounts with the same balance

Function Notation

To simplify writing out expressions and equations involving functions, a simplified notation is often used. We also use descriptive variables to help us remember the meaning of the quantities in the problem.

Rather than write "height is a function of age", we could use the descriptive variable h to represent height and we could use the descriptive variable a to represent age.

"height is a function of age" if we name the function f we write "h is f of a" or more simply $h = f(a)$ we could instead name the function h and write $h(a)$ which is read "h of a"

Remember we can use any variable to name the function; the notation $h(a)$ shows us that h depends on a. The value "a" must be put into the function "h" to get a result. Be careful - the parentheses indicate that age is input into the function (Note: do not confuse these parentheses with multiplication!).

Definition: Function Notation

The notation output = f(input) defines a function named f. This would be read "output is f of input"

Example 1.1.4

Introduce function notation to represent a function that takes as input the name of a month, and gives as output the number of days in that month.

Solution

The number of days in a month is a function of the name of the month, so if we name the function f, we could write "days = f(month)" or $d = f(m)$. If we simply name the function d, we could write $d(m)$

For example, d(March) = 31, since March has 31 days. The notation $d(m)$ reminds us that the number of days, d (the output) is dependent on the name of the month, m (the input)

Example 1.1.5

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does f(2005) = 300 tell us?

Solution

When we read f(2005) = 300, we see the input quantity is 2005, which is a value for the input quantity of the function, the year (y). The output value is 300, the number of police officers (N), a value for the output quantity. Remember $N = f(y)$. This tells us that in the year 2005 there were 300 police officers in the town.

Tables as Functions

Functions can be represented in many ways: Words (as we did in the last few examples), tables of values, graphs, or formulas. Represented as a table, we are presented with a list of input and output values. In some cases, these values represent everything we know about the relationship, while in other cases the table is simply providing us a few select values from a more complete relationship.

Table 1: This table represents the input, number of the month (January = 1, February = 2, and so on) while the output is the number of days in that month. This represents everything we know about the months & days for a given year (that is not a leap year)

(input) Month number, m	1	2	3	4	5	6	7	8	9	10	11	12
(output) Days in month, D	31	28	31	30	31	30	31	31	30	31	30	31

Table 2: The table below defines a function $Q = g(n)$. Remember this notation tells us g is the name of the function that takes the input n and gives the output Q.

n	1	2	3	4	5
Q	8	6	7	6	8

Table 3: This table represents the age of children in years and their corresponding heights. This represents just some of the data available for height and ages of children.

(input) a, age in years	5	5	6	7	8	9	10
(output) h, height inches	40	42	44	47	50	52	54

Example 1.1.6

Which of these tables define a function (if any)? Are any of them one-to-one?

Input	Output
2	1
5	3
8	6

Input	Output
-3	5
0	1
4	5

Input	Output
1	0

Input	Output
5	2
5	4

Solution

The first and second tables define functions. In both, each input corresponds to exactly one output. The third table does not define a function since the input value of 5 corresponds with two different output values.

Only the first table is one-to-one; it is both a function, and each output corresponds to exactly one input. Although table 2 is a function, because each input corresponds to exactly one output, each output does not correspond to exactly one input so this function is not one-to-one. Table 3 is not even a function and so we don't even need to consider if it is a one-to-one function.

Exercise 1.1.2

If each percentage earned translated to one letter grade, would this be a function? Is it one-to-one?

Answer

Yes it's a function; No, it's not one-to-one (several percents give the same letter grade)

Solving and Evaluating Functions

When we work with functions, there are two typical things we do: evaluate and solve. Evaluating a function is what we do when we know an input, and use the function to determine the corresponding output. Evaluating will always produce one result, since each input of a function corresponds to exactly one output.

Solving equations involving a function is what we do when we know an output, and use the function to determine the inputs that would produce that output. Solving a function could produce more than one solution, since different inputs can produce the same output.

Example 1.1.7

Using the table shown, where $Q = g(n)$

n	1	2	3	4	5
Q	8	6	7	6	8

a. Evaluate $g(3)$
b. Solve $g(n) = 6$

Solution

Evaluating $g(3)$ (read: "g of 3") means that we need to determine the output value, Q, of function g given the input value of $n = 3$. Looking at the table, we see the output corresponding to $n = 3$ is $Q = 7$, allowing us to conclude $g(3) = 7$.

(b) Solve $g(n) = 6$

Solving $g(n) = 6$ means we need to determine what input values, n, produce an output value of 6. Looking at the table we see there are two solutions: $n = 2$ and $n = 4$.

- When we input 2 into the function g, our output is $Q = 6$
- When we input 4 into the function g, our output is also $Q = 6$

Exercise 1.1.3

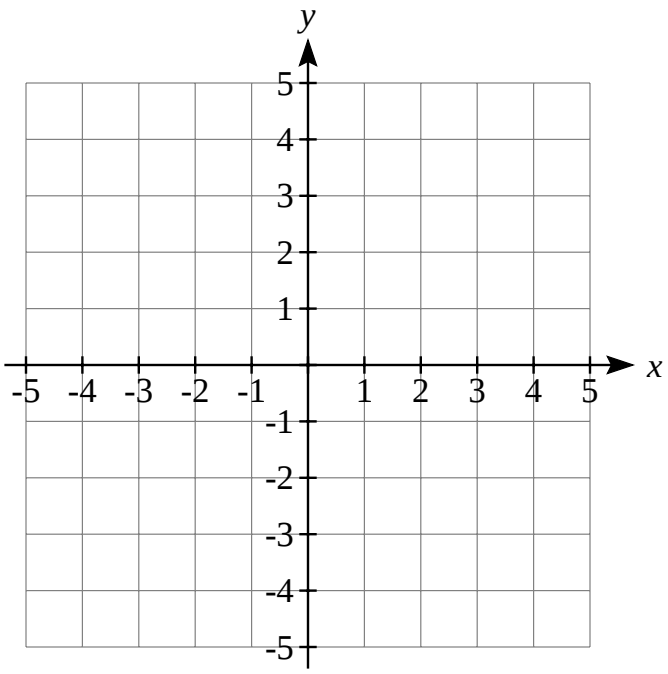
Using the function in Example 1.1.7, evaluate $g(4)$

Answer

When $n = 4$, $Q = g(4) = 6$

Graphs as Functions

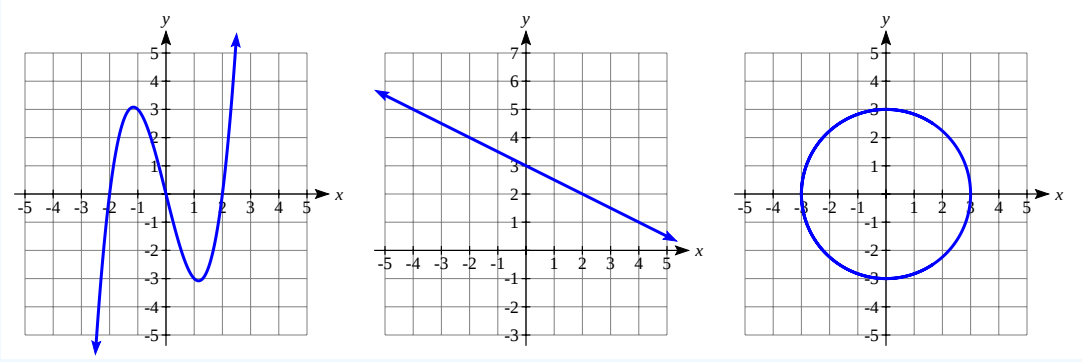
Oftentimes a graph of a relationship can be used to define a function. By convention, graphs are typically created with the input quantity along the horizontal axis and the output quantity along the vertical.



The most common graph has y on the vertical axis and x on the horizontal axis, and we say y is a function of x, or $y = f(x)$ when the function is named f.

Example 1.1.8

Which of these graphs defines a function $y = f(x)$? Which of these graphs defines a one-to-one function?



Solution

Looking at the three graphs above, the first two define a function $y=f(x)$, since for each input value along the horizontal axis there is exactly one output value corresponding, determined by the y-value of the graph. The 3 rd graph does not define a function $y=f(x)$ since some input values, such as $x=2$, correspond with more than one output value.

- Graph 1 is not a one-to-one function. For example, the output value 3 has two corresponding input values, -1 and 2.3
- Graph 2 is a one-to-one function; each input corresponds to exactly one output, and every output corresponds to exactly one input.
- Graph 3 is not even a function so there is no reason to even check to see if it is a one-to-one function.

Vertical Line Test

The **vertical line test** is a handy way to think about whether a graph defines the vertical output as a function of the horizontal input. Imagine drawing vertical lines through the graph. If any vertical line would cross the graph more than once, then the graph does not define only one vertical output for each horizontal input.

Horizontal Line Test

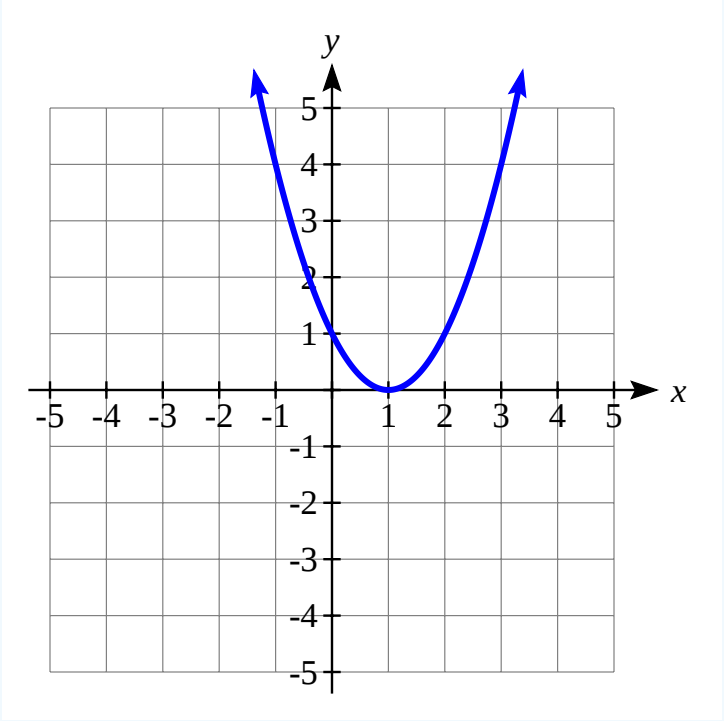
Once you have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line crosses the graph more than once, then the graph does not define a one-to-one function.

Evaluating a function using a graph requires taking the given input and using the graph to look up the corresponding output. Solving a function equation using a graph requires taking the given output and looking on the graph to determine the corresponding input.

Example 1.1.9

Given the graph of $f(x)$

a. Evaluate $f(2)$
b. Solve $f(x)=4$



Solution

(a) To evaluate $f(2)$, we find the input of $x = 2$ on the horizontal axis. Moving up to the graph gives the point (2, 1), giving an output of $y = 1$. $f(2) = 1$.

(b) To solve $f(x) = 4$, we find the value 4 on the vertical axis because if $f(x) = 4$ then 4 is the output. Moving horizontally across the graph gives two points with the output of 4: (-1, 4) and (3, 4). These give the two solution to $f(x) = 4$: $x = -1$ or $x = 3$ This means $f(-1) = 4$ and $f(3) = 4$, or when the input is -1 or 3, the output is 4.

Notice that while the graph in the previous example is a function, getting two input values for the output value of 4 shows us that this function is not one-to-one.

Exercise 1.1.4

Using the graph from example 1.1.9, solve $f(x) = 1$

Answer

There are two points where the output is 1: $x = 0$ or $x = 2$

Formulas as Functions

When possible, it is very convenient to define relationships using formulas. If it is possible to express the output as a formula involving the input quantity, then we can define a function.

Example 1.1.10

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$ if possible.

Solution

To express the relationship in this form, we need to be able to write the relationship where *p* is a function of *n*, which means writing it as *p* = [something involving n].

$$2n + 6p = 12$$

subtract $2n$ from both sides

$$6p = 12 - 2n$$

divide both sides by 6 and simplify

$$p = \frac{12 - 2n}{6} = \frac{12}{6} - \frac{2n}{6} = 2 - \frac{1}{3}n$$

Having rewritten the formula as $p =$, we can now express p as a function:

$$p = f(n) = 2 - \frac{1}{3}n$$

It is important to note that not every relationship can be expressed as a function with a formula.

Note the important feature of an equation written as a function is that the output value can be determined directly from the input by doing evaluations - no further solving is required. This allows the relationship to act as a magic box that takes an input, processes it, and returns an output. Modern technology and computers rely on these functional relationships, since the evaluation of the function can be programmed into machines, whereas solving things is much more challenging.

Example 1.1.11

Express the relationship $x^2 + y^2 = 1$ as a function $y = f(x)$ if possible.

Solution

If we try to solve for *y* in this equation:

$$y^2 = 1 - x^2$$
$$= \pm\sqrt{1 - x^2}$$

We end up with two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$.

As with tables and graphs, it is common to evaluate and solve functions involving formulas. Evaluating will require replacing the input variable in the formula with the value provided and calculating. Solving will require replacing the output variable in the formula with the value provided, and solving for the input(s) that would produce that output.

Example 1.1.12

Given the function $k(t) = t^3 + 2$

a. Evaluate $k(2)$
b. Solve $k(t) = 1$

Solution

(a) To evaluate $k(2)$, we plug in the input value 2 into the formula wherever we see the input variable t, then simplify

$$k(2) = 2^3 + 2$$
$$k(2) = 8 + 2$$

So

$$k(2) = 10$$

(b) To solve $k(t) = 1$, we set the formula for $k(t)$ equal to 1, and solve for the input value that will produce that output

$$k(t) = 1$$

substitute the original formula

$$k(t) = t^3 + 2$$
$$t^3 + 2 = 1$$

subtract 2 from each side

$$t^3 = -1$$

take the cube root of each side

$$t = -1$$

When solving an equation using formulas, you can check your answer by using your solution in the original equation to see if your calculated answer is correct.

We want to know if $k(t) = 1$ is true when $t = -1$.

$$\begin{aligned} (k(-1) &= (-1)^3 + 2 \\ &= -1 + 2 \\ &= 1 \text{ which was the desired result.} \end{aligned}$$

Example 1.1.13

Given the function $h(p) = p^2 + 2p$

a. Evaluate $h(4)$
b. Solve $h(p) = 3$

Solution

To evaluate $h(4)$ we substitute the value 4 for the input variable p in the given function.

(a)

$$\begin{aligned} h(4) &= (4)^2+2(4) \\ &= 16+8 \\ &= 24 \end{aligned}$$

(b) $h(p)=3$ Substitute the original function

$$h(p)=p^2+2p$$

$$p^2+2p=3$$

This is quadratic, so we can rearrange the equation to get it = 0

$$p^2+2p-3=0$$

subtract 3 from each side

$$p^2+2p-3=0$$

this is factorable, so we factor it

$$(p+3)(p-1)=0$$

By the zero factor theorem since $(p+3)(p-1)=0$, either $(p+3)=0$ or $(p-1)=0$ (or both of them equal 0) and so we solve both equations for p, finding $p=-3$ from the first equation and $p=1$ from the second equation.

This gives us the solution: $h(p)=3$ when $p=1$ or $p=-3$

We found two solutions in this case, which tells us this function is not one-to-one.

Exercise 1.1.5

Given the function $g(m)=\sqrt{m-4}$

a. Evaluate $g(5)$
b. Solve $g(m)=2$

Answer

a. $g(5)=\sqrt{5-4}=1$
b. $\sqrt{m-4}=2$. Square both sides to get $m-4=4$. $m=8$

Basic Toolkit Functions

In this text, we will be exploring functions – the shapes of their graphs, their unique features, their equations, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of elements to build from. We call these our "toolkit of functions" – a set of basic named functions for which we know the graph, equation, and special features.

For these definitions we will use *x* as the input variable and $f(x)$ as the output variable.

Toolkit Functions

Linear

- Constant: $f(x)=c$, where *c* is a constant (number)
- Identity: $f(x)=x$
- Absolute Value: $f(x)=|x|$

Power

- Quadratic: $f(x) = x^2$
- Cubic: $f(x) = x^3$
- Reciprocal: $f(x) = \dfrac{1}{x}$
- Reciprocal squared: $f(x) = \dfrac{1}{x^2}$
- Square root: $f(x) = \sqrt[2]{x} = \sqrt{x}$
- Cube root: $f(x) = \sqrt[3]{x}$

You will see these toolkit functions, combinations of toolkit functions, their graphs and their transformations frequently throughout this book. In order to successfully follow along later in the book, it will be very helpful if you can recognize these toolkit functions and their features quickly by name, equation, graph and basic table values.

Not every important equation can be written as $y = f(x)$. An example of this is the equation of a circle. Recall the distance formula for the distance between two points:

$$\text{dist} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1.1.1}$$

A circle with radius r with center at (h, k) can be described as all points (x, y) a distance of r from the center, so using the distance formula,

$$r = \sqrt{(x - h)^2 + (y - k)^2}$$

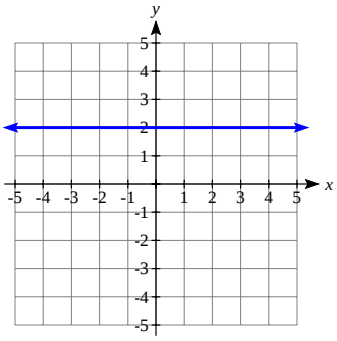
giving the equation of a circle.

Equation of a Circle

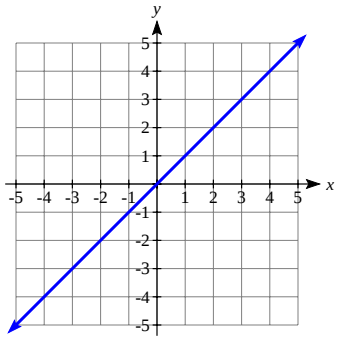
A circle with radius r with center (h, k) has equation

$$r^2 = (x - h)^2 + (y - k)^2 \tag{1.1.2}$$

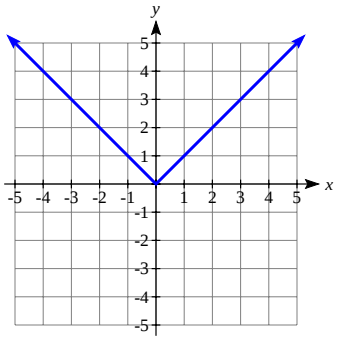
Graphs of the Toolkit Functions



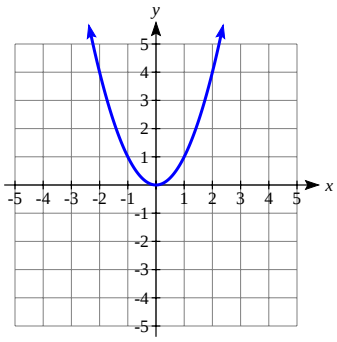
Constant Function: $f(x) = 2$



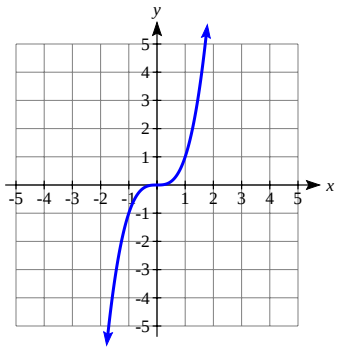
Identity: $f(x) = x$



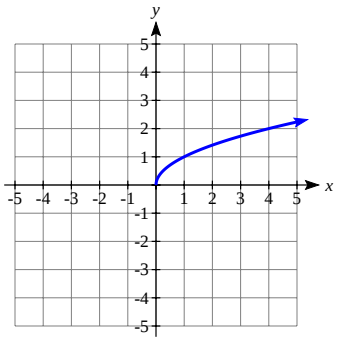
Absolute Value: $f(x) = |x|$



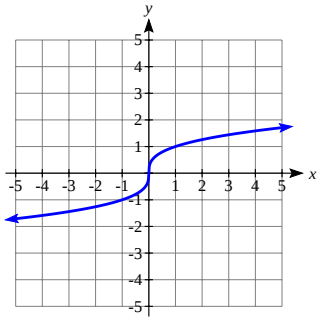
Quadratic: $f(x) = x^2$



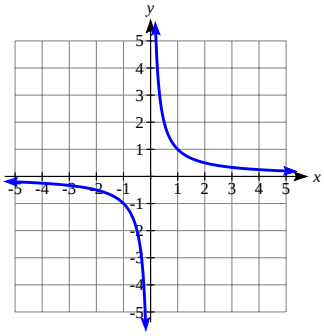
Cubic: $f(x) = x^3$



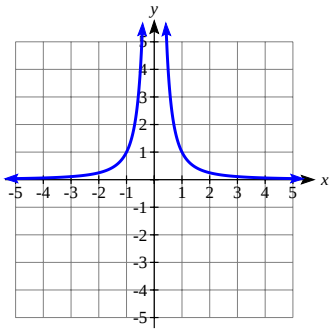
Square root: $f(x) = \sqrt{x}$



Cube root: $f(x) = \sqrt[3]{x}$



Reciprocal: $f(x) = \frac{1}{x}$



Reciprocal squared: $f(x) = \frac{1}{x^2}$

Important Topics of this Section

- Definition of a function
- Input (independent variable)
- Output (dependent variable)
- Definition of a one-to-one function
- Function notation
- Descriptive variables
- Functions in words, tables, graphs & formulas
- Vertical line test
- Horizontal line test
- Evaluating a function at a specific input value
- Solving a function given a specific output value
- Toolkit Functions

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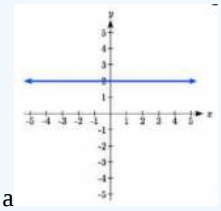
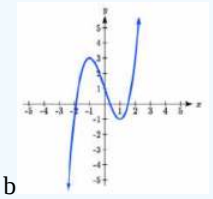
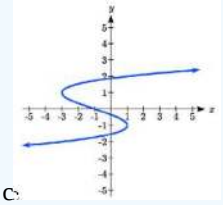
1.1E: Functions and Function Notation

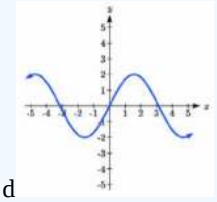
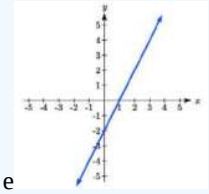
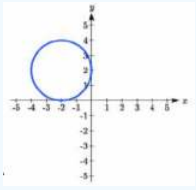
Exercise 1.1

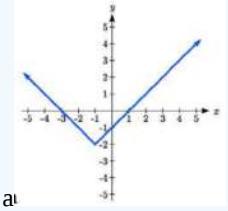
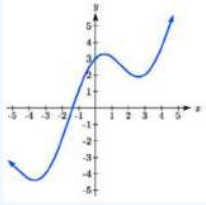
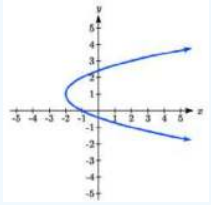
1. The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people. a. The town of Tola has a population of 40,000 and produces 13 tons of garbage each week. Express this information in terms of the function f.
   b. Explain the meaning of the statement $f(5) = 2$.
2. The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$. a. A garden with area 5000 ft ^{2} requires 50 cubic yards of dirt. Express this information in terms of the function g.
   b. Explain the meaning of the statement $g(100) = 1$.
3. Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:

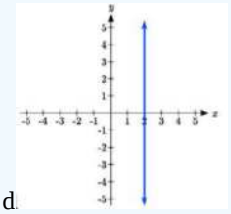
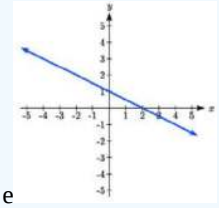
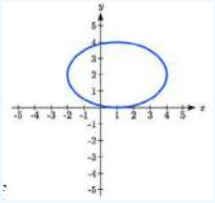
   a. $f(5) = 30$ b. $f(10) = 40$
4. Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:

   a. $h(1) = 200$ b. $h(2) = 350$
5. Select all of the following graphs which represent y as a function of x.

   a  b  c 

   d  e  f 
6. Select all of the following graphs which represent y as a function of x.

   a  b  c 

   d  e  f 
7. Select all of the following tables which represent y as a function of x.

   a.

x	5	10	15
y	3	8	14

   b.

x	5	10	15
y	3	8	8

   c.

x	5	10	10
y	3	8	14

8. Select all of the following tables which represent y as a function of x.

   a.

x	2	6	13
y	3	10	10

   b.

x	2	6	6
y	3	10	14

   c.

x	2	6	13
y	3	10	14

9. Select all of the following tables which represent y as a function of x.

a.

x	y
0	-2
3	1
4	6
8	9
3	1

b.

x	y
-1	-4
2	3
5	4
8	7
12	11

c.

x	y
0	-5
3	1
3	4
9	8
16	13

d.

x	y
-1	-4
1	2
4	2
9	7
12	13

10. Select all of the following tables which represent y as a function of x.

a.

x	y
-4	-2
3	2
6	4
9	7
12	16

b.

x	y
-5	-3
2	1
2	4
7	9
11	10

c.

x	y
-1	-3
1	2
5	4
9	8
1	2

d.

x	y
-1	-5
3	1
5	1
8	7
14	12

11. Select all of the following tables which represent y as a function of x **and** are one-to-one.

a.

x	3	8	12
y	4	7	7

b.

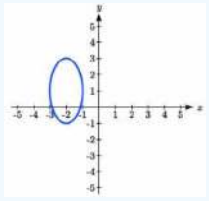
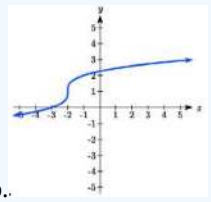
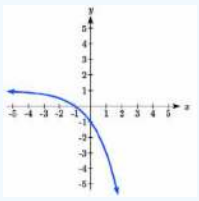
x	3	8	12
y	4	7	13

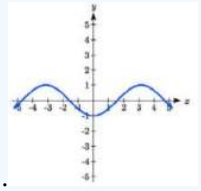
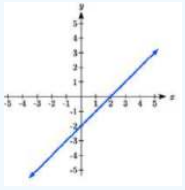
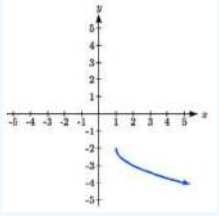
c.

x	3	8	8
y	4	7	13

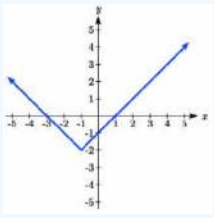
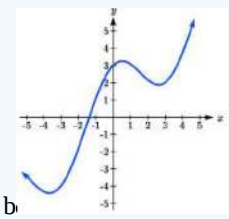
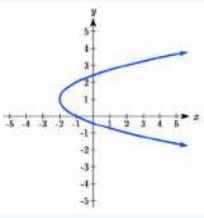
12. Select all of the following tables which represent y as a function of x **and** are one-to-one.

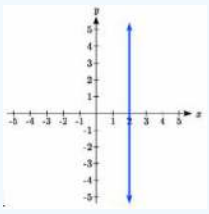
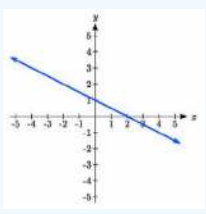
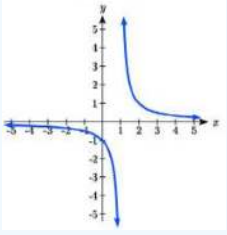
13. Select all of the following graphs which are **one-to-one functions**.

a.  b.  c. 

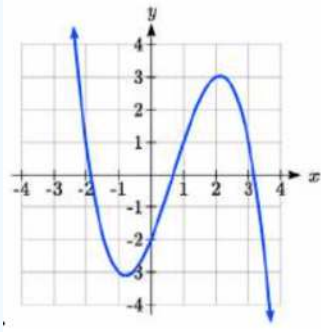
d.  e.  f. 

14. Select all of the following graphs which are **one-to-one functions**.

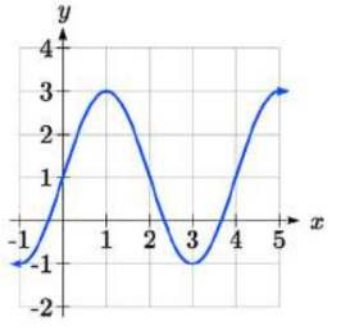
a  b  c 

d  e  f 

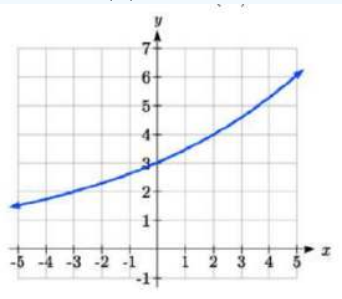
15. Given function $f(x)$ graphed, evaluate $f(1)$ and $f(3)$



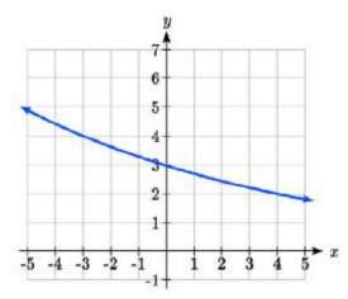
16. Given function $f(x)$ graphed, evaluate $f(1)$ and $f(3)$



17. Given the function $g(x)$ graphed here,
    a. Evaluate $g(2)$
    b. Solve $g(x) = 2$



18. Given the function $f(x)$ graphed here,
    a. Evaluate $f(4)$
    b. Solve $f(x) = 4$



19. Based on the table below,
    a. Evaluate $f(3)$ b. Solve $f(x) = 1$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	74	28	1	53	56	3	36	45	14	47

20. Based on the table below,
    a. Evaluate $f(8)$ b. Solve $f(x)=1$

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	62	8	7	38	86	73	70	39	75	34

    For 21-34 functions, evaluate: $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$

21. $f(x)=4-2x$
22. $f(x)=8-3x$
23. $f(x)=8x^2-7x+3$
24. $f(x)=6x^2-7x+4$
25. $f(x)=-x^3+2x$
26. $f(x)=5x^4+x^2$
27. $f(x)=3+\sqrt{x+3}$
28. $f(x)=4-\sqrt[3]{x-2}$
29. $f(x)=(x-2)(x+3)$
30. $f(x)=(x+3)(x-1)^2$
31. $f(x)=\dfrac{x-3}{x+1}$
32. $f(x)=\dfrac{x-2}{x+2}$
33. $f(x)=2^x$
34. $f(x)=3^x$
35. Suppose $f(x)=x^2+8x-4$. Compute the following:

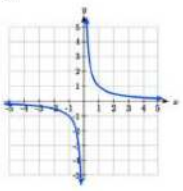
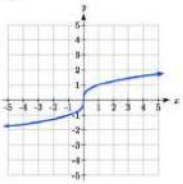
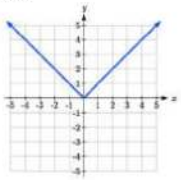
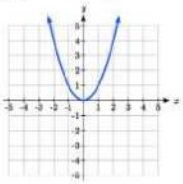
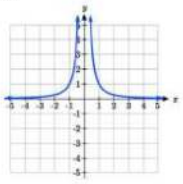
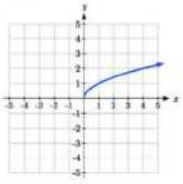
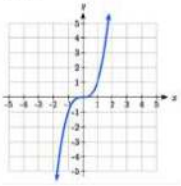
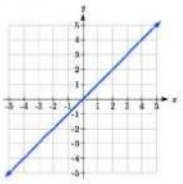
    a. $f(-1)+f(1)$
    b. $f(-1)-f(1)$
36. Suppose $f(x)=x^2+x+3$. Compute the following:

    a. $f(-2)+f(4)$
    b. $f(-2)-f(4)$
37. Let $f(t)=3t+5$

    a. Evaluate $f(0)$
    b. Solve $f(t)=0$
38. Let $g(p)=6-2p$

    a. Evaluate $g(0)$
    b. Solve $g(p)=0$
39. Match each function name with its equation.
    a. $y=x$ i. Cube root
    b. $y=x^3$ ii. Reciprocal
    c. $y=\sqrt[3]{x}$ iii. Linear
    d. $y=\dfrac{1}{x}$ iv. Square Root
    e. $y=x^2$ v. Absolute Value
    f. $y=\sqrt{x}$ vi. Quadratic
    g. $y=|x|$
    h. $y=\dfrac{1}{x^2}$
40. Match each graph with its equation.
    a. $y=x$
    b. $y=x^3$

c. $y = \sqrt[3]{x}$
d. $y = \dfrac{1}{x}$
e. $y = x^2$
f. $y = \sqrt{x}$
g. $y = |x|$
h. $y = \dfrac{1}{x^2}$

i. 
ii. 
iii. 
iv. 
v. 
vi. 
vii. 
viii. 

41. Match each table with its equation.

a. $y = x^2$
b. $y = x$
c. $y = \sqrt{x}$
d. $y = 1/x$
e. $y = |x|$
f. $y = x^3$

i.

In	Out
-2	-0.5
-1	-1
0	_
1	1
2	0.5
3	0.33

ii.

In	Out
-2	-2
-1	-1
0	0
1	1
2	2
3	3

iii.

In	Out
-2	-8
-1	-1
0	0
1	1
2	8
3	27

iv.

In	Out
-2	4
-1	1
0	0
1	1
2	4
3	9

v.

In	Out
-2	_
-1	_
0	0
1	1
4	2
9	3

vi.

In	Out
-2	2
-1	1
0	0
1	1
2	2
3	3

42. Match each equation with its table

a. Quadratic
b. Absolute Value
c. Square Root

d. Linear
e. Cubic
f. Reciprocal

i.

In	Out
-2	-0.5
-1	-1
0	_
1	1
2	0.5
3	0.33

ii.

In	Out
-2	-2
-1	-1
0	0
1	1
2	2
3	3

iii.

In	Out
-2	-8
-1	-1
0	0
1	1
2	8
3	27

iv.

In	Out
-2	4
-1	1
0	0
1	1
2	4
3	9

v.

In	Out
-2	_
-1	_
0	0
1	1
4	2
9	3

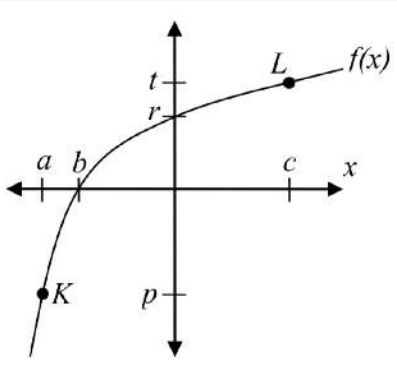
vi.

In	Out
-2	2
-1	1
0	0
1	1
2	2
3	3

43. Write the equation of the circle centered at (3, -9) with radius 6.
44. Write the equation of the circle centered at (9, -8) with radius 11.
45. Sketch a reasonable graph for each of the following functions. [UW] a. Height of a person depending on age.
    b. Height of the top of your head as you jump on a pogo stick for 5 seconds.
    c. The amount of postage you must put on a first class letter, depending on the weight of the letter.
46. Sketch a reasonable graph for each of the following functions. [UW]

    a. Distance of your big toe from the ground as you ride your bike for 10 seconds.
    b. Your height above the water level in a swimming pool after you dive off the high board.
    c. The percentage of dates and names you'll remember for a history test, depending on the time you study.
47. Using the graph shown,

    a. Evaluate $f(c)$
    b. Solve $f(x) = p$
    c. Suppose $f(b) = z$. Find $f(z)$
    d. What are the coordinates of points L and K?



48. Dave leaves his office in Padelford Hall on his way to teach in Gould Hall. Below are several different scenarios. In each case, sketch a plausible (reasonable) graph of the function $s = d(t)$ which keeps track of Dave's distance s from Padelford Hall at time t. Take distance units to be "feet" and time units to be "minutes." Assume Dave's path to Gould Hall is long a straight line which is 2400 feet long. [UW]



a. Dave leaves Padelford Hall and walks at a constant spend until he reaches Gould Hall 10 minutes later.
b. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute. He then continues on to Gould Hall at the same constant speed he had when he originally left Padelford Hall.
c. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave then continues on to Gould Hall at twice the constant speed he had when he originally left Padelford Hall.
d. Dave leaves Padelford Hall and walks at a constant speed. It takes him 6 minutes to reach the half-way point. Then he gets confused and stops for 1 minute to figure out where he is. Dave is totally lost, so he simply heads back to his office, walking the same constant speed he had when he originally left Padelford Hall.
e. Dave leaves Padelford heading for Gould Hall at the same instant Angela leaves Gould Hall heading for Padelford Hall. Both walk at a constant speed, but Angela walks twice as fast as Dave. Indicate a plot of "distance from Padelford" vs. "time" for the both Angela and Dave.
f. Suppose you want to sketch the graph of a new function $s = g(t)$ that keeps track of Dave's distance s from Gould Hall at time t. How would your graphs change in (a) - (e)?

Answer

1. (a) $f(40) = 13$, because the input 40 (in thousands of people) gives the output 13 (in tons of garbage)

(b) $f(5) = 2$, means that 5000 people produce 2 tons of garbage per week.

3. (a) In 1995 (5 years after 1990) there were 30 ducks in the lake
b. In 2000 (10 years after 1990) there were 40 ducks in the lake

5. Graphs (a) (b) (d) and (e) represent y as a function of x because for every value of x there is only one value for y. Graphs (c) and (f) are not functions because they contain points that have more than one output for a given input, or values for x that have 2 or more values for y.

7. Tables (a) and (b) represent y as a function of x because for every value of x there is only one value for y. Table (c) is not a function because for the input $x = 10$, there are two different outputs for y.

9. Tables (a) (b) and (d) represent y as a function of x because for every value of x there is only one value for y. Table (c) is not a function because for the input $x = 3$, there are two different outputs for y.

11. Table (b) represents y as a function of x and is one-to-one because there is a unique output for every input, and a unique input for every output. Table (a) is not one-to-one because two different inputs give the same output, and table (c) is not a function because there are two different outputs for the same input $x = 8$.

13. Graphs (b) (c) (e) and (f) are one-to-one functions because there is a unique input for every output. Graph (a) is not a function, and graph (d) is not one-to-one because it contains points which have the same output for two different inputs.

15. (a) $f(1) = 1$, (b) $f(3) = 1$

17. (a) $g(2) = 4$, (b) $g(-3) = 2$

19. (a) $f(3) = 53$, (b) $f(2) = 1$

	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$
21.	8	6	4	2	0
23.	49	18	3	4	21
25.	4	-1	0	1	-4

27.	4	4.414	4.732	5	5.236
29.	-4	-6	-6	-4	0
31.	5	DNE	-3	-1	-1/3
33.	1/4	1/2	1	2	4

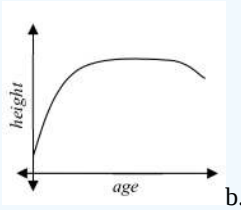
35. a. -6
b. -16

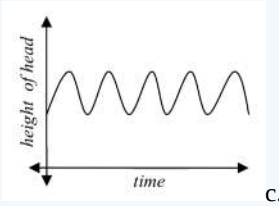
37. a. 5
b. $-\frac{5}{3}$

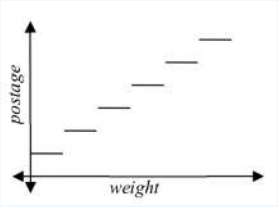
39. a. iii
b. viii
d. ii
e. vi
f. iv
g. v
h. vii

41. a. iv
b. ii
c. v
d. I
e. vi
f. iii

43. $(x-3)^2+(y+9)^2=36$

45. a. 

b. 

c. 

47. a. t
b. a
c. r
d. L: (c, t) and K: (a, p)

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1.2: Domain and Range

One of our main goals in mathematics is to model the real world with mathematical functions. In doing so, it is important to keep in mind the limitations of those models we create.

This table shows a relationship between circumference and height of a tree as it grows.

Circumference, c	1.7	2.5	5.5	8.2	13.7
Height, h	24.5	31	45.2	54.6	92.1

While there is a strong relationship between the two, it would certainly be ridiculous to talk about a tree with a circumference of -3 feet, or a height of 3000 feet. When we identify limitations on the inputs and outputs of a function, we are determining the domain and range of the function.

Definitions: Domain and Range

- **Domain:** The set of possible input values to a function
- **Range:** The set of possible output values of a function

Example 1.2.1

Using the tree table above, determine a reasonable domain and range.

Solution

We could combine the data provided with our own experiences and reason to approximate the domain and range of the function $h = f(c)$. For the domain, possible values for the input circumference c, it doesn't make sense to have negative values, so $c > 0$. We could make an educated guess at a maximum reasonable value, or look up that the maximum circumference measured is about 119 feet. With this information, we would say a reasonable domain is $0 < c \leq 119$ feet.

Similarly for the range, it doesn't make sense to have negative heights, and the maximum height of a tree could be looked up to be 379 feet, so a reasonable range is $0 < h \leq 379$ feet.

Example 1.2.2

When sending a letter through the United States Postal Service, the price depends upon the weight of the letter (https://pe.usps.com/text/dmm300/Notice123.htm, retrieved August 6, 2018), as shown in the table below. Determine the domain and range.

Table: Letters

Weight not Over	Price
1 ounce	\$0.50
2 ounces	\$0.71
3 ounces	\$0.92
3.5 ounces	\$1.13

Suppose we notate Weight by w and Price by p, and set up a function named P, where Price, p is a function of Weight, w. $p = P(w)$.

Since acceptable weights are 3.5 ounces or less, and negative weights don't make sense, the domain would be $0 < w \leq 3.5$. Technically 0 could be included in the domain, but logically it would mean we are mailing nothing, so it doesn't hurt to leave it out.

Since possible prices are from a limited set of values, we can only define the range of this function by listing the possible values. The range is p = \$0.50, \$0.71, \$0.92, or \$1.13.

? Try it now

The population of a small town in the year 1960 was 100 people. Since then the population has grown to 1400 people reported during the 2010 census. Choose descriptive variables for your input and output and use interval notation to write the domain and range.

Answer

Domain; $y =$ years [1960, 2010]

Range, $p =$ population, [100, 1400]

Notation

In the previous examples, we used inequalities to describe the domain and range of the functions. This is one way to describe intervals of input and output values, but is not the only way. Let us take a moment to discuss notation for domain and range.

Using inequalities, such as $0 < c \leq 163$, $0 < w \leq 3.5$, and $0 < h \leq 379$ imply that we are interested in all values between the low and high values, including the high values in these examples.

However, occasionally we are interested in a specific list of numbers like the range for the price to send letters, p = \$0.44, \$0.61, \$0.78, or \$0.95. These numbers represent a set of specific values: {0.44, 0.61, 0.78, 0.95}

Representing values as a set, or giving instructions on how a set is built, leads us to another type of notation to describe the domain and range.

Suppose we want to describe the values for a variable *x* that are 10 or greater, but less than 30. In inequalities, we would write $10 \leq x < 30$.

When describing domains and ranges, we sometimes extend this into **set-builder notation**, which would look like this: $\{x \mid 10 \leq x < 30\}$. The curly brackets {} are read as "the set of", and the vertical bar is read as "such that", so altogether we would read $\{x \mid 10 \leq x < 30\}$ as "the set of *x*-values such that 10 is less than or equal to x and x is less than 30."

When describing ranges in set-builder notation, we could similarly write something like $\{f(x) \mid 0 < f(x) < 30\}$, or if the output had its own variable, we could use it. So for our tree height example above, we could write for the range $\{h \mid 0 < h \leq 379\}$. In set-builder notation, if a domain or range is not limited, we could write $\{t \mid t$ is a real number$\}$, or $\{t \mid t \in \mathbb{R}\}$, read as "the set of *t*-values such that t is an element of the set of real numbers.

A more compact alternative to set-builder notation is **interval notation**, in which intervals of values are referred to by the starting and ending values. Curved parentheses are used for "strictly less than," and square brackets are used for "less than or equal to." Since infinity is not a number, we can't include it in the interval, so we always use curved parentheses with ∞ and - ∞. The table below will help you see how inequalities correspond to set-builder notation and interval notation:

Inequality	Set Builder Notation	Interval notation
$5 < h \leq 10$	$\{h \mid 5 < h \leq 10\}$	(5, 10]
$5 \leq h < 10$	$\{h \mid 5 \leq h < 10\}$	[5, 10)
$5 < h < 10$	$\{h \mid 5 < h < 10\}$	(5, 10)
$h < 10$	$\{h \mid h < 10\}$	$(-\infty, 10)$
$h \geq 10$	$\{h \mid h \geq 10\}$	$[10, \infty)$
all real numbers	$\{h \mid h \in \mathbb{R}\}$	$(-\infty, \infty)$

To combine two intervals together, using inequalities or set-builder notation we can use the word "or". In interval notation, we use the union symbol, $\cup$, to combine two unconnected intervals together.

Example 1.2.3

Describe the intervals of values shown on the line graph below using set builder and interval notations.



To describe the values, x, that lie in the intervals shown above we would say, "x is a real number greater than or equal to 1 and less than or equal to 3, or a real number greater than 5."

- As an inequality it is:

$$1 \leq x \leq 3 \quad \text{or} \quad x > 5$$

- In set builder notation:

$$\{x | 1 \leq x \leq 3 \quad \text{or} \quad x > 5\}$$

- In interval notation:

$$[1, 3] \cup (5, \infty)$$

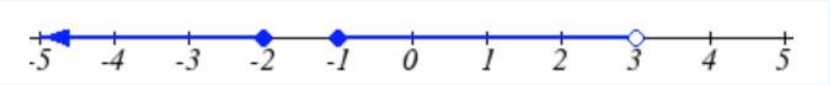
Remember when writing or reading interval notation:

Using a square bracket [means the start value is included in the set

Using a parenthesis (means the start value is not included in the set

Try it now

Given the following interval, write its meaning in words, set builder notation, and interval notation.



Answer

a. Values that are less than or equal to -2, or values that are greater than or equal to -1 and less than 3

b. $\{x \mid x \leq -2 \text{ or } -1 \leq x < 3\}$

c. $(-\infty, -2] \cup [-1, 3)$

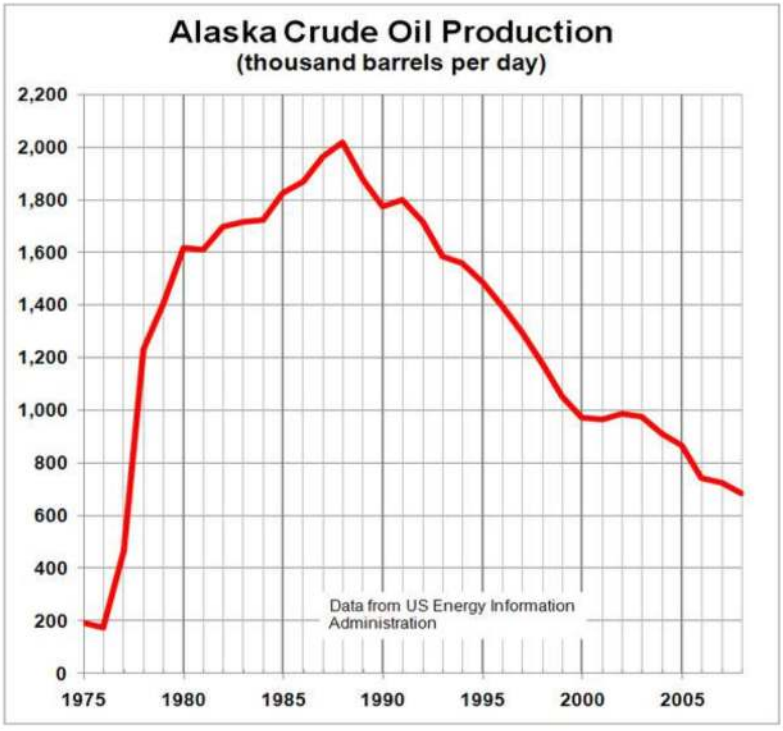
Domain and Range from Graphs

We can also talk about domain and range based on graphs. Since domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the graph. Remember that input values are almost always shown along the horizontal axis of the graph. Likewise, since range is the set of possible output values, the range of a graph we can see from the possible values along the vertical axis of the graph.

Be careful – if the graph continues beyond the window on which we can see the graph, the domain and range might be larger than the values we can see.

Example 1.2.4

Determine the domain and range of the graph below.



Solution

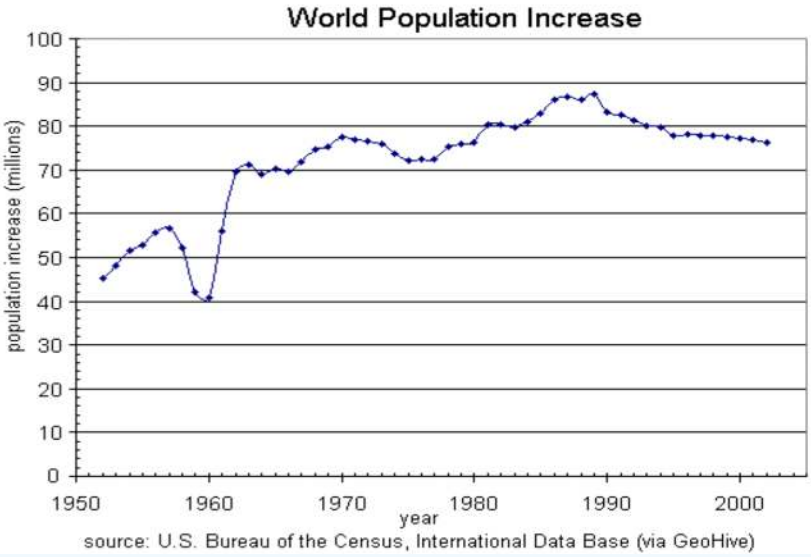
In the graph above (http://commons.wikimedia.org/wiki/Fi...Production.PNG, CC-BY-SA, July 19, 2010), the input quantity along the horizontal axis appears to be "year", which we could notate with the variable y. The output is "thousands of barrels of oil per day", which we might notate with the variable b, for barrels. The graph would likely continue to the left and right beyond what is shown, but based on the portion of the graph that is shown to us, we can determine the domain is $1975 \leq y \leq 2008$, and the range is approximately $180 \leq b \leq 2010$.

In interval notation, the domain would be [1975, 2008] and the range would be about [180, 2010]. For the range, we have to approximate the smallest and largest outputs since they don't fall exactly on the grid lines.

Remember that, as in the previous example, x and y are not always the input and output variables. Using descriptive variables is an important tool to remembering the context of the problem.

? Try it now

Given the graph below write the domain and range in interval notation



Answer

Domain; $y =$ years, [1952, 2002]; Range, $p =$ population in millions, [40, 88]

Domains and Ranges of the Toolkit Functions

We will now return to our set of toolkit functions to note the domain and range of each.

Constant Function: $f(x) = c$

- The domain here is not restricted; x can be anything. When this is the case we say the domain is all real numbers. The outputs are limited to the constant value of the function.
- Domain: $(-\infty, \infty)$
- Range: $[c]$
- *Since there is only one output value, we list it by itself in square brackets.*

Identity Function: $f(x) = x$

- Domain: $(-\infty, \infty)$
- Range: $(-\infty, \infty)$

Quadratic Function: $f(x) = x^2$

- Domain: $(-\infty, \infty)$
- Range: $[0, \infty)$
- *Multiplying a negative or positive number by itself can only yield a positive output.*

Cubic Function: $f(x) = x^3$

- Domain: $(-\infty, \infty)$
- Range: $(-\infty, \infty)$

Reciprocal: $f(x) = \dfrac{1}{x}$

- Domain: $(-\infty, 0) \cup (0, \infty)$
- Range: $(-\infty, 0) \cup (0, \infty)$
- *We cannot divide by 0 so we must exclude 0 from the domain.*
- *One divide by any value can never be 0, so the range will not include 0.*

Reciprocal Squared: $f(x) = \dfrac{1}{x^2}$

- Domain: $(-\infty, 0) \cup (0, \infty)$

- Range: $(0, \infty)$
- *We cannot divide by 0 so we must exclude 0 from the domain.*

Cube Root: $f(x) = \sqrt[3]{x}$

- Domain: $(-\infty, \infty)$
- Range: $(-\infty, \infty)$

Square Root: $f(x) = \sqrt[2]{x}$, commonly just written as, $f(x) = \sqrt{x}$

- Domain: $[0, \infty)$
- Range: $[0, \infty)$
- *When dealing with the set of real numbers we cannot take the square root of a negative number so the domain is limited to 0 or greater.*

Absolute Value Function: $f(x) = |x|$

- Domain: $(-\infty, \infty)$
- Range: $[0, \infty)$
- *Since absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.*

Example 1.2.5

Find the domain of each function:

a. $f(x) = 2\sqrt{x+4}$

b. $g(x) = \dfrac{3}{6-3x}$

Solution

a) Since we cannot take the square root of a negative number, we need the inside of the square root to be non-negative.

$x + 4 \geq 0$ when $x \geq -4$.

The domain of $f(x)$ is $[-4, \infty)$.

b) We cannot divide by zero, so we need the denominator to be non-zero.

$6 - 3x = 0$ when $x = 2$, so we must exclude 2 from the domain.

The domain of $g(x)$ is $(-\infty, 2) \cup (2, \infty)$.

Piecewise Functions

In the toolkit functions we introduced the absolute value function $f(x) = |x|$.

With a domain of all real numbers and a range of values greater than or equal to 0, the absolute value can be defined as the magnitude or modulus of a number, a real number value regardless of sign, the size of the number, or the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value the output is unchanged

$f(x) = x$ if $x \geq 0$

If we input a negative value the sign must change from negative to positive.

$f(x) = -x$ if $x < 0$, since multiplying a negative value by -1 makes it positive.

Since this requires two different processes or pieces, the absolute value function is often called the most basic piecewise defined function.

Definition: Piecewise Function

A **piecewise function** is a function in which the formula used depends upon the domain the input lies in. We notate this idea like:

$$f(x) = \begin{cases} \text{formula 1 if domain to use formula 1} \\ \text{formula 2 if domain to use formula 2} \\ \text{formula 3 if domain to use formula 3} \end{cases}$$

Example 1.2.6

A museum charges $5 per person for a guided tour with a group of 1 to 9 people, or a fixed $50 fee for 10 or more people in the group. Set up a function relating the number of people, n, to the cost, C.

Solution

To set up this function, two different formulas would be needed. $C = 5n$ would work for n values under 10, and $C = 50$ would work for values of n ten or greater. Notating this:

$$C(n) = \begin{cases} 5n & if \quad 0 < n < 10 \\ 50 & if \quad n \geq 10 \end{cases}$$

Example 1.2.7

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & if \quad 0 < g < 2 \\ 25 + 10(g-2) & if \quad g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data, and the cost of using 4 gigabytes of data.

Solution

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which piece of domain our input falls in. Since 1.5 is less than 2, we use the first formula, giving $C(1.5) = \$25$.

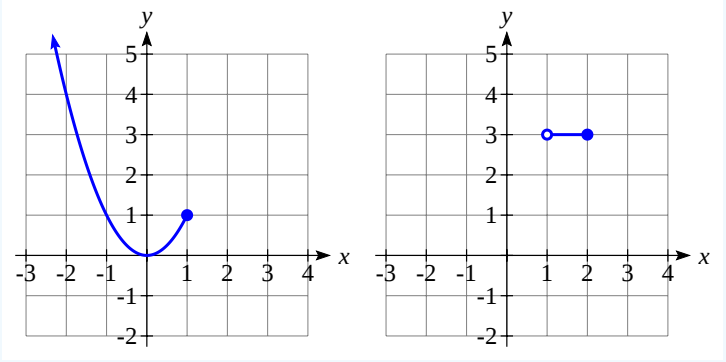
The find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we'll use the second formula. $C(4) = 25 + 10(4 - 2) = \$45$.

Example 1.2.8

Sketch a graph of the function $f(x) = \begin{cases} x^2 & if \quad x \leq 1 \\ 3 & if \quad 1 < x \leq 2 \\ 6 - x & if \quad x > 2 \end{cases}$

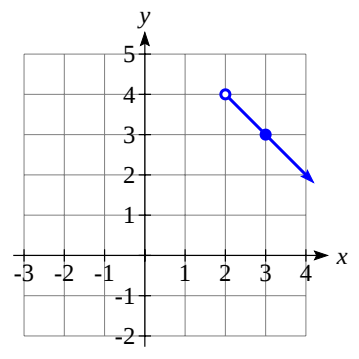
Solution

The first two component functions are from our library of Toolkit functions, so we know their shapes. We can imagine graphing each function, then limiting the graph to the indicated domain. At the endpoints of the domain, we put open circles to indicate where the endpoint is not included, due to a strictly-less-than inequality, and a closed circle where the endpoint is included, due to a less-than-or-equal-to inequality.

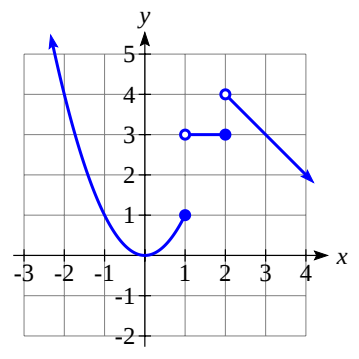


For the third function, you should recognize this as a linear equation from your previous coursework. If you remember how to graph a line using slope and intercept, you can do that. Otherwise, we could calculate a couple values, plot points, and connect them with a line.

- At $x = 2$, $f(2) = 6 - 2 = 4$. We place an open circle here.
- At $x = 3$, $f(3) = 6 - 3 = 3$. Connect these points with a line.



Now that we have each piece individually, we combine them onto the same graph:



Try It Now

At Pierce College during the 2009-2010 school year tuition rates for in-state residents were \$89.50 per credit for the first 10 credits, \$33 per credit for credits 11-18, and for over 18 credits the rate is \$73 per credit (www.pierce.ctc.edu/dist/tuit...ition_rate.pdf, retrieved August 6, 2010). Write a piecewise defined function for the total tuition, T, at Pierce College during 2009-2010 as a function of the number of credits taken, c. Be sure to consider a reasonable domain and range

Answer

$$T(c) = \begin{cases} 89.5c & if \quad c \leq 10 \\ 895 + 33(c-10) & if \quad 10 < c \leq 18 \\ 1159 + 73(c-18) & if \quad c > 18 \end{cases}$$ Tuition, T, as a function of credits, c.

Reasonable domain should be whole numbers 0 to (answers may vary), e.g. [0, 23]

Reasonable range should be \$0 – (answers may vary), e.g. [0,1524]

Important Topics of this Section

- Definition of domain
- Definition of range
- Inequalities
- Interval notation
- Set builder notation

- Domain and Range from graphs
- Domain and Range of toolkit functions
- Piecewise defined functions

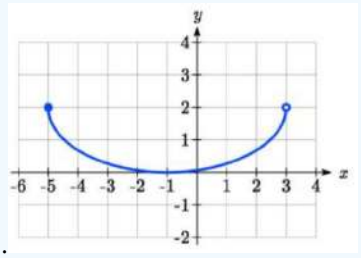
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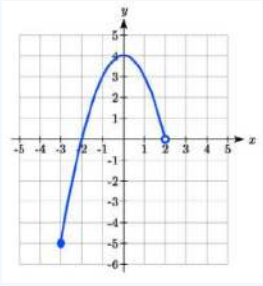
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1.2E: Domain and Range

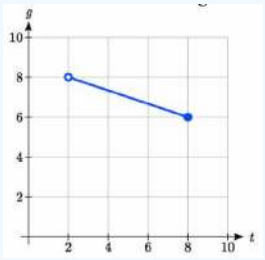
SECTION 1.2 EXERCISE

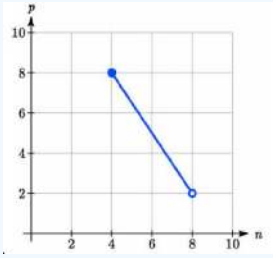
Write the domain and range of the function using interval notation.

1. 

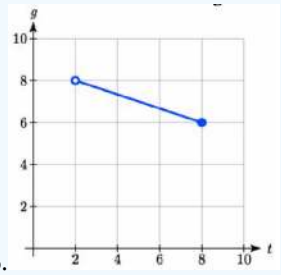
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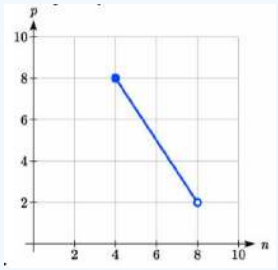
Write the domain and range of each graph as an inequality.

3. 

4. 

Suppose that you are holding your toy submarine under the water. You release it and it begins to ascend. The graph models the depth of the submarine as a function of time, stopping once the sub surfaces. What is the domain and range of the function in the graph?

5. 

6. 

Find the domain of each function

7. $f(x)=3\sqrt{x-2}$

8. $f(x)=5\sqrt{x+3}$

9. $f(x)=3-\sqrt{6-2x}$

10. $f(x)=5-\sqrt{10-2x}$

11. $f(x)=\dfrac{9}{x\ -\ 6}$

12. $f(x)=\dfrac{6}{x\ -\ 8}$

13. $f(x)=\dfrac{3x+1}{4x+2}$

14. $f(x)=\dfrac{5x+3}{4x-1}$

15. $f(x)=\dfrac{\sqrt{x+4}}{x-4}$

16. $f(x)=\dfrac{\sqrt{x+5}}{x-6}$

17. $f(x)=\dfrac{x\ -3}{x^2+\ 9x\ -22}$

18. $f(x)=\dfrac{x\ -8}{x^2+\ 8x\ -9}$

Given each function, evaluate: $f(-1)$, $f(0)$, $f(2)$, $f(4)$

19. $f(x)=\begin{cases}7x+3 & if \quad x<0\\ 7x+6 & if \quad x\geq 0\end{cases}$

20. $f(x)=\begin{cases}4x-9 & if \quad x<0\\ 4x-18 & if \quad x\geq 0\end{cases}$

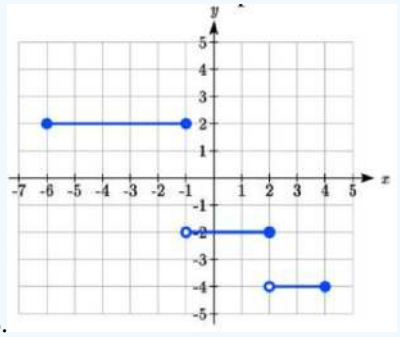
21. $f(x)=\begin{cases} x^2-2 & if \quad x<2 \\ 4+|x-5| & if \quad x\geq 2 \end{cases}$

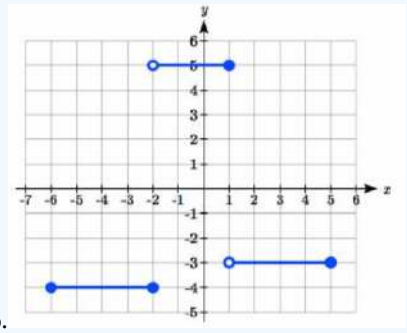
22. $f(x)=\begin{cases} 4-x^3 & if \quad x<1 \\ \sqrt{x+1} & if \quad x\geq 1 \end{cases}$

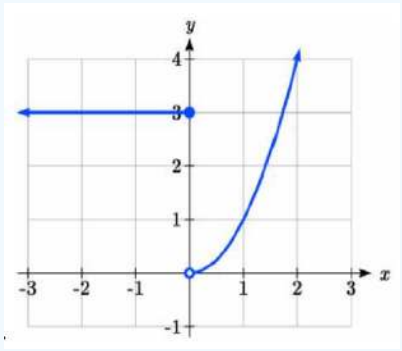
23. $f(x)=\begin{cases} 5x & if \quad x<0 \\ 3 & if \quad 0\leq x\leq 3 \\ x^2 & if \quad x>3 \end{cases}$

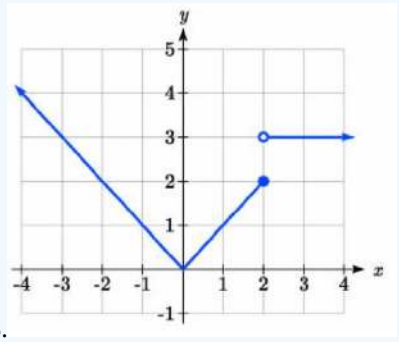
24. $f(x)=\begin{cases} x^3+1 & if \quad x<0 \\ 4 & if \quad 0\leq x\leq 3 \\ 3x+1 & if \quad x>3 \end{cases}$

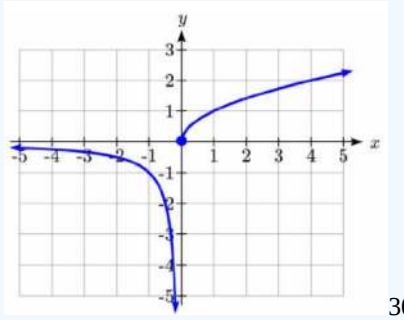
Write a formula for the piecewise function graphed below.

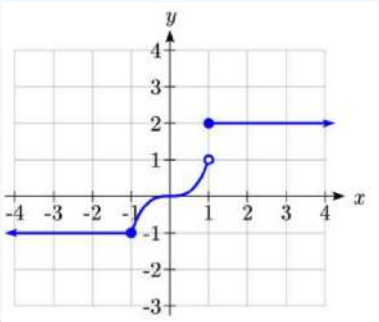
25. 

26. 

27. 

28. 

29. 

30. 

Sketch a graph of each piecewise function

31. $f(x)=\begin{cases} |x| & if \quad x<2 \\ 5 & if \quad x\geq 2 \end{cases}$

32. $f(x)=\begin{cases} 4 & if \quad x<0 \\ \sqrt{x} & if \quad x\geq 0 \end{cases}$

33. $f(x)=\begin{cases} x^2 & if \quad x<0 \\ x+2 & if \quad x\geq 0 \end{cases}$

34. $f(x)=\begin{cases} x+1 & if \quad x<1 \\ x^3 & if \quad x\geq 1 \end{cases}$

35. $f(x)=\begin{cases} 3 & if \quad x\leq -2 \\ -x+1 & if \quad -2<x\leq 1 \\ 3 & if \quad x>1 \end{cases}$

36. $f(x)=\begin{cases} -3 & if \quad x\leq -2 \\ x-1 & if \quad -2<x\leq 2 \\ 0 & if \quad x>2 \end{cases}$

Answer

1. D: [-5, 3) R: [0, 2]

3. D: $2<t\leq 8$ R: $6\leq g(t)<8$

5. D: [0, 4] R: [-3, 0]

7. $[2, \infty)$

9. $-\infty, 3]$

11. $(\infty, 6) \cup (6, \infty)$

13. $(-\infty, -\frac{1}{2}) \cup (-\frac{1}{2}, \infty)$

15. $[-4, 4) \cup (4, \infty)$

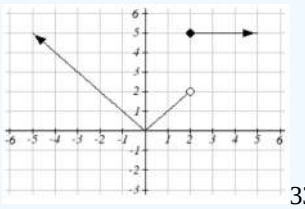
17. $(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$

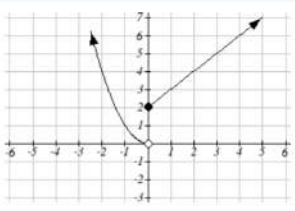
	$f(-1)$	$f(0)$	$f(2)$	$f(4)$
19.	-4	6	20	34
21.	-1	-2	7	5
23.	-5	3	3	16

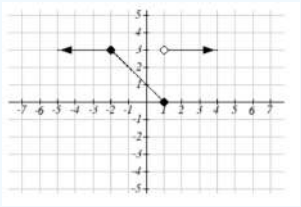
25. $f(x) = \begin{cases} 2 & if \quad -6 \leq x \leq -1 \\ -2 & if \quad -1 < x \leq 2 \\ -4 & if \quad 2 < x \leq 4 \end{cases}$

27. $f(x) = \begin{cases} 3 & if \quad x \leq 0 \\ x^2 & if \quad x > 0 \end{cases}$

29. $f(x) = \begin{cases} \frac{1}{x} & if \quad x < 0 \\ \sqrt{x} & if \quad x \geq 0 \end{cases}$

31. 

33. 

35. 

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1.3: Rates of Change and Behavior of Graphs

Since functions represent how an output quantity varies with an input quantity, it is natural to ask about the rate at which the values of the function are changing. For example, the function $C(t)$ below gives the average cost, in dollars, of a gallon of gasoline t years after 2000.

t	2	3	4	5	6	7	8	9
$C(t)$	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

If we were interested in how the gas prices had changed between 2002 and 2009, we could compute that the cost per gallon had increased from \$1.47 to \$2.14, an increase of \$0.67. While this is interesting, it might be more useful to look at how much the price changed *per year*. You are probably noticing that the price didn't change the same amount each year, so we would be finding the **average rate of change** over a specified amount of time.

The gas price increased by \$0.67 from 2002 to 2009, over 7 years, for an average of $\frac{\$0.67}{7 years} \approx 0.096$dollars per year. On average, the price of gas increased by about 9.6 cents each year.

Definition: Rate of Change

A **rate of change** describes how the output quantity changes in relation to the input quantity. The units on a rate of change are output units per input units.

Some other examples of rates of change would be quantities like:

- A population of rats increases by 40 rats per week
- A barista earns \$9 per hour (dollars per hour)
- A farmer plants 60,000 onions per acre
- A car can drive 27 miles per gallon
- A population of grey whales decreases by 8 whales per year
- The amount of money in your college account decreases by \$4,000 per quarter

Definition: Average Rate of Change

The **average rate of change** between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{1.3.1}$$

Example 1.3.1

Using the cost-of-gas function from earlier, find the average rate of change between 2007 and 2009.

Solution

From the table, in 2007 the cost of gas was \$2.64. In 2009 the cost was \$2.14.

The input (years) has changed by 2. The output has changed by \$2.14 - \$2.64 = -0.50.

The average rate of change is then $\frac{-\$0.50}{2 years}$ = -0.25 dollars per year

Exercise 1.3.1

Using the same cost-of-gas function, find the average rate of change between 2003 and 2008

Answer

$\frac{\$3.01 - \$1.69}{5\ years} = \frac{\$1.32}{5\ years} = 0.264$ dollars per year.

Notice that in the last example the change of output was *negative* since the output value of the function had decreased. Correspondingly, the average rate of change is negative.

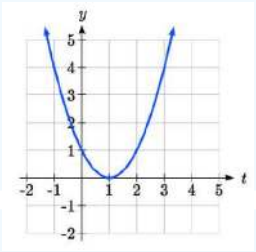
Example 1.3.2

Given the function $g(t)$ shown here, find the average rate of change on the interval [0, 3].

Solution

At t = 0, the graph shows $g(0) = 1$

At t = 3, the graph shows $g(3) = 4$



The output has changed by 3 while the input has changed by 3, giving an average rate of change of:

$$\frac{4-1}{3-0} = \frac{3}{3} = 1$$

Example 1.3.3

On a road trip, after picking up your friend who lives 10 miles away, you decide to record your distance from home over time. Find your average speed over the first 6 hours.

t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Solution

Here, your average speed is the average rate of change.

You traveled 282 miles in 6 hours, for an average speed of

$$\frac{292-10}{6-0} = \frac{282}{6} = 47 \text{ miles per hour}$$

We can more formally state the average rate of change calculation using function notation.

Definition: Average Rate of Change using Function Notation

Given a function $f(x)$, the average rate of change on the interval [a, b] is

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{f(b)-f(a)}{b-a} \tag{1.3.2}$$

Example 1.3.4

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval [2, 4].

Solution

We can start by computing the function values at each endpoint of the interval

$$f(2) = 2^2 - \frac{1}{2} = 4 - \frac{1}{2} = \frac{7}{2}$$

$$f(4) = 4^2 - \frac{1}{4} = 16 - \frac{1}{4} = \frac{63}{4}$$

Now computing the average rate of change

$$\text{Average rate of change} = \frac{f(4)-f(2)}{4-2} = \frac{\frac{63}{4} - \frac{7}{2}}{4-2} = \frac{\frac{49}{4}}{2} = \frac{49}{8}$$

Exercise 1.3.2

Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval [1, 9].

Answer

$$\text{Average rate of change} = \frac{f(9)-f(1)}{9-1} = \frac{(9-2\sqrt{9})-(1-2\sqrt{1})}{9-1} = \frac{(3)-(-1)}{9-1} = \frac{4}{8} = \frac{1}{2}$$

Example 1.3.5

The magnetic force F, measured in Newtons, between two magnets is related to the distance between the magnets d, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the magnets is increased from 2 cm to 6 cm.

Solution

We are computing the average rate of change of $F(d) = \frac{2}{d^2}$ on the interval [2, 6].

Average rate of change = $\frac{F(6) - F(2)}{6 - 2}$

Evaluating the function

$$\frac{F(6) - F(2)}{6 - 2} =$$

$$= \frac{\frac{2}{6^2} - \frac{2}{2^2}}{6 - 2}$$

Simplifying

$$= \frac{\frac{2}{36} - \frac{2}{4}}{4}$$

Combining the numerator terms

$$= \frac{\frac{-16}{36}}{4}$$

Simplifying further

$$= \frac{-1}{9}$$

Newtons per centimeter

This tells us the magnetic force decreases, on average, by 1/9 Newtons per centimeter over this interval.

Example 1.3.6

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. Your answer will be an expression involving a.

Solution

Using the average rate of change formula

$$\frac{g(a) - g(0)}{a - 0}$$

Evaluating the function

$$\frac{(a^2 + 3a + 1) - (0^2 + 3(0) + 1)}{a - 0}$$

Simplifying

$$\frac{a^2 + 3a + 1 - 1}{a}$$

Simplifying further, and factoring

$$\frac{a(a + 3)}{a}$$

Cancelling the common factor a

$$a + 3$$

This result tells us the average rate of change between t = 0 and any other point $t = a$. For example, on the interval [0, 5], the average rate of change would be 5+3 = 8.

Exercise 1.3.3

Find the average rate of change of $f(x) = x^3 + 2$ on the interval $[a, a + h]$.

Answer

$$\frac{f(a+h)-f(a)}{(a+h)-a} = \frac{((a+h)^3+2)-(a^3+2)}{h} = \frac{a^3+3a^2h+3ah^2+h^3+2-a^3-2}{h} = \frac{3a^2h+3ah^2+h^3}{h} = \frac{h(3a^2+3ah+h^2)}{h} = 3a^2 + 3ah + h^2$$

Graphical Behavior of Functions

As part of exploring how functions change, it is interesting to explore the graphical behavior of functions.

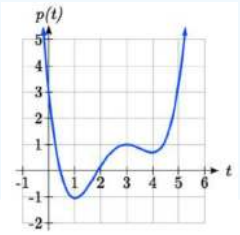
Definition: increasing/decreasing

A function is **increasing** on an interval if the function values increase as the inputs increase. More formally, a function is increasing if $f(b) > f(a)$ for any two input values a and b in the interval with $b > a$. The average rate of change of an increasing function is **positive.**

A function is **decreasing** on an interval if the function values decrease as the inputs increase. More formally, a function is decreasing if $f(b) < f(a)$ for any two input values a and b in the interval with $b > a$. The average rate of change of a decreasing function is **negative.**

Example 1.3.7

Given the function $p(t)$ graphed here, on what intervals does the function appear to be increasing?



The function appears to be increasing from $t = 1$ to $t = 3$, and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$and the interval $(4, \infty)$.

Solution

Add text here.

Notice in the last example that we used open intervals (intervals that don't include the endpoints) since the function is neither increasing nor decreasing at $t = 1$, 3, or 4.

Definition: Local Extrema

A point where a function changes from increasing to decreasing is called a **local maximum**.

A point where a function changes from decreasing to increasing is called a **local minimum**.

Together, local maxima and minima are called the **local extrema**, or local extreme values, of the function.

Example 1.3.8

Using the cost of gasoline function from the beginning of the section, find an interval on which the function appears to be decreasing. Estimate any local extrema using the table.

t	2	3	4	5	6	7	8	9
$C(t)$	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

It appears that the cost of gas increased from $t = 2$ to $t = 8$. It appears the cost of gas decreased from $t = 8$ to $t = 9$, so the function appears to be decreasing on the interval (8, 9).

Since the function appears to change from increasing to decreasing at $t = 8$, there is local maximum at $t = 8$.

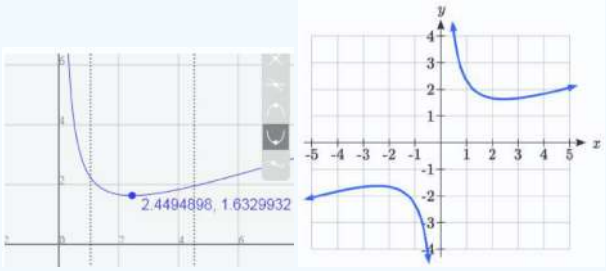
Example 1.3.9

Use a graph to estimate the local extrema of the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Use these to determine the intervals on which the function is increasing.

Solution

Using technology to graph the function, it appears there is a local minimum somewhere between $x = 2$ and $x = 3$, and a symmetric local maximum somewhere between $x = -3$ and $x = -2$.

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Below are screen images from two different technologies, showing the estimate for the local maximum and minimum.



Based on these estimates, the function is increasing on the intervals $(-\infty, -2.449)$and $(2.449, \infty)$. Notice that while we expect the extrema to be symmetric, the two different technologies agree only up to 4 decimals due to the differing approximation algorithms used by each.

? Exercise 1.3.4

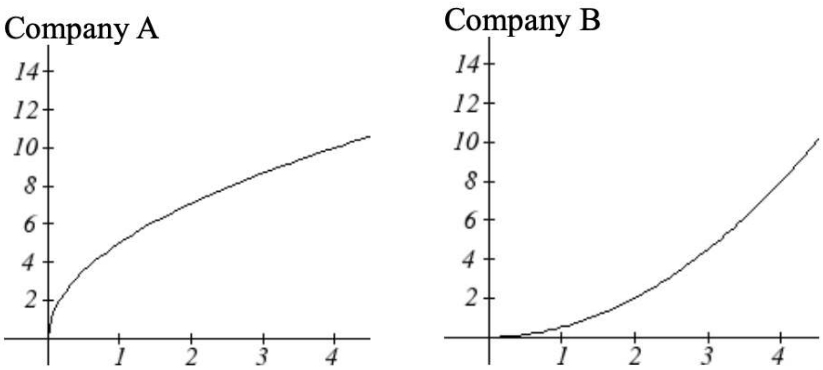
Use a graph of the function $f(x) = x^3 - 6x^2 - 15x + 20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Answer

Based on the graph, the local maximum appears to occur at (-1, 28), and the local minimum occurs at (5, -80). The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.

Concavity

The total sales, in thousands of dollars, for two companies over 4 weeks are shown.



As you can see, the sales for each company are increasing, but they are increasing in very different ways. To describe the difference in behavior, we can investigate how the average rate of change varies over different intervals. Using tables of values,

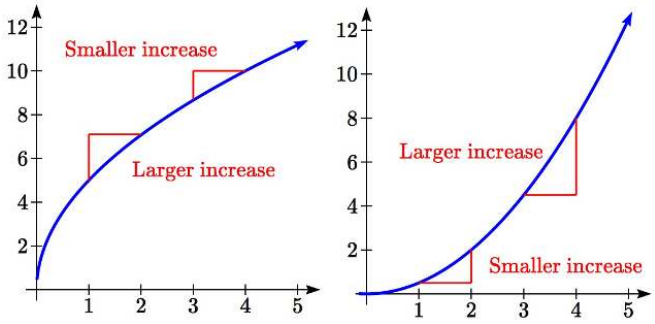
Company A

Week	Sales	Rate of Change
0	0	
		5
1	5	
		2.1
2	7.1	
		1.6
3	8.7	
		1.3
4	10	

Company B

Week	Sales	Rate of Change
0	0	
		0.5
1	0.5	
		1.5
2	2	
		2.5
3	4.5	
		3.5
4	8	

From the tables, we can see that the rate of change for company A is *decreasing*, while the rate of change for company B is *increasing*.



When the rate of change is getting smaller, as with Company A, we say the function is **concave down**. When the rate of change is getting larger, as with Company B, we say the function is **concave up**.

Definition: concavity

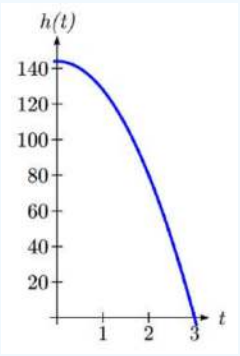
A function is **concave up** if the rate of change is increasing.

A function is **concave down** if the rate of change is decreasing.

A point where a function changes from concave up to concave down or vice versa is called an **inflection point**.

Example 1.3.10

An object is thrown from the top of a building. The object's height in feet above ground after t seconds is given by the function $h(t) = 144 - 16t^2$ for $0 \le t \le 3$. Describe the concavity of the graph.



Sketching a graph of the function, we can see that the function is decreasing. We can calculate some rates of change to explore the behavior.

t	$h(t)$	Rate of Change
0	144	
		-16
1	128	
		-48
2	80	
		-80
3	0	

Notice that the rates of change are becoming more negative, so the rates of change are *decreasing*. This means the function is concave down.

Example 1.3.11

The value, V, of a car after t years is given in the table below. Is the value increasing or decreasing? Is the function concave up or concave down?

t	0	2	4	6	8
$V(t)$	28000	24342	21162	18397	15994

Solution

can compute rates of change to determine concavity.

t	0	2	4	6	8
$V(t)$	28000	24342	21162	18397	15994
Rate of change	-1829	-1590	-1382.5	-1201.5	

Since these values are becoming less negative, the rates of change are *increasing*, so this function is concave up.

Exercise 1.3.5

Is the function described in the table below concave up or concave down?

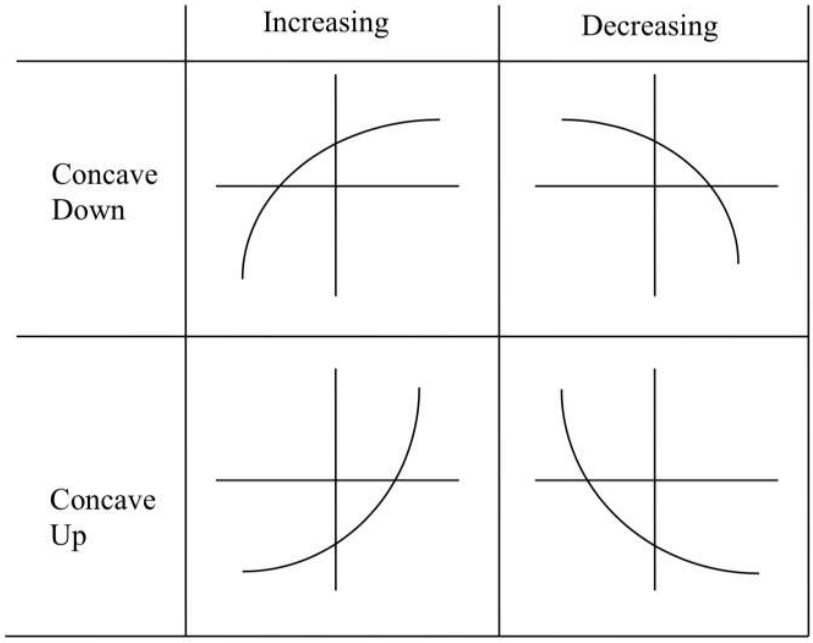
x	0	5	10	15	20
$g(x)$	10000	9000	7000	4000	0

Answer

Calculating the rates of change, we see the rates of change become *more* negative, so the rates of change are *decreasing*. This function is concave down.

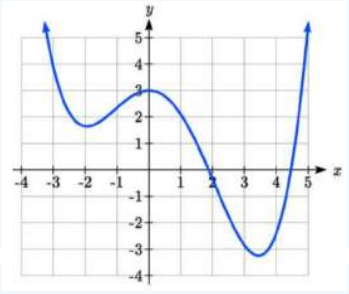
x	0	5	10	15	20
$g(x)$	10000	9000	7000	4000	0
Rate of change		-1000	-2000	-3000	-4000

Graphically, concave down functions bend downwards like a frown, and concave up function bend upwards like a smile.



Example 1.3.12

Estimate from the graph shown the intervals on which the function is concave down and concave up.



On the far left, the graph is decreasing but concave up, since it is bending upwards. It begins increasing at $x = -2$, but it continues to bend upwards until about $x = -1$.

From $x = -1$ the graph starts to bend downward, and continues to do so until about $x = 2$. The graph then begins curving upwards for the remainder of the graph shown.

From this, we can estimate that the graph is concave up on the intervals $(-\infty, -1)$ and $(2, \infty)$, and is concave down on the interval $(-1, 2)$. The graph has inflection points at $x = -1$ and $x = 2$.

Exercise 1.3.6

Using the graph from Try it Now 4, $f(x) = x^3 - 6x^2 - 15x + 20$, estimate the intervals on which the function is concave up and concave down.

Answer

Looking at the graph, it appears the function is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$.

Behaviors of the Toolkit Functions

We will now return to our toolkit functions and discuss their graphical behavior.

Function	Increasing/Decreasing	Concavity

Constant Function $f(x) = c$	Neither increasing nor decreasing	Neither concave up nor down
Identity Function $f(x) = x$	Increasing	Neither concave up nor down
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	Concave up $(-\infty, \infty)$
Cubic Function $f(x) = x^3$	Increasing	Concave down on $(-\infty, 0)$ Concave up on $(0, \infty)$ Inflection point at (0,0)
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	Concave down on $(-\infty, 0)$ Concave up on $(0, \infty)$
Function	Increasing/Decreasing	Concavity
Reciprocal squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	Concave up on $(-\infty, 0) \cup (0, \infty)$
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	Concave down on $(0, \infty)$ Concave up on $(-\infty, 0)$ Inflection point at (0,0)
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	Concave down on $(0, \infty)$
Absolute Value $f(x) = \lvert x \rvert$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$	Neither concave up or down

Important Topics of This Section

- Rate of Change
- Average Rate of Change
- Calculating Average Rate of Change using Function Notation
- Increasing/Decreasing
- Local Maxima and Minima (Extrema)
- Inflection points
- Concavity

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1.3E: Rates of Change and Behavior of Graphs

SECTION 1.3 EXERCISE

1. The table below gives the annual sales (in millions of dollars) of a product. What was the average rate of change of annual sales…

   a) Between 2001 and 2002?
   b) Between 2001 and 2004?

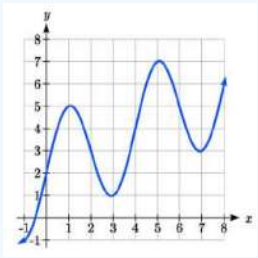
year	1998	1999	2000	2001	2002	2003	2004	2005	2006
sales	201	219	233	243	249	251	249	243	233

2. The table below gives the population of a town, in thousands. What was the average rate of change of population…

   a) Between 2002 and 2004?
   b) Between 2002 and 2006?

year	2000	2001	2002	2003	2004	2005	2006	2007	2008
population	87	84	83	80	77	76	75	78	81

3. Based on the graph shown, estimate the average rate of change from $x = 1$ to $x = 4$.
4. Based on the graph shown, estimate the average rate of change from $x = 2$ to $x = 5$.



Find the average rate of change of each function on the interval specified.

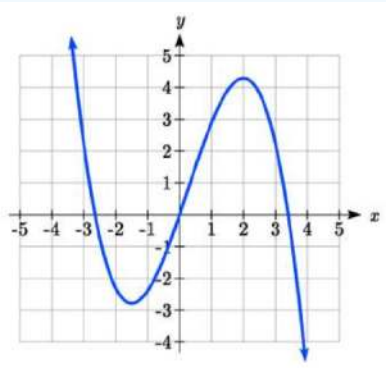
5. $f(x) = x^2$ on [1, 5]
6. $q(x) = x^3$ on [-4, 2]
7. $g(x) = 3x^3 - 1$ on [-3, 3]
8. $h(x) = 5 - 2x^2$ on [-2, 4]
9. $k(t) = 6t^2 + \frac{4}{t^3}$ on [-1, 3]
10. $p(t) = \frac{t^2 - 4t + 1}{t^2 + 3}$ on [-3, 1]

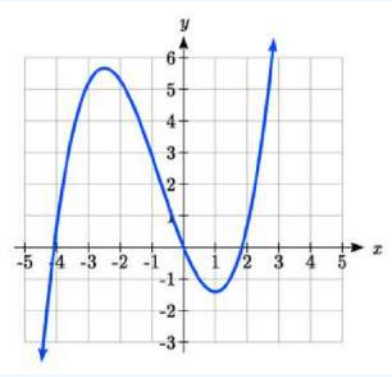
Find the average rate of change of each function on the interval specified. Your answers will be expressions involving a parameter (b or h).

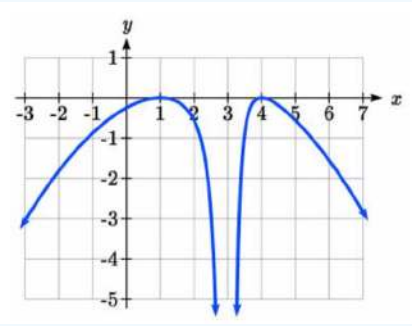
11. $f(x) = 4x^2 - 7$ on [1, b]
12. $g(x) = 2x^2 - 9$ on [4, b]
13. $h(x) = 3x + 4$ on [2, 2 + h]
14. $k(x) = 4x - 2$ on [3, 3 + h]
15. $a(t) = \frac{1}{t + 4}$ on [9, 9 + h]
16. $b(x) = \frac{1}{x + 3}$ on [1, 1 + h]
17. $j(x) = 3x^3$ on [1, 1 + h]
18. $r(t) = 4t^3$ on [2, 2 + h]

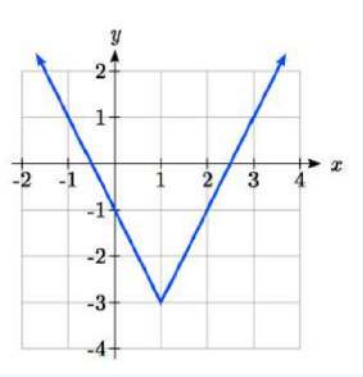
19. $f(x) = 2x^2 + 1$ on $[x, x+h]$
20. $g(x) = 3x^2 - 2$ on $[x, x+h]$

For each function graphed, estimate the intervals on which the function is increasing and decreasing.

21. 

22. 

23. 

24. 

For each table below, select whether the table represents a function that is increasing or decreasing, and whether the function is concave up or concave down.

25.

x	f(x)
1	2
2	4
3	8
4	16
5	32

26.

x	g(x)
1	90
2	80
3	75
4	72
5	70

27.

x	h(x)
1	300
2	290
3	270
4	240
5	200

28.

x	k(x)
1	0
2	15
3	25
4	32
5	35

29.

x	f(x)
1	-10
2	-25
3	-37
4	-47
5	-54

30.

x	g(x)
1	-200
2	-190
3	-160
4	-100
5	0

31.

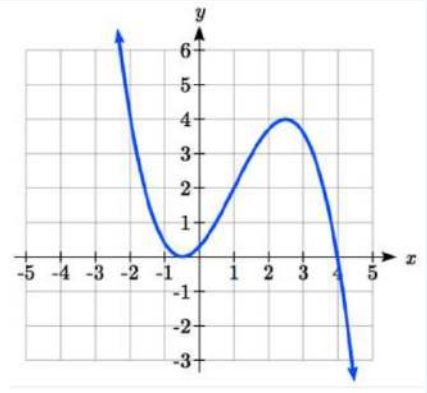
x	h(x)
1	- 100
2	-50
3	-25
4	-10
5	0

32.

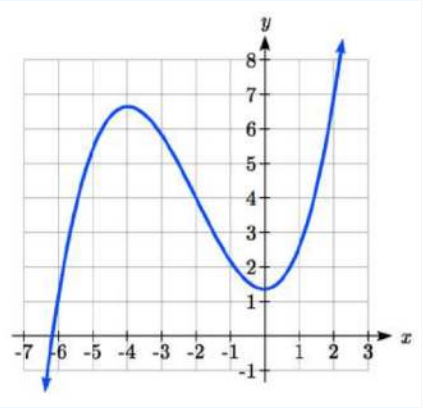
x	k(x)
1	-50
2	-100
3	-200
4	-400
5	-900

For each function graphed, estimate the intervals on which the function is concave up and concave down, and the location of any inflection points.

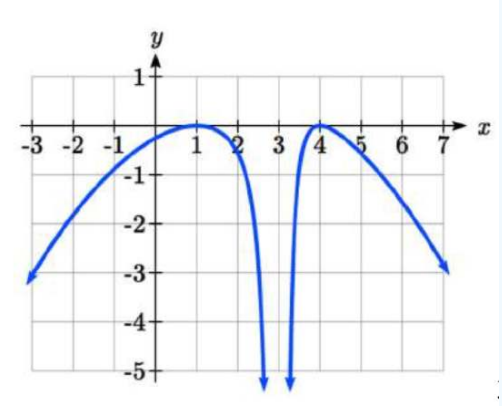
33.

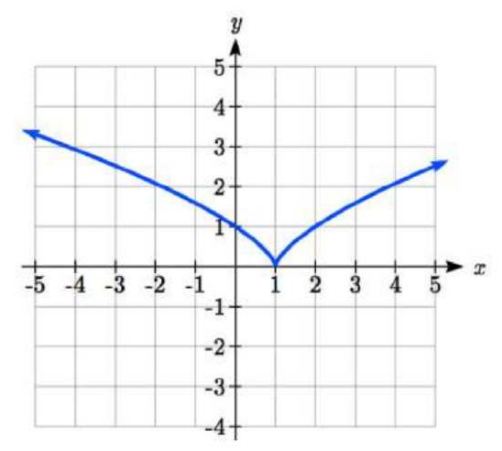


34.



35.



36. 

Use a graph to estimate the local extrema and inflection points of each function, and to estimate the intervals on which the function is increasing, decreasing, concave up, and concave down.

37. $f(x) = x^4 - 4x^3 + 5$

38. $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

39. $g(t) = t\sqrt{t+3}$

40. $k(t) = 3t^{\frac{2}{3}} - t$

41. $m(x) = x^4 + 2x^3 - 12x^2 - 10x + 4$

42. $n(x) = x^4 - 8x^3 + 18x^2 - 6x + 2$

Answer

1. a) 6 million dollars per year
   b) 2 million dollars per year

3. $\frac{4-5}{4-1} = -\frac{1}{3}$

5. 6

7. 27

9. $\frac{352}{27}$

11. $4b + 4$

13. 3

15. $-\frac{1}{13h + 169}$

17. $9 + 9h + 3h^2$

19. $4x + 2h$

21. Increasing: (-1.5, 2). Decreasing: $(-\infty, -1.4) \cup (2, \infty)$

23. Increasing: $(-\infty, 1) \cup (3, 4)$. Decreasing: $(1, 3) \cup (4, \infty)$

25. Increasing, concave up

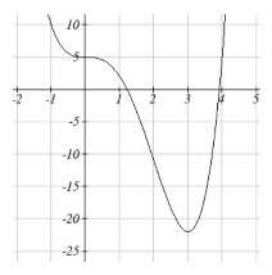
27. Decreasing, concave down

29. Decreasing, concave up

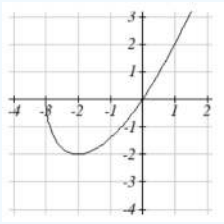
31. Increasing, concave down

33. Concave up $(-\infty, 1)$. Concave down $(1, \infty)$. Inflection point at (1, 2)

35. Concave down $-\infty, 3) \cup (3, \infty)$

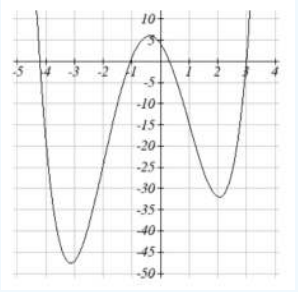


37. Local minimum at (3, -22).
Inflection points at (0, 5) and (2, -11).
Increasing on $3, \infty)$. Decreasing $(-\infty, 3)$
Concave up $(-\infty, 0) \cup (2, \infty)$. Concave down (0, 2)



39. Local minimum at $(-2, -2)$
Decreasing (-3, -2)
Increasing $(-2, \infty)$
Concave up $(-3, \infty)$

41. Local minimums at (-3.152, -47.626) and (2.041, -32.041)
Local maximum at (-0.389, 5.979)
Inflection points at (-2, -24) and (1, -15)
Increasing



$(-3.152, -0.389) \cup (2.041, \infty)$
Decreasing $(-\infty, -3.152) \cup (-0.389, 2.041)$
Concave up $(-\infty, -2) \cup (1, \infty)$
Concave down (-2, 1

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1.4: Composition of Functions

Suppose we wanted to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The temperature depends on the day, and the cost depends on the temperature. Using descriptive variables, we can notate these two functions.

The first function, $C(T)$, gives the cost C of heating a house when the average daily temperature is T degrees Celsius, and the second, $T(d)$, gives the average daily temperature on day d of the year in some city. If we wanted to determine the cost of heating the house on the 5th day of the year, we could do this by linking our two functions together, an idea called composition of functions. Using the function $T(d)$, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. We could then use that temperature as the input to the $C(T)$ function to find the cost to heat the house on the 5th day of the year: $C(T(5))$.

Definition: Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a **composition of functions**. We write $f(g(x))$, and read this as "f of g of x" or "f composed with g at x".

An alternate notation for composition uses the composition operator: $\circ$

$(f \circ g)(x)$ is read "f of g of x" or "f composed with g at x", just like $f(g(x))$.

Example 1.4.1

Suppose $c(s)$ gives the number of calories burned doing s sit-ups, and $s(t)$ gives the number of sit-ups a person can do in t minutes. Interpret $c(s(3))$.

Solution

When we are asked to interpret, we are being asked to explain the meaning of the expression in words. The inside expression in the composition is $s(3)$. Since the input to the s function is time, the 3 is representing 3 minutes, and $s(3)$ is the number of sit-ups that can be done in 3 minutes. Taking this output and using it as the input to the $c(s)$ function will gives us the calories that can be burned by the number of sit-ups that can be done in 3 minutes.

Note that it is not important that the same variable be used for the output of the inside function and the input to the outside function. However, it is essential that the units on the output of the inside function match the units on the input to the outside function, if the units are specified.

Example 1.4.2

Suppose $f(x)$ gives miles that can be driven in x hours, and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The expression $g(y)$ takes miles as the input and outputs a number of gallons. The function $f(x)$ is expecting a number of hours as the input; trying to give it a number of gallons as input does not make sense. Remember the units must match, and number of gallons does not match number of hours, so the expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and outputs a number of miles driven. The function $g(y)$ is expecting a number of miles as the input, so giving the output of the $f(x)$ function (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will give the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

Exercise 1.4.1

In a department store you see a sign that says 50% off clearance merchandise, so final cost C depends on the clearance price, p, according to the function $C(p)$. Clearance price, p, depends on the original discount, d, given to the clearance item, $p(d)$. Interpret $C(p(d))$.

Answer

The final cost, C, depends on the clearance price, p, which is based on the original discount, d. (Or the original discount d, determines the clearance price and the final cost is half of the clearance price.)

Composition of Functions using Tables and Graphs

When working with functions given as tables and graphs, we can look up values for the functions using a provided table or graph, as discussed in section 1.1. We start evaluation from the provided input, and first evaluate the inside function. We can then use the output of the inside function as the input to the outside function. To remember this, always work from the inside out.

Example 1.4.3

Using the tables below, evaluate $f(g(3))$and $g(f(4))$

x	$f(x)$
1	6
2	8
3	3
4	1

x	$g(x)$
1	3
2	5
3	2
4	7

Solution

To evaluate $f(g(3))$, we start from the inside with the value 3. We then evaluate the inside expression $g(3)$using the table that defines the function g : $g(3) = 2$.

We can then use that result as the input to the f function, so $g(3)$ is replaced by the equivalent value 2 and we can evaluate $f(2)$. Then using the table that defines the function f, we find that $f(2) = 8$.

$$f(g(3)) = f(2) = 8.$$

To evaluate $g(f(4))$, we first evaluate the inside expression $f(4)$using the first table: $f(4) = 1$. Then using the table for g we can evaluate:

$$g(f(4)) = g(1) = 3.$$

Exercise 1.4.2

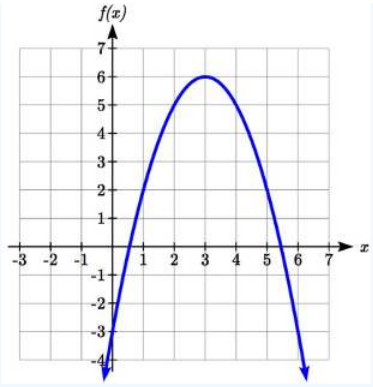
Using the tables from the example above, evaluate $f(g(1))$ and $g(f(3))$.

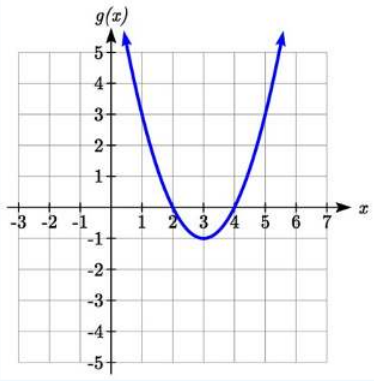
Answer

$f(g(1)) = f(3) = 3$ and $g(f(3)) = g(3) = 2$

Example 1.4.4

Using the graphs below, evaluate $f(g(1))$.





To evaluate $f(g(1))$, we again start with the inside evaluation. We evaluate $g(1)$ using the graph of the g(x) function, finding the input of 1 on the horizontal axis and finding the output value of the graph at that input. Here, $g(1) = 3$.

Using this value as the input to the f function, $f(g(1)) = f(3)$. We can then evaluate this by looking to the graph of the $f(x)$ function, finding the input of 3 on the horizontal axis, and reading the output value of the graph at this input.

$f(3) = 6$, so $f(g(1)) = 6$.

? Exercise 1.4.3

Using the graphs from the previous example, evaluate $g(f(2))$.

Answer

$g(f(2)) = g(5) = 3$

Composition using Formulas

When evaluating a composition of functions where we have either created or been given formulas, the concept of working from the inside out remains the same. First, we evaluate the inside function using the input value provided, then use the resulting output as the input to the outside function.

✓ Example 1.4.5

Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

Solution

Since the inside evaluation is $h(1)$ we start by evaluating the $h(x)$ function at 1:

$$h(1) = 3(1) + 2 = 5$$

Then $f(h(1)) = f(5)$, so we evaluate the f(t) function at an input of 5:

$$f(h(1)) = f(5) = 5^2 - 5 = 20$$

? Exercise 1.4.4

Using the functions from the example above, evaluate $h(f(-2))$.

Answer

$h(f(-2)) = h(6) = 20$ (*did you remember to insert your input values using parentheses?*)

While we can compose the functions as above for each individual input value, sometimes it would be really helpful to find a single formula which will calculate the result of a composition *f(g(x))*. To do this, we will extend our idea of function evaluation. Recall

that when we evaluate a function like $f(t) = t^2 - t$, we put whatever value is inside the parentheses after the function name into the formula wherever we see the input variable.

Example 1.4.6

Given $f(t) = t^2 - t$, evaluate $f(3)$ and $f(-2)$.

Solution

$$f(3) = 3^2 - 3$$
$$f(-2) = (-2)^2 - (-2)$$

We could simplify the results above if we wanted to

$$f(3) = 3^2 - 3 = 9 - 3 = 6$$
$$f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$$

We are not limited, however, to using a numerical value as the input to the function. We can put anything into the function: a value, a different variable, or even an algebraic expression, provided we use the input expression everywhere we see the input variable.

Example 1.4.7

Using the function from the previous example, evaluate f(a).

Solution

This means that the input value for t is some unknown quantity a. As before, we evaluate by replacing the input variable t with the input quantity, in this case a.

$$f(a) = a^2 - a$$

The same idea can then be applied to expressions more complicated than a single letter.

Example 1.4.8

Using the same $f(t)$ function from above, evaluate $f(x+2)$.

Solution

Everywhere in the formula for *f* where there was a *t*, we would replace it with the input $(x+2)$. Since in the original formula the input t was squared in the first term, the entire input $x+2$ needs to be squared when we substitute, so we need to use grouping parentheses. To avoid problems, it is advisable to always use parentheses around inputs.

$$f(x+2) = (x+2)^2 - (x+2)$$

We could simplify this expression further to $f(x+2) = x^2 + 3x + 2$ if we wanted to:

$$f(x+2) = (x+2)(x+2) - (x+2)$$

Use the “FOIL” technique (first, outside, inside, last)

$$f(x+2) = x^2 + 2x + 2x + 4 - (x+2)$$

distribute the negative sign

$$f(x+2) = x^2 + 2x + 2x + 4 - x - 2$$

combine like terms

$$f(x+2) = x^2 + 3x + 2$$

Example 1.4.9

Using the same function, evaluate $f(t^3)$.

Solution

Note that in this example, the same variable is used in the input expression and as the input variable of the function. This doesn't matter – we still replace the original input t in the formula with the new input expression, t^3.

$$f(t^3) = (t^3)^2 - (t^3) = t^6 - t^3$$

Exercise 1.4.5

Given $g(x) = 3x - \sqrt{x}$, evaluate $g(t-2)$.

Answer

$$g(t-2) = 3(t-2) - \sqrt{(t-2)}$$

This now allows us to find an expression for a composition of functions. If we want to find a formula for $f(g(x))$, we can start by writing out the formula for $g(x)$. We can then evaluate the function $f(x)$ at that expression, as in the examples above.

Example 1.4.10

Let $f(x) = x^2$ and $g(x) = \frac{1}{x} - 2x$, find $f(g(x))$ and $g(f(x))$.

Solution

To find $f(g(x))$, we start by evaluating the inside, writing out the formula for $g(x)$.

$$g(x) = \frac{1}{x} - 2x$$

We then use the expression $(\frac{1}{x} - 2x)$ as input for the function f.

$$f(g(x)) = f(\frac{1}{x} - 2x)$$

We then evaluate the function $f(x)$ using the formula for $g(x)$ as the input.

Since $f(x) = x^2$,

$$f(\frac{1}{x} - 2x) = (\frac{1}{x} - 2x)^2$$

This gives us the formula for the composition:

$$f(g(x)) = (\frac{1}{x} - 2x)^2$$

Likewise, to find g(f(x)), we evaluate the inside, writing out the formula for f(x)

$$g(f(x)) = g(x^2)$$

Now we evaluate the function g(x) using x^2 as the input.

$$g(f(x)) = \frac{1}{x^2} - 2x^2$$

Exercise 1.4.6

Let $f(x) = x^3 + 3x$ and $g(x) = \sqrt{x}$, find $f(g(x))$ and $g(f(x))$.

Answer

$$f(g(x) = f(\sqrt{x}) = (\sqrt{x})^3 + 3(\sqrt{x})$$

$$g(f(x) = g(x^3 + 3x) = \sqrt{(x^3 + 3x)}$$

Example 1.4.11

A city manager determines that the tax revenue, R, in millions of dollars collected on a population of p thousand people is given by the formula $R(p) = 0.03p + \sqrt{p}$, and that the city's population, in thousands, is predicted to follow the formula $p(t) = 60 + 2t + 0.3t^2$, where t is measured in years after 2010. Find a formula for the tax revenue as a function of the year.

Solution

Since we want tax revenue as a function of the year, we want year to be our initial input, and revenue to be our final output. To find revenue, we will first have to predict the city population, and then use that result as the input to the tax function. So we need to find $R(p(t))$. Evaluating this,

$$R(p(t)) = R(60 + 2t + 0.3t^2) = 0.03(60 + 2t + 0.3t^2) + \sqrt{60 + 2t + 0.3t^2}$$

This composition gives us a single formula which can be used to predict the tax revenue during a given year, without needing to find the intermediary population value.

For example, to predict the tax revenue in 2017, when t = 7 (because t is measured in years after 2010),

$$R(p(7)) = 0.03(60 + 2(7) + 0.3(7)^2) + \sqrt{60 + 2(7) + 0.3(7)^2} \approx 12.079 \text{ million dollars}$$

Domain of Compositions

When we think about the domain of a composition $h(x) = f(g(x))$, we must consider both the domain of the inner function and the domain of the composition itself. While it is tempting to only look at the resulting composite function, if the inner function were undefined at a value of x, the composition would not be possible.

Example 1.4.12

Let $f(x) = \dfrac{1}{x^2 - 1}$ and $g(x) = \sqrt{x - 2}$. Find the domain of $f(g(x))$.

Since we want to avoid the square root of negative numbers, the domain of $g(x)$ is the set of values where $x - 2 \geq 0$. The domain is $x \geq 2$.

The composition is

$$f(g(x)) = \frac{1}{(\sqrt{x-2})^2 - 1} = \frac{1}{(x-2) - 1} = \frac{1}{x - 3}$$

.

The composition is undefined when $x = 3$, so that value must also be excluded from the domain. Notice that the composition doesn't involve a square root, but we still have to consider the domain limitation from the inside function.

Combining the two restrictions, the domain is all values of *x* greater than or equal to 2, except $x = 3$.

In inequalities, the domain is: $2 \leq x < 3$ or $x > 3$

In interval notation, the domain is: $[2, 3) \cup (3, \infty)$.

Exercise 1.4.7

Let $f(x)=\dfrac{1}{x-2}$ and $g(x)=\dfrac{1}{x}$. Find the domain of $f(g(x))$.

Answer

$g(x)=\dfrac{1}{x}$ is undefined at $x=0$.

The composition,

$$f(g(x))=f(\frac{1}{x})=\frac{1}{\frac{1}{x}-2}=\frac{1}{\frac{1}{x}-\frac{2x}{x}}=\frac{1}{\frac{1-2x}{x}}=\frac{x}{1-2x}$$

is undefined when $1-2x=0$, when $x=\dfrac{1}{2}$.

Restricting these two values, the domain is $(-\infty,0)\cup(0,\frac{1}{2})\cup(\frac{1}{2},\infty)$

Decomposing Functions

In some cases, it is desirable to decompose a function – to write it as a composition of two simpler functions.

Example 1.4.13

Write $f(x)=3+\sqrt{5-x^2}$ as the composition of two functions.

Solution

We are looking for two functions, g and h, so $f(x)=g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that $5-x^2$ is the inside of the square root. We could then decompose the function as:

$$h(x)=5-x^2$$

$$g(x)=3+\sqrt{x}$$

We can check our answer by recomposing the functions:

$$g(h(x))=g(5-x^2)=3+\sqrt{5-x^2}$$

Note that this is not the only solution to the problem. Another non-trivial decomposition would be $h(x)=x^2$ and $g(x)=3+\sqrt{5-x}$

Important Topics of this Section

- Definition of Composition of Functions
- Compositions using:
- Words
- Tables
- Graphs
- Equations
- Domain of Compositions
- Decomposition of Functions

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1.4E: Composition of Functions

SECTION 1.4 EXERCISE

Given each pair of functions, calculate $f(g(0))$ and $g(f(0))$.

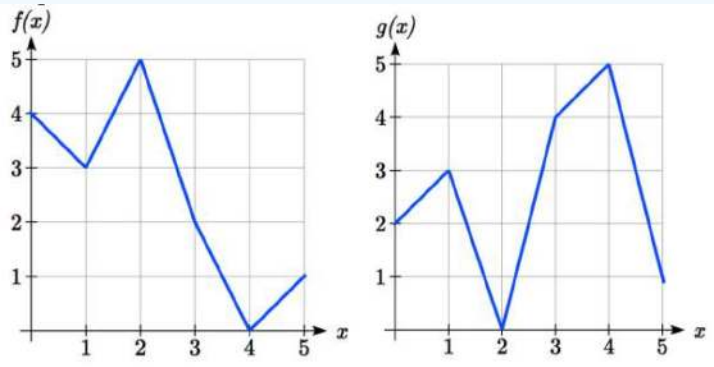
1. $f(x) = 4x + 8, g(x) = 7 - x^2$
2. $f(x) = 5x + 7, g(x) = 4 - 2x^2$
3. $f(x) = \sqrt{x+4}, g(x) = 12 - x^3$
4. $f(x) = \dfrac{1}{x+2}, g(x) = 4x + 3$

Use the table of values to evaluate each expression

x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

5. $f(g(8))$
6. $f(g(5))$
7. $g(f(5))$
8. $g(f(3))$
9. $f(f(4))$
10. $f(f(1))$
11. $g(g(2))$
12. $g(g(6))$

Use the graphs to evaluate the expressions below.



13. $f(g(3))$
14. $f(g(1))$
15. $g(f(1))$

16. $g(f(0))$

17. $f(f(5))$

18. $f(f(4))$

19. $g(g(2))$

20. $g(g(0))$

For each pair of functions, find $f(g(x))$ and $g(f(x))$. Simplify your answers.

21. $f(x)=\dfrac{1}{x-6}, g(x)=\dfrac{7}{x}+6$

22. $f(x)=\dfrac{1}{x-4}, g(x)=\dfrac{2}{x}+4$

23. $f(x)=x^2+1, g(x)=\sqrt{x+2}$

24. $f(x)=\sqrt{x}+2, g(x)=x^2+3$

25. $f(x)=|x|, g(x)=5x+1$

26. $f(x)=\sqrt[3]{x}, g(x)=\dfrac{x+1}{x^3}$

27. If $f(x)=x^4+6$, $g(x)=x-6$ and $h(x)=\sqrt{x}$, find $f(g(h(x)))$

28. If $f(x)=x^2+1$, $g(x)=\dfrac{1}{x}$ and $h(x)=x+3$, find $f(g(h(x)))$

29. The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost, $C(x)$ is the cost of producing x items. To determine the cost of production when the price is \$6, you would do which of the following:

a. Evaluate $D(C(6))$
b. Evaluate $C(D(6))$
c. Solve $D(C(x))=6$
d. Solve $C(D(p))=6$

20. The function $A(d)$ gives the pain level on a scale of 0-10 experienced by a patient with d milligrams of a pain reduction drug in their system. The milligrams of drug in the patient's system after t minutes is modeled by $m(t)$. To determine when the patient will be at a pain level of 4, you would need to:

a. Evaluate $A(m(4))$
b. Evaluate $m(A(4))$
c. Solve $A(m(t)=4$
d. Solve $m(A(d))=4$

31. The radius r, in inches, of a spherical balloon is related to the volume, V, by $r(V)=\sqrt[3]{\dfrac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t)=10+20t$.

a. Find the composite function $r(V(t))$

b. Find the radius after 20 seconds

32. The number of bacteria in a refrigerated food product is given by $N(T)=23T^2-56T+1$, $3<T<33$, where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t)=5t+1.5$, where t is the time in hours.

a. Find the composite function $N\left(T\left(t\right)\right)$
b. Find the bacteria count after 4 hours

33. Given $p(x)=\dfrac{1}{\sqrt{x}}$ and $m(x)=x^2-4$, find the domain of $m(p(x)$.

34. Given $p(x)=\dfrac{1}{\sqrt{x}}$ and $m(x)=9-x^2$, find the domain of $m(p(x))$.

35. Given $f(x)=\dfrac{1}{x+3}$ and $g(x)=\dfrac{2}{x-1}$, find the domain of $f(g(x))$.

36. Given $f(x)=\dfrac{x}{x+1}$ and $g(x)=\dfrac{4}{x}$, find the domain of $f(g(x))$.

37. Given $f(x)=\sqrt{x-2}$ and $g(x)=\dfrac{2}{x^2-3}$, find the domain of $g(f(x))$.

38. Given $f(x)=\sqrt{4-x}$ and $g(x)=\dfrac{1}{x^2-2}$, find the domain of $g(f(x))$.

Find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x)=f(g(x))$.

39. $h(x)=(x+2)^2$

40. $(x)=(x-5)^3$

41. $(x)=\dfrac{3}{x-5}$

42. $h(x)=\dfrac{4}{(x+2)^2}$

43. $h(x)=3+\sqrt{x-2}$

44. $h(x)=4+\sqrt[3]{x}$

45. Let $f(x)$ be a linear function, with form $f(x)=ax+b$ for constants a and b. [UW]

a. Show that $f\left(f\left(x\right)\right)$ is a linear function
b. Find a function $g(x)$ such that $g\left(g\left(x\right)\right)=6x-8$

46. Let $f(x)=\dfrac{1}{2}x+3$ [UW]

a. Sketch the graphs of $f(x)$, $f(f(x))$, $f(f(f(x)))$ on the interval $-2\le x\le 10$
b. Your graphs should all intersect at the point (6, 6). The value x = 6 is called a fixed point of the function $f(x)$ since $f(6)=6$; that is, 6 is fixed - it doesn't move when f is applied to it. Give an explanation for why 6 is a fixed point for any function $f(f(f(\ldots f(x)\ldots)))$.
c. Linear functions (with the exception of $f(x)=x$) can have at most one fixed point. Quadratic functions can have at most two. Find the fixed points of the function $g(x)=x^2-2$.
d. Give a quadratic function whose fixed points are $x=-2$ and $x=3$.

47. A car leaves Seattle heading east. The speed of the car in mph after m minutes is given by the function $C(m)=\dfrac{70m^2}{10+m^2}$. [UW]

a. Find a function $m=f(s)$ that converts seconds s into minutes m. Write out the formula for the new function $C(f(s))$; what does this function calculate?
b. Find a function $m=g(h)$ that converts hours h into minutes m. Write out the formula for the new function $C(g(h))$; what does this function calculate?
c. Find a function $z=v(s)$ that converts mph s into ft/sec z. Write out the formula for the new function $v(C(m)$; what does this function calculate?

Answer

1. $f(g(0))=36$. $g(f(0))=-57$

3. $f(g(0))=4$. $g(f(0))=4$

5. 4

7. 9

11. 7

13. 0

15. 4

17. 3

19. 2

21. $f(g(x)) = \frac{x}{7}$ $g(f(x)) = 7x - 36$

23. $f(g(x)) = x + 3$ $g(f(x)) = \sqrt{x^2 + 3}$

25. $f(g(x)) = |5x + 1|$ $g(f(x)) = 5|x| + 1$

27. $f(g(h(x))) = (\sqrt{x} - 6)^4 + 6$

29. b

31. a. $r(V(t)) = \sqrt[3]{\frac{3(10 + 20t)}{4\pi}}$

b. 4.609 in

33. $(0, \infty)$

35. $(-\infty, \frac{1}{3}) \cup (\frac{1}{3}, 1) \cup (1, \infty)$

37. $[2, 5) \cup (5, \infty)$

39. $g(x) = x + 2$, $f(x) = x^2$

41. $f(x) = \frac{3}{x}$, $g(x) = x - 5$

43. $f(x) = 3 + \sqrt{x}$, $g(x) = x - 2$, or $f(x) = 3 + x$, $g(x) = \sqrt{x - 2}$

45. a. $f(f(x)) = a(ax + b) + b = (a^2)x + (ab + b)$

b. $g(x) = \sqrt{6}x - \frac{8}{\sqrt{6} + 1}$ or $g(x) = -\sqrt{6}x - \frac{8}{1 - \sqrt{6}}$

47. a. $C(f(s)) = \frac{70(\frac{s}{60})^2}{10 + (\frac{s}{60})^2}$

b. $C(g(h)) = \frac{70(60h)^2}{10 + (60h)^2}$

c. $v(C(m)) = \frac{5280}{3600}(\frac{70m^2}{10 + m^2})$

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1.5: Transformation of Functions

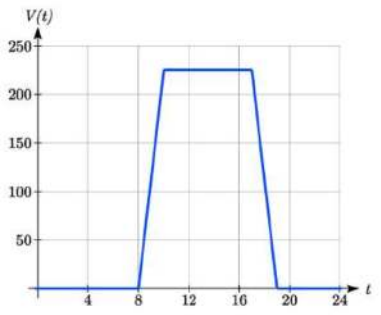
Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs and equations in order to explain or solve it. When building models, it is often helpful to build off of existing formulas or models. Knowing the basic graphs of your tool-kit functions can help you solve problems by being able to model new behavior by adapting something you already know. Unfortunately, the models and existing formulas we know are not always exactly the same as the ones presented in the problems we face.

Fortunately, there are systematic ways to shift, stretch, compress, flip and combine functions to help them become better models for the problems we are trying to solve. We can transform what we already know into what we need, hence the name, “Transformation of functions.” When we have a story problem, formula, graph, or table, we can then transform that function in a variety of ways to form new functions.

Shifts

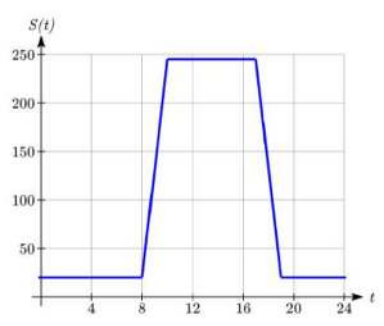
Example 1.5.1

To regulate temperature in a green building, air flow vents near the roof open and close throughout the day to allow warm air to escape. The graph below shows the open vents V (in square feet) throughout the day, t in hours after midnight. During the summer, the facilities staff decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day. Sketch a graph of this new function.



Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph up.



Notice that in the second graph, for each input value, the output value has increased by twenty, so if we call the new function $S(t)$, we could write $S(t) = V(t) + 20$.

Note that this notation tells us that for any value of t, $S(t)$ can be found by evaluating the V function at the same input, then adding twenty to the result.

This defines S as a transformation of the function V, in this case a vertical shift up 20 units.

Notice that with a vertical shift the input values stay the same and only the output values change.

Definition: Vertical shift

Given a function $f(x)$, if we define a new function $g(x)$ as

$$g(x) = f(x) + k \tag{1.5.1}$$

where k is a constant

then $g(x)$ is a **vertical shift** of Functions:Vertical Shifts of the function $f(x)$, where all the output values have been increased by k.

If k is positive, then the graph will shift up

If k is negative, then the graph will shift down

Example 1.5.2

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = f(x) - 3$

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x) - 3$ tells us that we can find the output values of the g function by subtracting 3 from the output values of the f function. For example,

$$f(2) = 1$$

is found from the given table

$$g(x) = f(x) - 3$$

is our given transformation

$$g(2) = f(2) - 3 = 1 - 3 = -2$$

Subtracting 3 from each $f(x)$ value, we can complete a table of values for $g(x)$

x	2	4	6	8
$g(x)$	-2	0	4	8

As with the earlier vertical shift, notice the input values stay the same and only the output values change.

Exercise 1.5.1

The function $h(t) = -4.9t^2 + 30t$ gives the height h of a ball (in meters) thrown upwards from the ground after t seconds. Suppose the ball was instead thrown from the top of a 10 meter building. Relate this new height function $b(t)$ to $h(t)$, then find a formula for $b(t)$.

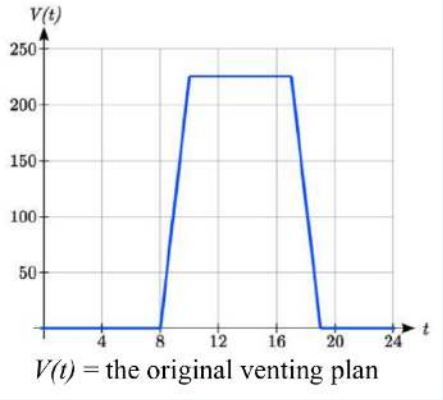
Answer

$$b(t) = h(t) + 10 = -4.9t^2 + 30t + 10$$

The vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning.

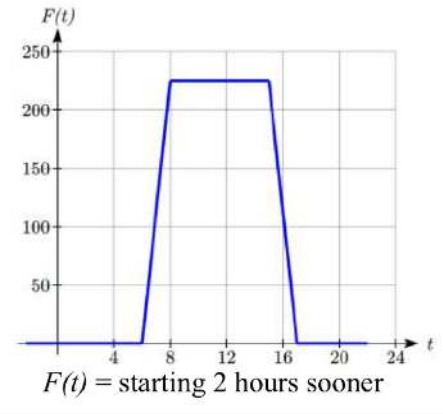
Example 1.5.3

Returning to our building air flow example from the beginning of the section, suppose that in Fall, the facilities staff decides that the original venting plan starts too late, and they want to move the entire venting program to start two hours earlier. Sketch a graph of the new function.



$V(t)$ = the original venting plan

Solution



$F(t)$ = starting 2 hours sooner

In the new graph, which we can call $F(t)$, at each time, the air flow is the same as the original function $V(t)$ was two hours later. For example, in the original function V, the air flow starts to change at 8 am, while for the function $F(t)$ the air flow starts to change at 6am. The comparable function values are $V(8) = F(6)$.

Notice also that the vents first opened to 220 sq. ft. at 10 a.m. under the original plan, while under the new plan the vents reach 220 sq. ft. at 8 a.m., so $V(10) = F(8)$.

In both cases we see that since $F(t)$ starts 2 hours sooner, the same output values are reached when, $F(t) = V(t+2)$

Note that $V(t+2)$ had the effect of shifting the graph to the $left$.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs two hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for F: $F(t) = V(t+2)$.

Definition: Horizontal shift

Given a function $f(x)$, if we define a new function $g(x)$ as

$$g(x) = f(x+k)$$

where k is a constant

then $g(x)$ is a **horizontal shift** of the function $f(x)$

If k is positive, then the graph will shift left

If k is negative, then the graph will shift right

Example 1.5.4

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = f(x-3)$.

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = f(x-3)$ tells us that the output values of g are the same as the output value of f with an input value three smaller. For example, we know that $f(2) = 1$. To get the same output from the g function, we will need an input value that is 3 *larger*: We input a value that is three larger for $g(x)$ because the function takes three away before evaluating the function f.

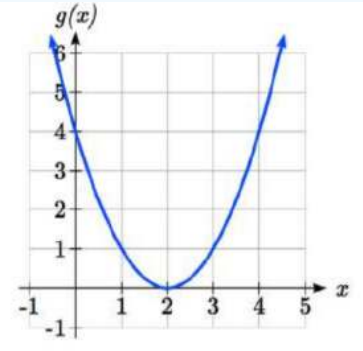
$$g(5) = f(5-3) = f(2) = 1$$

x	5	7	9	11
$g(x)$	1	3	7	11

The result is that the function $g(x)$ has been shifted to the right by 3. Notice the output values for $g(x)$ remain the same as the output values for $f(x)$ in the chart, but the corresponding input values, x, have shifted to the right by 3: 2 shifted to 5, 4 shifted to 7, 6 shifted to 9 and 8 shifted to 11.

Example 1.5.5

The graph shown is a transformation of the toolkit function $f(x) = x^2$. Relate this new function $g(x)$ to $f(x)$, and then find a formula for $g(x)$.



Solution

Notice that the graph looks almost identical in shape to the $f(x) = x^2$ function, but the x values are shifted to the right two units. The vertex used to be at (0, 0) but now the vertex is at (2, 0) . The graph is the basic quadratic function shifted two to the right, so

$$g(x) = f(x-2)$$

Notice how we must input the value $x = 2$, to get the output value $y = 0$; the x values must be two units larger, because of the shift to the right by 2 units.

We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$:

Since $f(x) = x^2$ and $g(x) = f(x-2)$

$$g(x) = f(x-2) = (x-2)^2$$

If you find yourself having trouble determining whether the shift is +2 or -2, it might help to consider a single point on the graph. For a quadratic, looking at the bottom-most point is convenient. In the original function, $f(0) = 0$. In our shifted

function, $g(2) = 0$. To obtain the output value of 0 from the f function, we need to decide whether a +2 or -2 will work to satisfy $g(2) = f(2 \ ? \ 2) = f(0) = 0$. For this to work, we will need to subtract 2 from our input values.

When thinking about horizontal and vertical shifts, it is good to keep in mind that vertical shifts are affecting the output values of the function, while horizontal shifts are affecting the input values of the function.

Example 1.5.6

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)+10$ and $G(m+10)$.

Solution

$G(m)+10$ is adding 10 to the output, gallons. This is 10 gallons of gas more than is required to drive m miles. So, this is the gas required to drive m miles, plus another 10 gallons of gas.

$G(m+10)$ is adding 10 to the input, miles. This is the number of gallons of gas required to drive 10 miles more than m miles.

Exercise 1.5.2

Given the function $f(x) = \sqrt{x}$ graph the original function $f(x)$ and the transformation $g(x) = f(x+2)$.

a. Is this a horizontal or a vertical change?

b. Which way is the graph shifted and by how many units?

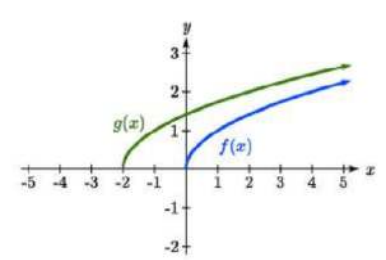
c. Graph $f(x)$ and $g(x)$ on the same axes.

Answer

a. Horizontal shift

b. The function is shifted to the LEFT by 2 units

c. Shown to the right



Now that we have two transformations, we can combine them together.

Remember:

Vertical Shifts are outside changes that affect the output (vertical) axis values shifting the transformed function up or down.

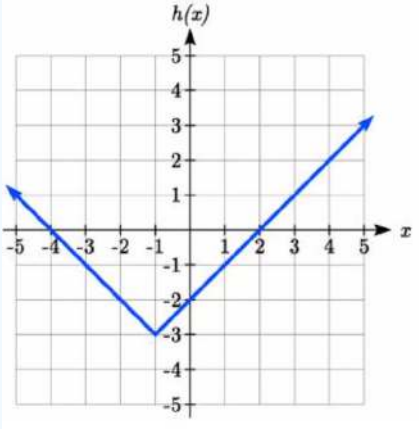
Horizontal Shifts are inside changes that affect the input (horizontal) axis values shifting the transformed function left or right.

Example 1.5.7

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x+1) - 3$.

Solution

The function f is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, then the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. Transforming the graph gives



Example 1.5.8

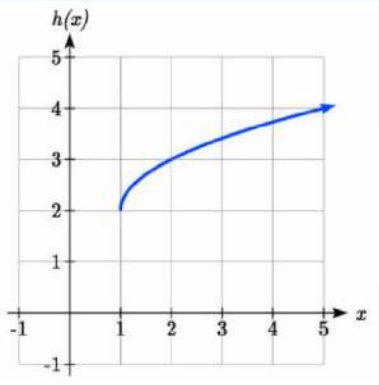
Write a formula for the graph shown, a transformation of the toolkit square root function.

Solution

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right, and up 2. In function notation, we could write that as $h(x) = f(x-1)+2$. Using the formula for the square root function we can write

$h(x) = \sqrt{x-1}+2$

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.



Reflections

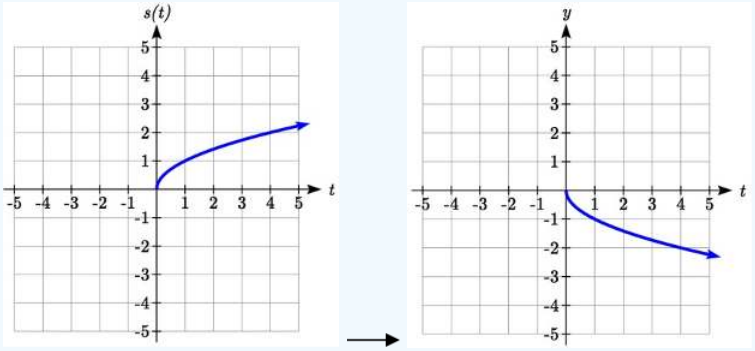
Another transformation that can be applied to a function is a reflection over the horizontal or vertical axis.

Example 1.5.9

Reflect the graph of $s(t) = \sqrt{t}$ both vertically and horizontally.

Solution

Reflecting the graph vertically, each output value will be reflected over the horizontal t axis:



Since each output value is the opposite of the original output value, we can write

$$V(t) = -s(t)$$

$$V(t) = -\sqrt{t}$$

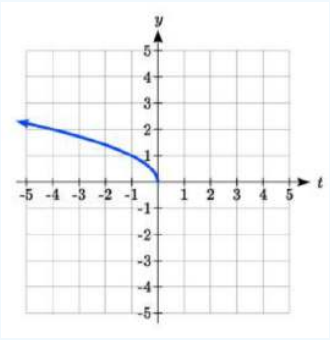
Notice this is an outside change or vertical change that affects the output *s(t)* values so the negative sign belongs outside of the function.

Reflecting horizontally, each input value will be reflected over the vertical axis.

Since each input value is the opposite of the original input value, we can write

$$H(t) = s(-t)$$

$$H(t) = \sqrt{-t}$$



Notice this is an inside change or horizontal change that affects the input values so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$, and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

Definition: reflections

Given a function $f(x)$, if we define a new function $g(x)$ as

$$g(x) = -f(x), \tag{1.5.2}$$

then $g(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about the x-axis

If we define a new function $g(x)$ as

$$g(x) = f(-x), \tag{1.5.3}$$

then $g(x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis

Example 1.5.10

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = -f(x)$ and $h(x) = f(-x)$

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

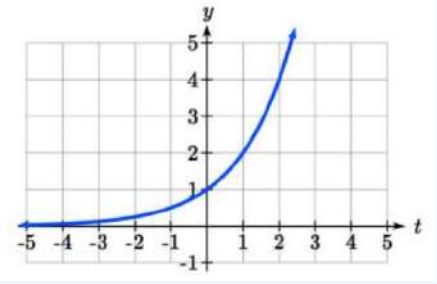
For $g(x)$, this is a vertical reflection, so the x values stay the same and each output value will be the opposite of the original output value

For $h(x)$, this is a horizontal reflection, and each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values:

x	-2	-4	-6	-8
$h(x)$	1	3	7	11

Example 1.5.11

A common model for learning has an equation similar to $k(t) = -2^{-t} + 1$, where k is the percentage of mastery that can be achieved after t practice sessions. This is a transformation of the function $f(t) = 2^t$ shown here. Sketch a graph of $k(t)$.



Solution

This equation combines three transformations into one equation.

A horizontal reflection: $f(-t)=2^{-t}$ combined with

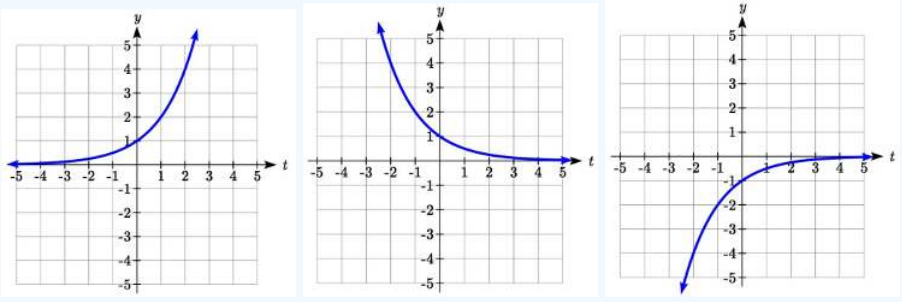
A vertical reflection: $-f(-t)=-2^{-t}$ combined with

A vertical shift up 1: $-f(-t)+1=-2^{-t}+1$

We can sketch a graph by applying these transformations one at a time to the original function:

The original graph Horizontally reflected Then vertically reflected

The Original Graph Horizontally Reflected The Vertically Reflected



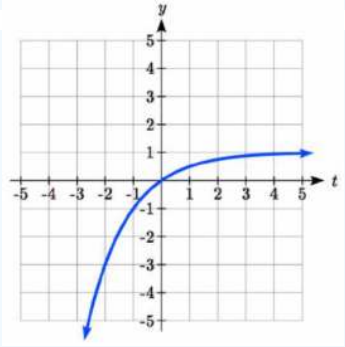
Then, after shifting up 1, we get the final graph.

$$k(t)=-f(-t)+1=-2^{-t}+1$$

Note: As a model for learning, this function would be limited to a domain of $t\geq 0$, with corresponding range $[0,1)$.

? exercise 1.5.3

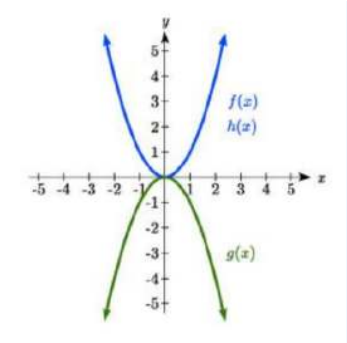
Given the toolkit function $f(x)=x^2$, graph $g(x)=-f(x)$ and $h(x)=f(-x)$. Do you notice anything surprising?



Answer

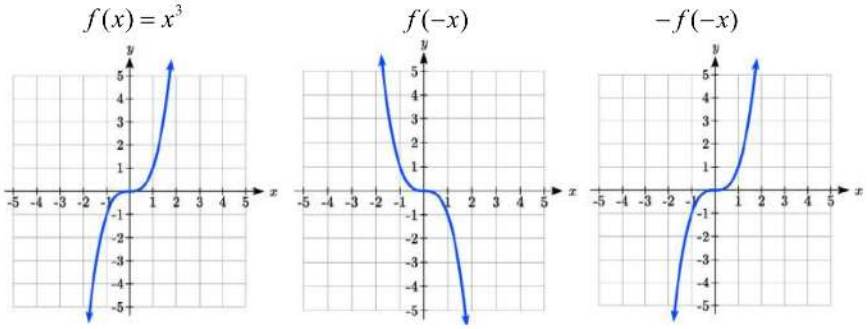
Shown in the graph

Notice: $g(x)=f(-x)$ looks the same as $f(x)$



Some functions exhibit symmetry, in which reflections result in the original graph. For example, reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ about the *y*-axis will result in the original graph. We call these types of graphs symmetric about the *y*-axis.

Likewise, if the graphs of $f(x) = x^3$ or $f(x) = \dfrac{1}{x}$ were reflected over both axes, the result would be the original graph:



We call these graphs symmetric about the origin.

Definition: even and odd functions

A function is called an **even function** if

$$f(x) = f(-x) \tag{1.5.4}$$

The graph of an even function is symmetric about the vertical axis

A function is called an **odd function** if

$$f(x) = -f(-x) \tag{1.5.5}$$

The graph of an odd function is symmetric about the origin

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, the $f(x) = 2^x$ function is neither even nor odd.

Example 1.5.12

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine this by finding formulas for the reflections, and seeing if they return us to the original function:

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even.

We can now try also applying a horizontal reflection:

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Since $-f(-x) = f(x)$, this is an odd function.

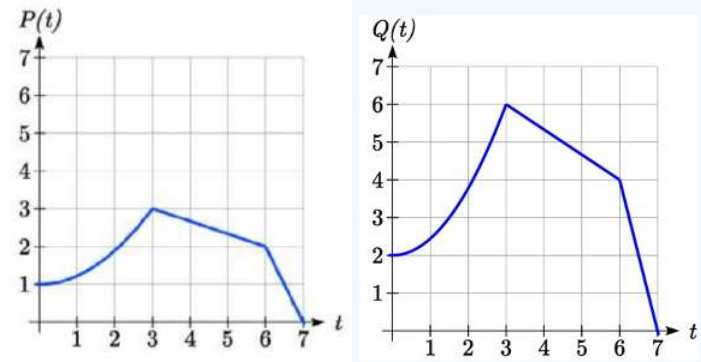
Stretches and Compressions

With shifts, we saw the effect of adding or subtracting to the inputs or outputs of a function. We now explore the effects of multiplying the inputs or outputs.

Remember, we can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Example 1.5.13

A function $P(t)$ models the growth of a population of fruit flies. The growth is shown in the graph. A scientist is comparing this to another population, Q, that grows the same way, but starts twice as large. Sketch a graph of this population.



Solution

Since the population is always twice as large, the new population's output values are always twice the original function output values. Graphically, this would look like the second graph shown.

Symbolically, $Q(t) = 2P(t)$

This means that for any input t, the value of the Q function is twice the value of the P function. Notice the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t, stay the same while the output values are twice as large as before.

Definition: Vertical stretch/compression

Given a function $f(x)$, if we define a new function $g(x)$ as

$g(x) = kf(x)$, where k is a constant

then $g(x)$ is a **vertical stretch or compression** of the function $f(x)$.

If $k > 1$, then the graph will be stretched

If $0 < k < 1$, then the graph will be compressed

If $k < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection

Example 1.5.14

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = \frac{1}{2} f(x)$

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

The formula $g(x) = \frac{1}{2} f(x)$ tells us that the output values of g are half of the output values of f with the same inputs. For example, we know that $f(4) = 3$. Then

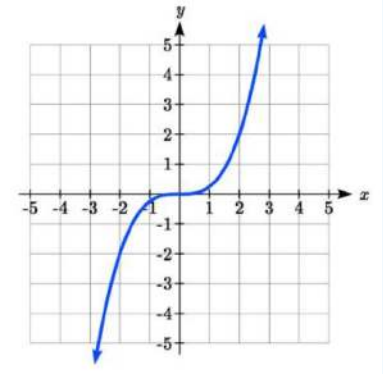
$$g(4) = \frac{1}{2} f(4) = \frac{1}{2}(3) = \frac{3}{2}$$

x	2	4	6	8
$g(x)$	1/2	3/2	7/2	11/2

The result is that the function $g(x)$ has been compressed vertically by 1/2. Each output value has been cut in half, so the graph would now be half the original height.

Example 1.5.15

The graph shown is a transformation of the toolkit function $f(x) = x^3$. Relate this new function $g(x)$ to $f(x)$, then find a formula for $g(x)$.



Solution

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input,

$$f(2) = 2^3 = 8$$

Based on that, it appears that the outputs of g are 1/4 the outputs of the function f, since

$$g(2) = \frac{1}{4} f(2)$$

From this we can fairly safely conclude that:

$$g(x) = \frac{1}{4} f(x)$$

We can write a formula for g by using the definition of the function f

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3$$

Now we consider changes to the inside of a function.

Example 1.5.16

Returning to the fruit fly population we looked at earlier, suppose the scientist is now comparing it to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount the original population did in 2 hours, and in 2 hours, will progress as much as the original population did in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$R(1) = P(2)$$

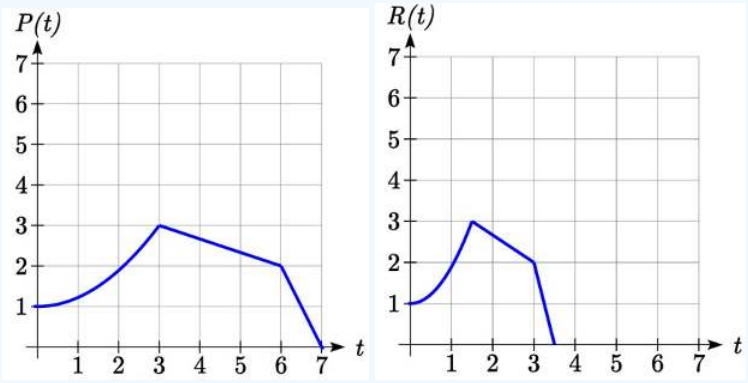
$$R(2) = P(4)$$

and in general,

$$R(t) = P(2t)$$

Graphing this,

Original Population, $P(t)$ Transformed, $R(t)$



Note the effect on the graph is a horizontal compression, where all input values are half their original distance from the vertical axis.

Definition: horizontal stretch/compression

Given a function $f(x)$, if we define a new function $g(x)$ as

$g(x) = f(kx)$, where k is a constant

then $g(x)$ is a **horizontal stretch or compression** of the function $f(x)$.

If $k > 1$, then the graph will be compressed by $1/k$

If $0 < k < 1$, then the graph will be stretched by $1/k$

If $k < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

Example 1.5.17

A function $f(x)$ is given as a table below. Create a table for the function $g(x) = f(\frac{1}{2}x)$

x	2	4	6	8
$f(x)$	1	3	7	11

Solution

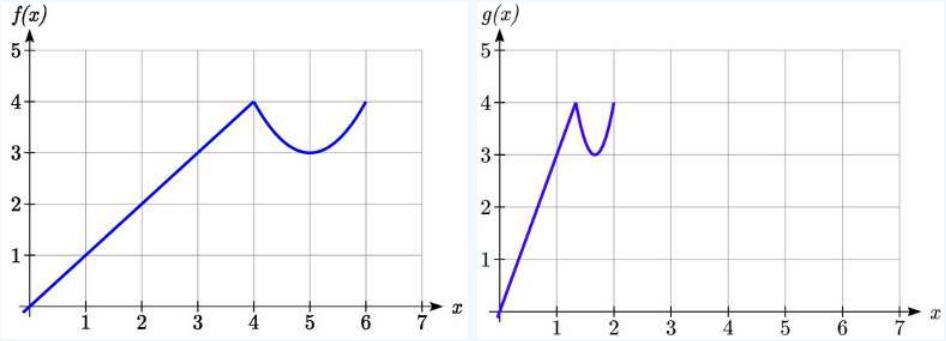
The formula $g(x) = f(\frac{1}{2}x)$ tells us that the output values for g are the same as the output values for the function f at an input half the size. Notice that we don't have enough information to determine $g(2)$ since $g(2) = f(\frac{1}{2} \cdot 2) = f(1)$, and we do not have a value for $f(1)$ in our table. Our input values to g will need to be twice as large to get inputs for f that we can evaluate. For example, we can determine $g(4)$ since $g(4) = f(\frac{1}{2} \cdot 4) = f(2) = 1$.

x	4	8	12	16
$g(x)$	1	3	7	11

Since each input value has been doubled, the result is that the function g(x) has been stretched horizontally by 2.

Example 1.5.18

Two graphs are shown below. Relate the function $g(x)$ to $f(x)$.



Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Since $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4) we can see that the x values have been compressed by 1/3, because 6(1/3) = 2. We might also notice that $g(2) = f(6)$, and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by 1/3.

Notice that the coefficient needed for a horizontal stretch or compression is the *reciprocal* of the stretch or compression. To stretch the graph horizontally by 4, we need a coefficient of 1/4 in our function: $f(\frac{1}{4}x)$. This means the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

exercise 1.5.4

Write a formula for the toolkit square root function horizontally stretched by three.

Answer

$$g(x) = f(\frac{1}{3}x$$

so using the square root function we get

$$g(x) = \sqrt{\frac{1}{3}x}$$

It is useful to note that for most toolkit functions, a horizontal stretch or vertical stretch can be represented in other ways. For example, a horizontal compression of the function $f(x)=x^2$ by ½ would result in a new function $g(x)=(2x)^2$, but this can also be written as $g(x)=4x^2$, a vertical stretch of $f(x)$ by 4. When writing a formula for a transformed toolkit, we only need to find one transformation that would produce the graph.

Combining Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 then vertically shifting by 3.

When we see an expression like $2f(x)+3$, which transformation should we start with? The answer here follows nicely from order of operations, for outside transformations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, then add 3, causing the vertical shift. (Multiplication before Addition)

Definition: combining vertical transformation

When combining vertical transformations written in the form $af(x)+k$,

first vertically stretch by a, then vertically shift by k.

Horizontal transformations are a little trickier to think about. When we write $g(x)=f(2x+3)$ for example, we have to think about how the inputs to the g function relate to the inputs to the f function. Suppose we know $f(7)=12$. What input to g would produce that output? In other words, what value of x will allow $g(x)=f(2x+3)=f(12)$? We would need $2x+3=12$. To solve for x, we would first subtract 3, resulting in horizontal shift, then divide by 2, causing a horizontal compression.

Definition:combining horizontal transformations

When **combining horizontal transformations** written in the form $f(bx+p)$,

first horizontally shift by p, then horizontally stretch by $1/b$.

This format ends up being very difficult to work with, since it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p)=f(b(x+\frac{p}{b})) \tag{1.5.6}$$

Factoring in this way allows us to horizontally stretch first, then shift horizontally.

Definition: combining horizontal transformations (factored form)

When **combining horizontal transformations** written in the form $f(b(x+h))$,

first horizontally stretch by $1/b$, then horizontally shift by h.

Definition: independence of horizontal and vertical transformations

Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are done first.

Example 1.5.19

Given the table of values for the function $f(x)$ below, create a table of values for the function $g(x)=2f(3x)+1$

x	6	12	18	24
$f(x)$	10	14	15	17

Solution

There are 3 steps to this transformation and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by 1/3, which means we multiply each x value by 1/3.

x	2	4	6	8
$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation.

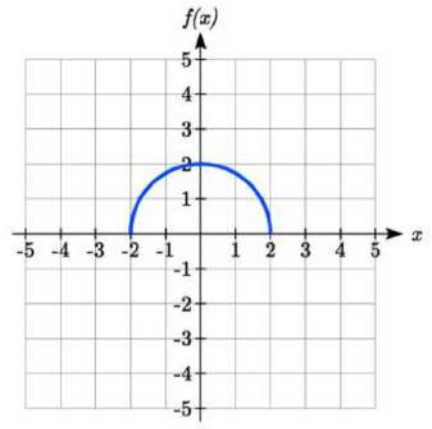
x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values.

x	2	4	6	8
$g(x)=2f(3x)+1$	21	29	31	35

Example 1.5.20

Using the graph of $f(x)$ below, sketch a graph of $k(x)=f(\frac{1}{2}x+1)-3$



Solution

To make things simpler, we'll start by factoring out the inside of the function

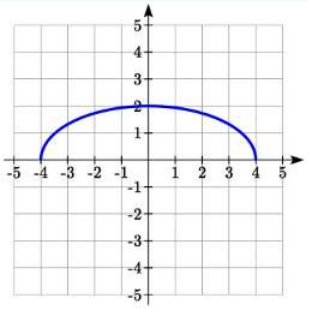
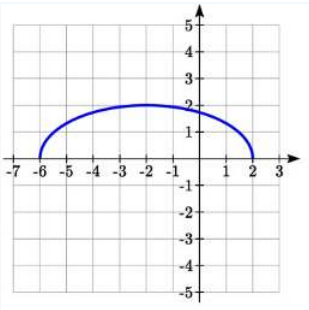
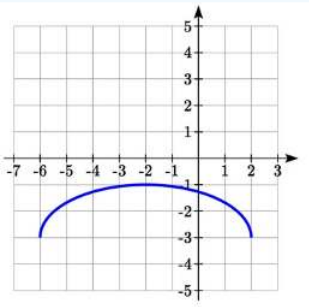
$$f(\frac{1}{2}x+1)-3=f(\frac{1}{2}(x+2))-3$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the 1/2 on the inside of the function. Remember twice the size of 0 is still 0, so the point (0,2) remains at (0,2) while the point (2,0) will stretch to (4,0).

Next, we horizontally shift left by 2 units, as indicated by the $x+2$.

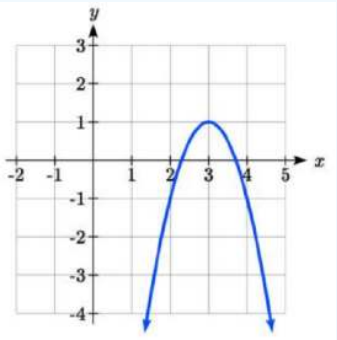
Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function.

Horizontal Stretch by 2 Horizontal Shift left by 2 Vertical Shift Down 3

Example 1.5.21

Write an equation for the transformed graph of the quadratic function shown.



Solution

Since this is a quadratic function, first consider what the basic quadratic tool kit function looks like and how this has changed. Observing the graph, we notice several transformations:

The original tool kit function has been flipped over the *x* axis, some kind of stretch or compression has occurred, and we can see a shift to the right 3 units and a shift up 1 unit.

In total there are four operations:

Vertical reflection, requiring a negative sign outside the function

Vertical Stretch *or* Horizontal Compression *

Horizontal Shift Right 3 units, which tells us to put *x*-3 on the inside of the function

Vertical Shift up 1 unit, telling us to add 1 on the outside of the function

* It is unclear from the graph whether it is showing a vertical stretch or a horizontal compression. For the quadratic, it turns out we could represent it either way, so we'll use a vertical stretch. You may be able to determine the vertical stretch by observation.

By observation, the basic tool kit function has a vertex at (0, 0) and symmetrical points at (1, 1) and (-1, 1). These points are one unit up and one unit over from the vertex. The new points on the transformed graph are one unit away horizontally but 2 units away vertically. They have been stretched vertically by two.

Not everyone can see this by simply looking at the graph. If you can, great, but if not, we can solve for it. First, we will write the equation for this graph, with an unknown vertical stretch.

$f(x) = x^2$ The original function

$-f(x) = -x^2$ Vertically reflected

$-af(x) = -ax^2$ Vertically stretched

$-af(x-3) = -a(x-3)^2$ Shifted right 3

$-af(x-3)+1=-a(x-3)^2+1$ Shifted up 1

We now know our graph is going to have an equation of the form $g(x)=-a(x-3)^2+1$. To find the vertical stretch, we can identify any point on the graph (other than the highest point), such as the point (2,-1), which tells us $g(2)=-1$. Using our general formula, and substituting 2 for x, and -1 for $g(x)$

$$\begin{aligned}-1&=-a(2-3)^2+1\\-1&=-a+1\\-2&=-a\\2&=a\end{aligned}$$

This tells us that to produce the graph we need a vertical stretch by two.

The function that produces this graph is therefore $g(x)=-2(x-3)^2+1$.

Exercise 1.5.5

Consider the linear function $g(x)=-2x+1$. Describe its transformation in words using the identity tool kit function $f(x)=x$ as a reference.

Answer

The identity tool kit function $f(x)=x$ has been transformed in 3 steps

a. Vertically stretched by 2.

b. Vertically reflected over the x axis.

c. Vertically shifted up by 1 unit.

Example 1.5.22

On what interval(s) is the function $g(x)=\dfrac{-2}{(x-1)^2}+3$ increasing and decreasing?

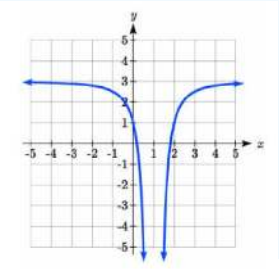
Solution

This is a transformation of the toolkit reciprocal squared function, $f(x)=\dfrac{1}{x^2}$:

$-2f(x)=\dfrac{-2}{x^2}$ A vertical flip and vertical stretch by 2

$-2f(x-1)=\dfrac{-2}{(x-1)^2}$ A shift right by 1

$-2f(x-1)+3=\dfrac{-2}{(x-1)^2}+3$ A shift up by 3



The basic reciprocal squared function is increasing on $(-\infty,0)$and decreasing on $(0,\infty)$. Because of the vertical flip, the $g(x)$ function will be decreasing on the left and increasing on the right. The horizontal shift right by 1 will also shift these intervals to the right one. From this, we can determine $g(x)$ will be increasing on $(1,\infty)$ and decreasing on $(-\infty,1)$. We also could graph the transformation to help us determine these intervals.

Exercise 1.5.6

On what interval(s) is the function $h(t)=(t-3)^3+2$ concave up and down?

Answer

$h(t)$ is concave down on $(-\infty,3)$ and concave up on $(3,\infty)$

Important Topics of This Section

Transformations

Vertical Shift (up & down)

Horizontal Shifts (left & right)

Reflections over the vertical & horizontal axis

Even & Odd functions

Vertical Stretches & Compressions

Horizontal Stretches & Compressions

Combinations of Transformation

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1.5E: Transformation of Functions

Section 1.5 Exercises

Describe how each function is a transformation of the original function $f(x)$

1. $f(x-49)$
2. $f(x+43)$
3. $f(x+3)$
4. $f(x-4)$
5. $f(x)+5$
6. $f(x)+8$
7. $f(x)-2$
8. $f(x)-7$
9. $f(x-2)+3$
10. $f(x+4)-1$
11. Write a formula for $f(x)=\sqrt{x}$ shifted up 1 unit and left 2 units.
12. Write a formula for $f(x)=|x|$ shifted down 3 units and right 1 unit.
13. Write a formula for $f(x)=\dfrac{1}{x}$ shifted down 4 units and right 3 units.
14. Write a formula for $f(x)=\dfrac{1}{x^2}$ shifted up 2 units and left 4 units.
15. Tables of values for $f(x)$, $g(x)$, and $h(x)$ are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	$f(x)$
-2	-2
-1	-1
0	-3
1	1
2	2

x	$g(x)$
-1	-2
0	-1
1	-3
2	1
3	2

x	$h(x)$
-2	-1
-1	0
0	-2
1	2
2	3

16. Tables of values for $f(x)$, $g(x)$, and $h(x)$ are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	$f(x)$
-2	-1
-1	-3
0	4
1	2
2	1

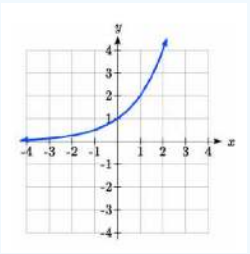
x	$g(x)$
-3	-1
-2	-3
-1	4
0	2
1	1

x	$h(x)$
-2	-2
-1	-4
0	3
1	1
2	0

The graph of $f(x)=2^x$ is shown. Sketch a graph of each transformation of $f(x)$.

17. $g(x)=2^x+1$
18. $h(x)=2^x-3$
19. $w(x)=2^{x-1}$
20. $q(x)=2^{x+3}$

Sketch a graph of each function as a transformation of a toolkit function.



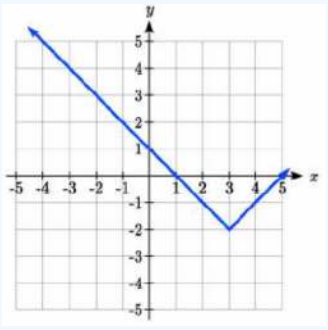
21. $f(t) = (t+1)^2 - 3$

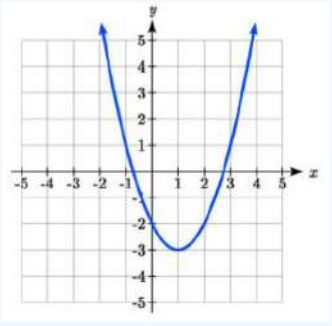
22. $h(x) = |x-1| + 4$

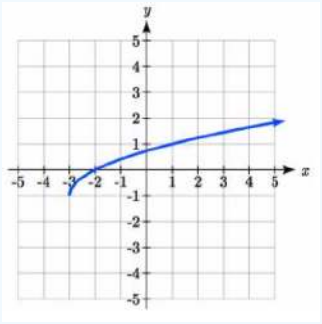
23. $k(x = (x-2)^3 - 1$

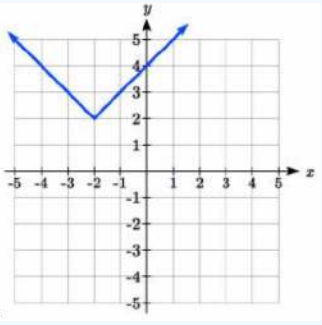
24. $m(t) = 3 + \sqrt{t+2}$

Write an equation for each function graphed below.

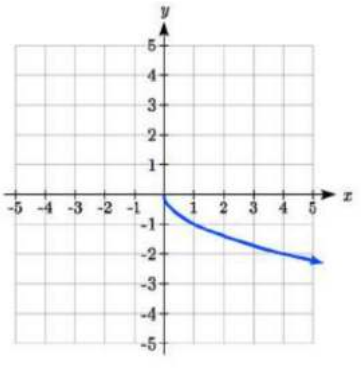
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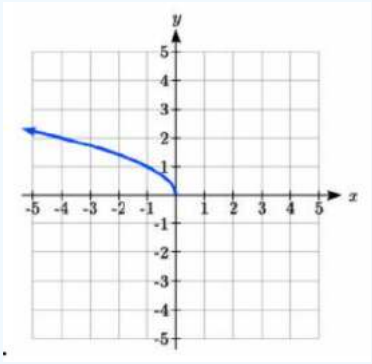
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27. 

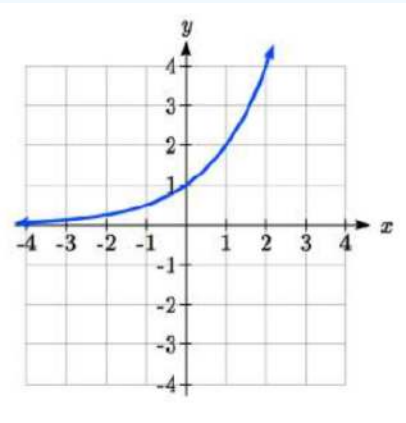
28. 

Find a formula for each of the transformations of the square root whose graphs are given below.

29. 

30. 

The graph of $f(x) = 2^x$ is shown. Sketch a graph of each transformation of $f(x)$



31. $g(x) = -2^x + 1$

32. $h(x) = 2^{-x}$

33. Starting with the graph of $f(x) = 6^x$ write the equation of the graph that results from

a. reflecting $f(x)$ about the x-axis and the y-axis

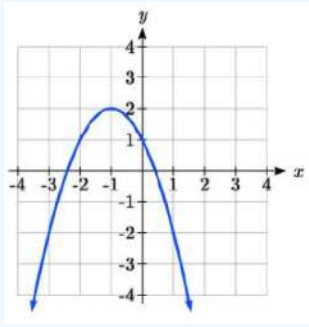
b. reflecting $f(x)$ about the x-axis, shifting left 2 units, and down 3 units

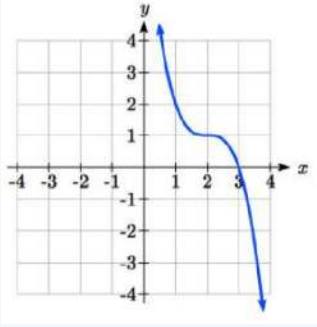
34. Starting with the graph of $f(x) = 4^x$ write the equation of the graph that results from

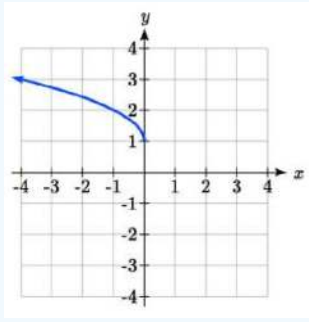
a. reflecting $f(x)$ about the x-axis

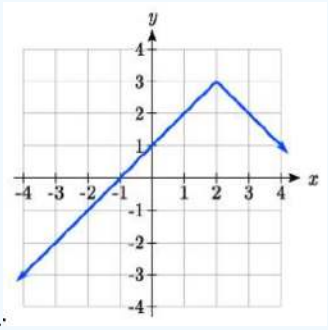
b. reflecting $f(x)$ about the y-axis, shifting right 4 units, and up 2 units

Write an equation for each function graphed below.

35. 

36. 

37. 

38. 

39. For each equation below, determine if the function is Odd, Even, or Neither.

a. $f(x) = 3x^4$

b. $g(x) = \sqrt{x}$

c. $h(x) = \dfrac{1}{x} + 3x$

40. For each equation below, determine if the function is Odd, Even, or Neither.

a. $f(x) = (x-2)^2$

b. $g(x) = 2x^4$

c. $h(x) = 2x - x^3$

Describe how each function is a transformation of the original function $f(x)$.

41. $-f(x)$

42. $f(-x)$

43. $4f(x)$

44. $6f(x)$

45. $f(5x)$

46. $f(2x)$

47. $f(\dfrac{1}{3}x)$

48. $f(\dfrac{1}{5}x)$

49. $3f(-x)$

50. $-f(3x)$

Write a formula for the function that results when the given toolkit function is transformed as described.

51. $f(x) = |x|$ reflected over the *y* axis and horizontally compressed by a factor of $\dfrac{1}{4}$.

52. $f(x)=\sqrt{x}$ reflected over the x axis and horizontally stretched by a factor of 2.

53. $f(x)=\dfrac{1}{x^2}$ vertically compressed by a factor of $\dfrac{1}{3}$, then shifted to the left 2 units and down 3 units.

54. $f(x)=\dfrac{1}{x}$ vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

55. $f(x)=x^2$ horizontally compressed by a factor of $\dfrac{1}{2}$, then shifted to the right 5 units and up 1 unit.

56. $f(x)=x^2$ horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

Describe how each formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

57. $f\,(x)=4(x+1)^2-5$

58. $g(x)=5(x+3)^2-2$

59. $h(x)=-2|x-4|+3$

60. $k(x)=-3\sqrt{x}-1$

61. $m(x)=\dfrac{1}{2}x^3$

62. $n(x)=\dfrac{1}{3}|x-2|$

63. $p(x)=(\dfrac{1}{3}x)^2-3$

64. $q(x)=(\dfrac{1}{4}x)^3+1$

65. $a(x)=\sqrt{-x+4}$

66. $b(x)=\sqrt[3]{-x-6}$

Determine the interval(s) on which the function is increasing and decreasing.

67. $f(x)=4(x+1)^2-5$

68. $g(x)=5(x+3)^2-2$

69. $a(x)=\sqrt{-x+4}$

70. $k(x)=-3\sqrt{x}-1$

Determine the interval(s) on which the function is concave up and concave down.

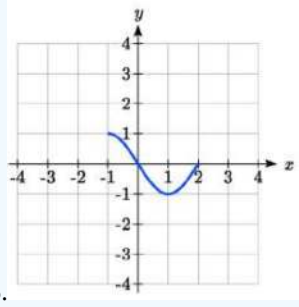
71. $m(x)=-2(x+3)^3+1$

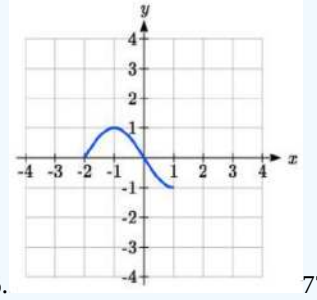
72. $b(x)=\sqrt[3]{-x-6}$

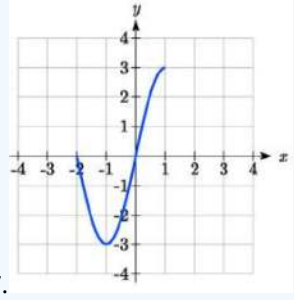
73. $p(x)=(\dfrac{1}{3}x)^2-3$

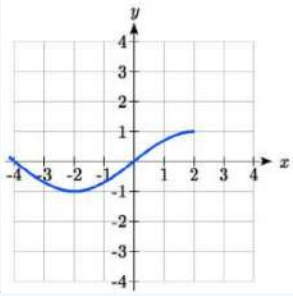
74. $k(x)=-3\sqrt{x}-1$

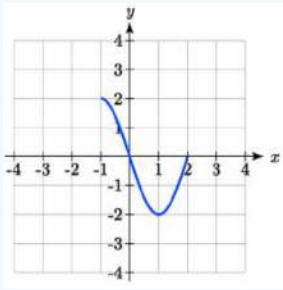
The function $f(x)$ is graphed here. Write an equation for each graph below as a transformation of $f(x)$.

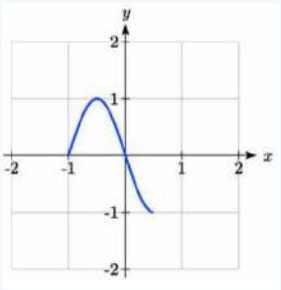
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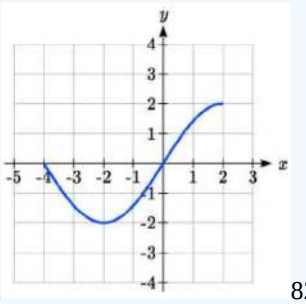
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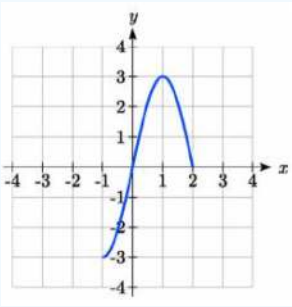
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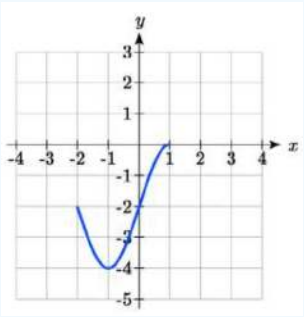
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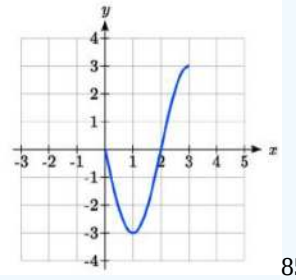
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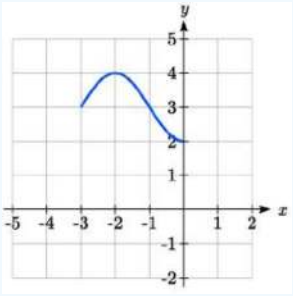
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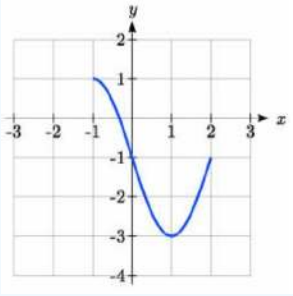
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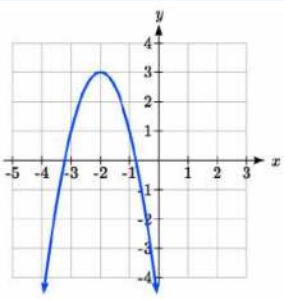
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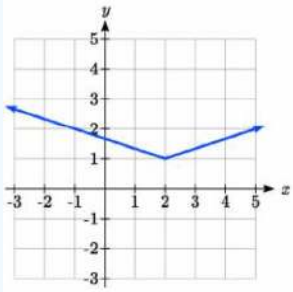
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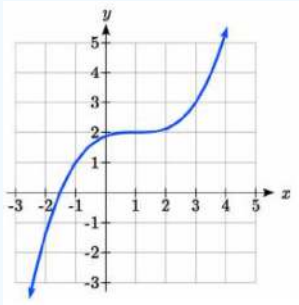
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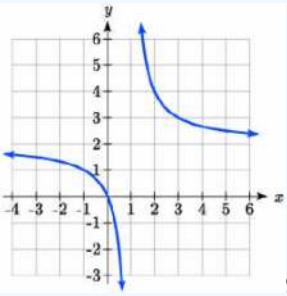
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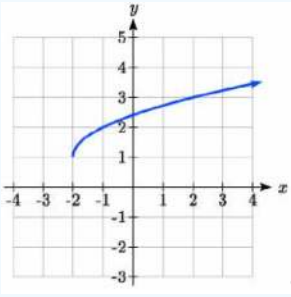
Write an equation for each transformed toolkit function graphed below.

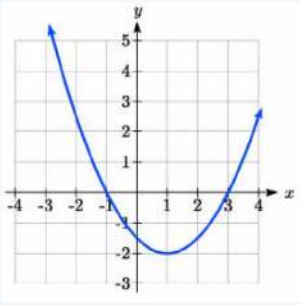
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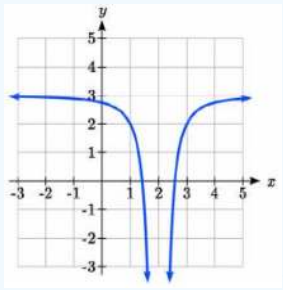
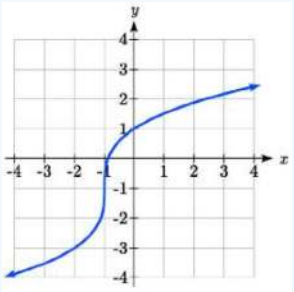
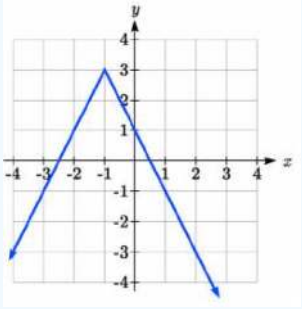
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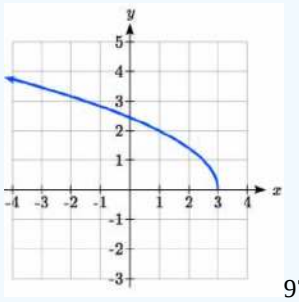
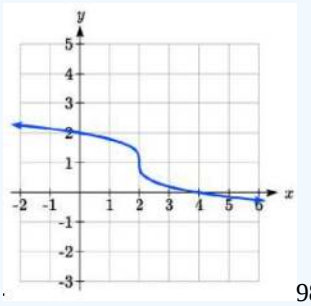
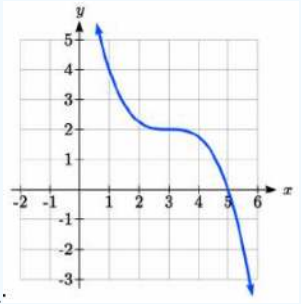
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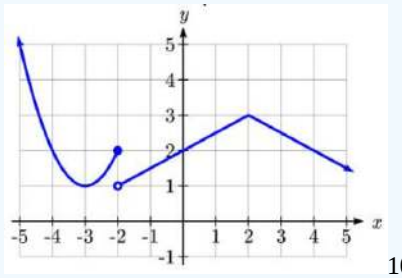
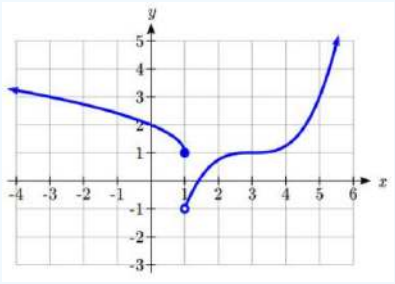
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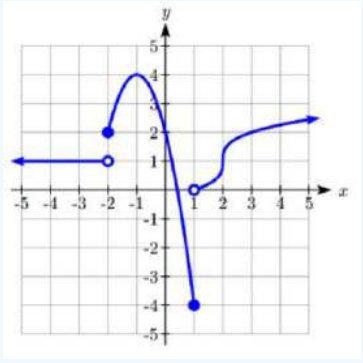
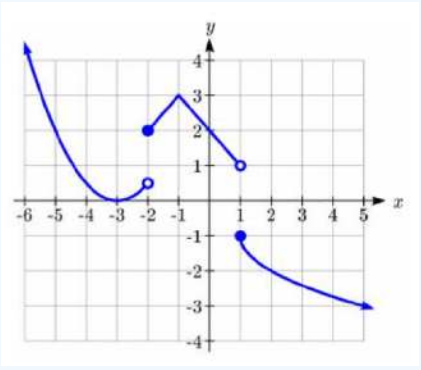
92. 

93. 
94. 
95. 

96. 
97. 
98. 

Write a formula for the piecewise function graphed below.

99. 
100. 

101. 
102. 

103. Suppose you have a function $y = f(x)$ such that the domain of $f(x)$ is $1 \leq x \leq 6$ and the range of $f(x)$ is (-3 \le y \le 5\). [UW]

a. What is the domain of $f(2(x-3))$?

b. What is the range of $f(2(x-3))$?

c. What is the domain of $2f(x)-3$?

d. What is the range of $2f(x)-3$?

e. Can you find constants B and C so that the domain of $f(B(x-C))$ is $8 \leq x \leq 9$?

f. Can you find constants A and D so that the range of $Af(x)+D$ is 0 $0 \leq y \leq 1$?

Answer

1. Horizontal shift right 49 units

3. Horizontal shift left 3 units

5. Vertical shift up 5 units

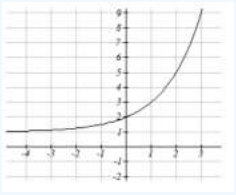
7. Vertical shift down 2 units

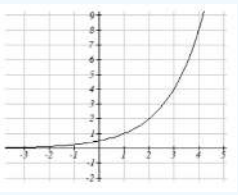
9. Horizontal shift right 2 units, Vertical shift up 3 units

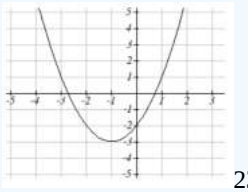
11. $f(x+2)+1=\sqrt{x+2}+1$

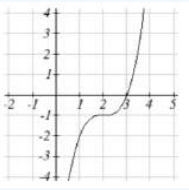
13. $f(x-3)-4=\dfrac{1}{x-3}-4$

15. $g(x)=f(x-1)$, $h(x)=f(x)+1$

17. 

19. 

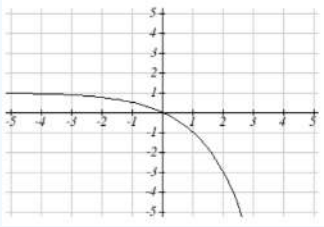
21. 

23. 

25. $y=|x-3|-2$

27. $y=\sqrt{x+3}-1$

29. $y=-\sqrt{x}$

31. 

33. a. $-f(-x)=-6^{-x}$
b. $-f(x+2)-3=-6^{x+2}-3$

35. $y=-(x+1)^2+2$

37. $y=\sqrt{-x}+1$

39. a. Even
b. Neither
c. Odd

41. Reflect $f(x)$ about the x-axis

43. Vertically stretch y values by 4

45. Horizontally compress x values by 1/5

47. Horizontally stretch x values by 3

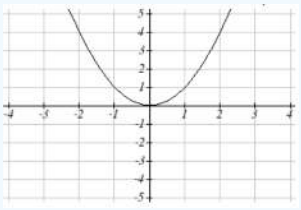
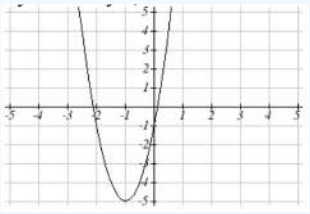
49. Reflect $f(x)$ about the y-axis and vertically stretch y values by 3

51. $f(-4x)=|-4x|$

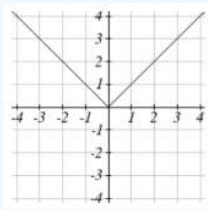
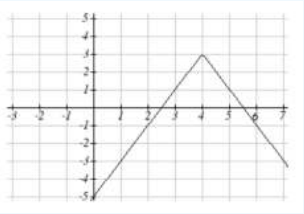
53. $\frac{1}{3}f(x+2)-3=\frac{1}{3(x+2)^2}-3$

55. $f(2(x-5))+1=(2(x-5))^2+1$

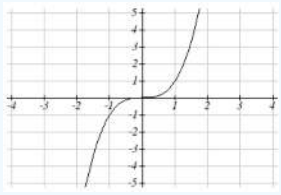
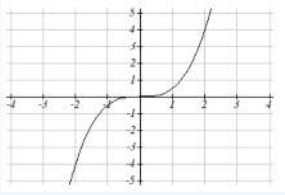
57. Horizontal shift left 1 unit, vertical stretch y values by 4, vertical shift down 5 units

 becomes 

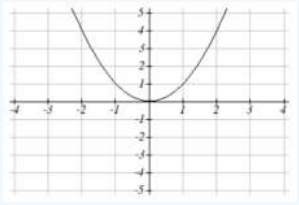
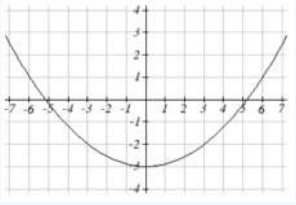
59. Horizontal shift right 4 units, vertical stretch y values by 2, reflect over x axis, vertically shift up 3 units.

 becomes 

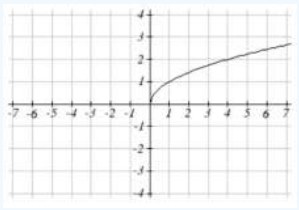
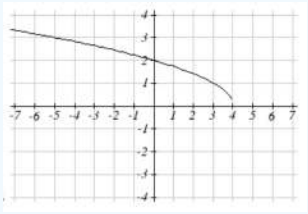
61. Vertically compress y values by 1/2

 becomes 

63. Horizontally stretch x values by 3, vertical shift down 3 units

 becomes 

65. Reflected over the y axis, horizontally shift right 4 units $a(x)=\sqrt{-(x-4)}$

 becomes 

67. This function is increasing on $(-1,\infty)$ and decreasing on $(-\infty,-1)$

69. This function is decreasing on $(-\infty,4)$

71. This function is concave down on $(-3,\infty)$ and concave up on $(-\infty,-3)$

73. This function is concave up everywhere

75. $f(-x)$

77. $3f(x)$

79. $2f(-x)$

81. $2f(\frac{1}{2}x)$

83. $2f(x)-2$

85. $-f(x+1)+3$

87. $y=-2(x+2)^2+3$

89. $y=(\frac{1}{2}(x-1))^3+2$

91. $y=\sqrt{2(x+2)}+1$

93. $y=\frac{-1}{(x-2)^2}+3$

95. $y=-2|x+1|+3$

97. $y=\sqrt[3]{-\frac{1}{2}(x-2)}+1$

99. $f(x)=\begin{cases}(x+3)^2+1 & if \quad x\leq -2\\ \frac{1}{2}|x-2|+3 & if \quad x>-2\end{cases}$

101. $f(x)=\begin{cases}1 & if \quad x<-2\\ -2(x+1)^2+4 & if \quad -2\leq x\leq 1\\ \sqrt[3]{x-2}+1 & if \quad x>1\end{cases}$

103a. Domain: $3.5\leq x\leq 6$
d. Range: $-9\leq y\leq 7$

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1.6: Inverse Functions

A fashion designer is travelling to Milan for a fashion show. He asks his assistant, Betty, what 75 degrees Fahrenheit is in Celsius, and after a quick search on Google, she finds the formula $C = \frac{5}{9}(F - 32)$. Using this formula, she calculates $\frac{5}{9}(75 - 32) \approx 24$ degrees Celsius. The next day, the designer sends his assistant the week's weather forecast for Milan, and asks her to convert the temperatures to Fahrenheit.



At first, Betty might consider using the formula she has already found to do the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write:

$$26 = \frac{5}{9}(F - 32)$$

$$26 \cdot \frac{9}{5} = F - 32$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79$$

After considering this option for a moment, she realizes that solving the equation for each of the temperatures would get awfully tedious, and realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one which takes the Celsius temperature and outputs the Fahrenheit temperature. This is the idea of an inverse function, where the input becomes the output and the output becomes the input.

Definition: Inverse function

If $f(a) = b$, then a function $g(x)$ is an **inverse** of f if $g(b) = a$.

The inverse of $f(x)$ is typically notated $f^{-1}(x)$, which is read "f inverse of x", so equivalently, if $f(a) = b$ then $f^{-1}(b) = a$.

Important: The raised -1 used in the notation for inverse functions is simply a notation, and does not designate an exponent or power of -1.

Example 1.6.1

If for a particular function, $f(2) = 4$, what do we know about the inverse?

Solution

The inverse function reverses which quantity is input and which quantity is output, so if $f(2) = 4$, then $f^{-1}(4) = 2$.

Alternatively, if you want to re-name the inverse function $g(x)$, then $g(4) = 2$

Exercise 1.6.1

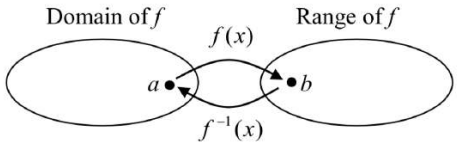
Given that $h^{-1}(6) = 2$, what do we know about the original function $h(x)$?

Answer

$g(2) = 6$

Notice that original function and the inverse function *undo* each other. If $f(a) = b$, then $f^{-1}(b) = a$, returning us to the original input. More simply put, if you compose these functions together you get the original input as your answer.

$$f^{-1}(f(a)) = a \text{ and } f(f^{-1}(b)) = b \tag{1.6.1}$$



Since the outputs of the function f are the inputs to f^{-1}, the range of f is also the domain of f^{-1}. Likewise, since the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}.

Basically, like how the input and output values switch, the domain & ranges switch as well. But be careful, because sometimes a function doesn't even have an inverse function, or only has an inverse on a limited domain. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, since a square "undoes" a square root, but it is only the inverse of $f(x)$ on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

Example 1.6.2

The function $f(x) = 2^x$ has domain $(-\infty, \infty)$ and range $(0, \infty)$, what would we expect the domain and range of f^{-1} to be?

Solution

We would expect f^{-1} to swap the domain and range of f, so f^{-1} would have domain $(0, \infty)$ and range $(-\infty, \infty)$.

Example 1.6.3

A function $f(t)$ is given as a table below, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Solution

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, the 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. Interpreting this, it means that to drive 70 miles, it took 90 minutes.

Alternatively, recall the definition of the inverse was that if $f(a) = b$ then $f^{-1}(b) = a$. By this definition, if you are given $f^{-1}(70) = a$ then you are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Exercise 1.6.2

Using the table below

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Find and interpret the following

a. $f(60)$

b. $f^{-1}(60)$

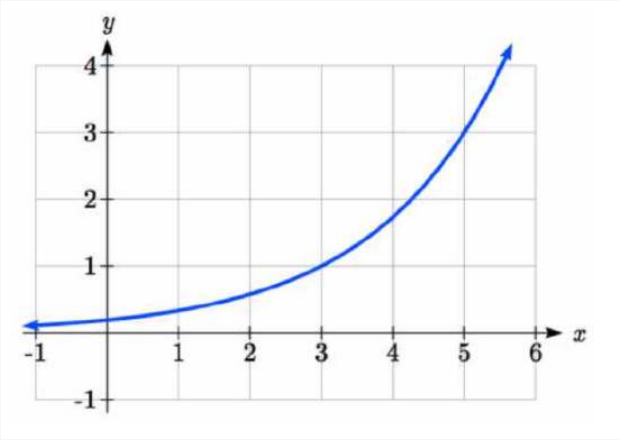
Answer

a. $f(60) = 50$. In 60 minutes, 50 miles are traveled.

b. $f^{-1}(60) = 70$. To travel 60 miles, it will take 70 minutes.

Example 1.6.4

A function $g(x)$ is given as a graph below. Find $g(3)$ and $g^{-1}(3)$



Solution

To evaluate $g(3)$, we find 3 on the horizontal axis and find the corresponding output value on the vertical axis. The point (3, 1) tells us that $g(3) = 1$

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means $g(x) = 3$. By looking for the output value 3 on the vertical axis we find the point (5, 3) on the graph, which means $g(5) = 3$, so by definition $g^{-1}(3) = 5$.

Exercise 1.6.3

Using the graph in Example 4 above

a. find $g^{-1}(1)$

b. estimate $g^{-1}(4)$

Answer

a. $g^{-1}(1) = 3$

b. $g^{-1}(4) = 5.5$ (this is an approximation - answers may vary slightly)

Example 1.6.5

Returning to our designer's assistant, find a formula for the inverse function that gives Fahrenheit temperature given a Celsius temperature.

Solution

A quick Google search would find the inverse function, but alternatively, Betty might look back at how she solved for the Fahrenheit temperature for a specific Celsius value, and repeat the process in general

$$C = \frac{5}{9}(F - 32)$$
$$C \cdot \frac{9}{5} = F - 32$$
$$F = \frac{9}{5}C + 32$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32)$$

Then

$$F = h^{-1}(C) = \frac{9}{5}C + 32$$

In this case, we introduced a function h to represent the conversion since the input and output variables are descriptive, and writing C^{-1} could get confusing.

It is important to note that not all functions will have an inverse function. Since the inverse $f^{-1}(x)$ takes an output of f and returns an input of f, in order for f^{-1} to itself be a function, then each output of f (input to f^{-1}) must correspond to exactly one input of f (output of f^{-1}) in order for f^{-1} to be a function. You might recall that this is the definition of a one-to-one function.

Definition: Properties of inverses

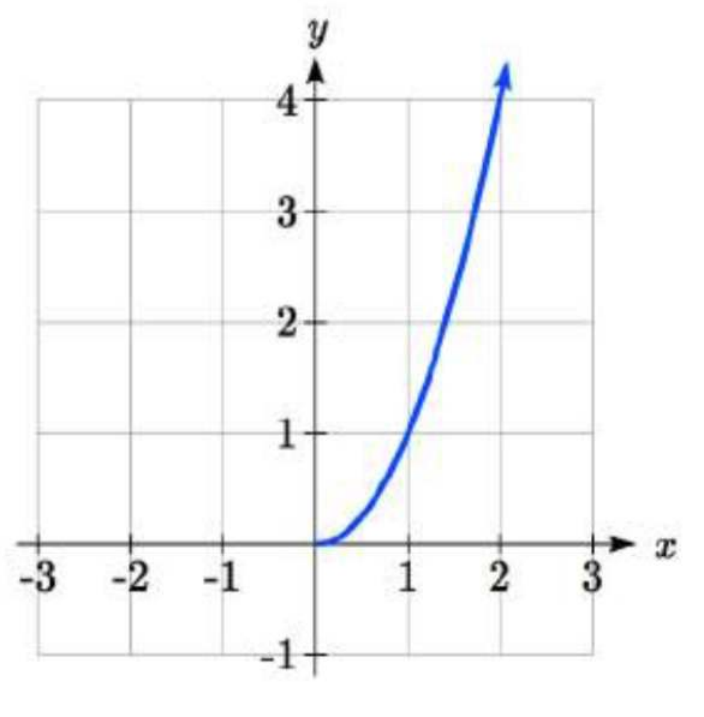
In order **for a function to have an inverse**, it must be a one-to-one function.

In some cases, it is desirable to have an inverse for a function even though the function is not one-to-one. In those cases, we can often limit the domain of the original function to an interval on which the function *is* one-to-one, then find an inverse only on that interval.

If you have not already done so, go back to the toolkit functions that were not one-to-one and limit or restrict the domain of the original function so that it is one-to-one. If you are not sure how to do this, proceed to Example 6.

Example 1.6.6

The quadratic function $h(x) = x^2$ is not one-to-one. Find a domain on which this function is one-to-one, and find the inverse on that domain.



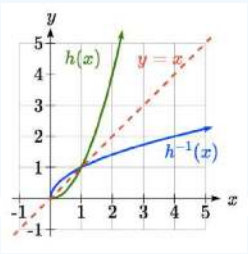
Solution

We can limit the domain to $[0, \infty)$to restrict the graph to a portion that is one-to-one, and find an inverse on this limited domain.

You may have already guessed that since we undo a square with a square root, the inverse of $h(x) = x^2$ on this domain is $h^{-1}(x) = \sqrt{x}$.

You can also solve for the inverse function algebraically. If $h(x) = x^2$, we can introduce the variable y to represent the output values, allowing us to write $y = x^2$. To find the inverse we solve for the input variable

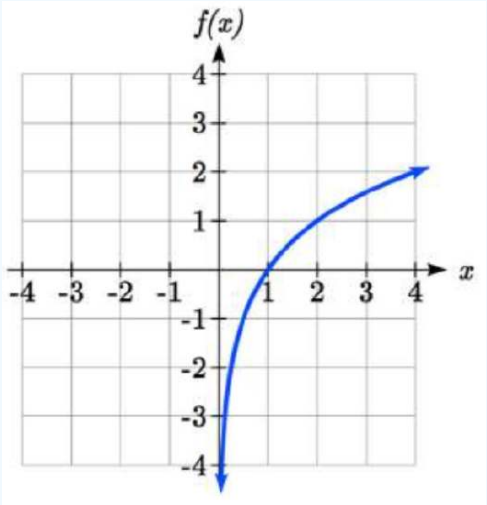
To solve for x we take the square root of each side. $\sqrt{y} = \sqrt{x^2}$ and get $\sqrt{y} = |x|$, so $x = \pm\sqrt{y}$. We have restricted x to being non-negative, so we'll use the positive square root, $x = \sqrt{y}$ or $h^{-1}(y) = \sqrt{y}$. In cases like this where the variables are not descriptive, it is common to see the inverse function rewritten with the variable x: $h^{-1}(x) = \sqrt{x}$. Rewriting the inverse using the variable x is often required for graphing inverse functions using calculators or computers.



Note that the domain and range of the square root function do correspond with the range and domain of the quadratic function on the limited domain. In fact, if we graph $h(x)$ on the restricted domain and $h^{-1}(x)$ on the same axes, we can notice symmetry: the graph of $h^{-1}(x)$ is the graph of h(x) reflected over the line $y = x$.

Example 1.6.7

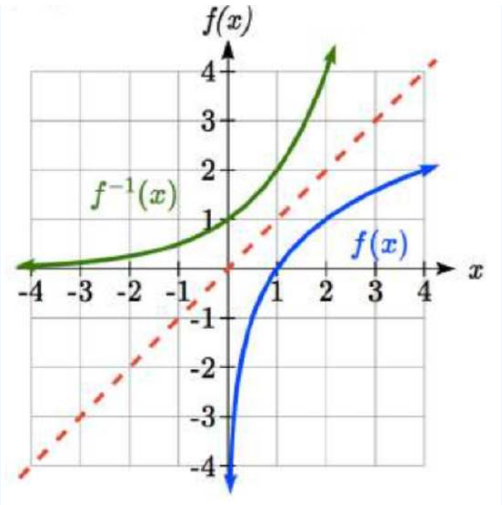
Given the graph of $f(x)$ shown, sketch a graph of $f^{-1}(x)$.



Solution

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

Reflecting this graph of the line $y = x$, the point (1, 0) reflects to (0, 1), and the point (4, 2) reflects to (2, 4). Sketching the inverse on the same axes as the original graph:



Important Topics of this Section

Definition of an inverse function

Composition of inverse functions yield the original input value

Not every function has an inverse function

To have an inverse a function must be one-to-one

Restricting the domain of functions that are not one-to-one.

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1.6E: Inverse Functions

section 1.6 exercises

Assume that the function f is a one-to-one function.

1. If $f(6) = 7$, find $f^{-1}(7)$

2. If $f(3) = 2$, find $f^{-1}(2)$

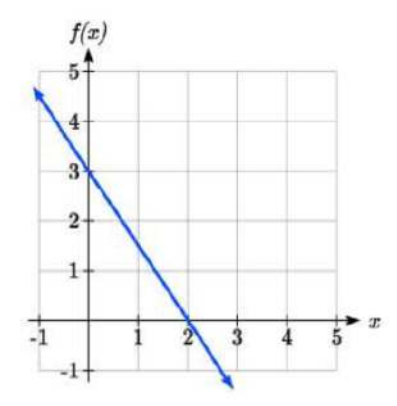
3. If $f^{-1}(-4) = -8$, find $f(-8)$

4. If $f^{-1}(-2) = -1$, find $f(-1)$

5. If $f(5) = 2$, find $(f(5))^{-1}$

6. If $f(1) = 4$, find $(f(1))^{-1}$

7. Using the graph of $f(x)$ shown



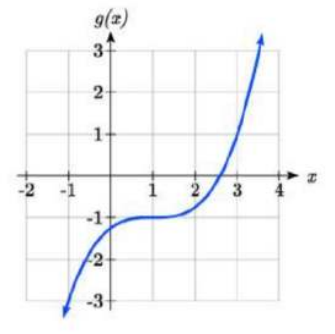
   a. Find $f(0)$

   b. Solve $f(x) = 0$

   c. Find $f^{-1}(0)$

   d. Solve $f^{-1}(x) = 0$

8. Using the graph shown



   a. Find $g(1)$

   b. Solve $g(x) = 1$

   c. Find $g^{-1}(1)$

   d. Solve $g^{-1}(x) = 1$

9. Use the table below to find the indicated quantities.

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	8	0	7	4	2	6	5	3	9	1

a. Find $f(1)$

b. Solve $f(x)=3$

c. Find $f^{-1}(0)$

d. Solve $f^{-1}(x)=7$

10. Use the table below to fill in the missing values.

t	0	1	2	3	4	5	6	7	8
$h(t)$	6	0	1	7	2	3	5	4	9

a. Find $h(6)$

b. Solve $h(t)=0$

c. Find $h^{-1}(5)$

d. Solve $h^{-1}(t)=1$

For each table below, create a table for $f^{-1}(x)$.

11.

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

For each function below, find $f^{-1}(x)$

13. $f(x)=x+3$

14. $f(x)=x+5$

15. $f(x)=2-x$

16. $f(x)=3-x$

17. $f(x)=11x+7$

18. $f(x)=9+10x$

For each function, find a domain on which f is one-to-one and non-decreasing, then find the inverse of f restricted to that domain.

19. $f(x)=(x+7)^2$

20. $f(x)=(x-6)^2$

21. $f(x)=x^2-5$

22. $f(x)=x^2+1$

23. If $f(x)=x^3-5$ and $g(x)=\sqrt[3]{x+5}$, find

a. $f(g(x))$

b. $g(f(x))$

c. What does this tell us about the relationship between $f(x)$ and $g(x)$?

24. If $f(x)=\dfrac{x}{2+x}$ and $g(x)=\dfrac{2x}{1-x}$, find

a. $f(g(x))$

b. $g(f(x))$

c. What does this tell us about the relationship between $f(x)$ and $g(x)$?

Answer

1. 6

3. -4

5. 1/2

7a. 3
b. 2
c. 2
d. 2

11.

x	1	4	7	12	16
$f^{-1}(x)$	3	6	9	13	14

13. $f^{-1}(x) = x - 3$

15. $f^{-1}(x) = -x + 2$

17. $f^{-1}(x) = \dfrac{x-7}{11}$

19. Restricted domain $x \geq -7$, $f^{-1}(x) = \sqrt{x} - 7$

21. Restricted domain $x \geq 0$, $f^{-1}(x) = \sqrt{x+5}$

23a. $f(g(x)) = (\sqrt[3]{x+5})^3 - 5 = x$
b. $g(f(x)) = \sqrt[3]{x^3 - 5 + 5} = x$
c. This means that they are inverse functions (of each other)

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CHAPTER OVERVIEW

2: Linear Functions

Chapter one was a window that gave us a peek into the entire course. Our goal was to understand the basic structure of functions and function notation, the toolkit functions, domain and range, how to recognize and understand composition and transformations of functions and how to understand and utilize inverse functions. With these basic components in hand we will further research the specific details and intricacies of each type of function in our toolkit and use them to model the world around us.

Definition: mathematical modeling

As we approach day to day life we often need to quantify the things around us, giving structure and numeric value to various situations. This ability to add structure enables us to make choices based on patterns we see that are weighted and systematic. With this structure in place we can model and even predict behavior to make decisions. Adding a numerical structure to a real world situation is called **Mathematical Modeling.**

When modeling real world scenarios, there are some common growth patterns that are regularly observed. We will devote this chapter and the rest of the book to the study of the functions used to model these growth patterns.

2.1: Linear Functions

2.1E: Linear Functions (Exercises)

2.2: Graphs of Linear Functions

2.2E: Graphs of Linear Functions (Exercises)

2.3: Modeling with Linear Functions

2.3E: Modeling with Linear Functions (Exercises)

2.4: Fitting Linear Models to Data

2.4E: Fitting Linear Models to Data (Exercises)

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2.1: Linear Functions

As you hop into a taxicab in Las Vegas, the meter will immediately read $3.50; this is the "drop" charge made when the taximeter is activated. After that initial fee, the taximeter will add $2.76 for each mile the taxi drives (Nevada Taxicab Authority, retrieved May 8, 2017. There is also a waiting fee assessed when the taxi is waiting at red lights, but we'll ignore that in this discussion. This chapter is part of *Precalculus: An Investigation of Functions* © Lippman & Rasmussen 2017. This material is licensed under a Creative Commons CC-BY-SA license.). In this scenario, the total taxi fare depends upon the number of miles ridden in the taxi, and we can ask whether it is possible to model this type of scenario with a function. Using descriptive variables, we choose m for miles and C for Cost in dollars as a function of miles: $C(m)$.

We know for certain that $C(0) = 3.50$, since the $3.50 drop charge is assessed regardless of how many miles are driven. Since $2.67 is added for each mile driven, then

$$C(1) = 3.50 + 2.67 = 6.17$$

If we then drove a second mile, another $2.67 would be added to the cost:

$$C(2) = 3.50 + 2.67 + 2.67 = 3.50 + 2.67(2) = 8.84$$

If we drove a third mile, another $2.67 would be added to the cost:

$$C(3) = 3.50 + 2.67 + 2.67 + 2.67 = 3.50 + 2.67(3) = 11.51$$

From this we might observe the pattern, and conclude that if m miles are driven,

$C(m) = 3.50 + 2.67m$ because we start with a $3.50 drop fee and then for each mile increase we add $2.67.

It is good to verify that the units make sense in this equation. The $3.50 drop charge is measured in dollars; the $2.67 charge is measured in dollars per mile.

$$C(m) = 3.50 \text{ dollars} + \left(2.67 \frac{\text{dollars}}{\text{mile}}\right)(m \text{ miles})$$

When dollars per mile are multiplied by a number of miles, the result is a number of dollars, matching the units on the 3.50, and matching the desired units for the *C* function.

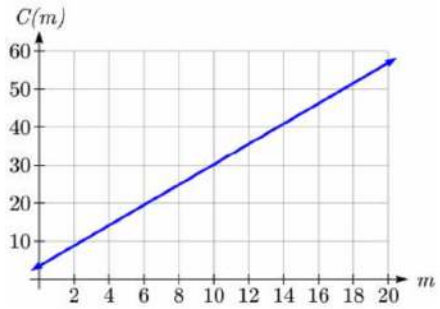
Notice this equation $C(m) = 3.50 + 2.67m$ consisted of two quantities. The first is the fixed $3.50 charge which does not change based on the value of the input. The second is the $2.67 dollars per mile value, which is a **rate of change**. In the equation, this rate of change is multiplied by the input value.

Looking at this same problem in table format we can also see the cost changes by $2.67 for every 1 mile increase.

m	0	1	2	3
$C(m)$	3.50	6.17	8.84	11.51

It is important here to note that in this equation, the **rate of change is constant**; over any interval, the rate of change is the same.

Graphing this equation, $C(m) = 3.50 + 2.67m$ we see the shape is a line, which is how these functions get their name: **linear functions**.



When the number of miles is zero the cost is $3.50, giving the point (0, 3.50) on the graph. This is the vertical or $C(m)$ intercept. The graph is increasing in a straight line from left to right because for each mile the cost goes up by $2.67; this rate remains consistent.

In this example, you have seen the taxicab cost modeled in words, an equation, a table and in graphical form. Whenever possible, ensure that you can link these four representations together to continually build your skills. It is important to note that you will not always be able to find all 4 representations for a problem and so being able to work with all 4 forms is very important.

Definition: Linear Function

A *linear function* is a function whose graph produces a line. Linear functions can always be written in the form

$f(x) = b + mx$ or $f(x) = mx + b$; they're equivalent

where

- b is the initial or starting value of the function (when input, x = 0), and
- m is the constant rate of change of the function

Many people like to write linear functions in the form $f(x) = b + mx$ because it corresponds to the way we tend to speak: "The output starts at b and increases at a rate of m."

For this reason alone we will use the $f(x) = b + mx$ form for many of the examples, but remember they are equivalent and can be written correctly both ways.

Definition: slope and increasing/decreasing

m is the constant rate of change of the function (also called **slope**). Slope The slope determines if the function is an increasing function or a decreasing function.

$f(x) = b + mx$ is an **increasing** function if $m > 0$

$f(x) = b + mx$ is a **decreasing** function if $m < 0$

If $m = 0$, the rate of change zero, and the function $f(x) = b + 0x = b$ is just a horizontal line passing through the point (0, b), neither increasing nor decreasing.

Example 2.1.1

Marcus currently owns 200 songs in his iTunes collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his iTunes collection as a function of the number of months, m. How many songs will he own in a year?

Solution

The initial value for this function is 200, since he currently owns 200 songs, so $N(0) = 200$. The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. With this information, we can write the formula:

$$N(m) = 200 + 15m$$

$N(m)$ is an increasing linear function. With this formula we can predict how many songs he will have in 1 year (12 months):

$$N(12) = 200 + 15(12) = 200 + 180 = 380$$

Marcus will have 380 songs in 12 months.

Exercise 2.1.1

If you earn $30,000 per year and you spend $29,000 per year write an equation for the amount of money you save after *y* years, if you start with nothing.

*"The most important thing, spend less than you earn!(*http://www.thesimpledollar.com/2009/...than-you-earn/*)"*

Answer

$S(y) = 30,000y - 29,000y = 1000y$ $1000 is saved each year.

Definition: Calculating rate of change

Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2, or a set of points, $(x_1,\ y_1)$ and $(x_2,\ y_2)$, if we wish to find a linear function that contains both points we can calculate the rate of change, m:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.1.1}$$

Rate of change of a linear function is also called the **slope** of the line.

Note in function notation, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{2.1.2}$$

Example 2.1.2

The population of a city increased from 23,400 to 27,800 between 2002 and 2006. Find the rate of change of the population during this time span.

Solution

The rate of change will relate the change in population to the change in time. The population increased by $27800 - 23400 = 4400$ people over the 4 year time interval. To find the rate of change, the number of people per year the population changed by:

$$\frac{4400 \text{ people}}{4 \text{ years}} = 1100 \frac{\text{people}}{\text{year}} = 1100 \text{ people per year}$$

Notice that we knew the population was increasing, so we would expect our value for m to be positive. This is a quick way to check to see if your value is reasonable.

Example 2.1.3

The pressure, P, in pounds per square inch (PSI) on a diver depends upon their depth below the water surface, d, in feet, following the equation $P(d) = 14.696 + 0.434d$. Interpret the components of this function.

Solution

The rate of change, or slope, 0.434 would have units $\frac{\text{output}}{\text{input}} = \frac{\text{pressure}}{\text{depth}} = \frac{\text{PSI}}{\text{ft}}$. This tells us the pressure on the diver increases by 0.434 PSI for each foot their depth increases.

The initial value, 14.696, will have the same units as the output, so this tells us that at a depth of 0 feet, the pressure on the diver will be 14.696 PSI.

Example 2.1.4

If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find the rate of change.

Solution

$f(3) = -2$ tells us that the input 3 corresponds with the output -2, and $f(8) = 1$ tells us that the input 8 corresponds with the output 1. To find the rate of change, we divide the change in output by the change in input:

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1 - (-2)}{8 - 3} = \frac{3}{5}$$

If desired we could also write this as $m = 0.6$

Note that it is not important which pair of values comes first in the subtractions so long as the first output value used corresponds with the first input value used.

? Exercise 2.1.2

Given the two points (2, 3) and (0, 4), find the rate of change. Is this function increasing or decreasing?

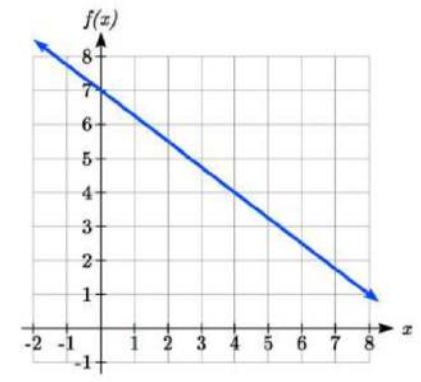
Answer

$m = \frac{4-3}{0-2} = \frac{1}{-2} = -\frac{1}{2}$; Decreasing because $m < 0$

We can now find the rate of change given two input-output pairs, and can write an equation for a linear function once we have the rate of change and initial value. If we have two input-output pairs and they do not include the initial value of the function, then we will have to solve for it.

✓ Example 2.1.5

Write an equation for the linear function graphed to the right.



Solution

Looking at the graph, we might notice that it passes through the points (0, 7) and (4, 4). From the first value, we know the initial value of the function is $b = 7$, so in this case we will only need to calculate the rate of change:

$$m = \frac{4-7}{4-0} = \frac{-3}{4}$$

This allows us to write the equation:

$$f(x) = 7 - \frac{3}{4}x$$

✓ Example 2.1.6

If $f(x)$is a linear function, $f(3) = -2$, and $f(8) = 1$, find an equation for the function.

Solution

In example 3, we computed the rate of change to be $m = \frac{3}{5}$. In this case, we do not know the initial value$f(0)$, so we will have to solve for it. Using the rate of change, we know the equation will have the form $f(x) = b + \frac{3}{5}x$. Since we know the value of the function when $x = 3$, we can evaluate the function at 3.

$$f(3) = b + \frac{3}{5}(3)$$

Since we know that $f(3) = -2$, we can substitute on the left side

$$-2 = b + \frac{3}{5}(3)$$

This leaves us with an equation we can solve for the initial value

$$b = -2 - \frac{9}{5} = \frac{-19}{5}$$

Combining this with the value for the rate of change, we can now write a formula for this function:

$$f(x) = \frac{-19}{5} + \frac{3}{5}x$$

Example 2.1.7

Working as an insurance salesperson, Ilya earns a base salary and a commission on each new policy, so Ilya's weekly income, I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies, and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

Solution

The given information gives us two input-output pairs: (3,760) and (5,920). We start by finding the rate of change.

$$m = \frac{920 - 760}{5 - 3} = \frac{160}{2} = 80$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy; Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value

$$I(n) = b + 80n$$

then when $n = 3$, $I(3) = 760$, giving

$$760 = b + 80(3)$$

this allows us to solve for b

$$b = 760 - 80(3) = 520$$

This value is the starting value for the function. This is Ilya's income when $n = 0$, which means no new policies are sold. We can interpret this as Ilya's base salary for the week, which does not depend upon the number of policies sold.

Writing the final equation:

$$I(n) = 520 + 80n$$

Our final interpretation is: Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold each week.

flashback

Looking at Example 7:

Determine the independent and dependent variables.

What is a reasonable domain and range?

Is this function one-to-one?

Answer

n (number of policies sold) is the independent variable

$I(n)$ (weekly income as a function of policies sold) is the dependent variable.

A reasonable domain is (0, 15)*

A reasonable range is ($540, $1740)*

*answers may vary given reasoning is stated; 15 is an arbitrary upper limit based on selling 3 policies per day in a 5 day work week and $1740 corresponds with the domain.

Yes this function is one-to-one

Exercise 2.1.3

The balance in your college payment account, C, is a function of the number of quarters, q, you attend. Interpret the function $C(a) = 20000 - 4000q$in words. How many quarters of college can you pay for until this account is empty?

Answer

Your College account starts with $20,000 in it and you withdraw $4,000 each quarter (or your account contains $20,000 and decreases by $4000 each quarter.) Solving $C(a) = 0$ gives $a = 5$. You can pay for 5 quarters before the money in this account is gone.

Example 2.1.8

Given the table below write a linear equation that represents the table values

w, number of weeks	0	2	4	6
$P(w)$, number of rats	1000	1080	1160	1240

Solution

We can see from the table that the initial value of rats is 1000 so in the linear format

$$P(w) = b + mw,\ b = 1000$$

Rather than solving for m, we can notice from the table that the population goes up by 80 for every 2 weeks that pass. This rate is consistent from week 0, to week 2, 4, and 6. The rate of change is 80 rats per 2 weeks. This can be simplified to 40 rats per week and we can write

$$P(w) = b + mw \text{ as } P(w) = 1000 + 40w$$

If you didn't notice this from the table you could still solve for the slope using any two points from the table. For example, using (2, 1080) and (6, 1240),

$$m = \frac{1240 - 1080}{6 - 2} = \frac{160}{4} = 40 \text{ rats per week}$$

Important Topics of this Section

- Definition of Modeling
- Definition of a linear function
- Structure of a linear function
- Increasing & Decreasing functions
- Finding the vertical intercept (0, *b*)
- Finding the slope/rate of change, *m*
- Interpreting linear functions

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2.1E: Linear Functions (Exercises)

Section 2.1 exercise

1. A town's population has been growing linearly. In 2003, the population was 45,000, and the population has been growing by 1700 people each year. Write an equation,

$P(t)$, for the population t years after 2003.

2. A town's population has been growing linearly. In 2005, the population was 69,000, and the population has been growing by 2500 people each year. Write an equation,

$P(t)$, for the population t years after 2005.

3. Sonya is currently 10 miles from home, and is walking further away at 2 miles per hour. Write an equation for her distance from home t hours from now.

4. A boat is 100 miles away from the marina, sailing directly towards it at 10 miles per hour. Write an equation for the distance of the boat from the marina after t hours.

5. Timmy goes to the fair with $40. Each ride costs $2. How much money will he have left after riding n rides?

6. At noon, a barista notices she has $20 in her tip jar. If she makes an average of $0.50 from each customer, how much will she have in her tip jar if she serves n more customers during her shift?

Determine if each function is increasing or decreasing

7. $f(x) = 4x + 3$

8. $g(x) = 5x + 6$

9. $a(x) = -2x + 4$

10. $b(x) = 8 - 3x$

11. $h(x) = -2x + 4$

12. $h(x) = -4x + 1$

13. $j(x) = \frac{1}{2}x - 3$

14. $p(x) = \frac{1}{4}x - 5$

15. $n(x) = -\frac{1}{3}x - 2$

16. $m(x) = -\frac{3}{8}x + 3$

Find the slope of the line that passes through the two given points

17. (2, 4) and (4, 10)

18. (1, 5) and (4, 11)

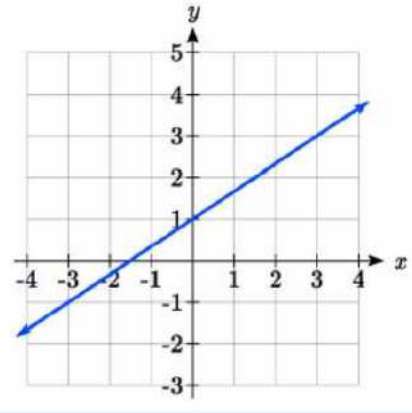
19. (-1, 4) and (5, 2)

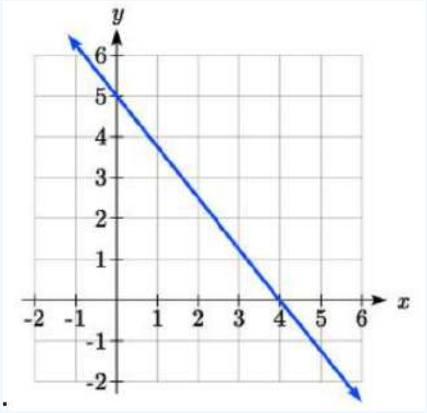
20. (-2, 8) and (4, 6)

21. (6, 11) and (-4, 3)

22. (9, 10) and (-6, -12)

Find the slope of the lines graphed

23. 

24. 

25. Sonya is walking home from a friend's house. After 2 minutes she is 1.4 miles from home. Twelve minutes after leaving, she is 0.9 miles from home. What is her rate?

26. A gym membership with two personal training sessions costs $125, while gym membership with 5 personal training sessions costs $260. What is the rate for personal training sessions?

27. A city's population in the year 1960 was 287,500. In 1989 the population was 275,900. Compute the slope of the population growth (or decline) and make a statement about the population rate of change in people per year.

28. A city's population in the year 1958 was 2,113,000. In 1991 the population was 2,099,800. Compute the slope of the population growth (or decline) and make a statement about the population rate of change in people per year.

29. A phone company charges for service according to the formula: $C(n) = 240. + 1n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

30. A phone company charges for service according to the formula: $C(n) = 260. + 04n$, where n is the number of minutes talked, and $C(n)$ is the monthly charge, in dollars. Find and interpret the rate of change and initial value.

31. Terry is skiing down a steep hill. Terry's elevation, $E(t)$, in feet after t seconds is given by $E(t) = 3000 - 70t$ .Write a complete sentence describing Terry's starting elevation and how it is changing over time.

32. Maria is climbing a mountain. Maria's elevation, $E(t)$, in feet after t minutes is given by $E(t) = 1200 + 40t$. Write a complete sentence describing Maria's starting elevation and how it is changing over time.

Given each set of information, find a linear equation satisfying the conditions, if possible

33. $f(-5) = -4$, and $f(5) = 2$

34. $f(-1) = 4$, and $f(5) = 1$

35. Passes through (2, 4) and (4, 10)

36. Passes through (1, 5) and (4, 11)

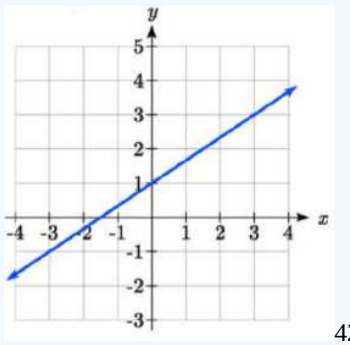
37. Passes through (-1, 4) and (5, 2)

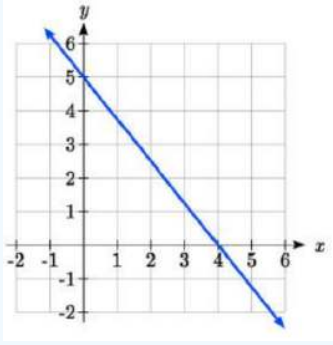
38. Passes through (-2, 8) and (4, 6)

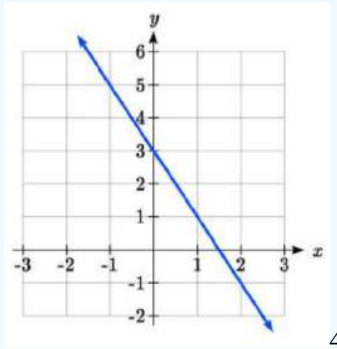
39. x intercept at (-2, 0) and y intercept at (0, -3)

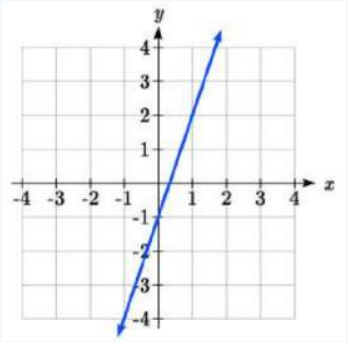
40. x intercept at (-5, 00 and y intercept at (0, 4)

Find an equation for the function graphed

41. 

42. 

43. 

44. 

45. A clothing business finds there is a linear relationship between the number of shirts, n, it can sell and the price, p, it can charge per shirt. In particular, historical data shows that 1000 shirts can be sold at a price of \$30, while 3000 shirts can be sold at a price of \$22 . Find a linear equation in the form $p = mn + b$ that gives the price p they can charge for n shirts.

46. A farmer finds there is a linear relationship between the number of bean stalks, n, she plants and the yield, y, each plant produces. When she plants 30 stalks, each plant yields 30 oz of beans. When she plants 34 stalks, each plant produces 28 oz of beans. Find a linear relationships in the form $y = mn + b$ that gives the yield when n stalks are planted.

47. Which of the following tables could represent a linear function? For each that could be linear, find a linear equation models the data.

x	*g(x)*
0	5
5	-10
10	-25
15	-40

x	*h(x)*
0	5
5	30
10	105
15	230

x	*f(x)*
0	-5
5	20
10	45
15	70

x	*k(x)*
5	13
10	28
20	58
25	73

48. Which of the following tables could represent a linear function? For each that could be linear, find a linear equation models the data.

x	*g(x)*
0	6
2	-19
4	-44
6	-69

x	*h(x)*
2	13
4	23
8	43
10	53

x	*f(x)*
2	-4
4	16
6	36
8	56

x	*k(x)*
0	6
2	31
6	106
8	231

49. While speaking on the phone to a friend in Oslo, Norway, you learned that the current temperature there was -23 Celsius (-23°C). After the phone conversation, you wanted to convert this temperature to Fahrenheit degrees, oF, but you could not find a reference with the correct formulas. You then remembered that the relationship between ^{o}F and ^{o}C is linear. [UW]

Using this and the knowledge that 32°F = 0 ^{o}C and 212 ^{o}F = 100 ^{o}C, find an equation that computes Celsius temperature in terms of Fahrenheit; i.e. an equation of the form C = “an expression involving only the variable F.”

Likewise, find an equation that computes Fahrenheit temperature in terms of Celsius temperature; i.e. an equation of the form F = “an expression involving only the variable C.”

How cold was it in Oslo in ^{o}F?

Answer

1. $P(t) = 1700t + 45000$

3. $D(t) = 10 + 2t$

5. $M(n) = 40 - 2n$

7. Increasing

9. Decreasing

11. Decreasing

13. Increasing

15. Decreasing

17. 3

19. $-\frac{1}{3}$

21. $\frac{4}{3}$

23. $\frac{2}{3}$

25. -0.05 mph (or 0.05 miles per hour toward her home)

27. Population is decreasing by 400 people per year

29. Monthly charge in dollars has an initial base charge of $24, and increases by $0.10 for each minute talked

31. Terry started at an elevation of 3,000 ft and is descending by 70ft per second.

33. $y = \frac{3}{5}x - 1$

35. $y = 3x - 2$

37. $y = -\frac{1}{3}x + \frac{11}{3}$

39. $y = -1.5x - 3$

41. $y = \frac{2}{3}x + 1$

43. $y = -2x + 3$

45. $P(n) = -0.004n + 34$

47. The 1st, 3rd & 4th tables are linear: respectively
1. $g(x) = -3x + 5$
3. $f(x) = 5x - 5$
4. $k(x) = 3x - 2$

49a. $C = \frac{5}{9}F - \frac{160}{9}$

b. $F = \frac{9}{5}C + 32$

c. $-9.4^{\circ}F$

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2.2: Graphs of Linear Functions

When we are working with a new function, it is useful to know as much as we can about the function: its graph, where the function is zero, and any other special behaviors of the function. We will begin this exploration of linear functions with a look at graphs.

When graphing a linear function, there are three basic ways to graph it:

1. By plotting points (at least 2) and drawing a line through the points
2. Using the initial value (output when $x = 0$) and rate of change (slope)
3. Using transformations of the identity function $f(x) = x$

Example 2.2.1

Graph $f(x) = 5 - \frac{2}{3}x$ by plotting points.

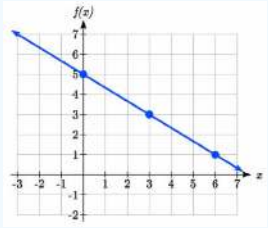
Solution

In general, we evaluate the function at two or more inputs to find at least two points on the graph. Usually it is best to pick input values that will "work nicely" in the equation. In this equation, multiples of 3 will work nicely due to the $\frac{2}{3}$ in the equation, and of course using $x = 0$ to get the vertical intercept. Evaluating $f(x)$ at $x = 0$, 3 and 6:

$$f(0) = 5 - \frac{2}{3}(0) = 5$$

$$f(3) = 5 - \frac{2}{3}(3) = 3$$

$$f(6) = 5 - \frac{2}{3}(6) = 1$$



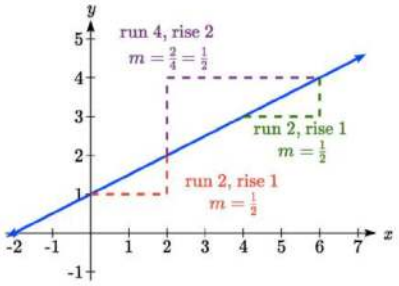
These evaluations tell us that the points (0,5), (3,3), and (6,1) lie on the graph of the line. Plotting these points and drawing a line through them gives us the graph.

When using the initial value and rate of change to graph, we need to consider the graphical interpretation of these values. Remember the initial value of the function is the output when the input is zero, so in the equation $f(x) = b + mx$, the graph includes the point $(0, b)$. On the graph, this is the vertical intercept – the point where the graph crosses the vertical axis.

For the rate of change, it is helpful to recall that we calculated this value as

$$m = \frac{change\ of\ output}{change\ of\ input}$$

From a graph of a line, this tells us that if we divide the vertical difference, or rise, of the function outputs by the horizontal difference, or run, of the inputs, we will obtain the rate of change, also called slope of the line.



$$m = \frac{change\ of\ output}{change\ of\ input} = \frac{rise}{run}$$

Notice that this ratio is the same regardless of which two points we use.

Definition: Graphical interpretation of a linear equation

Graphically, in the equation $f(x) = b + mx$,

b is the **vertical intercept** of the graph and tells us we can start our graph at (0, b)

m is the **slope of the line** and tells us how far to rise & run to get to the next point

Once we have at least 2 points, we can extend the graph of the line to the left and right.

Example 2.2.2

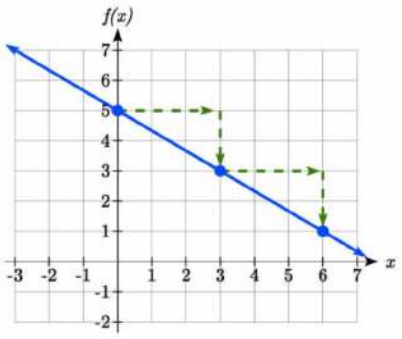
Graph $f(x) = 5 - \frac{2}{3}x$ using the vertical intercept and slope.

Solution

The vertical intercept of the function is (0, 5), giving us a point on the graph of the line.

The slope is $-\frac{2}{3}$. This tells us that for every 3 units the graph "runs" in the horizontal, the vertical "rise" decreases by 2 units.

In graphing, we can use this by first plotting our vertical intercept on the graph, then using the slope to find a second point. From the initial value (0, 5) the slope tells us that if we move to the right 3, we will move down 2, moving us to the point (3, 3). We can continue this again to find a third point at (6, 1). Finally, extend the line to the left and right, containing these points.



Exercise 2.2.1

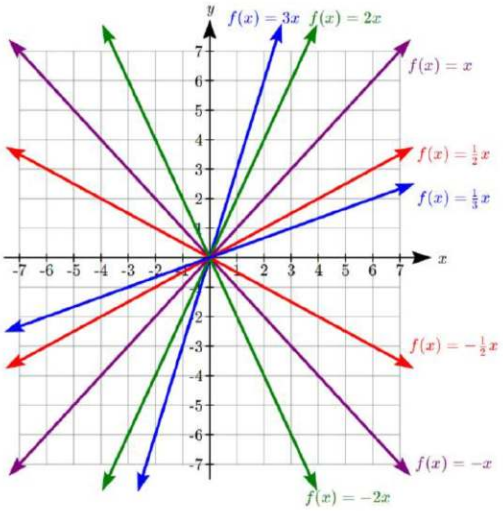
Consider that the slope $-\frac{2}{3}$ could also be written as $\frac{2}{-3}$. Using $\frac{2}{-3}$, find another point on the graph that has a negative x value.

Answer

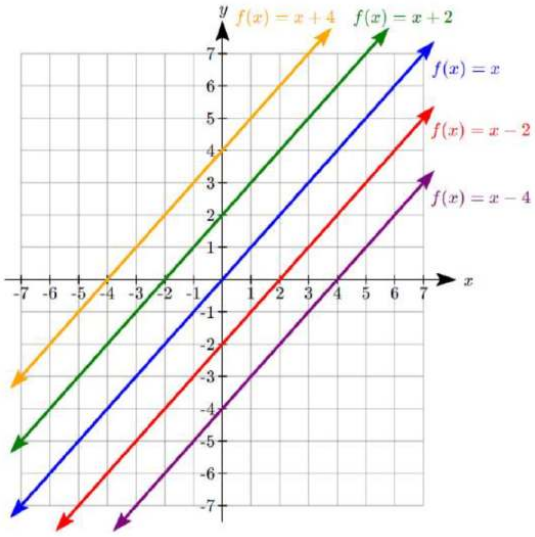
$(-3, 7)$ found by starting at the vertical intercept, going up 2 units and 3 in the negative horizontal direction. You could have also answered, $(-6, 9)$ or $(-9, 11)$ etc…

Another option for graphing is to use transformations of the identity function $f(x) = x$.

In the equation $f(x) = mx$, the m is acting as the vertical stretch of the identity function. When m is negative, there is also a vertical reflection of the graph. Looking at some examples:



In $f(x) = mx + b$, the b acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Some examples:



Using Vertical Stretches or Compressions along with Vertical Shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way for you to graph this type of function, make sure you practice each method.

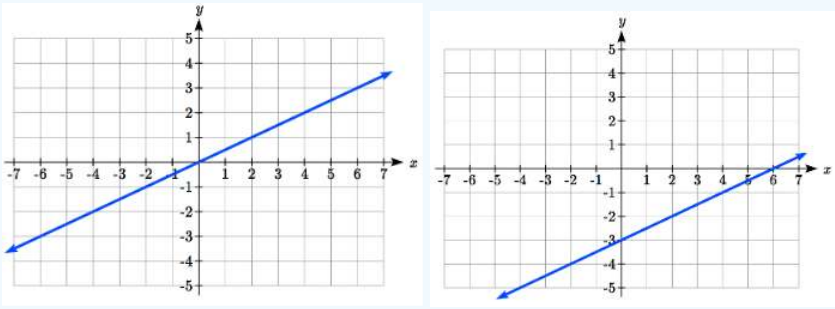
Example 2.2.3

Graph $f(x) = -3 + \frac{1}{2}x$ using transformations.

Solution

The equation is the graph of the identity function vertically compressed by ½ and vertically shifted down 3.

Vertical Compression combined with Vertical shift



Notice how this nicely compares to the other method where the vertical intercept is found at (0, -3) and to get to another point we rise (go up vertically) by 1 unit and run (go horizontally) by 2 units to get to the next point (2, -2), and the next one (4, -1). In these three points (0, -3), (2, -2), and (4, -1), the output values change by +1, and the x values change by +2, corresponding with the slope $m = 1/2$.

Example 2.2.4

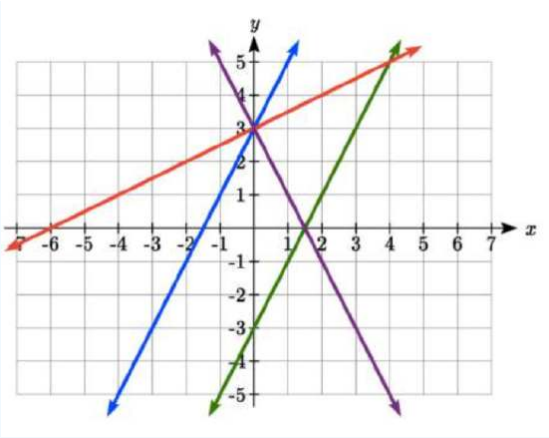
Match each equation with one of the lines in the graph below

$$f(x) = 2x + 3$$

$$g(x) = 2x - 3$$

$$(h(x) = -2x = 3$$

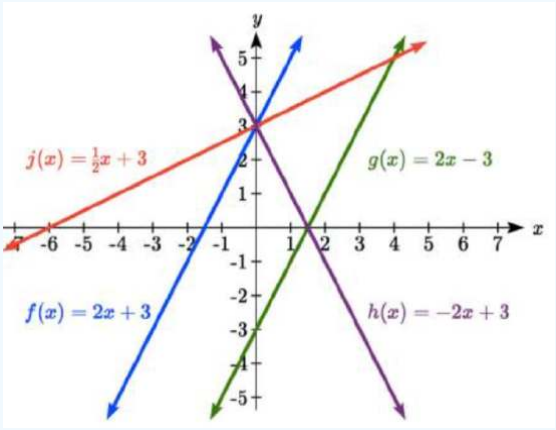
$$j(x) = \frac{1}{2}x + 3$$



Solution

Only one graph has a vertical intercept of - 3, so we can immediately match that graph with $g(x)$.

For the three graphs with a vertical intercept at 3, only one has a negative slope, so we can match that line with $h(x)$. Of the other two, the steeper line would have a larger slope, so we can match that graph with equation $f(x)$, and the flatter line with the equation $j(x)$.



In addition to understanding the basic behavior of a linear function (increasing or decreasing, recognizing the slope and vertical intercept), it is often helpful to know the horizontal intercept of the function – where it crosses the horizontal axis.

Definition: Finding Horizontal intercepts

The **horizontal intercept** of the function is where the graph crosses the horizontal axis. If a function has a horizontal intercept, you can always find it by solving $f(x) = 0$.

Example 2.2.5

Find the horizontal intercept of $f(x) = -3 + \frac{1}{2}x$

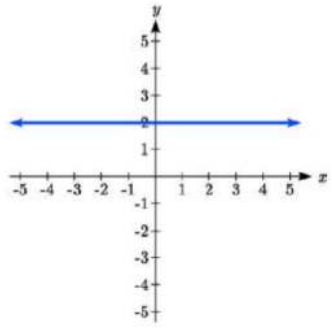
Solution

Setting the function equal to zero to find what input will put us on the horizontal axis,

$$\begin{aligned} 0 &= -3 + \frac{1}{2}x \\ 3 &= \frac{1}{2}x \\ x &= 6 \end{aligned}$$

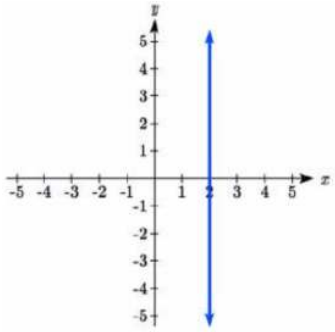
The graph crosses the horizontal axis at (6,0)

There are two special cases of lines: a horizontal line and a vertical line. In a horizontal line like the one graphed to the right, notice that between any two points, the change in the outputs is 0. In the slope equation, the numerator will be 0, resulting in a slope of 0. Using a slope of 0 in the $f(x) = b + mx$, the equation simplifies to $f(x) = b$.



Notice a horizontal line has a vertical intercept, but no horizontal intercept (unless it's the line $f(x) = 0$).

In the case of a vertical line, notice that between any two points, the change in the inputs is zero. In the slope equation, the denominator will be zero, and you may recall that we cannot divide by the zero; the slope of a vertical line is undefined. You might also notice that a vertical line is not a function. To write the equation of vertical line, we simply write input=value, like $x = b$.



Notice a vertical line has a horizontal intercept, but no vertical intercept (unless it's the line $x = 0$).

Definition: horizontal and vertical lines

Horizontal lines have equations of the form $f(x) = b$

Vertical lines have equations of the form $x = a$

Example 2.2.6

Write an equation for the horizontal line graphed above.

Solution

This line would have equation $f(x) = 2$

Example 2.2.7

Write an equation for the vertical line graphed above.

Solution

This line would have equation $x = 2$

Exercise 2.2.2

Describe the function $f(x) = 6 - 3x$ in terms of transformations of the identity function and find its horizontal intercept.

Answer

Vertically stretched by a factor of 3, Vertically flipped (flipped over the x axis),

Vertically shifted up by 6 units.

Horizontal intercept: $6 - 3x = 0$ when $x = 2$.

Parallel and Perpendicular Lines

When two lines are graphed together, the lines will be **parallel** if they are increasing at the same rate – if the rates of change are the same. In this case, the graphs will never cross (unless they're the same line).

Definition: Parallel lines

Two lines are **parallel** if the slopes are equal (or, if both lines are vertical).

In other words, given two linear equations $f(x) = b + m_1 x$ and $g(x) = b + m_2 x$, the lines will be parallel if $m_1 = m_2$.

Example 2.2.8

Find a line parallel to $f(x) = 6 + 3x$ that passes through the point (3, 0).

Solution

We know the line we're looking for will have the same slope as the given line, $m = 3$. Using this and the given point, we can solve for the new line's vertical intercept:

$$g(x) = b + 3x$$

then at (3, 0),

$$0 = b + 3(3)$$

$$b = -9$$

The line we're looking for is $g(x) = -9 + 3x$.

If two lines are not parallel, one other interesting possibility is that the lines are perpendicular, which means the lines form a right angle (90 degree angle – a square corner) where they meet. In this case, the slopes when multiplied together will equal -1. Solving for one slope leads us to the definition:

Definition: Perpendicular lines

Given two linear equations $f(x) = b + m_1 x$ and $g(x) = b + m_2 x$

The lines will be **perpendicular** if $m_1 m_2 = -1$, and so $m_2 = \dfrac{-1}{m_1}$

We often say the slope of a perpendicular line is the "negative reciprocal" of the other line's slope.

Example 2.2.9

Find the slope of a line perpendicular to a line with:

a) a slope of 2. b) a slope of -4. c) a slope of $\dfrac{2}{3}$.

Solution

If the original line had slope 2, the perpendicular line's slope would be $m_2 = \frac{-1}{2}$

If the original line had slope -4, the perpendicular line's slope would be $m_2 = \frac{-1}{-4} = \frac{1}{4}$

If the original line had slope $\frac{2}{3}$, the perpendicular line's slope would be $m_2 = \frac{-1}{2/3} = \frac{-3}{2}$

Example 2.2.10

Find the equation of a line perpendicular to $f(x) = 6 + 3x$ and passing through the point (3, 0).

Solution

The original line has slope m = 3. The perpendicular line will have slope $m = \frac{-1}{3}$. Using this and the given point, we can find the equation for the line.

$$g(x) = b - \frac{1}{3}x$$

then at (3, 0),

$$0 = b - \frac{1}{3}(3)$$

$$b = 1$$

The line we're looking for is $g(x) = 1 - \frac{1}{3}x$.

Exercise 2.2.3

Given the line $h(t) = -4 + 2t$, find an equation for the line passing through (0, 0) that is:

a. parallel to *h(t)*.
b. perpendicular to *h(t)*.

Answer

Parallel: $f(t) = 2t$

Perpendicular: $g(t) = -\frac{1}{2}t$

Example 2.2.12

A line passes through the points (-2, 6) and (4, 5). Find the equation of a perpendicular line that passes through the point (4, 5).

Solution

From the two given points on the reference line, we can calculate the slope of that line:

$$m_1 = \frac{5-6}{4-(-2)} = \frac{-1}{6}$$

The perpendicular line will have slope

$$m_2 = \frac{-1}{-1/6} = 6$$

We can then solve for the vertical intercept that makes the line pass through the desired point:

$$g(x) = b + 6x$$

then at (4, 5)

$$5 = b + 6(4)$$

$$b = -19$$

Giving the line $g(x) = -19 + 6x$

Intersections of Lines

The graphs of two lines will intersect if they are not parallel. They will intersect at the point that satisfies both equations. To find this point when the equations are given as functions, we can solve for an input value so that $f(x) = g(x)$. In other words, we can set the formulas for the lines equal, and solve for the input that satisfies the equation.

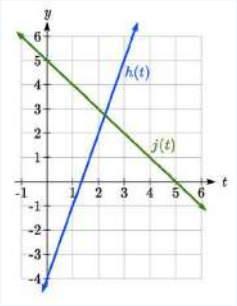
Example 2.2.13

Find the intersection of the lines $h(t) = 3t - 4$ and $j(t) = 5 - t$.

Solution

Setting $h(t) = j(t)$,

$$\begin{aligned} 3t - 4 &= 5 - t \\ 4t &= 9 \\ t &= \frac{9}{4} \end{aligned}$$



This tells us the lines intersect when the input is $\frac{9}{4}$.

We can then find the output value of the intersection point by evaluating either function at this input

$$j(\frac{9}{4}) = 5 - \frac{9}{4} = \frac{11}{4}$$

These lines intersect at the point $(\frac{9}{4}, \frac{11}{4})$. Looking at the graph, this result seems reasonable.

Two parallel lines can also intersect if they happen to be the same line. In that case, they intersect at every point on the lines.

Exercise 2.2.4

Look at the graph in example 13 above and answer the following for the function $h(t)$:

a. Vertical intercept coordinates

b. Horizontal intercepts coordinates

c. Slope

d. Is $j(t)$ parallel or perpendicular to $h(t)$ (or neither)

e. Is h(t) an Increasing or Decreasing function (or neither)

f. Write a transformation description from the identity toolkit function $f(x) = x$

Answer

4. Given $h(t) = 3t - 4$

a. (0,-4)

b. $(\frac{4}{3}, 0)$

c. Slope 3

d. Neither parallel nor perpendicular

e. Increasing function

f. Given the identity function, vertically stretch by 3 and shift down 4 units.

Finding the intersection allows us to answer other questions as well, such as discovering when one function is larger than another.

Example 2.2.14

Using the functions from the previous example, for what values of t is $h(t) > j(t)$.

Solution

To answer this question, it is helpful first to know where the functions are equal, since that is the point where $h(t)$ could switch from being greater to smaller than $j(t)$ or vice-versa. From the previous example, we know the functions are equal at $t = \frac{9}{4}$.

By examining the graph, we can see that $h(t)$, the function with positive slope, is going to be larger than the other function to the right of the intersection. So $h(t) > j(t)$ when $t > \frac{9}{4}$

Important Topics of this Section

- Methods for graphing linear functions
- Another name for slope = rise/run
- Horizontal intercepts (a,0)
- Horizontal lines
- Vertical lines
- Parallel lines
- Perpendicular lines
- Intersecting lines

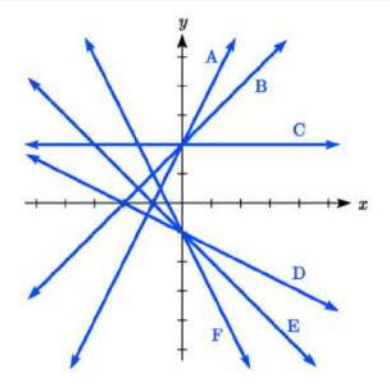
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2.2E: Graphs of Linear Functions (Exercises)

Section 2.2 exercise

Match each linear equation with its graph



1. $f(x) = -x - 1$
2. $f(x) = -2x - 1$
3. $f(x) = -\frac{1}{2}x - 1$
4. $f(x) = 2$
5. $f(x) = 2 + x$
6. $f(x) = 3x + 2$

Sketch a line with the given features

7. An *x*-intercept of (-4, 0) and *y*-intercept of (0, -2)
8. An *x*-intercept of (-2, 0) and *y*-intercept of (0, 4)
9. A vertical intercept of (0, 7) and slope $-\frac{3}{2}$
10. A vertical intercept of (0, 3) and slope $\frac{2}{5}$
11. Passing through the points (-6,-2) and (6,-6)
12. Passing through the points (-3,-4) and (3,0)

Sketch the graph of each equation

13. $f(x) = -2x - 1$
14. $g(x) = -3x + 2$
15. $h(x) = \frac{1}{3}x + 2$
16. $k(x) = \frac{2}{3}x - 3$
17. $k(t) = 3 + 2t$
18. $p(t) = -2 + 3t$
19. $x = 3$
20. $x = -2$
21. $r(x) = 4$

22. $q(x) = 3$

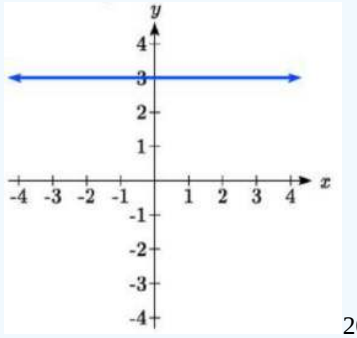
23. If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $3/4$, a shift left by 2, and a shift down by 4

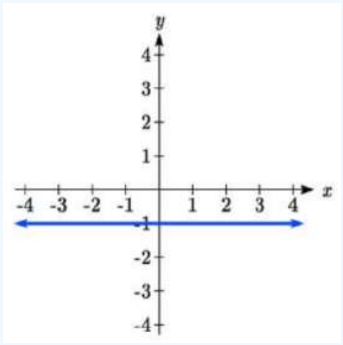
    a. Write an equation for $g(x)$
    b. What is the slope of this line?
    c. Find the vertical intercept of this line.

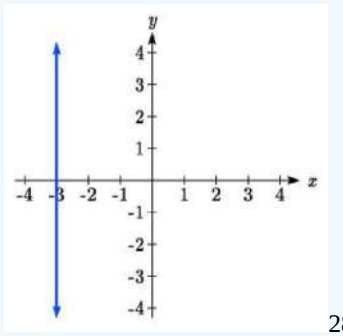
24. If $g(x)$ is the transformation of $f(x) = x$ after a vertical compression by $1/3$, a shift right by 1, and a shift up by 3

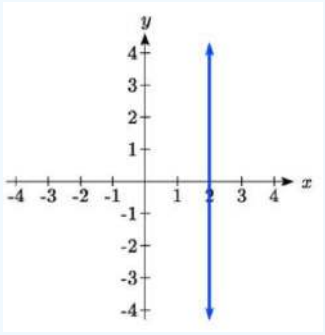
    a. Write an equation for $g(x)$
    b. What is the slope of this line?
    c. Find the vertical intercept of this line.

Write the equation of the line shown

25. 

26. 

27. 

28. 

Find the horizontal and vertical intercepts of each equation

29. $f(x) = -x + 2$

30. $g(x) = 2x + 4$

31. $h(x) = 3x - 5$

32. $k(x) = -5x + 1$

33. $-2x + 5y = 20$

34. $7x + 2y = 56$

Given below are descriptions of two lines. Find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular or neither?

35. Line 1: Passes through $(0, 6)$ and $(3, -24)$
Line 2: Passes through $(-1, 19)$ and $(8, -71)$

36. Line 1: Passes through $(-8, -55)$ and $(10, 89)$
Line 2: Passes through $(9, -44)$ and $(4, -14)$

37. Line 1: Passes through $(2, 3)$ and $(4, -1)$
Line 2: Passes through $(6, 3)$ and $(8, 5)$

38. Line 1: Passes through $(1, 7)$ and $(5, 5)$
Line 2: Passes through $(-1, -3)$ and $(1, 1)$

39. Line 1: Passes through $(0, 5)$ and $(3, 3)$
Line 2: Passes through $(1, -5)$ and $(3, -2)$

40. Line 1: Passes through $(2, 5)$ and $(5, -1)$
Line 2: Passes through $(-3, 7)$ and $(3, -5)$

41. Write an equation for a line parallel to $f(x) = -5x - 3$ and passing through the point (2,-12)

42. Write an equation for a line parallel to $g(x) = 3x - 1$ and passing through the point (4,9)

43. Write an equation for a line perpendicular to $h(t) = -2t + 4$ and passing through the point (-4,-1)

44. Write an equation for a line perpendicular to $p(t) = 3t + 4$ and passing through the point (3,1)

45. Find the point at which the line $f(x) = -2x - 1$ intersects the line $g(x) = -x$

46. Find the point at which the line $f(x) = 2x + 5$ intersects the line $g(x) = -3x - 5$

47. Use algebra to find the point at which the line $f(x) = -\frac{4}{5}x + \frac{274}{25}$ intersects the line $h(x) = \frac{9}{4}x + \frac{73}{10}$

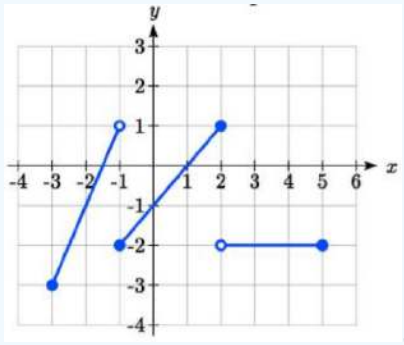
48. Use algebra to find the point at which the line $f(x) = \frac{7}{4}x + \frac{457}{60}$ intersects the line $g(x) = \frac{4}{3}x + \frac{31}{5}$

49. A car rental company offers two plans for renting a car.
Plan A: 30 dollars per day and 18 cents per mile
Plan B: 50 dollars per day with free unlimited mileage
How many miles would you need to drive for plan B to save you money?

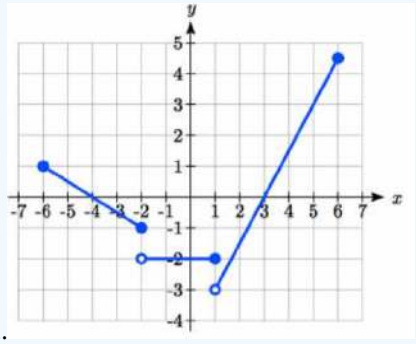
50. You're comparing two cell phone companies.
Company A: $20/month for unlimited talk and text, and $10/GB for data.
Company B: $65/month for unlimited talk, text, and data.
Under what circumstances will company A save you money?

Find a formula for each piecewise defined function.

51.



52.



53. Sketch an accurate picture of the line having equation $f(x) = 2 - \frac{1}{2}x$. Let c be an unknown constant. [UW]

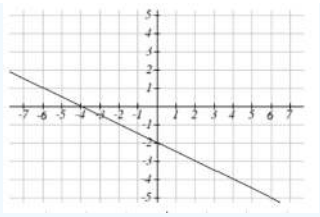
a. Find the point of intersection between the line you have graphed and the line $g(x) = 1 + cx$; your answer will be a point in the xy plane whose coordinates involve the unknown c.
b. Find c so that the intersection point in (a) has x-coordinate 10.
c. Find c so that the intersection point in (a) lies on the x-axis.

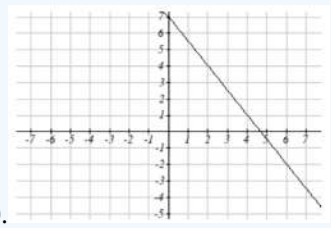
Answer

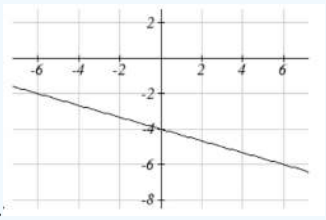
1. E

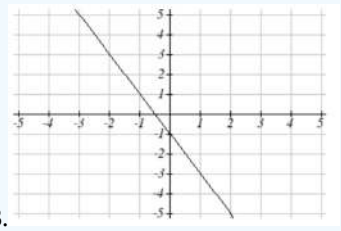
3. D

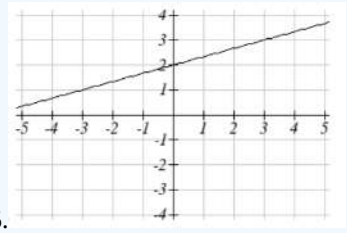
5. B

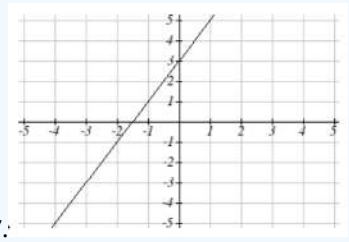
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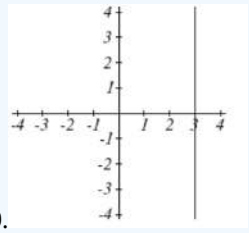
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11. 

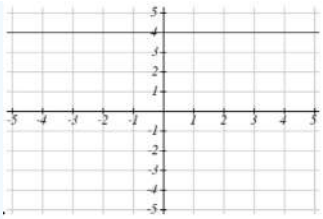
13. 

15. 

17. 

19. 

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21. 

23. a. $g(x) = \frac{3}{4}(x+2) - 4$

b. 3/4

c. -5/2

25. $y = 3$

27. $x = -3$

	Vertical Intercept	Horizontal Intercept
29.	(0, 2)	(2, 0)
31.	(0, -5)	(5/3, 0)
33.	(0, 4)	(-10, 0)

35. Line 1: $m = -10$ Line 2: $m = -10$ Parallel

37. Line 1: $m = -2$ Line 2: $m = 1$ Neither

39. Line 1: $m = -\frac{2}{3}$ Line 2: $m = \frac{3}{2}$ Perpendicular

41. $y = -5x - 2$

43. $y = \frac{1}{2}t + 1$

45. (-1, 1)

47. (1.2, 10)

49. Plan B saves money if the miles are > $111\frac{1}{9}$

51. $f(x) = \begin{cases} 2x+3 & if \quad -3 \leq x < -1 \\ x-1 & if \quad -1 \leq x \leq 2 \\ -2 & if \quad 2 < x \leq 5 \end{cases}$

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2.3: Modeling with Linear Functions

When modeling scenarios with a linear function and solving problems involving quantities changing linearly, we typically follow the same problem solving strategies that we would use for any type of function:

Problem Solving Strategy

1. Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2. Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
3. Carefully read the problem to identify what we are trying to find, identify, solve, or interpret.
4. Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the function being used to model the problem.
5. When needed, find a formula for the function.
6. Solve or evaluate using the formula you found for the desired quantities.
7. Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8. Clearly convey your result using appropriate units, and answer in full sentences when appropriate.

Example 2.3.1

Emily saved up $3500 for her summer visit to Seattle. She anticipates spending $400 each week on rent, food, and fun. Find and interpret the horizontal intercept and determine a reasonable domain and range for this function.

Solution

In the problem, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can define our variables, including units.

Output: M, money remaining, in dollars

Input: t, time, in weeks

Reading the problem, we identify two important values. The first, $3500, is the initial value for M. The other value appears to be a rate of change – the units of dollars per week match the units of our output variable divided by our input variable. She is spending money each week, so you should recognize that the amount of money remaining is decreasing each week and the slope is negative.

To answer the first question, looking for the horizontal intercept, it would be helpful to have an equation modeling this scenario. Using the intercept and slope provided in the problem, we can write the equation:

$$M(t) = 3500 - 400t$$

To find the horizontal intercept, we set the output to zero, and solve for the input:

$$\begin{aligned} 0 &= 3500 - 400t \\ t &= \frac{3500}{400} = 8.75 \end{aligned}$$

The horizontal intercept is 8.75 weeks. Since this represents the input value where the output will be zero, interpreting this, we could say: Emily will have no money left after 8.75 weeks.

When modeling any real life scenario with functions, there is typically a limited domain over which that model will be valid – almost no trend continues indefinitely. In this case, it certainly doesn't make sense to talk about input values less than zero. It is also likely that this model is not valid after the horizontal intercept (unless Emily's going to start using a credit card and go into debt).

The domain represents the set of input values and so the reasonable domain for this function is $0 \le t \le 8.75$.

However, in a real world scenario, the rental might be weekly or nightly. She may not be able to stay a partial week and so all options should be considered. Emily could stay in Seattle for 0 to 8 full weeks (and a couple of days), but would have to go

into debt to stay 9 full weeks, so restricted to whole weeks, a reasonable domain without going in to debt would be $0 \le t \le 8$, or $0 \le t \le 9$ if she went into debt to finish out the last week.

The range represents the set of output values and she starts with $3500 and ends with $0 after 8.75 weeks so the corresponding range is $0 \le M(t) \le 3500$. If we limit the rental to whole weeks, however, the range would change. If she left after 8 weeks because she didn't have enough to stay for a full 9 weeks, she would have $M(8) = 3500 - 400(8) = \$300$ dollars left after 8 weeks, giving a range of $300 \le M(t) \le 3500$. If she wanted to stay the full 9 weeks she would be $100 in debt giving a range of $-100 \le M(t) \le 3500$.

Most importantly remember that domain and range are tied together, and what ever you decide is most appropriate for the domain (the independent variable) will dictate the requirements for the range (the dependent variable).

Exercise 2.3.1

A database manager is loading a large table from backups. Getting impatient, she notices 1.2 million rows had been loaded. Ten minutes later, 2.5 million rows had been loaded. How much longer will she have to wait for all 80 million rows to load?

Answer

Letting t be the number of minutes since she got impatient, and N be the number rows loaded, in millions, we have two points: (0, 1.2) and (10, 2.5).

The slope is $m = \dfrac{2.5 - 1.2}{10 - 0} = \dfrac{1.3}{10} = 0.13$ million rows per minute.

We know the N intercept, so we can write the equation:

$$N = 0.13t + 1.2$$

To determine how long she will have to wait, we need to solve for when $N = 80$.

$$N = 0.13t + 1.2 = 80$$

$$0.13t = 78.8$$

$$t = \frac{78.8}{0.13} = 606$$

. She'll have to wait another 606 minutes, about 10 hours.

Example 2.3.2

Jamal is choosing between two moving companies. The first, U-Haul, charges an up-front fee of $20, then 59 cents a mile. The second, Budget, charges an up-front fee of $16, then 63 cents a mile(Rates retrieved Aug 2, 2010 from www.budgettruck.com and http://www.uhaul.com/). When will U-Haul be the better choice for Jamal?

Solution

The two important quantities in this problem are the cost, and the number of miles that are driven. Since we have two companies to consider, we will define two functions:

Input: m, miles driven

Outputs:

$Y(m)$: cost, in dollars, for renting from U-Haul

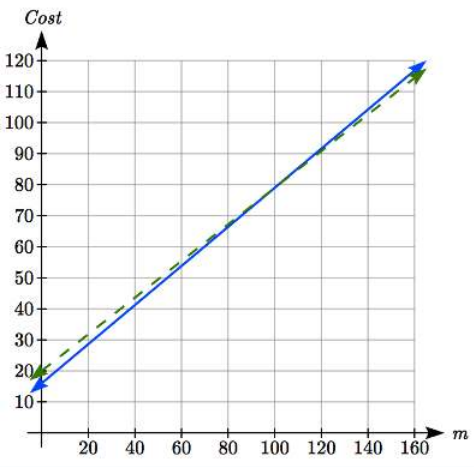
$B(m)$: cost, in dollars, for renting from Budget

Reading the problem carefully, it appears that we were given an initial cost and a rate of change for each company. Since our outputs are measured in dollars but the costs per mile given in the problem are in cents, we will need to convert these quantities to match our desired units: $0.59 a mile for U-Haul, and $0.63 a mile for Budget.

Looking to what we're trying to find, we want to know when U-Haul will be the better choice. Since all we have to make that decision from is the costs, we are looking for when U-Haul will cost less, or when $Y(m) < B(m)$. The solution pathway will lead us to find the equations for the two functions, find the intersection, then look to see where the $Y(m)$ function is smaller. Using the rates of change and initial charges, we can write the equations:

$$Y(m) = 20 + 0.59m$$

$$B(m) = 16 + 0.63m$$



These graphs are sketched to the right, with Y(m) drawn dashed.

To find the intersection, we set the equations equal and solve:

$$\begin{aligned} Y(m) &= B(m) \\ 20 + 0.59m &= 16 + 0.63m \\ 4 &= 0.04m \\ m &= 100 \end{aligned}$$

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that $Y(m)$ is growing at a slower rate, we can conclude that U-Haul will be the cheaper price when more than 100 miles are driven.

Example 2.3.3

A town's population has been growing linearly. In 2004 the population was 6,200. By 2009 the population had grown to 8,100. If this trend continues,

a. Predict the population in 2013
b. When will the population reach 15000?

Solution

The two changing quantities are the population and time. While we could use the actual year value as the input quantity, doing so tends to lead to very ugly equations, since the vertical intercept would correspond to the year 0, more than 2000 years ago!

To make things a little nicer, and to make our lives easier too, we will define our input as years since 2004:

Input: t, years since 2004

Output: $P(t)$, the town's population

The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to $t = 0$, giving the point (0, 6200). Notice that through our clever choice of variable definition, we have "given" ourselves the vertical intercept of the function. The year 2009 would correspond to $t = 5$, giving the point (5, 8100).

To predict the population in 2013 ($t = 9$), we would need an equation for the population. Likewise, to find when the population would reach 15000, we would need to solve for the input that would provide an output of 15000. Either way, we need an equation. To find it, we start by calculating the rate of change:

$$m = \frac{8100 - 6200}{5 - 0} = \frac{1900}{5} = 380 \text{ people per year}$$

Since we already know the vertical intercept of the line, we can immediately write the equation:

$$P(t) = 6200 + 380t$$

To predict the population in 2013, we evaluate our function at $t = 9$

$$P(9) = 6200 + 380(9) = 9620$$

If the trend continues, our model predicts a population of 9,620 in 2013.

To find when the population will reach 15,000, we can set $P(t) = 15000$ and solve for t.

$$\begin{aligned} 15000 &= 6200 + 380t \\ 8800 &= 380t \\ t &\approx 23.158 \end{aligned}$$

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

Example 2.3.4

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio with a range of 2 miles. How long after they start walking will they fall out of radio contact?

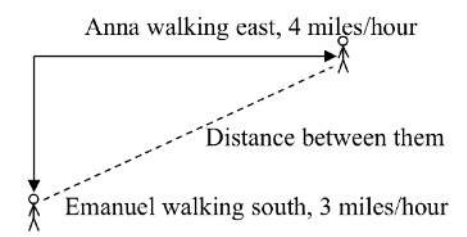
Solution

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question: how long will it take them to be 2 miles apart?

In this problem, our changing quantities are time and the two peoples' positions, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define

Input: t, time in hours.

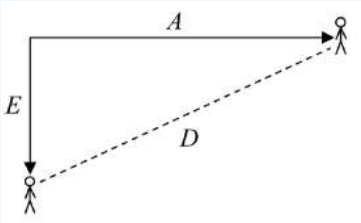
Since it is not obvious how to define our output variables, we'll start by drawing a picture.



Because of the complexity of this question, it may be helpful to introduce some intermediary variables. These are quantities that we aren't directly interested in, but seem important to the problem. For this problem, Anna's and Emanuel's distances from the starting point seem important. To notate these, we are going to define a coordinate system, putting the "starting point" at the intersection where they both started, then we're going to introduce a variable, A, to represent Anna's position, and define it to be a measurement from the starting point, in the eastward direction. Likewise, we'll introduce a variable, E, to represent Emanuel's position, measured from the starting point in the southward direction. Note that in defining the coordinate system we specified both the origin, or starting point, of the measurement, as well as the direction of measure.

While we're at it, we'll define a third variable, D, to be the measurement of the distance between Anna and Emanuel. Showing the variables on the picture is often helpful:

Looking at the variables on the picture, we remember we need to know how long it takes for D, the distance between them, to equal 2 miles.



Seeing this picture we remember that in order to find the distance between the two, we can use the Pythagorean Theorem, a property of right triangles.

From here, we can now look back at the problem for relevant information. Anna is walking 4 miles per hour, and Emanuel is walking 3 miles per hour, which are rates of change. Using those, we can write formulas for the distance each has walked.

They both start at the same intersection and so when $t = 0$, the distance travelled by each person should also be 0, so given the rate for each, and the initial value for each, we get:

$$A(t) = 4t$$

$$E(t) = 3t$$

Using the Pythagorean theorem we get:

$$D(t)^2 = A(t)^2 + E(t)^2$$

substitute in the function formulas

$$D(t)^2 = (4t)^2 + (3t)^2 = 16t^2 + 9t^2 = 25t^2$$

solve for $D(t)$ using the square root

$$D(t) = \pm\sqrt{25t^2} = \pm 5|t|$$

Since in this scenario we are only considering positive values of t and our distance $D(t)$ will always be positive, we can simplify this answer to $D(t) = 5t$

Interestingly, the distance between them is also a linear function. Using it, we can now answer the question of when the distance between them will reach 2 miles:

$$\begin{aligned} D(t) &= 2 \\ 5t &= 2 \\ t &= \frac{2}{5} = 0.4 \end{aligned}$$

They will fall out of radio contact in 0.4 hours, or 24 minutes.

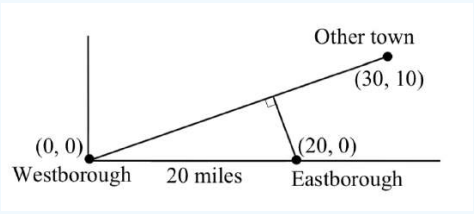
Example 2.3.5

There is currently a straight road leading from the town of Westborough to a town 30 miles east and 10 miles north. Partway down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

Solution

It might help here to draw a picture of the situation. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts the other town at coordinates (30, 10), and Eastborough at (20, 0).

Using this point along with the origin, we can find the slope of the line from Westborough to the other town: $m = \frac{10-0}{30-0} = \frac{1}{3}$. This gives the equation of the road from Westborough to the other town to be $W(x) = \frac{1}{3}x$.



From this, we can determine the perpendicular road to Eastborough will have slope $m = -3$. Since the town of Eastborough is at the point (20, 0), we can find the equation:

$$E(x) = -3x + b$$

plug in the point (20, 0)

$$0 = -3(20) + b$$

$$b = 60$$

$$E(x) = -3x + 60$$

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,

$$\frac{1}{3}x = -3x + 60$$

$$\frac{10}{3}x = 60$$

$$10x = 180$$

$$x = 18$$

Substituting this back into $W(x)$

$$y = W(18) = \frac{1}{3}(18) = 6$$

The roads intersect at the point (18, 6). Using the distance formula, we can now find the distance from Westborough to the junction:

$$dist = \sqrt{(18-0)^2 + (6-0)^2} \approx 18.934 \text{ miles}$$

Important Topics of this Section

The problem solving process

1. Identify changing quantities, and then carefully and clearly define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2. Carefully read the problem to identify important information. Look for information giving values for the variables, or values for parts of the functional model, like slope and initial value.
3. Carefully read the problem to identify what we are trying to find, identify, solve, or interpret.
4. Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table or even finding a formula for the function being used to model the problem.
5. When needed, find a formula for the function.

6. Solve or evaluate using the formula you found for the desired quantities.
7. Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8. Clearly convey your result using appropriate units, and answer in full sentences when appropriate.

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2.3E: Modeling with Linear Functions (Exercises)

? section 2.3 exercise

1. In 2004, a school population was 1001. By 2008 the population had grown to 1697. Assume the population is changing linearly.

a. How much did the population grow between the year 2004 and 2008?
b. How long did it take the population to grow from 1001 students to 1697 students?
c. What is the average population growth per year?
d. What was the population in the year 2000?
e. Find an equation for the population, P, of the school t years after 2000.
f. Using your equation, predict the population of the school in 2011.

2. In 2003, a town's population was 1431. By 2007 the population had grown to 2134. Assume the population is changing linearly.

a. How much did the population grow between the year 2003 and 2007?
b. How long did it take the population to grow from 1431 people to 2134?
c. What is the average population growth per year?
d. What was the population in the year 2000?
e. Find an equation for the population, P, of the town t years after 2000.
f. Using your equation, predict the population of the town in 2014.

3. A phone company has a monthly cellular plan where a customer pays a flat monthly fee and then a certain amount of money per minute used on the phone. If a customer uses 410 minutes, the monthly cost will be $71.50. If the customer uses 720 minutes, the monthly cost will be $118.

a. Find a linear equation for the monthly cost of the cell plan as a function of x, the number of monthly minutes used.
b. Interpret the slope and vertical intercept of the equation.
c. Use your equation to find the total monthly cost if 687 minutes are used.

4. A phone company has a monthly cellular data plan where a customer pays a flat monthly fee and then a certain amount of money per megabyte (MB) of data used on the phone. If a customer uses 20 MB, the monthly cost will be $11.20. If the customer uses 130 MB, the monthly cost will be $17.80.

a. Find a linear equation for the monthly cost of the data plan as a function of x, the number of MB used.
b. Interpret the slope and vertical intercept of the equation.
c. Use your equation to find the total monthly cost if 250 MB are used.

5. In 1991, the moose population in a park was measured to be 4360. By 1999, the population was measured again to be 5880. If the population continues to change linearly,

a. Find a formula for the moose population, P.
b. What does your model predict the moose population to be in 2003?

6. In 2003, the owl population in a park was measured to be 340. By 2007, the population was measured again to be 285. If the population continues to change linearly,

a. Find a formula for the owl population, P.
b. What does your model predict the owl population to be in 2012?

7. The Federal Helium Reserve held about 16 billion cubic feet of helium in 2010, and is being depleted by about 2.1 billion cubic feet each year.

a. Give a linear equation for the remaining federal helium reserves, R, in terms of t, the number of years since 2010.
b. In 2015, what will the helium reserves be?
c. If the rate of depletion doesn't change, when will the Federal Helium Reserve be depleted?

8. Suppose the world's current oil reserves are 1820 billion barrels. If, on average, the total reserves is decreasing by 25 billion barrels of oil each year:

a. Give a linear equation for the remaining oil reserves, R, in terms of t, the number of years since now.
b. Seven years from now, what will the oil reserves be?
c. If the rate of depletion isn't change, when will the world's oil reserves be depleted?

9. You are choosing between two different prepaid cell phone plans. The first plan charges a rate of 26 cents per minute. The second plan charges a monthly fee of \$19.95 $plus$ 11 cents per minute. How many minutes would you have to use in a month in order for the second plan to be preferable?

10. You are choosing between two different window washing companies. The first charges \$5 per window. The second charges a base fee of \$40 plus \$3 per window. How many windows would you need to have for the second company to be preferable?

11. When hired at a new job selling jewelry, you are given two pay options:

Option A: Base salary of \$17,000 a year, with a commission of 12% of your sales
Option B: Base salary of \$20,000 a year, with a commission of 5% of your sales

How much jewelry would you need to sell for option A to produce a larger income?

12. When hired at a new job selling electronics, you are given two pay options:

Option A: Base salary of \$14,000 a year, with a commission of 10% of your sales
Option B: Base salary of \$19,000 a year, with a commission of 4% of your sales

How much electronics would you need to sell for option A to produce a larger income?

13. Find the area of a triangle bounded by the y axis, the line $f(x) = 9 - \frac{6}{7}x$, and the line perpendicular to $f(x)$ that passes through the origin.

14. Find the area of a triangle bounded by the x axis, the line $f(x) = 12 - \frac{1}{3}x$, and the line perpendicular to $f(x)$ that passes through the origin.

15. Find the area of a parallelogram bounded by the y axis, the line $x = 3$, the line $f(x) = 1 + 2x$, and the line parallel to $f(x)$ passing through (2, 7)

16. Find the area of a parallelogram bounded by the x axis, the line $g(x) = 2$, the line $f(x) = 3x$, and the line parallel to $f(x)$ passing through (6, 1)

17. If $b > 0$ and $m < 0$, then the line $f(x) = b + mx$ cuts off a triangle from the first quadrant. Express the area of that triangle in terms of m and b. [UW]

18. Find the value of m so the lines $f(x) = mx + 5$ and $g(x) = x$ and the y-axis form a triangle with an area of 10. [UW]

19. The median home values in Mississippi and Hawaii (adjusted for inflation) are shown below. If we assume that the house values are changing linearly,

Year	Mississippi	Hawaii
1950	25200	74400
2000	71400	272700

a. In which state have home values increased at a higher rate?
b. If these trends were to continue, what would be the median home value in Mississippi in 2010?
c. If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd)

20. The median home value ins Indiana and Alabama (adjusted for inflation) are shown below. If we assume that the house values are changing linearly,

Year	Indiana	Alabama
1950	37700	27100

2000	94300	85100

a. In which state have home values increased at a higher rate?
b. If these trends were to continue, what would be the median home value in Indiana in 2010?
c. If we assume the linear trend existed before 1950 and continues after 2000, the two states' median house values will be (or were) equal in what year? (The answer might be absurd)

21. Pam is taking a train from the town of Rome to the town of Florence. Rome is located 30 miles due West of the town of Paris. Florence is 25 miles East, and 45 miles North of Rome. On her trip, how close does Pam get to Paris? [UW]

22. You're flying from Joint Base Lewis-McChord (JBLM) to an undisclosed location 226 km south and 230 km east. Mt. Rainier is located approximately 56 km east and 40 km south of JBLM. If you are flying at a constant speed of 800 km/hr, how long after you depart JBLM will you be the closest to Mt. Rainier?

Answer

1a. 696 people
b. 4 years
c. 174 people per year
d. 305 people
e. $P(t) = 305 + 174t$
f. 2219 people.

3a. $C(x) = 0.15x + 10$
b. The flat monthly fee is \$10 and there is an additional \$0.15 fee for each additional minute used
c. \$113.05

5a. $P(t) = 190t + 4170$
b. 6640 moose

7a. $R(t) = 16 - 2.1t$
b. 5.5 billion cubic feet
c. During the year 2017

9. More than 133 minutes

11. More than \$42857.14 worth of jewelry

13. 20.012 square units

15. 6 square units

17. $A = -\frac{b^2}{2m}$

19a. Hawaii
b. \$80640
c. During the year 1933

21. 26.225 miles

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2.4: Fitting Linear Models to Data

In the real world, rarely do things follow trends perfectly. When we expect the trend to behave linearly, or when inspection suggests the trend is behaving linearly, it is often desirable to find an equation to approximate the data. Finding an equation to approximate the data helps us understand the behavior of the data and allows us to use the linear model to make predictions about the data, inside and outside of the data range.

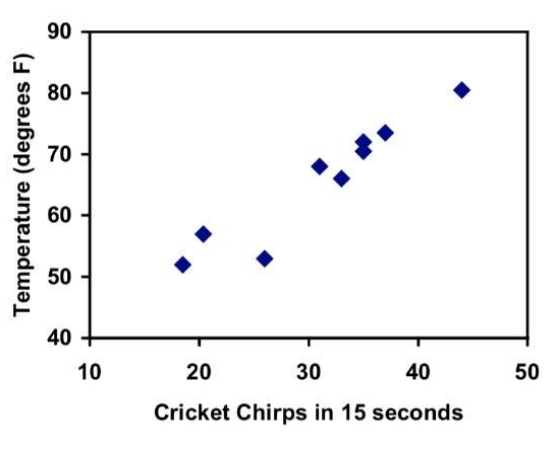
Example 2.4.1

The table below shows the number of cricket chirps in 15 seconds, and the air temperature, in degrees Fahrenheit (Selected data from classic.globe.gov/fsl/scientistsblog/2007/10/. Retrieved Aug 3, 2010). Plot this data, and determine whether the data appears to be linearly related.

chirps	44	35	20.4	33	31	35	18.5	37	26
Temp	80.5	70.5	57	66	68	72	52	73.5	53

Solution

Plotting this data, it appears there may be a trend, and that the trend appears roughly linear, though certainly not perfectly so.



The simplest way to find an equation to approximate this data is to try to “eyeball” a line that seems to fit the data pretty well, then find an equation for that line based on the slope and intercept.

You can see from the trend in the data that the number of chirps increases as the temperature increases. As you consider a function for this data you should know that you are looking at an increasing function or a function with a positive slope.

flashback

a. What descriptive variables would you choose to represent Temperature & Chirps?
b. Which variable is the independent variable and which is the dependent variable?
c. Based on this data and the graph, what is a reasonable domain & range?
d. Based on the data alone, is this function one-to-one, explain?

Answer

1. a. T = Temperature, C = Chirps (answers may vary)

   b. Independent (Chirps) , Dependent (Temperature)

   c. Reasonable Domain (18.5, 44) , Reasonable Range (52, 80.5) (answers may vary)

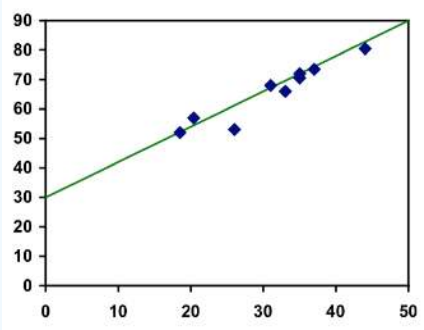
   d. NO, it is not one-to-one, there are two different output values for 35 chirps.

Example 2.4.2

Using the table of values from the previous example, find a linear function that fits the data by "eyeballing" a line that seems to fit.

Solution

On a graph, we could try sketching in a line. Note the scale on the axes have been adjusted to start at zero to include the vertical axis and vertical intercept in the graph.



Using the starting and ending points of our "hand drawn" line, points (0, 30) and (50, 90), this graph has a slope of $m = \frac{60}{50} = 1.2$ and a vertical intercept at 30, giving an equation of

$$T(c) = 30 + 1.2c$$

where c is the number of chirps in 15 seconds, and $T(c)$ is the temperature in degrees Fahrenheit.

This linear equation can then be used to approximate the solution to various questions we might ask about the trend. While the data does not perfectly fall on the linear equation, the equation is our best guess as to how the relationship will behave outside of the values we have data for. There is a difference, though, between making predictions inside the domain and range of values we have data for, and outside that domain and range.

Definition: Interpolation and Extrapolation

Interpolation: When we predict a value inside the domain and range of the data

Extrapolation: When we predict a value outside the domain and range of the data

For the Temperature as a function of chirps in our hand drawn model above,

- Interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44.
- Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

Example 2.4.3

a. Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.
b. Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss if it is reasonable.

Solution

With our cricket data, our number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model:

$$T(3) = 30 + 1.2(30) = 66 \text{ degrees}$$

Based on the data we have, this value seems reasonable.

The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation since 40 is outside the range of our data. Using our model:

$$\begin{aligned} 40 &= 30 + 1.2c \\ 10 &= 1.2c \\ c &\approx 8.33 \end{aligned}$$

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

When our model no longer applies after some point, it is sometimes called **model breakdown**.

? Exercise 2.4.1

What temperature would you predict if you counted 20 chirps in 15 seconds?

Answer

Add answer text here and it will automatically be hidden if you have a "AutoNum" template active on the page.

Fitting Lines with Technology

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values (Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.). This technique is called **least-square regression**, and can be computed by many graphing calculators, spreadsheet software like Excel or Google Docs, statistical software, and many web-based calculators(For example, http://www.shodor.org/unchem/math/lls/leastsq.html).

✓ Example 2.4.4

Find the least-squares regression line using the cricket chirp data from above.

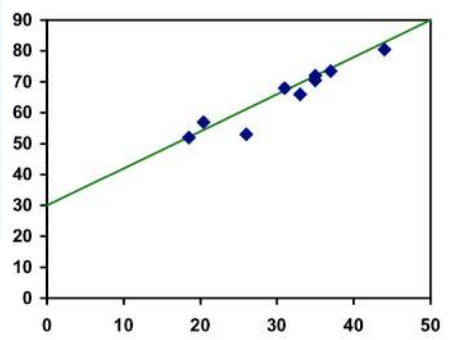
Solution

Using the cricket chirp data from earlier, with technology we obtain the equation:

$$T(c) = 30.281 + 1.143c$$

Notice that this line is quite similar to the equation we “eyeballed”, but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:

$$T(30) = 30.281 + 1.143(30) = 64.571 \approx 64.6 \text{ degrees}$$



Most calculators and computer software will also provide you with the **correlation coefficient**, a measure of how closely the line fits the data.

Definition: correlation coefficient

The **correlation coefficient** is a value, r, between -1 and 1.

$r > 0$ suggests a positive (increasing) relationship

$r < 0$ suggests a negative (decreasing) relationship

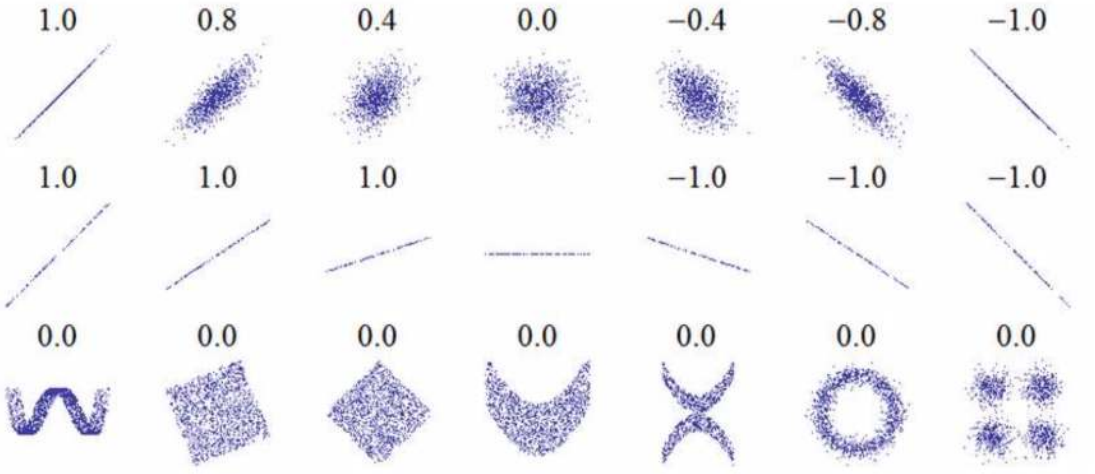
The closer the value is to 0, the more scattered the data

The closer the value is to 1 or -1, the less scattered the data is

The correlation coefficient provides an easy way to get some idea of how close to a line the data falls.

We should only compute the correlation coefficient for data that follows a linear pattern; if the data exhibits a non-linear pattern, the correlation coefficient is meaningless. To get a sense for the relationship between the value of r and the graph of the data, here are some large data sets with their correlation coefficients:

Examples of Correlation Coefficient Values



(http://en.Wikipedia.org/wiki/File:Co...n_examples.png)

Example 2.4.5

Calculate the correlation coefficient for our cricket data.

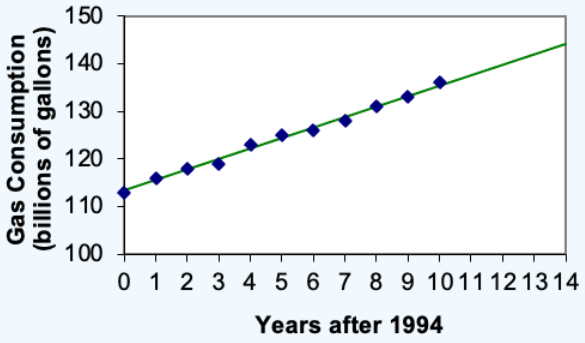
Solution

Because the data appears to follow a linear pattern, we can use technology to calculate $r = 0.9509$. Since this value is very close to 1, it suggests a strong increasing linear relationship.

Example 2.4.6

Gasoline consumption in the US has been increasing steadily. Consumption data from 1994 to 2004 is shown below. (www.bts.gov/publications/nati...ble_04_10.html) Determine if the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.

Year	'94	'95	'96	'97	'98	'99	'00	'01	'02	'03	'04
Consumption (billions of gallons)	113	116	118	119	123	125	126	128	131	133	136



Solution

To make things simpler, a new input variable is introduced, t, representing years since 1994.

Using technology, the correlation coefficient was calculated to be 0.9965, suggesting a very strong increasing linear trend.

The least-squares regression equation is:

$$C(t) = 113.318 + 2.209t$$

Using this to predict consumption in 2008 (t = 14),

$$C(14) = 113.318 + 2.209(14) = 144.244 \text{ billions of gallons}$$

The model predicts 144.244 billion gallons of gasoline will be consumed in 2008.

? Exercise 2.4.2

Use the model created by technology in example 6 to predict the gas consumption in 2011. Is this an interpolation or an extrapolation?

Answer

1. 54 degrees Fahrenheit
2. 150.871 billion gallons; extrapolation

Important Topics of this Section

- Fitting linear models to data by hand
- Fitting linear models to data using technology
- Interpolation
- Extrapolation
- Correlation coefficient

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2.4E: Fitting Linear Models to Data (Exercises)

? section 2.4 exercise

1. The following is data for the first and second quiz scores for 8 students in a class. Plot the points, then sketch a line that fits the data.

First Quiz	11	20	24	25	33	42	46	49
Second Quiz	10	16	23	28	30	39	40	49

2. Eight students were asked to estimate their score on a 10 point quiz. Their estimated and actual scores are given. Plot the points, then sketch a line that fits the data.

Predicted	5	7	6	8	10	9	10	7
Actual	6	6	7	8	9	9	10	6

Based on each set of data given, calculate the regression line using your calculator or other technology tool, and determine the correlation coefficient.

3.

x	y
5	4
7	12
10	17
12	22
15	24

4.

x	y
8	23
15	41
26	53
31	72
56	103

5.

x	y
3	21.9
4	22.22
5	22.74
6	22.26
7	20.78
8	17.6
9	16.52
10	18.54
11	15.76
12	13.68
13	14.1
14	14.02
15	11.94
16	12.76
17	11.28
18	9.1

6.

x	y
4	44.8
5	43.1
6	38.8
7	39
8	38
9	32.7
10	30.1
11	29.3
12	27
13	25.8
14	24.7
15	22
16	20.1
17	19.8
18	16.8

7. A regression was run to determine if there is a relationship between hours of TV watched per day (x) and number of situps a person can do (y). The results of the regression are given below. Use this to predict the number of situps a person who watches 11 hours of TV can do.

$y = ax + b$
$a = -1.341$
$b = 32.234$
$r^2 = 0.803$
$r = -0.896$

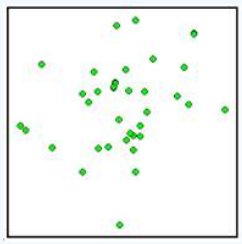
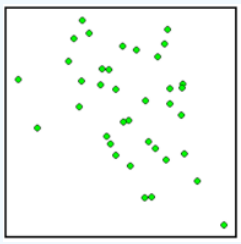
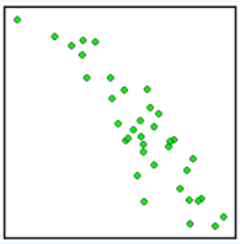
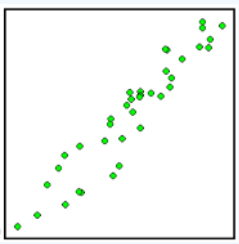
8. A regression was run to determine if there is a relationship between the diameter of a tree (x, in inches) and the tree's age (y, in years). The results of the regression are given below. Use this to predict the age of a tree with diameter 10 inches.

$y = ax + b$
$a = 6.301$
$b = -1.044$

$r^2 = 0.940$
$r = 0.970$

Match each scatterplot shown below with one of the four specified correlations.

9. $r = 0.95$ 10. $r = -0.89$ 11. $r = 0.26$ 12. $r = -0.39$

A  B  C  D 

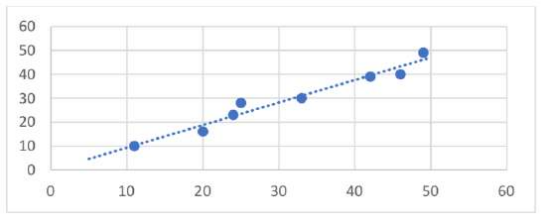
13. The US census tracks the percentage of persons 25 years or older who are college graduates. That data for several years is given below. Determine if the trend appears linear. If so and the trend continues, in what year will the percentage exceed 35%?

Year	1990	1992	1994	1996	1998	2000	2002	2004	2006	2008
Percent Graduates	21.3	21.4	22.2	23.6	24.4	25.6	26.7	27.7	28	29.4

14. The US import of wine (in hectoliters) for several years is given below. Determine if the trend appears linear. If so and the trend continues, in what year will imports exceed 12,000 hectoliters?

Year	1992	1994	1996	1998	2000	2002	2004	2006	2008	2009
Imports	2665	2688	3565	4129	4584	5655	6549	7950	8487	9462

Answer

1. 

3. $y = 1.971x - 3.519, r = 0.967$

5. $y = -0.901x + 26.04, r = -0.968$

7. $17.483 \approx 17$ *situps*

9. D

11. A

13. Yes, trend appears linear because $r = 0.994$ and will exceed 35% near the end of the year 2019.

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CHAPTER OVERVIEW

3: Quadratic and Polynomial Functions

- 3.1: Power Functions
  - 3.1E: Power Functions (Exercises)
- 3.2: Quadratic Functions
  - 3.2E: Quadratic Functions (Exercises)
- 3.3: Graphs of Polynomial Functions
  - 3.3E: Graphs of Polynomial Functions (Exercises)
- 3.4: Factor Theorem and Remainder Theorem
  - 3.4E: Factor Theorem and Remainder Theorem (Exercises)
- 3.5: Real Zeros of Polynomials
  - 3.5E: Real Zeros of Polynomials (Exercises)
- 3.6: Complex Zeros
  - 3.6E: Complex Zeros (Exercises)

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3.1: Power Functions

A square is cut out of cardboard, with each side having length L. If we wanted to write a function for the area of the square, with L as the input and the area as output, you may recall that the area of a rectangle can be found by multiplying the length times the width. Since our shape is a square, the length & the width are the same, giving the formula:

$$A(L) = L \cdot L = L^2 \tag{3.1.1}$$

Likewise, if we wanted a function for the volume of a cube with each side having some length L, you may recall volume of a rectangular box can be found by multiplying length by width by height, which are all equal for a cube, giving the formula:

$$V(L) = L \cdot L \cdot L = L^3 \tag{3.1.2}$$

These two functions are examples of **power functions**, functions that are some power of the variable.

Definition: Power Function

A **power function** is a function that can be represented in the form

$$f(x) = x^p \tag{3.1.3}$$

Where the base is a variable and the exponent, p, is a number.

Example 3.1.1

Which of our toolkit functions are power functions? (This chapter is part of *Precalculus: An Investigation of Functions* © Lippman & Rasmussen 2017, and contains content remixed with permission from *College Algebra* © Stitz & Zeager 2013. This material is licensed under a Creative Commons CC-BY-SA license.)

Solutions

The constant and identity functions are power functions, since they can be written as $f(x) = x^0$ and $f(x) = x^1$ respectively.

The quadratic and cubic functions are both power functions with whole number powers: $f(x) = x^2$ and $f(x) = x^3$.

The reciprocal and reciprocal squared functions are both power functions with negative whole number powers since they can be written as $f(x) = x^{-1}$ and $f(x) = x^{-2}$.

The square and cube root functions are both power functions with fractional powers since they can be written as $f(x) = x^{\frac{1}{2}}$ or $f(x) = x^{\frac{1}{3}}$

Exercise 3.1.1

What point(s) do the toolkit power functions have in common?

Answer

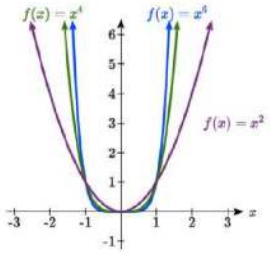
(0, 0) and (1, 1) are common to all power functions.

Characteristics of Power Functions

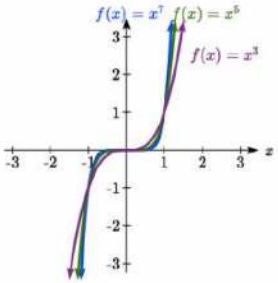
Shown to the right are the graphs of $f(x) = x^2$, $f(x) = x^4$, and $f(x) = x^6$, all even whole number powers. Notice that all these graphs have a fairly similar shape, very similar to the quadratic toolkit, but as the power increases the graphs flatten somewhat near the origin, and become steeper away from the origin.

To describe the behavior as numbers become larger and larger, we use the idea of infinity. The symbol for positive infinity is ∞, and $-\infty$ for negative infinity. When we say that "x approaches infinity", which can be symbolically written as $x \to \infty$, we are describing a behavior – we are saying that x is getting large in the positive direction.

With the even power functions, as the x becomes large in either the positive or negative direction, the output values become very large positive numbers. Equivalently, we could describe this by saying that as x approaches positive or negative infinity, the $f(x)$ values approach positive infinity. In symbolic form, we could write: as $x \to \pm\infty$, $f(x) \to \infty$.



Shown here are the graphs of $f(x) = x^3$, $f(x) = x^5$, and $f(x) = x^7$, all odd whole number powers. Notice all these graphs look similar to the cubic toolkit, but again as the power increases the graphs flatten near the origin and become steeper away from the origin.



For these odd power functions, as x approaches negative infinity, $f(x)$ approaches negative infinity. As x approaches positive infinity, $f(x)$ approaches positive infinity. In symbolic form we write: as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$.

Definition: long run behavior

The behavior of the graph of a function as the input takes on large negative values,$x \to -\infty$, and large positive values, $x \to \infty$, is referred to as the **long run behavior** of the function.

Example 3.1.2

Describe the long run behavior of the graph of $f(x) = x^8$.

Solution

Since $f(x) = x^8$ has a whole, even power, we would expect this function to behave somewhat like the quadratic function. As the input gets large positive or negative, we would expect the output to grow without bound in the positive direction. In symbolic form, as $x \to \pm\infty$, $f(x) \to \infty$.

Example 3.1.3

Describe the long run behavior of the graph of $f(x) = -x^9$

Solution

Since this function has a whole odd power, we would expect it to behave somewhat like the cubic function. The negative in front of the x^9 will cause a vertical reflection, so as the inputs grow large positive, the outputs will grow large in the negative direction, and as the inputs grow large negative, the outputs will grow large in the positive direction. In symbolic form, for the long run behavior we would write: as $x \to \infty$, $f(x) \to -\infty$and as $x \to -\infty$, $f(x) \to \infty$.

You may use words or symbols to describe the long run behavior of these functions.

Exercise 3.1.2

Describe in words and symbols the long run behavior of $f(x) = -x^4$

Answer

As x approaches positive and negative infinity, $f(x)$ approaches negative infinity: as $x \to \pm\infty$, $f(x) \to -\infty$ because of the vertical flip.

Treatment of the rational and radical forms of power functions will be saved for later.

Polynomials

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. If we wanted to write a formula for the area covered by the oil slick, we could do so by composing two functions together. The first is a formula for the radius, r, of the spill, which depends on the number of weeks, w, that have passed.

Hopefully you recognized that this relationship is linear:

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area, A, of a circle:

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks:

$$A(w) = A(r(w)) = A(24 + 8w) = \pi(24 + 8w)^2$$

Multiplying this out gives the formula

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a **polynomial**. A polynomial is simply the sum of terms each consisting of a vertically stretched or compressed power function with non-negative whole number power.

terminology of polynomial functions

A **polynomial** is function that can be written as

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \tag{3.1.4}$$

- Each of the a_i constants are called **coefficients** and can be positive, negative, or zero, and be whole numbers, decimals, or fractions.
- A **term** of the polynomial is any one piece of the sum, that is any $a_i x^i$. Each individual term is a transformed power function.
- The **degree** of the polynomial is the highest power of the variable that occurs in the polynomial.
- The **leading term** is the term containing the highest power of the variable: the term with the highest degree.
- The **leading coefficient** is the coefficient of the leading term.

Because of the definition of the "leading" term we often rearrange Equation 3.1.4 so that the powers are descending.

$$f(x) = a_n x^n + \ldots\ldots + a_2 x^2 + a_1 x + a_0 \tag{3.1.5}$$

Example 3.1.4

Identify the degree, leading term, and leading coefficient of these polynomials:

a. $f(x) = 3 + 2x^2 - 4x^3$
b. $g(t) = 5t^5 - 2t^3 + 7t$
c. $h(p) = 6p - p^3 - 2$

Solution

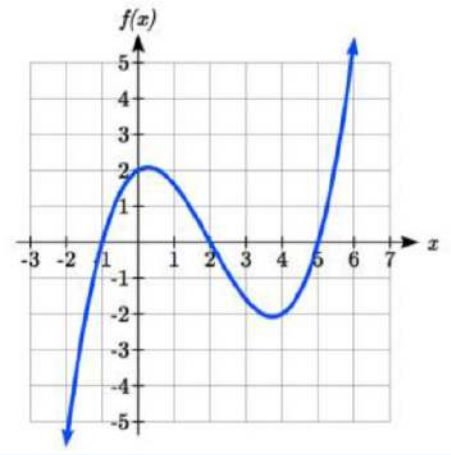
a. For the function $f(x)$, the degree is 3, the highest power on x. The leading term is the term containing that power, $-4x^3$. The leading coefficient is the coefficient of that term, -4.
b. For $g(t)$, the degree is 5, the leading term is $5t^5$, and the leading coefficient is 5.
c. For $h(p)$, the degree is 3, the leading term is $-p^3$, so the leading coefficient is -1.

Definition: long run behavior of polynomials

For any polynomial, the **long run behavior** of the polynomial will match the long run behavior of the leading term.

Example 3.1.5

What can we determine about the long run behavior and degree of the equation for the polynomial graphed here?



Solution

Since the output grows large and positive as the inputs grow large and positive, we describe the long run behavior symbolically by writing: as $x \to \infty$, $f(x) \to \infty$. Similarly, as $x \to -\infty$, $f(x) \to -\infty$.

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function which has not been reflected, so the degree of the polynomial creating this graph must be odd, and the leading coefficient would be positive.

Exercise 3.1.3

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$ use your algebra skills to write the function in standard polynomial form (as a sum of terms) and determine the leading term, degree, and long run behavior of the function.

Answer

The leading term is $0.2x^3$, so it is a degree 3 polynomial.As x approaches infinity (or gets very large in the positive direction) $f(x)$ approaches infinity; as x approaches negative infinity (or gets very large in the negative direction) $f(x)$ approaches negative infinity. (Basically the long run behavior is the same as the cubic function).

Short Run Behavior

Characteristics of the graph such as vertical and horizontal intercepts and the places the graph changes direction are part of the short run behavior of the polynomial.

Like with all functions, the vertical intercept is where the graph crosses the vertical axis, and occurs when the input value is zero. Since a polynomial is a function, there can only be one vertical intercept, which occurs at the point $(0, a_0)$. The horizontal intercepts occur at the input values that correspond with an output value of zero. It is possible to have more than one horizontal intercept.

Horizontal intercepts are also called **zeros**, or **roots** of the function.

Example 3.1.6

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form for your convenience, determine the vertical and horizontal intercepts.

Solution

The vertical intercept occurs when the input is zero.

$$f(0) = (0-2)(0+1)(0-4) = 8$$

The graph crosses the vertical axis at the point (0, 8).

The horizontal intercepts occur when the output is zero.

$$0 = (x-2)(x+1)(x-4)$$

when x = 2, -1, or 4.

$f(x)$ has zeros, or roots, at x = 2, -1, and 4.

The graph crosses the horizontal axis at the points (2, 0), (-1, 0), and (4, 0)

Notice that the polynomial in the previous example, which would be degree three if multiplied out, had three horizontal intercepts and two turning points – places where the graph changes direction. We will now make a general statement without justifying it – the reasons will become clear later in this chapter.

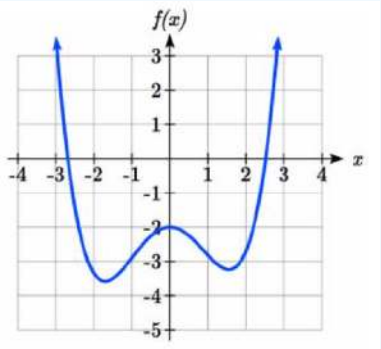
intercepts and turning points of polynomials

A polynomial of degree n will have:

- At most n horizontal intercepts. An odd degree polynomial will always have at least one.
- At most $n-1$ turning points

Example 3.1.7

What can we conclude about the graph of the polynomial shown here?



Solution

Based on the long run behavior, with the graph becoming large positive on both ends of the graph, we can determine that this is the graph of an even degree polynomial. The graph has 2 horizontal intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4, so it is probably a fourth degree polynomial.

Exercise 3.1.4

Given the function $f(x) = 0.2(x-2)(x+1)(x-5)$, determine the short run behavior.

Answer

Horizontal intercepts are (2, 0) (-1, 0) and (5, 0), the vertical intercept is (0, 2) and there are 2 turns in the graph.

Important Topics of this Section

- Power Functions
- Polynomials
- Coefficients
- Leading coefficient
- Term
- Leading Term
- Degree of a polynomial
- Long run behavior
- Short run behavior

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3.1E: Power Functions (Exercises)

section 3.1 EXERCISE

Find the long run behavior of each function as $x \to \infty$ and $x \to -\infty$

1. $f(x) = x^4$
2. $f(x) = x^6$
3. $f(x) = x^3$
4. $f(x) = x^5$
5. $f(x) = -x^2$
6. $f(x) = -x^4$
7. $f(x) = -x^7$
8. $f(x) = -x^9$

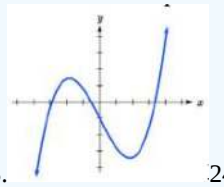
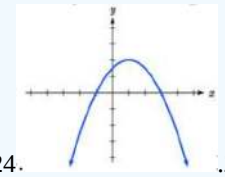
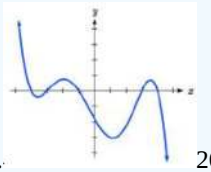
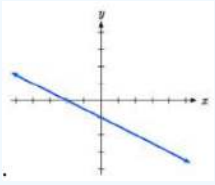
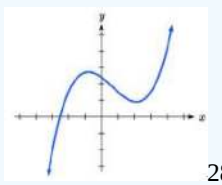
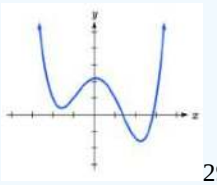
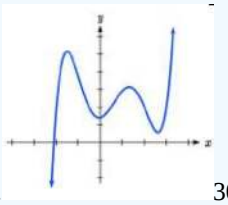
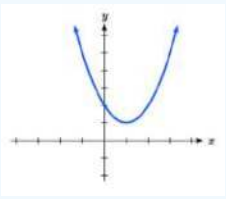
Find the degree and leading coefficient of each polynomial

9. $4x^7$
10. $5x^6$
11. $5 - x^2$
12. $6 + 3x - 4x^3$
13. $-2x^4 - 3x^2 + x - 1$
14. $6x^5 - 2x^4 + x^2 + 3$
15. $(2x + 3)(x - 4)(3x + 1)$
16. $(3x + 1)(x + 1)(4x + 3)$

Find the long run behavior of each function as $x \to \infty$ and $x \to -\infty$

17. $-2x^4 - 3x^2 + x - 1$
18. $6x^5 - 2x^4 + x^2 + 3$
19. $3x^2 + x - 2$
20. $-2x^3 + x^2 - x + 3$
21. What is the maximum number of x-intercepts and turning points for a polynomial of degree 5?
22. What is the maximum number of x-intercepts and turning points for a polynomial of degree 8?

What is the least possible degree of the polynomial function shown in each graph?

23. 
24. 
25. 
26. 
27. 
28. 
29. 
30. 

Find the vertical and horizontal intercepts of each function.

31. $f(t) = 2(t-1)(t+2)(t-3)$

32. $f(x) = 3(x+1)(x-4)(x+5)$

33. $g(n) = -2(3n-1)(2n+1)$

34. $k(u) = -3(4-n)(4n+3)$

Answer

1. As $x \to \infty$, $f(x) \to \infty$ As $x \to -\infty$, $f(x) \to \infty$
3. As $x \to \infty$, $f(x) \to \infty$ As $x \to -\infty$, $f(x) \to -\infty$
5. As $x \to \infty$, $f(x) \to -\infty$ As $x \to -\infty$, $f(x) \to -\infty$
7. As $x \to \infty$, $f(x) \to -\infty$ As $x \to -\infty$, $f(x) \to \infty$
9. 7th Degree, Leading coefficient 4
11. 2nd Degree, Leading coefficient -1
13. 4th Degree, Leading coefficient -2
15. 3rd Degree, Leading coefficient 6
17. As $x \to \infty$, $f(x) \to -\infty$ As $x \to -\infty$, $f(x) \to -\infty$
19. As $x \to \infty$, $f(x) \to \infty$ As $x \to -\infty$, $f(x) \to \infty$
21. intercepts: 5, turning points: 4
23. 3
25. 5
27. 3
29. 5
31. Horizontal Intercepts (1, 0), (-2, 0), (3, 0) Vertical Intercept (0, 12)
33. Horizontal Intercepts (1/3, 0) (-1/2. 0) Vertical Intercept (0, 2)

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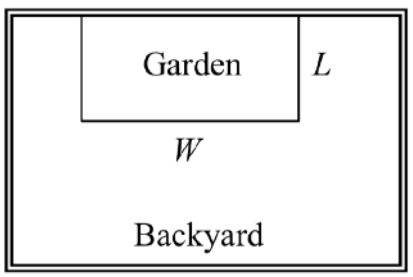
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3.2: Quadratic Functions

In this section, we will explore the family of 2nd degree polynomials, the quadratic functions. While they share many characteristics of polynomials in general, the calculations involved in working with quadratics is typically a little simpler, which makes them a good place to start our exploration of short run behavior. In addition, quadratics commonly arise from problems involving area and projectile motion, providing some interesting applications.

Example 3.2.1

A backyard farmer wants to enclose a rectangular space for a new garden. She has purchased 80 feet of wire fencing to enclose 3 sides, and will put the 4th side against the backyard fence. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.



Solution

In a scenario like this involving geometry, it is often helpful to draw a picture. It might also be helpful to introduce a temporary variable, W, to represent the side of fencing parallel to the 4th side or backyard fence.

Since we know we only have 80 feet of fence available, we know that $L+W+L=80$, or more simply, $2L+W=80$. This allows us to represent the width, W, in terms of L:

$$W=80-2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$A=LW=L(80-2L)$$

$$A(L)=80L-2L^2$$

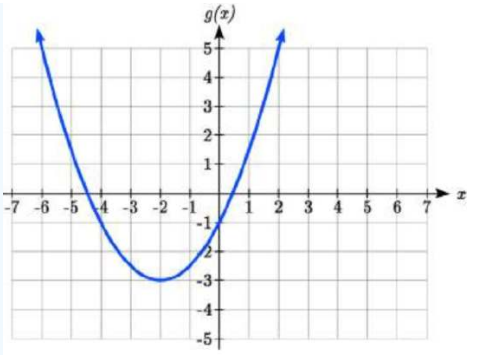
This formula represents the area of the fence in terms of the variable length L.

Short run Behavior: Vertex

We now explore the interesting features of the graphs of quadratics. In addition to intercepts, quadratics have an interesting feature where they change direction, called the **vertex**. You probably noticed that all quadratics are related to transformations of the basic quadratic function $f(x)=x^2$.

Example 3.2.2

Write an equation for the quadratic graphed below as a transformation of $f(x)=x^2$, then expand the formula and simplify terms to write the equation in standard polynomial form.



Solution

We can see the graph is the basic quadratic shifted to the left 2 and down 3, giving a formula in the form $g(x) = a(x+2)^2 - 3$. By plugging in a point that falls on the grid, such as (0,-1), we can solve for the stretch factor:

$$\begin{aligned} -1 &= a(0+2)^2 - 3 \\ 2 &= 4a \\ a &= \frac{1}{2} \end{aligned}$$

Written as a transformation, the equation for this formula is $g(x) = \frac{1}{2}(x+2)^2 - 3$. To write this in standard polynomial form, we can expand the formula and simplify terms:

$$\begin{aligned} g(x) &= \frac{1}{2}(x+2)^2 - 3 \\ g(x) &= \frac{1}{2}(x+2)(x+2) - 3 \\ g(x) &= \frac{1}{2}(x^2+4x+4) - 3 \\ g(x) &= \frac{1}{2}x^2 + 2x + 2 - 3 \\ g(x) &= \frac{1}{2}x^2 + 2x - 1 \end{aligned}$$

Notice that the horizontal and vertical shifts of the basic quadratic determine the location of the vertex of the parabola; the vertex is unaffected by stretches and compressions.

? Exercise 3.2.1

A coordinate grid has been superimposed over the quadratic path of a basketball(From http://blog.mrmeyer.com/?p=4778, © Dan Meyer, CC-BY). Find an equation for the path of the ball. Does he make the basket?



Answer

The path passes through the origin with vertex at (-4, 7).

$$h(x) = -\frac{7}{16}(x+4)^2 + 7$$

To make the shot, $h(-7.5)$ would need to be about 4. $h(-7.5) \approx 1.64$; he doesn't make it.



forms of quadratic function

The **standard form** of a quadratic function is

$$f(x) = ax^2 + bx + c \tag{3.2.1}$$

The **transformation form** of a quadratic function is

$$f(x) = a(x-h)^2 + k \tag{3.2.2}$$

The **vertex** of the quadratic function is located at (h, k), where h and k are the numbers in the transformation form of the function. Because the vertex appears in the transformation form, it is often called the **vertex form**.

In the previous example, we saw that it is possible to rewrite a quadratic function given in transformation form and rewrite it in standard form by expanding the formula. It would be useful to reverse this process since the transformation form reveals the vertex.

Expanding out the general transformation form of a quadratic gives:

$$f(x) = a(x-h)^2 + k = a(x-h)(x-h) + k$$
$$f(x) = a(x^2 - 2xh + h^2) + k = ax^2 - 2ahx + ah^2 + k$$

This should be equal to the standard form of the quadratic:

$$ax^2 - 2ahx + ah^2 + k = ax^2 + bx + c$$

The second degree terms are already equal. For the linear terms to be equal, the coefficients must be equal:

$$-2ah = b \text{ ,so } h = -\frac{b}{2a}$$

This provides us a method to determine the horizontal shift of the quadratic from the standard form. We could likewise set the constant terms equal to find:

$$ah^2 + k = c \text{ ,so } k = c - ah^2 = c - a\left(-\frac{b}{2a}\right)^2 = c - a\frac{b^2}{4a^2} = c - \frac{b^2}{4a}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $k = f(h)$.

finding the vertex of a quadratic

For a quadratic given in standard form, the vertex (h, k) is located at:

$$h = -\frac{b}{2a}, k = f(h) = f\left(\frac{-b}{2a}\right) \tag{3.2.3}$$

Example 3.2.3

Find the vertex of the quadratic $f(x) = 2x^2 - 6x + 7$. Rewrite the quadratic into transformation form (vertex form).

Solution

The horizontal coordinate of the vertex will be at

$$h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2}$$

The vertical coordinate of the vertex will be at

$$f(\frac{3}{2}) = 2(\frac{3}{2})^2 - 6(\frac{3}{2}) + 7 = \frac{5}{2}$$

Rewriting into transformation form, the stretch factor will be the same as the a in the original quadratic. Using the vertex to determine the shifts,

$$f(x) = 2(x - \frac{3}{2})^2 + \frac{5}{2}$$

Exercise 3.2.2

Given the equation $g(x) = 13 + x^2 - 6x$ write the equation in standard form and then in transformation/vertex form.

Answer

$g(x) = x^2 - 6x + 13$ in Standard form (Equation 3.2.1

Finding the vertex, $h = \frac{-(-6)}{2(1)} = 3$.

$$k = g(3) = 3^2 - 6(3) + 13 = 4$$

$g(x) = (x - 3)^2 + 4$ in Transformation form (Equation 3.2.2)

As an alternative to using a formula for finding the vertex, the equation can also be written into vertex form by **completing the square**. This process is most easily explained through example. In most cases, using the formula for finding the vertex will be quicker and easier than completing the square, but completing the square is a useful technique when faced with some other algebraic problems.

Example 3.2.4

Rewrite $f(x) = 2x^2 - 12x + 14$ into vertex form by completing the square.

Solution

We start by factoring the leading coefficient from the quadratic and linear terms.

$$2(x^2 - 6x) + 14$$

Next, we are going to add something inside the parentheses so that the quadratic inside the parentheses becomes a perfect square. In other words, we are looking for values p and q so that $(x^2 - 6x + p) = (x - q)^2$.

Notice that if multiplied out on the right, the middle term would be -2q, so q must be half of the middle term on the left; $q = -3$. In that case, p must be (-3)^2 = 9.

$$(x^2 - 6x + 9) = (x - 3)^2$$

Now, we can't just add 9 into the expression – that would change the value of the expression. In fact, adding 9 inside the parentheses actually adds 18 to the expression, since the 2 outside the parentheses will distribute. To keep the expression balanced, we can subtract 18.

$$2(x^2-6x+9)+14-18$$

Simplifying, we are left with vertex form.

$$2(x-3)^2-4$$

In addition to enabling us to more easily graph a quadratic written in standard form, finding the vertex serves another important purpose – it allows us to determine the maximum or minimum value of the function, depending on which way the graph opens.

Example 3.2.5

Returning to our backyard farmer from the beginning of the section, what dimensions should she make her garden to maximize the enclosed area?

Solution

Earlier we determined the area she could enclose with 80 feet of fencing on three sides was given by the equation $A(L)=80L-2L^2$. Notice that quadratic has been vertically reflected, since the coefficient on the squared term is negative, so the graph will open downwards, and the vertex will be a maximum value for the area.

In finding the vertex, we take care since the equation is not written in standard polynomial form with decreasing powers. But we know that a is the coefficient on the squared term, so $a=-2$, $b=80$, and $c=0$.

Finding the vertex:

$$h=-\frac{80}{2(-2)}=20, k=A(20)=80(20)-2(20)^2=800$$

The maximum value of the function is an area of 800 square feet, which occurs when $L=20$ feet. When the shorter sides are 20 feet, that leaves 40 feet of fencing for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet, and the longer side parallel to the existing fence has length 40 feet.

Example 3.2.6

A local newspaper currently has 84,000 subscribers, at a quarterly charge of $30. Market research has suggested that if they raised the price to $32, they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

Solution

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the charge per subscription times the number of subscribers. We can introduce variables, C for charge per subscription and S for the number subscribers, giving us the equation

$$\text{Revenue}=CS$$

Since the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently $S=84,000$ and $C=30$, and that if they raise the price to $32 they would lose 5,000 subscribers, giving a second pair of values, $C=32$ and $S=79,000$. From this we can find a linear equation relating the two quantities. Treating C as the input and S as the output, the equation will have form $S=mC+b$. The slope will be

$$m=\frac{79,000-84,000}{32-30}=\frac{-5,000}{2}=-2,500$$

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the vertical intercept

$$S=-2500C+b$$

Plug in the point $S=84,000$ and $C=30$

$$84,000=-2500(30)+b$$

Solve for b

$$b = 159,000$$

This gives us the linear equation $S = -2,500C + 159,000$ relating cost and subscribers. We now return to our revenue equation.

$$\text{Revenue} = CS$$

Substituting the equation for S from above

$$\text{Revenue} = C(-2,500C + 159,000)$$

Expanding

$$\text{Revenue} = -2,500C^2 + 159,000C$$

We now have a quadratic equation for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex:

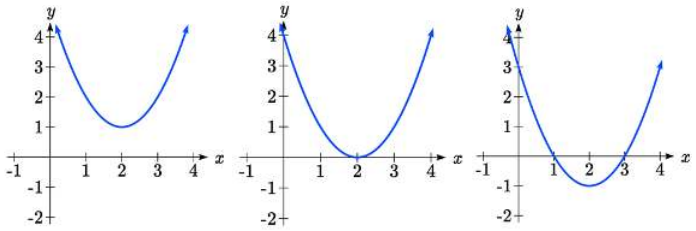
$$h = -\frac{159,000}{2(-2,500)} = 31.8$$

The model tells us that the maximum revenue will occur if the newspaper charges $31.80 for a subscription. To find what the maximum revenue is, we can evaluate the revenue equation:

$$\text{Maximum Revenue} = -2,500(31.8)^2 + 159,000(31.8) = \$2,528,100$$

Short run Behavior: Intercepts

As with any function, we can find the vertical intercepts of a quadratic by evaluating the function at an input of zero, and we can find the horizontal intercepts by solving for when the output will be zero. Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts.



Zero horizontal intercepts one horizontal intercept two horizontal intercepts

Example 3.2.7

Find the vertical and horizontal intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$

Solution

We can find the vertical intercept by evaluating the function at an input of zero:

$$f(0) = 3(0)^2 + 5(0) - 2 = -2$$

Vertical intercept at (0,-2)

For the horizontal intercepts, we solve for when the output will be zero

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution

$$0 = (3x - 1)(x + 2)$$

$$0 = 3x - 1 \quad \text{or} \quad 0 = x + 2$$
$$x = \frac{1}{3} \quad x = -2$$

Horizontal intercepts at $\left(\frac{1}{3}, 0\right)$ and (-2,0)

Notice that in the standard form of a quadratic, the constant term *c* reveals the vertical intercept of the graph.

Example 3.2.8

Find the horizontal intercepts of the quadratic $f(x) = 2x^2 + 4x - 4$

Solution

Again we will solve for when the output will be zero

$$0 = 2x^2 + 4x - 4$$

Since the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic into transformation form.

$$h = -\frac{b}{2a} = -\frac{4}{2(2)} = -1 \quad k = f(-1) = 2(-1)^2 + 4(-1) - 4 = -6$$

$$f(x) = 2(x+1)^2 - 6$$

Now we can solve for when the output will be zero

$$0 = 2(x+1)^2 - 6$$
$$6 = 2(x+1)^2$$
$$3 = (x+1)^2$$
$$x + 1 = \pm\sqrt{3}$$
$$x = -1 \pm \sqrt{3}$$

The graph has horizontal intercepts at $(-1 - \sqrt{3}, 0)$ and $(-1 + \sqrt{3}, 0)$

Exercise 3.2.3

In Try it Now problem 2 we found the standard & transformation form for the function $g(x) = 13 + x^2 - 6x$. Now find the Vertical & Horizontal intercepts (if any).

Answer

Vertical intercept at (0, 13), No horizontal intercepts since the vertex is above the x-axis and the graph opens upwards.

The process in the last example is done commonly enough that sometimes people find it easier to solve the problem once in general and remember the formula for the result, rather than repeating the process each time. Based on our previous work we showed that any quadratic in standard form can be written into transformation form as:

$$f(x) = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a} \tag{3.2.4}$$

Solving for the horizontal intercepts using this general equation gives:

$$0 = a(x + \frac{b}{2a})^2 + c - \frac{b^2}{4a}$$

start to solve for x by moving the constants to the other side

$$\frac{b^2}{4a} - c = a(x + \frac{b}{2a})^2$$

divide both sides by a

$$\frac{b^2}{4a^2} - \frac{c}{a} = (x + \frac{b}{2a})^2$$

find a common denominator to combine fractions

$$\frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

combine the fractions on the left side of the equation

$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

take the square root of both sides

$$\pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = x + \frac{b}{2a}$$

subtract $b/2a$ from both sides

$$-\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = x$$

combining the fractions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notice that this can yield two different answers for x

Quadratic Formula

For a quadratic function given in standard form $f(x) = ax^2 + bx + c$, the **quadratic formula** gives the horizontal intercepts of the graph of this function.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.2.5}$$

Example 3.2.9

A ball is thrown upwards from the top of a 40-foot-tall building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

What is the maximum height of the ball?

When does the ball hit the ground?

Solution

To find the maximum height of the ball, we would need to know the vertex of the quadratic.

$$h = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2}, k = H\left(\frac{5}{2}\right) = -16\left(\frac{5}{2}\right)^2 + 80\left(\frac{5}{2}\right) + 40 = 140$$

The ball reaches a maximum height of 140 feet after 2.5 seconds.

To find when the ball hits the ground, we need to determine when the height is zero – when $H(t) = 0$. While we could do this using the transformation form of the quadratic, we can also use the quadratic formula:

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} = \frac{-80 \pm \sqrt{8960}}{-32}$$

Since the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions:

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds.

? Exercise 3.2.4

For these two equations determine if the vertex will be a maximum value or a minimum value.

a. $g(x) = -8x + x^2 + 7$
b. $g(x) = -3(3 - x)^2 + 2$

Answer

a. Vertex is a minimum value, since $a > 0$ and the graph opens upwards

b. Vertex is a maximum value, since $a < 0$ and the graph opens downwards

Important Topics of this Section

- Quadratic functions
- Standard form
- Transformation form/Vertex form
- Vertex as a maximum / Vertex as a minimum
- Short run behavior
- Vertex / Horizontal & Vertical intercepts
- Quadratic formula

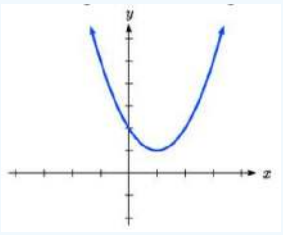
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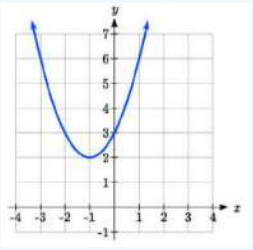
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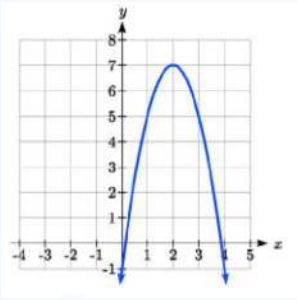
3.2E: Quadratic Functions (Exercises)

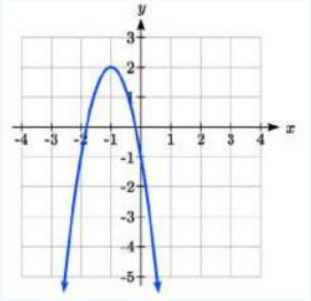
? section 3.2 exercise

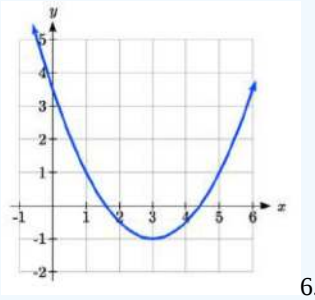
Write an equation for the quadratic function graphed.

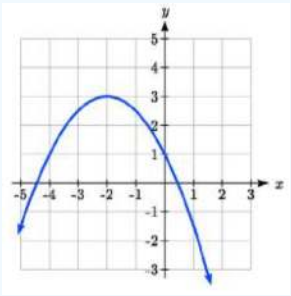
1. 

2. 

3. 

4. 

5. 

6. 

For each of the follow quadratic functions, find a) the vertex, b) the vertical intercept, and c) the horizontal intercepts.

7. $y(x) = 2x^2 + 10x + 12$

8. $z(p) = 3x^2 + 6x - 9$

9. $f(x) = 2x^2 - 10x + 4$

10. $g(x) = -2x^2 - 14x + 12$

11. $h(t) = -4t^2 + 6t - 1$

12. $k(t) = 2x^2 + 4x - 15$

Rewrite the quadratic function into vertex form.

13. $(x) = x^2 - 12x + 32$

14. $g(x) = x^2 + 2x - 3$

15. $h(x) = 2x^2 + 8x - 10$

16. $k(x) = 3x^2 - 6x - 9$

17. Find the values of b and c so $f(x) = -8x^2 + bx + c$ has vertex $2, -7)$

18. Find the values of b and c so $f(x) = 6x^2 + bx + c$ has vertex $(7, -9)$

Write an equation for a quadratic with the given features

19. x-intercepts (-3, 0) and (1, 0), and y intercept (0, 2)

20. x-intercepts (2, 0) and (-5, 0), and y intercept (0, 3)

21. x-intercepts (2, 0) and (5, 0), and y intercept (0, 6)

22. x-intercepts (1, 0) and (3, 0), and y intercept (0, 4)

23. Vertex at (4, 0), and y intercept (0, -4)

24. Vertex at (5, 6), and y intercept (0, -1)

25. Vertex at (-3, 2), and passing through (3, -2)

26. Vertex at (1, -3), and passing through (-2, 3)

27. A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$.

a. From what height was the rocket launched?
b. How high above sea level does the rocket reach its peak?
c. Assuming the rocket will splash down in the ocean, at what time does splashdown occur?

28. A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$.

a. From what height was the ball thrown?
b. How high above ground does the ball reach its peak?
c. When does the ball hit the ground?

29. The height of a ball thrown in the air is given by $h(x) = -\frac{1}{12}x^2 + 6x + 3$, where x is the horizontal distance in feet from the point at which the ball is thrown.

a. How high is the ball when it was thrown?
b. What is the maximum height of the ball?
c. How far from the thrower does the ball strike the ground?

30. A javelin is thrown in the air. Its height is given by $h(x) = -\frac{1}{20}x^2 + 8x + 6$, where x is the horizontal distance in feet from the point at which the javelin is thrown.

a. How high is the javelin when it was thrown?
b. What is the maximum height of the javelin?
c. How far from the thrower does the javelin strike the ground?

31. A box with a square base and no top is to be made from a square piece of cardboard by cutting 6 in. squares out of each corner and folding up the sides. The box needs to hold 1000 in^3. How big a piece of cardboard is needed?

32. A box with a square base and no top is to be made from a square piece of cardboard by cutting 4 in. squares out of each corner and folding up the sides. The box needs to hold 2700 in^3. How big a piece of cardboard is needed?

33. A farmer wishes to enclose two pens with fencing, as shown. If the farmer has 500 feet of fencing to work with, what dimensions will maximize the area enclosed?



34. A farmer wishes to enclose three pens with fencing, as shown. If the farmer has 700 feet of fencing to work with, what dimensions will maximize the area enclosed?



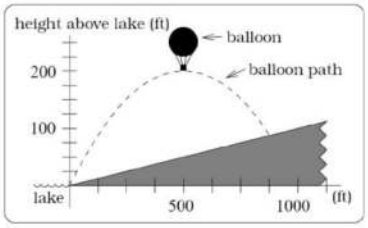
35. You have a wire that is 56 cm long. You wish to cut it into two pieces. One piece will be bent into the shape of a square. The other piece will be bent into the shape of a circle. Let A represent the total area enclosed by the square and the circle. What is the circumference of the circle when A is a minimum?

36. You have a wire that is 71 cm long. You wish to cut it into two pieces. One piece will be bent into the shape of a right triangle with legs of equal length. The other piece will be bent into the shape of a circle. Let A represent the total area enclosed by the triangle and the circle. What is the circumference of the circle when A is a minimum?

37. A soccer stadium holds 62,000 spectators. With a ticket price of $11, the average attendance has been 26,000. When the price dropped to $9, the average attendance rose to 31,000. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

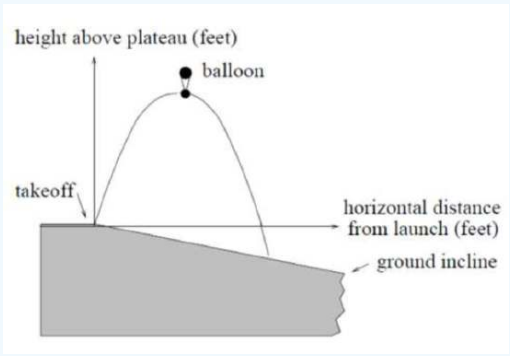
38. A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

39. A hot air balloon takes off from the edge of a mountain lake. Impose a coordinate system as pictured and assume that the path of the balloon follows the graph of $f(x) = -\frac{2}{2500}x^2 + \frac{4}{5}x$. The land rises at a constant incline from the lake at the rate of 2 vertical feet for each 20 horizontal feet. [UW]



a. What is the maximum height of the balloon above water level?
b. What is the maximum height of the balloon above ground level?
c. Where does the balloon land on the ground?
d. Where is the balloon 50 feet above the ground?

40. A hot air balloon takes off from the edge of a plateau. Impose a coordinate system as pictured below and assume that the path the balloon follows is the graph of the quadratic function $f(x) = -\frac{4}{2500}x^2 + \frac{4}{5}x$. The land drops at a constant incline from the plateau at the rate of 1 vertical foot for each 5 horizontal feet. [UW]



a. What is the maximum height of the balloon above plateau level?
b. What is the maximum height of the balloon above ground level?
c. Where does the balloon land on the ground?
d. Where is the balloon 50 feet above the ground?

Answer

1. $f(x) = (x-2)^2 - 3$
3. $f(x) = -2(x-2)^2 + 7$
5. $f(x) = \frac{1}{2}(x-3)^2 - 1$

	Vertex	Vertical Intercept	Horizontal Intercepts
7.	(-2.5, -0.5)	(0, 12)	(-2, 0). (-3, 0)
9.	(2.5, -8.5)	(0, 4)	(0.438, 0). (4.562, 0)
11.	(0.75, 1.25)	(0, -1)	(0.191, 0). (1.309, 0)

13. $f(x) = (x-6)^2 - 4$
15. $f(x_=2(x+2)^2 - 18$
17. $b = 32$ and $c = -39$
19. $f(x) - -\frac{2}{3}(x+3)(x-1)$
21. $f(x) = \frac{3}{5}(x-2)(x-5)$
23. $f(x) = -\frac{1}{4}(x-4)^2$
25. $f(x) = -\frac{1}{9}(x+3)^2 + 2$

27a. 234m
b. 2909.561 ft
c. 47.735 seconds

29a. 3 ft
b. 111 ft
c. 72.497 ft

31. 24.91 in by 24.91 in

33. 125 ft by $83\frac{1}{3}ft$

35. 24.6344 cm

37. $10.70

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3.3: Graphs of Polynomial Functions

In the previous section, we explored the short run behavior of quadratics, a special case of polynomials. In this section, we will explore the short run behavior of polynomials in general.

Short run Behavior: Intercepts

As with any function, the vertical intercept can be found by evaluating the function at an input of zero. Since this is evaluation, it is relatively easy to do it for a polynomial of any degree.

To find horizontal intercepts, we need to solve for when the output will be zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and 4 th degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases:

1. The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
2. The polynomial is given in factored form.
3. Technology is used to determine the intercepts.

Other techniques for finding the intercepts of general polynomials will be explored in the next section.

Example 3.3.1

Find the horizontal intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

$$x^6 - 3x^4 + 2x^2 = 0$$

Factoring out the greatest common factor

$$x^2(x^4 - 3x^2 + 2) = 0$$

Factoring the inside as a quadratic in x^2

$$x^2(x^2 - 1)(x^2 - 2) = 0$$

Then break apart to find solutions

$$\begin{array}{lll} x^2 = 0 & (x^2 - 1) = 0 & (x^2 - 2) = 0 \\ x = 0 \text{ or } & x^2 = 1 \text{ or } & x^2 = 2 \\ & x = \pm 1 & x = \pm\sqrt{2} \end{array}$$

This gives us 5 horizontal intercepts.

Example 3.3.2

Find the vertical and horizontal intercepts of $g(t) = (t-2)^2(2t+3)$

Solution

The vertical intercept can be found by evaluating $g(0)$.

$$g(0) = (0-2)^2(2(0)+3) = 12$$

The horizontal intercepts can be found by solving $g(t) = 0$

$$(t-2)^2(2t+3) = 0$$

Since this is already factored, we can break it apart:

$$(t-2)^2=0 \quad (2t+3)=0$$
$$t-2=0 \quad \text{or} \quad t=\frac{-3}{2}$$
$$t=2$$

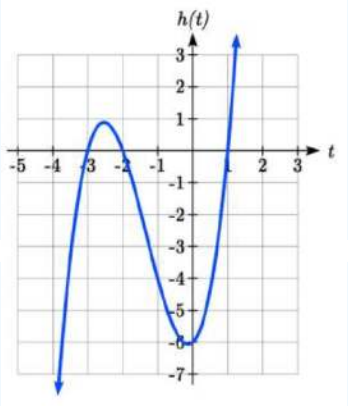
We can always check our answers are reasonable by graphing the polynomial.

Example 3.3.3

Find the horizontal intercepts of $h(t)=t^3+4t^2+t-6$

Solution

Since this polynomial is not in factored form, has no common factors, and does not appear to be factorable using techniques we know, we can turn to technology to find the intercepts.



Graphing this function, it appears there are horizontal intercepts at t = -3, -2, and 1.

We could check these are correct by plugging in these values for t and verifying that $h(-3)=h(-2)=h(1)=0$.

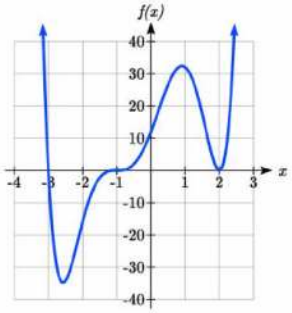
Exercise 3.3.1

Find the vertical and horizontal intercepts of the function $f(t)=t^4-4t^2$.

Answer

Vertical intercept (0, 0). $0=t^4-4t^2$ factors as $0=t^2\left(t^2-4\right)=t^2(t-2)(t+2)$ Horizontal intercepts (0, 0), (-2, 0), (2, 0).

Graphical Behavior at Intercepts



If we graph the function $f(x)=(x+3)(x-2)^2(x+1)^3$, notice that the behavior at each of the horizontal intercepts is different.

At the horizontal intercept x = -3, coming from the $(x+3)$ factor of the polynomial, the graph passes directly through the horizontal intercept.

The factor $(x+3)$ is linear (has a power of 1), so the behavior near the intercept is like that of a line - it passes directly through the intercept. We call this a single zero, since the zero corresponds to a single factor of the function.

At the horizontal intercept $x=2$, coming from the $(x-2)^2$ factor of the polynomial, the graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic – it bounces off the horizontal axis at the intercept. Since $(x-2)^2=(x-2)(x-2)$, the factor is repeated twice, so we call this a double zero. We could also say the zero has **multiplicity** 2.

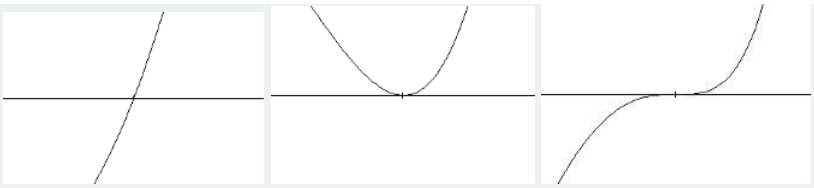
At the horizontal intercept $x=-1$, coming from the $(x+1)^3$ factor of the polynomial, the graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic, with the same "S" type shape near the intercept that the toolkit x^3 has. We call this a triple zero. We could also say the zero has multiplicity 3.

By utilizing these behaviors, we can sketch a reasonable graph of a factored polynomial function without needing technology.

graphical behavior of polynomials at horizontal intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the horizontal intercept h is determined by the power on the factor.

$p=1$ $p=2$ $p=3$



Single Zero Multiplicity 1 Double Zero Multiplicity 2 Triple Zero Multiplicity 3

For higher even powers 4,6,8 etc.… the graph will still bounce off the horizontal axis but the graph will appear flatter with each increasing even power as it approaches and leaves the axis.

For higher odd powers, 5,7,9 etc… the graph will still pass through the horizontal axis but the graph will appear flatter with each increasing odd power as it approaches and leaves the axis.

Example 3.3.4

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution

This graph has two horizontal intercepts. At $x = -3$, the factor is squared, indicating the graph will bounce at this horizontal intercept. At $x = 5$, the factor is not squared, indicating the graph will pass through the axis at this intercept.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the long-run behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs get large positive, and the inputs increasing as the inputs get large negative.

To sketch this we consider the following:

As $x \to -\infty$ the function $f(x) \to \infty$ so we know the graph starts in the 2nd quadrant and is decreasing toward the horizontal axis.

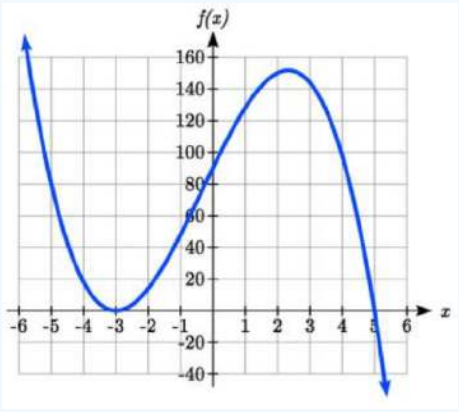
At (-3, 0) the graph bounces off the horizontal axis and so the function must start increasing.

At (0, 90) the graph crosses the vertical axis at the vertical intercept.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis since the graph passes through the next intercept at (5,0).

As $x \to \infty$ the function $f(x) \to -\infty$ so we know the graph continues to decrease and we can stop drawing the graph in the 4th quadrant.

Using technology we can verify the shape of the graph.



Exercise 3.3.2

Given the function $g(x) = x^3 - x^2 - 6x$ use the methods that we have learned so far to find the vertical & horizontal intercepts, determine where the function is negative and positive, describe the long run behavior and sketch the graph without technology.

Answer

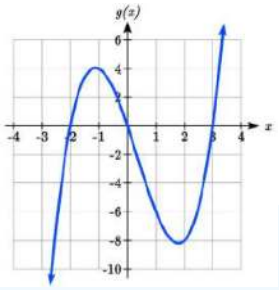
Vertical intercept (0, 0), Horizontal intercepts (-2, 0), (0, 0), (3, 0)

The function is negative on $(-\infty$, -2) and (0, 3)

The function is positive on (-2, 0) and (3,∞)

The leading term is x^3 so as $x \to -\infty$, $g(x) \to -\infty$ and as $x \to \infty$, $g(x) \to \infty$

Solving Polynomial Inequalities



One application of our ability to find intercepts and sketch a graph of polynomials is the ability to solve polynomial inequalities. It is a very common question to ask when a function will be positive and negative. We can solve polynomial inequalities by either utilizing the graph, or by using test values.

Example 3.3.5

Solve $(x+3)(x+1)^2(x-4)>0$

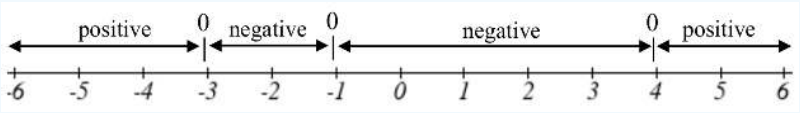
Solution

As with all inequalities, we start by solving the equality $(x+3)(x+1)^2(x-4)=0$, which has solutions at *x* = -3, -1, and 4. We know the function can only change from positive to negative at these values, so these divide the inputs into 4 intervals.

We could choose a test value in each interval and evaluate the function $f(x)=(x+3)(x+1)^2(x-4)$ at each test value to determine if the function is positive or negative in that interval

Interval	Test x in interval	f (test value)	> 0 or < 0?
$x<-3$	-4	72	> 0
$-3<x<-1$	-2	-6	< 0
$-1<x<4$	0	-12	< 0
$x>4$	5	288	> 0

On a number line this would look like:



From our test values, we can determine this function is positive when $x<-3$ or $x>4$, or in interval notation, $(-\infty,-3)\bigcup(4,\infty)$

We could have also determined on which intervals the function was positive by sketching a graph of the function. We illustrate that technique in the next example

Example 3.3.6

Find the domain of the function $v(t)=\sqrt{6-5t-t^2}$.

Solution

A square root is only defined when the quantity we are taking the square root of, the quantity inside the square root, is zero or greater. Thus, the domain of this function will be when $6-5t-t^2\geq 0$.

We start by solving the equality $6-5t-t^2=0$. While we could use the quadratic formula, this equation factors nicely to $(6+t)(1-t)=0$, giving horizontal intercepts $t=1$ and $t=-6$.

Sketching a graph of this quadratic will allow us to determine when it is positive.

From the graph we can see this function is positive for inputs between the intercepts. So $6-5t-t^2\geq 0$ for $-6\leq t\leq 1$, and this will be the domain of the $v(t)$ function.

Writing Equations using Intercepts

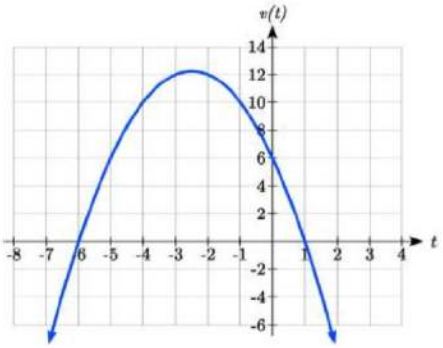
Since a polynomial function written in factored form will have a horizontal intercept where each factor is equal to zero, we can form a function that will pass through a set of horizontal intercepts by introducing a corresponding set of factors.

factored form of polynomials

If a polynomial has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, then the polynomial can be written in the factored form

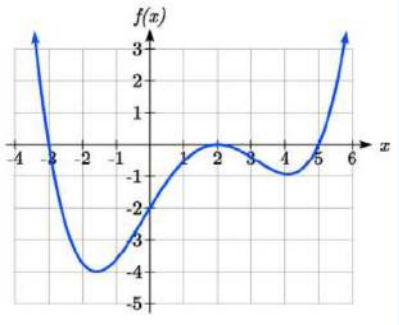
$$f(x) = a(x - x_1)^{p_1}(x - x_2)^{p_2} \cdots (x - x_n)^{p_n} \qquad (3.3.1)$$

where the powers p_i on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the horizontal intercept.



Example 3.3.7

Write a formula for the polynomial function graphed here.



Solution

This graph has three horizontal intercepts: x = -3, 2, and 5. At x = -3 and 5 the graph passes through the axis, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$ the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be 2nd degree (quadratic).

Together, this gives us:

$$f(x) = a(x+3)(x-2)^2(x-5)$$

To determine the stretch factor, we can utilize another point on the graph. Here, the vertical intercept appears to be (0,-2), so we can plug in those values to solve for a:

$$-2 = a(0+3)(0-2)^2(0-5)$$
$$-2 = -60a$$
$$a = \frac{1}{30}$$

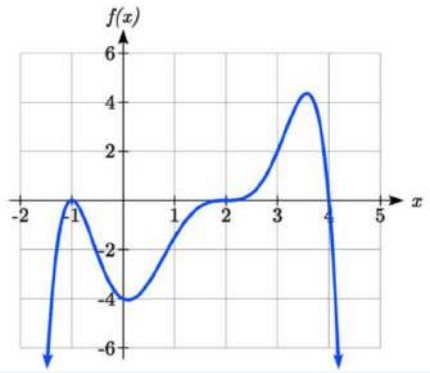
The graphed polynomial appears to represent the function

$$f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$$

Exercise 3.3.3

Given the graph, write a formula for the function shown.

https://math.libretexts.org/@go/page/127025



Answer

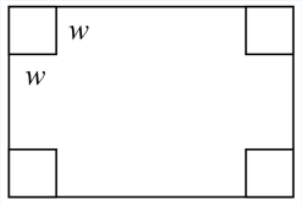
Double zero at $x=-1$, triple zero at $x=2$. Single zero at $x=4$. $f(x)=a(x-2)^3(x+1)^2(x-4)$. Substituting (0,-4) and solving for a, $f(x)=-\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

Estimating Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Example 3.3.8

An open-top box is to be constructed by cutting out squares from each corner of a 14cm by 20cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.



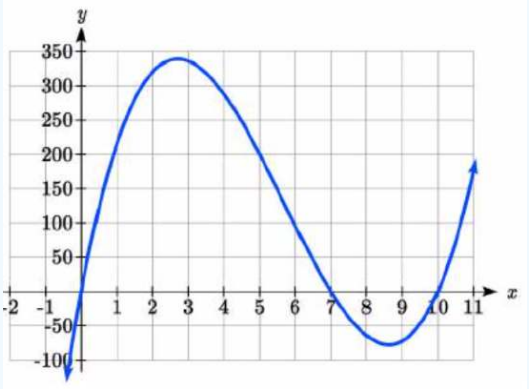
Solution

We will start this problem by drawing a picture, labeling the width of the cut-out squares with a variable, w.

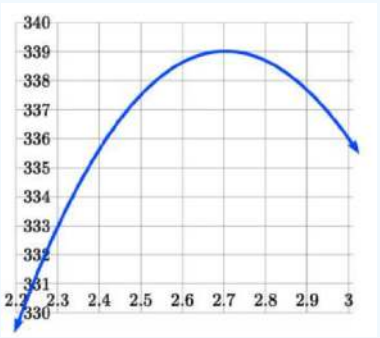
Notice that after a square is cut out from each end, it leaves a $(14-2w)$ cm by $(120-2w)$ cm rectangle for the base of the box, and the box will be w cm tall. This gives the volume:

$$V(w)=(14-2w)(20-2w)w=280w-68w^2+4w^3$$

Using technology to sketch a graph allows us to estimate the maximum value for the volume, restricted to reasonable values for w: values from 0 to 7.



From this graph, we can estimate the maximum value is around 340, and occurs when the squares are about 2.75cm square. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph.



From this zoomed-in view, we can refine our estimate for the max volume to about 339, when the squares are 2.7cm square.

? Exercise 3.3.4

Use technology to find the maximum and minimum values on the interval [-1, 4] of the function $f(x) = -0.2(x-2)^3(x+1)^2(x-4)$.

Answer

The minimum occurs at approximately the point (0, -6.5), and the maximum occurs at approximately the point (3.5, 7).

Important Topics of this Section

- Short Run Behavior
- Intercepts (Horizontal & Vertical)
- Methods to find Horizontal intercepts
- Factoring Methods
- Factored Forms
- Technology
- Graphical Behavior at intercepts
- Single, Double and Triple zeros (or multiplicity 1, 2, and 3 behaviors)
- Solving polynomial inequalities using test values & graphing techniques
- Writing equations using intercepts
- Estimating extrema

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3.3E: Graphs of Polynomial Functions (Exercises)

? section 3.3 exercise

Find the C and t intercepts of each function.

1. $C(t) = 2(t-4)(t+1)(t-6)$
2. $C(t) = 3(t+2)(t-3)(t+5)$
3. $C(t) = 4t(t-2)^2(t+1)$
4. $C(t) = 2t(t-3)(t+1)^2$
5. $C(t) = 2t^4 - 8t^3 + 6t^2$
6. $C(t) = 4t^4 + 12t^3 - 40t^2$

Use your calculator or other graphing technology to solve graphically for the zeros of the function.

7. $f(x) = x^3 - 7x^2 + 4x + 30$
8. $g(x) = x^3 - 6x^2 + x + 28$

Find the long run behavior of each function as $t \to \infty$ and $t \to -\infty$

9. $h(t) = 3(t-5)^3(t-3)^3(t-2)$
10. $k(t) = 2(t-3)^2(t+1)^3(t+2)$
11. $p(t) = -2t(t-1)(3-t)^2$
12. $q(t) = -4t(2-t)(t+1)^3$

Sketch a graph of each equation.

13. $f(x) = (x+3)^2(x-2)$
14. $g(x) = (x+4)(x-1)^2$
15. $h(x) = (x-1)^3(x+3)^2$
16. $k(x) = (x-3)^3(x-2)^2$
17. $m(x) = -2x(x-1)(x+3)$
18. $n(x) = -3x(x+2)(x-4)$

Solve each inequality.

19. $(x-3)(x-2)^2 > 0$
20. $(x-5)(x+1)^2 > 0$
21. $(x-1)(x+2)(x-3) < 0$
22. $(x-4)(x+3)(x+6) < 0$

Find the domain of each function.

23. $f(x) = \sqrt{-42 + 19x - 2x^2}$
24. $g(x) = \sqrt{28 - 17x - 3x^2}$
25. $h(x) = \sqrt{4 - 5x + x^2}$
26. $k(x) = \sqrt{2 + 7x + 3x^2}$
27. $n(x) = \sqrt{(x-3)(x+2)^2}$

28. $m(x) = \sqrt{(x-1)^2(x+3)}$

29. $p(t) = \dfrac{1}{t^2+2t-8}$

30. $q(t) = \dfrac{4}{x^2-4x-5}$

Write an equation for a polynomial the given features.

31. Degree 3. Zeros at x = -2, x = 1, and x = 3. Vertical intercept at (0, -4)

32. Degree 3. Zeros at x = -5, x = -2, and x = 1. Vertical intercept at (0, 6)

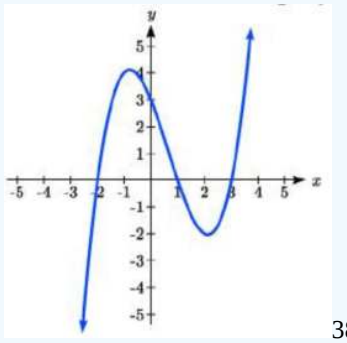
33. Degree 5. Roots of multiplicity 2 at x = 3 and x = 1, and a root of multiplicity 1 at x = -3. Vertical intercept at (0, 9)

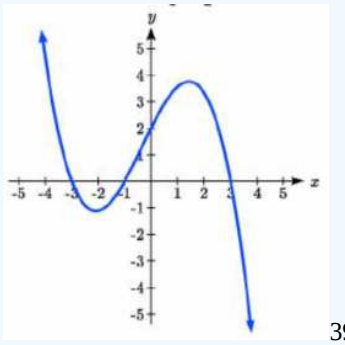
34. Degree 4. Root of multiplicity 2 at x = 4, and a roots of multiplicity 1 at x = 1 and x = -2. Vertical intercept at (0, -3)

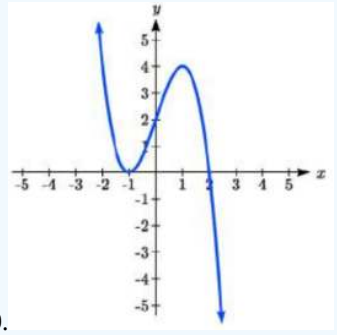
35. Degree 5. Double zero at x = 1, and triple zero at x = 3. Passes through the point (2, 15)

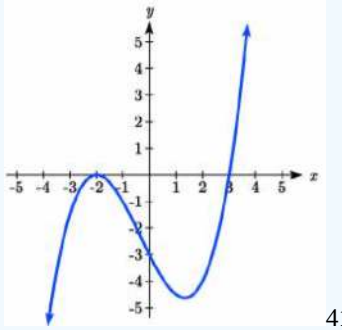
36. Degree 5. Single zero at x = -2 and x = 3, and triple zero at x = 1. Passes through the point (2, 4)

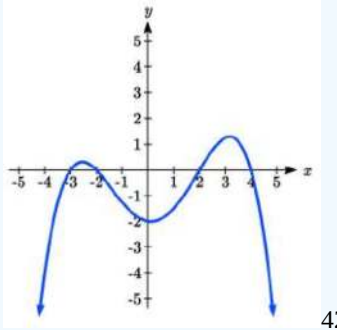
Write a formula for each polynomial function graphed.

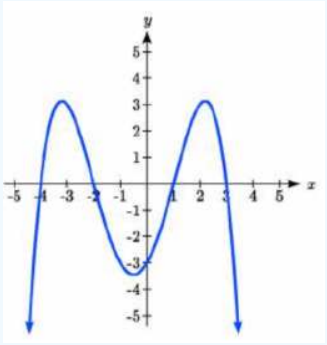
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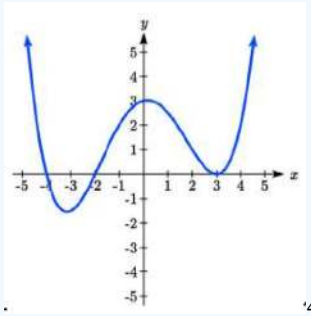
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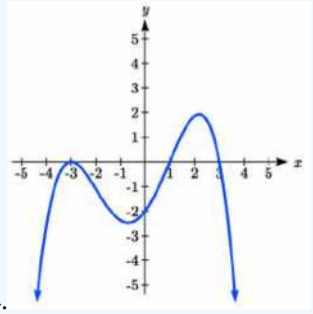
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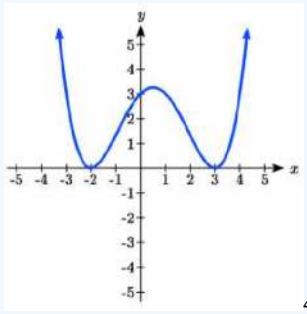
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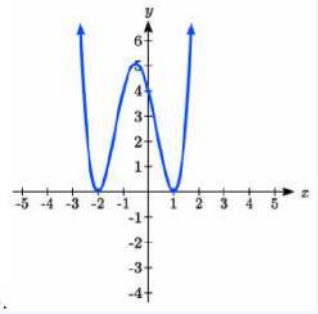
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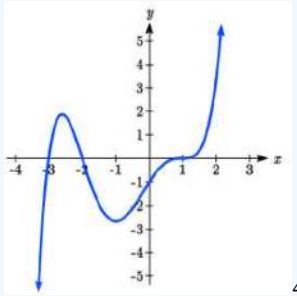
43. 

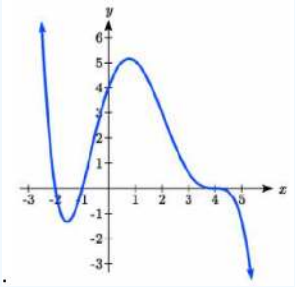
44. 

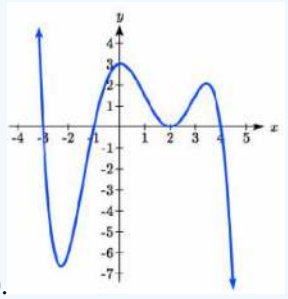
Write a formula for each polynomial function graphed.

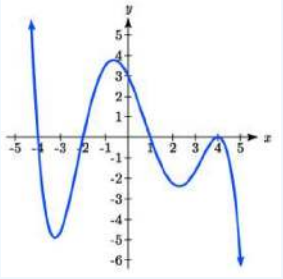
45. 

46. 

47. 

48. 

49. 

50. 

51. A rectangle is inscribed with its base on the x axis and its upper corners on the parabola $y = 5 - x^2$. What are the dimensions of such a rectangle that has the greatest possible area?

52. A rectangle is inscribed with its base on the x axis and its upper corners on the curve $y = 16 - x^4$. What are the dimensions of such a rectangle that has the greatest possible area?

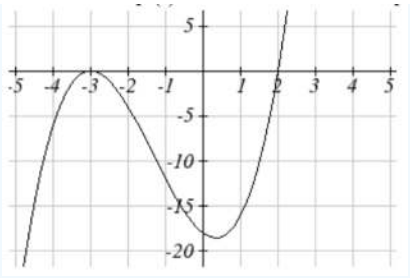
Answer

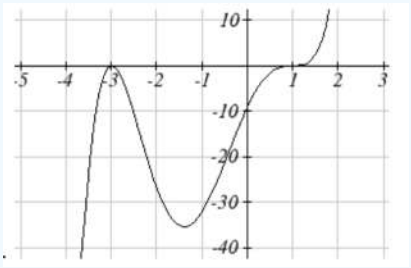
$C(t)$	C, intercepts	t, intercepts
1.	(0, 48)	(4, 0), (-1, 0), (6, 0)
3.	(0, 0)	(0, 0), (2, 0), (-1, 0)
5.	(0, 0)	(0, 0), (1, 0), (3, 0)

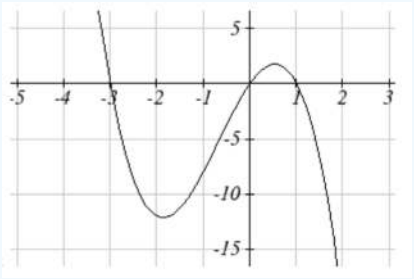
7. (-1.646, 0) (3.646, 0) (5, 0)

9. As $t \to \infty$, $h(t) \to \infty$ $t \to -\infty$, $h(t) \to -\infty$

11. As $t \to \infty$, $p(t) \to -\infty$ $t \to -\infty$, $p(t) \to -\infty$

13. 

15. 

17. 

19. $(3, \infty)$

21. $(-\infty, -2) \cup (1, 3)$

23. [3, 5, 6]

25. $(-\infty, 1] \cup [4, \infty)$

27. $[-2, -2] \cup [3, \infty)$

29. $(-\infty, -4) \cup (-4, 2) \cup (2, \infty)$

31. $y = -\frac{2}{3}(x+2)(x-1)(x-3)$

33. $y = \frac{1}{3}(x-1)^2(x-3)^2(x+3)$

35. $y = -15(x-1)^2(x-3)^2$

37. $y = \frac{1}{2}(x+2)(x-1)(x-3)$

39. $y = -(x+1)^2(x-2)$

41. $y = -\frac{1}{24}(x+3)(x+2)(x-2)(x-4)$

43. $y = \frac{1}{24}(x+4)(x+2)(x-3)^2$

45. $y = \frac{1}{12}(x+2)^2(x-3)^2$

47. $y = \frac{1}{6}(x+3)(x+2)(x-1)^3$

49. $y = -\frac{1}{16}(x+3)(x+1)(x-2)^2(x-4)$

51. Base 2.58, Height 3.336

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3.4: Factor Theorem and Remainder Theorem

In the last section, we limited ourselves to finding the intercepts, or zeros, of polynomials that factored simply, or we turned to technology. In this section, we will look at algebraic techniques for finding the zeros of polynomials like $h(t) = t^3 + 4t^2 + t - 6$.

Long Division

In the last section we saw that we could write a polynomial as a product of factors, each corresponding to a horizontal intercept. If we knew that $x = 2$ was an intercept of the polynomial $x^3 + 4x^2 - 5x - 14$, we might guess that the polynomial could be factored as $x^3 + 4x^2 - 5x - 14 = (x - 2)$ (something). To find that "something," we can use polynomial division.

Example 3.4.1

Divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$.

Solution

Start by writing the problem out in long division form

$$x-2\overline{)x^3+4x^2-5x-14}$$

Now we divide the leading terms: $x^3 \div x = x^2$. It is best to align it above the same-powered term in the dividend. Now, multiply that x^2 by $x - 2$ and write the result below the dividend.

$$\begin{array}{r} x^2 \\ x-2\overline{)x^3+4x^2-5x-14} \\ x^3-2x^2 \end{array}$$

Now subtract that expression from the dividend.

$$\begin{array}{r} x^2 \\ x-2\overline{)x^3+4x^2-5x-14} \\ \underline{-\left(x^3-2x^2\right)} \\ 6x^2-5x-14 \end{array}$$

Again, divide the leading term of the remainder by the leading term of the divisor. $6x^2 \div x = 6x$. We add this to the result, multiply 6*x* by $x - 2$, and subtract.

$$\begin{array}{r} x^2+6x \\ x-2\overline{)x^3+4x^2-5x-14} \\ \underline{-\left(x^3-2x^2\right)} \\ 6x^2-5x-14 \\ \underline{-\left(6x^2-12x\right)} \\ 7x-14 \end{array}$$

Repeat the process one last time.

$$\begin{array}{r} x^2+6x+7 \\ x-2\overline{)x^3+4x^2-5x-14} \\ \underline{-\left(x^3-2x^2\right)} \\ 6x^2-5x-14 \\ \underline{-\left(6x^2-12x\right)} \\ 7x-14 \\ \underline{-(7x-14)} \\ 0 \end{array}$$

This tells us $x^3+4x^2-5x-14$ divided by $x-2$ is x^2+6x+7, with a remainder of zero. This also means that we can factor $x^3+4x^2-5x-14$ as $(x-2)\left(x^2+6x+7\right)$.

This gives us a way to find the intercepts of this polynomial.

Example 3.4.2

Find the horizontal intercepts of $h(x)=x^3+4x^2-5x-14$.

Solution

To find the horizontal intercepts, we need to solve $h(x)=0$. From the previous example, we know the function can be factored as $h(x)=(x-2)\left(x^2+6x+7\right)$.

$h(x)=(x-2)\left(x^2+6x+7\right)=0$ when $x=2$ or when $x^2+6x+7=0$. This doesn't factor nicely, but we could use the quadratic formula to find the remaining two zeros.

$$x=\frac{-6\pm\sqrt{6^2-4(1)(7)}}{2(1)}=-3\pm\sqrt{2}$$

The horizontal intercepts will be at $(2,0)$, $\left(-3-\sqrt{2},0\right)$, and $\left(-3+\sqrt{2},0\right)$.

Exercise 3.4.1

Divide $2x^3-7x+3$ by $x+3$ using long division.

Answer

$$\begin{array}{r} 2x^2-6x+11 \\ x+3\overline{)\ 2x^3+0x^2-7x+3} \\ \underline{-\left(2x^3+6x^2\right)} \\ -6x^2-7x+3 \\ \underline{-\left(-6x^2-18x\right)} \\ 11x+3 \\ \underline{-(11x+33)} \\ -30 \end{array}$$

The quotient is $2x^2-6x+11$ with remainder -30.

The Factor and Remainder Theorems

When we divide a polynomial, $p(x)$ by some divisor polynomial $d(x)$, we will get a quotient polynomial $q(x)$ and possibly a remainder $r(x)$. In other words,

$$p(x)=d(x)q(x)+r(x) \tag{3.4.1}$$

Because of the division, the remainder will either be zero, or a polynomial of lower degree than d(x). Because of this, if we divide a polynomial by a term of the form $x-c$, then the remainder will be zero or a constant.

If $p(x)=(x-c)q(x)+r$, then $p(c)=(c-c)q(c)+r=0+r=r$, which establishes the Remainder Theorem.

The Remainder Theorem

If $p(x)$ is a polynomial of degree 1 or greater and c is a real number, then when p(x) is divided by $x-c$, the remainder is $p(c)$.

If $x-c$ is a factor of the polynomial p, then $p(x)=(x-c)q(x)$ for some polynomial q. Then $p(c)=(c-c)q(c)=0$, showing c is a zero of the polynomial. This shouldn't surprise us - we already knew that if the polynomial factors it reveals the roots.

If $p(c)=0$, then the remainder theorem tells us that if p is divided by $x-c$, then the remainder will be zero, which means $x-c$ is a factor of p.

the factor theorem

If $p(x)$ is a nonzero polynomial, then the real number c is a zero of $p(x)$ if and only if $x-c$ is a factor of $p(x)$.

Synthetic Division

Since dividing by $x-c$ is a way to check if a number is a zero of the polynomial, it would be nice to have a faster way to divide by $x-c$ than having to use long division every time. Happily, quicker ways have been discovered.

Let's look back at the long division we did in Example 1 and try to streamline it. First, let's change all the subtractions into additions by distributing through the negatives.

$$\begin{array}{r} x^2+6x+7 \\ x-2\overline{)\ x^3+4x^2-5x-14} \\ \underline{-x^3+2x^2} \\ 6x^2-5x-14 \\ \underline{-6x^2+12x} \\ 7x-14 \\ \underline{-7x+14} \\ 0 \end{array}$$

Next, observe that the terms $-x^3$, $-6x^2$, and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we can omit them without losing any information. Also note that the terms we 'bring down' (namely the -5x and -14) aren't really necessary to recopy, so we omit them, too.

$$\begin{array}{r} x^2+6x+7 \\ x-2\overline{)\ x^3+4x^2-5x-14} \\ \underline{\quad 2x^2} \\ 6x^2 \\ \underline{\quad\ 12x} \\ 7x \\ \underline{\quad\ 14} \\ 0 \end{array}$$

Now, let's move things up a bit and, for reasons which will become clear in a moment, copy the x^3 into the last row.

$$\begin{array}{r} x^2+6x+7 \\ x-2\overline{)x^3+4x^2-5x-14} \\ \underline{\quad 2x^2\ 12x\ \ 14} \\ x^3\quad 6x^2\ \ 7x\quad 0 \end{array}$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial.

This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$
\begin{array}{r|l}
x-2 & x^3+4x^2-5x-14 \\
 & \underline{\quad 2x^2\ 12x\ \ 14} \\
 & x^3\ \ 6x^2\ \ 7x\ \ \ 0
\end{array}
$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, 6x and 7, respectively, by the -2 in $x-2$, then by -1 when we changed the subtraction to addition. Multiplying by -2 then by -1 is the same as multiplying by 2, so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

We have constructed a **synthetic division** tableau for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in for the dividend. Then "bring down" the first coefficient of the dividend.

$$
\begin{array}{r|cccc}
2 & 1 & 4 & -5 & -14 \\
\hline
 & & & &
\end{array}
\qquad
\begin{array}{r|cccc}
2 & 1 & 4 & -5 & -14 \\
 & \downarrow & & & \\
\hline
 & 1 & & &
\end{array}
$$

Next, take the 2 from the divisor and multiply by the 1 that was "brought down" to get 2. Write this underneath the 4, then add to get 6.

$$
\begin{array}{r|cccc}
2 & 1 & 4 & -5 & -14 \\
 & \downarrow & 2 & & \\
\hline
 & 1 & & &
\end{array}
\qquad
\begin{array}{r|cccc}
2 & 1 & 4 & -5 & -14 \\
 & \downarrow & 2 & & \\
\hline
 & 1 & 6 & &
\end{array}
$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$
\begin{array}{r|cccc}
2 & 1 & 4 & -5 & -14 \\
 & \downarrow & 2 & 12 & \\
\hline
 & 1 & 6 & &
\end{array}
\qquad
\begin{array}{r|cccc}
2 & 1 & 4 & -5 & -14 \\
 & \downarrow & 2 & 12 & \\
\hline
 & 1 & 6 & 7 &
\end{array}
$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$
\begin{array}{r|cccc}
2 & 1 & 4 & -5 & -14 \\
 & \downarrow & 2 & 12 & 14 \\
\hline
 & 1 & 6 & 7 &
\end{array}
\qquad
\begin{array}{r|cccc}
2 & 1 & 4 & -5 & -14 \\
 & \downarrow & 2 & 12 & 14 \\
\hline
 & 1 & 6 & 7 & \boxed{0}
\end{array}
$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient is a second degree polynomial. Hence the quotient is x^2+6x+7. The number in the box is the remainder. Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x-c$. It is important to note that it works only for these kinds of divisors. Also take note that when a polynomial (of degree at least 1) is divided by $x-c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division.

Example 3.4.3

Use synthetic division to divide $5x^3 - 2x^2 + 1$ by $x - 3$.

Solution

When setting up the synthetic division tableau, we need to enter 0 for the coefficient of x in the dividend. Doing so gives

3	5	-2	0	1
	↓	15	39	117
	5	13	39	118

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$. This means

$$5x^3 - 2x^2 + 1 = (x - 3)(5x^2 + 13x + 39) + 118$$

It also means that $x - 3$ is not a factor of $5x^3 - 2x^2 + 1$.

Example 3.4.4

Divide $x^3 + 8$ by $x + 2$.

Solution

For this division, we rewrite $x + 2$ as $x - (-2)$ and proceed as before.

-2	1	0	0	8
	↓	-2	4	-8
	1	-2	4	0

The quotient is $x^2 - 2x + 4$ and the remainder is zero. Since the remainder is zero, $x + 2$ is a factor of $x^3 + 8$.

$$x^3 + 8 = (x + 2)\left(x^2 - 2x + 4\right)$$

Exercise 3.4.2

Divide $4x^4 - 8x^2 - 5x$ by $x - 3$ using synthetic division.

Answer

3	4	0	-8	-5	0
	↓	12	36	84	237
	4	12	28	79	237

$4x^4 - 8x^2 - 5x$ divided by $x - 3$ is $4x^3 + 12x^2 + 28x + 79$ with remainder 237

Using this process allows us to find the real zeros of polynomials, presuming we can figure out at least one root. We'll explore how to do that in the next section.

Example 3.4.5

The polynomial $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$ has a horizontal intercept at $x = \frac{1}{2}$ with multiplicity 2. Find the other intercepts of $p(x)$.

Solution

Since $x = \frac{1}{2}$ is an intercept with multiplicity 2, then $x - \frac{1}{2}$ is a factor twice. Use synthetic division to divide by $x - \frac{1}{2}$ twice.

1/2	4	-4	-11	12	-3
	↓	2	-1	-6	3
	4	-2	-1	-6	0
1/2	4	-2	-1	-6	
	↓	2	0	-6	
	4	0	-12	0	

From the first division, we get $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(4x^3 - 2x^2 - x - 6\right)$ The second division tells us

$$4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(4x^2 - 12\right)$$

To find the remaining intercepts, we set $4x^2 - 12 = 0$ and get $x = \pm\sqrt{3}$.

Note this also means $4x^4 - 4x^3 - 11x^2 + 12x - 3 = 4\left(x - \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$.

Important Topics of this Section

- Long division of polynomials
- Remainder Theorem
- Factor Theorem
- Synthetic division of polynomials

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3.4E: Factor Theorem and Remainder Theorem (Exercises)

? section 3.4 exercise

Use polynomial long division to perform the indicated division.

1. $\left(4x^2+3x-1\right)\div(x-3)$
2. $\left(2x^3-x+1\right)\div\left(x^2+x+1\right)$
3. $\left(5x^4-3x^3+2x^2-1\right)\div\left(x^2+4\right)$
4. $\left(-x^5+7x^3-x\right)\div\left(x^3-x^2+1\right)$
5. $\left(9x^3+5\right)\div(2x-3)$
6. $\left(4x^2-x-23\right)\div\left(x^2-1\right)$

Use synthetic division to perform the indicated division.

7. $\left(3x^2-2x+1\right)\div(x-1)$
8. $\left(x^2-5\right)\div(x-5)$
9. $\left(3-4x-2x^2\right)\div(x+1)$
10. $\left(4x^2-5x+3\right)\div(x+3)$
11. $\left(x^3+8\right)\div(x+2)$
12. $\left(4x^3+2x-3\right)\div(x-3)$
13. $\left(18x^2-15x-25\right)\div\left(x-\frac{5}{3}\right)$
14. $\left(4x^2-1\right)\div\left(x-\frac{1}{2}\right)$
15. $\left(2x^3+x^2+2x+1\right)\div\left(x+\frac{1}{2}\right)$
16. $\left(3x^3-x+4\right)\div\left(x-\frac{2}{3}\right)$
17. $\left(2x^3-3x+1\right)\div\left(x-\frac{1}{2}\right)$
18. $\left(4x^4-12x^3+13x^2-12x+9\right)\div\left(x-\frac{3}{2}\right)$
19. $\left(x^4-6x^2+9\right)\div\left(x-\sqrt{3}\right)$
20. $\left(x^6-6x^4+12x^2-8\right)\div\left(x+\sqrt{2}\right)$

Below you are given a polynomial and one of its zeros. Use the techniques in this section to find the rest of the real zeros and factor the polynomial.

21. $x^3-6x^2+11x-6,\quad c=1$
22. $x^3-24x^2+192x-512,\quad c=8$
23. $3x^3+4x^2-x-2,\quad c=\frac{2}{3}$
24. $2x^3-3x^2-11x+6,\quad c=\frac{1}{2}$
25. $x^3+2x^2-3x-6,\quad c=-2$

26. $2x^3 - x^2 - 10x + 5, \quad c = \frac{1}{2}$

27. $4x^4 - 28x^3 + 61x^2 - 42x + 9$, $c = \frac{1}{2}$ is a zero of multiplicity 2

28. $x^5 + 2x^4 - 12x^3 - 38x^2 - 37x - 12$, $c = -1$ is a zero of multiplicity 3

Answer

1. $4x^2 + 3x - 1 = (x - 3)(4x + 15) + 44$

3. $5x^4 - 3x^3 + 2x^2 - 1 = (x^2 + 4)(5x^2 - 3x - 18) + (12x + 71)$

5. $9x^3 + 5 = (2x - 3)(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}) + \frac{283}{8}$

7. $(3x^2 - 2x + 1) = (x - 1)(3x + 1) + 2$

9. $(3 - 4x - 2x^2) = (x + 1)(-2x - 2) + 5$

11. $(x^3 + 8) = (x + 2)(x^2 - 2x + 4) + 0$

13. $(18x^2 - 15x - 25) = (x - \frac{5}{3})(18x + 15) + 0$

15. $(2x^3 + x^2 + 2x + 1) = (x + \frac{1}{2})(2x^2 + 2) + 0$

17. $(2x^3 - 3x + 1) = (x - \frac{1}{2})(2x^2 + x - \frac{5}{2}) - \frac{1}{4}$

19. $(x^4 - 6x^2 + 9) = (x - \sqrt{3})(x^3 + \sqrt{3}x^2 - 3x - 3\sqrt{3}) + 0$

21. $x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$

23. $3x^3 + 4x^2 - x - 2 = 3(x - \frac{2}{3})(x + 1)^2$

25. $x^3 + 2x^2 - 3x - 6 = (x + 2)(x + \sqrt{3})(x - \sqrt{3})$

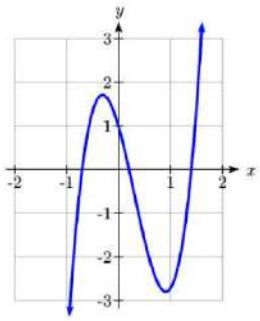
27. $4x^4 - 28x^3 + 61x^2 - 42x + 9 = 4(x - \frac{1}{2})^2(x - 3)^2$

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3.5: Real Zeros of Polynomials

In the last section, we saw how to determine if a real number was a zero of a polynomial. In this section, we will learn how to find good candidates to test using synthetic division. In the days before graphing technology was commonplace, mathematicians discovered a lot of clever tricks for determining the likely locations of zeros. Technology has provided a much simpler approach to narrow down potential candidates, but it is not always sufficient by itself. For example, the function shown to the right does not have any clear intercepts.



There are two results that can help us identify where the zeros of a polynomial are. The first gives us an interval on which all the real zeros of a polynomial can be found.

Definition: Cauchy's Bound

Given a polynomial

$$(f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \tag{3.5.1}$$

let M be the largest of the coefficients in absolute value. Then all the real zeros of $f(x)$ lie in the interval

$$\left[-\frac{M}{|a_n|} - 1, \quad \frac{M}{|a_n|} + 1\right] \tag{3.5.2}$$

Example 1

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Determine an interval which contains all the real zeros of *f*.

Solution

To find the M from Cauchy's Bound, we take the absolute value of the coefficients and pick the largest, in this case $|-6| = 6$. Divide this by the absolute value of the leading coefficient, 2, to get 3. All the real zeros of *f* lie in the interval

$$\left[-\frac{6}{|2|} - 1, \quad \frac{6}{|2|} + 1\right] = [-3 - 1, \quad 3 + 1] = [-4,\ 4].$$

Knowing this bound can be very helpful when using a graphing calculator, since we can use it to set the display bounds. This helps avoid missing a zero because it is graphed outside of the viewing window.

Exercise 3.5.1

Determine an interval which contains all the real zeros of $f(x) = 3x^3 - 12x^2 + 6x - 8$.

Answer

The maximum coefficient in absolute value is 12. Cauchy's Bound for all real zeros is

$$\left[-\frac{12}{|3|} - 1, \quad \frac{12}{|3|} + 1\right] = [-5, 5]$$

Now that we know *where* we can find the real zeros, we still need a list of *possible* real zeros. The Rational Roots Theorem provides us a list of potential integer and rational zeros.

rational roots theorem

Given a polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \tag{3.5.3}$$

with integer coefficients, if r is a rational zero of f, then r is of the form $r = \pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient, a_n.

This gives us a list of numbers to try in our synthetic division, which is a nicer place to start than simply guessing. If none of the numbers in the list are zeros, then either the polynomial has no real zeros at all, or all the real zeros are irrational numbers.

Example 3.5.2

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Roots Theorem to list all the possible rational zeros of $f(x)$.

Solution

To generate a complete list of rational zeros, we need to take each of the factors of the

constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$. The factors of -3 are ± 1 and ± 3. Since the Rational Roots Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Roots Theorem gives the list

$$\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}\right\} \text{ ,or } \left\{\pm 1, \pm\frac{1}{2}, \pm 3, \pm\frac{3}{2}\right\}$$

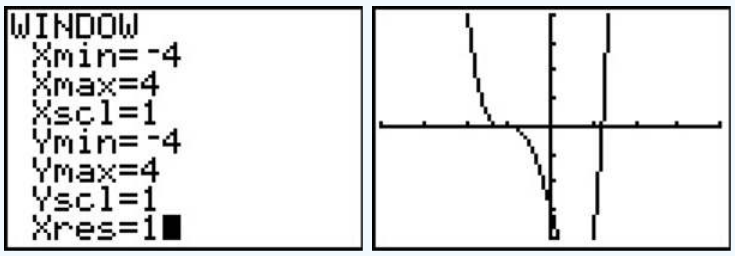
Now we can use synthetic division to test these possible zeros. To narrow the list first, we could use graphing technology to help us identify some good possibilities.

Example 3.5.3

Find the horizontal intercepts of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

Solution

From Example 1, we know that the real zeros lie in the interval [-4, 4]. Using a graphing calculator, we could set the window accordingly and get the graph below.



In Example 2, we learned that any rational zero must be on the list $\left\{\pm 1, \pm\frac{1}{2}, \pm 3, \pm\frac{3}{2}\right\}$. From the graph, it looks like -1 is a good possibility, so we try that using synthetic division.

-1	2	4	-1	-6	-3
	↓	-2	-2	3	3
	2	2	-3	-3	0

Success! Remembering that f was a fourth degree polynomial, we know that our quotient is a third degree polynomial. If we can do one more successful division, we will have knocked the quotient down to a quadratic, and, if all else fails, we can use the quadratic formula to find the last two zeros. Since there seems to be no other rational zeros to try, we continue with -1. Also, the shape of the crossing at $x = -1$ leads us to wonder if the zero $x = -1$ has multiplicity 3.

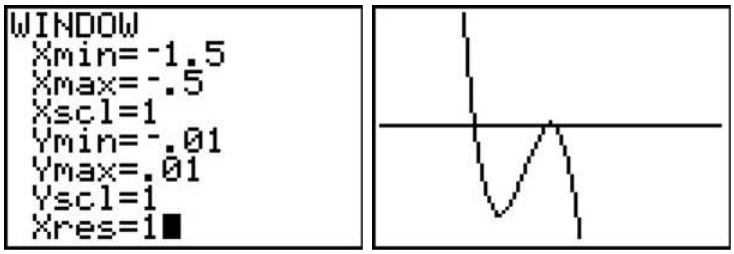
-1	2	2	-3	-3
	↓	-2	0	3
	2	0	-3	0

Success again! Our quotient polynomial is now $2x^2 - 3$. Setting this to zero gives $2x^2 - 3 = 0$, giving $x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$.

Since a fourth degree polynomial can have at most four zeros, including multiplicities, then the intercept x = -1 must only have multiplicity 2, which we had found through division, and not 3 as we had guessed.

It is interesting to note that we could greatly improve on the graph of $y = f(x)$in the previous example given to us by the calculator. For instance, from our determination of the zeros of f and their multiplicities, we know the graph crosses at $x = -\frac{\sqrt{6}}{2} \approx -1.22$ then turns back upwards to touch the x-axis at $x = -1$. This tells us that, despite what the calculator showed us the first time, there is a relative maximum occurring at $x = -1$ and not a "flattened crossing" as we originally believed.

After resizing the window, we see not only the relative maximum but also a relative minimum just to the left of $x = -1$



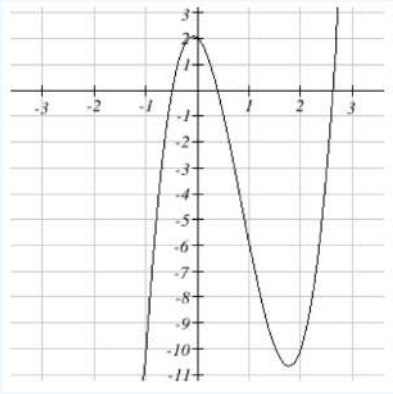
In this case, mathematics helped reveal something that was hidden in the initial graph.

Example 4

Find the real zeros of $f(x) = 4x^3 - 10x^2 - 2x + 2$.

Solution

Cauchy's Bound tells us that the real zeros lie in the interval $\left[-\frac{10}{|4|} - 1, \quad \frac{10}{|4|} + 1\right] = [-3.5,\ 3.5]$.



Graphing on this interval reveals no clear integer zeros. Turning to the rational roots theorem, we need to take each of the factors of the constant term, $a_0 = 2$, and divide them by each of the factors of the leading coefficient $a_3 = 4$. The factors of 2 are 1 and 2. The factors of 4 are 1, 2, and 4, so the Rational Roots Theorem gives the list

$$\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{2}{4}\right\} \text{ ,or } \left\{\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm 2\right\}$$

The two most likely candidates are $\pm\frac{1}{2}$.

Trying $\frac{1}{2}$,

1/2	4	-10	-2	2
	↓	2	-4	-3
	4	-8	-6	-1

The remainder is not zero, so this is not a zero. Trying $-\frac{1}{2}$,

-1/2	4	-10	-2	2
	↓	-2	6	-2
	4	-12	4	0

Success! This tells us $4x^3-10x^2-2x+2=\left(x+\frac{1}{2}\right)\left(4x^2-12x+4\right)$, and that the graph has a horizontal intercept at $x=-\frac{1}{2}$.

To find the remaining two intercepts, we can use the quadratic equation, setting $4x^2-12x+4=0$. First, we might pull out the common factor, $4\left(x^2-3x+1\right)=0$.

$$x=\frac{3\pm\sqrt{(-3)^2-4(1)(1)}}{2(1)}=\frac{3\pm\sqrt{5}}{2}\approx 2.618,\quad 0.382$$

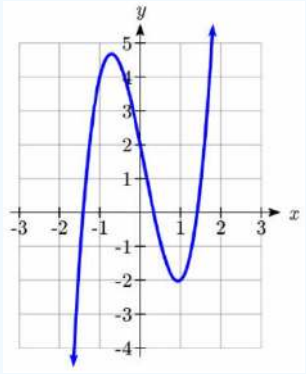
Exercise 3.5.2

Find the real zeros of $f(x)=3x^3-x^2-6x+2$

Answer

Cauchy's Bound tells us the zeros lie in the interval $\left[-\frac{6}{|3|}-1,\quad \frac{6}{|3|}+1\right]=[-3,3]$. The rational roots theorem tells us the possible rational zeros of the polynomial are on the list

$$\left\{\pm\frac{1}{1},\pm\frac{1}{3},\pm\frac{2}{1},\pm\frac{2}{3}\right\}=\left\{\pm1,\pm\frac{1}{3},\pm2,\pm\frac{2}{3}\right\}$$



Looking at a graph, the only likely candidate is $\frac{1}{3}$

Using synthetic division,

1/3	3	-1	-6	2
	↓	1	0	-2
	3	0	-6	0

$$3x^3-x^2-6x+2=\left(x-\frac{1}{3}\right)\left(3x^2-6\right)=3\left(x-\frac{1}{3}\right)\left(x^2-2\right)$$

Solving $x^2-2=0$ gives zeros $x=\pm\sqrt{2}$.

The real zeros of the polynomial are $x=\sqrt{2},\ -\sqrt{2},\ \frac{1}{3}$.

Important Topics of this Section

- Cauchy's Bound for all real zeros of a polynomial
- Rational Roots Theorem
- Finding real zeros of a polynomial

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3.5E: Real Zeros of Polynomials (Exercises)

? section 3.5 exercise

For each of the following polynomials, use Cauchy's Bound to find an interval containing all the real zeros, then use Rational Roots Theorem to make a list of possible rational zeros.

1. $f(x) = x^3 - 2x^2 - 5x + 6$
2. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
3. $f(x) = x^4 - 9x^2 - 4x + 12$
4. $f(x) = x^3 + 4x^2 - 11x + 6$
5. $f(x) = x^3 - 7x^2 + x - 7$
6. $f(x) = -2x^3 + 19x^2 - 49x + 20$
7. $f(x) = -17x^3 + 5x^2 + 34x - 10$
8. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
9. $f(x) = 3x^3 + 3x^2 - 11x - 10$
10. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

Find the real zeros of each polynomial.

11. $f(x) = x^3 - 2x^2 - 5x + 6$
12. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
13. $f(x) = x^4 - 9x^2 - 4x + 12$
14. $f(x) = x^3 + 4x^2 - 11x + 6$
15. $f(x) = x^3 - 7x^2 + x - 7$
16. $f(x) = -2x^3 + 19x^2 - 49x + 20$
17. $f(x) = -17x^3 + 5x^2 + 34x - 10$
18. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
19. $f(x) = 3x^3 + 3x^2 - 11x - 10$
20. $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
21. $f(x) = 9x^3 - 5x^2 - x$
22. $f(x) = 6x^4 - 5x^3 - 9x^2$
23. $f(x) = x^4 + 2x^2 - 15$
24. $f(x) = x^4 - 9x^2 + 14$
25. $f(x) = 3x^4 - 14x^2 - 5$
26. $f(x) = 2x^4 - 7x^2 + 6$
27. $f(x) = x^6 - 3x^3 - 10$
28. $f(x) = 2x^6 - 9x^3 + 10$
29. $f(x) = x^5 - 2x^4 - 4x + 8$
30. $f(x) = 2x^5 + 3x^4 - 18x - 27$
31. $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$
32. $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$

Answer

1. All the real zeros lie in the interval [-7, 7]

-Possible rational zeros are $\pm 1, \pm 2, \pm 3$

3. All of the real zeros lie in the interval [-13, 13]

-Possible rational zeros are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

5. All of the real zeors lie in the interval [-8, 8]

-Possible rational zeors are $\pm 1, \pm 7$

7. All of the real zeros lie in the interval [-3, 3]

-Possible rational zeros are $\pm\frac{1}{17}, \pm\frac{2}{17}, \pm\frac{5}{17}, \pm\frac{10}{17}, \pm 1, \pm 2, \pm 5, \pm 10$

9. All of the real zeros lie in the interval $[-\frac{14}{3}, \frac{14}{3}]$

-Possible rational zeros are $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}, \pm 1, \pm 2, \pm 5, \pm 10$

11. $x = -2, x = 1, x = 3$ (each has mult. 1)

13. $x = -2$ (mult. 2), $x = 1$ (mult. 1), $x = 3$ (mult. 1)

15. $x = 7$ (mult. 1)

17. $x = \frac{5}{17}, x = \pm\sqrt{2}$ (each has mult. 1)

19. $x = -2, x = \frac{3 \pm \sqrt{69}}{6}$ (each has mult. 1)

21. $x = 0, x = \frac{5 \pm \sqrt{61}}{18}$ (each has mult. 1)

23. $x = \pm\sqrt{3}$ (each has mult. 1)

25. $x = \pm\sqrt{5}$ (each has mult. 1)

27. $x = \sqrt[3]{-2} = -\sqrt[3]{2}, x = \sqrt[3]{5}$ (each has mult. 1)

29. $x = 2, x = \pm\sqrt{2}$ (each has mult. 1)

31. $x = -4$ (mult. 3), $x = 6$ (mult. 2)

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3.6: Complex Zeros

When finding the zeros of polynomials, at some point you're faced with the problem $x^2 = -1$. While there are clearly no real numbers that are solutions to this equation, leaving things there has a certain feel of incompleteness. To address that, we will need utilize the imaginary unit, i.

Definition: Imaginary number i

The most basic complex number is i, defined to be $i = \sqrt{-1}$, commonly called an **imaginary number**. Any real multiple of i is also an imaginary number.

Example 3.6.1

Simplify $\sqrt{-9}$.

Solution

We can separate $\sqrt{-9}$ as $\sqrt{9}\sqrt{-1}$. We can take the square root of 9, and write the square root of -1 as i.

$$\sqrt{-9} = \sqrt{9}\sqrt{-1} = 3i$$

A complex number is the sum of a real number and an imaginary number.

Definition: Complex Numbers

A **complex number** is a number $z = a + bi$, where a and b are real numbers

- a is the real part of the complex number
- b is the imaginary part of the complex number

$$i = \sqrt{-1} \tag{3.6.1}$$

Arithmetic on Complex Numbers

Before we dive into the more complicated uses of complex numbers, let's make sure we remember the basic arithmetic involved. To add or subtract complex numbers, we simply add the like terms, combining the real parts and combining the imaginary parts.

Example 3.6.2

Add $3 - 4i$ and $2 + 5i$.

Solution

Adding $(3 - 4i) + (2 + 5i)$, we add the real parts and the imaginary parts

$$3 + 2 - 4i + 5i$$

$$5 + i$$

Exercise 3.6.1

Subtract $2 + 5i$ from $3 - 4i$.

Answer

$$(3 - 4i) - (2 + 5i) = 1 - 9i$$

We can also multiply and divide complex numbers.

Example 3.6.3

Multiply: $4(2+5i)$.

Solution

To multiply the complex number by a real number, we simply distribute as we would when multiplying polynomials.

$$\begin{aligned} 4(2+5i) &= 4\cdot 2+4\cdot 5i \\ &= 8+20i \end{aligned}$$

Example 3.6.4

Divide $\dfrac{(2+5i)}{(4-i)}$.

Solution

To divide two complex numbers, we have to devise a way to write this as a complex number with a real part and an imaginary part.

We start this process by eliminating the complex number in the denominator. To do this, we multiply the numerator and denominator by a special complex number so that the result in the denominator is a real number. The number we need to multiply by is called the **complex conjugate**, in which the sign of the imaginary part is changed.

Here, 4+*i* is the complex conjugate of 4–*i*. Of course, obeying our algebraic rules, we must multiply by 4+*i* on both the top and bottom.

$$\frac{(2+5i)}{(4-i)}\cdot\frac{(4+i)}{(4+i)}$$

To multiply two complex numbers, we expand the product as we would with polynomials (the process commonly called FOIL – "first outer inner last"). In the numerator:

$$(2+5i)(4+i)$$

Expand

$$=8+20i+2i+5i^2$$

Since $i=\sqrt{-1}$, $i^2=-1$

$$=8+20i+2i+5(-1)$$

Simplify

$$=3+22i$$

Following the same process to multiply the denominator

$$(4-i)(4+i)$$

Expand

$$=(16-4i+4i-i^2)$$

Since $i=\sqrt{-1}$, $i^2=-1$

$$=(16-(-1))$$

$$=17$$

Combining this we get $\frac{3+22i}{17} = \frac{3}{17} + \frac{22i}{17}$

Exercise 3.6.2

Multiply $3-4i$ and $2+3i$.

Answer

$$(3-4i)(2+3i) = 18+i$$

In the last example, we used the conjugate of a complex number

Definition: Complex Conjugate

The **conjugate** of a complex number $a+bi$ is the number $a-bi$.

The notation commonly used for conjugation is a bar:

$$\overline{a+bi} = a-bi$$

Complex Zeros of Polynomials

Complex numbers allow us a way to write solutions to quadratic equations that do not have real solutions.

Example 3.6.5

Find the zeros of $f(x) = x^2 - 2x + 5$.

Solution

Using the quadratic formula,

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Exercise 3.6.3

Find the zeros of $f(x) = 2x^2 + 3x + 4$.

Answer

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(2)(4)}}{2(2)} = \frac{-3 \pm \sqrt{-23}}{4} = \frac{-3 \pm i\sqrt{23}}{4} = \frac{-3}{4} \pm \frac{\sqrt{23}}{4}i$$

Two things are important to note. First, the zeros $1+2i$ and $1-2i$ are complex conjugates. This will always be the case when we find non-real zeros to a quadratic function with real coefficients.

Second, we could write

$$f(x) = x^2 - 2x + 5 = (x - (1+2i))\,(x - (1-2i))$$

if we really wanted to, so the Factor and Remainder Theorems hold.

How do we know if a general polynomial has any complex zeros? We have seen examples of polynomials with no real zeros; can there be polynomials with no zeros at all? The answer to that last question, which comes from the Fundamental Theorem of Algebra, is "No."

A non-constant polynomial f with real or complex coefficients will have at least one real or complex zero.

This theorem is an example of an "existence" theorem in mathematics. It guarantees the existence of at least one zero, but provides no algorithm to use for finding it.

Now suppose we have a polynomial $f(x)$ of degree n. The Fundamental Theorem of Algebra guarantees at least one zero z_1, then the Factor Theorem guarantees that f can be factored as $f(x) = (x - z_1)\, q_1(x)$, where the quotient $q_1(x)$ will be of degree $n-1$.

If this function is non-constant, than the Fundamental Theorem of Algebra applies to it, and we can find another zero. This can be repeated n times.

If f is a polynomial f with real or complex coefficients with degree $n \geq 1$, then f has exactly n real or complex zeros, counting multiplicities.

If $z_1, z_2, \ldots, z_k$ are the distinct zero of f with multiplicities $m_1, m_2, \ldots, m_k$ respectively, then

$$f(x) = a(x - z_1)^{m_1}(x - z_2)^{m_2} \cdots (x - z_k)^{m_k} \tag{3.6.2}$$

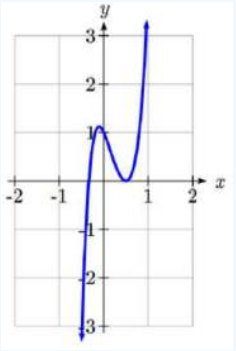
Example 3.6.6

Find all the real and complex zeros of $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$.

Solution

Using the Rational Roots Theorem, the possible real rational roots are

$$\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}\right\}$$



Testing $\frac{1}{2}$,

1/2	12	-20	19	-6	-2	1
	↓	6	-7	6	0	-1
	12	-14	12	0	-2	0

Success! Because the graph bounces at this intercept, it is likely that this zero has multiplicity 2. We can try synthetic division again to test that.

1/2	12	-14	12	0	-2
	↓	6	-4	4	2
	12	-8	8	-4	0

The other real root appears to be $-\frac{1}{3}$ or $-\frac{1}{4}$. Testing $-\frac{1}{3}$,

-1/3	12	-8	8	-4
	↓	-4	4	-4
	12	-12	12	0

Excellent! So far, we have factored the polynomial to

$$f(x) = \left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right)(12x^2 - 12x + 12) = 12\left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right)(x^2 - x + 1)$$

We can use the quadratic formula to find the two remaining zeros by setting $x^2 - x + 1 = 0$, which are likely complex zeros.

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

The zeros of the function are $x = \frac{1}{2}, -\frac{1}{3}, \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}$. We could write the function fully factored as $f(x) = 12\left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right) \left(x - \frac{1+i\sqrt{3}}{2}\right) \left(x - \frac{1-i\sqrt{3}}{2}\right)$.

When factoring a polynomial like we did at the end of the last example, we say that it is **factored completely over the complex numbers**, meaning it is impossible to factor the polynomial any further using complex numbers. If we wanted to factor the function over the **real numbers**, we would have stopped at $f(x) = 12\left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right) (x^2 - x + 1)$. Since the zeros of $x^2 - x + 1$ are nonreal, we call $x^2 - x + 1$ an **irreducible quadratic** meaning it is impossible to break it down any further using real numbers.

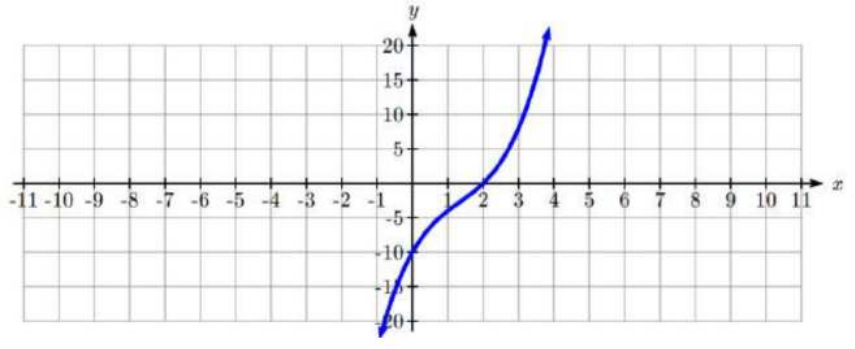
It turns out that a polynomial with real number coefficients can be factored into a product of linear factors corresponding to the real zeros of the function and irreducible quadratic factors which give the nonreal zeros of the function. Consequently, any nonreal zeros will come in conjugate pairs, so if z is a zero of the polynomial, so is $\bar{z}$.

? Exercise 3.6.4

Find the real and complex zeros of $f(x) = x^3 - 4x^2 + 9x - 10$.

Answer

Cauchy's Bound limits us to the interval [-11, 11]. The rational roots theorem gives a list of potential zeros: $\{\pm 1, \pm 2, \pm 5, \pm 10\}$. A quick graph shows that the likely rational root is $x = 2$.



Verifying this,

2	1	-4	9	-10
	↓	2	-4	10
	1	-2	5	0

So $f(x) = (x-2)(x^2 - 2x + 5)$

Using quadratic formula, we can find the complex roots from the irreducible quadratic.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

The zeros of this polynomial are $x = 2,\ -1 + 2i,\ -1 - 2i$

Important Topics of This Section

- Complex and Imaginary numbers
- Finding Complex zeros of polynomials

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3.6E: Complex Zeros (Exercises)

? section 3.6 exercise

Simplify each expression to a single complex number.

1. $\sqrt{-9}$
2. $\sqrt{-16}$
3. $\sqrt{-6}\sqrt{-24}$
4. $\sqrt{-3}\sqrt{-75}$
5. $\dfrac{2+\sqrt{-12}}{2}$
6. $\dfrac{4+\sqrt{-20}}{2}$

Simplify each expression to a single complex number.

7. $(3+2i)+(5-3i)$
8. $(-2-4i)+(1+6i)$
9. $(-5+3i)-(6-i)$
10. $(2-3i)-(3+2i)$
11. $(2+3i)(4i)$
12. $(5-2i)(3i)$
13. $(6-2i)(5)$
14. $(-2+4i)(8)$
15. $(2+3i)(4-i)$
16. $(-1+2i)(-2+3i)$
17. $(4-2i)(4+2i)$
18. $(3+4i)(3-4i)$
19. $\dfrac{3+4i}{2}$
20. $\dfrac{6-2i}{3}$
21. $\dfrac{-5+3i}{2i}$
22. $\dfrac{6+4i}{i}$
23. $\dfrac{2-3i}{4+3i}$
24. $\dfrac{3+4i}{2-i}$

Find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.

25. $f(x)=x^2-4x+13$
26. $f(x)=x^2-2x+5$
27. $f(x)=3x^2+2x+10$
28. $f(x)=x^3-2x^2+9x-18$
29. $f(x)=x^3+6x^2+6x+5$

30. $f(x)=3x^3-13x^2+43x-13$
31. $f(x)=x^3+3x^2+4x+12$
32. $f(x)=4x^3-6x^2-8x+15$
33. $f(x)=x^3+7x^2+9x-2$
34. $f(x)=9x^3+2x+1$
35. $f(x)=4x^4-4x^3+13x^2-12x+3$
36. $f(x)=2x^4-7x^3+14x^2-15x+6$
37. $f(x)=x^4+x^3+7x^2+9x-18$
38. $f(x)=6x^4+17x^3-55x^2+16x+12$
39. $f(x)=-3x^4-8x^3-12x^2-12x-5$
40. $f(x)=8x^4+50x^3+43x^2+2x-4$
41. $f(x)=x^4+9x^2+20$
42. $f(x)=x^4+5x^2-24$

Answer

1. $3i$
3. -12
5. $1+\sqrt{3}i$
7. $8-i$
9. $-11+4i$
11. $-12+8i$
13. $30-10i$
15. $11+10i$
17. 20
19. $\frac{3}{2}+2i$
21. $\frac{3}{2}+\frac{5}{2}i$
23. $-\frac{1}{25}-\frac{18}{25}i$
25. $f(x)=x^2-4x+13=(x-(2+3i))(x-(2-3i))$. Zeros: $x=2\pm3i$
27. $f(x)=3x^2+2x+10=3(x-(-\frac{1}{3}+\frac{\sqrt{29}}{3}i))(x-(-\frac{1}{3}-\frac{\sqrt{29}}{3}i))$. Zeros: $x=-\frac{1}{3}\pm\frac{\sqrt{29}}{3}i$
29. $f(x)=x^3+6x^2+6x+5=(x+5)(x^2+x+1)=(x+5)(x-(-\frac{1}{2}+\frac{\sqrt{3}}{2}i))(x-(-\frac{1}{2}-\frac{\sqrt{3}}{2}i))$ Zeros: $x=-5$, $x=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$
31. $f(x)=x^3+3x^2+4x+12=(x+3)(x^2+4)=(x+3)(x+2i)(x-2i)$. Zeros: $x=-3,\pm2i$
33. $f(x)=x^3+7x^2+9x-2=(x+2)(x-(-\frac{5}{2}+\frac{\sqrt{29}}{2}))(x-(-\frac{5}{2}-\frac{\sqrt{29}}{2}))$ Zeros: $x=-2$, $x=-\frac{5}{2}\pm\frac{\sqrt{29}}{2}$
35. $f(x)=4x^4-4x^3+13x^2-12x+3=(x-\frac{1}{2})^2(4x^2+12)=4(x-\frac{1}{2})^2(x+i\sqrt{3})(x-i\sqrt{3})$ Zeros: $x=\frac{1}{2}, x=\pm\sqrt{3}i$
37. $f(x)=x^4+x^3+7x^2+9x-18=(x+2)(x-1)(x^2+9)=(x+2)(x-1)(x+3i)(x-3i)$ Zeros: $x=-2,1,\pm3i$

39.

$f(x) = -3x^4 - 8x^3 - 12x^2 - 12x - 5 = (x+1)^2(-3x^2 - 2x - 5) = -3(x+1)^2(x - (-\frac{1}{3} + \frac{\sqrt{14}}{3}i))(x - (-\frac{1}{3} - \frac{\sqrt{14}}{3}i))$

Zeors: $x = -1$, $x = -\frac{1}{3} \pm \frac{\sqrt{14}}{3}i$

41. $f(x) = x^4 + 9x^2 + 20 = (x^2+4)(x^2+5) = (x-2i)(x+2i)(x-i\sqrt{5})(x+i\sqrt{5})$ Zeros: $x = \pm 2i, \pm i\sqrt{5}$

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CHAPTER OVERVIEW

4: Rational and Radical Functions

4.1: Rational Functions

4.1.1: Rational Functions (Exercises)

4.2: Further Topics in Functions

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4.1: Rational Functions

In the previous sections, we have built polynomials based on the positive whole number power functions. In this section, we explore functions based on power functions with negative integer powers, called rational functions.

Example 4.1.1

You plan to drive 100 miles. Find a formula for the time the trip will take as a function of the speed you drive.

Solution

You may recall that multiplying speed by time will give you distance. If we let t represent the drive time in hours, and v represent the velocity (speed or rate) at which we drive, then vt = distance. Since our distance is fixed at 100 miles, $vt = 100$. Solving this relationship for the time gives us the function we desired:

$$t(v) = \frac{100}{v} = 100v^{-1}$$

While this type of relationship can be written using the negative exponent, it is more common to see it written as a fraction.

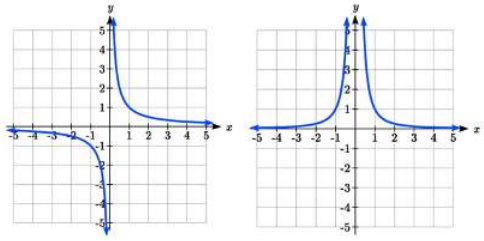
This particular example is one of an **inversely proportional** relationship – where one quantity is a constant divided by the other quantity, like $y = \frac{5}{x}$.

Notice that this is a transformation of the reciprocal toolkit function, $f(x) = \frac{1}{x}$

Several natural phenomena, such as gravitational force and volume of sound, behave in a manner **inversely proportional to the square** of another quantity. For example, the volume, V, of a sound heard at a distance d from the source would be related by $V = \frac{k}{d^2}$ for some constant value k.

These functions are transformations of the reciprocal squared toolkit function $f(x) = \frac{1}{x^2}$.

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. These graphs have several important features.



$f(x) = \frac{1}{x}$ $f(x) = \frac{1}{x^2}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. As you well know, dividing by zero is not allowed and therefore zero is not in the domain, and so the function is undefined at an input of zero.

Short run behavior:

As the input values approach zero from the left side (taking on very small, negative values), the function values become very large in the negative direction (in other words, they approach negative infinity).

We write: as$x \to 0^-$, $f(x) \to -\infty$.

As we approach zero from the right side (small, positive input values), the function values become very large in the positive direction (approaching infinity).

We write: as$x \to 0^+$, $f(x) \to \infty$.

This behavior creates a **vertical asymptote**. An asymptote is a line that the graph approaches. In this case the graph is approaching the vertical line $x = 0$ as the input becomes close to zero.

Long run behavior:

As the values of x approach infinity, the function values approach 0.

As the values of x approach negative infinity, the function values approach 0.

Symbolically: as$x \to \pm\infty, f(x) \to 0$

Based on this long run behavior and the graph we can see that the function approaches 0 but never actually reaches 0, it just "levels off" as the inputs become large. This behavior creates a **horizontal asymptote**. In this case the graph is approaching the horizontal line $f(x) = 0$ as the input becomes very large in the negative and positive directions.

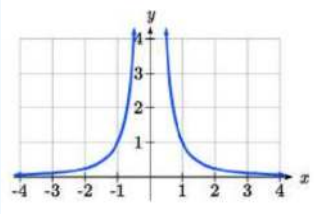
vertical and horizontal asymptotes

A **vertical asymptote** of a graph is a vertical line $x = a$ where the graph tends towards positive or negative infinity as the inputs approach a. As$x \to a, f(x) \to \pm\infty$.

A **horizontal asymptote** of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs get large. As $x \to \pm\infty, f(x) \to b$.

Exercise 4.1.1

Use symbolic notation to describe the long run behavior and short run behavior for the reciprocal squared function.



Answer

Long run behavior, as $x \to \pm\infty$, $f(x) \to 0$

Short run behavior, as $x \to 0$, $f(x) \to \infty$ (there are no horizontal or vertical intercepts)

Example 4.1.2

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.

Solution

Transforming the graph left 2 and up 3 would result in the function

$f(x) = \dfrac{1}{x+2} + 3$, or equivalently, by giving the terms a common denominator,

$f(x) = \dfrac{3x+7}{x+2}$.

Shifting the toolkit function would give us this graph. Notice that this equation is undefined at $x = -2$, and the graph also is showing a vertical asymptote at $x = -2$.

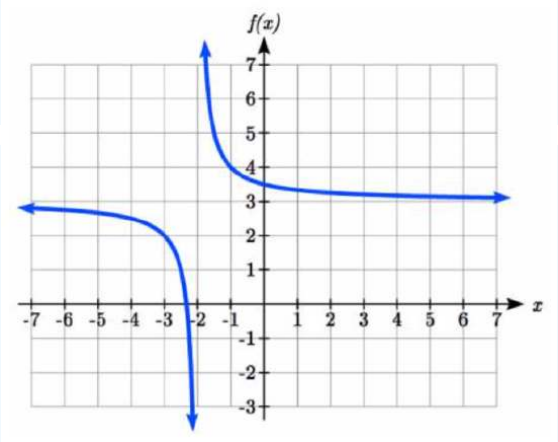
As$x \to -2^-, f(x) \to -\infty$, and

as $x \to -2^+, f(x) \to \infty$

As the inputs grow large, the graph appears to be leveling off at output values of 3, indicating a horizontal asymptote at $y = 3$.

As $x \to \pm\infty$, $f(x) \to 3$.

Notice that horizontal and vertical asymptotes get shifted left 2 and up 3 along with the function.



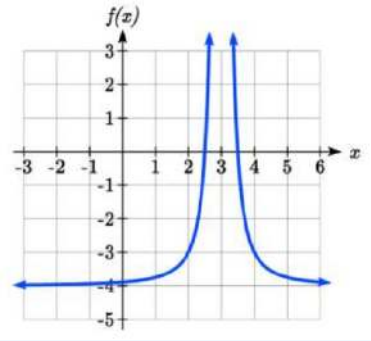
Exercise 4.1.2

Sketch the graph and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.

Answer

The function and the asymptotes are shifted 3 units right and 4 units down.

As $x \to 3$, $f(x) \to \infty$ and as $x \to \pm\infty$, $f(x) \to -4$



In the previous example, we shifted a toolkit function in a way that resulted in a function of the form $f(x) = \dfrac{3x+7}{x+2}$. This is an example of a more general rational function.

rational function

A **rational function** is a function that can be written as the ratio of two polynomials, $P(x)$ and $Q(x)$.

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_p x^p}{b_0 + b_1 x + b_2 x^2 + \cdots + b_q x^q} \tag{4.1.1}$$

Example 4.1.3

A large mixing tank currently contains 100 gallons of water, into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

Solution

Notice that the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\text{water} = 100 + 10t$$

$$\text{sugar} = 5 + 1t$$

The concentration, C, will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5+t}{100+10t}$$

Finding Asymptotes and Intercepts

Given a rational function, as part of investigating the short run behavior we are interested in finding any vertical and horizontal asymptotes, as well as finding any vertical or horizontal intercepts, as we have done in the past.

To find vertical asymptotes, we notice that the vertical asymptotes in our examples occur when the denominator of the function is undefined. With one exception, a vertical asymptote will occur whenever the denominator is undefined.

Example 4.1.4

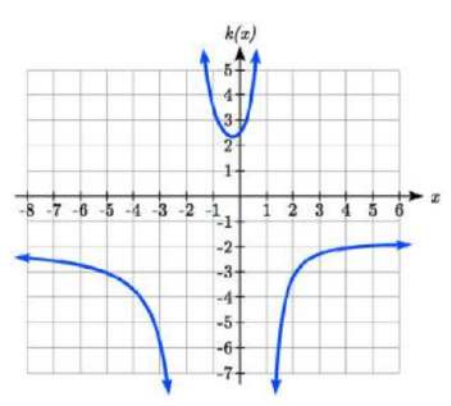
Find the vertical asymptotes of the function $k(x) = \dfrac{5+2x^2}{2-x-x^2}$

Solution

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$\begin{aligned} 2-x-x^2 &= 0 \\ (2+x)(1-x) &= 0 \\ x &= -2,\ 1 \end{aligned}$$

This indicates two vertical asymptotes, which a look at a graph confirms.



The exception to this rule can occur when both the numerator and denominator of a rational function are zero at the same input.

Example 4.1.5

Find the vertical asymptotes of the function $k(x) = \dfrac{x-2}{x^2-4}$.

Solution

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:

$$\begin{aligned} x^2-4 &= 0 \\ x^2 &= 4 \\ x &= -2,\ 2 \end{aligned}$$

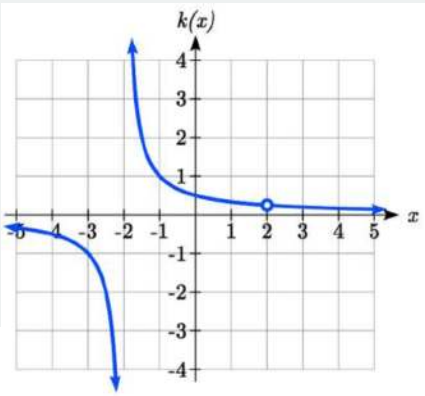
However, the numerator of this function is also equal to zero when $x = 2$. Because of this, the function will still be undefined at 2, since $\dfrac{0}{0}$ is undefined, but the graph will not have a vertical asymptote at $x = 2$.

The graph of this function will have the vertical asymptote at $x = -2$, but at $x = 2$ the graph will have a hole: a single point where the graph is not defined, indicated by an open circle.

vertical asymptotes and holes of rational functions

The **vertical asymptotes** of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero.

A **hole** occurs in the graph of a rational function if an input causes both numerator and denominator to be zero. In this case, factor the numerator and denominator and simplify; if the simplified expression still has a zero in the denominator at the original input the original function has a vertical asymptote at the input, otherwise it has a hole.



To find horizontal asymptotes, we are interested in the behavior of the function as the input grows large, so we consider long run behavior of the numerator and denominator separately. Recall that a polynomial's long run behavior will mirror that of the leading term. Likewise, a rational function's long run behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.

There are three distinct outcomes when this analysis is done:

Case 1: The degree of the denominator > degree of the numerator

Example 4.1.6

$$f(x)=\frac{3x+2}{x^2+4x-5}$$

In this case, the long run behavior is $f(x)\approx\frac{3x}{x^2}=\frac{3}{x}$. This tells us that as the inputs grow large, this function will behave similarly to the function $g(x)=\frac{3}{x}$. As the inputs grow large, the outputs will approach zero, resulting in a horizontal asymptote at $y=0$.

As $x\to\pm\infty$, $f(x)\to 0$

Case 2: The degree of the denominator < degree of the numerator

Example 4.1.7

$$f(x)=\frac{3x^2+2}{x-5}$$

In this case, the long run behavior is$f(x)\approx\frac{3x^2}{x}=3x$. This tells us that as the inputs grow large, this function will behave similarly to the function$g(x)=3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote.

As$x\to\pm\infty$,$f(x)\to\pm\infty$, respectively.

Case 3: The degree of the denominator = degree of the numerator

Example 4.1.8

$$f(x)=\frac{3x^2+2}{x^2+4x-5}$$

In this case, the long run behavior is $f(x)\approx\frac{3x^2}{x^2}=3$. This tells us that as the inputs grow large, this function will behave like the function $g(x)=3$, which is a horizontal line. As $x\to\pm\infty$, $f(x)\to 3$, resulting in a horizontal asymptote at $y=3$.

horizontal asymptote of rational functions

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

Degree of denominator > degree of numerator: Horizontal asymptote at $y = 0$

Degree of denominator < degree of numerator: No horizontal asymptote

Degree of denominator = degree of numerator: Horizontal asymptote at ratio of leading coefficients.

Example 4.1.9

In the sugar concentration problem from earlier, we created the equation

$$C(t) = \frac{5+t}{100+10t}$$

Find the horizontal asymptote and interpret it in context of the scenario.

Solution

Both the numerator and denominator are linear (degree 1), so since the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is $10t$, with coefficient 10. The horizontal asymptote will be at the ratio of these values: As $t \to \infty$, $C(t) \to \frac{1}{10}$. This function will have a horizontal asymptote at $y = \frac{1}{10}$.

This tells us that as the input gets large, the output values will approach 1/10. In context, this means that as more time goes by, the concentration of sugar in the tank will approach one tenth of a pound of sugar per gallon of water or 1/10 pounds per gallon.

Example 4.1.10

Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

Solution

First, note this function has no inputs that make both the numerator and denominator zero, so there are no potential holes. The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at x = 1, -2, and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than that of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x \to \pm\infty$, $f(x) \to 0$. This function will have a horizontal asymptote at $y = 0$.

Exercise 4.1.3

Find the vertical and horizontal asymptotes of the function $f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$

Answer

Vertical asymptotes at $x = 2$ and $x = -3$; horizontal asymptote at $y = 4$

Intercepts

As with all functions, a rational function will have a vertical intercept when the input is zero, if the function is defined at zero. It is possible for a rational function to not have a vertical intercept if the function is undefined at zero.

Likewise, a rational function will have horizontal intercepts at the inputs that cause the output to be zero (unless that input corresponds to a hole). It is possible there are no horizontal intercepts. Since a fraction is only equal to zero when the numerator is zero, horizontal intercepts will occur when the numerator of the rational function is equal to zero.

Example 4.1.11

Find the intercepts of $f(x)=\dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

Solution

We can find the vertical intercept by evaluating the function at zero

$$f(0)=\frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)}=\frac{-6}{10}=-\frac{3}{5}$$

The horizontal intercepts will occur when the function is equal to zero:

$$0=\frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$$

This is zero when the numerator is zero

$$\begin{aligned}0&=(x-2)(x+3)\\ x&=2,\ -3\end{aligned}$$

Exercise 4.1.4

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function and find the horizontal and vertical intercepts and the horizontal and vertical asymptotes.

Answer

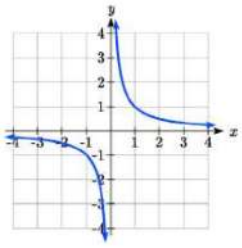
For the transformed reciprocal squared function, we find the rational form.

$$f(x)=\frac{1}{(x-3)^2}-4=\frac{1-4(x-3)^2}{(x-3)^2}=\frac{1-4(x^2-6x+9)}{(x-3)(x-3)}=\frac{-4x^2+24x-35}{x^2-6x+9}$$

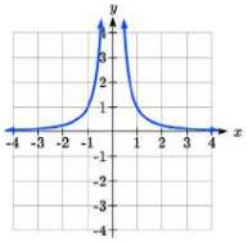
Since the numerator is the same degree as the denominator we know that as $x\to\pm\infty$, $f(x)\to-4$, $y=-4$ is the horizontal asymptote. Next, we set the denominator equal to zero to find the vertical asymptote at $x=3$, because as $x\to3$, $f(x)\to\infty$. We set the numerator equal to 0 and find the horizontal intercepts are at (2.5, 0) and (3.5, 0), then we evaluate at 0 and the vertical intercept is at $(0,\dfrac{-35}{9})$

From the previous example, you probably noticed that the numerator of a rational function reveals the horizontal intercepts of the graph, while the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Happily, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

When factors of the denominator have integer powers greater than one, the behavior at the corresponding vertical asymptote will mirror one of the two toolkit reciprocal functions.

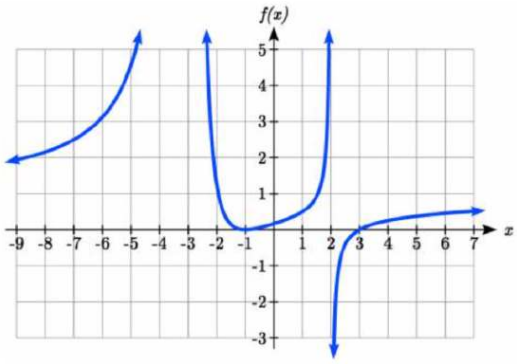


We get this behavior when the degree of the factor in the denominator is odd. The distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity.



We get this behavior when the degree of the factor in the denominator is even. The distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote, or heads toward negative infinity on both sides.

For example, the graph of $f(x)=\dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown here.



At the horizontal intercept $x=-1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph bounces at the intercept, consistent with the quadratic nature of the factor.

At the horizontal intercept $x=3$ corresponding to the $(x-3)$factor of the numerator, the graph passes through the axis as we'd expect from a linear factor.

At the vertical asymptote $x=-3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the $\dfrac{1}{x^2}$ toolkit.

At the vertical asymptote $x=2$ corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the $\dfrac{1}{x}$ toolkit.

Example 4.1.12

Sketch a graph of $f(x)=\dfrac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

Solution

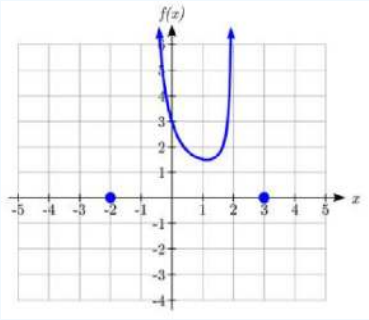
We can start our sketch by finding intercepts and asymptotes. Evaluating the function at zero gives the vertical intercept:

$$f(0)=\frac{(0+2)(0-3)}{(0+1)^2(0-2)}=3$$

Looking at when the numerator of the function is zero, we can determine the graph will have horizontal intercepts at $x=-2$ and $x=3$. At each, the behavior will be linear, with the graph passing through the intercept.

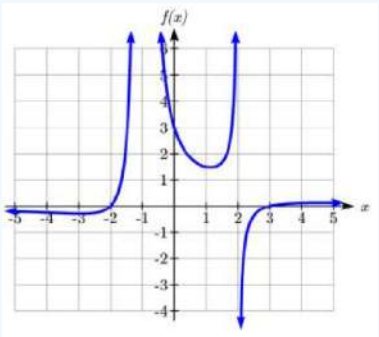
Looking at when the denominator of the function is zero, we can determine the graph will have vertical asymptotes at $x=-1$ and $x=2$.

Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at $y = 0$.



To sketch the graph, we might start by plotting the three intercepts. Since the graph has no horizontal intercepts between the vertical asymptotes, and the vertical intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph.

Since the factor associated with the vertical asymptote at $x = -1$ was squared, we know the graph will have the same behavior on both sides of the asymptote. Since the graph heads towards positive infinity as the inputs approach the asymptote on the right, the graph will head towards positive infinity on the left as well. For the vertical asymptote at $x = 2$, the factor was not squared, so the graph will have opposite behavior on either side of the asymptote.



After passing through the horizontal intercepts, the graph will then level off towards an output of zero, as indicated by the horizontal asymptote.

Exercise 4.1.5

Given the function $f(x) = \dfrac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Answer

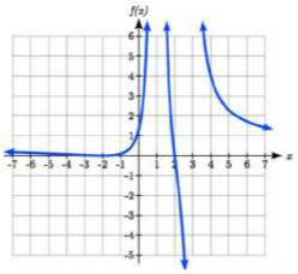
Horizontal asymptote at $y = 1/2$.

Vertical asymptotes are at $x = 1$, and $x = 3$.

Vertical intercept at $(0, 4/3)$.

Horizontal intercepts $(2, 0)$ and $(-2, 0)(-2, 0)$ is a double zero and the graph bounces off the axis at this point.

$(2, 0)$ is a single zero and crosses the axis at this point.



Since a rational function written in factored form will have a horizontal intercept where each factor of the numerator is equal to zero, we can form a numerator that will pass through a set of horizontal intercepts by introducing a corresponding set of factors. Likewise, since the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

writing rational functions from intercepts and asymptotes

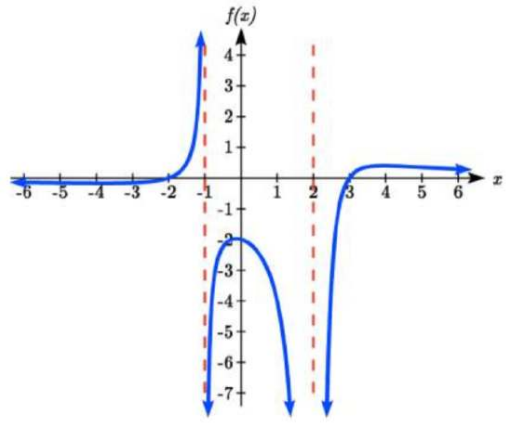
If a rational function has horizontal intercepts at $x = x_1, x_2, \ldots, x_n$, and vertical asymptotes at $x = v_1, v_2, \ldots, v_m$ then the function can be written in the form

$$f(x) = a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}} \tag{4.1.2}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the horizontal intercept, or by the horizontal asymptote if it is nonzero.

Example 4.1.13

Write an equation for the rational function graphed here.



Solution

The graph appears to have horizontal intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors.

The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\frac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other.

The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote.

Utilizing this information indicates an function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the vertical intercept (0,-2):

$$-2 = a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$
$$-2 = a\frac{-6}{4}$$
$$a = \frac{-8}{-6} = \frac{4}{3}$$

This gives us a final function of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$

Oblique Asymptotes

Earlier we saw graphs of rational functions that had no horizontal asymptote, which occurs when the degree of the numerator is larger than the degree of the denominator. We can, however, describe in more detail the long-run behavior of a rational function.

Example 4.1.14

Describe the long-run behavior of $f(x) = \frac{3x^2+2}{x-5}$

Solution

Earlier we explored this function when discussing horizontal asymptotes. We found the long-run behavior is $f(x) \approx \frac{3x^2}{x} = 3x$, meaning that $x \to \pm\infty, f(x) \to \pm\infty$, respectively, and there is no horizontal asymptote.

If we were to do polynomial long division, we could get a better understanding of the behavior as $x \to \pm\infty$.

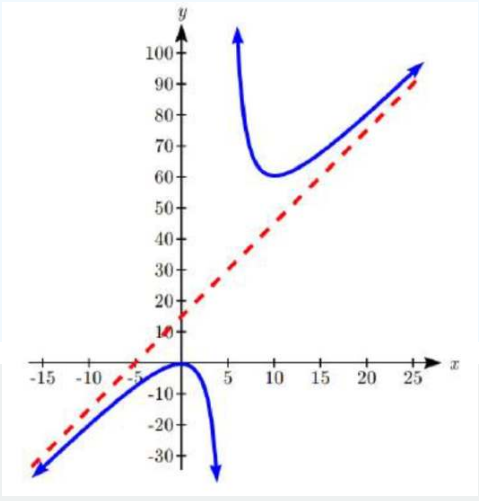
$$\begin{array}{r} 3x+15 \\ x-5\overline{)\ 3x^2+0x+2} \\ \underline{-(3x^2-15x)} \\ 15x+2 \\ \underline{-(15x-75)} \\ 77 \end{array}$$

This means $f(x) = \frac{3x^2+2}{x-5}$ can be rewritten as $f(x) = 3x+15+\frac{77}{x-5}$.

As $x \to \pm\infty$, the term $\frac{77}{x-5}$ will become very small and approach zero, becoming insignificant. The remaining $3x+15$then describes the long-run behavior of the function: as $x \to \pm\infty$, $f(x) \to 3x+15$.

We call this equation $y = 3x+15$ the **oblique asymptote** of the function.

In the graph, you can see how the function is approaching the line on the far left and far right.



oblique asymptotes

To explore the long-run behavior of a rational function,

1) Perform polynomial long division (or synthetic division)

2) The quotient will describe the asymptotic behavior of the function

When this result is a line, we call it an **oblique asymptote**, or **slant asymptote**.

Example 4.1.15

Find the oblique asymptote of $f(x)=\dfrac{-x^2+2x+1}{x+1}$

Solution

Performing polynomial long division:

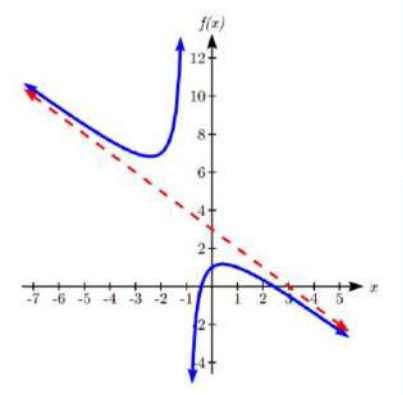
$$\begin{array}{r} 3x+15 \\ x-5\overline{)\ 3x^2+\ 0x+2} \\ \underline{-\left(3x^2-15x\right)} \\ 15x+2 \\ \underline{-\left(15x-75\right)} \\ 77 \end{array}$$

This allows us to rewrite the function as

$$f(x)=-x+3-\frac{2}{x+1}$$

.

The quotient, $y=-x+3$, is the oblique asymptote of $f(x)$. Just like functions we saw earlier approached their horizontal asymptote in the long run, this function will approach this oblique asymptote in the long run.



Exercise 4.1.6

Find the oblique asymptote of $f(x)=\dfrac{1+7x-2x^2}{x-2}$

Answer

Using long division:

$$\begin{array}{r} -2x+3 \\ x-2\overline{)\ -2x^2+7x+1} \\ \underline{-\left(-2x^2+4x\right)} \\ 3x+1 \\ \underline{-\left(3x-6\right)} \\ 7 \end{array}$$

$$f(x)=\frac{1+7x-2x^2}{x-2}=-2x+3+\frac{7}{x-2}$$

The oblique asymptote is $y=-2x+3$

While we primarily concern ourselves with oblique asymptotes, this same approach can describe other asymptotic behavior.

Example 4.1.16

Describe the long-run shape of $f(x)=\dfrac{-x^3-x^2+4x+2}{x+1}$

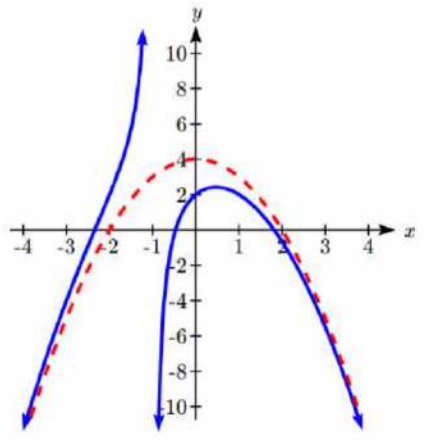
Solution

We could rewrite this using long division as

$$f(x) = -x^2 + 4 + \frac{2}{x+1}$$

Just looking at the quotient gives us the asymptote, $y = -x^2 + 4$.

This suggests that in the long run, the function will behave like a downwards opening parabola. The function will also have a vertical asymptote at $x = -1$.



Important Topics of this Section

Inversely proportional; Reciprocal toolkit function

Inversely proportional to the square; Reciprocal squared toolkit function

Horizontal Asymptotes

Vertical Asymptotes

Rational Functions

Finding intercepts, asymptotes, and holes.

Given equation sketch the graph

Identifying a function from its graph

Oblique Asymptotes

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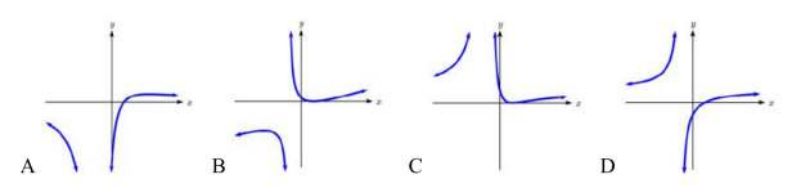
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4.1.1: Rational Functions (Exercises)

section 3.7 exercise

Match each equation form with one of the graphs.

1. $f(x)=\dfrac{x-A}{x-B}$ 2. $g(x)=\dfrac{(x-A)^2}{x-B}$ 3. $h(x)=\dfrac{x-A}{(x-B)^2}$ 4. $k(x)=\dfrac{(x-A)^2}{(x-B)^2}$



For each function, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal asymptote. Use that information to sketch a graph.

5. $p(x)=\dfrac{2x-3}{x+4}$

6. $q(x)=\dfrac{x-5}{3x-1}$

7. $s(x)=\dfrac{4}{(x-2)^2}$

8. $r(x)=\dfrac{5}{(x+1)^2}$

9. $f(x)=\dfrac{3x^2-14x-5}{3x^2+8x-16}$

10. $g(x)=\dfrac{2x^2+7x-15}{3x^2-14x+15}$

11. $a(x)=\dfrac{x^2+2x-3}{x^2-1}$

12. $b(x)=\dfrac{x^2-x-6}{x^2-4}$

13. $h(x)=\dfrac{2x^2+x-1}{x-4}$

14. $k(x)=\dfrac{2x^2-3x-20}{x-5}$

15. $n(x)=\dfrac{3x^2+4x-4}{x^3-4x^2}$

16. $m(x)=\dfrac{5-x}{2x^2+7x+3}$

17. $w(x)=\dfrac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

18. $z(x)=\dfrac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

Write an equation for a rational function with the given characteristics.

19. Vertical asymptotes at $x=5$ and $x=-5$
x intercepts at $(2,\ 0)$ and $(-1,\ 0)$ y intercept at $(0,\ 4)$

20. Vertical asymptotes at $x = -4$ and $x = -1$
x intercepts at $(1,\ 0)$ and $(5,\ 0)$ y intercept at $(0,\ 7)$

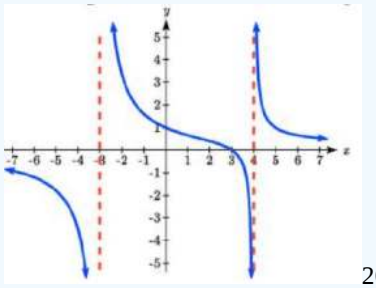
21. Vertical asymptotes at $x = -4$ and $x = -5$
x intercepts at $(4,\ 0)$ and $(-6,\ 0)$ Horizontal asymptote at $y = 7$

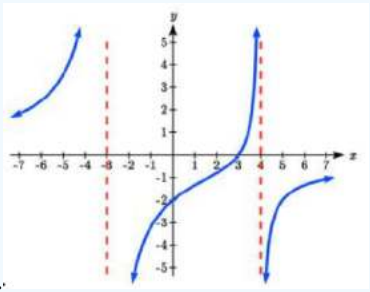
22. Vertical asymptotes at $x = -3$ and $x = 6$
x intercepts at $(-2,\ 0)$ and $(1,\ 0)$ Horizontal asymptote at $y = -2$

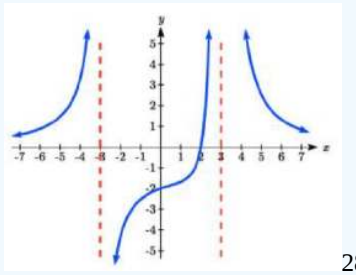
23. Vertical asymptote at $x = -1$
Double zero at $x = 2$ y intercept at $(0,\ 2)$

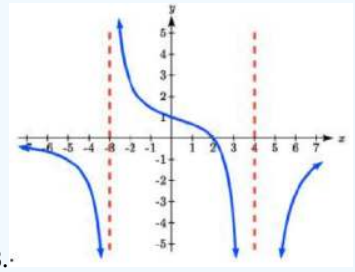
24. Vertical asymptote at $x = 3$
Double zero at $x = 1$ y intercept at $(0,\ 4)$

Write an equation for the function graphed.

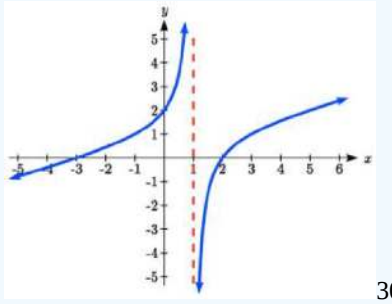
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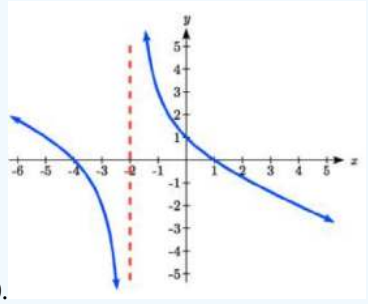
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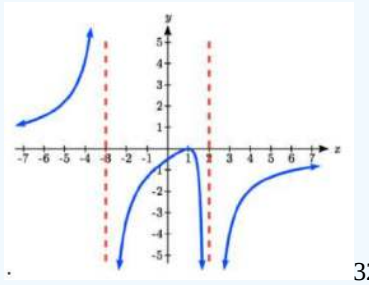
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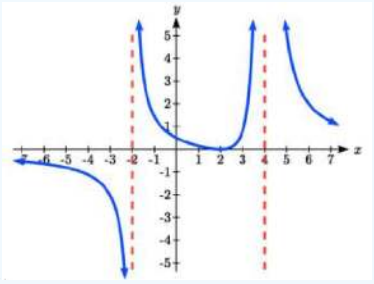
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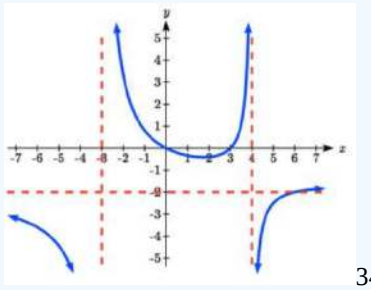
Write an equation for the function graphed.

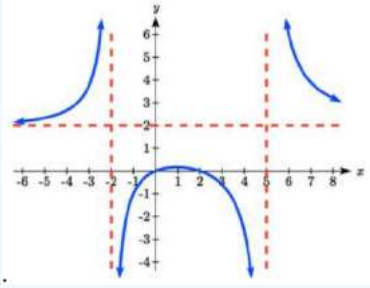
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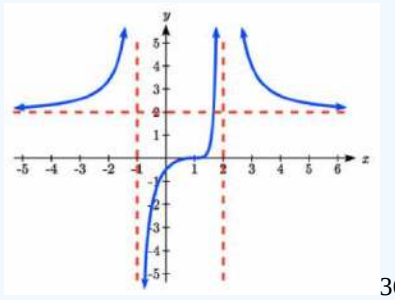
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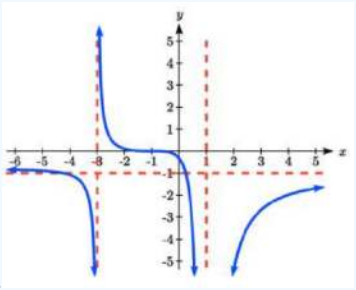
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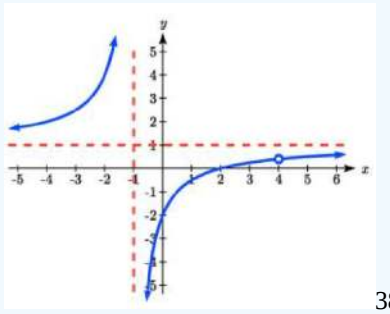
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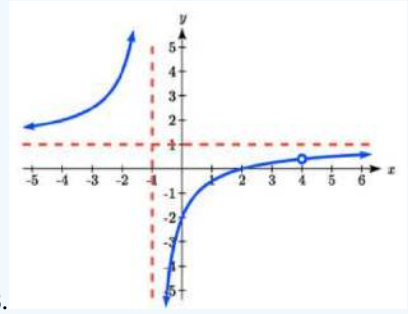
34. 

35. 

36. 

Write an equation for the function graphed.

37. 

38. 

Find the oblique asymptote of each function.

39. $f(x) = \dfrac{3x^2+4x}{x+2}$

40. $g(x) = \dfrac{2x^2+3x-8}{x-1}$

41. $h(x) = \dfrac{x^2-x-3}{2x-6}$

42. $k(x) = \dfrac{5+x-2x^2}{2x+1}$

43. $m(x) = \dfrac{-2x^3+x^2-6x+7}{x^2+3}$

44. $n(x) = \dfrac{2x^3+x^2+x}{x^2+x+1}$

45. A scientist has a beaker containing 20 mL of a solution containing 20% acid. To dilute this, she adds pure water.

    a. Write an equation for the concentration in the beaker after adding n mL of water.
    b. Find the concentration if 10 mL of water has been added.
    c. How many mL of water must be added to obtain a 4% solution?
    d. What is the behavior as $n \to \infty$, and what is the physical significance of this?

46. A scientist has a beaker containing 30 mL of a solution containing 3 grams of potassium hydroxide. To this, she mixes a solution containing 8 milligrams per mL of potassium hydroxide.

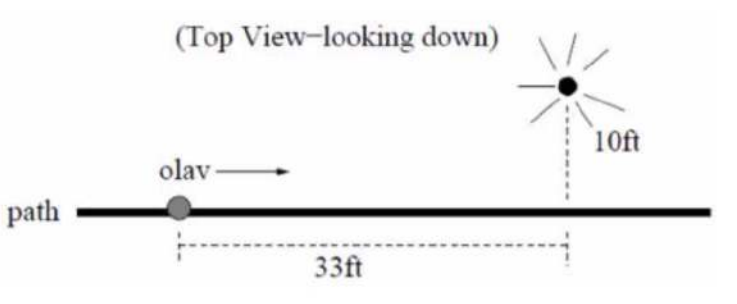
a. Write an equation for the concentration in the tank after adding n mL of the second solution.
b. Find the concentration if 10 mL of the second solution has been added.
c. How many mL of water must be added to obtain a 50 mg/mL solution?
d. What is the behavior as $n \to \infty$, and what is the physical significance of this?

47. Oscar is hunting magnetic fields with his gauss meter, a device for measuring the strength and polarity of magnetic fields. The reading on the meter will increase as Oscar gets closer to a magnet. Oscar is in a long hallway at the end of which is a room containing an extremely strong magnet. When he is far down the hallway from the room, the meter reads a level of 0.2. He then walks down the hallway and enters the room. When he has gone 6 feet into the room, the meter reads 2.3. Eight feet into the room, the meter reads 4.4. [UW]

a. Give a rational model of form $m(x) = \dfrac{ax+b}{cx+d}$ relating the meter reading $m(x)$ to how many feet x Oscar has gone into the room.
b. How far must he go for the meter to reach 10? 100?
c. Considering your function from part (a) and the results of part (b), how far into the room do you think the magnet is?

48. The more you study for a certain exam, the better your performance on it. If you study for 10 hours, your score will be 65%. If you study for 20 hours, your score will be 95%. You can get as close as you want to a perfect score just by studying long enough. Assume your percentage score, $p(n)$, is a function of the number of hours, n, that you study in the form $p(n) = \dfrac{an+b}{cn+d}$. If you want a score of 80%, how long do you need to study? [UW]

49. A street light is 10 feet north of a straight bike path that runs east-west. Olav is bicycling down the path at a rate of 15 miles per hour. At noon, Olav is 33 feet west of the point on the bike path closest to the street light. (See the picture). The relationship between the intensity C of light (in candlepower) and the distance d (in feet) from the light source is given by $C = \dfrac{k}{d^2}$, where k is a constant depending on the light source. [UW]



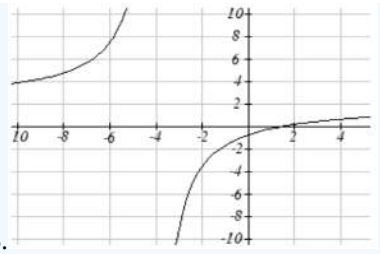
a. From 20 feet away, the street light has an intensity of 1 candle. What is k?
b. Find a function which gives the intensity of the light shining on Olav as a function of time, in seconds.
c. When will the light on Olav have maximum intensity?
d. When will the intensity of the light be 2 candles?

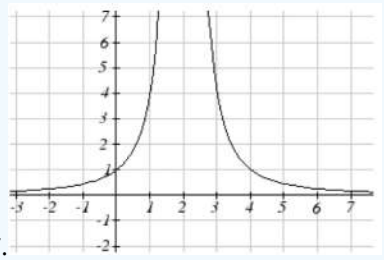
Answer

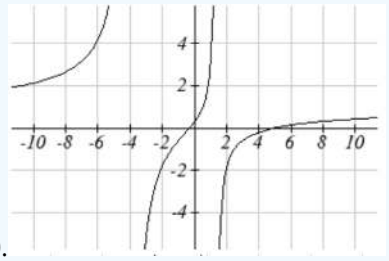
1. D

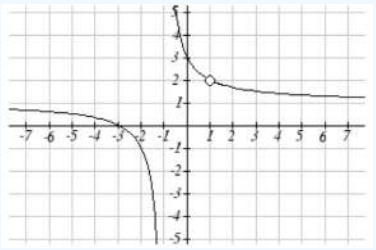
3. A

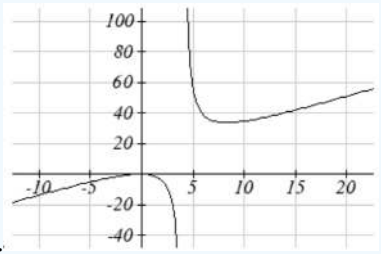
	Vertical asymptotes	Horizontal Asymptote	Vertical y-Intercept	Horizontal x- intercept
5.	$x = -4$	$y = 2$	(0, -3/4)	(3/2, 0)
7.	$x = 2$	$y = 0$	(0, 1)	DNE
9.	$x = -4, 1\frac{1}{3}$	$y = 1$	(0, 5/16)	(-1/3, 0), (5, 0)
11.	$x = -1$, hole at $x = 1$	$y = 1$	(0, 3)	(-3, 0)
13.	$x = 4$	none $y = 2x$ (obique)	(0, 1/4)	(-1, 0), (1/2, 0)
15.	$x = 0, 4$	$y = 0$	DNE	(-2, 0), (2/3, 0)
17.	$y = -2, 4$	$y = 1$	(0, -15/16)	(1, 0), (-3, 0), (5, 0)

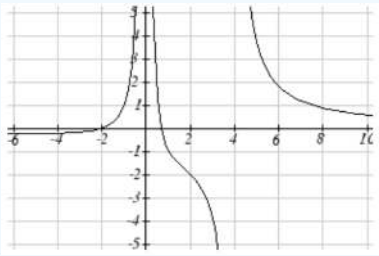
5. 

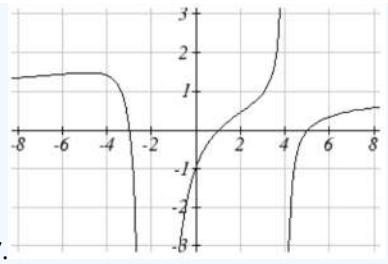
7. 

9. 

11. 

13. 

15. 

17. 

19. $y = \dfrac{50(x-2)(x+1)}{(x+5)(x-5)}$

21. $y = \dfrac{7(x-4)(x+6)}{(x+4)(x+5)}$

23. $y = \dfrac{1(x-2)^2}{2(x+1)}$

25. $y = \dfrac{4(x-3)}{(x+3)(x-4)}$

27. $y = \dfrac{27(x-2)}{(x+3)(x-3)^2}$

29. $y = \dfrac{1(x+3)(x-2)}{3(x-1)}$

31. $y = \dfrac{-6(x-1)^2}{(x+3)(x-2)^2}$

33. $y = -\dfrac{2(x)(x-3)}{(x+3)(x-4)}$

35. $y = \dfrac{2(x-1)^3}{(x+1)(x-2)^2}$

37. $y = \dfrac{(x-4)(x-2)}{(x-4)(x+1)}$

39. $y = 3x - 2$

41. $y = \dfrac{1}{2}x + 1$

43. $y = -2x + 1$

45. a. $C(n) = \dfrac{4}{20+n}$
b. $C(10) \approx 13.33$
c. 80 mL
d. as $n \to \infty$, $C \to 0$

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4.2: Further Topics in Functions

This section serves as a watershed for functions which are combinations of polynomial, and more generally, rational functions, with the operations of radicals. It is business of Calculus to discuss these functions in all the detail they demand so our aim in this section is to help shore up the requisite skills needed so that the reader can answer Calculus's call when the time comes. We briefly recall the definition and some of the basic properties of radicals from Intermediate Algebra.[1]

Definition 5.4. Principal Root

Let x be a real number and n a natural number.[a] If n is odd, the **principal n^{th} root** of x, denoted $\sqrt[n]{x}$ is the unique real number satisfying

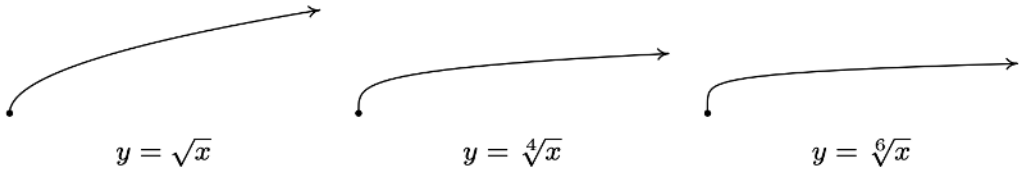
$$(\sqrt[n]{x})^n = x.$$

If n is even, $\sqrt[n]{x}$ is defined similarly[b] provided $x \geq 0$ and $\sqrt[n]{x} \geq 0$. The **index** is the number n and the **radicand** is the number x. For $n = 2$, we write $\sqrt{x}$ instead of $\sqrt[2]{x}$.

[a] Recall this means $n = 1, 2, 3, \ldots$.

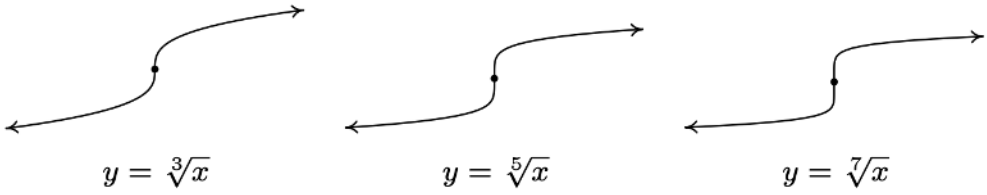
[b] Recall both $x = -2$ and $x = 2$ satisfy $x^4 = 16$, but $\sqrt[4]{16} = 2$, not -2.

It is worth remarking that, in light of Section 5.2, we could define $f(x) = \sqrt[n]{x}$ functionally as the inverse of $g(x) = x^n$ with the stipulation that when n is even, the domain of g is restricted to $[0, \infty)$. From what we know about $g(x) = x^n$ from Section 3.1 along with Theorem 5.3, we can produce the graphs of $f(x) = \sqrt[n]{x}$ by reflecting the graphs of $g(x) = x^n$ across the line $y = x$. Below are the graphs of $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$. The point $(0, 0)$ is indicated as a reference. The axes are hidden so we can see the vertical steepening near $x = 0$ and the horizontal flattening as $x \to \infty$.



$y = \sqrt{x}$ $\quad$ $y = \sqrt[4]{x}$ $\quad$ $y = \sqrt[6]{x}$

The odd-indexed radical functions also follow a predictable trend - steepening near $x = 0$ and flattening as $x \to \pm\infty$. In the exercises, you'll have a chance to graph some basic radical functions using the techniques presented in Section 1.7.



$y = \sqrt[3]{x}$ $\quad$ $y = \sqrt[5]{x}$ $\quad$ $y = \sqrt[7]{x}$

We have used all of the following properties at some point in the textbook for the case $n = 2$ (the square root), but we list them here in generality for completeness.

Theorem 5.6. Properties of Radicals

Let x and y be real numbers and m and n be natural numbers. If $\sqrt[n]{x}$, $\sqrt[n]{y}$ are real numbers, then

- **Product Rule:** $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$
- **Powers of Radicals:** $\sqrt[n]{x^m} = (\sqrt[n]{x})^m$
- **Quotient Rule:** $\sqrt[n]{\dfrac{x}{y}} = \dfrac{\sqrt[n]{x}}{\sqrt[n]{y}}$, provided $y \neq 0$.
- If n is odd, $\sqrt[n]{x^n} = x$; if n is even, $\sqrt[n]{x^n} = |x|$.

The proof of Theorem 5.6 is based on the definition of the principal roots and properties of exponents. To establish the product rule, consider the following. If n is odd, then by definition $\sqrt[n]{xy}$ is the unique real number such that $\left(\sqrt[n]{xy}\right)^n = xy$. Given that $\left(\sqrt[n]{x}\,\sqrt[n]{y}\right)^n = \left(\sqrt[n]{x}\right)^n\left(\sqrt[n]{y}\right)^n = xy$, it must be the case that $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$. If n is even, then $\sqrt[n]{xy}$ is the unique non-negative real number such that $\left(\sqrt[n]{xy}\right)^n = xy$. Also note that since n is even, $\sqrt[n]{x}$ and $\sqrt[n]{y}$ are also non-negative and hence so is $\sqrt[n]{x}\sqrt[n]{y}$. Proceeding as above, we find that $\sqrt[n]{xy} = \sqrt[n]{x}\,\sqrt[n]{y}$. The quotient rule is proved similarly and is left as an exercise. The power rule results from repeated application of the product rule, so long as $\sqrt[n]{x}$ is a real number to start with.[2] The last property is an application of the power rule when n is odd, and the occurrence of the absolute value when n is even is due to the requirement that $\sqrt[n]{x} \geq 0$ in Definition 5.4. For instance, $\sqrt[4]{(-2)^4} = \sqrt[4]{16} = 2 = |-2|$, not -2. It's this last property which makes compositions of roots and powers delicate. This is especially true when we use exponential notation for radicals. Recall the following definition.

Definition 5.5.

Let x be a real number, m an integer[a] and n a natural number.

- $x^{\frac{1}{n}} = \sqrt[n]{x}$ and is defined whenever $\sqrt[n]{x}$ is defined.
- $x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$, whenever $\left(\sqrt[n]{x}\right)^m$ is defined.

[a] Recall this means $m = 0, \pm 1, \pm 2, \ldots$

The rational exponents defined in Definition 5.5 behave very similarly to the usual integer exponents from Elementary Algebra with one critical exception. Consider the expression $\left(x^{2/3}\right)^{3/2}$. Applying the usual laws of exponents, we'd be tempted to simplify this as $\left(x^{2/3}\right)^{3/2} = x^{\frac{2}{3}\cdot\frac{3}{2}} = x^1 = x$. However, if we substitute $x = -1$ and apply Definition 5.5, we find $(-1)^{2/3} = \left(\sqrt[3]{-1}\right)^2 = (-1)^2 = 1$ so that $\left((-1)^{2/3}\right)^{3/2} = 1^{3/2} = \left(\sqrt{1}\right)^3 = 1^3 = 1$. We see in this case that $\left(x^{2/3}\right)^{3/2} \neq x$. If we take the time to rewrite $\left(x^{2/3}\right)^{3/2}$ with radicals, we see

$$\left(x^{2/3}\right)^{3/2} = \left(\left(\sqrt[3]{x}\right)^2\right)^{3/2} = \left(\sqrt{\left(\sqrt[3]{x}\right)^2}\right)^3 = \left(\left|\sqrt[3]{x}\right|\right)^3 = \left|\left(\sqrt[3]{x}\right)^3\right| = |x|$$

In the play-by-play analysis, we see that when we canceled the 2's in multiplying $\frac{2}{3}\cdot\frac{3}{2}$, we were, in fact, attempting to cancel a square with a square root. The fact that $\sqrt{x^2} = |x|$ and not simply x is the root[3] of the trouble. It may amuse the reader to know that $\left(x^{3/2}\right)^{2/3} = x$, and this verification is left as an exercise. The moral of the story is that when simplifying fractional exponents, it's usually best to rewrite them as radicals.[4] The last major property we will state, and leave to Calculus to prove, is that radical functions are continuous on their domains, so the Intermediate Value Theorem, Theorem 3.1, applies. This means that if we take combinations of radical functions with polynomial and rational functions to form what the authors consider the **algebraic functions**,[5] we can make sign diagrams using the procedure set forth in Section 4.2.

Steps for Constructing a Sign Diagram for an Algebraic Function

Suppose f is an algebraic function.

1. Place any values excluded from the domain of f on the number line with an '' above them.
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine the sign of $f(x)$ for each test value in step 3, and write that sign above the corresponding interval.

Our next example reviews quite a bit of Intermediate Algebra and demonstrates some of the new features of these graphs.

Example 5.3.1

For the following functions, state their domains and create sign diagrams. Check your answer graphically using your calculator

1. $f(x) = 3x\sqrt[3]{2-x}$
2. $g(x) = \sqrt{2-\sqrt[4]{x+3}}$
3. $h(x) = \sqrt[3]{\dfrac{8x}{x+1}}$

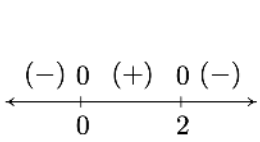
4. $k(x) = \dfrac{2x}{\sqrt{x^2-1}}$

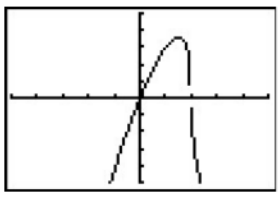
Solution.

1. As far as domain is concerned, $f(x)$ has no denominators and no even roots, which means its domain is $(-\infty, \infty)$. To create the sign diagram, we find the zeros of f.

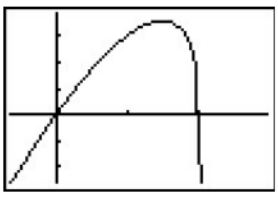
$$\begin{aligned} f(x) &= 0 \\ 3x\sqrt[3]{2-x} &= 0 \\ 3x = 0 \quad &\text{or} \quad \sqrt[3]{2-x} = 0 \\ x = 0 \quad &\text{or} \quad (\sqrt[3]{2-x})^3 = 0^3 \\ x = 0 \quad &\text{or} \quad 2-x = 0 \\ x = 0 \quad &\text{or} \quad x = 2 \end{aligned}$$

The zeros 0 and 2 divide the real number line into three test intervals. The sign diagram and accompanying graph are below. Note that the intervals on which f is $(+)$ correspond to where the graph of f is above the x-axis, and where the graph of f is below the x-axis we have that f is $(-)$. The calculator suggests something mysterious happens near $x = 2$. Zooming in shows the graph becomes nearly vertical there. You'll have to wait until Calculus to fully understand this phenomenon.



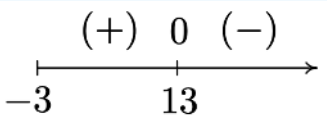


$y = f(x)$

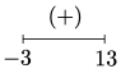


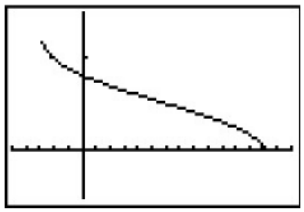
$y = f(x)$ near $x = 2$.

2. In $g(x) = \sqrt{2-\sqrt[4]{x+3}}$, we have two radicals both of which are even indexed. To satisfy $\sqrt[4]{x+3}$, we require $x+3 \geq 0$ or $x \geq -3$. To satisfy $\sqrt{2-\sqrt[4]{x+3}}$, we need $2-\sqrt[4]{x+3} \geq 0$. While it may be tempting to write this as $2 \geq \sqrt[4]{x+3}$ and take both sides to the fourth power, there are times when this technique will produce erroneous results.[6] Instead, we solve $2-\sqrt[4]{x+3} \geq 0$ using a sign diagram. If we let $r(x) = 2-\sqrt[4]{x+3}$, we know $x \geq -3$, so we concern ourselves with only this portion of the number line. To find the zeros of r we set $r(x) = 0$ and solve $2-\sqrt[4]{x+3} = 0$. We get $\sqrt[4]{x+3} = 2$ so that $(\sqrt[4]{x+3})^4 = 2^4$ from which we obtain $x+3 = 16$ or $x = 13$. Since we raised both sides of an equation to an even power, we need to check to see if $x = 13$ is an extraneous solution.[7] We find $x = 13$ does check since $2-\sqrt[4]{x+3} = 2-\sqrt[4]{13+3} = 2-\sqrt[4]{16} = 2-2 = 0$. Below is our sign diagram for r.



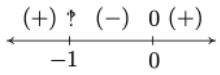
We find $2-\sqrt[4]{x+3} \geq 0$ on $[-3, 13]$ so this is the domain of g. To find a sign diagram for g, we look for the zeros of g. Setting $g(x) = 0$ is equivalent to $\sqrt{2-\sqrt[4]{x+3}} = 0$. After squaring both sides, we get $2-\sqrt[4]{x+3} = 0$, whose solution we have found to be $x = 13$. Since we squared both sides, we double check and find $g(13)$ is, in fact, 0. Our sign diagram and graph of g are below. Since the domain of g is $[-3, 13]$, what we have below is not just a *portion* of the graph of g, but the *complete* graph. It is always above or on the x-axis, which verifies our sign diagram.

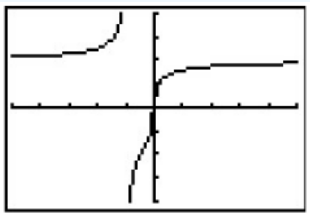




The complete graph of $y = g(x)$.

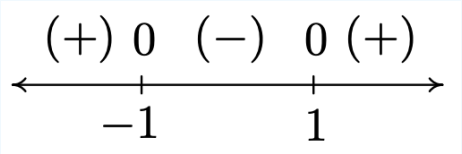
3. The radical in $h(x)$ is odd, so our only concern is the denominator. Setting $x+1=0$ gives $x=-1$, so our domain is $(-\infty,-1)\cup(-1,\infty)$. To find the zeros of h, we set $h(x)=0$. To solve $\sqrt[3]{\frac{8x}{x+1}}=0$, we cube both sides to get $\frac{8x}{x+1}=0$. We get $8x=0$, or $x=0$. Below is the resulting sign diagram and corresponding graph. From the graph, it appears as though $x=-1$ is a vertical asymptote. Carrying out an analysis as $x\to-1$ as in Section 4.2 confirms this. (We leave the details to the reader.) Near $x=0$, we have a situation similar to $x=2$ in the graph of f in number 1 above. Finally, it appears as if the graph of h has a horizontal asymptote $y=2$. Using techniques from Section 4.2, we find as $x\to\pm\infty$, $\frac{8x}{x+1}\to 8$. From this, it is hardly surprising that as $x\to\pm\infty$, $h(x)=\sqrt[3]{\frac{8x}{x+1}}\approx\sqrt[3]{8}=2$.





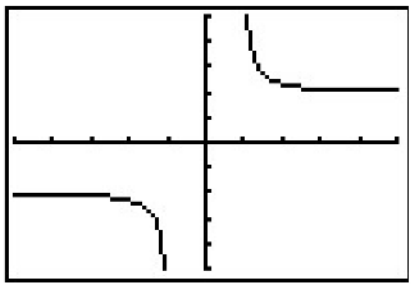
$y=h(x)$

4. To find the domain of k, we have both an even root and a denominator to concern ourselves with. To satisfy the square root, $x^2-1\geq 0$. Setting $r(x)=x^2-1$, we find the zeros of r to be $x=\pm 1$, and we find the sign diagram of r to be



We find $x^2-1\geq 0$ for $(-\infty,-1]\cup[1,\infty)$. To keep the denominator of $k(x)$ away from zero, we set $\sqrt{x^2-1}=0$. We leave it to the reader to verify the solutions are $x=\pm 1$, both of which must be excluded from the domain. Hence, the domain of k is $(-\infty,-1)\cup(1,\infty)$. To build the sign diagram for k, we need the zeros of k. Setting $k(x)=0$ results in $\frac{2x}{\sqrt{x^2-1}}=0$. We get $2x=0$ or $x=0$. However, $x=0$ isn't in the domain of k, which means k has no zeros. We construct our sign diagram on the domain of k below alongside the graph of k. It appears that the graph of k has two vertical asymptotes, one at $x=-1$ and one at $x=1$. The gap in the graph between the asymptotes is because of the gap in the domain of k. Concerning end behavior, there appear to be two horizontal asymptotes, $y=2$ and $y=-2$. To see why this is the case, we think of $x\to\pm\infty$. The radicand of the denominator $x^2-1\approx x^2$, and as such, $k(x)=\frac{2x}{\sqrt{x^2-1}}\approx\frac{2x}{\sqrt{x^2}}=\frac{2x}{|x|}$. As $x\to\infty$, we have $|x|=x$ so $k(x)\approx\frac{2x}{x}=2$. On the other hand, as $x\to-\infty$, $|x|=-x$, and as such $k(x)\approx\frac{2x}{-x}=-2$. Finally, it appears as though the graph of k passes the Horizontal Line Test which means k is one to one and k^{-1} exists. Computing k^{-1} is left as an exercise.





$y = k(x)$

As the previous example illustrates, the graphs of general algebraic functions can have features we've seen before, like vertical and horizontal asymptotes, but they can occur in new and exciting ways. For example, $k(x) = \frac{2x}{\sqrt{x^2-1}}$ had two distinct horizontal asymptotes. You'll recall that rational functions could have at most one horizontal asymptote. Also some new characteristics like 'unusual steepness'[8] and cusps[9] can appear in the graphs of arbitrary algebraic functions. Our next example first demonstrates how we can use sign diagrams to solve nonlinear inequalities. (Don't panic. The technique is very similar to the ones used in Chapters 2, 3 and 4.) We then check our answers graphically with a calculator and see some of the new graphical features of the functions in this extended family.

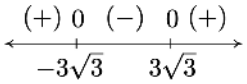
Example 5.3.2

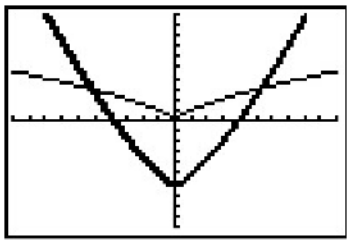
Solve the following inequalities. Check your answers graphically with a calculator.

1. $x^{2/3} < x^{4/3} - 6$
2. $3(2-x)^{1/3} \leq x(2-x)^{-2/3}$

Solution.

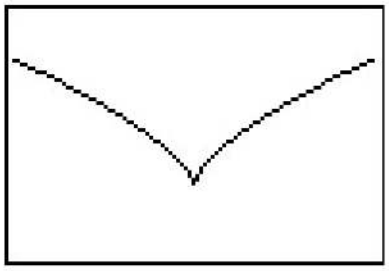
1. To solve $x^{2/3} < x^{4/3} - 6$, we get 0 on one side and attempt to solve $x^{4/3} - x^{2/3} - 6 > 0$. We set $r(x) = x^{4/3} - x^{2/3} - 6$ and note that since the denominators in the exponents are 3, they correspond to cube roots, which means the domain of r is $(-\infty, \infty)$. To find the zeros for the sign diagram, we set $r(x) = 0$ and attempt to solve $x^{4/3} - x^{2/3} - 6 = 0$. At this point, it may be unclear how to proceed. We could always try as a last resort converting back to radical notation, but in this case we can take a cue from Example 3.3.4. Since there are three terms, and the exponent on one of the variable terms, $x^{4/3}$, is exactly twice that of the other, $x^{2/3}$, we have ourselves a 'quadratic in disguise' and we can rewrite $x^{4/3} - x^{2/3} - 6 = 0$ as $\left(x^{2/3}\right)^2 - x^{2/3} - 6 = 0$. If we let $u = x^{2/3}$, then in terms of u, we get $u^2 - u - 6 = 0$. Solving for u, we obtain $u = -2$ or $u = 3$. Replacing $x^{2/3}$ back in for u, we get $x^{2/3} = -2$ or $x^{2/3} = 3$. To avoid the trouble we encountered in the discussion following Definition 5.5, we now convert back to radical notation. By interpreting $x^{2/3}$ as $\sqrt[3]{x^2}$ we have $\sqrt[3]{x^2} = -2$ or $\sqrt[3]{x^2} = 3$. Cubing both sides of these equations results in $x^2 = -8$, which admits no real solution, or $x^2 = 27$, which gives $x = \pm 3\sqrt{3}$. We construct a sign diagram and find $x^{4/3} - x^{2/3} - 6 > 0$ on $\left(-\infty, -3\sqrt{3}\right) \cup \left(3\sqrt{3}, \infty\right)$. To check our answer graphically, we set $f(x) = x^{2/3}$ and $g(x) = x^{4/3} - 6$. The solution to $x^{2/3} < x^{4/3} - 6$ corresponds to the inequality $f(x) < g(x)$, which means we are looking for the x values for which the graph of f is below the graph of g. Using the 'Intersect' command we confirm[10] that the graphs cross at $x = \pm 3\sqrt{3}$. We see that the graph of f is below the graph of g (the thicker curve) on $\left(-\infty, -3\sqrt{3}\right) \cup \left(3\sqrt{3}, \infty\right)$.



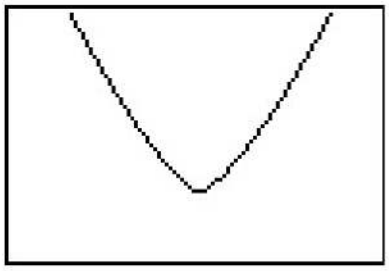


$y = f(x)$ and $y = g(x)$

As a point of interest, if we take a closer look at the graphs of f and g near $x = 0$ with the axes off, we see that despite the fact they both involve cube roots, they exhibit different behavior near $x = 0$. The graph of f has a sharp turn, or cusp, while g does not.[11]



$y = f(x)$ near $x = 0$



$y = g(x)$ near $x = 0$

2. To solve $3(2-x)^{1/3} \le x(2-x)^{-2/3}$, we gather all the nonzero terms on one side and obtain $3(2-x)^{1/3} - x(2-x)^{-2/3} \le 0$. We set $r(x) = 3(2-x)^{1/3} - x(2-x)^{-2/3}$. As in number 1, the denominators of the rational exponents are odd, which means there are no domain concerns there. However, the negative exponent on the second term indicates a denominator. Rewriting $r(x)$ with positive exponents, we obtain

$$r(x) = 3(2-x)^{1/3} - \frac{x}{(2-x)^{2/3}} \tag{4.2.1}$$

Setting the denominator equal to zero we get $(2-x)^{2/3} = 0$, or $\sqrt[3]{(2-x)^2} = 0$. After cubing both sides, and subsequently taking square roots, we get $2 - x = 0$, or $x = 2$. Hence, the domain of r is $(-\infty, 2) \cup (2, \infty)$. To find the zeros of r, we set $r(x) = 0$. There are two school of thought on how to proceed and we demonstrate both.

- $$\begin{aligned} r(x) &= 3(2-x)^{1/3} - x(2-x)^{-2/3} \\ &= (2-x)^{-2/3}\left[3(2-x)^{\frac{1}{3}-\left(-\frac{2}{3}\right)} - x\right] \\ [6pt] &= (2-x)^{-2/3}\left[3(2-x)^{3/3} - x\right] \\ &= (2-x)^{-2/3}\left[3(2-x)^{1} - x\right] \qquad \text{since } \sqrt[3]{u^3} = (\sqrt[3]{u})^3 = u \\ &= (2-x)^{-2/3}(6-4x) \\ &= (2-x)^{-2/3}(6-4x) \end{aligned}$$

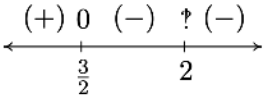
  To solve $r(x) = 0$, we set $(2-x)^{-2/3}(6-4x) = 0$, or $\frac{6-4x}{(2-x)^{2/3}} = 0$. We have $6 - 4x = 0$ or $x = \frac{3}{2}$.

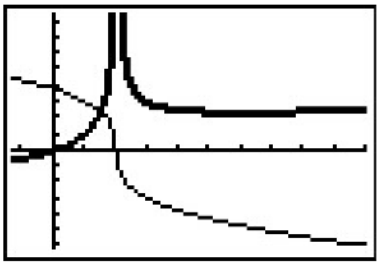
- *Common Denominator Approach.* We rewrite

$$
\begin{aligned}
r(x) &= 3(2-x)^{1/3} - x(2-x)^{-2/3} \\
&= 3(2-x)^{1/3} - \frac{x}{(2-x)^{2/3}} \\
[10pt] &= \frac{3(2-x)^{1/3}(2-x)^{2/3}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} && \text{common denominator} \\
[10pt] &= \frac{3(2-x)^{\frac{1}{3}+\frac{2}{3}}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} \\
[10pt] &= \frac{3(2-x)^{3/3}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} \\
[10pt] &= \frac{3(2-x)^{1}}{(2-x)^{2/3}} - \frac{x}{(2-x)^{2/3}} && \text{since } \sqrt[3]{u^3} = (\sqrt[3]{u})^3 = u \\
[10pt] &= \frac{3(2-x)-x}{(2-x)^{2/3}} \\
[10pt] &= \frac{6-4x}{(2-x)^{2/3}}
\end{aligned}
$$

As before, when we set $r(x) = 0$ we obtain $x = \frac{3}{2}$.

We now create our sign diagram and find $3(2-x)^{1/3} - x(2-x)^{-2/3} \leq 0$ on $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$. To check this graphically, we set $f(x) = 3(2-x)^{1/3}$ and $g(x) = x(2-x)^{-2/3}$ (the thicker curve). We confirm that the graphs intersect at $x = \frac{3}{2}$ and the graph of f is below the graph of g for $x \geq \frac{3}{2}$, with the exception of $x = 2$ where it appears the graph of g has a vertical asymptote.



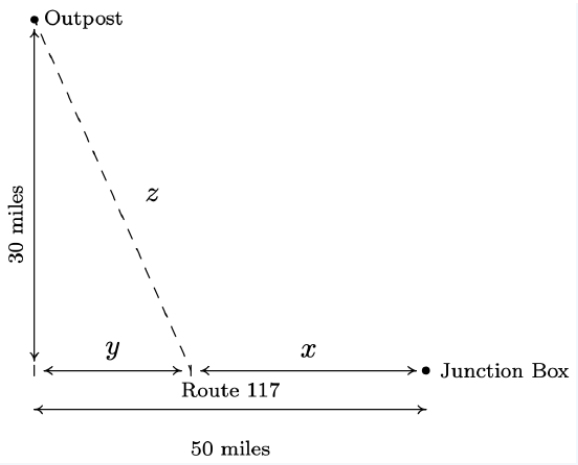


$y = f(x)$ and $\boldsymbol{y = g(x)}$

One application of algebraic functions was given in Example 1.6.6 in Section 1.1. Our last example is a more sophisticated application of distance.

Example 5.3.3

Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles downroad from the post, as indicated in the diagram below. Suppose it costs \$15 per mile to run cable along the road and \$20 per mile to run cable off of the road.



1. Express the total cost C of connecting the Junction Box to the Outpost as a function of x, the number of miles the cable is run along Route 117 before heading off road directly towards the Outpost. Determine a reasonable applied domain for the problem.
2. Use your calculator to graph $y = C(x)$ on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off of the road?

Solution.

1. The cost is broken into two parts: the cost to run cable along Route 117 at $\$15$ per mile, and the cost to run it off road at $\$20$ per mile. Since x represents the miles of cable run along Route 117, the cost for that portion is $15x$. From the diagram, we see that the number of miles the cable is run off road is z, so the cost of that portion is $20z$. Hence, the total cost is $C = 15x + 20z$. Our next goal is to determine z as a function of x. The diagram suggests we can use the Pythagorean Theorem to get $y^2 + 30^2 = z^2$. But we also see $x + y = 50$ so that $y = 50 - x$. Hence, $z^2 = (50 - x)^2 + 900$. Solving for z, we obtain $z = \pm\sqrt{(50 - x)^2 + 900}$. Since z represents a distance, we choose $z = \sqrt{(50 - x)^2 + 900}$ so that our cost as a function of x only is given by

$$C(x) = 15x + 20\sqrt{(50 - x)^2 + 900}$$

From the context of the problem, we have $0 \leq x \leq 50$.
2. Graphing $y = C(x)$ on a calculator and using the 'Minimum' feature, we find the relative minimum (which is also the absolute minimum in this case) to two decimal places to be $(15.98, 1146.86)$ Here the x-coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route 117 and then turn off of the road and head towards the outpost. The y-coordinate tells us that the minimum cost, in dollars, to do so is $\$1146.86$ The ability to stream live SasquatchCasts? Priceless.

5.3.1 Exercises

For each function in Exercises 1 - 10 below

- Find its domain.
- Create a sign diagram.
- Use your calculator to help you sketch its graph and identify any vertical or horizontal asymptotes, 'unusual steepness' or cusps.

1. $f(x) = \sqrt{1 - x^2}$
2. $f(x) = \sqrt{x^2 - 1}$
3. $f(x) = x\sqrt{1 - x^2}$
4. $f(x) = x\sqrt{x^2 - 1}$
5. $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$
6. $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$

7. $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$
8. $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$
9. $f(x) = \sqrt{x(x+5)(x-4)}$
10. $f(x) = \sqrt[3]{x^3+3x^2-6x-8}$

In Exercises 11 - 16, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using the transformations presented in Section 1.7.

11. $f(x) = \sqrt[3]{x}$, $g(x) = \sqrt[3]{x-1} - 2$
12. $f(x) = \sqrt[3]{x}$, $g(x) = -2\sqrt[3]{x+1} + 4$
13. $f(x) = \sqrt[4]{x}$, $g(x) = \sqrt[4]{x-1} - 2$
14. $f(x) = \sqrt[4]{x}$, $g(x) = 3\sqrt[4]{x-7} - 1$
15. $f(x) = \sqrt[5]{x}$, $g(x) = \sqrt[5]{x+2} + 3$
16. $f(x) = \sqrt[8]{x}$, $g(x) = \sqrt[8]{-x} - 2$
17. $x + 1 = \sqrt{3x+7}$
18. $2x + 1 = \sqrt{3-3x}$
19. $x + \sqrt{3x+10} = -2$
20. $3x + \sqrt{6-9x} = 2$
21. $2x - 1 = \sqrt{x+3}$
22. $x^{\frac{3}{2}} = 8$
23. $x^{\frac{2}{3}} = 4$
24. $\sqrt{x-2} + \sqrt{x-5} = 3$
25. $\sqrt{2x+1} = 3 + \sqrt{4-x}$
26. $5 - (4-2x)^{\frac{2}{3}} = 1$
27. $10 - \sqrt{x-2} \leq 11$
28. $\sqrt[3]{x} \leq x$
29. $2(x-2)^{-\frac{1}{3}} - \frac{2}{3}x(x-2)^{-\frac{4}{3}} \leq 0$
30. $-\frac{4}{3}(x-2)^{-\frac{4}{3}} + \frac{8}{9}x(x-2)^{-\frac{7}{3}} \geq 0$
31. $2x^{-\frac{1}{3}}(x-3)^{\frac{1}{3}} + x^{\frac{2}{3}}(x-3)^{-\frac{2}{3}} \geq 0$
32. $\sqrt[3]{x^3+3x^2-6x-8} > x+1$
33. $\frac{1}{3}x^{\frac{3}{4}}(x-3)^{-\frac{2}{3}} + \frac{3}{4}x^{-\frac{1}{4}}(x-3)^{\frac{1}{3}} < 0$
34. $x^{-\frac{1}{3}}(x-3)^{-\frac{2}{3}} - x^{-\frac{4}{3}}(x-3)^{-\frac{5}{3}}(x^2-3x+2) \geq 0$
35. $\frac{2}{3}(x+4)^{\frac{3}{5}}(x-2)^{-\frac{1}{3}} + \frac{3}{5}(x+4)^{-\frac{2}{5}}(x-2)^{\frac{2}{3}} \geq 0$
36. Rework Example 5.3.3 so that the outpost is 10 miles from Route 117 and the nearest junction box is 30 miles down the road for the post.
37. The volume V of a right cylindrical cone depends on the radius of its base r and its height h and is given by the formula $V = \frac{1}{3}\pi r^2 h$. The surface area S of a right cylindrical cone also depends on r and h according to the formula $S = \pi r\sqrt{r^2+h^2}$. Suppose a cone is to have a volume of 100 cubic centimeters.
    a. Use the formula for volume to find the height h as a function of r.
    b. Use the formula for surface area and your answer to 37a to find the surface area S as a function of r.
    c. Use your calculator to find the values of r and h which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.
38. The National Weather Service uses the following formula to calculate the wind chill:

$$W = 35.74 + 0.6215\,T_a - 35.75\,V^{0.16} + 0.4275\,T_a\,V^{0.16}$$

where W is the wind chill temperature in $^\circ$F, T_a is the air temperature in $^\circ$F, and V is the wind speed in miles per hour. Note that W is defined only for air temperatures at or lower than 50°F and wind speeds above 3 miles per hour.
    a. Suppose the air temperature is 42° and the wind speed is 7 miles per hour. Find the wind chill temperature. Round your answer to two decimal places.

b. Suppose the air temperature is 37°F and the wind chill temperature is 30°F. Find the wind speed. Round your answer to two decimal places.

39. As a follow-up to Exercise 38, suppose the air temperature is 28°F.
    a. Use the formula from Exercise 38 to find an expression for the wind chill temperature as a function of the wind speed, $W(V)$.
    b. Solve $W(V) = 0$, round your answer to two decimal places, and interpret.
    c. Graph the function W using your calculator and check your answer to part 39b.
40. The period of a pendulum in seconds is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

(for small displacements) where L is the length of the pendulum in meters and $g = 9.8$ meters per second per second is the acceleration due to gravity. My Seth-Thomas antique schoolhouse clock needs $T = \frac{1}{2}$ second and I can adjust the length of the pendulum via a small dial on the bottom of the bob. At what length should I set the pendulum?

41. The Cobb-Douglas production model states that the yearly total dollar value of the production output P in an economy is a function of labor x (the total number of hours worked in a year) and capital y (the total dollar value of all of the stuff purchased in order to make things). Specifically, $P = ax^b y^{1-b}$. By fixing P, we create what's known as an 'isoquant' and we can then solve for y as a function of x. Let's assume that the Cobb-Douglas production model for the country of Sasquatchia is $P = 1.23x^{0.4}y^{0.6}$.
    a. Let $P = 300$ and solve for y in terms of x. If $x = 100$, what is y?
    b. Graph the isoquant $300 = 1.23x^{0.4}y^{0.6}$. What information does an ordered pair (x, y) which makes $P = 300$ give you? With the help of your classmates, find several different combinations of labor and capital all of which yield $P = 300$. Discuss any patterns you may see.
42. According to Einstein's Theory of Special Relativity, the observed mass m of an object is a function of how fast the object is traveling. Specifically,

$$m(x) = \frac{m_r}{\sqrt{1 - \dfrac{x^2}{c^2}}}$$

where $m(0) = m_r$ is the mass of the object at rest, x is the speed of the object and c is the speed of light.
    a. Find the applied domain of the function.
    b. Compute $m(.1c)$, $m(.5c)$, $m(.9c)$and $m(.999c)$.
    c. As $x \to c^-$, what happens to $m(x)$?
    d. How slowly must the object be traveling so that the observed mass is no greater than 100 times its mass at rest?
43. Find the inverse of $k(x) = \dfrac{2x}{\sqrt{x^2 - 1}}$.
44. Suppose Fritzy the Fox, positioned at a point (x, y) in the first quadrant, spots Chewbacca the Bunny at $(0, 0)$. Chewbacca begins to run along a fence (the positive y-axis) towards his warren. Fritzy, of course, takes chase and constantly adjusts his direction so that he is always running directly at Chewbacca. If Chewbacca's speed is v_1 and Fritzy's speed is v_2, the path Fritzy will take to intercept Chewbacca, provided v_2 is directly proportional to, but not equal to, v_1 is modeled by

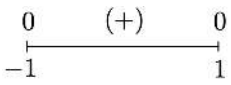
$$y = \frac{1}{2}\left(\frac{x^{1+v_1/v_2}}{1+v_1/v_2} - \frac{x^{1-v_1/v_2}}{1-v_1/v_2}\right) + \frac{v_1 v_2}{v_2^2 - v_1^2} \tag{4.2.2}$$

    a. Determine the path that Fritzy will take if he runs exactly twice as fast as Chewbacca; that is, $v_2 = 2v_1$. Use your calculator to graph this path for $x \geq 0$. What is the significance of the y-intercept of the graph?
    b. Determine the path Fritzy will take if Chewbacca runs exactly twice as fast as he does; that is, $v_1 = 2v_2$. Use your calculator to graph this path for $x > 0$. Describe the behavior of y as $x \to 0^+$ and interpret this physically.
    c. With the help of your classmates, generalize parts (a) and (b) to two cases: $v_2 > v_1$ and $v_2 < v_1$. We will discuss the case of $v_1 = v_2$ in Exercise 32 in Section 6.5.
45. Verify the Quotient Rule for Radicals in Theorem 5.6.

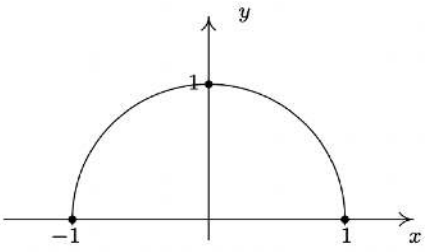
46. Show that $\left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = x$ for all $x \geq 0$.
47. Show that $\sqrt[3]{2}$ is an irrational number by first showing that it is a zero of $p(x) = x^3 - 2$ and then showing p has no rational zeros. (You'll need the Rational Zeros Theorem, Theorem 3.9, in order to show this last part.)
48. With the help of your classmates, generalize Exercise 47 to show that $\sqrt[n]{c}$ is an irrational number for any natural numbers $c \geq 2$ and $n \geq 2$ provided that $c \neq p^n$ for some natural number p.

5.3.2 Answers

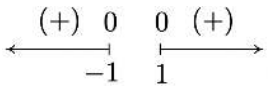
1. $f(x) = \sqrt{1 - x^2}$
   Domain: $[-1, 1]$

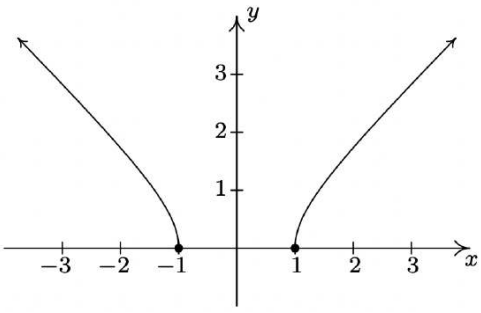
   No asymptotes
   Unusual steepness at $x = -1$ and $x = 1$
   No cusps

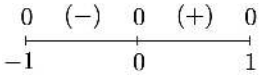
2. $f(x) = \sqrt{x^2 - 1}$
   Domain: $(-\infty, -1] \cup [1, \infty)$

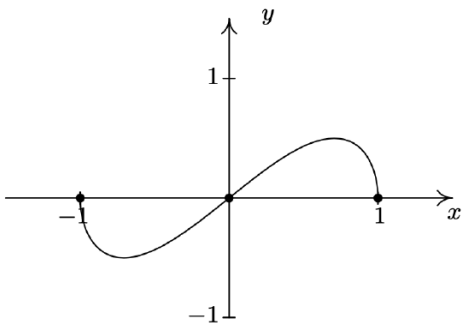
   No asymptotes
   Unusual steepness at $x = -1$ and $x = 1$
   No cusps

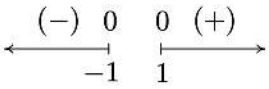
3. $f(x) = x\sqrt{1 - x^2}$
   Domain: $[-1, 1]$

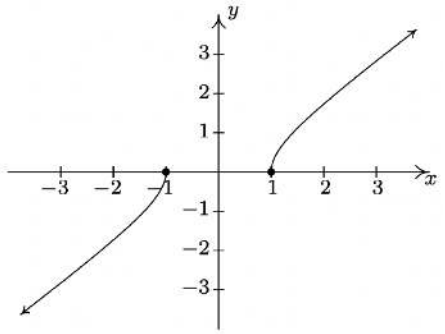
   No asymptotes
   Unusual steepness at $x = -1$ and $x = 1$
   No cusps



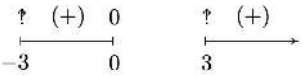
4. $f(x) = x\sqrt{x^2 - 1}$
   Domain: $(-\infty, -1] \cup [1, \infty)$



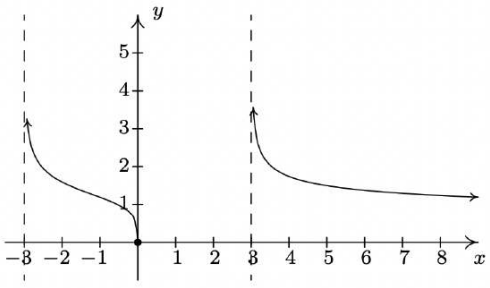
   No asymptotes
   Unusual steepness at $x = -1$ and $x = 1$
   No cusps



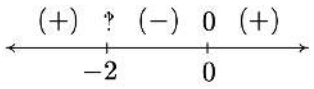
5. $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$
   Domain: $(-3, 0] \cup (3, \infty)$



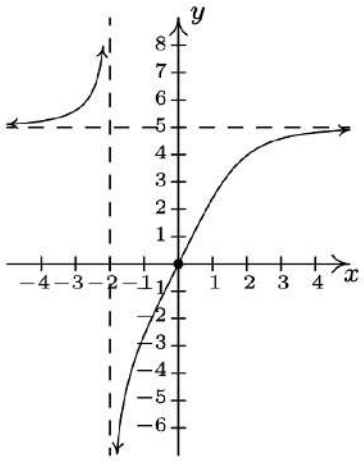
   Vertical asymptotes: $x = -3$ and $x = 3$
   Horizontal asymptote: $y = 0$
   Unusual steepness at $x = 0$
   No cusps



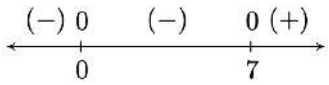
6. $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$
   Domain: $(-\infty, -2) \cup (-2, \infty)$



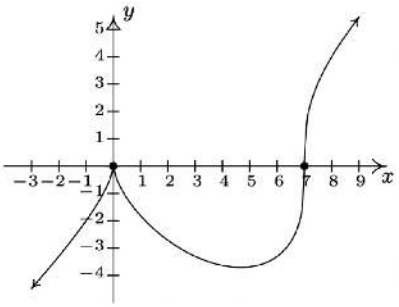
Vertical asymptote $x = -2$
Horizontal asymptote $y = 5$
No unusual steepness or cusps



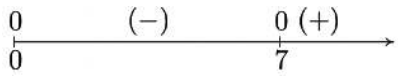
7. 2 $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$
Domain: $(-\infty, \infty)$



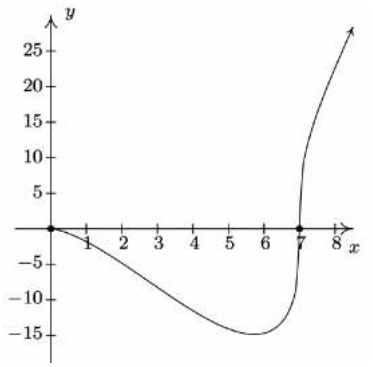
No vertical or horizontal asymptotes[13]
Unusual steepness at $x = 7$
Cusp at $x = 0$



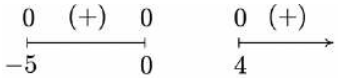
8. $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$
Domain: $[0, \infty)$



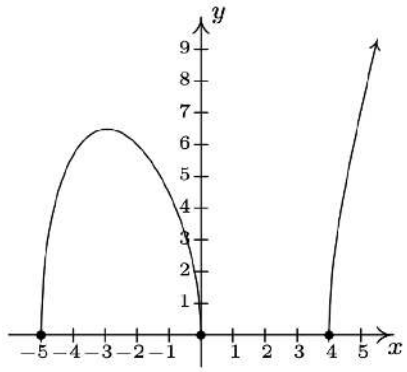
No asymptotes
Unusual steepness at $x = 7$
No cusps



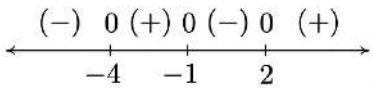
9. $f(x) = \sqrt{x(x+5)(x-4)}$
   Domain: $[-5, 0] \cup [4, \infty)$



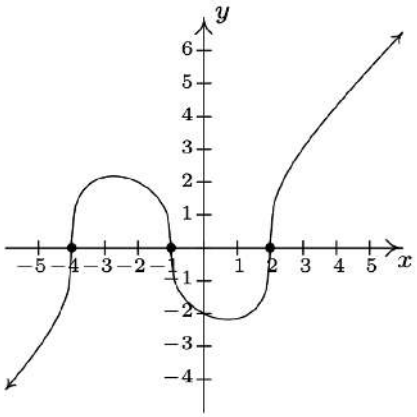
   No asymptotes
   Unusual steepness at $x = -5, x = 0$ and $x = 4$
   No cusps



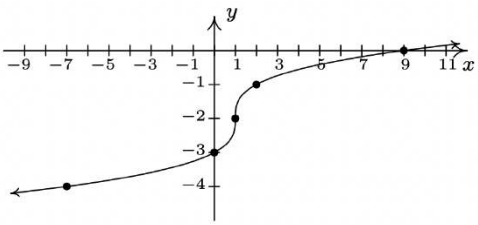
10. 2 $f(x) = \sqrt[3]{x^3 + 3x^2 - 6x - 8}$
    Domain: $(-\infty, \infty)$



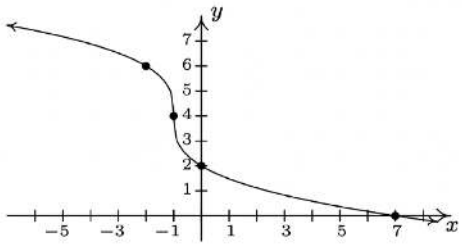
    No vertical or horizontal asymptotes[14]
    Unusual steepness at $x = -4, x = -1$ and $x = 2$
    No cusps



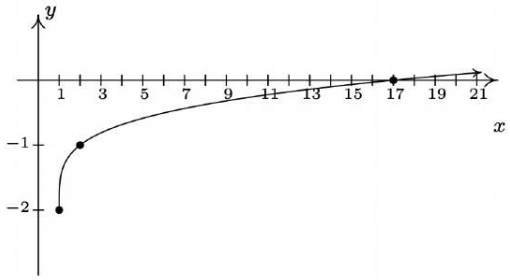
11. $g(x) = \sqrt[3]{x-1} - 2$



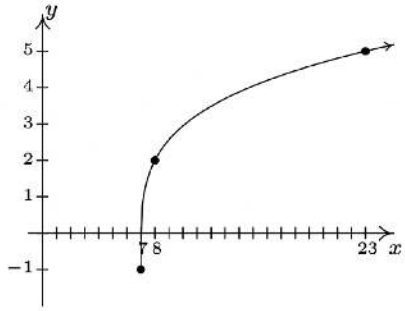
12. $g(x) = -2\sqrt[3]{x+1} + 4$



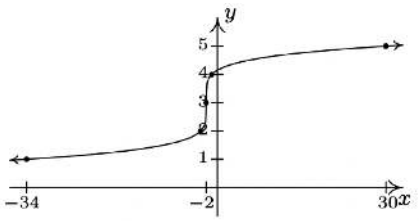
13. $g(x) = \sqrt[4]{x-1} - 2$



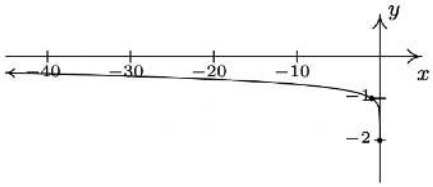
14. $g(x) = 3\sqrt[4]{x-7} - 1$



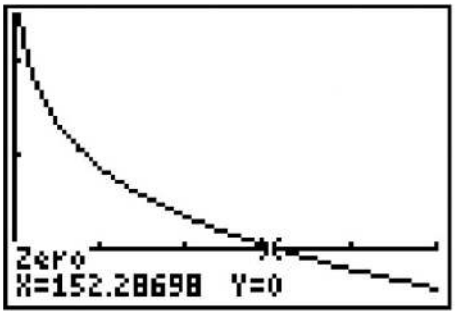
15. $g(x) = \sqrt[5]{x+2} + 3$



16. $g(x) = \sqrt[8]{-x} - 2$



17. $x = 3$
18. $x = \frac{1}{4}$
19. $x = -3$
20. $x = -\frac{1}{3}, \frac{2}{3}$
21. $x = \frac{5+\sqrt{57}}{8}$
22. $x = 4$
23. $x = \pm 8$
24. $x = 6$
25. $x = 4$
26. $x = -2, 6$
27. $[2, \infty)$
28. $[-1, 0] \cup [1, \infty)$
29. $(-\infty, 2) \cup (2, 3]$
30. $(2, 6]$
31. $(-\infty, 0) \cup [2, 3) \cup (3, \infty)$
32. $(-\infty, -1)$
33. $\left(0, \frac{27}{13}\right)$
34. $(-\infty, 0) \cup (0, 3)$
35. $(-\infty, -4) \cup \left(-4, -\frac{22}{19}\right] \cup (2, \infty)$
36. $C(x) = 15x + 20\sqrt{100 + (30-x)^2}$, $0 \le x \le 30$. The calculator gives the absolute minimum at $\approx (18.66, 582.29)$ This means to minimize the cost, approximately 18.66 miles of cable should be run along Route 117 before turning off the road and heading towards the outpost. The minimum cost to run the cable is approximately $\$582.29$
37. a. $h(r) = \frac{300}{\pi r^2}$, $r > 0$.
    b. $S(r) = \pi r \sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2} = \frac{\sqrt{\pi^2 r^6 + 90000}}{r}$, $r > 0$
    c. The calculator gives the absolute minimum at the point $\approx (4.07, 90.23)$ This means the radius should be (approximately) 4.07 centimeters and the height should be 5.76 centimeters to give a minimum surface area of 90.23 square centimeters.
38. a. $W \approx 37.55^\circ$ F.
    b. $V \approx 9.84$ miles per hour.
39. a. $W(V) = 53.142 - 23.78V^{0.16}$. Since we are told in Exercise 38 that wind chill is only effect for wind speeds of more than 3 miles per hour, we restrict the domain to $V > 3$.
    b. $W(V) = 0$ when $V \approx 152.29$. This means, according to the model, for the wind chill temperature to be 0°F, the wind speed needs to be 152.29miles per hour.
    c. The graph is below.



40. $9.8\left(\dfrac{1}{4\pi}\right)^2 \approx 0.062$ meters or 6.2 centimeters

41. a. First rewrite the model as $P = 1.23x^{\frac{2}{5}}y^{\frac{3}{5}}$. Then $300 = 1.23x^{\frac{2}{5}}y^{\frac{3}{5}}$ yields $y = \left(\dfrac{300}{1.23x^{\frac{2}{5}}}\right)^{\frac{5}{3}}$. If $x = 100$ then $y \approx 441.93687$.

42. a. $[0, c)$

b. $m(.1c) = \dfrac{m_r}{\sqrt{.99}} \approx 1.005m_r \quad m(.5c) = \dfrac{m_r}{\sqrt{.75}} \approx 1.155m_r$

$m(.9c) = \dfrac{m_r}{\sqrt{.19}} \approx 2.294m_r \quad m(.999c) = \dfrac{m_r}{\sqrt{.0.001999}} \approx 22.366m_r$

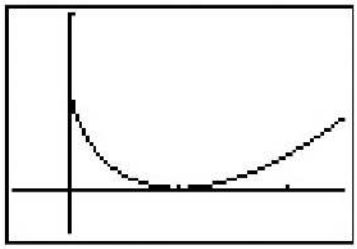
c. As $x \to c^-$, $m(x) \to \infty$

d. If the object is traveling no faster than approximately 0.99995 times the speed of light, then its observed mass will be no greater than $100m_r$.

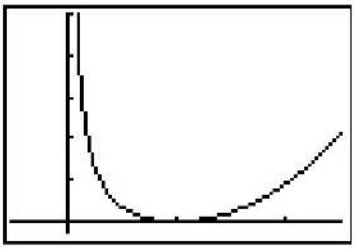
43. $k^{-1}(x) = \dfrac{x}{\sqrt{x^2-4}}$

44. a. $y = \frac{1}{3}x^{3/2} - \sqrt{x} + \frac{2}{3}$. The point $\left(0, \frac{2}{3}\right)$ is when Fritzy's path crosses Chewbacca's path - in other words, where Fritzy catches Chewbacca.

b. $y = \frac{1}{6}x^3 + \frac{1}{2x} - \frac{2}{3}$. Using the techniques from Chapter 4, we find as $x \to 0^+$, $y \to \infty$ which means, in this case, Fritzy's pursuit never ends; he never catches Chewbacca. This makes sense since Chewbacca has a head start and is running faster than Fritzy.



$y = \frac{1}{3}x^{3/2} - \sqrt{x} + \frac{2}{3}$



$y = \frac{1}{6}x^3 + \frac{1}{2x} - \frac{2}{3}$

Reference

[1] Although we discussed imaginary numbers in Section 3.4, we restrict our attention to real numbers in this section. See the epilogue on page 294 for more details.

[2] Otherwise we'd run into the same paradox we did in Section 3.4.

[3] Did you like that pun?

[4] In most other cases, though, rational exponents are preferred.

[5] As mentioned in Section 2.2, $f(x) = \sqrt{x^2} = |x|$ so that absolute value is also considered an algebraic function.

[6] For instance, $-2 \geq \sqrt[4]{x+3}$, which has no solution or $-2 \leq \sqrt[4]{x+3}$ 3 whose solution is $[-3, \infty)$.

[7] Recall, this means we have produced a candidate which doesn't satisfy the original equation. Do you remember how raising both sides of an equation to an even power could cause this.

[8] The proper Calculus term for this is 'vertical tangent', but for now we'll be okay calling it 'unusual steepness'.

[9] See page 241 for the first reference to this feature.

[10] Or at least confirm to several decimal places

[11] Again, we introduced this feature on page 241 as a feature which makes the graph of a function 'not smooth'.

[12] And we exercise special care when reducing the $\frac{3}{3}$ power to 1.

[13] Using Calculus it can be shown that $y = x - \frac{7}{3}$ is a slant asymptote of this graph.

[14] Using Calculus it can be shown that $y = x + 1$ is a slant asymptote of this graph.

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CHAPTER OVERVIEW

5: Exponential and Logarithmic Functions

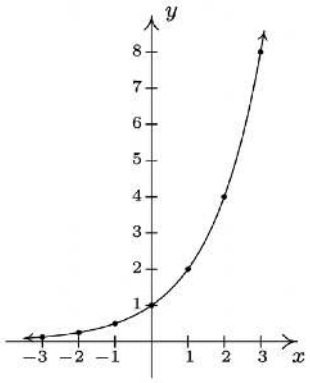
5.1: Introduction to Exponential and Logarithmic Functions
5.2: Properties of Logarithms
5.3: Exponential Equations and Inequalities
5.4: Logarithmic Equations and Inequalities
5.5: Applications of Exponential and Logarithmic Functions

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5.1: Introduction to Exponential and Logarithmic Functions

Of all of the functions we study in this text, exponential and logarithmic functions are possibly the ones which impact everyday life the most.[1] This section introduces us to these functions while the rest of the chapter will more thoroughly explore their properties. Up to this point, we have dealt with functions which involve terms like x^2 or $x^{2/3}$, in other words, terms of the form x^p where the base of the term, x, varies but the exponent of each term, p, remains constant. In this chapter, we study functions of the form $f(x) = b^x$ where the base b is a constant and the exponent x is the variable. We start our exploration of these functions with $f(x) = 2^x$. (Apparently this is a tradition. Every College Algebra book we have ever read starts with $f(x) = 2^x$.) We make a table of values, plot the points and connect the dots in a pleasing fashion.

x	$f(x)$	$(x, f(x))$
-3	$2^{-3} = \frac{1}{8}$	$\left(-3, \frac{1}{8}\right)$
-2	$2^{-2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
-1	$2^{-1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$
2	$2^2 = 4$	$(2, 4)$
3	$2^3 = 8$	$(3, 8)$



$y = f(x) = 2^x$

A few remarks about the graph $f(x) = 2^x$ which we have constructed are in order. As $x \to -\infty$ and attains values like $x = -100$ or $x = -1000$, the function $f(x) = 2^x$ takes on values like $f(-100) = 2^{-100} = \frac{1}{2^{100}}$ or $f(-1000) = 2^{-1000} = \frac{1}{2^{1000}}$. In other words, as $x \to -\infty$,

$$2^x \approx \frac{1}{\text{very big}(+)} \approx \text{ very small } (+)$$

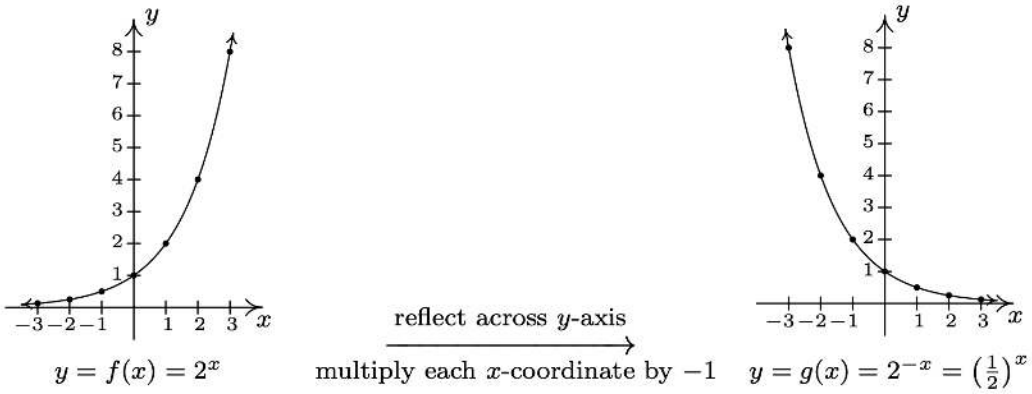
So as $x \to -\infty$, $2^x \to 0^+$. This is represented graphically using the x-axis (the line $y = 0$) as a horizontal asymptote. On the flip side, as $x \to \infty$, we find $f(100) = 2^{100}$, $f(1000) = 2^{1000}$, and so on, thus $2^x \to \infty$. As a result, our graph suggests the range of f is $(0, \infty)$. The graph of f passes the Horizontal Line Test which means f is one-to-one and hence invertible. We also note that when we 'connected the dots in a pleasing fashion', we have made the implicit assumption that $f(x) = 2^x$ is continuous[2] and has a domain of all real numbers. In particular, we have suggested that things like $2^{\sqrt{3}}$ exist as real numbers. We should take a moment to discuss what something like $2^{\sqrt{3}}$ might mean, and refer the interested reader to a solid course in Calculus for a more rigorous explanation. The number $\sqrt{3} = 1.73205\ldots$ is an irrational number[3] and as such, its decimal representation neither repeats nor terminates. We can, however, approximate $\sqrt{3}$ by terminating decimals, and it stands to reason[4] we can use these to approximate $2^{\sqrt{3}}$. For example, if we approximate $\sqrt{3}$ by 1.73, we can approximate $2^{\sqrt{3}} \approx 2^{1.73} = 2^{\frac{173}{100}} = \sqrt[100]{2^{173}}$. It is not, by any means, a pleasant number, but it is at least a number that we understand in terms of powers and roots. It also stands to reason that better and better approximations of $\sqrt{3}$ yield better and better approximations of $2^{\sqrt{3}}$, so the value of $2^{\sqrt{3}}$ should be the result of this sequence of approximations.[5]

Suppose we wish to study the family of functions $f(x) = b^x$. Which bases b make sense to study? We find that we run into difficulty if $b < 0$. For example, if $b = -2$, then the function $f(x) = (-2)^x$ has trouble, for instance, at $x = \frac{1}{2}$ since $(-2)^{1/2} = \sqrt{-2}$ is not a real number. In general, if x is any rational number with an even denominator, then $(-2)^x$ is not defined, so we must restrict our attention to bases $b \geq 0$. What about $b = 0$? The function $f(x) = 0^x$ is undefined for $x \leq 0$ because we cannot divide by 0 and 0^0 is an indeterminant form. For $x > 0$, $0^x = 0$ so the function $f(x) = 0^x$ is the same as the function $f(x) = 0$, $x > 0$. We know everything we can possibly know about this function, so we exclude it from our investigations. The only other base we exclude is $b = 1$, since the function $f(x) = 1^x = 1$ is, once again, a function we have already studied. We are now ready for our definition of exponential functions.

Definition 6.1

A function of the form $f(x) = b^x$ where b is a fixed real number, $b > 0$, $b \neq 1$ is called a base b exponential function.

We leave it to the reader to verify[6] that if $b > 1$, then the exponential function $f(x) = b^x$ will share the same basic shape and characteristics as $f(x) = 2^x$. What if $0 < b < 1$? Consider $g(x) = \left(\frac{1}{2}\right)^x$. We could certainly build a table of values and connect the points, or we could take a step back and note that $g(x) = \left(\frac{1}{2}\right)^x = \left(2^{-1}\right)^x = 2^{-x} = f(-x)$. Thinking back to Section 1.7, the graph of $f(-x)$ is obtained from the graph of $f(x)$ by reflecting it across the y-axis. We get



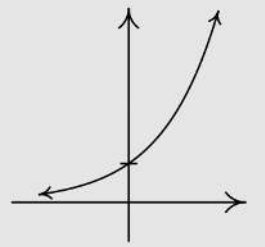
$y = f(x) = 2^x$ $\qquad$ $y = g(x) = 2^{-x} = \left(\frac{1}{2}\right)^x$

We see that the domain and range of g match that of f, namely $(-\infty, \infty)$ and $(0, \infty)$, Like f, g is also one-to-one. Whereas f is always increasing, g is always decreasing. As a result, as $x \to -\infty, g(x) \to \infty$, and on the flip side, as $x \to \infty, g(x) \to 0^+$. It shouldn't be too surprising that for all choices of the base $0 < b < 1$, the graph $y = b^x$ behaves similarly to the graph of g. We summarize the basic properties of exponential functions in the following theorem.[7]

Definition 6.1. Properties of Exponential Functions

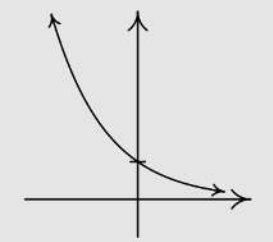
Support $f(x) = b^x$.

- ˆThe domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$.
- ˆ$(0, 1)$ is on the graph of f and $y = 0$ is a horizontal asymptote to the graph of f.
- ˆf is one-to-one, continuous and smooth[a]
- If $b > 1$:
  - f is always increasing
  - As $x \to -\infty$, $f(x) \to 0^+$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = b^x,\ b > 1$

- ^If $0 < b < 1$:
  - f is always decreasing
  - As $x \to -\infty$, $f(x) \to \infty$
  - As $x \to -\infty$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to 0^+$
  - The graph of f resembles:



$y = b^x,\ 0 < b < 1$

[a] Recall that this means the graph of f has no sharp turns or corners.

Of all of the bases for exponential functions, two occur the most often in scientific circles. The first, base 10, is often called the **common base**. The second base is an irrational number, $e \approx 2.718$, called the **natural base**. We will more formally discuss the origins of this number in Section 6.5. For now, it is enough to know that since $e > 1$, $f(x) = e^x$, is an increasing exponential function. The following examples give us an idea how these functions are used in the wild.

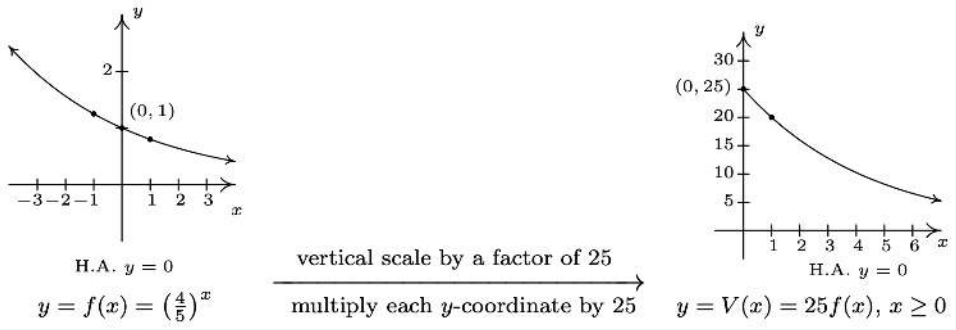
Example 6.1.1

The value of a car can be modeled by $V(x) = 25\left(\frac{4}{5}\right)^x$, where $x \geq 0$ is age of the car in years and $V(x)$ is the value in thousands of dollars.

1. Find and interpret $V(0)$.
2. Sketch the graph of $y = V(x)$ using transformations.
3. Find and interpret the horizontal asymptote of the graph you found in 2.

Solution

1. To find $V(0)$, we replace x with 0 to obtain $V(0) = 25\left(\frac{4}{5}\right)^0 = 25$. Since x represents the age of the car in years, $x = 0$ corresponds to the car being brand new. Since $V(x)$ is measured in thousands of dollars, $V(0) = 25$ corresponds to a value of \$25,000. Putting it all together, we interpret $V(0) = 25$ to mean the purchase price of the car was \$25,000.
2. To graph $y = 25\left(\frac{4}{5}\right)^x$, we start with the basic exponential function $f(x) = \left(\frac{4}{5}\right)^x$. Since the base $b = \frac{4}{5}$ is between 0 and 1, the graph of $y = f(x)$ is decreasing. We plot the y-intercept (0, 1) and two other points, $\left(-1, \frac{5}{4}\right)$ and $\left(1, \frac{4}{5}\right)$, and label the horizontal asymptote $y = 0$. To obtain $V(x) = 25\left(\frac{4}{5}\right)^x$, $x \geq 0$, we multiply the output from f by 25, in other words, $V(x) = 25f(x)$. In accordance with Theorem 1.5, this results in a vertical stretch by a factor of 25. We multiply all of the y values in the graph by 25 (including the y value of the horizontal asymptote) and obtain the $\left(-1, \frac{125}{4}\right)$, (0, 25) and (1, 20). The horizontal asymptote remains $y = 0$. Finally, we restrict the domain to $[0, \infty)$ to fit with the applied domain given to us. We have the result below.



3. We see from the graph of V that its horizontal asymptote is y = 0. (We leave it to reader to verify this analytically by thinking about what happens as we take larger and larger powers of $\frac{4}{5}$.) This means as the car gets older, its value diminishes to 0.

The function in the previous example is often called a 'decay curve'. Increasing exponential functions are used to model 'growth curves' and we shall see several different examples of those in Section 6.5. For now, we present another common decay curve which will serve as the basis for further study of exponential functions. Although it may look more complicated than the previous example, it is actually just a basic exponential function which has been modified by a few transformations from Section 1.7.

Example 6.1.2

According to Newton's Law of Cooling[8] the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$.

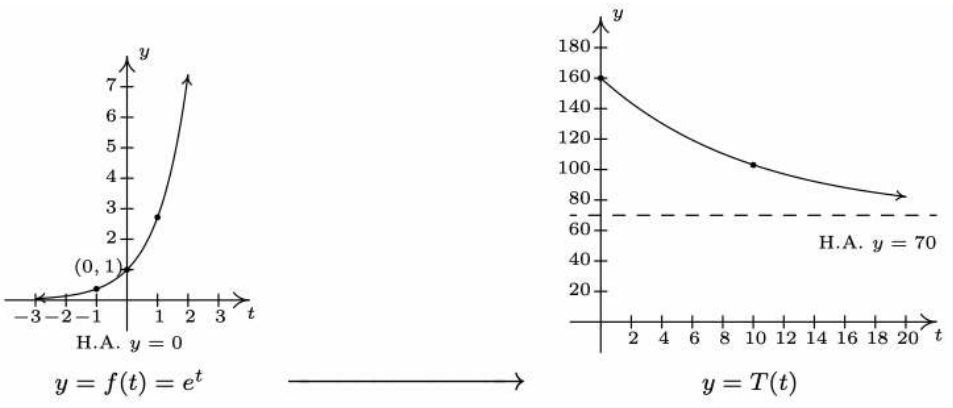
1. Find and interpret $T(0)$.
2. Sketch the graph of $y = T(t)$ using transformations.
3. Find and interpret the horizontal asymptote of the graph.

Solution

1. To find $T(0)$, we replace every occurrence of the independent variable t with 0 to obtain $T(0) = 70 + 90e^{-0.1(0)} = 160$. This means that the coffee was served at 160°F.
2. To graph $y = T(t)$ using transformations, we start with the basic function, $f(t) = e^t$. As we have already remarked, $e \approx 2.718 > 1$ so the graph of f is an increasing exponential with y-intercept (0, 1) and horizontal asymptote $y = 0$. The points $(-1, e^{-1}) \approx (-1, 0.37)$ and $(1, e) \approx (1, 2.72)$ are also on the graph. Since the formula $T(t)$ looks rather complicated, we rewrite $T(t)$ in the form presented in Theorem 1.7 and use that result to track the change to our three points and the horizontal asymptote. We have

$$T(t) = 70 + 90e^{-0.1t} = 90e^{-0.1t} + 70 = 90f(-0.1t) + 70$$

Multiplication of the input to f, t, by −0.1 results in a horizontal expansion by a factor of 10 as well as a reflection about the y-axis. We divide each of the x values of our points by −0. (which amounts to multiplying them by −10) to obtain $(10, e^{-1})$, $(0, 1)$, and $(-10, e)$. Since none of these changes affected the y values, the horizontal asymptote remains $y = 0$. Next, we see that the output from f is being multiplied by 90. This results in a vertical stretch by a factor of 90. We multiply the y-coordinates by 90 to obtain $(10, 90e^{-1})$, $(0, 90)$, $(-10, 90e)$. We also multiply the y value of the horizontal asymptote $y = 0$ by 90, and it remains $y = 0$. Finally, we add 70 to all of the y-coordinates, which shifts the graph upwards to obtain $(10, 90e^{-1} + 70) \approx (10, 103.11)$, $(0, 160)$, and $(-10, 90e + 70) \approx (-10, 314.64)$. Adding 70 to the horizontal asymptote shifts it upwards as well to $y = 70$. We connect these three points using the same shape in the same direction as in the graph of f and, last but not least, we restrict the domain to match the applied domain $[0, \infty)$. The result is below.



3. From the graph, we see that the horizontal asymptote is $y = 70$. It is worth a moment or two of our time to see how this happens analytically and to review some of the 'number sense' developed in Chapter 4. As $t \to \infty$, We get $T(t) = 70 + 90e^{-0.1t} \approx 70 + 90e^{\text{very big }(-)}$. Since $e > 1$,

$$e^{\text{very big }(-)} = \frac{1}{e^{\text{very big }(+)}} \approx \frac{1}{\text{very big }(+)} \approx \text{very small }(+)$$

The larger t becomes, the smaller $e^{-0.1t}$ becomes, so the term $90e^{-0.1t} \approx$ very small $(+)$. Hence, $T(t) \approx 70 +$ very small $(+)$ which means the graph is approaching the horizontal line $y = 70$ from above. This means that as time goes by, the temperature of the coffee is cooling to 70°F, presumably room temperature.

As we have already remarked, the graphs of $f(x) = b^x$ all pass the Horizontal Line Test. Thus the exponential functions are invertible. We now turn our attention to these inverses, the logarithmic functions, which are called 'logs' for short.

Definition 6.2

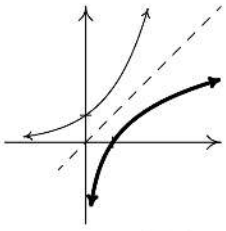
The inverse of the exponential function $f(x) = b^x$ is called the **base b logarithm function**, and is denoted $f^{-1}(x) = \log_b(x)$ We read '$\log_b(x)$' as 'log base b of x.'

We have special notations for the common base, $b = 10$, and the natural base, $b = e$.

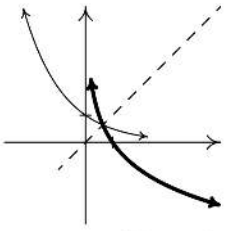
Definition 6.3

The **common logarithm** of a real number x is $\log_{10}(x)$ and is usually written $\log(x)$. The **natural logarithm** of a real number x is $\log_e(x)$ and is usually written $ln(x)$.

Since logs are defined as the inverses of exponential functions, we can use Theorems 5.2 and 5.3 to tell us about logarithmic functions. For example, we know that the domain of a log function is the range of an exponential function, namely $(0, \infty)$, and that the range of a log function is the domain of an exponential function, namely $(-\infty, \infty)$. Since we know the basic shapes of $y = f(x) = b^x$ for the different cases of b, we can obtain the graph of $y = f^{-1}(x) = \log_b(x)$ by reflecting the graph of f across the line $y = x$ as shown below. The y-intercept (0, 1) on the graph of f corresponds to an x-intercept of (1, 0) on the graph of f^{-1}. The horizontal asymptotes $y = 0$ on the graphs of the exponential functions become vertical asymptotes $x = 0$ on the log graphs.



$y = b^x$, $b > 1$
$\boldsymbol{y = \log_b(x)}$, $b > 1$



$y = b^x$, $0 < b < 1$
$\boldsymbol{y = \log_b(x)}$, $0 < b < 1$

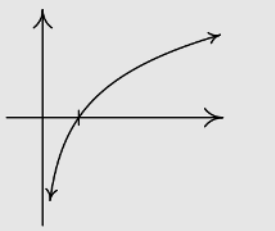
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On a procedural level, logs undo the exponentials. Consider the function $f(x) = 2^x$. When we evaluate $f(3) = 2^3 = 8$, the input 3 becomes the exponent on the base 2 to produce the real number 8. The function $f^{-1}(x) = \log_2(x)$ then takes the number 8 as its input and returns the exponent 3 as its output. In symbols, $\log_2(8) = 3$. More generally, $\log_2(x)$ is the exponent you put on 2 to get x. Thus, $\log_2(16) = 4$, because $2^4 = 16$. The following theorem summarizes the basic properties of logarithmic functions, all of which come from the fact that they are inverses of exponential functions.

Theorem 6.2. Properties of Logarithmic Functions

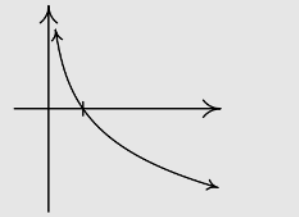
Suppose $f(x) = \log_b(x)$.

- The domain of f is $(0, \infty)$ and the range of f is $(-\infty, \infty)$.
- (1, 0) is on the graph of f and $x = 0$ is a vertical asymptote of the graph of f.
- f is one-to-one, continuous and smooth
- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b(b^x) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$
- If $b > 1$:
  - f is always increasing
  - As $x \to 0^+$, $f(x) \to -\infty$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = \log_b(x),\ b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to 0^+$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to -\infty$
  - The graph of f resembles:



$y = \log_b(x),\ 0 < b < 1$

As we have mentioned, Theorem 6.2 is a consequence of Theorems 5.2 and 5.3. However, it is worth the reader's time to understand Theorem 6.2 from an exponential perspective. For instance, we know that the domain of $g(x) = \log_2(x)$ is $(0, \infty)$. Why? Because the range of $f(x) = 2^x$ is $(0, \infty)$. In a way, this says everything, but at the same time, it doesn't. For example, if we try to find $\log_2(-1)$, we are trying to find the exponent we put on 2 to give us -1. In other words, we are looking for x that satisfies $2^x = -1$. There is no such real number, since all powers of 2 are positive. While what we have said is exactly the same thing as saying 'the domain of $g(x) = \log_2(x)$ is $(0, \infty)$ because the range of $f(x) = 2^x$ is $(0, \infty)$', we feel it is in a student's best interest to understand the statements in Theorem 6.2 at this level instead of just merely memorizing the facts.

Example 6.1.3

Simplify the following.

1. $\log_3(81)$
2. $\log_2\left(\frac{1}{8}\right)$
3. $\log_{\sqrt{5}}(25)$
4. $\ln\left(\sqrt[3]{e^2}\right)$
5. $\log(0.001)$
6. $2^{\log_2(8)}$
7. $117^{-\log_{117}(6)}$

Solution

1. The number $\log_3(81)$ is the exponent we put on 3 to get 81. As such, we want to write 81 as a power of 3. We find $81 = 3^4$, so that $\log_3(81) = 4$.
2. To find $\log_2\left(\frac{1}{8}\right)$, we need rewrit $\frac{1}{8}$ as a power of 2. We find $\frac{1}{8} = \frac{1}{2^3} = 2^{-3}$, so $\log_2\left(\frac{1}{8}\right) = -3$.
3. To determine $\log_{\sqrt{5}}(25)$, we need to express 25 as a power of $\sqrt{5}$. We know $25 = 5^2$, and $5 = (\sqrt{5})^2$, so we have $25 = \left((\sqrt{5})^2\right)^2 = (\sqrt{5})^4$. We get $\log_{\sqrt{5}}(25) = 4$.
4. First, recall that the not $\ln\left(\sqrt[3]{e^2}\right)$ means $\log_e\left(\sqrt[3]{e^2}\right)$, so we are looking for the exponent to put on e to obtain $\sqrt[3]{e^2}$. Rewriting $\sqrt[3]{e^2} = e^{2/3}$, we find $\ln\left(\sqrt[3]{e^2}\right) = \ln\left(e^{2/3}\right) = \frac{2}{3}$.
5. Rewriting $\log(0.001)$as $\log_{10}(0.001)$, we see that we need to write 0.001 as a power of 10. We have $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$. Hence, $\log(0.001) = \log\left(10^{-3}\right) = -3$.
6. We can use Theorem 6.2 directly to so simplify $2^{\log_2(8)} = 8$. We can also understand this problem by first finding $\log_2(8)$. By definition, $\log_2(8)$ is the exponent we put on 2 to get 8. Since $8 = 2^3$, we have $\log_2(8) = 3$. We now substitute to find $2^{\log_2(8)} = 2^3 = 8$.
7. $117^{-\log_{117}(6)} = \frac{1}{117^{\log_{117}(6)}} = \frac{1}{6}$

Up until this point, restrictions on the domains of functions came from avoiding division by zero and keeping negative numbers from beneath even radicals. With the introduction of logs, we now have another restriction. Since the domain of $f(x) = \log_b(x)$ is $(0, \infty)$, the argument[10] of the log must be strictly positive.

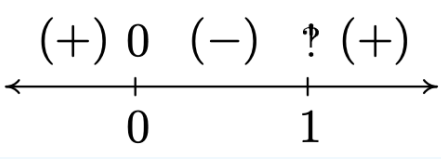
Example 6.1.4

Find the domain of the following functions. Check your answers graphically using the calculator.

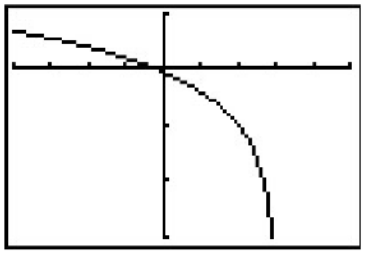
1. $f(x) = 2\log(3 - x) - 1$
2. $g(x) = \ln\left(\frac{x}{x-1}\right)$

Solution

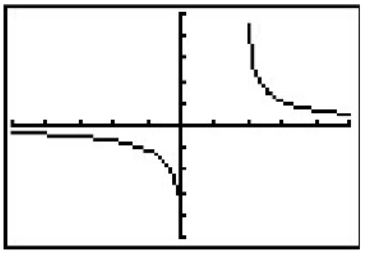
1. We set $3 - x > 0$ to obtain $x < 3$, or $(-\infty, 3)$. The graph from the calculator below verifies this. Note that we could have graphed f using transformations. Taking a cue from Theorem 1.7, we rewrite $f(x) = 2\log_{10}(-x + 3) - 1$ and find the main function involved is $y = h(x) = \log_{10}(x)$. We select three points to track, $\left(\frac{1}{10}, -1\right)$, $(1, 0)$ and $(10, 1)$, along with the vertical asymptote $x = 0$. Since $f(x) = 2h(-x + 3) - 1$, Theorem 1.7 tells us that to obtain the destinations of these points, we first subtract 3 from the x-coordinates (shifting the graph left 3 units), then divide (multiply) by the x-coordinates by -1 (causing a reflection across the y-axis). These transformations apply to the vertical asymptote $x = 0$ as well. Subtracting 3 gives us $x = -3$ as our asymptote, then multplying by -1 gives us the vertical asymptote $x = 3$. Next, we multiply the y-coordinates by 2 which results in a vertical stretch by a factor of 2, then we finish by subtracting 1 from the y-coordinates which shifts the graph down 1 unit. We leave it to the reader to perform the indicated arithmetic on the points themselves and to verify the graph produced by the calculator below.
2. To find the domain of g, we need to solve the inequality $\frac{x}{x-1} > 0$. As usual, we proceed using a sign diagram. If we define $r(x) = \frac{x}{x-1}$, we find r is undefined at $x = 1$ and $r(x) = 0$ when $x = 0$. Choosing some test values, we generate the sign diagram below.



We find $\frac{x}{x-1} > 0$ on $(-\infty, 0) \cup (1, \infty)$ to get the domain of g. The graph of $y = g(x)$ confirms this. We can tell from the graph of g that it is not the result of Section 1.7 transformation being applied to the graph $y = \ln(x)$, so barring a more detailed analysis using Calculus, the calculator graph is the best we can do. One thing worthy of note, however, is the end behavior of g. The graph suggests that as $x \to \pm\infty, g(x) \to 0$. We can verify this analytically. Using results from Chapter 4 and continuity, we know that as $x \to \pm\infty$, $\frac{x}{x-1} \approx 1$. Hence, it makes sense that $g(x) = \ln\left(\frac{x}{x-1}\right) \approx \ln(1) = 0$.



$y = f(x) = 2\log(3 - x) - 1$



$y = g(x) = \ln\left(\frac{x}{x-1}\right)$

While logarithms have some interesting applications of their own which you'll explore in the exercises, their primary use to us will be to undo exponential functions. (This is, after all, how they were defined.) Our last example solidifies this and reviews all of the material in the section.

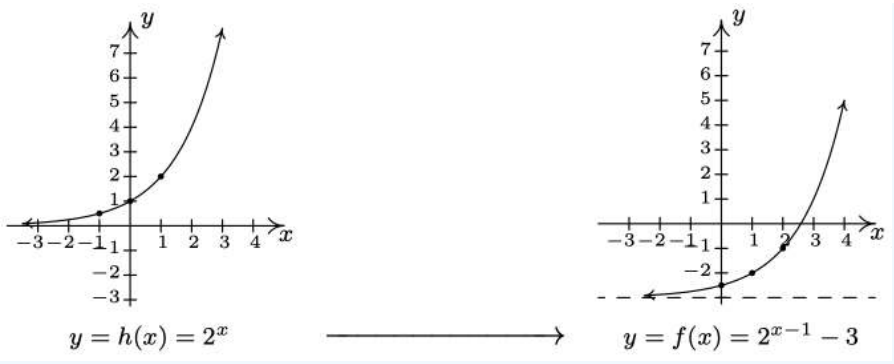
Example 6.1.5

Let $f(x) = 2^{x-1} - 3$.

1. Graph f using transformations and state the domain and range of f.
2. Explain why f is invertible and find a formula for $f^{-1}(x)$.
3. Graph f^{-1} using transformations and state the domain and range of f^{-1}.
4. Verify $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
5. Graph f and f^{-1} on the same set of axes and check the symmetry about the line $y = x$.

Solution

1. If we identify $g(x) = 2^x$, we see $f(x) = g(x - 1) - 3$. We pick the points $\left(-1, \frac{1}{2}\right)$, $(0, 1)$ and $(1, 2)$ 2) on the graph of g along with the horizontal asymptope $y = 0$ to track through the transformations. By Theorem 1.7 we first add 1 to the x-coordinates of the points on the graph of g (shifting g to the right 1 unit) to get $\left(0, \frac{1}{2}\right)$, $(1, 1)$ and $(2, 2)$. The horizontal asymptote remains $y = 0$. Next, we subtract 3 from the y-coordinates, shifting the graph down 3 units. We get the points $\left(0, -\frac{5}{2}\right)$, $(1, -2)$ and $(2, -1)$ with the horizontal asymptote now at $y = -3$. Connecting the dots in the order and manner as they were on the graph of g, we get the graph below. We see that the domain of f is the same as g, namely $(-\infty, \infty)$, but that the range of f is $(-3, \infty)$.



2. The graph of f passes the Horizontal Line Test so f is one-to-one, hence invertible. To find a formula for $f^{-1}(x)$, we normally set $y = f(x)$, interchange the x and y, then proceed to solve for y. Doing so in this situation leads us to the equation $x = 2^{y-1} - 3$. We have yet to discuss how to solve this kind of equation, so we will attempt to find the formula for f^{-1} from a procedural perspective. If we break $f(x) = 2^{x-1} - 3$ into a series of steps, we find f takes an input x and applies the steps

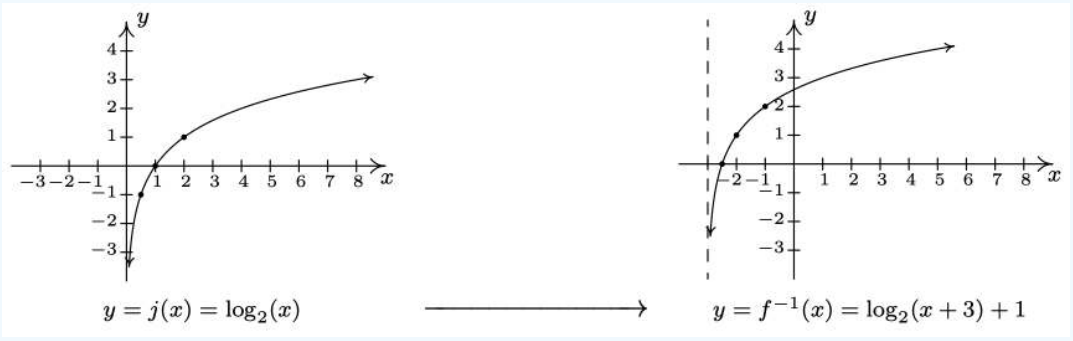
   a. subtract 1
   b. put as an exponent on 2
   c. subtract 3

   Clearly, to undo subtracting 1, we will add 1, and similarly we undo subtracting 3 by adding 3. How do we undo the second step? The answer is we use the logarithm. By definition, $\log_2(x)$ undoes exponentiation by 2. Hence, f^{-1} should

   a. add 3
   b. take the logarithm base 2
   c. add 1

   In symbols, $f^{-1}(x) = \log_2(x+3)+1$.

3. To graph $f^{-1}(x) = \log_2(x+3)+1$ using transformations, we start with $j(x) = \log_2(x)$. We track the points $\left(\frac{1}{2}, -1\right)$, $(1, 0)$ and $(2, 1)$ on the graph of j along with the vertical asymptote $x = 0$ through the transformations using Theorem 1.7. Since $f^{-1}(x) = j(x+3)+1$, we first subtract 3 from each of the x values (including the vertical asymptote) to obtain $\left(-\frac{5}{2}, -1\right)$, (−2, 0) and (−1, 1) with a vertical asymptote $x = -3$. Next, we add 1 to the y values on the graph and get $\left(-\frac{5}{2}, 0\right)$, (−2, 1) and (−1, 2). If you are experiencing d´ej`a vu, there is a good reason for it but we leave it to the reader to determine the source of this uncanny familiarity. We obtain the graph below. The domain of f^{-1} is $(-3, \infty)$, which matches the range of f, and the range of f^{-1} is $(-\infty, \infty)$, which matches the domain of f.



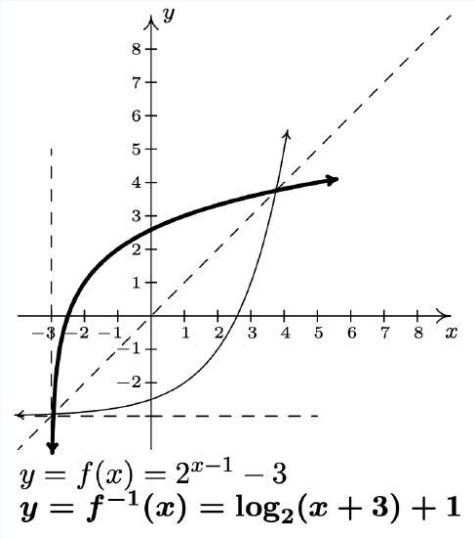
4. We now verify that $f(x) = 2^{x-1} - 3$ and $f^{-1}(x) = \log_2(x+3)+1$ satisfy the composition requirement for inverses. For all real numbers x,

$$\begin{aligned}\left(f^{-1}\circ f\right)(x) &= f^{-1}(f(x))\\ &= f^{-1}\left(2^{x-1}-3\right)\\ &= \log_2\left(\left[2^{x-1}-3\right]+3\right)+1\\ &= \log_2\left(2^{x-1}\right)+1\\ &= (x-1)+1 \quad \text{Since } \log_2(2^u)=u \text{ for all real numbers } u\\ &= x\checkmark\end{aligned}$$

For all real numbers $x > -3$, we have[11]

$$\begin{aligned}\left(f\circ f^{-1}\right)(x) &= f\left(f^{-1}(x)\right)\\ &= f\left(\log_2(x+3)+1\right)\\ &= 2^{(\log_2(x+3)+1)-1}-3\\ &= 2^{\log_2(x+3)}-3\\ &= (x+3)-3 \quad \text{Since } 2^{\log_2(u)}=u \text{ for all real numbers } u>0\\ &= x\checkmark\end{aligned}$$

5. Last, but certainly not least, we graph $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes and see the symmetry about the line $y = x$.



6.1.1 Exercises

In Exercises 1 - 15, use the property: $b^a = c$ if and only if $\log_b(c) = a$ from Theorem 6.2 to rewrite the given equation in the other form. That is, rewrite the exponential equations as logarithmic equations and rewrite the logarithmic equations as exponential equations.

1. $2^3 = 8$
2. $5^{-3} = \frac{1}{125}$
3. $4^{5/2} = 32$
4. $\left(\frac{1}{3}\right)^{-2} = 9$
5. $\left(\frac{4}{25}\right)^{-1/2} = \frac{5}{2}$
6. $10^{-3} = 0.001$
7. $e^0 = 1$
8. $\log_5(25) = 2$
9. $\log_{25}(5) = \frac{1}{2}$
10. $\log_3\left(\frac{1}{81}\right) = -4$
11. $\log_{\frac{4}{3}}\left(\frac{3}{4}\right) = -1$

12. $\log(100) = 2$
13. $\log(0.1) = -1$
14. $\ln(e) = 1$
15. $\ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2}$

In Exercises 16 - 42, evaluate the expression.

16. $\log_3(27)$
17. $\log_6(216)$
18. $\log_2(32)$
19. $\log_6\left(\frac{1}{36}\right)$
20. $\log_8(4)$
21. $\log_{36}(216)$
22. $\log_{\frac{1}{5}}(625)$
23. $\log_{\frac{1}{6}}(216)$
24. $\log_{36}(36)$
25. $\log\left(\frac{1}{1000000}\right)$
26. $\log(0.01)$
27. $\ln\left(e^3\right)$
28. $\log_4(8)$
29. $\log_6(1)$
30. $\log_{13}(\sqrt{13})$
31. $\log_{36}(\sqrt[4]{36})$
32. $7^{\log_7(3)}$
33. $36^{\log_{36}(216)}$
34. $\log_{36}\left(36^{216}\right)$
35. $\ln\left(e^5\right)$
36. $\log\left(\sqrt[9]{10^{11}}\right)$
37. $\log\left(\sqrt[3]{10^5}\right)$
38. $\ln\left(\frac{1}{\sqrt{e}}\right)$
39. $\log_5\left(3^{\log_3(5)}\right)$
40. $\log\left(e^{\ln(100)}\right)$
41. $\log_2\left(3^{-\log_3(2)}\right)$
42. $\ln\left(42^{6\log(1)}\right)$

In Exercises 43 - 57, find the domain of the function.

43. $f(x) = \ln\left(x^2+1\right)$
44. $f(x) = \log_7(4x+8)$
45. $f(x) = \ln(4x-20)$
46. $f(x) = \log\left(x^2+9x+18\right)$
47. $f(x) = \log\left(\frac{x+2}{x^2-1}\right)$
48. $f(x) = \log\left(\frac{x^2+9x+18}{4x-20}\right)$
49. $f(x) = \ln(7-x)+\ln(x-4)$
50. $f(x) = \ln(4x-20)+\ln\left(x^2+9x+18\right)$
51. $f(x) = \log\left(x^2+x+1\right)$
52. $f(x) = \sqrt[4]{\log_4(x)}$
53. $f(x) = \log_9(|x+3|-4)$
54. $f(x) = \ln(\sqrt{x-4}-3)$

55. $f(x) = \frac{1}{3-\log_5(x)}$
56. $f(x) = \frac{\sqrt{-1-x}}{\log_{\frac{1}{2}}(x)}$
57. $f(x) = \ln\left(-2x^3 - x^2 + 13x - 6\right)$

In Exercises 58 - 63, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice and the horizontal asymptote through the transformations. State the domain and range of g.

58. $f(x) = 2^x, g(x) = 2^x - 1$
59. $f(x) = \left(\frac{1}{3}\right)^x, g(x) = \left(\frac{1}{3}\right)^{x-1}$
60. $f(x) = 3^x, g(x) = 3^{-x} + 2$
61. $f(x) = 10^x, g(x) = 10^{\frac{x+1}{2}} - 20$
62. $f(x) = e^x, g(x) = 8 - e^{-x}$
63. $f(x) = e^x, g(x) = 10e^{-0.1x}$

In Exercises 64 - 69, sketch the graph of $y = g(x)$ by starting with the graph of $y = f(x)$ and using transformations. Track at least three points of your choice and the vertical asymptote through the transformations. State the domain and range of g.

64. $f(x) = \log_2(x), g(x) = \log_2(x+1)$
65. $f(x) = \log_{\frac{1}{3}}(x), g(x) = \log_{\frac{1}{3}}(x) + 1$
66. $f(x) = \log_3(x), g(x) = -\log_3(x-2)$
67. $f(x) = \log(x), g(x) = 2\log(x+20) - 1$
68. $f(x) = \ln(x), g(x) = -\ln(8-x)$
69. $f(x) = \ln(x), g(x) = -10\ln\left(\frac{x}{10}\right)$
70. Verify that each function in Exercises 64 - 69 is the inverse of the corresponding function in Exercises 58 - 63. (Match up #58 and #64, and so on.)

In Exercises 71 - 74, find the inverse of the function from the 'procedural perspective' discussed in Example 6.1.5 and graph the function and its inverse on the same set of axes.

71. $f(x) = 3^{x+2} - 4$
72. $f(x) = \log_4(x-1)$
73. $f(x) = -2^{-x} + 1$
74. $f(x) = 5\log(x) - 2$

(Logarithmic Scales) In Exercises 75 - 77, we introduce three widely used measurement scales which involve common logarithms: the Richter scale, the decibel scale and the pH scale. The computations involved in all three scales are nearly identical so pay attention to the subtle differences.

75. Earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a solid course in Geology[12] or the U.S. Geological Survey's Earthquake Hazards Program found here and present only a simplified version of the Richter scale. The Richter scale measures the magnitude of an earthquake by comparing the amplitude of the seismic waves of the given earthquake to those of a "magnitude 0 event", which was chosen to be a seismograph reading of 0.001 millimeters recorded on a seismometer 100 kilometers from the earthquake's epicenter. Specifically, the magnitude of an earthquake is given by

$$M(x) = \log\left(\frac{x}{0.001}\right)$$

where x is the seismograph reading in millimeters of the earthquake recorded 100 kilometers from the epicenter.

   a. Show that $M(0.001) = 0$.
   b. Compute $M(80,000)$.
   c. Show that an earthquake which registered 6.7 on the Richter scale had a seismograph reading ten times larger than one which measured 5.7.
   d. Find two news stories about recent earthquakes which give their magnitudes on the Richter scale. How many times larger was the seismograph reading of the earthquake with larger magnitude?

76. While the decibel scale can be used in many disciplines,[13] we shall restrict our attention to its use in acoustics, specifically its use in measuring the intensity level of sound.[14] The Sound Intensity Level L (measured in decibels) of a sound intensity I

(measured in watts per square meter) is given by

$$L(I) = 10 \log\left(\frac{I}{10^{-12}}\right).$$

Like the Richter scale, this scale compares I to baseline: $10^{-12}\,\frac{W}{m^2}$ is the threshold of human hearing.

a. Compute $L\left(10^{-6}\right)$.
b. Damage to your hearing can start with short term exposure to sound levels around 115 decibels. What intensity I is needed to produce this level?
c. Compute $L(1)$. How does this compare with the threshold of pain which is around 140 decibels?

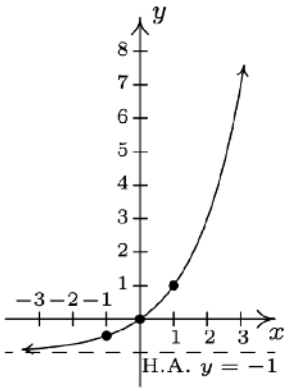
77. The pH of a solution is a measure of its acidity or alkalinity. Specifically, $\text{pH} = -\log\left[\text{H}^+\right]$ where $\left[\text{H}^+\right]$ is the hydrogen ion concentration in moles per liter. A solution with a pH less than 7 is an acid, one with a pH greater than 7 is a base (alkaline) and a pH of 7 is regarded as neutral.
    a. The hydrogen ion concentration of pure water is $\left[\text{H}^+\right] = 10^{-7}$. Find its pH.
    b. Find the pH of a solution with $\left[\text{H}^+\right] = 6.3 \times 10^{-13}$.
    c. The pH of gastric acid (the acid in your stomach) is about 0.7. What is the corresponding hydrogen ion concentration?
78. Show that $\log_b 1 = 0$ and $\log_b b = 1$ for every $b > 0, b \neq 1$.
79. (Crazy bonus question) Without using your calculator, determine which is larger: e^π or π^e.

6.1.2 Answers

1. $\log_2(8) = 3$
2. $\log_5\left(\frac{1}{125}\right) = -3$
3. $\log_4(32) = \frac{5}{2}$
4. $\log_{\frac{1}{3}}(9) = -2$
5. $\log_{\frac{4}{25}}\left(\frac{5}{2}\right) = -\frac{1}{2}$
6. $\log(0.001) = -3$
7. $\ln(1) = 0$
8. $5^2 = 25$
9. $(25)^{\frac{1}{2}} = 5$
10. $3^{-4} = \frac{1}{81}$
11. $\left(\frac{4}{3}\right)^{-1} = \frac{3}{4}$
12. $10^2 = 100$
13. $10^{-1} = 0.1$
14. $e^1 = e$
15. $e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$
16. $\log_3(27) = 3$
17. $\log_6(216) = 3$
18. $\log_2(32) = 5$
19. $\log_6\left(\frac{1}{36}\right) = -2$
20. $\log_8(4) = \frac{2}{3}$
21. $\log_{36}(216) = \frac{3}{2}$
22. $\log_{\frac{1}{5}}(625) = -4$
23. $\log_{\frac{1}{6}}(216) = -3$
24. $\log_{36}(36) = 1$
25. $\log \frac{1}{1000000} = -6$
26. $\log(0.01) = -2$
27. $\ln\left(e^3\right) = 3$
28. $\log_4(8) = \frac{3}{2}$
29. $\log_6(1) = 0$
30. $\log_{13}(\sqrt{13}) = \frac{1}{2}$

31. $\log_{36}\left(\sqrt[4]{36}\right)=\frac{1}{4}$
32. $7^{\log_7(3)}=3$
33. $36^{\log_{36}(216)}=216$
34. $\log_{36}\left(36^{216}\right)=216$
35. $\ln\left(e^5\right)=5$
36. $\log\left(\sqrt[9]{10^{11}}\right)=\frac{11}{9}$
37. $\log\left(\sqrt[3]{10^5}\right)=\frac{5}{3}$
38. $\ln\left(\frac{1}{\sqrt{e}}\right)=-\frac{1}{2}$
39. $\log_5\left(3^{\log_3 5}\right)=1$
40. $\log\left(e^{\ln(100)}\right)=2$
41. $\log_2\left(3^{-\log_3(2)}\right)=-1$
42. $\ln\left(42^{6\log(1)}\right)=0$
43. $(-\infty,\infty)$
44. $(-2,\infty)$
45. $(5,\infty)$
46. $(-\infty,-6)\cup(-3,\infty)$
47. $(-2,-1)\cup(1,\infty)$
48. $(-6,-3)\cup(5,\infty)$
49. $(4,7)$
50. $(5,\infty)$
51. $(-\infty,\infty)$
52. $[1,\infty)$
53. $(-\infty,-7)\cup(1,\infty)$
54. $(13,\infty)$
55. $(0,125)\cup(125,\infty)$
56. No domain
57. $(-\infty,-3)\cup\left(\frac{1}{2},2\right)$
58. Domain of g: $(-\infty,\infty)$

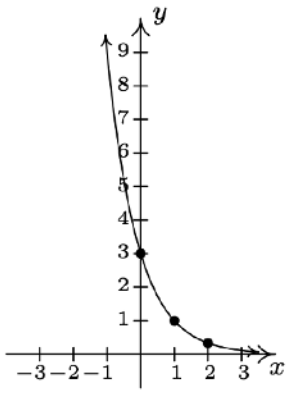
    Range of g: $(-1,\infty)$



$y=g(x)=2^x-1$

59. Domain of g: $(-\infty,\infty)$

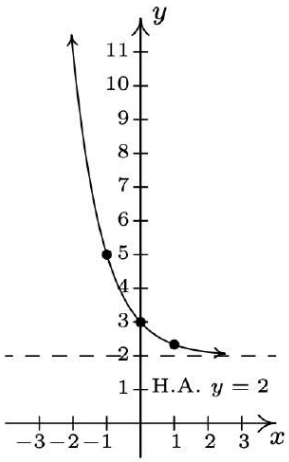
    Range of g: $(0,\infty)$



$y = g(x) = \left(\frac{1}{3}\right)^{x-1}$

60. Domain of g: $(-\infty, \infty)$

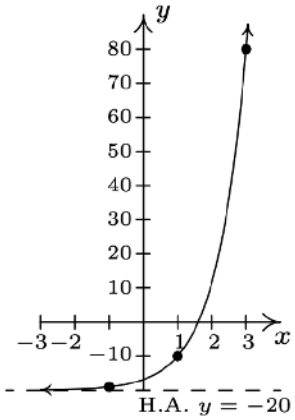
Range of g: $(-20, \infty)$



$y = g(x) = 3^{-x} + 2$

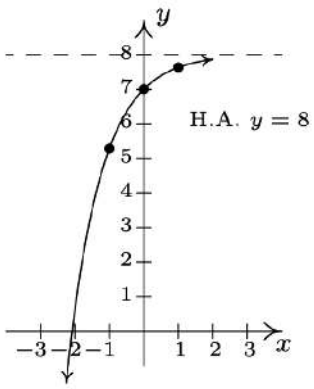
61. Domain of g: $(-\infty, \infty)$

Range of g: $(-\infty, 8)$



62. Domain of g: $(-\infty, \infty)$

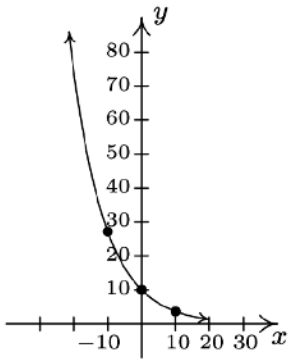
Range of g: $(0, \infty)$



$y = g(x) = 8 - e^{-x}$

63. Domain of g: $(-1, \infty)$

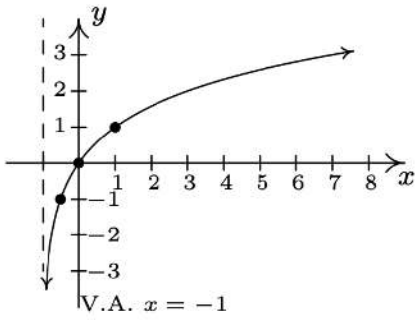
Range of g: $(-\infty, \infty)$



$y = g(x) = 10e^{-0.1x}$

64. Domain of g: $(-1, \infty)$

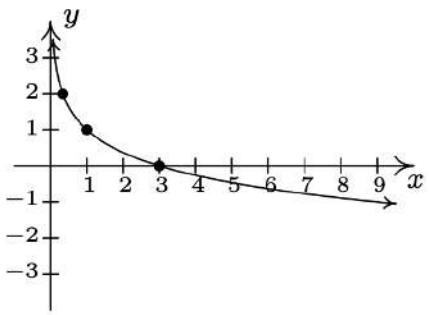
Range of g: $(-\infty, \infty)$



$y = g(x) = \log_2(x + 1)$

65. Domain of g: $(0, \infty)$

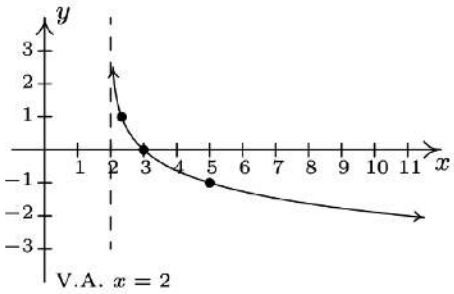
Range of g: $(-\infty, \infty)$



$$y = g(x) = \log_{\frac{1}{3}}(x) + 1$$

66. Domain of g: $(2, \infty)$

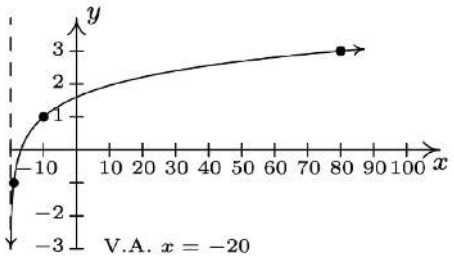
Range of g: $(-\infty, \infty)$



$$y = g(x) = -\log_3(x - 2)$$

67. Domain of g: $(-20, \infty)$

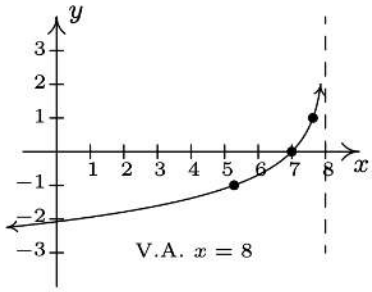
Range of g: $(-\infty, \infty)$



$$y = g(x) = 2\log(x + 20) - 1$$

68. Domain of g: $(-\infty, 8)$

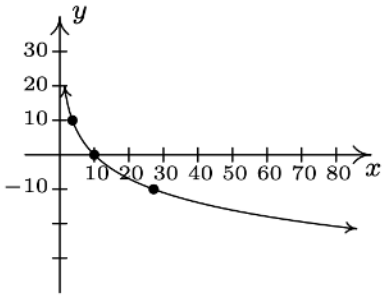
Range of g: $(-\infty, \infty)$



$$y = g(x) = -\ln(8 - x)$$

69. Domain of g: $(0, \infty)$

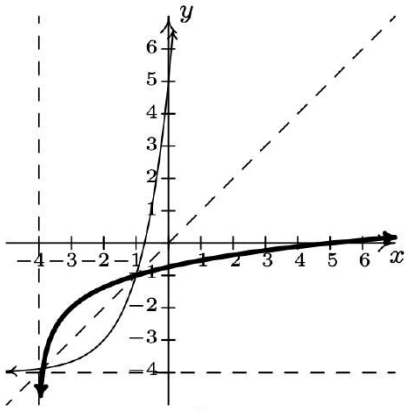
Range of g: $(-\infty, \infty)$



$y = g(x) = -10 \ln\left(\frac{x}{10}\right)$

70. $f(x) = 3^{x+2} - 4$

$f^{-1}(x) = \log_3(x+4) - 2$

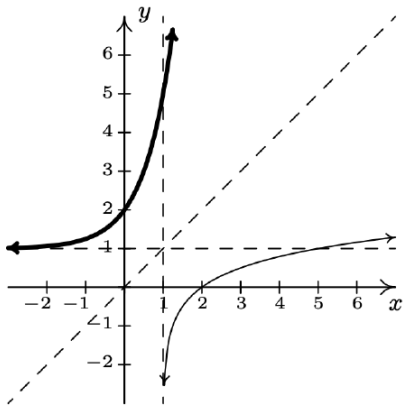


$y = f(x) = 3^{x+2} - 4$

$\boldsymbol{y = f^{-1}(x) = \log_3(x+4) - 2}$

71. $f(x) = \log_4(x-1)$

$f^{-1}(x) = 4^x + 1$

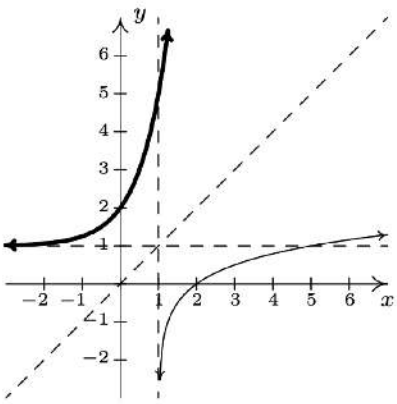


$y = f(x) = \log_4(x-1)$

$\boldsymbol{y = f^{-1}(x) = 4^x + 1}$

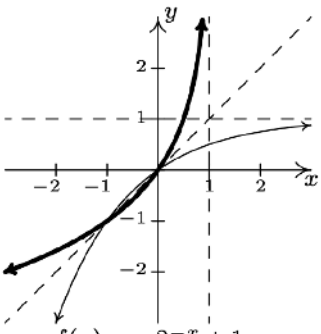
72. $f(x) = \log_4(x-1)$

$f^{-1}(x) = 4^x + 1$



73. $f(x) = -2^{-x} + 1$

$f^{-1}(x) = -\log_2(1-x)$

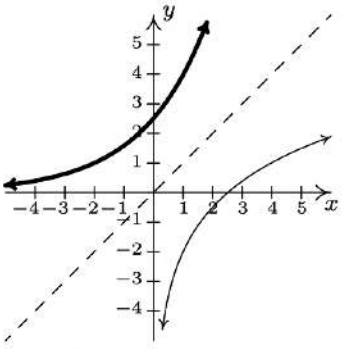


$y = f(x) = -2^{-x} + 1$

$\boldsymbol{y = f^{-1}(x) = -\log_2(1-x)}$

74. $f(x) = 5\log(x) - 2$

$f^{-1}(x) = 10^{\frac{x+2}{5}}$



$y = f(x) = 5\log(x) - 2$

$\boldsymbol{y = f^{-1}(x) = 10^{\frac{x+2}{5}}}$

75. a. $M(0.001) = \log\left(\frac{0.001}{0.001}\right) = \log(1) = 0.$

b. $M(80,000) = \log\left(\frac{80{,}000}{0.001}\right) = \log(80,000,000) \approx 7.9$

76. a. $L\left(10^{-6}\right) = 60$ decibels.

b. $I = 10^{-.5} \approx 0.316$ watts per square meter.

c. Since $L(1) = 120$ decibels and $L(100) = 140$ decibels, a sound with intensity level 140 decibels has an intensity 100 times greater than a sound with intensity level 120 decibels.

77. a. The pH of pure water is 7.

b. If $\left[\mathrm{H}^{+}\right] = 6.3 \times 10^{-13}$ then the solution has a pH of 12.2.

c. $\left[\mathrm{H}^{+}\right] = 10^{-0.7} \approx .1995$ moles per liter.

Reference

[1] Take a class in Differential Equations and you'll see why.

[2] Recall that this means there are no holes or other kinds of breaks in the graph.

[3] You can actually prove this by considering the polynomial $p(x) = x^2 - 3$ and showing it has no rational zeros by applying Theorem 3.9.

[4] This is where Calculus and continuity come into play.

[5] Want more information? Look up "convergent sequences" on the Internet.

[6] Meaning, graph some more examples on your own.

[7] The proof of which, like many things discussed in the text, requires Calculus.

[8] We will discuss this in greater detail in Section 6.5.

[9] It is worth a moment of your time to think your way through why $117^{\log_{117}(6)} = 6$. By definition, $\log_{117}(6)$ is the exponent we put on 117 to get 6. What are we doing with this exponent? We are putting it on 117. By definition we get 6. In other words, the exponential function $f(x) = 117^x$ undoes the logarithmic function $g(x) = \log_{117}(x)$.

[10] See page 55 if you've forgotten what this term means

[11] Pay attention - can you spot in which step below we need $x > -3$?

[12] Rock-solid, perhaps?

[13] See this webpage for more information.

[14] As of the writing of this exercise, the Wikipedia page given here states that it may not meet the "general notability guideline" nor does it cite any references or sources. I find this odd because it is this very usage of the decibel scale which shows up in every College Algebra book I have read. Perhaps those other books have been wrong all along and we're just blindly following tradition.

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5.2: Properties of Logarithms

In Section 6.1, we introduced the logarithmic functions as inverses of exponential functions and discussed a few of their functional properties from that perspective. In this section, we explore the algebraic properties of logarithms. Historically, these have played a huge role in the scientific development of our society since, among other things, they were used to develop analog computing devices called slide rules which enabled scientists and engineers to perform accurate calculations leading to such things as space travel and the moon landing. As we shall see shortly, logs inherit analogs of all of the properties of exponents you learned in Elementary and Intermediate Algebra. We first extract two properties from Theorem 6.2 to remind us of the definition of a logarithm as the inverse of an exponential function.

Theorem 6.3. Inverse Properties of Exponential and Logarithmic Functions

Let $b > 0, b \neq 1$.

- $b^a = c$ if and only if $\log_b(c) = a$
- $\log_b(b^x) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$

Next, we spell out what it means for exponential and logarithmic functions to be one-to-one.

Theorem 6.4. One-to-one Properties of Exponential and Logarithmic Functions

Let $f(x) = b^x$ and $g(x) = \log_b(x)$ where $b > 0, b \neq 1$. Then f and g are one-to-one and

- $b^u = b^w$ if and only if $u = w$ for all real numbers u and w.
- $\log_b(u) = \log_b(w)$ if and only if $u = w$ for all real numbers $u > 0, w > 0$.

We now state the algebraic properties of exponential functions which will serve as a basis for the properties of logarithms. While these properties may look identical to the ones you learned in Elementary and Intermediate Algebra, they apply to real number exponents, not just rational exponents. Note that in the theorem that follows, we are interested in the properties of exponential functions, so the base b is restricted to $b > 0$, $b \neq 1$. An added benefit of this restriction is that it eliminates the pathologies discussed in Section 5.3 when, for example, we simplified $\left(x^{2/3}\right)^{3/2}$ and obtained $|x|$ instead of what we had expected from the arithmetic in the exponents, $x^1 = x$.

Theorem 6.5. Algebraic Properties of Exponential Functions

Let $f(x) = b^x$ be an exponential function ($b > 0, b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $f(u+w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u-w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $(b^u)^w = b^{uw}$

While the properties listed in Theorem 6.5 are certainly believable based on similar properties of integer and rational exponents, the full proofs require Calculus. To each of these properties of exponential functions corresponds an analogous property of logarithmic functions. We list these below in our next theorem.

Theorem 6.6. Algebraic Properties of Logarithmic Functions

Let $g(x) = \log_b(x)$ be a logarithmic function ($b > 0, b \neq 1$) and let $u > 0$ and $w > 0$ be real numbers.

- **Product Rule:** $g(uw) = g(u) + g(w)$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$
- **Quotient Rule:** $g\left(\dfrac{u}{w}\right) = g(u) - g(w)$. In other words, $\log_b\left(\dfrac{u}{w}\right) = \log_b(u) - \log_b(w)$
- **Power Rule:** $g(u^w) = wg(u)$. In other words, $\log_b(u^w) = w\log_b(u)$

There are a couple of different ways to understand why Theorem 6.6 is true. Consider the product rule: $\log_b(uw) = \log_b(u) + \log_b(w)$. Let $a = \log_b(uw)$, $c = \log_b(u)$, and $d = \log_b(w)$. Then, by definition, $b^a = uw$, $b^c = u$ and

$b^d = w$. Hence, $b^a = uw = b^c b^d = b^{c+d}$, so that $b^a = b^{c+d}$. By the one-to-one property of b^x, we have $a = c + d$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$. The remaining properties are proved similarly. From a purely functional approach, we can see the properties in Theorem 6.6 as an example of how inverse functions interchange the roles of inputs in outputs. For instance, the Product Rule for exponential functions given in Theorem 6.5, $f(u+w) = f(u)f(w)$, says that adding inputs results in multiplying outputs. Hence, whatever f^{-1} is, it must take the products of outputs from f and return them to the sum of their respective inputs. Since the outputs from f are the inputs to f^{-1} and vice-versa, we have that that f^{-1} must take products of its inputs to the sum of their respective outputs. This is precisely what the Product Rule for Logarithmic functions states in Theorem 6.6: $g(uw) = g(u) + g(w)$. The reader is encouraged to view the remaining properties listed in Theorem 6.6 similarly. The following examples help build familiarity with these properties. In our first example, we are asked to 'expand' the logarithms. This means that we read the properties in Theorem 6.6 from left to right and rewrite products inside the log as sums outside the log, quotients inside the log as differences outside the log, and powers inside the log as factors outside the log.[1]

Example 6.2.1

1. $\log_2\left(\dfrac{8}{x}\right)$
2. $\log_{0.1}\left(10x^2\right)$
3. $\ln\left(\dfrac{3}{ex}\right)^2$
4. $\log\sqrt[3]{\dfrac{100x^2}{yz^5}}$
5. $\log_{117}\left(x^2-4\right)$

Solution.

1. To expand $\log_2\left(\frac{8}{x}\right)$, we use the Quotient Rule identifying $u = 8$ and $w = x$ and simplify.

$$\begin{aligned} \log_2\left(\frac{8}{x}\right) &= \log_2(8) - \log_2(x) && \text{Quotient Rule} \\ &= 3 - \log_2(x) && \text{Since } 2^3 = 8 \\ &= -\log_2(x) + 3 \end{aligned}$$

2. In the expression $\log_{0.1}\left(10x^2\right)$, we have a power (the x^2) and a product. In order to use the Product Rule, the *entire* quantity inside the logarithm must be raised to the same exponent. Since the exponent 2 applies only to the x, we first apply the Product Rule with $u = 10$ and $w = x^2$. Once we get the x^2 by itself inside the log, we may apply the Power Rule with $u = x$ and $w = 2$ and simplify.

$$\begin{aligned} \log_{0.1}\left(10x^2\right) &= \log_{0.1}(10) + \log_{0.1}\left(x^2\right) && \text{Product Rule} \\ &= \log_{0.1}(10) + 2\log_{0.1}(x) && \text{Power Rule} \\ &= -1 + 2\log_{0.1}(x) && \text{Since } (0.1)^{-1} = 10 \\ &= 2\log_{0.1}(x) - 1 \end{aligned}$$

3. We have a power, quotient and product occurring in $\ln\left(\frac{3}{ex}\right)^2$. Since the exponent 2 applies to the entire quantity inside the logarithm, we begin with the Power Rule with $u = \frac{3}{ex}$ and $w = 2$. Next, we see the Quotient Rule is applicable, with $u = 3$ and $w = ex$, so we replace $\ln\left(\frac{3}{ex}\right)$ with the quantity $\ln(3) - \ln(ex)$. Since $\ln\left(\frac{3}{ex}\right)$ is being multiplied by 2, the entire quantity $\ln(3) - \ln(ex)$ is multiplied by 2. Finally, we apply the Product Rule with $u = e$ and $w = x$, and replace $\ln(ex)$ with the quantity $\ln(e) + \ln(x)$, and simplify, keeping in mind that the natural log is log base e.

$$\begin{aligned}
\ln\left(\frac{3}{ex}\right)^2 &= 2\ln\left(\frac{3}{ex}\right) && \text{Power Rule}\\
&= 2\left[\ln(3)-\ln(ex)\right] && \text{Quotient Rule}\\
&= 2\ln(3)-2\ln(ex)\\
&= 2\ln(3)-2\left[\ln(e)+\ln(x)\right] && \text{Product Rule}\\
&= 2\ln(3)-2\ln(e)-2\ln(x)\\
&= 2\ln(3)-2-2\ln(x) && \text{Since } e^1=e\\
&= -2\ln(x)+2\ln(3)-2
\end{aligned}$$

4. In Theorem 6.6, there is no mention of how to deal with radicals. However, thinking back to Definition 5.5, we can rewrite the cube root as a $\frac{1}{3}$ exponent. We begin by using the Power Rule[2], and we keep in mind that the common log is log base 10.

$$\begin{aligned}
\log\sqrt[3]{\frac{100x^2}{yz^5}} &= \log\left(\frac{100x^2}{yz^5}\right)^{1/3}\\
[10pt] &= \tfrac{1}{3}\log\left(\frac{100x^2}{yz^5}\right) && \text{Power Rule}\\
&= \tfrac{1}{3}\left[\log\left(100x^2\right)-\log\left(yz^5\right)\right] && \text{Quotient Rule}\\
&= \tfrac{1}{3}\log\left(100x^2\right)-\tfrac{1}{3}\log\left(yz^5\right)\\
&= \tfrac{1}{3}\left[\log(100)+\log\left(x^2\right)\right]-\tfrac{1}{3}\left[\log(y)+\log\left(z^5\right)\right] && \text{Product Rule}\\
&= \tfrac{1}{3}\log(100)+\tfrac{1}{3}\log\left(x^2\right)-\tfrac{1}{3}\log(y)-\tfrac{1}{3}\log\left(z^5\right)\\
&= \tfrac{1}{3}\log(100)+\tfrac{2}{3}\log(x)-\tfrac{1}{3}\log(y)-\tfrac{5}{3}\log(z) && \text{Power Rule}\\
&= \tfrac{2}{3}+\tfrac{2}{3}\log(x)-\tfrac{1}{3}\log(y)-\tfrac{5}{3}\log(z) && \text{Since } 10^2=100\\
&= \tfrac{2}{3}\log(x)-\tfrac{1}{3}\log(y)-\tfrac{5}{3}\log(z)+\tfrac{2}{3}
\end{aligned}$$

5. At first it seems as if we have no means of simplifying $\log_{117}\left(x^2-4\right)$, since none of the properties of logs addresses the issue of expanding a difference *inside* the logarithm. However, we may factor $x^2-4=(x+2)(x-2)$ thereby introducing a product which gives us license to use the Product Rule.

$$\begin{aligned}
\log_{117}\left(x^2-4\right) &= \log_{117}[(x+2)(x-2)] && \text{Factor}\\
&= \log_{117}(x+2)+\log_{117}(x-2) && \text{Product Rule}
\end{aligned}$$

A couple of remarks about Example 6.2.1 are in order. First, while not explicitly stated in the above example, a general rule of thumb to determine which log property to apply first to a complicated problem is 'reverse order of operations.' For example, if we were to substitute a number for x into the expression $\log_{0.1}\left(10x^2\right)$, we would first square the x, then multiply by 10. The last step is the multiplication, which tells us the first log property to apply is the Product Rule. In a multi-step problem, this rule can give the required guidance on which log property to apply at each step. The reader is encouraged to look through the solutions to Example 6.2.1 to see this rule in action. Second, while we were instructed to assume when necessary that all quantities represented positive real numbers, the authors would be committing a sin of omission if we failed to point out that, for instance, the functions $f(x)=\log_{117}\left(x^2-4\right)$ and $g(x)=\log_{117}(x+2)+\log_{117}(x-2)$ have different domains, and, hence, are different functions. We leave it to the reader to verify the domain of f is $(-\infty,-2)\cup(2,\infty)$ whereas the domain of g is $(2,\infty)$. In general, when using log properties to expand a logarithm, we may very well be restricting the domain as we do so. One last comment before we move to reassembling logs from their various bits and pieces. The authors are well aware of the propensity for some students to become overexcited and invent their own properties of logs like $\log_{117}\left(x^2-4\right)=\log_{117}\left(x^2\right)-\log_{117}(4)$, which simply isn't true, in general. The unwritten[3] property of logarithms is that if it isn't written in a textbook, it probably isn't true.

Example 6.2.2

Use the properties of logarithms to write the following as a single logarithm.

1. $\log_3(x-1)-\log_3(x+1)$
2. $\log(x)+2\log(y)-\log(z)$
3. $4\log_2(x)+3$
4. $-\ln(x)-\frac{1}{2}$

Solution

Whereas in Example 6.2.1 we read the properties in Theorem 6.6 from left to right to expand logarithms, in this example we read them from right to left.

1. The difference of logarithms requires the Quotient Rule: $\log_3(x-1)-\log_3(x+1)=\log_3\left(\frac{x-1}{x+1}\right)$.
2. In the expression, $\log(x)+2\log(y)-\log(z)$, we have both a sum and difference of logarithms. However, before we use the product rule to combine $\log(x)+2\log(y)$, we note that we need to somehow deal with the coefficient 2 on $\log(y)$. This can be handled using the Power Rule. We can then apply the Product and Quotient Rules as we move from left to right. Putting it all together, we have

$$\begin{aligned} \log(x)+2\log(y)-\log(z) &= \log(x)+\log\left(y^2\right)-\log(z) && \text{Power Rule} \\ [6pt] &= \log\left(xy^2\right)-\log(z) && \text{Product Rule} \\ [10pt] &= \log\left(\frac{xy^2}{z}\right) && \text{Quotient Rule} \end{aligned} \tag{5.2.1}$$

3. We can certainly get started rewriting $4\log_2(x)+3$ by applying the Power Rule to $4\log_2(x)$ to obtain $\log_2\left(x^4\right)$, but in order to use the Product Rule to handle the addition, we need to rewrite 3 as a logarithm base 2. From Theorem 6.3, we know $3=\log_2\left(2^3\right)$, so we get

$$\begin{aligned} 4\log_2(x)+3 &= \log_2\left(x^4\right)+3 && \text{Power Rule} \\ &= \log_2\left(x^4\right)+\log_2\left(2^3\right) && \text{Since } 3=\log_2\left(2^3\right) \\ &= \log_2\left(x^4\right)+\log_2(8) \\ &= \log_2\left(8x^4\right) && \text{Product Rule} \end{aligned} \tag{5.2.2}$$

4. To get started with $-\ln(x)-\frac{1}{2}$, we rewrite $-\ln(x)$ as $(-1)\ln(x)$. We can then use the Power Rule to obtain $(-1)\ln(x)=\ln\left(x^{-1}\right)$. In order to use the Quotient Rule, we need to write $\frac{1}{2}$ as a natural logarithm. Theorem 6.3 gives us $\frac{1}{2}=\ln\left(e^{1/2}\right)=\ln(\sqrt{e})$. We have

$$\begin{aligned} -\ln(x)-\tfrac{1}{2} &= (-1)\ln(x)-\tfrac{1}{2} \\ &= \ln\left(x^{-1}\right)-\tfrac{1}{2} && \text{Power Rule} \\ &= \ln\left(x^{-1}\right)-\ln\left(e^{1/2}\right) && \text{Since } \tfrac{1}{2}=\ln\left(e^{1/2}\right) \\ &= \ln\left(x^{-1}\right)-\ln(\sqrt{e}) \\ [6pt] &= \ln\left(\frac{x^{-1}}{\sqrt{e}}\right) && \text{Quotient Rule} \\ [10pt] &= \ln\left(\frac{1}{x\sqrt{e}}\right) \end{aligned}$$

As we would expect, the rule of thumb for re-assembling logarithms is the opposite of what it was for dismantling them. That is, if we are interested in rewriting an expression as a single logarithm, we apply log properties following the usual order of operations: deal with multiples of logs first with the Power Rule, then deal with addition and subtraction using the Product and Quotient Rules, respectively. Additionally, we find that using log properties in this fashion can increase the domain of the expression. For example, we leave it to the reader to verify the domain of $f(x)=\log_3(x-1)-\log_3(x+1)$ is $(1,\infty)$ but the domain of

$g(x) = \log_3\left(\frac{x-1}{x+1}\right)$ is $(-\infty, -1) \cup (1, \infty)$. We will need to keep this in mind when we solve equations involving logarithms in Section 6.4 - it is precisely for this reason we will have to check for extraneous solutions.

The two logarithm buttons commonly found on calculators are the 'LOG' and 'LN' buttons which correspond to the common and natural logs, respectively. Suppose we wanted an approximation to $\log_2(7)$. The answer should be a little less than 3, (Can you explain why?) but how do we coerce the calculator into telling us a more accurate answer? We need the following theorem.

Theorem 6.7. Change of Base Formulas

Let $a, b > 0$, $a, b \neq 1$.

- $a^x = b^{x \log_b(a)}$ for all real numbers x.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x > 0$.

The proofs of the Change of Base formulas are a result of the other properties studied in this section. If we start with $b^{x \log_b(a)}$ and use the Power Rule in the exponent to rewrite $x \log_b(a)$ as $\log_b(a^x)$ and then apply one of the Inverse Properties in Theorem 6.3, we get

$$b^{x \log_b(a)} = b^{\log_b(a^x)} = a^x,$$

as required. To verify the logarithmic form of the property, we also use the Power Rule and an Inverse Property. We note that

$$\log_a(x) \cdot \log_b(a) = \log_b\left(a^{\log_a(x)}\right) = \log_b(x),$$

and we get the result by dividing through by $\log_b(a)$. Of course, the authors can't help but point out the inverse relationship between these two change of base formulas. To change the base of an exponential expression, we *multiply* the *input* by the factor $\log_b(a)$. To change the base of a logarithmic expression, we *divide* the *output* by the factor $\log_b(a)$. While, in the grand scheme of things, both change of base formulas are really saying the same thing, the logarithmic form is the one usually encountered in Algebra while the exponential form isn't usually introduced until Calculus.[4] What Theorem 6.7 really tells us is that all exponential and logarithmic functions are just scalings of one another. Not only does this explain why their graphs have similar shapes, but it also tells us that we could do all of mathematics with a single base - be it 10, e, 42, or 117. Your Calculus teacher will have more to say about this when the time comes.

Example 6.2.3

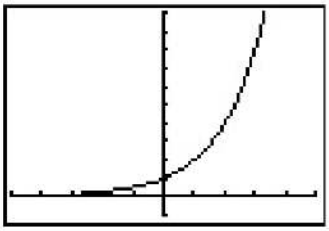
Use an appropriate change of base formula to convert the following expressions to ones with the indicated base. Verify your answers using a calculator, as appropriate.

1. 3^2 to base 10
2. 2^x to base e
3. $\log_4(5)$ to base e
4. $\ln(x)$ to base 10

Solution.

1. We apply the Change of Base formula with $a = 3$ and $b = 10$ to obtain $3^2 = 10^{2 \log(3)}$. Typing the latter in the calculator produces an answer of 9 as required.
2. Here, $a = 2$ and $b = e$ so we have $2^x = e^{x \ln(2)}$. To verify this on our calculator, we can graph $f(x) = 2^x$ and $g(x) = e^{x \ln(2)}$. Their graphs are indistinguishable which provides evidence that they are the same function.

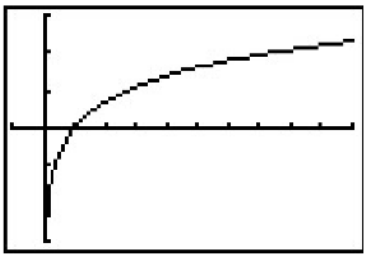




$y = f(x) = 2^x$ and $y = g(x) = e^{x\ln(2)}$

3. Applying the change of base with $a = 4$ and $b = e$ leads us to write $\log_4(5) = \frac{\ln(5)}{\ln(4)}$. Evaluating this in the calculator gives $\frac{\ln(5)}{\ln(4)} \approx 1.16$. How do we check this really is the value of $\log_4(5)$? By definition, $\log_4(5)$ is the exponent we put on 4 to get 5. The calculator confirms this.[5]
4. We write $\ln(x) = \log_e(x) = \frac{\log(x)}{\log(e)}$. We graph both $f(x) = \ln(x)$ and $g(x) = \frac{\log(x)}{\log(e)}$ and find both graphs appear to be identical.





$y = f(x) = \ln(x)$ and $y = g(x) = \frac{\log(x)}{\log(e)}$

6.2.1. Exercises

In Exercises 1 - 15, expand the given logarithm and simplify. Assume when necessary that all quantities represent positive real numbers.

1. $\ln(x^3y^2)$
2. $\log_2\left(\frac{128}{x^2+4}\right)$
3. $\log_5\left(\frac{z}{25}\right)^3$
4. $\log(1.23 \times 10^{37})$
5. $\ln\left(\frac{\sqrt{z}}{xy}\right)$
6. $\log_5\left(x^2-25\right)$
7. $\log_{\sqrt{2}}\left(4x^3\right)$
8. $\log_{\frac{1}{3}}\left(9x(y^3-8)\right)$
9. $\log\left(1000x^3y^5\right)$
10. $\log_3\left(\frac{x^2}{81y^4}\right)$
11. $\ln\left(\sqrt[4]{\frac{xy}{ez}}\right)$
12. $\log_6\left(\frac{216}{x^3y}\right)^4$
13. $\log\left(\frac{100x\sqrt{y}}{\sqrt[3]{10}}\right)$
14. $\log_{\frac{1}{2}}\left(\frac{4\sqrt[3]{x^2}}{y\sqrt{z}}\right)$

15. $\ln\left(\dfrac{\sqrt[3]{x}}{10\sqrt{yz}}\right)$

In Exercises 16 - 29, use the properties of logarithms to write the expression as a single logarithm.

16. $4\ln(x)+2\ln(y)$
17. $\log_2(x)+\log_2(y)-\log_2(z)$
18. $\log_3(x)-2\log_3(y)$
19. $\frac{1}{2}\log_3(x)-2\log_3(y)-\log_3(z)$
20. $2\ln(x)-3\ln(y)-4\ln(z)$
21. $\log(x)-\frac{1}{3}\log(z)+\frac{1}{2}\log(y)$
22. $-\frac{1}{3}\ln(x)-\frac{1}{3}\ln(y)+\frac{1}{3}\ln(z)$
23. $\log_5(x)-3$
24. $3-\log(x)$
25. $\log_7(x)+\log_7(x-3)-2$
26. $\ln(x)+\frac{1}{2}$
27. $\log_2(x)+\log_4(x)$
28. $\log_2(x)+\log_4(x-1)$
29. $\log_2(x)+\log_{\frac{1}{2}}(x-1)$

In Exercises 30 - 33, use the appropriate change of base formula to convert the given expression to an expression with the indicated base.

30. 7^{x-1} to base e
31. $\log_3(x+2)$ to base 10
32. $\left(\dfrac{2}{3}\right)^x$ to base e
33. $\log(x^2+1)$ to base e

In Exercises 34 - 39, use the appropriate change of base formula to approximate the logarithm.

34. $\log_3(12)$
35. $\log_5(80)$
36. $\log_6(72)$
37. $\log_4\left(\dfrac{1}{10}\right)$
38. $\log_{\frac{3}{5}}(1000)$
39. $\log_{\frac{2}{3}}(50)$
40. Compare and contrast the graphs of $y=\ln(x^2)$ and $y=2\ln(x)$.
41. Prove the Quotient Rule and Power Rule for Logarithms.
42. Give numerical examples to show that, in general,
    a. $\log_b(x+y)\neq\log_b(x)+\log_b(y)$
    b. $\log_b(x-y)\neq\log_b(x)-\log_b(y)$
    c. $\log_b\left(\dfrac{x}{y}\right)\neq\dfrac{\log_b(x)}{\log_b(y)}$
43. [HendersonHasselbalch] The Henderson-Hasselbalch Equation: Suppose HA represents a weak acid. Then we have a reversible chemical reaction

$$HA\rightleftharpoons H^+ + A^-.$$

The acid disassociation constant, K_a, is given by

$$K_\alpha=\frac{[H^+][A^-]}{[HA]}=[H^+]\frac{[A^-]}{[HA]},$$

where the square brackets denote the concentrations just as they did in Exercise 77 in Section 6.1. The symbol $\mathrm{p}K_a$ is defined similarly to pH in that $\mathrm{p}K_a = -\log(K_a)$. Using the definition of pH from Exercise 77 and the properties of logarithms, derive the Henderson-Hasselbalch Equation which states

$$\mathrm{pH} = \mathrm{p}K_a + \log\frac{[A^-]}{[HA]}$$

44. Research the history of logarithms including the origin of the word 'logarithm' itself. Why is the abbreviation of natural log 'ln' and not 'nl'?
45. There is a scene in the movie 'Apollo 13' in which several people at Mission Control use slide rules to verify a computation. Was that scene accurate? Look for other pop culture references to logarithms and slide rules.

6.2.2. Answers

1. $3\ln(x)+2\ln(y)$
2. $7-\log_2(x^2+4)$
3. $3\log_5(z)-6$
4. $\log(1.23)+37$
5. $\frac{1}{2}\ln(z)-\ln(x)-\ln(y)$
6. $\log_5(x-5)+\log_5(x+5)$
7. $3\log_{\sqrt{2}}(x)+4$
8. $-2+\log_{\frac{1}{3}}(x)+\log_{\frac{1}{3}}(y-2)+\log_{\frac{1}{3}}(y^2+2y+4)$
9. $3+3\log(x)+5\log(y)$
10. $2\log_3(x)-4-4\log_3(y)$
11. $\frac{1}{4}\ln(x)+\frac{1}{4}\ln(y)-\frac{1}{4}-\frac{1}{4}\ln(z)$
12. $12-12\log_6(x)-4\log_6(y)$
13. $\frac{5}{3}+\log(x)+\frac{1}{2}\log(y)$
14. $-2+\frac{2}{3}\log_{\frac{1}{2}}(x)-\log_{\frac{1}{2}}(y)-\frac{1}{2}\log_{\frac{1}{2}}(z)$
15. $\frac{1}{3}\ln(x)-\ln(10)-\frac{1}{2}\ln(y)-\frac{1}{2}\ln(z)$
16. $\ln(x^4y^2)$
17. $\log_2\left(\frac{xy}{z}\right)$
18. $\log_3\left(\frac{x}{y^2}\right)$
19. $\log_3\left(\frac{\sqrt{x}}{y^2z}\right)$
20. $\ln\left(\frac{x^2}{y^3z^4}\right)$
21. $\log\left(\frac{x\sqrt{y}}{\sqrt[3]{z}}\right)$
22. $\ln\left(\sqrt[3]{\frac{z}{xy}}\right)$
23. $\log_5\left(\frac{x}{125}\right)$
24. $\log\left(\frac{1000}{x}\right)$
25. $\log_7\left(\frac{x(x-3)}{49}\right)$
26. $\ln(x\sqrt{e})$
27. $\log_2\left(x^{3/2}\right)$
28. $\log_2(x\sqrt{x-1})$
29. $\log_2\left(\frac{x}{x-1}\right)$
30. $7^{x-1} = e^{(x-1)\ln(7)}$
31. $\log_3(x+2) = \frac{\log(x+2)}{\log(3)}$
32. $\left(\frac{2}{3}\right)^x = e^{x\ln\left(\frac{2}{3}\right)}$
33. $\log(x^2+1) = \frac{\ln(x^2+1)}{\ln(10)}$

34. $\log_3(12) \approx 2.26186$
35. $\log_5(80) \approx 2.72271$
36. $\log_6(72) \approx 2.38685$
37. $\log_4\left(\frac{1}{10}\right) \approx -1.66096$
38. $\log_{\frac{3}{5}}(1000) \approx -13.52273$
39. $\log_{\frac{2}{3}}(50) \approx -9.64824$

Reference

[1] Interestingly enough, it is the exact opposite process (which we will practice later) that is most useful in Algebra, the utility of expanding logarithms becomes apparent in Calculus.

[2] At this point in the text, the reader is encouraged to carefully read through each step and think of which quantity is playing the role of u and which is playing the role of w as we apply each property.

[3] The authors relish the irony involved in writing what follows.

[4] The authors feel so strongly about showing students that every property of logarithms comes from and corresponds to a property of exponents that we have broken tradition with the vast majority of other authors in this field. This isn't the first time this happened, and it certainly won't be the last.

[5] Which means if it is lying to us about the first answer it gave us, at least it is being consistent.

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5.3: Exponential Equations and Inequalities

In this section we will develop techniques for solving equations involving exponential functions. Suppose, for instance, we wanted to solve the equation $2^x = 128$. After a moment's calculation, we find $128 = 2^7$, so we have $2^x = 2^7$. The one-to-one property of exponential functions, detailed in Theorem 6.4, tells us that $2^x = 2^7$ if and only if $x = 7$. This means that not only is $x = 7$ a solution to $2^x = 2^7$, it is the *only* solution. Now suppose we change the problem ever so slightly to $2^x = 129$. We could use one of the inverse properties of exponentials and logarithms listed in Theorem 6.3 to write $129 = 2^{\log_2(129)}$. We'd then have $2^x = 2^{\log_2(129)}$, which means our solution is $x = \log_2(129)$. This makes sense because, after all, the definition of $\log_2(129)$ is 'the exponent we put on 2 to get 129.' Indeed we could have obtained this solution directly by rewriting the equation $2^x = 129$ in its logarithmic form $\log_2(129) = x$. Either way, in order to get a reasonable decimal approximation to this number, we'd use the change of base formula, Theorem 6.7, to give us something more calculator friendly,[1] say $\log_2(129) = \frac{\ln(129)}{\ln(2)}$. Another way to arrive at this answer is as follows

$$\begin{aligned} 2^x &= 129 \\ \ln(2^x) &= \ln(129) && \text{Take the natural log of both sides.} \\ x\ln(2) &= \ln(129) && \text{Power Rule} \\ x &= \frac{\ln(129)}{\ln(2)} \end{aligned}$$

'Taking the natural log' of both sides is akin to squaring both sides: since $f(x) = \ln(x)$ is a *function*, as long as two quantities are equal, their natural logs are equal.[2] Also note that we treat $\ln(2)$ as any other non-zero real number and divide it through[3] to isolate the variable x. We summarize below the two common ways to solve exponential equations, motivated by our examples.

Steps for Solving an Equation involving Exponential Functions

1. Isolate the exponential function.
2. a. If convenient, express both sides with a common base and equate the exponents.
   b. Otherwise, take the natural log of both sides of the equation and use the Power Rule.

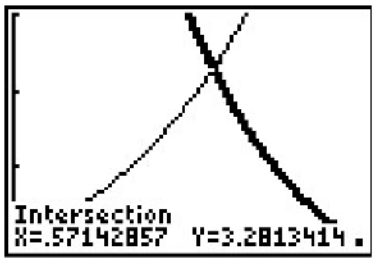
Example 6.3.1.

Solve the following equations. Check your answer graphically using a calculator.

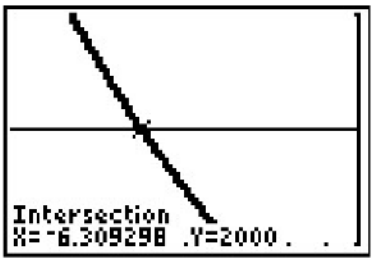
1. $2^{3x} = 16^{1-x}$
2. $2000 = 1000 \cdot 3^{-0.1t}$
3. $9 \cdot 3^x = 7^{2x}$
4. $75 = \frac{100}{1+3e^{-2t}}$
5. $25^x = 5^x + 6$
6. $\frac{e^x - e^{-x}}{2} = 5$

Solution.

1. Since 16 is a power of 2, we can rewrite $2^{3x} = 16^{1-x}$ as $2^{3x} = \left(2^4\right)^{1-x}$. Using properties of exponents, we get $2^{3x} = 2^{4(1-x)}$. Using the one-to-one property of exponential functions, we get $3x = 4(1-x)$ which gives $x = \frac{4}{7}$. To check graphically, we set $f(x) = 2^{3x}$ and $g(x) = 16^{1-x}$ and see that they intersect at $x = \frac{4}{7} \approx 0.5714$.
2. We begin solving $2000 = 1000 \cdot 3^{-0.1t}$ by dividing both sides by 1000 to isolate the exponential which yields $3^{-0.1t} = 2$. Since it is inconvenient to write 2 as a power of 3, we use the natural log to get $\ln\left(3^{-0.1t}\right) = \ln(2)$. Using the Power Rule, we get $-0.1t\ln(3) = \ln(2)$, so we divide both sides by $-0.1\ln(3)$ to get $t = -\frac{\ln(2)}{0.1\ln(3)} = -\frac{10\ln(2)}{\ln(3)}$. On the calculator, we graph $f(x) = 2000$ and $g(x) = 1000 \cdot 3^{-0.1x}$ and find that they intersect at $x = -\frac{10\ln(2)}{\ln(3)} \approx -6.3093$.



$y = f(x) = 2^{3x}$ and
$y = g(x) = 16^{1-x}$



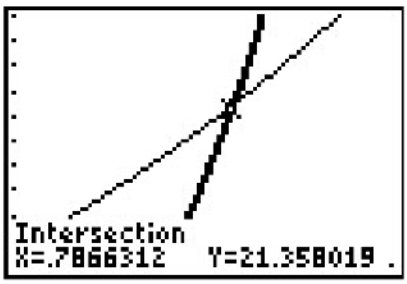
$y = f(x) = 2000$ and
$y = g(x) = 1000 \cdot 3^{-0.1x}$

3. We first note that we can rewrite the equation $9 \cdot 3^x = 7^{2x}$ as $3^2 \cdot 3^x = 7^{2x}$ to obtain $3^{x+2} = 7^{2x}$. Since it is not convenient to express both sides as a power of 3 (or 7 for that matter) we use the natural log: $\ln\left(3^{x+2}\right) = \ln\left(7^{2x}\right)$. The power rule gives $(x+2)\ln(3) = 2x\ln(7)$. Even though this equation appears very complicated, keep in mind that $\ln(3)$ and $\ln(7)$ are just constants. The equation $(x+2)\ln(3) = 2x\ln(7)$ is actually a linear equation and as such we gather all of the terms with x on one side, and the constants on the other. We then divide both sides by the coefficient of x, which we obtain by factoring.

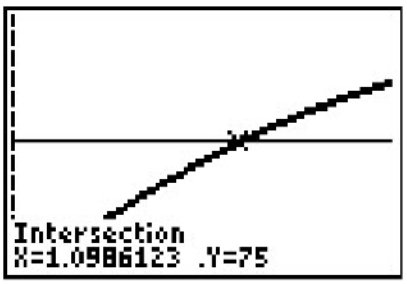
$$\begin{aligned} (x+2)\ln(3) &= 2x\ln(7) \\ x\ln(3)+2\ln(3) &= 2x\ln(7) \\ 2\ln(3) &= 2x\ln(7) - x\ln(3) \\ 2\ln(3) &= x(2\ln(7) - \ln(3)) \quad \text{Factor.} \\ x &= \frac{2\ln(3)}{2\ln(7)-\ln(3)} \end{aligned}$$

Graphing $f(x) = 9 \cdot 3^x$ and $g(x) = 7^{2x}$ on the calculator, we see that these two graphs intersect at $x = \frac{2\ln(3)}{2\ln(7)-\ln(3)} \approx 0.7866$.

4. Our objective in solving $75 = \frac{100}{1+3e^{-2t}}$ is to first isolate the exponential. To that end, we clear denominators and get $75\left(1+3e^{-2t}\right) = 100$. From this we get $75 + 225e^{-2t} = 100$, which leads to $225e^{-2t} = 25$, and finally, $e^{-2t} = \frac{1}{9}$. Taking the natural log of both sides gives $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$. Since natural log is log base e, $\ln\left(e^{-2t}\right) = -2t$. We can also use the Power Rule to write $\ln\left(\frac{1}{9}\right) = -\ln(9)$. Putting these two steps together, we simplify $\ln\left(e^{-2t}\right) = \ln\left(\frac{1}{9}\right)$ to $-2t = -\ln(9)$. We arrive at our solution, $t = \frac{\ln(9)}{2}$ which simplifies to $t = \ln(3)$. (Can you explain why?) The calculator confirms the graphs of $f(x) = 75$ and $g(x) = \frac{100}{1+3e^{-2x}}$ intersect at $x = \ln(3) \approx 1.099$.



$y = f(x) = 9 \cdot 3^x$ and
$y = g(x) = 7^{2x}$

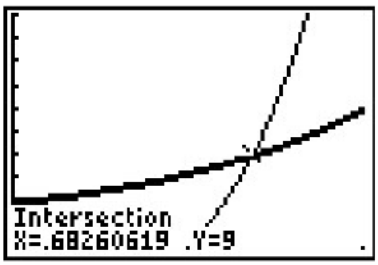


$y = f(x) = 75$ and
$y = g(x) = \frac{100}{1+3e^{-2x}}$

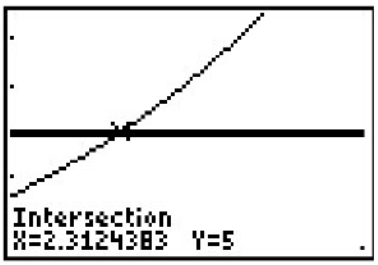
5. We start solving $25^x = 5^x + 6$ by rewriting $25 = 5^2$ so that we have $\left(5^2\right)^x = 5^x + 6$, or $5^{2x} = 5^x + 6$. Even though we have a common base, having two terms on the right hand side of the equation foils our plan of equating exponents or taking logs. If we stare at this long enough, we notice that we have three terms with the exponent on one term exactly twice that of another. To our surprise and delight, we have a 'quadratic in disguise'. Letting $u = 5^x$, we have $u^2 = \left(5^x\right)^2 = 5^{2x}$ so the equation $5^{2x} = 5^x + 6$ becomes $u^2 = u + 6$. Solving this as $u^2 - u - 6 = 0$ gives $u = -2$ or $u = 3$. Since $u = 5^x$, we have $5^x = -2$ or $5^x = 3$. Since $5^x = -2$ has no real solution, (Why not?) we focus on $5^x = 3$. Since it isn't convenient to

express 3 as a power of 5, we take natural logs and get $\ln(5^x) = \ln(3)$ so that $x\ln(5) = \ln(3)$ or $x = \frac{\ln(3)}{\ln(5)}$. On the calculator, we see the graphs of $f(x) = 25^x$ and $g(x) = 5^x + 6$ intersect at $x = \frac{\ln(3)}{\ln(5)} \approx 0.6826$.

6. At first, it's unclear how to proceed with $\frac{e^x - e^{-x}}{2} = 5$, besides clearing the denominator to obtain $e^x - e^{-x} = 10$. Of course, if we rewrite $e^{-x} = \frac{1}{e^x}$, we see we have another denominator lurking in the problem: $e^x - \frac{1}{e^x} = 10$. Clearing this denominator gives us $e^{2x} - 1 = 10e^x$, and once again, we have an equation with three terms where the exponent on one term is exactly twice that of another - a 'quadratic in disguise.' If we let $u = e^x$, then $u^2 = e^{2x}$ so the equation $e^{2x} - 1 = 10e^x$ can be viewed as $u^2 - 1 = 10u$. Solving $u^2 - 10u - 1 = 0$, we obtain by the quadratic formula $u = 5 \pm \sqrt{26}$. From this, we have $e^x = 5 \pm \sqrt{26}$. Since $5 - \sqrt{26} < 0$, we get no real solution to $e^x = 5 - \sqrt{26}$, but for $e^x = 5 + \sqrt{26}$, we take natural logs to obtain $x = \ln\left(5 + \sqrt{26}\right)$. If we graph $f(x) = \frac{e^x - e^{-x}}{2}$ and $g(x) = 5$, we see that the graphs intersect at $x = \ln\left(5 + \sqrt{26}\right) \approx 2.312$



$y = f(x) = 25^x$ and
$y = g(x) = 5^x + 6$



$y = f(x) = \frac{e^x - e^{-x}}{2}$ and
$y = g(x) = 5$

The authors would be remiss not to mention that Example 6.3.1 still holds great educational value. Much can be learned about logarithms and exponentials by verifying the solutions obtained in Example 6.3.1 analytically. For example, to verify our solution to $2000 = 1000 \cdot 3^{-0.1t}$, we substitute $t = -\frac{10\ln(2)}{\ln(3)}$ and obtain

$$\begin{aligned}
2000 &\stackrel{?}{=} 1000 \cdot 3^{-0.1\left(-\frac{10\ln(2)}{\ln(3)}\right)} \\
2000 &\stackrel{?}{=} 1000 \cdot 3^{\frac{\ln(2)}{\ln(3)}} \\
2000 &\stackrel{?}{=} 1000 \cdot 3^{\log_3(2)} && \text{Change of Base} \\
2000 &\stackrel{?}{=} 1000 \cdot 2 && \text{Inverse Property} \\
2000 &\stackrel{\checkmark}{=} 2000
\end{aligned}$$

The other solutions can be verified by using a combination of log and inverse properties. Some fall out quite quickly, while others are more involved. We leave them to the reader.

Since exponential functions are continuous on their domains, the Intermediate Value Theorem 3.1 applies. As with the algebraic functions in Section 5.3, this allows us to solve inequalities using sign diagrams as demonstrated below.

Example 6.3.2

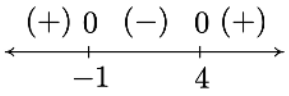
Solve the following inequalities. Check your answer graphically using a calculator.

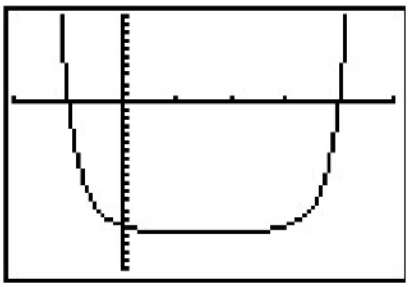
1. $2^{x^2-3x} - 16 \geq 0$
2. $\dfrac{e^x}{e^x - 4} \leq 3$
3. $xe^{2x} < 4x$

Solution.

1. Since we already have 0 on one side of the inequality, we set $r(x) = 2^{x^2-3x} - 16$. The domain of r is all real numbers, so in order to construct our sign diagram, we need to find the zeros of r. Setting $r(x) = 0$ gives $2^{x^2-3x} - 16 = 0$ or $2^{x^2-3x} = 16$. Since $16 = 2^4$ we have $2^{x^2-3x} = 2^4$, so by the one-to-one property of exponential functions, $x^2 - 3x = 4$.

Solving $x^2 - 3x - 4 = 0$ gives $x = 4$ and $x = -1$. From the sign diagram, we see $r(x) \geq 0$ on $(-\infty, -1] \cup [4, \infty)$, which corresponds to where the graph of $y = r(x) = 2^{x^2-3x} - 16$, is on or above the x-axis.





$y = r(x) = 2^{x^2-3x} - 16$

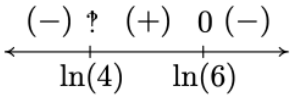
2. The first step we need to take to solve $\frac{e^x}{e^x-4} \leq 3$ is to get 0 on one side of the inequality. To that end, we subtract 3 from both sides and get a common denominator

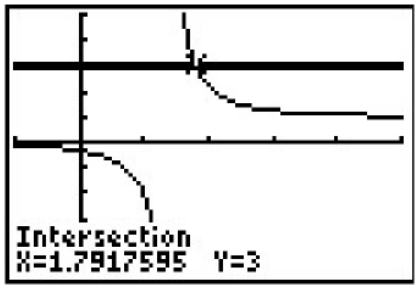
$$\begin{aligned} \frac{e^x}{e^x-4} &\leq 3 \\ \frac{e^x}{e^x-4} - 3 &\leq 0 \\ \frac{e^x}{e^x-4} - \frac{3(e^x-4)}{e^x-4} &\leq 0 \quad \text{Common denomintors.} \\ \frac{12-2e^x}{e^x-4} &\leq 0 \end{aligned}$$

We set $r(x) = \frac{12-2e^x}{e^x-4}$ and we note that r is undefined when its denominator $e^x - 4 = 0$, or when $e^x = 4$. Solving this gives $x = \ln(4)$, so the domain of r is $(-\infty, \ln(4)) \cup (\ln(4), \infty)$. To find the zeros of r, we solve $r(x) = 0$ and obtain $12 - 2e^x = 0$. Solving for e^x, we find $e^x = 6$, or $x = \ln(6)$. When we build our sign diagram, finding test values may be a little tricky since we need to check values around $\ln(4)$ and $\ln(6)$. Recall that the function $\ln(x)$ is increasing[4] which means $\ln(3) < \ln(4) < \ln(5) < \ln(6) < \ln(7)$. While the prospect of determining the sign of $r(\ln(3))$ may be very unsettling, remember that $e^{\ln(3)} = 3$, so

$$r(\ln(3)) = \frac{12 - 2e^{\ln(3)}}{e^{\ln(3)} - 4} = \frac{12 - 2(3)}{3 - 4} = -6$$

We determine the signs of $r(\ln(5))$ and $r(\ln(7))$ similarly.[5] From the sign diagram, we find our answer to be $(-\infty, \ln(4)) \cup [\ln(6), \infty)$. Using the calculator, we see the graph of $f(x) = \frac{e^x}{e^x-4}$ is below the graph of $g(x) = 3$ on $(-\infty, \ln(4)) \cup (\ln(6), \infty)$, and they intersect at $x = \ln(6) \approx 1.792$.





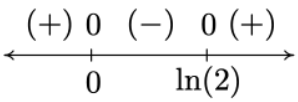
$y = f(x) = \frac{e^x}{e^x-4}$

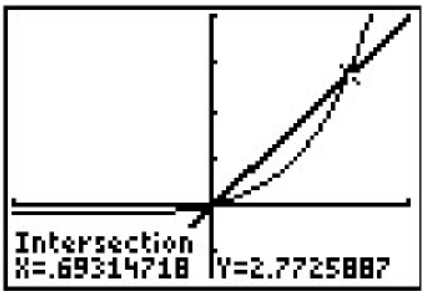
$\boldsymbol{y = g(x) = 3}$

3. As before, we start solving $xe^{2x} < 4x$ by getting 0 on one side of the inequality, $xe^{2x} - 4x < 0$. We set $r(x) = xe^{2x} - 4x$ and since there are no denominators, even-indexed radicals, or logs, the domain of r is all real numbers. Setting $r(x) = 0$ produces $xe^{2x} - 4x = 0$. We factor to get $x\left(e^{2x} - 4\right) = 0$ which gives $x = 0$ or $e^{2x} - 4 = 0$. To solve the latter, we isolate the exponential and take logs to get $2x = \ln(4)$, or $x = \frac{\ln(4)}{2} = \ln(2)$. (Can you explain the last equality using properties of logs?) As in the previous example, we need to be careful about choosing test values. Since $\ln(1) = 0$, we choose $\ln\left(\frac{1}{2}\right)$, $\ln\left(\frac{3}{2}\right)$ and $\ln(3)$. Evaluating,[6] we get

$$\begin{aligned} r\left(\ln\left(\tfrac{1}{2}\right)\right) &= \ln\left(\tfrac{1}{2}\right)e^{2\ln\left(\frac{1}{2}\right)} - 4\ln\left(\tfrac{1}{2}\right) \\ &= \ln\left(\tfrac{1}{2}\right)e^{\ln\left(\frac{1}{2}\right)^2} - 4\ln\left(\tfrac{1}{2}\right) \qquad \text{Power Rule} \\ &= \ln\left(\tfrac{1}{2}\right)e^{\ln\left(\frac{1}{4}\right)} - 4\ln\left(\tfrac{1}{2}\right) \\ &= \tfrac{1}{4}\ln\left(\tfrac{1}{2}\right) - 4\ln\left(\tfrac{1}{2}\right) = -\tfrac{15}{4}\ln\left(\tfrac{1}{2}\right) \end{aligned}$$

Since $\frac{1}{2} < 1$, $\ln\left(\frac{1}{2}\right) < 0$ and we get $r(\ln\left(\frac{1}{2}\right))$ is $(+)$, so $r(x) < 0$ on $(0, \ln(2))$. The calculator confirms that the graph of $f(x) = xe^{2x}$ is below the graph of $g(x) = 4x$ on these intervals.[7]





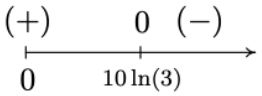
$y = f(x) = xe^{2x}$ and $\boldsymbol{y = g(x) = 4x}$

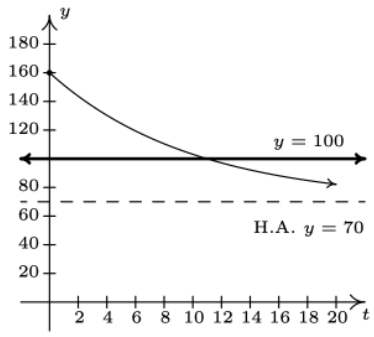
Example 6.3.3

Recall from Example 6.1.2 that the temperature of coffee T (in degrees Fahrenheit) t minutes after it is served can be modeled by $T(t) = 70 + 90e^{-0.1t}$. When will the coffee be warmer than 100°F?

Solution

We need to find when $T(t) > 100$, or in other words, we need to solve the inequality $70 + 90e^{-0.1t} > 100$. Getting 0 on one side of the inequality, we have $90e^{-0.1t} - 30 > 0$, and we set $r(t) = 90e^{-0.1t} - 30$. The domain of r is artificially restricted due to the context of the problem to $[0, \infty)$, so we proceed to find the zeros of r. Solving $90e^{-0.1t} - 30 = 0$ results in $e^{-0.1t} = \frac{1}{3}$ so that $t = -10\ln\left(\frac{1}{3}\right)$ which, after a quick application of the Power Rule leaves us with $t = 10\ln(3)$. If we wish to avoid using the calculator to choose test values, we note that since $1 < 3$, $0 = \ln(1) < \ln(3)$ so that $10\ln(3) > 0$. So we choose $t = 0$ as a test value in $[0, 10\ln(3))$. Since $3 < 4$, $10\ln(3) < 10\ln(4)$, so the latter is our choice of a test value for the interval $(10\ln(3), \infty)$. Our sign diagram is below, and next to it is our graph of $y = T(t)$ from Example 6.1.2 with the horizontal line $y = 100$.





$y = T(t)$

In order to interpret what this means in the context of the real world, we need a reasonable approximation of the number $10\ln(3) \approx 10.986$. This means it takes approximately 11 minutes for the coffee to cool to 100°F. Until then, the coffee is warmer than that.[8]

We close this section by finding the inverse of a function which is a composition of a rational function with an exponential function.

Example 6.3.4

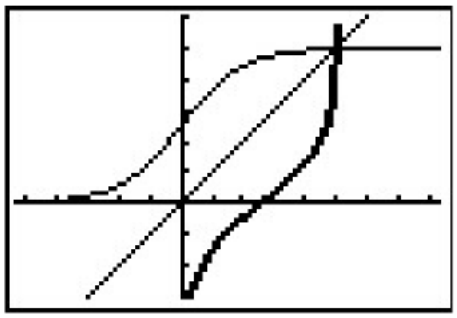
The function $f(x) = \dfrac{5e^x}{e^x + 1}$ is one-to-one. Find a formula for $f^{-1}(x)$ and check your answer graphically using your calculator.

Solution

We start by writing $y = f(x)$, and interchange the roles of x and y. To solve for y, we first clear denominators and then isolate the exponential function.

$$\begin{aligned} y &= \frac{5e^x}{e^x+1} \\ [12pt] x &= \frac{5e^y}{e^y+1} \qquad \text{Switch } x \text{ and } y \\ [12pt] x\left(e^y+1\right) &= 5e^y \\ xe^y + x &= 5e^y \\ x &= 5e^y - xe^y \\ x &= e^y(5-x) \\ e^y &= \frac{x}{5-x} \\ \ln(e^y) &= \ln\left(\frac{x}{5-x}\right) \\ y &= \ln\left(\frac{x}{5-x}\right) \end{aligned}$$

We claim $f^{-1}(x) = \ln\left(\frac{x}{5-x}\right)$. To verify this analytically, we would need to verify the compositions $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}. We leave this to the reader. To verify our solution graphically, we graph $y = f(x) = \frac{5e^x}{e^x+1}$ and $y = g(x) = \ln\left(\frac{x}{5-x}\right)$ on the same set of axes and observe the symmetry about the line $y = x$. Note the domain of f is the range of g and vice-versa.



$y = f(x) = \frac{5e^x}{e^x+1}$ and $y = g(x) = \ln\left(\frac{x}{5-x}\right)$

6.3.1. Exercises

In Exercises 1 - 33, solve the equation analytically.

1. $2^{4x} = 8$ [expeqnfirst]
2. $3^{(x-1)} = 27$
3. $5^{2x-1} = 125$
4. $4^{2x} = \frac{1}{2}$
5. $8^x = \frac{1}{128}$
6. $2^{(x^3-x)} = 1$
7. $3^{7x} = 81^{4-2x}$
8. $9 \cdot 3^{7x} = \left(\frac{1}{9}\right)^{2x}$
9. $3^{2x} = 5$
10. $5^{-x} = 2$
11. $5^x = -2$
12. $3^{(x-1)} = 29$
13. $(1.005)^{12x} = 3$
14. $e^{-5730k} = \frac{1}{2}$
15. $2000e^{0.1t} = 4000$
16. $500\left(1 - e^{2x}\right) = 250$
17. $70 + 90e^{-0.1t} = 75$
18. $30 - 6e^{-0.1x} = 20$
19. $\dfrac{100e^x}{e^x + 2} = 50$
20. $\dfrac{5000}{1 + 2e^{-3t}} = 2500$
21. $\dfrac{150}{1 + 29e^{-0.8t}} = 75$
22. $25\left(\frac{4}{5}\right)^x = 10$
23. $e^{2x} = 2e^x$
24. $7e^{2x} = 28e^{-6x}$
25. $3^{(x-1)} = 2^x$
26. $3^{(x-1)} = \left(\frac{1}{2}\right)^{(x+5)}$
27. $7^{3+7x} = 3^{4-2x}$
28. $e^{2x} - 3e^x - 10 = 0$
29. $e^{2x} = e^x + 6$
30. $4^x + 2^x = 12$
31. $e^x - 3e^{-x} = 2$
32. $e^x + 15e^{-x} = 8$
33. $3^x + 25 \cdot 3^{-x} = 10$ [expeqnlast]

In Exercises 34 - 39, solve the inequality analytically.

34. $e^x > 53$ [expineqfirst]
35. $1000(1.005)^{12t} \geq 3000$
36. $2^{(x^3-x)} < 1$
37. $25\left(\frac{4}{5}\right)^x \geq 10$
38. $\dfrac{150}{1+29e^{-0.8t}} \leq 130$
39. $70 + 90e^{-0.1t} \leq 75$ [expineqlast]

In Exercises 40 - 45, use your calculator to help you solve the equation or inequality.

40. $2^x = x^2$ [calcexpineqfirst]
41. $e^x = \ln(x) + 5$
42. $e^{\sqrt{x}} = x + 1$
43. $e^{-x} - xe^{-x} \geq 0$
44. $3^{(x-1)} < 2^x$
45. $e^x < x^3 - x$ [calcexpineqlast]
46. [onetoonelogexercise] Since $f(x) = \ln(x)$ is a strictly increasing function, if $0 < a < b$ then $\ln(a) < \ln(b)$. Use this fact to solve the inequality $e^{(3x-1)} > 6$ without a sign diagram. Use this technique to solve the inequalities in Exercises 34 - 39. (NOTE: Isolate the exponential function first!)
47. [hyperbolicsine] Compute the inverse of $f(x) = \dfrac{e^x - e^{-x}}{2}$. State the domain and range of both f and f^{-1}.
48. In Example 6.3.4, we found that the inverse of $f(x) = \dfrac{5e^x}{e^x + 1}$ was $f^{-1}(x) = \ln\left(\dfrac{x}{5-x}\right)$ but we left a few loose ends for you to tie up.
    1. Show that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
    2. Find the range of f by finding the domain of f^{-1}.
    3. Let $g(x) = \dfrac{5x}{x+1}$ and $h(x) = e^x$. Show that $f = g \circ h$ and that $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$. (We know this is true in general by Exercise 31 in Section 5.2, but it's nice to see a specific example of the property.)
49. With the help of your classmates, solve the inequality $e^x > x^n$ for a variety of natural numbers n. What might you conjecture about the "speed" at which $f(x) = e^x$ grows versus any polynomial?

6.3.2. Answers

1. $x = \frac{3}{4}$
2. $x = 4$
3. $x = 2$
4. $x = -\frac{1}{4}$
5. $x = -\frac{7}{3}$
6. $x = -1,\ 0,\ 1$
7. $x = \frac{16}{15}$
8. $x = -\frac{2}{11}$
9. $x = \dfrac{\ln(5)}{2\ln(3)}$
10. $x = -\dfrac{\ln(2)}{\ln(5)}$
11. No solution.
12. $x = \dfrac{\ln(29)+\ln(3)}{\ln(3)}$
13. $x = \dfrac{\ln(3)}{12\ln(1.005)}$
14. $k = \dfrac{\ln\left(\frac{1}{2}\right)}{-5730} = \dfrac{\ln(2)}{5730}$
15. $t = \dfrac{\ln(2)}{0.1} = 10\ln(2)$

16. $x = \frac{1}{2}\ln\left(\frac{1}{2}\right) = -\frac{1}{2}\ln(2)$
17. $t = \frac{\ln\left(\frac{1}{18}\right)}{-0.1} = 10\ln(18)$
18. $x = -10\ln\left(\frac{5}{3}\right) = 10\ln\left(\frac{3}{5}\right)$
19. $x = \ln(2)$
20. $t = \frac{1}{3}\ln(2)$
21. $t = \frac{\ln\left(\frac{1}{29}\right)}{-0.8} = \frac{5}{4}\ln(29)$
22. $x = \frac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)} = \frac{\ln(2)-\ln(5)}{\ln(4)-\ln(5)}$
23. $x = \ln(2)$
24. $x = -\frac{1}{8}\ln\left(\frac{1}{4}\right) = \frac{1}{4}\ln(2)$
25. $x = \frac{\ln(3)}{\ln(3)-\ln(2)}$
26. $x = \frac{\ln(3)+5\ln\left(\frac{1}{2}\right)}{\ln(3)-\ln\left(\frac{1}{2}\right)} = \frac{\ln(3)-5\ln(2)}{\ln(3)+\ln(2)}$
27. $x = \frac{4\ln(3)-3\ln(7)}{7\ln(7)+2\ln(3)}$
28. $x = \ln(5)$
29. $x = \ln(3)$
30. $x = \frac{\ln(3)}{\ln(2)}$
31. $x = \ln(3)$
32. $x = \ln(3), \ln(5)$
33. $x = \frac{\ln(5)}{\ln(3)}$
34. $(\ln(53), \infty)$
35. $\left[\frac{\ln(3)}{12\ln(1.005)}, \infty\right)$
36. $(-\infty, -1) \cup (0, 1)$
37. $\left(-\infty, \frac{\ln\left(\frac{2}{5}\right)}{\ln\left(\frac{4}{5}\right)}\right] = \left(-\infty, \frac{\ln(2)-\ln(5)}{\ln(4)-\ln(5)}\right]$
38. $\left(-\infty, \frac{\ln\left(\frac{2}{377}\right)}{-0.8}\right] = \left(-\infty, \frac{5}{4}\ln\left(\frac{377}{2}\right)\right]$
39. $\left[\frac{\ln\left(\frac{1}{18}\right)}{-0.1}, \infty\right) = [10\ln(18), \infty)$
40. $x \approx -0.76666,\ x = 2,\ x = 4$
41. $x \approx 0.01866,\ x \approx 1.7115$
42. $x = 0$
43. $(-\infty, 1]$
44. $\approx (-\infty, 2.7095)$
45. $\approx (2.3217, 4.3717)$
46. $x > \frac{1}{3}(\ln(6)+1)$
47. $f^{-1} = \ln\left(x + \sqrt{x^2+1}\right)$. Both f and f^{-1} have domain $(-\infty, \infty)$ and range $(-\infty, \infty)$.

Reference

[1] You can use natural logs or common logs. We choose natural logs. (In Calculus, you'll learn these are the most 'mathy' of the logarithms.)

[2] This is also the 'if' part of the statement $\log_b(u) = \log_b(w)$ if and only if $u = w$ in Theorem 6.4.

[3] Please resist the temptation to divide both sides by 'ln' instead of ln(2). Just like it wouldn't make sense to divide both sides by the square root symbol '$\sqrt{}$' when solving $x\sqrt{2} = 5$, it makes no sense to divide by 'ln'.

[4] This is because the base of $\ln(x)$ is $e > 1$. If the base b were in the interval $0 < b < 1$, then $\log_b(x)$ would decreasing.

[5] We could, of course, use the calculator, but what fun would that be?

[6] A calculator can be used at this point. As usual, we proceed without apologies, with the analytical method.

[7] Note: $\ln(2) \approx 0.693$.

[8] Critics may point out that since we needed to use the calculator to interpret our answer anyway, why not use it earlier to simplify the computations? It is a fair question which we answer unfairly: it's our book

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5.4: Logarithmic Equations and Inequalities

In Section 6.3 we solved equations and inequalities involving exponential functions using one of two basic strategies. We now turn our attention to equations and inequalities involving logarithmic functions, and not surprisingly, there are two basic strategies to choose from. For example, suppose we wish to solve $\log_2(x) = \log_2(5)$. Theorem 6.4 tells us that the *only* solution to this equation is $x = 5$. Now suppose we wish to solve $\log_2(x) = 3$. If we want to use Theorem 6.4, we need to rewrite 3 as a logarithm base 2. We can use Theorem 6.3 to do just that: $3 = \log_2\left(2^3\right) = \log_2(8)$. Our equation then becomes $\log_2(x) = \log_2(8)$ so that $x = 8$. However, we could have arrived at the same answer, in fewer steps, by using Theorem 6.3 to rewrite the equation $\log_2(x) = 3$ as $2^3 = x$, or $x = 8$. We summarize the two common ways to solve log equations below.

Steps for Solving an Equation involving Logarithmic Functions

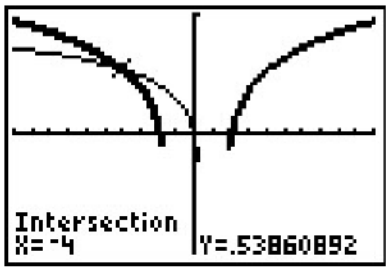
1. Isolate the logarithmic function.
2. a. If convenient, express both sides as logs with the same base and equate the arguments of the log functions.
   b. Otherwise, rewrite the log equation as an exponential equation.

Example 6.4.1

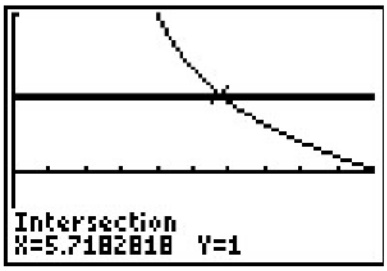
1. $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$
2. $2 - \ln(x-3) = 1$
3. $\log_6(x+4) + \log_6(3-x) = 1$
4. $\log_7(1-2x) = 1 - \log_7(3-x)$
5. $\log_2(x+3) = \log_2(6-x) + 3$
6. $1 + 2\log_4(x+1) = 2\log_2(x)$

Solution.

1. Since we have the same base on both sides of the equation $\log_{117}(1-3x) = \log_{117}\left(x^2-3\right)$, we equate what's inside the logs to get $1-3x = x^2-3$. Solving $x^2+3x-4=0$ gives $x=-4$ and $x=1$. To check these answers using the calculator, we make use of the change of base formula and graph $f(x) = \frac{\ln(1-3x)}{\ln(117)}$ and $g(x) = \frac{\ln(x^2-3)}{\ln(117)}$ and we see they intersect only at $x=-4$. To see what happened to the solution $x=1$, we substitute it into our original equation to obtain $\log_{117}(-2) = \log_{117}(-2)$. While these expressions look identical, neither is a real number,[1] which means $x=1$ is not in the domain of the original equation, and is not a solution.
2. Our first objective in solving $2 - \ln(x-3) = 1$ is to isolate the logarithm. We get $\ln(x-3) = 1$, which, as an exponential equation, is $e^1 = x-3$. We get our solution $x = e+3$. On the calculator, we see the graph of $f(x) = 2 - \ln(x-3)$ intersects the graph of $g(x) = 1$ at $x = e+3 \approx 5.718$.

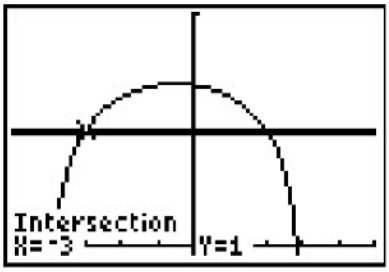


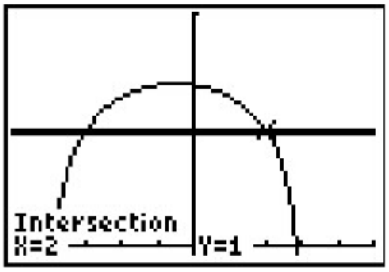
$y = f(x) = \log_{117}(1-3x)$ and $\boldsymbol{y = g(x) = \log_{117}\left(x^2-3\right)}$



$y = f(x) = 2 - \ln(x-3)$ and $\boldsymbol{y = g(x) = 1}$

3. We can start solving $\log_6(x+4) + \log_6(3-x) = 1$ by using the Product Rule for logarithms to rewrite the equation as $\log_6[(x+4)(3-x)] = 1$. Rewriting this as an exponential equation, we get $6^1 = (x+4)(3-x)$. This reduces to $x^2 + x - 6 = 0$, which gives $x=-3$ and $x=2$. Graphing $y = f(x) = \frac{\ln(x+4)}{\ln(6)} + \frac{\ln(3-x)}{\ln(6)}$ and $y = g(x) = 1$, we see they intersect twice, at $x=-3$ and $x=2$.



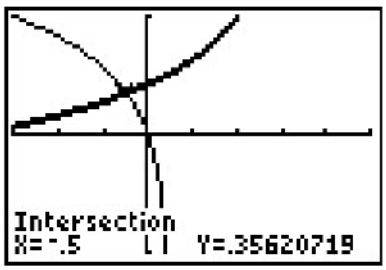


$y = f(x) = \log_6(x+4) + \log_6(3-x)$ and $\boldsymbol{y = g(x) = 1}$

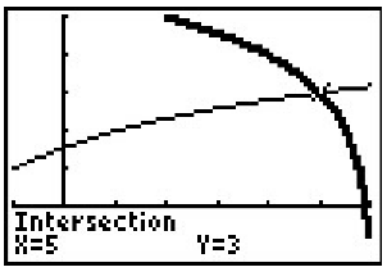
4. Taking a cue from the previous problem, we begin solving $\log_7(1-2x) = 1 - \log_7(3-x)$ by first collecting the logarithms on the same side, $\log_7(1-2x) + \log_7(3-x) = 1$, and then using the Product Rule to get $\log_7[(1-2x)(3-x)] = 1$. Rewriting this as an exponential equation gives $7^1 = (1-2x)(3-x)$ which gives the quadratic equation $2x^2 - 7x - 4 = 0$. Solving, we find $x = -\frac{1}{2}$ and $x = 4$. Graphing, we find $y = f(x) = \frac{\ln(1-2x)}{\ln(7)}$ and $y = g(x) = 1 - \frac{\ln(3-x)}{\ln(7)}$ intersect only at $x = -\frac{1}{2}$. Checking $x = 4$ in the original equation produces $\log_7(-7) = 1 - \log_7(-1)$, which is a clear domain violation.
5. Starting with $\log_2(x+3) = \log_2(6-x) + 3$, we gather the logarithms to one side and get $\log_2(x+3) - \log_2(6-x) = 3$. We then use the Quotient Rule and convert to an exponential equation

$$\log_2\left(\frac{x+3}{6-x}\right) = 3 \iff 2^3 = \frac{x+3}{6-x}$$

This reduces to the linear equation $8(6-x) = x+3$, which gives us $x = 5$. When we graph $f(x) = \frac{\ln(x+3)}{\ln(2)}$ and $g(x) = \frac{\ln(6-x)}{\ln(2)} + 3$, we find they intersect at $x = 5$.



$y = f(x) = \log_7(1-2x)$ and
$\boldsymbol{y = g(x) = 1 - \log_7(3-x)}$



$y = f(x) = \log_2(x+3)$ and
$\boldsymbol{y = g(x) = \log_2(6-x) + 3}$

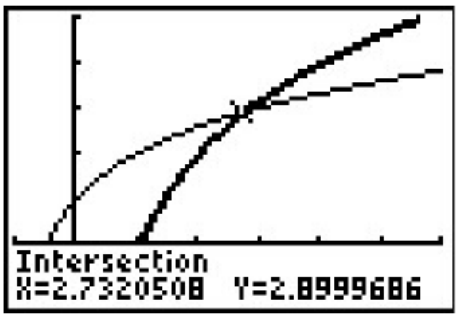
6. Starting with $1 + 2\log_4(x+1) = 2\log_2(x)$, we gather the logs to one side to get the equation $1 = 2\log_2(x) - 2\log_4(x+1)$. Before we can combine the logarithms, however, we need a common base. Since 4 is a power of 2, we use change of base to convert

$$\log_4(x+1) = \frac{\log_2(x+1)}{\log_2(4)} = \frac{1}{2}\log_2(x+1)$$

Hence, our original equation becomes

$$\begin{aligned} 1 &= 2\log_2(x) - 2\left(\tfrac{1}{2}\log_2(x+1)\right) \\ 1 &= 2\log_2(x) - \log_2(x+1) \\ 1 &= \log_2(x^2) - \log_2(x+1) && \text{Power Rule} \\ 1 &= \log_2\left(\frac{x^2}{x+1}\right) && \text{Quotient Rule} \end{aligned}$$

Rewriting this in exponential form, we get $\frac{x^2}{x+1}=2$ or $x^2-2x-2=0$. Using the quadratic formula, we get $x=1\pm\sqrt{3}$. Graphing $f(x)=1+\frac{2\ln(x+1)}{\ln(4)}$ and $g(x)=\frac{2\ln(x)}{\ln(2)}$, we see the graphs intersect only at $x=1+\sqrt{3}\approx 2.732$. The solution $x=1-\sqrt{3}<0$, which means if substituted into the original equation, the term $2\log_2\left(1-\sqrt{3}\right)$ is undefined.



$y=f(x)=1+2\log_4(x+1)$ and $y=g(x)=2\log_2(x)$

If nothing else, Example 6.4.1 demonstrates the importance of checking for extraneous solutions[2] when solving equations involving logarithms. Even though we checked our answers graphically, extraneous solutions are easy to spot - any supposed solution which causes a negative number inside a logarithm needs to be discarded. As with the equations in Example 6.3.1, much can be learned from checking all of the answers in Example 6.4.1 analytically. We leave this to the reader and turn our attention to inequalities involving logarithmic functions. Since logarithmic functions are continuous on their domains, we can use sign diagrams.

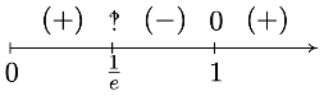
Example 6.4.2

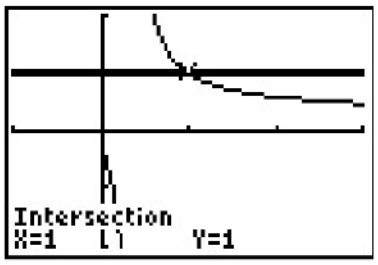
Solve the following inequalities. Check your answer graphically using a calculator.

1. $\dfrac{1}{\ln(x)+1}\le 1$
2. $(\log_2(x))^2<2\log_2(x)+3$
3. $x\log(x+1)\ge x$

Solution.

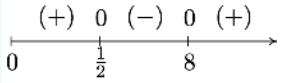
1. We start solving $\frac{1}{\ln(x)+1}\le 1$ by getting 0 on one side of the inequality: $\frac{1}{\ln(x)+1}-1\le 0$. Getting a common denominator yields $\frac{1}{\ln(x)+1}-\frac{\ln(x)+1}{\ln(x)+1}\le 0$ which reduces to $\frac{-\ln(x)}{\ln(x)+1}\le 0$, or $\frac{\ln(x)}{\ln(x)+1}\ge 0$. We define $r(x)=\frac{\ln(x)}{\ln(x)+1}$ and set about finding the domain and the zeros of r. Due to the appearance of the term $\ln(x)$, we require $x>0$. In order to keep the denominator away from zero, we solve $\ln(x)+1=0$ so $\ln(x)=-1$, so $x=e^{-1}=\frac{1}{e}$. Hence, the domain of r is $\left(0,\frac{1}{e}\right)\cup\left(\frac{1}{e},\infty\right)$. To find the zeros of r, we set $r(x)=\frac{\ln(x)}{\ln(x)+1}=0$ so that $\ln(x)=0$, and we find $x=e^0=1$. In order to determine test values for r without resorting to the calculator, we need to find numbers between 0, $\frac{1}{e}$, and 1 which have a base of e. Since $e\approx 2.718>1$, $0<\frac{1}{e^2}<\frac{1}{e}<\frac{1}{\sqrt{e}}<1<e$. To determine the sign of $r\left(\frac{1}{e^2}\right)$, we use the fact that $\ln\left(\frac{1}{e^2}\right)=\ln\left(e^{-2}\right)=-2$, and find $r\left(\frac{1}{e^2}\right)=\frac{-2}{-2+1}=2$, which is $(+)$. The rest of the test values are determined similarly. From our sign diagram, we find the solution to be $\left(0,\frac{1}{e}\right)\cup[1,\infty)$. Graphing $f(x)=\frac{1}{\ln(x)+1}$ and $g(x)=1$, we see the graph of f is below the graph of g on the solution intervals, and that the graphs intersect at $x=1$.

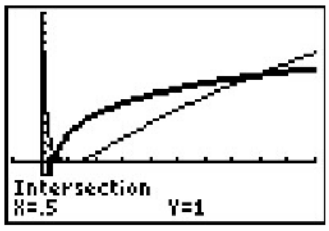




$y = f(x) = \frac{1}{\ln(x)+1}$ and $\boldsymbol{y = g(x) = 1}$

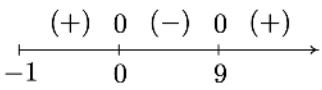
2. Moving all of the nonzero terms of $(\log_2(x))^2 < 2\log_2(x) + 3$ to one side of the inequality, we have $(\log_2(x))^2 - 2\log_2(x) - 3 < 0$. Defining $r(x) = (\log_2(x))^2 - 2\log_2(x) - 3$, we get the domain of r is $(0, \infty)$, due to the presence of the logarithm. To find the zeros of r, we set $r(x) = (\log_2(x))^2 - 2\log_2(x) - 3 = 0$ which results in a 'quadratic in disguise.' We set $u = \log_2(x)$ so our equation becomes $u^2 - 2u - 3 = 0$ which gives us $u = -1$ and $u = 3$. Since $u = \log_2(x)$, we get $\log_2(x) = -1$, which gives us $x = 2^{-1} = \frac{1}{2}$, and $\log_2(x) = 3$, which yields $x = 2^3 = 8$. We use test values which are powers of 2: $0 < \frac{1}{4} < \frac{1}{2} < 1 < 8 < 16$, and from our sign diagram, we see $r(x) < 0$ on $\left(\frac{1}{2}, 8\right)$. Geometrically, we see the graph of $f(x) = \left(\frac{\ln(x)}{\ln(2)}\right)^2$ is below the graph of $y = g(x) = \frac{2\ln(x)}{\ln(2)} + 3$ on the solution interval.

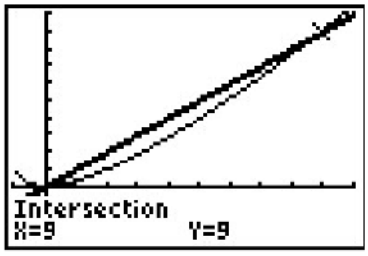




$y = f(x) = (\log_2(x))^2$ and $\boldsymbol{y = g(x) = 2\log_2(x) + 3}$

3. We begin to solve $x\log(x+1) \geq x$ by subtracting x from both sides to get $x\log(x+1) - x \geq 0$. We define $r(x) = x\log(x+1) - x$ and due to the presence of the logarithm, we require $x + 1 > 0$, or $x > -1$. To find the zeros of r, we set $r(x) = x\log(x+1) - x = 0$. Factoring, we get $x\,(\log(x+1) - 1) = 0$, which gives $x = 0$ or $\log(x+1) - 1 = 0$. The latter gives $\log(x+1) = 1$, or $x + 1 = 10^1$, which admits $x = 9$. We select test values x so that $x + 1$ is a power of 10, and we obtain $-1 < -0.9 < 0 < \sqrt{10} - 1 < 9 < 99$. Our sign diagram gives the solution to be $(-1, 0] \cup [9, \infty)$. The calculator indicates the graph of $y = f(x) = x\log(x+1)$ is above $y = g(x) = x$ on the solution intervals, and the graphs intersect at $x = 0$ and $x = 9$.





$y = f(x) = x\log(x+1)$ and $\boldsymbol{y = g(x) = x}$

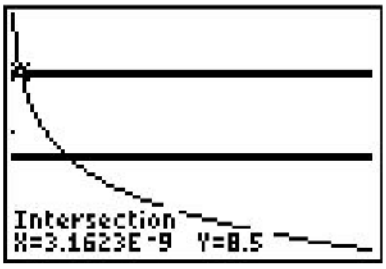
Our next example revisits the concept of pH first seen in Exercise 77 in Section 6.1.

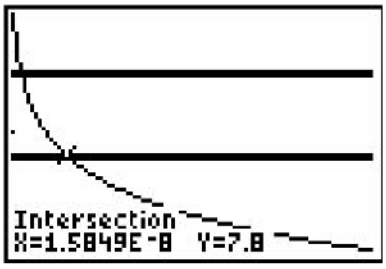
Example 6.4.3

In order to successfully breed Ippizuti fish the pH of a freshwater tank must be at least 7.8 but can be no more than 8.5. Determine the corresponding range of hydrogen ion concentration, and check your answer using a calculator.

Solution

Recall from Exercise 77 in Section 6.1 that $\text{pH} = -\log[\text{H}^+]$ where $[\text{H}^+]$ is the hydrogen ion concentration in moles per liter. We require $7.8 \leq -\log[\text{H}^+] \leq 8.5$ or $-7.8 \geq \log[\text{H}^+] \geq -8.5$. To solve this compound inequality we solve $-7.8 \geq \log[\text{H}^+]$ and $\log[\text{H}^+] \geq -8.5$ and take the intersection of the solution sets.[3] The former inequality yields $0 < [\text{H}^+] \leq 10^{-7.8}$ and the latter yields $[\text{H}^+] \geq 10^{-8.5}$. Taking the intersection gives us our final answer $10^{-8.5} \leq [\text{H}^+] \leq 10^{-7.8}$. (Your Chemistry professor may want the answer written as $3.16 \times 10^{-9} \leq [\text{H}^+] \leq 1.58 \times 10^{-8}$.) After carefully adjusting the viewing window on the graphing calculator we see that the graph of $f(x) = -\log(x)$ lies between the lines $y = 7.8$ and $y = 8.5$ on the interval $[3.16 \times 10^{-9}, 1.58 \times 10^{-8}]$.





The graphs of $y = f(x) = -\log(x)$, $\boldsymbol{y = 7.8}$ and $\boldsymbol{y = 8.5}$

The graphs of $y = f(x) = -\log(x)$, $y = 7.8$ and $y = 8.5$

We close this section by finding an inverse of a one-to-one function which involves logarithms.

Example 6.4.4

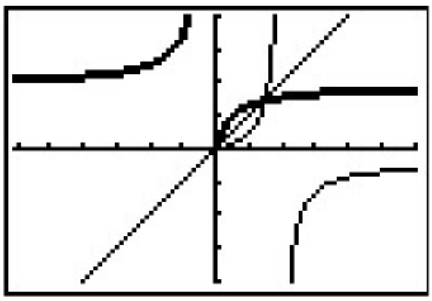
The function $f(x) = \dfrac{\log(x)}{1 - \log(x)}$ is one-to-one. Find a formula for $f^{-1}(x)$ and check your answer graphically using your calculator.

Solution

We first write $y = f(x)$ then interchange the x and y and solve for y.

$$\begin{aligned} y &= f(x) \\ y &= \frac{\log(x)}{1 - \log(x)} \\ x &= \frac{\log(y)}{1 - \log(y)} \qquad \text{Interchange } x \text{ and } y. \\ x\,(1 - \log(y)) &= \log(y) \\ x - x\log(y) &= \log(y) \\ x &= x\log(y) + \log(y) \\ x &= (x+1)\log(y) \\ \frac{x}{x+1} &= \log(y) \\ y &= 10^{\frac{x}{x+1}} \qquad \text{Rewrite as an exponential equation.} \end{aligned}$$

We have $f^{-1}(x) = 10^{\frac{x}{x+1}}$. Graphing f and f^{-1} on the same viewing window yields



$y = f(x) = \dfrac{\log(x)}{1 - \log(x)}$ and $y = g(x) = 10^{\frac{x}{x+1}}$

6.4.1. Exercises

In Exercises 1 - 24, solve the equation analytically.

1. $\log(3x-1) = \log(4-x)$
2. $\log_2\left(x^3\right) = \log_2(x)$
3. $\ln\left(8-x^2\right) = \ln(2-x)$
4. $\log_5\left(18-x^2\right) = \log_5(6-x)$
5. $\log_3(7-2x) = 2$
6. $\log_{\frac{1}{2}}(2x-1) = -3$
7. $\ln\left(x^2-99\right) = 0$
8. $\log(x^2-3x) = 1$
9. $\log_{125}\left(\dfrac{3x-2}{2x+3}\right) = \dfrac{1}{3}$
10. $\log\left(\dfrac{x}{10^{-3}}\right) = 4.7$
11. $-\log(x) = 5.4$ [sixfourpHequ]
12. $10\log\left(\dfrac{x}{10^{-12}}\right) = 150$
13. $6 - 3\log_5(2x) = 0$
14. $3\ln(x) - 2 = 1 - \ln(x)$
15. $\log_3(x-4) + \log_3(x+4) = 2$
16. $\log_5(2x+1) + \log_5(x+2) = 1$
17. $\log_{169}(3x+7) - \log_{169}(5x-9) = \dfrac{1}{2}$
18. $\ln(x+1) - \ln(x) = 3$
19. $2\log_7(x) = \log_7(2) + \log_7(x+12)$
20. $\log(x) - \log(2) = \log(x+8) - \log(x+2)$
21. $\log_3(x) = \log_{\frac{1}{3}}(x) + 8$
22. $\ln(\ln(x)) = 3$
23. $(\log(x))^2 = 2\log(x) + 15$
24. $\ln(x^2) = (\ln(x))^2$

In Exercises 25 - 30, solve the inequality analytically.

25. $\dfrac{1-\ln(x)}{x^2} < 0$
26. $x\ln(x) - x > 0$
27. $10\log\left(\dfrac{x}{10^{-12}}\right) \geq 90$
28. $5.6 \leq \log\left(\dfrac{x}{10^{-3}}\right) \leq 7.1$

29. $2.3 < -\log(x) < 5.4$
30. $\ln(x^2) \leq (\ln(x))^2$

In Exercises 31 - 34, use your calculator to help you solve the equation or inequality.

31. $\ln(x) = e^{-x}$
32. $\ln(x) = \sqrt[4]{x}$
33. $\ln(x^2+1) \geq 5$
34. $\ln(-2x^3 - x^2 + 13x - 6) < 0$
35. Since $f(x) = e^x$ is a strictly increasing function, if $a < b$ then $e^a < e^b$. Use this fact to solve the inequality $\ln(2x+1) < 3$ without a sign diagram. Use this technique to solve the inequalities in Exercises 27 - 29. (Compare this to Exercise 46 in Section 6.3.)
36. Solve $\ln(3-y) - \ln(y) = 2x + \ln(5)$ for y.
37. In Example 6.4.4 we found the inverse of $f(x) = \dfrac{\log(x)}{1-\log(x)}$ to be $f^{-1}(x) = 10^{\frac{x}{x+1}}$.
    1. Show that $\left(f^{-1} \circ f\right)(x) = x$ for all x in the domain of f and that $\left(f \circ f^{-1}\right)(x) = x$ for all x in the domain of f^{-1}.
    2. Find the range of f by finding the domain of f^{-1}.
    3. Let $g(x) = \dfrac{x}{1-x}$ and $h(x) = \log(x)$. Show that $f = g \circ h$ and $(g \circ h)^{-1} = h^{-1} \circ g^{-1}$.
    (We know this is true in general by Exercise 31 in Section 5.2, but it's nice to see a specific example of the property.)
38. Let $f(x) = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$. Compute $f^{-1}(x)$ and find its domain and range.
39. Explain the equation in Exercise 10 and the inequality in Exercise 28 above in terms of the Richter scale for earthquake magnitude. (See Exercise 75 in Section 6.1.)
40. Explain the equation in Exercise 12 and the inequality in Exercise 27 above in terms of sound intensity level as measured in decibels. (See Exercise 76 in Section 6.1.)
41. Explain the equation in Exercise 11 and the inequality in Exercise 29 above in terms of the pH of a solution. (See Exercise 77 in Section 6.1.)
42. With the help of your classmates, solve the inequality $\sqrt[n]{x} > \ln(x)$ for a variety of natural numbers n. What might you conjecture about the "speed" at which $f(x) = \ln(x)$ grows versus any principal n^{th} root function?

6.4.2 Answers

1. $x = \frac{5}{4}$
2. $x = 1$
3. $x = -2$
4. $x = -3,\ 4$
5. $x = -1$
6. $x = \frac{9}{2}$
7. $x = \pm 10$
8. $x = -2,\ 5$
9. $x = -\frac{17}{7}$
10. $x = 10^{1.7}$
11. $x = 10^{-5.4}$
12. $x = 10^3$
13. $x = \frac{25}{2}$
14. $x = e^{3/4}$
15. $x = 5$
16. $x = \frac{1}{2}$
17. $x = 2$
18. $x = \frac{1}{e^3 - 1}$
19. $x = 6$
20. $x = 4$
21. $x = 81$

22. $x = e^{e^3}$
23. $x = 10^{-3},\ 10^5$
24. $x = 1,\ x = e^2$
25. (e, ∞)
26. (e, ∞)
27. $\left[10^{-3}, \infty\right)$
28. $\left[10^{2.6}, 10^{4.1}\right]$
29. $\left(10^{-5.4}, 10^{-2.3}\right)$
30. $(0, 1] \cup [e^2, \infty)$
31. $x \approx 1.3098$
32. $x \approx 4.177,\ x \approx 5503.665$
33. $\approx (-\infty, -12.1414) \cup (12.1414, \infty)$
34. $\approx (-3.0281, -3) \cup (0.5, 0.5991) \cup (1.9299, 2)$
35. $-\dfrac{1}{2} < x < \dfrac{e^3 - 1}{2}$
36. $y = \dfrac{3}{5e^{2x} + 1}$
38. $f^{-1}(x) = \dfrac{e^{2x} - 1}{e^{2x} + 1} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. (To see why we rewrite this in this form, see Exercise 51 in Section 11.10.) The domain of f^{-1} is $(-\infty, \infty)$ and its range is the same as the domain of f, namely $(-1, 1)$.

Reference

[1] They do, however, represent the same **family** of complex numbers. We stop ourselves at this point and refer the reader to a good course in Complex Variables.

[2] Recall that an extraneous solution is an answer obtained analytically which does not satisfy the original equation.

[3] Refer to page 4 for a discussion of what this means.

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5.5: Applications of Exponential and Logarithmic Functions

As we mentioned in Section 6.1, exponential and logarithmic functions are used to model a wide variety of behaviors in the real world. In the examples that follow, note that while the applications are drawn from many different disciplines, the mathematics remains essentially the same. Due to the applied nature of the problems we will examine in this section, the calculator is often used to express our answers as decimal approximations.

6.5.1. Applications of Exponential Functions

Perhaps the most well-known application of exponential functions comes from the financial world. Suppose you have $\$100$ to invest at your local bank and they are offering a whopping $5\,\%$ annual percentage interest rate. This means that after one year, the bank will pay *you* 5% of that $\$100$, or $\$100(0.05) = \$5$ in interest, so you now have $\$105$.[1] This is in accordance with the formula for *simple interest* which you have undoubtedly run across at some point before.

Equation 6.1. Simple Interest

The amount of interest I accrued at an annual rate r on an investment[a] P after t years is

$$I = Prt$$

The amount A in the account after t years is given by

$$A = P + I = P + Prt = P(1 + rt)$$

[a] Called the **principal**

Suppose, however, that six months into the year, you hear of a better deal at a rival bank.[2] Naturally, you withdraw your money and try to invest it at the higher rate there. Since six months is one half of a year, that initial $\$100$ yields $\$100(0.05)\left(\frac{1}{2}\right) = \$2.50$ in interest. You take your $\$102.50$ off to the competitor and find out that those restrictions which *may* apply actually apply to you, and you return to your bank which happily accepts your $\$102.50$ for the remaining six months of the year. To your surprise and delight, at the end of the year your statement reads $\$105.06$, not $\$105$ as you had expected.[3] Where did those extra six cents come from? For the first six months of the year, interest was earned on the original principal of $\$100$, but for the second six months, interest was earned on $\$102.50$, that is, you earned interest on your interest. This is the basic concept behind **compound interest**. In the previous discussion, we would say that the interest was compounded twice, or semiannually.[4] If more money can be earned by earning interest on interest already earned, a natural question to ask is what happens if the interest is compounded more often, say 4 times a year, which is every three months, or 'quarterly.' In this case, the money is in the account for three months, or $\frac{1}{4}$ of a year, at a time. After the first quarter, we have $A = P(1 + rt) = \$100\left(1 + 0.05 \cdot \frac{1}{4}\right) = \$101.25$. We now invest the $\$101.25$ for the next three months and find that at the end of the second quarter, we have $A = \$101.25\left(1 + 0.05 \cdot \frac{1}{4}\right) \approx \$102.51$. Continuing in this manner, the balance at the end of the third quarter is $\$103.79$, and, at last, we obtain $\$105.08$. The extra two cents hardly seems worth it, but we see that we do in fact get more money the more often we compound. In order to develop a formula for this phenomenon, we need to do some abstract calculations. Suppose we wish to invest our principal P at an annual rate r and compound the interest n times per year. This means the money sits in the account $\frac{1}{n}^{\text{\tiny th}}$ of a year between compoundings. Let A_k denote the amount in the account after the $k^{\text{\tiny th}}$ compounding. Then $A_1 = P\left(1 + r\left(\frac{1}{n}\right)\right)$ which simplifies to $A_1 = P\left(1 + \frac{r}{n}\right)$. After the second compounding, we use A_1 as our new principal and get $A_2 = A_1\left(1 + \frac{r}{n}\right) = \left[P\left(1 + \frac{r}{n}\right)\right]\left(1 + \frac{r}{n}\right) = P\left(1 + \frac{r}{n}\right)^2$. Continuing in this fashion, we get $A_3 = P\left(1 + \frac{r}{n}\right)^3$, $A_4 = P\left(1 + \frac{r}{n}\right)^4$, and so on, so that $A_k = P\left(1 + \frac{r}{n}\right)^k$. Since we compound the interest n times per year, after t years, we have nt compoundings. We have just derived the general formula for compound interest below.

Equation 6.2. Compounded Interest

If an initial principal P is invested at an annual rate r and the interest is compounded n times per year, the amount A in the account after t years is

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

If we take $P = 100$, $r = 0.05$, and $n = 4$, Equation 6.2 becomes $A(t) = 100\left(1 + \frac{0.05}{4}\right)^{4t}$ which reduces to $A(t) = 100(1.0125)^{4t}$. To check this new formula against our previous calculations, we find $A\left(\frac{1}{4}\right) = 100(1.0125)^{4\left(\frac{1}{4}\right)} = 101.25$, $A\left(\frac{1}{2}\right) \approx \$102.51$, $A\left(\frac{3}{4}\right) \approx \$103.79$, and $A(1) \approx \$105.08$.

Example 6.5.1

Suppose $\$2000$ is invested in an account which offers 7.125% compounded monthly.

1. Express the amount A in the account as a function of the term of the investment t in years.
2. How much is in the account after 5 years?
3. How long will it take for the initial investment to double?
4. Find and interpret the average rate of change[5] of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year.

Solution.

1. Substituting $P = 2000$, $r = 0.07125$, and $n = 12$ (since interest is compounded *monthly*) into Equation 6.2 yields $A(t) = 2000\left(1 + \frac{0.07125}{12}\right)^{12t} = 2000(1.0059375)^{12t}$.
2. Since t represents the length of the investment in years, we substitute $t = 5$ into $A(t)$ to find $A(5) = 2000(1.0059375)^{12(5)} \approx 2852.92$ After 5 years, we have approximately $\$2852.92$
3. Our initial investment is $\$2000$, so to find the time it takes this to double, we need to find t when $A(t) = 4000$. We get $2000(1.0059375)^{12t} = 4000$ or $(1.0059375)^{12t} = 2$ Taking natural logs as in Section 6.3, we get $t = \frac{\ln(2)}{12\ln(1.0059375)} \approx 9.75$. Hence, it takes approximately 9 years 9 months for the investment to double.
4. To find the average rate of change of A from the end of the fourth year to the end of the fifth year, we compute $\frac{A(5)-A(4)}{5-4} \approx 195.63$. Similarly, the average rate of change of A from the end of the thirty-fourth year to the end of the thirty-fifth year is $\frac{A(35)-A(34)}{35-34} \approx 1648.21$. This means that the value of the investment is increasing at a rate of approximately $\$195.63$ per year between the end of the fourth and fifth years, while that rate jumps to $\$1648.21$ per year between the end of the thirty-fourth and thirty-fifth years. So, not only is it true that the longer you wait, the more money you have, but also the longer you wait, the faster the money increases.[6]

We have observed that the more times you compound the interest per year, the more money you will earn in a year. Let's push this notion to the limit.[7] Consider an investment of $\$1$ invested at 100% interest for 1 year compounded n times a year. Equation 6.2 tells us that the amount of money in the account after 1 year is $A = \left(1 + \frac{1}{n}\right)^n$. Below is a table of values relating n and A.

n	A
1	2
2	2.25
4	≈ 2.4414
12	≈ 2.6130
360	≈ 2.7145
1000	≈ 2.7169
10000	≈ 2.7181
100000	≈ 2.7182

As promised, the more compoundings per year, the more money there is in the account, but we also observe that the increase in money is greatly diminishing. We are witnessing a mathematical 'tug of war'. While we are compounding more times per year, and hence getting interest on our interest more often, the amount of time between compoundings is getting smaller and smaller, so there is less time to build up additional interest. With Calculus, we can show[8] that as $n \to \infty$, $A = \left(1 + \frac{1}{n}\right)^n \to e$, where e is the natural base first presented in Section 6.1. Taking the number of compoundings per year to infinity results in what is called **continuously** compounded interest.

Theorem 6.8.

If you invest $\$1$ at 100% interest compounded continuously, then you will have $\$e$ at the end of one year.

Using this definition of e and a little Calculus, we can take Equation 6.2 and produce a formula for continuously compounded interest.

Equation 6.3. Continuously Compounded Interest

If an initial principal P is invested at an annual rate r and the interest is compounded continuously, the amount A in the account after t years is

$$A(t) = Pe^{rt}$$

If we take the scenario of Example 6.5.1 and compare monthly compounding to continuous compounding over 35 years, we find that monthly compounding yields $A(35) = 2000(1.0059375)^{12(35)}$ which is about $\$24{,}035.28$ whereas continuously compounding gives $A(35) = 2000e^{0.07125(35)}$which is about $\$24{,}213.18$- a difference of less than 1%.

Equations 6.2 and 6.3 both use exponential functions to describe the growth of an investment. Curiously enough, the same principles which govern compound interest are also used to model short term growth of populations. In Biology, **The Law of Uninhibited Growth** states as its premise that the *instantaneous* rate at which a population increases at any time is directly proportional to the population at that time.[9] In other words, the more organisms there are at a given moment, the faster they reproduce. Formulating the law as stated results in a differential equation, which requires Calculus to solve. Its solution is stated below.

Equation 6.4. Uninhibited Growth

If a population increases according to The Law of Uninhibited Growth, the number of organisms N at time t is given by the formula

$$N(t) = N_0 e^{kt},$$

where $N(0) = N_0$ (read 'N nought') is the initial number of organisms and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } N(t) \text{ at time } t) = k\,N(t)$$

It is worth taking some time to compare Equations 6.3 and 6.4. In Equation 6.3, we use P to denote the initial investment; in Equation 6.4, we use N_0 to denote the initial population. In Equation 6.3, r denotes the annual interest rate, and so it shouldn't be too surprising that the k in Equation 6.4 corresponds to a growth rate as well. While Equations 6.3 and 6.4 look entirely different, they both represent the same mathematical concept.

Example 6.5.2

In order to perform arthrosclerosis research, epithelial cells are harvested from discarded umbilical tissue and grown in the laboratory. A technician observes that a culture of twelve thousand cells grows to five million cells in one week. Assuming that the cells follow The Law of Uninhibited Growth, find a formula for the number of cells, N, in thousands, after t days.

Solution

We begin with $N(t) = N_0 e^{kt}$. Since N is to give the number of cells *in thousands*, we have $N_0 = 12$, so $N(t) = 12e^{kt}$. In order to complete the formula, we need to determine the growth rate k. We know that after one week, the number of cells has grown to five million. Since t measures days and the units of N are in thousands, this translates mathematically to $N(7) = 5000$. We get the equation $12e^{7k} = 5000$ which gives $k = \frac{1}{7}\ln\left(\frac{1250}{3}\right)$. Hence, $N(t) = 12e^{\frac{t}{7}\ln\left(\frac{1250}{3}\right)}$. Of course, in practice, we would approximate k to some desired accuracy, say $k \approx 0.8618$, which we can interpret as an 86.18% daily growth rate for the cells.

Whereas Equations 6.3 and 6.4 model the growth of quantities, we can use equations like them to describe the decline of quantities. One example we've seen already is Example 6.1.1 in Section 6.1. There, the value of a car declined from its purchase price of $\$25{,}000$ to nothing at all. Another real world phenomenon which follows suit is radioactive decay. There are elements which are unstable and emit energy spontaneously. In doing so, the amount of the element itself diminishes. The assumption behind this model is that the rate of decay of an element at a particular time is directly proportional to the amount of the element present at that time. In other words, the more of the element there is, the faster the element decays. This is precisely the same kind of hypothesis which drives The Law of Uninhibited Growth, and as such, the equation governing radioactive decay is hauntingly similar to Equation 6.4 with the exception that the rate constant k is negative.

Equation 6.5. Radioactive Decay

The amount of a radioactive element A at time t is given by the formula

$$A(t) = A_0 e^{kt},$$

where $A(0) = A_0$ is the initial amount of the element and $k < 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } A(t) \text{ at time } t) = k\,A(t)$$

Example 6.5.3

Iodine-131 is a commonly used radioactive isotope used to help detect how well the thyroid is functioning. Suppose the decay of Iodine-131 follows the model given in Equation 6.5, and that the half-life[10] of Iodine-131 is approximately 8 days. If 5 grams of Iodine-131 is present initially, find a function which gives the amount of Iodine-131, A, in grams, t days later.

Solution

Since we start with 5 grams initially, Equation 6.5 gives $A(t) = 5e^{kt}$. Since the half-life is 8 days, it takes 8 days for half of the Iodine-131 to decay, leaving half of it behind. Hence, $A(8) = 2.5$ which means $5e^{8k} = 2.5$. Solving, we get $k = \frac{1}{8}\ln\left(\frac{1}{2}\right) = -\frac{\ln(2)}{8} \approx -0.08664$, which we can interpret as a loss of material at a rate of 8.664% daily. Hence, $A(t) = 5e^{-\frac{t\ln(2)}{8}} \approx 5e^{-0.08664t}$.

We now turn our attention to some more mathematically sophisticated models. One such model is Newton's Law of Cooling, which we first encountered in Example 6.1.2 of Section 6.1. In that example we had a cup of coffee cooling from 160°F to room temperature 70°F according to the formula $T(t) = 70 + 90e^{-0.1t}$, where t was measured in minutes. In this situation, we know the physical limit of the temperature of the coffee is room temperature,[11] and the differential equation which gives rise to our formula for $T(t)$ takes this into account. Whereas the radioactive decay model had a rate of decay at time t directly proportional to the amount of the element which remained at time t, Newton's Law of Cooling states that the rate of cooling of the coffee at a given time t is directly proportional to how much of a temperature exists between the coffee at time t and room temperature, not the temperature of the coffee itself. In other words, the coffee cools faster when it is first served, and as its temperature nears room temperature, the coffee cools ever more slowly. Of course, if we take an item from the refrigerator and let it sit out in the kitchen, the object's temperature will rise to room temperature, and since the physics behind warming and cooling is the same, we combine both cases in the equation below.

Equation 6.6. Newton's Law of Cooling (Warming)

The temperature T of an object at time t is given by the formula

$$T(t) = T_a + (T_0 - T_a)\,e^{-kt},$$

where $T(0) = T_0$ is the initial temperature of the object, T_a is the ambient temperature[a] and $k > 0$ is the constant of proportionality which satisfies the equation

$$(\text{instantaneous rate of change of } T(t) \text{ at time } t) = k\,(T(t) - T_a)$$

[a] That is, the temperature of the surroundings.

If we re-examine the situation in Example 6.1.2 with $T_0 = 160$, $T_a = 70$, and $k = 0.1$, we get, according to Equation 6.6, $T(t) = 70 + (160 - 70)e^{-0.1t}$ which reduces to the original formula given. The rate constant $k = 0.1$ indicates the coffee is cooling at a rate equal to 10% of the difference between the temperature of the coffee and its surroundings. Note in Equation 6.6 that the constant k is positive for both the cooling and warming scenarios. What determines if the function $T(t)$ is increasing or decreasing is if T_0 (the initial temperature of the object) is greater than T_a (the ambient temperature) or vice-versa, as we see in our next example.

Example 6.5.4.

A 40°F roast is cooked in a 350°F oven. After 2 hours, the temperature of the roast is 125°F.

1. Assuming the temperature of the roast follows Newton's Law of Warming, find a formula for the temperature of the roast T as a function of its time in the oven, t, in hours.
2. The roast is done when the internal temperature reaches 165°F. When will the roast be done?

Solution.

1. The initial temperature of the roast is 40°F, so $T_0 = 40$. The environment in which we are placing the roast is the 350°F oven, so $T_a = 350$. Newton's Law of Warming tells us $T(t) = 350 + (40 - 350)e^{-kt}$, or $T(t) = 350 - 310e^{-kt}$. To determine k, we use the fact that after 2 hours, the roast is 125°F, which means $T(2) = 125$. This gives rise to the equation $350 - 310e^{-2k} = 125$ which yields $k = -\frac{1}{2}\ln\left(\frac{45}{62}\right) \approx 0.1602$. The temperature function is
$$T(t) = 350 - 310e^{\frac{t}{2}\ln\left(\frac{45}{62}\right)} \approx 350 - 310e^{-0.1602t}.$$
2. To determine when the roast is done, we set $T(t) = 165$. This gives $350 - 310e^{-0.1602t} = 165$ whose solution is $t = -\frac{1}{0.1602}\ln\left(\frac{37}{62}\right) \approx 3.22$. It takes roughly 3 hours and 15 minutes to cook the roast completely.

If we had taken the time to graph $y = T(t)$ in Example 6.5.4, we would have found the horizontal asymptote to be $y = 350$, which corresponds to the temperature of the oven. We can also arrive at this conclusion by applying a bit of 'number sense'. As t $\to \infty$, $-0.1602t \approx$ very big $(-)$ so that $e^{-0.1602t} \approx$ very small $(+)$. The larger the value of t, the smaller $e^{-0.1602t}$ becomes so that $T(t) \approx 350 -$ very small $(+)$, which indicates the graph of $y = T(t)$ is approaching its horizontal asymptote $y = 350$ from below. Physically, this means the roast will eventually warm up to 350°F.[12] The function T is sometimes called a **limited** growth model, since the function T remains bounded as $t \to \infty$. If we apply the principles behind Newton's Law of Cooling to a biological example, it says the growth rate of a population is directly proportional to how much room the population has to grow. In other words, the more room for expansion, the faster the growth rate. The **logistic** growth model combines The Law of Uninhibited Growth with limited growth and states that the rate of growth of a population varies jointly with the population itself as well as the room the population has to grow.

Equation 6.7. Logistic Growth

If a population behaves according to the assumptions of logistic growth, the number of organisms N at time t is given by the equation
$$N(t) = \frac{L}{1 + Ce^{-kLt}},$$
where $N(0) = N_0$ is the initial population, L is the limiting population,[a] C is a measure of how much room there is to grow given by
$$C = \frac{L}{N_0} - 1.$$
and $k > 0$ is the constant of proportionality which satisfies the equation
$$(\text{instantaneous rate of change of } N(t) \text{ at time } t) = k\,N(t)\,(L - N(t))$$

[a] That is, a $t \to \infty$, $N(t) \to L$

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The logistic function is used not only to model the growth of organisms, but is also often used to model the spread of disease and rumors.[13]

Example 6.5.5

The number of people N, in hundreds, at a local community college who have heard the rumor 'Carl is afraid of Virginia Woolf' can be modeled using the logistic equation

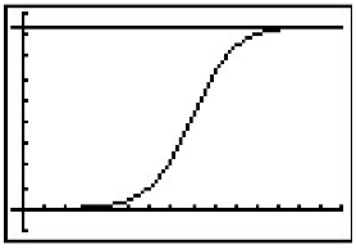
$$N(t) = \frac{84}{1 + 2799e^{-t}},$$

where $t \geq 0$ is the number of days after April 1, 2009.

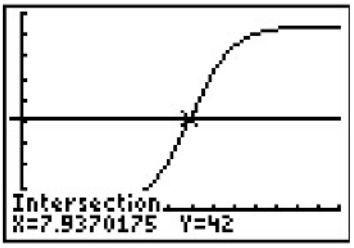
1. Find and interpret $N(0)$.
2. Find and interpret the end behavior of $N(t)$.
3. How long until 4200 people have heard the rumor?
4. Check your answers to 2 and 3 using your calculator.

Solution.

1. We find $N(0) = \frac{84}{1+2799e^{0}} = \frac{84}{2800} = \frac{3}{100}$. Since $N(t)$ measures the number of people who have heard the rumor in hundreds, $N(0)$ corresponds to 3 people. Since $t = 0$ corresponds to April 1, 2009, we may conclude that on that day, 3 people have heard the rumor.[14]
2. We could simply note that $N(t)$ is written in the form of Equation 6.7, and identify $L = 84$. However, to see why the answer is 84, we proceed analytically. Since the domain of N is restricted to $t \geq 0$, the only end behavior of significance is $t \to \infty$. As we've seen before,[15] as $t \to \infty$, we have $1997e^{-t} \to 0^{+}$ and so $N(t) \approx \frac{84}{1+\text{ very small }(+)} \approx 84$. Hence, as $t \to \infty$, $N(t) \to 84$. This means that as time goes by, the number of people who will have heard the rumor approaches 8400.
3. To find how long it takes until 4200 people have heard the rumor, we set $N(t) = 42$. Solving $\frac{84}{1+2799e^{-t}} = 42$ gives $t = \ln(2799) \approx 7.937$. It takes around 8 days until 4200 people have heard the rumor.
4. We graph $y = N(x)$ using the calculator and see that the line $y = 84$ is the horizontal asymptote of the graph, confirming our answer to part 2, and the graph intersects the line $y = 42$ at $x = \ln(2799) \approx 7.937$, which confirms our answer to part 3.

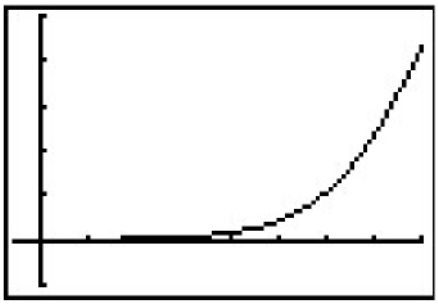


$y = f(x) = \frac{84}{1+2799e^{-x}}$ and $y = 84$

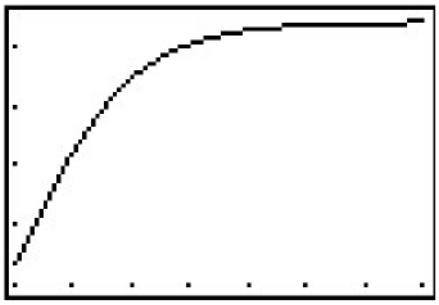


$y = f(x) = \frac{84}{1+2799e^{-x}}$ and $y = 42$

If we take the time to analyze the graph of $y = N(x)$ above, we can see graphically how logistic growth combines features of uninhibited and limited growth. The curve seems to rise steeply, then at some point, begins to level off. The point at which this happens is called an **inflection point** or is sometimes called the 'point of diminishing returns'. At this point, even though the function is still increasing, the rate at which it does so begins to decline. It turns out the point of diminishing returns always occurs at half the limiting population. (In our case, when $y = 42$.) While these concepts are more precisely quantified using Calculus, below are two views of the graph of $y = N(x)$, one on the interval $[0, 8]$, the other on $[8, 15]$. The former looks strikingly like uninhibited growth; the latter like limited growth.



$y = f(x) = \frac{84}{1+2799e^{-x}}$ for $0 \le x \le 8$



$y = f(x) = \frac{84}{1+2799e^{-x}}$ for $8 \le x \le 16$

6.5.2 Applications of Logarithms

Just as many physical phenomena can be modeled by exponential functions, the same is true of logarithmic functions. In Exercises 75, 76 and 77 of Section 6.1, we showed that logarithms are useful in measuring the intensities of earthquakes (the Richter scale), sound (decibels) and acids and bases (pH). We now present yet a different use of the a basic logarithm function, password strength.

Example 6.5.6

The information entropy H, in bits, of a randomly generated password consisting of L characters is given by $H = L\log_2(N)$, where N is the number of possible symbols for each character in the password. In general, the higher the entropy, the stronger the password.

1. If a 7 character case-sensitive[16] password is comprised of letters and numbers only, find the associated information entropy.
2. How many possible symbol options per character is required to produce a 7 character password with an information entropy of 50 bits?

Solution.

1. There are 26 letters in the alphabet, 52 if upper and lower case letters are counted as different. There are 10 digits (0 through 9) for a total of $N = 62$ symbols. Since the password is to be 7 characters long, $L = 7$. Thus, $H = 7\log_2(62) = \frac{7\ln(62)}{\ln(2)} \approx 41.68$.
2. We have $L = 7$ and $H = 50$ and we need to find N. Solving the equation $50 = 7\log_2(N)$ gives $N = 2^{50/7} \approx 141.323$, so we would need 142 different symbols to choose from.[17]

Chemical systems known as buffer solutions have the ability to adjust to small changes in acidity to maintain a range of pH values. Buffer solutions have a wide variety of applications from maintaining a healthy fish tank to regulating the pH levels in blood. Our next example shows how the pH in a buffer solution is a little more complicated than the pH we first encountered in Exercise 77 in Section 6.1.

Example 6.5.7

Blood is a buffer solution. When carbon dioxide is absorbed into the bloodstream it produces carbonic acid and lowers the pH. The body compensates by producing bicarbonate, a weak base to partially neutralize the acid. The equation[18] which models blood pH in this situation is $\text{pH} = 6.1 + \log\left(\frac{800}{x}\right)$, where x is the partial pressure of carbon dioxide in arterial blood, measured in torr. Find the partial pressure of carbon dioxide in arterial blood if the pH is 7.4.

Solution

We set $\text{pH} = 7.4$ and get $7.4 = 6.1 + \log\left(\frac{800}{x}\right)$, or $\log\left(\frac{800}{x}\right) = 1.3$. Solving, we find $x = \frac{800}{10^{1.3}} \approx 40.09$. Hence, the partial pressure of carbon dioxide in the blood is about 40 torr.

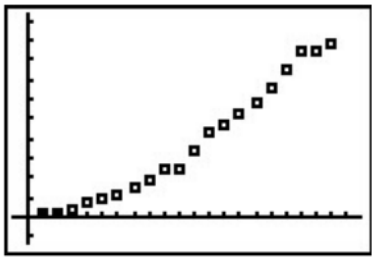
Another place logarithms are used is in data analysis. Suppose, for instance, we wish to model the spread of influenza A (H1N1), the so-called 'Swine Flu'. Below is data taken from the World Health Organization (WHO) where t represents the number of days

since April 28, 2009, and N represents the number of confirmed cases of H1N1 virus worldwide.

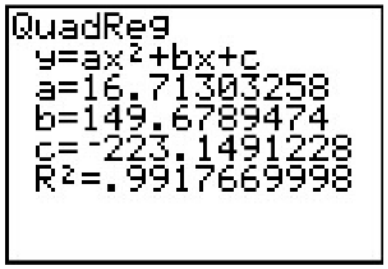
t	1	2	3	4	5	6	7	8	9	10	11	12	13
N	148	257	367	658	898	1085	1490	1893	2371	2500	3440	4379	4694

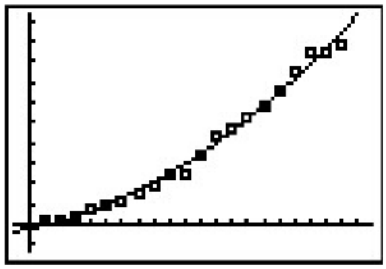
t	14	15	16	17	18	19	20
N	5251	5728	6497	7520	8451	8480	8829

Making a scatter plot of the data treating t as the independent variable and N as the dependent variable gives



Which models are suggested by the shape of the data? Thinking back Section 2.5, we try a Quadratic Regression, with pretty good results.



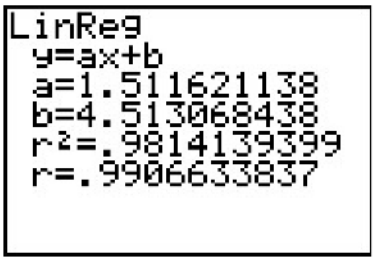


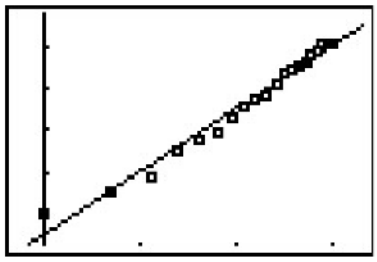
However, is there any scientific reason for the data to be quadratic? Are there other models which fit the data equally well, or better? Scientists often use logarithms in an attempt to 'linearize' data sets - in other words, transform the data sets to produce ones which result in straight lines. To see how this could work, suppose we guessed the relationship between N and t was some kind of power function, not necessarily quadratic, say $N = Bt^A$. To try to determine the A and B, we can take the natural log of both sides and get $\ln(N) = \ln(Bt^A)$. Using properties of logs to expand the right hand side of this equation, we get $\ln(N) = A\ln(t) + \ln(B)$. If we set $X = \ln(t)$ and $Y = \ln(N)$, this equation becomes $Y = AX + \ln(B)$. In other words, we have a line with slope A and Y-intercept $\ln(B)$. So, instead of plotting N versus t, we plot $\ln(N)$ versus $\ln(t)$.

$\ln(t)$	0	0.693	1.099	1.386	1.609	1.792	1.946	2.079	2.197	2.302	2.398	2.485	2.565
$\ln(N)$	4.997	5.549	5.905	6.489	6.800	6.989	7.306	7.546	7.771	7.824	8.143	8.385	8.454

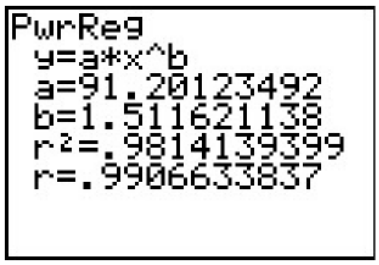
$\ln(t)$	2.639	2.708	2.773	2.833	2.890	2.944	2.996
$\ln(N)$	8.566	8.653	8.779	8.925	9.042	9.045	9.086

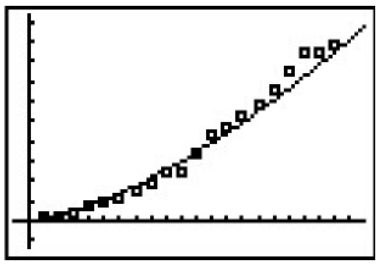
Running a linear regression on the data gives



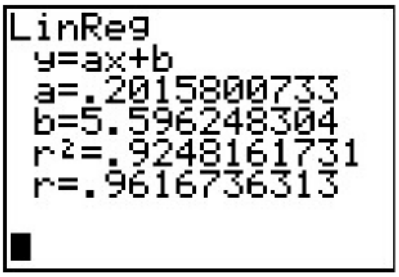


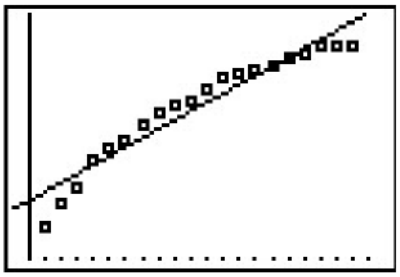
The slope of the regression line is $a \approx 1.512$ which corresponds to our exponent A. The y-intercept $b \approx 4.513$ corresponds to $\ln(B)$, so that $B \approx 91.201$. Hence, we get the model $N = 91.201t^{1.512}$, something from Section 5.3. Of course, the calculator has a built-in 'Power Regression' feature. If we apply this to our original data set, we get the same model we arrived at before.[19]



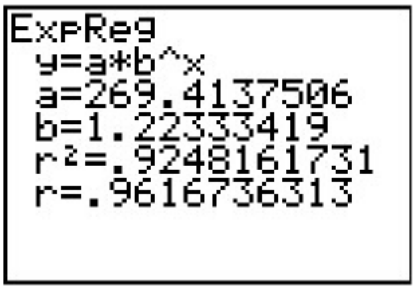


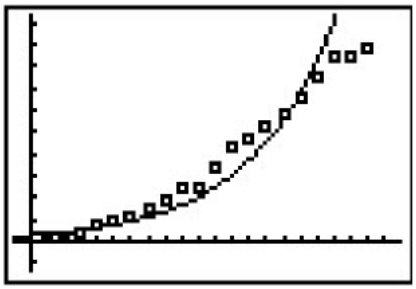
This is all well and good, but the quadratic model appears to fit the data better, and we've yet to mention any scientific principle which would lead us to believe the actual spread of the flu follows any kind of power function at all. If we are to attack this data from a scientific perspective, it does seem to make sense that, at least in the early stages of the outbreak, the more people who have the flu, the faster it will spread, which leads us to proposing an uninhibited growth model. If we assume $N = Be^{At}$ then, taking logs as before, we get $\ln(N) = At + \ln(B)$. If we set $X = t$ and $Y = \ln(N)$, then, once again, we get $Y = AX + \ln(B)$, a line with slope A and Y-intercept $\ln(B)$. Plotting $\ln(N)$ versus t gives the following linear regression.



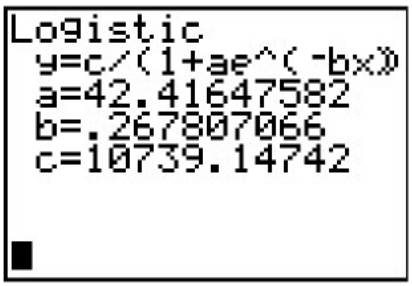


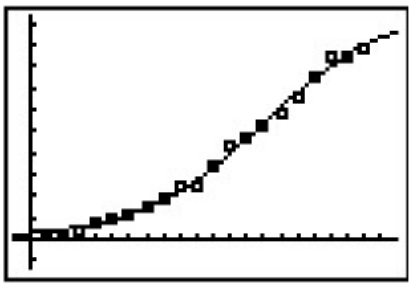
We see the slope is $a \approx 0.202$ and which corresponds to A in our model, and the y-intercept is $b \approx 5.596$ which corresponds to $\ln(B)$. We get $B \approx 269.414$, so that our model is $N = 269.414e^{0.202t}$. Of course, the calculator has a built-in 'Exponential Regression' feature which produces what appears to be a different model $N = 269.414(1.22333419)^t$. Using properties of exponents, we write $e^{0.202t} = \left(e^{0.202}\right)^t \approx (1.223848)^t$, which, had we carried more decimal places, would have matched the base of the calculator model exactly.





The exponential model didn't fit the data as well as the quadratic or power function model, but it stands to reason that, perhaps, the spread of the flu is not unlike that of the spread of a rumor and that a logistic model can be used to model the data. The calculator does have a 'Logistic Regression' feature, and using it produces the model $N = \frac{10739.147}{1+42.416e^{0.268t}}$.





This appears to be an excellent fit, but there is no friendly coefficient of determination, R^2, by which to judge this numerically. There are good reasons for this, but they are far beyond the scope of the text. Which of the models, quadratic, power, exponential, or logistic is the 'best model'? If by 'best' we mean 'fits closest to the data,' then the quadratic and logistic models are arguably the winners with the power function model a close second. However, if we think about the science behind the spread of the flu, the logistic model gets an edge. For one thing, it takes into account that only a finite number of people will ever get the flu (according to our model, $10{,}739$), whereas the quadratic model predicts no limit to the number of cases. As we have stated several times before in the text, mathematical models, regardless of their sophistication, are just that: models, and they all have their limitations.[20]

6.5.3. Exercises

For each of the scenarios given in Exercises 1 - 6,

- Find the amount A in the account as a function of the term of the investment t in years.
- Determine how much is in the account after 5 years, 10 years, 30 years and 35 years. Round your answers to the nearest cent.
- Determine how long will it take for the initial investment to double. Round your answer to the nearest year.
- Find and interpret the average rate of change of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year. Round your answer to two decimal places.

1. $\$500$ is invested in an account which offers 0.75%, compounded monthly.
2. $\$500$ is invested in an account which offers 0.75%, compounded continuously.
3. $\$1000$ is invested in an account which offers 1.25%, compounded monthly.
4. $\$1000$ is invested in an account which offers 1.25%, compounded continuously.
5. $\$5000$ is invested in an account which offers 2.125%, compounded monthly.
6. $\$5000$ is invested in an account which offers 2.125%, compounded continuously.
7. Look back at your answers to Exercises 1 - 6. What can be said about the difference between monthly compounding and continuously compounding the interest in those situations? With the help of your classmates, discuss scenarios where the difference between monthly and continuously compounded interest would be more dramatic. Try varying the interest rate, the term of the investment and the principal. Use computations to support your answer.
8. How much money needs to be invested now to obtain $\$2000$ in 3 years if the interest rate in a savings account is 0.25%, compounded continuously? Round your answer to the nearest cent.
9. How much money needs to be invested now to obtain $\$5000$ in 10 years if the interest rate in a CD is 2.25%, compounded monthly? Round your answer to the nearest cent.
10. On May, 31, 2009, the Annual Percentage Rate listed at Jeff's bank for regular savings accounts was 0.25% compounded monthly. Use Equation 6.2 to answer the following.
    a. If $P = 2000$ what is $A(8)$?
    b. Solve the equation $A(t) = 4000$ for t.
    c. What principal P should be invested so that the account balance is $2000 is three years?
11. Jeff's bank also offers a 36-month Certificate of Deposit (CD) with an APR of 2.25%.
    a. If $P = 2000$ what is $A(8)$?
    b. Solve the equation $A(t) = 4000$ for t.
    c. What principal P should be invested so that the account balance is $2000 in three years?
    d. The Annual Percentage Yield is the interest rate that returns the same amount of interest after one year as the compound interest does. With the help of your classmates, compute the APY for this investment.

12. A finance company offers a promotion on $\$5000$ loans. The borrower does not have to make any payments for the first three years, however interest will continue to be charged to the loan at 29.9% compounded continuously. What amount will be due at the end of the three year period, assuming no payments are made? If the promotion is extended an additional three years, and no payments are made, what amount would be due?
13. Use Equation 6.2 to show that the time it takes for an investment to double in value does depend on the principal P, but rather, depends only on the APR and the number of compoundings per year. Let $n = 12$ and with the help of your classmates compute the doubling time for a variety of rates r. Then look up the Rule of 72 and compare your answers to what that rule says. If you're really interested[21] in Financial Mathematics, you could also compare and contrast the Rule of 72 with the Rule of 70 and the Rule of 69.

In Exercises 14 - 18, we list some radioactive isotopes and their associated half-lives. Assume that each decays according to the formula $A(t) = A_0 e^{kt}$ where A_0 is the initial amount of the material and k is the decay constant. For each isotope:

- Find the decay constant k. Round your answer to four decimal places.
- Find a function which gives the amount of isotope A which remains after time t. (Keep the units of A and t the same as the given data.)
- Determine how long it takes for 90% of the material to decay. Round your answer to two decimal places. (HINT: If 90% of the material decays, how much is left?)

14. Cobalt 60, used in food irradiation, initial amount 50 grams, half-life of 5.27 years.
15. Phosphorus 32, used in agriculture, initial amount 2 milligrams, half-life 14 days.
16. Chromium 51, used to track red blood cells, initial amount 75 milligrams, half-life 27.7 days.
17. Americium 241, used in smoke detectors, initial amount 0.29 micrograms, half-life 432.7 years.
18. Uranium 235, used for nuclear power, initial amount 1 kg grams, half-life 704 million years.
19. With the help of your classmates, show that the time it takes for 90% of each isotope listed in Exercises 14 - 18 to decay does not depend on the initial amount of the substance, but rather, on only the decay constant k. Find a formula, in terms of k only, to determine how long it takes for 90% of a radioactive isotope to decay.
20. In Example 6.1.1 in Section 6.1, the exponential function $V(x) = 25\left(\frac{4}{5}\right)^x$ was used to model the value of a car over time. Use the properties of logs and/or exponents to rewrite the model in the form $V(t) = 25e^{kt}$.
21. The Gross Domestic Product (GDP) of the US (in billions of dollars) t years after the year 2000 can be modeled by:

$$G(t) = 9743.77e^{0.0514t}$$

    a. Find and interpret $G(0)$.
    b. According to the model, what should have been the GDP in 2007? In 2010? (According to the US Department of Commerce, the 2007 GDP was $\$14,369.1$ billion and the 2010 GDP was $\$14,657.8$ billion.)

22. The diameter D of a tumor, in millimeters, t days after it is detected is given by:

$$D(t) = 15e^{0.0277t}$$

    a. What was the diameter of the tumor when it was originally detected?
    b. How long until the diameter of the tumor doubles?

23. Under optimal conditions, the growth of a certain strain of *E. Coli* is modeled by the Law of Uninhibited Growth $N(t) = N_0 e^{kt}$ where N_0 is the initial number of bacteria and t is the elapsed time, measured in minutes. From numerous experiments, it has been determined that the doubling time of this organism is 20 minutes. Suppose 1000 bacteria are present initially.
    a. Find the growth constant k. Round your answer to four decimal places.
    b. Find a function which gives the number of bacteria $N(t)$ after t minutes.
    c. How long until there are 9000 bacteria? Round your answer to the nearest minute.
24. Yeast is often used in biological experiments. A research technician estimates that a sample of yeast suspension contains 2.5 million organisms per cubic centimeter (cc). Two hours later, she estimates the population density to be 6 million organisms per cc. Let t be the time elapsed since the first observation, measured in hours. Assume that the yeast growth follows the Law of Uninhibited Growth $N(t) = N_0 e^{kt}$.
    a. Find the growth constant k. Round your answer to four decimal places.
    b. Find a function which gives the number of yeast (in millions) per cc $N(t)$ after t hours.

c. What is the doubling time for this strain of yeast?

25. The Law of Uninhibited Growth also applies to situations where an animal is re-introduced into a suitable environment. Such a case is the reintroduction of wolves to Yellowstone National Park. According to the National Park Service, the wolf population in Yellowstone National Park was 52 in 1996 and 118 in 1999. Using these data, find a function of the form $N(t) = N_0 e^{kt}$ which models the number of wolves t years after 1996. (Use $t = 0$ to represent the year 1996. Also, round your value of k to four decimal places.) According to the model, how many wolves were in Yellowstone in 2002? (The recorded number is 272.)
26. During the early years of a community, it is not uncommon for the population to grow according to the Law of Uninhibited Growth. According to the Painesville Wikipedia entry, in 1860, the Village of Painesville had a population of 2649. In 1920, the population was 7272. Use these two data points to fit a model of the form $N(t) = N_0 e^{kt}$ were $N(t)$ is the number of Painesville Residents t years after 1860. (Use $t = 0$ to represent the year 1860. Also, round the value of k to four decimal places.) According to this model, what was the population of Painesville in 2010? (The 2010 census gave the population as 19,563) What could be some causes for such a vast discrepancy? For more on this, see Exercise 37.
27. The population of Sasquatch in Bigfoot county is modeled by

$$P(t) = \frac{120}{1 + 3.167e^{-0.05t}}$$

where $P(t)$ is the population of Sasquatch t years after 2010.

a. Find and interpret $P(0)$.
b. Find the population of Sasquatch in Bigfoot county in 2013. Round your answer to the nearest Sasquatch.
c. When will the population of Sasquatch in Bigfoot county reach 60? Round your answer to the nearest year.
d. Find and interpret the end behavior of the graph of $y = P(t)$. Check your answer using a graphing utility.

28. The half-life of the radioactive isotope Carbon-14 is about 5730 years.

a. Use Equation 6.5 to express the amount of Carbon-14 left from an initial N milligrams as a function of time t in years.
b. What percentage of the original amount of Carbon-14 is left after 20,000 years?
c. If an old wooden tool is found in a cave and the amount of Carbon-14 present in it is estimated to be only 42% of the original amount, approximately how old is the tool?
d. Radiocarbon dating is not as easy as these exercises might lead you to believe. With the help of your classmates, research radiocarbon dating and discuss why our model is somewhat over-simplified.

29. Carbon-14 cannot be used to date inorganic material such as rocks, but there are many other methods of radiometric dating which estimate the age of rocks. One of them, Rubidium-Strontium dating, uses Rubidium-87 which decays to Strontium-87 with a half-life of 50 billion years. Use Equation 6.5 to express the amount of Rubidium-87 left from an initial 2.3 micrograms as a function of time t in *billions* of years. Research this and other radiometric techniques and discuss the margins of error for various methods with your classmates.
30. Use Equation 6.5 to show that $k = -\dfrac{\ln(2)}{h}$ where h is the half-life of the radioactive isotope.
31. A pork roast[22] was taken out of a hardwood smoker when its internal temperature had reached 180°F and it was allowed to rest in a 75°F house for 20 minutes after which its internal temperature had dropped to 170°F. Assuming that the temperature of the roast follows Newton's Law of Cooling (Equation 6.6),

a. Express the temperature T (in $^\circ$F) as a function of time t (in minutes).
b. Find the time at which the roast would have dropped to 140°F had it not been carved and eaten.

32. In reference to Exercise 44 in Section 5.3, if Fritzy the Fox's speed is the same as Chewbacca the Bunny's speed, Fritzy's pursuit curve is given by

$$y(x) = \frac{1}{4}x^2 - \frac{1}{4}\ln(x) - \frac{1}{4}$$

Use your calculator to graph this path for $x > 0$. Describe the behavior of y as $x \to 0^+$ and interpret this physically.

33. The current i measured in amps in a certain electronic circuit with a constant impressed voltage of 120 volts is given by $i(t) = 2 - 2e^{-10t}$ where $t \geq 0$ is the number of seconds after the circuit is switched on. Determine the value of i as $t \to \infty$. (This is called the **steady state** current.)
34. If the voltage in the circuit in Exercise 33 above is switched off after 30 seconds, the current is given by the piecewise-defined function

$$i(t) = \begin{cases} 2 - 2e^{-10t} & \text{if} \quad 0 \le t < 30 \\ [6pt] \left(2 - 2e^{-300}\right) e^{-10t+300} & \text{if} \quad t \ge 30 \end{cases}$$

With the help of your calculator, graph $y = i(t)$ and discuss with your classmates the physical significance of the two parts of the graph $0 \le t < 30$ and $t \ge 30$.

35. In Exercise 6a in Section 2.5, we stated that the cable of a suspension bridge formed a parabola but that a free hanging cable did not. A free hanging cable forms a and its basic shape is given by $y = \frac{1}{2}(e^x + e^{-x})$. Use your calculator to graph this function. What are its domain and range? What is its end behavior? Is it invertible? How do you think it is related to the function given in Exercise 47 in Section 6.3 and the one given in the answer to Exercise 38 in Section 6.4? When flipped upside down, the catenary makes an arch. The Gateway Arch in St. Louis, Missouri has the shape

$$y = 757.7 - \frac{127.7}{2}\left(e^{\frac{x}{127.7}} + e^{-\frac{x}{127.7}}\right)$$

where x and y are measured in feet and $-315 \le x \le 315$. Find the highest point on the arch.

36. In Exercise 6a in Section 2.5, we examined the data set given below which showed how two cats and their surviving offspring can produce over 80 million cats in just ten years. It is virtually impossible to see this data plotted on your calculator, so plot x versus $\ln(x)$ as was done on page 480. Find a linear model for this new data and comment on its goodness of fit. Find an exponential model for the original data and comment on its goodness of fit.

Year x	1	2	3	4	5	6	7	8		9
Number of Cats $N(x)$	12	66	382	2201	12680	73041	420715	2423316	13968290	80399780

37. This exercise is a follow-up to Exercise 26 which more thoroughly explores the population growth of Painesville, Ohio. According to Wikipedia, the population of Painesville, Ohio is given by

Year t	1860	1870	1880	1890	1900	1910	1920	1930	1940	1950
Population	2649	3728	3841	4755	5024	5501	7272	10944	12235	14432

Year t	1960	1970	1980	1990	2000
Population	16116	16536	16351	15699	17503

a. Use a graphing utility to perform an exponential regression on the data from 1860 through 1920 only, letting $t = 0$ represent the year 1860 as before. How does this calculator model compare with the model you found in Exercise 26? Use the calculator's exponential model to predict the population in 2010. (The 2010 census gave the population as 19,563)

b. The logistic model fit to *all* of the given data points for the population of Painesville t years after 1860 (again, using $t = 0$ as 1860) is

$$P(t) = \frac{18691}{1 + 9.8505e^{-0.03617t}}$$

According to this model, what should the population of Painesville have been in 2010? (The 2010 census gave the population as 19,563.) What is the population limit of Painesville?

38. According to OhioBiz , the census data for Lake County, Ohio is as follows:

Year t	1860	1870	1880	1890	1900	1910	1920	1930	1940	1950
Population	15576	15935	16326	18235	21680	22927	28667	41674	50020	75979

Year t	1960	1970	1980	1990	2000
Population	148700	197200	212801	215499	227511

a. Use your calculator to fit a logistic model to these data, using $x = 0$ to represent the year 1860.

b. Graph these data and your logistic function on your calculator to judge the reasonableness of the fit.

c. Use this model to estimate the population of Lake County in 2010. (The 2010 census gave the population to be 230,041.)

d. According to your model, what is the population limit of Lake County, Ohio?

39. According to facebook , the number of active users of facebook has grown significantly since its initial launch from a Harvard dorm room in February 2004. The chart below has the approximate number $U(x)$ of active users, in , x months after February 2004. For example, the first entry $(10, 1)$ means that there were 1 million active users in December 2004 and the last entry $(77, 500)$ means that there were 500 million active users in July 2010.

    With the help of your classmates, find a model for this data.

40. Each Monday during the registration period before the Fall Semester at LCCC, the Enrollment Planning Council gets a report prepared by the data analysts in Institutional Effectiveness and Planning.[23] While the ongoing enrollment data is analyzed in many different ways, we shall focus only on the overall headcount. Below is a chart of the enrollment data for Fall Semester 2008. It starts 21 weeks before "Opening Day" and ends on "Day 15" of the semester, but we have relabeled the top row to be $x = 1$ through $x = 24$ so that the math is easier. (Thus, $x = 22$ is Opening Day.)

Week x	1	2	3	4	5	6	7	8
Total Headcount	1194	1564	2001	2475	2802	3141	3527	3790

Week x	9	10	11	12	13	14	15	16
Total Headcount	4065	4371	4611	4945	5300	5657	6056	6478

Week x	17	18	19	20	21	22	23	24
Total Headcount	7161	7772	8505	9256	10201	10743	11102	11181

    With the help of your classmates, find a model for this data. Unlike most of the phenomena we have studied in this section, there is no single differential equation which governs the enrollment growth. Thus there is no scientific reason to rely on a logistic function even though the data plot may lead us to that model. What are some factors which influence enrollment at a community college and how can you take those into account mathematically?

41. When we wrote this exercise, the Enrollment Planning Report for Fall Semester 2009 had only 10 data points for the first 10 weeks of the registration period. Those numbers are given below.

Week x	17	18	19	20	21	22	23	24
Total Headcount	7161	7772	8505	9256	10201	10743	11102	11181

    With the help of your classmates, find a model for this data and make a prediction for the Opening Day enrollment as well as the Day 15 enrollment. (WARNING: The registration period for 2009 was one week shorter than it was in 2008 so Opening Day would be $x = 21$ and Day 15 is $x = 23$.)

6.5.4. Answers

1. - $A(t) = 500\left(1 + \frac{0.0075}{12}\right)^{12t}$
   - $A(5) \approx \$519.10$, $A(10) \approx \$538.93$, $A(30) \approx \$626.12$, $A(35) \approx \$650.03$
   - It will take approximately 92 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 3.88. This means that the investment is growing at an average rate of $\$3.88$ per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 4.85. This means that the investment is growing at an average rate of $\$4.85$ per year at this point.
2. - $A(t) = 500e^{0.0075t}$
   - $A(5) \approx \$519.11$, $A(10) \approx \$538.94$, $A(30) \approx \$626.16$, $A(35) \approx \$650.09$
   - It will take approximately 92 years for the investment to double.
   - The average rate of change from the end of the fourth year to the end of the fifth year is approximately 3.88. This means that the investment is growing at an average rate of $\$3.88$ per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 4.86. This means that the investment is growing at an average rate of $\$4.86$ per year at this point.

3. ○ $A(t) = 1000\left(1 + \frac{0.0125}{12}\right)^{12t}$
   ○ $A(5) \approx \$1064.46$, $A(10) \approx \$1133.07$, $A(30) \approx \$1454.71$, $A(35) \approx \$1548.48$
   ○ It will take approximately 55 years for the investment to double.
   ○ The average rate of change from the end of the fourth year to the end of the fifth year is approximately 13.22. This means that the investment is growing at an average rate of $\$13.22$per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 19.23. This means that the investment is growing at an average rate of $\$19.23$per year at this point.
4. ○ $A(t) = 1000e^{0.0125t}$
   ○ $A(5) \approx \$1064.49$, $A(10) \approx \$1133.15$, $A(30) \approx \$1454.99$, $A(35) \approx \$1548.83$
   ○ It will take approximately 55 years for the investment to double.
   ○ The average rate of change from the end of the fourth year to the end of the fifth year is approximately 13.22. This means that the investment is growing at an average rate of $\$13.22$per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 19.24. This means that the investment is growing at an average rate of $\$19.24$per year at this point.
5. ○ $A(t) = 5000\left(1 + \frac{0.02125}{12}\right)^{12t}$
   ○ $A(5) \approx \$5559.98$, $A(10) \approx \$6182.67$, $A(30) \approx \$9453.40$, $A(35) \approx \$10512.13$
   ○ It will take approximately 33 years for the investment to double.
   ○ The average rate of change from the end of the fourth year to the end of the fifth year is approximately 116.80. This means that the investment is growing at an average rate of $\$116.80$per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 220.83. This means that the investment is growing at an average rate of $\$220.83$per year at this point.
6. ○ $A(t) = 5000e^{0.02125t}$
   ○ $A(5) \approx \$5560.50$, $A(10) \approx \$6183.83$, $A(30) \approx \$9458.73$, $A(35) \approx \$10519.05$
   ○ It will take approximately 33 years for the investment to double.
   ○ The average rate of change from the end of the fourth year to the end of the fifth year is approximately 116.91. This means that the investment is growing at an average rate of $\$116.91$per year at this point. The average rate of change from the end of the thirty-fourth year to the end of the thirty-fifth year is approximately 221.17. This means that the investment is growing at an average rate of $\$221.17$per year at this point.

8. $P = \frac{2000}{e^{0.0025 \cdot 3}} \approx \$1985.06$
9. $P = \frac{5000}{\left(1 + \frac{0.0225}{12}\right)^{12 \cdot 10}} \approx \$3993.42$
10. a. $A(8) = 2000\left(1 + \frac{0.0025}{12}\right)^{12 \cdot 8} \approx \$2040.40$
    b. $t = \frac{\ln(2)}{12\ln\left(1 + \frac{0.0025}{12}\right)} \approx 277.29$ years
    c. $P = \frac{2000}{\left(1 + \frac{0.0025}{12}\right)^{36}} \approx \$1985.06$
11. a. $A(8) = 2000\left(1 + \frac{0.0225}{12}\right)^{12 \cdot 8} \approx \$2394.03$
    b. $t = \frac{\ln(2)}{12\ln\left(1 + \frac{0.0225}{12}\right)} \approx 30.83$ years
    c. $P = \frac{2000}{\left(1 + \frac{0.0225}{12}\right)^{36}} \approx \$1869.57$
    d. $\left(1 + \frac{0.0225}{12}\right)^{12} \approx 1.0227$ so the APY is 2.27%
12. $A(3) = 5000e^{0.299 \cdot 3} \approx \$12,226.18$, $A(6) = 5000e^{0.299 \cdot 6} \approx \$30,067.29$
14. ○ $k = \frac{\ln(1/2)}{5.27} \approx -0.1315$
    ○ $A(t) = 50e^{-0.1315t}$
    ○ $t = \frac{\ln(0.1)}{-0.1315} \approx 17.51$ years.
15. ○ $k = \frac{\ln(1/2)}{14} \approx -0.0495$

- $A(t) = 2e^{-0.0495t}$
- $t = \frac{\ln(0.1)}{-0.0495} \approx 46.52$ days.

16. - $k = \frac{\ln(1/2)}{27.7} \approx -0.0250$
    - $A(t) = 75e^{-0.0250t}$
    - $t = \frac{\ln(0.1)}{-0.025} \approx 92.10$ days.
17. - $k = \frac{\ln(1/2)}{432.7} \approx -0.0016$
    - $A(t) = 0.29e^{-0.0016t}$
    - $t = \frac{\ln(0.1)}{-0.0016} \approx 1439.11$ years.
18. - $k = \frac{\ln(1/2)}{704} \approx -0.0010$
    - $A(t) = e^{-0.0010t}$
    - $t = \frac{\ln(0.1)}{-0.0010} \approx 2302.58$ million years, or 2.30 billion years.
19. $t = \frac{\ln(0.1)}{k} = -\frac{\ln(10)}{k}$
20. $V(t) = 25e^{\ln\left(\frac{4}{5}\right)t} \approx 25e^{-0.22314355t}$
21. a. $G(0) = 9743.77$ This means that the GDP of the US in 2000 was $\$9743.77$billion dollars.
    b. $G(7) = 13963.24$and $G(10) = 16291.25$, so the model predicted a GDP of $\$13,963.24$billion in 2007 and $\$16,291.25$ billion in 2010.
22. a. $D(0) = 15$, so the tumor was 15 millimeters in diameter when it was first detected.
    b. $t = \frac{\ln(2)}{0.0277} \approx 25$ days.
23. a. $k = \frac{\ln(2)}{20} \approx 0.0346$
    b. $N(t) = 1000e^{0.0346t}$
    c. $t = \frac{\ln(9)}{0.0346} \approx 63$ minutes
24. a. $k = \frac{1}{2}\frac{\ln(6)}{2.5} \approx 0.4377$
    b. $N(t) = 2.5e^{0.4377t}$
    c. $t = \frac{\ln(2)}{0.4377} \approx 1.58$ hours
25. $N_0 = 52, k = \frac{1}{3}\ln\left(\frac{118}{52}\right) \approx 0.2731, N(t) = 52e^{0.2731t}\cdot N(6) \approx 268$.
26. $N_0 = 2649, k = \frac{1}{60}\ln\left(\frac{7272}{2649}\right) \approx 0.0168, N(t) = 2649e^{0.0168t}\cdot N(150) \approx 32923$, so the population of Painesville in 2010 based on this model would have been 32,923.
27. a. $P(0) = \frac{120}{4.167} \approx 29$. There are 29 Sasquatch in Bigfoot County in 2010.
    b. $P(3) = \frac{120}{1+3.167e^{-0.05(3)}} \approx 32$ Sasquatch.
    c. $t = 20\ln(3.167) \approx 23$ years.
    d. As $t \to \infty$, $P(t) \to 120$. As time goes by, the Sasquatch Population in Bigfoot County will approach 120. Graphically, $y = P(x)$ has a horizontal asymptote $y = 120$.
28. a. $A(t) = Ne^{-\left(\frac{\ln(2)}{5730}\right)t} \approx Ne^{-0.00012097t}$
    b. $A(20000) \approx 0.088978 \cdot N$ so about 8.9% remains
    c. $t \approx \dfrac{\ln(.42)}{-0.00012097} \approx 7171$ years old
29. $A(t) = 2.3e^{-0.0138629t}$
31. a. $T(t) = 75 + 105e^{-0.005005t}$
    b. The roast would have cooled to 140°F in about 95 minutes.
32. From the graph, it appears that as $x \to 0^+$, $y \to \infty$. This is due to the presence of the $\ln(x)$ term in the function. This means that Fritzy will never catch Chewbacca, which makes sense since Chewbacca has a head start and Fritzy only runs as fast as he does.



$y(x) = \frac{1}{4}x^2 - \frac{1}{4}\ln(x) - \frac{1}{4}$

33. The steady state current is 2 amps.

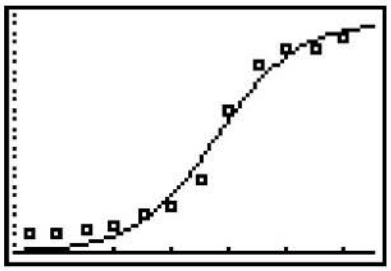
36. The linear regression on the data below is $y = 1.74899x + 0.70739$ with $r^2 \approx 0.999995$. This is an excellent fit.

x	1	2	3	4	5	6	7	8	9	10
$\ln(N(x))$	2.4849	4.1897	5.9454	7.6967	9.4478	11.1988	12.9497	14.7006	16.4523	18.2025

$N(x) = 2.02869(5.74879)^x = 2.02869e^{1.74899x}$ with $r^2 \approx 0.999995$. This is also an excellent fit and corresponds to our linearized model because $\ln(2.02869) \approx 0.70739$

37. a. The calculator gives: $y = 2895.06(1.0147)^x$. Graphing this along with our answer from Exercise 26 over the interval $[0, 60]$ shows that they are pretty close. From this model, $y(150) \approx 25840$ which once again overshoots the actual data value.
    b. $P(150) \approx 18717$, so this model predicts 17,914 people in Painesville in 2010, a more conservative number than was recorded in the 2010 census. As $t \to \infty$, $P(t) \to 18691$. So the limiting population of Painesville based on this model is 18,691 people.

38. a. $y = \dfrac{242526}{1 + 874.62e^{-0.07113x}}$, where x is the number of years since 1860.
    b. The plot of the data and the curve is below.



    c. $y(140) \approx 232889$, so this model predicts 232,889 people in Lake County in 2010.
    d. As $x \to \infty$, $y \to 242526$, so the limiting population of Lake County based on this model is 242,526 people.

Reference

[1] How generous of the

[2] Some restrictions may apply.

[3] Actually, the final balance should be $105.0625.

[4] Using this convention, simple interest after one year is the same as compounding the interest only once.

[5] See Definition 2.3 in section 2.1.

[6] In fact, the rate of increase of the amount in the account is exponential as well. This is the quality that really defines exponential functions and we refer the reader to a course in Calculus.

[7] Once you've had a semester of Calculus, you'll be able to fully appreciate this very lame pun.

[8] Or define, depending on your point of view.

[9] The average rate of change of a function over an interval was first introduced in Section 2.1. Instantaneous rates of change are the business of Calculus, as is mentioned on Page 161.

[10] The time it takes for half of the substance to decay.

[11] The Second Law of Thermodynamics states that heat can spontaneously flow from a hotter object to a colder one, but not the other way around. Thus, the coffee could not continue to release heat into the air so as to cool below room temperature.

[12] at which point it would be more toast than roast.

[13] Which can be just as damaging as diseases.

[14] Or, more likely, three people started the rumor. I'd wager Jeff, Jamie, and Jason started it. So much for telling your best friends something in confidence!

[15] See, for example, Example 6.1.2.

[16] That is, upper and lower case letters are treated as different characters.

[17] Since there are only 94 distinct ASCII keyboard characters, to achieve this strength, the number of characters in the password should be increase.

[18] Derived from the Henderson-Hasselbalch Equation. See Exercise 43 in Section 6.2. Hasselbalch himself was studying carbon dioxide dissolving in blood - a process called metabolic acidosis.

[19] Critics may question why the authors of the book have chosen to even discuss linearization of data when the calculator has a Power Regression built-in and ready to go. Our response: talk to your science faculty

[20] Speaking of limitations, as of June 3, 2009, there were 19,273 confirmed cases of influenza A (H1N1). This is well above our prediction of 10,739. Each time a new report is issued, the data set increases and the model must be recalculated. We leave this recalculation to the reader.

[21] Awesome pun!

[22] This roast was enjoyed by Jeff and his family on June 10, 2009. This is real data, folks!

[23] The authors thank Dr. Wendy Marley and her staff for this data and Dr. Marcia Ballinger for the permission to use it in this problem.

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CHAPTER OVERVIEW

6: Conic Sections

In this chapter, we study the **Conic Sections** - literally 'sections of a cone'.

6.1: Introduction to Conics
6.2: Circles
6.3: Ellipses
6.4: Parabolas
6.5: Hyperbolas

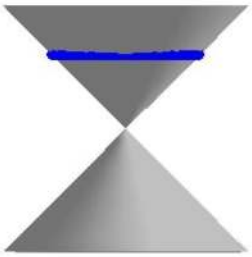
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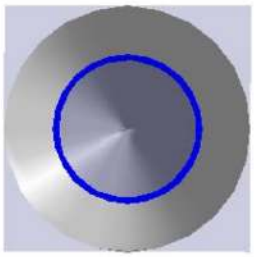
6.1: Introduction to Conics

In this chapter, we study the **Conic Sections** - literally ‘sections of a cone’. Imagine a double-napped cone as seen below being ‘sliced’ by a plane.

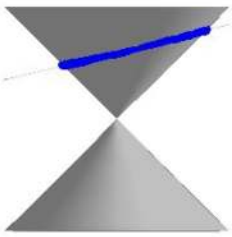


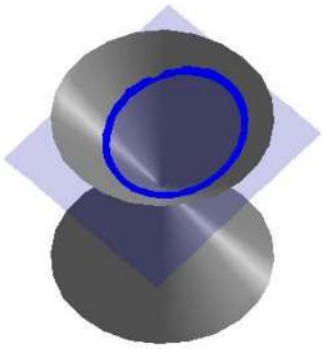
If we slice the cone with a horizontal plane the resulting curve is a **circle**.



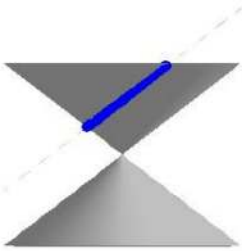


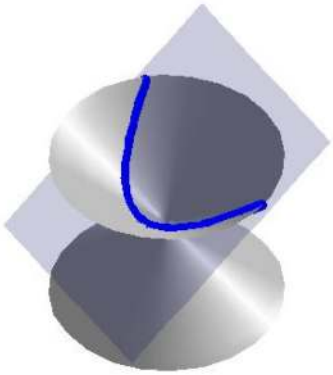
Tilting the plane ever so slightly produces an **ellipse**.



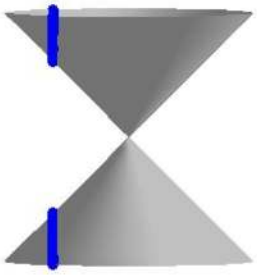


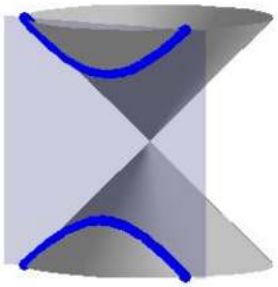
If the plane cuts parallel to the cone, we get a **parabola**.





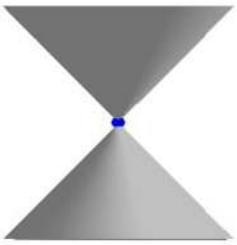
If we slice the cone with a vertical plane, we get a **hyperbola**.

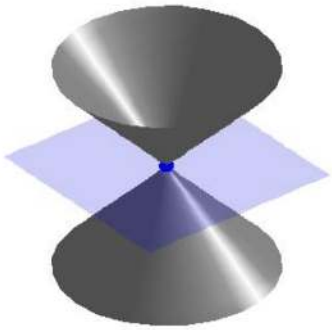


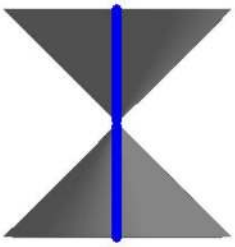


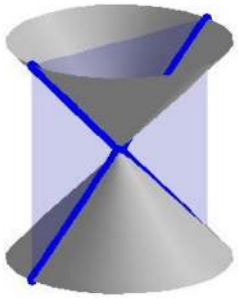
For a wonderful animation describing the conics as intersections of planes and cones, see Dr. Louis Talman's Mathematics Animated Website.

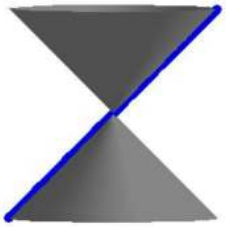
If the slicing plane contains the vertex of the cone, we get the so-called 'degenerate' conics: a point, a line, or two intersecting lines.

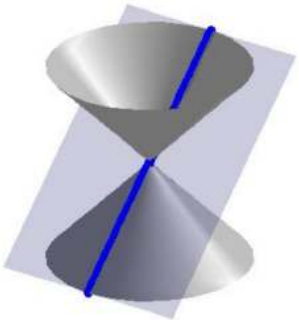












We will focus the discussion on the non-degenerate cases: circles, parabolas, ellipses, and hyperbolas, in that order. To determine equations which describe these curves, we will make use of their definitions in terms of distances.

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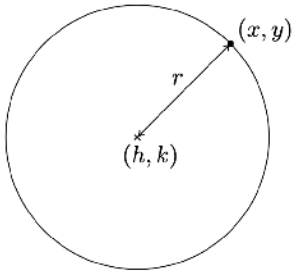
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6.2: Circles

Recall from Geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

Definition 7.1

A **circle** with center (h, k) and radius $r > 0$ is the set of all points (x, y) in the plane whose distance to (h, k) is r.



From the picture, we see that a point (x, y) is on the circle if and only if its distance to (h, k) is r. We express this relationship algebraically using the Distance Formula, Equation 1.1, as

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r > 0$) which gives us the standard equation of a circle.

Equation 7.1. The Standard Equation of a Circle

The Standard Equation of a Circle: The equation of a circle with center (h, k) and radius $r > 0$ is $(x-h)^2 + (y-k)^2 = r^2$.

Example 7.2.1

Write the standard equation of the circle with center $(-2, 3)$ and radius 5.

Solution

Here, $(h, k) = (-2, 3)$ and $r = 5$, so we get

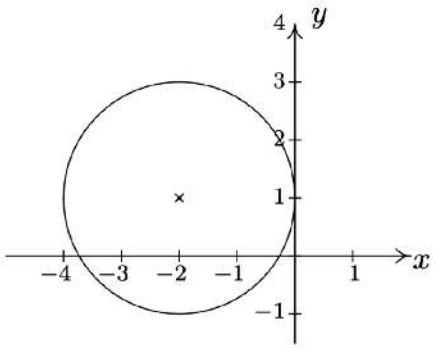
$$\begin{aligned} (x-(-2))^2 + (y-3)^2 &= (5)^2 \\ (x+2)^2 + (y-3)^2 &= 25 \end{aligned}$$

Example 7.2.2

Graph $(x+2)^2 + (y-1)^2 = 4$. Find the center and radius.

Solution

From the standard form of a circle, Equation 7.1, we have that $x+2$ is $x-h$, so $h = -2$ and $y-1$ is $y-k$ so $k = 1$. This tells us that our center is $(-2, 1)$. Furthermore, $r^2 = 4$, so $r = 2$. Thus we have a circle centered at $(-2, 1)$ with a radius of 2. Graphing gives us



If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2+(y-1)^2=4$, we'd be contending with $x^2+4x+y^2-2y+1=0$. If we're given such an equation, we can complete the square in each of the variables to see if it fits the form given in Equation 7.1 by following the steps given below.

To Write the Equation of a Circle in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square on both variables as needed.
3. Divide both sides by the coefficient of the squares. (For circles, they will be the same.)

Example 7.2.3

Complete the square to find the center and radius of $3x^2-6x+3y^2+4y-4=0$.

Solution.

$$\begin{aligned} 3x^2-6x+3y^2+4y-4 &= 0 && \\ 3x^2-6x+3y^2+4y &= 4 && \text{add 4 to both sides} \\ 3\left(x^2-2x\right)+3\left(y^2+\frac{4}{3}y\right) &= 4 && \text{factor out leading coefficients} \\ [10pt]3\left(x^2-2x+\underline{1}\right)+3\left(y^2+\frac{4}{3}y+\underline{\underline{\frac{4}{9}}}\right) &= 4+3\underline{(1)}+3\underline{\underline{\left(\frac{4}{9}\right)}} && \text{complete the square in } x, y \\ [10pt]3(x-1)^2+3\left(y+\frac{2}{3}\right)^2 &= \frac{25}{3} && \text{factor} \\ [10pt](x-1)^2+\left(y+\frac{2}{3}\right)^2 &= \frac{25}{9} && \text{divide both sides by 3} \end{aligned}$$

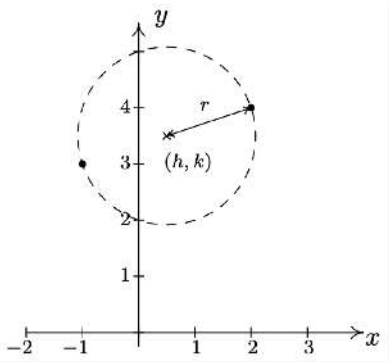
From Equation 7.1, we identify $x-1$ as $x-h$, so $h=1$, and $y+\frac{2}{3}$ as $y-k$, so $k=-\frac{2}{3}$. Hence, the center is $(h,k)=\left(1,-\frac{2}{3}\right)$. Furthermore, we see that $r^2=\frac{25}{9}$ so the radius is $r=\frac{5}{3}$.

It is possible to obtain equations like $(x-3)^2+(y+1)^2=0$ or $(x-3)^2+(y+1)^2=-1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations. The next example uses the Midpoint Formula, Equation 1.2, in conjunction with the ideas presented so far in this section.

Example 7.2.4

Write the standard equation of the circle which has $(-1,3)$ and $(2,4)$ as the endpoints of a diameter.

We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields



Since the given points are endpoints of a diameter, we know their midpoint (h,k) is the center of the circle. Equation 1.2 gives us

$$\begin{aligned}(h,k) &= \left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)\\ [8pt] &= \left(\frac{-1+2}{2},\frac{3+4}{2}\right)\\ [8pt] &= \left(\frac{1}{2},\frac{7}{2}\right)\end{aligned}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\ [8pt] &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ [8pt] &= \frac{1}{2}\sqrt{3^2+1^2}\\ [8pt] &= \frac{\sqrt{10}}{2}\end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2=\frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2=\frac{10}{4}$

We close this section with the most important[1] circle in all of mathematics: the **Unit Circle**.

Definition 7.2

The **Unit Circle** is the circle centered at $(0,0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2+y^2=1$.

Example 7.2.5

Find the points on the unit circle with y-coordinate $\frac{\sqrt{3}}{2}$.

Solution

We replace y with $\frac{\sqrt{3}}{2}$ in the equation $x^2+y^2=1$ to get

$$\begin{aligned} x^2+y^2 &= 1 \\ x^2+\left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\ [13pt]\frac{3}{4}+x^2 &= 1 \\ x^2 &= \frac{1}{4} \\ x &= \pm\sqrt{\frac{1}{4}} \\ [7pt]x &= \pm\frac{1}{2} \end{aligned}$$

Our final answers are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

7.2.1. Exercises

In Exercises 1 - 6, find the standard equation of the circle and then graph it.

1. Center $(-1, -5)$, radius 10
2. Center $(4, -2)$, radius 3
3. Center $\left(-3, \frac{7}{13}\right)$, radius $\frac{1}{2}$
4. Center $(5, -9)$, radius $\ln(8)$
5. Center $\left(-e, \sqrt{2}\right)$, radius π
6. Center (π, e^2), radius $\sqrt[3]{91}$

In Exercises 7 - 12, complete the square in order to put the equation into standard form. Identify the center and the radius or explain why the equation does not represent a circle.

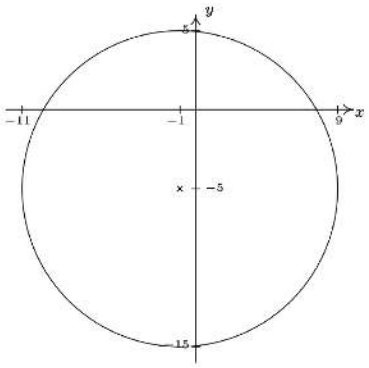
7. $x^2-4x+y^2+10y=-25$
8. $-2x^2-36x-2y^2-112=0$
9. $x^2+y^2+8x-10y-1=0$
10. $x^2+y^2+5x-y-1=0$
11. $4x^2+4y^2-24y+36=0$
12. $x^2+x+y^2-\frac{6}{5}y=1$

In Exercises 13 - 16, find the standard equation of the circle which satisfies the given criteria.

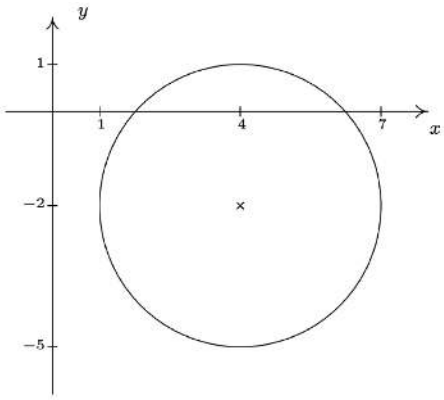
13. center $(3, 5)$, passes through $(-1, -2)$
14. center $(3, 6)$, passes through $(-1, 4)$
15. endpoints of a diameter: $(3, 6)$ and $(-1, 4)$
16. endpoints of a diameter: $\left(\frac{1}{2}, 4\right)$, $\left(\frac{3}{2}, -1\right)$
17. The Giant Wheel at Cedar Point is a circle with diameter 128 feet which sits on an 8 foot tall platform making its overall height is 136 feet.[2] Find an equation for the wheel assuming that its center lies on the y-axis and that the ground is the x-axis.
18. Verify that the following points lie on the Unit Circle: $(\pm 1, 0)$, $(0, \pm 1)$, $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}\right)$, $\left(\pm\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ and $\left(\pm\frac{\sqrt{3}}{2}, \pm\frac{1}{2}\right)$
19. Discuss with your classmates how to obtain the standard equation of a circle, Equation 7.1, from the equation of the Unit Circle, $x^2+y^2=1$ using the transformations discussed in Section 1.7. (Thus every circle is just a few transformations away from the Unit Circle.)
20. Find an equation for the function represented graphically by the top half of the Unit Circle. Explain how the transformations is Section 1.7 can be used to produce a function whose graph is either the top or bottom of an arbitrary circle.
21. Find a one-to-one function whose graph is half of a circle. (Hint: Think piecewise.)

7.2.2. Answers

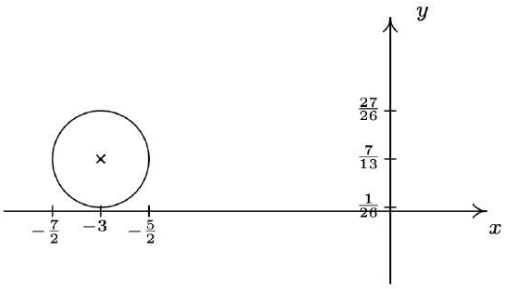
1. $(x+1)^2+(y+5)^2=100$



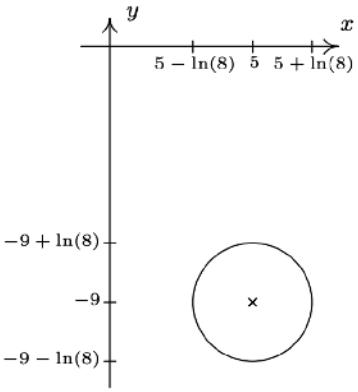
2. $(x-4)^2+(y+2)^2=9$



3. $(x+3)^2+\left(y-\frac{7}{13}\right)^2=\frac{1}{4}$

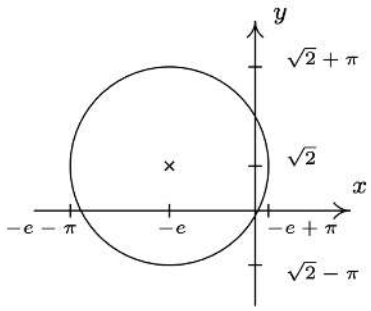


4. $(x-5)^2+(y+9)^2=(\ln(8))^2$

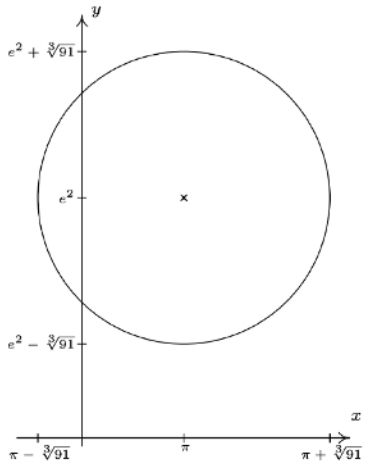


5. $(x+e)^2+\left(y-\sqrt{2}\right)^2=\pi^2$

https://math.libretexts.org/@go/page/127045



6. $(x-\pi)^2+\left(y-e^2\right)^2=91^{\frac{2}{3}}$



7. $(x-2)^2+(y+5)^2=4$
   Center $(2,-5)$, radius $r=2$
8. $(x+9)^2+y^2=25$
   Center $(-9,0)$, radius $r=5$
9. $(x+4)^2+(y-5)^2=42$
   Center $(-4,5)$, radius $r=\sqrt{42}$
10. $\left(x+\frac{5}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{30}{4}$
    Center $\left(-\frac{5}{2},\frac{1}{2}\right)$, radius $r=\frac{\sqrt{30}}{2}$
11. $x^2+(y-3)^2=0$
    This is not a circle.
12. $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{3}{5}\right)^2=\frac{161}{100}$
    Center $\left(-\frac{1}{2},\frac{3}{5}\right)$, radius $r=\frac{\sqrt{161}}{10}$
13. $(x-3)^2+(y-5)^2=65$
14. $(x-3)^2+(y-6)^2=20$
15. $(x-1)^2+(y-5)^2=5$
16. $(x-1)^2+\left(y-\frac{3}{2}\right)^2=\frac{13}{2}$
17. $x^2+(y-72)^2=4096$

Reference

[1] While this may seem like an opinion, it is indeed a fact. See Chapters 10 and 11 for details.

[2] Source: Cedar Point's webpage.

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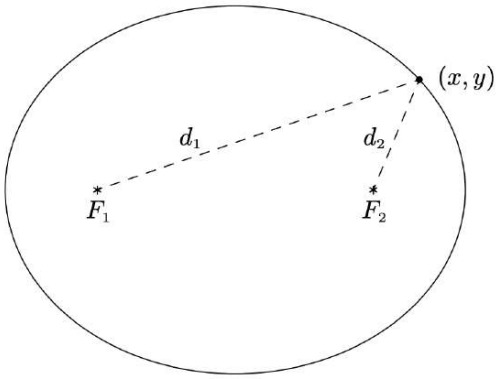
6.3: Ellipses

In the definition of a circle, Definition 7.1, we fixed a point called the **center** and considered all of the points which were a fixed distance r from that one point. For our next conic section, the ellipse, we fix two distinct points and a distance d to use in our definition.

Definition 7.4

Given two distinct points F_1 and F_2 in the plane and a fixed distance d, an **ellipse** is the set of all points (x, y) in the plane such that the sum of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci**[a] of the ellipse.

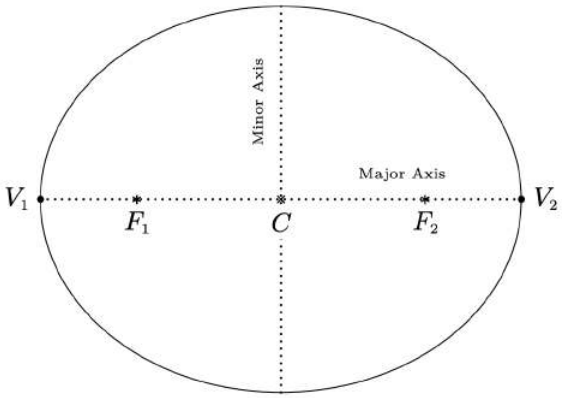
[a] the plural of 'focus'



$d_1 + d_2 = d$ for all (x, y) on the ellipse

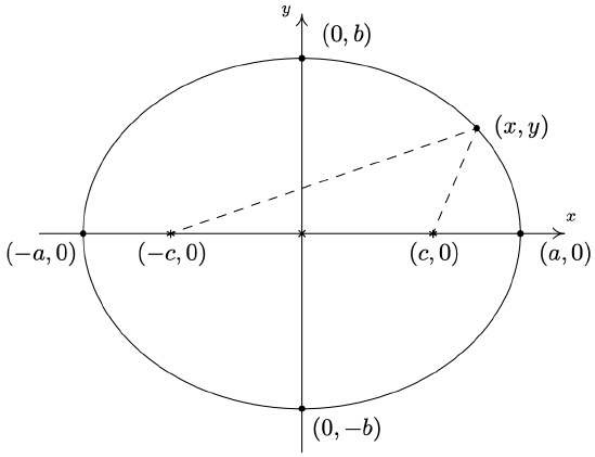
We may imagine taking a length of string and anchoring it to two points on a piece of paper. The curve traced out by taking a pencil and moving it so the string is always taut is an ellipse.

The **center** of the ellipse is the midpoint of the line segment connecting the two foci. The **major axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which also contains the center and foci. The **minor axis** of the ellipse is the line segment connecting two opposite ends of the ellipse which contains the center but is perpendicular to the major axis. The **vertices** of an ellipse are the points of the ellipse which lie on the major axis. Notice that the center is also the midpoint of the major axis, hence it is the midpoint of the vertices. In pictures we have,



An ellipse with center C; foci F_1, F_2; and vertices V_1, V_2

Note that the major axis is the longer of the two axes through the center, and likewise, the minor axis is the shorter of the two. In order to derive the standard equation of an ellipse, we assume that the ellipse has its center at $(0, 0)$, its major axis along the x-axis, and has foci $(c, 0)$ and $(-c, 0)$ and vertices $(-a, 0)$ and $(a, 0)$. We will label the y-intercepts of the ellipse as $(0, b)$ and $(0, -b)$ (We assume a, b, and c are all positive numbers.) Schematically,



Note that since $(a,0)$ is on the ellipse, it must satisfy the conditions of Definition 7.4. That is, the distance from $(-c,0)$ to $(a,0)$ plus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all of these points lie on the x-axis, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) + \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a+c)+(a-c) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance d mentioned in the definition of the ellipse is none other than the length of the major axis. We now use that fact $(0,b)$ is on the ellipse, along with the fact that $d=2a$ to get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (0,b) + \text{distance from } (c,0) \text{ to } (0,b) &= 2a \\ \sqrt{(0-(-c))^2+(b-0)^2}+\sqrt{(0-c)^2+(b-0)^2} &= 2a \\ \sqrt{b^2+c^2}+\sqrt{b^2+c^2} &= 2a \\ 2\sqrt{b^2+c^2} &= 2a \\ \sqrt{b^2+c^2} &= a \end{aligned}$$

From this, we get $a^2=b^2+c^2$, or $b^2=a^2-c^2$, which will prove useful later. Now consider a point (x,y) on the ellipse. Applying Definition 7.4, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (x,y) + \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\ \sqrt{(x-(-c))^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2} &= 2a \\ \sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} &= 2a \end{aligned}$$

In order to make sense of this situation, we need to make good use of Intermediate Algebra.

$$\begin{aligned} \sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2} &= 2a \\ \sqrt{(x+c)^2+y^2} &= 2a-\sqrt{(x-c)^2+y^2} \\ \left(\sqrt{(x+c)^2+y^2}\right)^2 &= \left(2a-\sqrt{(x-c)^2+y^2}\right)^2 \\ (x+c)^2+y^2 &= 4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2 \\ 4a\sqrt{(x-c)^2+y^2} &= 4a^2+(x-c)^2-(x+c)^2 \\ 4a\sqrt{(x-c)^2+y^2} &= 4a^2-4cx \\ a\sqrt{(x-c)^2+y^2} &= a^2-cx \\ \left(a\sqrt{(x-c)^2+y^2}\right)^2 &= \left(a^2-cx\right)^2 \\ a^2\left((x-c)^2+y^2\right) &= a^4-2a^2cx+c^2x^2 \\ a^2x^2-2a^2cx+a^2c^2+a^2y^2 &= a^4-2a^2cx+c^2x^2 \\ a^2x^2-c^2x^2+a^2y^2 &= a^4-a^2c^2 \\ \left(a^2-c^2\right)x^2+a^2y^2 &= a^2\left(a^2-c^2\right) \end{aligned}$$

We are nearly finished. Recall that $b^2 = a^2 - c^2$ so that

$$\begin{aligned} \left(a^2 - c^2\right) x^2 + a^2 y^2 &= a^2\left(a^2 - c^2\right) \\ b^2 x^2 + a^2 y^2 &= a^2 b^2 \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \end{aligned}$$

This equation is for an ellipse centered at the origin. To get the formula for the ellipse centered at (h, k), we could use the transformations from Section 1.7 or re-derive the equation using Definition 7.4 and the distance formula to obtain the formula below.

Equation 7.4. The Standard Equation of an Ellipse

The Standard Equation of an Ellipse: For positive unequal numbers a and b, the equation of an ellipse with center (h, k) is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Some remarks about Equation 7.4 are in order. First note that the values a and b determine how far in the x and y directions, respectively, one counts from the center to arrive at points on the ellipse. Also take note that if $a > b$, then we have an ellipse whose major axis is horizontal, and hence, the foci lie to the left and right of the center. In this case, as we've seen in the derivation, the distance from the center to the focus, c, can be found by $c = \sqrt{a^2 - b^2}$. If $b > a$, the roles of the major and minor axes are reversed, and the foci lie above and below the center. In this case, $c = \sqrt{b^2 - a^2}$. In either case, c is the distance from the center to each focus, and $c = \sqrt{\text{bigger denominator} - \text{smaller denominator}}$. Finally, it is worth mentioning that if we take the standard equation of a circle, Equation 7.1, and divide both sides by r^2, we get

Equation 7.5. The Alternate Standard Equation of a Circle

The Alternate Standard Equation of a Circle: The equation of a circle with center (h, k) and radius $r > 0$ is

$$\frac{(x-h)^2}{r^2} + \frac{(y-k)^2}{r^2} = 1$$

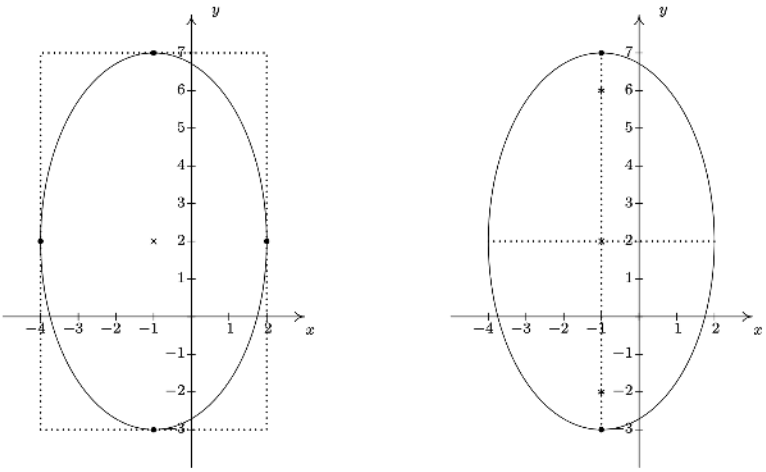
Notice the similarity between Equation 7.4 and Equation 7.5. Both equations involve a sum of squares equal to 1; the difference is that with a circle, the denominators are the same, and with an ellipse, they are different. If we take a transformational approach, we can consider both Equations 7.4 and 7.5 as shifts and stretches of the Unit Circle $x^2 + y^2 = 1$ in Definition 7.2. Replacing x with $(x - h)$ and y with $(y - k)$ causes the usual horizontal and vertical shifts. Replacing x with $\frac{x}{a}$ and y with $\frac{y}{b}$ causes the usual vertical and horizontal stretches. In other words, it is perfectly fine to think of an ellipse as the deformation of a circle in which the circle is stretched farther in one direction than the other.[1]

Example 7.4.1

Graph $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{25} = 1$. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.

Solution

We see that this equation is in the standard form of Equation 7.4. Here $x - h$ is $x + 1$ so $h = -1$, and $y - k$ is $y - 2$ so $k = 2$. Hence, our ellipse is centered at $(-1, 2)$. We see that $a^2 = 9$ so $a = 3$, and $b^2 = 25$ so $b = 5$. This means that we move 3 units left and right from the center and 5 units up and down from the center to arrive at points on the ellipse. As an aid to sketching, we draw a rectangle matching this description, called a **guide rectangle**, and sketch the ellipse inside this rectangle as seen below on the left.



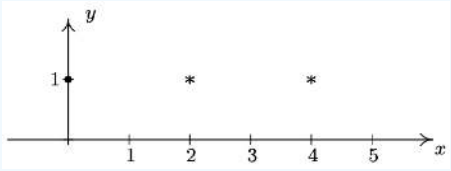
Since we moved farther in the y direction than in the x direction, the major axis will lie along the vertical line $x=-1$, which means the minor axis lies along the horizontal line, $y=2$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1,7)$ and $(-1,-3)$, and the endpoints of the minor axis are $(-4,2)$ and $(2,2)$. (Notice these points are the four points we used to draw the guide rectangle.) To find the foci, we find $c=\sqrt{25-9}=\sqrt{16}=4$, which means the foci lie 4 units from the center. Since the major axis is vertical, the foci lie 4 units above and below the center, at $(-1,-2)$ and $(-1,6)$. Plotting all this information gives the graph seen above on the right.

Example 7.4.2

Find the equation of the ellipse with foci $(2,1)$ and $(4,1)$ and vertex $(0,1)$.

Solution

Plotting the data given to us, we have



From this sketch, we know that the major axis is horizontal, meaning $a>b$. Since the center is the midpoint of the foci, we know it is $(3,1)$. Since one vertex is $(0,1)$ we have that $a=3$, so $a^2=9$. All that remains is to find b^2. Since the foci are 1 unit away from the center, we know $c=1$. Since $a>b$, we have $c=\sqrt{a^2-b^2}$, or $1=\sqrt{9-b^2}$, so $b^2=8$. Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, we get our final answer to be $\frac{(x-3)^2}{9}+\frac{(y-1)^2}{8}=1$.

As with circles and parabolas, an equation may be given which is an ellipse, but isn't in the standard form of Equation 7.4. In those cases, as with circles and parabolas before, we will need to massage the given equation into the standard form.

To Write the Equation of an Ellipse in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side.
2. Complete the square in both variables as needed.
3. Divide both sides by the constant term so that the constant on the other side of the equation becomes 1.

Example 7.4.3

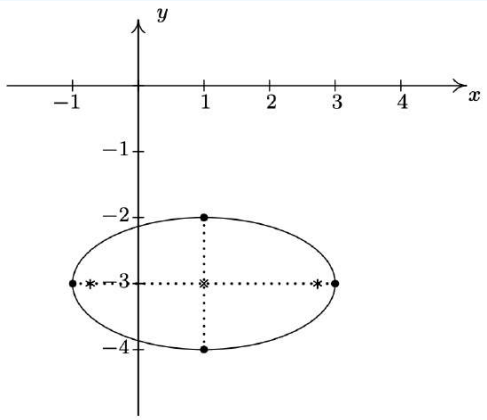
Graph $x^2+4y^2-2x+24y+33=0$. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, and the foci.

Solution

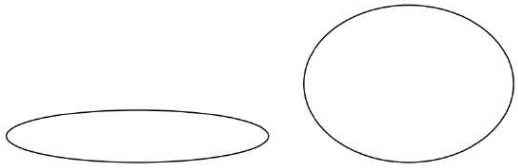
Since we have a sum of squares and the squared terms have unequal coefficients, it's a good bet we have an ellipse on our hands.[2] We need to complete both squares, and then divide, if necessary, to get the right-hand side equal to 1.

$$\begin{aligned} x^2+4y^2-2x+24y+33 &= 0 \\ x^2-2x+4y^2+24y &= -33 \\ x^2-2x+4\left(y^2+6y\right) &= -33 \\ \left(x^2-2x+1\right)+4\left(y^2+6y+9\right) &= -33+1+4(9) \\ (x-1)^2+4(y+3)^2 &= 4 \\ \frac{(x-1)^2+4(y+3)^2}{4} &= \frac{4}{4} \\ \frac{(x-1)^2}{4}+(y+3)^2 &= 1 \\ \frac{(x-1)^2}{4}+\frac{(y+3)^2}{1} &= 1 \end{aligned}$$

Now that this equation is in the standard form of Equation 7.4, we see that $x-h$ is $x-1$ so $h=1$, and $y-k$ is $y+3$ so $k=-3$. Hence, our ellipse is centered at $(1,-3)$. We see that $a^2=4$ so $a=2$, and $b^2=1$ so $b=1$. This means we move 2 units left and right from the center and 1 unit up and down from the center to arrive at points on the ellipse. Since we moved farther in the x direction than in the y direction, the major axis will lie along the horizontal line $y=-3$, which means the minor axis lies along the vertical line $x=1$. The vertices are the points on the ellipse which lie along the major axis so in this case, they are the points $(-1,-3)$ and $(3,-3)$, and the endpoints of the minor axis are $(1,-2)$ and $(1,-4)$. To find the foci, we find $c=\sqrt{4-1}=\sqrt{3}$, which means the foci lie $\sqrt{3}$ units from the center. Since the major axis is horizontal, the foci lie $\sqrt{3}$ units to the left and right of the center, at $(1-\sqrt{3},-3)$ and $(1+\sqrt{3},-3)$. Plotting all of this information gives



As you come across ellipses in the homework exercises and in the wild, you'll notice they come in all shapes in sizes. Compare the two ellipses below.



Certainly, one ellipse is more round than the other. This notion of 'roundness' is quantified below.

Definition 7.5

The **eccentricity** of an ellipse, denoted e, is the following ratio:

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

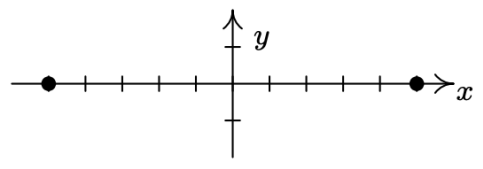
In an ellipse, the foci are closer to the center than the vertices, so $0 < e < 1$. The ellipse above on the left has eccentricity $e \approx 0.98$; for the ellipse above on the right, $e \approx 0.66$. In general, the closer the eccentricity is to 0, the more 'circular' the ellipse; the closer the eccentricity is to 1, the more 'eccentric' the ellipse.

Example 7.4.4

Find the equation of the ellipse whose vertices are $(\pm 5, 0)$ with eccentricity $e = \frac{1}{4}$.

Solution

As before, we plot the data given to us

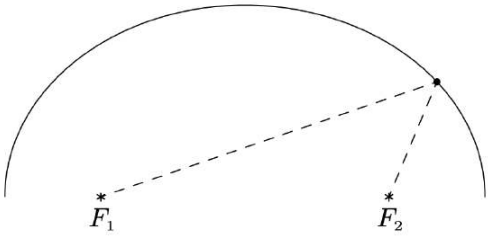


From this sketch, we know that the major axis is horizontal, meaning $a > b$. With the vertices located at $(\pm 5, 0)$, we get $a = 5$ so $a^2 = 25$. We also know that the center is $(0, 0)$ because the center is the midpoint of the vertices. All that remains is to find b^2. To that end, we use the fact that the eccentricity $e = \frac{1}{4}$ which means

$$e = \frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}} = \frac{c}{a} = \frac{c}{5} = \frac{1}{4}$$

from which we get $c = \frac{5}{4}$. To get b^2, we use the fact that $c = \sqrt{a^2 - b^2}$, so $\frac{5}{4} = \sqrt{25 - b^2}$ from which we get $b^2 = \frac{375}{16}$. Substituting all of our findings into the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, yields our final answer $\frac{x^2}{25} + \frac{16y^2}{375} = 1$.

As with parabolas, ellipses have a reflective property. If we imagine the dashed lines below representing sound waves, then the waves emanating from one focus reflect off the top of the ellipse and head towards the other focus.



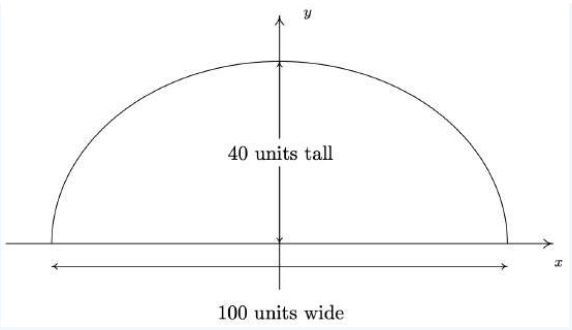
Such geometry is exploited in the construction of so-called 'Whispering Galleries'. If a person whispers at one focus, a person standing at the other focus will hear the first person as if they were standing right next to them. We explore the Whispering Galleries in our last example.

Example 7.4.5

Jamie and Jason want to exchange secrets (terrible secrets) from across a crowded whispering gallery. Recall that a whispering gallery is a room which, in cross section, is half of an ellipse. If the room is 40 feet high at the center and 100 feet wide at the floor, how far from the outer wall should each of them stand so that they will be positioned at the foci of the ellipse?

Solution

Graphing the data yields



It's most convenient to imagine this ellipse centered at $(0,0)$. Since the ellipse is 100 units wide and 40 units tall, we get $a=50$ and $b=40$. Hence, our ellipse has the equation $\frac{x^2}{50^2}+\frac{y^2}{40^2}=1$. We're looking for the foci, and we get $c=\sqrt{50^2-40^2}=\sqrt{900}=30$, so that the foci are 30 units from the center. That means they are $50-30=20$ units from the vertices. Hence, Jason and Jamie should stand 20 feet from opposite ends of the gallery.

7.4.1. Exercises

In Exercises 1 - 8, graph the ellipse. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.

1. $\frac{x^2}{169}+\frac{y^2}{25}=1$
2. $\frac{x^2}{9}+\frac{y^2}{25}=1$
3. $\frac{(x-2)^2}{4}+\frac{(y+3)^2}{9}=1$
4. $\frac{(x+5)^2}{16}+\frac{(y-4)^2}{1}=1$
5. $\frac{(x-1)^2}{10}+\frac{(y-3)^2}{11}=1$
6. $\frac{(x-1)^2}{9}+\frac{(y+3)^2}{4}=1$
7. $\frac{(x+2)^2}{16}+\frac{(y-5)^2}{20}=1$
8. $\frac{(x-4)^2}{8}+\frac{(y-2)^2}{18}=1$

In Exercises 9 - 14, put the equation in standard form. Find the center, the lines which contain the major and minor axes, the vertices, the endpoints of the minor axis, the foci and the eccentricity.

9. $9x^2+25y^2-54x-50y-119=0$
10. $12x^2+3y^2-30y+39=0$
11. $5x^2+18y^2-30x+72y+27=0$
12. $x^2-2x+2y^2-12y+3=0$
13. $9x^2+4y^2-4y-8=0$
14. $6x^2+5y^2-24x+20y+14=0$

In Exercises 15 - 20, find the standard form of the equation of the ellipse which has the given properties.

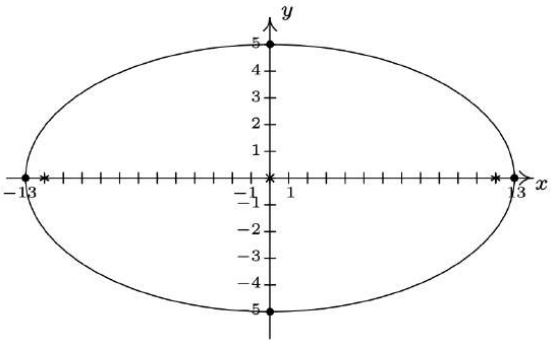
15. Center $(3,7)$, Vertex $(3,2)$, Focus $(3,3)$
16. Foci $(0,\pm5)$, Vertices $(0,\pm8)$.
17. Foci $(\pm3,0)$, length of the Minor Axis 10
18. Vertices $(3,2)$, $(13,2)$; Endpoints of the Minor Axis $(8,4)$, $(8,0)$
19. Center $(5,2)$, Vertex $(0,2)$, eccentricity $\frac{1}{2}$
20. All points on the ellipse are in Quadrant IV except $(0,-9)$ and $(8,0)$. (One might also say that the ellipse is "tangent to the axes" at those two points.)

21. Repeat Example 7.4.5 for a whispering gallery 200 feet wide and 75 feet tall.
22. An elliptical arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch. Compare your result with your answer to Exercise 21 in Section 7.3.
23. The Earth's orbit around the sun is an ellipse with the sun at one focus and eccentricity $e \approx 0.0167$. The length of the semimajor axis (that is, half of the major axis) is defined to be 1 astronomical unit (AU). The vertices of the elliptical orbit are given special names: 'aphelion' is the vertex farthest from the sun, and 'perihelion' is the vertex closest to the sun. Find the distance in AU between the sun and aphelion and the distance in AU between the sun and perihelion.
24. The graph of an ellipse clearly fails the Vertical Line Test, Theorem 1.1, so the equation of an ellipse does not define y as a function of x. However, much like with circles and horizontal parabolas, we can split an ellipse into a top half and a bottom half, each of which would indeed represent y as a function of x. With the help of your classmates, use your calculator to graph the ellipses given in Exercises 1 - 8 above. What difficulties arise when you plot them on the calculator?
25. Some famous examples of whispering galleries include St. Paul's Cathedral in London, England, National Statuary Hall in Washington, D.C., and The Cincinnati Museum Center. With the help of your classmates, research these whispering galleries. How does the whispering effect compare and contrast with the scenario in Example Example 7.4.5?
26. With the help of your classmates, research "extracorporeal shock-wave lithotripsy". It uses the reflective property of the ellipsoid to dissolve kidney stones.

Answers

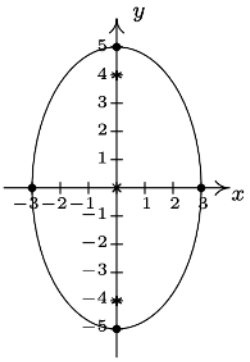
1. $\frac{x^2}{169} + \frac{y^2}{25} = 1$

   Center $(0, 0)$
   Major axis along $y = 0$
   Minor axis along $x = 0$
   Vertices $(13, 0),\ (-13, 0)$
   Endpoints of Minor Axis $(0, -5),\ (0, 5)$
   Foci $(12, 0),\ (-12, 0)$
   $e = \frac{12}{13}$



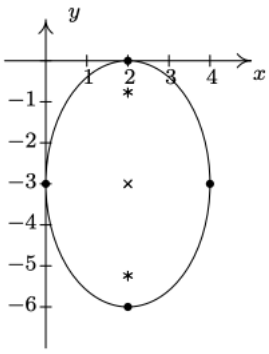
2. $\frac{x^2}{9} + \frac{y^2}{25} = 1$

   Center $(0, 0)$
   Major axis along $x = 0$
   Minor axis along $y = 0$
   Vertices $(0, 5),\ (0, -5)$
   Endpoints of Minor Axis $(-3, 0),\ (3, 0)$
   Foci $(0, -4),\ (0, 4)$
   $e = \frac{4}{5}$



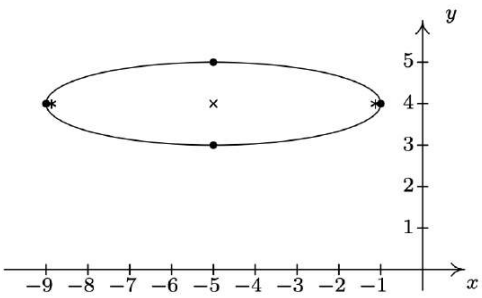
3. $\dfrac{(x-2)^2}{4}+\dfrac{(y+3)^2}{9}=1$

Center $(2,-3)$
Major axis along $x=2$
Minor axis along $y=-3$
Vertices $(2,0),\ (2,-6)$
Endpoints of Minor Axis $(0,-3),\ (4,-3)$
Foci $(2,-3+\sqrt{5}),\ (2,-3-\sqrt{5})$
$e=\frac{\sqrt{5}}{3}$



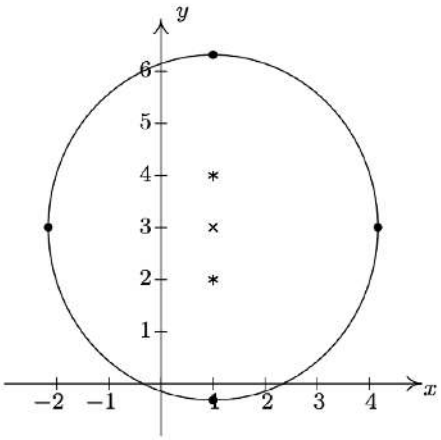
4. $\dfrac{(x+5)^2}{16}+\dfrac{(y-4)^2}{1}=1$

Center $(-5,4)$
Major axis along $y=4$
Minor axis along $x=-5$
Vertices $(-9,4),\ (-1,4)$
Endpoints of Minor Axis $(-5,3),\ (-5,5)$
Foci $(-5+\sqrt{15},4),\ (-5-\sqrt{15},4)$
$e=\frac{\sqrt{15}}{4}$



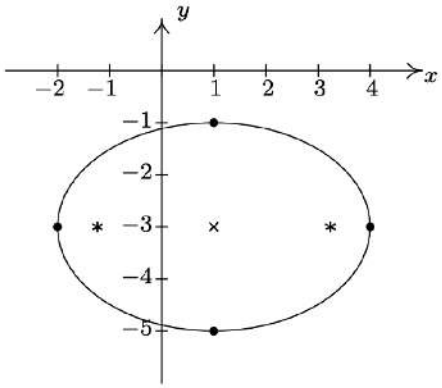
5. $\frac{(x-1)^2}{10}+\frac{(y-3)^2}{11}=1$

Center $(1,3)$
Major axis along $x=1$
Minor axis along $y=3$
Vertices $(1,3+\sqrt{11}),\ (1,3-\sqrt{11})$
Endpoints of the Minor Axis
$(1-\sqrt{10},3),\ (1+\sqrt{10},3)$
Foci $(1,2),\ (1,4)$
$e=\frac{\sqrt{11}}{11}$



6. $\frac{(x-1)^2}{9}+\frac{(y+3)^2}{4}=1$

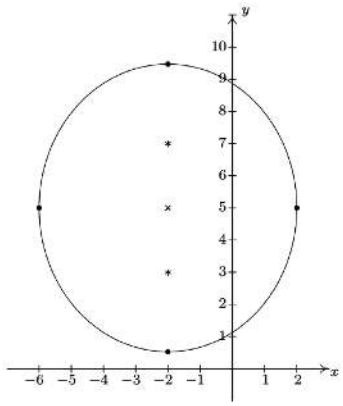
Center $(1,-3)$
Major axis along $y=-3$
Minor axis along $x=1$
Vertices $(4,-3),\ (-2,-3)$
Endpoints of the Minor Axis $(1,-1),\ (1,-5)$
Foci $(1+\sqrt{5},-3),\ (1-\sqrt{5},-3)$
$e=\frac{\sqrt{5}}{3}$



7. $\frac{(x+2)^2}{16}+\frac{(y-5)^2}{20}=1$

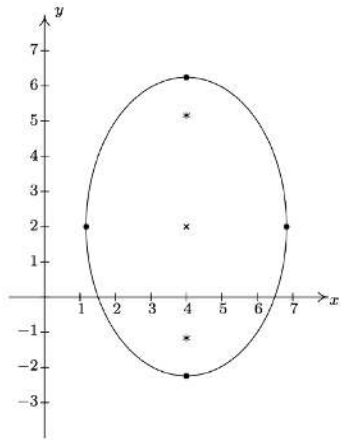
Center $(-2,5)$
Major axis along $x=-2$
Minor axis along $y=5$

Vertices $(-2, 5+2\sqrt{5}),\ (-2, 5-2\sqrt{5})$
Endpoints of the Minor Axis $(-6, 5),\ (2, 5)$ Foci $(-2, 7),\ (-2, 3)$
$e = \frac{\sqrt{5}}{5}$



8. $\frac{(x-4)^2}{8} + \frac{(y-2)^2}{18} = 1$

Center $(4, 2)$
Major axis along $x = 4$
Minor axis along $y = 2$
Vertices $(4, 2+3\sqrt{2}),\ (4, 2-3\sqrt{2})$
Endpoints of the Minor Axis
$(4-2\sqrt{2}, 2), (4+2\sqrt{2}, 2)$
Foci $(4, 2+\sqrt{10}),\ (4, 2-\sqrt{10})$
$e = \frac{\sqrt{5}}{3}$



9. $\frac{(x-3)^2}{25} + \frac{(y-1)^2}{9} = 1$

Center $(3, 1)$
Major Axis along $y = 1$
Minor Axis along $x = 3$
Vertices $(8, 1), (-2, 1)$
Endpoints of Minor Axis $(3, 4), (3, -2)$
Foci $(7, 1), (-1, 1)$
$e = \frac{4}{5}$

10. $\frac{x^2}{3}+\frac{(y-5)^2}{12}=1$
    Center $(0,5)$
    Major axis along $x=0$
    Minor axis along $y=5$
    Vertices $(0,5-2\sqrt{3}),(0,5+2\sqrt{3})$
    Endpoints of Minor Axis $(-\sqrt{3},5),(\sqrt{3},5)$
    Foci $(0,2),(0,8)$
    $e=\frac{\sqrt{3}}{2}$
11. $\frac{(x-3)^2}{18}+\frac{(y+2)^2}{5}=1$
    Center $(3,-2)$
    Major axis along $y=-2$
    Minor axis along $x=3$
    Vertices $(3-3\sqrt{2},-2),(3+3\sqrt{2},-2)$
    Endpoints of Minor Axis $(3,-2+\sqrt{5}),(3,-2-\sqrt{5})$
    Foci $(3-\sqrt{13},-2),(3+\sqrt{13},-2)$
    $e=\frac{\sqrt{26}}{6}$
12. $\frac{(x-1)^2}{16}+\frac{(y-3)^2}{8}=1$
    Center $(1,3)$
    Major Axis along $y=3$
    Minor Axis along $x=1$
    Vertices $(5,3),(-3,3)$
    Endpoints of Minor Axis $(1,3+2\sqrt{2}),(1,3-2\sqrt{2})$
    Foci $(1+2\sqrt{2},3),(1-2\sqrt{2},3)$
    $e=\frac{\sqrt{2}}{2}$
13. $\frac{x^2}{1}+\frac{4\left(y-\frac{1}{2}\right)^2}{9}=1$
    Center $\left(0,\frac{1}{2}\right)$
    Major Axis along $x=0$ (the y-axis)
    Minor Axis along $y=\frac{1}{2}$
    Vertices $(0,2),(0,-1)$
    Endpoints of Minor Axis $\left(-1,\frac{1}{2}\right),\left(1,\frac{1}{2}\right)$
    Foci $\left(0,\frac{1+\sqrt{5}}{2}\right),\left(0,\frac{1-\sqrt{5}}{2}\right)$
    $e=\frac{\sqrt{5}}{3}$
14. $\frac{(x-2)^2}{5}+\frac{(y+2)^2}{6}=1$
    Center $(2,-2)$
    Major Axis along $x=2$
    Minor Axis along $y=-2$
    Vertices $\left(2,-2+\sqrt{6}\right),(2,-2-\sqrt{6})$
    Endpoints of Minor Axis $\left(2-\sqrt{5},-2\right),\left(2+\sqrt{5},-2\right)$
    Foci $(2,-1),(2,-3)$
    $e=\frac{\sqrt{6}}{6}$
15. $\frac{(x-3)^2}{9}+\frac{(y-7)^2}{25}=1$
16. $\frac{x^2}{39}+\frac{y^2}{64}=1$

17. $\frac{x^2}{34}+\frac{y^2}{25}=1$
18. $\frac{(x-8)^2}{25}+\frac{(y-2)^2}{4}=1$
19. $\frac{(x-5)^2}{25}+\frac{4(y-2)^2}{75}=1$
20. $\frac{(x-8)^2}{64}+\frac{(y+9)^2}{81}=1$
21. Jamie and Jason should stand $100-25\sqrt{7}\approx 33.86$ feet from opposite ends of the gallery.
22. The arch can be modeled by the top half of $\frac{x^2}{9}+\frac{y^2}{81}=1$. One foot in from the base of the arch corresponds to either $x=\pm 2$. Plugging in $x=\pm 2$ gives $y=\pm 3\sqrt{5}$ and since y represents a height, we choose $y=3\sqrt{5}\approx 6.71$ feet.
23. Distance from the sun to aphelion ≈ 1.0167 AU.
    Distance from the sun to perihelion ≈ 0.9833 AU.

Reference

[1] This was foreshadowed in Exercise 19 in Section 7.2.

[2] The equation of a parabola has only one squared variable and the equation of a circle has two squared variables with identical coefficients.

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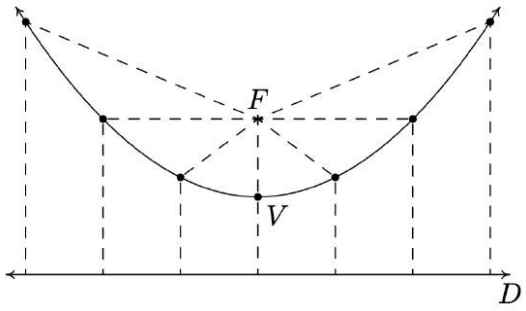
6.4: Parabolas

We have already learned that the graph of a quadratic function $f(x) = ax^2 + bx + c \ \ (a \neq 0)$ is called a **parabola**. To our surprise and delight, we may also define parabolas in terms of distance.

Definition 7.3

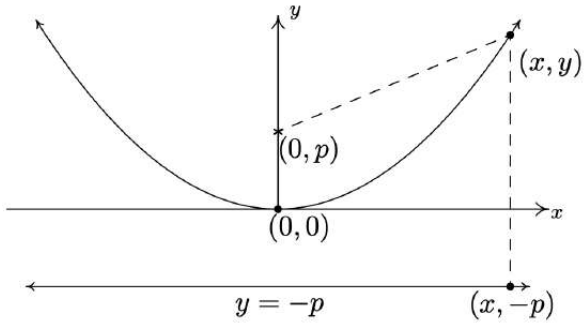
Let F be a point in the plane and D be a line not containing F. A **parabola** is the set of all points equidistant from F and D. The point F is called the **focus** of the parabola and the line D is called the **directrix** of the parabola.

Schematically, we have the following.



Each dashed line from the point F to a point on the curve has the same length as the dashed line from the point on the curve to the line D. The point suggestively labeled V is, as you should expect, the **vertex**. The vertex is the point on the parabola closest to the focus.

We want to use only the distance definition of parabola to derive the equation of a parabola and, if all is right with the universe, we should get an expression much like those studied in Section 2.3. Let p denote the directed[1] distance from the vertex to the focus, which by definition is the same as the distance from the vertex to the directrix. For simplicity, assume that the vertex is $(0, 0)$ and that the parabola opens upwards. Hence, the focus is $(0, p)$ and the directrix is the line $y = -p$. Our picture becomes



From the definition of parabola, we know the distance from $(0, p)$ to (x, y) is the same as the distance from $(x, -p)$ to (x, y). Using the Distance Formula, Equation 1.1, we get

$$\begin{aligned}
\sqrt{(x-0)^2+(y-p)^2} &= \sqrt{(x-x)^2+(y-(-p))^2} \\
\sqrt{x^2+(y-p)^2} &= \sqrt{(y+p)^2} \\
x^2+(y-p)^2 &= (y+p)^2 && \text{square both sides} \\
x^2+y^2-2py+p^2 &= y^2+2py+p^2 && \text{expand quantities} \\
x^2 &= 4py && \text{gather like terms}
\end{aligned}$$

Solving for y yields $y = \frac{x^2}{4p}$, which is a quadratic function of the form found in Equation 2.4 with $a = \frac{1}{4p}$ and vertex $(0, 0)$.

We know from previous experience that if the coefficient of x^2 is negative, the parabola opens downwards. In the equation $y = \frac{x^2}{4p}$ this happens when $p < 0$. In our formulation, we say that p is a 'directed distance' from the vertex to the focus: if $p > 0$, the focus

is above the vertex; if $p < 0$, the focus is below the vertex. The **focal length** of a parabola is $|p|$.

If we choose to place the vertex at an arbitrary point (h, k), we arrive at the following formula using either transformations from Section 1.7 or re-deriving the formula from Definition 7.3.

Equation 7.2. The Standard Equation of a Vertical[a] Parabola

The equation of a (vertical) parabola with vertex (h, k) and focal length $|p|$ is

$$(x-h)^2 = 4p(y-k)$$

If $p > 0$, the parabola opens upwards; if $p < 0$, it opens downwards.

[a] That is, a parabola which opens either upwards or downwards.

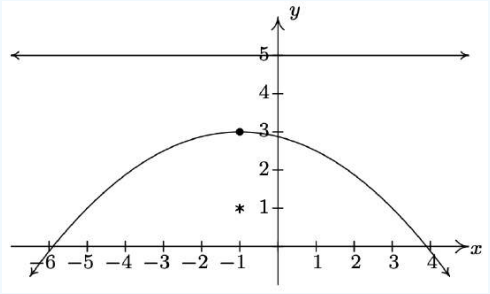
Notice that in the standard equation of the parabola above, only one of the variables, x, is squared. This is a quick way to distinguish an equation of a parabola from that of a circle because in the equation of a circle, both variables are squared.

Example 7.3.1

Graph $(x+1)^2 = -8(y-3)$. Find the vertex, focus, and directrix.

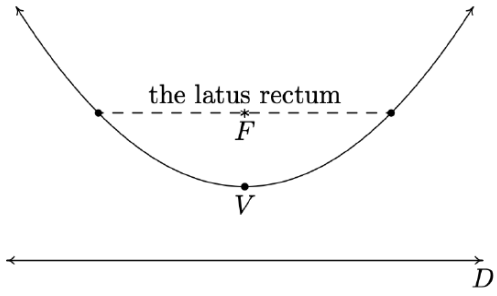
Solution

We recognize this as the form given in Equation 7.2. Here, $x-h$ is $x+1$ so $h=-1$, and $y-k$ is $y-3$ so $k=3$. Hence, the vertex is $(-1, 3)$. We also see that $4p = -8$ so $p = -2$. Since $p < 0$, the focus will be below the vertex and the parabola will open downwards.



The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units below the vertex. From $(-1, 3)$, we move down 2 units and find the focus at $(-1, 1)$. The directrix, then, is 2 units above the vertex, so it is the line $y = 5$.

Of all of the information requested in the previous example, only the vertex is part of the graph of the parabola. So in order to get a sense of the actual shape of the graph, we need some more information. While we could plot a few points randomly, a more useful measure of how wide a parabola opens is the length of the parabola's latus rectum.[2] The **latus rectum** of a parabola is the line segment parallel to the directrix which contains the focus. The endpoints of the latus rectum are, then, two points on 'opposite' sides of the parabola. Graphically, we have the following.



It turns out[3] that the length of the latus rectum, called the **focal diameter** of the parabola is $|4p|$, which, in light of Equation 7.2, is easy to find. In our last example, for instance, when graphing $(x+1)^2 = -8(y-3)$, we can use the fact that the focal diameter is

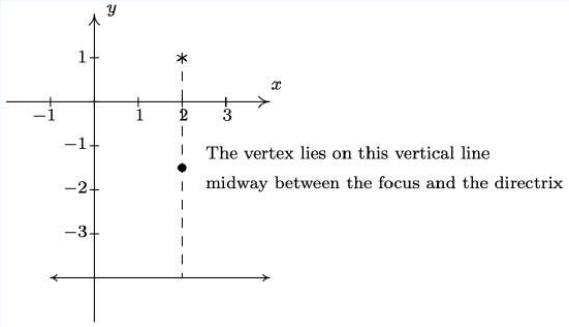
$|-8| = 8$, which means the parabola is 8 units wide at the focus, to help generate a more accurate graph by plotting points 4 units to the left and right of the focus.

Example 7.3.2

Find the standard form of the parabola with focus $(2, 1)$ and directrix $y = -4$.

Solution

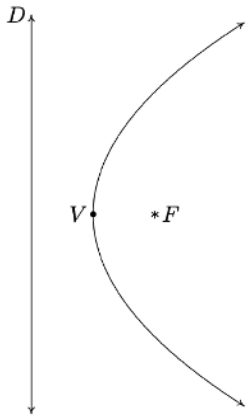
Sketching the data yields,



From the diagram, we see the parabola opens upwards. (Take a moment to think about it if you don't see that immediately.) Hence, the vertex lies below the focus and has an x-coordinate of 2. To find the y-coordinate, we note that the distance from the focus to the directrix is $1 - (-4) = 5$, which means the vertex lies $\frac{5}{2}$ units (halfway) below the focus. Starting at $(2, 1)$ and moving down $5/2$ units leaves us at $(2, -3/2)$, which is our vertex. Since the parabola opens upwards, we know p is positive. Thus $p = 5/2$. Plugging all of this data into Equation 7.2 give us

$$\begin{aligned} (x-2)^2 &= 4\left(\frac{5}{2}\right)\left(y - \left(-\frac{3}{2}\right)\right) \\ (x-2)^2 &= 10\left(y + \frac{3}{2}\right) \end{aligned}$$

If we interchange the roles of x and y, we can produce 'horizontal' parabolas: parabolas which open to the left or to the right. The directrices[4] of such animals would be vertical lines and the focus would either lie to the left or to the right of the vertex, as seen below.



Equation 7.3. The Standard Equation of a Horizontal Parabola

The equation of a (horizontal) parabola with vertex (h, k) and focal length $|p|$ is

$$(y-k)^2 = 4p(x-h)$$

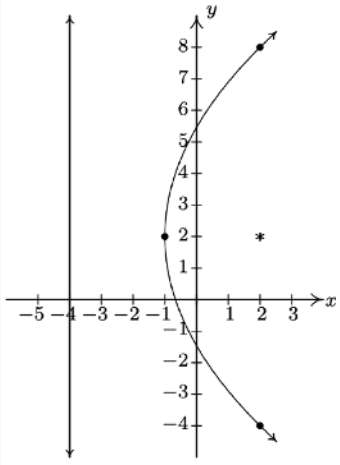
If $p > 0$, the parabola opens to the right; if $p < 0$, it opens to the left.

Example 7.3.3

Graph $(y-2)^2 = 12(x+1)$. Find the vertex, focus, and directrix.

Solution

We recognize this as the form given in Equation 7.3. Here, $x-h$ is $x+1$ so $h=-1$, and $y-k$ is $y-2$ so $k=2$. Hence, the vertex is $(-1,2)$. We also see that $4p=12$ so $p=3$. Since $p>0$, the focus will be the right of the vertex and the parabola will open to the right. The distance from the vertex to the focus is $|p|=3$, which means the focus is 3 units to the right. If we start at $(-1,2)$ and move right 3 units, we arrive at the focus $(2,2)$. The directrix, then, is 3 units to the left of the vertex and if we move left 3 units from $(-1,2)$, we'd be on the vertical line $x=-4$. Since the focal diameter is $|4p|=12$, the parabola is 12 units wide at the focus, and thus there are points 6 units above and below the focus on the parabola.



As with circles, not all parabolas will come to us in the forms in Equations 7.2 or 7.3. If we encounter an equation with two variables in which exactly one variable is squared, we can attempt to put the equation into a standard form using the following steps.

To Write the Equation of a Parabola in Standard Form

1. Group the variable which is squared on one side of the equation and position the non-squared variable and the constant on the other side.
2. Complete the square if necessary and divide by the coefficient of the perfect square.
3. Factor out the coefficient of the non-squared variable from it and the constant.

Example 7.3.4

Consider the equation $y^2+4y+8x=4$. Put this equation into standard form and graph the parabola. Find the vertex, focus, and directrix.

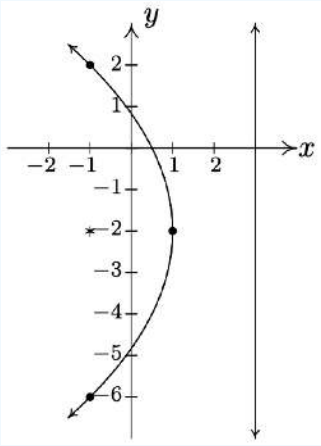
Solution

We need a perfect square (in this case, using y) on the left-hand side of the equation and factor out the coefficient of the non-squared variable (in this case, the x) on the other.

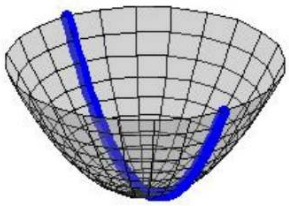
$$\begin{aligned} y^2+4y+8x &= 4 \\ y^2+4y &= -8x+4 \\ y^2+4y+4 &= -8x+4+4 \quad \text{complete the square in } y \text{ only} \\ (y+2)^2 &= -8x+8 \quad \text{factor} \\ (y+2)^2 &= -8(x-1) \end{aligned}$$

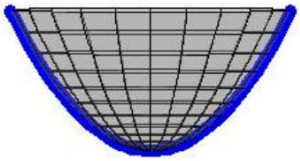
Now that the equation is in the form given in Equation 7.3, we see that $x-h$ is $x-1$ so $h=1$, and $y-k$ is $y+2$ so $k=-2$. Hence, the vertex is $(1,-2)$. We also see that $4p=-8$ so that $p=-2$. Since $p<0$, the focus will be the left of the

vertex and the parabola will open to the left. The distance from the vertex to the focus is $|p| = 2$, which means the focus is 2 units to the left of 1, so if we start at $(1, -2)$ and move left 2 units, we arrive at the focus $(-1, -2)$. The directrix, then, is 2 units to the right of the vertex, so if we move right 2 units from $(1, -2)$, we'd be on the vertical line $x = 3$. Since the focal diameter is $|4p|$ is 8, the parabola is 8 units wide at the focus, so there are points 4 units above and below the focus on the parabola.

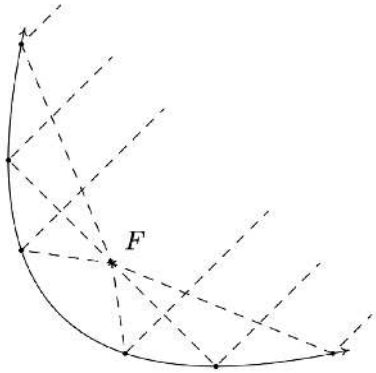


In studying quadratic functions, we have seen parabolas used to model physical phenomena such as the trajectories of projectiles. Other applications of the parabola concern its 'reflective property' which necessitates knowing about the focus of a parabola. For example, many satellite dishes are formed in the shape of a **paraboloid of revolution** as depicted below.





Every cross section through the vertex of the paraboloid is a parabola with the same focus. To see why this is important, imagine the dashed lines below as electromagnetic waves heading towards a parabolic dish. It turns out that the waves reflect off the parabola and concentrate at the focus which then becomes the optimal place for the receiver. If, on the other hand, we imagine the dashed lines as emanating from the focus, we see that the waves are reflected off the parabola in a coherent fashion as in the case in a flashlight. Here, the bulb is placed at the focus and the light rays are reflected off a parabolic mirror to give directional light.

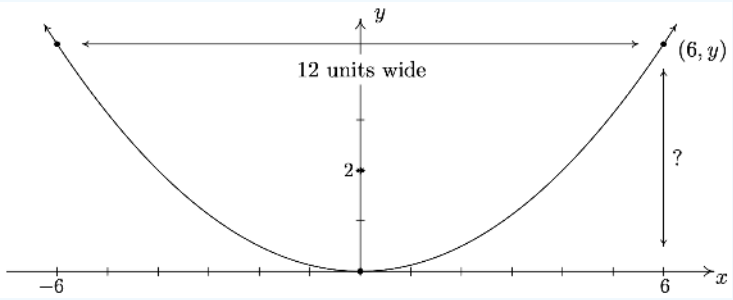


Example 7.3.5

A satellite dish is to be constructed in the shape of a paraboloid of revolution. If the receiver placed at the focus is located 2 ft above the vertex of the dish, and the dish is to be 12 feet wide, how deep will the dish be?

Solution

One way to approach this problem is to determine the equation of the parabola suggested to us by this data. For simplicity, we'll assume the vertex is $(0,0)$ and the parabola opens upwards. Our standard form for such a parabola is $x^2 = 4py$. Since the focus is 2 units above the vertex, we know $p = 2$, so we have $x^2 = 8y$. Visually,



Since the parabola is 12 feet wide, we know the edge is 6 feet from the vertex. To find the depth, we are looking for the y value when $x = 6$. Substituting $x = 6$ into the equation of the parabola yields $6^2 = 8y$ or $y = \frac{36}{8} = \frac{9}{2} = 4.5$. Hence, the dish will be 4.5 feet deep.

7.3.1. Exercises

In Exercises 1 - 8, sketch the graph of the given parabola. Find the vertex, focus and directrix. Include the endpoints of the latus rectum in your sketch.

1. $(x-3)^2 = -16y$
2. $\left(x+\frac{7}{3}\right)^2 = 2\left(y+\frac{5}{2}\right)$
3. $(y-2)^2 = -12(x+3)$
4. $(y+4)^2 = 4x$
5. $(x-1)^2 = 4(y+3)$
6. $(x+2)^2 = -20(y-5)$
7. $(y-4)^2 = 18(x-2)$
8. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$

In Exercises 9 - 14, put the equation into standard form and identify the vertex, focus and directrix.

9. $y^2 - 10y - 27x + 133 = 0$
10. $25x^2 + 20x + 5y - 1 = 0$
11. $x^2 + 2x - 8y + 49 = 0$
12. $2y^2 + 4y + x - 8 = 0$
13. $x^2 - 10x + 12y + 1 = 0$
14. $3y^2 - 27y + 4x + \frac{211}{4} = 0$

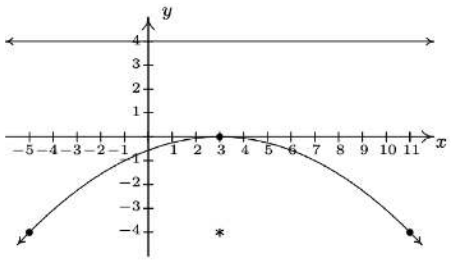
In Exercises 15 - 18, find an equation for the parabola which fits the given criteria.

15. Vertex $(7,0)$, focus $(0,0)$
16. Focus $(10,1)$, directrix $x = 5$
17. Vertex $(-8,-9)$; $(0,0)$ and $(-16,0)$ are points on the curve
18. The endpoints of latus rectum are $(-2,-7)$ and $(4,-7)$
19. The mirror in Carl's flashlight is a paraboloid of revolution. If the mirror is 5 centimeters in diameter and 2.5 centimeters deep, where should the light bulb be placed so it is at the focus of the mirror?
20. A parabolic Wi-Fi antenna is constructed by taking a flat sheet of metal and bending it into a parabolic shape.[5] If the cross section of the antenna is a parabola which is 45 centimeters wide and 25 centimeters deep, where should the receiver be placed to maximize reception?
21. [parabolaarch] A parabolic arch is constructed which is 6 feet wide at the base and 9 feet tall in the middle. Find the height of the arch exactly 1 foot in from the base of the arch.

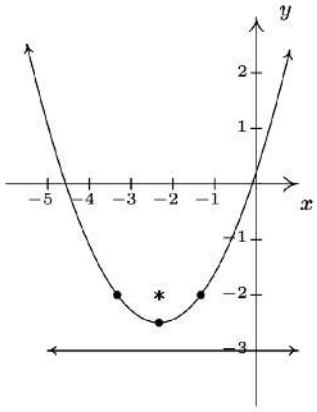
22. A popular novelty item is the ‘mirage bowl.’ Follow this link to see another startling application of the reflective property of the parabola.
23. With the help of your classmates, research spinning liquid mirrors. To get you started, check out this website.

7.3.2. Answers

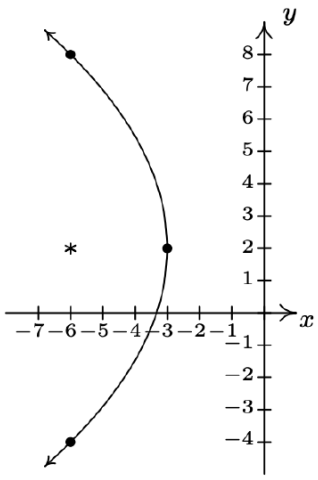
1. $(x-3)^2=-16y$
   Vertex $(3,0)$
   Focus $(3,-4)$
   Directrix $y=4$
   Endpoints of latus rectum $(-5,-4)$, $(11,-4)$



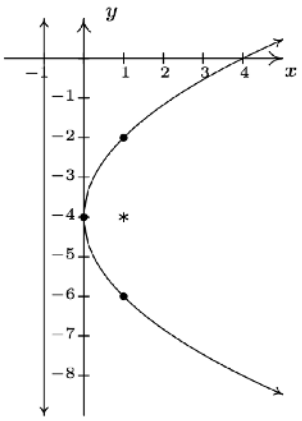
2. $\left(x+\frac{7}{3}\right)^2=2\left(y+\frac{5}{2}\right)$
   Vertex $\left(-\frac{7}{3},-\frac{5}{2}\right)$
   Focus $\left(-\frac{7}{3},-2\right)$
   Directrix $y=-3$
   Endpoints of latus rectum $\left(-\frac{10}{3},-2\right)$, $\left(-\frac{4}{3},-2\right)$



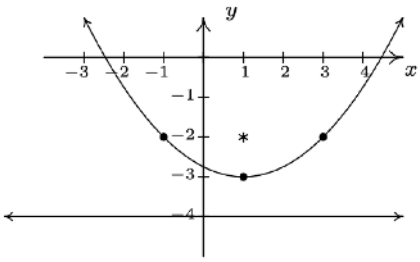
3. $(y-2)^2=-12(x+3)$
   Vertex $(-3,2)$
   Focus $(-6,2)$
   Directrix $x=0$
   Endpoints of latus rectum $(-6,8)$, $(-6,-4)$



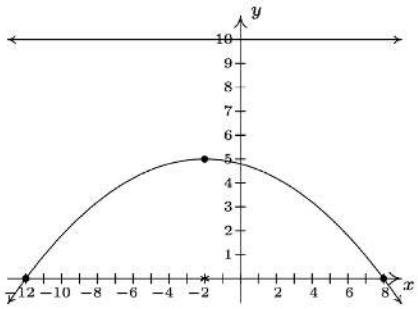
4. $(y+4)^2 = 4x$
   Vertex $(0, -4)$
   Focus $(1, -4)$
   Directrix $x = -1$
   Endpoints of latus rectum $(1, -2)$, $(1, -6)$



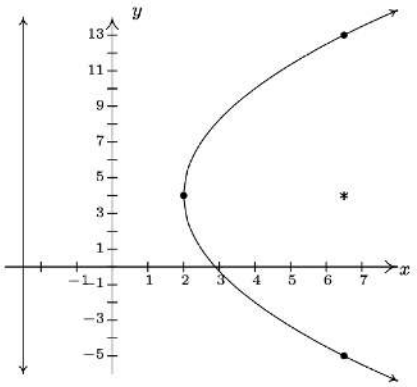
5. $(x-1)^2 = 4(y+3)$
   Vertex $(1, -3)$
   Focus $(1, -2)$
   Directrix $y = -4$
   Endpoints of latus rectum $(3, -2)$, $(-1, -2)$



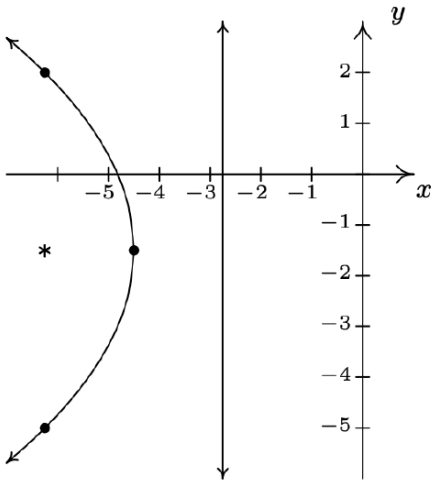
6. $(x+2)^2 = -20(y-5)$
   Vertex $(-2, 5)$
   Focus $(-2, 0)$
   Directrix $y = 10$
   Endpoints of latus rectum $(-12, 0)$, $(8, 0)$



7. $(y-4)^2 = 18(x-2)$
   Vertex $(2,4)$
   Focus $\left(\frac{13}{2},4\right)$
   Directrix $x = -\frac{5}{2}$
   Endpoints of latus rectum $\left(\frac{13}{2},-5\right)$, $\left(\frac{13}{2},13\right)$



8. $\left(y+\frac{3}{2}\right)^2 = -7\left(x+\frac{9}{2}\right)$
   Vertex $\left(-\frac{9}{2},-\frac{3}{2}\right)$
   Focus $\left(-\frac{25}{4},-\frac{3}{2}\right)$
   Directrix $x = -\frac{11}{4}$
   Endpoints of latus rectum $\left(-\frac{25}{4},2\right)$, $\left(-\frac{25}{4},-5\right)$



9. $(y-5)^2 = 27(x-4)$
   Vertex $(4,5)$

https://math.libretexts.org/@go/page/127047

Focus $\left(\frac{43}{4}, 5\right)$
Directrix $x = -\frac{11}{4}$

10. $\left(x + \frac{2}{5}\right)^2 = -\frac{1}{5}(y - 1)$
    Vertex $\left(-\frac{2}{5}, 1\right)$
    Focus $\left(-\frac{2}{5}, \frac{19}{20}\right)$
    Directrix $y = \frac{21}{20}$
11. $(x+1)^2 = 8(y-6)$
    Vertex $(-1, 6)$
    Focus $(-1, 8)$
    Directrix $y = 4$
12. $(y+1)^2 = -\frac{1}{2}(x - 10)$
    Vertex $(10, -1)$
    Focus $\left(\frac{79}{8}, -1\right)$
    Directrix $x = \frac{81}{8}$
13. $(x-5)^2 = -12(y-2)$
    Vertex $(5, 2)$
    Focus $(5, -1)$
    Directrix $y = 5$
14. $\left(y - \frac{9}{2}\right)^2 = -\frac{4}{3}(x-2)$
    Vertex $\left(2, \frac{9}{2}\right)$
    Focus $\left(\frac{5}{3}, \frac{9}{2}\right)$
    Directrix $x = \frac{7}{3}$
15. $y^2 = -28(x-7)$
16. $(y-1)^2 = 10\left(x - \frac{15}{2}\right)$
17. $(x+8)^2 = \frac{64}{9}(y+9)$
18. $(x-1)^2 = 6\left(y + \frac{17}{2}\right)$ or
    $(x-1)^2 = -6\left(y + \frac{11}{2}\right)$
19. The bulb should be placed 0.625 centimeters above the vertex of the mirror. (As verified by Carl himself!)
20. The receiver should be placed 5.0625 centimeters from the vertex of the cross section of the antenna.
21. The arch can be modeled by $x^2 = -(y-9)$ or $y = 9 - x^2$. One foot in from the base of the arch corresponds to either $x = \pm 2$, so the height is $y = 9 - (\pm 2)^2 = 5$ feet.

Reference

[1] We'll talk more about what 'directed' means later.

[2] No, I'm not making this up.

[3] Consider this an exercise to show what follows.

[4] Plural of 'directrix'

[5] This shape is called a 'parabolic cylinder.'

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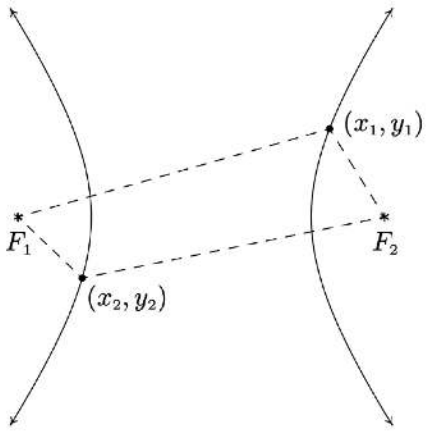
- **7.3: Parabolas** by Carl Stitz & Jeff Zeager is licensed CC BY-NC-SA 3.0. Original source: https://www.stitz-zeager.com/latex-source-code.html.

6.5: Hyperbolas

In the definition of an ellipse, Definition 7.4, we fixed two points called foci and looked at points whose distances to the foci always **added** to a constant distance d. Those prone to syntactical tinkering may wonder what, if any, curve we'd generate if we replaced **added** with **subtracted**. The answer is a hyperbola.

Definition 7.6

Given two distinct points F_1 and F_2 in the plane and a fixed distance d, a **hyperbola** is the set of all points (x, y) in the plane such that the absolute value of the difference of each of the distances from F_1 and F_2 to (x, y) is d. The points F_1 and F_2 are called the **foci** of the hyperbola.



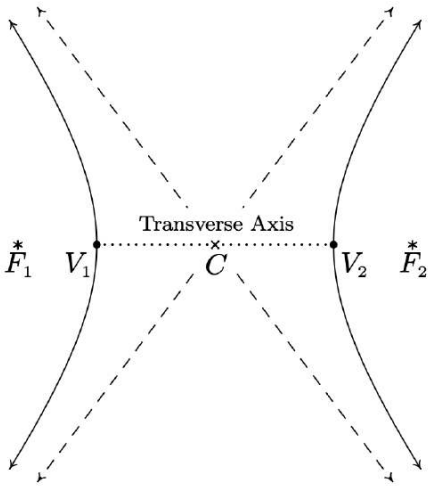
In the figure above:

$$\text{the distance from } F_1 \text{ to } (x_1, y_1) - \text{the distance from } F_2 \text{ to } (x_1, y_1) \quad = \quad d$$

and

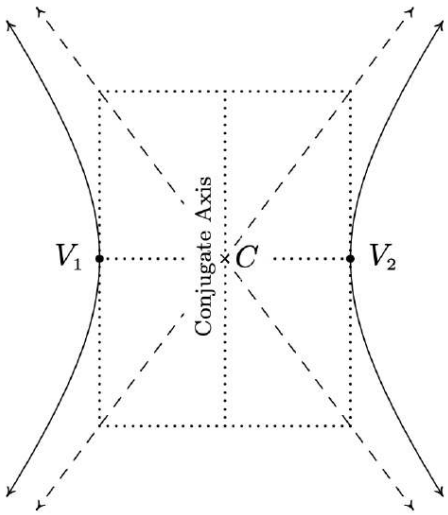
$$\text{the distance from } F_2 \text{ to } (x_2, y_2) - \text{the distance from } F_1 \text{ to } (x_2, y_2) \quad = \quad d$$

Note that the hyperbola has two parts, called **branches**. The **center** of the hyperbola is the midpoint of the line segment connecting the two foci. The **transverse axis** of the hyperbola is the line segment connecting two opposite ends of the hyperbola which also contains the center and foci. The **vertices** of a hyperbola are the points of the hyperbola which lie on the transverse axis. In addition, we will show momentarily that there are lines called **asymptotes** which the branches of the hyperbola approach for large x and y values. They serve as guides to the graph. In pictures,



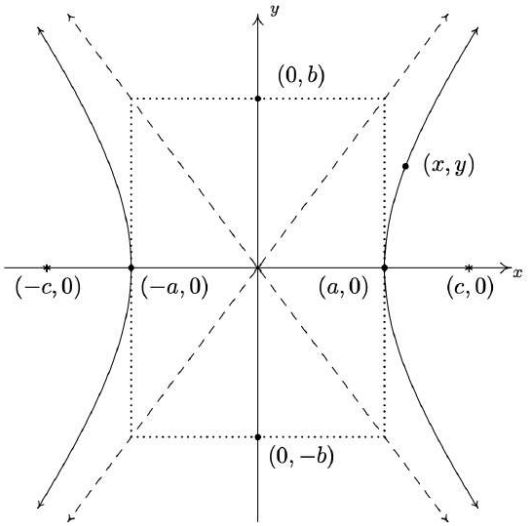
A hyperbola with center C; foci F_1, F_2; and vertices V_1, V_2 and asymptotes (dashed)

Before we derive the standard equation of the hyperbola, we need to discuss one further parameter, the **conjugate axis** of the hyperbola. The conjugate axis of a hyperbola is the line segment through the center which is perpendicular to the transverse axis and has the same length as the line segment through a vertex which connects the asymptotes. In pictures we have



Note that in the diagram, we can construct a rectangle using line segments with lengths equal to the lengths of the transverse and conjugate axes whose center is the center of the hyperbola and whose diagonals are contained in the asymptotes. This **guide rectangle**, much akin to the one we saw Section 7.4 to help us graph ellipses, will aid us in graphing hyperbolas.

Suppose we wish to derive the equation of a hyperbola. For simplicity, we shall assume that the center is $(0,0)$, the vertices are $(a,0)$ and $(-a,0)$ and the foci are $(c,0)$ and $(-c,0)$. We label the endpoints of the conjugate axis $(0,b)$ and $(0,-b)$. (Although b does not enter into our derivation, we will have to justify this choice as you shall see later.) As before, we assume a, b, and c are all positive numbers. Schematically we have



Since $(a,0)$ is on the hyperbola, it must satisfy the conditions of Definition 7.6. That is, the distance from $(-c,0)$ to $(a,0)$ minus the distance from $(c,0)$ to $(a,0)$ must equal the fixed distance d. Since all these points lie on the x-axis, we get

$$\begin{aligned} \text{distance from } (-c,0) \text{ to } (a,0) - \text{distance from } (c,0) \text{ to } (a,0) &= d \\ (a+c)-(c-a) &= d \\ 2a &= d \end{aligned}$$

In other words, the fixed distance d from the definition of the hyperbola is actually the length of the transverse axis! (Where have we seen that type of coincidence before?) Now consider a point (x,y) on the hyperbola. Applying Definition 7.6, we get

$$\begin{aligned}\text{distance from } (-c,0) \text{ to } (x,y) - \text{distance from } (c,0) \text{ to } (x,y) &= 2a \\ \sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} &= 2a \\ \sqrt{(x+c)^2+y^2} - \sqrt{(x-c)^2+y^2} &= 2a\end{aligned}$$

Using the same arsenal of Intermediate Algebra weaponry we used in deriving the standard formula of an ellipse, Equation 7.4, we arrive at the following.[1]

$$\left(a^2-c^2\right)x^2+a^2y^2 = a^2\left(a^2-c^2\right)$$

What remains is to determine the relationship between a, b and c. To that end, we note that since a and c are both positive numbers with $a<c$, we get $a^2<c^2$ so that a^2-c^2 is a negative number. Hence, c^2-a^2 is a positive number. For reasons which will become clear soon, we re-write the equation by solving for y^2/x^2 to get

$$\begin{aligned}\left(a^2-c^2\right)x^2+a^2y^2 &= a^2\left(a^2-c^2\right) \\ -\left(c^2-a^2\right)x^2+a^2y^2 &= -a^2\left(c^2-a^2\right) \\ a^2y^2 &= \left(c^2-a^2\right)x^2-a^2\left(c^2-a^2\right) \\ \frac{y^2}{x^2} &= \frac{\left(c^2-a^2\right)}{a^2}-\frac{\left(c^2-a^2\right)}{x^2}\end{aligned}$$

As x and y attain very large values, the quantity $\frac{(c^2-a^2)}{x^2}\to 0$ so that $\frac{y^2}{x^2}\to\frac{(c^2-a^2)}{a^2}$. By setting $b^2=c^2-a^2$ we get $\frac{y^2}{x^2}\to\frac{b^2}{a^2}$. This shows that $y\to\pm\frac{b}{a}x$ as $|x|$ grows large. Thus $y=\pm\frac{b}{a}x$ are the asymptotes to the graph as predicted and our choice of labels for the endpoints of the conjugate axis is justified. In our equation of the hyperbola we can substitute $a^2-c^2=-b^2$ which yields

$$\begin{aligned}\left(a^2-c^2\right)x^2+a^2y^2 &= a^2\left(a^2-c^2\right) \\ -b^2x^2+a^2y^2 &= -a^2b^2 \\ \frac{x^2}{a^2}-\frac{y^2}{b^2} &= 1\end{aligned}$$

The equation above is for a hyperbola whose center is the origin and which opens to the left and right. If the hyperbola were centered at a point (h,k), we would get the following.

Equation 7.6. The Standard Equation of a Horizontal[a] Hyperbola

The Standard Equation of a Horizontal[a] Hyperbola For positive numbers a and b, the equation of a horizontal hyperbola with center (h,k) is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

[a] That is, a hyperbola whose branches open to the left and right

If the roles of x and y were interchanged, then the hyperbola's branches would open upwards and downwards and we would get a 'vertical' hyperbola.

Equation 7.7. The Standard Equation of a Vertical Hyperbola

The Standard Equation of a Vertical Hyperbola For positive numbers a and b, the equation of a vertical hyperbola with center (h,k) is:

$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

The values of a and b determine how far in the x and y directions, respectively, one counts from the center to determine the rectangle through which the asymptotes pass. In both cases, the distance from the center to the foci, c, as seen in the derivation, can be found by the formula $c=\sqrt{a^2+b^2}$. Lastly, note that we can quickly distinguish the equation of a hyperbola from that of a

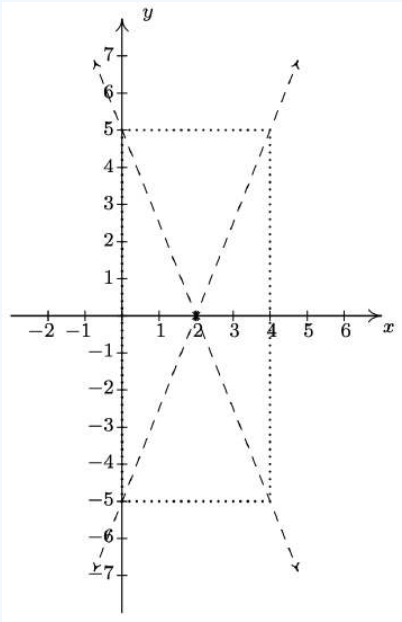
circle or ellipse because the hyperbola formula involves a **difference** of squares where the circle and ellipse formulas both involve the **sum** of squares.

Example 7.5.1

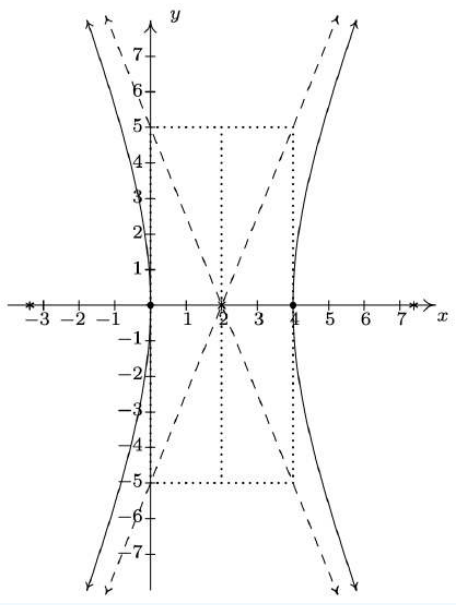
Graph the equation $\frac{(x-2)^2}{4} - \frac{y^2}{25} = 1$. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

Solution

We first see that this equation is given to us in the standard form of Equation 7.6. Here $x-h$ is $x-2$ so $h=2$, and $y-k$ is y so $k=0$. Hence, our hyperbola is centered at $(2,0)$. We see that $a^2=4$ so $a=2$, and $b^2=25$ so $b=5$. This means we move 2 units to the left and right of the center and 5 units up and down from the center to arrive at points on the guide rectangle. The asymptotes pass through the center of the hyperbola as well as the corners of the rectangle. This yields the following set up.



Since the y^2 term is being subtracted from the x^2 term, we know that the branches of the hyperbola open to the left and right. This means that the transverse axis lies along the x-axis. Hence, the conjugate axis lies along the vertical line $x=2$. Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are 2 units to the left and right of $(2,0)$ at $(0,0)$ and $(4,0)$. To find the foci, we need $c=\sqrt{a^2+b^2}=\sqrt{4+25}=\sqrt{29}$. Since the foci lie on the transverse axis, we move $\sqrt{29}$ units to the left and right of $(2,0)$ to arrive at $(2-\sqrt{29},0)$ (approximately $(-3.39,0)$) and $(2+\sqrt{29},0)$ (approximately $(7.39,0)$). To determine the equations of the asymptotes, recall that the asymptotes go through the center of the hyperbola, $(2,0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a}=\pm\frac{5}{2}$. Using the point-slope equation of a line, Equation 2.2, yields $y-0=\pm\frac{5}{2}(x-2)$, so we get $y=\frac{5}{2}x-5$ and $y=-\frac{5}{2}x+5$. Putting it all together, we get

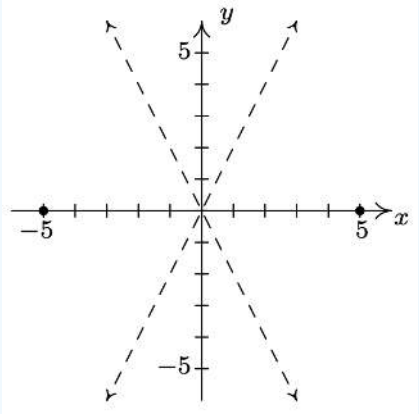


Example 7.5.2

Find the equation of the hyperbola with asymptotes $y = \pm 2x$ and vertices $(\pm 5, 0)$.

Solution

Plotting the data given to us, we have



This graph not only tells us that the branches of the hyperbola open to the left and to the right, it also tells us that the center is $(0, 0)$. Hence, our standard form is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Since the vertices are $(\pm 5, 0)$, we have $a = 5$ so $a^2 = 25$. In order to determine b^2, we recall that the slopes of the asymptotes are $\pm \frac{b}{a}$. Since $a = 5$ and the slope of the line $y = 2x$ is 2, we have that $\frac{b}{5} = 2$, so $b = 10$. Hence, $b^2 = 100$ and our final answer is $\frac{x^2}{25} - \frac{y^2}{100} = 1$.

As with the other conic sections, an equation whose graph is a hyperbola may not be given in either of the standard forms. To rectify that, we have the following.

To Write the Equation of a Hyperbola in Standard Form

1. Group the same variables together on one side of the equation and position the constant on the other side
2. Complete the square in both variables as needed
3. Divide both sides by the constant term so that the constant on the other side of the equation becomes 1

Example 7.5.3

Consider the equation $9y^2 - x^2 - 6x = 10$. Put this equation in to standard form and graph. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci, and the equations of the asymptotes.

Solution

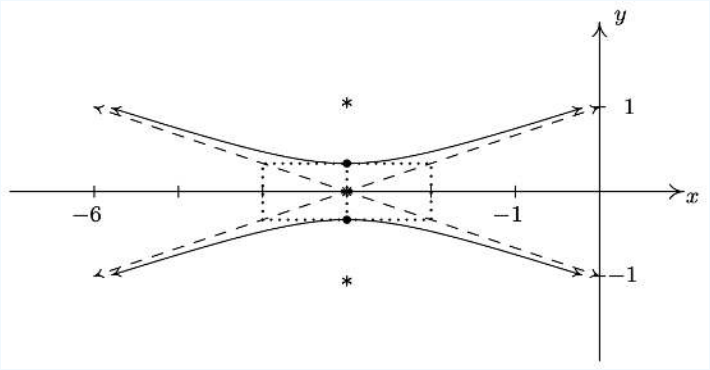
We need only complete the square on x:

$$\begin{aligned} 9y^2 - x^2 - 6x &= 10 \\ 9y^2 - 1\left(x^2 + 6x\right) &= 10 \\ 9y^2 - \left(x^2 + 6x + 9\right) &= 10 - 1(9) \\ 9y^2 - (x+3)^2 &= 1 \\ \frac{y^2}{\frac{1}{9}} - \frac{(x+3)^2}{1} &= 1 \end{aligned}$$

Now that this equation is in the standard form of Equation 7.7, we see that $x - h$ is $x + 3$ so $h = -3$, and $y - k$ is y so $k = 0$. Hence, our hyperbola is centered at $(-3, 0)$. We find that $a^2 = 1$ so $a = 1$, and $b^2 = \frac{1}{9}$ so $b = \frac{1}{3}$. This means that we move 1 unit to the left and right of the center and $\frac{1}{3}$ units up and down from the center to arrive at points on the guide rectangle. Since the x^2 term is being subtracted from the y^2 term, we know the branches of the hyperbola open upwards and downwards. This means the transverse axis lies along the vertical line $x = -3$ and the conjugate axis lies along the x-axis. Since the vertices of the hyperbola are where the hyperbola intersects the transverse axis, we get that the vertices are $\frac{1}{3}$ of a unit above and below $(-3, 0)$ at $\left(-3, \frac{1}{3}\right)$ and $\left(-3, -\frac{1}{3}\right)$. To find the foci, we use

$$c = \sqrt{a^2 + b^2} = \sqrt{\frac{1}{9} + 1} = \frac{\sqrt{10}}{3}$$

Since the foci lie on the transverse axis, we move $\frac{\sqrt{10}}{3}$ units above and below $(-3, 0)$ to arrive at $\left(-3, \frac{\sqrt{10}}{3}\right)$ and $\left(-3, -\frac{\sqrt{10}}{3}\right)$. To determine the asymptotes, recall that the asymptotes go through the center of the hyperbola, $(-3, 0)$, as well as the corners of guide rectangle, so they have slopes of $\pm\frac{b}{a} = \pm\frac{1}{3}$. Using the point-slope equation of a line, Equation 2.2, we get $y = \frac{1}{3}x + 1$ and $y = -\frac{1}{3}x - 1$. Putting it all together, we get



Hyperbolas can be used in so-called 'trilateration,' or 'positioning' problems. The procedure outlined in the next example is the basis of the (now virtually defunct) LOng Range Aid to Navigation (LORAN for short) system.[2]

Example 7.5.4

Jeff is stationed 10 miles due west of Carl in an otherwise empty forest in an attempt to locate an elusive Sasquatch. At the stroke of midnight, Jeff records a Sasquatch call 9 seconds earlier than Carl. If the speed of sound that night is 760 miles per hour, determine a hyperbolic path along which Sasquatch must be located.

Solution

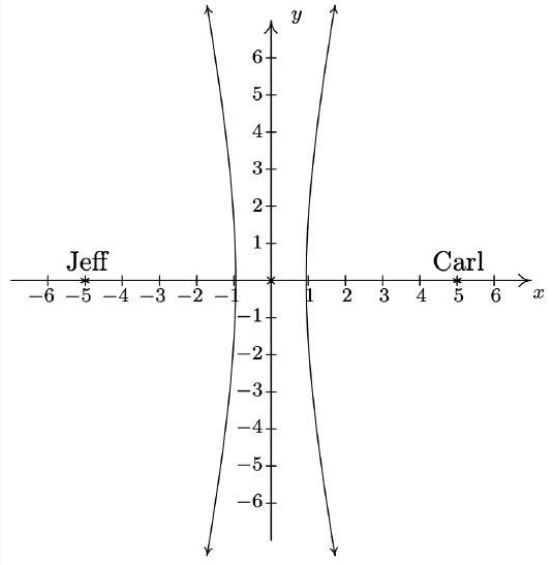
Since Jeff hears Sasquatch sooner, it is closer to Jeff than it is to Carl. Since the speed of sound is 760 miles per hour, we can determine how much closer Sasquatch is to Jeff by multiplying

$$760\ \frac{\text{miles}}{\text{hour}} \times \frac{1\ \text{hour}}{3600\ \text{seconds}} \times 9\ \text{seconds} = 1.9\ \text{miles}$$

This means that Sasquatch is 1.9 miles closer to Jeff than it is to Carl. In other words, Sasquatch must lie on a path where

$$(\text{the distance to Carl}) - (\text{the distance to Jeff}) = 1.9$$

This is exactly the situation in the definition of a hyperbola, Definition 7.6. In this case, Jeff and Carl are located at the foci,[3] and our fixed distance d is 1.9. For simplicity, we assume the hyperbola is centered at $(0,0)$ with its foci at $(-5,0)$ and $(5,0)$. Schematically, we have



We are seeking a curve of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in which the distance from the center to each focus is $c = 5$. As we saw in the derivation of the standard equation of the hyperbola, Equation 7.6, $d = 2a$, so that $2a = 1.9$, or $a = 0.95$ and $a^2 = 0.9025$. All that remains is to find b^2. To that end, we recall that $a^2 + b^2 = c^2$ so $b^2 = c^2 - a^2 = 25 - 0.9025 = 24.0975$. Since Sasquatch is closer to Jeff than it is to Carl, it must be on the western (left hand) branch of $\frac{x^2}{0.9025} - \frac{y^2}{24.0975} = 1$.

In our previous example, we did not have enough information to pin down the exact location of Sasquatch. To accomplish this, we would need a third observer.

Example 7.5.5

By a stroke of luck, Kai was also camping in the woods during the events of the previous example. He was located 6 miles due north of Jeff and heard the Sasquatch call 18 seconds after Jeff did. Use this added information to locate Sasquatch.

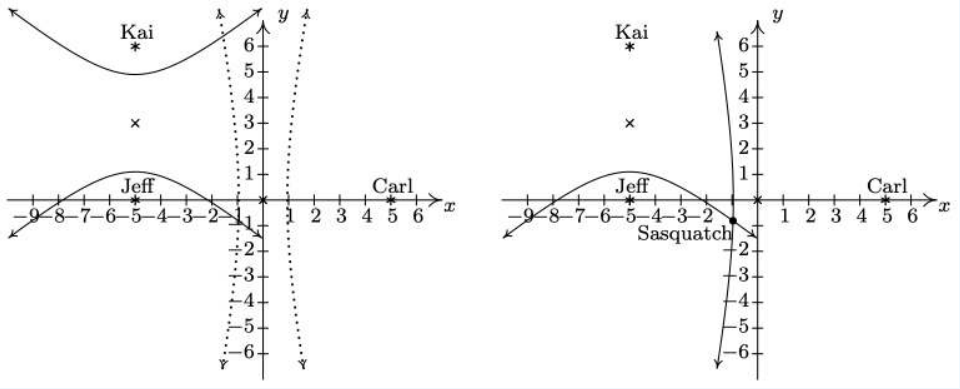
Solution

Kai and Jeff are now the foci of a second hyperbola where the fixed distance d can be determined as before

$$760\ \frac{\text{miles}}{\text{hour}} \times \frac{1\ \text{hour}}{3600\ \text{seconds}} \times 18\ \text{seconds} = 3.8\ \text{miles}$$

Since Jeff was positioned at $(-5,0)$, we place Kai at $(-5,6)$. This puts the center of the new hyperbola at $(-5,3)$. Plotting Kai's position and the new center gives us the diagram below on the left. The second hyperbola is vertical, so it must be of the form $\frac{(y-3)^2}{b^2} - \frac{(x+5)^2}{a^2} = 1$. As before, the distance d is the length of the major axis, which in this case is $2b$. We get $2b = 3.8$

so that $b = 1.9$ and $b^2 = 3.61$. With Kai 6 miles due North of Jeff, we have that the distance from the center to the focus is $c = 3$. Since $a^2 + b^2 = c^2$, we get $a^2 = c^2 - b^2 = 9 - 3.61 = 5.39$. Kai heard the Sasquatch call after Jeff, so Kai is farther from Sasquatch than Jeff. Thus Sasquatch must lie on the southern branch of the hyperbola $\frac{(y-3)^2}{3.61} - \frac{(x+5)^2}{5.39} = 1$. Looking at the western branch of the hyperbola determined by Jeff and Carl along with the southern branch of the hyperbola determined by Kai and Jeff, we see that there is exactly one point in common, and this is where Sasquatch must have been when it called.



To determine the coordinates of this point of intersection exactly, we would need techniques for solving systems of non-linear equations (which we won't see until Section 8.7), so we use the calculator[4] Doing so, we get Sasquatch is approximately at $(-0.9629, -0.8113)$

Each of the conic sections we have studied in this chapter result from graphing equations of the form $Ax^2 + Cy^2 + Dx + Ey + F = 0$ for different choices of A, C, D, E, and[5] F. While we've seen examples[6] demonstrate *how* to convert an equation from this general form to one of the standard forms, we close this chapter with some advice about *which* standard form to choose.[7]

Strategies for Identifying Conic Sections

Suppose the graph of equation $Ax^2 + Cy^2 + Dx + Ey + F = 0$ is a non-degenerate conic section.[a]

- If just *one* variable is squared, the graph is a parabola. Put the equation in the form of Equation 7.2 (if x is squared) or Equation 7.3 (if y is squared).

If *both* variables are squared, look at the coefficients of x^2 and y^2, A and B.

- If $A = B$, the graph is a circle. Put the equation in the form of Equation 7.1.
- If $A \neq B$ but A and B have the *same sign*, the graph is an ellipse. Put the equation in the form of Equation 7.4.
- If A and B have the *different signs*, the graph is a hyperbola. Put the equation in the form of either Equation 7.6 or Equation 7.7.

[a] That is, a parabola, circle, ellipse, or hyperbola – see Section 7.1.

7.5.1. Exercises

In Exercises 1 - 8, graph the hyperbola. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

1. $\frac{x^2}{16} - \frac{y^2}{9} = 1$
2. $\frac{y^2}{9} - \frac{x^2}{16} = 1$
3. $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$
4. $\frac{(y-3)^2}{11} - \frac{(x-1)^2}{10} = 1$
5. $\frac{(x+4)^2}{16} - \frac{(y-4)^2}{1} = 1$

6. $\frac{(x+1)^2}{9}-\frac{(y-3)^2}{4}=1$
7. $\frac{(y+2)^2}{16}-\frac{(x-5)^2}{20}=1$
8. $\frac{(x-4)^2}{8}-\frac{(y-2)^2}{18}=1$

In Exercises 9 - 12, put the equation in standard form. Find the center, the lines which contain the transverse and conjugate axes, the vertices, the foci and the equations of the asymptotes.

9. $12x^2-3y^2+30y-111=0$
10. $18y^2-5x^2+72y+30x-63=0$
11. $9x^2-25y^2-54x-50y-169=0$
12. $-6x^2+5y^2-24x+40y+26=0$

In Exercises 13 - 18, find the standard form of the equation of the hyperbola which has the given properties.

13. Center $(3,7)$, Vertex $(3,3)$, Focus $(3,2)$ [buildhypfirst]
14. Vertex $(0,1)$, Vertex $(8,1)$, Focus $(-3,1)$
15. Foci $(0,\pm 8)$, Vertices $(0,\pm 5)$.
16. Foci $(\pm 5,0)$, length of the Conjugate Axis 6
17. Vertices $(3,2)$, $(13,2)$; Endpoints of the Conjugate Axis $(8,4)$, $(8,0)$
18. Vertex $(-10,5)$, Asymptotes $y=\pm\frac{1}{2}(x-6)+5$ [buildhyplast]

In Exercises 19 - 28, find the standard form of the equation using the guidelines on page and then graph the conic section.

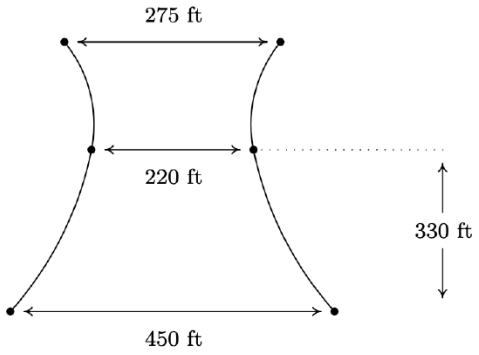
19. $x^2-2x-4y-11=0$
20. $x^2+y^2-8x+4y+11=0$
21. $9x^2+4y^2-36x+24y+36=0$
22. $9x^2-4y^2-36x-24y-36=0$
23. $y^2+8y-4x+16=0$
24. $4x^2+y^2-8x+4=0$
25. $4x^2+9y^2-8x+54y+49=0$
26. $x^2+y^2-6x+4y+14=0$
27. $2x^2+4y^2+12x-8y+25=0$
28. $4x^2-5y^2-40x-20y+160=0$
29. The graph of a vertical or horizontal hyperbola clearly fails the Vertical Line Test, Theorem 1.1, so the equation of a vertical of horizontal hyperbola does not define y as a function of x.[8] However, much like with circles, horizontal parabolas and ellipses, we can split a hyperbola into pieces, each of which would indeed represent y as a function of x. With the help of your classmates, use your calculator to graph the hyperbolas given in Exercises 1 - 8 above. How many pieces do you need for a vertical hyperbola? How many for a horizontal hyperbola?
30. The location of an earthquake's epicenter − the point on the surface of the Earth directly above where the earthquake actually occurred − can be determined by a process similar to how we located Sasquatch in Example 7.5.5. (As we said back in Exercise 75 in Section 6.1, earthquakes are complicated events and it is not our intent to provide a complete discussion of the science involved in them. Instead, we refer the interested reader to a course in Geology or the U.S. Geological Survey's Earthquake Hazards Program found here.) Our technique works only for relatively small distances because we need to assume that the Earth is flat in order to use hyperbolas in the plane.[9] The P-waves ("P" stands for Primary) of an earthquake in Sasquatchia travel at 6 kilometers per second.[10] Station A records the waves first. Then Station B, which is 100 kilometers due north of Station A, records the waves 2 seconds later. Station C, which is 150 kilometers due west of Station A records the waves 3 seconds after that (a total of 5 seconds after Station A). Where is the epicenter?
31. The notion of eccentricity introduced for ellipses in Definition 7.5 in Section 7.4 is the same for hyperbolas in that we can define the eccentricity e of a hyperbola as

$$e=\frac{\text{distance from the center to a focus}}{\text{distance from the center to a vertex}}$$

    a. With the help of your classmates, explain why $e>1$ for any hyperbola.
    b. Find the equation of the hyperbola with vertices $(\pm 3,0)$ and eccentricity $e=2$.

c. With the help of your classmates, find the eccentricity of each of the hyperbolas in Exercises 1 - 8. What role does eccentricity play in the shape of the graphs?

32. On page 510 in Section 7.3, we discussed paraboloids of revolution when studying the design of satellite dishes and parabolic mirrors. In much the same way, 'natural draft' cooling towers are often shaped as **hyperboloids of revolution**. Each vertical cross section of these towers is a hyperbola. Suppose the a natural draft cooling tower has the cross section below. Suppose the tower is 450 feet wide at the base, 275 feet wide at the top, and 220 feet at its narrowest point (which occurs 330 feet above the ground.) Determine the height of the tower to the nearest foot.



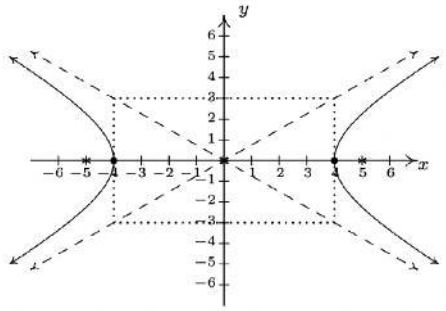
33. With the help of your classmates, research the Cassegrain Telescope. It uses the reflective property of the hyperbola as well as that of the parabola to make an ingenious telescope.
34. [conicsclassificationnoxytermex] With the help of your classmates show that if $Ax^2+Cy^2+Dx+Ey+F=0$ determines a non-degenerate conic[11] then
    - $AC<0$ means that the graph is a hyperbola
    - $AC=0$ means that the graph is a parabola
    - $AC>0$ means that the graph is an ellipse or circle

    NOTE: This result will be generalized in Theorem 11.11 in Section 11.6.1.

7.5.2. Answers

1. $\frac{x^2}{16}-\frac{y^2}{9}=1$

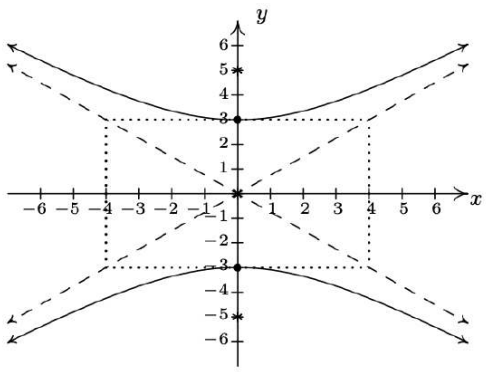
   Center $(0,0)$
   Transverse axis on $y=0$
   Conjugate axis on $x=0$
   Vertices $(4,0),(-4,0)$
   Foci $(5,0),(-5,0)$
   Asymptotes $y=\pm\frac{3}{4}x$



2. $\frac{y^2}{9}-\frac{x^2}{16}=1$

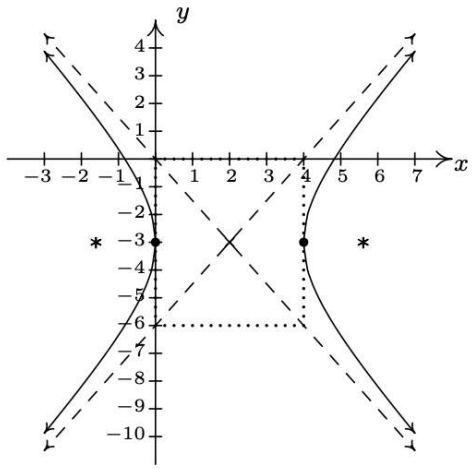
   Center $(0,0)$
   Transverse axis on $x=0$
   Conjugate axis on $y=0$

Vertices $(0,3),(0,-3)$
Foci $(0,5),(0,-5)$
Asymptotes $y=\pm\frac{3}{4}x$



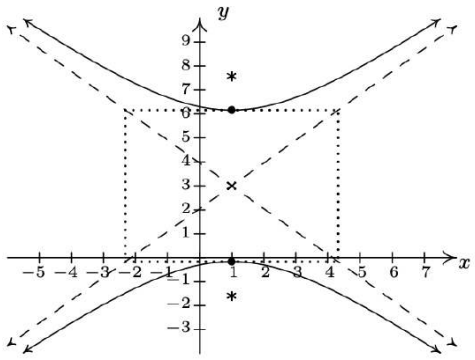
3. $\dfrac{(x-2)^2}{4}-\dfrac{(y+3)^2}{9}=1$

Center $(2,-3)$
Transverse axis on $y=-3$
Conjugate axis on $x=2$
Vertices $(0,-3),(4,-3)$
Foci $(2+\sqrt{13},-3),(2-\sqrt{13},-3)$
Asymptotes $y=\pm\frac{3}{2}(x-2)-3$



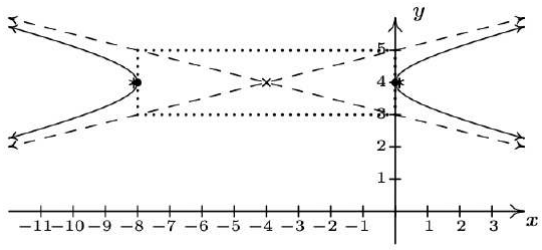
4. $\dfrac{(y-3)^2}{11}-\dfrac{(x-1)^2}{10}=1$

Center $(1,3)$
Transverse axis on $x=1$
Conjugate axis on $y=3$
Vertices $(1,3+\sqrt{11}),(1,3-\sqrt{11})$
Foci $(1,3+\sqrt{21}),(1,3-\sqrt{21})$
Asymptotes $y=\pm\frac{\sqrt{110}}{10}(x-1)+3$



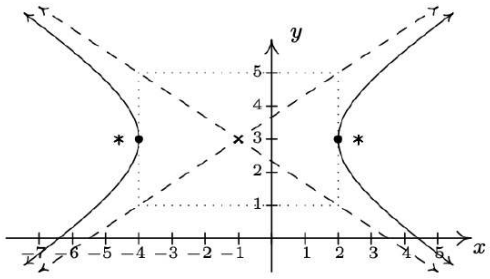
5. $\frac{(x+4)^2}{16}-\frac{(y-4)^2}{1}=1$

Center $(-4,4)$
Transverse axis on $y=4$
Conjugate axis on $x=-4$
Vertices $(-8,4),(0,4)$
Foci $(-4+\sqrt{17},4),(-4-\sqrt{17},4)$
Asymptotes $y=\pm\frac{1}{4}(x+4)+4$



6. $\frac{(x+1)^2}{9}-\frac{(y-3)^2}{4}=1$

Center $(-1,3)$
Transverse axis on $y=3$
Conjugate axis on $x=-1$
Vertices $(2,3),(-4,3)$
Foci $\left(-1+\sqrt{13},3\right),\left(-1-\sqrt{13},3\right)$
Asymptotes $y=\pm\frac{2}{3}(x+1)+3$

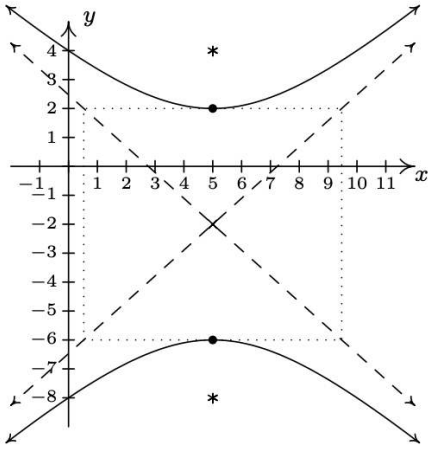


7. $\frac{(y+2)^2}{16}-\frac{(x-5)^2}{20}=1$

Center $(5,-2)$
Transverse axis on $x=5$
Conjugate axis on $y=-2$
Vertices $(5,2),(5,-6)$

Foci $(5,4),(5,-8)$

Asymptotes $y=\pm\frac{2\sqrt{5}}{5}(x-5)-2$



8. $\frac{(x-4)^2}{8}-\frac{(y-2)^2}{18}=1$

Center $(4,2)$

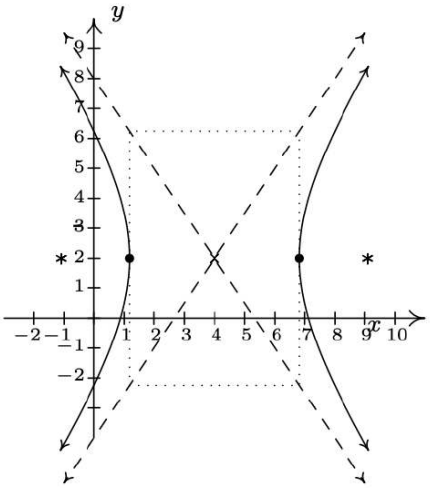
Transverse axis on $y=2$

Conjugate axis on $x=4$

Vertices $\left(4+2\sqrt{2},2\right),\left(4-2\sqrt{2},2\right)$

Foci $\left(4+\sqrt{26},2\right),\left(4-\sqrt{26},2\right)$

Asymptotes $y=\pm\frac{3}{2}(x-4)+2$



9. $\frac{x^2}{3}-\frac{(y-5)^2}{12}=1$

Center $(0,5)$

Transverse axis on $y=5$

Conjugate axis on $x=0$

Vertices $(\sqrt{3},5),(-\sqrt{3},5)$

Foci $(\sqrt{15},5),(-\sqrt{15},5)$

Asymptotes $y=\pm 2x+5$

10. $\frac{(y+2)^2}{5}-\frac{(x-3)^2}{18}=1$

Center $(3,-2)$
Transverse axis on $x=3$
Conjugate axis on $y=-2$
Vertices $(3,-2+\sqrt{5}),(3,-2-\sqrt{5})$
Foci $(3,-2+\sqrt{23}),(3,-2-\sqrt{23})$
Asymptotes $y=\pm\frac{\sqrt{10}}{6}(x-3)-2$

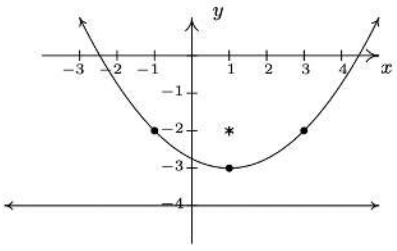
11. $\frac{(x-3)^2}{25}-\frac{(y+1)^2}{9}=1$

Center $(3,-1)$
Transverse axis on $y=-1$
Conjugate axis on $x=3$
Vertices $(8,-1),(-2,-1)$
Foci $\left(3+\sqrt{34},-1\right),\left(3-\sqrt{34},-1\right)$
Asymptotes $y=\pm\frac{3}{5}(x-3)-1$

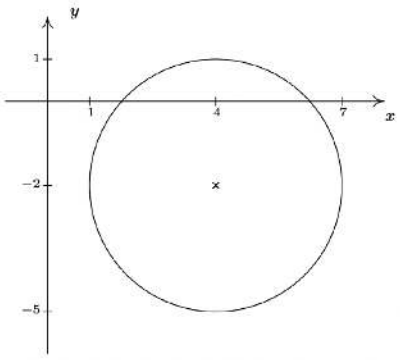
12. $\frac{(y+4)^2}{6}-\frac{(x+2)^2}{5}=1$

Center $(-2,-4)$
Transverse axis on $x=-2$
Conjugate axis on $y=-4$
Vertices $\left(-2,-4+\sqrt{6}\right),\left(-2,-4-\sqrt{6}\right)$
Foci $\left(-2,-4+\sqrt{11}\right),\left(-2,-4-\sqrt{11}\right)$
Asymptotes $y=\pm\frac{\sqrt{30}}{5}(x+2)-4$

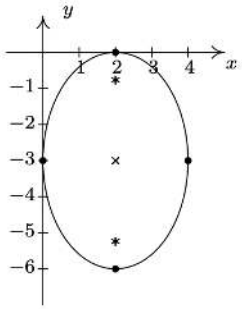
13. $\frac{(y-7)^2}{16}-\frac{(x-3)^2}{9}=1$
14. $\frac{(x-4)^2}{16}-\frac{(y-1)^2}{33}=1$
15. $\frac{y^2}{25}-\frac{x^2}{39}=1$
16. $\frac{x^2}{16}-\frac{y^2}{9}=1$
17. $\frac{(x-8)^2}{25}-\frac{(y-2)^2}{4}=1$
18. $\frac{(x-6)^2}{256}-\frac{(y-5)^2}{64}=1$
19. $(x-1)^2=4(y+3)$



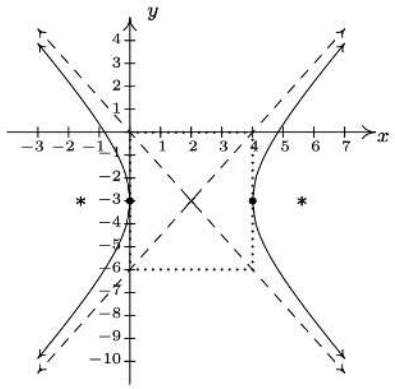
20. $(x-4)^2+(y+2)^2=9$



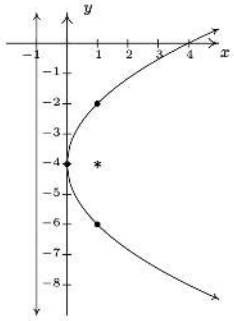
21. $\frac{(x-2)^2}{4}+\frac{(y+3)^2}{9}=1$



22. $\frac{(x-2)^2}{4}-\frac{(y+3)^2}{9}=1$



23. $(y+4)^2=4x$

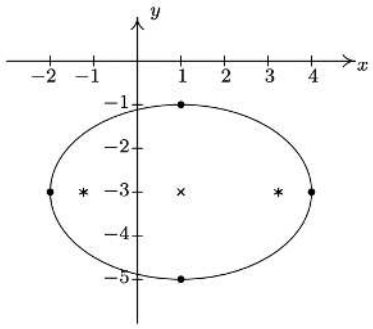


24. $\frac{(x-1)^2}{1}+\frac{y^2}{4}=0$

The graph is the point $(1,0)$ only.

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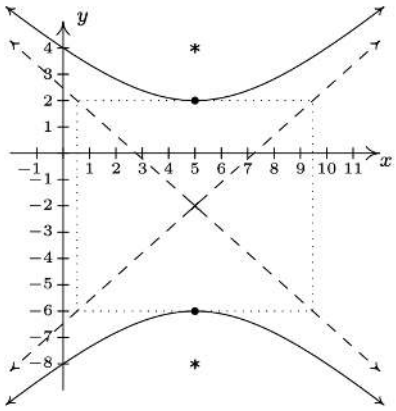
25. $\dfrac{(x-1)^2}{9}+\dfrac{(y+3)^2}{4}=1$



26. $(x-3)^2+(y+2)^2=-1$
There is no graph.

27. $\dfrac{(x+3)^2}{2}+\dfrac{(y-1)^2}{1}=-\dfrac{3}{4}$
There is no graph.

28. $\dfrac{(y+2)^2}{16}-\dfrac{(x-5)^2}{20}=1$



29. By placing Station A at $(0,-50)$ and Station B at $(0,50)$, the two second time difference yields the hyperbola $\frac{y^2}{36}-\frac{x^2}{2464}=1$ with foci A and B and center $(0,0)$. Placing Station C at $(-150,-50)$ and using foci A and C gives us a center of $(-75,-50)$ and the hyperbola $\frac{(x+75)^2}{225}-\frac{(y+50)^2}{5400}=1$. The point of intersection of these two hyperbolas which is closer to A than B and closer to A than C is $(-57.8444,-9.21336)$ so that is the epicenter.

30. $\dfrac{x^2}{9}-\dfrac{y^2}{27}=1$.

31. The tower may be modeled (approximately)[12] by $\frac{x^2}{12100}-\frac{(y-330)^2}{34203}=1$. To find the height, we plug in $x=137.5$ which yields $y\approx 191$ or $y\approx 469$. Since the top of the tower is above the narrowest point, we get the tower is approximately 469 feet tall.

Reference

[1] It is a good exercise to actually work this out.

[2] GPS now rules the positioning kingdom. Is there still a place for LORAN and other land-based systems? Do satellites ever malfunction?

[3] We usually like to be the center of attention, but being the focus of attention works equally well.

[4] First solve each hyperbola for y, and choose the correct equation (branch) before proceeding.

[5] See Section 11.6 to see why we skip B.

[6] Examples 7.2.3, 7.3.4, 7.4.3, and 7.5.3, in particular.

[7] We formalize this in Exercise 34.

[8] We will see later in the text that the graphs of certain rotated hyperbolas pass the Vertical Line Test.

[9] Back in the Exercises in Section 1.1 you were asked to research people who believe the world is flat. What did you discover?

[10] Depending on the composition of the crust at a specific location, P-waves can travel between 5 kps and 8 kps.

[11] Recall that this means its graph is either a circle, parabola, ellipse or hyperbola.

[12] The exact value underneath $(y-330)^2$ is $\frac{52707600}{1541}$ in case you need more precision.

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CHAPTER OVERVIEW

7: Systems

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions.

7.1: Prelude to Systems of Equations and Inequalities
7.2: Systems of Linear Equations - Two Variables
7.3: Systems of Linear Equations - Three Variables
7.4: Systems of Nonlinear Equations and Inequalities - Two Variables
7.5: Partial Fractions
7.6: Matrices and Matrix Operations
7.7: Solving Systems with Gaussian Elimination
7.8: Solving Systems with Inverses
7.9: Solving Systems with Cramer's Rule
7.E: Systems of Equations and Inequalities (Exercises)
7.R: Systems of Equations and Inequalities (Review)

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7.1: Prelude to Systems of Equations and Inequalities



Figure 7.1.1: Enigma machines like this one, once owned by Italian dictator Benito Mussolini, were used by government and military officials for enciphering and deciphering top-secret communications during World War II. (credit: Dave Addey, Flickr)

By 1943, it was obvious to the Nazi regime that defeat was imminent unless it could build a weapon with unlimited destructive power, one that had never been seen before in the history of the world. In September, Adolf Hitler ordered German scientists to begin building an atomic bomb. Rumors and whispers began to spread from across the ocean. Refugees and diplomats told of the experiments happening in Norway. However, Franklin D. Roosevelt wasn't sold, and even doubted British Prime Minister Winston Churchill's warning. Roosevelt wanted undeniable proof. Fortunately, he soon received the proof he wanted when a group of mathematicians cracked the "Enigma" code, proving beyond a doubt that Hitler was building an atomic bomb. The next day, Roosevelt gave the order that the United States begin work on the same.

The Enigma is perhaps the most famous cryptographic device ever known. It stands as an example of the pivotal role cryptography has played in society. Now, technology has moved cryptanalysis to the digital world.

Many ciphers are designed using invertible matrices as the method of message transference, as finding the inverse of a matrix is generally part of the process of decoding. In addition to knowing the matrix and its inverse, the receiver must also know the key that, when used with the matrix inverse, will allow the message to be read.

In this chapter, we will investigate matrices and their inverses, and various ways to use matrices to solve systems of equations. First, however, we will study systems of equations on their own: linear and nonlinear, and then partial fractions. We will not be breaking any secret codes here, but we will lay the foundation for future courses.

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7.2: Systems of Linear Equations - Two Variables

Learning Objectives

- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables.

A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section, we will consider linear equations with two variables to answer these and similar questions.



Figure 7.2.1

Introduction to Systems of Equations

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. **A system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.

$$\begin{aligned} 2x+y &= 15 \\ 3x-y &= 5 \end{aligned}$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair $(4,7)$ is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.

$$\begin{aligned} 2(4)+(7) &= 15 \text{ True} \\ 3(4)-(7) &= 5 \text{ True} \end{aligned}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A consistent system of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same y-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an **inconsistent system**, which is one in which the equations represent two parallel lines. The lines have the same slope and different y-intercepts. There are no points common to both lines; hence, there is no solution to the system.

TYPES OF LINEAR SYSTEMS

There are three types of systems of linear equations in two variables, and three types of solutions.

- An independent system has exactly one solution pair (x, y). The point where the two lines intersect is the only solution.
- An inconsistent system has no solution. Notice that the two lines are parallel and will never intersect.
- A dependent system has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

Figure $7.2.2$ compares graphical representations of each type of system.

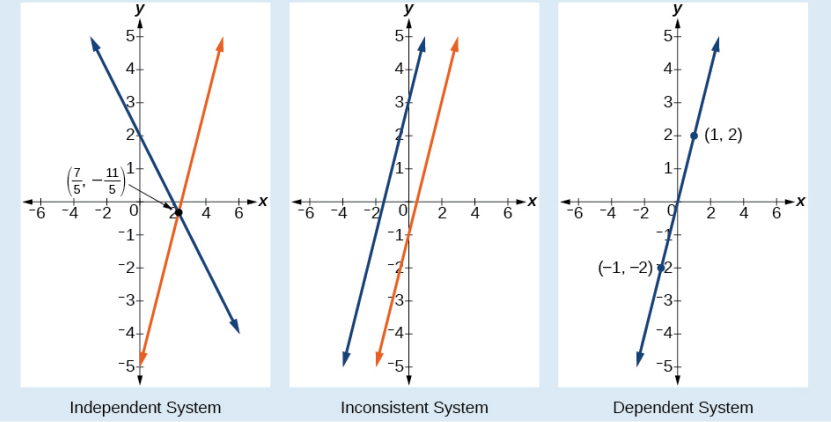


Figure 7.2.2

Given a system of linear equations and an ordered pair, determine whether the ordered pair is a solution

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Example 7.2.1: Determining Whether an Ordered Pair Is a Solution to a System of Equations

Determine whether the ordered pair $(5, 1)$ is a solution to the given system of equations.

$$\begin{aligned} x + 3y &= 8 \\ 2x - 9 &= y \end{aligned}$$

Solution

Substitute the ordered pair $(5, 1)$ into both equations.

$$\begin{aligned} (5) + 3(1) &= 8 \\ 8 &= 8 \text{ True} \\ 2(5) - 9 &= (1) \\ 1 &= 1 \text{ True} \end{aligned}$$

The ordered pair $(5, 1)$ satisfies both equations, so it is the solution to the system.

Analysis

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines. See Figure $7.2.3$.

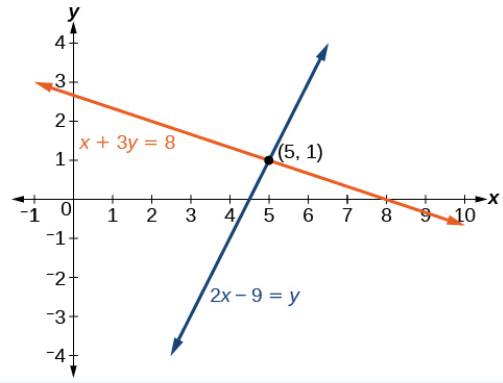


Figure 7.2.3

? Exercise 7.2.1

Determine whether the ordered pair $(8, 5)$ is a solution to the following system.

$$\begin{aligned} 5x - 4y &= 20 \\ 2x + 1 &= 3y \end{aligned}$$

Answer

Not a solution.

Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

✓ Example 7.2.2: Solving a System of Equations in Two Variables by Graphing

Solve the following system of equations by graphing. Identify the type of system.

$$\begin{aligned} 2x + y &= -8 \\ x - y &= -1 \end{aligned}$$

Solution

Solve the first equation for y.

$$\begin{aligned} 2x + y &= -8 \\ y &= -2x - 8 \end{aligned}$$

Solve the second equation for y.

$$\begin{aligned} x - y &= -1 \\ y &= x + 1 \end{aligned}$$

Graph both equations on the same set of axes as in Figure 7.2.4.

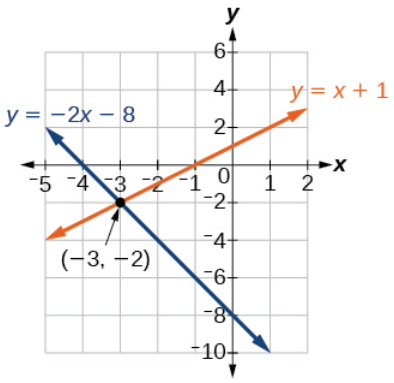


Figure 7.2.4

The lines appear to intersect at the point $(-3,-2)$. We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.

$$\begin{aligned} 2(-3)+(-2) &= -8 \\ -8 &= -8 \text{ True} \\ (-3)-(-2) &= -1 \\ -1 &= -1 \text{ True} \end{aligned}$$

The solution to the system is the ordered pair $(-3,-2)$,so the system is independent.

Exercise 7.2.2

Solve the following system of equations by graphing.

$$\begin{aligned} 2x-5y &= -25 \\ -4x+5y &= 35 \end{aligned}$$

Answer

The solution to the system is the ordered pair $(-5,3)$.

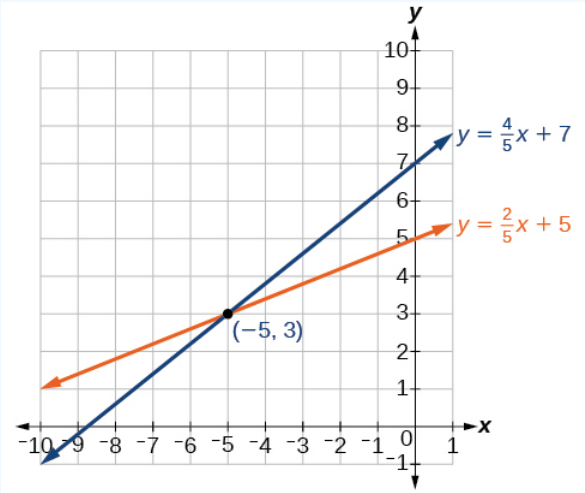


Figure 7.2.5

Q&A

Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent

system.

Solving Systems of Equations by Substitution

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a **system of linear equations** that are more precise than graphing. One such method is solving a system of equations by the **substitution method**, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

How to: Given a system of two equations in two variables, solve using the substitution method.

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

Example 7.2.3: Solving a System of Equations in Two Variables by Substitution

Solve the following system of equations by substitution.

$$\begin{aligned} -x+y &= -5 \\ 2x-5y &= 1 \end{aligned}$$

Solution

First, we will solve the first equation for y.

$$\begin{aligned} -x+y &= -5 \\ y &= x-5 \end{aligned}$$

Now we can substitute the expression $x-5$ for y in the second equation.

$$\begin{aligned} 2x-5y &= 1 \\ 2x-5(x-5) &= 1 \\ 2x-5x+25 &= 1 \\ -3x &= -24 \\ x &= 8 \end{aligned}$$

Now, we substitute $x=8$ into the first equation and solve for y.

$$\begin{aligned} -(8)+y &= -5 \\ y &= 3 \end{aligned}$$

Our solution is $(8,3)$.

Check the solution by substituting $(8,3)$ into both equations.

$$\begin{aligned} -x+y &= -5 \\ -(8)+(3) &= -5 \text{ True} \\ 2x-5y &= 1 \\ 2(8)-5(3) &= 1 \text{ True} \end{aligned}$$

Exercise 7.2.3

Solve the following system of equations by substitution.

$$x = y + 3$$
$$4 = 3x - 2y$$

Answer

$(-2, -5)$

Q&A

Can the substitution method be used to solve any linear system in two variables?

Yes, but the method works best if one of the equations contains a coefficient of 1 *or* -1 *so that we do not have to deal with fractions.*

Solving Systems of Equations in Two Variables by the Addition Method

A third method of solving systems of linear equations is the addition method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

How to: Given a system of equations, solve using the addition method.

1. Write both equations with *x*- and *y*-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

Example 7.2.4: Solving a System by the Addition Method

Solve the given system of equations by addition.

$$x + 2y = -1$$
$$-x + y = 3$$

Solution

Both equations are already set equal to a constant. Notice that the coefficient of x in the second equation, -1, is the opposite of the coefficient of x in the first equation, 1. We can add the two equations to eliminate x without needing to multiply by a constant.

$$\begin{aligned} x + 2y &= -1 \\ \underline{-x + y} &= \underline{3} \\ 3y &= 2 \end{aligned}$$

Now that we have eliminated x, we can solve the resulting equation for y.

$$3y = 2$$
$$y = \frac{2}{3}$$

Then, we substitute this value for y into one of the original equations and solve for x.

$$\begin{aligned} -x+y &= 3 \\ -x+\frac{2}{3} &= 3 \\ -x &= 3-\frac{2}{3} \\ -x &= \frac{7}{3} \\ x &= -\frac{7}{3} \end{aligned}$$

The solution to this system is $\left(-\frac{7}{3}, \frac{2}{3}\right)$.

Check the solution in the first equation.

$$\begin{aligned} x+2y &= -1 \\ \left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right) &= \\ -\frac{7}{3}+\frac{4}{3} &= -\frac{3}{3} \\ -1 &= -1 \qquad \text{True} \end{aligned}$$

Analysis

We gain an important perspective on systems of equations by looking at the graphical representation. See Figure 7.2.6 to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.

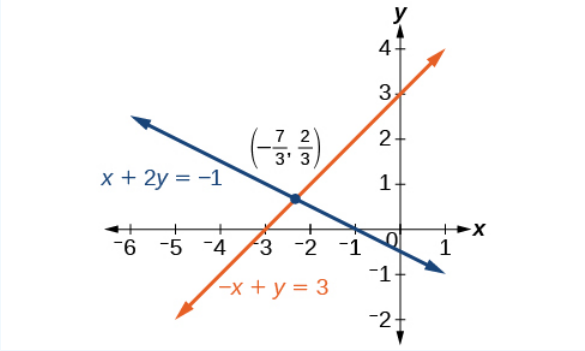


Figure 7.2.6

Example 7.2.5: Using the Addition Method When Multiplication of One Equation Is Required

Solve the given system of equations by the addition method.

$$\begin{aligned} 3x+5y &= -11 \\ x-2y &= 11 \end{aligned}$$

Solution

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has $3x$ in it and the second equation has x. So if we multiply the second equation by -3,the x-terms will add to zero.

$$\begin{aligned} x-2y &= 11 \\ -3(x-2y) &= -3(11) \qquad \text{Multiply both sides by } -3. \\ -3x+6y &= -33 \qquad \text{Use the distributive property.} \end{aligned}$$

Now, let's add them.

$$\begin{aligned} 3x+5y &= -11 \\ \underline{-3x+6y} &= \underline{-33} \\ 11y &= -44 \\ y &= -4 \end{aligned}$$

For the last step, we substitute $y=-4$ into one of the original equations and solve for x.

$$\begin{aligned} 3x+5y &= -11 \\ 3x+5(-4) &= -11 \\ 3x-20 &= -11 \\ 3x &= 9 \\ x &= 3 \end{aligned}$$

Our solution is the ordered pair $(3,-4)$. See Figure 7.2.7. Check the solution in the original second equation.

$$\begin{aligned} x-2y &= 11 \\ (3)-2(-4) &= 3+8 \\ &= 11 \qquad \text{True} \end{aligned}$$

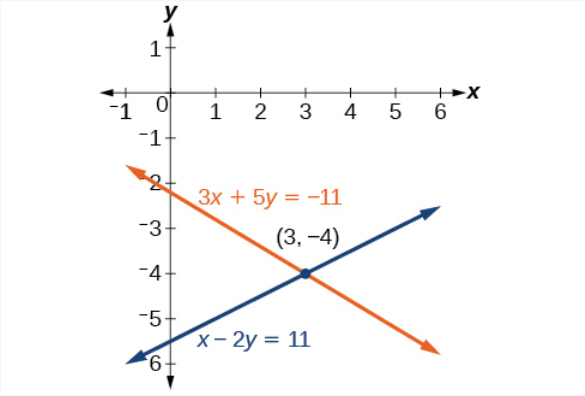


Figure 7.2.7

Exercise 7.2.4

Solve the system of equations by addition.

$$\begin{aligned} 2x-7y &= 2 \\ 3x+y &= -20 \end{aligned}$$

Answer

$(-6,-2)$

Example 7.2.6: Using the Addition Method When Multiplication of Both Equations Is Required

Solve the given system of equations in two variables by addition.

$$\begin{aligned} 2x+3y &= -16 \\ 5x-10y &= 30 \end{aligned}$$

Solution

One equation has $2x$ and the other has $5x$. The least common multiple is $10x$ so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate x by multiplying the first equation by -5 and the second equation by 2.

$$\begin{aligned} -5(2x+3y) &= -5(-16) \\ -10x-15y &= 80 \\ 2(5x-10y) &= 2(30) \\ 10x-20y &= 60 \end{aligned}$$

Then, we add the two equations together.

$$\begin{aligned} -10x-15y &= 80 \\ \underline{10x-20y} &= \underline{60} \\ -35y &= 140 \\ y &= -4 \end{aligned}$$

Substitute $y=-4$ into the original first equation.

$$\begin{aligned} 2x+3(-4) &= -16 \\ 2x-12 &= -16 \\ 2x &= -4 \\ x &= -2 \end{aligned}$$

The solution is $(-2,-4)$. Check it in the other equation.

$$\begin{aligned} 5x-10y &= 30 \\ 5(-2)-10(-4) &= 30 \\ -10+40 &= 30 \\ 30 &= 30 \end{aligned}$$

See Figure 7.2.8.

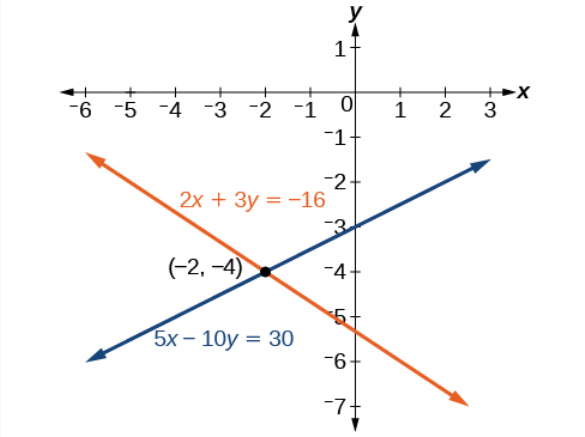


Figure 7.2.8

Example 7.2.7: Using the Addition Method in Systems of Equations Containing Fractions

Solve the given system of equations in two variables by addition.

$$\begin{aligned} \frac{x}{3}+\frac{y}{6} &= 3 \\ \frac{x}{2}-\frac{y}{4} &= 1 \end{aligned}$$

Solution

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.

$$6\left(\frac{x}{3}+\frac{y}{6}\right)=6(3)$$
$$2x+y=18$$
$$4\left(\frac{x}{2}-\frac{y}{4}\right)=4(1)$$
$$2x-y=4$$

Now multiply the second equation by -1 so that we can eliminate the x-variable.

$$-1(2x-y)=-1(4)$$
$$-2x+y=-4$$

Add the two equations to eliminate the x-variable and solve the resulting equation.

$$2x+y=18$$
$$-2x+y=-4$$
$$2y=14$$
$$y=7$$

Substitute $y=7$ into the first equation.

$$2x+(7)=18$$
$$2x=11$$
$$x=\frac{11}{2}$$
$$=7.5$$

The solution is $\left(\frac{11}{2},7\right)$. Check it in the other equation.

$$\frac{x}{2}-\frac{y}{4}=1$$
$$\frac{\frac{11}{2}}{2}-\frac{7}{4}=1$$
$$\frac{11}{4}-\frac{7}{4}=1$$
$$\frac{4}{4}=1$$

? Exercise 7.2.5

Solve the system of equations by addition.

$$2x+3y=8$$
$$3x+5y=10$$

Answer

$(10,-4)$

Identifying Inconsistent Systems of Equations Containing Two Variables

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different y-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as $12=0$.

Example 7.2.8: Solving an Inconsistent System of Equations

Solve the following system of equations.

$$\begin{aligned} x &= 9-2y \\ x+2y &= 13 \end{aligned}$$

Solution

We can approach this problem in two ways. Because one equation is already solved for x, the most obvious step is to use substitution.

$$\begin{aligned} x+2y &= 13 \\ (9-2y)+2y &= 13 \\ 9+0y &= 13 \\ 9 &= 13 \end{aligned}$$

Clearly, this statement is a contradiction because $9 \neq 13$. Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$\begin{aligned} x &= 9-2y \\ 2y &= -x+9 \\ y &= -\frac{1}{2}x+\frac{9}{2} \end{aligned}$$

We then convert the second equation expressed to slope-intercept form.

$$\begin{aligned} x+2y &= 13 \\ 2y &= -x+13 \\ y &= -\frac{1}{2}x+\frac{13}{2} \end{aligned}$$

Comparing the equations, we see that they have the same slope but different y-intercepts. Therefore, the lines are parallel and do not intersect.

$$\begin{aligned} y &= -\frac{1}{2}x+\frac{9}{2} \\ y &= -\frac{1}{2}x+\frac{13}{2} \end{aligned}$$

Analysis

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown in Figure 7.2.9.

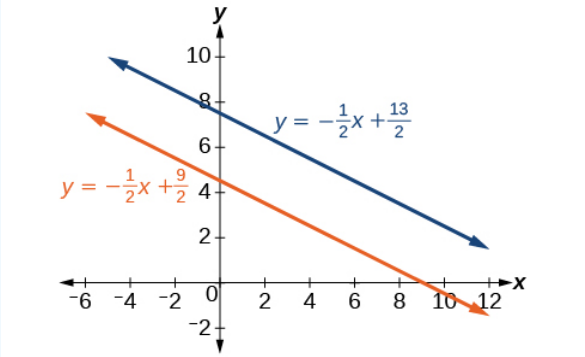


Figure 7.2.9

? Exercise 7.2.6

Solve the following system of equations in two variables.

$$\begin{aligned} 2y-2x &= 2 \\ 2y-2x &= 6 \end{aligned}$$

Answer

No solution. It is an inconsistent system.

Expressing the Solution of a System of Dependent Equations Containing Two Variables

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as $0=0$.

✓ Example 7.2.9: Finding a Solution to a Dependent System of Linear Equations

Find a solution to the system of equations using the **addition method**.

$$\begin{aligned} x+3y &= 2 \\ 3x+9y &= 6 \end{aligned}$$

Solution

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating x. If we multiply both sides of the first equation by -3, then we will be able to eliminate the x-variable.

$$\begin{aligned} x+3y &= 2 \\ (-3)(x+3y) &= (-3)(2) \\ -3x-9y &= -6 \end{aligned}$$

Now add the equations.

$$\begin{aligned} -3x-9y &= -6 \\ \underline{+\,3x+9y} &= \underline{6} \\ 0 &= 0 \end{aligned}$$

We can see that there will be an infinite number of solutions that satisfy both equations.

Analysis

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.

$$\begin{aligned} x+3y &= 2 \\ 3y &= -x+2 \\ y &= -\frac{1}{3}x+\frac{2}{3} \\ 3x+9y &= 6 \\ 9y &= -3x+6 \\ y &= -\frac{3}{9}x+\frac{6}{9} \\ y &= -\frac{1}{3}x+\frac{2}{3} \end{aligned}$$

See Figure 7.2.10. Notice the results are the same. The general solution to the system is $\left(x, -\frac{1}{3}x+\frac{2}{3}\right)$.

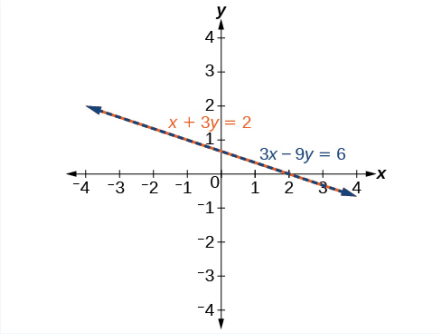


Figure 7.2.10

Exercise 7.2.7

Solve the following system of equations in two variables.

$$\begin{aligned} y-2x &= 5 \\ -3y+6x &= -15 \end{aligned}$$

Answer

The system is dependent so there are infinite solutions of the form $(x, 2x+5)$.

Using Systems of Equations to Investigate Profits

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's revenue function is the function used to calculate the amount of money that comes into the business. It can be represented by the equation $R = xp$, where x=quantity and p=price. The revenue function is shown in orange in Figure 7.2.11.

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in Figure 7.2.11. The x-axis represents quantity in hundreds of units. The y-axis represents either cost or revenue in hundreds of dollars.

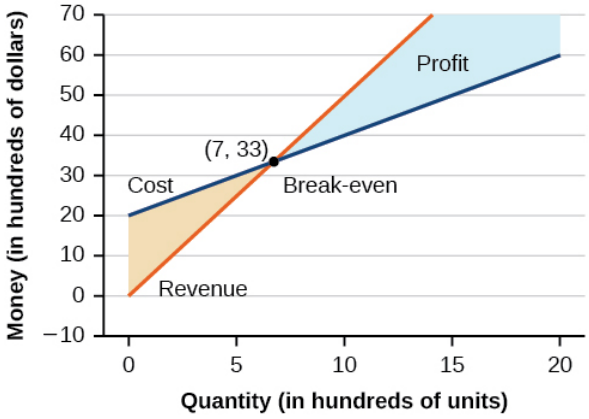


Figure 7.2.11

The point at which the two lines intersect is called the break-even point. We can see from the graph that if 700 units are produced, the cost is $\$3,300$ and the revenue is also $\$3,300$. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss. The profit function is the revenue function minus the cost function, written as $P(x) = R(x) - C(x)$. Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

Example 7.2.10: Finding the Break-Even Point and the Profit Function Using Substitution

Given the cost function $C(x) = 0.85x + 35,000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

Solution

Write the system of equations using y to replace function notation.

$$\begin{aligned} y &= 0.85x + 35,000 \\ y &= 1.55x \end{aligned}$$

Substitute the expression $0.85x + 35,000$ from the first equation into the second equation and solve for x.

$$\begin{aligned} 0.85x + 35,000 &= 1.55x \\ 35,000 &= 0.7x \\ 50,000 &= x \end{aligned}$$

Then, we substitute $x = 50,000$ into either the cost function or the revenue function.

$$1.55(50,000) = 77,500$$

The break-even point is $(50,000, 77,500)$.

The profit function is found using the formula $P(x) = R(x) - C(x)$.

$$\begin{aligned} P(x) &= 1.55x - (0.85x + 35,000) \\ &= 0.7x - 35,000 \end{aligned}$$

The profit function is $P(x) = 0.7x - 35,000$.

Analysis

The cost to produce $50,000$ units is $\$77,500$, and the revenue from the sales of $50,000$ units is also $\$77,500$. To make a profit, the business must produce and sell more than $50,000$ units. See Figure 7.2.12.

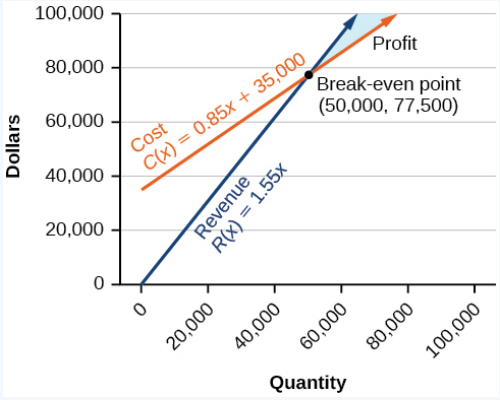


Figure 7.2.12

We see from the graph in Figure 7.2.13 that the profit function has a negative value until $x = 50,000$, when the graph crosses the x-axis. Then, the graph emerges into positive y-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0. The area to the left of the break-even point represents operating at a loss.

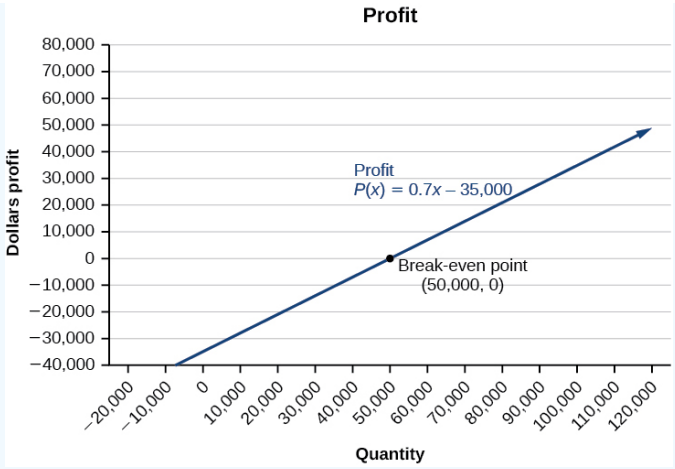


Figure 7.2.13

Example 7.2.11: Writing and Solving a System of Equations in Two Variables

The cost of a ticket to the circus is $\$25.00$ for children and $\$50.00$ for adults. On a certain day, attendance at the circus is $2,000$ and the total gate revenue is $\$70,000$. How many children and how many adults bought tickets?

Solution

Let c = the number of children and a = the number of adults in attendance.

The total number of people is $2,000$. We can use this to write an equation for the number of people at the circus that day.

$$c+a=2,000$$

The revenue from all children can be found by multiplying $\$25.00$ by the number of children, $25c$. The revenue from all adults can be found by multiplying $\$50.00$ by the number of adults, $50a$. The total revenue is $\$70,000$. We can use this to write an equation for the revenue.

$$25c+50a=70,000$$

We now have a system of linear equations in two variables.

$$c+a=2,000$$
$$25c+50a=70,000$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either c or a. We will solve for a.

$$\begin{aligned} c+a &= 2,000 \\ a &= 2,000-c \end{aligned}$$

Substitute the expression $2,000-c$ in the second equation fora a and solve for c.

$$\begin{aligned} 25c+50(2,000-c) &= 70,000 \\ 25c+100,000-50c &= 70,000 \\ -25c &= -30,000 \\ c &= 1,200 \end{aligned}$$

Substitute $c=1,200$ into the first equation to solve for a.

$$\begin{aligned} 1,200+a &= 2,000 \\ a &= 800 \end{aligned}$$

We find that $1,200$ children and 800 adults bought tickets to the circus that day.

Exercise 7.2.8

Meal tickets at the circus cost $\$4.00$ for children and $\$12.00$ for adults. If $1,650$ meal tickets were bought for a total of, $\$14,200$, how many children and how many adults bought meal tickets?

Answer

700 children, 950 adults

Media

Access these online resources for additional instruction and practice with systems of linear equations.

- Solving Systems of Equations Using Substitution
- Solving Systems of Equations Using Elimination
- Applications of Systems of Equations

Key Concepts

- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. See Example $7.2.1$.
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes. See Example $7.2.2$.
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation. See Example $7.2.3$.
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables. See Example $7.2.4$.
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together. See Example $7.2.5$, Example $7.2.6$, and Example $7.2.7$.
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect. See Example $7.2.8$.
- The solution to a system of dependent equations will always be true because both equations describe the same line. See Example $7.2.9$.
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit. See Example $7.2.10$ and Example $7.2.11$.

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7.3: Systems of Linear Equations - Three Variables

Learning Objectives

- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables.

John received an inheritance of $\$12,000$ that he divided into three parts and invested in three ways: in a money-market fund paying 3% annual interest; in municipal bonds paying 4% annual interest; and in mutual funds paying 7% annual interest. John invested $\$4,000$ more in municipal funds than in municipal bonds. He earned $\$670$ in interest the first year. How much did John invest in each type of fund?



Figure 7.3.1: (credit: "Elembis," Wikimedia Commons)

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

Solving Systems of Three Equations in Three Variables

In order to solve systems of equations in three variables, known as three-by-three systems, the primary tool we will be using is called Gaussian elimination, named after the prolific German mathematician Karl Friedrich Gauss. While there is no definitive order in which operations are to be performed, there are specific guidelines as to what type of moves can be made. We may number the equations to keep track of the steps we apply. The goal is to eliminate one variable at a time to achieve upper triangular form, the ideal form for a three-by-three system because it allows for straightforward back-substitution to find a solution (x, y, z), which we call an ordered triple. A system in upper triangular form looks like the following:

$$\begin{aligned} Ax + By + Cz &= D \\ Ey + Fz &= G \\ Hz &= K \end{aligned}$$

The third equation can be solved for z,and then we back-substitute to find y and x. To write the system in upper triangular form, we can perform the following operations:

1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

The **solution set** to a three-by-three system is an ordered triple (x, y, z). Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

NUMBER OF POSSIBLE SolutionS

Figure 7.3.2 and Figure 7.3.3 illustrate possible solution scenarios for three-by-three systems.

- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple (x, y, z). Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as $0 = 0$. Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as $3 = 0$. Graphically, a system with no solution is represented by three planes with no point in common.

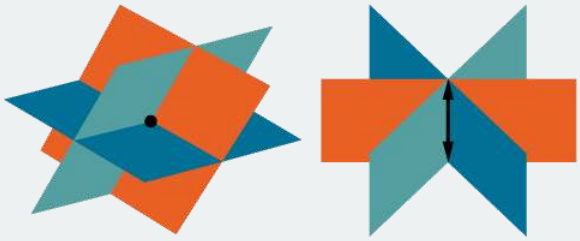


Figure 7.3.2: (a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.



Figure 7.3.3: All three figures represent three-by-three systems with no solution. (a) The three planes intersect with each other, but not at a common point. (b) Two of the planes are parallel and intersect with the third plane, but not with each other. (c) All three planes are parallel, so there is no point of intersection.

Example 7.3.1: Determining Whether an Ordered Triple Is a Solution to a System

Determine whether the ordered triple $(3, -2, 1)$ is a solution to the system.

$$\begin{aligned} x + y + z &= 2 \\ 6x - 4y + 5z &= 31 \\ 5x + 2y + 2z &= 13 \end{aligned}$$

Solution

We will check each equation by substituting in the values of the ordered triple for x, y, and z.

$x + y + z = 2$	$6x - 4y + 5z = 31$	$5x + 2y + 2z = 13$
$(3) + (-2) + (1) = 2$	$6(3) - 4(-2) + 5(1) = 31$	$5(3) + 2(-2) + 2(1) = 13$
True	$18 + 8 + 5 = 31$	$15 - 4 + 2 = 13$
	True	True

The ordered triple $(3, -2, 1)$ is indeed a solution to the system.

How to: Given a linear system of three equations, solve for three unknowns

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.

4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Example 7.3.2: Solving a System of Three Equations in Three Variables by Elimination

Find a solution to the following system:

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ -x+3y-z &= -6 \quad (2)\\ 2x-5y+5z &= 17 \quad (3) \end{aligned}$$

Solution

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate x by adding equations (1) and (2).

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ \underline{-x+3y-z} &\underline{= -6 \quad (2)}\\ y+2z &= 3 \quad (4) \end{aligned}$$

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable x.

$$\begin{aligned} -2x+4y-6z &= -18 \quad (1) \quad \text{multiplied by } -2\\ \underline{2x-5y+5z} &\underline{= 17 \quad (3)}\\ -y-z &= -1 \quad (5) \end{aligned}$$

In equations (4) and (5), we have created a new two-by-two system. We can solve for z by adding the two equations.

$$\begin{aligned} y+2z &= 3 \quad (4)\\ \underline{-y-z} &\underline{= -1 \quad (5)}\\ z &= 2 \quad (6) \end{aligned}$$

Choosing one equation from each new system, we obtain the upper triangular form:

$$\begin{aligned} x-2y+3z &= 9 \quad (1)\\ y+2z &= 3 \quad (4)\\ z &= 2 \quad (6) \end{aligned}$$

Next, we back-substitute $z=2$ into equation (4) and solve for y.

$$\begin{aligned} y+2(2) &= 3\\ y+4 &= 3\\ y &= -1 \end{aligned}$$

Finally, we can back-substitute $z=2$ and $y=-1$ into equation (1). This will yield the solution for x.

$$\begin{aligned} x-2(-1)+3(2) &= 9\\ x+2+6 &= 9\\ x &= 1 \end{aligned}$$

The solution is the ordered triple $(1,-1,2)$. See Figure 7.3.4.

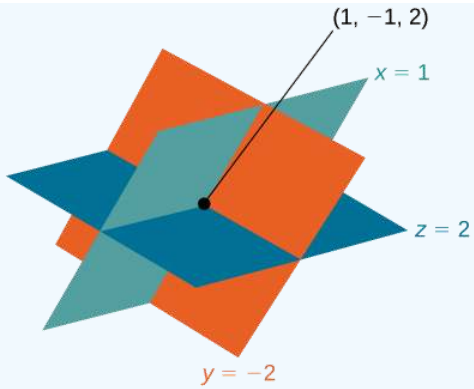


Figure 7.3.4

Example 7.3.3: Solving a Real-World Problem Using a System of Three Equations in Three Variables

In the problem posed at the beginning of the section, John invested his inheritance of $\$12,000$ in three different funds: part in a money-market fund paying 3% interest annually; part in municipal bonds paying 4% annually; and the rest in mutual funds paying 7% annually. John invested $\$4,000$ more in mutual funds than he invested in municipal bonds. The total interest earned in one year was $\$670$. How much did he invest in each type of fund?

Solution

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:

$$x = \text{amount invested in money-market fund}$$
$$y = \text{amount invested in municipal bonds}$$
$$z = \text{amount invested in mutual funds}$$

The first equation indicates that the sum of the three principal amounts is $\$12,000$.

$$x + y + z = 12,000$$

We form the second equation according to the information that John invested $\$4,000$ more in mutual funds than he invested in municipal bonds.

$$z = y + 4,000$$

The third equation shows that the total amount of interest earned from each fund equals $\$670$.

$$0.03x + 0.04y + 0.07z = 670$$

Then, we write the three equations as a system.

$$\begin{aligned} x + y + z &= 12,000 \\ -y + z &= 4,000 \\ 0.03x + 0.04y + 0.07z &= 670 \end{aligned}$$

To make the calculations simpler, we can multiply the third equation by 100. Thus,

$$\begin{aligned} x + y + z &= 12,000 \quad (1) \\ -y + z &= 4,000 \quad (2) \\ 3x + 4y + 7z &= 67,000 \quad (3) \end{aligned}$$

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.

$$\begin{aligned} x+y+z &= 12,000 \\ 3x+4y+7z &= 67,000 \\ -y+z &= 4,000 \end{aligned}$$

Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.

$$\begin{aligned} x+y+z &= 12,000 \\ y+4z &= 31,000 \\ -y+z &= 4,000 \end{aligned}$$

Step 3. Add equation (2) to equation (3) and write the result as equation (3).

$$\begin{aligned} x+y+z &= 12,000 \\ y+4z &= 31,000 \\ 5z &= 35,000 \end{aligned}$$

Step 4. Solve for z in equation (3). Back-substitute that value in equation (2) and solve for y. Then, back-substitute the values for z and y into equation (1) and solve for x.

$$\begin{aligned} 5z &= 35,000 \\ z &= 7,000 \end{aligned}$$

$$\begin{aligned} y+4(7,000) &= 31,000 \\ y &= 3,000 \end{aligned}$$

$$\begin{aligned} x+3,000+7,000 &= 12,000 \\ x &= 2,000 \end{aligned}$$

John invested $\$2,000$ in a money-market fund, $\$3,000$ in municipal bonds, and $\$7,000$ in mutual funds.

? Exercise 7.3.1

Solve the system of equations in three variables.

$$\begin{aligned} 2x+y-2z &= -1 \\ 3x-3y-z &= 5 \\ x-2y+3z &= 6 \end{aligned}$$

Answer

$(1,-1,1)$

Identifying Inconsistent Systems of Equations Containing Three Variables

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as $3=7$ or some other contradiction.

✓ Example 7.3.4: Solving an Inconsistent System of Three Equations in Three Variables

Solve the following system.

$$x-3y+z=4 \quad (7.3.1)$$
$$-x+2y-5z=3 \quad (7.3.2)$$
$$5x-13y+13z=8 \quad (7.3.3)$$

Solution

Looking at the coefficients of x, we can see that we can eliminate x by adding Equation 7.3.1 to Equation 7.3.2.

$$\begin{aligned} x-3y+z&=4\,(1)\\ -x+2y-5z&=3\,(2)\\ \hline -y-4z&=7\,(4)\end{aligned}$$

Next, we multiply equation (1) by -5 and add it to equation (3).

$$\begin{aligned} -5x+15y-5z&=-20\,(1) \quad \text{multiplied by } -5\\ 5x-13y+13z&=8\,(3)\\ \hline 2y+8z&=-12\,(5)\end{aligned}$$

Then, we multiply equation (4) by 2 and add it to equation (5).

$$\begin{aligned} -2y-8z&=14\,(4) \quad \text{multiplied by } 2\\ 2y+8z&=-12\,(5)\\ \hline 0&=2\end{aligned}$$

The final equation $0=2$ is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

Analysis

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

? Exercise 7.3.2

Solve the system of three equations in three variables.

$$x+y+z=2$$
$$y-3z=1$$
$$2x+y+5z=0$$

Answer

No solution.

Expressing the Solution of a System of Dependent Equations Containing Three Variables

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

✓ Example 7.3.5: Finding the Solution to a Dependent System of Equations

Find the solution to the given system of three equations in three variables.

$$\begin{aligned} 2x+y-3z &= 0 \quad (1) \\ 4x+2y-6z &= 0 \quad (2) \\ x-y+z &= 0 \quad (3) \end{aligned}$$

Solution

First, we can multiply equation (1) by -2 and add it to equation (2).

$$\begin{aligned} -4x-2y+6z &= 0 \ (1) \quad \text{multiplied by } -2 \\ \underline{4x+2y-6z} &= \underline{0 \ (2)} \\ 0 &= 0 \end{aligned}$$

We do not need to proceed any further. The result we get is an identity, $0=0$, which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2, and adding it to equation (1). We then perform the same steps as above and find the same result, $0=0$.

When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have

$$\begin{aligned} 2x+y-3z &= 0 \\ x-y+z &= 0 \\ 3x-2z &= 0 \end{aligned}$$

We then solve the resulting equation for z.

$$\begin{aligned} 3x-2z &= 0 \\ z &= \frac{3}{2}x \end{aligned}$$

We back-substitute the expression for z into one of the equations and solve for y.

$$\begin{aligned} 2x+y-3\left(\frac{3}{2}x\right) &= 0 \\ 2x+y-\frac{9}{2}x &= 0 \\ y &= \frac{9}{2}x-2x \\ y &= \frac{5}{2}x \end{aligned}$$

So the general solution is $\left(x, \frac{5}{2}x, \frac{3}{2}x\right)$. In this solution, x can be any real number. The values of y and z are dependent on the value selected for x.

Analysis

As shown in Figure 7.3.5, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.

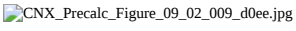


Figure 7.3.5

Q&A: Does the generic solution to a dependent system always have to be written in terms of x?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of x and if needed x and y.

Exercise 7.3.3:

Solve the following system.

$$\begin{aligned} x+y+z &= 7 \\ 3x-2y-z &= 4 \\ x+6y+5z &= 24 \end{aligned}$$

Answer

Infinite number of solutions of the form $(x, 4x-11, -5x+18)$.

Media

Access these online resources for additional instruction and practice with systems of equations in three variables.

- Ex 1: System of Three Equations with Three Unknowns Using Elimination
- Ex. 2: System of Three Equations with Three Unknowns Using Elimination

Key Concepts

- A solution set is an ordered triple {(x,y,z)} that represents the intersection of three planes in space. See Example 7.3.1.
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation. See Example 7.3.2.
- Systems of three equations in three variables are useful for solving many different types of real-world problems. See Example 7.3.3.
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction. See Example 7.3.4.
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity. See Example 7.3.5.
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

Contributors

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7.4: Systems of Nonlinear Equations and Inequalities - Two Variables

Learning Objectives

- Solve a system of nonlinear equations using substitution.
- Solve a system of nonlinear equations using elimination.
- Graph a nonlinear inequality.
- Graph a system of nonlinear inequalities.

Halley's Comet (Figure 7.4.1) orbits the sun about once every 75 years. Its path can be considered to be a very elongated ellipse. Other comets follow similar paths in space. These orbital paths can be studied using systems of equations. These systems, however, are different from the ones we considered in the previous section because the equations are not linear.



Figure 7.4.1: Halley's Comet (credit: "NASA Blueshift"/Flickr)

In this section, we will consider the intersection of a parabola and a line, a circle and a line, and a circle and an ellipse. The methods for solving systems of nonlinear equations are similar to those for linear equations.

Solving a System of Nonlinear Equations Using Substitution

A system of nonlinear equations is a system of two or more equations in two or more variables containing at least one equation that is not linear. Recall that a linear equation can take the form $Ax + By + C = 0$. Any equation that cannot be written in this form in nonlinear. The substitution method we used for linear systems is the same method we will use for nonlinear systems. We solve one equation for one variable and then substitute the result into the second equation to solve for another variable, and so on. There is, however, a variation in the possible outcomes.

Intersection of a Parabola and a Line

There are three possible types of solutions for a system of nonlinear equations involving a parabola and a line.

Possible Types of Solutions For Points of Intersection of a Parabola and a Line

Figure 7.4.2 illustrates possible solution sets for a system of equations involving a parabola and a line.

- No solution - The line will never intersect the parabola.
- One solution - The line is tangent to the parabola and intersects the parabola at exactly one point.
- Two solutions - The line crosses on the inside of the parabola and intersects the parabola at two points.

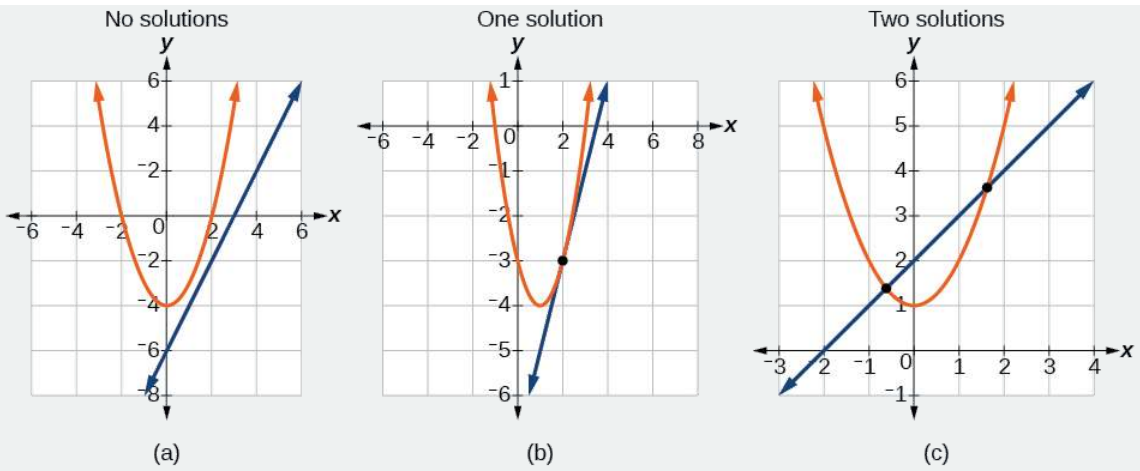


Figure 7.4.2

How to: Given a system of equations containing a line and a parabola, find the solution

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the parabola equation.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

Example 7.4.1: Solving a System of Nonlinear Equations Representing a Parabola and a Line

Solve the system of equations.

$$\begin{aligned} x-y &= -1 \\ y &= x^2+1 \end{aligned}$$

Solution

Solve the first equation for x and then substitute the resulting expression into the second equation.

$$\begin{aligned} x-y &= -1 \\ x &= y-1 \qquad \text{Solve for } x. \end{aligned}$$

$$\begin{aligned} y &= x^2+1 \\ y &= (y-1)^2+1 \qquad \text{Substitute expression for } x. \end{aligned}$$

Expand the equation and set it equal to zero.

$$\begin{aligned} y &= (y-1)^2+1 \\ &= (y^2-2y+1)+1 \\ &= y^2-2y+2 \\ 0 &= y^2-3y+2 \\ &= (y-2)(y-1) \end{aligned}$$

Solving for y gives $y=2$ and $y=1$. Next, substitute each value for y into the first equation to solve for x. Always substitute the value into the linear equation to check for extraneous solutions.

$$\begin{aligned} x-y &= -1 \\ x-(2) &= -1 \\ x &= 1 \\ x-(1) &= -1 \\ x &= 0 \end{aligned}$$

The solutions are $(1, 2)$ and $(0, 1)$,which can be verified by substituting these (x, y) values into both of the original equations (Figure 7.4.3).

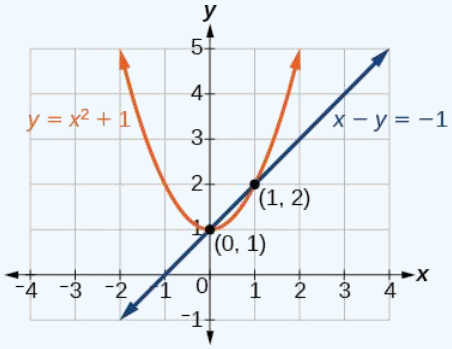


Figure 7.4.3

Q&A: Could we have substituted values for y into the second equation to solve for x in last example?

Yes, but because x is squared in the second equation this could give us extraneous solutions for x.

For $y = 1$

$$\begin{aligned} y &= x^2 + 1 \\ y &= x^2 + 1 \\ x^2 &= 0 \\ x &= \pm\sqrt{0} = 0 \end{aligned}$$

This gives us the same value as in the solution.

For $y = 2$

$$\begin{aligned} y &= x^2 + 1 \\ 2 &= x^2 + 1 \\ x^2 &= 1 \\ x &= \pm\sqrt{1} = \pm 1 \end{aligned}$$

Notice that -1 is an extraneous solution.

? Exercise 7.4.1

Solve the given system of equations by substitution.

$$\begin{aligned} 3x - y &= -2 \\ 2x^2 - y &= 0 \end{aligned}$$

Answer

$\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(2, 8)$

Intersection of a Circle and a Line

Just as with a parabola and a line, there are three possible outcomes when solving a system of equations representing a circle and a line.

POSSIBLE TYPES OF SOLUTIONS FOR THE POINTS OF INTERSECTION OF A CIRCLE AND A LINE

Figure 7.4.4 illustrates possible solution sets for a system of equations involving a circle and a line.

- No solution - The line does not intersect the circle.

- One solution - The line is tangent to the circle and intersects the circle at exactly one point.
- Two solutions - The line crosses the circle and intersects it at two points.

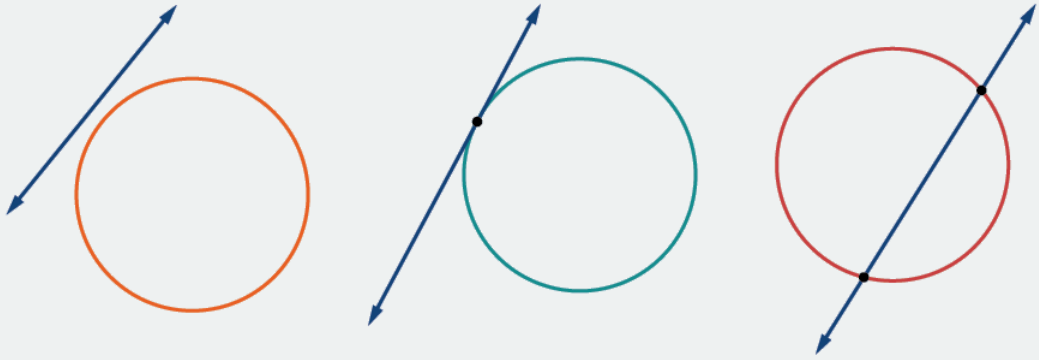


Figure 7.4.4

How to: Given a system of equations containing a line and a circle, find the solution

1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the equation for the circle.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

Example 7.4.2: Finding the Intersection of a Circle and a Line by Substitution

Find the intersection of the given circle and the given line by substitution.

$$\begin{aligned} x^2+y^2 &= 5 \\ y &= 3x-5 \end{aligned}$$

Solution

One of the equations has already been solved for y. We will substitute $y=3x-5$ into the equation for the circle.

$$\begin{aligned} x^2+(3x-5)^2 &= 5 \\ x^2+9x^2-30x+25 &= 5 \\ 10x^2-30x+20 &= 0 \end{aligned}$$

Now, we factor and solve for x.

$$\begin{aligned} 10(x2-3x+2) &= 0 \\ 10(x-2)(x-1) &= 0 \\ x &= 2 \\ x &= 1 \end{aligned}$$

Substitute the two x-values into the original linear equation to solve for y.

$$\begin{aligned} y &= 3(2)-5 \\ &= 1 \\ y &= 3(1)-5 \\ &= -2 \end{aligned}$$

The line intersects the circle at $(2,1)$ and $(1,-2)$,which can be verified by substituting these (x,y) values into both of the original equations (Figure 7.4.5).

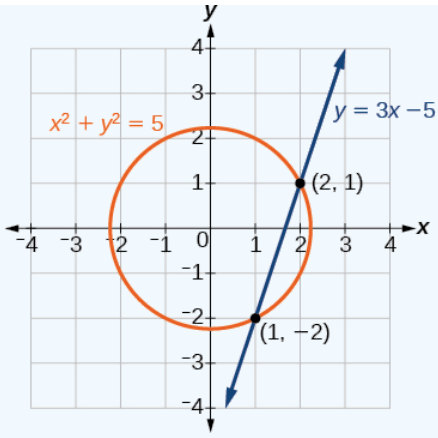


Figure 7.4.5

? Exercise 7.4.2

Solve the system of nonlinear equations.

$$\begin{aligned} x^2 + y^2 &= 10 \\ x - 3y &= -10 \end{aligned}$$

Answer

$(-1, 3)$

Solving a System of Nonlinear Equations Using Elimination

We have seen that substitution is often the preferred method when a system of equations includes a linear equation and a nonlinear equation. However, when both equations in the system have like variables of the second degree, solving them using elimination by addition is often easier than substitution. Generally, elimination is a far simpler method when the system involves only two equations in two variables (a two-by-two system), rather than a three-by-three system, as there are fewer steps. As an example, we will investigate the possible types of solutions when solving a system of equations representing a circle and an ellipse.

POSSIBLE TYPES OF SOLUTIONS FOR THE POINTS OF INTERSECTION OF A CIRCLE AND AN ELLIPSE

Figure 7.4.6 illustrates possible solution sets for a system of equations involving a circle and an ellipse.

- No solution - The circle and ellipse do not intersect. One shape is inside the other or the circle and the ellipse are a distance away from the other.
- One solution - The circle and ellipse are tangent to each other, and intersect at exactly one point.
- Two solutions - The circle and the ellipse intersect at two points.
- Three solutions - The circle and the ellipse intersect at three points.
- Four solutions - The circle and the ellipse intersect at four points.

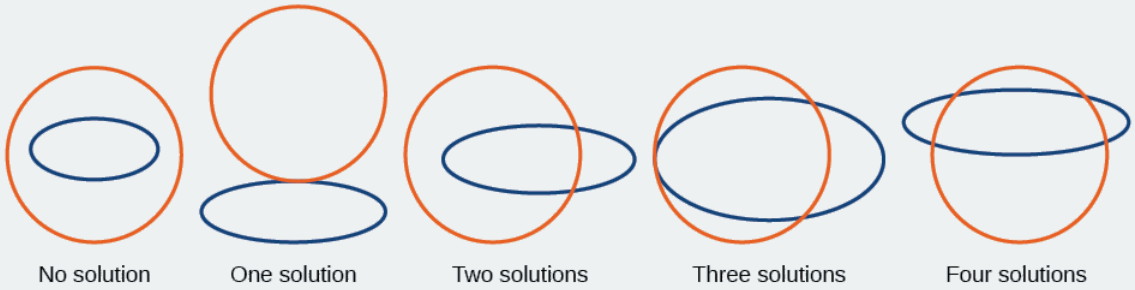


Figure 7.4.6

Example 7.4.3: Solving a System of Nonlinear Equations Representing a Circle and an Ellipse

Solve the system of nonlinear equations.

$$\begin{aligned} x^2+y^2 &= 26 \quad (1)\\ 3x^2+25y^2 &= 100 \quad (2)\end{aligned}$$

Solution

Let's begin by multiplying equation (1) by -3,and adding it to equation (2).

$$\begin{aligned} (-3)(x^2+y^2) &= (-3)(26)\\ -3x^2-3y^2 &= -78\\ 3x^2+25y^2 &= 100\\ \hline 22y^2 &= 22\end{aligned}$$

After we add the two equations together, we solve for y.

$$\begin{aligned} y^2 &= 1\\ y &= \pm\sqrt{1} = \pm 1\end{aligned}$$

Substitute $y=\pm 1$ into one of the equations and solve for x.

$$\begin{aligned} x^2+(1)^2 &= 26\\ x^2+1 &= 26\\ x^2 &= 25\\ x &= \pm\sqrt{25} = \pm 5\\ x^2+(-1)^2 &= 26\\ x^2+1 &= 26\\ x^2 &= \pm\sqrt{25} = \pm 5\end{aligned}$$

There are four solutions: $(5,1)$, $(-5,1)$, $(5,-1)$,and $(-5,-1)$. See Figure 7.4.7.

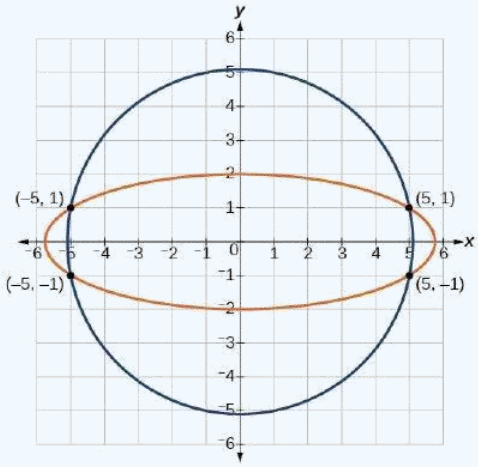


Figure 7.4.7

Exercise 7.4.3

Find the solution set for the given system of nonlinear equations.

$$\begin{aligned} 4x^2+y^2 &= 13\\ x^2+y^2 &= 10\end{aligned}$$

Answer

$(1,3),(1,-3),(-1,3),(-1,-3)$

Graphing a Nonlinear Inequality

All of the equations in the systems that we have encountered so far have involved equalities, but we may also encounter systems that involve inequalities. We have already learned to graph linear inequalities by graphing the corresponding equation, and then shading the region represented by the inequality symbol. Now, we will follow similar steps to graph a nonlinear **inequality** so that we can learn to solve systems of nonlinear inequalities. A **nonlinear inequality** is an inequality containing a nonlinear expression. Graphing a nonlinear inequality is much like graphing a linear inequality.

Recall that when the inequality is greater than, $y > a$,or less than, $y < a$,the graph is drawn with a dashed line. When the inequality is greater than or equal to, $y \geq a$,or less than or equal to, $y \leq a$,the graph is drawn with a solid line. The graphs will create regions in the plane, and we will test each region for a solution. If one point in the region works, the whole region works. That is the region we shade (Figure 7.4.8).

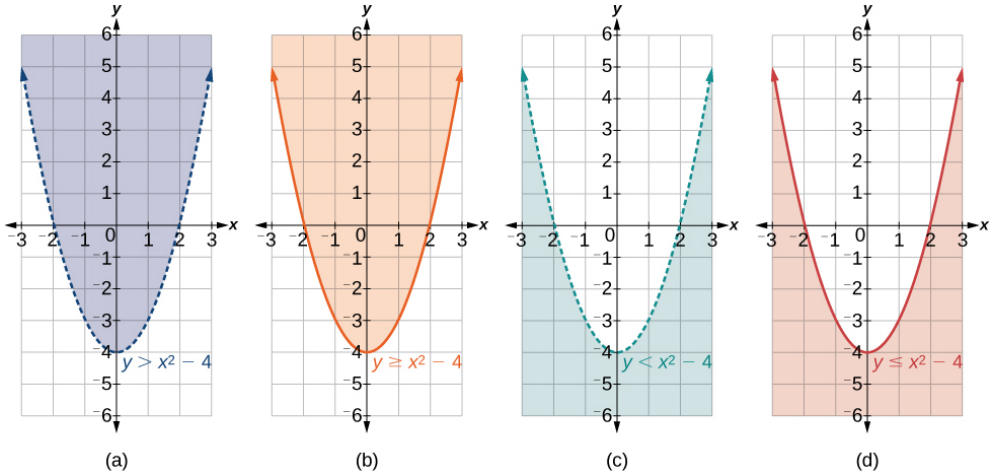


Figure 7.4.8: (a) an example of $y > a$; (b) an example of $y \geq a$; (c) an example of $y < a$; (d) an example of $y \leq a$

How to: Given an inequality bounded by a parabola, sketch a graph

1. Graph the parabola as if it were an equation. This is the boundary for the region that is the solution set.
2. If the boundary is included in the region (the operator is $\leq$ or $\geq$), the parabola is graphed as a solid line.
3. If the boundary is not included in the region (the operator is $<$ or $>$), the parabola is graphed as a dashed line.
4. Test a point in one of the regions to determine whether it satisfies the inequality statement. If the statement is true, the solution set is the region including the point. If the statement is false, the solution set is the region on the other side of the boundary line.
5. Shade the region representing the solution set.

Example 7.4.4: Graphing an Inequality for a Parabola

Graph the inequality $y > x^2 + 1$.

Solution

First, graph the corresponding equation $y = x^2 + 1$. Since $y > x^2 + 1$ has a greater than symbol, we draw the graph with a dashed line. Then we choose points to test both inside and outside the parabola. Let's test the points

$(0, 2)$ and $(2, 0)$. One point is clearly inside the parabola and the other point is clearly outside.

$$y > x^2 + 1$$
$$2 > (0)^2 + 1$$
$$2 > 1 \qquad \text{True}$$

$$0 > (2)^2 + 1$$
$$0 > 5 \qquad \text{False}$$

The graph is shown in Figure 7.4.9. We can see that the solution set consists of all points inside the parabola, but not on the graph itself.

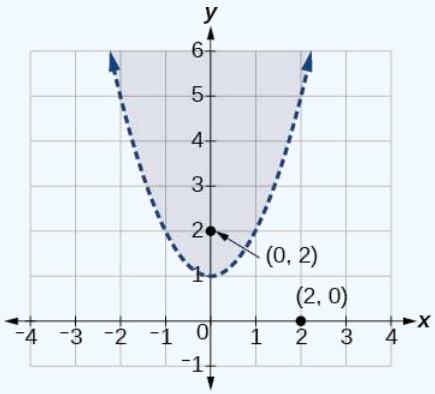


Figure 7.4.9

Graphing a System of Nonlinear Inequalities

Now that we have learned to graph nonlinear inequalities, we can learn how to graph systems of nonlinear inequalities. A **system of nonlinear inequalities** is a system of two or more inequalities in two or more variables containing at least one inequality that is not linear. Graphing a system of nonlinear inequalities is similar to graphing a system of linear inequalities. The difference is that our graph may result in more shaded regions that represent a solution than we find in a system of linear inequalities. The solution to a nonlinear system of inequalities is the region of the graph where the shaded regions of the graph of each inequality overlap, or where the regions intersect, called the **feasible region**.

How to: Given a system of nonlinear inequalities, sketch a graph

1. Find the intersection points by solving the corresponding system of nonlinear equations.
2. Graph the nonlinear equations.
3. Find the shaded regions of each inequality.
4. Identify the feasible region as the intersection of the shaded regions of each inequality or the set of points common to each inequality.

Example 7.4.5: Graphing a System of Inequalities

Graph the given system of inequalities.

$$x^2 - y \leq 0$$
$$2x^2 + y \leq 12$$

Solution

These two equations are clearly parabolas. We can find the points of intersection by the elimination process: Add both equations and the variable y will be eliminated. Then we solve for x.

$$\begin{aligned} x^2 - y &= 0 \\ 2x^2 + y &= 12 \\ \hline 3x^2 &= 12 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Substitute the x-values into one of the equations and solve for y.

$$\begin{aligned} x^2 - y &= 0 \\ (2)^2 - y &= 0 \\ 4 - y &= 0 \\ y &= 4 \end{aligned}$$

$$\begin{aligned} (-2)^2 - y &= 0 \\ 4 - y &= 0 \\ y &= 4 \end{aligned}$$

The two points of intersection are $(2, 4)$ and $(-2, 4)$. Notice that the equations can be rewritten as follows.

$$\begin{aligned} x^2 - y &\leq 0 \\ x^2 &\leq y \\ y &\geq x^2 \end{aligned}$$

$$\begin{aligned} 2x^2 + y &\leq 12 \\ y &\leq -2x^2 + 12 \end{aligned}$$

Graph each inequality. See Figure 7.4.10. The feasible region is the region between the two equations bounded by $2x^2 + y \leq 12$ on the top and $x^2 - y \leq 0$ on the bottom.

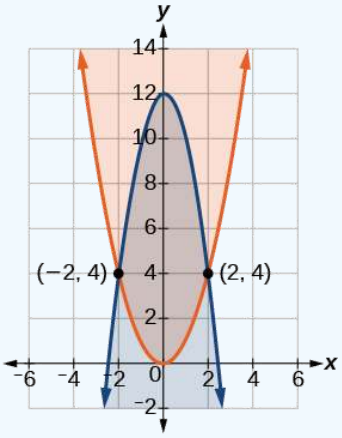


Figure 7.4.10

Exercise 7.4.5

Graph the given system of inequalities.

$$\begin{aligned} y &\geq x^2 - 1 \\ x - y &\geq -1 \end{aligned}$$

Answer

Shade the area bounded by the two curves, above the quadratic and below the line.

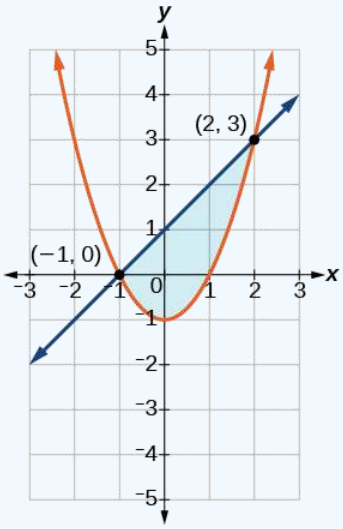


Figure 7.4.11

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Access these online resources for additional instruction and practice with nonlinear equations.

- Solve a System of Nonlinear Equations Using Substitution
- Solve a System of Nonlinear Equations Using Elimination

Key Concepts

- There are three possible types of solutions to a system of equations representing a line and a parabola: (1) no solution, the line does not intersect the parabola; (2) one solution, the line is tangent to the parabola; and (3) two solutions, the line intersects the parabola in two points. See Example 7.4.1.
- There are three possible types of solutions to a system of equations representing a circle and a line: (1) no solution, the line does not intersect the circle; (2) one solution, the line is tangent to the parabola; (3) two solutions, the line intersects the circle in two points. See Example 7.4.2.
- There are five possible types of solutions to the system of nonlinear equations representing an ellipse and a circle: (1) no solution, the circle and the ellipse do not intersect; (2) one solution, the circle and the ellipse are tangent to each other; (3) two solutions, the circle and the ellipse intersect in two points; (4) three solutions, the circle and ellipse intersect in three places; (5) four solutions, the circle and the ellipse intersect in four points. See Example 7.4.3.
- An inequality is graphed in much the same way as an equation, except for > or <, we draw a dashed line and shade the region containing the solution set. See Example 7.4.4.
- Inequalities are solved the same way as equalities, but solutions to systems of inequalities must satisfy both inequalities. See Example 7.4.5.

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7.5: Partial Fractions

Learning Objectives

Decompose $\frac{P(x)}{Q(x)}$, where

- $Q(x)$ has only nonrepeated linear factors.
- $Q(x)$ has repeated linear factors.
- $Q(x)$ has a nonrepeated irreducible quadratic factor.
- $Q(x)$ has a repeated irreducible quadratic factor.

Earlier in this chapter, we studied systems of two equations in two variables, systems of three equations in three variables, and nonlinear systems. Here we introduce another way that systems of equations can be utilized—the decomposition of rational expressions. Fractions can be complicated; adding a variable in the denominator makes them even more so. The methods studied in this section will help simplify the concept of a rational expression.

Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Only Nonrepeated Linear Factors

Recall the algebra regarding adding and subtracting rational expressions. These operations depend on finding a common denominator so that we can write the sum or difference as a single, simplified rational expression. In this section, we will look at **partial fraction decomposition**, which is the undoing of the procedure to add or subtract rational expressions. In other words, it is a return from the single simplified **rational expression** to the original expressions, called the **partial fractions**.

For example, suppose we add the following fractions:

$$\frac{2}{x-3}+\frac{-1}{x+2}$$

We would first need to find a common denominator: $(x+2)(x-3)$.

Next, we would write each expression with this common denominator and find the sum of the terms.

$$\begin{aligned}\frac{2}{x-3}\left(\frac{x+2}{x+2}\right)+\frac{-1}{x+2}\left(\frac{x-3}{x-3}\right) &= \frac{2x+4-x+3}{(x+2)(x-3)}\\ &= \frac{x+7}{x^2-x-6}\end{aligned}$$

Partial fraction **decomposition** is the reverse of this procedure. We would start with the solution and rewrite (decompose) it as the sum of two fractions.

$$\underbrace{\frac{x+7}{x^2-x-6}}_{\text{Simplified sum}}=\underbrace{\frac{2}{x-3}+\frac{-1}{x+2}}_{\text{Partial fraction decomposition}}$$

We will investigate rational expressions with linear factors and quadratic factors in the denominator where the degree of the numerator is less than the degree of the denominator. Regardless of the type of expression we are decomposing, the first and most important thing to do is factor the denominator.

When the denominator of the simplified expression contains distinct linear factors, it is likely that each of the original rational expressions, which were added or subtracted, had one of the linear factors as the denominator. In other words, using the example above, the factors of x^2-x-6 are $(x-3)(x+2)$, the denominators of the decomposed rational expression. So we will rewrite the simplified form as the sum of individual fractions and use a variable for each numerator. Then, we will solve for each numerator using one of several methods available for partial fraction decomposition.

PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS NONREPEATED LINEAR FACTORS

The *partial fraction decomposition* of $\dfrac{P(x)}{Q(x)}$ when $Q(x)$ has nonrepeated linear factors and the degree of $P(x)$ is less than the degree of $Q(x)$ is

$$\frac{P(x)}{Q(x)}=\frac{A_1}{(a_1x+b_1)}+\frac{A_2}{(a_2x+b_2)}+\frac{A_3}{(a_3x+b_3)}+\cdots+\frac{A_n}{(a_nx+b_n)} \tag{7.5.1}$$

How to: Given a rational expression with distinct linear factors in the denominator, decompose it

1. Use a variable for the original numerators, usually A, B, or C, depending on the number of factors, placing each variable over a single factor. For the purpose of this definition, we use A_n for each numerator
$$\frac{P(x)}{Q(x)}=\frac{A_1}{(a_1x+b_1)}+\frac{A_2}{(a_2x+b_2)}+\frac{A_3}{(a_3x+b_3)}+\cdots+\frac{A_n}{(a_nx+b_n)}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.5.1: Decomposing a Rational Function with Distinct Linear Factors

Decompose the given rational expression with distinct linear factors.

$$\frac{3x}{(x+2)(x-1)}$$

Solution

We will separate the denominator factors and give each numerator a symbolic label, like A, B ,or C.

$$\frac{3x}{(x+2)(x-1)}=\frac{A}{(x+2)}+\frac{B}{(x-1)}$$

Multiply both sides of the equation by the common denominator to eliminate the fractions:

$$(x+2)(x-1)\left[\frac{3x}{(x+2)(x-1)}\right]=(x+2)(x-1)\left[\frac{A}{(x+2)}\right]+(x+2)(x-1)\left[\frac{B}{(x-1)}\right]$$

The resulting equation is

$$3x=A(x-1)+B(x+2)$$

Expand the right side of the equation and collect like terms.

$$\begin{aligned}3x&=Ax-A+Bx+2B\\3x&=(A+B)x-A+2B\end{aligned}$$

Set up a system of equations associating corresponding coefficients.

$$\begin{aligned}3&=A+B\\0&=-A+2B\end{aligned}$$

Add the two equations and solve for B.

$$\begin{aligned}3&=A+B\\\underline{0}&=\underline{-A+2B}\\3&=0+3B\\1&=B\end{aligned}$$

Substitute $B=1$ into one of the original equations in the system.

$$\begin{aligned} 3 &= A+1 \\ 2 &= A \end{aligned}$$

Thus, the partial fraction decomposition is

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Another method to use to solve for A or B is by considering the equation that resulted from eliminating the fractions and substituting a value for x that will make either the $A-$ or $B-$term equal 0. If we let $x=1$, the

$A-$ term becomes 0 and we can simply solve for B.

$$\begin{aligned} 3x &= A(x-1)+B(x+2) \\ 3(1) &= A[(1)-1]+B[(1)+2] \\ 3 &= 0+3B \\ 1 &= B \end{aligned}$$

Next, either substitute $B=1$ into the equation and solve for A, or make the $B-$term 0 by substituting $x=-2$ into the equation.

$$\begin{aligned} 3x &= A(x-1)+B(x+2) \\ 3(-2) &= A[(-2)-1]+B[(-2)+2] \\ -6 &= -3A+0 \\ \frac{-6}{-3} &= A \\ 2 &= A \end{aligned}$$

We obtain the same values for A and B using either method, so the decompositions are the same using either method.

$$\frac{3x}{(x+2)(x-1)} = \frac{2}{(x+2)} + \frac{1}{(x-1)}$$

Although this method is not seen very often in textbooks, we present it here as an alternative that may make some partial fraction decompositions easier. It is known as the **Heaviside method**, named after Charles Heaviside, a pioneer in the study of electronics.

? Exercise 7.5.1

Find the partial fraction decomposition of the following expression.

$$\frac{x}{(x-3)(x-2)}$$

Answer

$$\frac{3}{x-3} - \frac{2}{x-2}$$

Decomposing $\frac{P(x)}{Q(x)}$ where $Q(x)$ Has Repeated Linear Factors

Some fractions we may come across are special cases that we can decompose into partial fractions with repeated linear factors. We must remember that we account for repeated factors by writing each factor in increasing powers.

PARTIAL FRACTION DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS REPEATED LINEAR FACTORS

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated linear factor occurringn n times and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{7.5.2}$$

Write the denominator powers in increasing order.

How to: decompose a rational expression with repeated linear factors

1. Use a variable like A, B, or C for the numerators and account for increasing powers of the denominators.

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(a_1x+b_1)} + \frac{A_2}{(a_2x+b_2)} + \frac{A_3}{(a_3x+b_3)} + \cdots + \frac{A_n}{(a_nx+b_n)} \tag{7.5.3}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.5.2: Decomposing with Repeated Linear Factors

Decompose the given rational expression with repeated linear factors.

$$\frac{-x^2+2x+4}{x^3-4x^2+4x}$$

Solution

The denominator factors are $x(x-2)^2$. To allow for the repeated factor of $(x-2)$, the decomposition will include three denominators: x, $(x-2)$, and $(x-2)^2$. Thus,

$$\frac{-x^2+2x+4}{x^3-4x^2+4x} = \frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

Next, we multiply both sides by the common denominator.

$$x(x-2)^2\left[\frac{-x^2+2x+4}{x(x-2)^2}\right] = \left[\frac{A}{x} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}\right] x(x-2)^2$$

$$-x^2+2x+4 = A(x-2)^2 + Bx(x-2) + Cx$$

On the right side of the equation, we expand and collect like terms.

$$\begin{aligned} -x^2+2x+4 &= A(x^2-4x+4) + B(x^2-2x) + Cx \\ &= Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx \\ &= (A+B)x^2 + (-4A-2B+C)x + 4A \end{aligned}$$

Next, we compare the coefficients of both sides. This will give the system of equations in three variables:

$$-x^2+2x+4 = (A+B)x^2 + (-4A-2B+C)x + 4A \tag{7.5.4}$$

$$A+B = -1 \tag{7.5.5}$$

$$-4A-2B+C = 2 \tag{7.5.6}$$

$$4A = 4 \tag{7.5.7}$$

Solving for A in Equation 7.5.7, we have

$$\begin{aligned} 4A &= 4 \\ A &= 1 \end{aligned}$$

Substitute $A = 1$ into Equation 7.5.5.

$$\begin{aligned} A + B &= -1 \\ (1) + B &= -1 \\ B &= -2 \end{aligned}$$

Then, to solve for C, substitute the values for A and B into Equation 7.5.6.

$$\begin{aligned} -4A - 2B + C &= 2 \\ -4(1) - 2(-2) + C &= 2 \\ -4 + 4 + C &= 2 \\ C &= 2 \end{aligned}$$

Thus,

$$\frac{-x^2 + 2x + 4}{x^3 - 4x^2 + 4x} = \frac{1}{x} - \frac{2}{(x-2)} + \frac{2}{(x-2)^2}$$

? Exercise 7.5.2

Find the partial fraction decomposition of the expression with repeated linear factors.

$$\frac{6x - 11}{(x-1)^2}$$

Answer

$$\frac{6}{x-1} - \frac{5}{(x-1)^2}$$

Decomposing $\frac{P(x)}{Q(x)}$, where $Q(x)$ Has a Nonrepeated Irreducible Quadratic Factor

So far, we have performed partial fraction decomposition with expressions that have had linear factors in the denominator, and we applied numerators A, B, or C representing constants. Now we will look at an example where one of the factors in the denominator is a quadratic expression that does not factor. This is referred to as an irreducible quadratic factor. In cases like this, we use a linear numerator such as $Ax + B$, $Bx + C$, etc.

DECOMPOSITION OF $\frac{P(x)}{Q(x)}$: $Q(x)$ HAS A NONREPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\dfrac{P(x)}{Q(x)}$ such that $Q(x)$ has a nonrepeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$ is written as

$$\frac{P(x)}{Q(x)} = \frac{A_1 x + B_1}{(a_1 x^2 + b1_x + c_1)} + \frac{A_2 x + B_2}{(a_2 x^2 + b_2 x + c_2)} + \cdots + \frac{A_n x + B_n}{(a_n x^2 + b_n x + c_n)} \tag{7.5.8}$$

The decomposition may contain more rational expressions if there are linear factors. Each linear factor will have a different constant numerator: A, B, C, and so on.

Howto: decompose a rational expression where the factors of the denominator are distinct, irreducible quadratic factors

1. Use variables such as A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator.

$$\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{A_1x+B_1}{(a_1x^2+b1_x+c_1)}+\frac{A_2x+B_2}{(a_2x^2+b_2x+c_2)}+\cdots+\frac{A_nx+B_n}{(a_nx^2+b_nx+c_n)}$$

2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.5.3: Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Contains a Nonrepeated Irreducible Quadratic Factor

Find a partial fraction decomposition of the given expression.

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}$$

Solution

We have one linear factor and one irreducible quadratic factor in the denominator, so one numerator will be a constant and the other numerator will be a linear expression. Thus,

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=\frac{A}{(x+3)}+\frac{Bx+C}{(x^2+x+2)}$$

We follow the same steps as in previous problems. First, clear the fractions by multiplying both sides of the equation by the common denominator.

$$\begin{aligned}(x+3)(x^2+x+2)\left[\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}\right] &= \left[\frac{A}{(x+3)}+\frac{Bx+C}{(x^2+x+2)}\right](x+3)(x^2+x+2)\\ 8x^2+12x-20 &= A(x^2+x+2)+(Bx+C)(x+3)\end{aligned}$$

Notice we could easily solve for A by choosing a value for x that will make the $Bx+C$ term equal 0. Let $x=-3$ and substitute it into the equation.

$$\begin{aligned}8x^2+12x-20 &= A(x^2+x+2)+(Bx+C)(x+3)\\ 8(-3)^2+12(-3)-20 &= A((-3)^2+(-3)+2)+(B(-3)+C)((-3)+3)\\ 16 &= 8A\\ A &= 2\end{aligned}$$

Now that we know the value of A, substitute it back into the equation. Then expand the right side and collect like terms.

$$\begin{aligned}8x^2+12x-20 &= 2(x^2+x+2)+(Bx+C)(x+3)\\ 8x^2+12x-20 &= 2x^2+2x+4+Bx^2+3B+Cx+3C\\ 8x^2+12x-20 &= (2+B)x^2+(2+3B+C)x+(4+3C)\end{aligned}$$

Setting the coefficients of terms on the right side equal to the coefficients of terms on the left side gives the system of equations.

$$2+B=8 \tag{7.5.9}$$

$$2+3B+C=12 \tag{7.5.10}$$

$$4+3C=-20 \tag{7.5.11}$$

Solve for B using Equation 7.5.9

\[

$$2+B=8$$

$$B=6$$

and solve for C using Equation 7.5.11.

$$4+3C=-20$$

$$3C=-24$$

$$C=-8$$

Thus, the partial fraction decomposition of the expression is

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}=\frac{2}{(x+3)}+\frac{6x-8}{(x^2+x+2)}$$

Q&A: Could we have just set up a system of equations to solve the example above?

Yes, we could have solved it by setting up a system of equations without solving for A first. The expansion on the right would be:

$$8x^2+12x-20 = Ax^2+Ax+2A+Bx^2+3B+Cx+3C$$

$$8x^2+12x-20 = (A+B)x^2+(A+3B+C)x+(2A+3C)$$

So the system of equations would be:

$$A+B=8$$

$$A+3B+C=12$$

$$2A+3C=-20$$

Exercise 7.5.3

Find the partial fraction decomposition of the expression with a nonrepeating irreducible quadratic factor.

$$\frac{5x^2-6x+7}{(x-1)(x^2+1)}$$

Answer

$$\frac{3}{x-1}+\frac{2x-4}{x^2+1}$$

Decomposing $\frac{P(x)}{Q(x)}$ When $Q(x)$ Has a Repeated Irreducible Quadratic Factor

Now that we can decompose a simplified rational expression with an irreducible quadratic factor, we will learn how to do partial fraction decomposition when the simplified rational expression has repeated irreducible quadratic factors. The decomposition will consist of partial fractions with linear numerators over each irreducible quadratic factor represented in increasing powers.

DECOMPOSITION OF $\frac{P(x)}{Q(x)}$ WHEN $Q(X)$ HAS A REPEATED IRREDUCIBLE QUADRATIC FACTOR

The partial fraction decomposition of $\frac{P(x)}{Q(x)}$, when $Q(x)$ has a repeated irreducible quadratic factor and the degree of $P(x)$ is less than the degree of $Q(x)$, is

$$\frac{P(x)}{(ax^2+bx+c)^n}=\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\frac{A_3x+B_3}{(ax^2+bx+c)^3}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n} \quad (7.5.12)$$

Write the denominators in increasing powers.

How to: decompose a rational expression that has a repeated irreducible factor

1. Use variables like A, B, or C for the constant numerators over linear factors, and linear expressions such as A_1x+B_1, A_2x+B_2, etc., for the numerators of each quadratic factor in the denominator written in increasing powers, such as
$$\frac{P(x)}{Q(x)}=\frac{A}{ax+b}+\frac{A_1x+B_1}{(ax^2+bx+c)}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

Example 7.5.4: Decomposing a Rational Function with a Repeated Irreducible Quadratic Factor in the Denominator

Decompose the given expression that has a repeated irreducible factor in the denominator.
$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}$$

Solution

The factors of the denominator are x, (x^2+1), and $(x^2+1)^2$. Recall that, when a factor in the denominator is a quadratic that includes at least two terms, the numerator must be of the linear form $Ax+B$. So, let's begin the decomposition.
$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{(x^2+1)}+\frac{Dx+E}{(x^2+1)^2}$$

We eliminate the denominators by multiplying each term by $x(x^2+1)^2$. Thus,
$$\begin{aligned} x^4+x^3+x^2-x+1 &= A(x^2+1)^2+(Bx+C)(x)(x^2+1)+(Dx+E)(x) \\ x^4+x^3+x^2-x+1 &= A(x^4+2x^2+1)+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex \qquad \text{Expand the right side.} \\ &= Ax^4+2Ax^2+A+Bx^4+Bx^2+Cx^3+Cx+Dx^2+Ex \end{aligned}$$

Now we will collect like terms.
$$x^4+x^3+x^2-x+1=(A+B)x^4+(C)x^3+(2A+B+D)x^2+(C+E)x+A$$

Set up the system of equations matching corresponding coefficients on each side of the equal sign.
$$\begin{aligned} A+B &= 1 \\ C &= 1 \\ 2A+B+D &= 1 \\ C+E &= -1 \\ A &= 1 \end{aligned}$$

We can use substitution from this point. Substitute $A=1$ into the first equation.
$$\begin{aligned} 1+B &= 1 \\ B &= 0 \end{aligned}$$

Substitute $A=1$ and $B=0$ into the third equation.
$$\begin{aligned} 2(1)+0+D &= 1 \\ D &= -1 \end{aligned}$$

Substitute $C = 1$ into the fourth equation.

$$1 + E = -1$$
$$E = -2$$

Now we have solved for all of the unknowns on the right side of the equal sign. We have $A = 1$, $B = 0$, $C = 1$, $D = -1$, and $E = -2$. We can write the decomposition as follows:

$$\frac{x^4 + x^3 + x^2 - x + 1}{x(x^2+1)^2} = \frac{1}{x} + \frac{1}{(x^2+1)} - \frac{x+2}{(x^2+1)^2}$$

Exercise 7.5.4

Find the partial fraction decomposition of the expression with a repeated irreducible quadratic factor.

$$\frac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2}$$

Answer

$$\frac{x-2}{x^2-2x+3} + \frac{2x+1}{(x^2-2x+3)^2}$$

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Access these online resources for additional instruction and practice with partial fractions.

- Partial Fraction Decomposition
- Partial Fraction Decomposition With Repeated Linear Factors
- Partial Fraction Decomposition With Linear and Quadratic Factors

Key Concepts

- Decompose $\frac{P(x)}{Q(x)}$ by writing the partial fractions as

  $$\frac{A}{a_1 x + b_1} + \frac{B}{a_2 x + b_2}.$$

  Solve by clearing the fractions, expanding the right side, collecting like terms, and setting corresponding coefficients equal to each other, then setting up and solving a system of equations (see Example 7.5.1).
- The decomposition of $\frac{P(x)}{Q(x)}$ with repeated linear factors must account for the factors of the denominator in increasing powers (see Example 7.5.2).
- The decomposition of $\frac{P(x)}{Q(x)}$ with a nonrepeated irreducible quadratic factor needs a linear numerator over the quadratic factor, as in $\frac{A}{x} + \frac{Bx + C}{(ax^2 + bx + c)}$ (see Example 7.5.3).
- In the decomposition of $\frac{P(x)}{Q(x)}$, where $Q(x)$ has a repeated irreducible quadratic factor, when the irreducible quadratic factors are repeated, powers of the denominator factors must be represented in increasing powers as

  $$\frac{A_1 x + B_1}{ax^2 + bx + c} + \frac{A_2 x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_n x + B_n}{(ax^2 + bx + c)^n}$$

  See Example 7.5.4.

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7.6: Matrices and Matrix Operations

Learning Objectives

- Find the sum and difference of two matrices.
- Find scalar multiples of a matrix.
- Find the product of two matrices.

Two club soccer teams, the Wildcats and the Mud Cats, are hoping to obtain new equipment for an upcoming season. Table 7.6.1 shows the needs of both teams.

Table 7.6.1

	Wildcats	Mud Cats
Goals	6	10
Balls	30	24
Jerseys	14	20

A goal costs $300; a ball costs $10; and a jersey costs $30. How can we find the total cost for the equipment needed for each team? In this section, we discover a method in which the data in the soccer equipment table can be displayed and used for calculating other information. Then, we will be able to calculate the cost of the equipment.



Figure 7.6.1

Finding the Sum and Difference of Two Matrices

To solve a problem like the one described for the soccer teams, we can use a matrix, which is a rectangular array of numbers. A row in a matrix is a set of numbers that are aligned horizontally. A column in a matrix is a set of numbers that are aligned vertically. Each number is an entry, sometimes called an element, of the matrix. Matrices (plural) are enclosed in [ ] or ( ), and are usually named with capital letters. For example, three matrices named A, B, and C are shown below.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & -5 & 6 \\ 7 & 8 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 3 \\ 0 & 2 \\ 3 & 1 \end{bmatrix}$$

A matrix is often referred to by its size or dimensions: $m \times n$ indicating m rows and n columns. Matrix entries are defined first by row and then by column. For example, to locate the entry in matrix A identified as a_{ij},we look for the entry in row i,column j. In matrix A, shown below, the entry in row 2, column 3 is a_{23}.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- A **square matrix** is a matrix with dimensions $n \times n$, meaning that it has the same number of rows as columns. The 3×3 matrix above is an example of a square matrix.
- A **row matrix** is a matrix consisting of one row with dimensions $1 \times n$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}$$

- A **column matrix** is a matrix consisting of one column with dimensions $m \times 1$.

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

A matrix may be used to represent a system of equations. In these cases, the numbers represent the coefficients of the variables in the system. Matrices often make solving systems of equations easier because they are not encumbered with variables. We will investigate this idea further in the next section, but first we will look at basic **matrix operations**.

Definition: MATRICES

A *matrix* is a rectangular array of numbers that is usually named by a capital letter: A, B, C,and so on. Each entry in a matrix is referred to as a_{ij},such that i represents the row and j represents the column. Matrices are often referred to by their dimensions: $m \times n$ indicating m rows and n columns.

Example 7.6.1: Finding the Dimensions of the Given Matrix and Locating Entries

Given matrix A:

a. What are the dimensions of matrix A?
b. What are the entries at a_{31} and a_{22}?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 4 & 7 \\ 3 & 1 & -2 \end{bmatrix}$$

Solution

a. The dimensions are 3×3 because there are three rows and three columns.
b. Entry a_{31} is the number at row 3, column 1, which is 3. The entry a_{22} is the number at row 2, column 2, which is 4. Remember, the row comes first, then the column.

Adding and Subtracting Matrices

We use matrices to list data or to represent systems. Because the entries are numbers, we can perform operations on matrices. We add or subtract matrices by adding or subtracting corresponding entries. To do this, the entries must correspond. Therefore, *addition and subtraction of matrices is only possible when the matrices have the same dimensions*. We can add or subtract a 3×3 matrix and another 3×3 matrix, but we cannot add or subtract a 2×3 matrix and a 3×3 matrix because some entries in one matrix will not have a corresponding entry in the other matrix.

ADDING AND SUBTRACTING MATRICES

Given matrices A and B of like dimensions, addition and subtraction of A and B will produce matrix C or matrix D of the same dimension.

$$A+B=C \tag{7.6.1}$$

such that $a_{ij}+b_{ij}=c_{ij}$

$$A-B=D \tag{7.6.2}$$

such that $a_{ij}-b_{ij}=d_{ij}$

Matrix addition is *commutative*.

$$A+B=B+A \tag{7.6.3}$$

It is also *associative*.

$$(A+B)+C=A+(B+C) \tag{7.6.4}$$

Example 7.6.2A: Finding the Sum of Matrices

Find the sum of A and B, given

$$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and

$$B=\begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Solution

Add corresponding entries.

$$\begin{aligned} A+B &= \begin{bmatrix} a & b \\ c & d \end{bmatrix}+\begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ &= \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix} \end{aligned}$$

Example 7.6.2B: Adding Matrix A and Matrix B

Find the sum of A and B.

$$A=\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

and

$$B=\begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix}$$

Solution

Add corresponding entries. Add the entry in row 1, column 1, a_{11}, of matrix A to the entry in row 1, column 1, b_{11}, of B. Continue the pattern until all entries have been added.

$$\begin{aligned} A+B &= \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}+\begin{bmatrix} 5 & 9 \\ 0 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 4+5 & 1+9 \\ 3+0 & 2+7 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 10 \\ 3 & 9 \end{bmatrix} \end{aligned}$$

Example 7.6.2C: Finding the Difference of Two Matrices

Find the difference of A and B.

$$A=\begin{bmatrix}-2 & 3\\ 0 & 1\end{bmatrix} \text{ and } B=\begin{bmatrix}8 & 1\\ 5 & 4\end{bmatrix}$$

Solution

We subtract the corresponding entries of each matrix.

$$\begin{aligned} A-B &= \begin{bmatrix}-2 & 3\\ 0 & 1\end{bmatrix}-\begin{bmatrix}8 & 1\\ 5 & 4\end{bmatrix}\\ &= \begin{bmatrix}-2-8 & 3-1\\ 0-5 & 1-4\end{bmatrix}\\ &= \begin{bmatrix}-10 & 2\\ -5 & -3\end{bmatrix}\end{aligned}$$

Example 7.6.2D: Finding the Sum and Difference of Two 3 x 3 Matrices

Given A and B:

a. Find the sum.
b. Find the difference.

$$A=\begin{bmatrix}2 & -10 & -2\\ 14 & 12 & 10\\ 4 & -2 & 2\end{bmatrix}$$

and

$$B=\begin{bmatrix}6 & 10 & -2\\ 0 & -12 & -4\\ -5 & 2 & -2\end{bmatrix}$$

Solution

a. Add the corresponding entries.

$$\begin{aligned} A+B &= \begin{bmatrix}2 & -10 & -2\\ 14 & 12 & 10\\ 4 & -2 & 2\end{bmatrix}+\begin{bmatrix}6 & 10 & -2\\ 0 & -12 & -4\\ -5 & 2 & -2\end{bmatrix}\\ &= \begin{bmatrix}2+6 & -10+10 & -2-2\\ 14+0 & 12-12 & 10-4\\ 4-5 & -2+2 & 2-2\end{bmatrix}\\ &= \begin{bmatrix}8 & 0 & -4\\ 14 & 0 & 6\\ -1 & 0 & 0\end{bmatrix}\end{aligned}$$

b. Subtract the corresponding entries.

$$A-B=\begin{bmatrix}2&-10&-2\\14&12&10\\4&-2&2\end{bmatrix}-\begin{bmatrix}6&10&-2\\0&-12&-4\\-5&2&-2\end{bmatrix}$$
$$=\begin{bmatrix}2-6&-10-10&-2+2\\14-0&12+12&10+4\\4+5&-2-2&2+2\end{bmatrix}$$
$$=\begin{bmatrix}-4&-20&0\\14&24&14\\9&-4&4\end{bmatrix}$$

? Exercise 7.6.1

Add matrix A and matrix B.

$$A=\begin{bmatrix}2&6\\1&0\\1&-3\end{bmatrix}$$

and

$$B=\begin{bmatrix}3&-2\\1&5\\-4&3\end{bmatrix}$$

Answer

$$A+B=\begin{bmatrix}2&6\\1&0\\1&-3\end{bmatrix}+\begin{bmatrix}3&-2\\1&5\\-4&3\end{bmatrix}$$
$$=\begin{bmatrix}2+3&6+(-2)\\1+1&0+5\\1+(-4)&-3+3\end{bmatrix}$$
$$=\begin{bmatrix}5&4\\2&5\\-3&0\end{bmatrix}$$

Finding Scalar Multiples of a Matrix

Besides adding and subtracting whole matrices, there are many situations in which we need to multiply a matrix by a constant called a scalar. Recall that a scalar is a real number quantity that has magnitude, but not direction. For example, time, temperature, and distance are scalar quantities. The process of scalar multiplication involves multiplying each entry in a matrix by a scalar. A scalar multiple is any entry of a matrix that results from scalar multiplication.

Consider a real-world scenario in which a university needs to add to its inventory of computers, computer tables, and chairs in two of the campus labs due to increased enrollment. They estimate that 15 more equipment is needed in both labs. The school's current inventory is displayed in Table 7.6.2.

Table 7.6.2

	Lab A	Lab B
Computers	15	27
Computer Tables	16	34
Chairs	16	34

Converting the data to a matrix, we have

$$C_{2013} = \begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix}$$

To calculate how much computer equipment will be needed, we multiply all entries in matrix C by 0.15.

$$(0.15)C_{2013} = \begin{bmatrix} (0.15)15 & (0.15)27 \\ (0.15)16 & (0.15)34 \\ (0.15)16 & (0.15)34 \end{bmatrix} = \begin{bmatrix} 2.25 & 4.05 \\ 2.4 & 5.1 \\ 2.4 & 5.1 \end{bmatrix}$$

We must round up to the next integer, so the amount of new equipment needed is

$$\begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix}$$

Adding the two matrices as shown below, we see the new inventory amounts.

$$\begin{bmatrix} 15 & 27 \\ 16 & 34 \\ 16 & 34 \end{bmatrix} + \begin{bmatrix} 3 & 5 \\ 3 & 6 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix}$$

This means

$$C_{2014} = \begin{bmatrix} 18 & 32 \\ 19 & 40 \\ 19 & 40 \end{bmatrix}$$

Thus, Lab A will have 18 computers, 19 computer tables, and 19 chairs; Lab B will have 32 computers, 40 computer tables, and 40 chairs.

SCALAR MULTIPLICATION

Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the scalar multiple cA is

$$cA = c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$
$$= \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

Scalar multiplication is distributive. For the matrices A, B,and C with scalars a and b,

$$a(A+B) = aA + aB \tag{7.6.5}$$
$$(a+b)A = aA + bA \tag{7.6.6}$$

Example 7.6.3: Multiplying the Matrix by a Scalar

Multiply matrix A by the scalar 3.

$$A = \begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix}$$

Solution

Multiply each entry in A by the scalar 3.

$$\begin{aligned} 3A &= 3\begin{bmatrix} 8 & 1 \\ 5 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 3\cdot 8 & 3\cdot 1 \\ 3\cdot 5 & 3\cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 24 & 3 \\ 15 & 12 \end{bmatrix} \end{aligned}$$

Exercise 7.6.2

Given matrix B, find $-2B$ where

$$B = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$$

Answer

$$-2B = \begin{bmatrix} -8 & -2 \\ -6 & -4 \end{bmatrix}$$

Example 7.6.4: Finding the Sum of Scalar Multiples

Find the sum $3A + 2B$.

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 1 & -4 \end{bmatrix}$$

Solution

First, find $3A$, then $2B$.

$$\begin{aligned} 3A &= \begin{bmatrix} 3\cdot 1 & 3(-2) & 3\cdot 0 \\ 3\cdot 0 & 3(-1) & 3\cdot 2 \\ 3\cdot 4 & 3\cdot 3 & 3(-6) \end{bmatrix} \\ &= \begin{bmatrix} 3 & -6 & 0 \\ 0 & -3 & 6 \\ 12 & 9 & -18 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} 2B &= \begin{bmatrix} 2(-1) & 2\cdot 2 & 2\cdot 1 \\ 2\cdot 0 & 2(-3) & 2\cdot 2 \\ 2\cdot 0 & 2\cdot 1 & 2(-4) \end{bmatrix} \\ &= \begin{bmatrix} -2 & 4 & 2 \\ 0 & -6 & 4 \\ 0 & 2 & -8 \end{bmatrix} \end{aligned}$$

Now, add $3A + 2B$.

$$3A+2B=\begin{bmatrix}3&-6&0\\0&-3&6\\12&9&-18\end{bmatrix}+\begin{bmatrix}-2&4&2\\0&-6&4\\0&2&-8\end{bmatrix}$$
$$=\begin{bmatrix}3-2&-6+4&0+2\\0+0&-3-6&6+4\\12+0&9+2&-18-8\end{bmatrix}$$
$$=\begin{bmatrix}1&-2&2\\0&-9&10\\12&11&-26\end{bmatrix}$$

Finding the Product of Two Matrices

In addition to multiplying a matrix by a scalar, we can multiply two matrices. Finding the product of two matrices is only possible when the inner dimensions are the same, meaning that the number of columns of the first matrix is equal to the number of rows of the second matrix. If A is an $m\times r$ matrix and B is an $r\times n$ matrix, then the product matrix AB is an $m\times n$ matrix. For example, the product AB is possible because the number of columns in A is the same as the number of rows in B. If the inner dimensions do not match, the product is not defined.

$$\underset{2\times 3}{A}\quad\cdot\quad\underbrace{\underset{3\times 3}{B}}_{\text{same}}$$

We multiply entries of A with entries of B according to a specific pattern as outlined below. The process of **matrix multiplication** becomes clearer when working a problem with real numbers.

To obtain the entries in row i of AB, we multiply the entries in row i of A by column j in B and add. For example, given matrices A and B, where the dimensions of A are 2×3 and the dimensions of B are 3×3 ,the product of AB will be a 2×3 matrix.

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\end{bmatrix}$$

and

$$B=\begin{bmatrix}b_{11}&b_{12}&b_{13}\\b_{21}&b_{22}&b_{23}\\b_{31}&b_{32}&b_{33}\end{bmatrix}$$

Multiply and add as follows to obtain the first entry of the product matrix AB.

1. To obtain the entry in row 1, column 1 of AB,multiply the first row in A by the first column in B,and add.

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\end{bmatrix}\cdot\begin{bmatrix}b_{11}\\b_{21}\\b_{31}\end{bmatrix}=a_{11}\cdot b_{11}+a_{12}\cdot b_{21}+a_{13}\cdot b_{31}$$

2. To obtain the entry in row 1, column 2 of AB,multiply the first row of A by the second column in B,and add.

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\end{bmatrix}\cdot\begin{bmatrix}b_{12}\\b_{22}\\b_{32}\end{bmatrix}=a_{11}\cdot b_{12}+a_{12}\cdot b_{22}+a_{13}\cdot b_{32}$$

3. To obtain the entry in row 1, column 3 of AB,multiply the first row of A by the third column in B,and add.

$$\begin{bmatrix}a_{11}&a_{12}&a_{13}\end{bmatrix}\cdot\begin{bmatrix}b_{13}\\b_{23}\\b_{33}\end{bmatrix}=a_{11}\cdot b_{13}+a_{12}\cdot b_{23}+a_{13}\cdot b_{33}$$

We proceed the same way to obtain the second row of AB. In other words, row 2 of A times column 1 of B; row 2 of A times column 2 of B; row 2 of A times column 3 of B. When complete, the product matrix will be

$$AB = \begin{bmatrix} a_{11}\cdot b_{11}+a_{12}\cdot b_{21}+a_{13}\cdot b_{31} & a_{11}\cdot b_{12}+a_{12}\cdot b_{22}+a_{13}\cdot b_{32} & a_{11}\cdot b_{13}+a_{12}\cdot b_{23}+a_{13}\cdot b_{33} \\ a_{21}\cdot b_{11}+a_{22}\cdot b_{21}+a_{23}\cdot b_{31} & a_{21}\cdot b_{12}+a_{22}\cdot b_{22}+a_{23}\cdot b_{32} & a_{21}\cdot b_{13}+a_{22}\cdot b_{23}+a_{23}\cdot b_{33} \end{bmatrix}$$

PROPERTIES OF MATRIX MULTIPLICATION

For the matrice A, B,and C the following properties hold.

- Matrix multiplication is *associative*:

$$(AB)C = A(BC). \tag{7.6.7}$$

- Matrix multiplication is *distributive*:

$$C(A+B) = CA+CB \tag{7.6.8}$$

$$(A+B)C = AC+BC. \tag{7.6.9}$$

Note that matrix multiplication is not commutative.

Example 7.6.5A: Multiplying Two Matrices

Multiply matrix A and matrix B.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution

First, we check the dimensions of the matrices. Matrix A has dimensions 2×2 and matrix B has dimensions 2×2. The inner dimensions are the same so we can perform the multiplication. The product will have the dimensions 2×2.

We perform the operations outlined previously.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\cdot\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ &= \begin{bmatrix} 1(5)+2(7) & 1(6)+2(8) \\ 3(5)+4(7) & 3(6)+4(8) \end{bmatrix} \\ &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \end{aligned}$$

Example 7.6.5B: Multiplying Two Matrices

Given A and B:

a. Find AB.
b. Find BA.

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix}$$

Solution

a. As the dimensions of A are 2×3 and the dimensions of B are 3×2,these matrices can be multiplied together because the number of columns in A matches the number of rows in B. The resulting product will be a 2×2 matrix, the number of rows in A by the number of columns in B.

$$\begin{aligned} AB &= \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} -1(5)+2(-4)+3(2) & -1(-1)+2(0)+3(3) \\ 4(5)+0(-4)+5(2) & 4(-1)+0(0)+5(3) \end{bmatrix} \\ &= \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \end{aligned}$$

b. The dimensions of B are 3×2 and the dimensions of A are 2×3. The inner dimensions match so the product is defined and will be a 3×3 matrix.

$$\begin{aligned} BA &= \begin{bmatrix} 5 & -1 \\ -4 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 5(-1)+-1(4) & 5(2)+-1(0) & 5(3)+-1(5) \\ -4(-1)+0(4) & -4(2)+0(0) & -4(3)+0(5) \\ 2(-1)+3(4) & 2(2)+3(0) & 2(3)+3(5) \end{bmatrix} \\ &= \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} \end{aligned}$$

Analysis

Notice that the products AB and BA are not equal.

$$AB = \begin{bmatrix} -7 & 10 \\ 30 & 11 \end{bmatrix} \neq \begin{bmatrix} -9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21 \end{bmatrix} = BA$$

This illustrates the fact that matrix multiplication is **not** commutative.

Q&A: Is it possible for AB to be defined but not BA?

Yes, consider a matrix A with dimension 3×4 and matrix B with dimension 4×2. For the product AB the inner dimensions are 4 and the product is defined, but for the product BA the inner dimensions are 2 and 3 so the product is undefined.

Example 7.6.6: Using Matrices in Real-World Problems

Let's return to the problem presented at the opening of this section. We have Table $7.6.3$, representing the equipment needs of two soccer teams.

Table 7.6.3

	Wildcats	Mud Cats
Goals	6	10

	Wildcats	Mud Cats
Balls	30	24
Jerseys	14	20

We are also given the prices of the equipment, as shown in Table 7.6.4.

Table 7.6.4

Goal	$300
Ball	$10
Jersey	$30

We will convert the data to matrices. Thus, the equipment need matrix is written as

$$E = \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix}$$

The cost matrix is written as

$$C = \begin{bmatrix} 300 & 10 & 30 \end{bmatrix}$$

We perform matrix multiplication to obtain costs for the equipment.

$$\begin{aligned} CE &= \begin{bmatrix} 300 & 10 & 30 \end{bmatrix} \cdot \begin{bmatrix} 6 & 10 \\ 30 & 24 \\ 14 & 20 \end{bmatrix} \\ &= \begin{bmatrix} 300(6)+10(30)+30(14) & 300(10)+10(24)+30(20) \end{bmatrix} \\ &= \begin{bmatrix} 2,520 & 3,840 \end{bmatrix} \end{aligned}$$

The total cost for equipment for the Wildcats is $\$2,520$, and the total cost for equipment for the Mud Cats is $\$3,840$.

How to: Given a matrix operation, evaluate using a calculator

1. Save each matrix as a matrix variable $[A], [B], [C], \ldots$
2. Enter the operation into the calculator, calling up each matrix variable as needed.
3. If the operation is defined, the calculator will present the solution matrix; if the operation is undefined, it will display an error message.

Example 7.6.7: Using a Calculator to Perform Matrix Operations

Find $AB - C$ given

$$A = \begin{bmatrix} -15 & 25 & 32 \\ 41 & -7 & -28 \\ 10 & 34 & -2 \end{bmatrix}, B = \begin{bmatrix} 45 & 21 & -37 \\ -24 & 52 & 19 \\ 6 & -48 & -31 \end{bmatrix}, \text{ and } C = \begin{bmatrix} -100 & -89 & -98 \\ 25 & -56 & 74 \\ -67 & 42 & -75 \end{bmatrix}$$

Solution

On the matrix page of the calculator, we enter matrix A above as the matrix variable $[A]$, matrix B above as the matrix variable $[B]$, and matrix C above as the matrix variable $[C]$.

On the home screen of the calculator, we type in the problem and call up each matrix variable as needed.

$$[A] \times [B] - [C]$$

The calculator gives us the following matrix.

$$\begin{bmatrix} -983 & -462 & 136 \\ 1,820 & 1,897 & -856 \\ -311 & 2,032 & 413 \end{bmatrix}$$

Media

Access these online resources for additional instruction and practice with matrices and matrix operations.

- Dimensions of a Matrix
- Matrix Addition and Subtraction
- Matrix Operations
- Matrix Multiplication

Key Concepts

- A matrix is a rectangular array of numbers. Entries are arranged in rows and columns.
- The dimensions of a matrix refer to the number of rows and the number of columns. A 3×2 matrix has three rows and two columns. See Example $7.6.1$.
- We add and subtract matrices of equal dimensions by adding and subtracting corresponding entries of each matrix. See Example $7.6.2$, Example $7.6.3$, Example $7.6.4$, and Example $7.6.5$.
- Scalar multiplication involves multiplying each entry in a matrix by a constant. See Example $7.6.6$.
- Scalar multiplication is often required before addition or subtraction can occur. See Example $7.6.7$.
- Multiplying matrices is possible when inner dimensions are the same—the number of columns in the first matrix must match the number of rows in the second.
- The product of two matrices, A and B,is obtained by multiplying each entry in row 1 of A by each entry in column 1 of B; then multiply each entry of row 1 of A by each entry in columns 2 of B,and so on. See Example $7.6.8$ and Example $7.6.9$.
- Many real-world problems can often be solved using matrices. See Example $7.6.10$.
- We can use a calculator to perform matrix operations after saving each matrix as a matrix variable. See Example $7.6.11$.

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7.7: Solving Systems with Gaussian Elimination

Learning Objectives

- Write the augmented matrix of a system of equations.
- Write the system of equations from an augmented matrix.
- Perform row operations on a matrix.
- Solve a system of linear equations using matrices.

Carl Friedrich Gauss lived during the late 18th century and early 19th century, but he is still considered one of the most prolific mathematicians in history. His contributions to the science of mathematics and physics span fields such as algebra, number theory, analysis, differential geometry, astronomy, and optics, among others. His discoveries regarding matrix theory changed the way mathematicians have worked for the last two centuries.



Figure 7.7.1: German mathematician Carl Friedrich Gauss (1777–1855).

We first encountered Gaussian elimination in Systems of Linear Equations: Two Variables. In this section, we will revisit this technique for solving systems, this time using matrices.

Writing the Augmented Matrix of a System of Equations

A matrix can serve as a device for representing and solving a system of equations. To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs. When a system is written in this form, we call it an **augmented matrix**.

For example, consider the following 2×2 system of equations.

$$\begin{aligned} 3x+4y &= 7 \\ 4x-2y &= 5 \end{aligned}$$

We can write this system as an augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 4 & 7 \\ 4 & -2 & 5 \end{array}\right]$$

We can also write a matrix containing just the coefficients. This is called the **coefficient matrix.**

$$\begin{bmatrix} 3 & 4 \\ 4 & -2 \end{bmatrix}$$

A three-by-three **system of equations** such as

$$\begin{aligned} 3x-y-z &= 0 \\ x+y &= 5 \\ 2x-3z &= 2 \end{aligned}$$

has a coefficient matrix

$$\begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & -3 \end{bmatrix}$$

and is represented by the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 2 \end{array}\right]$$

Notice that the matrix is written so that the variables line up in their own columns: x-terms go in the first column, y-terms in the second column, and z-terms in the third column. It is very important that each equation is written in standard form $ax+by+cz=d$ so that the variables line up. When there is a missing variable term in an equation, the coefficient is 0.

How to: Given a system of equations, write an augmented matrix

1. Write the coefficients of the x-terms as the numbers down the first column.
2. Write the coefficients of the y-terms as the numbers down the second column.
3. If there are z-terms, write the coefficients as the numbers down the third column.
4. Draw a vertical line and write the constants to the right of the line.

Example 7.7.1: Writing the Augmented Matrix for a System of Equations

Write the augmented matrix for the given system of equations.

$$\begin{aligned} x+2y-z &= 3 \\ 2x-y+2z &= 6 \\ x-3y+3z &= 4 \end{aligned}$$

Solution

The augmented matrix displays the coefficients of the variables, and an additional column for the constants.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -1 & 2 & 6 \\ 1 & -3 & 3 & 4 \end{array}\right]$$

Exercise 7.7.1

Write the augmented matrix of the given system of equations.

$$\begin{aligned} 4x-3y &= 11 \\ 3x+2y &= 4 \end{aligned}$$

Answer

$$\left[\begin{array}{cc|c} 4 & -3 & 11 \\ 3 & 2 & 4 \end{array}\right]$$

Writing a System of Equations from an Augmented Matrix

We can use augmented matrices to help us solve systems of equations because they simplify operations when the systems are not encumbered by the variables. However, it is important to understand how to move back and forth between formats in order to make finding solutions smoother and more intuitive. Here, we will use the information in an augmented matrix to write the system of equations in standard form.

Example 7.7.2: Writing a System of Equations from an Augmented Matrix Form

Find the system of equations from the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{array}\right]$$

Solution

When the columns represent the variables x, y, and z,

$$\left[\begin{array}{ccc|c} 1 & -3 & -5 & -2 \\ 2 & -5 & -4 & 5 \\ -3 & 5 & 4 & 6 \end{array}\right] \rightarrow \begin{array}{r} x-3y-5z=-2 \\ 2x-5y-4z=5 \\ -3x+5y+4z=6 \end{array}$$

Exercise 7.7.2

Write the system of equations from the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & -1 & 3 & 1 \\ 0 & 1 & 1 & -9 \end{array}\right]$$

Answer

$$\begin{aligned} x-y+z &= 5 \\ 2x-y+3z &= 1 \\ y+z &= -9 \end{aligned}$$

Performing Row Operations on a Matrix

Now that we can write systems of equations in augmented matrix form, we will examine the various **row operations** that can be performed on a matrix, such as addition, multiplication by a constant, and interchanging rows.

Performing row operations on a matrix is the method we use for solving a system of equations. In order to solve the system of equations, we want to convert the matrix to **row-echelon** form, in which there are ones down the **main diagonal** from the upper left corner to the lower right corner, and zeros in every position below the main diagonal as shown.

$$\text{Row-echelon form } \begin{bmatrix} 1 & a & b \\ 0 & 1 & d \\ 0 & 0 & 1 \end{bmatrix}$$

We use row operations corresponding to equation operations to obtain a new matrix that is **row-equivalent** in a simpler form. Here are the guidelines to obtaining row-echelon form.

1. In any nonzero row, the first nonzero number is a 1. It is called a *leading* 1.
2. Any all-zero rows are placed at the bottom on the matrix.
3. Any leading 1 is below and to the right of a previous leading 1.
4. Any column containing a leading 1 has zeros in all other positions in the column.

To solve a system of equations we can perform the following row operations to convert the **coefficient matrix** to row-echelon form and do back-substitution to find the solution.

1. Interchange rows. (Notation: $R_i \leftrightarrow R_j$)
2. Multiply a row by a constant. (Notation: cR_i)
3. Add the product of a row multiplied by a constant to another row. (Notation: $R_i + cR_j$)

Each of the row operations corresponds to the operations we have already learned to solve systems of equations in three variables. With these operations, there are some key moves that will quickly achieve the goal of writing a matrix in row-echelon form. To

obtain a matrix in row-echelon form for finding solutions, we use Gaussian elimination, a method that uses row operations to obtain a 1 as the first entry so that row 1 can be used to convert the remaining rows.

GAUSSIAN ELIMINATION

The Gaussian elimination method refers to a strategy used to obtain the row-echelon form of a matrix. The goal is to write matrix A with the number 1 as the entry down the main diagonal and have all zeros below.

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{After\ Gaussian\ elimination} A=\begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

The first step of the Gaussian strategy includes obtaining a 1 as the first entry, so that row 1 may be used to alter the rows below.

How to: Given an augmented matrix, perform row operations to achieve row-echelon form

1. The first equation should have a leading coefficient of 1. Interchange rows or multiply by a constant, if necessary.
2. Use row operations to obtain zeros down the first column below the first entry of 1.
3. Use row operations to obtain a 1 in row 2, column 2.
4. Use row operations to obtain zeros down column 2, below the entry of 1.
5. Use row operations to obtain a 1 in row 3, column 3.
6. Continue this process for all rows until there is a \(1 in every entry down the main diagonal and there are only zeros below.
7. If any rows contain all zeros, place them at the bottom.

Example 7.7.3: Solving a 2×2 System by Gaussian Elimination

Solve the given system by Gaussian elimination.

$$\begin{aligned} 2x+3y &= 6 \\ x-y &= \frac{1}{2} \end{aligned}$$

Solution

First, we write this as an augmented matrix.

$$\left[\begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{array}\right]$$

We want a 1 in row 1, column 1. This can be accomplished by interchanging row 1 and row 2.

$$R_1 \leftrightarrow R_2 \rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{array}\right]$$

We now have a 1 as the first entry in row 1, column 1. Now let's obtain a 0 in row 2, column 1. This can be accomplished by multiplying row 1 by -2, and then adding the result to row 2.

$$-2R_1+R_2=R_2 \rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{array}\right]$$

We only have one more step, to multiply row 2 by $\frac{1}{5}$.

$$\frac{1}{5}R_2=R_2 \rightarrow \left[\begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 1 \end{array}\right]$$

Use back-substitution. The second row of the matrix represents $y = 1$. Back-substitute $y = 1$ into the first equation.

$$x - (1) = \frac{1}{2}$$
$$x = \frac{3}{2}$$

The solution is the point $\left(\frac{3}{2}, 1\right)$.

Exercise 7.7.3

Solve the given system by Gaussian elimination.

$$4x + 3y = 11$$
$$x - 3y = -1$$

Answer

$(2, 1)$

Example 7.7.4: Using Gaussian Elimination to Solve a System of Equations

Use **Gaussian elimination** to solve the given 2×2 **system of equations**.

$$2x + y = 1$$
$$4x + 2y = 6$$

Solution

Write the system as an **augmented matrix**.

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 6 \end{array}\right]$$

Obtain a 1 in row 1, column 1. This can be accomplished by multiplying the first row by $\frac{1}{2}$.

$$\frac{1}{2}R_1 = R_1 \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 4 & 2 & 6 \end{array}\right]$$

Next, we want a 0 in row 2, column 1. Multiply row 1 by -4 and add row 1 to row 2.

$$-4R_1 + R_2 = R_2 \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 4 \end{array}\right]$$

The second row represents the equation $0 = 4$. Therefore, the system is inconsistent and has no solution.

Example 7.7.5: Solving a Dependent System

Solve the system of equations.

$$3x + 4y = 12$$
$$6x + 8y = 24$$

Solution

Perform **row operations** on the augmented matrix to try and achieve **row-echelon form**.

$$A = \left[\begin{array}{cc|c} 3 & 4 & 12 \\ 6 & 8 & 24 \end{array}\right]$$

$$-\frac{1}{2}R_2+R_1=R_1\rightarrow\left[\begin{array}{cc|c}0&0&0\\6&8&24\end{array}\right]$$

$$R_1\leftrightarrow R_2=\left[\begin{array}{cc|c}6&8&24\\0&0&0\end{array}\right]$$

The matrix ends up with all zeros in the last row: $0y=0$. Thus, there are an infinite number of solutions and the system is classified as dependent. To find the generic solution, return to one of the original equations and solve for y.

$$\begin{aligned}3x+4y&=12\\4y&=12-3x\\y&=3-\frac{3}{4}x\end{aligned}$$

So the solution to this system is $\left(x,3-\frac{3}{4}x\right)$.

Example 7.7.6: Performing Row Operations on a 3×3 Augmented Matrix to Obtain Row-Echelon Form

Perform row operations on the given matrix to obtain row-echelon form.

$$\left[\begin{array}{ccc|c}1&-3&4&3\\2&-5&6&6\\-3&3&4&6\end{array}\right]$$

Solution

The first row already has a 1 in row 1, column 1. The next step is to multiply row 1 by -2 and add it to row 2. Then replace row 2 with the result.

$$-2R_1+R_2=R_2\left[\begin{array}{ccc|c}1&-3&4&3\\0&1&-2&0\\-3&3&4&6\end{array}\right]$$

Next, obtain a zero in row 3, column 1.

$$3R_1+R_3=R_3\left[\begin{array}{ccc|c}1&-3&4&3\\0&1&-2&0\\0&-6&16&15\end{array}\right]$$

Next, obtain a zero in row 3, column 2.

$$6R_2+R_3=R_3\left[\begin{array}{ccc|c}1&-3&4&3\\0&1&-2&0\\0&0&4&15\end{array}\right]$$

The last step is to obtain a 1 in row 3, column 3.

$$\frac{1}{4}R_3=R_3\left[\begin{array}{ccc|c}1&-3&4&3\\0&1&-2&0\\0&0&1&\frac{15}{4}\end{array}\right]$$

Exercise 7.7.4

Write the system of equations in row-echelon form.

$$\begin{aligned} x-2y+3z &= 9 \\ -x+3y &= -4 \\ 2x-5y+5z &= 17 \end{aligned}$$

Answer

$$\left[\begin{array}{cccc} 1 & -\frac{5}{2} & \frac{5}{2} & \frac{17}{2} \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Solving a System of Linear Equations Using Matrices

We have seen how to write a **system of equations** with an **augmented matrix**, and then how to use row operations and back-substitution to obtain row-echelon form. Now, we will take **row-echelon form** a step farther to solve a 3 by 3 system of linear equations. The general idea is to eliminate all but one variable using row operations and then back-substitute to solve for the other variables.

Example 7.7.7: Solving a System of Linear Equations Using Matrices

Solve the system of linear equations using matrices.

$$\begin{aligned} x-y+z &= 8 \\ 2x+3y-z &= -2 \\ 3x-2y-9z &= 9 \end{aligned}$$

Solution

First, we write the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array}\right]$$

Next, we perform row operations to obtain row-echelon form.

$$-2R_1+R_2=R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 3 & -2 & -9 & 9 \end{array}\right]$$

$$-3R_1+R_3=R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 0 & 1 & -12 & -15 \end{array}\right]$$

The easiest way to obtain a 1 in row 2 of column 1 is to interchange R_2 and R_3.

$$Interchange\ R_2\ and\ R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18 \end{array}\right]$$

Then

$$-5R_2+R_3=R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{array}\right]$$

$$-\frac{1}{57}R_3=R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

The last matrix represents the equivalent system.

$$\begin{aligned} x-y+z &= 8 \\ y-12z &= -15 \\ z &= 1 \end{aligned}$$

Using back-substitution, we obtain the solution as $(4,-3,1)$.

Example 7.7.8: Solving a Dependent System of Linear Equations Using Matrices

Solve the following system of linear equations using matrices.

$$\begin{aligned} -x-2y+z &= -1 \\ 2x+3y &= 2 \\ y-2z &= 0 \end{aligned}$$

Solution

Write the augmented matrix.

$$\left[\begin{array}{ccc|c} -1 & -2 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

First, multiply row 1 by -1 to get a 1 in row 1, column 1. Then, perform **row operations** to obtain row-echelon form.

$$-R_1 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 2 & 3 & 0 & 2 \end{array}\right]$$

$$-2R_1+R_3=R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$R_2+R_3=R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix represents the following system.

$$\begin{aligned} x+2y-z &= 1 \\ y-2z &= 0 \\ 0 &= 0 \end{aligned}$$

We see by the identity $0=0$ that this is a dependent system with an infinite number of solutions. We then find the generic solution. By solving the second equation for y and substituting it into the first equation we can solve for z in terms of x.

$$\begin{aligned} x+2y-z &= 1 \\ y &= 2z \\ x+2(2z)-z &= 1 \\ x+3z &= 1 \\ z &= \frac{1-x}{3} \end{aligned}$$

Now we substitute the expression for z into the second equation to solve for y in terms of x.

$$y-2z=0$$
$$z=\frac{1-x}{3}$$
$$y-2\left(\frac{1-x}{3}\right)=0$$
$$y=\frac{2-2x}{3}$$

The generic solution is $\left(x,\frac{2-2x}{3},\frac{1-x}{3}\right)$.

? Exercise 7.7.5

Solve the system using matrices.

$$\begin{aligned}x+4y-z&=4\\2x+5y+8z&=1\\5x+3y-3z&=1\end{aligned}$$

Answer

$(1,1,1)$

Q&A: Can any system of linear equations be solved by Gaussian elimination?

Yes, a system of linear equations of any size can be solved by Gaussian elimination.

How to: Given a system of equations, solve with matrices using a calculator

1. Save the augmented matrix as a matrix variable $[A]$, $[B]$, $[C]$, ...
2. Use the **ref(** function in the calculator, calling up each matrix variable as needed.

Example 7.7.9A: Solving Systems of Equations with Matrices Using a Calculator

Solve the system of equations.

$$\begin{aligned}5x+3y+9z&=-1\\-2x+3y-z&=-2\\-x-4y+5z&=1\end{aligned}$$

Solution

Write the augmented matrix for the system of equations.

$$\left[\begin{array}{ccc|c}5&3&9&-1\\-2&3&-1&-2\\-1&-4&5&1\end{array}\right]$$

On the matrix page of the calculator, enter the augmented matrix above as the matrix variable $[A]$.

$$[A]=\left[\begin{array}{ccc|c}5&3&9&-1\\-2&3&-1&-2\\-1&-4&5&1\end{array}\right]$$

Use the **ref(** function in the calculator, calling up the matrix variable $[A]$.

ref([A])

Evaluate

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{9}{5} & \frac{1}{5} \\ 0 & 1 & \frac{13}{21} & -\frac{4}{7} \\ 0 & 0 & 1 & -\frac{24}{187} \end{array}\right] \rightarrow \begin{aligned} x+\frac{3}{5}y+\frac{9}{5}z &= -\frac{1}{5} \\ y+\frac{13}{21}z &= -\frac{4}{7} \\ z &= -\frac{24}{187} \end{aligned}$$

Using back-substitution, the solution is $\left(\frac{61}{187}, -\frac{92}{187}, -\frac{24}{187}\right)$.

Example 7.7.9B: Applying 2×2 Matrices to Finance

Carolyn invests a total of $\$12,000$ in two municipal bonds, one paying 10.5 interest and the other paying 12 interest. The annual interest earned on the two investments last year was $\$1,335$. How much was invested at each rate?

Solution

We have a system of two equations in two variables. Let $x =$ the amount invested at 10.5 interest, and $y =$ the amount invested at 12 interest.

$$\begin{aligned} x+y &= 12,000 \\ 0.105x+0.12y &= 1,335 \end{aligned}$$

As a matrix, we have

$$\left[\begin{array}{cc|c} 1 & 1 & 12,000 \\ 0.105 & 0.12 & 1,335 \end{array}\right]$$

Multiply row 1 by -0.105 and add the result to row 2.

$$\left[\begin{array}{cc|c} 1 & 1 & 12,000 \\ 0 & 0.015 & 75 \end{array}\right]$$

Then,

$$\begin{aligned} 0.015y &= 75 \\ y &= 5,000 \end{aligned}$$

So $12,000-5,000=7,000$.

Thus, $\$5,000$ was invested at 12 interest and $\$7,000$ at 10.5 interest.

Example 7.7.10: Applying 3×3 Matrices to Finance

Ava invests a total of $\$10,000$ in three accounts, one paying 5 interest, another paying 8 interest, and the third paying 9 interest. The annual interest earned on the three investments last year was $\$770$. The amount invested at 9 was twice the amount invested at 5. How much was invested at each rate?

Solution

We have a system of three equations in three variables. Let x be the amount invested at 5 interest, let y be the amount invested at 8 interest, and let z be the amount invested at 9 interest. Thus,

$$\begin{aligned} x+y+z &= 10,000 \\ 0.05x+0.08y+0.09z &= 770 \\ 2x-z &= 0 \end{aligned}$$

As a matrix, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10,000 \\ 0.05 & 0.08 & 0.09 & 770 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

Now, we perform Gaussian elimination to achieve row-echelon form.

$$-0.05R_1 + R_2 = R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10,000 \\ 0 & 0.03 & 0.04 & 270 \\ 2 & 0 & -1 & 0 \end{array}\right]$$

$$-2R_1 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10,000 \\ 0 & 0.03 & 0.04 & 270 \\ 0 & -2 & -3 & -20,000 \end{array}\right]$$

$$\frac{1}{0.03}R_2 = R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10,000 \\ 0 & 1 & \frac{4}{3} & 9,000 \\ 0 & -2 & -3 & -20,000 \end{array}\right]$$

$$2R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10,000 \\ 0 & 1 & \frac{4}{3} & 9,000 \\ 0 & 0 & -\frac{1}{3} & -2,000 \end{array}\right]$$

The third row tells us $-\frac{1}{3}z = -2,000$; thus $z = 6,000$.

The second row tells us $y + \frac{4}{3}z = 9,000$. Substituting $z = 6,000$,we get

$$\begin{aligned} y + \frac{4}{3}(6,000) &= 9,000 \\ y + 8,000 &= 9,000 \\ y &= 1,000 \end{aligned}$$

The first row tells us $x + y + z = 10,000$. Substituting $y = 1,000$ and $z = 6,000$,we get

$$\begin{aligned} x + 1,000 + 6,000 &= 10,000 \\ x &= 3,000 \end{aligned}$$

The answer is $\$3,000$invested at 5 interest, $\$1,000$invested at 8, and $\$6,000$invested at 9 interest.

? Exercise 7.7.6

A small shoe company took out a loan of $\$1,500,000$to expand their inventory. Part of the money was borrowed at 7, part was borrowed at 8, and part was borrowed at 10. The amount borrowed at 10 was four times the amount borrowed at 7, and the annual interest on all three loans was $\$130,500$ Use matrices to find the amount borrowed at each rate.

Answer

$\$150,000$at 7, $\$750,000$at 8, $\$600,000$at 10

Media

Access these online resources for additional instruction and practice with solving systems of linear equations using Gaussian elimination.

- Solve a System of Two Equations Using an Augmented Matrix
- Solve a System of Three Equations Using an Augmented Matrix

- Augmented Matrices on the Calculator

Key Concepts

- An augmented matrix is one that contains the coefficients and constants of a system of equations. See Example 7.7.1.
- A matrix augmented with the constant column can be represented as the original system of equations. See Example 7.7.2.
- Row operations include multiplying a row by a constant, adding one row to another row, and interchanging rows.
- We can use Gaussian elimination to solve a system of equations. See Example 7.7.3, Example 7.7.4, and Example 7.7.5.
- Row operations are performed on matrices to obtain row-echelon form. See Example 7.7.6.
- To solve a system of equations, write it in augmented matrix form. Perform row operations to obtain row-echelon form. Back-substitute to find the solutions. See Example 7.7.7 and Example 7.7.8.
- A calculator can be used to solve systems of equations using matrices. See Example 7.7.9.
- Many real-world problems can be solved using augmented matrices. See Example 7.7.10 and Example 7.7.11.

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7.8: Solving Systems with Inverses

Learning Objectives

- Find the inverse of a matrix.
- Solve a system of linear equations using an inverse matrix

Nancy plans to invest $\$10,500$ into two different bonds to spread out her risk. The first bond has an annual return of 10, and the second bond has an annual return of 6. To receive an 8.5 return from the two bonds, how much should Nancy invest in each bond? What is the best method to solve this problem? There are several ways we can solve this problem. As we have seen in previous sections, systems of equations and matrices are useful in solving real-world problems involving finance. After studying this section, we will have the tools to solve the bond problem using the inverse of a matrix.

Finding the Inverse of a Matrix

We know that the multiplicative inverse of a real number a is a^{-1}, so

$$aa^{-1} = a^{-1}a = \left(\frac{1}{a}\right)a = 1 \tag{7.8.1}$$

For example, consider the scalar multiplication situation

$$2^{-1} = \frac{1}{2}$$

therefore from Equation 7.8.1

$$\left(\frac{1}{2}\right)2 = 1.$$

The **multiplicative inverse of a matrix** is similar in concept, except that the product of matrix A and its inverse A^{-1} equals the **identity matrix.** The identity matrix is a square matrix containing ones down the main diagonal and zeros everywhere else. We identify identity matrices by I_n where n represents the dimension of the matrix. Equations 7.8.2 and 7.8.3 are the identity matrices for a 2×2 matrix and a 3×3 matrix, respectively:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.8.2}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.8.3}$$

The identity matrix acts as a 1 in matrix algebra. For example,

$$AI = IA = A$$

A matrix that has a multiplicative inverse has the properties

$$AA^{-1} = I \tag{7.8.4}$$

$$A^{-1}A = I \tag{7.8.5}$$

A matrix that has a multiplicative inverse is called an **invertible matrix**. Only a square matrix may have a multiplicative inverse, as the reversibility,

$$AA^{-1} = A^{-1}A = I \tag{7.8.6}$$

is a requirement. Not all square matrices have an inverse, but if A is invertible, then A^{-1} is unique. We will look at two methods for finding the inverse of a 2×2 matrix and a third method that can be used on both 2×2 and 3×3 matrices.

Definitions: THE IDENTITY MATRIX AND MULTIPLICATIVE INVERSE

The *identity matrix*, I_n, is a square matrix containing ones down the main diagonal and zeros everywhere else.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

as for the 2×2 identity matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as for the 3×3 identity matrix

If A is an $n \times n$ matrix and B is an $n \times n$ matrix such that $AB = BA = I_n$, then $B = A - 1$, the **multiplicative inverse of a matrix A.**

Example 7.8.1: Showing That the Identity Matrix Acts as a 1

Given matrix A, show that $AI = IA = A$.

$$A = \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$$

Solution

Use matrix multiplication to show that the product of A and the identity matric is equal to the product of the identity matrix and A.

$$\begin{aligned} AI &= \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \cdot 1 + 4 \cdot 0 & 3 \cdot 0 + 4 \cdot 1 \\ -2 \cdot 1 + 5 \cdot 0 & -2 \cdot 0 + 5 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} AI &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 3 + 0 \cdot (-2) & 1 \cdot 4 + 0 \cdot 5 \\ 0 \cdot 3 + 1 \cdot (-2) & 0 \cdot 4 + 1 \cdot 5 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix} \end{aligned}$$

How to: Given two matrices, show that one is the multiplicative inverse of the other

- Given matrix A of order $n \times n$ and matrix B of order $n \times n$ multiply AB.
- If $AB = I$, then find the product BA. If $BA = I$, then $B = A^{-1}$ and $A = B^{-1}$.

Example 7.8.2: Showing That Matrix A Is the Multiplicative Inverse of Matrix B

Show that the given matrices are multiplicative inverses of each other.

$$A = \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix}$$

Solution

Multiply AB and BA. If both products equal the identity, then the two matrices are inverses of each other.

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \cdot \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-9)+5(2) & 1(-5)+5(1) \\ -2(-9)-9(2) & -2(-5)-9(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} BA &= \begin{bmatrix} -9 & -5 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ -2 & -9 \end{bmatrix} \\ &= \begin{bmatrix} -9(1)-5(-2) & -9(5)-5(-9) \\ 2(1)+1(-2) & 2(-5)+1(-9) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

A and B are inverses of each other.

? Exercise 7.8.1

Show that the following two matrices are inverses of each other.

$$A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}$$

Answer

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1(-3)+4(1) & 1(-4)+4(1) \\ -1(-3)+-3(1) & -1(-4)+-3(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -3(1)+-4(-1) & -3(4)+-4(-3) \\ 1(1)+1(-1) & 1(4)+1(-3) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Finding the Multiplicative Inverse Using Matrix Multiplication

We can now determine whether two matrices are inverses, but how would we find the inverse of a given matrix? Since we know that the product of a matrix and its inverse is the identity matrix, we can find the inverse of a matrix by setting up an equation using **matrix multiplication**.

Example 7.8.3: Finding the Multiplicative Inverse Using Matrix Multiplication

Use matrix multiplication to find the inverse of the given matrix.

$$A=\begin{bmatrix}1 & -2\\ 2 & -3\end{bmatrix}$$

Solution

For this method, we multiply A by a matrix containing unknown constants and set it equal to the identity.

$$\begin{bmatrix}1 & -2\\ 2 & -3\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Find the product of the two matrices on the left side of the equal sign.

$$\begin{bmatrix}1 & -2\\ 2 & -3\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}1a-2c & 1b-2d\\ 2a-3c & 2b-3d\end{bmatrix}$$

Next, set up a system of equations with the entry in row 1, column 1 of the new matrix equal to the first entry of the identity, 1. Set the entry in row 2, column 1 of the new matrix equal to the corresponding entry of the identity, which is 0.

$$1a-2c=1\ R_1$$

$$2a-3c=0\ R_2$$

Using row operations, multiply and add as follows: $(-2)R_1+R_2\rightarrow R_2$. Add the equations, and solve for c.

$$\begin{aligned}1a-2c&=1\\ 0+1c&=-2\\ c&=-2\end{aligned}$$

Back-substitute to solve for a.

$$\begin{aligned}a-2(-2)&=1\\ a+4&=1\\ a&=-3\end{aligned}$$

Write another system of equations setting the entry in row 1, column 2 of the new matrix equal to the corresponding entry of the identity, 0. Set the entry in row 2, column 2 equal to the corresponding entry of the identity.

$$1b-2d=0\ R_1$$

$$2b-3d=1\ R_2$$

Using row operations, multiply and add as follows: $(-2)R_1+R_2=R_2$. Add the two equations and solve for d.

$$\begin{aligned}1b-2d&=0\\ 0+1d&=1\\ d&=1\end{aligned}$$

Once more, back-substitute and solve for b.

$$b-2(1)=0$$
$$b-2=0$$
$$b=2$$
$$A^{-1}=\begin{bmatrix}-3 & 2\\ -2 & 1\end{bmatrix}$$

Finding the Multiplicative Inverse by Augmenting with the Identity

Another way to find the **multiplicative inverse** is by augmenting with the identity. When matrix A is transformed into I, the augmented matrix I transforms into A^{-1}.

For example, given

$$A=\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}$$

augment A with the identity

$$\left[\begin{array}{cc|cc}2 & 1 & 1 & 0\\ 5 & 3 & 0 & 1\end{array}\right]$$

Perform row operations with the goal of turning A into the identity.

1. Switch row 1 and row 2.

$$\left[\begin{array}{cc|cc}5 & 3 & 0 & 1\\ 2 & 1 & 1 & 0\end{array}\right]$$

2. Multiply row 2 by −2 and add to row 1.

$$\left[\begin{array}{cc|cc}1 & 1 & -2 & 1\\ 2 & 1 & 1 & 0\end{array}\right]$$

3. Multiply row 1 by −2 and add to row 2.

$$\left[\begin{array}{cc|cc}1 & 1 & -2 & 1\\ 0 & -1 & 5 & -2\end{array}\right]$$

4. Add row 2 to row 1.

$$\left[\begin{array}{cc|cc}1 & 0 & 3 & -1\\ 0 & -1 & 5 & -2\end{array}\right]$$

5. Multiply row 2 by−1. −1.

$$\left[\begin{array}{cc|cc}1 & 0 & 3 & -1\\ 0 & 1 & -5 & 2\end{array}\right]$$

The matrix we have found is A^{-1}.

$$A^{-1}=\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}$$

Finding the Multiplicative Inverse of 2×2 Matrices Using a Formula

When we need to find the **multiplicative inverse** of a 2×2 matrix, we can use a special formula instead of using matrix multiplication or augmenting with the identity.

If A is a 2×2 matrix, such as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the multiplicative inverse of A is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $ad - bc \neq 0$. If $ad - bc = 0$, then A has no inverse.

Example 7.8.4: Using the Formula to Find the Multiplicative Inverse of Matrix A

Use the formula to find the multiplicative inverse of

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

Solution

We can check that our formula works by using one of the other methods to calculate the inverse. Let's augment A with the identity.

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{array}\right]$$

Perform row operations with the goal of turning A into the identity.

1. Multiply row 1 by -2 and add to row 2.

$$\left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

2. Multiply row 1 by 2 and add to row 1.

$$\left[\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

So, we have verified our original solution.

$$A^{-1} = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

Exercise 7.8.2

Use the formula to find the inverse of matrix A. Verify your answer by augmenting with the identity matrix.

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$$

Answer

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$$

Example 7.8.5: Finding the Inverse of the Matrix, If It Exists

Find the inverse, if it exists, of the given matrix.

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$$

Solution

We will use the method of augmenting with the identity.

$$\left[\begin{array}{cc|cc} 3 & 6 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right]$$

1. Switch row 1 and row 2.

$$\left[\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 3 & 6 & 1 & 0 \end{array}\right]$$

2. Multiply row 1 by -3 and add it to row 2.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{array}\right]$$

3. There is nothing further we can do. The zeros in row 2 indicate that this matrix has no inverse.

Finding the Multiplicative Inverse of 3×3 Matrices

Unfortunately, we do not have a formula similar to the one for a 2×2 matrix to find the inverse of a 3×3 matrix. Instead, we will augment the original matrix with the identity matrix and use row operations to obtain the inverse.

Given a 3×3 matrix

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

augment A with the identity matrix

$$A \left| I \right. = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

To begin, we write the **augmented matrix** with the identity on the right and A on the left. Performing elementary **row operations** so that the **identity matrix** appears on the left, we will obtain the **inverse matrix** on the right. We will find the inverse of this matrix in the next example.

How to: Given a 3×3 matrix, find the inverse

1. Write the original matrix augmented with the identity matrix on the right.
2. Use elementary row operations so that the identity appears on the left.
3. What is obtained on the right is the inverse of the original matrix.
4. Use matrix multiplication to show that $AA^{-1} = I$ and $A^{-1}A = I$.

Example 7.8.6: Finding the Inverse of a 3×3 Matrix

Given the 3×3 matrix A, find the inverse.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

Solution

Augment A with the identity matrix, and then begin row operations until the identity matrix replaces A. The matrix on the right will be the inverse of A.

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Interchange } R_2 \text{ and } R_1} \left[\begin{array}{ccc|ccc} 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-R_2 + R_1 = R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$-R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$-2R_1 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0 \end{array}\right]$$

$$-3R_2 + R_3 = R_3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3 \end{array}\right]$$

Thus,

$$A^{-1} = B = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

Analysis

To prove that $B = A^{-1}$, let's multiply the two matrices together to see if the product equals the identity, if $AA^{-1} = I$ and $A^{-1}A = I$.

$$AA^{-1} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)+3(-1)+1(6) & 2(1)+3(0)+1(-2) & 2(0)+3(1)+1(-3) \\ 3(-1)+3(-1)+1(6) & 3(1)+3(0)+1(-2) & 3(0)+3(1)+1(-3) \\ 2(-1)+4(-1)+1(6) & 2(1)+4(0)+1(-2) & 2(0)+4(1)+1(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & & & \end{bmatrix}$$

$$A^{-1}A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \begin{bmatrix} & 2 & 31 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1(2)+1(3)+0(2) & -1(3)+1(3)+0(4) & -1(1)+1(1)+0(1) \\ -1(2)+0(3)+1(2) & -1(3)+0(3)+1(4) & -1(1)+0(1)+1(1) \\ 6(2)+-2(3)+-3(2) & 6(3)+-2(3)+-3(4) & 6(1)+-2(1)+-3(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

? Exercise 7.8.3

Find the inverse of the 3×3 matrix.

$$A = \begin{bmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{bmatrix}$$

Answer

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}$$

Solving a System of Linear Equations Using the Inverse of a Matrix

Solving a system of linear equations using the inverse of a matrix requires the definition of two new matrices: X is the matrix representing the variables of the system, and B is the matrix representing the constants. Using **matrix multiplication**, we may define a system of equations with the same number of equations as variables as

$$AX = B$$

To solve a system of linear equations using an inverse matrix, let A be the coefficient matrix, let X be the variable matrix, and let B be the constant matrix. Thus, we want to solve a system $AX = B$. For example, look at the following system of equations.

$$a_1x + b_1y = c_1$$

$$a_2x + b_2y = c_2$$

From this system, the coefficient matrix is

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The variable matrix is

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$

And the constant matrix is

$$B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Then $AX = B$ looks like

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Recall the discussion earlier in this section regarding multiplying a real number by its inverse, $(2^{-1})2 = \left(\frac{1}{2}\right) 2 = 1$. To solve a single linear equation $ax = b$ for x, we would simply multiply both sides of the equation by the multiplicative inverse (reciprocal) of a. Thus,

$$\begin{aligned} ax &= b \\ \left(\frac{1}{a}\right) ax &= \left(\frac{1}{a}\right) b \\ \left(a^{-1}\right) ax &= \left(a^{-1}\right) b \\ \left[\left(a^{-1}\right) a\right] x &= \left(a^{-1}\right) b \\ 1x &= \left(a^{-1}\right) b \\ x &= \left(a^{-1}\right) b \end{aligned}$$

The only difference between a solving a linear equation and a system of equations written in matrix form is that finding the inverse of a matrix is more complicated, and matrix multiplication is a longer process. However, the goal is the same—to isolate the variable.

We will investigate this idea in detail, but it is helpful to begin with a 2×2 system and then move on to a 3×3 system.

SOLVING A SYSTEM OF EQUATIONS USING THE INVERSE OF A MATRIX

Given a system of equations, write the coefficient matrix A, the variable matrix X, and the constant matrix B. Then

$$AX = B$$

Multiply both sides by the inverse of A to obtain the solution.

$$\begin{aligned} \left(A^{-1}\right) AX &= \left(A^{-1}\right) B \\ \left[\left(A^{-1}\right) A\right] X &= \left(A^{-1}\right) B \\ IX &= \left(A^{-1}\right) B \\ X &= \left(A^{-1}\right) B \end{aligned}$$

Q&A: If the coefficient matrix does not have an inverse, does that mean the system has no solution?

No, if the coefficient matrix is not invertible, the system could be inconsistent and have no solution, or be dependent and have infinitely many solutions.

Example 7.8.7: Solving a 2×2 System Using the Inverse of a Matrix

Solve the given system of equations using the inverse of a matrix.

$$\begin{aligned} 3x+8y &= 5 \\ 4x+11y &= 7 \end{aligned}$$

Solution

Write the system in terms of a coefficient matrix, a variable matrix, and a constant matrix.

$$A=\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}, B=\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

First, we need to calculate A^{-1}. Using the formula to calculate the inverse of a 2 by 2 matrix, we have:

$$\begin{aligned} A^{-1} &= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3(11)-8(4)}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \\ &= \frac{1}{1}\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix} \end{aligned}$$

So,

$$A^{-1}=\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}$$

Now we are ready to solve. Multiply both sides of the equation by A^{-1}.

$$\left(A^{-1}\right)AX=\left(A^{-1}\right)B$$

$$\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ 4 & 11 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 11(5)+(-8)7 \\ -4(5)+3(7) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The solution is $(-1,1)$.

Q&A: Can we solve for X by finding the product BA^{-1}?

No, recall that matrix multiplication is not commutative, so $A^{-1}B \neq BA^{-1}$. Consider our steps for solving the matrix equation.

$$\begin{aligned} \left(A^{-1}\right)AX &= \left(A^{-1}\right)B \\ \left[\left(A^{-1}\right)A\right]X &= \left(A^{-1}\right)B \\ IX &= \left(A^{-1}\right)B \\ X &= \left(A^{-1}\right)B \end{aligned}$$

Notice in the first step we multiplied both sides of the equation by A^{-1}, but the A^{-1} was to the left of A on the left side and to the left of B on the right side. Because matrix multiplication is not commutative, order matters.

Example 7.8.8: Solving a 3 × 3 System Using the Inverse of a Matrix

Solve the following system using the inverse of a matrix.

$$\begin{aligned} 5x+15y+56z &= 35 \\ -4x-11y-41z &= -26 \\ -x-3y-11z &= -7 \end{aligned}$$

Solution

Write the equation $AX = B$.

$$\begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

First, we will find the inverse of A by augmenting with the identity.

$$\left[\begin{array}{ccc|ccc} 5 & 15 & 56 & 1 & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by $\frac{1}{5}$.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Multiply row 1 by 4 and add to row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1 \end{array}\right]$$

Add row 1 to row 3.

$$\left[\begin{array}{ccc|ccc} 1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array}\right]$$

Multiply row 2 by -3 and add to row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1 \end{array}\right]$$

Multiply row 3 by 5.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

Multiply row 3 by $\frac{1}{5}$ and add to row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

Multiply row 3 by $-\frac{19}{5}$ and add to row 2.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & 1 & -19 \\ 0 & 0 & 1 & 1 & 0 & 5 \end{array}\right]$$

So,

$$A^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix}$$

Multiply both sides of the equation by A^{-1}. We want $A^{-1}AX = A^{-1}B$:

$$\begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 35 \\ -26 \\ -7 \end{bmatrix}$$

Thus,

$$A^{-1}B = \begin{bmatrix} -70+78-7 \\ -105-26+133 \\ 35+0-35 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

The solution is $(1, 2, 0)$.

Exercise 7.8.4

Solve the system using the inverse of the coefficient matrix.

$$\begin{aligned} 2x - 17y + 11z &= 0 \\ -x + 11y - 7z &= 8 \\ 3y - 2z &= -2 \end{aligned}$$

Answer

$$X = \begin{bmatrix} 4 \\ 38 \\ 58 \end{bmatrix}$$

How to: Given a system of equations, solve with matrix inverses using a calculator

1. Save the coefficient matrix and the constant matrix as matrix variables $[A]$ and $[B]$.
2. Enter the multiplication into the calculator, calling up each matrix variable as needed.
3. If the coefficient matrix is invertible, the calculator will present the solution matrix; if the coefficient matrix is not invertible, the calculator will present an error message.

Example 7.8.9: Using a Calculator to Solve a System of Equations with Matrix Inverses

Solve the system of equations with matrix inverses using a calculator

$$\begin{aligned} 2x+3y+z &= 32 \\ 3x+3y+z &= -27 \\ 2x+4y+z &= -2 \end{aligned}$$

Solution

On the matrix page of the calculator, enter the coefficient matrix as the matrix variable $[A]$, and enter the constant matrix as the matrix variable $[B]$.

$$[A] = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}, [B] = \begin{bmatrix} 32 \\ -27 \\ -2 \end{bmatrix}$$

On the home screen of the calculator, type in the multiplication to solve for X, calling up each matrix variable as needed.

$$[A]^{-1} \times [B]$$

Evaluate the expression.

$$\begin{bmatrix} -59 \\ -34 \\ 252 \end{bmatrix}$$

Media

Access these online resources for additional instruction and practice with solving systems with inverses.

- The Identity Matrix
- Determining Inverse Matrices
- Using a Matrix Equation to Solve a System of Equations

Key Equations

Identity matrix for a 2×2 matrix	$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Identity matrix for a 3×3 matrix	$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Multiplicative inverse of a 2×2 matrix	$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, where $ad - bc \neq 0$

Key Concepts

- An identity matrix has the property $AI = IA = A$. See Example $7.8.1$.
- An invertible matrix has the property $AA^{-1} = A^{-1}A = I$. See Example $7.8.2$.
- Use matrix multiplication and the identity to find the inverse of a 2×2 matrix. See Example $7.8.3$.
- The multiplicative inverse can be found using a formula. See Example $7.8.4$.
- Another method of finding the inverse is by augmenting with the identity. See Example $7.8.5$.
- We can augment a 3×3 matrix with the identity on the right and use row operations to turn the original matrix into the identity, and the matrix on the right becomes the inverse. See Example $7.8.6$.
- Write the system of equations as $AX = B$, and multiply both sides by the inverse of A: $A^{-1}AX = A^{-1}B$. See Example $7.8.7$ and Example $7.8.8$.
- We can also use a calculator to solve a system of equations with matrix inverses. See Example $7.8.9$.

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7.9: Solving Systems with Cramer's Rule

Learning Objectives

- Evaluate 2 × 2 determinants.
- Use Cramer's Rule to solve a system of equations in two variables.
- Evaluate 3 × 3 determinants.
- Use Cramer's Rule to solve a system of three equations in three variables.
- Know the properties of determinants.

We have learned how to solve systems of equations in two variables and three variables, and by multiple methods: substitution, addition, Gaussian elimination, using the inverse of a matrix, and graphing. Some of these methods are easier to apply than others and are more appropriate in certain situations. In this section, we will study two more strategies for solving systems of equations.

Evaluating the Determinant of a 2 × 2 Matrix

A determinant is a real number that can be very useful in mathematics because it has multiple applications, such as calculating area, volume, and other quantities. Here, we will use determinants to reveal whether a matrix is invertible by using the entries of a square matrix to determine whether there is a solution to the system of equations. Perhaps one of the more interesting applications, however, is their use in cryptography. Secure signals or messages are sometimes sent encoded in a matrix. The data can only be decrypted with an invertible matrix and the determinant. For our purposes, we focus on the determinant as an indication of the invertibility of the matrix. Calculating the determinant of a matrix involves following the specific patterns that are outlined in this section.

FIND THE DETERMINANT OF A 2 × 2 MATRIX

The *determinant* of a 2 × 2 matrix, given

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is defined as

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

Notice the change in notation. There are several ways to indicate the determinant, including $\det(A)$ and replacing the brackets in a matrix with straight lines, $|A|$.

Example 7.9.1: Finding the Determinant of a 2×2 Matrix

Find the determinant of the given matrix.

$$A = \begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}$$

Solution

$$\begin{aligned} \det(A) &= \begin{vmatrix} 5 & 2 \\ -6 & 3 \end{vmatrix} \\ &= 5(3) - (-6)(2) \\ &= 27 \end{aligned}$$

Using Cramer's Rule to Solve a System of Two Equations in Two Variables

We will now introduce a final method for solving systems of equations that uses determinants. Known as **Cramer's Rule**, this technique dates back to the middle of the 18th century and is named for its innovator, the Swiss mathematician Gabriel Cramer (1704-1752), who introduced it in 1750 in *Introduction à l'Analyse des lignes Courbes algébriques*. Cramer's Rule is a viable and

efficient method for finding solutions to systems with an arbitrary number of unknowns, provided that we have the same number of equations as unknowns.

Cramer's Rule will give us the unique solution to a system of equations, if it exists. However, if the system has no solution or an infinite number of solutions, this will be indicated by a determinant of zero. To find out if the system is inconsistent or dependent, another method, such as elimination, will have to be used.

To understand Cramer's Rule, let's look closely at how we solve systems of linear equations using basic row operations. Consider a system of two equations in two variables.

$$a_1x+b_1y=c_1 (1) \tag{7.9.1}$$
$$a_2x+b_2y=c_2 (2) \tag{7.9.2}$$

We eliminate one variable using row operations and solve for the other. Say that we wish to solve for x. If Equation 7.9.2 is multiplied by the opposite of the coefficient of y in Equation 7.9.1, Equation 7.9.1 is multiplied by the coefficient of y in Equation 7.9.2, and we add the two equations, the variable y will be eliminated.

$$\begin{array}{rll} b_2a_1x+b_2b_1y=b_2c_1 & & \text{Multiply } R_1 \text{ by } b_2 \\ \underline{-b_1a_2x-b_1b_2y=-b_1c_2} & & \text{Multiply } R_2 \text{ by } -b_1 \\ b_2a_1x-b_1a_2x=b_2c_1-b_1c_2 & & \end{array}$$

Now, solve for x.

$$\begin{aligned} b_2a_1x-b_1a_2x &= b_2c_1-b_1c_2 \\ x(b_2a_1-b_1a_2) &= b_2c_1-b_1c_2 \end{aligned}$$
$$x=\frac{b_2c_1-b_1c_2}{b_2a_1-b_1a_2}=\frac{\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}$$

Similarly, to solve for y,we will eliminate x.

$$\begin{array}{rll} a_2a_1x+a_2b_1y=a_2c_1 & & \text{Multiply } R_1 \text{ by } a_2 \\ \underline{-a_1a_2x-a_1b_2y=-a_1c_2} & & \text{Multiply } R_2 \text{ by } -a_1 \\ a_2b_1y-a_1b_2y=a_2c_1-a_1c_2 & & \end{array}$$

Solving for y gives

$$\begin{aligned} a_2b_1y-a_1b_2y &= a_2c_1-a_1c_2 \\ y(a_2b_1-a_1b_2) &= a_2c_1-a_1c_2 \end{aligned}$$
$$y=\frac{a_2c_1-a_1c_2}{a_2b_1-a_1b_2}=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}=\frac{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}$$

Notice that the denominator for both x and y is the determinant of the coefficient matrix.

We can use these formulas to solve for x and y, but Cramer's Rule also introduces new notation:

- D:determinant of the coefficient matrix
- D_x:determinant of the numerator in the solution of x

$$x=\frac{D_x}{D} \tag{7.9.3}$$

- D_y:determinant of the numerator in the solution of y

$$y=\frac{D_y}{D} \tag{7.9.4}$$

The key to Cramer's Rule is replacing the variable column of interest with the constant column and calculating the determinants. We can then express x and y as a quotient of two determinants.

CRAMER'S RULE FOR 2×2 SYSTEMS

Cramer's Rule is a method that uses determinants to solve systems of equations that have the same number of equations as variables.

Consider a system of two linear equations in two variables.

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$

The solution using Cramer's Rule is given as

$$x = \frac{D_x}{D} = \frac{\begin{bmatrix} c_1 & b_1 \\ c_2 & b_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}, D \neq 0 \tag{7.9.5}$$

$$y = \frac{D_y}{D} = \frac{\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix}}{\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}}, D \neq 0 \tag{7.9.6}$$

If we are solving for x, the x column is replaced with the constant column. If we are solving for y, the y column is replaced with the constant column.

Example 7.9.2: Using Cramer's Rule to Solve a 2×2 System

Solve the following 2×2 system using Cramer's Rule.

$$\begin{aligned} 12x + 3y &= 15 \\ 2x - 3y &= 13 \end{aligned}$$

Solution

Solve for x.

$$\begin{aligned} x &= \frac{D_x}{D} \\ &= \frac{\begin{bmatrix} 15 & 3 \\ 13 & -3 \end{bmatrix}}{\begin{bmatrix} 12 & 3 \\ 2 & -3 \end{bmatrix}} \\ &= \frac{-45 - 39}{-36 - 6} \\ &= \frac{-84}{-42} \\ &= 2 \end{aligned}$$

Solve for y.

$$
\begin{aligned}
y &= \frac{D_y}{D} \\
&= \frac{\begin{bmatrix} 12 & 15 \\ 2 & 13 \end{bmatrix}}{\begin{bmatrix} 12 & 3 \\ 2 & -3 \end{bmatrix}} \\
&= \frac{156-30}{-36-6} \\
&= -\frac{126}{42} \\
&= -3
\end{aligned}
$$

The solution is $(2, -3)$.

? Exercise 7.9.1

Use Cramer's Rule to solve the 2×2 system of equations.

$$
\begin{aligned}
x + 2y &= -11 \\
-2x + y &= -13
\end{aligned}
$$

Answer

$(3, -7)$

Evaluating the Determinant of a 3 × 3 Matrix

Finding the determinant of a 2×2 matrix is straightforward, but finding the determinant of a 3×3 matrix is more complicated. One method is to augment the 3×3 matrix with a repetition of the first two columns, giving a 3×5 matrix. Then we calculate the sum of the products of entries *down* each of the three diagonals (upper left to lower right), and subtract the products of entries *up* each of the three diagonals (lower left to upper right). This is more easily understood with a visual and an example.

Find the determinant of the 3×3 matrix.

$$
A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}
$$

1. Augment A with the first two columns.

$$
\det(A) = \left| \begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array} \right|
$$

2. From upper left to lower right: Multiply the entries down the first diagonal. Add the result to the product of entries down the second diagonal. Add this result to the product of the entries down the third diagonal.
3. From lower left to upper right: Subtract the product of entries up the first diagonal. From this result subtract the product of entries up the second diagonal. From this result, subtract the product of entries up the third diagonal.

$$
\det(A) = \left| \begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array} \right|
$$

The algebra is as follows:

$$
|A| = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1
$$

Example 7.9.3: Finding the Determinant of a 3 × 3 Matrix

Find the determinant of the 3×3 matrix given

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & -1 & 1 \\ 4 & 0 & 1 \end{bmatrix}$$

Solution

Augment the matrix with the first two columns and then follow the formula. Thus,

$$\begin{aligned} |A| &= \left|\begin{array}{ccc|cc} 0 & 2 & 1 & 0 & 2 \\ 3 & -1 & 1 & 3 & -1 \\ 4 & 0 & 1 & 4 & 0 \end{array}\right| \\ &= 0(-1)(1)+2(1)(4)+1(3)(0)-4(-1)(1)-0(1)(0)-1(3)(2) \\ &= 0+8+0+4-0-6 \\ &= 6 \end{aligned}$$

Exercise 7.9.2

Find the determinant of the 3 × 3 matrix.

$$\det(A) = \begin{vmatrix} 1 & -3 & 7 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

Answer

-10

Q&A: Can we use the same method to find the determinant of a larger matrix?

No, this method only works for 2 × 2 and 3 × 3 matrices. For larger matrices it is best to use a graphing utility or computer software.

Using Cramer's Rule to Solve a System of Three Equations in Three Variables

Now that we can find the determinant of a 3×3 matrix, we can apply Cramer's Rule to solve a system of three equations in three variables. Cramer's Rule is straightforward, following a pattern consistent with Cramer's Rule for 2×2 matrices. As the order of the matrix increases to 3×3, however, there are many more calculations required.

When we calculate the determinant to be zero, Cramer's Rule gives no indication as to whether the system has no solution or an infinite number of solutions. To find out, we have to perform elimination on the system.

Consider a 3×3 system of equations.

$$a_1x + b_1y + c_1z = d_1 \tag{7.9.7}$$
$$a_2x + b_2y + c_2z = d_2 \tag{7.9.8}$$
$$a_3x + b_3y + c_3z = d_3 \tag{7.9.9}$$

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}, D \neq 0$$

where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \tag{7.9.10}$$

If we are writing the determinant D_x,we replace the x column with the constant column. If we are writing the determinant D_y,we replace they y column with the constant column. If we are writing the determinant D_z,we replace the z column with the constant column. Always check the answer.

Example 7.9.4: Solving a 3×3 System Using Cramer's Rule

Find the solution to the given 3×3 system using Cramer's Rule.

$$\begin{aligned} x+y-z &= 6 \\ 3x-2y+z &= -5 \\ x+3y-2z &= 14 \end{aligned}$$

Solution

Use Cramer's Rule.

$$D = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -2 & 1 \\ 1 & 3 & -2 \end{vmatrix}, D_x = \begin{vmatrix} 6 & 1 & -1 \\ -5 & -2 & 1 \\ 14 & 3 & -2 \end{vmatrix}, D_y = \begin{vmatrix} 1 & 6 & -1 \\ 3 & -5 & 1 \\ 1 & 14 & -2 \end{vmatrix}, D_z = \begin{vmatrix} 1 & 1 & 6 \\ 3 & -2 & -5 \\ 1 & 3 & 14 \end{vmatrix}$$

Then,

$$\begin{aligned} x = \frac{D_x}{D} &= \frac{-3}{-3} = 1 \\ y = \frac{D_y}{D} &= \frac{-9}{-3} = 3 \\ z = \frac{D_z}{D} &= \frac{6}{-3} = -2 \end{aligned}$$

The solution is $(1, 3, -2)$.

Exercise 7.9.3

Use Cramer's Rule to solve the 3×3 matrix.

$$\begin{aligned} x-3y+7z &= 13 \\ x+y+z &= 1 \\ x-2y+3z &= 4 \end{aligned}$$

Answer

$$\left(-2, \frac{3}{5}, \frac{12}{5}\right)$$

Example 7.9.5A: Using Cramer's Rule to Solve an Inconsistent System

Solve the system of equations using Cramer's Rule.

$$3x-2y = 4 \quad (7.9.11)$$
$$6x-4y = 0 \quad (7.9.12)$$

Solution

We begin by finding the determinants D, D_x,and D_y.

$$D = \begin{vmatrix} 3 & -2 \\ 6 & -4 \end{vmatrix} = 3(-4) - 6(-2) = 0$$

We know that a determinant of zero means that either the system has no solution or it has an infinite number of solutions. To see which one, we use the process of elimination. Our goal is to eliminate one of the variables.

1. Multiply Equation 7.9.11 by -2.

2. Add the result to Equation 7.9.12.

$$\begin{aligned} -6x+4y &= -8 \\ \underline{6x-4y} &= \underline{0} \\ 0 &= -8 \end{aligned}$$

We obtain the equation $0=-8$, which is false. Therefore, the system has no solution. Graphing the system reveals two parallel lines. See Figure 7.9.1.

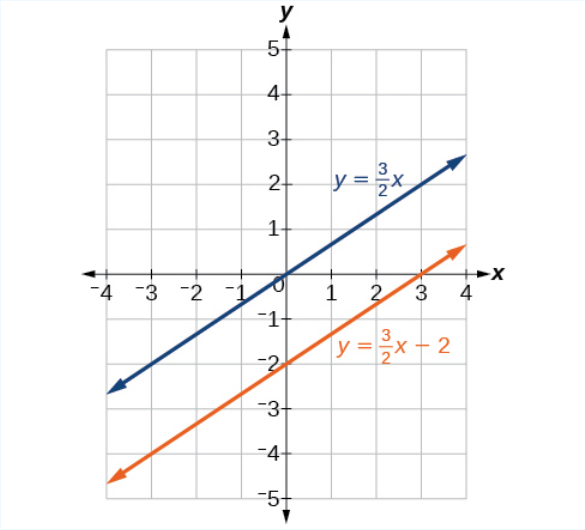


Figure 7.9.1

Example 7.9.5B: Use Cramer's Rule to Solve a Dependent System

Solve the system with an infinite number of solutions.

$$x-2y+3z=0 \quad (7.9.13)$$
$$3x+y-2z=0 \quad (7.9.14)$$
$$2x-4y+6z=0 \quad (7.9.15)$$

Solution

Let's find the determinant first. Set up a matrix augmented by the first two columns.

$$\left|\begin{array}{ccc|cc} 1 & -2 & 3 & 1 & -2 \\ 3 & 1 & -2 & 3 & 1 \\ 2 & -4 & 6 & 2 & -4 \end{array}\right|$$

Then,

$$1(1)(6)+(-2)(-2)(2)+3(3)(-4)-2(1)(3)-(-4)(-2)(1)-6(3)(-2)=0$$

As the determinant equals zero, there is either no solution or an infinite number of solutions. We have to perform elimination to find out.

1. Multiply Equation 7.9.13 by -2 and add the result to Equation 7.9.15:

$$\begin{aligned} -2x+4y-6x &= 0 \\ \underline{2x-4y+6z} &= \underline{0} \\ 0 &= 0 \end{aligned}$$

2. Obtaining an answer of $0=0$, a statement that is always true, means that the system has an infinite number of solutions. Graphing the system, we can see that two of the planes are the same and they both intersect the third plane on a line. See Figure 7.9.2.

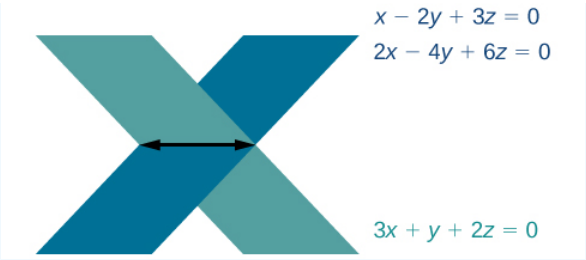


Figure 7.9.2

Understanding Properties of Determinants

There are many properties of determinants. Listed here are some properties that may be helpful in calculating the determinant of a matrix.

PROPERTIES OF DETERMINANTS

1. If the matrix is in upper triangular form, the determinant equals the product of entries down the main diagonal.
2. When two rows are interchanged, the determinant changes sign.
3. If either two rows or two columns are identical, the determinant equals zero.
4. If a matrix contains either a row of zeros or a column of zeros, the determinant equals zero.
5. The determinant of an inverse matrix A^{-1} is the reciprocal of the determinant of the matrix A.
6. If any row or column is multiplied by a constant, the determinant is multiplied by the same factor.

Example 7.9.6: Illustrating Properties of Determinants

Illustrate each of the properties of determinants.

Solution

Property 1 states that if the matrix is in upper triangular form, the determinant is the product of the entries down the main diagonal.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Augment A with the first two columns.

$$A = \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 0 & 2 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 \end{array}\right]$$

Then

$$\begin{aligned} \det(A) &= 1(2)(-1)+2(1)(0)+3(0)(0)-0(2)(3)-0(1)(1)+1(0)(2) \\ &= -2 \end{aligned}$$

Property 2 states that interchanging rows changes the sign. Given

$$\begin{aligned} A &= \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix} \\ \det(A) &= (-1)(-3)-(4)(5) \\ &= 3-20 \\ &= -17 \end{aligned}$$

$$\begin{aligned} B &= \begin{bmatrix} 4 & -3 \\ -1 & 5 \end{bmatrix} \\ \det(B) &= (4)(5)-(-1)(-3) \\ &= 20-3 \\ &= 17 \end{aligned}$$

Property 3 states that if two rows or two columns are identical, the determinant equals zero.

$$\begin{aligned} A &= \left[\begin{array}{ccc|cc} 1 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ -1 & 2 & 2 & -1 & 2 \end{array}\right] \\ \det(A) &= 1(2)(2)+2(2)(-1)+2(2)(2)+1(2)(2)-2(2)(1)-2(2)(2) \\ &= 4-4+8+4-4-8 \\ &= 0 \end{aligned}$$

Property 4 states that if a row or column equals zero, the determinant equals zero. Thus,

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \\ \det(A) &= 1(0)-2(0) \\ &= 0 \end{aligned}$$

Property 5 states that the determinant of an inverse matrix A^{-1} is the reciprocal of the determinant A. Thus,

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \det(A) &= 1(4)-3(2) \\ &= -2 \end{aligned}$$

$$\begin{aligned} A^{-1} &= \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \\ \det(A^{-1}) &= -2\left(-\frac{1}{2}\right)-\frac{3}{2}(1) \\ &= -\frac{1}{2} \end{aligned}$$

Property 6 states that if any row or column of a matrix is multiplied by a constant, the determinant is multiplied by the same factor. Thus,

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ \det(A) &= 1(4)-2(3) \\ &= -2 \end{aligned}$$

$$\begin{aligned} B &= \begin{bmatrix} 2(1) & 2(2) \\ 3 & 4 \end{bmatrix} \\ \det(B) &= 2(4)-3(4) \\ &= -4 \end{aligned}$$

Example 7.9.7: Using Cramer's Rule and Determinant Properties to Solve a System

Find the solution to the given 3×3 system.

$$2x+4y+4z = 2 \tag{7.9.16}$$
$$3x+7y+7z = -5 \tag{7.9.17}$$
$$x+2y+2z = 4 \tag{7.9.18}$$

Solution

Using Cramer's Rule, we have

$$D = \begin{bmatrix} 2 & 4 & 4 \\ 3 & 7 & 7 \\ 1 & 2 & 2 \end{bmatrix}$$

Notice that the second and third columns are identical. According to Property 3, the determinant will be zero, so there is either no solution or an infinite number of solutions. We have to perform elimination to find out.

1. Multiply Equation 7.9.18 by -2 and add the result to Equation 7.9.16.

$$\begin{aligned} -2x - 4y - 4x &= -8 \\ 2x + 4y + 4z &= 2 \\ 0 &= -6 \end{aligned}$$

Obtaining a statement that is a contradiction means that the system has no solution.

Media

Access these online resources for additional instruction and practice with Cramer's Rule.

- Solve a System of Two Equations Using Cramer's Rule
- Solve a Systems of Three Equations using Cramer's Rule

Key Concepts

- The determinant for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$. See Example 7.9.1.
- Cramer's Rule replaces a variable column with the constant column. Solutions are $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$. See Example 7.9.2.
- To find the determinant of a 3×3 matrix, augment with the first two columns. Add the three diagonal entries (upper left to lower right) and subtract the three diagonal entries (lower left to upper right). See Example 7.9.3.
- To solve a system of three equations in three variables using Cramer's Rule, replace a variable column with the constant column for each desired solution: $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$. See Example 7.9.4.
- Cramer's Rule is also useful for finding the solution of a system of equations with no solution or infinite solutions. See Example 7.9.5 and Example 7.9.6.
- Certain properties of determinants are useful for solving problems. For example:
  - If the matrix is in upper triangular form, the determinant equals the product of entries down the main diagonal.
  - When two rows are interchanged, the determinant changes sign.
  - If either two rows or two columns are identical, the determinant equals zero.
  - If a matrix contains either a row of zeros or a column of zeros, the determinant equals zero.
  - The determinant of an inverse matrix A^{-1} is the reciprocal of the determinant of the matrix A.
  - If any row or column is multiplied by a constant, the determinant is multiplied by the same factor. See Example 7.9.7 and Example 7.9.8.

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7.E: Systems of Equations and Inequalities (Exercises)

9.1: Systems of Linear Equations: Two Variables

Verbal

1) Can a system of linear equations have exactly two solutions? Explain why or why not.

Answer

No, you can either have zero, one, or infinitely many. Examine graphs.

2) If you are performing a break-even analysis for a business and their cost and revenue equations are dependent, explain what this means for the company's profit margins.

3) If you are solving a break-even analysis and get a negative break-even point, explain what this signifies for the company?

Answer

This means there is no realistic break-even point. By the time the company produces one unit they are already making profit.

4) If you are solving a break-even analysis and there is no break-even point, explain what this means for the company. How should they ensure there is a break-even point?

5) Given a system of equations, explain at least two different methods of solving that system.

Answer

You can solve by substitution (isolating x or y), graphically, or by addition.

Algebraic

For the exercises 6-10, determine whether the given ordered pair is a solution to the system of equations.

6) $\begin{aligned} 5x-y &= 4 \\ x+6y &= 2 \end{aligned}$ and $(4,0)$

7) $\begin{aligned} -3x-5y &= 13 \\ -x+4y &= 10 \end{aligned}$ and $(-6,1)$

Answer

Yes

8) $\begin{aligned} 3x+7y &= 1 \\ 2x+4y &= 0 \end{aligned}$ and $(2,3)$

9) $\begin{aligned} -2x+5y &= 7 \\ 2x+9y &= 7 \end{aligned}$ and $(-1,1)$

Answer

Yes

10) $\begin{aligned} x+8y &= 43 \\ 3x-2y &= -1 \end{aligned}$ and $(3,5)$

For the exercises 11-20, solve each system by substitution.

11) $\begin{aligned} x+5y &= 5 \\ 2x+3y &= 4 \end{aligned}$

Answer

$(-1,2)$

12) $\begin{aligned} 3x-2y &= 18 \\ 5x+10y &= -10 \end{aligned}$

13) $\begin{aligned} 4x+2y &= -10 \\ 3x+9y &= 0 \end{aligned}$

Answer

$(-3, 1)$

14) $\begin{aligned} 2x+4y &= -3.8 \\ 9x-5y &= 1.3 \end{aligned}$

15) $\begin{aligned} -2x+3y &= 1.2 \\ -3x-6y &= 1.8 \end{aligned}$

Answer

$\left(-\frac{3}{5}, 0\right)$

16) $\begin{aligned} x-0.2y &= 1 \\ -10x+2y &= 5 \end{aligned}$

17) $\begin{aligned} 3x+5y &= 9 \\ 30x+50y &= -90 \end{aligned}$

Answer

No solutions exist

18) $\begin{aligned} -3x+y &= 2 \\ 12x-4y &= -8 \end{aligned}$

19) $\begin{aligned} \frac{1}{2}x+\frac{1}{3}y &= 16 \\ \frac{1}{6}x+\frac{1}{4}y &= 9 \end{aligned}$

Answer

$\left(\frac{72}{5}, \frac{132}{5}\right)$

20) $\begin{aligned} -\frac{1}{4}x+\frac{3}{2}y &= 11 \\ -\frac{1}{8}x+\frac{1}{3}y &= 3 \end{aligned}$

For the exercises 21-30, solve each system by addition.

21) $\begin{aligned} -2x+5y &= -42 \\ 7x+2y &= 30 \end{aligned}$

Answer

$(6, -6)$

22) $\begin{aligned} 6x-5y &= -34 \\ 2x+6y &= 4 \end{aligned}$

23) $\begin{aligned} 5x-y &= -2.6 \\ -4x-6y &= 1.4 \end{aligned}$

Answer

$$\left(-\frac{1}{2}, \frac{1}{10}\right)$$

24) $\begin{aligned} 7x-2y &= 3 \\ 4x+5y &= 3.25 \end{aligned}$

25) $\begin{aligned} -x+2y &= -1 \\ 5x-10y &= 6 \end{aligned}$

Answer

No solutions exist

26) $\begin{aligned} 7x+6y &= 2 \\ -28x-24y &= -8 \end{aligned}$

27) $\begin{aligned} \frac{5}{6}x+\frac{1}{4}y &= 0 \\ \frac{1}{8}x-\frac{1}{2}y &= -\frac{43}{120} \end{aligned}$

Answer

$$\left(-\frac{1}{5}, \frac{2}{3}\right)$$

28) $\begin{aligned} \frac{1}{3}x+\frac{1}{9}y &= \frac{2}{9} \\ -\frac{1}{2}x+\frac{4}{5}y &= -\frac{1}{3} \end{aligned}$

29) $\begin{aligned} -0.2x+0.4y &= 0.6 \\ x-2y &= -3 \end{aligned}$

Answer

$$\left(x, \frac{x+3}{2}\right)$$

30) $\begin{aligned} -0.1x+0.2y &= 0.6 \\ 5x-10y &= 1 \end{aligned}$

For the exercises 31-40, solve each system by any method.

31) $\begin{aligned} 5x+9y &= 16 \\ x+2y &= 4 \end{aligned}$

Answer

$(-4, 4)$

32) $\begin{aligned} 6x-8y &= -0.6 \\ 3x+2y &= 0.9 \end{aligned}$

33) $\begin{aligned} 5x-2y &= 2.25 \\ 7x-4y &= 3 \end{aligned}$

Answer

$$\left(\frac{1}{2}, \frac{1}{8}\right)$$

34) $$\begin{aligned} x-\frac{5}{12}y &= -\frac{55}{12} \\ -6x+\frac{5}{2}y &= \frac{55}{2} \end{aligned}$$

35) $$\begin{aligned} 7x-4y &= \frac{7}{6} \\ 2x+4y &= \frac{1}{3} \end{aligned}$$

Answer

$$\left(\frac{1}{6}, 0\right)$$

36) $$\begin{aligned} 3x+6y &= 11 \\ 2x+4y &= 9 \end{aligned}$$

37) $$\begin{aligned} \frac{7}{3}x-\frac{1}{6}y &= 2 \\ -\frac{21}{6}x+\frac{3}{12}y &= -3 \end{aligned}$$

Answer

$$(x, 2(7x-6))$$

38) $$\begin{aligned} \frac{1}{2}x+\frac{1}{3}y &= \frac{1}{3} \\ \frac{3}{2}x+\frac{1}{4}y &= -\frac{1}{8} \end{aligned}$$

39) $$\begin{aligned} 2.2x+1.3y &= -0.1 \\ 4.2x+4.2y &= 2.1 \end{aligned}$$

Answer

$$\left(-\frac{5}{6}, \frac{4}{3}\right)$$

40) $$\begin{aligned} 0.1x+0.2y &= 2 \\ 0.35x-0.3y &= 0 \end{aligned}$$

Graphical

For the exercises 41-45, graph the system of equations and state whether the system is consistent, inconsistent, or dependent and whether the system has one solution, no solution, or infinite solutions.

41) $$\begin{aligned} 3x-y &= 0.6 \\ x-2y &= 1.3 \end{aligned}$$

Answer

Consistent with one solution

42) $$\begin{aligned} -x+2y &= 4 \\ 2x-4y &= 1 \end{aligned}$$

43) $$\begin{aligned} x+2y &= 7 \\ 2x+6y &= 12 \end{aligned}$$

Answer

Consistent with one solution

44) $\begin{aligned} 3x-5y &= 7 \\ x-2y &= 3 \end{aligned}$

45) $\begin{aligned} 3x-2y &= 5 \\ -9x+6y &= -15 \end{aligned}$

Answer

Dependent with infinitely many solutions

Technology

For the exercises 46-50, use the intersect function on a graphing device to solve each system. Round all answers to the nearest hundredth.

46) $\begin{aligned} 0.1x+0.2y &= 0.3 \\ -0.3x+0.5y &= 1 \end{aligned}$

47) $\begin{aligned} -0.01x+0.12y &= 0.62 \\ 0.15x+0.20y &= 0.52 \end{aligned}$

Answer

$(-3.08, 4.91)$

48) $\begin{aligned} 0.5x+0.3y &= 4 \\ 0.25x-0.9y &= 0.46 \end{aligned}$

49) $\begin{aligned} 0.15x+0.27y &= 0.39 \\ -0.34x+0.56y &= 1.8 \end{aligned}$

Answer

$(-1.52, 2.29)$

50) $\begin{aligned} -0.71x+0.92y &= 0.13 \\ 0.83x+0.05y &= 2.1 \end{aligned}$

Extensions

For the exercises 51-55, solve each system in terms of $A, B, C, D,$ and F where $A-F$ are nonzero numbers. Note that $A \neq B$ and $AE \neq BD$.

51) $\begin{aligned} x+y &= A \\ x-y &= B \end{aligned}$

Answer

$\left(\frac{A+B}{2}, \frac{A-B}{2}\right)$

52) $\begin{aligned} x+Ay &= 1 \\ x+By &= 1 \end{aligned}$

53) $\begin{aligned} Ax+y &= 0 \\ Bx+y &= 1 \end{aligned}$

Answer

$\left(\frac{-1}{A-B}, \frac{A}{A-B}\right)$

54) $\begin{aligned} Ax+By &= C \\ x+y &= 1 \end{aligned}$

55) $\begin{aligned} Ax+By &= C \\ Dx+Ey &= F \end{aligned}$

Answer

$$\left(\frac{CE-BF}{BD-AE}, \frac{AF-CD}{BD-AE}\right)$$

Real-World Applications

For the exercises 56-60, solve for the desired quantity.

56) A stuffed animal business has a total cost of production $C = 12x + 30$ and a revenue function $R = 20x$. Find the break-even point.

57) A fast-food restaurant has a cost of production $C(x) = 11x + 120$ and a revenue function $R(x) = 5x$. When does the company start to turn a profit?

Answer

They never turn a profit.

58) A cell phone factory has a cost of productiona $C(x) = 150x + 10,000$ and a revenue function $R(x) = 200x$. What is the break-even point?

59) A musician charges $C(x) = 64x + 20,000$, where x is the total number of attendees at the concert. The venue charges $\$80$ per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

Answer

$(1,250, 100,000)$

60) A guitar factory has a cost of production $C(x) = 75x + 50,000$. If the company needs to break even after 150 units sold, at what price should they sell each guitar? Round up to the nearest dollar, and write the revenue function.

For the exercises 61-77, use a system of linear equations with two variables and two equations to solve.

61) Find two numbers whose sum is 28 and difference is 13.

Answer

The numbers are 7.5 and 20.5

62) A number is 9 more than another number. Twice the sum of the two numbers is 10. Find the two numbers.

63) The startup cost for a restaurant is $\$120,000$ and each meal costs $\$10$ for the restaurant to make. If each meal is then sold for $\$15$, after how many meals does the restaurant break even?

Answer

$24,000$

64) A moving company charges a flat rate of $\$150$, and an additional $\$5$ for each box. If a taxi service would charge $\$20$ for each box, how many boxes would you need for it to be cheaper to use the moving company, and what would be the total cost?

65) A total of $1,595$ first- and second-year college students gathered at a pep rally. The number of freshmen exceeded the number of sophomores by 15. How many freshmen and sophomores were in attendance?

Answer

790 sophomores, 805 freshman

66) 276 students enrolled in a freshman-level chemistry class. By the end of the semester, 5 times the number of students passed as failed. Find the number of students who passed, and the number of students who failed.

67) There were 130 faculty at a conference. If there were 18 more women than men attending, how many of each gender attended the conference?

Answer

56 men, 74 women

68) A jeep and BMW enter a highway running east-west at the same exit heading in opposite directions. The jeep entered the highway 30 minutes before the BMW did, and traveled 7 mph slower than the BMW. After 2 hours from the time the BMW entered the highway, the cars were 306.5 miles apart. Find the speed of each car, assuming they were driven on cruise control.

69) If a scientist mixed 10% saline solution with 60% saline solution to get 25 gallons of 40% saline solution, how many gallons of 10% and 60% solutions were mixed?

Answer

10 gallons of 10% solution, 15 gallons of 60% solution

70) An investor earned triple the profits of what she earned last year. If she made $500,000.48 total for both years, how much did she earn in profits each year?

71) An investor who dabbles in real estate invested 1.1 million dollars into two land investments. On the first investment, Swan Peak, her return was a 110% increase on the money she invested. On the second investment, Riverside Community, she earned 50% over what she invested. If she earned $1 million in profits, how much did she invest in each of the land deals?

Answer

Swan Peak: $750,000, Riverside: $350,000

72) If an investor invests a total of $25,000 into two bonds, one that pays 3% simple interest, and the other that pays $2\frac{7}{8}\%$ interest, and the investor earns $737.50 annual interest, how much was invested in each account?

73) If an investor invests $23,000 into two bonds, one that pays 4% in simple interest, and the other paying 2% simple interest, and the investor earns $710.00 annual interest, how much was invested in each account?

Answer

$12,500 in the first account, $10,500 in the second account.

74) CDs cost $5.96 more than DVDs at All Bets Are Off Electronics. How much would 6 CDs and 2 DVDs cost if 5 CDs and 2 DVDs cost $127.73?

75) A store clerk sold 60 pairs of sneakers. The high-tops sold for $98.99 and the low-tops sold for $129.99. If the receipts for the two types of sales totaled $6,404.40, how many of each type of sneaker were sold?

Answer

High-tops: 45, Low-tops: 15

76) A concert manager counted 350 ticket receipts the day after a concert. The price for a student ticket was $12.50, and the price for an adult ticket was $16.00. The register confirms that $5,075 was taken in. How many student tickets and adult tickets were sold?

77) Admission into an amusement park for 4 children and 2 adults is $116.90. For 6 children and 3 adults, the admission is $175.35. Assuming a different price for children and adults, what is the price of the child's ticket and the price of the adult ticket?

Answer

Infinitely many solutions. We need more information.

9.2: Systems of Linear Equations: Three Variables

Verbal

1) Can a linear system of three equations have exactly two solutions? Explain why or why not

Answer

No, there can be only one, zero, or infinitely many solutions.

2) If a given ordered triple solves the system of equations, is that solution unique? If so, explain why. If not, give an example where it is not unique.

3) If a given ordered triple does not solve the system of equations, is there no solution? If so, explain why. If not, give an example.

Answer

Not necessarily. There could be zero, one, or infinitely many solutions. For example, $(0,0,0)$ is not a solution to the system below, but that does not mean that it has no solution.

$$\begin{aligned} 2x+3y-6z &= 1 \\ -4x-6y+12z &= -2 \\ x+2y+5z &= 10 \end{aligned}$$

4) Using the method of addition, is there only one way to solve the system?

5) Can you explain whether there can be only one method to solve a linear system of equations? If yes, give an example of such a system of equations. If not, explain why not.

Answer

Every system of equations can be solved graphically, by substitution, and by addition. However, systems of three equations become very complex to solve graphically so other methods are usually preferable.

Algebraic

For the exercises 6-10, determine whether the ordered triple given is the solution to the system of equations.

6) $\begin{aligned} 2x-6y+6z &= -12 \\ x+4y+5z &= -1 \\ -x+2y+3z &= -1 \end{aligned}$ and $(0,1,-1)$

7) $\begin{aligned} 6x-y+3z &= 6 \\ 3x+5y+2z &= 0 \\ x+y &= 0 \end{aligned}$ and $(3,-3,-5)$

Answer

No

8) $\begin{aligned} 6x-7y+z &= 2 \\ -x-y+3z &= 4 \\ 2x+y-z &= 1 \end{aligned}$ and $(4,2,-6)$

9) $\begin{aligned} x-y &= 0 \\ x-z &= 5 \\ x-y+z &= -1 \end{aligned}$ and $(4,4,-1)$

Answer

Yes

10) $\begin{aligned} -x-y+2z &= 3 \\ 5x+8y-3z &= 4 \\ -x+3y-5z &= -5 \end{aligned}$ and $(4,1,-7)$

For the exercises 11-16, solve each system by substitution.

11) $\begin{aligned} 3x-4y+2z &= -15 \\ 2x+4y+z &= 16 \\ 2x+3y+5z &= 20 \end{aligned}$

Answer

$(-1,4,2)$

12) $\begin{aligned} 5x-2y+3z &= 20 \\ 2x-4y-3z &= -9 \\ x+6y-8z &= 21 \end{aligned}$

13) $\begin{aligned} 5x+2y+4z &= 9 \\ -3x+2y+z &= 10 \\ 4x-3y+5z &= -3 \end{aligned}$

Answer

$\left(-\frac{85}{107}, \frac{312}{107}, \frac{191}{107}\right)$

14) $\begin{aligned} 4x-3y+5z &= 31 \\ -x+2y+4z &= 20 \\ x+5y-2z &= -29 \end{aligned}$

15) $\begin{aligned} 5x-2y+3z &= 4 \\ -4x+6y-7z &= -1 \\ 3x+2y-z &= 4 \end{aligned}$

Answer

$\left(1, \frac{1}{2}, 0\right)$

16) $\begin{aligned} 4x+6y+9z &= 4 \\ -5x+2y-6z &= 3 \\ 7x-4y+3z &= -3 \end{aligned}$

For the exercises 17-45, solve each system by Gaussian elimination.

17) $\begin{aligned} 2x-y+3z &= 17 \\ -5x+4y-2z &= -46 \\ 2y+5z &= -7 \end{aligned}$

Answer

$(4,-6,1)$

18) $\begin{aligned} 5x-6y+3z &= 50 \\ -x+4y &= 10 \\ 2x-z &= 10 \end{aligned}$

19) $\begin{aligned} 2x+3y-6z &= 1 \\ -4x-6y+12z &= -2 \\ x+2y+5z &= 10 \end{aligned}$

Answer

$$\left(x, \frac{1}{27}(65-16x), \frac{x+28}{27}\right)$$

20) $$\begin{aligned} 4x+6y-2z &= 8 \\ 6x+9y-3z &= 12 \\ -2x-3y+z &= -4 \end{aligned}$$

21) $$\begin{aligned} 2x+3y-4z &= 5 \\ -3x+2y+z &= 11 \\ -x+5y+3z &= 4 \end{aligned}$$

Answer

$$\left(-\frac{45}{13}, \frac{17}{13}, -2\right)$$

22) $$\begin{aligned} 10x+2y-14z &= 8 \\ -x-2y-4z &= -1 \\ -12x-6y+6z &= -12 \end{aligned}$$

23) $$\begin{aligned} x+y+z &= 14 \\ 2y+3z &= -14 \\ -16y-24z &= -112 \end{aligned}$$

Answer

No solutions exist

24) $$\begin{aligned} 5x-3y+4z &= -1 \\ -4x+2y-3z &= 0 \\ -x+5y+7z &= -11 \end{aligned}$$

25) $$\begin{aligned} x+y+z &= 0 \\ 2x-y+3z &= 0 \\ x-z &= 0 \end{aligned}$$

Answer

$$(0,0,0)$$

26) $$\begin{aligned} 3x+2y-5z &= 6 \\ 5x-4y+3z &= -12 \\ 4x+5y-2z &= 15 \end{aligned}$$

27) $$\begin{aligned} x+y+z &= 0 \\ 2x-y+3z &= 0 \\ x-z &= 1 \end{aligned}$$

Answer

$$\left(\frac{4}{7}, -\frac{1}{7}, -\frac{3}{7}\right)$$

28) $$\begin{aligned} 3x-\frac{1}{2}y-z &= -\frac{1}{2} \\ 4x+z &= 3 \\ -x+\frac{3}{2}y &= \frac{5}{2} \end{aligned}$$

29) $$\begin{aligned} 6x-5y+6z &= 38 \\ \frac{1}{5}x-\frac{1}{2}y+\frac{3}{5}z &= 1 \\ -4x-\frac{3}{2}y-z &= -74 \end{aligned}$$

Answer

$(7, 20, 16)$

30) $$\begin{aligned} \frac{1}{2}x-\frac{1}{5}y+\frac{2}{5}z &= -\frac{13}{10} \\ \frac{1}{4}x-\frac{2}{5}y-\frac{1}{5}z &= -\frac{7}{20} \\ -\frac{1}{2}x-\frac{3}{4}y-\frac{1}{2}z &= -\frac{5}{4} \end{aligned}$$

31) $$\begin{aligned} -\frac{1}{3}x-\frac{1}{2}y-\frac{1}{4}z &= \frac{3}{4} \\ -\frac{1}{2}x-\frac{1}{4}y-\frac{1}{2}z &= 2 \\ -\frac{1}{4}x-\frac{3}{4}y-\frac{1}{2}z &= -\frac{1}{2} \end{aligned}$$

Answer

$(-6, 2, 1)$

32) $$\begin{aligned} \frac{1}{2}x-\frac{1}{4}y+\frac{3}{4}z &= 0 \\ \frac{1}{4}x-\frac{1}{10}y+\frac{2}{5}z &= -2 \\ \frac{1}{8}x+\frac{1}{5}y-\frac{1}{8}z &= 2 \end{aligned}$$

33) $$\begin{aligned} \frac{4}{5}x-\frac{7}{8}y+\frac{1}{2}z &= 1 \\ -\frac{4}{5}x-\frac{3}{4}y+\frac{1}{3}z &= -8 \\ -\frac{2}{5}x-\frac{7}{8}y+\frac{1}{2}z &= -5 \end{aligned}$$

Answer

$(5, 12, 15)$

34) $$\begin{aligned} -\frac{1}{3}x-\frac{1}{8}y+\frac{1}{6}z &= -\frac{4}{3} \\ -\frac{2}{3}x-\frac{7}{8}y+\frac{1}{3}z &= -\frac{23}{3} \\ -\frac{1}{3}x-\frac{5}{8}y+\frac{5}{6}z &= 0 \end{aligned}$$

35) $$\begin{aligned} -\frac{1}{4}x-\frac{5}{4}y+\frac{5}{2}z &= -5 \\ -\frac{1}{2}x-\frac{5}{3}y+\frac{5}{4}z &= \frac{55}{12} \\ -\frac{1}{3}x-\frac{1}{3}y+\frac{1}{3}z &= \frac{5}{3} \end{aligned}$$

Answer

$(-5, -5, -5)$

36) $$\begin{aligned} \frac{1}{40}x+\frac{1}{60}y+\frac{1}{80}z &= \frac{1}{100} \\ -\frac{1}{2}x-\frac{1}{3}y-\frac{1}{4}z &= -\frac{1}{5} \\ \frac{3}{8}x+\frac{3}{12}y+\frac{3}{16}z &= \frac{3}{20} \end{aligned}$$

37) $$\begin{aligned} 0.1x-0.2y+0.3z &= 2 \\ 0.5x-0.1y+0.4z &= 8 \\ 0.7x-0.2y+0.3z &= 8 \end{aligned}$$

Answer

$(10, 10, 10)$

38) $$\begin{aligned} 0.2x+0.1y-0.3z &= 0.2 \\ 0.8x+0.4y-1.2z &= 0.1 \\ 1.6x+0.8y-2.4z &= 0.2 \end{aligned}$$

39) $$\begin{aligned} 1.1x+0.7y-3.1z &= -1.79 \\ 2.1x+0.5y-1.6z &= -0.13 \\ 0.5x+0.4y-0.5z &= -0.07 \end{aligned}$$

Answer

$\left(\frac{1}{2}, \frac{1}{5}, \frac{4}{5}\right)$

40) $$\begin{aligned} 0.5x-0.5y+0.5z &= 10 \\ 0.2x-0.2y+0.2z &= 4 \\ 0.1x-0.1y+0.1z &= 2 \end{aligned}$$

41) $$\begin{aligned} 0.1x+0.2y+0.3z &= 0.37 \\ 0.1x-0.2y-0.3z &= -0.27 \\ 0.5x-0.1y-0.3z &= -0.03 \end{aligned}$$

Answer

$\left(\frac{1}{2}, \frac{2}{5}, \frac{4}{5}\right)$

42) $$\begin{aligned} 0.5x-0.5y-0.3z &= 0.13 \\ 0.4x-0.1y-0.3z &= 0.11 \\ 0.2x-0.8y-0.9z &= -0.32 \end{aligned}$$

43) $$\begin{aligned} 0.5x+0.2y-0.3z &= 1 \\ 0.4x-0.6y+0.7z &= 0.8 \\ 0.3x-0.1y-0.9z &= 0.6 \end{aligned}$$

Answer

$(2, 0, 0)$

44) $$\begin{aligned} 0.3x+0.3y+0.5z &= 0.6 \\ 0.4x+0.4y+0.4z &= 1.8 \\ 0.4x+0.2y+0.1z &= 1.6 \end{aligned}$$

45) $$\begin{aligned} 0.8x+0.8y+0.8z &= 2.4 \\ 0.3x-0.5y+0.2z &= 0 \\ 0.1x+0.2y+0.3z &= 0.6 \end{aligned}$$

Answer

$(1,1,1)$

Extensions

For the exercises 46-50, solve the system for x, y, and z.

46)
$$\begin{aligned} x+y+z &= 3 \\ \frac{x-1}{2}+\frac{y-3}{2}+\frac{z+1}{2} &= 0 \\ \frac{x-2}{3}+\frac{y+4}{3}+\frac{z-3}{3} &= \frac{2}{3} \end{aligned}$$

47)
$$\begin{aligned} 5x-3y-\frac{z+1}{2} &= \frac{1}{2} \\ 6x+\frac{y-9}{2}+2z &= -3 \\ \frac{x+8}{2}-4y+z &= 4 \end{aligned}$$

Answer

$$\left(\frac{128}{557}, \frac{23}{557}, \frac{428}{557}\right)$$

48)
$$\begin{aligned} \frac{x+4}{7}-\frac{y-1}{6}+\frac{z+2}{3} &= 1 \\ \frac{x-2}{4}+\frac{y+1}{8}-\frac{z+8}{2} &= 0 \\ \frac{x+6}{3}-\frac{y+2}{3}+\frac{z+4}{2} &= 3 \end{aligned}$$

49)
$$\begin{aligned} \frac{x-3}{6}+\frac{y+2}{2}-\frac{z-3}{3} &= 2 \\ \frac{x+2}{4}+\frac{y-5}{2}+\frac{z+4}{2} &= 1 \\ \frac{x+6}{2}-\frac{y-3}{3}+z+1 &= 9 \end{aligned}$$

Answer

$(6,-1,0)$

50)
$$\begin{aligned} \frac{x-1}{3}+\frac{y+3}{4}+\frac{z+2}{6} &= 1 \\ 4x+3y-2z &= 11 \\ 0.02x+0.015y-0.01z &= 0.065 \end{aligned}$$

Real-World Applications

51) Three even numbers sum up to 108. The smaller is half the larger and the middle number is $\frac{3}{4}$ the larger. What are the three numbers?

Answer

$24, 36, 48$

52) Three numbers sum up to 147. The smallest number is half the middle number, which is half the largest number. What are the three numbers?

53) At a family reunion, there were only blood relatives, consisting of children, parents, and grandparents, in attendance. There were 400 people total. There were twice as many parents as grandparents, and 50 more children than parents. How many children, parents, and grandparents were in attendance?

Answer

70 grandparents, 140 parents, 190 children

54) An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?

55) Your roommate, Sarah, offered to buy groceries for you and your other roommate. The total bill was $82. She forgot to save the individual receipts but remembered that your groceries were $0.05 cheaper than half of her groceries, and that your other roommate's groceries were $2.10 more than your groceries. How much was each of your share of the groceries?

Answer

Your share was $19.95, Sarah's share was $40, and your other roommate's share was $22.05

56) Your roommate, John, offered to buy household supplies for you and your other roommate. You live near the border of three states, each of which has a different sales tax. The total amount of money spent was $100.75. Your supplies were bought with 5% tax, John's with 8% tax, and your third roommate's with 9% sales tax. The total amount of money spent without taxes is $93.50. If your supplies before tax were $1 more than half of what your third roommate's supplies were before tax, how much did each of you spend? Give your answer both with and without taxes.

57) Three coworkers work for the same employer. Their jobs are warehouse manager, office manager, and truck driver. The sum of the annual salaries of the warehouse manager and office manager is $82,000. The office manager makes $4,000 more than the truck driver annually. The annual salaries of the warehouse manager and the truck driver total $78,000. What is the annual salary of each of the co-workers?

Answer

There are infinitely many solutions; we need more information

58) At a carnival, $2,914.25 in receipts were taken at the end of the day. The cost of a child's ticket was $20.50, an adult ticket was $29.75, and a senior citizen ticket was $15.25. There were twice as many senior citizens as adults in attendance, and 20 more children than senior citizens. How many children, adult, and senior citizen tickets were sold?

59) A local band sells out for their concert. They sell all 1,175 tickets for a total purse of $28,112.50 The tickets were priced at $20 for student tickets, $22.50 for children, and $29 for adult tickets. If the band sold twice as many adult as children tickets, how many of each type was sold?

Answer

500 students, 225 children, and 450 adults

60) In a bag, a child has 325 coins worth $19.50. There were three types of coins: pennies, nickels, and dimes. If the bag contained the same number of nickels as dimes, how many of each type of coin was in the bag?

61) Last year, at Haven's Pond Car Dealership, for a particular model of BMW, Jeep, and Toyota, one could purchase all three cars for a total of $140,000. This year, due to inflation, the same cars would cost $151,830. The cost of the BMW increased by 8%, the Jeep by 5%, and the Toyota by 12%. If the price of last year's Jeep was $7,000 less than the price of last year's BMW, what was the price of each of the three cars last year?

Answer

The BMW was $49,636, the Jeep was $42,636, and the Toyota was $47,727

62) A recent college graduate took advantage of his business education and invested in three investments immediately after graduating. He invested $80,500 into three accounts, one that paid 4% simple interest, one that paid $3\frac{1}{8}\%$ simple interest, and one that paid $2\frac{1}{2}\%$ simple interest. He earned $2,670 interest at the end of one year. If the amount of the money invested in the second account was four times the amount invested in the third account, how much was invested in each account?

63) You inherit one million dollars. You invest it all in three accounts for one year. The first account pays 3% compounded annually, the second account pays 4% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $\$34,000$ in interest. If you invest four times the money into the account that pays 3% compared to 2%, how much did you invest in each account?

Answer

$\$400,000$ in the account that pays 3% interest, $\$500,000$ in the account that pays 4% interest, and $\$100,000$ in the account that pays 2% interest.

64) You inherit one hundred thousand dollars. You invest it all in three accounts for one year. The first account pays 4% compounded annually, the second account pays 3% compounded annually, and the third account pays 2% compounded annually. After one year, you earn $\$3,650$ in interest. If you invest five times the money in the account that pays 4% compared to 3%, how much did you invest in each account?

65) The top three countries in oil consumption in a certain year are as follows: the United States, Japan, and China. In millions of barrels per day, the three top countries consumed 39.8% of the world's consumed oil. The United States consumed 0.7% more than four times China's consumption. The United States consumed 5% more than triple Japan's consumption. What percent of the world oil consumption did the United States, Japan, and China consume?

Answer

The United States consumed 26.3%, Japan 7.1%, and China 6.4% of the world's oil.

66) The top three countries in oil production in the same year are Saudi Arabia, the United States, and Russia. In millions of barrels per day, the top three countries produced 31.4% of the world's produced oil. Saudi Arabia and the United States combined for 22.1% of the world's production, and Saudi Arabia produced 2% more oil than Russia. What percent of the world oil production did Saudi Arabia, the United States, and Russia produce?

67) The top three sources of oil imports for the United States in the same year were Saudi Arabia, Mexico, and Canada. The three top countries accounted for 47% of oil imports. The United States imported 1.8% more from Saudi Arabia than they did from Mexico, and 1.7% more from Saudi Arabia than they did from Canada. What percent of the United States oil imports were from these three countries?

Answer

Saudi Arabia imported 16.8%, Canada imported 15.1%, and Mexico 15.0%

68) The top three oil producers in the United States in a certain year are the Gulf of Mexico, Texas, and Alaska. The three regions were responsible for 64% of the United States oil production. The Gulf of Mexico and Texas combined for 47% of oil production. Texas produced 3% more than Alaska. What percent of United States oil production came from these regions?

69) At one time, in the United States, 398 species of animals were on the endangered species list. The top groups were mammals, birds, and fish, which comprised 55% of the endangered species. Birds accounted for 0.7% more than fish, and fish accounted for 1.5% more than mammals. What percent of the endangered species came from mammals, birds, and fish?

Answer

Birds were 19.3%, fish were 18.6%, and mammals were 17.1% of endangered species

70) Meat consumption in the United States can be broken into three categories: red meat, poultry, and fish. If fish makes up 4% less than one-quarter of poultry consumption, and red meat consumption is 18.2% higher than poultry consumption, what are the percentages of meat consumption?

9.3: Systems of Nonlinear Equations and Inequalities: Two Variables

Verbal

1) Explain whether a system of two nonlinear equations can have exactly two solutions. What about exactly three? If not, explain why not. If so, give an example of such a system, in graph form, and explain why your choice gives two or three answers.

Answer

A nonlinear system could be representative of two circles that overlap and intersect in two locations, hence two solutions. A nonlinear system could be representative of a parabola and a circle, where the vertex of the parabola meets the circle and the branches also intersect the circle, hence three solutions.

2) When graphing an inequality, explain why we only need to test one point to determine whether an entire region is the solution?

3) When you graph a system of inequalities, will there always be a feasible region? If so, explain why. If not, give an example of a graph of inequalities that does not have a feasible region. Why does it not have a feasible region?

Answer

No. There does not need to be a feasible region. Consider a system that is bounded by two parallel lines. One inequality represents the region above the upper line; the other represents the region below the lower line. In this case, no points in the plane are located in both regions; hence there is no feasible region.

4) If you graph a revenue and cost function, explain how to determine in what regions there is profit.

5) If you perform your break-even analysis and there is more than one solution, explain how you would determine which x-values are profit and which are not.

Answer

Choose any number between each solution and plug into $C(x)$ and $R(x)$. If $C(x) < R(x)$, then there is profit.

Algebraic

For the exercises 6-10, solve the system of nonlinear equations using substitution.

6) $\begin{aligned} x+y &= 4 \\ x^2+y^2 &= 9 \end{aligned}$

7) $\begin{aligned} y &= x-3 \\ x^2+y^2 &= 9 \end{aligned}$

Answer

$(0,-3),(3,0)$

8) $\begin{aligned} y &= x \\ x^2+y^2 &= 9 \end{aligned}$

9) $\begin{aligned} y &= -x \\ x^2+y^2 &= 9 \end{aligned}$

Answer

$\left(-\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right),\left(\frac{3\sqrt{2}}{2},-\frac{3\sqrt{2}}{2}\right)$

10) $\begin{aligned} x &= 2 \\ x^2-y^2 &= 9 \end{aligned}$

For the exercises 11-15, solve the system of nonlinear equations using elimination.

11) $\begin{aligned} 4x^2-9y^2 &= 36 \\ 4x^2+9y^2 &= 36 \end{aligned}$

Answer

$(-3,0),(3,0)$

12) $\begin{aligned} x^2+y^2 &= 25 \\ x^2-y^2 &= 1 \end{aligned}$

13) $\begin{aligned} 2x^2+4y^2 &= 4 \\ 2x^2-4y^2 &= 25x-10 \end{aligned}$

Answer

$\left(\frac{1}{4},-\frac{\sqrt{62}}{8}\right),\left(\frac{1}{4},\frac{\sqrt{62}}{8}\right)$

14) $\begin{aligned} y^2-x^2 &= 9 \\ 3x^2+2y^2 &= 8 \end{aligned}$

15) $\begin{aligned} x^2+y^2+\frac{1}{16} &= 2500 \\ y &= 2x^2 \end{aligned}$

Answer

$\left(-\frac{\sqrt{398}}{4},\frac{199}{4}\right),\left(\frac{\sqrt{398}}{4},\frac{199}{4}\right)$

For the exercises 16-23, use any method to solve the system of nonlinear equations.

16) $\begin{aligned} -2x^2+y &= -5 \\ 6x-y &= 9 \end{aligned}$

17) $\begin{aligned} -x^2+y &= 2 \\ -x+y &= 2 \end{aligned}$

Answer

$(0,2),(1,3)$

18) $\begin{aligned} x^2+y^2 &= 1 \\ y &= 20x^2-1 \end{aligned}$

19) $\begin{aligned} x^2+y^2 &= 1 \\ y &= -x^2 \end{aligned}$

Answer

$\left(-\sqrt{\frac{1}{2}(\sqrt{5}-1)},\frac{1}{2}(1-\sqrt{5})\right),\left(\sqrt{\frac{1}{2}(\sqrt{5}-1)},\frac{1}{2}(1-\sqrt{5})\right)$

20) $\begin{aligned} 2x^3-x^2 &= y \\ y &= \frac{1}{2}-x \end{aligned}$

21) $\begin{aligned} 9x^2+25y^2 &= 225 \\ (x-6)^2+y^2 &= 1 \end{aligned}$

Answer

$(5,0)$

22) $\begin{aligned} x^4-x^2 &= y \\ x^2+y &= 0 \end{aligned}$

23) $\begin{aligned} 2x^3 - x^2 &= y \\ x^2 + y &= 0 \end{aligned}$

Answer

$(0, 0)$

For the exercises 24-38, use any method to solve the nonlinear system.

24) $\begin{aligned} x^2 + y^2 &= 9 \\ y &= 3 - x^2 \end{aligned}$

25) $\begin{aligned} x^2 - y^2 &= 9 \\ x &= 3 \end{aligned}$

Answer

$(3, 0)$

26) $\begin{aligned} x^2 - y^2 &= 9 \\ y &= 3 \end{aligned}$

27) $\begin{aligned} x^2 - y^2 &= 9 \\ x - y &= 0 \end{aligned}$

Answer

No Solutions Exist

28) $\begin{aligned} -x^2 + y &= 2 \\ -4x + y &= -1 \end{aligned}$

29) $\begin{aligned} -x^2 + y &= 2 \\ 2y &= -x \end{aligned}$

Answer

No Solutions Exist

30) $\begin{aligned} x^2 + y^2 &= 25 \\ x^2 - y^2 &= 36 \end{aligned}$

31) $\begin{aligned} x^2 + y^2 &= 1 \\ y^2 &= x^2 \end{aligned}$

Answer

$\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

32) $\begin{aligned} 16x^2 - 9y^2 + 144 &= 0 \\ y^2 + x^2 &= 16 \end{aligned}$

33) $\begin{aligned} 3x^2 - y^2 &= 12 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$

Answer

$(2, 0)$

34) $\begin{aligned} 3x^2 - y^2 &= 12 \\ (x-1)^2 + y^2 &= 4 \end{aligned}$

35) $\begin{aligned} 3x^2 - y^2 &= 12 \\ x^2 + y^2 &= 16 \end{aligned}$

Answer

$(-\sqrt{7}, -3), (-\sqrt{7}, 3), (\sqrt{7}, -3), (\sqrt{7}, 3)$

36) $\begin{aligned} x^2 - y^2 - 6x - 4y - 11 &= 0 \\ -x^2 + y^2 &= 5 \end{aligned}$

37) $\begin{aligned} x^2 + y^2 - 6y &= 7 \\ x^2 + y &= 1 \end{aligned}$

Answer

$\left(-\sqrt{\frac{1}{2}(\sqrt{73}-5)}, \frac{1}{2}(7-\sqrt{73})\right), \left(\sqrt{\frac{1}{2}(\sqrt{73}-5)}, \frac{1}{2}(7-\sqrt{73})\right)$

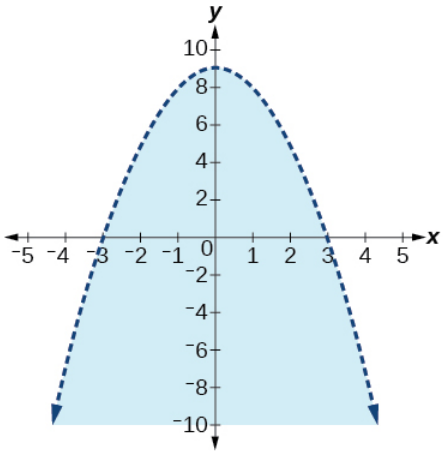
38) $\begin{aligned} x^2 + y^2 &= 6 \\ xy &= 1 \end{aligned}$

Graphical

For the exercises 39-40, graph the inequality.

39) $x^2 + y < 9$

Answer

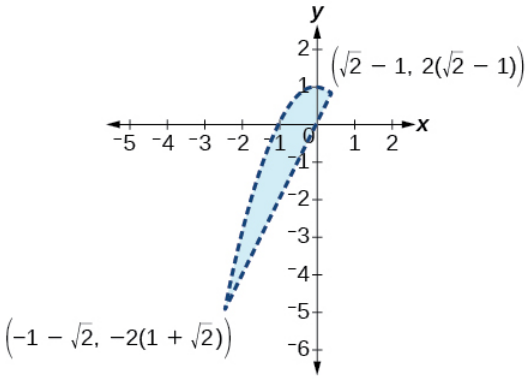


40) $x^2 + y^2 < 4$

For the exercises 41-45, graph the system of inequalities. Label all points of intersection.

41) $\begin{aligned} x^2 + y &< 1 \\ y &> 2x \end{aligned}$

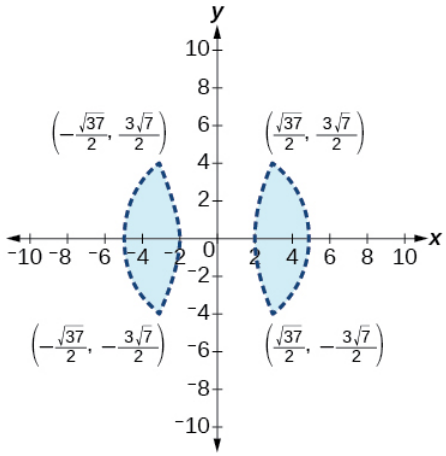
Answer



42) $\begin{aligned} x^2+y &< -5 \\ y &> 5x+10 \end{aligned}$

43) $\begin{aligned} x^2+y^2 &< 25 \\ 3x^2-y^2 &> 12 \end{aligned}$

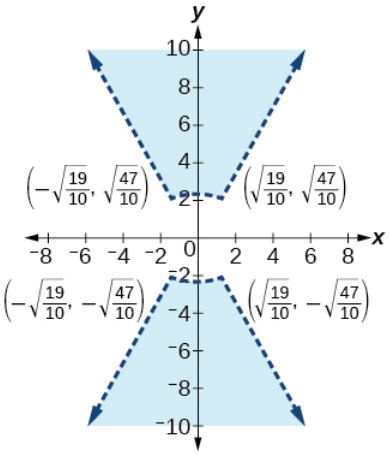
Answer



44) $\begin{aligned} x^2-y^2 &> -4 \\ x^2+y^2 &< 12 \end{aligned}$

45) $\begin{aligned} x^2+3y^2 &> 16 \\ 3x^2-y^2 &< 1 \end{aligned}$

Answer



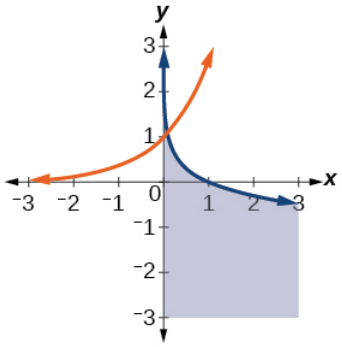
Extensions

For the exercises 46-47, graph the inequality.

46) $y \geq e^x$
$y \leq \ln(x) + 5$

47) $y \leq -\log(x)$
$y \leq e^x$

Answer



For the exercises 48-52, find the solutions to the nonlinear equations with two variables.

48) $\frac{4}{x^2} + \frac{1}{y^2} = 24$
$\frac{5}{x^2} - \frac{2}{y^2} + 4 = 0$

49) $\frac{6}{x^2} - \frac{1}{y^2} = 8$
$\frac{1}{x^2} - \frac{6}{y^2} = \frac{1}{8}$

Answer

$\left(-2\sqrt{\frac{70}{383}}, -2\sqrt{\frac{35}{29}}\right), \left(-2\sqrt{\frac{70}{383}}, 2\sqrt{\frac{35}{29}}\right), \left(2\sqrt{\frac{70}{383}}, -2\sqrt{\frac{35}{29}}\right), \left(2\sqrt{\frac{70}{383}}, 2\sqrt{\frac{35}{29}}\right)$

50) $x^2 - xy + y^2 - 2 = 0$
$x + 3y = 4$

51) $\begin{aligned} x^2 - xy - 2y^2 - 6 &= 0 \\ x^2 + y^2 &= 1 \end{aligned}$

Answer

No Solution Exists

52) $\begin{aligned} x^2 + 4xy - 2y^2 - 6 &= 0 \\ x &= y + 2 \end{aligned}$

Technology

For the exercises 53-54, solve the system of inequalities. Use a calculator to graph the system to confirm the answer.

53) $\begin{aligned} xy &< 1 \\ y &> \sqrt{x} \end{aligned}$

Answer

$x = 0, y > 0$ and $0 < x < 1$, $\sqrt{x} < y < \dfrac{1}{x}$

54) $\begin{aligned} x^2 + y &< 3 \\ y &> 2x \end{aligned}$

Real-World Applications

For the exercises 55-, construct a system of nonlinear equations to describe the given behavior, then solve for the requested solutions.

55) Two numbers add up to 300. One number is twice the square of the other number. What are the numbers?

Answer

$12, 288$

56) The squares of two numbers add to 360. The second number is half the value of the first number squared. What are the numbers?

57) A laptop company has discovered their cost and revenue functions for each day: $C(x) = 3x^2 - 10x + 200$ and $R(x) = -2x^2 + 100x + 50$. If they want to make a profit, what is the range of laptops per day that they should produce? Round to the nearest number which would generate profit.

Answer

2 - 20 computers

58) A cell phone company has the following cost and revenue functions: $C(x) = 8x^2 - 600x + 21,500$ and $R(x) = -3x^2 + 480x$. What is the range of cell phones they should produce each day so there is profit? Round to the nearest number that generates profit.

9.4: Partial Fractions

Verbal

1) Can any quotient of polynomials be decomposed into at least two partial fractions? If so, explain why, and if not, give an example of such a fraction.

Answer

No, a quotient of polynomials can only be decomposed if the denominator can be factored. For example, $\dfrac{1}{x^2+1}$ cannot be decomposed because the denominator cannot be factored.

2) Can you explain why a partial fraction decomposition is unique? (**Hint**: Think about it as a system of equations.)

3) Can you explain how to verify a partial fraction decomposition graphically?

Answer

Graph both sides and ensure they are equal.

4) You are unsure if you correctly decomposed the partial fraction correctly. Explain how you could double-check your answer.

5) Once you have a system of equations generated by the partial fraction decomposition, can you explain another method to solve it? For example if you had $\frac{7x+13}{3x^2+8x+15}=\frac{A}{x+1}+\frac{B}{3x+5}$ we eventually simplify to $7x+13=A(3x+5)+B(x+1)$. Explain how you could intelligently choose an x-value that will eliminate either A or B and solve for A and B.

Answer

If we choose $x=-1$, then the B-term disappears, letting us immediately know that $A=3$. We could alternatively plug in $x=-\frac{5}{3}$, giving us a B-value of -2.

Algebraic

For the exercises 6-19, find the decomposition of the partial fraction for the nonrepeating linear factors.

6) $\frac{5x+16}{x^2+10x+24}$

7) $\frac{3x-79}{x^2-5x-24}$

Answer

$\frac{8}{x+3}-\frac{5}{x-8}$

8) $\frac{-x-24}{x^2-2x-24}$

9) $\frac{10x+47}{x^2+7x+10}$

Answer

$\frac{1}{x+5}+\frac{9}{x+2}$

10) $\frac{x}{6x^2+25x+25}$

11) $\frac{32x-11}{20x^2-13x+2}$

Answer

$\frac{3}{5x-2}+\frac{4}{4x-1}$

12) $\frac{x+1}{x^2+7x+10}$

13) $\frac{5x}{x^2-9}$

Answer

$\frac{5}{2(x+3)}+\frac{5}{2(x-3)}$

14) $\frac{10x}{x^2-25}$

15) $\frac{6x}{x^2-4}$

Answer

$\frac{3}{x+2}+\frac{3}{x-2}$

16) $\frac{2x-3}{x^2-6x+5}$

17) $\frac{4x-1}{x^2-x-6}$

Answer

$\frac{9}{5(x+2)}+\frac{11}{5(x-3)}$

18) $\frac{4x+3}{x^2+8x+15}$

19) $\frac{3x-1}{x^2-5x+6}$

Answer

$\frac{8}{x-3}-\frac{5}{x-2}$

For the exercises 20-30, find the decomposition of the partial fraction for the repeating linear factors.

20) $\frac{-5x-19}{(x+4)^2}$

21) $\frac{x}{(x-2)^2}$

Answer

$\frac{1}{x-2}-\frac{2}{(x-2)^2}$

22) $\frac{7x+14}{(x+3)^2}$

23) $\frac{-24x-27}{(4x+5)^2}$

Answer

$-\frac{6}{4x+5}+\frac{3}{(4x+5)^2}$

24) $\frac{-24x-27}{(6x-7)^2}$

25) $\frac{5-x}{(x-7)^2}$

Answer

$-\frac{1}{x-7}-\frac{2}{(x-7)^2}$

26) $\dfrac{5x+14}{2x^2+12x+18}$

27) $\dfrac{5x^2+20x+8}{2x(x+1)^2}$

Answer

$$\frac{4}{x}-\frac{3}{2(x+1)}+\frac{7}{2(x+1)^2}$$

28) $\dfrac{4x^2+55x+25}{5x(3x+5)^2}$

29) $\dfrac{54x^3+127x^2+80x+16}{2x^2(3x+2)^2}$

Answer

$$\frac{4}{x}+\frac{2}{x^2}-\frac{3}{3x+2}+\frac{7}{2(3x+2)^2}$$

30) $\dfrac{x^3-5x^2+12x+144}{x^2(x^2+12x+36)}$

For the exercises 31-43, find the decomposition of the partial fraction for the irreducible nonrepeating quadratic factor.

31) $\dfrac{4x^2+6x+11}{(x+2)(x^2+x+3)}$

Answer

$$\frac{x+1}{x^2+x+3}+\frac{3}{(x+2)}$$

32) $\dfrac{4x^2+9x+23}{(x-1)(x^2+6x+11)}$

33) $\dfrac{-2x^2+10x+4}{(x-1)(x^2+3x+8)}$

Answer

$$\frac{4-3x}{x^2+3x+8}+\frac{1}{(x-1)}$$

34) $\dfrac{x^2+3x+1}{(x+1)(x^2+5x-2)}$

35) $\dfrac{4x^2+17x-1}{(x+3)(x^2+6x+1)}$

Answer

$$\frac{2x-1}{x^2+6x+1}+\frac{2}{(x+3)}$$

36) $\dfrac{4x^2}{(x+5)(x^2+7x-5)}$

37) $\dfrac{4x^2+x+3}{x^3-1}$

Answer

$$\frac{1}{x^2+x+1}+\frac{4}{(x-1)}$$

38) $\frac{-5x^2+18x-4}{x^3+8}$

39) $\frac{3x^2-7x+33}{x^3+27}$

Answer

$$\frac{2}{x^2-3x+9}+\frac{3}{(x+3)}$$

40) $\frac{x^2+2x+40}{x^3-125}$

41) $\frac{4x^2+4x+12}{8x^3-27}$

Answer

$$-\frac{1}{4x^2+6x+9}+\frac{1}{(2x-3)}$$

42) $\frac{-50x^2+5x-3}{125x^3-1}$

43) $\frac{-2x^3-30x^2+36x+216}{x^4+216x}$

Answer

$$\frac{1}{x}+\frac{1}{x+6}-\frac{4x}{x^2-6x+36}$$

For the exercises 44-54, find the decomposition of the partial fraction for the irreducible repeating quadratic factor.

44) $\frac{3x^3+2x^2+14x+15}{(x^2+4)^2}$

45) $\frac{x^3+6x^2+5x+9}{(x^2+1)^2}$

Answer

$$\frac{x+6}{x^2+1}+\frac{4x+3}{(x^2+1)^2}$$

46) $\frac{x^3-x^2+x-1}{(x^2-3)^2}$

47) $\frac{x^2+5x+5}{(x+2)^2}$

Answer

$$\frac{x+1}{x+2}+\frac{2x+3}{(x+2)^2}$$

48) $\frac{x^3+2x^2+4x}{(x^2+2x+9)^2}$

49) $\frac{x^2+25}{(x^2+3x+25)^2}$

Answer

$$\frac{1}{x^2+3x+25}-\frac{3x}{(x^2+3x+25)^2}$$

50) $\frac{2x^3+11x+7x+70}{(2x^2+x+14)^2}$

51) $\frac{5x+2}{x(x^2+4)^2}$

Answer

$$\frac{1}{8x}-\frac{x}{8(x^2+4)}+\frac{10-x}{8(x^2+4)^2}$$

52) $\frac{x^4+x^3+8x^2+6x+36}{x(x^2+6)^2}$

53) $\frac{2x-9}{(x^2-x)^2}$

Answer

$$-\frac{16}{x}-\frac{9}{x^2}+\frac{16}{x-1}-\frac{7}{(x-1)^2}$$

54) $\frac{5x^3-2x+1}{(x^2+2x)^2}$

Extensions

For the exercises 55-56, find the partial fraction expansion.

55) $\frac{x^2+4}{(x+1)^3}$

Answer

$$\frac{1}{x+1}-\frac{2}{(x+1)^2}+\frac{5}{(x+1)^3}$$

56) $\frac{x^3-4x^2+5x+4}{(x-2)^3}$

For the exercises 57-59, perform the operation and then find the partial fraction decomposition.

57) $\frac{7}{x+8}+\frac{5}{x-2}-\frac{x-1}{x^2-6x-16}$

Answer

$$\frac{5}{x-2}-\frac{3}{10(x+2)}+\frac{7}{x+8}-\frac{7}{10(x-8)}$$

58) $\frac{1}{x-4}-\frac{3}{x+6}-\frac{2x+7}{x^2+2x-24}$

59) $\frac{2x}{x^2-16}-\frac{1-2x}{x^2+6x+8}-\frac{x-5}{x^2-4x}$

Answer

$$-\frac{5}{4x}-\frac{5}{2(x+2)}+\frac{11}{2(x+4)}+\frac{5}{4(x+4)}$$

9.5: Matrices and Matrix Operations

Verbal

1) Can we add any two matrices together? If so, explain why; if not, explain why not and give an example of two matrices that cannot be added together.

Answer

No, they must have the same dimensions. An example would include two matrices of different dimensions. One cannot add the following two matrices because the first is a 2×2 matrix and the second is a 2×3. $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}+\begin{bmatrix}6 & 5 & 4\\3 & 2 & 1\end{bmatrix}$ has no sum.

2) Can we multiply any column matrix by any row matrix? Explain why or why not.

3) Can both the products AB and BA be defined? If so, explain how; if not, explain why.

Answer

Yes, if the dimensions of A are $m\times n$ and the dimensions of B are $n\times m$ both products will be defined.

4) Can any two matrices of the same size be multiplied? If so, explain why, and if not, explain why not and give an example of two matrices of the same size that cannot be multiplied together.

5) Does matrix multiplication commute? That is, does $AB=BA$? If so, prove why it does. If not, explain why it does not.

Answer

Not necessarily. To find AB, we multiply the first row of A by the first column of B to get the first entry of AB. To find BA, we multiply the first row of B by the first column of A to get the first entry of BA. Thus, if those are unequal, then the matrix multiplication does not commute.

Algebraic

For the exercises 6-11, use the matrices below and perform the matrix addition or subtraction. Indicate if the operation is undefined.

$$A=\begin{bmatrix}1 & 3\\0 & 7\end{bmatrix}, B=\begin{bmatrix}2 & 14\\22 & 6\end{bmatrix}, C=\begin{bmatrix}1 & 5\\8 & 92\\12 & 6\end{bmatrix}, D=\begin{bmatrix}10 & 14\\7 & 2\\5 & 61\end{bmatrix}, E=\begin{bmatrix}6 & 12\\14 & 5\end{bmatrix}, F=\begin{bmatrix}0 & 9\\78 & 17\\15 & 4\end{bmatrix}$$

6) $A+B$

7) $C+D$

Answer

$$\begin{bmatrix}11 & 19\\15 & 94\\17 & 67\end{bmatrix}$$

8) $A+C$

9) $B-E$

Answer

$$\begin{bmatrix}-4 & 2\\8 & 1\end{bmatrix}$$

10) $C+F$

11) $D-B$

Answer

Undefined; dimensions do not match

For the exercises 12-17, use the matrices below to perform scalar multiplication.

$$A=\begin{bmatrix}4 & 6\\ 13 & 12\end{bmatrix}, B=\begin{bmatrix}3 & 9\\ 21 & 12\\ 0 & 64\end{bmatrix}, C=\begin{bmatrix}16 & 3 & 7 & 18\\ 90 & 5 & 3 & 29\end{bmatrix}, D=\begin{bmatrix}18 & 12 & 13\\ 8 & 14 & 6\\ 7 & 4 & 21\end{bmatrix}$$

12) $5A$

13) $3B$

Answer

$$\begin{bmatrix}9 & 27\\ 63 & 36\\ 0 & 192\end{bmatrix}$$

14) $-2B$

15) $-4C$

Answer

$$\begin{bmatrix}-64 & -12 & -28 & -72\\ -360 & -20 & -12 & -116\end{bmatrix}$$

16) $\frac{1}{2}C$

17) $100D$

Answer

$$\begin{bmatrix}1,800 & 1,200 & 1,300\\ 800 & 1,400 & 600\\ 700 & 400 & 2,100\end{bmatrix}$$

For the exercises 18-23, use the matrices below to perform matrix multiplication.

$$A=\begin{bmatrix}-1 & 5\\ 3 & 2\end{bmatrix}, B=\begin{bmatrix}3 & 6 & 4\\ -8 & 0 & 12\end{bmatrix}, C=\begin{bmatrix}4 & 10\\ -2 & 6\\ 5 & 9\end{bmatrix}, D=\begin{bmatrix}2 & -3 & 12\\ 9 & 3 & 1\\ 0 & 8 & -10\end{bmatrix}$$

18) AB

19) BC

Answer

$$\begin{bmatrix}20 & 102\\ 28 & 28\end{bmatrix}$$

20) CA

21) BD

Answer

$$\begin{bmatrix} 60 & 41 & 2 \\ -16 & 120 & -216 \end{bmatrix}$$

22) DC

23) CB

Answer

$$\begin{bmatrix} -68 & 24 & 136 \\ -54 & -12 & 64 \\ -57 & 30 & 128 \end{bmatrix}$$

For the exercises 24-29, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed.

$$A=\begin{bmatrix} 2 & -5 \\ 6 & 7 \end{bmatrix}, B=\begin{bmatrix} -9 & 6 \\ -4 & 2 \end{bmatrix}, C=\begin{bmatrix} 0 & 9 \\ 7 & 1 \end{bmatrix}, D=\begin{bmatrix} -8 & 7 & -5 \\ 4 & 3 & 2 \\ 0 & 9 & 2 \end{bmatrix}, E=\begin{bmatrix} 4 & 5 & 3 \\ 7 & -6 & -5 \\ 1 & 0 & 9 \end{bmatrix}$$

24) $A+B-C$

25) $4A+5D$

Answer

Undefined; dimensions do not match.

26) $2C+B$

27) $3D+4E$

Answer

$$\begin{bmatrix} -8 & 41 & -3 \\ 40 & -15 & -14 \\ 4 & 27 & 42 \end{bmatrix}$$

28) $C-0.5D$

29) $100D-10E$

Answer

$$\begin{bmatrix} -840 & 650 & -530 \\ 330 & 360 & 250 \\ -10 & 900 & 110 \end{bmatrix}$$

For the exercises 30-40, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2=A\cdot A$)

$$A=\begin{bmatrix} -10 & 20 \\ 5 & 25 \end{bmatrix}, B=\begin{bmatrix} 40 & 10 \\ -20 & 30 \end{bmatrix}, C=\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}$$

30) AB

31) BA

Answer

$$\begin{bmatrix} -350 & 1,050 \\ 350 & 350 \end{bmatrix}$$

32) CA

33) BC

Answer

Undefined; inner dimensions do not match.

34) A^2

35) B^2

Answer

$$\begin{bmatrix} 1,400 & 700 \\ -1,400 & 700 \end{bmatrix}$$

36) C^2

37) B^2A^2

Answer

$$\begin{bmatrix} 332,500 & 927,500 \\ -227,500 & 87,500 \end{bmatrix}$$

38) A^2B^2

39) $(AB)^2$

Answer

$$\begin{bmatrix} 490,000 & 0 \\ 0 & 490,000 \end{bmatrix}$$

40)$(BA)^2$

For the exercises 41-49, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. (Hint: $A^2 = A \cdot A$)

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} -2 & 3 & 4 \\ -1 & 1 & -5 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 0.1 \\ 1 & 0.2 \\ -0.5 & 0.3 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & -1 \\ -6 & 7 & 5 \\ 4 & 2 & 1 \end{bmatrix}$$

41) AB

Answer

$$\begin{bmatrix} -2 & 3 & 4 \\ -7 & 9 & -7 \end{bmatrix}$$

42) BA

43) BD

Answer

$$\begin{bmatrix} -4 & 29 & 21 \\ -27 & -3 & 1 \end{bmatrix}$$

44) DC

45) D^2

Answer

$$\begin{bmatrix} -3 & -2 & -2 \\ -28 & 59 & 46 \\ -4 & 16 & 7 \end{bmatrix}$$

46) A^2

47) D^3

Answer

$$\begin{bmatrix} 1 & -18 & -9 \\ -198 & 505 & 369 \\ -72 & 126 & 91 \end{bmatrix}$$

48) $(AB)C$

49) $A(BC)$

Answer

$$\begin{bmatrix} 0 & 1.6 \\ 9 & -1 \end{bmatrix}$$

Technology

For the exercises 50-54, use the matrices below to perform the indicated operation if possible. If not possible, explain why the operation cannot be performed. Use a calculator to verify your solution.

$$A = \begin{bmatrix} -2 & 0 & 9 \\ 1 & 8 & -3 \\ 0.5 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 0.5 & 3 & 0 \\ -4 & 1 & 6 \\ 8 & 7 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

50) AB

51) BA

Answer

$$\begin{bmatrix} 2 & 24 & -4.5 \\ 12 & 32 & -9 \\ -8 & 64 & 61 \end{bmatrix}$$

52) CA

53) BC

Answer

$$\begin{bmatrix} 0.5 & 3 & 0.5 \\ 2 & 1 & 2 \\ 10 & 7 & 10 \end{bmatrix}$$

54) ABC

Extensions

For the exercises 55-, use the matrix below to perform the indicated operation on the given matrix.

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

55) B^2

Answer

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

56) B^3

57) B^4

Answer

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

58) B^5

59) Using the above questions, find a formula for B^n. Test the formula for B^{201} and B^{202}, using a calculator.

Answer

$$B^n = \begin{cases} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, & n \text{ even} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, & n \text{ odd} \end{cases}$$

9.6: Solving Systems with Gaussian Elimination

Verbal

1) Can any system of linear equations be written as an augmented matrix? Explain why or why not. Explain how to write that augmented matrix.

Answer

Yes. For each row, the coefficients of the variables are written across the corresponding row, and a vertical bar is placed; then the constants are placed to the right of the vertical bar.

2) Can any matrix be written as a system of linear equations? Explain why or why not. Explain how to write that system of equations.

3) Is there only one correct method of using row operations on a matrix? Try to explain two different row operations possible to solve the augmented matrix $\left[\begin{array}{cc|c} 9 & 3 & 0 \\ 1 & -2 & 6 \end{array}\right]$.

Answer

No, there are numerous correct methods of using row operations on a matrix. Two possible ways are the following:

i. Interchange rows 1 and 2. Then $R_2 = R_2 - 9R_1$.
ii. $R_2 = R_1 - 9R_2$. Then divide row 1 by 9.

4) Can a matrix whose entry is 0 on the diagonal be solved? Explain why or why not. What would you do to remedy the situation?

5) Can a matrix that has 0 entries for an entire row have one solution? Explain why or why not.

Answer

No. A matrix with 0 entries for an entire row would have either zero or infinitely many solutions.

Algebraic

For the exercises 6-10, write the augmented matrix for the linear system.

6) $\begin{aligned} 8x-37y &= 8 \\ 2x+12y &= 3 \end{aligned}$

7) $\begin{aligned} 16y &= 4 \\ 9x-y &= 2 \end{aligned}$

Answer

$$\left[\begin{array}{cc|c} 0 & 16 & 4 \\ 9 & -1 & 2 \end{array}\right]$$

8) $\begin{aligned} 3x+2y+10z &= 3 \\ -6x+2y+5z &= 13 \\ 4x+z &= 18 \end{aligned}$

9) $\begin{aligned} x+5y+8z &= 19 \\ 12x+3y &= 4 \\ 3x+4y+9z &= -7 \end{aligned}$

Answer

$$\left[\begin{array}{ccc|c} 1 & 5 & 8 & 16 \\ 12 & 3 & 0 & 4 \\ 3 & 4 & 9 & -7 \end{array}\right]$$

10) $\begin{aligned} 6x+12y+16z &= 4 \\ 19x-5y+3z &= -9 \\ x+2y &= -8 \end{aligned}$

For the exercises 11-15, write the linear system from the augmented matrix.

11) $\left[\begin{array}{cc|c} -2 & 5 & 5 \\ 6 & -18 & 26 \end{array}\right]$

Answer

$$\begin{aligned} -2x+5y &= 5 \\ 6x-18y &= 26 \end{aligned}$$

12) $\left[\begin{array}{cc|c} 3 & 4 & 10 \\ 10 & 17 & 439 \end{array}\right]$

13) $\left[\begin{array}{ccc|c} 3 & 2 & 0 & 3 \\ -1 & -9 & 4 & -1 \\ 8 & 5 & 7 & 8 \end{array}\right]$

Answer

$$\begin{aligned} 3x+2y &= 13 \\ -x-9y+4z &= 53 \\ 8x+5y+7z &= 80 \end{aligned}$$

14) $\left[\begin{array}{ccc|c} 8 & 29 & 1 & 43 \\ -1 & 7 & 5 & 38 \\ 0 & 0 & 3 & 10 \end{array}\right]$

15) $\left[\begin{array}{ccc|c} 4 & 5 & -2 & 12 \\ 0 & 1 & 58 & 2 \\ 8 & 7 & -3 & -5 \end{array}\right]$

Answer

$$\begin{aligned} 4x+5y-2z &= 12 \\ y+58z &= 2 \\ 8x+7y-3z &= -5 \end{aligned}$$

For the exercises 16-46, solve the system by Gaussian elimination.

16) $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 0 \end{array}\right]$

17) $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 0 & 2 \end{array}\right]$

Answer

No solutions

18) $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$

19) $\left[\begin{array}{cc|c} -1 & 2 & -3 \\ 4 & -5 & 6 \end{array}\right]$

Answer

$(-1,-2)$

20) $\left[\begin{array}{cc|c} -2 & 0 & 1 \\ 0 & 2 & -1 \end{array}\right]$

21) $\begin{aligned} 2x-3y &= -9 \\ 5x+4y &= 58 \end{aligned}$

Answer

$(6,7)$

22) $\begin{aligned} 6x+2y &= -4 \\ 3x+4y &= -17 \end{aligned}$

23) $\begin{aligned} 2x+3y &= 12 \\ 4x+y &= 14 \end{aligned}$

Answer

$(3,2)$

24) $\begin{aligned} -4x-3y &= -2 \\ 3x-5y &= -13 \end{aligned}$

25) $\begin{aligned} -5x+8y &= 3 \\ 10x+6y &= 5 \end{aligned}$

Answer

$$\left(\frac{1}{5},\frac{1}{2}\right)$$

26) $\begin{aligned} 3x+4y &= 12 \\ -6x-8y &= -24 \end{aligned}$

27) $\begin{aligned} -60x+45y &= 12 \\ 20x-15y &= -4 \end{aligned}$

Answer

$$\left(x,\frac{4}{15}(5x+1)\right)$$

28) $\begin{aligned} 11x+10y &= 43 \\ 15x+20y &= 65 \end{aligned}$

29) $\begin{aligned} 2x-y &= 2 \\ 3x+2y &= 17 \end{aligned}$

Answer

$$(3,4)$$

30) $\begin{aligned} -1.06x-2.25y &= 5.51 \\ -5.03x-1.08y &= 5.40 \end{aligned}$

31) $\begin{aligned} \frac{3}{4}x-\frac{3}{5}y &= 4 \\ \frac{1}{4}x+\frac{2}{3}y &= 1 \end{aligned}$

Answer

$$\left(\frac{196}{39},-\frac{5}{13}\right)$$

32) $\begin{aligned} \frac{1}{4}x-\frac{2}{3}y &= -1 \\ \frac{1}{2}x+\frac{1}{3}y &= 3 \end{aligned}$

33) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 31 \\ 0 & 1 & 1 & 45 \\ 0 & 0 & 1 & 87 \end{array}\right]$

Answer

$$(31,-42,87)$$

34) $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 50 \\ 1 & 1 & 0 & 20 \\ 0 & 1 & 1 & -90 \end{array}\right]$

35) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{array}\right]$

Answer

$$\left(\frac{21}{40}, \frac{1}{20}, \frac{9}{8}\right)$$

36) $$\left[\begin{array}{ccc|c} -0.1 & 0.3 & -0.1 & 0.2 \\ -0.4 & 0.2 & 0.1 & 0.8 \\ 0.6 & 0.1 & 0.7 & -0.8 \end{array}\right]$$

37) $$\begin{aligned} -2x+3y-2z &= 3 \\ 4x+2y-z &= 9 \\ 4x-8y+2z &= -6 \end{aligned}$$

Answer

$$\left(\frac{18}{13}, \frac{15}{13}, -\frac{15}{13}\right)$$

38) $$\begin{aligned} x+y-4z &= -4 \\ 5x-3y-2z &= 0 \\ 2x+6y+7z &= 30 \end{aligned}$$

39) $$\begin{aligned} 2x+3y+2z &= 1 \\ -4x-6y-4z &= -2 \\ 10x+15y+10z &= 5 \end{aligned}$$

Answer

$$\left(x, y, \frac{1}{2}(1-2x-3y)\right)$$

40) $$\begin{aligned} x+2y-z &= 1 \\ -x-2y+2z &= -2 \\ 3x+6y-3z &= 5 \end{aligned}$$

41) $$\begin{aligned} x+2y-z &= 1 \\ -x-2y+2z &= -2 \\ 3x+6y-3z &= 3 \end{aligned}$$

Answer

$$\left(x, -\frac{x}{2}, -1\right)$$

42) $$\begin{aligned} x+y &= 2 \\ x+z &= 1 \\ -y-z &= -3 \end{aligned}$$

43) $$\begin{aligned} x+y+z &= 100 \\ x+2z &= 125 \\ -y+2z &= 25 \end{aligned}$$

Answer

$$(125, -25, 0)$$

44) $$\begin{aligned} \frac{1}{4}x-\frac{2}{3}z &= -\frac{1}{2} \\ \frac{1}{5}x+\frac{1}{3}y &= \frac{4}{7} \\ \frac{1}{5}y-\frac{1}{3}z &= \frac{2}{9} \end{aligned}$$

45)
$$\begin{aligned} -\frac{1}{2}x+\frac{1}{2}y+\frac{1}{7}z &= -\frac{53}{14} \\ \frac{1}{2}x-\frac{1}{2}y+\frac{1}{4}z &= 3 \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{3}z &= \frac{23}{15} \end{aligned}$$

Answer

$(8, 1, -2)$

46)
$$\begin{aligned} -\frac{1}{2}x-\frac{1}{3}y+\frac{1}{4}z &= -\frac{29}{6} \\ \frac{1}{5}x+\frac{1}{6}y-\frac{1}{7}z &= \frac{431}{210} \\ -\frac{1}{8}x+\frac{1}{9}y+\frac{1}{10}z &= -\frac{49}{45} \end{aligned}$$

Extensions

For the exercises 47-51, use Gaussian elimination to solve the system.

47)
$$\begin{aligned} \frac{x-1}{7}+\frac{y-2}{8}+\frac{z-3}{4} &= 0 \\ x+y+z &= 6 \\ \frac{x+2}{3}+2y+\frac{z-3}{3} &= 5 \end{aligned}$$

Answer

$(1, 2, 3)$

48)
$$\begin{aligned} \frac{x-1}{4}-\frac{y+1}{4}+3z &= -1 \\ \frac{x+5}{2}+\frac{y+7}{4}-z &= 4 \\ x+y-\frac{z-2}{2} &= 1 \end{aligned}$$

49)
$$\begin{aligned} \frac{x-3}{4}-\frac{y-1}{3}+2z &= -1 \\ \frac{x+5}{2}+\frac{y+5}{2}+\frac{z+5}{2} &= 8 \\ x+y+z &= 1 \end{aligned}$$

Answer

$$\left(x, \frac{31}{28}-\frac{3x}{4}, \frac{1}{28}(-7x-3)\right)$$

50)
$$\begin{aligned} \frac{x-3}{10}+\frac{y+3}{2}-2z &= 3 \\ \frac{x+5}{4}-\frac{y-1}{8}+z &= \frac{3}{2} \\ \frac{x-1}{4}+\frac{y+4}{2}+3z &= \frac{3}{2} \end{aligned}$$

51)
$$\begin{aligned} \frac{x-3}{4}-\frac{y-1}{3}+2z &= -1 \\ \frac{x+5}{2}+\frac{y+5}{2}+\frac{z+5}{2} &= 7 \\ x+y+z &= 1 \end{aligned}$$

Answer

No solutions exist.

Real-World Applications

For the exercises 52-61, set up the augmented matrix that describes the situation, and solve for the desired solution.

52) Every day, a cupcake store sells $5,000$ cupcakes in chocolate and vanilla flavors. If the chocolate flavor is 3 times as popular as the vanilla flavor, how many of each cupcake sell per day?

53) At a competing cupcake store, $\$4,520$ worth of cupcakes are sold daily. The chocolate cupcakes cost $\$2.25$ and the red velvet cupcakes cost $\$1.75$. If the total number of cupcakes sold per day is $2,200$, how many of each flavor are sold each day?

Answer

860 red velvet, $1,340$ chocolate

54) You invested $\$10,000$ into two accounts: one that has simple 3% interest, the other with 2.5% interest. If your total interest payment after one year was $\$283.50$, how much was in each account after the year passed?

55) You invested $\$2,300$ into account 1, and $\$2,700$ into account 2. If the total amount of interest after one year is $\$254$, and account 2 has 1.5 times the interest rate of account 1, what are the interest rates? Assume simple interest rates.

Answer

4% for account 1, 6% for account 2

56) Bikes'R'Us manufactures bikes, which sell for $\$250$. It costs the manufacturer $\$180$ per bike, plus a startup fee of $\$3,500$. After how many bikes sold will the manufacturer break even?

57) A major appliance store is considering purchasing vacuums from a small manufacturer. The store would be able to purchase the vacuums for $\$86$ each, with a delivery fee of $\$9,200$, regardless of how many vacuums are sold. If the store needs to start seeing a profit after 230 units are sold, how much should they charge for the vacuums?

Answer

$\$126$

58) The three most popular ice cream flavors are chocolate, strawberry, and vanilla, comprising 83% of the flavors sold at an ice cream shop. If vanilla sells 1% more than twice strawberry, and chocolate sells 11% more than vanilla, how much of the total ice cream consumption are the vanilla, chocolate, and strawberry flavors?

59) At an ice cream shop, three flavors are increasing in demand. Last year, banana, pumpkin, and rocky road ice cream made up 12% of total ice cream sales. This year, the same three ice creams made up 16.9% of ice cream sales. The rocky road sales doubled, the banana sales increased by 50%, and the pumpkin sales increased by 20%. If the rocky road ice cream had one less percent of sales than the banana ice cream, find out the percentage of ice cream sales each individual ice cream made last year.

Answer

Banana was 3%, pumpkin was 7%, and rocky road was 2%

60) A bag of mixed nuts contains cashews, pistachios, and almonds. There are $1,000$ total nuts in the bag, and there are 100 less almonds than pistachios. The cashews weigh 3 g, pistachios weigh 4 g, and almonds weigh 5 g. If the bag weighs 3.7 kg, find out how many of each type of nut is in the bag.

61) A bag of mixed nuts contains cashews, pistachios, and almonds. Originally there were 900 nuts in the bag. 30% of the almonds, 20% of the cashews, and 10% of the pistachios were eaten, and now there are 770 nuts left in the bag. Originally, there were 100 more cashews than almonds. Figure out how many of each type of nut was in the bag to begin with.

Answer

100 almonds, 200 cashews, 600 pistachios

9.7: Solving Systems with Inverses

Verbal

1) In a previous section, we showed that matrix multiplication is not commutative, that is, $AB \neq BA$ in most cases. Can you explain why matrix multiplication is commutative for matrix inverses, that is, $A^{-1}A = AA^{-1}$?

Answer

If A^{-1} is the inverse of A, then $AA^{-1} = I$, the identity matrix. Since A is also the inverse of A^{-1}, $A^{-1}A = I$. You can also check by proving this for a 2×2 matrix.

2) Does every 2×2 matrix have an inverse? Explain why or why not. Explain what condition is necessary for an inverse to exist.

3) Can you explain whether a 2×2 matrix with an entire row of zeros can have an inverse?

Answer

No, because ad and bc are both 0, so $ad - bc = 0$, which requires us to divide by 0 in the formula.

4) Can a matrix with an entire column of zeros have an inverse? Explain why or why not.

5) Can a matrix with zeros on the diagonal have an inverse? If so, find an example. If not, prove why not. For simplicity, assume a 2×2 matrix.

Answer

Yes. Consider the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The inverse is found with the following calculation:

$$A^{-1} = \frac{1}{0(0) - 1(1)} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Algebraic

In the exercises 6-12, show that matrix A is the inverse of matrix B.

6) $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

7) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Answer

$$AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

8) $A = \begin{bmatrix} 4 & 5 \\ 7 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & \frac{1}{7} \\ \frac{1}{5} & -\frac{4}{35} \end{bmatrix}$

9) $A = \begin{bmatrix} -2 & \frac{1}{2} \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} -2 & -1 \\ -6 & -4 \end{bmatrix}$

Answer

$$AB = BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

10) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, B = \frac{1}{2} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$

11) $A=\begin{bmatrix}1&0&1\\0&1&-1\\0&1&1\end{bmatrix}, B=\frac{1}{2}\begin{bmatrix}2&1&-1\\0&1&1\\0&-1&1\end{bmatrix}$

Answer

$$AB=BA=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$$

12) $A=\begin{bmatrix}3&8&2\\1&1&1\\5&6&12\end{bmatrix}, B=\frac{1}{36}\begin{bmatrix}-6&84&-6\\7&-26&1\\-1&-22&5\end{bmatrix}$

For the exercises 13-26, find the multiplicative inverse of each matrix, if it exists.

13) $\begin{bmatrix}3&-2\\1&9\end{bmatrix}$

Answer

$\frac{1}{29}\begin{bmatrix}9&2\\-1&3\end{bmatrix}$

14) $\begin{bmatrix}-2&2\\3&1\end{bmatrix}$

15) $\begin{bmatrix}-3&7\\9&2\end{bmatrix}$

Answer

$\frac{1}{69}\begin{bmatrix}-2&7\\9&3\end{bmatrix}$

16) $\begin{bmatrix}-4&-3\\-5&8\end{bmatrix}$

17) $\begin{bmatrix}1&1\\2&2\end{bmatrix}$

Answer

There is no inverse

18) $\begin{bmatrix}0&1\\1&0\end{bmatrix}$

19) $\begin{bmatrix}0.5&1.5\\1&-0.5\end{bmatrix}$

Answer

$\frac{4}{7}\begin{bmatrix}0.5&1.5\\1&-0.5\end{bmatrix}$

20) $\begin{bmatrix}1&0&6\\-2&1&7\\3&0&2\end{bmatrix}$

21) $\begin{bmatrix} 0 & 1 & -3 \\ 4 & 1 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

Answer

$\frac{1}{17}\begin{bmatrix} -5 & 5 & -3 \\ 20 & -3 & 12 \\ 1 & -1 & 4 \end{bmatrix}$

22) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 1 \\ -2 & -4 & -5 \end{bmatrix}$

23) $\begin{bmatrix} 1 & 9 & -3 \\ 2 & 5 & 6 \\ 4 & -2 & -7 \end{bmatrix}$

Answer

$\frac{1}{209}\begin{bmatrix} 47 & -57 & 69 \\ 10 & 19 & -12 \\ -24 & 38 & -13 \end{bmatrix}$

24) $\begin{bmatrix} 1 & -2 & 3 \\ -4 & 8 & -12 \\ 1 & 4 & 2 \end{bmatrix}$

25) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \end{bmatrix}$

Answer

$\begin{bmatrix} 18 & 60 & -168 \\ -56 & -140 & 448 \\ 40 & 80 & -280 \end{bmatrix}$

26) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

For the exercises 27-34, solve the system using the inverse of a 2×2 matrix.

27) $\begin{aligned} 5x - 6y &= -61 \\ 4x + 3y &= -2 \end{aligned}$

Answer

$(-5, 6)$

28) $\begin{aligned} 8x + 4y &= -100 \\ 3x - 4y &= 1 \end{aligned}$

29) $\begin{aligned} 3x - 2y &= 6 \\ -x + 5y &= -2 \end{aligned}$

Answer

$(2, 0)$

30) $\begin{aligned} 5x-4y &= -5 \\ 4x+y &= 2.3 \end{aligned}$

31) $\begin{aligned} -3x-4y &= 9 \\ 12x+4y &= -6 \end{aligned}$

Answer

$\left(\frac{1}{3}, -\frac{5}{2}\right)$

32) $\begin{aligned} -2x+3y &= \frac{3}{10} \\ -x+5y &= \frac{1}{2} \end{aligned}$

33) $\begin{aligned} \frac{8}{5}x-\frac{4}{5}y &= \frac{2}{5} \\ -\frac{8}{5}x+\frac{1}{5}y &= \frac{7}{10} \end{aligned}$

Answer

$\left(-\frac{2}{3}, -\frac{11}{6}\right)$

34) $\begin{aligned} \frac{1}{2}x+\frac{1}{5}y &= -\frac{1}{4} \\ \frac{1}{2}x-\frac{3}{5}y &= -\frac{9}{4} \end{aligned}$

For the exercises 35-42, solve a system using the inverse of a 3×3 matrix.

35) $\begin{aligned} 3x-2y+5z &= 21 \\ 5x+4y &= 37 \\ x-2y-5z &= 5 \end{aligned}$

Answer

$\left(7, \frac{1}{2}, \frac{1}{5}\right)$

36) $\begin{aligned} 4x+4y+4z &= 40 \\ 2x-3y+4z &= -12 \\ -x+3y+4z &= 9 \end{aligned}$

37) $\begin{aligned} 6x-5y-z &= 31 \\ -x+2y+z &= -6 \\ 3x+3y+2z &= 13 \end{aligned}$

Answer

$(5, 0, -1)$

38) $\begin{aligned} 6x-5y+2z &= -4 \\ 2x+5y-z &= 12 \\ 2x+5y+z &= 12 \end{aligned}$

39) $\begin{aligned} 4x-2y+3z &= -12 \\ 2x+2y-9z &= 33 \\ 6y-4z &= 1 \end{aligned}$

Answer

$$\frac{1}{34}(-35,-97,-154)$$

40) $$\begin{aligned} \frac{1}{10}x-\frac{1}{5}y+4z &= \frac{-41}{2} \\ \frac{1}{5}x-20y+\frac{2}{5}z &= -101 \\ \frac{3}{10}x+4y-\frac{3}{10}z &= 23 \end{aligned}$$

41) $$\begin{aligned} \frac{1}{2}x-\frac{1}{5}y+\frac{1}{5}z &= \frac{31}{100} \\ -\frac{3}{4}x-\frac{1}{4}y+\frac{1}{2}z &= \frac{7}{40} \\ -\frac{4}{5}x-\frac{1}{2}y+\frac{3}{2}z &= \frac{1}{4} \end{aligned}$$

Answer

$$\frac{1}{690}(65,-1136,-229)$$

42) $$\begin{aligned} 0.1x+0.2y+0.3z &= -1.4 \\ 0.1x-0.2y+0.3z &= 0.6 \\ 0.4y+0.9z &= -2 \end{aligned}$$

Technology

For the exercises 43-46, use a calculator to solve the system of equations with matrix inverses.

43) $$\begin{aligned} 2x-y &= -3 \\ -x+2y &= 2.3 \end{aligned}$$

Answer

$$\left(-\frac{37}{30},\frac{8}{15}\right)$$

44) $$\begin{aligned} -\frac{1}{2}x-\frac{3}{2}y &= -\frac{43}{20} \\ \frac{5}{2}x+\frac{11}{5}y &= \frac{31}{4} \end{aligned}$$

45) $$\begin{aligned} 12.3x-2y-2.5z &= 2 \\ 36.9x+7y-7.5z &= -7 \\ 8y-5z &= -10 \end{aligned}$$

Answer

$$\left(\frac{10}{123},-1,\frac{2}{5}\right)$$

46) $$\begin{aligned} 0.5x-3y+6z &= -0.8 \\ 0.7x-2y &= -0.06 \\ 0.5x+4y+5z &= 0 \end{aligned}$$

Extensions

For the exercises 47-51, find the inverse of the given matrix.

47) $$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Answer

$$\frac{1}{2}\begin{bmatrix} 2 & 1 & -1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

48) $\begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 2 \\ 0 & 2 & -1 & 0 \\ 1 & -3 & 0 & 1 \end{bmatrix}$

49) $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 4 & -2 & 3 \\ -5 & 0 & 1 & 1 \end{bmatrix}$

Answer

$$\frac{1}{39}\begin{bmatrix} 3 & 2 & 1 & -7 \\ 18 & -53 & 32 & 10 \\ 24 & -36 & 21 & 9 \\ -9 & 46 & -16 & -5 \end{bmatrix}$$

50) $\begin{bmatrix} 1 & 2 & 0 & 2 & 3 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$

51) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

Answer

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & 1 \end{bmatrix}$$

Real-World Applications

For the exercises 52-61, write a system of equations that represents the situation. Then, solve the system using the inverse of a matrix.

52) $2,400$ tickets were sold for a basketball game. If the prices for floor 1 and floor 2 were different, and the total amount of money brought in is $\$64,000$, how much was the price of each ticket?

53) In the previous exercise, if you were told there were 400 more tickets sold for floor 2 than floor 1, how much was the price of each ticket?

Answer

Infinite solutions.

54) A food drive collected two different types of canned goods, green beans and kidney beans. The total number of collected cans was 350 and the total weight of all donated food was 348 lb, 12 oz. If the green bean cans weigh 2 oz less than the kidney bean cans, how many of each can was donated?

55) Students were asked to bring their favorite fruit to class. 95% of the fruits consisted of banana, apple, and oranges. If oranges were twice as popular as bananas, and apples were 5% less popular than bananas, what are the percentages of each individual fruit?

Answer

50% oranges, 25% bananas, 20% apples

56) A sorority held a bake sale to raise money and sold brownies and chocolate chip cookies. They priced the brownies at $\$1$ and the chocolate chip cookies at $\$0.75$. They raised $\$700$ and sold 850 items. How many brownies and how many cookies were sold?

57) A clothing store needs to order new inventory. It has three different types of hats for sale: straw hats, beanies, and cowboy hats. The straw hat is priced at $\$13.99$, the beanie at $\$7.99$, and the cowboy hat at $\$14.49$. If 100 hats were sold this past quarter, $\$1,119$ was taken in by sales, and the amount of beanies sold was 10 more than cowboy hats, how many of each should the clothing store order to replace those already sold?

Answer

10 straw hats, 50 beanies, 40 cowboy hats

58) Anna, Ashley, and Andrea weigh a combined 370 lb. If Andrea weighs 20 lb more than Ashley, and Anna weighs 1.5 times as much as Ashley, how much does each girl weigh?

59) Three roommates shared a package of 12 ice cream bars, but no one remembers who ate how many. If Tom ate twice as many ice cream bars as Joe, and Albert ate three less than Tom, how many ice cream bars did each roommate eat?

Answer

Tom ate 6, Joe ate 3, and Albert ate 3.

60) A farmer constructed a chicken coop out of chicken wire, wood, and plywood. The chicken wire cost $\$2$ per square foot, the wood $\$10$ per square foot, and the plywood $\$5$ per square foot. The farmer spent a total of $\$51$, and the total amount of materials used was 14 ft^2. He used 3 ft^2 more chicken wire than plywood. How much of each material in did the farmer use?

61) Jay has lemon, orange, and pomegranate trees in his backyard. An orange weighs 8 oz, a lemon 5 oz, and a pomegranate 11 oz. Jay picked 142 pieces of fruit weighing a total of 70 lb, 10 oz. He picked 15.5 times more oranges than pomegranates. How many of each fruit did Jay pick?

Answer

124 oranges, 10 lemons, 8 pomegranates

9.8: Solving Systems with Cramer's Rule

Verbal

1) Explain why we can always evaluate the determinant of a square matrix.

Answer

A determinant is the sum and products of the entries in the matrix, so you can always evaluate that product—even if it does end up being 0.

2) Examining Cramer's Rule, explain why there is no unique solution to the system when the determinant of your matrix is 0. For simplicity, use a 2×2 matrix.

3) Explain what it means in terms of an inverse for a matrix to have a 0 determinant.

Answer

The inverse does not exist.

4) The determinant of 2×2 matrix A is 3. If you switch the rows and multiply the first row by 6 and the second row by 2, explain how to find the determinant and provide the answer.

Algebraic

For the exercises 5-24, find the determinant.

5) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$

Answer

-2

6) $\begin{vmatrix} -1 & 2 \\ 3 & -4 \end{vmatrix}$

7) $\begin{vmatrix} 2 & -5 \\ -1 & 6 \end{vmatrix}$

Answer

7

8) $\begin{vmatrix} -8 & 4 \\ -1 & 5 \end{vmatrix}$

9) $\begin{vmatrix} 1 & 0 \\ 3 & -4 \end{vmatrix}$

Answer

-4

10) $\begin{vmatrix} 10 & 20 \\ 0 & -10 \end{vmatrix}$

11) $\begin{vmatrix} 10 & 0.2 \\ 5 & 0.1 \end{vmatrix}$

Answer

0

12) $\begin{vmatrix} 6 & -3 \\ 8 & 4 \end{vmatrix}$

13) $\begin{vmatrix} -2 & -3 \\ 3.1 & 4,000 \end{vmatrix}$

Answer

$-7,990.7$

14) $\begin{vmatrix} -1.1 & 0.6 \\ 7.2 & -0.5 \end{vmatrix}$

15) $\begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{vmatrix}$

Answer

3

16) $\begin{vmatrix} -1 & 4 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix}$

17) $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$

Answer

-1

18) $\begin{vmatrix} 2 & -3 & 1 \\ 3 & -4 & 1 \\ -5 & 6 & 1 \end{vmatrix}$

19) $\begin{vmatrix} -2 & 1 & 4 \\ -4 & 2 & -8 \\ 2 & -8 & -3 \end{vmatrix}$

Answer

224

20) $\begin{vmatrix} 6 & -1 & 2 \\ -4 & -3 & 5 \\ 1 & 9 & -1 \end{vmatrix}$

21) $\begin{vmatrix} 5 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -6 & -3 \end{vmatrix}$

Answer

15

22) $\begin{vmatrix} 1.1 & 2 & -1 \\ -4 & 0 & 0 \\ 4.1 & -0.4 & 2.5 \end{vmatrix}$

23) $\begin{vmatrix} 2 & -1.6 & 3.1 \\ 1.1 & 3 & -8 \\ -9.3 & 0 & 2 \end{vmatrix}$

Answer

-17.03

24) $\begin{vmatrix} -\frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & -\frac{1}{6} & \frac{1}{7} \\ 0 & 0 & \frac{1}{8} \end{vmatrix}$

For the exercises 25-34, solve the system of linear equations using Cramer's Rule.

25) $\begin{aligned} 2x-3y &= -1 \\ 4x+5y &= 9 \end{aligned}$

Answer

$(1,1)$

26) $\begin{aligned} 5x-4y &= 2 \\ -4x+7y &= 6 \end{aligned}$

27) $\begin{aligned} 6x-3y &= 2 \\ -8x+9y &= -1 \end{aligned}$

Answer

$\left(\frac{1}{2},\frac{1}{3}\right)$

28) $\begin{aligned} 2x+6y &= 12 \\ 5x-2y &= 13 \end{aligned}$

29) $\begin{aligned} 4x+3y &= 23 \\ 2x-y &= -1 \end{aligned}$

Answer

$(2,5)$

30) $\begin{aligned} 10x-6y &= 2 \\ -5x+8y &= -1 \end{aligned}$

31) $\begin{aligned} 4x-3y &= -3 \\ 2x+6y &= -4 \end{aligned}$

Answer

$\left(-1,-\frac{1}{3}\right)$

32) $\begin{aligned} 4x-5y &= 7 \\ -3x+9y &= 0 \end{aligned}$

33) $\begin{aligned} 4x+10y &= 180 \\ -3x-5y &= -105 \end{aligned}$

Answer

$(15,12)$

34) $\begin{aligned} 8x-2y &= -3 \\ -4x+6y &= 4 \end{aligned}$

For the exercises 35-44, solve the system of linear equations using Cramer's Rule.

35) $\begin{aligned} x+2y-4z &= -1 \\ 7x+3y+5z &= 26 \\ -2x-6y+7z &= -6 \end{aligned}$

Answer

$(1,3,2)$

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36)
$$\begin{aligned} -5x+2y-4z &= -47 \\ 4x-3y-z &= -94 \\ 3x-3y+2z &= 94 \end{aligned}$$

37)
$$\begin{aligned} 4x+5y-z &= -7 \\ -2x-9y+2z &= 8 \\ 5y+7z &= 21 \end{aligned}$$

Answer

$(-1, 0, 3)$

38)
$$\begin{aligned} 4x-3y+4z &= 10 \\ 5x-2z &= -2 \\ 3x+2y-5z &= -9 \end{aligned}$$

39)
$$\begin{aligned} 4x-2y+3z &= 6 \\ -6x+y &= -2 \\ 2x+7y+8z &= 24 \end{aligned}$$

Answer

$\left(\frac{1}{2}, 1, 2\right)$

40)
$$\begin{aligned} 5x+2y-z &= 1 \\ -7x-8y+3z &= 1.5 \\ 6x-12y+z &= 7 \end{aligned}$$

41)
$$\begin{aligned} 13x-17y+16z &= 73 \\ -11x+15y+17z &= 61 \\ 46x+10y-30z &= -18 \end{aligned}$$

Answer

$(2, 1, 4)$

42)
$$\begin{aligned} -4x-3y-8z &= -7 \\ 2x-9y+5z &= 0.5 \\ 5x-6y-5z &= -2 \end{aligned}$$

43)
$$\begin{aligned} 4x-6y+8z &= 10 \\ -2x+3y-4z &= -5 \\ x+y+z &= 1 \end{aligned}$$

Answer

Infinite solutions

44)
$$\begin{aligned} 4x-6y+8z &= 10 \\ -2x+3y-4z &= -5 \\ 12x+18y-24z &= -30 \end{aligned}$$

Technology

For the exercises 45-48, use the determinant function on a graphing utility.

45)
$$\begin{vmatrix} 1 & 0 & 8 & 9 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 2 & 4 & 3 \end{vmatrix}$$

Answer

24

46) $\begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & -9 & 1 & 3 \\ 3 & 0 & -2 & -1 \\ 0 & 1 & 1 & -2 \end{vmatrix}$

47) $\begin{vmatrix} \frac{1}{2} & 1 & 7 & 4 \\ 0 & \frac{1}{2} & 100 & 5 \\ 0 & 0 & 2 & 2,000 \\ 0 & 0 & 0 & 2 \end{vmatrix}$

Answer

1

48) $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{vmatrix}$

Real-World Applications

For the exercises 49-52, create a system of linear equations to describe the behavior. Then, calculate the determinant. Will there be a unique solution? If so, find the unique solution.

49) Two numbers add up to 56. One number is 20 less than the other.

Answer

Yes; 18, 38

50) Two numbers add up to 104. If you add two times the first number plus two times the second number, your total is 208

51) Three numbers add up to 106. The first number is 3 less than the second number. The third number is 4 more than the first number.

Answer

Yes; 33, 36, 37

52) Three numbers add to 216. The sum of the first two numbers is 112. The third number is 8 less than the first two numbers combined.

For the exercises 53-65, create a system of linear equations to describe the behavior. Then, solve the system for all solutions using Cramer's Rule.

53) You invest $\$10,000$ into two accounts, which receive 8% interest and 5% interest. At the end of a year, you had $\$10,710$ in your combined accounts. How much was invested in each account?

Answer

$\$7,000$in first account, $\$3,000$in second account.

54) You invest $\$80,000$ into two accounts, $\$22,000$ in one account, and $\$58,000$ in the other account. At the end of one year, assuming simple interest, you have earned $\$2,470$ in interest. The second account receives half a percent less than twice the interest on the first account. What are the interest rates for your accounts?

55) A movie theater needs to know how many adult tickets and children tickets were sold out of the $1,200$ total tickets. If children's tickets are $\$5.95$, adult tickets are $\$11.15$, and the total amount of revenue was $\$12,756$, how many children's tickets and adult tickets were sold?

Answer

120 children, 1,080 adult

56) A concert venue sells single tickets for $40 each and couple's tickets for $65. If the total revenue was $18,090 and the 321 tickets were sold, how many single tickets and how many couple's tickets were sold?

57) You decide to paint your kitchen green. You create the color of paint by mixing yellow and blue paints. You cannot remember how many gallons of each color went into your mix, but you know there were 10 gal total. Additionally, you kept your receipt, and know the total amount spent was $29.50. If each gallon of yellow costs $2.59, and each gallon of blue costs $3.19, how many gallons of each color go into your green mix?

Answer

4 gal yellow, 6 gal blue

58) You sold two types of scarves at a farmers' market and would like to know which one was more popular. The total number of scarves sold was 56, the yellow scarf cost $10, and the purple scarf cost $11. If you had total revenue of $583, how many yellow scarves and how many purple scarves were sold?

59) Your garden produced two types of tomatoes, one green and one red. The red weigh 10 oz, and the green weigh 4 oz. You have 30 tomatoes, and a total weight of 13 lb, 14 oz. How many of each type of tomato do you have?

Answer

13 green tomatoes, 17 red tomatoes

60) At a market, the three most popular vegetables make up 53% of vegetable sales. Corn has 4% higher sales than broccoli, which has 5% more sales than onions. What percentage does each vegetable have in the market share?

61) At the same market, the three most popular fruits make up 37% of the total fruit sold. Strawberries sell twice as much as oranges, and kiwis sell one more percentage point than oranges. For each fruit, find the percentage of total fruit sold.

Answer

Strawberries 18%, oranges 9%, kiwi 10%

62) Three bands performed at a concert venue. The first band charged $15 per ticket, the second band charged $45 per ticket, and the final band charged $22 per ticket. There were 510 tickets sold, for a total of $12,700. If the first band had 40 more audience members than the second band, how many tickets were sold for each band?

63) A movie theatre sold tickets to three movies. The tickets to the first movie were $5, the tickets to the second movie were $11, and the third movie was $12. 100 tickets were sold to the first movie. The total number of tickets sold was 642, for a total revenue of $6,774. How many tickets for each movie were sold?

Answer

100 for movie 1, 230 for movie 2, 312 for movie 3

64) Men aged 20–29, 30–39, and 40–49 made up 78% of the population at a prison last year. This year, the same age groups made up 82.08% of the population. The 20–29 age group increased by 20, the 30–39 age group increased by 2%, and the 40–49 age group decreased to $\frac{3}{4}$ of their previous population. Originally, the 30–39 age group had 2% more prisoners than the 20–29 age group. Determine the prison population percentage for each age group last year.

65) At a women's prison down the road, the total number of inmates aged 20–49 totaled 5,525. This year, the 20–29 age group increased by 10%, the 30–39 age group decreased by 20%, and the 40–49 age group doubled. There are now 6,040 prisoners. Originally, there were 500 more in the 30–39 age group than the 20–29 age group. Determine the prison population for each age group last year.

Answer

20–29 : 2,100 30–39 : 2,600 40–49 : 825

For the exercises 66-68, use this scenario: A health-conscious company decides to make a trail mix out of almonds, dried cranberries, and chocolate-covered cashews. The nutritional information for these items is shown in the Table below.

	Fat (g)	Protein (g)	Carbohydrates (g)
Almonds (10)	6	2	3
Cranberries (10)	0.02	0	8
Cashews (10)	7	3.5	5.5

66) For the special "low-carb"trail mix, there are $1,000$ pieces of mix. The total number of carbohydrates is 425 g, and the total amount of fat is 570.2 g. If there are 200 more pieces of cashews than cranberries, how many of each item is in the trail mix?

67) For the "hiking" mix, there are $1,000$ pieces in the mix, containing 390.8 g of fat, and 165 g of protein. If there is the same amount of almonds as cashews, how many of each item is in the trail mix?

Answer

300 almonds, 400 cranberries, 300 cashews

68) For the "energy-booster" mix, there are $1,000$ pieces in the mix, containing 145 g of protein and 625 g of carbohydrates. If the number of almonds and cashews summed together is equivalent to the amount of cranberries, how many of each item is in the trail mix?

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7.R: Systems of Equations and Inequalities (Review)

9.1: Systems of Linear Equations: Two Variables

For the exercises 1-2, determine whether the ordered pair is a solution to the system of equations.

1) $\begin{aligned} 3x-y &= 4 \\ x+4y &= -3 \end{aligned}$ and $(-1,1)$

Answer

No

2) $\begin{aligned} 6x-2y &= 24 \\ -3x+3y &= 18 \end{aligned}$ and $(9,15)$

For the exercises 3-5, use substitution to solve the system of equations.

3) $\begin{aligned} 10x+5y &= -5 \\ 3x-2y &= -12 \end{aligned}$

Answer

$(-2,3)$

4) $\begin{aligned} \frac{4}{7}x+\frac{1}{5}y &= \frac{43}{70} \\ \frac{5}{6}x-\frac{1}{3}y &= -\frac{2}{3} \end{aligned}$

5) $\begin{aligned} 5x+6y &= 14 \\ 4x+8y &= 8 \end{aligned}$

Answer

$(4,-1)$

For the exercises 6-8, use addition to solve the system of equations.

6) $\begin{aligned} 3x+2y &= -7 \\ 2x+4y &= 6 \end{aligned}$

7) $\begin{aligned} 3x+4y &= 2 \\ 9x+12y &= 3 \end{aligned}$

Answer

No solutions exist.

8) $\begin{aligned} 8x+4y &= 2 \\ 6x-5y &= 0.7 \end{aligned}$

For the exercises 9-10, write a system of equations to solve each problem. Solve the system of equations.

9) A factory has a cost of production $C(x)=150x+15,000$ and a revenue function $R(x)=200x$. What is the break-even point?

Answer

$(300,60,000)$

10) A performer charges $C(x)=50x+10,000$, where x is the total number of attendees at a show. The venue charges $\$75$ per ticket. After how many people buy tickets does the venue break even, and what is the value of the total tickets sold at that point?

Answer

$(400,30,000)$

9.2: Systems of Linear Equations: Three Variables

For the exercises 1-8, solve the system of three equations using substitution or addition.

1) $$\begin{aligned} 0.5x-0.5y &= 10 \\ -0.2y+0.2x &= 4 \\ 0.1x+0.1z &= 2 \end{aligned}$$

Answer

$(10, -10, 10)$

2) $$\begin{aligned} 5x+3y-z &= 5 \\ 3x-2y+4z &= 13 \\ 4x+3y+5z &= 22 \end{aligned}$$

3) $$\begin{aligned} x+y+z &= 1 \\ 2x+2y+2z &= 1 \\ 3x+3y &= 2 \end{aligned}$$

Answer

No solutions exist.

4) $$\begin{aligned} 2x-3y+z &= -1 \\ x+y+z &= -4 \\ 4x+2y-3z &= 33 \end{aligned}$$

5) $$\begin{aligned} 3x+2y-z &= -10 \\ x-y+2z &= 7 \\ -x+3y+z &= -2 \end{aligned}$$

Answer

$(-1, -2, 3)$

6) $$\begin{aligned} 3x+4z &= -11 \\ x-2y &= 5 \\ 4y-z &= -10 \end{aligned}$$

7) $$\begin{aligned} 2x-3y+z &= 0 \\ 2x+4y-3z &= 0 \\ 6x-2y-z &= 0 \end{aligned}$$

Answer

$\left(x, \frac{8x}{5}, \frac{14x}{5}\right)$

8) $$\begin{aligned} 6x-4y-2z &= 2 \\ 3x+2y-5z &= 4 \\ 6y-7z &= 5 \end{aligned}$$

For the exercises 9-10, write a system of equations to solve each problem. Solve the system of equations.

9) Three odd numbers sum up to 61. The smaller is one-third the larger and the middle number is 16 less than the larger. What are the three numbers?

Answer

$11, 17, 33$

10) A local theatre sells out for their show. They sell all 500 tickets for a total purse of $\$8,070.00$ The tickets were priced at $\$15$ for students, $\$12$ for children, and $\$18$ for adults. If the band sold three times as many adult tickets as children's tickets, how many of each type was sold?

9.3: Systems of Nonlinear Equations and Inequalities: Two Variables

For the exercises 1-5, solve the system of nonlinear equations.

1) $\begin{aligned} y &= x^2 - 7 \\ y &= 5x - 13 \end{aligned}$

Answer

$(2, -3), (3, 2)$

2) $\begin{aligned} y &= x^2 - 4 \\ y &= 5x + 10 \end{aligned}$

3) $\begin{aligned} x^2 + y^2 &= 16 \\ y &= x - 8 \end{aligned}$

Answer

No solution

4) $\begin{aligned} x^2 + y^2 &= 25 \\ y &= x^2 + 5 \end{aligned}$

5) $\begin{aligned} x^2 + y^2 &= 4 \\ y - x^2 &= 3 \end{aligned}$

Answer

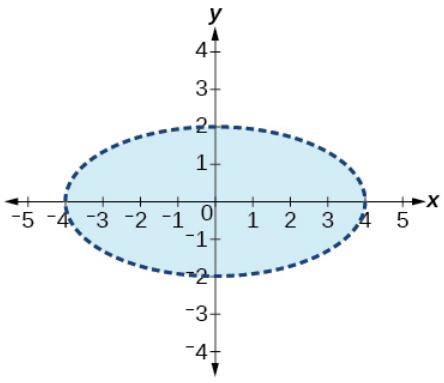
No solution

For the exercises 6-7, graph the inequality.

6) $y > x^2 - 1$

7) $\frac{1}{4}x^2 + y^2 < 4$

Answer

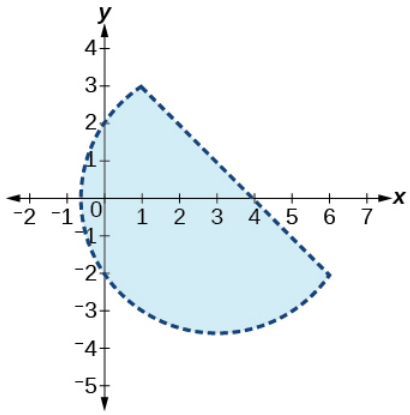


For the exercises 8-10, graph the system of inequalities.

8) $\begin{aligned} x^2 + y^2 + 2x &< 3 \\ y &> -x^2 - 3 \end{aligned}$

9) $\begin{aligned} x^2-2x+y^2-4x &< 4 \\ y &< -x+4 \end{aligned}$

Answer



10) $\begin{aligned} x^2+y^2 &< 1 \\ y^2 &< x \end{aligned}$

9.4: Partial Fractions

For the exercises 1-8, decompose into partial fractions.

1) $\dfrac{-2x+6}{x^2+3x+2}$

Answer

$\dfrac{2}{x+2}, \dfrac{-4}{x+1}$

2) $\dfrac{10x+2}{4x^2+4x+1}$

3) $\dfrac{7x+20}{x^2+10x+25}$

Answer

$\dfrac{7}{x+5}, \dfrac{-15}{(x+5)^2}$

4) $\dfrac{x-18}{x^2-12x+36}$

5) $\dfrac{-x^2+36x+70}{x^3-125}$

Answer

$\dfrac{3}{x-5}, \dfrac{-4x+1}{x^2+5x+25}$

6) $\dfrac{-5x^2+6x-2}{x^3+27}$

7) $\dfrac{x^3-4x^2+3x+11}{(x^2-2)^2}$

Answer

$$\frac{x-4}{(x^2-2)}, \frac{5x+3}{(x^2-2)^2}$$

8) $\dfrac{4x^4-2x^3+22x^2-6x+48}{x(x^2+4)^2}$

9.5: Matrices and Matrix Operations

For the exercises 1-12, perform the requested operations on the given matrices.

$$A=\begin{bmatrix}4 & -2\\ 1 & 3\end{bmatrix}, \begin{bmatrix}6 & 7 & -3\\ 11 & -2 & 4\end{bmatrix}, C=\begin{bmatrix}6 & 7\\ 11 & -2\\ 14 & 0\end{bmatrix} D=\begin{bmatrix}1 & -4 & 9\\ 10 & 5 & -7\\ 2 & 8 & 5\end{bmatrix} E=\begin{bmatrix}7 & -14 & 3\\ 2 & -1 & 3\\ 0 & 1 & 9\end{bmatrix}$$

1) $-4A$

Answer

$$\begin{bmatrix}-16 & 8\\ -4 & -12\end{bmatrix}$$

2) $10D-6E$

3) $B+C$

Answer

undefined; dimensions do not match

4) AB

5) BA

Answer

undefined; inner dimensions do not match

6) BC

7) CB

Answer

$$\begin{bmatrix}113 & 28 & 10\\ 44 & 81 & -41\\ 84 & 98 & -42\end{bmatrix}$$

8) DE

9) ED

Answer

$$\begin{bmatrix}-127 & -74 & 176\\ -2 & 11 & 40\\ 28 & 77 & 38\end{bmatrix}$$

10) EC

11) CE

Answer

undefined; inner dimensions do not match

12) A^3

9.6: Solving Systems with Gaussian Elimination

For the exercises 1-2, write the system of linear equations from the augmented matrix. Indicate whether there will be a unique solution.

1) $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 7 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Answer

$\begin{aligned} x-3z &= 7 \\ y+2z &= -5 \end{aligned}$ with infinite solutions

2) $\left[\begin{array}{ccc|c} 1 & 0 & 5 & -9 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right]$

For the exercises 3-5, write the augmented matrix from the system of linear equations.

3) $\begin{aligned} -2x+2y+z &= 7 \\ 2x-8y+5z &= 0 \\ 19x-10y+22z &= 3 \end{aligned}$

Answer

$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 7 \\ 2 & -8 & 5 & 0 \\ 19 & -10 & 22 & 3 \end{array}\right]$

4) $\begin{aligned} 4x+2y-3z &= 14 \\ -12x+3y+z &= 100 \\ 9x-6y+2z &= 31 \end{aligned}$

5) $\begin{aligned} x+3z &= 12 \\ -x+4y &= 0 \\ y+2z &= -7 \end{aligned}$

Answer

$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 12 \\ -1 & 4 & 0 & 0 \\ 0 & 1 & 2 & -7 \end{array}\right]$

For the exercises 6-10, solve the system of linear equations using Gaussian elimination.

6) $\begin{aligned} 3x-4y &= -7 \\ -6x+8y &= 14 \end{aligned}$

7) $\begin{aligned} 3x-4y &= 1 \\ -6x+8y &= 6 \end{aligned}$

Answer

No solutions exist.

8) $\begin{aligned} -1.1x-2.3y &= 6.2 \\ -5.2x-4.1y &= 4.3 \end{aligned}$

9) $\begin{aligned} 2x+3y+2z &= 1 \\ -4x-6y-4z &= -2 \\ 10x+15y+10z &= 0 \end{aligned}$

Answer

No solutions exist.

10) $\begin{aligned} -x+2y-4z &= 8 \\ 3y+8z &= -4 \\ -7x+y+2z &= 1 \end{aligned}$

9.7: Solving Systems with Inverses

For the exercises 1-4, find the inverse of the matrix.

1) $\begin{bmatrix} -0.2 & 1.4 \\ 1.2 & -0.4 \end{bmatrix}$

Answer

$\frac{1}{8}\begin{bmatrix} 2 & 7 \\ 6 & 1 \end{bmatrix}$

2) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$

3) $\begin{bmatrix} 12 & 9 & -6 \\ -1 & 3 & 2 \\ -4 & -3 & 2 \end{bmatrix}$

Answer

No inverse exists.

4) $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

For the exercises 5-8, find the solutions by computing the inverse of the matrix.

5) $\begin{aligned} 0.3x-0.1y &= -10 \\ -0.1x+0.3y &= 14 \end{aligned}$

Answer

$(-20, 40)$

6) $\begin{aligned} 0.4x-0.2y &= -0.6 \\ -0.1x+0.05y &= 0.3 \end{aligned}$

7) $\begin{aligned} 4x+3y-3z &= -4.3 \\ 5x-4y-z &= -6.1 \\ x+z &= -0.7 \end{aligned}$

Answer

$(-1, 0.2, 0.3)$

8) $\begin{aligned} -2x-3y+2z &= 3 \\ -x+2y+4z &= -5 \\ -2y+5z &= -3 \end{aligned}$

For the exercises 9-10, write a system of equations to solve each problem. Solve the system of equations.

9) Students were asked to bring their favorite fruit to class. 90% of the fruits consisted of banana, apple, and oranges. If oranges were half as popular as bananas and apples were 5% more popular than bananas, what are the percentages of each individual fruit?

Answer

17% oranges, 34% bananas, 39% apples

10) A sorority held a bake sale to raise money and sold brownies and chocolate chip cookies. They priced the brownies at $\$2$ and the chocolate chip cookies at $\$1$. They raised $\$250$ and sold 175 items. How many brownies and how many cookies were sold?

9.8: Solving Systems with Cramer's Rule

For the exercises 1-4, find the determinant.

1) $\begin{vmatrix} 100 & 0 \\ 0 & 0 \end{vmatrix}$

Answer

0

2) $\begin{vmatrix} 0.2 & -0.6 \\ 0.7 & -1.1 \end{vmatrix}$

3) $\begin{vmatrix} -1 & 4 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & -3 \end{vmatrix}$

Answer

6

4) $\begin{vmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{vmatrix}$

For the exercises 5-10, use Cramer's Rule to solve the linear systems of equations.

5) $\begin{aligned} 4x-2y &= 23 \\ -5x-10y &= -35 \end{aligned}$

Answer

$\left(6, \frac{1}{2}\right)$

6) $\begin{aligned} 0.2x-0.1y &= 0 \\ -0.3x+0.3y &= 2.5 \end{aligned}$

7) $\begin{aligned} -0.5x+0.1y &= 0.3 \\ -0.25x+0.05y &= 0.15 \end{aligned}$

Answer

$x, 5x+3$

8) $$\begin{aligned} x+6y+3z &= 4 \\ 2x+y+2z &= 3 \\ 3x-2y+z &= 0 \end{aligned}$$

9) $$\begin{aligned} 4x-3y+5z &= -\frac{5}{2} \\ 7x-9y-3z &= \frac{3}{2} \\ x-5y-5z &= \frac{5}{2} \end{aligned}$$

Answer

$$\left(0, 0, -\frac{1}{2}\right)$$

10) $$\begin{aligned} \frac{3}{10}x-\frac{1}{5}y-\frac{3}{10}z &= -\frac{1}{50} \\ \frac{1}{10}x-\frac{1}{10}y-\frac{1}{2}z &= -\frac{9}{50} \\ \frac{2}{5}x-\frac{1}{2}y-\frac{3}{5}z &= -\frac{1}{5} \end{aligned}$$

Practice Test

1) Is the following ordered pair a solution to the system of equations?

$$\begin{aligned} -5x-y &= 12 \text{ with } (-3, 3) \\ x+4y &= 9 \end{aligned}$$

Answer

Yes

For the exercises 2-9, solve the systems of linear and nonlinear equations using substitution or elimination. Indicate if no solution exists.

2) $$\begin{aligned} \frac{1}{2}x-\frac{1}{3}y &= 4 \\ \frac{3}{2}x-y &= 0 \end{aligned}$$

3) $$\begin{aligned} -\frac{1}{2}x-4y &= 4 \\ 2x+16y &= 2 \end{aligned}$$

Answer

No solutions exist.

4) $$\begin{aligned} 5x-y &= 1 \\ -10x+2y &= -2 \end{aligned}$$

5) $$\begin{aligned} 4x-6y-2z &= \frac{1}{10} \\ x-7y+5z &= -\frac{1}{4} \\ 3x+6y-9z &= \frac{6}{5} \end{aligned}$$

Answer

$$\frac{1}{20}(10, 5, 4)$$

$$x + z = 20$$

6) $$\begin{aligned} x + y + z &= 20 \\ x + 2y + z &= 10 \end{aligned}$$

$$\begin{aligned} 5x - 4y - 3z &= 0 \\ 7)\quad 2x + y + 2z &= 0 \\ x - 6y - 7z &= 0 \end{aligned}$$

Answer

$$\left(x, \frac{16x}{5} - \frac{13x}{5}\right)$$

8) $$\begin{aligned} y &= x^2 + 2x - 3 \\ y &= x - 1 \end{aligned}$$

9) $$\begin{aligned} y^2 + x^2 &= 25 \\ y^2 - 2x^2 &= 1 \end{aligned}$$

Answer

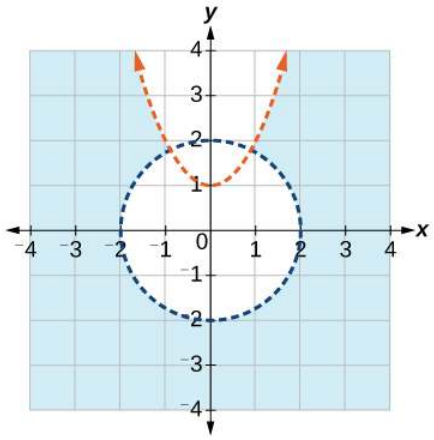
$$(-2\sqrt{2}, -\sqrt{17}), (-2\sqrt{2}, \sqrt{17}), (2\sqrt{2}, -\sqrt{17}), (2\sqrt{2}, \sqrt{17})$$

For the exercises 10-11, graph the following inequalities.

10) $y < x^2 + 9$

11) $$\begin{aligned} x^2 + y^2 &> 4 \\ y &< x^2 + 1 \end{aligned}$$

Answer



For the exercises 12-14, write the partial fraction decomposition.

12) $\dfrac{-8x - 30}{x^2 + 10x + 25}$

13) $\dfrac{13x + 2}{(3x + 1)^2}$

Answer

$$\frac{5}{3x + 1} - \frac{2x + 3}{(3x + 1)^2}$$

14) $\dfrac{x^4 - x^3 + 2x - 1}{x(x^2 + 1)^2}$

For the exercises 15-21, perform the given matrix operations.

15) $5\begin{bmatrix} 4 & 9 \\ -2 & 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -6 & 12 \\ 4 & -8 \end{bmatrix}$

Answer

$\begin{bmatrix} 17 & 51 \\ -8 & 11 \end{bmatrix}$

16) $\begin{bmatrix} 1 & 4 & -7 \\ -2 & 9 & 5 \\ 12 & 0 & -4 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & 3 \\ 5 & 10 \end{bmatrix}$

17) $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{5} \end{bmatrix}^{-1}$

Answer

$\begin{bmatrix} 12 & -20 \\ -15 & 30 \end{bmatrix}$

18) $\mathbf{det}\begin{vmatrix} 0 & 0 \\ 400 & 4,000 \end{vmatrix}$

19) $\mathbf{det}\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{vmatrix}$

Answer

$-\frac{1}{8}$

20) If $\mathbf{det}(A) = -6$, what would be the determinant if you switched rows 1 and 3, multiplied the second row by 12, and took the inverse?

21) Rewrite the system of linear equations as an augmented matrix.

$$\begin{aligned} 14x - 2y - 13z &= 140 \\ -2x + 3y - 6z &= -1 \\ x - 5y + 12z &= 11 \end{aligned}$$

Answer

$\left[\begin{array}{ccc|c} 14 & -2 & 13 & 140 \\ -2 & 3 & -6 & -1 \\ 1 & -5 & 12 & 11 \end{array}\right]$

22) Rewrite the augmented matrix as a system of linear equations.

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 12 \\ -2 & 4 & 9 & -5 \\ -6 & 1 & 2 & 8 \end{array}\right]$$

For the exercises 23-24, use Gaussian elimination to solve the systems of equations.

23) $\begin{aligned} x - 6y &= 4 \\ 2x - 12y &= 0 \end{aligned}$

Answer

No solutions exist.

24) $$\begin{aligned} 2x+y+z &= -3 \\ x-2y+3z &= 6 \\ x-y-z &= 6 \end{aligned}$$

For the exercises 25-26, use the inverse of a matrix to solve the systems of equations.

25) $$\begin{aligned} 4x-5y &= -50 \\ -x+2y &= 80 \end{aligned}$$

Answer

$(100, 90)$

26) $$\begin{aligned} \frac{1}{100}x-\frac{3}{100}y+\frac{1}{20}z &= -49 \\ \frac{3}{100}x-\frac{7}{100}y-\frac{1}{100}z &= 13 \\ \frac{9}{100}x-\frac{9}{100}y-\frac{9}{100}z &= 99 \end{aligned}$$

For the exercises 27-28, use Cramer's Rule to solve the systems of equations.

27) $$\begin{aligned} 200x-300y &= 2 \\ 400x+715y &= 4 \end{aligned}$$

Answer

$\left(\frac{1}{100}, 0\right)$

28) $$\begin{aligned} 0.1x+0.1y-0.1z &= -1.2 \\ 0.1x-0.2y+0.4z &= -1.2 \\ 0.5x-0.3y+0.8z &= -5.9 \end{aligned}$$

For the exercises 29-30, solve using a system of linear equations.

29) A factory producing cell phones has the following cost and revenue functions: $C(x)=x^2+75x+2,688$ and $R(x)=x^2+160x$. What is the range of cell phones they should produce each day so there is profit? Round to the nearest number that generates profit.

Answer

32 or more cell phones per day

30) A small fair charges $\$1.50$ for students, $\$1$ for children, and $\$2$ for adults. In one day, three times as many children as adults attended. A total of 800 tickets were sold for a total revenue of $\$1,050$. How many of each type of ticket was sold?

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