MATH C120: TRIGONOMETRY

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Math C120: Trigonometry (Tran)

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TABLE OF CONTENTS

Licensing

1: Trigonometric Functions

- 1.1: Prelude to Trigonometric Functions
- 1.2: Angles
- 1.3: Unit Circle - Sine and Cosine Functions
- 1.4: The Other Trigonometric Functions
- 1.5: Right Triangle Trigonometry
- 1.E: Trigonometric Functions (Exercises)
- 1.R: Trigonometric Functions (Review)

2: Periodic Functions

- 2.1: Prelude to Periodic Functions
- 2.2: Graphs of the Sine and Cosine Functions
- 2.3: Graphs of the Other Trigonometric Functions
- 2.4: Inverse Trigonometric Functions
- 2.E: Periodic Functions (Exercises)
- 2.R: Periodic Functions (Review)

3: Trigonometric Identities and Equations

- 3.1: Prelude to Trigonometric Identities and Equations
- 3.2: Solving Trigonometric Equations with Identities
- 3.3: Sum and Difference Identities
- 3.4: Double-Angle, Half-Angle, and Reduction Formulas
- 3.5: Sum-to-Product and Product-to-Sum Formulas
- 3.6: Solving Trigonometric Equations
- 3.7: Modeling with Trigonometric Equations
- 3.E: Trigonometric Identities and Equations (Exercises)
- 3.R: Trigonometric Identities and Equations (Review)

4: Further Applications of Trigonometry

- 4.1: Prelude to Further Applications of Trigonometry
- 4.2: Non-right Triangles - Law of Sines
- 4.3: Non-right Triangles - Law of Cosines
- 4.4: Polar Coordinates
- 4.5: Polar Coordinates - Graphs
- 4.6: Polar Form of Complex Numbers
- 4.7: Parametric Equations
- 4.8: Parametric Equations - Graphs
- 4.9: Vectors
- 4.E: Further Applications of Trigonometry (Exercises)
- 4.R: Further Applications of Trigonometry (Review)

Index

Glossary

Detailed Licensing

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Trigonometric Functions

The trigonometric functions are functions of an angle. and relate the angles of a triangle to the lengths of its sides. They are important in the study of triangles and modeling periodic phenomena, among many other applications.

1.1: Prelude to Trigonometric Functions
1.2: Angles
1.3: Unit Circle - Sine and Cosine Functions
1.4: The Other Trigonometric Functions
1.5: Right Triangle Trigonometry
1.E: Trigonometric Functions (Exercises)
1.R: Trigonometric Functions (Review)

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1.1: Prelude to Trigonometric Functions

Life is dense with phenomena that repeat in regular intervals. Each day, for example, the tides rise and fall in response to the gravitational pull of the moon. Similarly, the progression from day to night occurs as a result of Earth's rotation, and the pattern of the seasons repeats in response to Earth's revolution around the sun. Outside of nature, many stocks that mirror a company's profits are influenced by changes in the economic business cycle.

The tide rises and falls at regular, predictable intervals. (credit: Andrea Schaffer, Flickr)

In mathematics, a function that repeats its values in regular intervals is known as a periodic function. The graphs of such functions show a general shape reflective of a pattern that keeps repeating. This means the graph of the function has the same output at exactly the same place in every cycle. And this translates to all the cycles of the function having exactly the same length. So, if we know all the details of one full cycle of a true periodic function, then we know the state of the function's outputs at all times, future and past. In this chapter, we will investigate various examples of periodic functions.

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1.2: Angles

Learning Objectives

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

A golfer swings to hit a ball over a sand trap and onto the green. An airline pilot maneuvers a plane toward a narrow runway. A dress designer creates the latest fashion. What do they all have in common? They all work with angles, and so do all of us at one time or another. Sometimes we need to measure angles exactly with instruments. Other times we estimate them or judge them by eye. Either way, the proper angle can make the difference between success and failure in many undertakings. In this section, we will examine properties of angles.

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a *ray*. A ray consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its *endpoint* and any other point on it. The ray in Figure 1.2.1 can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

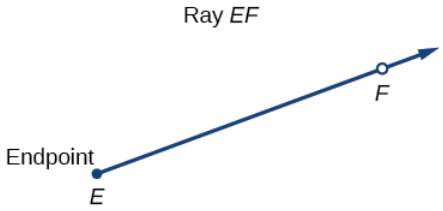

Figure 1.2.1

An *angle* is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in Figure 1.2.2 is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.

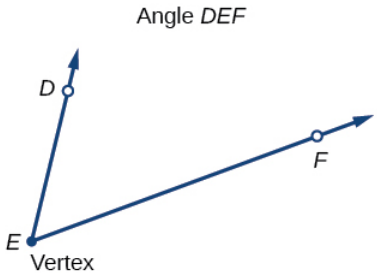

Figure 1.2.2

Greek letters are often used as variables for the measure of an angle. Table 1.2.1 is a list of Greek letters commonly used to represent angles, and a sample angle is shown in Figure 1.2.3.

Table 1.2.1

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

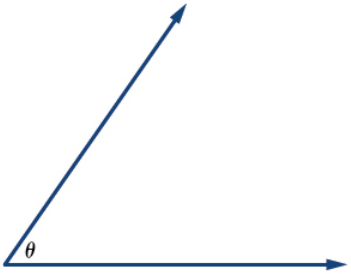

***Figure* 1.2.3**: *Angle theta, shown as* $\angle\theta$

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure 1.2.4.

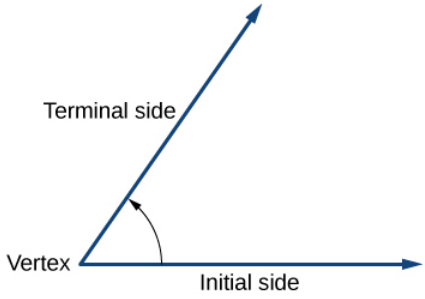

Figure 1.2.4

As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol °. For example, 90 degrees = 90°.

To formalize our work, we will begin by drawing angles on an *x*-*y* coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive *x*-axis. See Figure 1.2.5.

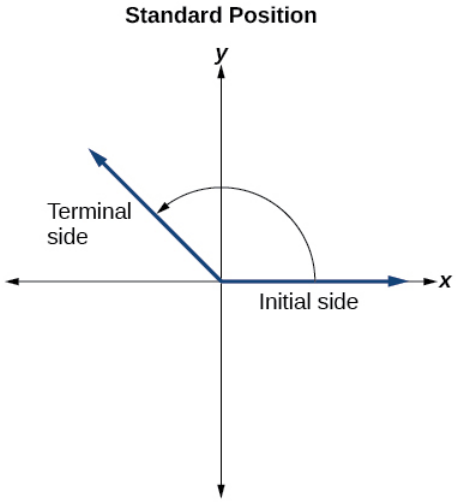

Figure 1.2.5

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Drawing an angle in standard position always starts the same way—draw the initial side along the positive *x*-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90^\circ}{360^\circ} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive *x*-axis. To draw a 360° angle, we calculate that $\frac{360^\circ}{360^\circ} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive *x*-axis. In this case, the initial side and the terminal side overlap. See Figure 1.2.6.

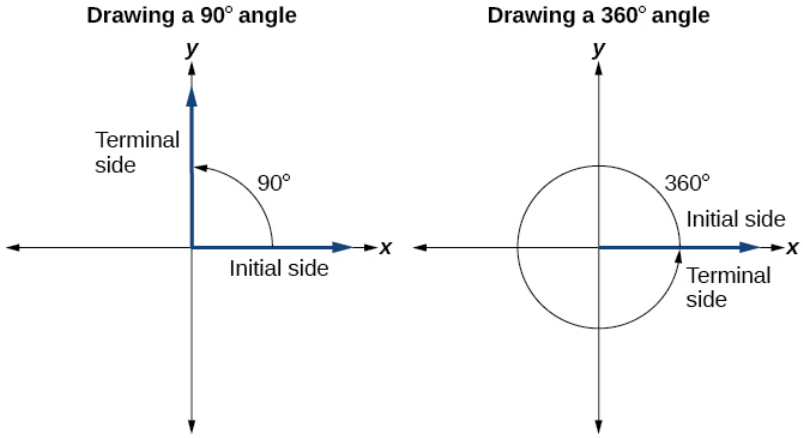

Figure 1.2.6

Since we define an angle in **standard position** by its initial side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270° or 360°. See Figure 1.2.7.

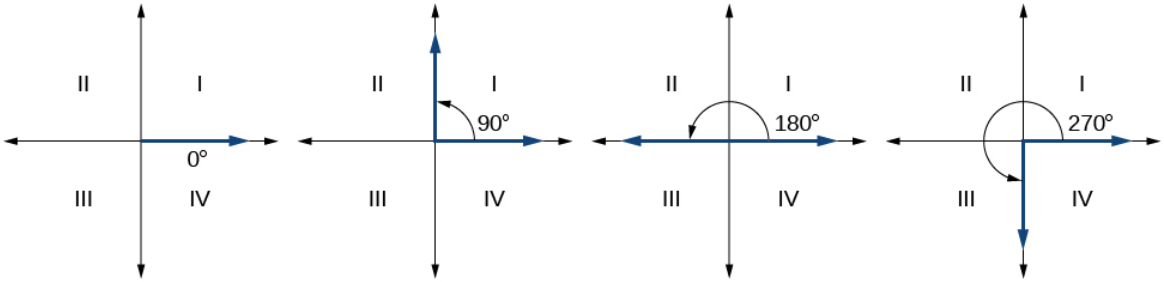

Figure 1.2.7: Quadrantal angles are angles in standard position whose terminal side lies along an axis. Examples are shown.

QUADRANTAL ANGLES

Quadrantal angles are angles in standard position whose terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.

Given an angle measure in degrees, draw the angle in standard position

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive *x*-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 1.2.1: Drawing an Angle in Standard Position Measured in Degrees

1. Sketch an angle of 30° in standard position.
2. Sketch an angle of −135° in standard position.

Solution

1. Divide the angle measure by 360°.

$$\frac{30^\circ}{360^\circ} = \frac{1}{12} \tag{1.2.1}$$

To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}(\frac{1}{4}) \tag{1.2.2}$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in Figure 1.2.8.

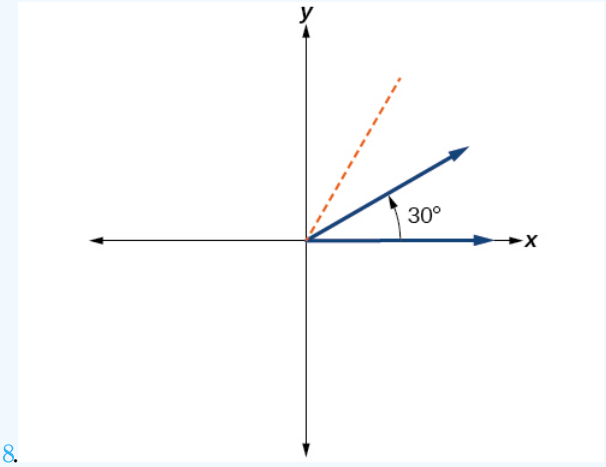

***Figure* 1.2.8**

2. Divide the angle measure by 360°.

$$\frac{-135^\circ}{360^\circ} = -\frac{3}{8} \tag{1.2.3}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}(\frac{1}{4}) \tag{1.2.4}$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in Figure 1.2.9.

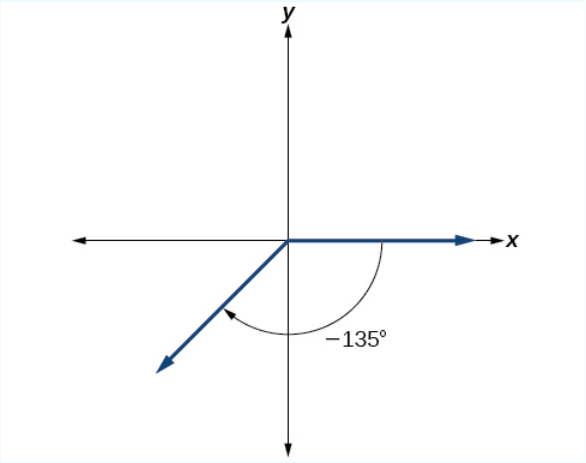

***Figure* 1.2.9**

1.2.1:

Show an angle of 240° on a circle in standard position.

Solution

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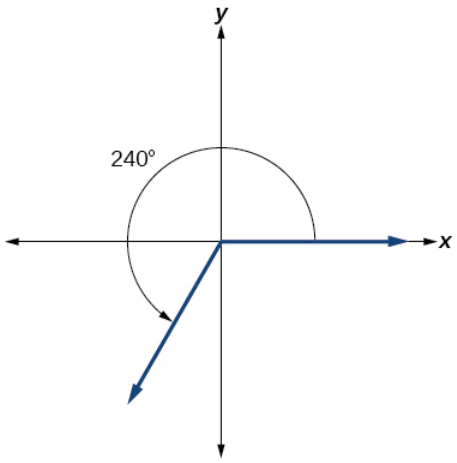

Figure 1.2.10

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An **arc** may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the **circumference** of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in Figure 1.2.11.

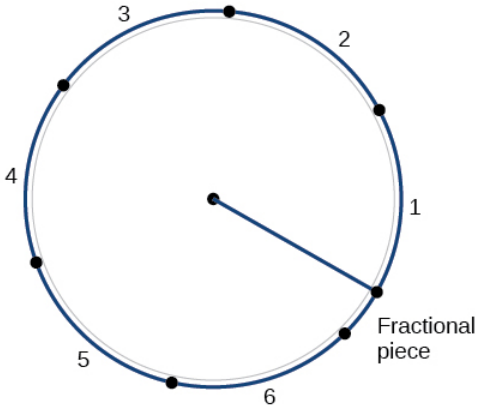

Figure 1.2.11

This brings us to our new angle measure. One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians } = 360^\circ \tag{1.2.5}$$

$$\pi \text{ radians } = \frac{360^\circ}{2} = 180^\circ \tag{1.2.6}$$

$$1 \text{ radian } = \frac{180^\circ}{\pi} \approx 57.3^\circ \tag{1.2.7}$$

See Figure 1.2.12. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

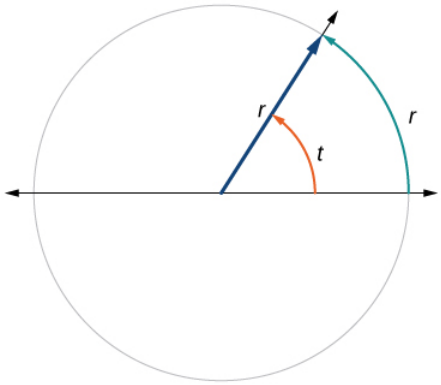

Figure 1.2.12: The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the **radian measure**, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s s to the radius r. See Figure 1.2.13.

$$s = r\theta \tag{1.2.8}$$

$$\theta = \frac{s}{r} \tag{1.2.9}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$ radian.

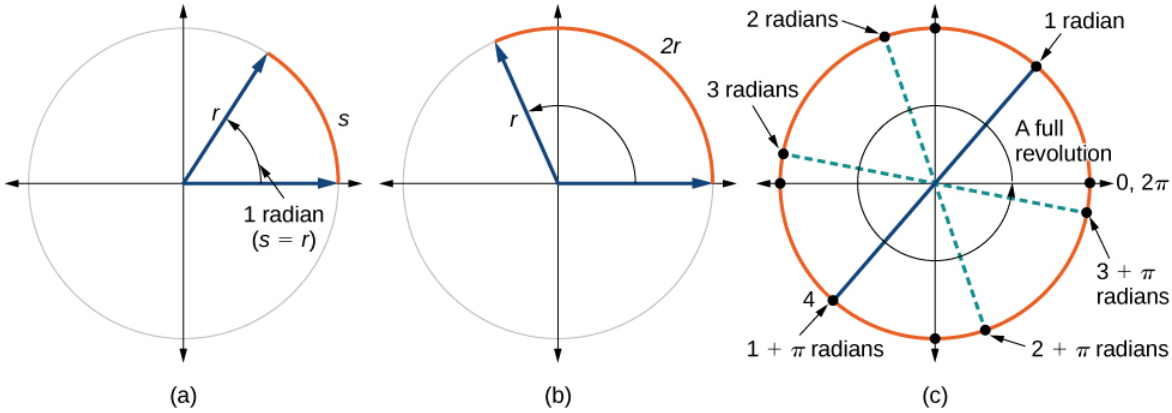

Figure 1.2.13: (a) In an angle of 1 radian, the arc length s equals the radius r. (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$,where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45° angle on the two circles, as in Figure 1.2.14.

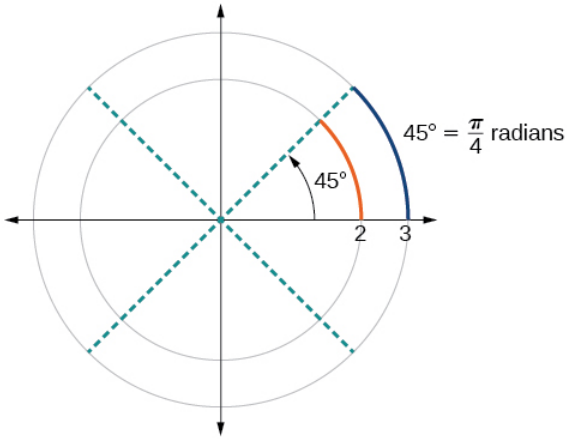

Figure 1.2.14: A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi \tag{1.2.10}$$

$$\text{Larger circle: } \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi \tag{1.2.11}$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

RADIANS

One **radian** is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.

The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

A measure of 1 radian looks to be about 60°. Is that correct?

Yes. It is approximately 57.3°. Because 2π radians equals 360°, 1 radian equals $\frac{360^\circ}{2\pi} \approx 57.3^\circ$.

Using Radians

Because **radian** measure is the ratio of two lengths, it is a unitless measure. For example, in Figure 1.2.14, suppose the radius were 2 inches and the distance along the arc were also 2 inches. When we calculate the radian measure of the angle, the "inches" cancel, and we have a result without units. Therefore, it is not necessary to write the label "radians" after a radian measure, and if we see an angle that is not labeled with "degrees" or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the **unit circle** (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$\begin{aligned} 1\text{rotation} &= 360^\circ &&= 2\pi &&\text{radians} \\ \frac{1}{2}\text{rotation} &= 180^\circ &&= \pi &&\text{radians} \\ \frac{1}{4}\text{rotation} &= 90^\circ &&= \frac{\pi}{2} &&\text{radians} \end{aligned} \tag{1.2.12}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in Figure 1.2.15. Memorizing these angles will be very useful as we study the properties associated with angles.

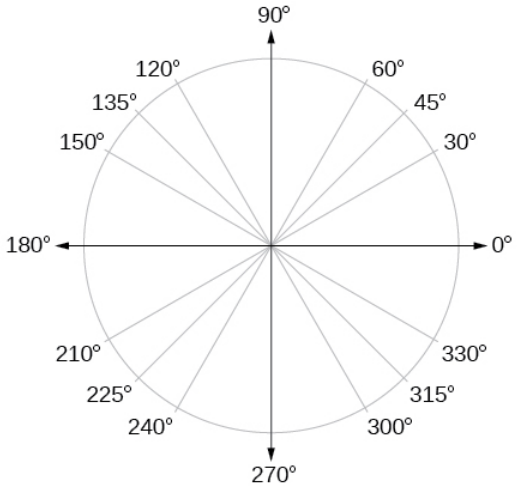

Figure 1.2.15: Commonly encountered angles measured in degrees

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in Figure 1.2.15, which are shown in Figure 1.2.16. Be sure you can verify each of these measures.

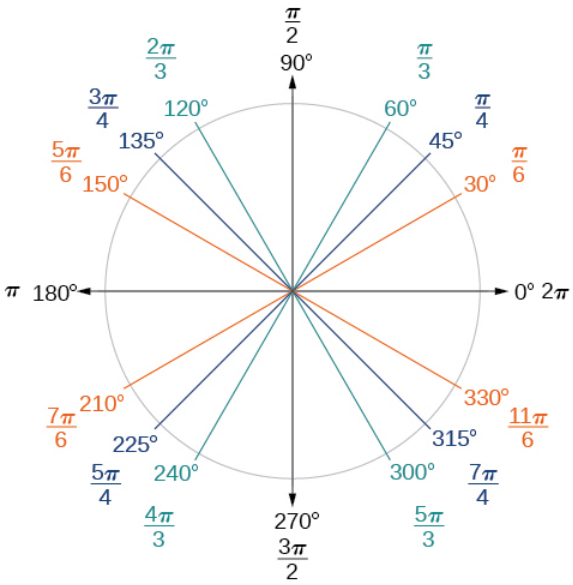

Figure 1.2.16: Commonly encountered angles measured in radians

Example 1.2.2: Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Solution

For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$$1 \text{ rotation} = 2\pi r \tag{1.2.13}$$

So,

$$s = \frac{1}{3}(2\pi r) \tag{1.2.14}$$

$$= \frac{2\pi r}{3} \tag{1.2.15}$$

The radian measure would be the arc length divided by the radius.

$$\text{radian measure} = \frac{\frac{2\pi r}{3}}{r} \tag{1.2.16}$$

$$= \frac{2\pi r}{3r} \tag{1.2.17}$$

$$= \frac{2\pi}{3} \tag{1.2.18}$$

1.2.2

Find the radian measure of three-fourths of a full rotation.

Solution

$$\frac{3\pi}{2} \tag{1.2.19}$$

Converting between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{1.2.20}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{Radians}{\pi} \tag{1.2.21}$$

CONVERTING BETWEEN RADIANS AND DEGREES

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{1.2.22}$$

Example 1.2.3: Converting Radians to Degrees

Convert each radian measure to degrees.

1. $\frac{\pi}{6}$
2. 3

Solution

Because we are given radians and we want degrees, we should set up a proportion and solve it.

1. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{1.2.23}$$

$$\frac{\theta}{180} = \frac{\frac{\pi}{6}}{\pi} \tag{1.2.24}$$

$$\theta = \frac{180}{6} \tag{1.2.25}$$

$$\theta = 30^\circ \tag{1.2.26}$$

2. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{1.2.27}$$

$$\frac{\theta}{180} = \frac{3}{\pi} \tag{1.2.28}$$

$$\theta = \frac{3(180)}{\pi} \tag{1.2.29}$$

$$\theta \approx 172^\circ \tag{1.2.30}$$

1.2.3

Convert $-\frac{3\pi}{4}$ radians to degrees.

Solution

$-135°$

Example 1.2.4: Converting Degrees to Radians

Convert 15 degrees to radians.

Solution

In this example, we start with degrees and want radians, so we again set up a proportion and solve it, but we substitute the given information into a different part of the proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{1.2.31}$$

$$\frac{15}{180} = \frac{\theta^R}{\pi} \tag{1.2.32}$$

$$\frac{15\pi}{180} = \theta^R \tag{1.2.33}$$

$$\frac{\pi}{12} = \theta^R \tag{1.2.34}$$

Analysis

Another way to think about this problem is by remembering that $30^\circ = \frac{\pi}{6}$. Because $15^\circ = \frac{1}{2}(30^\circ)$, ,we can find that $\frac{1}{2}(\frac{\pi}{6})$ is $\frac{\pi}{12}$.

1.2.4

Convert 126° to radians.

Solution

$\frac{7\pi}{10}$

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at Figure 1.2.17. The angle of 140° is a **positive angle**, measured counterclockwise. The angle of –220° is a **negative angle**, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are **coterminal** angles. Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.

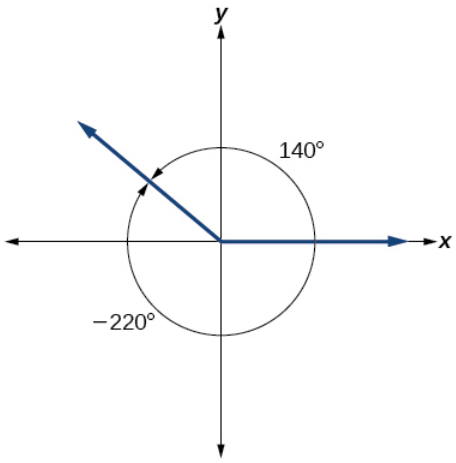

Figure 1.2.17: An angle of 140° and an angle of –220° are coterminal angles.

Any angle has infinitely many **coterminal angles** because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is −260°. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See Figure 1.2.18 for examples of reference angles for angles in different quadrants.

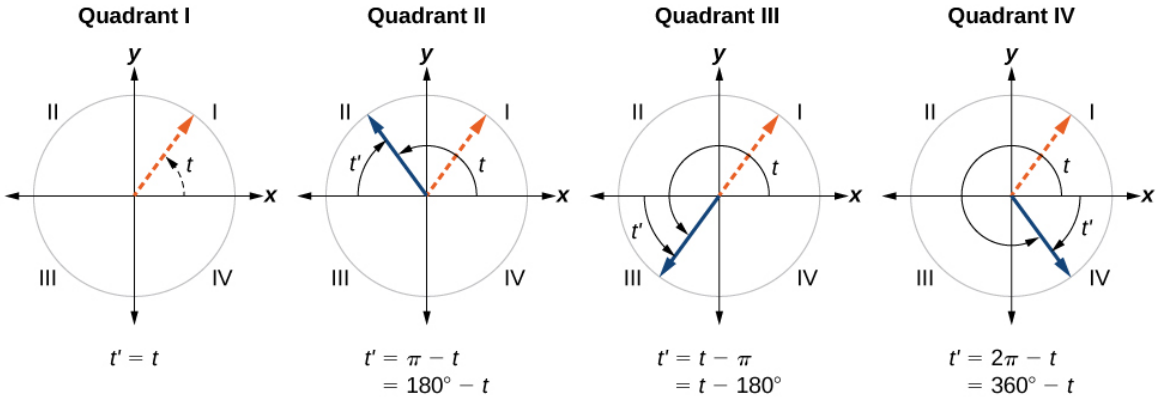

Figure 1.2.18

COTERMINAL AND REFERENCE ANGLES

- *Coterminal angles* are two angles in standard position that have the same terminal side.
- An angle's *reference angle* is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

Given an angle greater than 360°, find a coterminal angle between 0° and 360°

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 1.2.5: Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0^\circ \le \theta < 360^\circ$.

Solution

An angle with measure 800° is coterminal with an angle with measure 800 − 360 = 440°, but 440° is still greater than 360°, so we subtract 360° again to find another coterminal angle: 440 − 360 = 80°.

The angle $\theta = 80^\circ$ is coterminal with 800°. To put it another way, 800° equals 80° plus two full rotations, as shown in Figure 1.2.19.

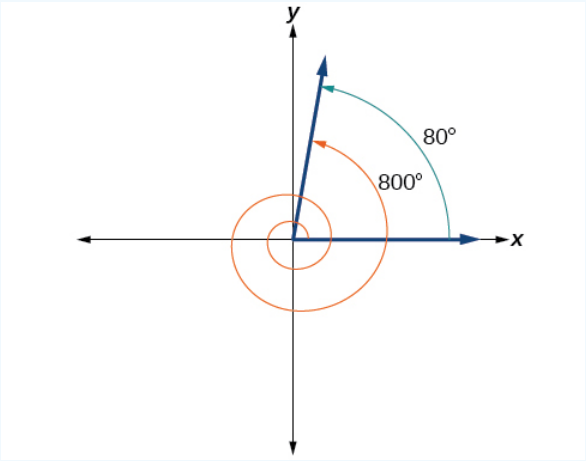

Figure 1.2.19

1.2.5

Find an angle α that is coterminal with an angle measuring 870°, where $0^\circ \le \alpha < 360^\circ$.

Solution

$\alpha = 150^\circ$

Given an angle with measure less than 0°, find a coterminal angle having a measure between 0° and 360°.

1. Add 360° to the given angle.
2. If the result is still less than 0°, add 360° again until the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example 1.2.6: Finding an Angle Coterminal with an Angle Measuring Less Than 0°

Show the angle with measure −45° on a circle and find a positive coterminal angle α such that 0° ≤ α < 360°.

Solution

Since 45° is half of 90°, we can start at the positive horizontal axis and measure clockwise half of a 90° angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360°, we can find a positive coterminal angle here by adding 360°:

$$-45^\circ + 360^\circ = 315^\circ \tag{1.2.35}$$

We can then show the angle on a circle, as in Figure 1.2.20.

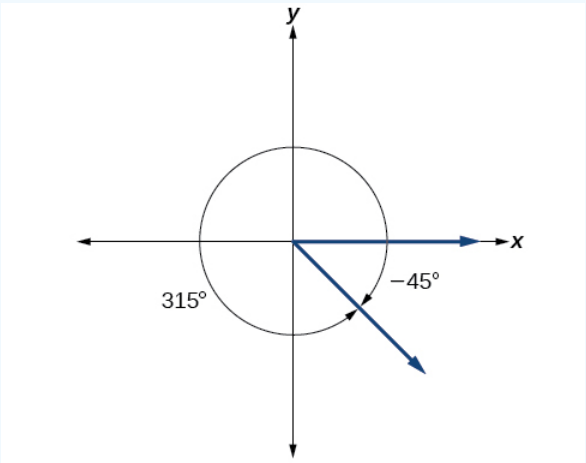

Figure 1.2.20

1.2.6

Find an angle β that is coterminal with an angle measuring −300° such that $0^\circ \le \beta < 360^\circ$.

Solution

$\beta = 60^\circ$

Finding Coterminal Angles Measured in Radians

We can find **coterminal** angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Example 1.2.7: Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \le \beta < 2\pi$.

Solution

When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} \tag{1.2.36}$$

$$= \frac{11\pi}{4} \tag{1.2.37}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} \tag{1.2.38}$$

$$= \frac{3\pi}{4} \tag{1.2.39}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown in Figure 1.2.21.

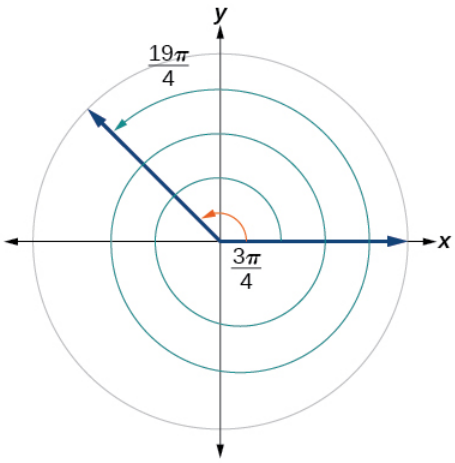

Figure 1.2.21

1.2.7

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

Solution

$\frac{7\pi}{6}$

Determining the Length of an Arc

Recall that the **radian measure** θ of an angle was defined as the ratio of the **arc length** s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

ARC LENGTH ON A CIRCLE

In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians, shown in Figure 1.2.22, is

$$s = r\theta \tag{1.2.40}$$

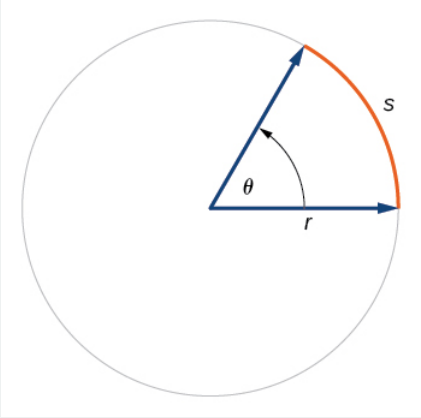

Figure 1.2.22

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of θ : $s = r\theta$.

Example 1.2.8: Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Solution

1. Let's begin by finding the circumference of Mercury's orbit.

$$C = 2\pi r \tag{1.2.41}$$
$$= 2\pi(36 \text{ million miles}) \tag{1.2.42}$$
$$\approx 226 \text{ million miles} \tag{1.2.43}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled:

$$(0.0114)226 \text{ million miles} = 2.58 \text{ million miles} \tag{1.2.44}$$

2. Now, we convert to radians:

$$\text{radian} = \frac{\text{arc length}}{\text{radius}} \tag{1.2.45}$$
$$= \frac{2.58 \text{ million miles}}{36 \text{ million miles}} \tag{1.2.46}$$
$$= 0.0717 \tag{1.2.47}$$

1.2.8

Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Solution

$$\frac{215\pi}{18} = 37.525 \text{ units} \tag{1.2.48}$$

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a **sector of a circle**. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A = \pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\text{Area of sector} = (\frac{\theta}{2\pi})\pi r^2 \tag{1.2.49}$$
$$= \frac{\theta \pi r^2}{2\pi} \tag{1.2.50}$$
$$= \frac{1}{2}\theta r^2 \tag{1.2.51}$$

AREA OF A SECTOR

The area of a **sector of a circle** with radius r subtended by an angle θ, measured in radians, is

$$A = \frac{1}{2}\theta r^2 \tag{1.2.52}$$

See Figure 1.2.23.

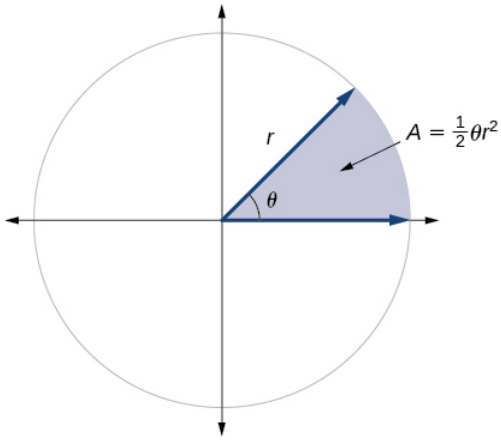

Figure 1.2.23: The area of the sector equals half the square of the radius times the central angle measured in radians.

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r : $A = \frac{1}{2}\theta r^2$.

Example 1.2.9: Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in Figure 1.2.24. What is the area of the sector of grass the sprinkler waters?

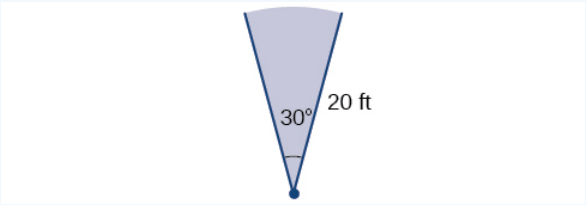

Figure 1.2.24: The sprinkler sprays 20 ft within an arc of 30°.

Solution

First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$30 \text{ degrees} = 30 \cdot \frac{\pi}{180} \tag{1.2.53}$$

$$= \frac{\pi}{6} \text{ radians} \tag{1.2.54}$$

The area of the sector is then

$$\text{Area} = \frac{1}{2}\left(\frac{\pi}{6}\right)(20)^2 \tag{1.2.55}$$

$$\approx 104.72 \tag{1.2.56}$$

So the area is about 104.72 ft^2.

1.2.9

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

Solution

1.88

Use Linear and Angular Speed to Describe Motion on a Circular Path

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its **displacement**) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10π inches, every second. So the linear speed of the point is 10π in./s. The equation for linear speed is as follows where v is linear speed, s is displacement, and t is time.

$$v = \frac{s}{t} \tag{1.2.57}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, angular speed is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}} = 90$ degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega = \frac{\theta}{t} \tag{1.2.58}$$

Combining the definition of angular speed with the arc length equation, $s = r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arc length equation gives:

$$s = r\theta \tag{1.2.59}$$
$$= r\omega t \tag{1.2.60}$$

Substituting this into the linear speed equation gives:

$$v = \frac{s}{t} \qquad = \frac{r\omega t}{t} = r\omega \tag{1.2.61}$$

ANGULAR AND LINEAR SPEED

As a point moves along a circle of radius r, its **angular speed**, ω, is the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t} \tag{1.2.62}$$

The **linear speed.** v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$$v = \frac{s}{t} \tag{1.2.63}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v = r\omega \tag{1.2.64}$$

This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

Given the amount of angle rotation and the time elapsed, calculate the angular speed

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

Example 1.2.10: Finding Angular Speed

A water wheel, shown in Figure 1.2.25, completes 1 rotation every 5 seconds. Find the angular speed in radians per second.

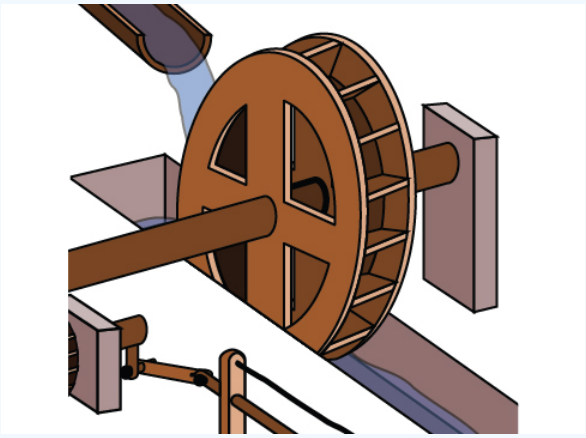

Figure 1.2.25

Solution

The wheel completes 1 rotation, or passes through an angle of 2π radians in 5 seconds, so the angular speed would be $\omega = \frac{2\pi}{5} \approx 1.257$ radians per second.

1.2.10

An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

Solution

$-\frac{3\pi}{2}$ rad/s

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega = \frac{\theta}{t}$.
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply $v = r\omega$.

Example 1.2.11: Finding a Linear Speed

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Solution

Here, we have an angular speed and need to find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road.

We begin by converting from rotations per minute to radians per minute. It can be helpful to utilize the units to make this conversion:

$$180 \frac{\text{rotations}}{\text{minute}} \cdot \frac{2\pi \text{ radians}}{\text{rotation}} = 360\pi \frac{\text{radians}}{\text{minute}} \tag{1.2.65}$$

Using the formula from above along with the radius of the wheels, we can find the linear speed:

$$v = (14 \text{ inches})(360\pi \frac{\text{radians}}{\text{minute}}) \tag{1.2.66}$$

$$= 5040\pi \frac{\text{inches}}{\text{minute}} \tag{1.2.67}$$

Remember that radians are a unitless measure, so it is not necessary to include them.unitless measure, so it is not necessary to include them.

Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour.

$$5040\pi \frac{\cancel{\text{inches}}}{\cancel{\text{minute}}} \cdot \frac{1 \ \cancel{\text{feet}}}{12 \ \cancel{\text{inches}}} \cdot \frac{1 \ \cancel{\text{mile}}}{5280 \ \cancel{\text{feet}}} \cdot \frac{60 \ \cancel{\text{minutes}}}{1 \text{ hour}} \approx 14.99 \text{ miles per hour (mph)} \tag{1.2.68}$$

Try It 1.2.11

A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

Solution

1655 kilometers per hour

Media

Access these online resources for additional instruction and practice with angles, arc length, and areas of sectors.

- Angles in Standard Position
- Angle of Rotation
- Coterminal Angles
- Determining Coterminal Angles
- Positive and Negative Coterminal Angles
- Radian Measure
- Coterminal Angles in Radians
- Arc Length and Area of a Sector

Key Equations

arc length	$s = r\theta$
area of a sector	$A = \frac{1}{2}\theta r^2$
angular speed	$\omega = \frac{\theta}{t}$
linear speed	$v = \frac{s}{t}$
linear speed related to angular speed	$v = r\omega$

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive *x*-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive *x*-axis and then place the terminal side according to the fraction of a full rotation the angle represents. See **Example**.
- In addition to degrees, the measure of an angle can be described in radians. See **Example**.
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{\theta^R}{\pi}$. See Example and Example.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting 360° or 2π. See Example and Example.

- Coterminal angles can be found using radians just as they are for degrees. See **Example**.
- The length of a circular arc is a fraction of the circumference of the entire circle. See **Example**.
- The area of sector is a fraction of the area of the entire circle. See **Example**.
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time. See **Example**.
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time. See **Example**.

Glossary

angle
the union of two rays having a common endpoint

angular speed
the angle through which a rotating object travels in a unit of time

arc length
the length of the curve formed by an arc

area of a sector
area of a portion of a circle bordered by two radii and the intercepted arc; the fraction $\frac{\theta}{2\pi}$ multiplied by the area of the entire circle

coterminal angles
description of positive and negative angles in standard position sharing the same terminal side

degree
a unit of measure describing the size of an angle as one-360th of a full revolution of a circle

initial side
the side of an angle from which rotation begins

linear speed
the distance along a straight path a rotating object travels in a unit of time; determined by the arc length

measure of an angle
the amount of rotation from the initial side to the terminal side

negative angle
description of an angle measured clockwise from the positive x-axis

positive angle
description of an angle measured counterclockwise from the positive x-axis

quadrantal angle
an angle whose terminal side lies on an axis

radian measure
the ratio of the arc length formed by an angle divided by the radius of the circle

radian
the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle

ray
one point on a line and all points extending in one direction from that point; one side of an angle

reference angle

the measure of the acute angle formed by the terminal side of the angle and the horizontal axis

standard position

the position of an angle having the vertex at the origin and the initial side along the positive *x*-axis

terminal side

the side of an angle at which rotation ends

vertex

the common endpoint of two rays that form an angle

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1.3: Unit Circle - Sine and Cosine Functions

Learning Objectives

- Find function values for the sine and cosine of $30°$ or $\left(\frac{\pi}{6}\right)$,$45°$ or $\left(\frac{\pi}{4}\right)$,and $60°$ or $\left(\frac{\pi}{3}\right)$.
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Looking for a thrill? Then consider a ride on the Singapore Flyer, the world's tallest Ferris wheel. Located in Singapore, the Ferris wheel soars to a height of 541 feet—a little more than a tenth of a mile! Described as an observation wheel, riders enjoy spectacular views as they travel from the ground to the peak and down again in a repeating pattern. In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.

Figure 1.3.1: The Singapore Flyer is the world's tallest Ferris wheel. (credit: "Vibin JK"/Flickr)

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in Figure 1.3.2. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a **unit circle**, $s = t$.

Recall that the *x*- and *y*-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

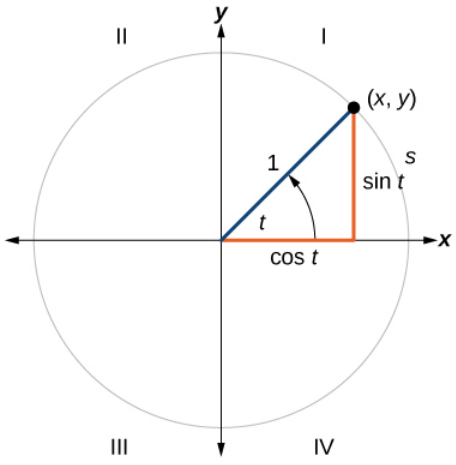

Figure 1.3.2: Unit circle where the central angle is t radians

UNIT CIRCLE

A unit circle has a center at $(0, 0)$ and radius 1. The length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions

Now that we have our unit circle labeled, we can learn how the (x, y) coordinates relate to the **arc length** and **angle**. The **sine function** relates a real number t to the y-coordinate of the point where the corresponding angle intercepts the unit circle. More precisely, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t. In Figure 1.3.3, the sine is equal to y. Like all functions, the sine function has an input and an output. Its input is the measure of the angle; its output is the y-coordinate of the corresponding point on the unit circle.

The **cosine function** of an angle t equals the x-value of the endpoint on the unit circle of an arc of length t. In Figure 1.3.1, the cosine is equal to x.

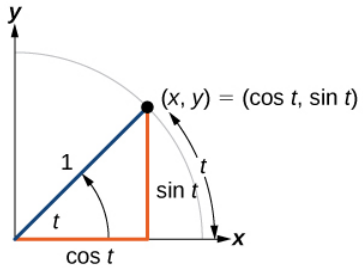

Figure 1.3.3

Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. Likewise, $\cos^2 t$ is a commonly used shorthand notation for $(\cos(t))^2$. Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

SINE AND COSINE FUNCTIONS

If t is a real number and a point (x, y) on the unit circle corresponds to an angle of t, then

$$\cos t = x \tag{1.3.1}$$
$$\sin t = y \tag{1.3.2}$$

How To: Given a point $P(x, y)$ on the unit circle corresponding to an angle of t, find the sine and cosine

1. The sine of t is equal to the y-coordinate of point P: $\sin t = y$.
2. The cosine of t is equal to the x-coordinate of point P: $\cos t = x$.

Example 1.3.1: Finding Function Values for Sine and Cosine

Point P is a point on the unit circle corresponding to an angle of t, as shown in Figure 1.3.4. Find $\cos(t)$ and $\sin(t)$.

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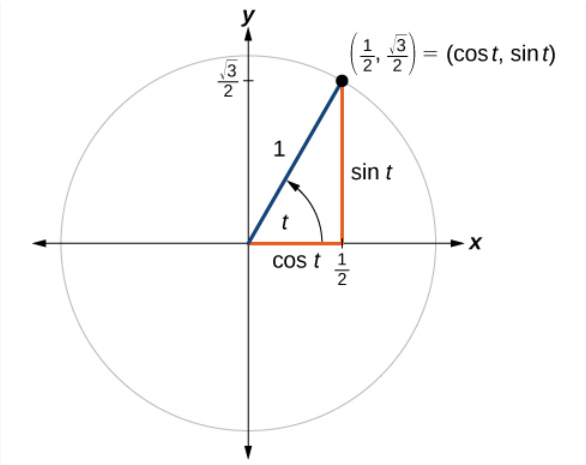

Figure 1.3.4

Solution

We know that $\cos\, t$ is the x-coordinate of the corresponding point on the unit circle and $\sin\, t$ is the y-coordinate of the corresponding point on the unit circle. So:

$$x = \cos\, t = \frac{1}{2}$$
$$y = \sin\, t = \frac{\sqrt{3}}{2}$$

Try It 1.3.1

A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in Figure 1.3.5. Find $\cos\, t$ and $\sin\, t$.

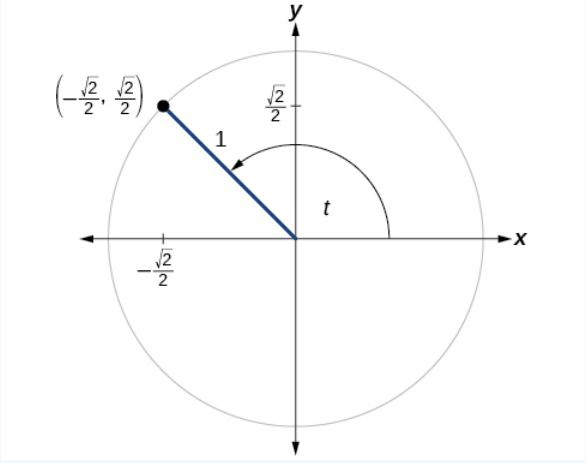

Figure 1.3.5

Answer

$$\cos(t) = -\frac{\sqrt{2}}{2}, \sin(t) = \frac{\sqrt{2}}{2}$$

Finding Sines and Cosines of Angles on an Axis

For quadrantral angles, the corresponding point on the unit circle falls on the x- or y-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Example 1.3.2: Calculating Sines and Cosines along an Axis

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.

Solution

Moving 90° counterclockwise around the unit circle from the positive x-axis brings us to the top of the circle, where the (x, y) coordinates are (0, 1), as shown in Figure 1.3.6.

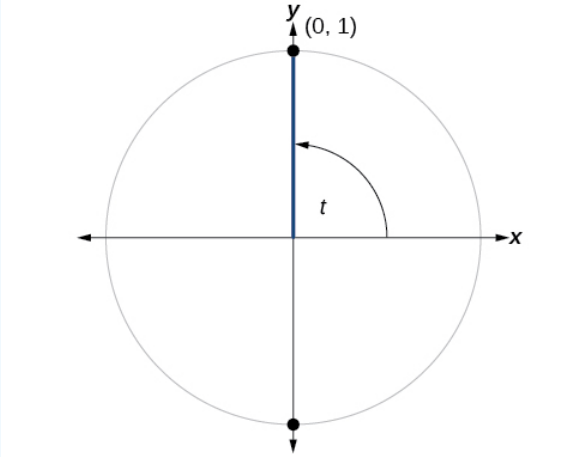

Figure 1.3.6

Using our definitions of cosine and sine,

$$x = \cos t = \cos(90^\circ) = 0$$
$$y = \sin t = \sin(90^\circ) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

Try It 1.3.2

Find cosine and sine of the angle π.

Answer

$\cos(\pi) = -1,\ \sin(\pi) = 0$

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See Figure 1.3.7.

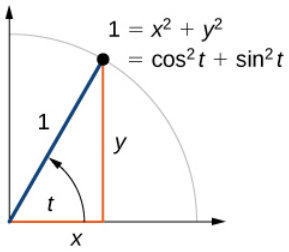

Figure 1.3.7

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1 \tag{1.3.3}$$

How To: Given the sine of some angle t and its quadrant location, find the cosine of t

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

Example 1.3.3: Finding a Cosine from a Sine or a Sine from a Cosine

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution

If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem. See Figure 1.3.8.

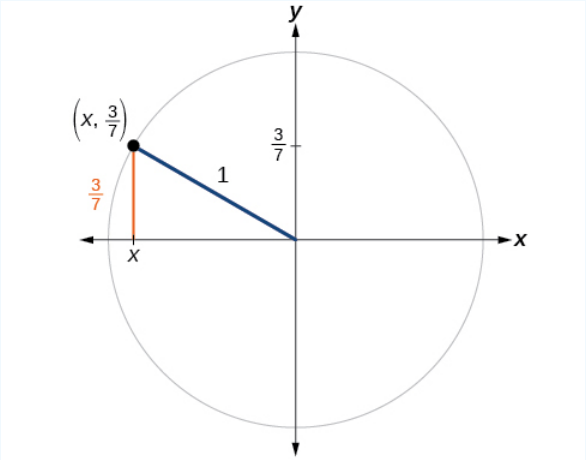

Figure 1.3.8

Substituting the known value for sine into the Pythagorean Identity,

$$\begin{aligned} \cos^2(t) + \sin^2(t) &= 1 \\ \cos^2(t) + \frac{9}{49} &= 1 \\ \cos^2(t) &= \frac{40}{49} \\ \cos(t) &= \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7} \end{aligned}$$

Because the angle is in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative. So

$$\cos(t) = -\frac{2\sqrt{10}}{7}$$

Try It 1.3.3

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Answer

$$\sin(t) = -\frac{7}{25}$$

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the **Pythagorean Identity** and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in Figure 1.3.9. A $45^\circ - 45^\circ - 90^\circ$ triangle is an isosceles triangle, so the x- and y-coordinates of the corresponding point on the circle are the same. Because the x- and y-values are the same, the sine and cosine values will also be equal.

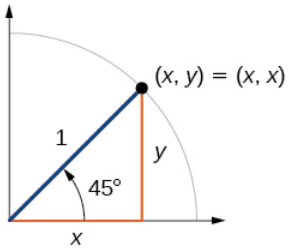

Figure 1.3.9

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first **quadrantal angle**. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y (with $y = x$), and a radius = 1. See Figure 1.3.10.

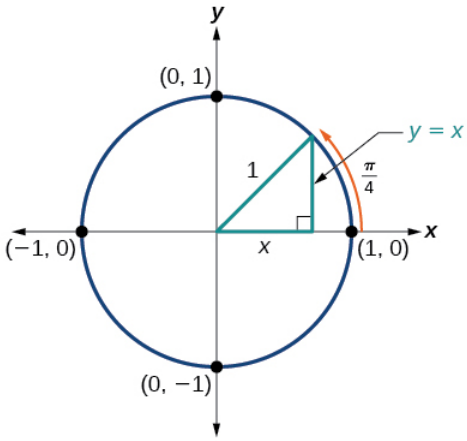

Figure 1.3.10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

Substituting $y = x$, we get

$$x^2 + x^2 = 1$$

Combining like terms we get

$$2x^2 = 1$$

And solving for x, we get

$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

In quadrant I, $x = \frac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$(x, y) = (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$$
$$\cos t = \frac{1}{\sqrt{2}},\ \sin t = \frac{1}{\sqrt{2}}$$

If we then rationalize the denominators, we get

$$\cos t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\sin t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30°, or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of 30°, and another at an angle of -30°, as shown in Figure 1.3.11. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60°, as shown in Figure 1.3.12

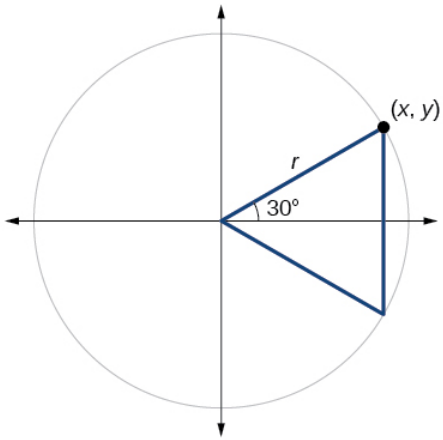

Figure 1.3.11

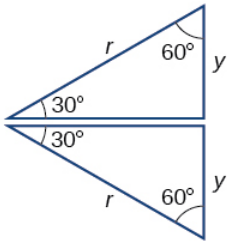

Figure 1.3.12

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

And since $r = 1$ in our **unit circle**,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1)\\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\begin{aligned}\cos^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{6}\right) &= 1\\ \cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 &= 1\\ \cos^2\left(\frac{\pi}{6}\right) &= \frac{3}{4} && \text{Use the square root property.}\\ \cos\left(\frac{\pi}{6}\right) &= \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} && \text{Since } y \text{ is positive, choose the positive root.}\end{aligned}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30° are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t = \frac{\pi}{3}$ (60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in Figure 1.3.13. Angle A has measure 60°. 60°. At point B, we draw an angle ABC with measure of 60°. We know the angles in a triangle sum to 180°, so the measure of angle C is also 60°. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

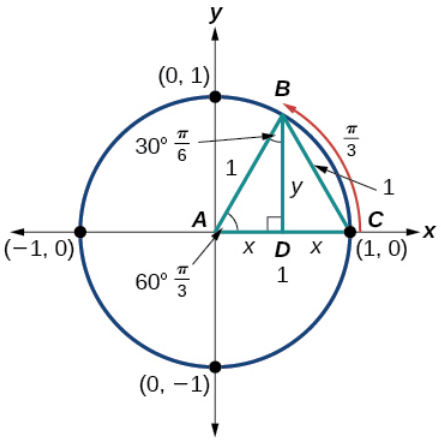

Figure 1.3.13

The measure of angle ABD is 30°. So, if double, angle ABC is 60°. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x-coordinate of point B, which is at the intersection of the 60°

angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2+y^2=1$$

Substituting $x=\frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2+y^2=1$$

Solving for y, we get

$$\begin{aligned}\frac{1}{4}+y^2&=1\\ y^2&=1-\frac{1}{4}\\ y^2&=\frac{3}{4}\\ y&=\pm\frac{\sqrt{3}}{2}\end{aligned}$$

Since $t=\frac{\pi}{3}$ has the terminal side in quadrant I where the y-coordinate is positive, we choose $y=\frac{\sqrt{3}}{2}$, the positive value.

At $t=\frac{\pi}{3}$ (60°), the (x,y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$(x,y)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$$

$$x=\frac{1}{2},\ y=\frac{\sqrt{3}}{2}$$

$$\cos t=\frac{1}{2},\ \sin t=\frac{\sqrt{3}}{2}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. Table 1.3.1 summarizes these values.

Table 1.3.1

Angle	0	$\frac{\pi}{6}$, or 30	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Figure 1.3.14 shows the common angles in the first quadrant of the unit circle.

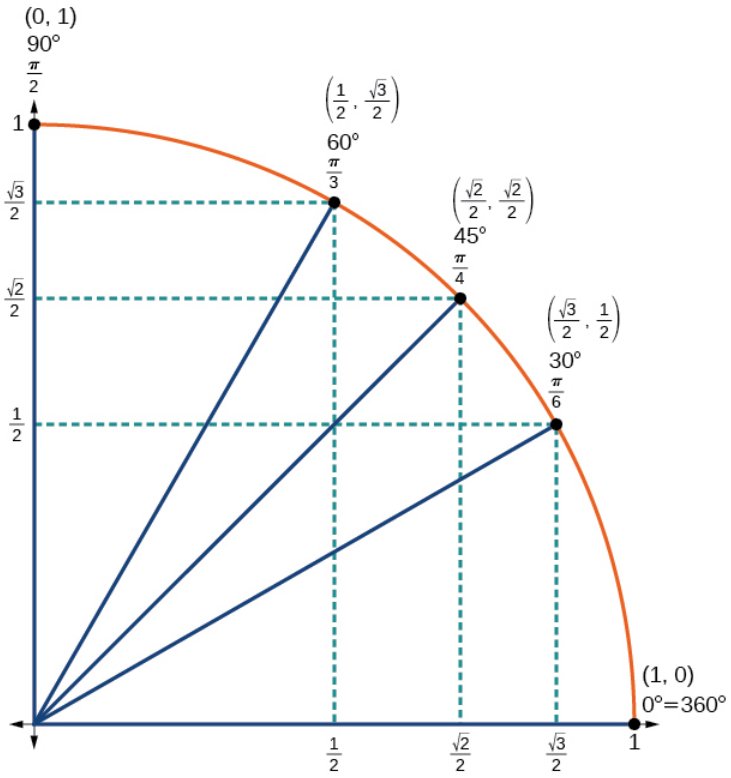

Figure 1.3.14

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the **special angles**, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To: Given an angle in radians, use a graphing calculator to find the cosine

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Example 1.3.4: Using a Graphing Calculator to Find Sine and Cosine

Evaluate $\cos\left(\frac{5\pi}{3}\right)$ using a graphing calculator or computer.

Solution

Enter the following keystrokes:

$$\text{COS}(5 \times \pi \div 3) \text{ ENTER}$$

$$\cos\left(\frac{5\pi}{3}\right) = 0.5$$

Analysis

We can find the cosine or sine of an angle in degrees directly on a calculator with degree mode. For calculators or software that use only radian mode, we can find the sign of 20°, for example, by including the conversion factor to radians as part of the input:

$$\text{SIN}(20 \times \pi \div 180) \text{ ENTER}$$

Try It 1.3.4

Evaluate $\sin\left(\frac{\pi}{3}\right)$.

Answer

approximately 0.866025403

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the **unit circle**, as shown in Figure 1.3.15. The bounds of the x-coordinate are $[-1, 1]$. The bounds of the y-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

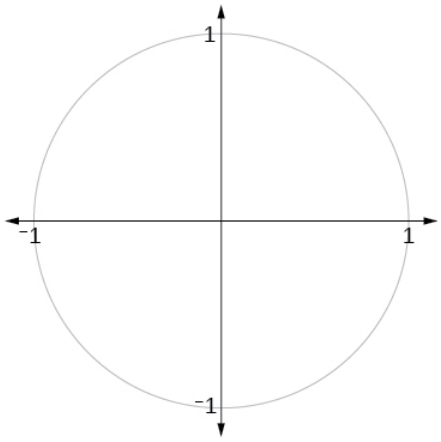

Figure 1.3.15

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y-coordinate on the unit circle, the other angle with the same sine will share the same y-value, but have the opposite x-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x-value but will have the opposite y-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in Figure 1.3.16, angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\sin(t) = \sin(\alpha) \quad \text{and} \quad \cos(t) = -\cos(\alpha)$$
$$\sin(t) = -\sin(\beta) \quad \text{and} \quad \cos(t) = \cos(\beta)$$

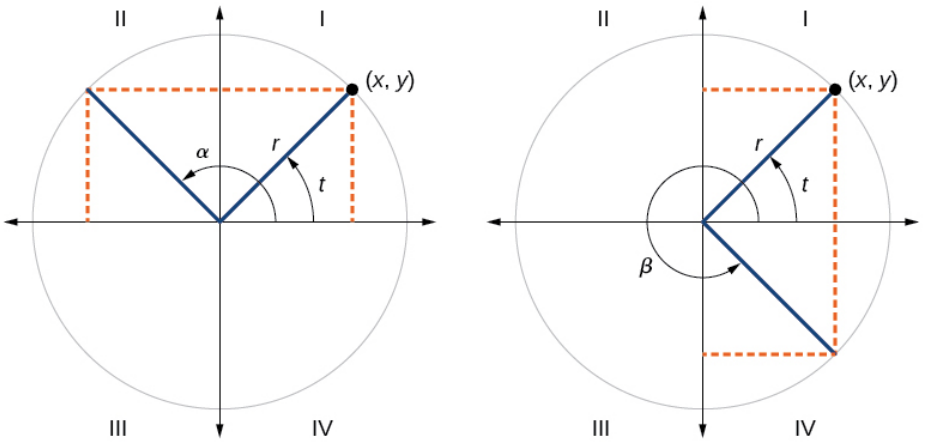

Figure 1.3.16

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90°, or 0 and $\frac{\pi}{2}$ radians. As we can see from Figure $1.3.17$, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

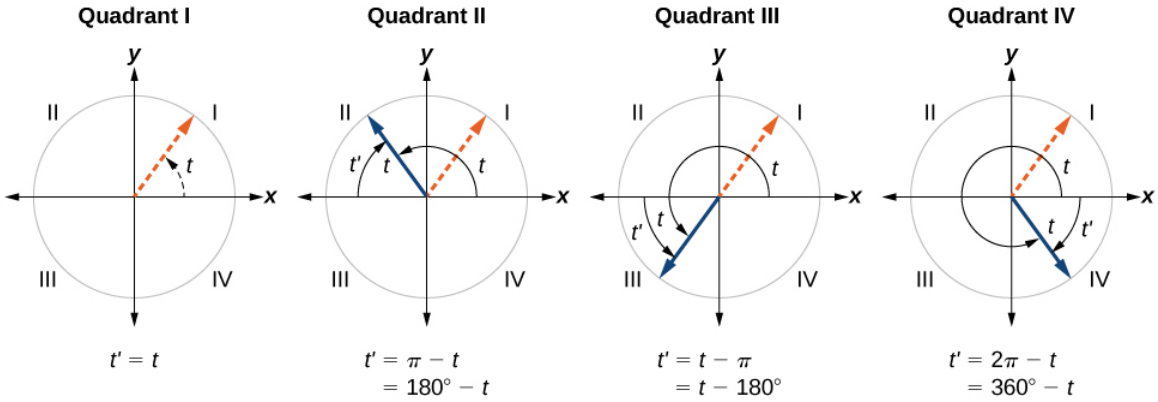

Figure 1.3.17

How To: Given an angle between 0 and 2π, find its reference angle

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180^\circ - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360^\circ - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Example $1.3.5$: Finding a Reference Angle

Find the reference angle of 225° as shown in Figure $1.3.18$

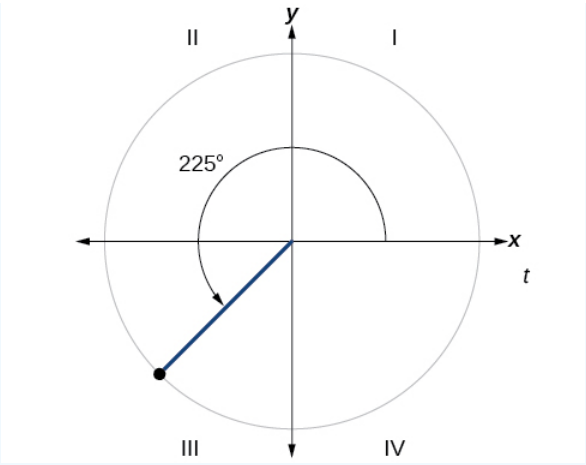

Figure 1.3.18

Solution

Because 225° is in the third quadrant, the reference angle is

$$|(180^\circ - 225^\circ)| = |-45^\circ| = 45^\circ$$

Try It 1.3.5

Find the reference angle of $\frac{5\pi}{3}$.

Answer

(\dfrac{π}{3}\)

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a **negative angle**. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the **reference angle** of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

USING REFERENCE ANGLES TO FIND COSINE AND SINE

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To: Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Example 1.3.6: Using Reference Angles to Find Sine and Cosine

1. Using a reference angle, find the exact value of $\cos(150^\circ)$ and $\sin(150^\circ)$.
2. Using the reference angle, find $\cos\ \dfrac{5\pi}{4}$ and $\sin\ \frac{5\pi}{4}$.

Solution

This tells us that 150° has the same sine and cosine values as 30°, except for the sign. We know that

Since 150° is in the second quadrant, the x-coordinate of the point on the circle is negative, so the cosine value is negative. The y-coordinate is positive, so the sine value is positive.

$\dfrac{5\pi}{4}$ is in the third quadrant. Its reference angle is $\left|\dfrac{5\pi}{4}-\pi\right|=\dfrac{\pi}{4}$. The cosine and sine of $\dfrac{\pi}{4}$ are both $\dfrac{\sqrt{2}}{2}$. In the third quadrant, both x and y are negative, so:

$$\cos\ \frac{5\pi}{4}=-\frac{\sqrt{2}}{2}\ \text{and}\ \sin\ \frac{5\pi}{4}=-\frac{\sqrt{2}}{2}$$

Try It 1.3.6

1. Use the reference angle of 315° to find $\cos(315^\circ)$ and $\sin(315^\circ)$.
2. Use the reference angle of $-\dfrac{\pi}{6}$ to find $\cos\left(-\dfrac{\pi}{6}\right)$ and $\sin\left(-\dfrac{\pi}{6}\right)$

Answer

1. $\cos(315^\circ)=\dfrac{\sqrt{2}}{2},\sin(315^\circ)=\dfrac{-\sqrt{2}}{2}$
2. $\cos\left(-\dfrac{\pi}{6}\right)=\dfrac{\sqrt{3}}{2},\sin\left(-\dfrac{\pi}{6}\right)=-\frac{1}{2}$

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in Figure 1.3.19. Take time to learn the (x,y) coordinates of all of the major angles in the first quadrant.

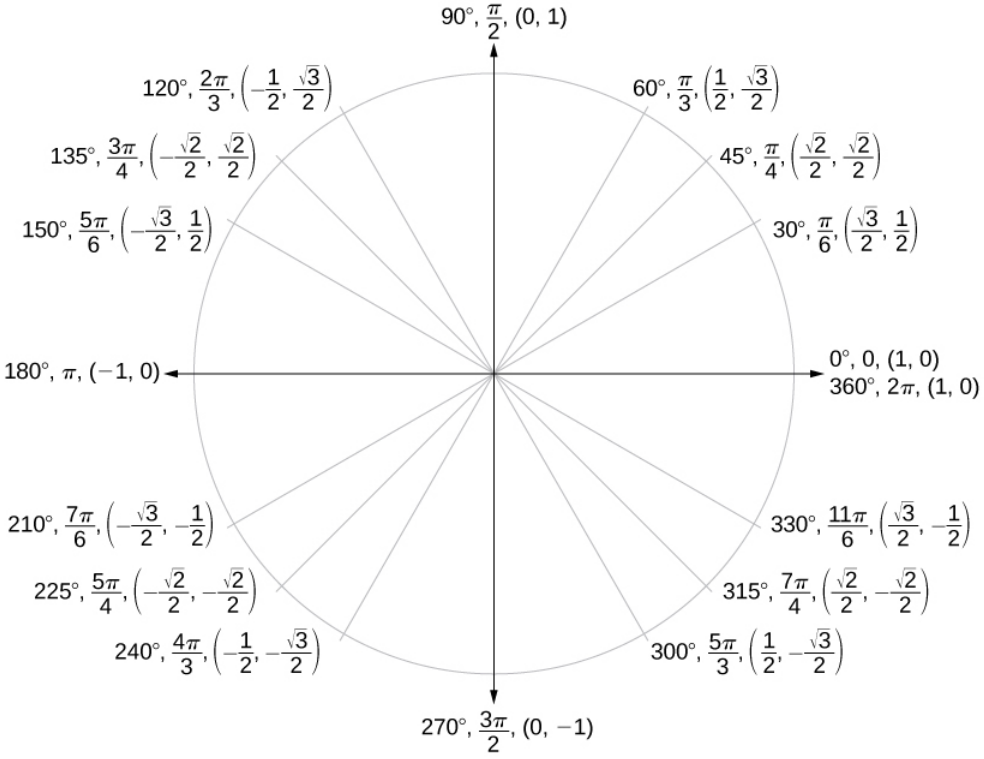

Figure 1.3.19: Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the **identities**

$$x = \cos t$$
$$y = \sin t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the **reference angle**, and give them the signs corresponding to the y- and x-values of the quadrant.

How To: Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point

1. Find the reference angle by measuring the smallest angle to the x-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example 1.3.7: Using the Unit Circle to Find Coordinates

Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.

Solution

We know that the angle $\frac{7\pi}{6}$ is in the third quadrant.

First, let's find the reference angle by measuring the angle to the x-axis. To find the reference angle of an angle whose terminal side is in quadrant III, we find the difference of the angle and π.

$$\frac{7\pi}{6} - \pi = \frac{\pi}{6}$$

Next, we will find the cosine and sine of the reference angle:

$$\cos\left(\frac{\pi}{6}\right) = \frac{3}{2} \qquad \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

We must determine the appropriate signs for x and y in the given quadrant. Because our original angle is in the third quadrant, where both x and y are negative, both cosine and sine are negative.

$$\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$
$$\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}$$

Now we can calculate the (x, y) coordinates using the identities $x = \cos\,\theta$ and $y = \sin\,\theta$.

The coordinates of the point are $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ on the unit circle.

Try It 1.3.7

Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Answer

$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

media

Access these online resources for additional instruction and practice with sine and cosine functions.

- Trigonometric Functions Using the Unit Circle
- Sine and Cosine from the Unit Circle
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Six
- Sine and Cosine from the Unit Circle and Multiples of Pi Divided by Four
- Trigonometric Functions Using Reference Angles

Key Equations

Cosine	$\cos\,t = x$
Sine	$\sin\,t = y$
Pythagorean Identity	$\cos^2 t + \sin^2 t = 1$

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x-value of the endpoint. See Example.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis. See Example.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles. See Example.
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known. See Example.
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the *x*- and y-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis. See Example.

- Reference angles can be used to find the sine and cosine of the original angle. See Example.
- Reference angles can also be used to find the coordinates of a point on a circle. See Example.

Glossary

cosine function

the x-value of the point on a unit circle corresponding to a given angle

Pythagorean Identity

a corollary of the Pythagorean Theorem stating that the square of the cosine of a given angle plus the square of the sine of that angle equals 1

sine function

the y-value of the point on a unit circle corresponding to a given angle

unit circle

a circle with a center at $(0, 0)$ and radius 1.

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1.4: The Other Trigonometric Functions

Learning Objectives

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}$, $\frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

A wheelchair ramp that meets the standards of the Americans with Disabilities Act must make an angle with the ground whose tangent is $\frac{1}{12}$ or less, regardless of its length. A tangent represents a ratio, so this means that for every 1 inch of rise, the ramp must have 12 inches of run. Trigonometric functions allow us to specify the shapes and proportions of objects independent of exact dimensions. We have already defined the sine and cosine functions of an angle. Though sine and cosine are the trigonometric functions most often used, there are four others. Together they make up the set of six trigonometric functions. In this section, we will investigate the remaining functions.

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

To define the remaining functions, we will once again draw a unit circle with a point (x, y) corresponding to an angle of t,as shown in Figure 1.4.1. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

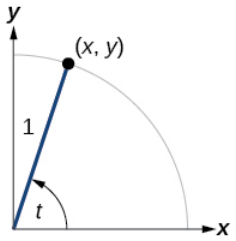

Figure 1.4.1

The first function we will define is the tangent. The **tangent** of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle. In Figure 1.4.1, the tangent of angle t is equal to $\frac{y}{x}, x \neq 0$. Because the *y*-value is equal to the sine of t, and the *x*-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}, \cos t \neq 0$. The tangent function is abbreviated as tan. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In Figure 1.4.1, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}, x \neq 0$. The secant function is abbreviated as sec.
- The **cotangent** function is the reciprocal of the tangent function. In Figure 1.4.1, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as cot.
- The **cosecant** function is the reciprocal of the sine function. In Figure 1.4.1, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as csc.

TANGENT, SECANT, COSECANT, AND COTANGENT FUNCTIONS

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0 \tag{1.4.1}$$

$$\sec t = \frac{1}{x}, x \neq 0 \tag{1.4.2}$$

$$\csc t = \frac{1}{y}, y \neq 0 \tag{1.4.3}$$

$$\cot t = \frac{x}{y}, y \neq 0 \tag{1.4.4}$$

Example 1.4.1: Finding Trigonometric Functions from a Point on the Unit Circle

The point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the unit circle, as shown in Figure 1.4.2. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

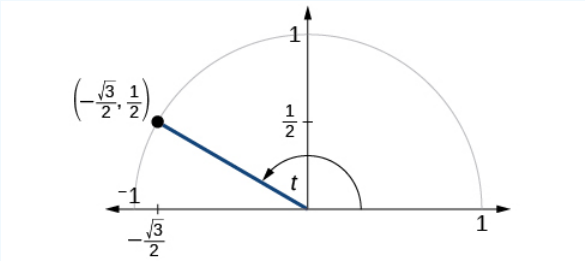

Figure 1.4.2

Solution

Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

$$\sin t = y = \frac{1}{2}$$

$$\cos t = x = -\frac{\sqrt{3}}{2}$$

$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\sec t = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot t = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

Exercise 1.4.1:

The point $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ is on the unit circle, as shown in Figure 1.4.3. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

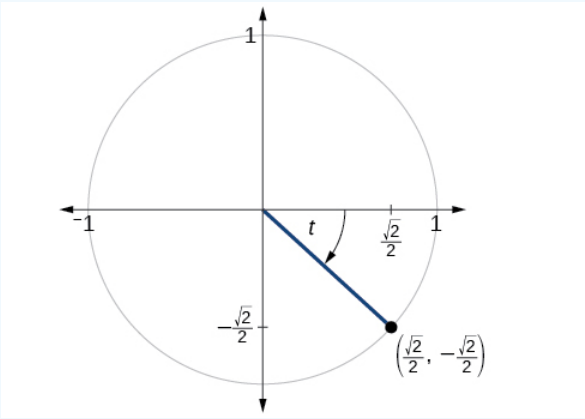

Figure 1.4.3

Solution

$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = -1, \sec t = \sqrt{2}, \csc t = -\sqrt{2}, \cot t = -1$

Example 1.4.2: Finding the Trigonometric Functions of an Angle

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{6}$.

Solution

We have previously used the properties of equilateral triangles to demonstrate that $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\begin{aligned}
\tan\frac{\pi}{6} &= \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} \\
&= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \\
\sec\frac{\pi}{6} &= \frac{1}{\cos\frac{\pi}{6}} \\
&= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \\
\csc\frac{\pi}{6} &= \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2 \\
\cot\frac{\pi}{6} &= \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} \\
&= \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}
\end{aligned}$$

Exercise 1.4.2:

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{3}$.

Solution

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \tag{1.4.5}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} \tag{1.4.6}$$

$$\tan\frac{\pi}{3} = \sqrt{3} \tag{1.4.7}$$

$$\sec\frac{\pi}{3} = 2 \tag{1.4.8}$$

$$\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3} \tag{1.4.9}$$

$$\cot\frac{\pi}{3} = \frac{\sqrt{3}}{3} \tag{1.4.10}$$

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x x equal to the cosine and y y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in Table 1.4.1.

Table 1.4.1

Angle	**0**	$\frac{\pi}{6}$, or **30**$^\circ$	$\frac{\pi}{4}$, or **45**$^\circ$	$\frac{\pi}{3}$, or **60**$^\circ$	$\frac{\pi}{2}$, or **90**$^\circ$
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the **reference angle** formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by *x*- and *y*-values in the original quadrant. Figure 1.4.4 shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.

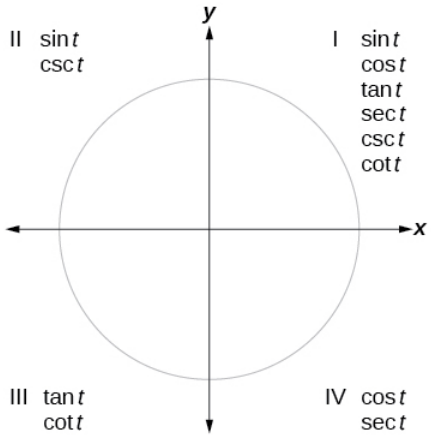

Figure 1.4.4

HOWTO: Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example 1.4.3: Using Reference Angles to Find Trigonometric Functions

Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

Solution

The angle between this angle's terminal side and the *x*-axis is $\frac{\pi}{6}$, so that is the reference angle. Since $-\frac{5\pi}{6}$ is in the third quadrant, where both x and y are negative, cosine, sine, secant, and cosecant will be negative, while tangent and cotangent will be positive.

$$\cos(-\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}, \sin(-\frac{5\pi}{6}) = -\frac{1}{2}, \tan(-\frac{5\pi}{6}) = \frac{\sqrt{3}}{3} \quad (1.4.11)$$

$$\sec(-\frac{5\pi}{6}) = -\frac{2\sqrt{3}}{3}, \csc(-\frac{5\pi}{6}) = -2, \cot(-\frac{5\pi}{6}) = \sqrt{3} \quad (1.4.12)$$

Exercise 1.4.3

Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

Solution

$\sin(-\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{-7\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{-7\pi}{4}) = 1,$

$\sec(\frac{-7\pi}{4}) = \sqrt{2}, \csc(\frac{-7\pi}{4}) = \sqrt{2}, \cot(\frac{-7\pi}{4}) = 1$

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

Consider the function $f(x) = x^2$, shown in Figure 1.4.5. The graph of the function is symmetrical about the *y*-axis. All along the curve, any two points with opposite *x*-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2, (-5)^2 = (5)^2$, and so on. So $f(x) = x^2$ is an **even function**, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.

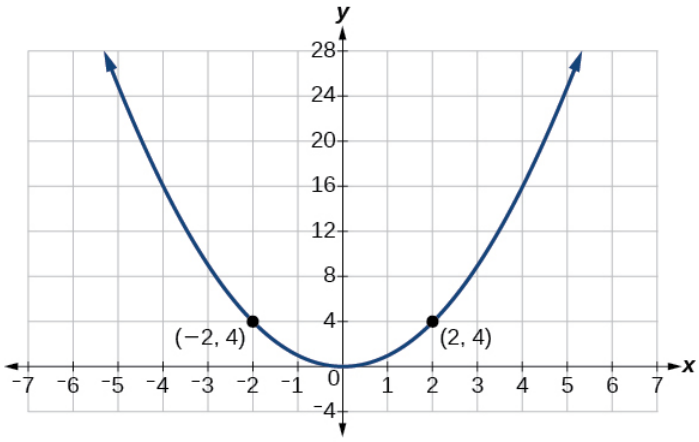

Figure 1.4.5: The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in Figure 1.4.6. The graph is not symmetrical about the *y*-axis. All along the graph, any two points with opposite *x*-values also have opposite *y*-values. So $f(x) = x^3$ is an **odd function**, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.

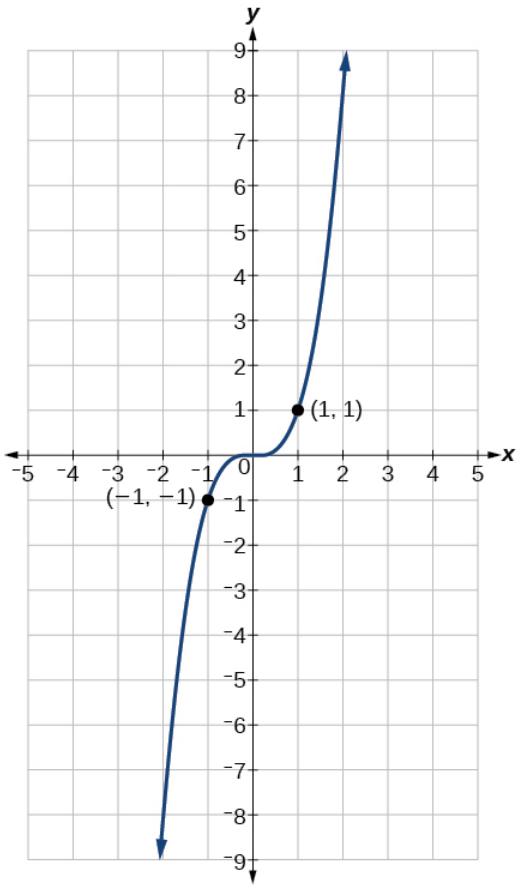

Figure 1.4.6: The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a **unit circle** with a positive and a negative angle, as in Figure 1.4.7. The sine of the positive angle is y. The sine of the negative angle is $-y$. The **sine function**, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in Table 1.4.2.

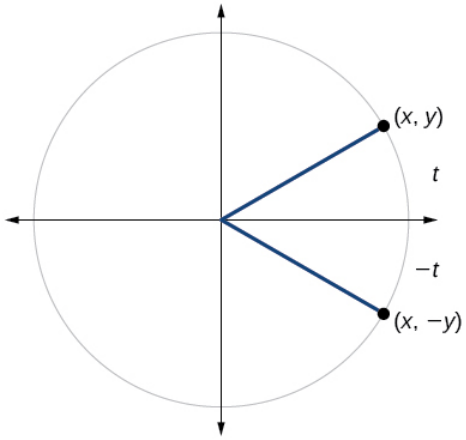

Figure 1.4.7

Table 1.4.2

$$\sin t = y \quad (1.4.13)$$
$$\sin(-t) = -y \quad (1.4.14)$$
$$\sin t \neq sin(-t) \quad (1.4.15)$$

$$\cos t = x \quad (1.4.16)$$
$$\cos(-t) = x \quad (1.4.17)$$
$$\cos t = \cos(-t) \quad (1.4.18)$$

$$\tan(t) = \frac{y}{x} \quad (1.4.19)$$
$$\tan(-t) = -\frac{y}{x} \quad (1.4.20)$$
$$\tan t \neq \tan(-t) \quad (1.4.21)$$

$$\sec t = \frac{1}{x} \quad (1.4.22)$$
$$\sec(-t) = \frac{1}{x} \quad (1.4.23)$$
$$\sec t = \sec(-t) \quad (1.4.24)$$
$$\csc t = \frac{1}{y} \quad (1.4.25)$$
$$\csc(-t) = \frac{1}{-y} \quad (1.4.26)$$
$$\csc t \neq \csc(-t) \quad (1.4.27)$$
$$\cot t = \frac{x}{y} \quad (1.4.28)$$
$$\cot(-t) = \frac{x}{-y} \quad (1.4.29)$$
$$\cot t \neq \cot(-t) \quad (1.4.30)$$

EVEN AND ODD TRIGONOMETRIC FUNCTIONS

- An **even function** is one in which $f(-x) = f(x)$.
- An **odd function** is one in which $f(-x) = -f(x)$.

Cosine and secant are even:

$$\cos(-t) = \cos t \quad (1.4.31)$$
$$\sec(-t) = \sec t \quad (1.4.32)$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t \quad (1.4.33)$$
$$\tan(-t) = -\tan t \quad (1.4.34)$$
$$\csc(-t) = -\csc t \quad (1.4.35)$$
$$\cot(-t) = -\cot t \quad (1.4.36)$$

Example 1.4.4: Using Even and Odd Properties of Trigonometric Functions

If the secant of angle t is 2, what is the secant of $-t$?

Solution

Secant is an even function. The secant of an angle is the same as the secant of its opposite. So if the secant of angle t is 2, the secant of $-t$ is also 2.

Exercise 1.4.4:

If the cotangent of angle t is $\sqrt{3}$, what is the cotangent of $-t$?

Solution

$-\sqrt{3}$

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

FUNDAMENTAL IDENTITIES

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \quad (1.4.37)$$
$$\sec t = \frac{1}{\cos t} \quad (1.4.38)$$
$$\csc t = \frac{1}{\sin t} \quad (1.4.39)$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t} \tag{1.4.40}$$

Example 1.4.5: Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45^\circ) = \frac{\sqrt{2}}{2}, \cos(45^\circ) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45^\circ)$.

b. Given $\sin(\frac{5\pi}{6}) = \frac{1}{2}, \cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$, evaluate $\sec(\frac{5\pi}{6})$.

Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.
$$\begin{aligned} \tan(45^\circ) &= \frac{\sin(45^\circ)}{\cos(45^\circ)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1 \end{aligned}$$

b.
$$\begin{aligned} \sec(\frac{5\pi}{6}) &= \frac{1}{\cos(\frac{5\pi}{6})} \\ &= \frac{1}{-\frac{\sqrt{3}}{2}} \\ &= \frac{-2\sqrt{3}}{1} \\ &= \frac{-2}{\sqrt{3}} \\ &= -\frac{2\sqrt{3}}{3} \end{aligned}$$

Exercise 1.4.5

Evaluate $\csc(\frac{7\pi}{6})$.

Solution

-2

Example 1.4.6: Using Identities to Simplify Trigonometric Expressions

Simplify $\frac{\sec t}{\tan t}$.

Solution

We can simplify this by rewriting both functions in terms of sine and cosine.

$$\begin{aligned} \frac{\sec t}{\tan t} &= \frac{1/\cos t}{\sin t/\cos t} && \text{To divide the functions, we multiply by the reciprocal.} \\ &= \frac{1}{\cos t}\frac{\cos t}{\sin t} && \text{Divide out the cosines.} \\ &= \frac{1}{\sin t} && \text{Simplify and use the identity.} \\ &= \csc t \end{aligned} \tag{1.4.41}$$

By showing that $\frac{\sec t}{\tan t}$ can be simplified to $\csc t$,we have, in fact, established a new identity.

$$\frac{\sec t}{\tan t} = \csc t$$

Exercise 1.4.6

Simplify $(\tan t)(\cos t)$.

Solution

$\sin t$

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the **Pythagorean Identity**, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \tag{1.4.42}$$

$$1 + \tan^2 t = \sec^2 t \tag{1.4.43}$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \tag{1.4.44}$$

$$\cot^2 t + 1 = \csc^2 t \tag{1.4.45}$$

ALTERNATE FORMS OF THE PYTHAGOREAN IDENTITY

$$1 + \tan^2 t = \sec^2 t \tag{1.4.46}$$

$$\cot^2 t + 1 = \csc^2 t \tag{1.4.47}$$

Example 1.4.7: Using Identities to Relate Trigonometric Functions

If cos(t)=1213 cos(t)=1213 and t t is in quadrant IV, as shown in Figure 1.4.8, find the values of the other five trigonometric functions.

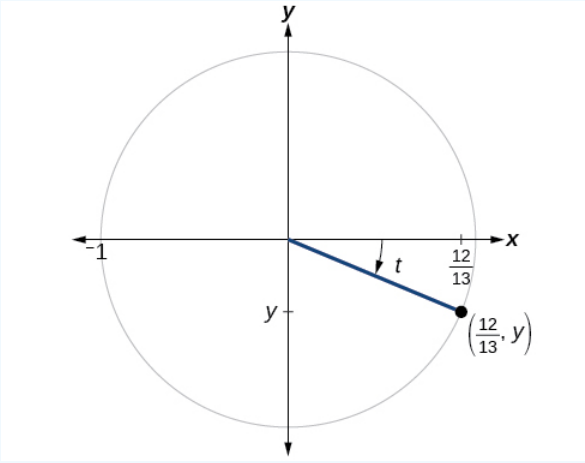

Figure 1.4.8

Solution

We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$(\frac{12}{13})^2+\sin^2 t=1 \tag{1.4.48}$$

$$\sin^2 t=1-(\frac{12}{13})^2 \tag{1.4.49}$$

$$\sin^2 t=1-\frac{144}{169} \tag{1.4.50}$$

$$\sin^2 t=\frac{25}{169} \tag{1.4.51}$$

$$\sin t=\pm\sqrt{\frac{25}{169}} \tag{1.4.52}$$

$$\sin t=\pm\frac{\sqrt{25}}{\sqrt{169}} \tag{1.4.53}$$

$$\sin t=\pm\frac{5}{13} \tag{1.4.54}$$

The sign of the sine depends on the *y*-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the *y*-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t=\frac{\sin t}{\cos t}=\frac{-\frac{5}{13}}{\frac{12}{13}}=-\frac{5}{12} \tag{1.4.55}$$

$$\sec t=\frac{1}{\cos t}=\frac{1}{\frac{12}{13}}=\frac{13}{12} \tag{1.4.56}$$

$$\csc t=\frac{1}{\sin t}=\frac{1}{-\frac{5}{13}}=-\frac{13}{5} \tag{1.4.57}$$

$$\cot t=\frac{1}{\tan t}=\frac{1}{-\frac{5}{12}}=-\frac{12}{5} \tag{1.4.58}$$

Exercise 1.4.7:

If $\sec(t)=-\frac{17}{8}$ and $0<t<\pi,$ find the values of the other five functions.

Solution

$\cos t=-\frac{8}{17}, \sin t=\frac{15}{17}, \tan t=-\frac{15}{8}$

$\csc t=\frac{17}{15}, \cot t=-\frac{8}{15}$

As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a **periodic function**. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π,will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x+4)=f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

PERIOD OF A FUNCTION

The **period** P of a repeating function f f is the number representing the interval such that $f(x+P)=f(x)$ for any value of x.

The period of the cosine, sine, secant, and cosecant functions is 2π.

The period of the tangent and cotangent functions is π.

Example 1.4.8: Finding the Values of Trigonometric Functions

Find the values of the six trigonometric functions of angle t based on Figure 1.4.9.

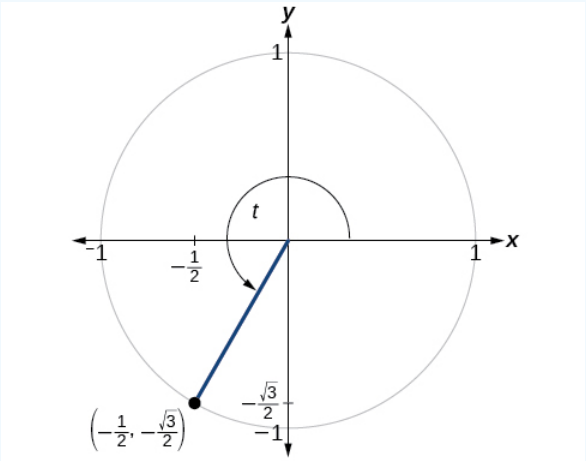

Figure 1.4.9

Solution

$$\sin t = y = -\frac{\sqrt{3}}{2}$$

$$\cos t = x = -\frac{1}{2}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 1.4.8

Find the values of the six trigonometric functions of angle t based on Figure 1.4.10.

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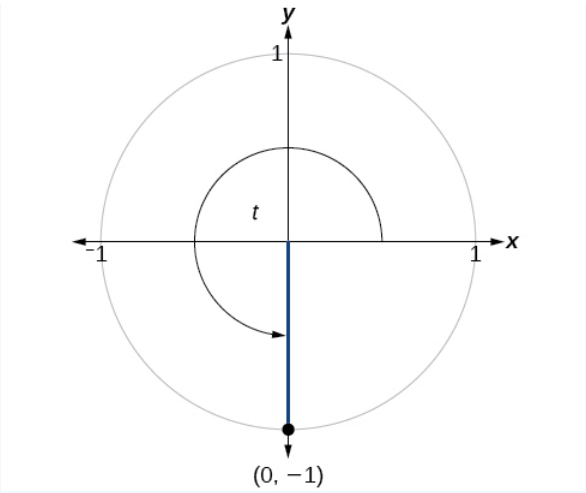

Figure 1.4.10

Solution

$$\sin t = -1, \cos t = 0, \tan t = \text{Undefined} \tag{1.4.59}$$

$$\tag{1.4.60}$$

$$sect = \text{Undefined}, \csc t = -1, \cot t = 0 \tag{1.4.61}$$

Example 1.4.9: Finding the Value of Trigonometric Functions

If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$,find $\sec(t), \csc(t), \tan(t), \cot(t)$.

Solution

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2 \tag{1.4.62}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} - \frac{2\sqrt{3}}{3} \tag{1.4.63}$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3} \tag{1.4.64}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3} \tag{1.4.65}$$

Exercise 1.4.9:

If $\sin(t) = \frac{\sqrt{2}}{2}$ and $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sec(t), \csc(t), \tan(t)$, and $\cot(t)$.

Solution

$\sec t = \sqrt{2}, \csc t = \sqrt{2}, \tan t = 1, \cot t = 1$

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of 30°, we could press

$$(\text{for a scientific calculator}): \frac{1}{30 \times \frac{\pi}{180}} \text{COS} \tag{1.4.66}$$

or

$$(\text{for a graphing calculator}): \frac{1}{\cos(\frac{30\pi}{180})} \tag{1.4.67}$$

how to: Given an angle measure in radians, use a scientific calculator to find the cosecant

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

how to: Given an angle measure in radians, use a graphing utility/calculator to find the cosecant

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Press the SIN key.
4. Enter the value of the angle inside parentheses.
5. Press the ENTER key.

Example 1.4.10: Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Solution

For a scientific calculator, enter information as follows:

$$1/(5 \times \pi/7)\text{SIN} = \tag{1.4.68}$$

$$\csc(\frac{5\pi}{7}) \approx 1.279 \tag{1.4.69}$$

Exercise 1.4.10:

Evaluate the cotangent of $-\frac{\pi}{8}$.

≈ -2.414

media

Access these online resources for additional instruction and practice with other trigonometric functions.

- Determining Trig Function Values
- More Examples of Determining Trig Functions
- Pythagorean Identities
- Trig Functions on a Calculator

Key Equations

Tangent function	$\tan t = \frac{\sin t}{\cos t}$
Secant function	$\sec t = \frac{1}{\cos t}$

Cosecant function	$\csc t = \frac{1}{\sin t}$
Cotangent function	$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$

Key Concepts

- The tangent of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle.
- The secant, cotangent, and cosecant are all reciprocals of other functions. The secant is the reciprocal of the cosine function, the cotangent is the reciprocal of the tangent function, and the cosecant is the reciprocal of the sine function.
- The six trigonometric functions can be found from a point on the unit circle. See Example.
- Trigonometric functions can also be found from an angle. See Example.
- Trigonometric functions of angles outside the first quadrant can be determined using reference angles. See Example.
- A function is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$.
- Cosine and secant are even; sine, tangent, cosecant, and cotangent are odd.
- Even and odd properties can be used to evaluate trigonometric functions. See Example.
- The Pythagorean Identity makes it possible to find a cosine from a sine or a sine from a cosine.
- Identities can be used to evaluate trigonometric functions. See Example and Example.
- Fundamental identities such as the Pythagorean Identity can be manipulated algebraically to produce new identities. See Example.
- The trigonometric functions repeat at regular intervals.
- The period P of a repeating function f f is the smallest interval such that $f(x + P) = f(x)$ for any value of x.
- The values of trigonometric functions of special angles can be found by mathematical analysis.
- To evaluate trigonometric functions of other angles, we can use a calculator or computer software. See Example.

Glossary

cosecant

the reciprocal of the sine function: on the unit circle, $\csc t = \frac{1}{y}, y \neq 0$

cotangent

the reciprocal of the tangent function: on the unit circle, $\cot t = \frac{x}{y}, y \neq 0$

identities

statements that are true for all values of the input on which they are defined

period

the smallest interval P of a repeating function f such that $f(x + P) = f(x)$

secant

the reciprocal of the cosine function: on the unit circle, $\sec t = \frac{1}{x}, x \neq 0$

tangent

the quotient of the sine and cosine: on the unit circle, $\tan t = \frac{y}{x}, x \neq 0$

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1.5: Right Triangle Trigonometry

Learning Objectives

- Use right triangles to evaluate trigonometric functions.
- Find function values for $30°(\frac{\pi}{6})$,$45°(\frac{\pi}{4})$,and $60°(\frac{\pi}{3})$.
- Use equal cofunctions of complementary angles.
- Use the definitions of trigonometric functions of any angle.
- Use right-triangle trigonometry to solve applied problems.

Mt. Everest, which straddles the border between China and Nepal, is the tallest mountain in the world. Measuring its height is no easy task and, in fact, the actual measurement has been a source of controversy for hundreds of years. The measurement process involves the use of triangles and a branch of mathematics known as trigonometry. In this section, we will define a new group of functions known as trigonometric functions, and find out how they can be used to measure heights, such as those of the tallest mountains.

We have previously defined the sine and cosine of an angle in terms of the coordinates of a point on the unit circle intersected by the terminal side of the angle:

$$\cos t = x$$
$$\sin t = y$$

In this section, we will see another way to define trigonometric functions using properties of *right triangles*.

Using Right Triangles to Evaluate Trigonometric Functions

In earlier sections, we used a unit circle to define the *trigonometric functions*. In this section, we will extend those definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure $1.5.1$ shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the *x*-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

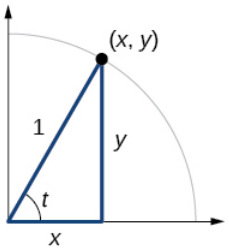


Figure 1.5.1

We know

$$\cos t = \frac{x}{1} = x \tag{1.5.1}$$

Likewise, we know

$$\sin t = \frac{y}{1} = y \tag{1.5.2}$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x,we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means "next to.") Instead of y,we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1,we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure $1.5.2$.

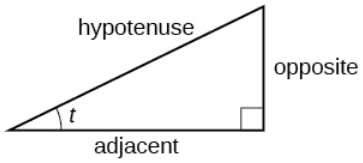


Figure 1.5.2: The sides of a right triangle in relation to angle t.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{1.5.3}$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{1.5.4}$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} \tag{1.5.5}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

how to: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example 1.5.1: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure 1.5.3, find the value of $\cos\alpha$.

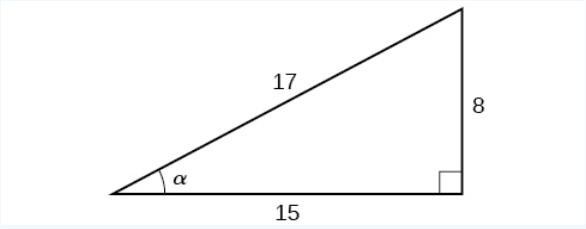


Figure 1.5.3

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so via Equation 1.5.4:

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$= \frac{15}{17}$$

Exercise 1.5.1

Given the triangle shown in Figure 1.5.4, find the value of $\sin t$.

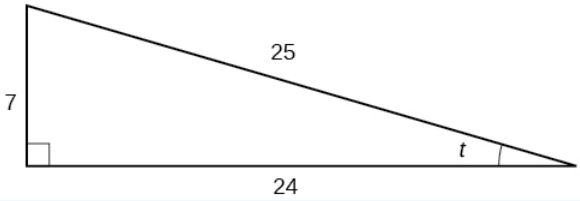


Figure 1.5.4

Answer

$\frac{7}{25}$

Relating Angles and Their Functions

When working with right triangles, the same rules apply regardless of the orientation of the triangle. In fact, we can evaluate the six trigonometric functions of either of the two acute angles in the triangle in Figure 1.5.5. The side opposite one acute angle is the side adjacent to the other acute angle, and vice versa.

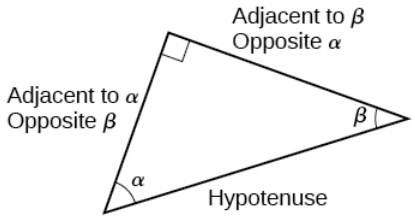


Figure 1.5.5: The side adjacent to one angle is opposite the other.

We will be asked to find all six trigonometric functions for a given angle in a triangle. Our strategy is to find the sine, cosine, and tangent of the angles first. Then, we can find the other trigonometric functions easily because we know that the reciprocal of sine is cosecant, the reciprocal of cosine is secant, and the reciprocal of tangent is cotangent.

how to: Given the side lengths of a right triangle, evaluate the six trigonometric functions of one of the acute angles

1. If needed, draw the right triangle and label the angle provided.
2. Identify the angle, the adjacent side, the side opposite the angle, and the hypotenuse of the right triangle.
3. Find the required function:
   - sine as the ratio of the opposite side to the hypotenuse
   - cosine as the ratio of the adjacent side to the hypotenuse
   - tangent as the ratio of the opposite side to the adjacent side
   - secant as the ratio of the hypotenuse to the adjacent side
   - cosecant as the ratio of the hypotenuse to the opposite side
   - cotangent as the ratio of the adjacent side to the opposite side

Example 1.5.2: Evaluating Trigonometric Functions of Angles Not in Standard Position

Using the triangle shown in Figure 1.5.6, evaluate $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$, and $\cot\alpha$.

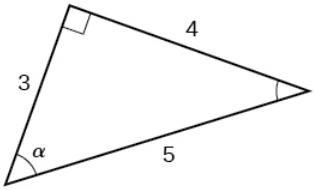


Figure 1.5.6

Solution

$$\sin\alpha = \frac{\text{opposite } \alpha}{\text{hypotenuse}} = \frac{4}{5}$$
$$\cos\alpha = \frac{\text{adjacent to } \alpha}{\text{hypotenuse}} = \frac{3}{5}$$
$$\tan\alpha = \frac{\text{opposite } \alpha}{\text{adjacent to } \alpha} = \frac{4}{3}$$
$$\sec\alpha = \frac{\text{hypotenuse}}{\text{adjacent to } \alpha} = \frac{5}{3}$$
$$\csc\alpha = \frac{\text{hypotenuse}}{\text{opposite } \alpha} = \frac{5}{4}$$
$$\cot\alpha = \frac{\text{adjacent to } \alpha}{\text{opposite } \alpha} = \frac{3}{4}$$

Exercise 1.5.2

Using the triangle shown in Figure 1.5.7, evaluate $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$, and $\cot t$.

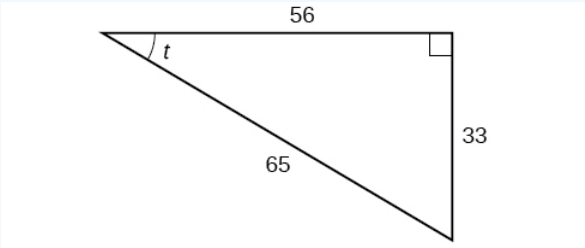


Figure 1.5.7

Answer

$$\sin t = \frac{33}{65}, \cos t = \frac{56}{65}, \tan t = \frac{33}{56},$$

$$\sec t = \frac{65}{56}, \csc t = \frac{65}{33}, \cot t = \frac{56}{33}$$

Finding Trigonometric Functions of Special Angles Using Side Lengths

We have already discussed the trigonometric functions as they relate to the **special angles** on the unit circle. Now, we can use those relationships to evaluate triangles that contain those special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of $30^\circ, 60^\circ,$ and 45°, however, remember that when dealing with right triangles, we are limited to angles between 0° and 90°.

Suppose we have a $30^\circ, 60^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45^\circ, 45^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure 1.5.8.

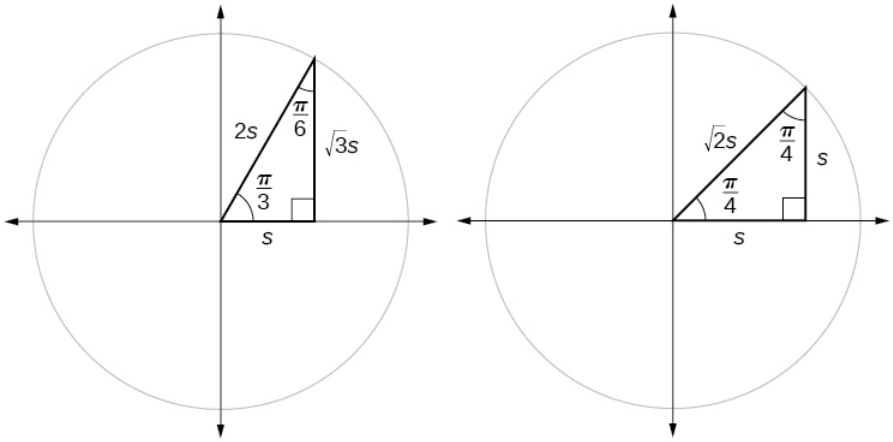


Figure 1.5.8: Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure 1.5.8 for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 1.5.3: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin(\frac{\pi}{3}) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\cos(\frac{\pi}{3}) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$
$$\tan(\frac{\pi}{3}) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$
$$\sec(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$
$$\csc(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot(\frac{\pi}{3}) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Exercise 1.5.3

Find the exact value of the trigonometric functions of $\frac{\pi}{4}$ using side lengths.

Answer

$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}, \tan(\frac{\pi}{4}) = 1,$
$\sec(\frac{\pi}{4}) = \sqrt{2}, \csc(\frac{\pi}{4}) = \sqrt{2}, \cot(\frac{\pi}{4}) = 1$

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the

sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure 1.5.9).

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

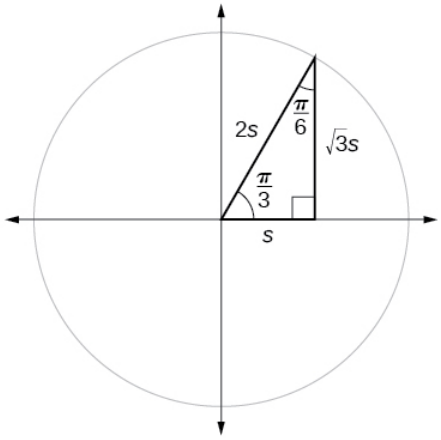
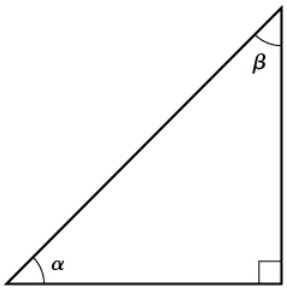


Figure 1.5.9: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from Figure 1.5.9, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure 1.5.10.



$\sin\alpha = \cos\beta$
$\sin\beta = \cos\alpha$

Figure 1.5.10: Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

COFUNCTION IDENTITIES

The **cofunction identities** in radians are listed in Table 1.5.1.

Table 1.5.1

$\cos t = \sin(\frac{\pi}{2} - t)$	$\sin t = \cos(\frac{\pi}{2} - t)$
$\tan t = \cot(\frac{\pi}{2} - t)$	$\cot t = \tan(\frac{\pi}{2} - t)$

$\sec t = \csc(\frac{\pi}{2} - t)$	$\csc t = \sec(\frac{\pi}{2} - t)$

how to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 1.5.4: Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $(\cos\frac{\pi}{2} - t)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t = \cos(\frac{\pi}{2} - t).$$

So

$$\cos(\frac{\pi}{2} - t) = \frac{5}{12}.$$

Exercise 1.5.4

If $\csc(\frac{\pi}{6}) = 2$, find $\sec(\frac{\pi}{3})$.

Solution

2

Using Trigonometric Functions

In previous examples, we evaluated the sine and cosine in triangles where we knew all three sides. But the real power of right-triangle trigonometry emerges when we look at triangles in which we know an angle but do not know all the sides.

how to: Given a right triangle, the length of one side, and the measure of one acute angle, find the remaining sides

1. For each side, select the trigonometric function that has the unknown side as either the numerator or the denominator. The known side will in turn be the denominator or the numerator.
2. Write an equation setting the function value of the known angle equal to the ratio of the corresponding sides.
3. Using the value of the trigonometric function and the known side length, solve for the missing side length.

Example 1.5.5: Finding Missing Side Lengths Using Trigonometric Ratios

Find the unknown sides of the triangle in Figure 1.5.11.

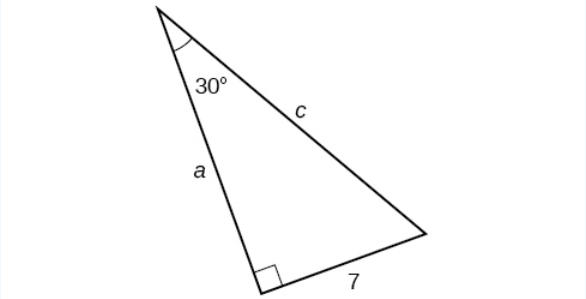


Figure 1.5.11

Solution

We know the angle and the opposite side, so we can use the tangent to find the adjacent side.

$$\tan(30^\circ) = \frac{7}{a}$$

We rearrange to solve for a.

$$a = \frac{7}{\tan(30^\circ)} \tag{1.5.6}$$

$$= 12.1 \tag{1.5.7}$$

We can use the sine to find the hypotenuse.

$$\sin(30^\circ) = \frac{7}{c}$$

Again, we rearrange to solve for c.

$$c = \frac{7}{\sin(30^\circ)} = 14$$

Exercise 1.5.5:

A right triangle has one angle of $\frac{\pi}{3}$ and a hypotenuse of 20. Find the unknown sides and angle of the triangle.

Answer

$\text{adjacent} = 10; \text{opposite} = 10\sqrt{3}$; missing angle is $\frac{\pi}{6}$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure 1.5.12

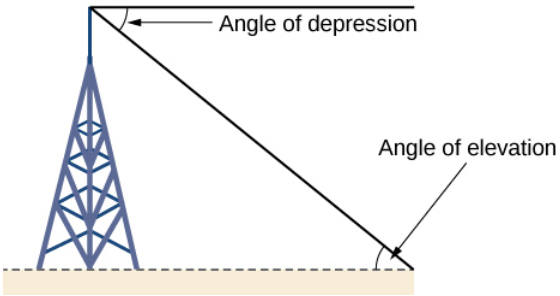


Figure 1.5.12

how to: Given a tall object, measure its height indirectly

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.

4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 1.5.6: Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° 57° between a line of sight to the top of the tree and the ground, as shown in Figure 1.5.13. Find the height of the tree.

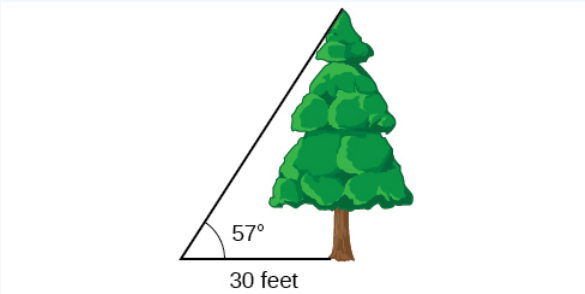


Figure 1.5.13

Solution

We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57°, letting h be the unknown height.

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(57^\circ) &= \frac{h}{30} && \text{Solve for } h. \\ h &= 30\tan(57^\circ) && \text{Multiply.} \\ h &\approx 46.2 && \text{Use a calculator.} \end{aligned} \tag{1.5.8}$$

The tree is approximately 46 feet tall.

Exercise 1.5.6:

How long a ladder is needed to reach a windowsill 50 feet above the ground if the ladder rests against the building making an angle of $\frac{5\pi}{12}$ with the ground? Round to the nearest foot.

Answer

About 52 ft

media:

Access these online resources for additional instruction and practice with right triangle trigonometry.

- Finding Trig Functions on Calculator
- Finding Trig Functions Using a Right Triangle
- Relate Trig Functions to Sides of a Right Triangle
- Determine Six Trig Functions from a Triangle
- Determine Length of Right Triangle Side

Visit this website for additional practice questions from Learningpod.

Key Equations

Cofunction Identities

$$\begin{aligned}\cos t &= \sin\left(\frac{\pi}{2} - t\right)\\ \sin t &= \cos\left(\frac{\pi}{2} - t\right)\\ \tan t &= \cot\left(\frac{\pi}{2} - t\right)\\ \cot t &= \tan\left(\frac{\pi}{2} - t\right)\\ \sec t &= \csc\left(\frac{\pi}{2} - t\right)\\ \csc t &= \sec\left(\frac{\pi}{2} - t\right)\end{aligned}$$

Key Concepts

- We can define trigonometric functions as ratios of the side lengths of a right triangle. See Example.
- The same side lengths can be used to evaluate the trigonometric functions of either acute angle in a right triangle. See Example.
- We can evaluate the trigonometric functions of special angles, knowing the side lengths of the triangles in which they occur. See Example.
- Any two complementary angles could be the two acute angles of a right triangle.
- If two angles are complementary, the cofunction identities state that the sine of one equals the cosine of the other and vice versa. See Example.
- We can use trigonometric functions of an angle to find unknown side lengths.
- Select the trigonometric function representing the ratio of the unknown side to the known side. See Example.
- Right-triangle trigonometry permits the measurement of inaccessible heights and distances.
- The unknown height or distance can be found by creating a right triangle in which the unknown height or distance is one of the sides, and another side and angle are known. See Example.

Glossary

adjacent side
in a right triangle, the side between a given angle and the right angle

angle of depression
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned lower than the observer

angle of elevation
the angle between the horizontal and the line from the object to the observer's eye, assuming the object is positioned higher than the observer

opposite side
in a right triangle, the side most distant from a given angle

hypotenuse
the side of a right triangle opposite the right angle

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1.E: Trigonometric Functions (Exercises)

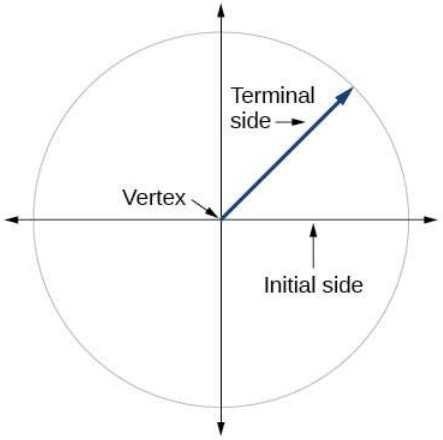
5.1: Angles

In this section, we will examine properties of angles.

Verbal

1) Draw an angle in standard position. Label the vertex, initial side, and terminal side.

Answer



2) Explain why there are an infinite number of angles that are coterminal to a certain angle.

3) State what a positive or negative angle signifies, and explain how to draw each.

Answer

Whether the angle is positive or negative determines the direction. A positive angle is drawn in the counterclockwise direction, and a negative angle is drawn in the clockwise direction.

4) How does radian measure of an angle compare to the degree measure? Include an explanation of 1 radian in your paragraph.

5) Explain the differences between linear speed and angular speed when describing motion along a circular path.

Answer

Linear speed is a measurement found by calculating distance of an arc compared to time. Angular speed is a measurement found by calculating the angle of an arc compared to time.

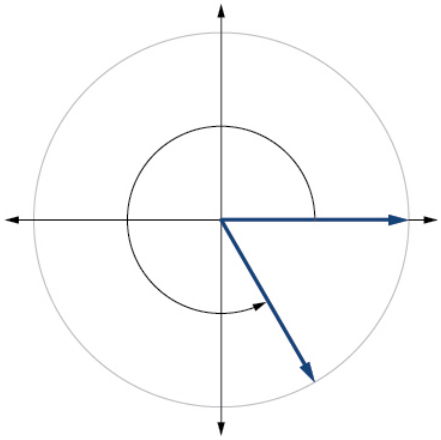
Graphical

For the exercises 6-21, draw an angle in standard position with the given measure.

6) 30°

7) 300°

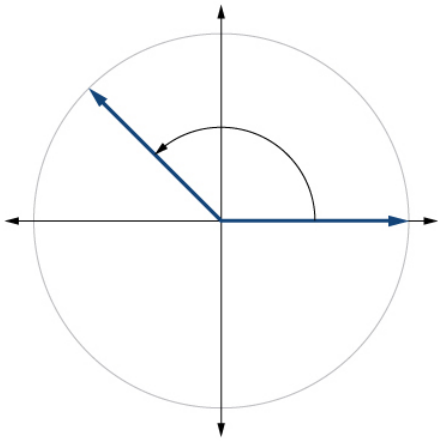
Answer



8) -80°

9) 135°

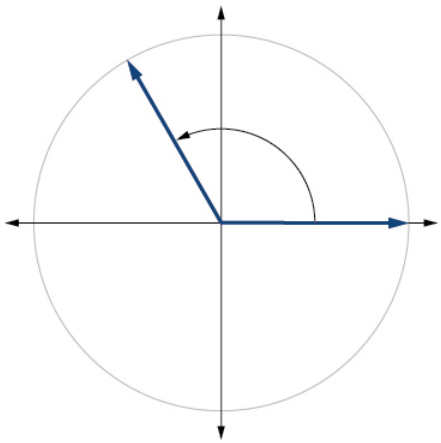
Answer



10) -150°

11) $\frac{2\pi}{3}$

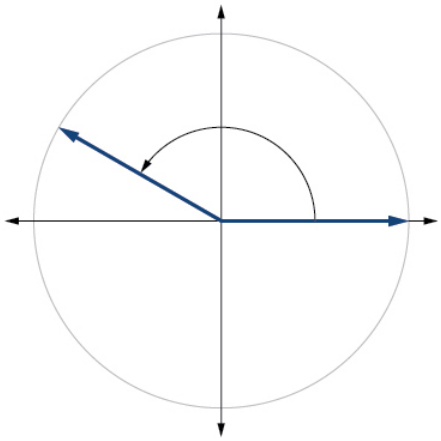
Answer



12) $\frac{7\pi}{4}$

13) $\frac{5\pi}{6}$

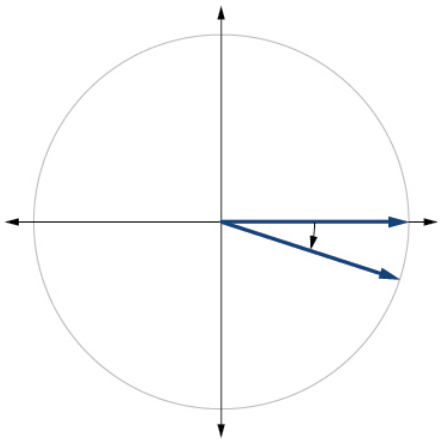
Answer



14) $\frac{\pi}{2}$

15) $-\frac{\pi}{10}$

Answer



16) 415°

17) -120°

Answer

240°

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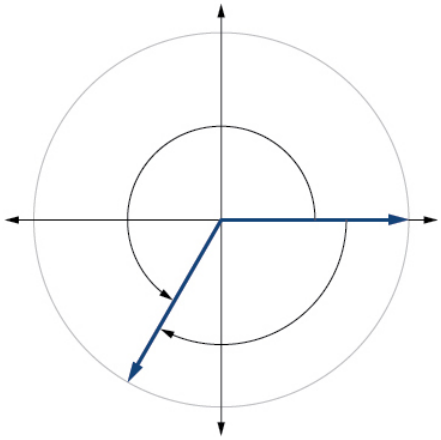
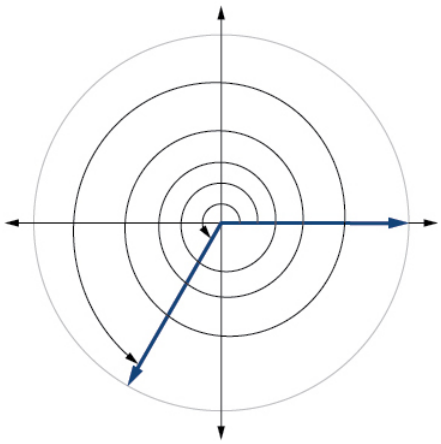


18) -315°

19) $\frac{22\pi}{3}$

Answer

$\frac{4\pi}{3}$

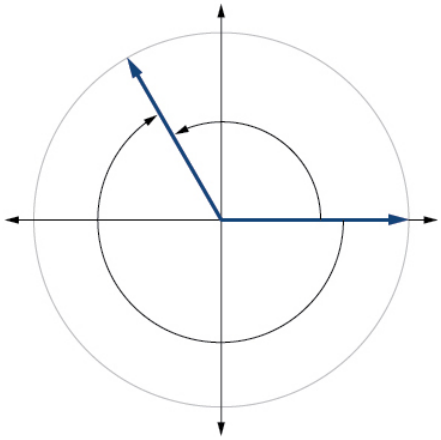


20) $-\frac{\pi}{6}$

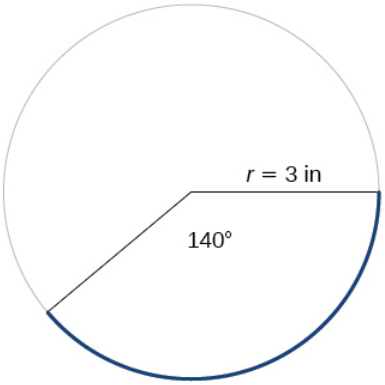
21) $-\frac{4\pi}{3}$

Answer

$\frac{2\pi}{3}$



For the exercises 22-23, refer to Figure below. Round to two decimal places.



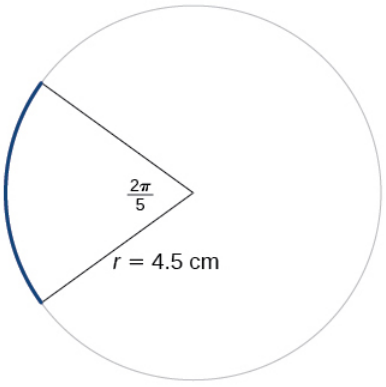
22) Find the arc length.

23) Find the area of the sector.

Answer

$\frac{27\pi}{2} \approx 11.00 \text{ in}^2$

For the exercises 24-25, refer to Figure below. Round to two decimal places.



24) Find the arc length.

25) Find the area of the sector.

Answer

$\frac{81\pi}{20} \approx 12.72 \text{ cm}^2$

Algebraic

For the exercises 26-32, convert angles in radians to degrees.

26) $\frac{3\pi}{4}$ radians

27) $\frac{\pi}{9}$ radians

Answer

20°

28) $-\frac{5\pi}{4}$ radians

29) $\frac{\pi}{3}$ radians

Answer

60°

30) $-\frac{7\pi}{3}$ radians

31) $-\frac{5\pi}{12}$ radians

Answer

-75°

32) $\frac{11\pi}{6}$ radians

For the exercises 33-39, convert angles in degrees to radians.

33) 90°

Answer

$\frac{\pi}{2}$ radians

34) 100°

35) -540°

Answer

-3π radians

36) -120°

37) 180°

Answer

π radians

38) -315°

39) 150°

Answer

$\frac{5\pi}{6}$ radians

For the exercises 40-45, use to given information to find the length of a circular arc. Round to two decimal places.

40) Find the length of the arc of a circle of radius 12 inches subtended by a central angle of $\frac{\pi}{4}$ radians.

41) Find the length of the arc of a circle of radius 5.02 miles subtended by the central angle of $\frac{\pi}{3}$.

Answer

$\frac{5.02\pi}{3} \approx 5.26$ miles

42) Find the length of the arc of a circle of diameter 14 meters subtended by the central angle of $\frac{5\pi}{6}$.

43) Find the length of the arc of a circle of radius 10 centimeters subtended by the central angle of 50°.

Answer

$\frac{25\pi}{9} \approx 8.73$ centimeters

44) Find the length of the arc of a circle of radius 5 inches subtended by the central angle of 220^{circ}.

45) Find the length of the arc of a circle of diameter 12 meters subtended by the central angle is 63^{circ}.

Answer

$\frac{21\pi}{10} \approx 6.60$ meters

For the exercises 46-49, use the given information to find the area of the sector. Round to four decimal places.

46) A sector of a circle has a central angle of 45° and a radius 6 cm.

47) A sector of a circle has a central angle of 30° and a radius of 20 cm.

Answer

$104.7198\ cm^2$

48) A sector of a circle with diameter 10 feet and an angle of $\frac{\pi}{2}$ radians.

49) A sector of a circle with radius of 0.7 inches and an angle of π radians.

Answer

$0.7697\ in^2$

For the exercises 50-53, find the angle between 0° and 360° that is coterminal to the given angle.

50) -40°

51) -110°

Answer

250°

52) 700°

53) 1400°

Answer

320°

For the exercises 54-57, find the angle between 0 and 2π in radians that is coterminal to the given angle.

54) $-\frac{\pi}{9}$

55) $\frac{10\pi}{3}$

Answer

$\frac{4\pi}{3}$

56) $\frac{13\pi}{6}$

57) $\frac{44\pi}{9}$

Answer

$\frac{8\pi}{9}$

Real-World Applications

58) A truck with 32-inch diameter wheels is traveling at 60 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

59) A bicycle with 24-inch diameter wheels is traveling at 15 mi/h. Find the angular speed of the wheels in rad/min. How many revolutions per minute do the wheels make?

Answer

$1320\text{rad}\ 210.085\text{RPM}$

60) A wheel of radius 8 inches is rotating $15^\circ/s$. What is the linear speed v, the angular speed in RPM, and the angular speed in rad/s?

61) A wheel of radius 14 inches is rotating 0.5rad/s. What is the linear speed v, the angular speed in RPM, and the angular speed in deg/s?

Answer

7 in./s, 4.77RPM, 28.65deg/s

62) A CD has diameter of 120 millimeters. When playing audio, the angular speed varies to keep the linear speed constant where the disc is being read. When reading along the outer edge of the disc, the angular speed is about 200 RPM (revolutions per minute). Find the linear speed.

63) When being burned in a writable CD-R drive, the angular speed of a CD is often much faster than when playing audio, but the angular speed still varies to keep the linear speed constant where the disc is being written. When writing along the outer edge of the disc, the angular speed of one drive is about 4800 RPM (revolutions per minute). Find the linear speed if the CD has diameter of 120 millimeters.

Answer

$1,809,557.37\ \text{mm/min} = 30.16\ \text{m/s}$

64) A person is standing on the equator of Earth (radius 3960 miles). What are his linear and angular speeds?

65) Find the distance along an arc on the surface of Earth that subtends a central angle of 5 minutes $(1 \text{ minute} = \frac{1}{60} \text{ degree})$. The radius of Earth is 3960 miles.

Answer

5.76 miles

66) Find the distance along an arc on the surface of Earth that subtends a central angle of 7 minutes $(1 \text{ minute} = \frac{1}{60} \text{ degree})$. The radius of Earth is 3960 miles.

67) Consider a clock with an hour hand and minute hand. What is the measure of the angle the minute hand traces in 20 minutes?

Answer

120°

Extensions

68) Two cities have the same longitude. The latitude of city A is 9.00 degrees north and the latitude of city B is 30.00 degree north. Assume the radius of the earth is 3960 miles. Find the distance between the two cities.

69) A city is located at 40 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

Answer

794 miles per hour

70) A city is located at 75 degrees north latitude. Assume the radius of the earth is 3960 miles and the earth rotates once every 24 hours. Find the linear speed of a person who resides in this city.

71) Find the linear speed of the moon if the average distance between the earth and moon is 239,000 miles, assuming the orbit of the moon is circular and requires about 28 days. Express answer in miles per hour.

Answer

2,234 miles per hour

72) A bicycle has wheels 28 inches in diameter. A tachometer determines that the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

73) A car travels 3 miles. Its tires make 2640 revolutions. What is the radius of a tire in inches?

Answer

11.5 inches

74) A wheel on a tractor has a 24-inch diameter. How many revolutions does the wheel make if the tractor travels 4 miles?

5.2: Unit Circle - Sine and Cosine Functions

Verbal

1) Describe the unit circle.

Answer

The unit circle is a circle of radius 1 centered at the origin.

2) What do the x- and y-coordinates of the points on the unit circle represent?

3) Discuss the difference between a coterminal angle and a reference angle.

Answer

Coterminal angles are angles that share the same terminal side. A reference angle is the size of the smallest acute angle, t, formed by the terminal side of the angle t and the horizontal axis.

4) Explain how the cosine of an angle in the second quadrant differs from the cosine of its reference angle in the unit circle.

5) Explain how the sine of an angle in the second quadrant differs from the sine of its reference angle in the unit circle.

Answer

The sine values are equal.

Algebraic

For the exercises 6-9, use the given sign of the sine and cosine functions to find the quadrant in which the terminal point determined by t lies.

6) $\sin(t) < 0$ and $\cos(t) < 0$

7) $\sin(t) > 0$ and $\cos(t) > 0$

Answer

I

8) $\sin(t) > 0$ and $\cos(t) < 0$

9) $\sin(t) < 0$ and $\cos(t) > 0$

Answer

IV

For the exercises 10-22, find the exact value of each trigonometric function.

10) $\sin\frac{\pi}{2}$

11) $\sin\frac{\pi}{3}$

Answer

$\frac{\sqrt{3}}{2}$

12) $\cos\frac{\pi}{2}$

13) $\cos\frac{\pi}{3}$

Answer

$\frac{1}{2}$

14) $\sin\frac{\pi}{4}$

15) $\cos\frac{\pi}{4}$

Answer

$\frac{\sqrt{2}}{2}$

16) $\sin\frac{\pi}{6}$

17) $\sin\pi$

Answer

0

18) $\sin \frac{3\pi}{2}$

19) $\cos \pi$

Answer

-1

20) $\cos 0$

21) $cos \frac{\pi}{6}$

Answer

$\frac{\sqrt{3}}{2}$

22) $\sin 0$

Numeric

For the exercises 23-33, state the reference angle for the given angle.

23) 240°

Answer

60°

24) -170°

25) 100°

Answer

80°

26) -315°

27) 135°

Answer

45°

28) $\frac{5\pi}{4}$

29) $\frac{2\pi}{3}$

Answer

$\frac{\pi}{3}$

30) $\frac{5\pi}{6}$

31) $-\frac{11\pi}{3}$

Answer

$\frac{\pi}{3}$

32) $\frac{-7\pi}{4}$

33) $\frac{-\pi}{8}$

Answer

$\frac{\pi}{8}$

For the exercises 34-49, find the reference angle, the quadrant of the terminal side, and the sine and cosine of each angle. If the angle is not one of the angles on the unit circle, use a calculator and round to three decimal places.

34) 225°

35) 300°

Answer

60°, Quadrant IV, $\sin(300^\circ) = -\frac{\sqrt{3}}{2}, \cos(300^\circ) = \frac{1}{2}$

36) 320°

37) 135°

Answer

45°, Quadrant II, $\sin(135^\circ) = \frac{\sqrt{2}}{2}, \cos(135^\circ) = -\frac{\sqrt{2}}{2}$

38) 210°

39) 120°

Answer

60°, Quadrant II, $\sin(120^\circ) = \frac{\sqrt{3}}{2}, \cos(120^\circ) = -\frac{1}{2}$

40) 250°

41) 150°

Answer

30°, Quadrant II, $\sin(150^\circ) = \frac{1}{2}, \cos(150^\circ) = -\frac{\sqrt{3}}{2}$

42) $\frac{5\pi}{4}$

43) $\frac{7\pi}{6}$

Answer

$\frac{\pi}{6}$, Quadrant III, $\sin\left(\frac{7\pi}{6}\right) = -\frac{1}{2}, \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

44) $\frac{5\pi}{3}$

45) $\frac{3\pi}{4}$

Answer

$\frac{\pi}{4}$, Quadrant II, $\sin\left(\frac{3\pi}{4}\right)=\frac{\sqrt{2}}{2}$, $\cos\left(\frac{4\pi}{3}\right)=-\frac{\sqrt{2}}{2}$

46) $\frac{4\pi}{3}$

47) $\frac{2\pi}{3}$

Answer

$\frac{\pi}{3}$, Quadrant II, $\sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$, $\cos\left(\frac{2\pi}{3}\right)=-\frac{1}{2}$

48) $\frac{5\pi}{6}$

49) $\frac{7\pi}{4}$

Answer

$\frac{\pi}{4}$, Quadrant IV, $\sin\left(\frac{7\pi}{4}\right)=-\frac{\sqrt{2}}{2}$, $\cos\left(\frac{7\pi}{4}\right)=\frac{\sqrt{2}}{2}$

For the exercises 50-59, find the requested value.

50) If $\cos(t)=\frac{1}{7}$ and t is in the 4th quadrant, find $\sin(t)$.

51) If $\cos(t)=\frac{2}{9}$ and t is in the 1st quadrant, find $\sin(t)$.

Answer

$\frac{\sqrt{77}}{9}$

52) If $\sin(t)=\frac{3}{8}$ and t is in the 2nd quadrant, find $\cos(t)$.

53) If $\sin(t)=-\frac{1}{4}$ and t is in the 3rd quadrant, find $\cos(t)$.

Answer

$-\frac{\sqrt{15}}{4}$

54) Find the coordinates of the point on a circle with radius 15 corresponding to an angle of 220°.

55) Find the coordinates of the point on a circle with radius 20 corresponding to an angle of 120°.

Answer

$(-10, 10\sqrt{3})$

56) Find the coordinates of the point on a circle with radius 8 corresponding to an angle of $\frac{7\pi}{4}$.

57) Find the coordinates of the point on a circle with radius 16 corresponding to an angle of $\frac{5\pi}{9}$.

Answer

$(-2.778, 15.757)$

58) State the domain of the sine and cosine functions.

59) State the range of the sine and cosine functions.

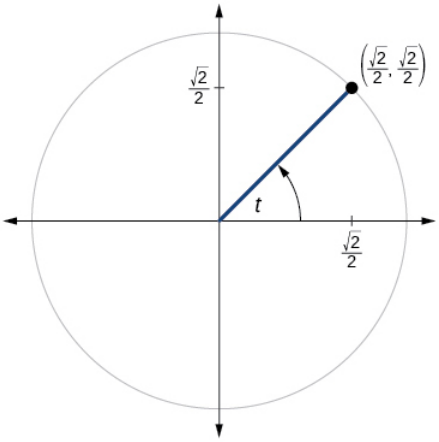
Answer

$[-1, 1]$

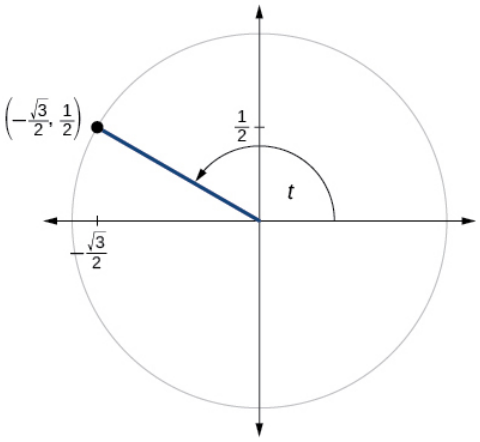
Graphical

For the exercises 60-79, use the given point on the unit circle to find the value of the sine and cosine of t.

60)



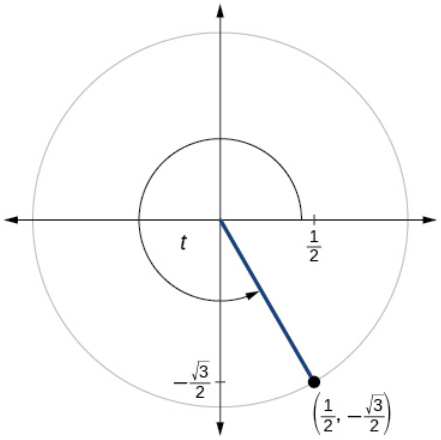
61)



Answer

$\sin t = \frac{1}{2}, \cos t = -\frac{\sqrt{3}}{2}$

62)



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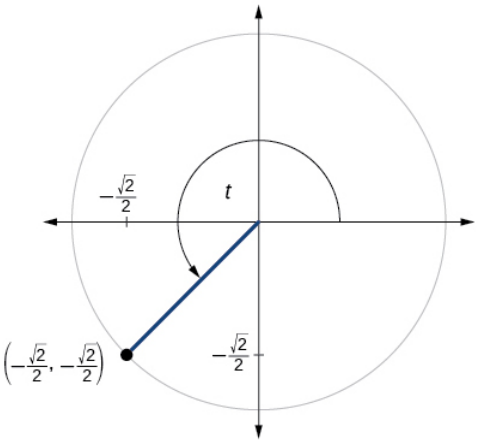
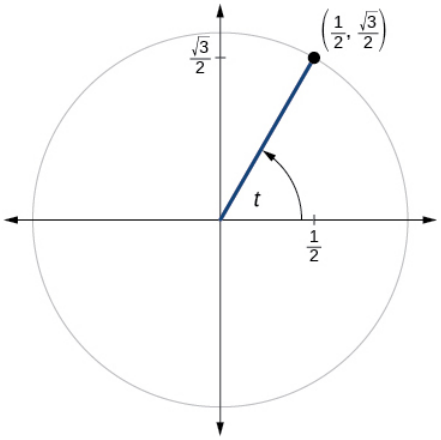


Answer

$$\sin t = -\frac{\sqrt{2}}{2}, \cos t = -\frac{\sqrt{2}}{2}$$

64)



65)

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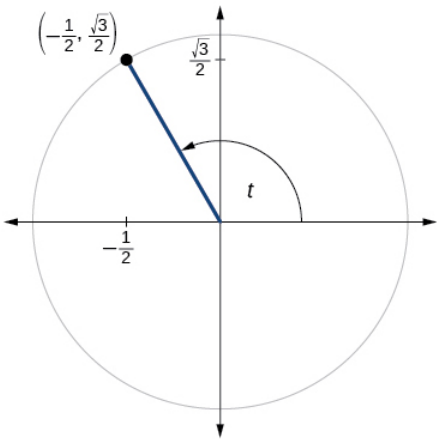
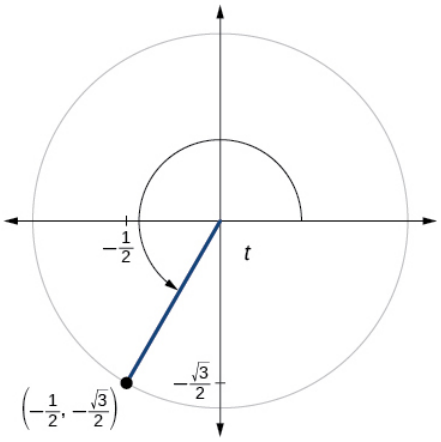


Answer

$$\sin t = \frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}$$

66)



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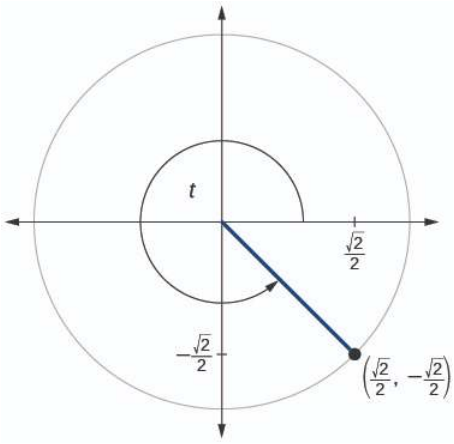

Answer

$$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}$$

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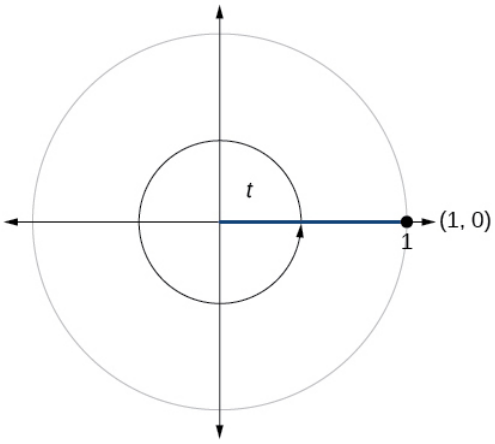
68)



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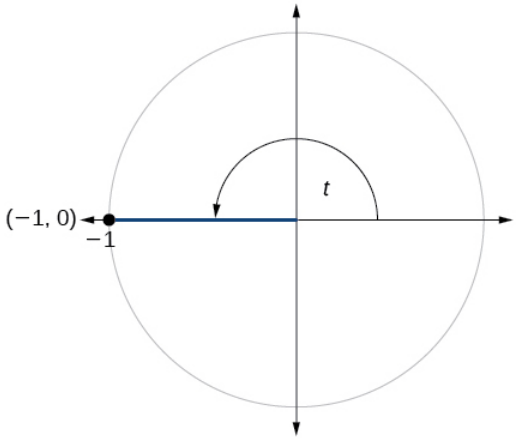
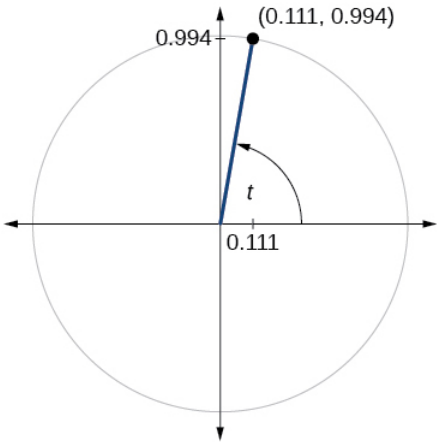


Answer

$\sin t = 0, \cos t = -1$

70)



71)

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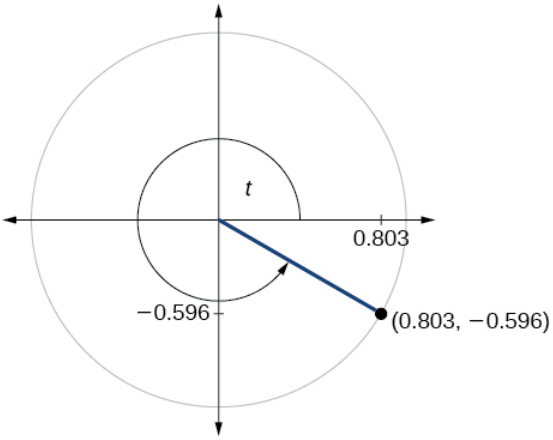
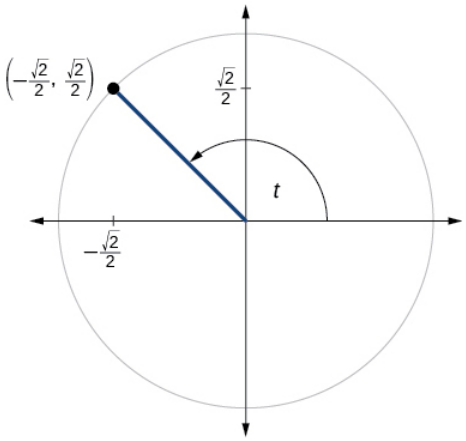


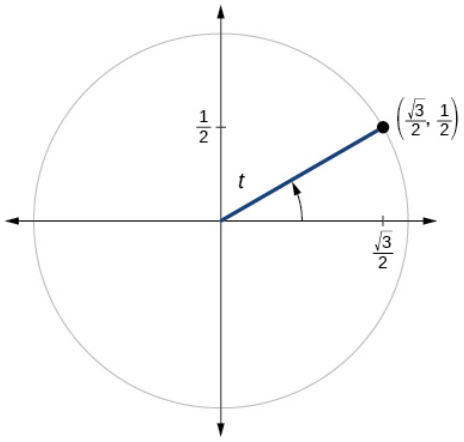
Answer

$\sin t = -0.596, \cos t = 0.803$

72)



73)



Answer

$\sin t = \frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$

74)

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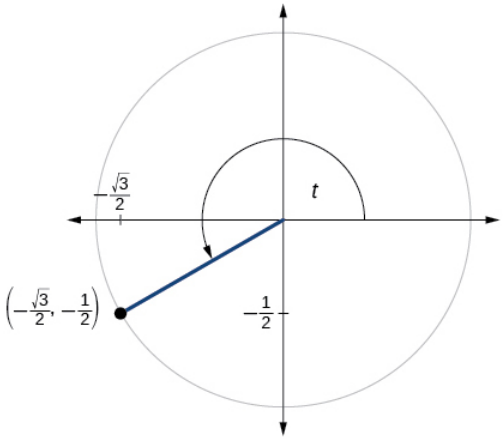
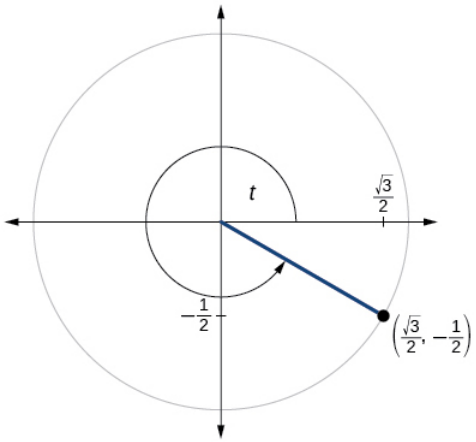


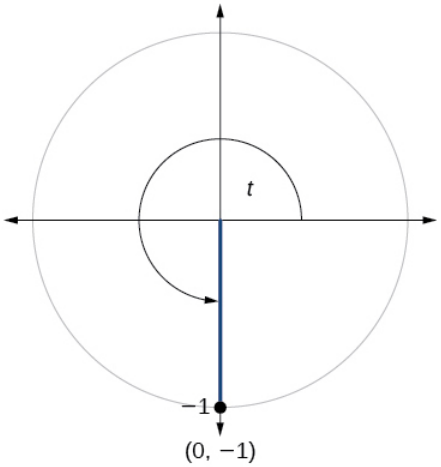
75)



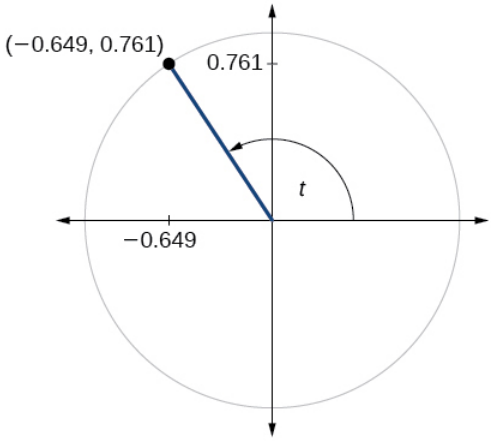
Answer

$\sin t = -\frac{1}{2}, \cos t = \frac{\sqrt{3}}{2}$

76)



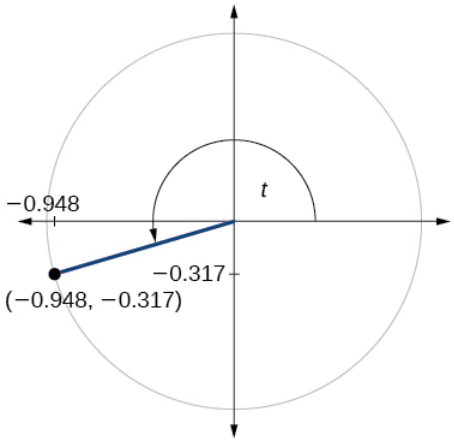
77)



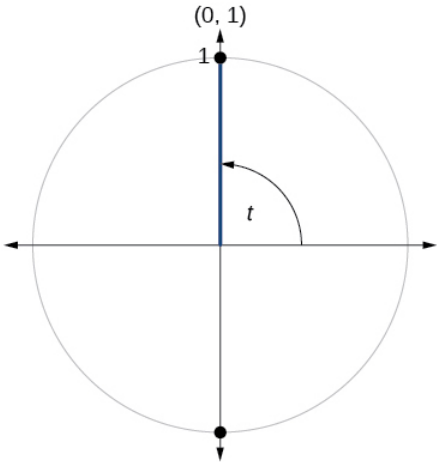
Answer

$\sin t = 0.761, \cos t = -0.649$

78)



79)



Answer

$\sin t = 1, \cos t = 0$

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Technology

For the exercises 80-89, use a graphing calculator to evaluate.

80) $\sin\frac{5\pi}{9}$

81) $cos\frac{5\pi}{9}$

Answer

-0.1736

82) $\sin\frac{\pi}{10}$

83) $\cos\frac{\pi}{10}$

Answer

0.9511

84) $\sin\frac{3\pi}{4}$

85) $\cos\frac{3\pi}{4}$

Answer

-0.7071

86) $\sin 98^\circ$

87) $\cos 98^\circ$

Answer

-0.1392

88) $\cos 310^\circ$

89) $\sin 310^\circ$

Answer

-0.7660

Extensions

For the exercises 90-99, evaluate.

90) $\sin\left(\frac{11\pi}{3}\right)\cos\left(\frac{-5\pi}{6}\right)$

91) $\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right)$

Answer

$\frac{\sqrt{2}}{4}$

92) $\sin\left(-\frac{4\pi}{3}\right)\cos\left(\frac{\pi}{2}\right)$

93) $\sin\left(\frac{-9\pi}{4}\right)\cos\left(\frac{-\pi}{6}\right)$

Answer

$-\frac{\sqrt{6}}{4}$

94) $\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{-\pi}{3}\right)$

95) $\sin\left(\frac{7\pi}{4}\right)\cos\left(\frac{-2\pi}{3}\right)$

Answer

$\frac{\sqrt{2}}{4}$

96) $\cos\left(\frac{5\pi}{6}\right)\cos\left(\frac{2\pi}{3}\right)$

97) $\cos\left(\frac{-\pi}{3}\right)\cos\left(\frac{\pi}{4}\right)$

Answer

$\frac{\sqrt{2}}{4}$

98) $\sin\left(\frac{-5\pi}{4}\right)\sin\left(\frac{11\pi}{6}\right)$

99) $\sin(\pi)\sin\left(\frac{\pi}{6}\right)$

Answer

0

Real-World Applications

For the exercises 100-104, use this scenario: A child enters a carousel that takes one minute to revolve once around. The child enters at the point $(0,1)$, that is, on the due north position. Assume the carousel revolves counter clockwise.

100) What are the coordinates of the child after 45 seconds?

101) What are the coordinates of the child after 90 seconds?

Answer

$(0,-1)$

102) What is the coordinates of the child after 125 seconds?

103) When will the child have coordinates $(0.707,-0.707)$if the ride lasts 6 minutes? (There are multiple answers.)

Answer

37.5seconds, 97.5seconds, 157.5seconds, 217.5seconds, 277.5seconds, 337.5seconds

104) When will the child have coordinates $(-0.866,-0.5)$if the ride last 6 minutes?

5.3: The Other Trigonometric Functions

Verbal

1) On an interval of $[0, 2\pi)$, can the sine and cosine values of a radian measure ever be equal? If so, where?

Answer

Yes, when the reference angle is $\frac{\pi}{4}$ and the terminal side of the angle is in quadrants I and III. Thus, at $x = \frac{\pi}{4}, \frac{5\pi}{4}$, the sine and cosine values are equal.

2) What would you estimate the cosine of π degrees to be? Explain your reasoning.

3) For any angle in quadrant II, if you knew the sine of the angle, how could you determine the cosine of the angle?

Answer

Substitute the sine of the angle in for y in the Pythagorean Theorem $x^2 + y^2 = 1$. Solve for x and take the negative solution.

4) Describe the secant function.

5) Tangent and cotangent have a period of π. What does this tell us about the output of these functions?

Answer

The outputs of tangent and cotangent will repeat every π units.

Algebraic

For the exercises 6-17, find the exact value of each expression.

6) $\tan\frac{\pi}{6}$

7) $\sec\frac{\pi}{6}$

Answer

$\frac{2\sqrt{3}}{3}$

8) $\csc\frac{\pi}{6}$

9) $\cot\frac{\pi}{6}$

Answer

$\sqrt{3}$

10) $\tan\frac{\pi}{4}$

11) $\sec\frac{\pi}{4}$

Answer

$\sqrt{2}$

12) $\csc\frac{\pi}{4}$

13) $\cot\frac{\pi}{4}$

Answer

1

14) $\tan\frac{\pi}{3}$

15) $\sec\frac{\pi}{3}$

Answer

2

16) $\csc\frac{\pi}{3}$

17) $\cot\frac{\pi}{3}$

Answer

$\frac{\sqrt{3}}{3}$

For the exercises 18-48, use reference angles to evaluate the expression.

18) $\tan\frac{5\pi}{6}$

19) $\sec\frac{7\pi}{6}$

Answer

$-\frac{2\sqrt{3}}{3}$

20) $\csc\frac{11\pi}{6}$

21) $\cot\frac{13\pi}{6}$

Answer

$\sqrt{3}$

22) $\tan\frac{7\pi}{4}$

23) $\sec\frac{3\pi}{4}$

Answer

$-\sqrt{2}$

24) $\csc\frac{5\pi}{4}$

25) $\cot\frac{11\pi}{4}$

Answer

-1

26) $\tan\frac{8\pi}{3}$

27) $\sec \frac{4\pi}{3}$

Answer

-2

28) $\csc \frac{2\pi}{3}$

29) $\cot \frac{5\pi}{3}$

Answer

$-\frac{\sqrt{3}}{3}$

30) $\tan 225^\circ$

31) $\sec 300^\circ$

Answer

2

32) $\csc 150^\circ$

33) $\cot 240^\circ$

Answer

$\frac{\sqrt{3}}{3}$

34) $\tan 330^\circ$

35) $\sec 120^\circ$

Answer

-2

36) $\csc 210^\circ$

37) $\cot 315^\circ$

Answer

-1

38) If $\sin t = \frac{3}{4}$, and t is in quadrant II, find $\cos t, \sec t, \csc t, \tan t, \cot t$.

39) If $\cos t = -\frac{1}{3}$, and t is in quadrant III, find $\sin t, \sec t, \csc t, \tan t, \cot t$.

Answer

If $\sin t = -\frac{2\sqrt{2}}{3}, \sec t = -3, \csc t = -\csc t = -\frac{3\sqrt{2}}{4}, \tan t = 2\sqrt{2}, \cot t = \frac{\sqrt{2}}{4}$

40) If $\tan t = \frac{12}{5}$, and $0 \le t < \frac{\pi}{2}$, find $\sin t, \cos t, \sec t, \csc t$, and $\cot t$.

41) If $\sin t = \frac{\sqrt{3}}{2}$ and $\cos t = \frac{1}{2}$, find $\sec t, \csc t, \tan t$, and $\cot t$.

Answer

$\sec t = 2, \csc t = \csc t = \dfrac{2\sqrt{3}}{3}, \tan t = \sqrt{3}, \cot t = \dfrac{\sqrt{3}}{3}$

42) If $\sin 40^\circ \approx 0.643 \ \cos 40^\circ \approx 0.766 \ \sec 40^\circ, \csc 40^\circ, \tan 40^\circ,$ and $\cot 40.^\circ$

43) If $\sin t = \dfrac{\sqrt{2}}{2},$ what is the $\sin(-t)$?

Answer

$-\dfrac{\sqrt{2}}{2}$

44) If $\cos t = \dfrac{1}{2},$ what is the $\cos(-t)$?

45) If $\sec t = 3.1,$ what is the $\sec(-t)$?

Answer

3.1

46) If $\csc t = 0.34,$ what is the $\csc(-t)$?

47) If $\tan t = -1.4,$ what is the $\tan(-t)$?

Answer

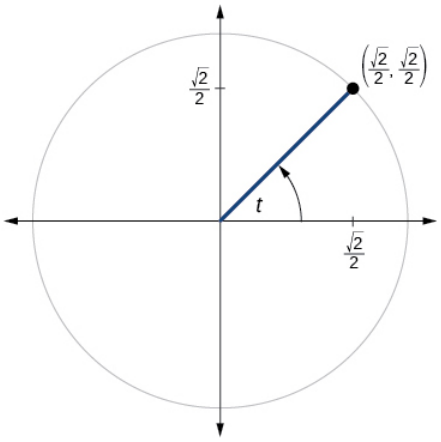
1.4

48) If $\cot t = 9.23,$ what is the $\cot(-t)$?

Graphical

For the exercises 49-51, use the angle in the unit circle to find the value of the each of the six trigonometric functions.

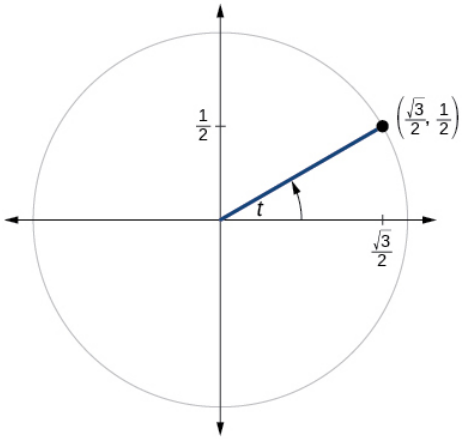
49)



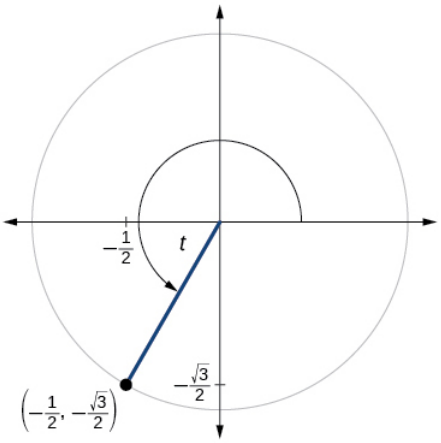
Answer

$\sin t = \dfrac{\sqrt{2}}{2}, \cos t = \dfrac{\sqrt{2}}{2}, \tan t = 1, \cot t = 1, \sec t = \sqrt{2}, \csc t = \csc t = \sqrt{2}$

50)



51)



Answer

$$\sin t = -\frac{\sqrt{3}}{2}, \cos t = -\frac{1}{2}, \tan t = \sqrt{3}, \cot t = \frac{\sqrt{3}}{3}, \sec t = -2, \csc t = -\csc t = -\frac{2\sqrt{3}}{3}$$

Technology

For the exercises 52-61, use a graphing calculator to evaluate.

52) $\csc \frac{5\pi}{9}$

53) $\cot \frac{4\pi}{7}$

Answer

-0.228

54) $\sec \frac{\pi}{10}$

55) $\tan \frac{5\pi}{8}$

Answer

-2.414

56) $\sec \frac{3\pi}{4}$

57) $\csc\frac{\pi}{4}$

Answer

1.414

58) $\tan 98^\circ$

59) $\cot 33^\circ$

Answer

1.540

60) $\cot 140^\circ$

61) $\sec 310^\circ$

Answer

1.556

Extensions

For the exercises 62-69, use identities to evaluate the expression.

62) If $\tan(t) \approx 2.7$, and $\sin(t) \approx 0.94$, find $\cos(t)$.

63) If $\tan(t) \approx 1.3$, and $\cos(t) \approx 0.61$, find $\sin(t)$.

Answer

$\sin(t) \approx 0.79$

64) If $\csc(t) \approx 3.2$, and $\csc(t) \approx 3.2$, and $\cos(t) \approx 0.95$, find $\tan(t)$.

65) If $\cot(t) \approx 0.58$, and $\cos(t) \approx 0.5$, find $\csc(t)$.

Answer

$\csc(t) \approx 1.16$

66) Determine whether the function $f(x) = 2\sin x \cos x$ is even, odd, or neither.

67) Determine whether the function $f(x) = 3\sin^2 x \cos x + \sec x$ is even, odd, or neither.

Answer

even

68) Determine whether the function $f(x) = \sin x - 2\cos^2 x$ is even, odd, or neither.

69) Determine whether the function $f(x) = \csc^2 x + \sec x$ is even, odd, or neither.

Answer

even

For the exercises 70-71, use identities to simplify the expression.

70) $\csc t \tan t$

71) $\dfrac{\sec t}{\csc t}$

Answer

$$\frac{\sin t}{\cos t} = \tan t$$

Real-World Applications

72) The amount of sunlight in a certain city can be modeled by the function $h = 15\cos\left(\frac{1}{600}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on February 10, the 42nd day of the year. State the period of the function.

73) The amount of sunlight in a certain city can be modeled by the function $h = 16\cos\left(\frac{1}{500}d\right)$, where h represents the hours of sunlight, and d is the day of the year. Use the equation to find how many hours of sunlight there are on September 24, the 267th day of the year. State the period of the function.

Answer

13.77hours, period: 1000π

74) The equation $P = 20\sin(2\pi t) + 100$ models the blood pressure, P, where t represents time in seconds.

a. Find the blood pressure after 15 seconds.
b. What are the maximum and minimum blood pressures?

75) The height of a piston, h, in inches, can be modeled by the equation $y = 2\cos x + 6$, where x represents the crank angle. Find the height of the piston when the crank angle is 55°.

Answer

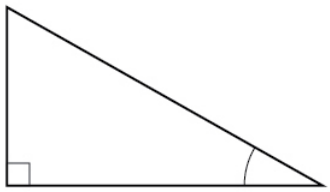
7.73inches

76) The height of a piston, h,in inches, can be modeled by the equation $y = 2\cos x + 5$, where x represents the crank angle. Find the height of the piston when the crank angle is 55°.

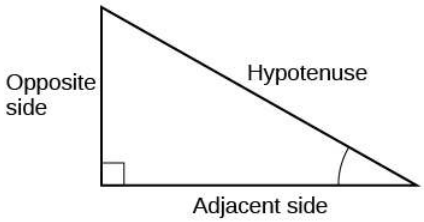
5.4: Right Triangle Trigonometry

Verbal

1) For the given right triangle, label the adjacent side, opposite side, and hypotenuse for the indicated angle.



Answer



2) When a right triangle with a hypotenuse of 1 is placed in the unit circle, which sides of the triangle correspond to the x- and y-coordinates?

3) The tangent of an angle compares which sides of the right triangle?

Answer

The tangent of an angle is the ratio of the opposite side to the adjacent side.

4) What is the relationship between the two acute angles in a right triangle?

5) Explain the cofunction identity.

Answer

For example, the sine of an angle is equal to the cosine of its complement; the cosine of an angle is equal to the sine of its complement.

Algebraic

For the exercises 6-9, use cofunctions of complementary angles.

6) $\cos(34^\circ) = \sin(\_\_^\circ)$

7) $\cos(\frac{\pi}{3}) = \sin(\_\_\_)$

Answer

$\frac{\pi}{6}$

8) $\csc(21^\circ) = \sec(\_\_\_^\circ)$

9) $\tan(\frac{\pi}{4}) = \cot(\_\_)$

Answer

$\frac{\pi}{4}$

For the exercises 10-16, find the lengths of the missing sides if side a is opposite angle A, side b is opposite angle B, and side c is the hypotenuse.

10) $\cos B = \frac{4}{5}, a = 10$

11) $\sin B = \frac{1}{2}, a = 20$

Answer

$b = \frac{20\sqrt{3}}{3}, c = \frac{40\sqrt{3}}{3}$

12) $\tan A = \frac{5}{12}, b = 6$

13) $\tan A = 100, b = 100$

Answer

$a = 10,000, c = 10,000.5$

14) $\sin B = \frac{1}{\sqrt{3}}, a = 2$

15) $a = 5, \measuredangle A = 60^\circ$

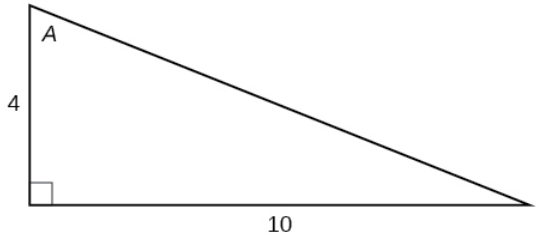
Answer

$b = \frac{5\sqrt{3}}{3}, c = \frac{10\sqrt{3}}{3}$

16) $c = 12, \measuredangle A = 45^\circ$

Graphical

For the exercises 17-22, use Figure below to evaluate each trigonometric function of angle A.



17) $\sin A$

Answer

$\frac{5\sqrt{29}}{29}$

18) $\cos A$

19) $\tan A$

Answer

$\frac{5}{2}$

20) $\csc A$

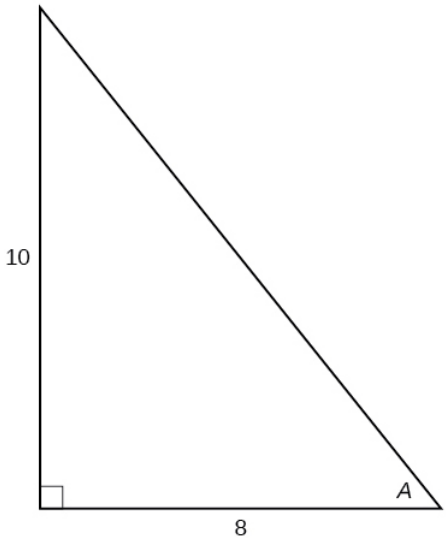
21) $\sec A$

Answer

$\frac{\sqrt{29}}{2}$

22) $\cot A$

For the exercises 23-,28 use Figure below to evaluate each trigonometric function of angle A.



23) $\sin A$

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Answer

$\frac{5\sqrt{41}}{41}$

24) $\cos A$

25) $\tan A$

Answer

$\frac{5}{4}$

26) $\csc A$

27) $\sec A$

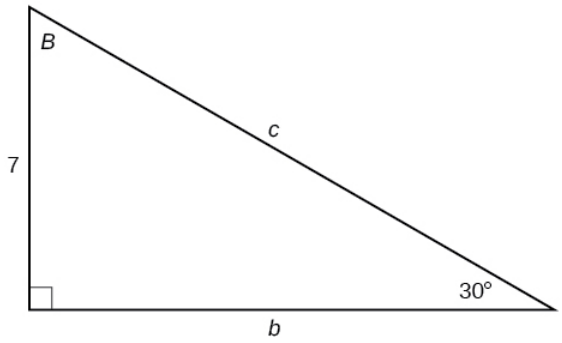
Answer

$\frac{\sqrt{41}}{4}$

28) $\cot A$

For the exercises 29-31, solve for the unknown sides of the given triangle.

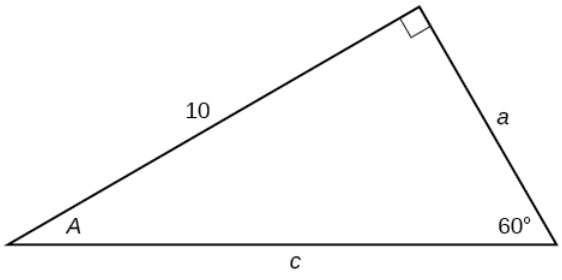
29)



Answer

$c = 14, b = 7\sqrt{3}$

30)



31)

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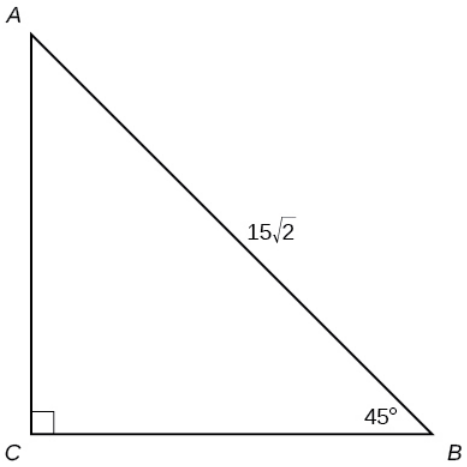


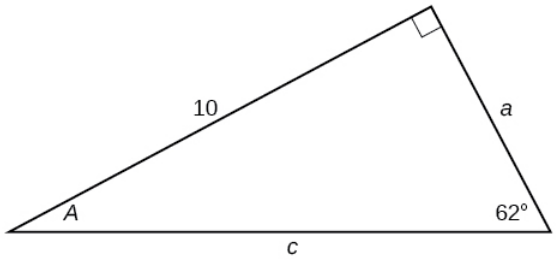
Answer

$a = 15, b = 15$

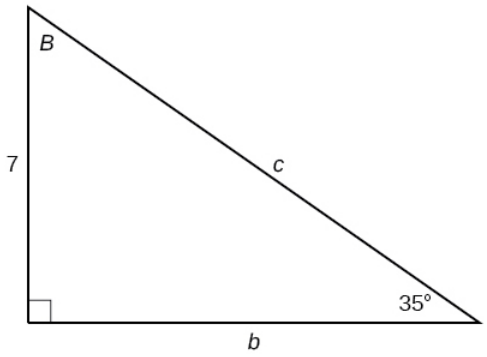
Technology

For the exercises 32-41, use a calculator to find the length of each side to four decimal places.

32)



33)



Answer

$b = 9.9970, c = 12.2041$

34)

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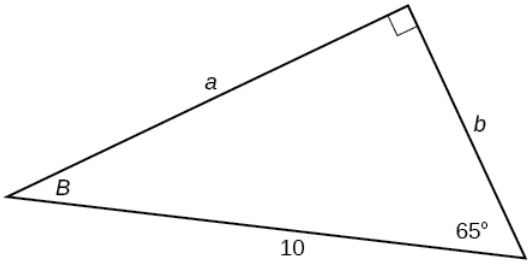

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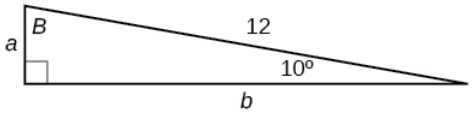
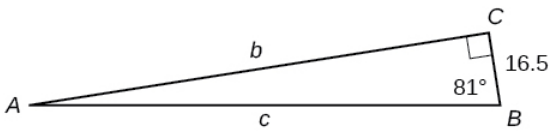


Answer

$a = 2.0838, b = 11.8177$

36)



37) $b = 15, \angle B = 15^\circ$

Answer

$a = 55.9808, c = 57.9555$

38) $c = 200, \angle B = 5^\circ$

39) $c = 50, \angle B = 21^\circ$

Answer

$a = 46.6790, b = 17.9184$

40) $a = 30, \angle A = 27^\circ$

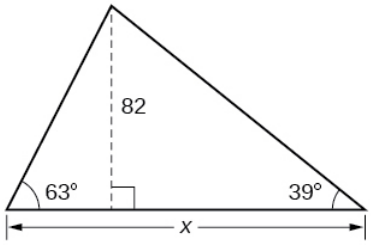
41) $b = 3.5, \angle A = 78^\circ$

Answer

$a = 16.4662, c = 16.8341$

Extensions

42) Find x.



43) Find x.

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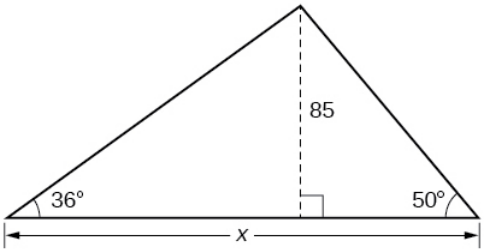
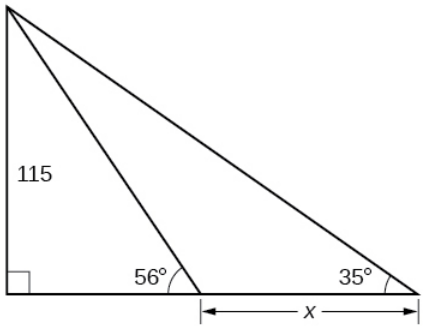


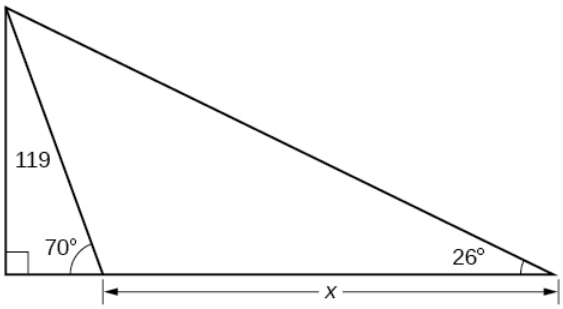
Answer

188.3159

44) Find x.



45) Find x.



Answer

200.6737

46) A radio tower is located 400 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 36°, and that the angle of depression to the bottom of the tower is 23°. How tall is the tower?

47) A radio tower is located 325 feet from a building. From a window in the building, a person determines that the angle of elevation to the top of the tower is 43°, and that the angle of depression to the bottom of the tower is 31°. How tall is the tower?

Answer

498.3471ft

48) A 200-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 15°, and that the angle of depression to the bottom of the tower is 2°. How far is the person from the monument?

49) A 400-foot tall monument is located in the distance. From a window in a building, a person determines that the angle of elevation to the top of the monument is 18°, and that the angle of depression to the bottom of the monument is 3°. How far is the person from the monument?

Answer

1060.09ft

50) There is an antenna on the top of a building. From a location 300 feet from the base of the building, the angle of elevation to the top of the building is measured to be 40°. From the same location, the angle of elevation to the top of the antenna is measured to be 43°. Find the height of the antenna.

51) There is lightning rod on the top of a building. From a location 500 feet from the base of the building, the angle of elevation to the top of the building is measured to be 36°. From the same location, the angle of elevation to the top of the lightning rod is measured to be 38°. Find the height of the lightning rod.

Answer

27.372ft

Real-World Applications

52) A 33-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

53) A 23-ft ladder leans against a building so that the angle between the ground and the ladder is 80°. How high does the ladder reach up the side of the building?

Answer

22.6506ft

54) The angle of elevation to the top of a building in New York is found to be 9 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

55) The angle of elevation to the top of a building in Seattle is found to be 2 degrees from the ground at a distance of 2 miles from the base of the building. Using this information, find the height of the building.

Answer

368.7633ft

56) Assuming that a 370-foot tall giant redwood grows vertically, if I walk a certain distance from the tree and measure the angle of elevation to the top of the tree to be 60°, how far from the base of the tree am I?

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1.R: Trigonometric Functions (Review)

5.1: Review Exercises

For the exercises 1-2, convert the angle measures to degrees.

1) $\frac{\pi}{4}$

Answer

45°

2) $-\frac{5\pi}{3}$

For the exercises 3-6, convert the angle measures to radians.

3) -210°

Answer

$-\frac{7\pi}{6}$

4) 180°

5) Find the length of an arc in a circle of radius 7 meters subtended by the central angle of 85°.

Answer

10.385 meters

6) Find the area of the sector of a circle with diameter 32 feet and an angle of $\frac{3\pi}{5}$ radians.

For the exercises 7-8, find the angle between 0° and 360° that is coterminal with the given angle.

7) 420°

Answer

60°

8) -80°

For the exercises 9-10, find the angle between 0 and 2π in radians that is coterminal with the given angle.

9) $-\frac{20\pi}{11}$

Answer

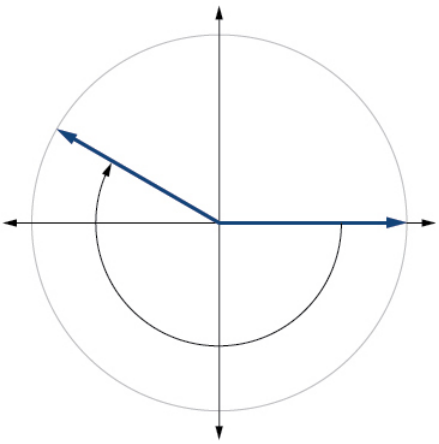
$\frac{2\pi}{11}$

10) $\frac{14\pi}{5}$

For the exercises 11-, draw the angle provided in standard position on the Cartesian plane.

11) -210°

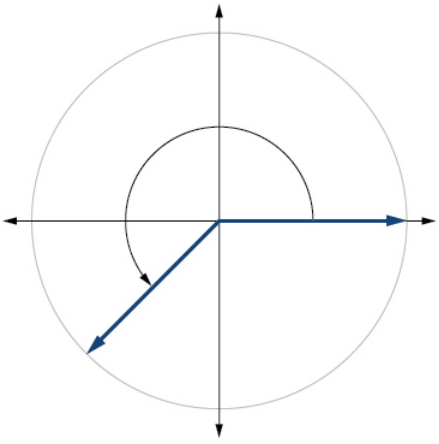
Answer



12) 75°

13) $\frac{5\pi}{4}$

Answer



14) $-\frac{\pi}{3}$

15) Find the linear speed of a point on the equator of the earth if the earth has a radius of $3,960$ miles and the earth rotates on its axis every 24 hours. Express answer in miles per hour.

Answer

1036.73 miles per hour

16) A car wheel with a diameter of 18 inches spins at the rate of 10 revolutions per second. What is the car's speed in miles per hour?

5.2: Review Exercises

1) Find the exact value of $\sin\frac{\pi}{3}$.

Answer

$\frac{\sqrt{3}}{2}$

2) Find the exact value of $\cos\frac{\pi}{4}$.

3) Find the exact value of $\cos\pi$.

Answer

-1

4) State the reference angle for 300°.

5) State the reference angle for $\frac{3\pi}{4}$.

Answer

$\frac{\pi}{4}$

6) Compute cosine of 330°.

7) Compute sine of $\frac{5\pi}{4}$.

Answer

$-\frac{\sqrt{2}}{2}$

8) State the domain of the sine and cosine functions.

9) State the range of the sine and cosine functions.

Answer

$[-1, 1]$

For the exercises 1-4, find the exact value of the given expression.

1) $\cos\frac{\pi}{6}$

2) $\tan\frac{\pi}{4}$

Answer

1

3) $\csc\frac{\pi}{3}$

4) $\sec\frac{\pi}{4}$

Answer

$\sqrt{2}$

For the exercises 4-12, use reference angles to evaluate the given expression.

5) $\sec\frac{11\pi}{3}$

6) $\sec 315^\circ$

Answer

$\sqrt{2}$

7) If $\sec(t) = -2.5$, what is the $\sec(-t)$?

8) If $\tan(t) = -0.6$, what is the $\tan(-t)$?

Answer

0.6

9) If $\tan(t) = \frac{1}{3}$, find $\tan(t - \pi)$.

10) If $\cos(t) = \frac{\sqrt{2}}{2}$, find $\sin(t + 2\pi)$.

Answer

$\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$

11) Which trigonometric functions are even?

12) Which trigonometric functions are odd?

Answer

sine, cosecant, tangent, cotangent

5.4: Review Exercises

For the exercises 1-5, use side lengths to evaluate.

1) $\cos\frac{\pi}{4}$

2) $\cot\frac{\pi}{3}$

Answer

$\frac{\sqrt{3}}{3}$

3) $\tan\frac{\pi}{6}$

4) $\cos(\frac{\pi}{2}) = \sin(\_\_^\circ)$

Answer

0

5) $\csc(18^\circ) = \sec(\_\_^\circ)$

For the exercises 6-7, use the given information to find the lengths of the other two sides of the right triangle.

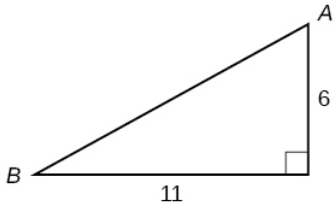
6) $\cos B = \frac{3}{5}, a = 6$

Answer

$b = 8, c = 10$

7) $\tan A = \frac{5}{9}, b = 6$

For the exercises 8-9, use Figure below to evaluate each trigonometric function.



8) $\sin A$

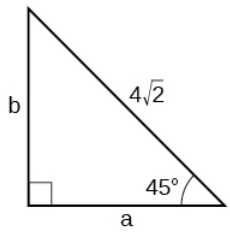
Answer

$$\frac{11\sqrt{157}}{157}$$

9) $\tan B$

For the exercises 10-11, solve for the unknown sides of the given triangle.

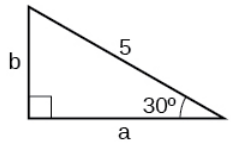
10)



Answer

$a = 4, b = 4$

11)



12) A 15-ft ladder leans against a building so that the angle between the ground and the ladder is 70°. How high does the ladder reach up the side of the building?

Answer

$14.0954\,\text{ft}$

13) The angle of elevation to the top of a building in Baltimore is found to be 4 degrees from the ground at a distance of 1 mile from the base of the building. Using this information, find the height of the building.

Practice Test

1) Convert $\frac{5\pi}{6}$ radians to degrees.

Answer

150°

2) Convert -620° to radians.

3) Find the length of a circular arc with a radius 12 centimeters subtended by the central angle of 30°.

Answer

6.283 centimeters

4) Find the area of the sector with radius of 8 feet and an angle of $\frac{5\pi}{4}$ radians.

5) Find the angle between 0° and 360° that is coterminal with 375°.

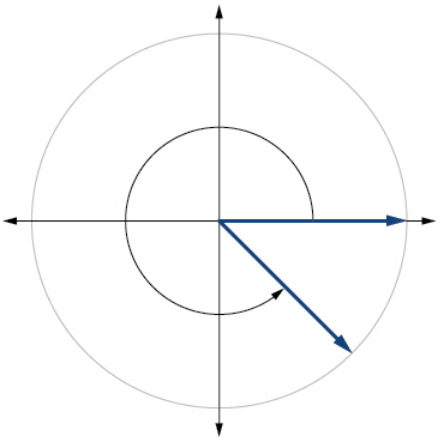
Answer

15°

6) Find the angle between 0 and 2π in radians that is coterminal with $-\frac{4\pi}{7}$.

7) Draw the angle 315° in standard position on the Cartesian plane.

Answer



8) Draw the angle $-\frac{\pi}{6}$ in standard position on the Cartesian plane.

9) A carnival has a Ferris wheel with a diameter of 80 feet. The time for the Ferris wheel to make one revolution is 75 seconds. What is the linear speed in feet per second of a point on the Ferris wheel? What is the angular speed in radians per second?

Answer

3.351 feet per second, $\frac{2\pi}{75}$ radians per second

10) Find the exact value of $\sin\frac{\pi}{6}$.

11) Compute sine of 240°.

Answer

$$-\frac{\sqrt{3}}{2}$$

12) State the domain of the sine and cosine functions.

13) State the range of the sine and cosine functions.

Answer

$$[-1, 1]$$

14) Find the exact value of $\cot\frac{\pi}{4}$.

15) Find the exact value of $\tan\frac{\pi}{3}$.

Answer

$$\sqrt{3}$$

16) Use reference angles to evaluate $\csc\frac{7\pi}{4}$.

17) Use reference angles to evaluate $\tan 210^\circ$.

Answer

$$\frac{\sqrt{3}}{3}$$

18) If $\csc t = 0.68$, what is the $\csc(-t)$?

19) If $\cos t = \frac{\sqrt{3}}{2}$, find $\cos(t - 2\pi)$.

Answer

$$\frac{\sqrt{3}}{2}$$

20) Which trigonometric functions are even?

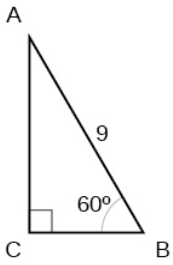
21) Find the missing angle: $\cos\left(\frac{\pi}{6}\right) = \sin(\ )$

Answer

$$\frac{\pi}{3}$$

22) Find the missing sides of the triangle ABC : $\sin B = \frac{3}{4}, c = 12$

23) Find the missing sides of the triangle.



Answer

$$a = \frac{9}{2}, b = \frac{9\sqrt{3}}{2}$$

24) The angle of elevation to the top of a building in Chicago is found to be 9 degrees from the ground at a distance of 2000 feet from the base of the building. Using this information, find the height of the building.

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CHAPTER OVERVIEW

2: Periodic Functions

In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

2.1: Prelude to Periodic Functions
2.2: Graphs of the Sine and Cosine Functions
2.3: Graphs of the Other Trigonometric Functions
2.4: Inverse Trigonometric Functions
2.E: Periodic Functions (Exercises)
2.R: Periodic Functions (Review)

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2.1: Prelude to Periodic Functions

Each day, the sun rises in an easterly direction, approaches some maximum height relative to the celestial equator, and sets in a westerly direction. The celestial equator is an imaginary line that divides the visible universe into two halves in much the same way Earth's equator is an imaginary line that divides the planet into two halves. The exact path the sun appears to follow depends on the exact location on Earth, but each location observes a predictable pattern over time.

Figure 2.1.1: (credit: "Maxxer_", Flickr)

The pattern of the sun's motion throughout the course of a year is a periodic function. Creating a visual representation of a periodic function in the form of a graph can help us analyze the properties of the function. In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

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2.2: Graphs of the Sine and Cosine Functions

Learning Objectives

- Graph variations of $y = \sin(x)$ and $y = \cos(x)$.
- Use phase shifts of sine and cosine curves.

White light, such as the light from the sun, is not actually white at all. Instead, it is a composition of all the colors of the rainbow in the form of waves. The individual colors can be seen only when white light passes through an optical prism that separates the waves according to their wavelengths to form a rainbow.

Figure 2.2.1: Light can be separated into colors because of its wavelike properties. (credit: "wonderferret"/ Flickr)

Light waves can be represented graphically by the sine function. In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions.

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x- and y-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. Table 2.2.1 lists some of the values for the sine function on a unit circle.

Table 2.2.1

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Plotting the points from the table and continuing along the x-axis gives the shape of the sine function. See Figure 2.2.2.

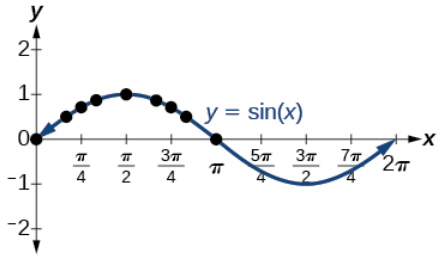

Figure 2.2.2: The sine function

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in

quadrants III and IV on the unit circle. See Figure 2.2.3.

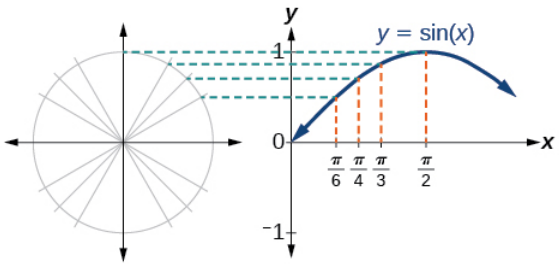

Figure 2.2.3: Plotting values of the sine function

Now let's take a similar look at the **cosine function**. Again, we can create a table of values and use them to sketch a graph. Table 2.2.2 lists some of the values for the cosine function on a unit circle.

Table 2.2.2

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

As with the sine function, we can plots points to create a graph of the cosine function as in Figure 2.2.4.

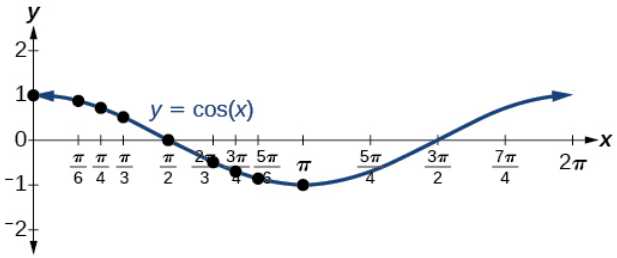

Figure 2.2.4: The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A **periodic function** is a function for which a specific **horizontal shift**, P, results in a function equal to the original function: $f(x+P) = f(x)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P > 0$ the **period** of the function. Figure 2.2.5 shows several periods of the sine and cosine functions.

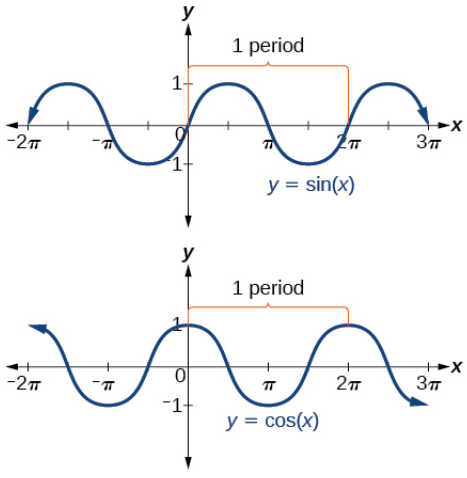

Figure 2.2.5

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in Figure 2.2.6, the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin\ x$. Now we can clearly see this property from the graph.

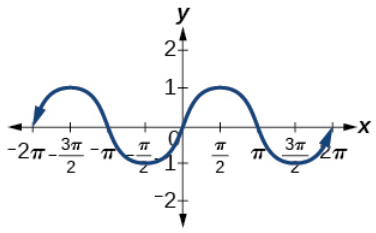

Figure 2.2.6: Odd symmetry of the sine function

Figure 2.2.7 shows that the cosine function is symmetric about the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos\ x$.

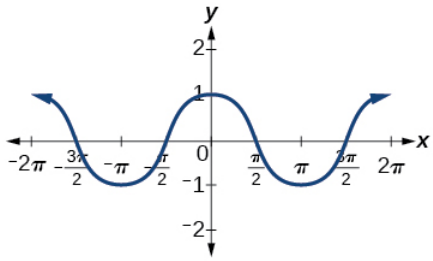

Figure 2.2.7: Even symmetry of the cosine function

CHARACTERISTICS OF SINE AND COSINE FUNCTIONS

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos\ x$ is symmetric about they- y-axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or **cosine function** is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D \tag{2.2.1}$$

and

$$y = A\cos(Bx - C) + D \tag{2.2.2}$$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \dfrac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π,which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π

and the graph is compressed. If $f(x) = \sin\left(\frac{x}{2}\right)$, then $B = \frac{1}{2}$, so the period is 4π and the graph is stretched. Notice in Figure 2.2.8 how the period is indirectly related to $|B|$.

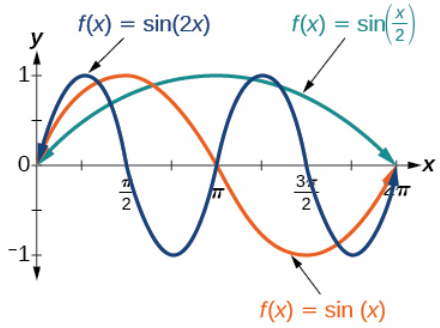

Figure 2.2.8

PERIOD OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

- $y = A\sin(Bx)$
- $y = A\cos(Bx)$

The period is $\frac{2\pi}{|B|}$.

Example 2.2.1: Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x) = \sin\left(\frac{\pi}{6}x\right)$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx)$.

In the given equation, $B = \frac{\pi}{6}$, so the period will be

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{6}} \\ &= 2\pi \cdot \frac{6}{\pi} \\ &= 12 \end{aligned}$$

Exercise 2.2.1

Determine the period of the function $g(x) = \cos(\frac{x}{3})$.

Answer

6π

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical **midline** of the

graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the x-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4sinx$ is twice the amplitude of

$$f(x) = 2\sin x$$

If $|A| < 1$, the function is compressed. Figure $2.2.9$ compares several sine functions with different amplitudes.

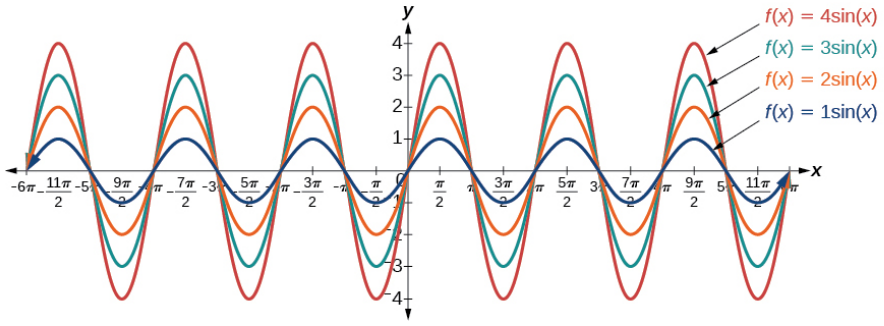

Figure 2.2.9

AMPLITUDE OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A\sin(Bx) \text{ and } y = A\cos(Bx) \tag{2.2.3}$$

The amplitude is A, and the vertical height from the midline is $|A|$. In addition, notice in the example that

$$|A| = amplitude = \frac{1}{2}\,|\,maximum - minimum| \tag{2.2.4}$$

Example 2.2.2: Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Solution

Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis

The negative value of A results in a reflection across the x-axis of the sine function, as shown in Figure $2.2.10$

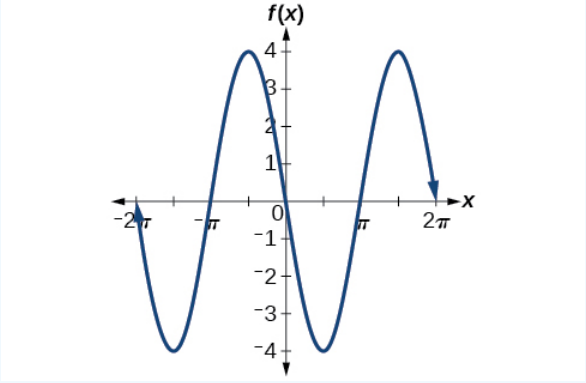

Figure 2.2.10

Exercise 2.2.2

What is the amplitude of the sinusoidal function $f(x) = \frac{1}{2}\sin(x)$? Is the function stretched or compressed vertically?

Answer

$\frac{1}{2}$ compressed

Analyzing Graphs of Variations of $y = \sin\ x$ and $y = \cos\ x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y = A\sin(Bx - C) + D \qquad \text{and} \qquad y = A\cos(Bx - C) + D \tag{2.2.5}$$

or

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \qquad \text{and} \qquad y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D \tag{2.2.6}$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. Figure 2.2.11 shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is more than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

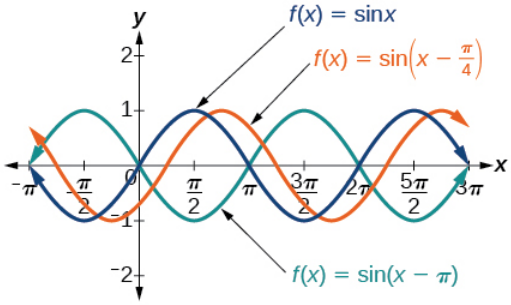

Figure 2.2.11

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See Figure 2.2.12 The function $y = \cos(x) + D$ has its midline at $y = D$.

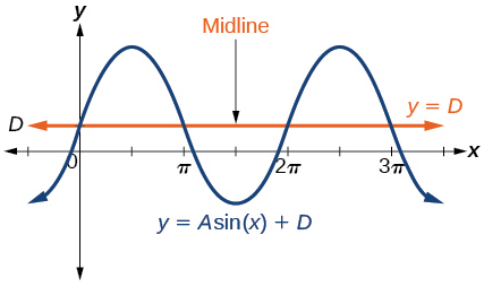

Figure 2.2.12

Any value of D other than zero shifts the graph up or down. Figure 2.2.13compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

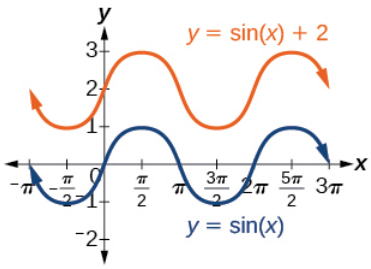

Figure 2.2.13

VARIATIONS OF SINE AND COSINE FUNCTIONS

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{B}$ is the **phase shift** and D is the **vertical shift.**

Example 2.2.3: Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

In the given equation, notice that $B = 1$ and $C = -\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned} \frac{C}{B} &= -\frac{\frac{\pi}{6}}{1} \\ &= -\frac{\pi}{6} \end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f(x) = \sin(x + \frac{\pi}{6}) - 2$ can be rewritten as $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2$. If the value of C is negative, the shift is to the left.

Exercise 2.2.3

Determine the direction and magnitude of the phase shift for $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$.

Answer

$\frac{\pi}{2}$; right

Example 2.2.4: Identifying the Vertical Shift of a Function

Determine the direction and magnitude of the vertical shift for $f(x) = \cos(x) - 3$.

Solution

Let's begin by comparing the equation to the general form $y = A\cos(Bx - C) + D$.

In the given equation, $D = -3$ so the shift is 3 units downward.

Exercise 2.2.4

Determine the direction and magnitude of the vertical shift for $f(x) = 3\sin(x) + 2$.

Answer

2 units up

How to: Given a sinusoidal function in the form $f(x) = A\sin(Bx - C) + D$,identify the midline, amplitude, period, and phase shift

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y = D$.

Example 2.2.5: Identifying the Variations of a Sinusoidal Function from an Equation

Determine the midline, amplitude, period, and phase shift of the function $y = 3\sin(2x) + 1$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

$A = 3$, so the amplitude is $|A| = 3$.

Next, $B = 2$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{2} = \pi$.

There is no added constant inside the parentheses, so $C = 0$ and the phase shift is $\dfrac{C}{B} = \dfrac{0}{2} = 0$.

Finally, $D = 1$, so the midline is $y = 1$.

Analysis

Inspecting the graph, we can determine that the period is π, the midline is $y = 1$, and the amplitude is 3. See Figure 2.2.14

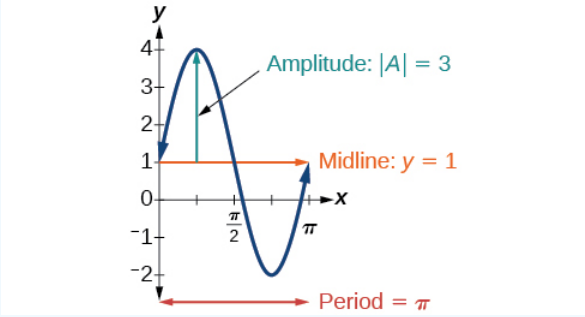

Figure 2.2.14

Exercise 2.2.5

Determine the midline, amplitude, period, and phase shift of the function $y = \frac{1}{2}\cos\left(\frac{x}{3} - \frac{\pi}{3}\right)$.

Answer

midline: $y = 0$; amplitude: $|A| = \frac{1}{2}$; period: $P = \frac{2\pi}{|B|} = 6\pi$; phase shift: $\frac{C}{B} = \pi$

Example 2.2.6: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the formula for the cosine function in Figure 2.2.15.

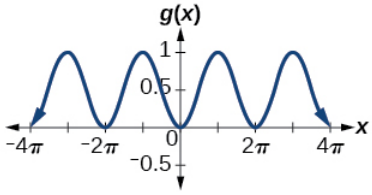

Figure 2.2.15

Solution

To determine the equation, we need to identify each value in the general form of a sinusoidal function.

$$y = A\sin(Bx - C) + D$$

$$y = A\cos(Bx - C) + D$$

The graph could represent either a sine or a **cosine function** that is shifted and/or reflected. When $x = 0$, the graph has an extreme point, $(0, 0)$. Since the cosine function has an extreme point for $x = 0$, let us write our equation in terms of a cosine function.

Let's start with the midline. We can see that the graph rises and falls an equal distance above and below $y = 0.5$. This value, which is the midline, is D in the equation, so $D = 0.5$.

The greatest distance above and below the midline is the amplitude. The maxima are 0.5 units above the midline and the minima are 0.5 units below the midline. So $|A| = 0.5$. Another way we could have determined the amplitude is by recognizing that the difference between the height of local maxima and minima is 1, so $|A| = \frac{1}{2} = 0.5$. Also, the graph is reflected about the x-axis so that $A = -0.5$.

The graph is not horizontally stretched or compressed, so $B = 1$; and the graph is not shifted horizontally, so $C = 0$.

Putting this all together,

$$g(x) = -0.5\cos(x) + 0.5$$

Exercise 2.2.6

Determine the formula for the sine function in Figure 2.2.16.

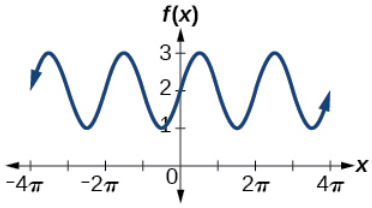

Figure 2.2.16

Answer

$f(x) = \sin(x) + 2$

Example 2.2.7: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in Figure 2.2.17.

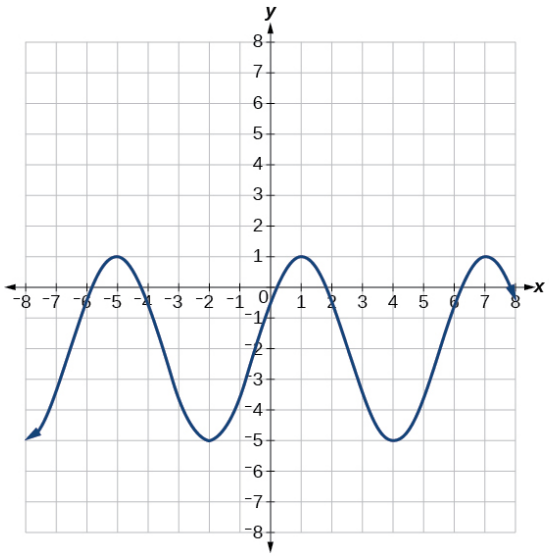

Figure 2.2.17

Solution

With the highest value at 1 and the lowest value at -5, the midline will be halfway between at -2. So $D = -2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A| = 3$.

The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$,or from the distance between the lowest points. Therefore, $P = \dfrac{2\pi}{|B|} = 6$. Using the positive value for B,we find that

$$\begin{aligned} B &= \frac{2\pi}{P} \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

So far, our equation is either $y = 3\sin\left(\dfrac{\pi}{3}x - C\right) - 2$ or $y = 3\cos\left(\dfrac{\pi}{3}x - C\right) - 2$ .For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes

$$\begin{aligned} y &= 3\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 2 \qquad \text{or} \\ y &= -3\cos\left(\frac{\pi}{3}x + \frac{2\pi}{3}\right) - 2 \end{aligned}$$

Again, these functions are equivalent, so both yield the same graph.

? Exercise 2.2.7

Write a formula for the function graphed in Figure 2.2.18.

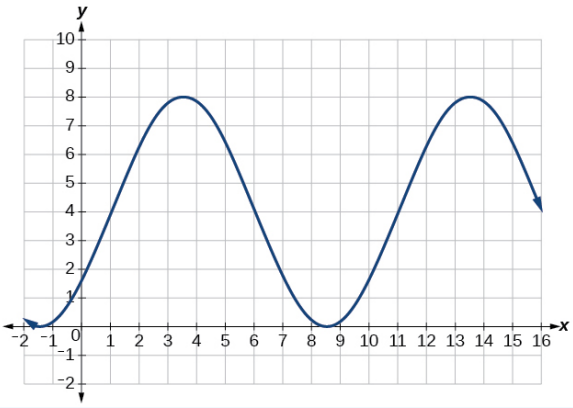

Figure 2.2.18

Answer

two possibilities: $y = 4\sin\left(\frac{\pi}{5}x - \frac{\pi}{5}\right) + 4$ or $y = -4\sin\left(\frac{\pi}{5}x + \frac{4\pi}{5}\right) + 4$

Graphing Variations of $y = \sin\ x$ and $y = \cos\ x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

Given the function $y = A\sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \frac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \frac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the *x*-axis at $x = \frac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \frac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the *x*-axis at $x = \frac{\pi}{2|B|}$.

Example 2.2.8: Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\frac{\pi x}{2}\right)$.

Solution

Let's begin by comparing the equation to the form $y = A\sin(Bx)$.

- *Step 1.* We can see from the equation that $A = -2$, so the amplitude is 2.

  $|A| = 2$

- *Step 2.* The equation shows that $B = \frac{\pi}{2}$, so the period is

$$P = \frac{2\pi}{\frac{\pi}{2}}$$
$$= 2\pi \cdot \frac{2}{\pi}$$
$$= 4$$

- *Step 3.* Because A is negative, the graph descends as we move to the right of the origin.
- *Step 4.* The x-intercepts are at the beginning of one period, $x = 0$, the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline, at $x = 1$, and a local maximum will occur at 2 units above the midline, at $x = 3$. Figure 2.2.19 shows the graph of the function.

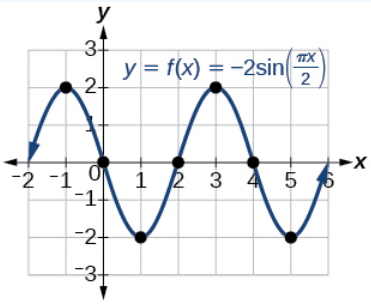

Figure 2.2.19

Exercise 2.2.8

Sketch a graph of $g(x) = -0.8\cos(2x)$. Determine the midline, amplitude, period, and phase shift.

Answer

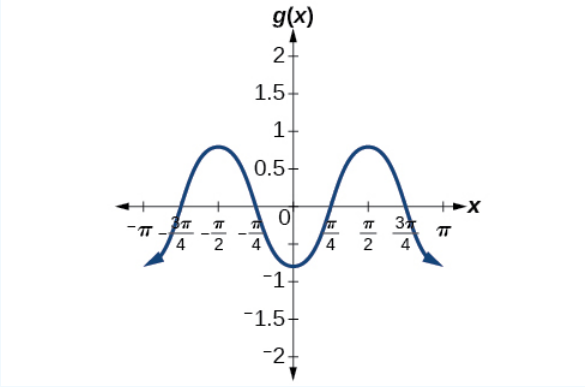

Figure 2.2.20

midline: $y = 0$; amplitude: $|A| = 0.8$; period: $P = \frac{2\pi}{|B|} = \pi$; phase shift: $\frac{C}{B} = 0$ or none

How to: Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \frac{2\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.

5. Draw the graph of $f(x) = A\sin(Bx)$ shifted to the right or left by $\frac{C}{B}$ and up or down by D.

Example 2.2.9: Graphing a Transformed Sinusoid

Sketch a graph of $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.

Solution

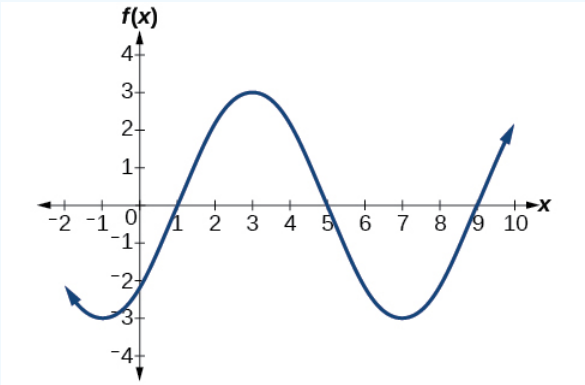

Figure 2.2.21: A horizontally compressed, vertically stretched, and horizontally shifted sinusoid

- *Step 1.* The function is already written in general form: $f(x) = 3\sin\left(\frac{\pi}{4}x - \frac{\pi}{4}\right)$.This graph will have the shape of a sine function, starting at the midline and increasing to the right.
- *Step 2.* $|A| = |3| = 3$. The amplitude is 3.
- *Step 3.* Since $|B| = |\frac{\pi}{4}| = \frac{\pi}{4}$, we determine the period as follows.

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{4}} \\ &= 2\pi \cdot \frac{4}{\pi} \\ &= 8 \end{aligned}$$

The period is 8.

- *Step 4.* Since $C = \frac{\pi}{4}$, the phase shift is

$$\frac{C}{B} = \frac{\frac{\pi}{4}}{\frac{\pi}{4}} = 1 \tag{2.2.7}$$

.

The phase shift is 1 unit.

- *Step 5.* Figure 2.2.21shows the graph of the function.

Exercise 2.2.9

Draw a graph of $g(x) = -2\cos\left(\frac{\pi}{3}x + \frac{\pi}{6}\right)$. Determine the midline, amplitude, period, and phase shift.

Answer

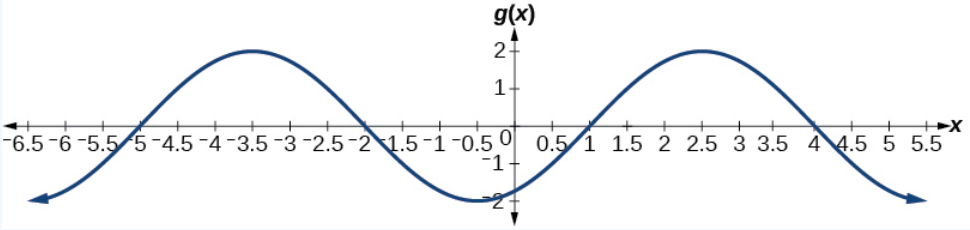

Figure 2.2.22

midline: $y = 0$; amplitude: $|A| = 2$; period: $P = \dfrac{2\pi}{|B|} = 6$; phase shift: $\dfrac{C}{B} = -\dfrac{1}{2}$

Example 2.2.10: Identifying the Properties of a Sinusoidal Function

Given $y = -2cos\left(\dfrac{\pi}{2}x + \pi\right) + 3$, determine the amplitude, period, phase shift, and horizontal shift. Then graph the function.

Solution

Begin by comparing the equation to the general form.

$$y = A\cos(Bx - C) + D$$

- *Step 1.* The function is already written in general form.
- *Step 2.* Since $A = -2$, the amplitude is $|A| = 2$.
- *Step 3.* $|B| = \dfrac{\pi}{2}$, so the period is $P = \dfrac{2\pi}{|B|} = \dfrac{2\pi}{\frac{pi}{2}} = 2\pi \cdot \dfrac{2}{\pi} = 4$. The period is 4.
- *Step 4.* $C = -\pi$,so we calculate the phase shift as $\dfrac{C}{B} = -\dfrac{\pi}{\frac{\pi}{2}} = -\pi \cdot \dfrac{2}{\pi} = -2$. The phase shift is -2.
- *Step 5.* $D = 3$,so the midline is $y = 3$, and the vertical shift is up 3.

Since A is negative, the graph of the cosine function has been reflected about the x-axis.

Figure 2.2.23shows one cycle of the graph of the function.

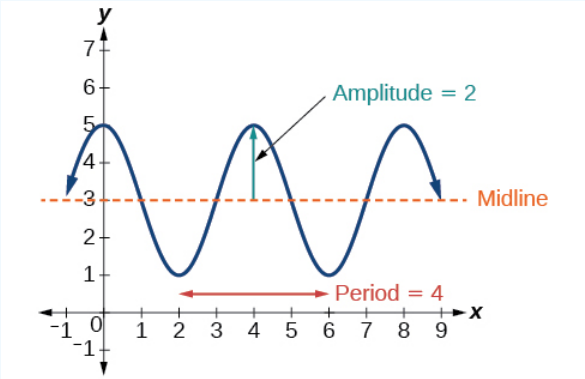

Figure 2.2.23

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter,**circular motion** can be modeled using either the sine or **cosine function**.

Example 2.2.11: Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution

Recall that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in Figure 2.2.24

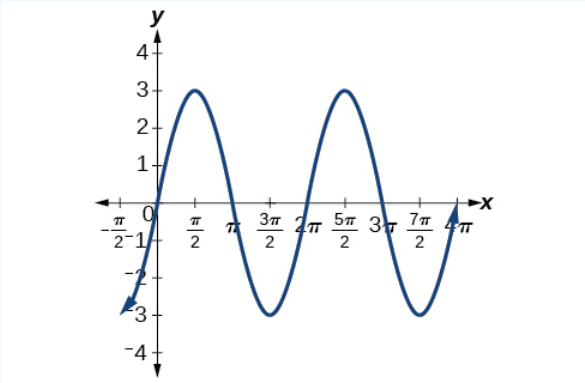

Figure 2.2.24

Analysis

Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi$,...Because the outputs of the graph will now oscillate between -3 and 3, the amplitude of the sine wave is 3.

? Exercise 2.2.10

What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Answer

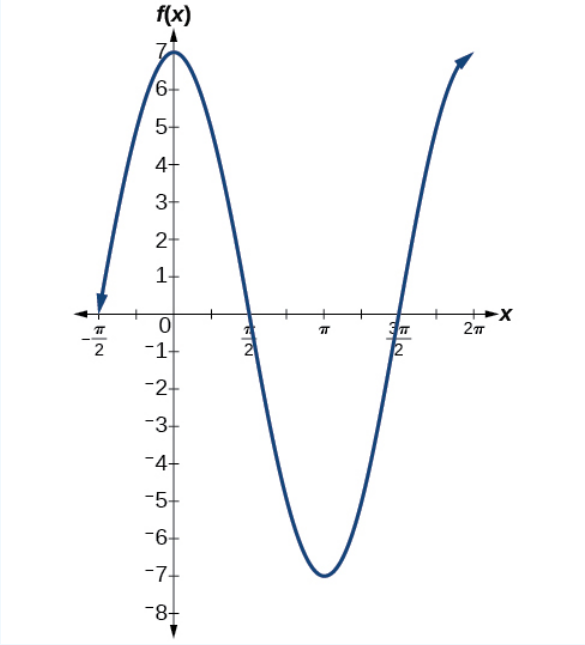

Figure 2.2.25

✓ Example 2.2.12: Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in Figure 2.2.26. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

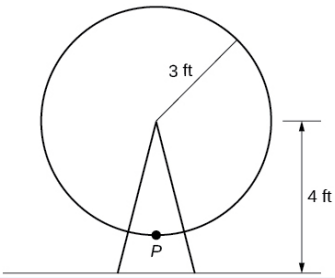

Figure 2.2.26

Solution

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in Figure 2.2.27.

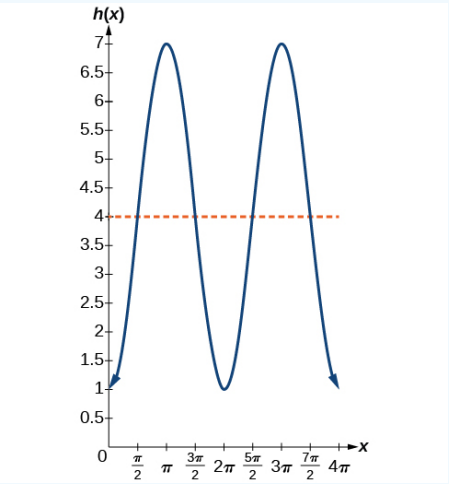

Figure 2.2.27

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a **sine function** starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

? Exercise 2.2.11

A weight is attached to a spring that is then hung from a board, as shown in Figure 2.2.28. As the spring oscillates up and down, the position y of the weight relative to the board ranges from -1 in. (at time $x = 0$) to -7 in. (at time $x = \pi$) below the board. Assume the position of y is given as a sinusoidal function of x. Sketch a graph of the function, and then find a cosine function that gives the position y in terms of x.

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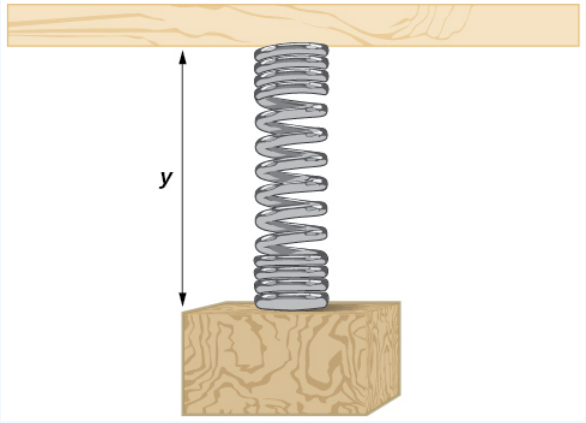

Figure 2.2.28

Answer

$y = 3\cos(x) - 4$

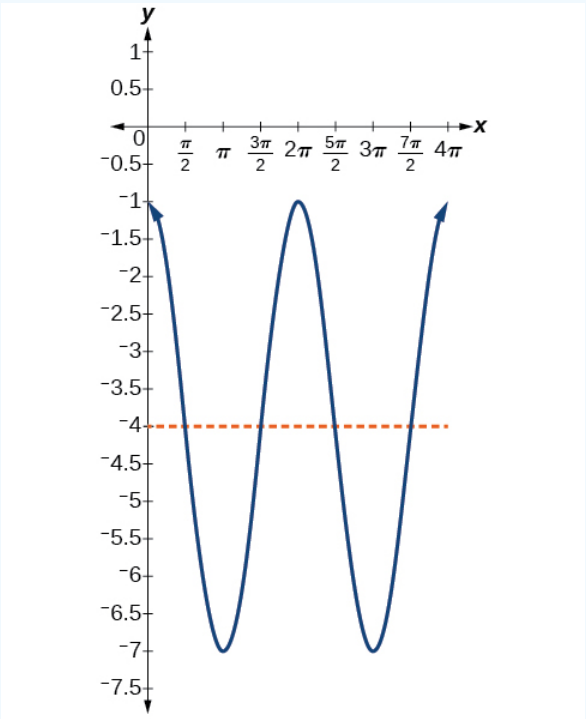

Figure 2.2.29

Example 2.2.13: Determining a Rider's Height on a Ferris Wheel

The London Eye is a huge Ferris wheel with a diameter of 135 meters (443 feet). It completes one rotation every 30 minutes. Riders board from a platform 2 meters above the ground. Express a rider's height above ground as a function of time in minutes.

Solution

With a diameter of 135 m, the wheel has a radius of 67.5 m. The height will oscillate with amplitude 67.5 m above and below the center.

Passengers board 2 m above ground level, so the center of the wheel must be located $67.5 + 2 = 69.5$ m above ground level. The midline of the oscillation will be at 69.5 m.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with a period of 30 minutes.

Lastly, because the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a vertically reflected cosine curve.

- Amplitude: 67.5, so $A = 67.5$
- Midline: 69.5, so $D = 69.5$
- Period:30, so $B = \dfrac{2\pi}{30} = \dfrac{\pi}{15}$
- Shape: $-\cos(t)$

An equation for the rider's height would be

$$y = -67.5\cos\left(\frac{\pi}{15}t\right) + 69.5$$

where t is in minutes and y is measured in meters.

Media

Access these online resources for additional instruction and practice with graphs of sine and cosine functions.

- Amplitude and Period of Sine and Cosine
- Translations of Sine and Cosine
- Graphing Sine and Cosine Transformations
- Graphing the Sine Function

Key Equations

Sinusoidal functions	$f(x) = A\sin(Bx - C) + D$ $f(x) = A\cos(Bx - C) + D$

Key Concepts

- Periodic functions repeat after a given value. The smallest such value is the period. The basic sine and cosine functions have a period of 2π.
- The function $\sin x$ is odd, so its graph is symmetric about the origin. The function $\cos x$ is even, so its graph is symmetric about the y-axis.
- The graph of a sinusoidal function has the same general shape as a sine or cosine function.
- In the general formula for a sinusoidal function, the period is $P = \dfrac{2\pi}{|B|}$. See Example $2.2.1$.
- In the general formula for a sinusoidal function, $|A|$ represents amplitude. If $|A| > 1$, the function is stretched, whereas if $|A| < 1$, the function is compressed. See Example $2.2.2$.
- The value $\dfrac{C}{B}$ in the general formula for a sinusoidal function indicates the phase shift. See Example $2.2.3$.
- The value D in the general formula for a sinusoidal function indicates the vertical shift from the midline. See Example $2.2.4$.
- Combinations of variations of sinusoidal functions can be detected from an equation. See Example $2.2.5$.
- The equation for a sinusoidal function can be determined from a graph. See Example $2.2.6$ and Example $2.2.7$.
- A function can be graphed by identifying its amplitude and period. See Example $2.2.8$ and Example $2.2.9$.
- A function can also be graphed by identifying its amplitude, period, phase shift, and horizontal shift. See Example $2.2.10$.
- Sinusoidal functions can be used to solve real-world problems. See Example $2.2.11$, Example $2.2.12$, and Example $2.2.13$.

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2.3: Graphs of the Other Trigonometric Functions

Learning Objectives

- Analyze the graph of $y = \tan x$.
- Graph variations of $y = \tan x$.
- Analyze the graphs of $y = \sec x$ and $y = \csc x$.
- Graph variations of $y = \sec x$ and $y = \csc x$.
- Analyze the graph of $y = \cot x$.
- Graph variations of $y = \cot x$.

We know the tangent function can be used to find distances, such as the height of a building, mountain, or flagpole. But what if we want to measure repeated occurrences of distance? Imagine, for example, a police car parked next to a warehouse. The rotating light from the police car would travel across the wall of the warehouse in regular intervals. If the input is time, the output would be the distance the beam of light travels. The beam of light would repeat the distance at regular intervals. The tangent function can be used to approximate this distance. Asymptotes would be needed to illustrate the repeated cycles when the beam runs parallel to the wall because, seemingly, the beam of light could appear to extend forever. The graph of the tangent function would clearly illustrate the repeated intervals. In this section, we will explore the graphs of the tangent and other trigonometric functions.

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

$$\tan x = \frac{\sin x}{\cos x} \tag{2.3.1}$$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{aligned} \tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Definition of tangent} \\ &= \frac{-\sin x}{\cos x} && \text{Sine is an odd function, cosine is even} \\ &= -\frac{\sin x}{\cos x} && \text{The quotient of an odd and an even function is odd} \\ &= -\tan x && \text{Definition of tangent} \end{aligned}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in Table 2.3.1.

Table 2.3.1

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in Table 2.3.2.

Table 2.3.2

x	1.3	1.5	1.55	1.56
$\tan x$	3.6	14.1	48.1	92.6

As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in Table 2.3.3.

Table 2.3.3

x	−1.3	−1.5	−1.55	−1.56
$\tan x$	−3.6	−14.1	−48.1	−92.6

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos x = 0$. For example, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. Figure 2.3.1 represents the graph of $y = \tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

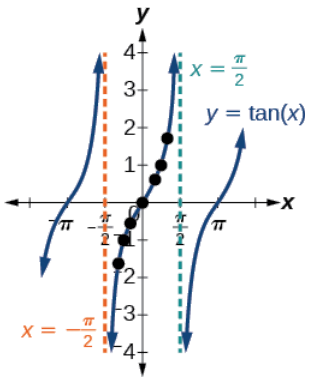

Figure 2.3.1: Graph of the tangent function

Graphing Variations of $y = \tan x$

As with the sine and cosine functions, the *tangent function* can be described by a general equation.

$$y = A\tan(Bx)$$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

FEATURES OF THE GRAPH OF $Y = A\tan(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \frac{\pi}{|B|}$.
- The domain is all real numbers x,where $x \neq \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ such that k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \frac{\pi}{2|B|} + \frac{\pi}{|B|}k$ where k is an integer.
- $y = A\tan(Bx)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the **tangent function** to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x) = A\tan(Bx)$. We focus on a single **period** of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is

then the interval $\left(-\frac{P}{2}, \frac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\frac{P}{2}$ where $P = \frac{\pi}{B}$. On $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the graph will come up from the left asymptote at $x = -\frac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x = \frac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right) = A\tan\left(B\frac{P}{4}\right) = A\tan\left(B\frac{\pi}{4B}\right) = A$$

because $\tan\left(\frac{\pi}{4}\right) = 1$.

Howto: Given the function $f(x) = A\tan(Bx)$, graph one period.

1. Identify the stretching factor, $|A|$.
2. Identify B and determine the period, $P = \frac{\pi}{|B|}$.
3. Draw vertical asymptotes at $x = -\frac{P}{2}$ and $x = \frac{P}{2}$.
4. For $A > 0$, the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for $A < 0$).
5. Plot reference points at $\left(\frac{P}{4}, A\right)$, $(0, 0)$, and $\left(-\frac{P}{4}, -A\right)$, and draw the graph through these points.

Example 2.3.1: Sketching a Compressed Tangent

Sketch a graph of one period of the function $y = 0.5\tan\left(\frac{\pi}{2}x\right)$.

Solution

First, we identify A and B.

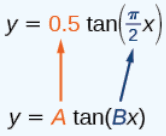

Figure 2.3.2

Because $A = 0.5$ and $B = \frac{\pi}{2}$, we can find the stretching/compressing factor and period. The period is $\frac{\pi}{\frac{\pi}{2}} = 2$, so the asymptotes are at $x = \pm 1$. At a quarter period from the origin, we have

$$\begin{aligned} f(0.5) &= 0.5\tan\left(\frac{0.5\pi}{2}\right) \\ &= 0.5\tan\left(\frac{\pi}{4}\right) \\ &= 0.5 \end{aligned}$$

This means the curve must pass through the points $(0.5, 0.5)$, $(0, 0)$,and $(-0.5, -0.5)$. The only inflection point is at the origin. Figure 2.3.3 shows the graph of one period of the function.

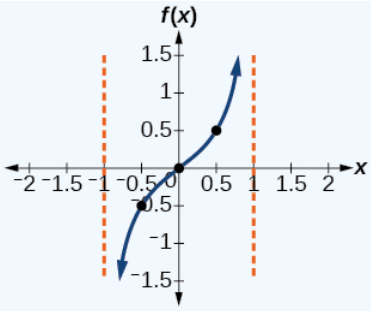

Figure 2.3.3

? Exercise 2.3.1

Sketch a graph of $f(x) = 3\tan\left(\frac{\pi}{6}x\right)$.

Answer

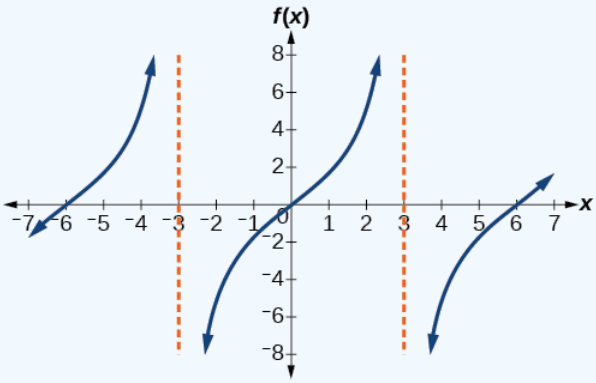

Figure 2.3.4

Graphing One Period of a Shifted Tangent Function

Now that we can graph a **tangent function** that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.

$$f(x) = A\tan(Bx - C) + D$$

The graph of a transformed tangent function is different from the basic tangent function $\tan x$ in several ways:

FEATURES OF THE GRAPH OF $Y = A\tan(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\frac{\pi}{|B|}$.
- The domain is $x \neq \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an odd integer.
- There is no amplitude.
- $y = A\tan(Bx)$ is and odd function because it is the qoutient of odd and even functions(sin and cosine perspectively).

Howto: Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the **stretching/compressing** factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Example 2.3.2: Graphing One Period of a Shifted Tangent Function

Graph one period of the function $y = -2\tan(\pi x + \pi) - 1$.

Solution

- *Step 1.* The function is already written in the form $y = A\tan(Bx - C) + D$.
- *Step 2.*\(A=−2\), so the stretching factor is $|A| = 2$.
- *Step 3.* \(B=\pi\), so the period is $P = \dfrac{\pi}{|B|} = \dfrac{\pi}{pi} = 1$.
- *Step 4.* \(C=−\pi\), so the phase shift is $CB = \dfrac{-\pi}{\pi} = -1$.
- *Step 5-7.* The asymptotes are at $x = -\dfrac{3}{2}$ and $x = -\dfrac{1}{2}$ and the three recommended reference points are $(-1.25, 1)$, $(-1, -1)$, and $(-0.75, -3)$. The graph is shown in Figure 2.3.5.

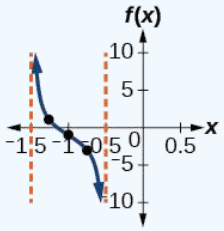

Figure 2.3.5

Analysis

Note that this is a decreasing function because $A < 0$.

Exercise 2.3.2

How would the graph in Example 2.3.2 look different if we made $A = 2$ instead of -2?

Answer

It would be reflected across the line $y = -1$, becoming an increasing function.

Howto: Given the graph of a tangent function, identify horizontal and vertical stretches.

1. Find the period P from the spacing between successive vertical asymptotes or *x*-intercepts.
2. Write $f(x) = A\tan\left(\dfrac{\pi}{P}x\right)$.
3. Determine a convenient point $(x, f(x))$ on the given graph and use it to determine A.

Example 2.3.3: Identifying the Graph of a Stretched Tangent

Find a formula for the function graphed in Figure 2.3.6.

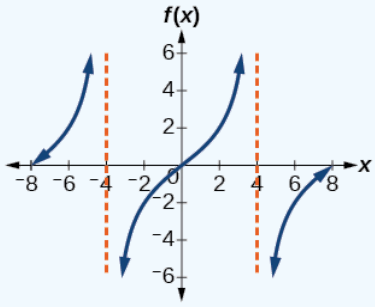

Figure 2.3.6: A stretched tangent function

Solution

The graph has the shape of a tangent function.

- *Step 1.* One cycle extends from -4 to 4, so the period is $P = 8$. Since $P = \frac{\pi}{|B|}$, we have $B = \frac{\pi}{P} = \frac{\pi}{8}$.
- *Step 2.* The equation must have the form $f(x) = A\tan\left(\frac{\pi}{8}x\right)$.
- *Step 3.* To find the vertical stretch A,we can use the point $(2, 2)$.

$$\begin{aligned} 2 &= A\tan\left(\frac{\pi}{8}\cdot 2\right) \\ &= A\tan\left(\frac{\pi}{4}\right) \end{aligned}$$

Because $\tan\left(\frac{\pi}{4}\right) = 1$, $A = 2$.

This function would have a formula $f(x) = 2\tan\left(\frac{\pi}{8}x\right)$.

Exercise 2.3.3

Find a formula for the function in Figure 2.3.7.

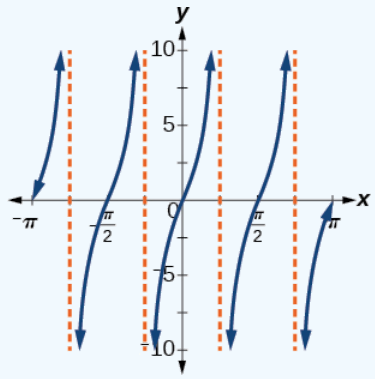

Figure 2.3.7

Answer

$g(x) = 4\tan(2x)$

Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

The secant was defined by the reciprocal identity $sec\, x = \dfrac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\dfrac{\pi}{2}$, $\dfrac{3\pi}{2}$ etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See Figure 2.3.8. The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the x-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

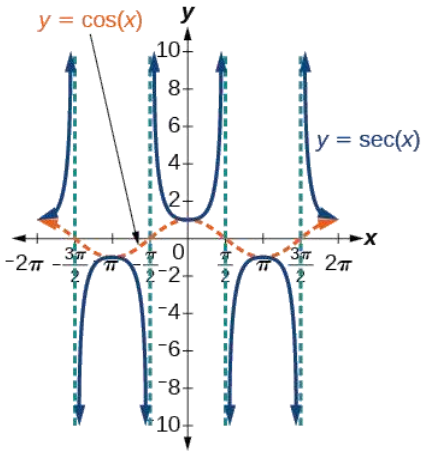

Figure 2.3.8: Graph of the secant function, $f(x) = \sec x = \dfrac{1}{\cos x}$

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

FEATURES OF THE GRAPH OF $Y = A\sec(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

Similar to the secant, the **cosecant** is defined by the reciprocal identity $\csc x = \dfrac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See Figure 2.3.7. The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the **cosecant function** decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in Figure 2.3.9, is similar to the graph of secant.

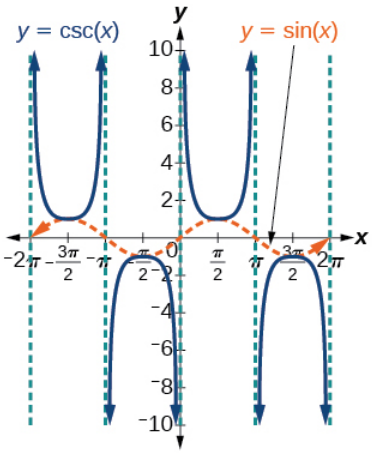

Figure 2.3.9: The graph of the cosecant function, $f(x) = \csc x = \frac{1}{\sin x}$

FEATURES OF THE GRAPH OF $Y = A\csc(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

Graphing Variations of $y = \sec x$ and $y = \csc x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions.The equations become the following.

$$y = A\sec(Bx - C) + D \tag{2.3.2}$$

$$y = A\csc(Bx - C) + D \tag{2.3.3}$$

FEATURES OF THE GRAPH OF $Y = A\sec(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- There is no amplitude.

- $y = A\sec(Bx)$ is an even function because cosine is an even function.

FEATURES OF THE GRAPH OF $Y = A\csc(Bx - C) + D$

1. The stretching factor is $|A|$.
2. The period is $\dfrac{2\pi}{|B|}$.
3. The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an integer.
4. The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
5. The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
6. There is no amplitude.
7. $y = A\csc(Bx)$ is an odd function because sine is an odd function.

HOWTO: Given a function of the form $y = A\sec(Bx)$, graph one period

1. Express the function given in the form $y = A\sec(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Sketch the graph of $y = A\cos(Bx)$.
5. Use the reciprocal relationship between $y = \cos\ x$ and $y = \sec\ x$ to draw the graph of $y = A\sec(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 2.3.4: Graphing a Variation of the Secant Function

Graph one period of $f(x) = 2.5\sec(0.4x)$

Solution

- *Step 1.* The given function is already written in the general form, $y = A\sec(Bx)$.
- *Step 2.* $A = 2.5$ so the stretching factor is 2.5.
- *Step 3.* $B = 0.4$ so $P = \dfrac{2\pi}{0.4} = 5\pi$. The period is 5π units.
- *Step 4.* Sketch the graph of the function $g(x) = 2.5\cos(0.4x)$.
- *Step 5.* Use the reciprocal relationship of the cosine and secant functions to draw the cosecant function.
- *Steps 6–7.* Sketch two asymptotes at $x = 1.25\pi$ and $x = 3.75\pi$. We can use two reference points, the local minimum at $(0, 2.5)$ and the local maximum at $(2.5\pi, -2.5)$ Figure $2.3.10$ shows the graph.

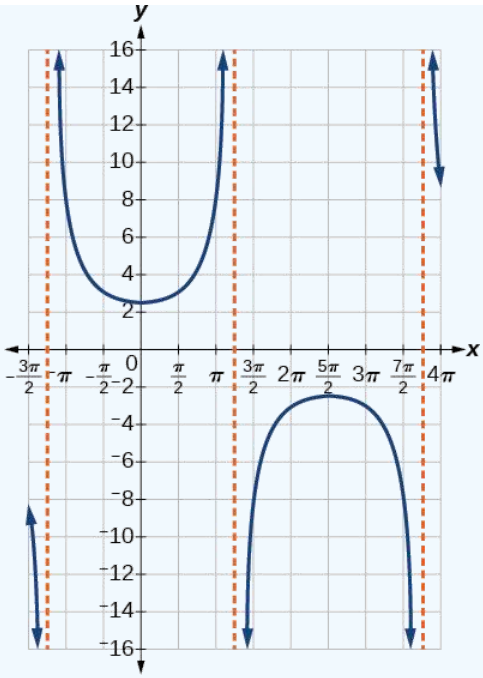

Figure 2.3.10

Exercise 2.3.4

Graph one period of $f(x) = -2.5\sec(0.4x)$.

Answer

This is a vertical reflection of the preceding graph because A is negative.

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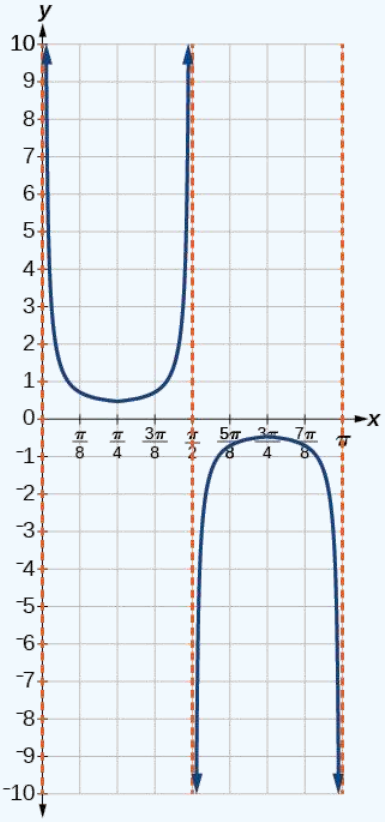

Figure 2.3.11

Q&A: Do the vertical shift and stretch/compression affect the secant's range?

Yes. The range of $f(x) = A\sec(Bx - C) + D$ is $(-\infty, -|A| + D] \cup [|A| + D, \infty)$.

Howto: Given a function of the form $f(x) = A\sec(Bx - C) + D$, graph one period.

1. Express the function given in the form $y = A\sec(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\dfrac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\sec(Bx)$. but shift it to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$, where k is an odd integer.

Example 2.3.5: Graphing a Variation of the Secant Function

Graph one period of $y = 4\sec\left(\dfrac{\pi}{3x} - \dfrac{\pi}{2}\right) + 1$.

Solution

- *Step 1.* Express the function given in the form $y = 4\sec\left(\dfrac{\pi}{3x} - \dfrac{\pi}{2}\right) + 1$.
- *Step 2.* The stretching/compressing factor is $|A| = 4$.
- *Step 3.* The period is

$$\begin{aligned}\frac{2\pi}{|B|} &= \frac{2\pi}{\frac{\pi}{3}} \\ &= 2\pi \cdot \frac{3}{\pi} \\ &= 6\end{aligned}$$

- *Step 4.* The phase shift is

$$\begin{aligned}\frac{C}{B} &= \frac{\frac{\pi}{2}}{\frac{\pi}{3}} \\ &= \frac{\pi}{2} \cdot \frac{3}{\pi} \\ &= 1.5\end{aligned}$$

- *Step 5.* Draw the graph of $y = A\sec(Bx)$, but shift it to the right by $\frac{C}{B} = 1.5$ and up by $D = 6$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 3$, and $x = 6$. There is a local minimum at $(1.5, 5)$ and a local maximum at $(4.5, -3)$. Figure 2.3.12 shows the graph.

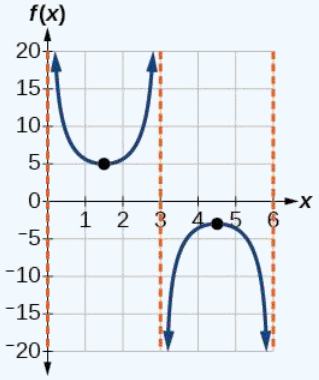

Figure 2.3.12

Exercise 2.3.5

Graph one period of $f(x) = -6\sec(4x + 2) - 8$.

Answer

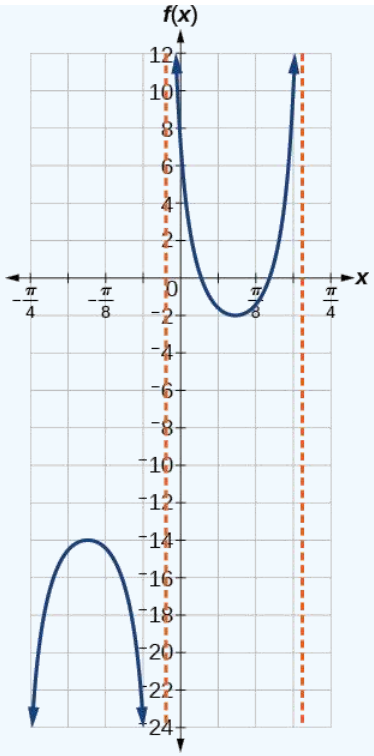

Figure 2.3.13

Q&A: The domain of $\csc\ x$ was given to be all x such that $x \neq k\pi$ for any integer k. Would the domain of $y = A\csc(Bx - C) + D$ be $x \neq \dfrac{C + k\pi}{B}$?

Yes. The excluded points of the domain follow the vertical asymptotes. Their locations show the horizontal shift and compression or expansion implied by the transformation to the original function's input.

Howto: Given a function of the form $y = A\csc(Bx)$, graph one period.

1. Express the function given in the form $y = A\csc(Bx)$.
2. $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Draw the graph of $y = A\sin(Bx)$.
5. Use the reciprocal relationship between $y = sin\ x$ and $y = \csc\ x$ to draw the graph of $y = A\csc(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Example 2.3.6: Graphing a Variation of the Cosecant Function

Graph one period of $f(x) = -3\csc(4x)$.

Solution

- *Step 1.* The given function is already written in the general form, $y = A\csc(Bx)$.
- *Step 2.* $|A| = |-3| = 3$,so the stretching factor is 3.
- *Step 3.* $B = 4$,so $P = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$. The period is $\dfrac{\pi}{2}$ units.
- *Step 4.* Sketch the graph of the function $g(x) = -3\sin(4x)$.
- *Step 5.* Use the reciprocal relationship of the sine and cosecant functions to draw the cosecant function.

- *Steps 6–7.* Sketch three asymptotes at $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{\pi}{2}$. We can use two reference points, the local maximum at $\left(\frac{\pi}{8}, -3\right)$ and the local minimum at $\left(\frac{3\pi}{8}, 3\right)$. Figure 2.3.14 shows the graph.

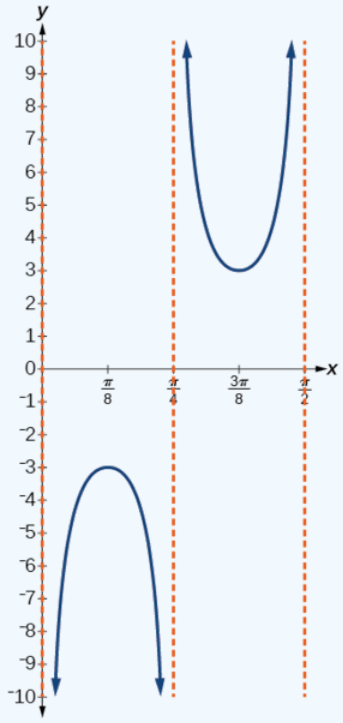

Figure 2.3.14

Exercise 2.3.6

Graph one period of $f(x) = 0.5 \csc(2x)$.

Answer

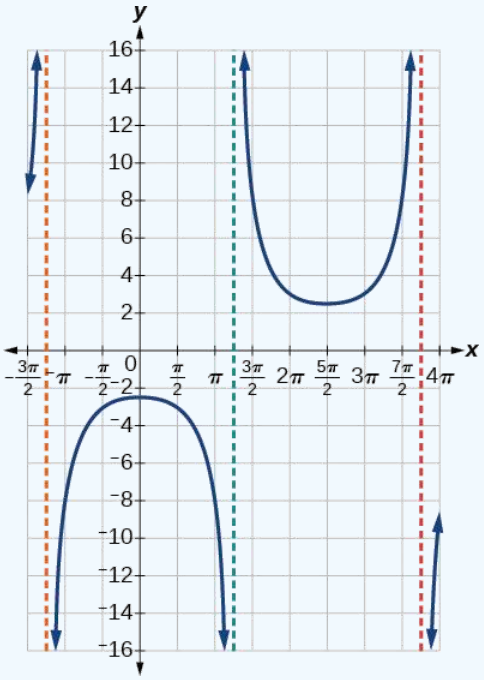

Figure 2.3.15

Howto: Given a function of the form $f(x) = A\csc(Bx - C) + D$, graph one period

1. Express the function given in the form $y = A\csc(Bx - C) + D$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $\frac{2\pi}{|B|}$.
4. Identify C and determine the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\csc(Bx)$ but shift it to the right by and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.

Example 2.3.7: Graphing a Vertically Stretched, Horizontally Compressed, and Vertically Shifted Cosecant

Sketch a graph of $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$. What are the domain and range of this function?

Solution

- *Step 1.* Express the function given in the form $y = 2\csc\left(\frac{\pi}{2}x\right) + 1$.
- *Step 2.* Identify the stretching/compressing factor, $|A| = 2$.
- *Step 3.* The period is $\frac{2\pi}{|B|} = \frac{2\pi}{\frac{\pi}{2}} = 2\pi \cdot \frac{2}{\pi} = 4$.
- *Step 4.* The phase shift is $\frac{0}{\frac{\pi}{2}} = 0$.
- *Step 5.* Draw the graph of $y = A\csc(Bx)$ but shift it up $D = 1$.
- *Step 6.* Sketch the vertical asymptotes, which occur at $x = 0$, $x = 2$, $x = 4$.

The graph for this function is shown in Figure 2.3.16.

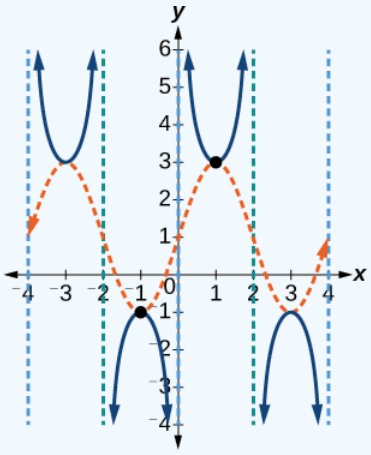

Figure 2.3.16: A transformed cosecant function

Analysis

The vertical asymptotes shown on the graph mark off one period of the function, and the local extrema in this interval are shown by dots. Notice how the graph of the transformed cosecant relates to the graph of $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$,shown as the orange dashed wave.

Exercise 2.3.7

Given the graph of $f(x) = 2\cos\left(\frac{\pi}{2}x\right) + 1$ shown in Figure 2.3.17, sketch the graph of $g(x) = 2\sec\left(\frac{\pi}{2}x\right) + 1$ on the same axes.

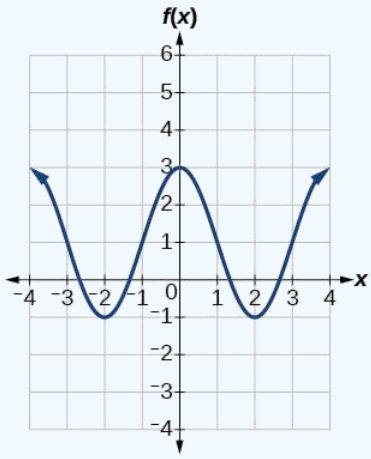

Figure 2.3.17

Answer

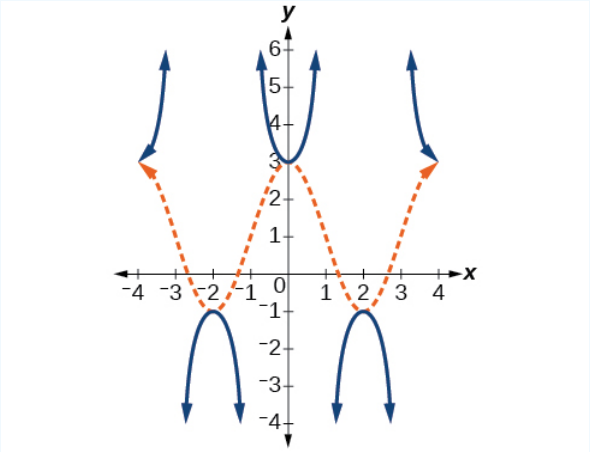

Figure 2.3.18

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $cot\, x = \dfrac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See Figure 2.3.19. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

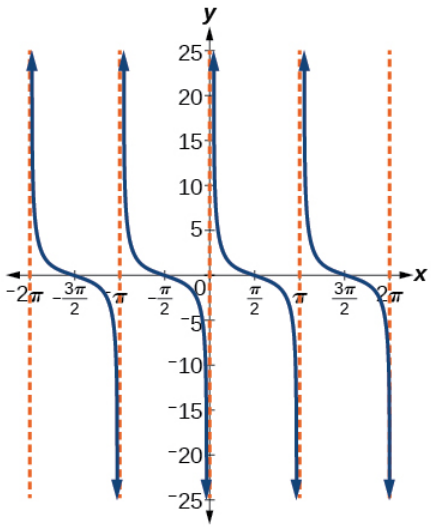

Figure 2.3.19: The cotangent function

FEATURES OF THE GRAPH OF $Y = A\cot(BX)$

- The stretching factor is $|A|$.
- The period is $P = \dfrac{\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Graphing Variations of $y = \cot x$

We can transform the graph of the cotangent in much the same way as we did for the tangent. The equation becomes the following.

$$y = A\cot(Bx - C) + D \tag{2.3.4}$$

PROPERTIES OF THE GRAPH OF $Y = A\cot(Bx - c) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{\pi}{|B|}$
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
- There is no amplitude.
- $y = A\cot(Bx)$ is an odd function because it is the quotient of even and odd functions (cosine and sine, respectively)

Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx)$,graph one period.

1. Express the function in the form $f(x) = A\cot(Bx)$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \dfrac{\pi}{|B|}$.
4. Draw the graph of $y = A\tan(Bx)$.
5. Plot any two reference points.

6. Use the reciprocal relationship between tangent and cotangent to draw the graph of $y = Acot(Bx)$.
7. Sketch the asymptotes.

Example 2.3.8: Graphing Variations of the Cotangent Function

Determine the stretching factor, period, and phase shift of $y = 3\cot(4x)$, and then sketch a graph.

Solution

- *Step 1.* Expressing the function in the form $f(x) = A\cot(Bx)$ gives $f(x) = 3\cot(4x)$.
- *Step 2.* The stretching factor is $|A| = 3$.
- *Step 3.* The period is $P = \frac{\pi}{4}$.
- *Step 4.* Sketch the graph of $y = 3\tan(4x)$.
- *Step 5.* Plot two reference points. Two such points are $\left(\frac{\pi}{16}, 3\right)$ and $\left(\frac{3\pi}{16}, -3\right)$.
- *Step 6.* Use the reciprocal relationship to draw $y = 3\cot(4x)$.
- *Step 7.* Sketch the asymptotes, $x = 0$, $x = \frac{\pi}{4}$.

The orange graph in Figure 2.3.20 shows $y = 3\tan(4x)$ and the blue graph shows $y = 3\cot(4x)$.

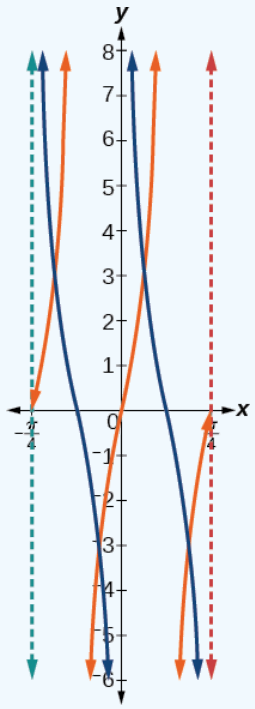

Figure 2.3.20

Howto: Given a modified cotangent function of the form $f(x) = A\cot(Bx - C) + D$, graph one period.

1. Express the function in the form $f(x) = A\cot(Bx - C) + D$.
2. Identify the stretching factor, $|A|$.
3. Identify the period, $P = \frac{\pi}{|B|}$.
4. Identify the phase shift, $\frac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\frac{C}{B}$ and up by D.
6. Sketch the asymptotes $x = \frac{C}{B} + \frac{\pi}{|B|}k$,where k is an integer.
7. Plot any three reference points and draw the graph through these points.

Example 2.3.9: Graphing a Modified Cotangent

Sketch a graph of one period of the function $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.

Solution

- *Step 1.* The function is already written in the general form $f(x) = A\cot(Bx - C) + D$.
- *Step 2.* $A = 4$,so the stretching factor is 4.
- *Step 3.* $B = \frac{\pi}{8}$, so the period is $P = \frac{\pi}{|B|} = \frac{\pi}{\frac{\pi}{8}} = 8$.
- *Step 4.* $C = \frac{\pi}{2}$,so the phase shift is $CB = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$.
- *Step 5.* We draw $f(x) = 4\tan\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 6-7.* Three points we can use to guide the graph are $(6, 2)$, $(8, -2)$, and $(10, -6)$. We use the reciprocal relationship of tangent and cotangent to draw $f(x) = 4\cot\left(\frac{\pi}{8}x - \frac{\pi}{2}\right) - 2$.
- *Step 8.* The vertical asymptotes are $x = 4$ and $x = 12$.

The graph is shown in Figure 2.3.21.

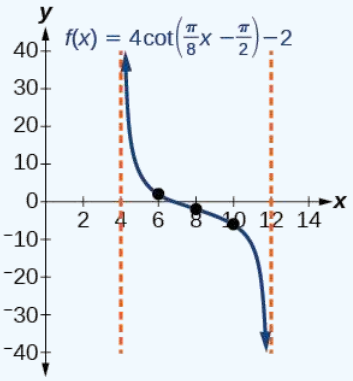

Figure 2.3.21: One period of a modified cotangent function

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the **tangent function**.

Example 2.3.10: Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan(\frac{\pi}{4}t)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.
b. Graph on the interval $[0, 5]$.
c. Evaluate $f(1)$ and discuss the function's value at that input.

Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the stretching factor and $\frac{\pi}{B}$ is the period.

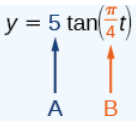

Figure 2.3.22

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is $\frac{\pi}{\frac{\pi}{4}} = \frac{\pi}{1} \cdot \frac{4}{\pi} = 4$. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the stretching factor and period. See Figure 2.3.23

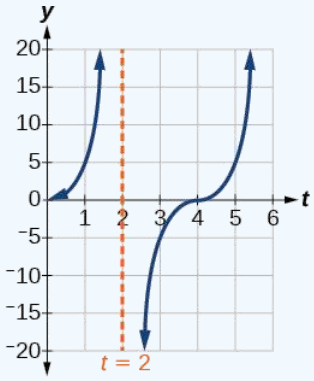

Figure 2.3.23

c. period: $f(1) = 5\tan(\frac{\pi}{4}(1)) = 5(1) = 5$; after 1 second, the beam of has moved 5 ft from the spot across from the police car.

Media

Access these online resources for additional instruction and practice with graphs of other trigonometric functions.

- Graphing the Tangent
- Graphing Cosecant and Secant
- Graphing the Cotangent

Key Equations

Shifted, compressed, and/or stretched tangent function	$y = A\tan(Bx - C) + D$
Shifted, compressed, and/or stretched secant function	$y = A\sec(Bx - C) + D$
Shifted, compressed, and/or stretched cosecant function	$y = A\csc(Bx - C) + D$
Shifted, compressed, and/or stretched cotangent function	$y = A\cot(Bx - C) + D$

Key Concepts

- The tangent function has period π.
- $f(x) = A\tan(Bx - C) + D$ is a tangent with vertical and/or horizontal stretch/compression and shift. See Example 2.3.1, Example 2.3.2, and Example 2.3.3.
- The secant and cosecant are both periodic functions with a period of 2π. $f(x) = A\sec(Bx - C) + D$ gives a shifted, compressed, and/or stretched secant function graph. See Example 2.3.4 and Example 2.3.5.
- $f(x) = A\csc(Bx - C) + D$ gives a shifted, compressed, and/or stretched cosecant function graph. See Example 2.3.6 and Example 2.3.7.
- The cotangent function has period π and vertical asymptotes at $0, \pm\pi, \pm 2\pi, ...$.
- The range of cotangent is $(-\infty, \infty)$, and the function is decreasing at each point in its range.

- The cotangent is zero at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$,....
- $f(x) = A\cot(Bx - C) + D$ is a cotangent with vertical and/or horizontal stretch/compression and shift. See Example $2.3.8$ and Example $2.3.9$.
- Real-world scenarios can be solved using graphs of trigonometric functions. See Example $2.3.10$.

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2.4: Inverse Trigonometric Functions

Learning Objectives

- Understand and use the inverse sine, cosine, and tangent functions.
- Find the exact value of expressions involving the inverse sine, cosine, and tangent functions.
- Use a calculator to evaluate inverse trigonometric functions.
- Find exact values of composite functions with inverse trigonometric functions.

For any right triangle, given one other angle and the length of one side, we can figure out what the other angles and sides are. But what if we are given only two sides of a right triangle? We need a procedure that leads us from a ratio of sides to an angle. This is where the notion of an inverse to a trigonometric function comes into play. In this section, we will explore the inverse trigonometric functions.

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in Figure $2.4.1$.

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

Figure 2.4.1

For example, if $f(x) = \sin\ x$, then we would write $f^{-1}(x) = \sin^{-1} x$. Be aware that $\sin^{-1} x$ does not mean $\dfrac{1}{\sin\ x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$, then $\dfrac{\pi}{6} = \sin^{-1}\left(\dfrac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\dfrac{\pi}{4}\right) = 1$, then $\dfrac{\pi}{4} = \tan^{-1}(1)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the **domain** of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. Figure $2.4.2$ shows the graph of the sine function limited to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ and the graph of the cosine function limited to $[0, \pi]$.

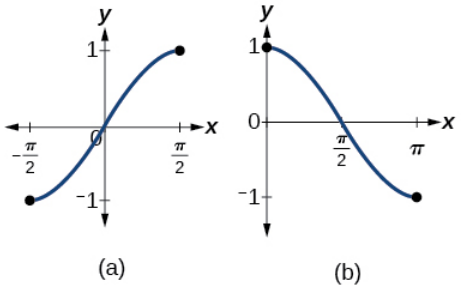

Figure 2.4.2: (a) Sine function on a restricted domain of $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$; (b) Cosine function on a restricted domain of $[0, \pi]$

Figure 2.4.3 shows the graph of the tangent function limited to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

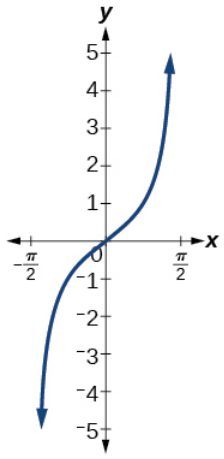

Figure 2.4.3: Tangent function on a restricted domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one **vertical asymptote** to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin\ y$. The inverse sine function is sometimes called the **arcsine** function, and notated $\arcsin\ x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos\ y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos\ x$.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The inverse tangent function $y = \tan^{-1} x$ means $x = \tan\ y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan\ x$.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

The graphs of the inverse functions are shown in Figures 2.4.4 - 2.4.6. Notice that the output of each of these inverse functions is a *number*, an angle in radian measure. We see that $\sin^{-1} x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1} x$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

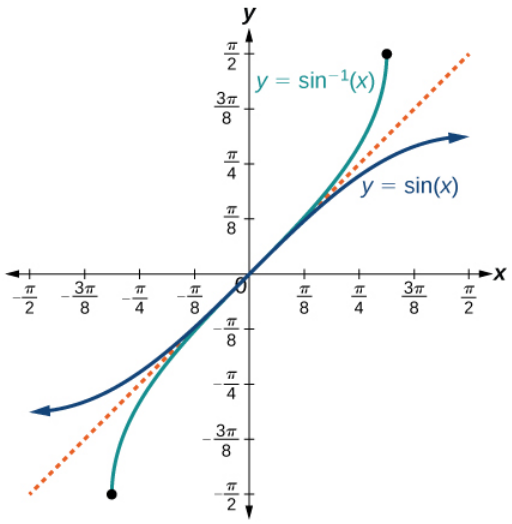

Figure 2.4.4: The sine function and inverse sine (or arcsine) function

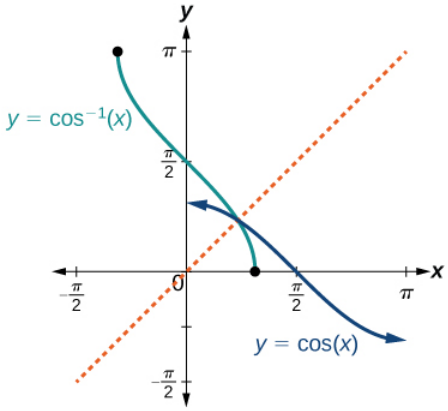

Figure 2.4.5: The cosine function and inverse cosine (or arccosine) function

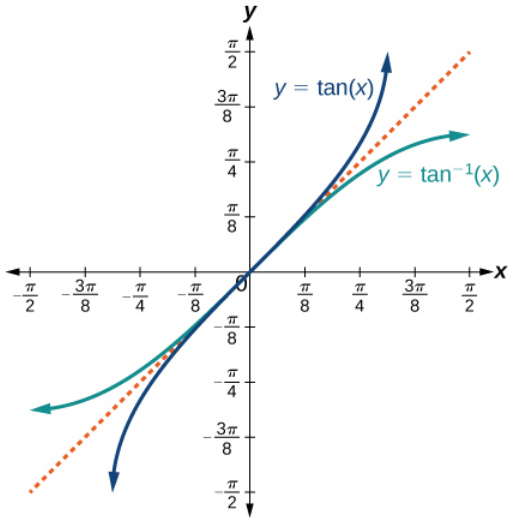

Figure 2.4.6: The tangent function and inverse tangent (or arctangent) function

RELATIONS FOR INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$,then $\tan^{-1} x = y$.

Example 2.4.1: Writing a Relation for an Inverse Function

Given $\sin\left(\frac{5\pi}{12}\right) \approx 0.96593$, write a relation involving the inverse sine.

Solution

Use the relation for the inverse sine. If $\sin y = x$, then $\sin^{-1} x = y$.

In this problem, $x = 0.96593$, and $y = \frac{5\pi}{12}$.

$\sin^{-1}(0.96593) \approx \frac{5\pi}{12}$

Exercise 2.4.1

Given $\cos(0.5) \approx 0.8776$,write a relation involving the inverse cosine.

Answer

$\arccos(0.8776) \approx 0.5$

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}(30°)$, $\frac{\pi}{4}(45°)$, and $\frac{\pi}{3}(60°)$, and their reflections into other quadrants.

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x,depending on which corresponds to the given inverse function.

Example 2.4.2: Evaluating Inverse Trigonometric Functions for Special Input Values

Evaluate each of the following.

a. $\sin^{-1}\left(\frac{1}{2}\right)$

b. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

c. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

d. $\tan^{-1}(1)$

Solution

a. Evaluating $\sin^{-1}\left(\frac{1}{2}\right)$ is the same as determining the angle that would have a sine value of $\frac{1}{2}$. In other words, what angle x would satisfy $\sin(x) = \frac{1}{2}$? There are multiple values that would satisfy this relationship, such as $\frac{\pi}{6}$ and $\frac{5\pi}{6}$, but we know we need the angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the answer will be $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$. Remember that the inverse is a

function, so for each input, we will get exactly one output.

b. To evaluate $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$, we know that $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$ both have a sine value of $-\frac{\sqrt{2}}{2}$, but neither is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For that, we need the negative angle coterminal with $\frac{7\pi}{4}$: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$.

c. To evaluate $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$, we are looking for an angle in the interval $[0, \pi]$ with a cosine value of $-\frac{\sqrt{3}}{2}$. The angle that satisfies this is $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

d. Evaluating $\tan^{-1}(1)$, we are looking for an angle in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with a tangent value of 1. The correct angle is $\tan^{-1}(1) = \frac{\pi}{4}$.

? Exercise 2.4.2

Evaluate each of the following.

a. $\sin^{-1}(-1)$
b. $\tan^{-1}(-1)$
c. $\cos^{-1}(-1)$
d. $\cos^{-1}\left(\frac{1}{2}\right)$

Answer a

$-\frac{\pi}{2}$

Answer b

$-\frac{\pi}{4}$

Answer c

π

Answer d

$\frac{\pi}{3}$

Using a Calculator to Evaluate Inverse Trigonometric Functions

To evaluate **inverse trigonometric functions** that do not involve the special angles discussed previously, we will need to use a calculator or other type of technology. Most scientific calculators and calculator-emulating applications have specific keys or buttons for the inverse sine, cosine, and tangent functions. These may be labeled, for example, SIN-1, ARCSIN, or ASIN.

In the previous chapter, we worked with trigonometry on a right triangle to solve for the sides of a triangle given one side and an additional angle. Using the inverse trigonometric functions, we can solve for the angles of a right triangle given two sides, and we can use a calculator to find the values to several decimal places.

In these examples and exercises, the answers will be interpreted as angles and we will use θ as the independent variable. The value displayed on the calculator may be in degrees or radians, so be sure to set the mode appropriate to the application.

✓ Example 2.4.3: Evaluating the Inverse Sine on a Calculator

Evaluate $\sin^{-1}(0.97)$ using a calculator.

Solution

Because the output of the inverse function is an angle, the calculator will give us a degree value if in degree mode and a radian value if in radian mode. Calculators also use the same domain restrictions on the angles as we are using.

In radian mode, $\sin^{-1}(0.97) \approx 1.3252$. In degree mode, $\sin^{-1}(0.97) \approx 75.93^\circ$. Note that in calculus and beyond we will use radians in almost all cases.

Exercise 2.4.3

Evaluate $\cos^{-1}(-0.4)$ using a calculator.

Answer

$1.9823 \text{or } 113.578^\circ$

Given two sides of a right triangle like the one shown in Figure 8.4.7, find an angle.

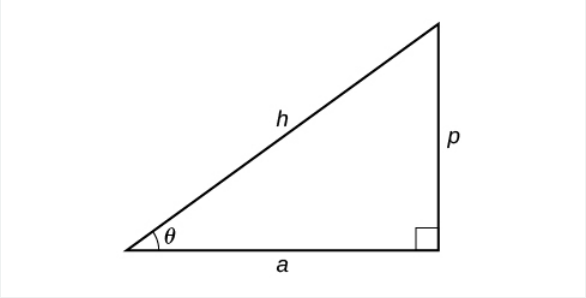

Figure 2.4.7

1. If one given side is the hypotenuse of length h and the side of length a adjacent to the desired angle is given, use the equation $\theta = \cos^{-1}\left(\frac{a}{h}\right)$.
2. If one given side is the hypotenuse of length h and the side of length p opposite to the desired angle is given, use the equation $\theta = \sin^{-1}\left(\frac{p}{h}\right)$.
3. If the two legs (the sides adjacent to the right angle) are given, then use the equation $\theta = \tan^{-1}\left(\frac{p}{a}\right)$.

Example 2.4.4: Applying the Inverse Cosine to a Right Triangle

Solve the triangle in Figure 2.4.8 for the angle θ.

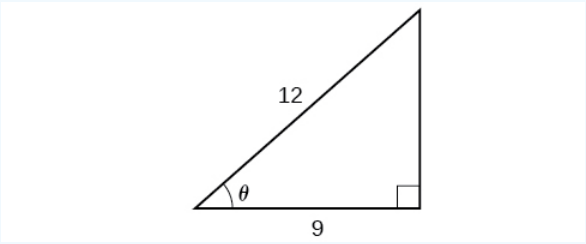

Figure 2.4.8

Solution

Because we know the hypotenuse and the side adjacent to the angle, it makes sense for us to use the cosine function.

$$\cos\theta = \frac{9}{12}$$
$$\theta = \cos^{-1}\left(\frac{9}{12}\right) \quad \text{Apply definition of the inverse}$$
$$\theta \approx 0.7227 \quad \text{or about } 41.4096^\circ \text{ Evaluate}$$

Exercise 2.4.4

Solve the triangle in Figure 2.4.9 for the angle θ.

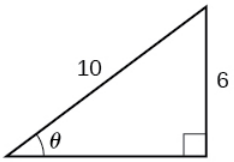

Figure 2.4.9

Answer

$\sin^{-1}(0.6) = 36.87^\circ = 0.6435$radians

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set$\{\ \sin(x), \cos(x), \tan(x)\ \}$ and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function,$f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

COMPOSITIONS OF A TRIGONOMETRIC FUNCTION AND ITS INVERSE

$$\begin{aligned}
\sin(\sin^{-1}x) &= x \quad && \text{for } -1 \le x \le 1 \\
\cos(\cos^{-1}x) &= x \quad && \text{for } -1 \le x \le 1 \\
\tan(\tan^{-1}x) &= x \quad && \text{for } -\infty < x < \infty \\
\sin^{-1}(\sin x) &= x \quad && \text{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\
\cos^{-1}(\cos x) &= x \quad && \text{only for } 0 \le x \le \pi \\
\tan^{-1}(\tan x) &= x \quad && \text{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{aligned}$$

Q&A

Is it correct that $\sin^{-1}(\sin x) = x$?

No. This equation is correct ifx x belongs to the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but sine is defined for all real input values, and for x outside the restricted interval, the equation is not correct because its inverse always returns a value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The situation is similar for cosine and tangent and their inverses. For example, $\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \frac{\pi}{4}$.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain off f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

Example 2.4.5: Using Inverse Trigonometric Functions

Evaluate the following:

a. $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

b. $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$

c. $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$

d. $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right)$

Solution

a. $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so $\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

b. $\frac{2\pi}{3}$ is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, but $sin\left(\frac{2\pi}{3}\right) = sin\left(\frac{\pi}{3}\right)$, so $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$.

c. $\frac{2\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$.

d. $-\frac{\pi}{3}$ is not in $[0, \pi]$, but $\cos\left(-\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)$ because cosine is an even function. $\frac{\pi}{3}$ is in $[0, \pi]$, so $\cos^{-1}\left(\cos\left(-\frac{\pi}{3}\right)\right) = \frac{\pi}{3}$.

Exercise 2.4.5

Evaluate $\tan^{-1}\left(\tan\left(\frac{\pi}{8}\right)\right)$ and $\tan^{-1}\left(\tan\left(\frac{11\pi}{9}\right)\right)$.

Answer

$\frac{\pi}{8}; \frac{2\pi}{9}$

Evaluating Compositions of the Form $f^{-1}(g(x))$

Now that we can compose a trigonometric function with its inverse, we can explore how to evaluate a composition of a trigonometric function and the inverse of another trigonometric function. We will begin with compositions of the form $f^{-1}(g(x))$. For special values of x,we can exactly evaluate the inner function and then the outer, inverse function. However, we can find a more general approach by considering the relation between the two acute angles of a right triangle where one is θ, making the other $\frac{\pi}{2} - \theta$.Consider the sine and cosine of each angle of the right triangle in Figure 2.4.10

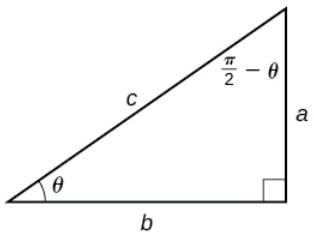

Figure 2.4.10: Right triangle illustrating the cofunction relationships

Because $\cos\theta = \frac{b}{c} = sin\left(\frac{\pi}{2} - \theta\right)$, we have $\sin^{-1}(\cos\theta) = \frac{\pi}{2} - \theta$ if $0 \le \theta \le \pi$. If θ is not in this domain, then we need to find another angle that has the same cosine as θ and does belong to the restricted domain; we then subtract this angle from $\frac{\pi}{2}$.Similarly, $\sin\theta = \frac{a}{c} = \cos\left(\frac{\pi}{2} - \theta\right)$, so $\cos^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$ if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. These are just the function-cofunction relationships presented in another way.

Given functions of the form $\sin^{-1}(\cos x)$ and $\cos^{-1}(\sin x)$, evaluate them.

1. If x is in $[0, \pi]$, then $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$.
2. If x is not in $[0, \pi]$, then find another angle y in $[0, \pi]$ such that $\cos y = \cos x$.

$$\sin^{-1}(\cos x) = \frac{\pi}{2} - y \tag{2.4.1}$$

3. If x is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$.
4. If x is not in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then find another angle y in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y = \sin x$.

$$\cos^{-1}(\sin x) = \frac{\pi}{2} - y \tag{2.4.2}$$

Example 2.4.6: Evaluating the Composition of an Inverse Sine with a Cosine

Evaluate $\sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$

a. by direct evaluation.
b. by the method described previously.

Solution

a. Here, we can directly evaluate the inside of the composition.

$$\begin{aligned} \cos\left(\frac{13\pi}{6}\right) &= \cos\left(\frac{\pi}{6} + 2\pi\right) \\ &= \cos\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

Now, we can evaluate the inverse function as we did earlier.

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} \tag{2.4.3}$$

b. We have $x = \frac{13\pi}{6}$, $y = \frac{\pi}{6}$, and

$$\begin{aligned} \sin^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right) &= \frac{\pi}{2} - \frac{\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

Exercise 2.4.6

Evaluate $\cos^{-1}\left(\sin\left(-\frac{11\pi}{4}\right)\right)$.

Answer

$\frac{3\pi}{4}$

Evaluating Compositions of the Form $f(g^{-1}(x))$

To evaluate compositions of the form $f(g^{-1}(x))$, where f and g are any two of the functions sine, cosine, or tangent and x is any input in the domain of g^{-1}, we have exact formulas, such as $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$. When we need to use them, we can derive these formulas by using the trigonometric relations between the angles and sides of a right triangle, together with the use of

Pythagoras's relation between the lengths of the sides. We can use the Pythagorean identity, $\sin^2 x + \cos^2 x = 1$, to solve for one when given the other. We can also use the inverse trigonometric functions to find compositions involving algebraic expressions.

Example 2.4.7: Evaluating the Composition of a Sine with an Inverse Cosine

Find an exact value for $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$.

Solution

Beginning with the inside, we can say there is some angle such that $\theta = \cos^{-1}\left(\frac{4}{5}\right)$, which means $\cos\theta = \frac{4}{5}$, and we are looking for $\sin\theta$. We can use the Pythagorean identity to do this.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 && \text{Use our known value for cosine} \\ \sin^2\theta + \left(\frac{4}{5}\right)^2 &= 1 && \text{Solve for sine} \\ \sin^2\theta &= 1 - \frac{16}{25} \\ \sin\theta &= \pm\frac{9}{25} \\ &= \pm\frac{3}{5} \end{aligned}$$

Since $\theta = \cos^{-1}\left(\frac{4}{5}\right)$ is in quadrant I, $\sin\theta$ must be positive, so the solution is 35. See Figure 2.4.11.

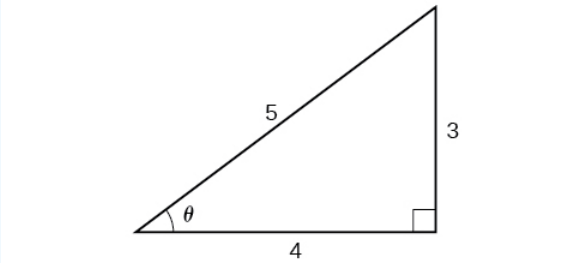

Figure 2.4.11: Right triangle illustrating that if $\cos\theta = \frac{4}{5}$, then $\sin\theta = \frac{3}{5}$

We know that the inverse cosine always gives an angle on the interval $[0, \pi]$, so we know that the sine of that angle must be positive; therefore $\sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) = \sin\theta = \frac{3}{5}$.

Exercise 2.4.7

Evaluate $\cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right)$.

Answer

$\frac{12}{13}$

Example 2.4.8: Evaluating the Composition of a Sine with an Inverse Tangent

Find an exact value for $\sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right)$.

Solution

While we could use a similar technique as in Example 2.4.6, we will demonstrate a different technique here. From the inside, we know there is an angle such that $\tan\theta = \frac{7}{4}$. We can envision this as the opposite and adjacent sides on a right triangle, as shown in Figure 2.4.12

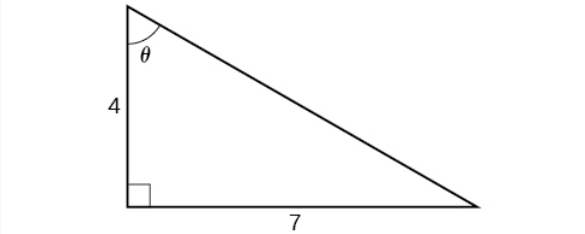

Figure 2.4.12: A right triangle with two sides known

Using the Pythagorean Theorem, we can find the hypotenuse of this triangle.

$$\begin{aligned} 4^2 + 7^2 &= hypotenuse^2 \\ hypotenuse &= \sqrt{65} \end{aligned}$$

Now, we can evaluate the sine of the angle as the opposite side divided by the hypotenuse.

$$\sin\theta = \frac{7}{\sqrt{65}}$$

This gives us our desired composition.

$$\begin{aligned} \sin\left(\tan^{-1}\left(\frac{7}{4}\right)\right) &= \sin\theta \\ &= \frac{7}{\sqrt{65}} \\ &= \frac{7\sqrt{65}}{65} \end{aligned}$$

? Exercise 2.4.8

Evaluate $\cos\left(\sin^{-1}\left(\frac{7}{9}\right)\right)$.

Answer

$\frac{4\sqrt{2}}{9}$

✓ Example 2.4.9: Finding the Cosine of the Inverse Sine of an Algebraic Expression

Find a simplified expression for $\cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right)$ for $-3 \le x \le 3$.

Solution

We know there is an angle θ such that $\sin\theta = \frac{x}{3}$.

$$\begin{aligned} \sin^2\theta + \cos^2\theta &= 1 \qquad \text{Use the Pythagorean Theorem} \\ \left(\frac{x}{3}\right)^2 + \cos^2\theta &= 1 \qquad \text{Solve for cosine} \\ \cos^2\theta &= 1 - \frac{x^2}{9} \\ \cos\theta &= \pm\sqrt{\frac{9-x^2}{9}} \\ &= \pm\sqrt{\frac{9-x^2}{3}} \end{aligned}$$

Because we know that the inverse sine must give an angle on the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, we can deduce that the cosine of that angle must be positive.

$cos\left(\sin^{-1}\left(\frac{x}{3}\right)\right) = \sqrt{\frac{9-x^2}{3}}$

Exercise 2.4.9

Find a simplified expression for $\sin(\tan^{-1}(4x))$ for $-\frac{1}{4} \leq x \leq \frac{1}{4}$.

Answer

$$\frac{4x}{\sqrt{16x^2+1}}$$

Media

Access this online resource for additional instruction and practice with inverse trigonometric functions.

- Evaluate Expressions Involving Inverse Trigonometric Functions

Visit this website for additional practice questions from Learningpod.

Key Concepts

- An inverse function is one that "undoes" another function. The domain of an inverse function is the range of the original function and the range of an inverse function is the domain of the original function.
- Because the trigonometric functions are not one-to-one on their natural domains, inverse trigonometric functions are defined for restricted domains.
- For any trigonometric function $f(x)$, if $x = f^{-1}(y)$, then $f(x) = y$. However, $f(x) = y$ only implies $x = f^{-1}(y)$ if x is in the restricted domain of f. See Example 2.4.1.
- Special angles are the outputs of inverse trigonometric functions for special input values; for example, $\frac{\pi}{4} = \tan^{-1}(1)$ and $\frac{\pi}{6} = \sin^{-1}(\frac{1}{2})$ .See Example 2.4.2.
- A calculator will return an angle within the restricted domain of the original trigonometric function. See Example 2.4.3.
- Inverse functions allow us to find an angle when given two sides of a right triangle. See Example 2.4.4.
- In function composition, if the inside function is an inverse trigonometric function, then there are exact expressions; for example,$\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$. See Example 2.4.5.
- If the inside function is a trigonometric function, then the only possible combinations are $\sin^{-1}(\cos x) = \frac{\pi}{2} - x$ if $0 \leq x \leq \pi$ and $\cos^{-1}(\sin x) = \frac{\pi}{2} - x$ if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. See Example 2.4.6 and Example 2.4.7.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, draw a reference triangle to assist in determining the ratio of sides that represents the output of the trigonometric function. See Example 2.4.8.
- When evaluating the composition of a trigonometric function with an inverse trigonometric function, you may use trig identities to assist in determining the ratio of sides. See Example 2.4.9.

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2.E: Periodic Functions (Exercises)

6.1: Graphs of the Sine and Cosine Functions

In the chapter on Trigonometric Functions, we examined trigonometric functions such as the sine function. In this section, we will interpret and create graphs of sine and cosine functions

Verbal

1) Why are the sine and cosine functions called periodic functions?

Answer

The sine and cosine functions have the property that $f(x+P)=f(x)$ for a certain P. This means that the function values repeat for every P units on the x-axis.

2) How does the graph of $y=\sin x$ compare with the graph of $y=\cos x$? Explain how you could horizontally translate the graph of $y=\sin x$ to obtain $y=\cos x$.

3) For the equation $A\cos(Bx+C)+D$, what constants affect the range of the function and how do they affect the range?

Answer

The absolute value of the constant A (amplitude) increases the total range and the constant D (vertical shift) shifts the graph vertically.

4) How does the range of a translated sine function relate to the equation $y=A\sin(Bx+C)+D$?

5) How can the unit circle be used to construct the graph of $f(t)=\sin t$?

Answer

At the point where the terminal side of t intersects the unit circle, you can determine that the $\sin t$ equals the y-coordinate of the point.

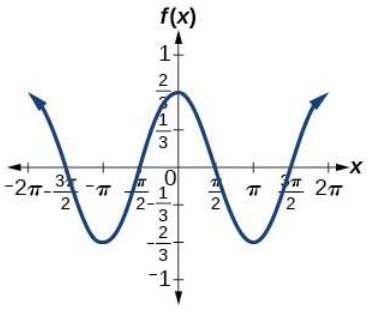
Graphical

For the following exercises, graph two full periods of each function and state the amplitude, period, and midline. State the maximum and minimum y-values and their corresponding x-values on one period for $x>0$. Round answers to two decimal places if necessary.

6) $f(x)=2\sin x$

7) $f(x)=\frac{2}{3}\cos x$

Answer

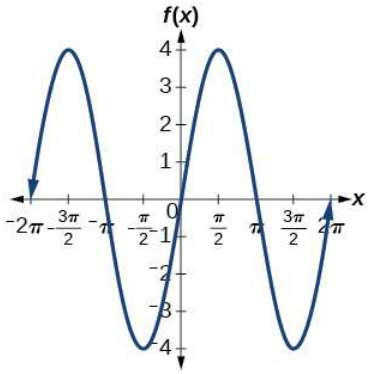


amplitude: $\frac{2}{3}$; period: 2π; midline: $y=0$; maximum: $y=\frac{2}{3}$ occurs at $x=0$; minimum: $y=-\frac{2}{3}$ occurs at $x=\pi$; for one period, the graph starts at 0 and ends at 2π.

8) $f(x) = -3\sin x$

9) $f(x) = 4\sin x$

Answer

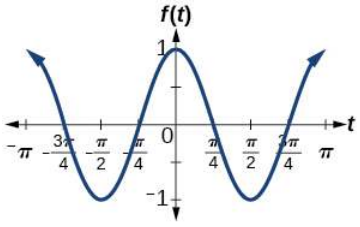


amplitude: 4; period: 2π; midline: $y = 0$; maximum $y = 4$ occurs at $x = \frac{\pi}{2}$; minimum: $y = -4$ occurs at $x = \frac{3\pi}{2}$; one full period occurs from $x = 0$ to $x = 2\pi$

10) $f(x) = 2\cos x$

11) $f(x) = \cos(2x)$

Answer

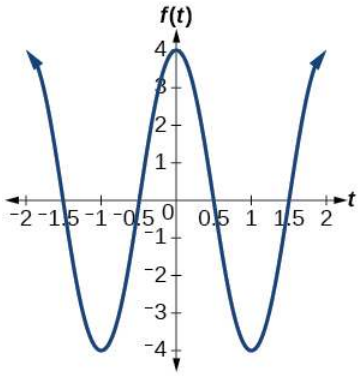


amplitude: 1; period: π; midline: $y = 0$; maximum: $y = 1$ occurs at $x = \pi$; minimum: $y = -1$ occurs at $x = \frac{\pi}{2}$; one full period is graphed from $x = 0$ to $x = \pi$

12) $f(x) = 2\sin\left(\frac{1}{2}x\right)$

13) $f(x) = 4\cos(\pi x)$

Answer

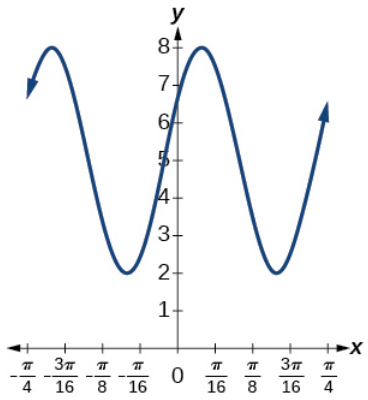


amplitude: 4; period: 2; midline: $y = 0$; maximum: $y = 4$ occurs at $x = 0$; minimum: $y = -4$ occurs at $x = 1$

14) $f(x) = 3\cos\left(\frac{6}{5}x\right)$

15) $y = 3\sin(8(x+4)) + 5$

Answer

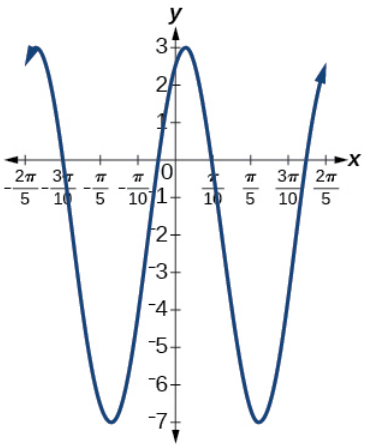


amplitude: 3; period: $\frac{\pi}{4}$; midline: $y = 5$;
maximum: $y = 8$ occurs at $x = -4 + \frac{21\pi}{16} \approx 0.123$;
minimum: $y = 2$ occurs at $x = -4 + \frac{23\pi}{16} \approx 0.516$;
horizontal shift: -4; vertical translation 5;
one period occurs from $x = -4 + \frac{22\pi}{16} \approx 0.320$ to $x = -4 + \frac{26\pi}{16} \approx 1.105$

16) $y = 2\sin(3x - 21) + 4$

17) $y = 5\sin(5x + 20) - 2$

Answer



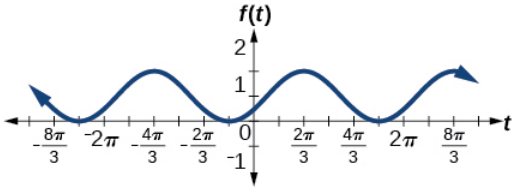
amplitude: 5; period: $\frac{2\pi}{5}$; midline: $y = -2$;
maximum: $y = 3$ occurs at $x = -4 + \frac{13\pi}{10} \approx 0.084$;
minimum: $y = -7$ occurs at $x = -4 + \frac{15\pi}{10} \approx 0.712$;
phase shift: -4; vertical translation: -2;
one full period can be graphed on $x = -4 + \frac{7\pi}{5} \approx 0.398$ to$x = -4 + \frac{9\pi}{5} \approx 1.655$

For the following exercises, graph one full period of each function, starting at $x = 0$.
For each function, state the amplitude, period, and midline.
State the maximum and minimum y-values and their corresponding x-values on one period for $x > 0$.
State the phase shift and vertical translation, if applicable.
Round answers to two decimal places if necessary.

18) $f(t) = 2\sin\left(t - \frac{5\pi}{6}\right)$

19) $f(t) = -\cos\left(t + \frac{\pi}{3}\right) + 1$

Answer

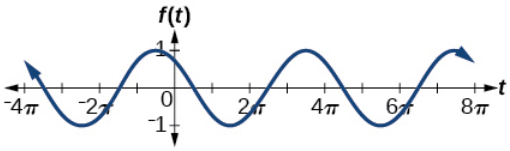


amplitude: 1; period: 2π; midline: $y = 1$;
maximum: $y = 2$ occurs at $t = \frac{2\pi}{3} \approx 2.094$;
minimum: $y = 0$ occurs at $t = \frac{2\pi}{3} \approx 5.24$;
phase shift: $-\frac{\pi}{3}$; vertical translation: 1;
one full period is from $t = \frac{2\pi}{3} \approx 2.094$ to $t = \frac{8\pi}{3} \approx 8.378$

20) $f(t) = 4\cos\left(2\left(t + \frac{\pi}{4}\right)\right) - 3$

21) $f(t) = -\sin\left(\frac{1}{2}t + \frac{5\pi}{3}\right)$

Answer



amplitude: 1; period: 4π; midline: $y = 0$;
maximum: $y = 1$ occurs at $t = \frac{11\pi}{3} \approx 11.52$;
minimum: $y = -1$ occurs at $t = \frac{5\pi}{3} \approx 5.24$;
phase shift: $-\dfrac{10\pi}{3}$; vertical shift: 0;
one full period is from $t = \frac{2\pi}{3} \approx 2.094$ to $t = \frac{14\pi}{3} \approx 14.661$

22) $f(x) = 4\sin\left(\dfrac{\pi}{2}(x-3)\right) + 7$

23) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.

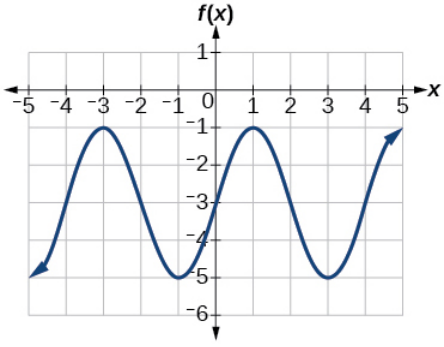


Figure $2.E.23$

Answer

23. amplitude: 2; midline: $y = -3$ period: 4; equation: $f(x) = 2\sin\left(\dfrac{\pi}{2}x\right) - 3$

24) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.

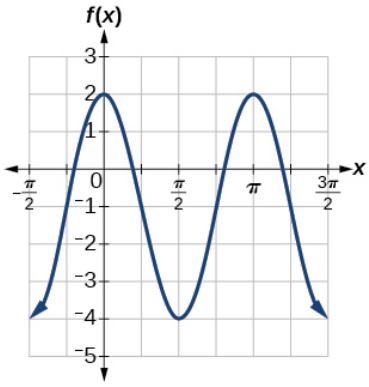


Figure $2.E.24$

25) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.

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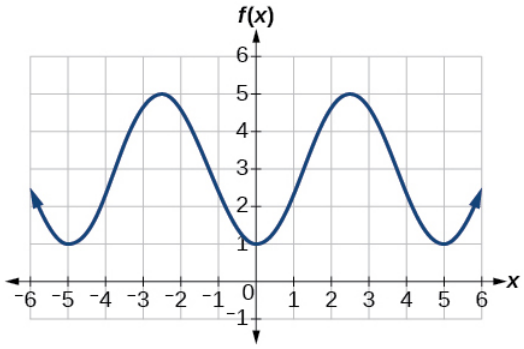


Figure $2.E.25$

Answer

25. amplitude: 2; period: 5; midline: $y = 3$ equation: $f(x) = -2\cos\left(\frac{2\pi}{5}x\right) + 3$

26) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.

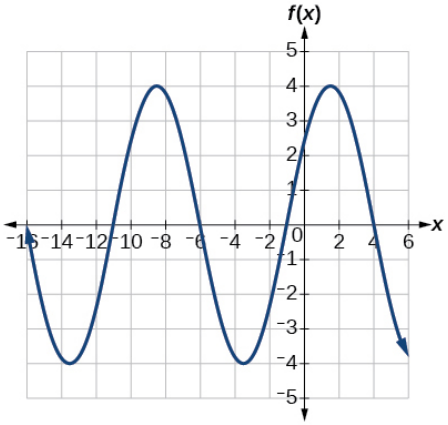


Figure $2.E.26$

27) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.

Figure $2.E.27$

Answer

27. amplitude: 4; period: 2; midline: $y = 0$; equation: $f(x) = -4\cos\left(\pi\left(x - \frac{\pi}{2}\right)\right)$

28) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.

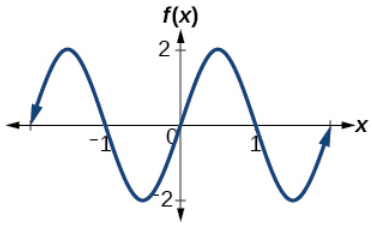


Figure 2.E.28

29) Determine the amplitude, midline, period, and an equation involving the cosine function for the graph shown in Figure below.

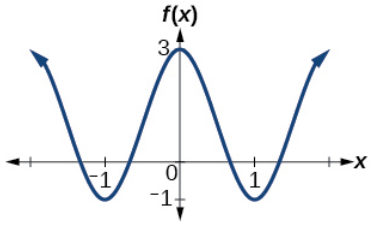


Figure 2.E.29

Answer

29. amplitude: 2; period: 2; midline $y = 1$ equation: $f(x) = 2\cos(\pi x) + 1$

30) Determine the amplitude, midline, period, and an equation involving the sine function for the graph shown in Figure below.

Figure 2.E.30

Algebraic

For the following exercises, let $f(x) = \sin x$.

31) On $[0, 2\pi)$, solve $f(x) = 0$.

32) On $[0, 2\pi)$, solve $f(x) = \dfrac{1}{2}$.

Answer

$\dfrac{\pi}{6}, \dfrac{5\pi}{6}$

33) Evaluate $f\left(\dfrac{\pi}{2}\right)$.

34) On $[0, 2\pi)$, $f(x) = \dfrac{\sqrt{2}}{2}$. Find all values of x.

Answer

$\dfrac{\pi}{4}, \dfrac{3\pi}{4}$

35) On $[0, 2\pi)$, the maximum value(s) of the function occur(s) at what x-value(s)?

36) On $[0, 2\pi)$, the minimum value(s) of the function occur(s) at what x-value(s)?

Answer

$\dfrac{3\pi}{2}$

37) Show that $f(-x) = -f(x)$. This means that $f(x) = \sin x$ is an odd function and possesses symmetry with respect to _____.

For the following exercises, let $f(x) = \cos x$

38) On $[0, 2\pi)$, solve the equation $f(x) = \cos x = 0$

Answer

$\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

39) On $[0, 2\pi)$, solve $f(x) = \dfrac{1}{2}$.

40) On $[0, 2\pi)$, find the x-intercepts of $f(x) = \cos x$.

Answer

$\dfrac{\pi}{2}, \dfrac{3\pi}{2}$

41) On $[0, 2\pi)$, find the x-values at which the function has a maximum or minimum value.

42) On $[0, 2\pi)$, solve the equation $f(x) = \dfrac{\sqrt{3}}{2}$.

Answer

$\dfrac{\pi}{6}, \dfrac{11\pi}{6}$

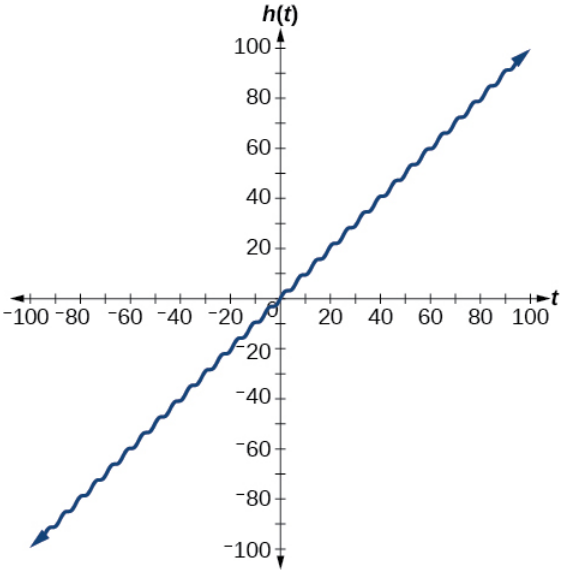
Technology

43) Graph $h(x) = x + \sin x$ on $[0, 2\pi]$. Explain why the graph appears as it does.

44) Graph $h(x) = x + \sin x$ on $[-100, 100]$. Did the graph appear as predicted in the previous exercise?

Answer

The graph appears linear. The linear functions dominate the shape of the graph for large values of x.



45) Graph $f(x) = x \sin x$ on $[0, 2\pi]$ and verbalize how the graph varies from the graph of $f(x) = x \sin x$.

46) Graph $f(x) = x \sin x$ on the window $[-10, 10]$ and explain what the graph shows.

Answer

The graph is symmetric with respect to the y-axis and there is no amplitude because the function is not periodic.

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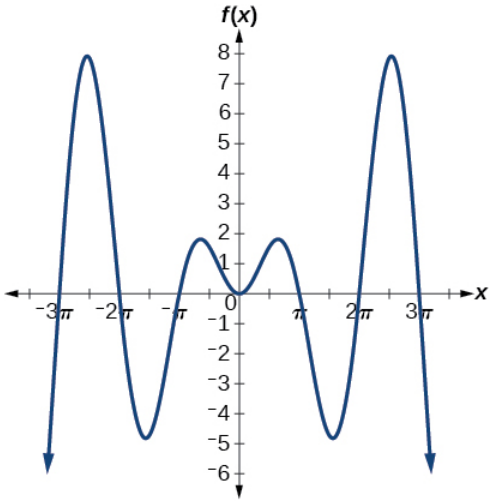


47) Graph $f(x) = \dfrac{\sin x}{x}$ on the window $[-5\pi, 5\pi]$and explain what the graph shows.

Real-World Applications

48) A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h(t)$ gives a person's height in meters above the ground t minutes after the wheel begins to turn

a. Find the amplitude, midline, and period of $h(t)$.
b. Find a formula for the height function $h(t)$.
c. How high off the ground is a person after 5 minutes?

Answer

a. Amplitude: 12.5; period: 10; midline: $y = 13.5$
b. $h(t) = 12.5 \sin\left(\dfrac{\pi}{5}(t - 2.5)\right) + 13.5$
c. 26 ft

6.2: Graphs of the Other Trigonometric Functions

This section addresses the graphing of the Tangent, Cosecant, Secant, and Cotangent curves.

Verbal

1) Explain how the graph of the sine function can be used to graph $y = \csc x$.

Answer

Since $y = \csc x$ is the reciprocal function of $y = \sin x$, you can plot the reciprocal of the coordinates on the graph of $y = \sin x$ obtain the y-coordinates of $y = \csc x$. The x-intercepts of the graph $y = \sin x$ are the vertical asymptotes for the graph of $y = \csc x$.

2) How can the graph of $y = \cos x$ be used to construct the graph of $y = \sec x$?

3) Explain why the period of $y = \tan x$ is equal to π.

Answer

Answers will vary. Using the unit circle, one can show that $y = \tan(x + \pi) = \tan x$.

4) Why are there no intercepts on the graph of $y = \csc x$?

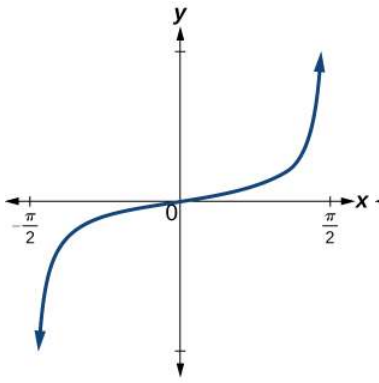
5) How does the period of $y = \csc x$ compare with the period of $y = \sin x$?

Answer

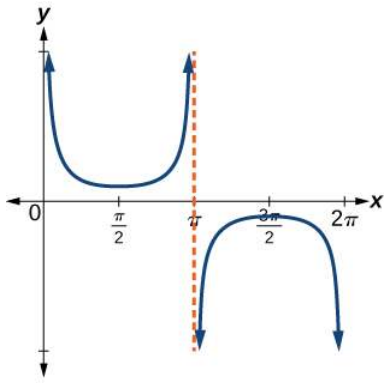
The period is the same: 2π

Algebraic

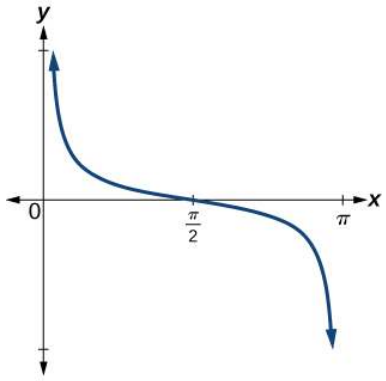
For the exercises 6-9, match each trigonometric function with one of the following graphs.



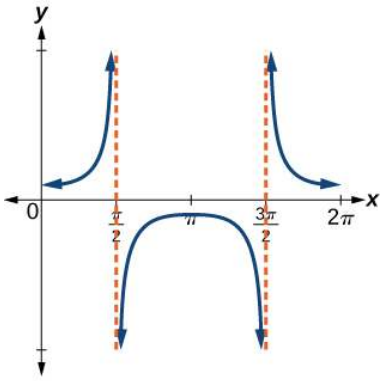
I



II



III



IV

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6) $f(x) = \tan x$

7) $f(x) = \sec x$

Answer

IV

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8) $f(x) = \csc x$

9) $f(x) = \cot x$

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Answer

III

For the exercises 10-16, find the period and horizontal shift of each of the functions.

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10) $f(x) = 2\tan(4x - 32)$

11) $h(x) = 2\sec\left(\frac{\pi}{4}(x + 1)\right)$

Answer

period: 8; horizontal shift: 1 unit to left

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12) $m(x) = 6\csc\left(\frac{\pi}{3}x + \pi\right)$

13) If $\tan x = -1.5$, find $\tan(-x)$.

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Answer

1.5

14) If $\sec x = 2$, find $\sec(-x)$.

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15) If $\csc x = -5$, find $\csc(-x)$.

Answer

5

16) If $x \sin x = 2$, find $(-x)\sin(-x)$.

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For the exercises 17-18, rewrite each expression such that the argument x is positive.

17) $\cot(-x)\cos(-x)+\sin(-x)$

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Answer

$-\cot x \cos x - \sin x$

18) $\cos(-x) + \tan(-x)\sin(-x)$

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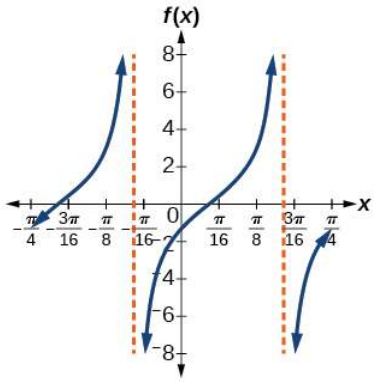
Graphical

For the exercises 19-36, sketch two periods of the graph for each of the following functions. Identify the stretching factor, period, and asymptotes.

19) $f(x) = 2\tan(4x - 32)$

Answer



stretching factor: 2; period: $\frac{\pi}{3}$; asymptotes: $x = \frac{1}{4}\left(\frac{\pi}{2} + \pi k\right) + 8$, where k is an integer

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20) $h(x) = 2\sec\left(\frac{\pi}{4}(x+1)\right)$

21) $m(x) = 6\csc\left(\frac{\pi}{3}x+\pi\right)$

Answer

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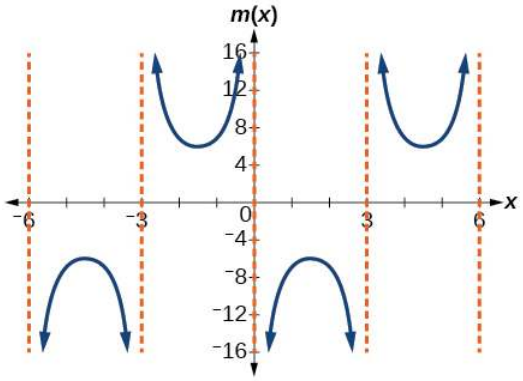

stretching factor: 6; period: 6; asymptotes: $x = k$, where k is an integer

22) $j(x) = \tan\left(\frac{\pi}{2}x\right)$

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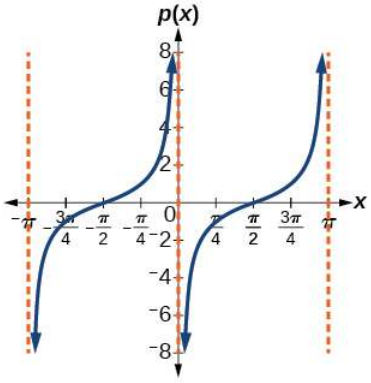
23) $p(x) = \tan\left(x - \frac{\pi}{2}\right)$

Answer



stretching factor: 1; period: π; asymptotes: $x = \pi k$, where k is an integer

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24) $f(x) = 4\tan(x)$

25) $f(x) = \tan\left(x + \frac{\pi}{4}\right)$

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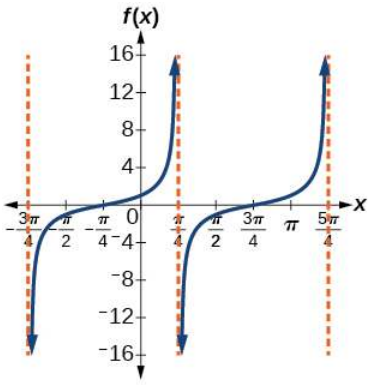
Answer



Stretching factor: 1; period: π; asymptotes: $x = \frac{\pi}{4} + \pi k$, where k is an integer

26) $f(x) = \pi \tan(\pi x - \pi) - \pi$

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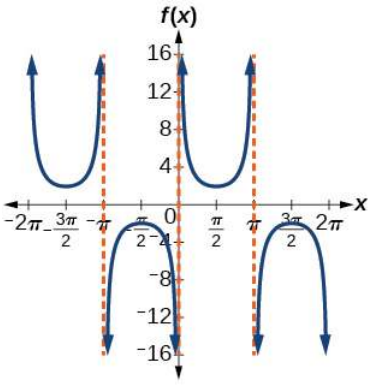
27) $f(x) = 2\csc(x)$

Answer



stretching factor: 2; period: 2π; asymptotes: $x = \pi k$, where k is an integer

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28) $f(x) = -\frac{1}{4}\csc(x)$

29) $f(x) = 4\sec(3x)$

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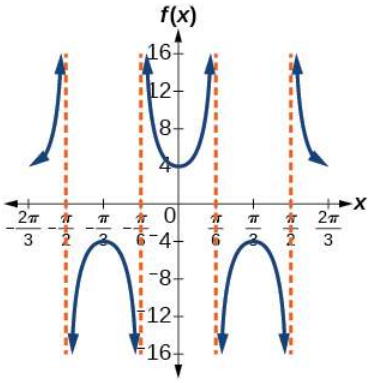
Answer



stretching factor: 4; period: $\frac{2\pi}{3}$; asymptotes: $x = \frac{\pi}{6}k$, where k is an odd integer

30) $f(x) = -3\cot(2x)$

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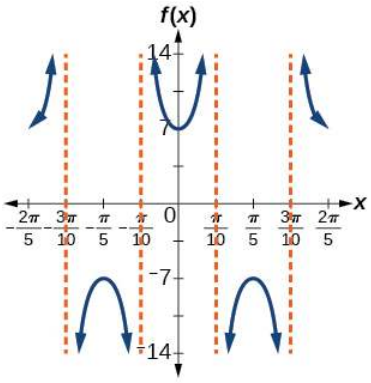
31) $f(x) = 7\sec(5x)$

Answer



stretching factor: 7; period: $\frac{2\pi}{5}$; asymptotes: $x = \frac{\pi}{10}k$, where k is an odd integer

32) $f(x) = \frac{9}{10}\csc(\pi x)$

33) $f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 1$

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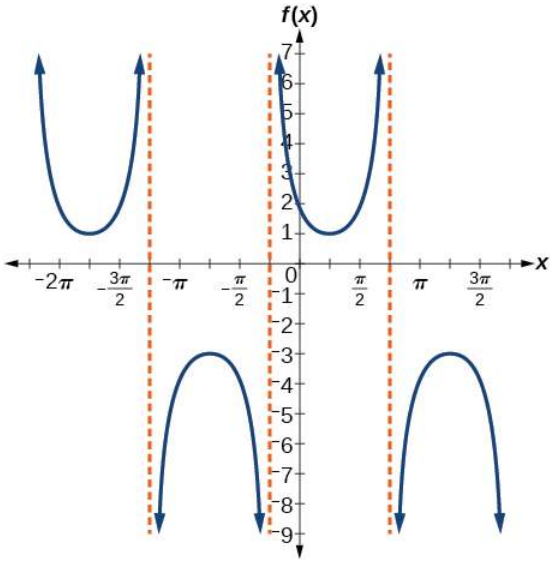
Answer



Stretching factor: 2; period: 2π ; asymptotes: $x = -\frac{\pi}{4} + \pi k$, where k is an integer

34) $f(x) = -\sec\left(x - \frac{\pi}{3}\right) - 2$

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35) $f(x) = \frac{7}{5}\csc\left(x - \frac{\pi}{4}\right)$

Answer

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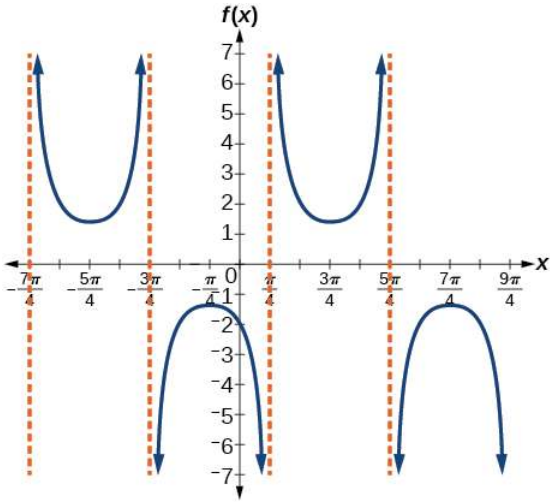

Stretching factor: $\frac{7}{5}$; period: 2π ; asymptotes: $x = \frac{\pi}{4} + \pi k$, where k is an integer

36) $f(x) = 5\left(\cot\left(x + \frac{\pi}{2}\right) - 3\right)$

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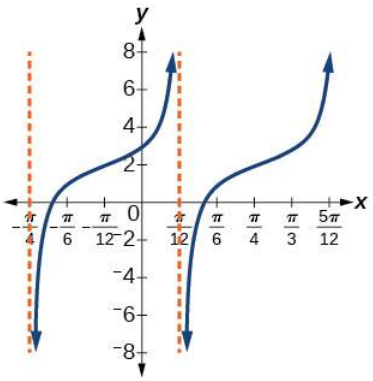
37) A tangent curve, $A = 1$, period of $\frac{\pi}{3}$; and phase shift $(h, k) = \left(\frac{\pi}{4}, 2\right)$

Answer

$$y = \tan\left(3\left(x - \frac{\pi}{4}\right)\right) + 2$$



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38) A tangent curve, $A = -2$, period of $\frac{\pi}{4}$; and phase shift $(h, k) = \left(-\frac{\pi}{4}, -2\right)$

For the exercises 39-45, find an equation for the graph of each function.

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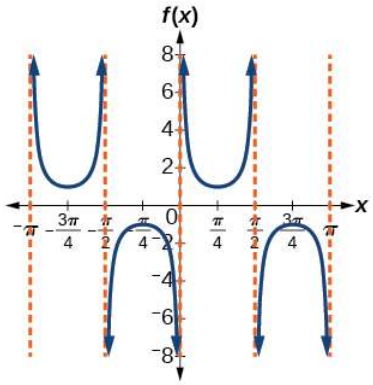
39)



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Answer

$f(x) = \csc(2x)$

40)

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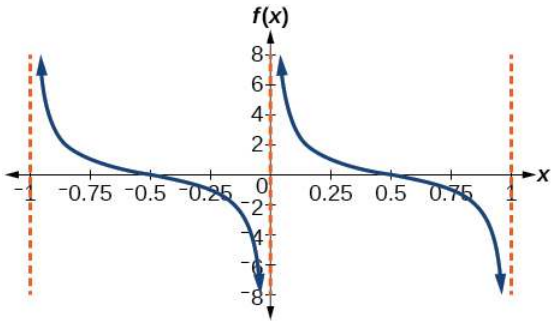

41)

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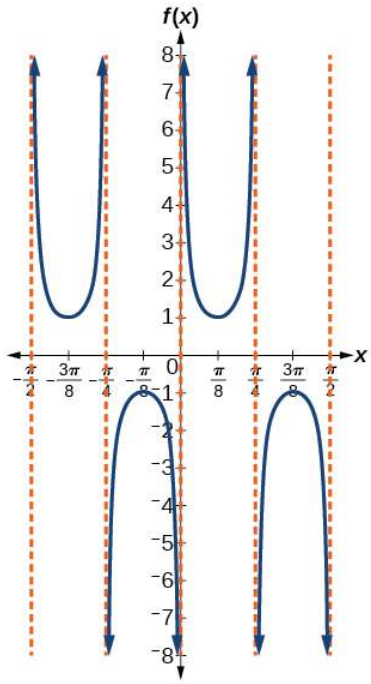

Answer

$f(x) = \csc(4x)$

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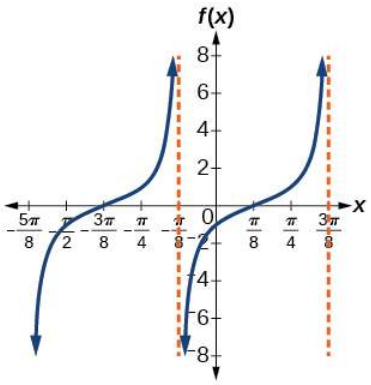
42)



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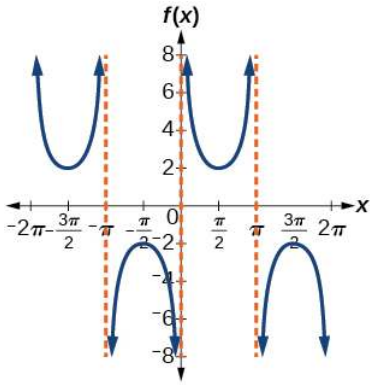
43)



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Answer

$f(x) = 2 \csc x$

44)

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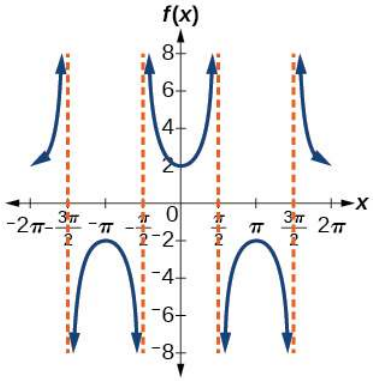

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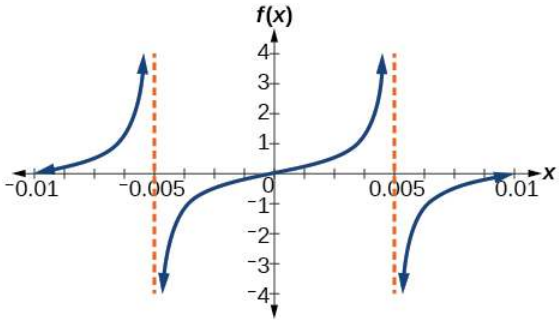

Answer

$$f(x) = \frac{1}{2}\tan(100\pi x)$$

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Technology

For the exercises 46-53, use a graphing calculator to graph two periods of the given function. Note: most graphing calculators do not have a cosecant button; therefore, you will need to input $\csc x$ as $\frac{1}{\sin x}$

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46) $f(x) = |\csc(x)|$

47) $f(x) = |\cot(x)|$

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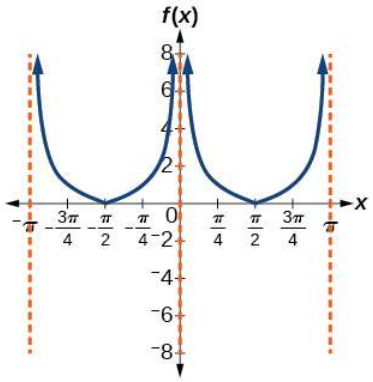
Answer



48) $f(x) = 2^{\csc(x)}$

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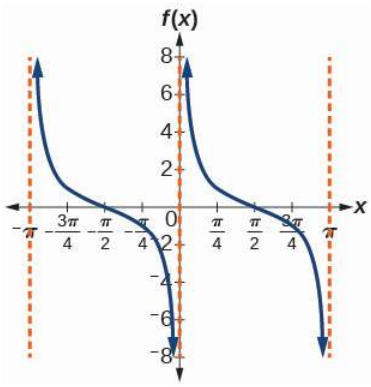
49) $f(x) = \frac{\csc(x)}{\sec(x)}$

Answer



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50) Graph $f(x) = 1 + \sec^2(x) - \tan^2(x)$. What is the function shown in the graph?

51) $f(x) = \sec(0.001x)$

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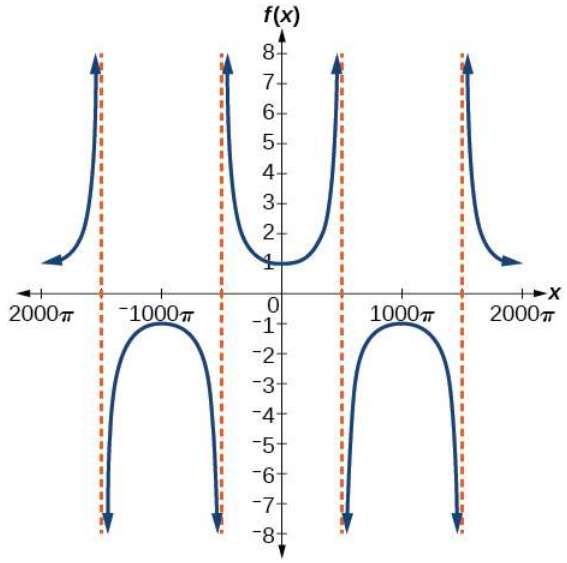
Answer



52) $f(x) = \cot(100\pi x)$

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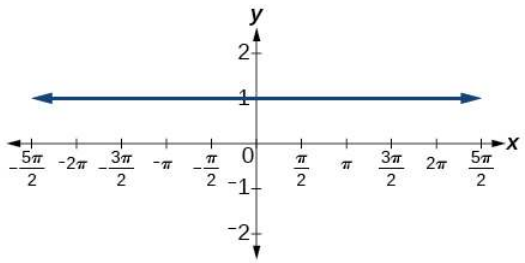
53) $f(x) = \sin^2 x + \cos^2 x$

Answer



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Real-World Applications

54) The function $f(x) = 20\tan\left(\frac{\pi}{10}x\right)$ marks the distance in the movement of a light beam from a police car across a wall for time x, in seconds, and distance $f(x)$, in feet.

a. Graph on the interval $[0, 5]$
b. Find and interpret the stretching factor, period, and asymptote.
c. Evaluate $f(10)$ and $f(2.5)$ and discuss the function's values at those inputs.

55) Standing on the shore of a lake, a fisherman sights a boat far in the distance to his left. Let x, measured in radians, be the angle formed by the line of sight to the ship and a line due north from his position. Assume due north is 0 and x is measured negative to the left and positive to the right. (See Figure below.) The boat travels from due west to due east and, ignoring the curvature of the Earth, the distance $d(x)$, in kilometers, from the fisherman to the boat is given by the function $d(x) = 1.5 \sec(x)$.

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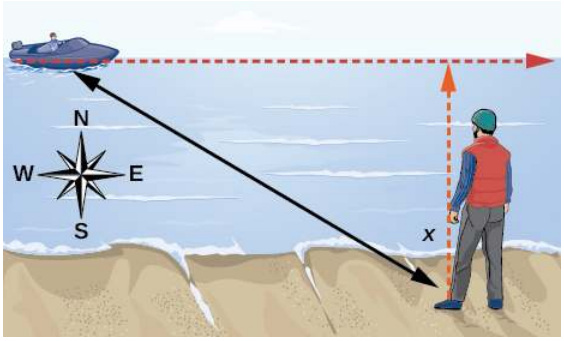
a. What is a reasonable domain for $d(x)$?
b. Graph $d(x)$ on this domain.
c. Find and discuss the meaning of any vertical asymptotes on the graph of $d(x)$.
d. Calculate and interpret $d\left(-\frac{\pi}{3}\right)$. Round to the second decimal place.
e. Calculate and interpret $d\left(\frac{\pi}{6}\right)$. Round to the second decimal place.
f. What is the minimum distance between the fisherman and the boat? When does this occur?



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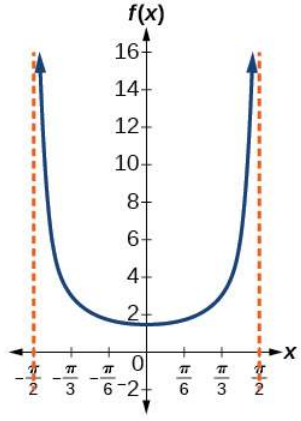
Answer

a. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b. 

c. $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$; the distance grows without bound as $|x|$ approaches $\frac{\pi}{2}$—i.e., at right angles to the line representing due north, the boat would be so far away, the fisherman could not see it;

d. 3; when $x = -\frac{\pi}{3}$, the boat is 3 km away;

e. 1.73; when $x = \frac{\pi}{6}$, the boat is about 1.73km away;

f. 1.5 km; when $x = 0$

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56) A laser rangefinder is locked on a comet approaching Earth. The distance $g(x)$, in kilometers, of the comet after x days, for x in the interval 0 to 30 days, is given by $g(x) = 250,000 \csc\left(\frac{\pi}{30}x\right)$.

a. Graph $g(x)$ on the interval $[0, 35]$.
b. Evaluate $g(5)$ and interpret the information.
c. What is the minimum distance between the comet and Earth? When does this occur? To which constant in the equation does this correspond?
d. Find and discuss the meaning of any vertical asymptotes.

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57) A video camera is focused on a rocket on a launching pad 2 miles from the camera. The angle of elevation from the ground to the rocket after x seconds is $\frac{\pi}{120}x$.

a. Write a function expressing the altitude $h(x)$, in miles, of the rocket above the ground after x seconds. Ignore the curvature of the Earth.
b. Graph $h(x)$ on the interval $(0, 60)$.
c. Evaluate and interpret the values $h(0)$ and $h(30)$.
d. What happens to the values of $h(x)$ as x approaches 60 seconds? Interpret the meaning of this in terms of the problem.

Answer

a. $h(x) = 2\tan\left(\frac{\pi}{120}x\right)$

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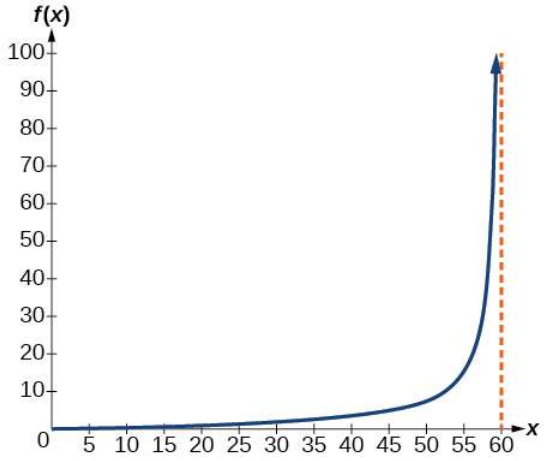
b. 

c. $h(0)=0$: after 0 seconds, the rocket is 0 mi above the ground; $h(30)=2$: after 30 seconds, the rockets is 2 mi high;

d. As x approaches 60 seconds, the values of $h(x)$ grow increasingly large. The distance to the rocket is growing so large that the camera can no longer track it.

6.3: Inverse Trigonometric Functions

In this section, we will explore the inverse trigonometric functions. Inverse trigonometric functions "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa.

Verbal

1) Why do the functions $f(x)=\sin^{-1}x$ and $g(x)=\cos^{-1}x$ have different ranges?

Answer

The function $y=\sin x$ is one-to-one on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$; thus, this interval is the range of the inverse function of $y=\sin x$, $f(x)=\sin^{-1}x$ The function $y=\cos x$ is one-to-one on $[0,\pi]$; thus, this interval is the range of the inverse function of $y=\cos x$, $f(x)=\cos^{-1}x$

2) Since the functions $y=\cos x$ and $y=\cos^{-1}x$ are inverse functions, why is $\cos^{-1}\left(\cos\left(-\frac{\pi}{6}\right)\right)$ not equal to $-\frac{\pi}{6}$?

3) Explain the meaning of $\frac{\pi}{6}=\arcsin(0.5)$.

Answer

$\frac{\pi}{6}$ is the radian measure of an angle between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose sine is 0.5.

4) Most calculators do not have a key to evaluate $\sec^{-1}(2)$. Explain how this can be done using the cosine function or the inverse cosine function.

5) Why must the domain of the sine function, $\sin x$, be restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for the inverse sine function to exist?

Answer

In order for any function to have an inverse, the function must be one-to-one and must pass the horizontal line test. The regular sine function is not one-to-one unless its domain is restricted in some way. Mathematicians have agreed to restrict the sine function to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that it is one-to-one and possesses an inverse.

6) Discuss why this statement is incorrect: $\arccos(\cos x) = x$ for all x.

7) Determine whether the following statement is true or false and explain your answer: $\arccos(-x) = \pi - \arccos x$

Answer

True . The angle, θ_1 that equals $\arccos(-x)$, $x > 0$, will be a second quadrant angle with reference angle, θ_2, where θ_2 equals $\arccos x$, $x > 0$. Since θ_2 is the reference angle for θ_1, $\theta_2 = \pi - \theta_1$ and $\arccos(-x) = \pi - \arccos x -$

Algebraic

For the exercises 8-16, evaluate the expressions.

8) $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$

9) $\sin^{-1}\left(-\frac{1}{2}\right)$

Answer

$-\frac{\pi}{6}$

10) $\cos^{-1}\left(-\frac{1}{2}\right)$

11) $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Answer

$\frac{3\pi}{4}$

12) $\tan^{-1}(1)$

13) $\tan^{-1}(-\sqrt{3})$

Answer

$-\frac{\pi}{3}$

14) $\tan^{-1}(-1)$

15) $\tan^{-1}(\sqrt{3})$

Answer

$\frac{\pi}{3}$

16) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$

For the exercises 17-21, use a calculator to evaluate each expression. Express answers to the nearest hundredth.

17) $\cos^{-1}(-0.4)$

Answer

1.98

18) $\arcsin(0.23)$

19) $\arccos\left(\frac{3}{5}\right)$

Answer

0.93

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20) $\cos^{-1}(-0.8)$

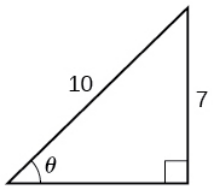
21) $\tan^{-1}(6)$

Answer

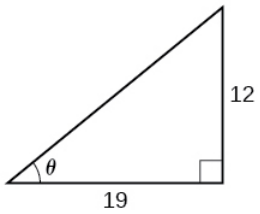
1.41

For the exercises 22-23, find the angle θ in the given right triangle. Round answers to the nearest hundredth.

22)



23)



Answer

0.56radians

For the exercises 24-36, find the exact value, if possible, without a calculator. If it is not possible, explain why.

24) $\sin^{-1}(\cos(\pi))$

25) $\tan^{-1}(\sin(\pi))$

Answer

0

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26) $\cos^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

27) $\tan^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

Answer

0.71

28) $\sin^{-1}\left(\cos\left(\frac{-\pi}{2}\right)\right)$

29) $\tan^{-1}\left(\sin\left(\frac{4\pi}{3}\right)\right)$

Answer

-0.71

30) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

31) $\tan^{-1}\left(\sin\left(\frac{-5\pi}{2}\right)\right)$

Answer

$-\frac{\pi}{4}$

32) $\cos\left(\sin^{-1}\left(\frac{4}{5}\right)\right)$

33) $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$

Answer

0.8

34) $\sin\left(\tan^{-1}\left(\frac{4}{3}\right)\right)$

35) $\cos\left(\tan^{-1}\left(\frac{12}{5}\right)\right)$

Answer

$\frac{5}{13}$

36) $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)\right)$

For the exercises 37-41, find the exact value of the expression in terms of x with the help of a reference triangle.

37) $\tan\left(\sin^{-1}(x-1)\right)$

Answer

$\frac{x-1}{\sqrt{-x^2+2x}}$

38) $\sin\left(\sin^{-1}(1-x)\right)$

39) $\cos\left(\sin^{-1}\left(\frac{1}{x}\right)\right)$

Answer

$\frac{\sqrt{x^2-1}}{x}$

40) $\cos\left(\tan^{-1}(3x-1)\right)$

41) $\tan\left(\sin^{-1}\left(x+\frac{1}{2}\right)\right)$

Answer

$\frac{x+0.5}{\sqrt{-x^2-x+\frac{3}{4}}}$

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Extensions

For the exercise 42, evaluate the expression without using a calculator. Give the exact value.

2) $\dfrac{\sin^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\cos^{-1}(1)}{\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)+\cos^{-1}\left(\frac{1}{2}\right)-\sin^{-1}(0)}$

For the exercises 43-47, find the function if $\sin t = \dfrac{x}{x+1}$

43) $\cos t$

Answer

$\dfrac{\sqrt{2x+1}}{x+1}$

44) $\sec t$

45) $\cot t$

Answer

$\dfrac{\sqrt{2x+1}}{x}$

46) $\cos\left(\sin^{-1}\left(\dfrac{x}{x+1}\right)\right)$

47) $\tan^{-1}\left(\dfrac{x}{\sqrt{2x+1}}\right)$

Answer

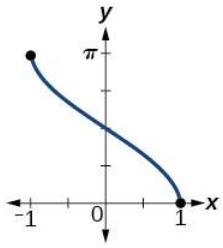
t

Graphical

48) Graph $y=\sin^{-1}x$ and state the domain and range of the function.

49) Graph $y=\arccos x$ and state the domain and range of the function.

Answer



domain $[-1, 1]$; range $[0, \pi]$

50) Graph one cycle of $y = \tan^{-1} x$ and state the domain and range of the function.

51) For what value of x does $\sin x = \sin^{-1} x$? Use a graphing calculator to approximate the answer.

Answer

approximately $x = 0.00$

52) For what value of x does $\cos x = \cos^{-1} x$? Use a graphing calculator to approximate the answer.

Real-World Applications

53) Suppose a 13-foot ladder is leaning against a building, reaching to the bottom of a second-floor window 12 feet above the ground. What angle, in radians, does the ladder make with the building?

Answer

0.395radians

54) Suppose you drive 0.6 miles on a road so that the vertical distance changes from 0 to 150 feet. What is the angle of elevation of the road?

55) An isosceles triangle has two congruent sides of length 9 inches. The remaining side has a length of 8 inches. Find the angle that a side of 9 inches makes with the 8-inch side.

Answer

1.11radians

56) Without using a calculator, approximate the value of $\arctan(10,000)$. Explain why your answer is reasonable.

57) A truss for the roof of a house is constructed from two identical right triangles. Each has a base of 12 feet and height of 4 feet. Find the measure of the acute angle adjacent to the 4-foot side.

Answer

1.25radians

58) The line $y = \frac{3}{5}x$ passes through the origin in the x, y-plane. What is the measure of the angle that the line makes with the positive x-axis?

59) The line $y = \frac{-3}{7}x$ passes through the origin in the x, y-plane. What is the measure of the angle that the line makes with the negative x-axis?

Answer

0.405radians

60) What percentage grade should a road have if the angle of elevation of the road is 4 degrees? (The percentage grade is defined as the change in the altitude of the road over a 100-foot horizontal distance. For example a 5% grade means that the road rises 5 feet for every 100 feet of horizontal distance.)

61) A 20-foot ladder leans up against the side of a building so that the foot of the ladder is 10 feet from the base of the building. If specifications call for the ladder's angle of elevation to be between 35 and 45 degrees, does the placement of this ladder satisfy safety specifications?

Answer

No. The angle the ladder makes with the horizontal is 60 degrees.

62) Suppose a 15-foot ladder leans against the side of a house so that the angle of elevation of the ladder is 42 degrees. How far is the foot of the ladder from the side of the house?

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2.R: Periodic Functions (Review)

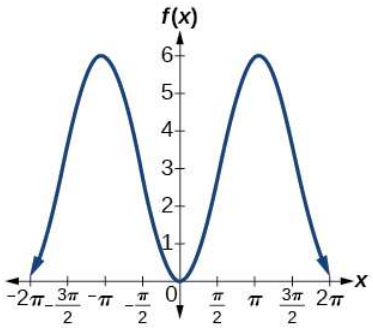
6.1: Graphs of the Sine and Cosine Functions

For the exercises 1-8, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

1) $f(x) = -3\cos x + 3$

Answer

amplitude: 3; period: 2π; midline: $y = 3$; no asymptotes

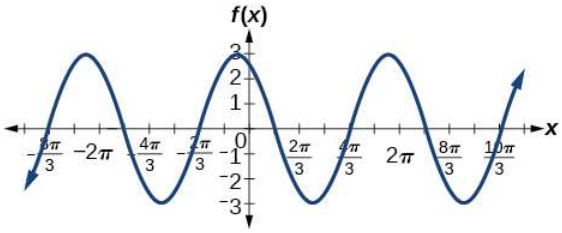


2) $f(x) = \frac{1}{4}\sin x$

3) $f(x) = 3\cos\left(x + \frac{\pi}{6}\right)$

Answer

amplitude: 3; period: 2π; midline: $y = 0$; no asymptotes

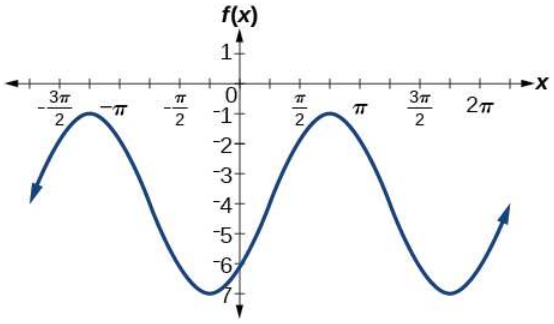


4) $f(x) = -2\sin\left(x - \frac{2\pi}{3}\right)$

5) $f(x) = 3\sin\left(x - \frac{\pi}{4}\right) - 4$

Answer

amplitude: 3; period: 2π; midline: $y = -4$; no asymptotes

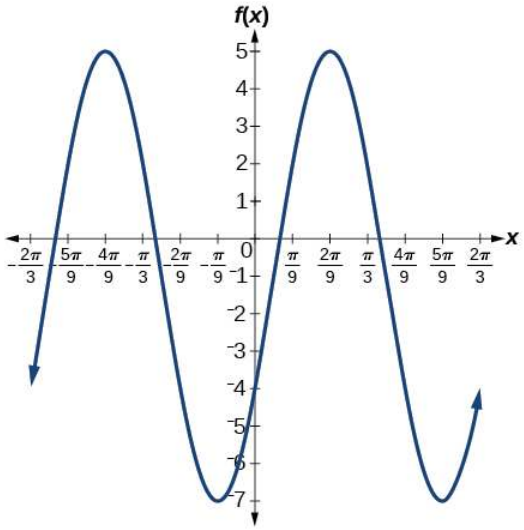


6) $f(x) = 2\left(\cos\left(x - \frac{4\pi}{3}\right) + 1\right)$

7) $f(x) = 6\sin\left(3x - \frac{\pi}{6}\right) - 1$

Answer

amplitude: 6; period: $dfrac2\pi3$; midline: $y = -1$; no asymptotes



8) $f(x) = -100\sin(50x - 20)$

6.2: Graphs of the Other Trigonometric Functions

For the exercises 1-4, graph the functions for two periods and determine the amplitude or stretching factor, period, midline equation, and asymptotes.

1) $f(x) = \tan x - 4$

Answer

stretching factor: none; period: π; midline: $y = -4$; asymptotes: $x = \frac{\pi}{2} + \pi k$, where k is an integer

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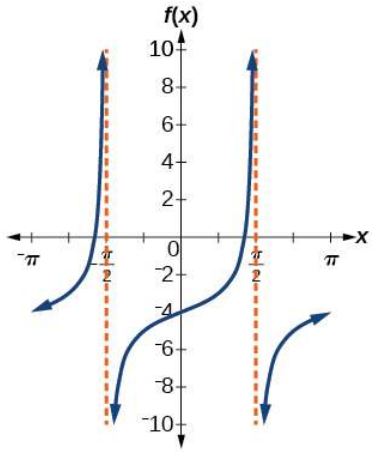
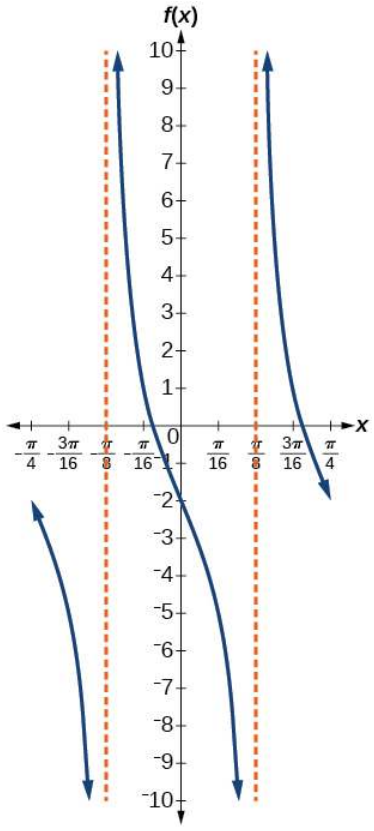


2) $f(x) = 2\tan\left(x - \frac{\pi}{6}\right)$

3) $f(x) = -3\tan(4x) - 2$

Answer

stretching factor: 3; period: $\frac{\pi}{4}$; midline: $y = -2$; asymptotes: $x = \frac{\pi}{8} + \frac{\pi}{4}k$, where k is an integer



4) $f(x) = 0.2\cos(0.1x) + 0.3$

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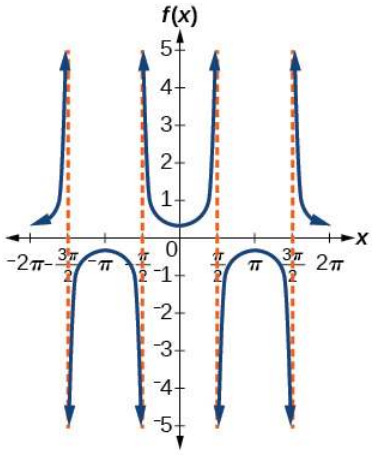
For the exercises 5-10, graph two full periods. Identify the period, the phase shift, the amplitude, and asymptotes.

5) $f(x) = \frac{1}{3}\sec x$

Answer

amplitude: none; period: 2π; no phase shift; asymptotes: $x = \frac{\pi}{2}k$, where k is an integer

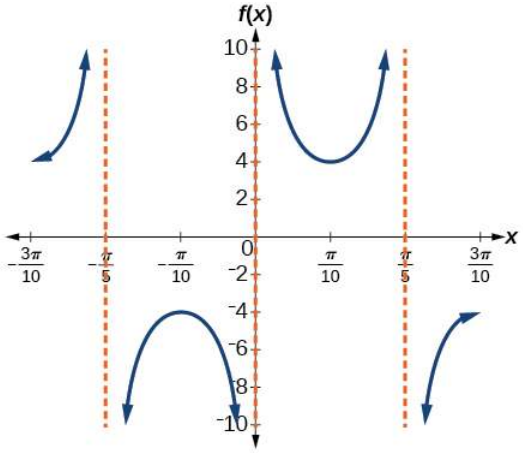


6) $f(x) = 3\cot x$

7) $f(x) = 4\csc(5x)$

Answer

amplitude: none; period: $\frac{2\pi}{5}$; no phase shift; asymptotes: $x = \frac{\pi}{5}k$, where k is an integer

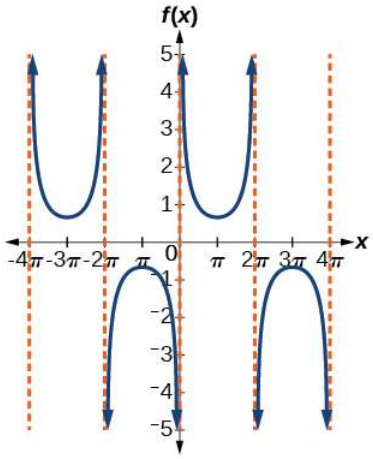


8) $f(x) = 8\sec\left(\frac{1}{4}x\right)$

9) $f(x) = \frac{2}{3}\csc\left(\frac{1}{2}x\right)$

Answer

amplitude: none; period: 4π; no phase shift; asymptotes: $x = 2\pi k$, where k is an integer



10) $f(x) = -\csc(2x + \pi)$

For the exercises 11-15, use this scenario: The population of a city has risen and fallen over a 20-year interval. Its population may be modeled by the following function: $y = 12{,}000 + 8{,}000\sin(0.628x)$, where the domain is the years since 1980 and the range is the population of the city.

11) What is the largest and smallest population the city may have?

Answer

largest: $20,000$; smallest: $4,000$

12) Graph the function on the domain of $[0, 40]$.

13) What are the amplitude, period, and phase shift for the function?

Answer

amplitude: $8,000$; period: 10; phase shift: 0

14) Over this domain, when does the population reach $18,000$? $13,000$?

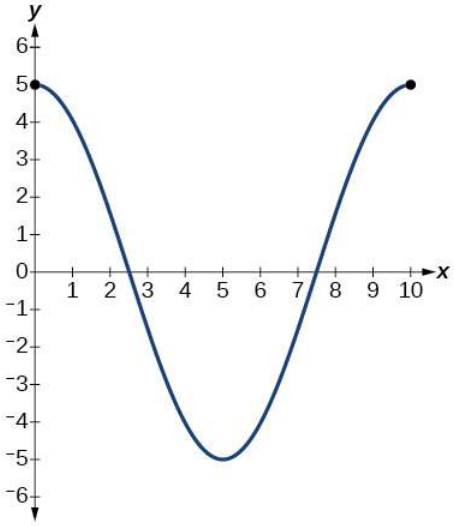
15) What is the predicted population in 2007? 2010?

Answer

In 2007, the predicted population is $4,413$. In 2010, the population will be $11,924$.

For the exercises 16a-16d, suppose a weight is attached to a spring and bobs up and down, exhibiting symmetry.

16) Suppose the graph of the displacement function is shown in the Figure below, where the values on the x-axis represent the time in seconds and the y-axis represents the displacement in inches.



a. Give the equation that models the vertical displacement of the weight on the spring.
b. At $\text{time} = 0$, what is the displacement of the weight?

Answer

5 in.

c. At what time does the displacement from the equilibrium point equal zero?
d. What is the time required for the weight to return to its initial height of 5 inches? In other words, what is the period for the displacement function?

Answer

10 seconds

6.3: Inverse Trigonometric Functions

For the exercises 1-11, find the exact value without the aid of a calculator.

1) $\sin^{-1}(1)$

2) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Answer

$\frac{\pi}{6}$

3) $\tan^{-1}(-1)$

4) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

Answer

$\frac{\pi}{4}$

5) $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right)$

6) $\sin^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)$

Answer

$\frac{\pi}{3}$

7) $\cos^{-1}\left(\tan\left(\frac{3\pi}{4}\right)\right)$

8) $\sin\left(\sec^{-1}\left(\frac{3}{5}\right)\right)$

Answer

No solution

9) $\cot\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$

10) $\tan\left(\cos^{-1}\left(\frac{5}{13}\right)\right)$

Answer

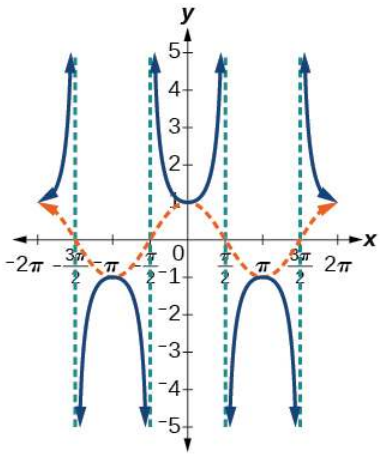
$\frac{12}{5}$

11) $\sin\left(\cos^{-1}\left(\frac{x}{x+1}\right)\right)$

12) Graph $f(x)=\cos x$ and $f(x)=\sec x$ on the interval $[0,2\pi)$ and explain any observations.

Answer

The graphs are not symmetrical with respect to the line $y=x$. They are symmetrical with respect to the y-axis.



13) Graph $f(x) = \sin x$ and $f(x) = \csc x$ and explain any observations.

14) Graph the function $f(x) = \dfrac{x}{1} - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!}$ on the interval $[-1, 1]$ and compare the graph to the graph of $f(x) = \sin x$ on the same interval. Describe any observations.

Answer

The graphs appear to be identical.

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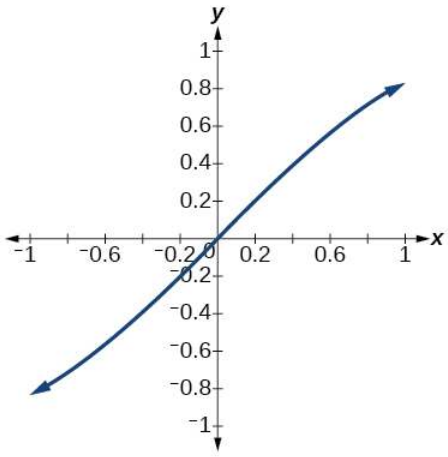
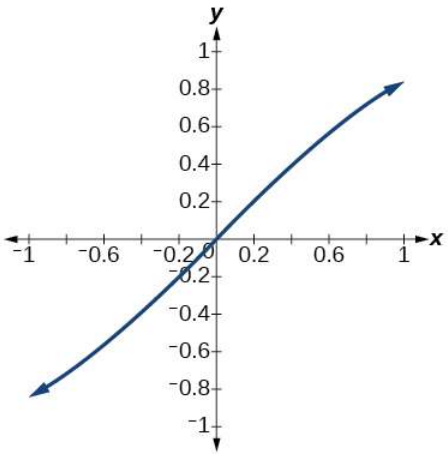




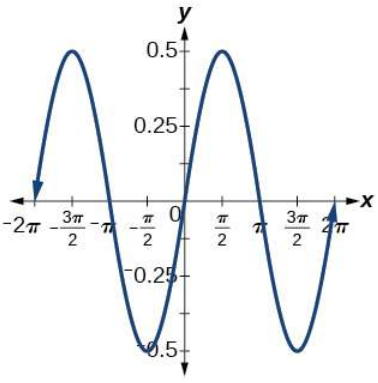
Practice Test

For the exercises 1-13, sketch the graph of each function for two full periods. Determine the amplitude, the period, and the equation for the midline.

1) $f(x) = 0.5 \sin x$

Answer

amplitude: 0.5; period: 2π; midline $y = 0$



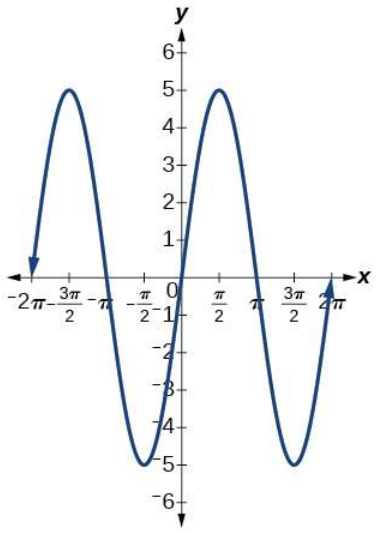
y=0 y=0

2) $f(x) = 5\cos x$

3) $f(x) = 5\sin x$

Answer

amplitude: 0.5; period: 2π; midline $y = 0$

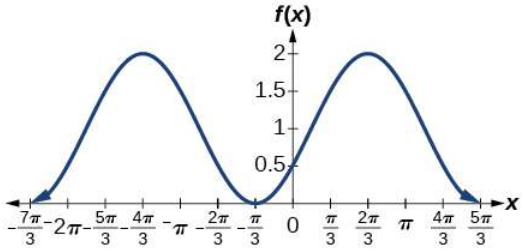


4) $f(x) = \sin(3x)$

5) $f(x) = -\cos\left(x + \dfrac{\pi}{3}\right) + 1$

Answer

amplitude: 1; period: 2π; midline $y = 1$

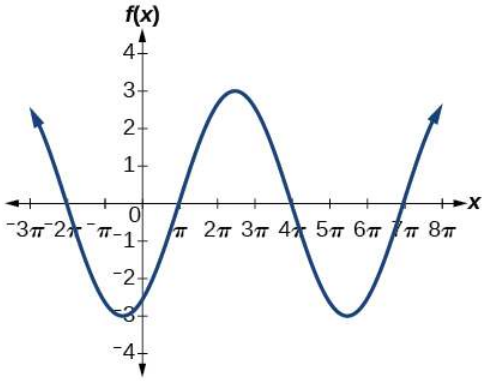


6) $f(x) = 5\sin\left(3\left(x - \dfrac{\pi}{6}\right)\right) + 4$

7) $f(x) = 3\cos\left(\dfrac{1}{3}x - \dfrac{5\pi}{6}\right)$

Answer

amplitude: 3; period: 6π; midline $y = 0$

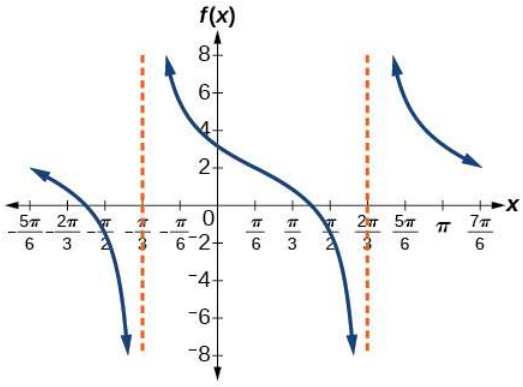


8) $f(x) = \tan(4x)$

9) $f(x) = -2\tan\left(x - \frac{7\pi}{6}\right) + 2$

Answer

amplitude: none; period: π; midline $y = 0$, asymptotes: $x = \frac{2\pi}{3} + \pi k$, where k is an integer



10) $f(x) = \pi\cos(3x + \pi)$

11) $f(x) = 5\csc(3x)$

Answer

amplitude: none; period: $\frac{2\pi}{3}$; midline $y = 0$, asymptotes: $x = \frac{\pi}{3}k$, where k is an integer

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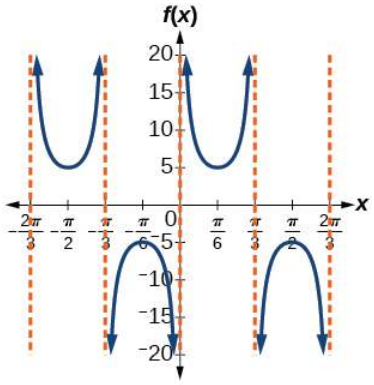
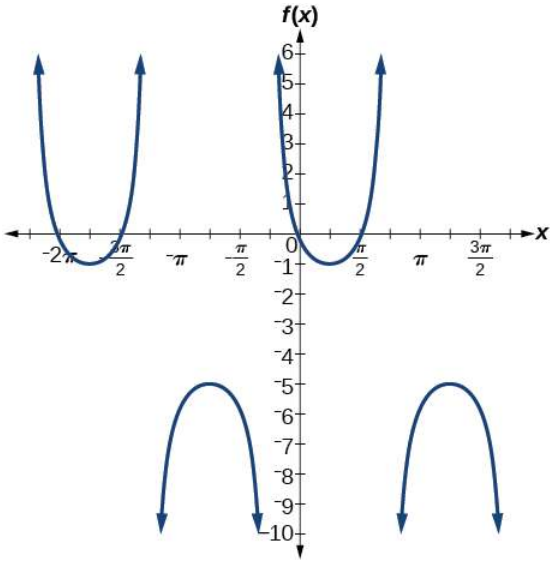


12) $f(x) = \pi \sec\left(\frac{\pi}{2}x\right)$

13) $f(x) = 2\csc\left(x + \frac{\pi}{4}\right) - 3$

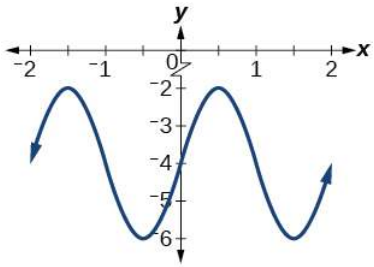
Answer

amplitude: none; period: 2π; midline $y = -3$

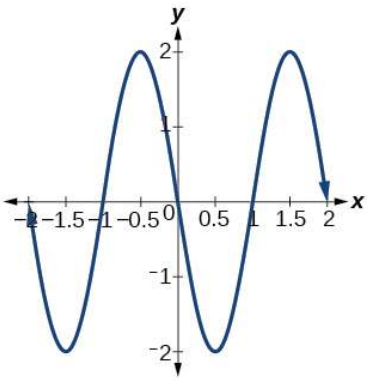


For the exercises 14-16, determine the amplitude, period, and midline of the graph, and then find a formula for the function.

14) Give in terms of a sine function.



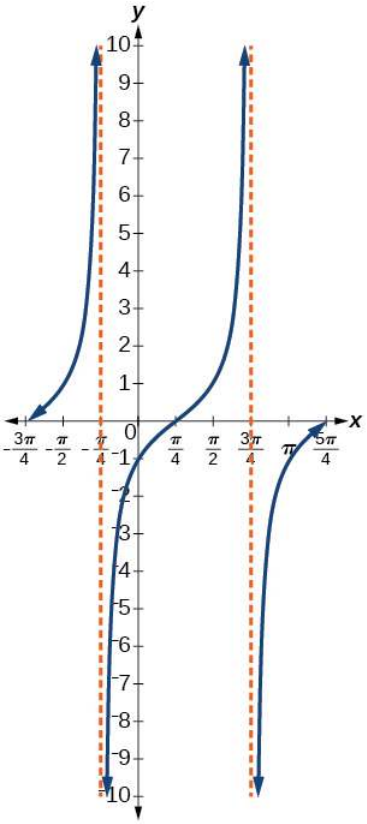
15) Give in terms of a sine function.



Answer

amplitude: 2; period: 2; midline: $y = 0$; $f(x) = 2\sin(\pi(x-1))$

16) Give in terms of a tangent function.



For the exercises 17-20, find the amplitude, period, phase shift, and midline.

17) $y = \sin\left(\frac{\pi}{6}x + \pi\right) - 3$

Answer

amplitude: 1; period: 12; phase shift: -6; midline: $y = -3$

18) $y = 8\sin\left(\frac{7\pi}{6}x + \frac{7\pi}{2}\right) + 6$

19) The outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 68°F at midnight and the high and low temperatures during the day are 80°F and 56°F, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

Answer

$D(t) = 68 - 12\sin\left(\frac{\pi}{12}x\right)$

20) Water is pumped into a storage bin and empties according to a periodic rate. The depth of the water is 3 feet at its lowest at 2:00 a.m. and 71 feet at its highest, which occurs every 5 hours. Write a cosine function that models the depth of the water as a function of time, and then graph the function for one period.

For the exercises 21-25, find the period and horizontal shift of each function.

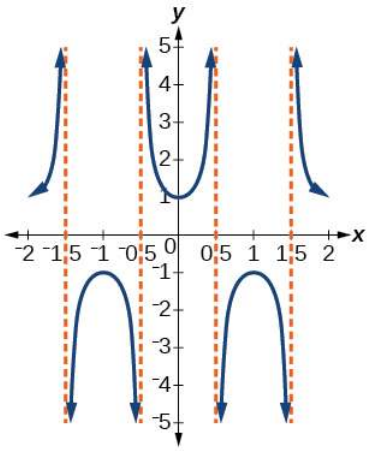
21) $g(x) = 3\tan(6x + 42)$

Answer

period: $\frac{\pi}{6}$; horizontal shift: -7

22) $n(x) = 4\csc\left(\frac{5\pi}{3}x - \frac{20\pi}{3}\right)$

23) Write the equation for the graph in the Figure below in terms of the secant function and give the period and phase shift.



Answer

$f(x) = \sec(\pi x)$; period: 2; phase shift: 0

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24) If $\tan x = 3$, find $\tan(-x)$.

25) If $\sec x = 4$, find $\sec(-x)$.

Answer

4

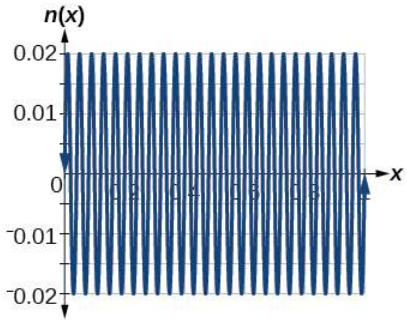
For the exercises 26-28, graph the functions on the specified window and answer the questions.

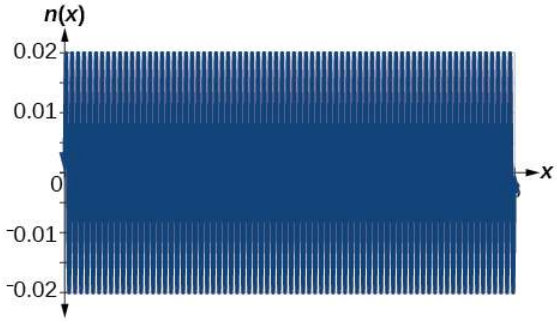
26) Graph $m(x) = \sin(2x) + \cos(3x)$ on the viewing window $[-10, 10]$ by $[-3, 3]$. Approximate the graph's period.

27) Graph $n(x) = 0.02\sin(50\pi x)$ on the following domains in x : $[0, 1]$ and$[0, 3]$. Suppose this function models sound waves. Why would these views look so different?

Answer

The views are different because the period of the wave is 125. Over a bigger domain, there will be more cycles of the graph.





28) Graph $f(x) = \dfrac{\sin x}{x}$ on $[-0.5, 0.5]$ and explain any observations.

For the exercises 29-31, let $f(x) = \dfrac{3}{5}\cos(6x)$.

29) What is the largest possible value for $f(x)$?

Answer

$\frac{3}{5}$

30) What is the smallest possible value for $f(x)$?

31) Where is the function increasing on the interval $[0, 2\pi]$?

Answer

On the approximate intervals $(0.5, 1), (1.6, 2.1), (2.6, 3.1), (3.7, 4.2), (4.7, 5.2), (5.6, 6.28)$

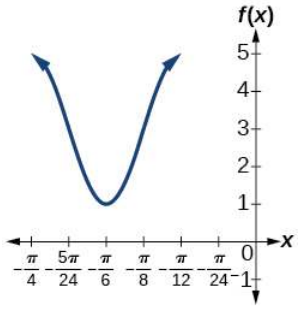
For the exercises 32-33, find and graph one period of the periodic function with the given amplitude, period, and phase shift.

32) Sine curve with amplitude 3, period $\frac{\pi}{3}$, and phase shift $(h, k) = \left(\frac{\pi}{4}, 2\right)$

33) Cosine curve with amplitude 2, period \(\dfrac{\pi }{6}\), and phase shift $(h, k) = \left(-\frac{\pi}{4}, 3\right)$

Answer

$f(x) = 2\cos\left(12\left(x + \frac{\pi}{4}\right)\right) + 3$



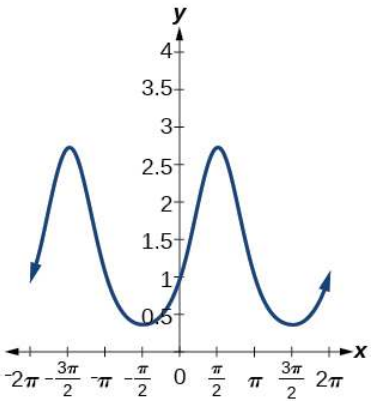
For the exercises 34-35, graph the function. Describe the graph and, wherever applicable, any periodic behavior, amplitude, asymptotes, or undefined points.

34) $f(x) = 5\cos(3x) + 4\sin(2x)$

35) $f(x) = e^{(sint)}$

Answer

This graph is periodic with a period of 2π



For the exercises 36-43, find the exact value.

36) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

37) $\tan^{-1}\left(\sqrt{3}\right)$

Answer

$\frac{\pi}{3}$

38) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

39) $\cos^{-1}(\sin(\pi))$

Answer

$\frac{\pi}{2}$

40) $\cos^{-1}\left(\tan\left(\frac{7\pi}{4}\right)\right)$

41) $\cos(\sin^{-1}(1-2x))$

Answer

$\sqrt{1-(1-2x)^2}$

42) $\cos^{-1}(-0.4)$

43) $\cos\left(\tan^{-1}\left(x^2\right)\right)$

Answer

$\frac{1}{\sqrt{1+x^4}}$

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For the exercises 44-46, suppose $\sin t = \dfrac{x}{x+1}$. Evaluate the following expressions.

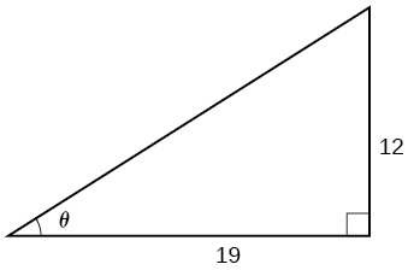
44) $\tan t$

45) $csct$

Answer

$\dfrac{x+1}{x}$

46) Given Figure, find the measure of angle θ to three decimal places. Answer in radians.



For the exercises 47-49, determine whether the equation is true or false.

47) $\arcsin\left(\sin\left(\dfrac{5\pi}{6}\right)\right) = \dfrac{5\pi}{6}$

Answer

False

48) $\arccos\left(\cos\left(\dfrac{5\pi}{6}\right)\right) = \dfrac{5\pi}{6}$

49) The grade of a road is 7%. This means that for every horizontal distance of 100 feet on the road, the vertical rise is 7 feet. Find the angle the road makes with the horizontal in radians.

Answer

approximately 0.07 radians

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CHAPTER OVERVIEW

3: Trigonometric Identities and Equations

In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

3.1: Prelude to Trigonometric Identities and Equations
3.2: Solving Trigonometric Equations with Identities
3.3: Sum and Difference Identities
3.4: Double-Angle, Half-Angle, and Reduction Formulas
3.5: Sum-to-Product and Product-to-Sum Formulas
3.6: Solving Trigonometric Equations
3.7: Modeling with Trigonometric Equations
3.E: Trigonometric Identities and Equations (Exercises)
3.R: Trigonometric Identities and Equations (Review)

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3.1: Prelude to Trigonometric Identities and Equations

Math is everywhere, even in places we might not immediately recognize. For example, mathematical relationships describe the transmission of images, light, and sound. The sinusoidal graph in Figure 3.1.1 models music playing on a phone, radio, or computer. Such graphs are described using trigonometric equations and functions. In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

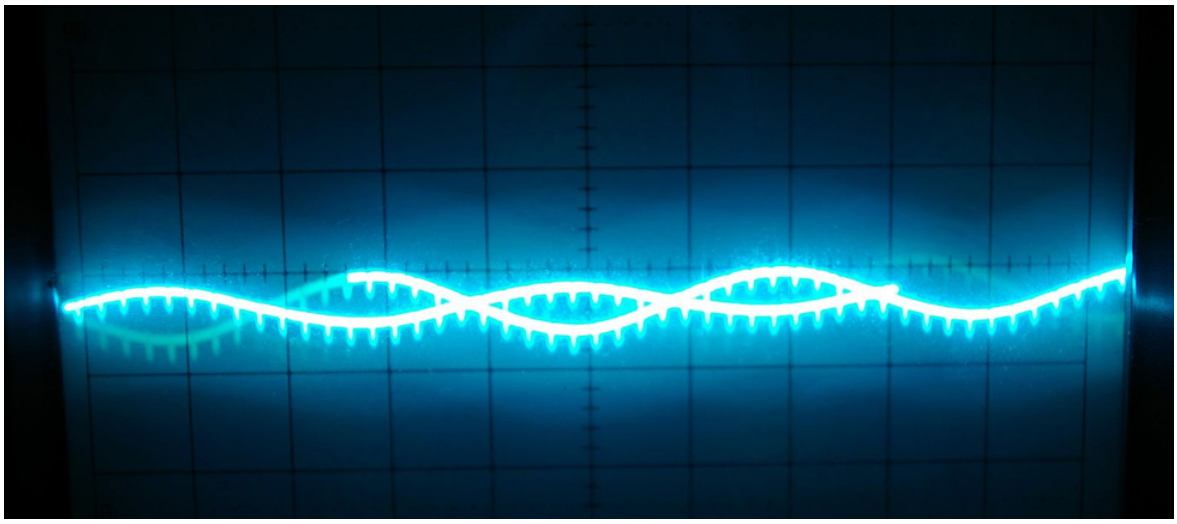

Figure 3.1.1: A sine wave models disturbance. (credit: modification of work by Mikael Altemark, Flickr).

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3.2: Solving Trigonometric Equations with Identities

Learning Objectives

- Verify the fundamental trigonometric identities.
- Simplify trigonometric expressions using algebra and the identities.

In espionage movies, we see international spies with multiple passports, each claiming a different identity. However, we know that each of those passports represents the same person. The trigonometric identities act in a similar manner to multiple passports—there are many ways to represent the same trigonometric expression. Just as a spy will choose an Italian passport when traveling to Italy, we choose the identity that applies to the given scenario when solving a trigonometric equation.

Figure 3.2.1: International passports and travel documents

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Verifying the Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the **Pythagorean identities**, the even-odd identities, the reciprocal identities, and the quotient identities.

We will begin with the **Pythagorean identities** (Table 3.2.1), which are equations involving trigonometric functions based on the properties of a right triangle. We have already seen and used the first of these identifies, but now we will also use additional identities.

Table 3.2.1: Pythagorean Identities

$\sin^2\theta+\cos^2\theta=1$	$1+\cot^2\theta=\csc^2\theta$	$1+\tan^2\theta=\sec^2\theta$

The second and third identities can be obtained by manipulating the first. The identity $1+\cot^2\theta=\csc^2\theta$ is found by rewriting the left side of the equation in terms of sine and cosine.

Prove: $1+\cot^2\theta=\csc^2\theta$

$$\begin{aligned} 1+\cot^2\theta &= (1+\frac{\cos^2}{\sin^2}) && \text{Rewrite the left side} \\ &= \left(\frac{\sin^2}{\sin^2}\right)+\left(\frac{\cos^2}{\sin^2}\right) && \text{Write both terms with the common denominator} \\ &= \frac{\sin^2+\cos^2}{\sin^2} \\ &= \frac{1}{\sin^2} \\ &= \csc^2 \end{aligned}$$

Similarly,$1+\tan^2\theta=\sec^2\theta$ can be obtained by rewriting the left side of this identity in terms of sine and cosine. This gives

$$\begin{aligned} 1+\tan^2\theta &= 1+\left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Rewrite left side} \\ &= \left(\frac{\cos\theta}{\cos\theta}\right)^2+\left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Write both terms with the common denominator} \\ &= \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} \\ &= \frac{1}{\cos^2\theta} \\ &= \sec^2\theta \end{aligned}$$

Recall that we determined which trigonometric functions are odd and which are even. The next set of fundamental identities is the set of **even-odd identities.** The **even-odd identities** relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle (Table 3.2.2).

Table 3.2.2: Even-Odd Identities

$\tan(-\theta)=-\tan\theta$	$\sin(-\theta)=-\sin\theta$	$\cos(-\theta)=\cos\theta$
$\cot(-\theta)=-\cot\theta$	$\csc(-\theta)=-\csc\theta$	$\sec(-\theta)=\sec\theta$

Recall that an **odd function** is one in which $f(-x)=-f(x)$ for all x in the domain off. f. The sine function is an odd function because $\sin(-\theta)=-\sin\theta$. The graph of an odd function is symmetric about the origin. For example, consider corresponding inputs of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. The output of $\sin\left(\frac{\pi}{2}\right)$ is opposite the output of $\sin\left(-\frac{\pi}{2}\right)$. Thus,

$$\begin{aligned} \sin\left(\frac{\pi}{2}\right) &= 1 \\ \sin\left(-\frac{\pi}{2}\right) &= -\sin\left(\frac{\pi}{2}\right) \\ &= -1 \end{aligned}$$

This is shown in Figure 3.2.2.

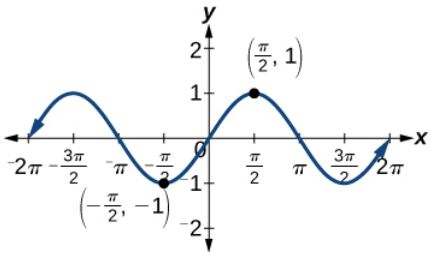

Figure 3.2.2: Graph of $y=\sin\theta$

Recall that an **even function** is one in which

$f(-x)=f(x)$ for all x in the domain of f

The graph of an even function is symmetric about the *y*-axis. The cosine function is an even function because $\cos(-\theta) = \cos\theta$. For example, consider corresponding inputs $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. The output of $\cos\left(\frac{\pi}{4}\right)$ is the same as the output of $\cos\left(-\frac{\pi}{4}\right)$. Thus,

$$\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$
$$\approx 0.707$$

See Figure 3.2.3.

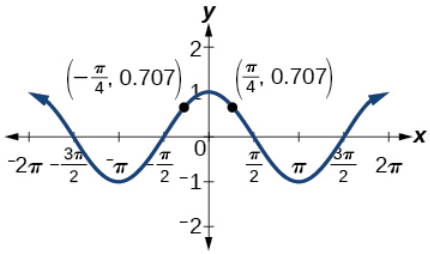

Figure 3.2.3: Graph of $y = \cos\theta$

For all θ in the domain of the sine and cosine functions, respectively, we can state the following:

- Since $\sin(-\theta) = -\sin\theta$,sine is an odd function.
- Since $\cos(-\theta) = \cos\theta$,cosine is an even function.

The other even-odd identities follow from the even and odd nature of the sine and cosine functions. For example, consider the tangent identity,$\tan(-\theta) = -\tan\theta$. We can interpret the tangent of a negative angle as

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta.$$

Tangent is therefore an odd function, which means that $\tan(-\theta) = -\tan(\theta)$ for all θ in the domain of the **tangent function.**

The cotangent identity, $\cot(-\theta) = -\cot\theta$,also follows from the sine and cosine identities. We can interpret the cotangent of a negative angle as

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta.$$

Cotangent is therefore an odd function, which means that $\cot(-\theta) = -\cot(\theta)$ for all θ in the domain of the **cotangent function.**

The **cosecant function** is the reciprocal of the sine function, which means that the cosecant of a negative angle will be interpreted as

$$\csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta.$$

The cosecant function is therefore odd.

Finally, the secant function is the reciprocal of the cosine function, and the secant of a negative angle is interpreted as

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta.$$

The secant function is therefore even.

To sum up, only two of the trigonometric functions, cosine and secant, are even. The other four functions are odd, verifying the even-odd identities.

The next set of fundamental identities is the set of reciprocal identities, which, as their name implies, relate trigonometric functions that are reciprocals of each other. (Table 3.2.3). Recall that we first encountered these identities when defining trigonometric functions from right angles in Right Angle Trigonometry.

Table 3.2.3: Reciprocal Identities

$\sin\theta = \dfrac{1}{\csc\theta}$	$\csc\theta = \dfrac{1}{\sin\theta}$
$\cos\theta = \dfrac{1}{\sec\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$
$\tan\theta = \dfrac{1}{\cot\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$

The final set of identities is the set of **quotient identities,** which define relationships among certain trigonometric functions and can be very helpful in verifying other identities (Table 3.2.4).

Table 3.2.4: Quotient Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

The reciprocal and quotient identities are derived from the definitions of the basic trigonometric functions.

SUMMARIZING TRIGONOMETRIC IDENTITIES

The Pythagorean identities are based on the properties of a right triangle.

$$\cos^2\theta + \sin^2\theta = 1 \tag{3.2.1}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{3.2.2}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{3.2.3}$$

The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle.

$$\tan(-\theta) = -\tan\theta \tag{3.2.4}$$

$$\cot(-\theta) = -\cot\theta \tag{3.2.5}$$

$$\sin(-\theta) = -\sin\theta \tag{3.2.6}$$

$$\csc(-\theta) = -\csc\theta \tag{3.2.7}$$

$$\cos(-\theta) = \cos\theta \tag{3.2.8}$$

$$\sec(-\theta) = \sec\theta \tag{3.2.9}$$

The reciprocal identities define reciprocals of the trigonometric functions.

$$\sin\theta = \frac{1}{\csc\theta} \tag{3.2.10}$$

$$\cos\theta = \frac{1}{\sec\theta} \tag{3.2.11}$$

$$\tan\theta = \frac{1}{\cot\theta} \tag{3.2.12}$$

$$\csc\theta = \frac{1}{\sin\theta} \tag{3.2.13}$$

$$\sec\theta = \frac{1}{\cos\theta} \tag{3.2.14}$$

$$\cot\theta = \frac{1}{\tan\theta} \tag{3.2.15}$$

The **quotient identities** define the relationship among the trigonometric functions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{3.2.16}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{3.2.17}$$

Example 3.2.1: Graphing the Equations of an Identity

Graph both sides of the identity $\cot\theta = \dfrac{1}{\tan\theta}$. In other words, on the graphing calculator, graph $y = \cot\theta$ and $y = \dfrac{1}{\tan\theta}$.

Solution

See Figure 3.2.4.

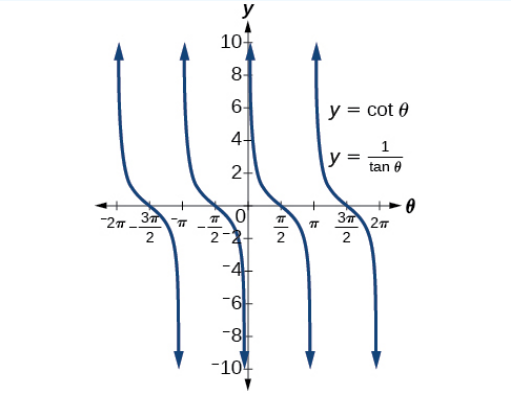

Figure 3.2.4

Analysis

We see only one graph because both expressions generate the same image. One is on top of the other. This is a good way to prove any identity. If both expressions give the same graph, then they must be identities.

How to: Given a trigonometric identity, verify that it is true.

1. Work on one side of the equation. It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. Noting which functions are in the final expression, look for opportunities to use the identities and make the proper substitutions.
4. If these steps do not yield the desired result, try converting all terms to sines and cosines.

Example 3.2.2: Verifying a Trigonometric Identity

Verify $\tan\theta\cos\theta = \sin\theta$.

Solution

We will start on the left side, as it is the more complicated side:

$$\begin{aligned}\tan\theta\cos\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta \\ &= \sin\theta\end{aligned}$$

Analysis

This identity was fairly simple to verify, as it only required writing $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

? Exercise 3.2.1

Verify the identity $\csc\theta\cos\theta\tan\theta = 1$.

Answer

$$\begin{aligned}\csc\theta\cos\theta\tan\theta &= \left(\frac{1}{\sin\theta}\right)\cos\theta\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\cos\theta}{\sin\theta}\left(\frac{\sin\theta}{\cos\theta}\right)\\ &= \frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}\\ &= 1\end{aligned}$$

✓ Example 3.2.3A: Verifying the Equivalency Using the Even-Odd Identities

Verify the following equivalency using the even-odd identities:

$$(1+\sin x)[1+\sin(-x)] = \cos^2 x$$

Solution

Working on the left side of the equation, we have

$$(1+\sin x)[1+\sin(-x)] = (1+\sin x)(1-\sin x)$$

Since

$$\begin{aligned}\sin(-x) &= -\sin x\\ [5pt] &= 1-\sin^2 x \qquad \text{Difference of squares}\\ [5pt] &= \cos^2 x\\ \cos^2 x &= 1-\sin^2 x\end{aligned}$$

✓ Example 3.2.3B: Verifying a Trigonometric Identity Involving $\sec^2\theta$

Verify the identity $\dfrac{\sec^2\theta-1}{\sec^2\theta} = \sin^2\theta$

Solution

As the left side is more complicated, let's begin there.

$$\begin{aligned}\frac{\sec^2\theta-1}{\sec^2\theta} &= \frac{(\tan^2\theta+1)-1}{\sec^2\theta}\\ \sec^2\theta &= \tan^2\theta+1\\ &= \frac{\tan^2\theta}{\sec^2\theta}\\ &= \tan^2\theta\left(\frac{1}{\sec^2\theta}\right)\\ &= \tan^2\theta\left(\cos^2\theta\right)\\ \cos^2\theta &= \frac{1}{\sec^2\theta}\\ &= \left(\frac{\sin^2\theta}{\cos^2\theta}\right)\\ \tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta}\\ &= \sin^2\theta\end{aligned}$$

There is more than one way to verify an identity. Here is another possibility. Again, we can start with the left side.

$$\begin{aligned}\frac{\sec^2\theta-1}{\sec^2\theta} &= \frac{\sec^2\theta}{\sec^2\theta}-\frac{1}{\sec^2\theta}\\ &= 1-\cos^2\theta\\ &= \sin^2\theta\end{aligned}$$

Analysis

In the first method, we used the identity $\sec^2\theta=\tan^2\theta+1$ and continued to simplify. In the second method, we split the fraction, putting both terms in the numerator over the common denominator. This problem illustrates that there are multiple ways we can verify an identity. Employing some creativity can sometimes simplify a procedure. As long as the substitutions are correct, the answer will be the same.

Exercise 3.2.2

Show that $\dfrac{\cot\theta}{\csc\theta}=\cos\theta$.

Answer

$$\begin{aligned}\frac{\cot\theta}{\csc\theta} &= \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}}\\ &= \frac{\cos\theta}{\sin\theta}\cdot\frac{\sin\theta}{1}\\ &= \cos\theta\end{aligned}$$

Example 3.2.4: Creating and Verifying an Identity

Create an identity for the expression $2\tan\theta\sec\theta$ by rewriting strictly in terms of sine.

Solution

There are a number of ways to begin, but here we will use the quotient and reciprocal identities to rewrite the expression:

$$\begin{aligned}2\tan\theta\sec\theta &= 2\left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{1}{\cos\theta}\right)\\ &= \frac{2\sin\theta}{\cos^2\theta}\\ &= \frac{2\sin\theta}{1-\sin^2\theta} \qquad \text{Substitute } 1-\sin^2\theta \text{ for } \cos^2\theta\end{aligned}$$

Thus,

$$2\tan\theta\sec\theta=\frac{2\sin\theta}{1-\sin^2\theta}$$

Example 3.2.5: Verifying an Identity Using Algebra and Even/Odd Identities

Verify the identity:

$$\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}=\cos\theta-\sin\theta$$

Solution

Let's start with the left side and simplify:

$$\begin{aligned}\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)} &= \frac{[\sin(-\theta)]^2-[\cos(-\theta)]^2}{\sin(-\theta)-\cos(-\theta)} \\ &= \frac{(-\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta} \quad ,\sin(-x)=-\sin\ x \text{ and } \cos(-x)=\cos\ x \\ &= \frac{(\sin\theta)^2-(\cos\theta)^2}{-\sin\theta-\cos\theta} \quad \text{Difference of squares} \\ &= \frac{(\sin\theta-\cos\theta)(\sin\theta+\cos\theta)}{-(\sin\theta+\cos\theta)} \\ &= \cos\theta-\sin\theta\end{aligned}$$

? Exercise 3.2.3

Verify the identity $\dfrac{\sin^2\theta-1}{\tan\theta\sin\theta-\tan\theta}=\dfrac{\sin\theta+1}{\tan\theta}$.

Answer

$$\begin{aligned}\frac{\sin^2\theta-1}{\tan\theta\sin\theta-\tan\theta} &= \frac{(\sin\theta+1)(\sin\theta-1)}{\tan\theta(\sin\theta-1)} \\ &= \frac{\sin\theta+1}{\tan\theta}\end{aligned}$$

✓ Example 3.2.6: Verifying an Identity Involving Cosines and Cotangents

Verify the identity: $(1-\cos^2x)(1+\cot^2x)=1$.

Solution

$$\begin{aligned}(1-\cos^2x)(1+\cot^2x) &= (1-\cos^2x)\left(1+\frac{\cos^2x}{\sin^2x}\right) \\ &= (1-\cos^2x)\left(\frac{\sin^2x}{\sin^2x}+\frac{\cos^2x}{\sin^2x}\right) \quad \text{Find the common denominator} \\ &= (1-\cos^2x)\left(\frac{\sin^2x+\cos^2x}{\sin^2x}\right) \\ &= (\sin^2x)\left(\frac{1}{\sin^2x}\right) \\ &= 1\end{aligned}$$

Using Algebra to Simplify Trigonometric Expressions

We have seen that algebra is very important in verifying trigonometric identities, but it is just as critical in simplifying trigonometric expressions before solving. Being familiar with the basic properties and formulas of algebra, such as the difference of squares formula, the perfect square formula, or substitution, will simplify the work involved with trigonometric expressions and equations.

For example, the equation $(\sin x+1)(\sin x-1)=0$ resembles the equation $(x+1)(x-1)=0$, which uses the factored form of the difference of squares. Using algebra makes finding a solution straightforward and familiar. We can set each factor equal to zero and solve. This is one example of recognizing algebraic patterns in trigonometric expressions or equations.

Another example is the difference of squares formula, $a^2-b^2=(a-b)(a+b)$, which is widely used in many areas other than mathematics, such as engineering, architecture, and physics. We can also create our own identities by continually expanding an expression and making the appropriate substitutions. Using algebraic properties and formulas makes many trigonometric equations easier to understand and solve.

Example 3.2.7A: Writing the Trigonometric Expression as an Algebraic Expression

Write the following trigonometric expression as an algebraic expression: $2\cos^2\theta+\cos\theta-1$.

Solution

Notice that the pattern displayed has the same form as a standard quadratic expression,ax^2+bx+c . Letting $\cos\theta=x$,we can rewrite the expression as follows:

$$2x^2+x-1$$

This expression can be factored as $(2x+1)(x-1)$. If it were set equal to zero and we wanted to solve the equation, we would use the zero factor property and solve each factor for x. At this point, we would replace x with $\cos\theta$ and solve for θ.

Example 3.2.7B: Rewriting a Trigonometric Expression Using the Difference of Squares

Rewrite the trigonometric expression using the difference of squares: $4cos^2\theta-1$.

Solution

Notice that both the coefficient and the trigonometric expression in the first term are squared, and the square of the number 1 is 1. This is the difference of squares.

$$\begin{aligned}4\cos^2\theta-1 &= (2\cos\theta)^2-1\\ &= (2\cos\theta-1)(2\cos\theta+1)\end{aligned}$$

Analysis

If this expression were written in the form of an equation set equal to zero, we could solve each factor using the zero factor property. We could also use substitution like we did in the previous problem and let $\cos\theta=x$, rewrite the expression as $4x^2-1$, and factor $(2x-1)(2x+1)$. Then replace x with $\cos\theta$ and solve for the angle.

Exercise 3.2.4

Rewrite the trigonometric expression using the difference of squares: $25-9\sin^2\theta$.

Answer

This is a difference of squares formula: $25-9\sin^2\theta=(5-3\sin\theta)(5+3\sin\theta)$.

Example 3.2.8: Simplify by Rewriting and Using Substitution

Simplify the expression by rewriting and using identities:

$$\csc^2\theta-\cot^2\theta$$

Solution

We can start with the Pythagorean identity.

$$1+\cot^2\theta=\csc^2\theta$$

Now we can simplify by substituting $1+\cot^2\theta$ for $\csc^2\theta$

$$\begin{aligned}\csc^2\theta-\cot^2\theta &= 1+\cot^2\theta-\cot^2\theta\\ &= 1\end{aligned}$$

Exercise 3.2.5

Use algebraic techniques to verify the identity: $\dfrac{\cos\theta}{1+\sin\theta}=\dfrac{1-\sin\theta}{\cos\theta}$.

(Hint: Multiply the numerator and denominator on the left side by $1-\sin\theta$.)

Answer

$$\begin{aligned}\frac{\cos\theta}{1+\sin\theta}\left(\frac{1-\sin\theta}{1-\sin\theta}\right) &= \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta} \\ &= \frac{\cos\theta(1-\sin\theta)}{\cos^2\theta} \\ &= \frac{1-\sin\theta}{\cos\theta}\end{aligned}$$

Media

Access these online resources for additional instruction and practice with the fundamental trigonometric identities.

- Fundamental Trigonometric Identities
- Verifying Trigonometric Identities

Key Equations

Pythagorean identities	$\cos^2\theta+\sin^2\theta=1$ $1+\cot^2\theta=\csc^2\theta$ $1+\tan^2\theta=\sec^2\theta$
Even-odd identities	$\tan(-\theta)=-\tan\theta$ $\cot(-\theta)=-\cot\theta$ $\sin(-\theta)=-\sin\theta$ $\csc(-\theta)=-\csc\theta$ $\cos(-\theta)=\cos\theta$ $\sec(-\theta)=\sec\theta$
Reciprocal identities	$\sin\theta=\frac{1}{\csc\theta}$ $\cos\theta=\frac{1}{\sec\theta}$ $\tan\theta=\frac{1}{\cot\theta}$ $\csc\theta=\frac{1}{\sin\theta}$ $\sec\theta=\frac{1}{\cos\theta}$ $\cot\theta=\frac{1}{\tan\theta}$
Quotient identities	$\tan\theta=\frac{\sin\theta}{\cos\theta}$ $\cot\theta=\frac{\cos\theta}{\sin\theta}$

Key Concepts

- There are multiple ways to represent a trigonometric expression. Verifying the identities illustrates how expressions can be rewritten to simplify a problem.
- Graphing both sides of an identity will verify it. See Example 3.2.1.
- Simplifying one side of the equation to equal the other side is another method for verifying an identity. See Example 3.2.2 and Example 3.2.3.
- The approach to verifying an identity depends on the nature of the identity. It is often useful to begin on the more complex side of the equation. See Example 3.2.4.
- We can create an identity and then verify it. See Example 3.2.5.
- Verifying an identity may involve algebra with the fundamental identities. See Example 3.2.6 and Example 3.2.7.

- Algebraic techniques can be used to simplify trigonometric expressions. We use algebraic techniques throughout this text, as they consist of the fundamental rules of mathematics. See Example 3.2.8, Example 3.2.9, and Example 3.2.10.

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3.3: Sum and Difference Identities

Learning Objectives

- Use sum and difference formulas for cosine.
- Use sum and difference formulas for sine.
- Use sum and difference formulas for tangent.
- Use sum and difference formulas for cofunctions.
- Use sum and difference formulas to verify identities.

How can the height of a mountain be measured? What about the distance from Earth to the sun? Like many seemingly impossible problems, we rely on mathematical formulas to find the answers. The trigonometric identities, commonly used in mathematical proofs, have had real-world applications for centuries, including their use in calculating long distances.

The trigonometric identities we will examine in this section can be traced to a Persian astronomer who lived around 950 AD, but the ancient Greeks discovered these same formulas much earlier and stated them in terms of chords. These are special equations or postulates, true for all values input to the equations, and with innumerable applications.

Figure 3.3.1: Mount McKinley, in Denali National Park, Alaska, rises 20,237 feet (6,168 m) above sea level. It is the highest peak in North America. (credit: Daniel A. Leifheit, Flickr)

In this section, we will learn techniques that will enable us to solve problems such as the ones presented above. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word *identity*.

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we can review in the unit circle shown in Figure 3.3.2.

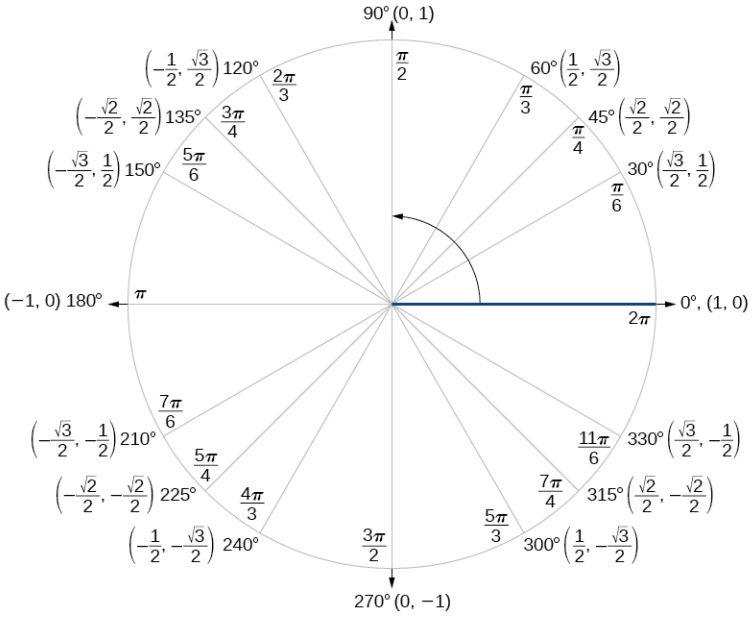

Figure 3.3.2: The Unit Circle

We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles (Table 3.3.1).

Table 3.3.1

Sum formula for cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference formula for cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

First, we will prove the difference formula for cosines. Let's consider two points on the unit circle (Figure 3.3.3). Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha,\sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta,\sin\beta)$. Note the measure of angle POQ is $\alpha-\beta$.

Label two more points: A at an angle of $(\alpha-\beta)$ from the positive x-axis with coordinates $(\cos(\alpha-\beta),\sin(\alpha-\beta))$; and point B with coordinates $(1,0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B.

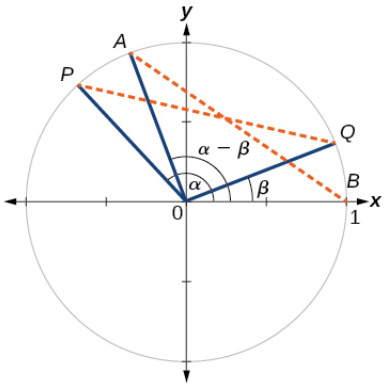

Figure 3.3.3

We can find the distance from P to Q using the distance formula.

$$\begin{aligned} d_{PQ} &= \sqrt{(\cos\alpha-\cos\beta)^2+(sin\alpha-sin\beta)^2} \\ &= \sqrt{\cos^2\alpha-2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha-2\sin\alpha\sin\beta+\sin^2\beta} && \text{Apply Pythagorean identity and simplify.} \\ &= \sqrt{(\cos^2\alpha+\sin^2\alpha)+(\cos^2\beta+\sin^2\beta)-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \\ &= \sqrt{1+1-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \\ &= \sqrt{2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta} \end{aligned}$$

Similarly, using the distance formula we can find the distance from A to B.

$$\begin{aligned} d_{AB} &= \sqrt{(\cos(\alpha-\beta)-1)^2+(\sin(\alpha-\beta)-0)^2} \\ &= \sqrt{\cos^2(\alpha-\beta)-2\cos(\alpha-\beta)+1+\sin^2(\alpha-\beta)} && \text{Apply Pythagorean identity and simplify} \\ &= \sqrt{(\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta))-2\cos(\alpha-\beta)+1} \\ &= \sqrt{1-2\cos(\alpha-\beta)+1} \\ &= \sqrt{2-2\cos(\alpha-\beta)} && \text{Subtract 2 from both sides and divide both sides by } -2. \end{aligned}$$

$$\cos\alpha\cos\beta+\sin\alpha\sin\beta = \cos(\alpha-\beta)$$

Thus, we have the difference formula for cosine. We can use similar methods to derive the cosine of the sum of two angles.

SUM AND DIFFERENCE FORMULAS FOR COSINE

These formulas can be used to calculate the cosine of sums and differences of angles.

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \tag{3.3.1}$$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \tag{3.3.2}$$

How to: Given two angles, find the cosine of the difference between the angles

1. Write the difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

Example 3.3.1: Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right)$.

Solution

Begin by writing the formula for the cosine of the difference of two angles. Then substitute the given values.

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ \cos\left(\frac{5\pi}{4}-\frac{\pi}{6}\right) &= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right)+\sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)\\ &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\\ &= -\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{-\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in radian mode.

Exercise 3.3.1:

Find the exact value of $\cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$.

Answer

$\frac{\sqrt{2}+\sqrt{6}}{4}$

Example 3.3.2: Finding the Exact Value Using the Formula for the Sum of Two Angles for Cosine

Find the exact value of $\cos(75^\circ)$.

Solution

As $75^\circ = 45^\circ + 30^\circ$,we can evaluate $\cos(75^\circ)$ as $\cos(45^\circ+30^\circ)$.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\ \cos(45^\circ+30^\circ) &= \cos(45^\circ)\cos(30^\circ)-\sin(45^\circ)\sin(30^\circ)\\ &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Keep in mind that we can always check the answer using a graphing calculator in degree mode.

Analysis

Note that we could have also solved this problem using the fact that $75^\circ = 135^\circ - 60^\circ$.

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ \cos(135^\circ-60^\circ) &= \cos(135^\circ)\cos(60^\circ)+\sin(135^\circ)\sin(60^\circ)\\ &= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= -\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

? Exercise 3.3.2

Find the exact value of $\cos(105^\circ)$.

Answer

$$\frac{\sqrt{2}-\sqrt{6}}{4}$$

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

SUM AND DIFFERENCE FORMULAS FOR SINE

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \tag{3.3.3}$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \tag{3.3.4}$$

How to: Given two angles, find the sine of the difference between the angles

1. Write the difference formula for sine.
2. Substitute the given angles into the formula.
3. Simplify.

Example 3.3.3: Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

a. $\sin(45^\circ-30^\circ)$
b. $\sin(135^\circ-120^\circ)$

Solution

a. Let's begin by writing the formula and substitute the given angles.

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \sin(45^\circ-30^\circ) &= \sin(45^\circ)\cos(30^\circ)-\cos(45^\circ)\sin(30^\circ)\end{aligned}$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45^\circ)=\tfrac{\sqrt{2}}{2},\qquad \cos(30^\circ)=\tfrac{\sqrt{3}}{2},\qquad \cos(45^\circ)=\tfrac{\sqrt{2}}{2},\qquad \sin(30^\circ)=\tfrac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(45^\circ-30^\circ) &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b. Again, we write the formula and substitute the given angles.

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \sin(135^\circ-120^\circ) &= \sin(135^\circ)\cos(120^\circ)-\cos(135^\circ)\sin(120^\circ)\end{aligned}$$

Next, we find the values of the trigonometric expressions.

$$\sin(135^\circ)=\tfrac{\sqrt{2}}{2},\qquad \cos(120^\circ)=-\tfrac{1}{2},\qquad \cos(135^\circ)=\tfrac{\sqrt{2}}{2},\qquad \sin(120^\circ)=\tfrac{\sqrt{3}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(135^\circ-120^\circ) &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example 3.3.4: Finding the Exact Value of an Expression Involving an Inverse Trigonometric Function

Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2}+\sin^{-1}\frac{3}{5}\right)$. Then check the answer with a graphing calculator.

Solution

The pattern displayed in this problem is $\sin(\alpha+\beta)$. Let $\alpha=\cos^{-1}\frac{1}{2}$ and $\beta=\sin^{-1}\frac{3}{5}$. Then we can write

$$\begin{aligned}\cos\alpha &= \frac{1}{2}, \quad 0\le\alpha\le\pi\\ \sin\beta &= \frac{3}{5}, \quad -\frac{\pi}{2}\le\beta\le\frac{\pi}{2}\end{aligned}$$

We will use the Pythagorean identities to find $\sin\alpha$ and $\cos\beta$

$$\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha}\\ &= \sqrt{1-\frac{1}{4}}\\ &= \sqrt{\frac{3}{4}}\\ &= \frac{\sqrt{3}}{2}\\ \cos\beta &= \sqrt{1-\sin^2\beta}\\ &= \sqrt{1-\frac{9}{25}}\\ &= \sqrt{\frac{16}{25}}\\ &= \frac{4}{5}\end{aligned}$$

Using the sum formula for sine,

$$\begin{aligned}\sin\left(\cos^{-1}\tfrac{1}{2}+\sin^{-1}\tfrac{3}{5}\right) &= \sin(\alpha+\beta)\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \frac{\sqrt{3}}{2}\cdot\frac{4}{5}+\frac{1}{2}\cdot\frac{3}{5}\\ &= \frac{4\sqrt{3}+3}{10}\end{aligned}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for sine and cosine and simplifying. Recall, $\tan x=\dfrac{\sin x}{\cos x}$, when $\cos x\neq 0$.

Let's derive the sum formula for tangent.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\ &= \frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\end{aligned}$$

We can derive the difference formula for tangent in a similar way.

SUM AND DIFFERENCE FORMULAS FOR TANGENT

The sum and difference formulas for tangent are:

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \tag{3.3.5}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{3.3.6}$$

How to: Given two angles, find the tangent of the sum of the angles

1. Write the sum formula for tangent.
2. Substitute the given angles into the formula.
3. Simplify.

Example 3.3.5: Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$.

Solution

Let's first write the sum formula for tangent and then substitute the given angles into the formula.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\tan\left(\frac{\pi}{6}\right)+\tan\left(\frac{\pi}{4}\right)}{1-\left(\tan\left(\frac{\pi}{6}\right)\right)\left(\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual function values within the formula:

$$\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}},\quad \text{and}\quad \tan\left(\frac{\pi}{4}\right)=1 \tag{3.3.7}$$

So we have,

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\frac{1}{\sqrt{3}}+1}{1-\left(\frac{1}{\sqrt{3}}\right)(1)}$$

$$=\frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$

$$=\frac{1+\sqrt{3}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}-1}$$

$$=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

? Exercise 3.3.5:

Find the exact value of $\tan\left(\frac{2\pi}{3}+\frac{\pi}{4}\right)$.

Answer

$\frac{1-\sqrt{3}}{1+\sqrt{3}}$

✓ Example 3.3.6: Finding Multiple Sums and Differences of Angles

Given $\sin\alpha=\frac{3}{5}, \quad 0<\alpha<\frac{\pi}{2}$, and $\cos\beta=-\frac{5}{13}, \quad \pi<\beta<\frac{3\pi}{2}$,

find

a. $\sin(\alpha+\beta)$
b. $\cos(\alpha+\beta)$
c. $\tan(\alpha+\beta)$
d. $\tan(\alpha-\beta)$

Solution

We can use the sum and difference formulas to identify the sum or difference of angles when the ratio of sine, cosine, or tangent is provided for each of the individual angles. To do so, we construct what is called a reference triangle to help find each component of the sum and difference formulas.

a. To find $\sin(\alpha+\beta)$, we begin with $\sin\alpha=\frac{3}{5}$ and $0<\alpha<\frac{\pi}{2}$. The side opposite α has length 3, the hypotenuse has length 5, and α is in the first quadrant. See Figure 3.3.4. Using the Pythagorean Theorem, we can find the length of side a:

$$a^2+3^2=5^2$$
$$a^2=16$$
$$a=4$$

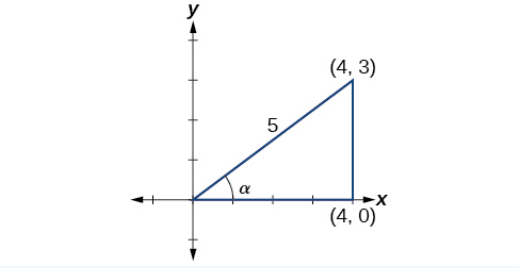

Figure 3.3.4

Since $\cos\beta=-\frac{5}{13}$ and $\pi<\beta<\frac{3\pi}{2}$,the side adjacent to β is -5,the hypotenuse is 13, and β is in the third quadrant. See Figure 3.3.5. Again, using the Pythagorean Theorem, we have

$$\begin{aligned}(-5)^2+a^2 &= 13^2\\ 25+a^2 &= 169\\ a^2 &= 144\\ a &= \pm 12\end{aligned}$$

Since β is in the third quadrant, $a = -12$.

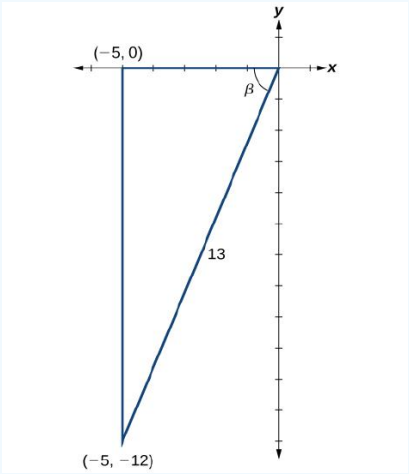

Figure 3.3.5

The next step is finding the cosine of α and the sine of β. The cosine ofα α is the adjacent side over the hypotenuse. We can find it from the triangle in Figure 3.3.5: $\cos\alpha = \frac{4}{5}$. We can also find the sine of β from the triangle in Figure 3.3.5, as opposite side over the hypotenuse: $\sin\beta = -\frac{12}{13}$. Now we are ready to evaluate $\sin(\alpha+\beta)$.

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)+\left(\frac{4}{5}\right)\left(-\frac{12}{13}\right)\\ &= -\frac{15}{65}-\frac{48}{65}\\ &= -\frac{63}{65}\end{aligned}$$

b. We can find $\cos(\alpha+\beta)$ in a similar manner. We substitute the values according to the formula.

$$\begin{aligned}\cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\ &= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right)-\left(\frac{3}{5}\right)\left(-\frac{12}{13}\right)\\ &= -\frac{20}{65}+\frac{36}{65}\\ &= \frac{16}{65}\end{aligned}$$

c. For $\tan(\alpha+\beta)$,if $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, then

$$\tan\alpha = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

If $\sin\beta = -\frac{12}{13}$ and $\cos\beta = -\frac{5}{13}$, then

$$\tan\beta = \frac{-\frac{12}{13}}{-\frac{5}{13}} = \frac{12}{5}$$

Then,

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\ &= \frac{\frac{3}{4}+\frac{12}{5}}{1-\frac{3}{4}\left(\frac{12}{5}\right)}\\ &= \frac{\frac{63}{20}}{-\frac{16}{20}}\\ &= -\frac{63}{16}\end{aligned}$$

d. To find $\tan(\alpha-\beta)$, we have the values we need. We can substitute them in and evaluate.

$$\begin{aligned}\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\\ &= \frac{\frac{3}{4}-\frac{12}{5}}{1+\frac{3}{4}\left(\frac{12}{5}\right)}\\ &= \frac{-\frac{33}{20}}{\frac{56}{20}}\\ &= -\frac{33}{56}\end{aligned}$$

Analysis

A common mistake when addressing problems such as this one is that we may be tempted to think that α and β are angles in the same triangle, which of course, they are not. Also note that

$$\tan(\alpha+\beta) = \sin(\alpha+\beta)\cos(\alpha+\beta)$$

Using Sum and Difference Formulas for Cofunctions

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from Right Triangle Trigonometry that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In Figure 3.3.6, notice that if one of the acute angles is labeled as θ, then the other acute angle must be labeled $\frac{\pi}{2}-\theta$.

Notice also that $\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$,which is opposite over hypotenuse. Thus, when two angles are complimentary, we can say that the sine of θ equals the cofunction of the complement of θ. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

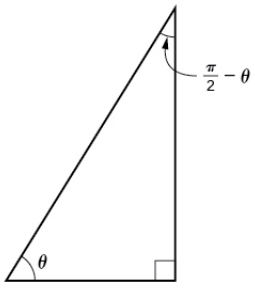

Figure 3.3.6

From these relationships, the cofunction identities are formed. Recall that you first encountered these identities in The Unit Circle: Sine and Cosine Functions.

COFUNCTION IDENTITIES

The cofunction identities are summarized in Table 3.3.2.

Table 3.3.2

$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$
$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$	$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$
$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right)$	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$

Notice that the formulas in the table may also justified algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

We can write

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \theta\right) &= \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta \\ &= (0)\cos\theta + (1)\sin\theta \\ &= \sin\theta\end{aligned}$$

Example 3.3.7: Finding a Cofunction with the Same Value as the Given Expression

Write $\tan\frac{\pi}{9}$ in terms of its cofunction.

Solution

The cofunction of $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$. Thus,

$$\begin{aligned}\tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{\pi}{2} - \frac{\pi}{9}\right) \\ &= \cot\left(\frac{9\pi}{18} - \frac{2\pi}{18}\right) \\ &= \cot\left(\frac{7\pi}{18}\right)\end{aligned}$$

Exercise 3.3.7

Write $\sin\frac{\pi}{7}$ in terms of its cofunction.

Answer

$\cos\left(\frac{5\pi}{14}\right)$

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules presented earlier may help simplify the process of verifying an identity.

How to: Given an identity, verify using sum and difference formulas

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign. Occasionally, we might have to alter both sides, but working on only one side is the most efficient.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

Example 3.3.8A: Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

Solution

We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

We can rewrite each using the sum and difference formulas.

$$\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta\\ &= 2\sin\alpha\cos\beta\end{aligned}$$

We see that the identity is verified.

Example 3.3.8B: Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\tan\alpha-\tan\beta$$

Solution

We can begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned}\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} &= \frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\\ &= \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} && \text{Rewrite using a common denominator}\\ &= \frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta} && \text{Cancel}\\ &= \tan\alpha-\tan\beta && \text{Rewrite in terms of tangent}\end{aligned}$$

Exercise 3.3.8:

Verify the identity: $\tan(\pi-\theta)=-\tan\theta$.

Answer

$$\begin{aligned}\tan(\pi-\theta) &= \frac{\tan(\pi)-\tan\theta}{1+\tan(\pi)\tan\theta}\\ &= \frac{0-\tan\theta}{1+0\cdot\tan\theta}\\ &= -\tan\theta\end{aligned}$$

Example 3.3.9A: Using Sum and Difference Formulas to Solve an Application Problem

Let L_1 and L_2 denote two non-vertical intersecting lines, and let θ denote the acute angle between L_1 and L_2. See Figure 3.3.7. Show that

$$\tan\theta=\frac{m_2-m_1}{1+m_1m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2 respectively. (**Hint:** Use the fact that $\tan\theta_1=m_1$ and $\tan\theta_2=m_2$.)

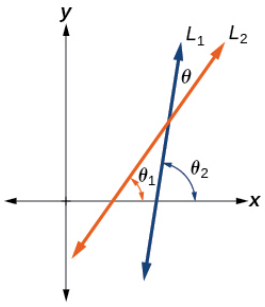

Figure 3.3.7

Solution

Using the difference formula for tangent, this problem does not seem as daunting as it might.

$$\begin{aligned}\tan\theta &= \tan(\theta_2 - \theta_1)\\ &= \frac{\tan\theta_2 - \tan\theta_1}{1+\tan\theta_1\tan\theta_2}\\ &= \frac{m_2 - m_1}{1+m_1 m_2}\end{aligned}$$

Example 3.3.9B: Investigating a Guy-wire Problem

For a climbing wall, a guy-wire R is attached 47 feet high on a vertical pole. Added support is provided by another guy-wire S attached 40 feet above ground on the same pole. If the wires are attached to the ground 50 feet from the pole, find the angle α between the wires. See Figure 3.3.8.

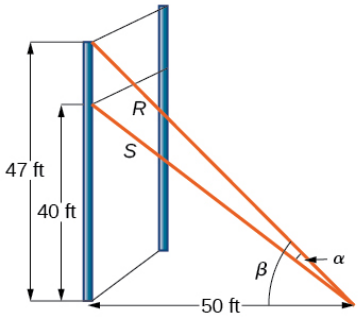

Figure 3.3.8

Solution

Let's first summarize the information we can gather from the diagram. As only the sides adjacent to the right angle are known, we can use the tangent function. Notice that $\tan\beta = \frac{47}{50}$, and $\tan(\beta-\alpha) = \frac{40}{50} = \frac{4}{5}$. We can then use difference formula for tangent.

$$\tan(\beta-\alpha) = \frac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha} \tag{3.3.8}$$

Now, substituting the values we know into the formula, we have,

$$\frac{4}{5} = \frac{\frac{47}{50} - \tan\alpha}{1+\frac{47}{50}\tan\alpha}$$

$$4\left(1+\tfrac{47}{50}\tan\alpha\right) = 5\left(\tfrac{47}{50} - \tan\alpha\right)$$

Use the distributive property, and then simplify the functions.

$$\begin{aligned} 4(1)+4\left(\tfrac{47}{50}\right)\tan\alpha &= 5\left(\tfrac{47}{50}\right)-5\tan\alpha \\ 4+3.76\tan\alpha &= 4.7-5\tan\alpha \\ 5\tan\alpha+3.76\tan\alpha &= 0.7 \\ 8.76\tan\alpha &= 0.7 \\ \tan\alpha &\approx 0.07991 \\ \tan^{-1}(0.07991) &\approx .079741 \end{aligned}$$

Now we can calculate the angle in degrees.

$$\begin{aligned} \alpha &\approx 0.079741\left(\frac{180}{\pi}\right) \\ &\approx 4.57^\circ \end{aligned}$$

Analysis

Occasionally, when an application appears that includes a right triangle, we may think that solving is a matter of applying the Pythagorean Theorem. That may be partially true, but it depends on what the problem is asking and what information is given.

Media

Access these online resources for additional instruction and practice with sum and difference identities.

- Sum and Difference Identities for Cosine
- Sum and Difference Identities for Sine
- Sum and Difference Identities for Tangent

Key Equations

Sum Formula for Cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference Formula for Cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
Sum Formula for Sine	$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
Difference Formula for Sine	$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
Sum Formula for Tangent	$\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
Difference Formula for Tangent	$\tan(\alpha-\beta)=\dfrac{\tan(\alpha)-\tan(\beta)}{1+\tan(\alpha)\tan(\beta)}$
Cofunction identities	$\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$ $\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$ $\tan\theta=\cot\left(\frac{\pi}{2}-\theta\right)$ $\cot\theta=\tan\left(\frac{\pi}{2}-\theta\right)$ $\sec\theta=\csc\left(\frac{\pi}{2}-\theta\right)$ $\csc\theta=\sec\left(\frac{\pi}{2}-\theta\right)$

Key Concepts

- The sum formula for cosines states that the cosine of the sum of two angles equals the product of the cosines of the angles minus the product of the sines of the angles. The difference formula for cosines states that the cosine of the difference of two angles equals the product of the cosines of the angles plus the product of the sines of the angles.
- The sum and difference formulas can be used to find the exact values of the sine, cosine, or tangent of an angle. See Example 3.3.1 and Example 3.3.2.
- The sum formula for sines states that the sine of the sum of two angles equals the product of the sine of the first angle and cosine of the second angle plus the product of the cosine of the first angle and the sine of the second angle. The difference formula for sines states that the sine of the difference of two angles equals the product of the sine of the first angle and cosine of the second angle minus the product of the cosine of the first angle and the sine of the second angle. See Example 3.3.3.
- The sum and difference formulas for sine and cosine can also be used for inverse trigonometric functions. See Example 3.3.4.

- The sum formula for tangent states that the tangent of the sum of two angles equals the sum of the tangents of the angles divided by 1 minus the product of the tangents of the angles. The difference formula for tangent states that the tangent of the difference of two angles equals the difference of the tangents of the angles divided by 1 plus the product of the tangents of the angles. See Example 3.3.5.
- The Pythagorean Theorem along with the sum and difference formulas can be used to find multiple sums and differences of angles. See Example 3.3.6.
- The cofunction identities apply to complementary angles and pairs of reciprocal functions. See Example 3.3.7.
- Sum and difference formulas are useful in verifying identities. See Example 3.3.8 and Example 3.3.9.
- Application problems are often easier to solve by using sum and difference formulas. See Example 3.3.10 and Example 3.3.11.

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3.4: Double-Angle, Half-Angle, and Reduction Formulas

Learning Objectives

- Use double-angle formulas to find exact values
- Use double-angle formulas to verify identities
- Use reduction formulas to simplify an expression
- Use half-angle formulas to find exact values

Bicycle ramps made for competition (see Figure 3.4.1) must vary in height depending on the skill level of the competitors. For advanced competitors, the angle formed by the ramp and the ground should be θ such that $\tan\theta = \frac{5}{3}$. The angle is divided in half for novices. What is the steepness of the ramp for novices? In this section, we will investigate three additional categories of identities that we can use to answer questions such as this one.

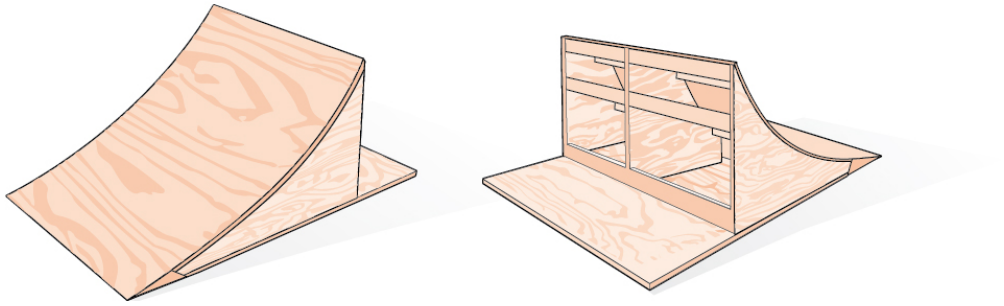

Figure 3.4.1: Bicycle ramps for advanced riders have a steeper incline than those designed for novices.

Using Double-Angle Formulas to Find Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The **double-angle formulas** are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{3.4.1}$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned}\sin(\theta+\theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ \sin(2\theta) &= 2\sin\theta\cos\theta\end{aligned}$$

Deriving the double-angle for cosine gives us three options. First, starting from the sum formula, $\cos(\alpha+\beta) = \cos\,\alpha\,\cos\,\beta - \sin\,\alpha\,\sin\,\beta$,and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned}\cos(\theta+\theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more variations. The first variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= (1-\sin^2\theta) - \sin^2\theta\end{aligned}$$

The second variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1 - \cos^2\theta) \\ &= 2\cos^2\theta - 1\end{aligned}$$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

DOUBLE-ANGLE FORMULAS

The *double-angle formulas* are summarized as follows:

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{3.4.2}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{3.4.3}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} \tag{3.4.4}$$

How to: Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 3.4.1: Using a Double-Angle Formula to Find the Exact Value Involving Tangent

Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$
b. $\cos(2\theta)$
c. $\tan(2\theta)$

Solution

If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan\theta = -\frac{3}{4}$,such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the x-axis and is negative. Use the **Pythagorean Theorem** to find the length of the hypotenuse:

$$\begin{aligned}(-4)^2 + (3)^2 &= c^2 \\ 16 + 9 &= c^2 \\ 25 &= c^2 \\ c &= 5\end{aligned}$$

Now we can draw a triangle similar to the one shown in Figure 3.4.2.

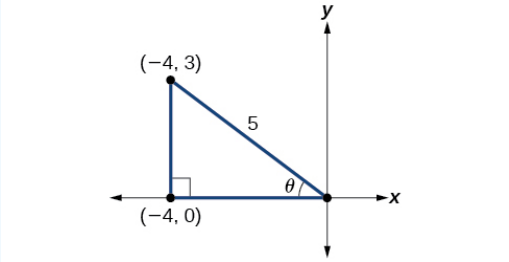

Figure 3.4.2

a. Let's begin by writing the double-angle formula for sine.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

We see that we to need to find $\sin\theta$ and $\cos\theta$. Based on Figure 3.4.2, we see that the hypotenuse equals 5, so $\sin\theta = 35$, $\sin\theta = 35$, and $\cos\theta = -45$. Substitute these values into the equation, and simplify.

Thus,

$$\begin{aligned}\sin(2\theta) &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

b. Write the double-angle formula for cosine.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Again, substitute the values of the sine and cosine into the equation, and simplify.

$$\begin{aligned}\cos(2\theta) &= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2\\ &= \frac{16}{25} - \frac{9}{25}\\ &= \frac{7}{25}\end{aligned}$$

c. Write the double-angle formula for tangent.

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

In this formula, we need the tangent, which we were given as $\tan\theta = -\frac{3}{4}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2}\\ &= \frac{-\frac{3}{2}}{1-\frac{9}{16}}\\ &= -\frac{3}{2}\left(\frac{16}{7}\right)\\ &= -\frac{24}{7}\end{aligned}$$

Exercise 3.4.1

Given $\sin\alpha = \frac{5}{8}$,with θ in quadrant I, find $\cos(2\alpha)$.

Answer

$$\cos(2\alpha) = \frac{7}{32}$$

Example 3.4.2: Using the Double-Angle Formula for Cosine without Exact Values

Use the double-angle formula for cosine to write $\cos(6x)$ in terms of $cos(3x)$.

Solution

$$\begin{aligned}\cos(6x) &= \cos(3x+3x)\\ &= \cos 3x\cos 3x - \sin 3x\sin 3x\\ &= \cos^2 3x - \sin^2 3x\end{aligned}$$

Analysis

This example illustrates that we can use the double-angle formula without having exact values. It emphasizes that the pattern is what we need to remember and that identities are true for all values in the domain of the trigonometric function.

Using Double-Angle Formulas to Verify Identities

Establishing identities using the double-angle formulas is performed using the same steps we used to derive the sum and difference formulas. Choose the more complicated side of the equation and rewrite it until it matches the other side.

Example 3.4.3: Using the Double-Angle Formulas to Verify an Identity

Verify the following identity using double-angle formulas:

$$1+\sin(2\theta) = (\sin\theta+\cos\theta)^2$$

Solution

We will work on the right side of the equal sign and rewrite the expression until it matches the left side.

$$\begin{aligned}(\sin\theta+\cos\theta)^2 &= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta\\ &= (\sin^2\theta+\cos^2\theta) + 2\sin\theta\cos\theta\\ &= 1 + 2\sin\theta\cos\theta\\ &= 1+\sin(2\theta)\end{aligned}$$

Analysis

This process is not complicated, as long as we recall the perfect square formula from algebra:

$$(a\pm b)^2 = a^2 \pm 2ab + b^2$$

where $a = \sin\theta$ and $b = \cos\theta$. Part of being successful in mathematics is the ability to recognize patterns. While the terms or symbols may change, the algebra remains consistent.

Exercise 3.4.2

Verify the identity: $\cos^4\theta - \sin^4\theta = \cos(2\theta)$.

Answer

$$\cos^4\theta - \sin^4\theta = (\cos^2\theta + \sin^2\theta)(\cos^2\theta - \sin^2\theta) = \cos(2\theta)$$

Example 3.4.4: Verifying a Double-Angle Identity for Tangent

Verify the identity: $\tan(2\theta) = 2\cot\theta - \tan\theta$

Solution

In this case, we will work with the left side of the equation and simplify or rewrite until it equals the right side of the equation.

$$\begin{aligned}
\tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta} && \text{Double-angle formula} \\
&= \frac{2\tan\theta\left(\frac{1}{\tan\theta}\right)}{(1-\tan^2\theta)\left(\frac{1}{\tan\theta}\right)} && \text{Multiply by a term that results in desired numerator} \\
&= \frac{2}{\frac{1}{\tan\theta} - \frac{\tan^2\theta}{\tan\theta}} \\
&= \frac{2}{\cot\theta - \tan\theta} && \text{Use reciprocal identity for } \frac{1}{\tan\theta}
\end{aligned}$$

Analysis

Here is a case where the more complicated side of the initial equation appeared on the right, but we chose to work the left side. However, if we had chosen the left side to rewrite, we would have been working backwards to arrive at the equivalency. For example, suppose that we wanted to show

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{2}{\cot\theta - \tan\theta}$$

Lets work on the right side

$$\begin{aligned}
\frac{2}{\cot\theta - \tan\theta} &= \frac{2}{\frac{1}{\tan\theta} - \tan\theta}\left(\frac{\tan\theta}{\tan\theta}\right) \\
&= \frac{2\tan\theta}{\frac{1}{\tan\theta}(\tan\theta) - \tan\theta(\tan\theta)} \\
&= \frac{2\tan\theta}{1-\tan^2\theta}
\end{aligned}$$

When using the identities to simplify a trigonometric expression or solve a trigonometric equation, there are usually several paths to a desired result. There is no set rule as to what side should be manipulated. However, we should begin with the guidelines set forth earlier.

Exercise 3.4.3

Verify the identity: $\cos(2\theta)\cos\theta = \cos^3\theta - \cos\theta\sin^2\theta$.

Answer

$$\cos(2\theta)\cos\,\theta = (\cos^2\theta - \sin^2\theta)\cos\,\theta = \cos^3\theta - \cos\,\theta\sin^2\theta$$

Use Reduction Formulas to Simplify an Expression

The double-angle formulas can be used to derive the **reduction formulas,** which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\,\sin^2\theta$. Solve for $\sin^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 1 - 2\sin^2\theta \\ 2\sin^2\theta &= 1 - \cos(2\theta) \\ \sin^2\theta &= \frac{1-\cos(2\theta)}{2}\end{aligned}$$

Next, we use the formula $\cos(2\theta) = 2\,\cos^2\theta - 1$. Solve for $\cos^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 2\cos^2\theta - 1 \\ 1+\cos(2\theta) &= 2\cos^2\theta \\ \frac{1+\cos(2\theta)}{2} &= \cos^2\theta\end{aligned}$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned}\tan^2\,\theta &= \frac{\sin^2\,\theta}{\cos^2\,\theta} \\ &= \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} && \text{(Substitute the reduction formulas)} \\ &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{2}{1+\cos(2\theta)}\right) \\ &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)}\end{aligned}$$

REDUCTION FORMULAS

The reduction formulas are summarized as follows:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \tag{3.4.5}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \tag{3.4.6}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \tag{3.4.7}$$

Example 3.4.5: Writing an Equivalent Expression Not Containing Powers Greater Than 1

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Solution

We will apply the reduction formula for cosine twice.

$$\begin{aligned}\cos^4 x &= \left(\cos^2 x\right)^2 \\ &= \left(\frac{1+\cos(2x)}{2}\right)^2 \qquad \text{Substitute reduction formula} \\ &= \frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \qquad \text{Substitute reduction formula for } \cos^2 x \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{8}+\frac{1}{8}\cos(4x) \\ &= \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\end{aligned}$$

Analysis

The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.

Example 3.4.6: Using the Power-Reducing Formulas to Prove an Identity

Use the power-reducing formulas to prove $\sin^3(2x) = \left[\frac{1}{2}\sin(2x)\right][1-\cos(4x)]$

Solution

We will work on simplifying the left side of the equation:

$$\begin{aligned}\sin^3(2x) &= [\sin(2x)][\sin^2(2x)] \\ &= \sin(2x)\left[\frac{1-\cos(4x)}{2}\right] \qquad \text{Substitute the power-reduction formula.} \\ &= \sin(2x)\left(\frac{1}{2}\right)[1-\cos(4x)] \\ &= \frac{1}{2}[\sin(2x)][1-\cos(4x)]\end{aligned}$$

Analysis

Note that in this example, we substituted $\frac{1-\cos(4x)}{2}$ for $\sin^2(2x)$. The formula states $\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

We let $\theta = 2x$, so $2\theta = 4x$.

Exercise 3.4.4

Use the power-reducing formulas to prove that $10\cos^4 x = \frac{15}{4} + 5\cos(2x) + \frac{5}{4}\cos(4x)$.

Answer

$$
\begin{aligned}
10\cos^4 x &= 10(\cos^2 x)^2 \\
&= 10\left[\frac{1+\cos(2x)}{2}\right]^2 \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}[1+2\cos(2x)+\cos^2(2x)] \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{4}\left(\frac{1+\cos^2(2x)}{2}\right) \qquad \text{Substitute reduction formula for } \cos^2 x \\
&= \frac{10}{4}+\frac{10}{2}\cos(2x)+\frac{10}{8}+\frac{10}{8}\cos(4x) \\
&= \frac{30}{8}+5\cos(2x)+\frac{10}{8}\cos(4x) \\
&= \frac{15}{4}+5\cos(2x)+\frac{5}{4}\cos(4x)
\end{aligned}
$$

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$,the half-angle formula for sine is found by simplifying the equation and solving for $\sin\left(\frac{\alpha}{2}\right)$. Note that the half-angle formulas are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\frac{\alpha}{2}$ terminates.

The half-angle formula for sine is derived as follows:

$$
\begin{aligned}
\sin^2\theta &= \frac{1-\cos(2\theta)}{2} \\
\sin^2\left(\frac{\alpha}{2}\right) &= \frac{1-\left(\cos 2\cdot\frac{\alpha}{2}\right)}{2} \\
&= \frac{1-\cos\alpha}{2} \\
\sin\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1-\cos\alpha}{2}}
\end{aligned}
$$

To derive the half-angle formula for cosine, we have

$$
\begin{aligned}
\cos^2\theta &= \frac{1+\cos(2\theta)}{2} \\
\cos^2\left(\frac{\alpha}{2}\right) &= \frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2} \\
&= \frac{1+\cos\alpha}{2} \\
\cos\left(\frac{\alpha}{2}\right) &= \pm\sqrt{\frac{1+\cos\alpha}{2}}
\end{aligned}
$$

For the tangent identity, we have

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\tan^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

HALF-ANGLE FORMULAS

The **half-angle formulas** are as follows:

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}} \tag{3.4.8}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \tag{3.4.9}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} \tag{3.4.10}$$

Example 3.4.7

Using a Half-Angle Formula to Find the Exact Value of a Sine Function. Find $\sin(15^\circ)$ using a half-angle formula.

Solution

Since $15^\circ = \frac{30^\circ}{2}$,we use the half-angle formula for sine (Equation 3.4.8):

$$\begin{aligned}\sin\frac{30^\circ}{2} &= \sqrt{\frac{1-\cos 30^\circ}{2}}\\ &= \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}\\ &= \sqrt{\frac{2-\sqrt{3}}{4}}\\ &= \frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

Remember that we can check the answer with a graphing calculator.

Analysis

Notice that we used only the positive root because $\sin(15^\circ)$ is positive.

Howto: Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 3.4.8: Finding Exact Values Using Half-Angle Identities

Given that $\tan\alpha = \frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$
b. $\cos\left(\frac{\alpha}{2}\right)$
c. $\tan\left(\frac{\alpha}{2}\right)$

Solution

Using the given information, we can draw the triangle shown in Figure 3.4.3. Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin\alpha = -\frac{8}{17}$ and $\cos\alpha = -\frac{15}{17}$.

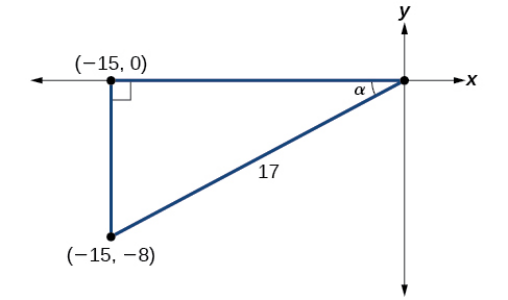

Figure 3.4.3

a. Before we start, we must remember that if α is in quadrant III, then $180^\circ < \alpha < 270^\circ$,so $\frac{180^\circ}{2} < \frac{\alpha}{2} < \frac{270^\circ}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90^\circ < \frac{\alpha}{2} < 135^\circ$. To find $\sin\frac{\alpha}{2}$,we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle in Figure 3.4.3 and simplify.

$$\begin{aligned}
\sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
&= \pm\sqrt{\frac{1-(-\frac{15}{17})}{2}} \\
&= \pm\sqrt{\frac{\frac{32}{17}}{2}} \\
&= \pm\sqrt{\frac{32}{17}\cdot\frac{1}{2}} \\
&= \pm\sqrt{\frac{16}{17}} \\
&= \pm\frac{4}{\sqrt{17}} \\
&= \frac{4\sqrt{17}}{17}
\end{aligned}$$

We choose the positive value of $\sin\frac{\alpha}{2}$ because the angle terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos\frac{\alpha}{2}$,we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle in Figure 3.4.3, and simplify.

$$\begin{aligned}\cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\ &= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}} \\ &= \pm\sqrt{\frac{\frac{2}{17}}{2}} \\ &= \pm\sqrt{\frac{2}{17}\cdot\frac{1}{2}} \\ &= \pm\sqrt{\frac{1}{17}} \\ &= -\frac{\sqrt{17}}{17}\end{aligned}$$

We choose the negative value of $\cos\frac{\alpha}{2}$ because the angle is in quadrant II because cosine is negative in quadrant II.

c. To find $\tan\frac{\alpha}{2}$,we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle in Figure 3.4.3 and simplify.

$$\begin{aligned}\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\ &= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}} \\ &= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}} \\ &= \pm\sqrt{\frac{32}{2}} \\ &= -\sqrt{16} \\ &= -4\end{aligned}$$

We choose the negative value of $\tan\frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

? Exercise 3.4.5

Given that $\sin\alpha = -\frac{4}{5}$ and α lies in quadrant IV, find the exact value of $\cos\left(\frac{\alpha}{2}\right)$.

Answer

$$-\frac{2}{\sqrt{5}}$$

Example 3.4.9: Finding the Measurement of a Half Angle

Now, we will return to the problem posed at the beginning of the section. A bicycle ramp is constructed for high-level competition with an angle of θ formed by the ramp and the ground. Another ramp is to be constructed half as steep for novice competition. If $tan\ \theta = 53$ for higher-level competition, what is the measurement of the angle for novice competition?

Solution

Since the angle for novice competition measures half the steepness of the angle for the high level competition, and $\tan\theta = \frac{5}{3}$ for high competition, we can find $\cos\theta$ from the right triangle and the Pythagorean theorem so that we can use the half-angle identities. See Figure 3.4.4.

$$3^2 + 5^2 = 34$$
$$c = \sqrt{34}$$

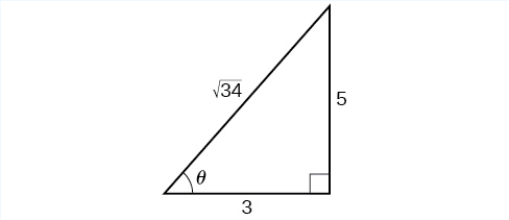

Figure 3.4.4

We see that $\cos\theta = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34}$. We can use the half-angle formula for tangent: $\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$. Since $\tan\theta$ is in the first quadrant, so is $\tan\frac{\theta}{2}$.

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\frac{3\sqrt{34}}{34}}{1+\frac{3\sqrt{34}}{34}}}$$
$$= \sqrt{\frac{\frac{34-3\sqrt{34}}{34}}{\frac{34+3\sqrt{34}}{34}}}$$
$$= \sqrt{\frac{34-3\sqrt{34}}{34+3\sqrt{34}}}$$
$$\approx 0.57$$

We can take the inverse tangent to find the angle: $\tan^{-1}(0.57) \approx 29.7^\circ$. So the angle of the ramp for novice competition is $\approx 29.7^\circ$.

Media

Access these online resources for additional instruction and practice with double-angle, half-angle, and reduction formulas.

- Double-Angle Identities
- Half-Angle Identities

Key Equations

Double-angle formulas	$\sin(2\theta) = 2\sin\,\theta\,\cos\,\theta$ $\cos(2\theta) = \cos^2\theta - \sin^2\theta$ $= 1 - 2\sin^2\theta$ $= 2\cos^2\theta - 1$ $\tan(2\theta) = \dfrac{2\tan\,\theta}{1-\tan^2\theta}$
Reduction formulas	$\sin^2\theta = \dfrac{1-\cos(2\theta)}{2}$ $\cos^2\theta = \dfrac{1+\cos(2\theta)}{2}$ $\tan^2\theta = \dfrac{1-\cos(2\theta)}{1+\cos(2\theta)}$
Half-angle formulas	$\sin\,\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\,\alpha}{2}}$ $\cos\,\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1+\cos\,\alpha}{2}}$ $\tan\,\dfrac{\alpha}{2} = \pm\sqrt{\dfrac{1-\cos\,\alpha}{1+\cos\,\alpha}}$ $= \dfrac{\sin\,\alpha}{1+\cos\,\alpha}$ $= \dfrac{1-\cos\,\alpha}{\sin\,\alpha}$

Key Concepts

- Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent. See Example 3.4.1, Example 3.4.2, Example 3.4.3, and Example 3.4.4.
- Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term. See Example 3.4.5 and Example 3.4.6.
- Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not. See Example 3.4.7, Example 3.4.8, and Example 3.4.9.

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3.5: Sum-to-Product and Product-to-Sum Formulas

Learning Objectives

- Express products as sums.
- Express sums as products.

A band marches down the field creating an amazing sound that bolsters the crowd. That sound travels as a wave that can be interpreted using trigonometric functions.

Figure 3.5.1: The UCLA marching band (credit: Eric Chan, Flickr).

For example, Figure 3.5.2 represents a sound wave for the musical note A. In this section, we will investigate trigonometric identities that are the foundation of everyday phenomena such as sound waves.

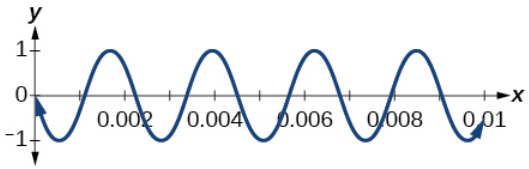

Figure 3.5.2

Expressing Products as Sums

We have already learned a number of formulas useful for expanding or simplifying trigonometric expressions, but sometimes we may need to express the product of cosine and sine as a sum. We can use the **product-to-sum formulas,** which express products of trigonometric functions as sums. Let's investigate the cosine identity first and then the sine identity.

Expressing Products as Sums for Cosine

We can derive the product-to-sum formula from the sum and difference identities for **cosine**. If we add the two equations, we get:

$$\begin{aligned} \cos\alpha\cos\beta+\sin\alpha\sin\beta &= \cos(\alpha-\beta) \\ +\cos\alpha\cos\beta-\sin\alpha\sin\beta &= \cos(\alpha+\beta) \\ \hline 2\cos\alpha\cos\beta &= \cos(\alpha-\beta)+\cos(\alpha+\beta) \end{aligned}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{3.5.1}$$

How to: Given a product of cosines, express as a sum

1. Write the formula for the product of cosines.
2. Substitute the given angles into the formula.
3. Simplify.

Example 3.5.1: Writing the Product as a Sum Using the Product-to-Sum Formula for Cosine

Write the following product of cosines as a sum: $2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right)$.

Solution

We begin by writing the formula for the product of cosines (Equation 3.5.1):

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

We can then substitute the given angles into the formula and simplify.

$$\begin{aligned}2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right) &= 2\left(\frac{1}{2}\right)\left[\cos\left(\frac{7x}{2}-\frac{3x}{2}\right)+\cos\left(\frac{7x}{2}+\frac{3x}{2}\right)\right]\\ &=\cos\left(\frac{4x}{2}\right)+\cos\left(\frac{10x}{2}\right)\\ &=\cos 2x+\cos 5x\end{aligned}$$

Exercise 3.5.1

Use the product-to-sum formula (Equation 3.5.1) to write the product as a sum or difference: $\cos(2\theta)\cos(4\theta)$.

Answer

$\frac{1}{2}(\cos 6\theta+\cos 2\theta)$

Expressing the Product of Sine and Cosine as a Sum

Next, we will derive the product-to-sum formula for sine and cosine from the sum and difference formulas for **sine**. If we add the sum and difference identities, we get:

$$\begin{aligned}\cos\alpha\cos\beta+\sin\alpha\sin\beta &= \cos(\alpha-\beta)\\ +\cos\alpha\cos\beta-\sin\alpha\sin\beta &= \cos(\alpha+\beta)\\ \hline 2\cos\alpha\cos\beta &= \cos(\alpha-\beta)+\cos(\alpha+\beta)\end{aligned}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

Example 3.5.2: Writing the Product as a Sum Containing only Sine or Cosine

Express the following product as a sum containing only sine or cosine and no products: $\sin(4\theta)\cos(2\theta)$.

Solution

Write the formula for the product of sine and cosine. Then substitute the given values into the formula and simplify.

$$\begin{aligned}\sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]\\ \sin(4\theta)\cos(2\theta) &= \frac{1}{2}[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)]\\ &= \frac{1}{2}[\sin(6\theta)+\sin(2\theta)]\end{aligned}$$

Exercise 3.5.2

Use the product-to-sum formula to write the product as a sum: $\sin(x+y)\cos(x-y)$.

Answer

$\frac{1}{2}(\sin 2x+\sin 2y)$

Expressing Products of Sines in Terms of Cosine

Expressing the product of sines in terms of **cosine** is also derived from the sum and difference identities for cosine. In this case, we will first subtract the two cosine formulas:

$$\begin{aligned}\cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta\\ -\cos(\alpha+\beta) &= -(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\\ \cos(\alpha-\beta)-\cos(\alpha+\beta) &= 2\sin\alpha\sin\beta\end{aligned}$$

Then, we divide by 2 to isolate the product of sines:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

Similarly we could express the product of cosines in terms of sine or derive other product-to-sum formulas.

THE PRODUCT-TO-SUM FORMULAS

The **product-to-sum** formulas are as follows:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{3.5.2}$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \tag{3.5.3}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \tag{3.5.4}$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)] \tag{3.5.5}$$

Example 3.5.3: Express the Product as a Sum or Difference

Write $\cos(3\theta)\cos(5\theta)$ as a sum or difference.

Solution

We have the product of cosines, so we begin by writing the related formula. Then we substitute the given angles and simplify.

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\cos(3\theta)\cos(5\theta) = \frac{1}{2}[\cos(3\theta-5\theta)+\cos(3\theta+5\theta)]$$

$$= \frac{1}{2}[\cos(2\theta)+\cos(8\theta)] \qquad \text{Use even-odd identity}$$

? Exercise 3.5.3

Use the product-to-sum formula to evaluate $\cos\frac{11\pi}{12}\cos\frac{\pi}{12}$.

Answer

$$\frac{-2-\sqrt{3}}{4}$$

Expressing Sums as Products

Some problems require the reverse of the process we just used. The **sum-to-product formulas** allow us to express sums of sine or cosine as products. These formulas can be derived from the product-to-sum identities. For example, with a few substitutions, we can derive the sum-to-product identity for **sine**. Let $\frac{u+v}{2}=\alpha$ and $\frac{u-v}{2}=\beta$.

Then,

$$\alpha+\beta = \frac{u+v}{2}+\frac{u-v}{2}$$

$$= \frac{2u}{2}$$

$$= u$$

$$\alpha-\beta = \frac{u+v}{2}-\frac{u-v}{2}$$

$$= \frac{2v}{2}$$

$$= v$$

Thus, replacing α and β in the product-to-sum formula with the substitute expressions, we have

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \frac{1}{2}[\sin u+\sin v] \qquad \text{Substitute for } (\alpha+\beta) \text{ and } (\alpha\beta)$$

$$2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) = \sin u+\sin v$$

The other sum-to-product identities are derived similarly.

SUM-TO-PRODUCT FORMULAS

The **sum-to-product formulas** are as follows:

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{3.5.6}$$

$$\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{3.5.7}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{3.5.8}$$

$$\cos\alpha + \cos\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{3.5.9}$$

Example 3.5.4: Writing the Difference of Sines as a Product

Write the following difference of sines expression as a product: $\sin(4\theta) - \sin(2\theta)$.

Solution

We begin by writing the formula for the difference of sines.

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

Substitute the values into the formula, and simplify.

$$\begin{aligned}\sin(4\theta) - \sin(2\theta) &= 2\sin\left(\frac{4\theta-2\theta}{2}\right)\cos\left(\frac{4\theta+2\theta}{2}\right)\\ &= 2\sin\left(\frac{2\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right)\\ &= 2\sin\theta\cos(3\theta)\end{aligned}$$

Exercise 3.5.4

Use the sum-to-product formula to write the sum as a product: $\sin(3\theta) + \sin(\theta)$.

Answer

$2\sin(2\theta)\cos(\theta)$

Example 3.5.5: Evaluating Using the Sum-to-Product Formula

Evaluate $\cos(15^\circ) - \cos(75^\circ)$. Check the answer with a graphing calculator.

Solution

We begin by writing the formula for the difference of cosines.

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

Then we substitute the given angles and simplify.

$$\begin{aligned}\cos(15^\circ) - \cos(75^\circ) &= -2\sin\left(\frac{15^\circ+75^\circ}{2}\right)\sin\left(\frac{15^\circ-75^\circ}{2}\right)\\ &= -2\sin(45^\circ)\sin(-30^\circ)\\ &= -2\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right)\\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Example 3.5.6: Proving an Identity

Prove the identity:

$$\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)}=-\tan t \tag{3.5.10}$$

Solution

We will start with the left side, the more complicated side of the equation, and rewrite the expression until it matches the right side.

$$\begin{aligned}\frac{\cos(4t)-\cos(2t)}{\sin(4t)+\sin(2t)} &= \frac{-2\sin\left(\frac{4t+2t}{2}\right)\sin\left(\frac{4t-2t}{2}\right)}{2\sin\left(\frac{4t+2t}{2}\right)\cos\left(\frac{4t-2t}{2}\right)}\\ &= \frac{-2\sin(3t)\sin t}{2\sin(3t)\cos t}\\ &= -\frac{\sin t}{\cos t}\\ &= -\tan t\end{aligned}$$

Analysis

Recall that verifying trigonometric identities has its own set of rules. The procedures for solving an equation are not the same as the procedures for verifying an identity. When we prove an identity, we pick one side to work on and make substitutions until that side is transformed into the other side.

Example 3.5.7: Verifying the Identity Using Double-Angle Formulas and Reciprocal Identities

Verify the identity $\csc^2\theta-2=\cos(2\theta)\sin 2\theta$.

Solution

For verifying this equation, we are bringing together several of the identities. We will use the double-angle formula and the reciprocal identities. We will work with the right side of the equation and rewrite it until it matches the left side.

$$\begin{aligned}\cos(2\theta)\sin 2\theta &= \frac{1-2\sin^2\theta}{\sin^2\theta}\\ &= \frac{1}{\sin^2\theta}-\frac{2\sin^2\theta}{\sin^2\theta}\\ &= \csc^2\theta-2\end{aligned}$$

Exercise 3.5.5

Verify the identity $\tan\theta\cot\theta-\cos^2\theta=\sin^2\theta$.

Answer

$$\begin{aligned}\tan\theta\cot\theta-\cos^2\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right)-\cos^2\theta\\ &= 1-\cos^2\theta\\ &= \sin^2\theta\end{aligned}$$

Media

Access these online resources for additional instruction and practice with the product-to-sum and sum-to-product identities.

- Sum to Product Identities
- Sum to Product and Product to Sum Identities

Key Equations

Product-to-sum Formulas

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

Sum-to-product Formulas

$$\sin\alpha+\sin\beta = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$$

$$\sin\alpha-\sin\beta = 2\sin(\frac{\alpha-\beta}{2})\cos(\frac{\alpha+\beta}{2})$$

$$\cos\alpha-\cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$$

$$\cos\alpha+\cos\beta = 2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$$

Key Concepts

- From the sum and difference identities, we can derive the product-to-sum formulas and the sum-to-product formulas for sine and cosine.
- We can use the product-to-sum formulas to rewrite products of sines, products of cosines, and products of sine and cosine as sums or differences of sines and cosines. See Example 3.5.1, Example 3.5.2, and Example 3.5.3.
- We can also derive the sum-to-product identities from the product-to-sum identities using substitution.
- We can use the sum-to-product formulas to rewrite sum or difference of sines, cosines, or products sine and cosine as products of sines and cosines. See Example 3.5.4.
- Trigonometric expressions are often simpler to evaluate using the formulas. See Example 3.5.5.
- The identities can be verified using other formulas or by converting the expressions to sines and cosines. To verify an identity, we choose the more complicated side of the equals sign and rewrite it until it is transformed into the other side. See Example 3.5.6 and Example 3.5.7.

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3.6: Solving Trigonometric Equations

Learning Objectives

- Solve linear trigonometric equations in sine and cosine.
- Solve equations involving a single trigonometric function.
- Solve trigonometric equations using a calculator.
- Solve trigonometric equations that are quadratic in form.
- Solve trigonometric equations using fundamental identities.
- Solve trigonometric equations with multiple angles.
- Solve right triangle problems.

Thales of Miletus (circa 625–547 BC) is known as the founder of geometry. The legend is that he calculated the height of the Great Pyramid of Giza in Egypt using the theory of *similar triangles*, which he developed by measuring the shadow of his staff. Based on proportions, this theory has applications in a number of areas, including fractal geometry, engineering, and architecture. Often, the angle of elevation and the angle of depression are found using similar triangles.

Figure 3.6.1: Egyptian pyramids standing near a modern city. (credit: Oisin Mulvihill)

In earlier sections of this chapter, we looked at trigonometric identities. Identities are true for all values in the domain of the variable. In this section, we begin our study of trigonometric equations to study real-world scenarios such as the finding the dimensions of the pyramids.

Solving Linear Trigonometric Equations in Sine and Cosine

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is valid. The **period** of both the sine function and the cosine function is 2π. In other words, every 2π units, the *y*-values repeat. If we need to find all possible solutions, then we must add $2\pi k$,where k is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is 2π:

$$\sin\theta = \sin(\theta \pm 2k\pi) \tag{3.6.1}$$

There are similar rules for indicating all possible solutions for the other trigonometric functions. Solving trigonometric equations requires the same techniques as solving algebraic equations. We read the equation from left to right, horizontally, like a sentence. We look for known patterns, factor, find common denominators, and substitute certain expressions with a variable to make solving a more straightforward process. However, with trigonometric equations, we also have the advantage of using the identities we developed in the previous sections.

Example 3.6.1A: Solving a Linear Trigonometric Equation Involving the Cosine Function

Find all possible exact solutions for the equation $\cos\theta = \frac{1}{2}$.

Solution

From the unit circle, we know that

$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3},\ \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\theta = \frac{\pi}{3} \pm 2k\pi \quad \text{and} \quad \theta = \frac{5\pi}{3} \pm 2k\pi$$

where k is an integer.

Example 3.6.1B: Solving a Linear Equation Involving the Sine Function

Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.

Solution

Solving for all possible values of t means that solutions include angles beyond the period of 2π. From the section on Sum and Difference Identities, we can see that the solutions are $t = \frac{\pi}{6}$ and $t = \frac{5\pi}{6}$. But the problem is asking for all possible values that solve the equation. Therefore, the answer is

$$t = \frac{\pi}{6} \pm 2\pi k \quad \text{and} \quad t = \frac{5\pi}{6} \pm 2\pi k$$

where k is an integer.

Howto: Given a trigonometric equation, solve using algebra

1. Look for a pattern that suggests an algebraic property, such as the difference of squares or a factoring opportunity.
2. Substitute the trigonometric expression with a single variable, such as x or u.
3. Solve the equation the same way an algebraic equation would be solved.
4. Substitute the trigonometric expression back in for the variable in the resulting expressions.
5. Solve for the angle.

Example 3.6.2: Solve the Linear Trigonometric Equation

Solve the equation exactly: $2\cos\theta - 3 = -5$, $0 \le \theta < 2\pi$.

Solution

Use algebraic techniques to solve the equation.

$$\begin{aligned} 2\cos\theta - 3 &= -5 \\ 2\cos\theta &= -2 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Exercise 3.6.2

Solve exactly the following linear equation on the interval $[0, 2\pi)$: $2\sin x + 1 = 0$.

Answer

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}$$

Solving Equations Involving a Single Trigonometric Function

When we are given equations that involve only one of the six trigonometric functions, their solutions involve using algebraic techniques and the unit circle (see [link]). We need to make several considerations when the equation involves trigonometric functions other than sine and cosine. Problems involving the reciprocals of the primary trigonometric functions need to be viewed from an algebraic perspective. In other words, we will write the reciprocal function, and solve for the angles using the function. Also, an equation involving the tangent function is slightly different from one containing a sine or cosine function. First, as we know, the period of tangent is π,not 2π. Further, the domain of tangent is all real numbers with the exception of odd integer multiples of $\frac{\pi}{2}$,unless, of course, a problem places its own restrictions on the domain.

Solving a Problem Involving a Single Trigonometric Function

Solve the problem exactly: $2\sin^2\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Solution

As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

$$\begin{aligned} 2\sin^2\theta - 1 &= 0 \\ 2\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{2} \\ \sqrt{\sin^2\theta} &= \pm\sqrt{\frac{1}{2}} \\ \sin\theta &= \pm\frac{1}{\sqrt{2}} \\ &= \pm\frac{\sqrt{2}}{2} \\ \theta &= \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

Example 3.6.3B: Solving a Trigonometric Equation Involving Cosecant

Solve the following equation exactly: $\csc\theta = -2$, $0 \le \theta < 4\pi$.

Solution

We want all values of θ for which $\csc\theta = -2$ over the interval $0 \le \theta < 4\pi$.

$$\begin{aligned} \csc\theta &= -2 \\ \frac{1}{\sin\theta} &= -2 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6} \end{aligned}$$

Analysis

As $\sin\theta = -\frac{1}{2}$, notice that all four solutions are in the third and fourth quadrants.

Example 3.6.3C: Solving an Equation Involving Tangent

Solve the equation exactly: $\tan\left(\theta - \frac{\pi}{2}\right) = 1$, $0 \le \theta < 2\pi$.

Solution

Recall that the tangent function has a period of π. On the interval $[0, \pi)$, and at the angle of $\frac{\pi}{4}$, the tangent has a value of 1. However, the angle we want is $\left(\theta - \frac{\pi}{2}\right)$. Thus, if $\tan\left(\frac{\pi}{4}\right) = 1$, then

$$\begin{aligned} \theta - \frac{\pi}{2} &= \frac{\pi}{4} \\ \theta &= \frac{3\pi}{4} \pm k\pi \end{aligned}$$

Over the interval $[0, 2\pi)$, we have two solutions:

$\theta = \frac{3\pi}{4}$ and $\theta = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$

Exercise 3.6.3

Find all solutions for $\tan x = \sqrt{3}$.

Answer

$\frac{\pi}{3} \pm \pi k$

Example 3.6.4: Identify all Solutions to the Equation Involving Tangent

Identify all exact solutions to the equation $2(\tan x + 3) = 5 + \tan x$, $0 \le x < 2\pi$.

Solution

We can solve this equation using only algebra. Isolate the expression $\tan x$ on the left side of the equals sign.

$$\begin{aligned} 2(\tan x) + 2(3) &= 5 + \tan x \\ 2\tan x + 6 &= 5 + \tan x \\ 2\tan x - \tan x &= 5 - 6 \\ \tan x &= -1 \end{aligned}$$

There are two angles on the unit circle that have a tangent value of -1: $\theta = \frac{3\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

Solve Trigonometric Equations Using a Calculator

Not all functions can be solved exactly using only the unit circle. When we must solve an equation involving an angle other than one of the special angles, we will need to use a calculator. Make sure it is set to the proper mode, either degrees or radians, depending on the criteria of the given problem.

Example 3.6.5A: Using a Calculator to Solve a Trigonometric Equation Involving Sine

Use a calculator to solve the equation $\sin\theta = 0.8$, where θ is in radians.

Solution

Make sure mode is set to radians. To find θ, use the inverse sine function. On most calculators, you will need to push the 2ND button and then the SIN button to bring up the $\sin^{-1}$ function. What is shown on the screen is $\sin^{-1}$. The calculator is ready for

the input within the parentheses. For this problem, we enter $\sin^{-1}(0.8)$, and press ENTER. Thus, to four decimals places,

$\sin^{-1}(0.8) \approx 0.9273$

The solution is

$\theta \approx 0.9273 \pm 2\pi k$

The angle measurement in degrees is

$$\begin{aligned} \theta &\approx 53.1^\circ \\ \theta &\approx 180^\circ - 53.1^\circ \\ &\approx 126.9^\circ \end{aligned}$$

Analysis

Note that a calculator will only return an angle in quadrants I or IV for the sine function, since that is the range of the inverse sine. The other angle is obtained by using $\pi - \theta$.

Example $3.6.5B$: Using a Calculator to Solve a Trigonometric Equation Involving Secant

Use a calculator to solve the equation $\sec\theta = -4$, giving your answer in radians.

Solution

We can begin with some algebra.

$$\begin{aligned} \sec\theta &= -4 \\ \frac{1}{\cos\theta} &= -4 \\ \cos\theta &= -\frac{1}{4} \end{aligned}$$

Check that the MODE is in radians. Now use the inverse cosine function

$$\begin{aligned} \cos^{-1}\left(-\frac{1}{4}\right) &\approx 1.8235 \\ \theta &\approx 1.8235 + 2\pi k \end{aligned}$$

Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$,1.8235 is between these two numbers, thus $\theta \approx 1.8235$ is in quadrant II. Cosine is also negative in quadrant III. Note that a calculator will only return an angle in quadrants I or II for the cosine function, since that is the range of the inverse cosine. See Figure $3.6.2$.

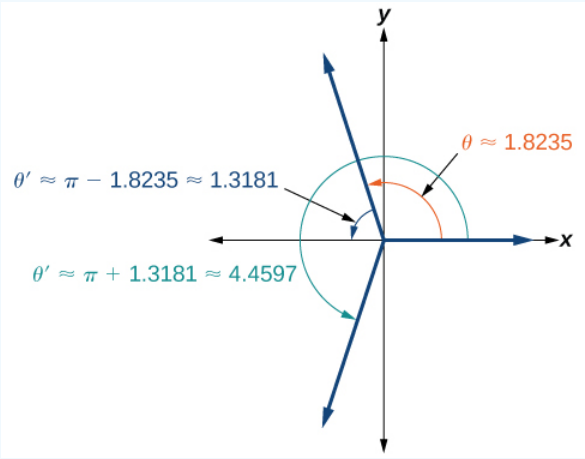

Figure 3.6.2

So, we also need to find the measure of the angle in quadrant III. In quadrant III, the reference angle is $\theta' \approx \pi - 1.8235 \approx 1.3181$. The other solution in quadrant III is $\theta' \approx \pi + 1.3181 \approx 4.4597$.

The solutions are $\theta \approx 1.8235 \pm 2\pi k$ and $\theta \approx 4.4597 \pm 2\pi k$.

Exercise 3.6.5

Solve $\cos\theta = -0.2$.

Answer

$\theta \approx 1.7722 \pm 2\pi k$ and $\theta \approx 4.5110 \pm 2\pi k$

Solving Trigonometric Equations in Quadratic Form

Solving a *quadratic equation* may be more complicated, but once again, we can use algebra as we would for any quadratic equation. Look at the pattern of the equation. Is there more than one trigonometric function in the equation, or is there only one? Which trigonometric function is squared? If there is only one function represented and one of the terms is squared, think about the standard form of a quadratic. Replace the trigonometric function with a variable such as x or u. If substitution makes the equation look like a quadratic equation, then we can use the same methods for solving quadratics to solve the trigonometric equations.

Example 3.6.6A: Solving a Trigonometric Equation in Quadratic Form

Solve the equation exactly: $\cos^2\theta + 3\cos\theta - 1 = 0$, $0 \le \theta < 2\pi$.

Solution

We begin by using substitution and replacing $\cos\theta$ with x. It is not necessary to use substitution, but it may make the problem easier to solve visually. Let $\cos\theta = x$. We have

$$x^2 + 3x - 1 = 0$$

The equation cannot be factored, so we will use the **quadratic formula:** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$\begin{aligned} x &= \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(-1)}}{2} \\ &= \frac{-3 \pm \sqrt{13}}{2} \end{aligned}$$

Replace x with $\cos\theta$ and solve.

$$\begin{aligned} \cos\theta &= \frac{-3 \pm \sqrt{13}}{2} \\ \theta &= \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \end{aligned}$$

Note that only the + sign is used. This is because we get an error when we solve $\theta = \cos^{-1}\left(\dfrac{-3 - \sqrt{13}}{2}\right)$ on a calculator, since the domain of the inverse cosine function is $[-1, 1]$. However, there is a second solution:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \\ &\approx 1.26 \end{aligned}$$

This terminal side of the angle lies in quadrant I. Since cosine is also positive in quadrant IV, the second solution is

$$\begin{aligned} \theta &= 2\pi - \cos^{-1}\left(\frac{-3 + \sqrt{13}}{2}\right) \\ &\approx 5.02 \end{aligned}$$

Example 3.6.6B: Solving a Trigonometric Equation in Quadratic Form by Factoring

Solve the equation exactly: $2\sin^2\theta - 5\sin\theta + 3 = 0$, $0 \le \theta \le 2\pi$.

Solution

Using grouping, this quadratic can be factored. Either make the real substitution, $\sin\theta = u$,or imagine it, as we factor:

$$\begin{aligned} 2\sin^2\theta - 5\sin\theta + 3 &= 0 \\ (2\sin\theta - 3)(\sin\theta - 1) &= 0 \qquad \text{Now set each factor equal to zero.} \\ 2\sin\theta - 3 &= 0 \\ 2\sin\theta &= 3 \\ \sin\theta &= \frac{3}{2} \\ \sin\theta - 1 &= 0 \\ \sin\theta &= 1 \end{aligned}$$

Next solve for θ: $\sin\theta \neq \frac{3}{2}$, as the range of the sine function is $[-1, 1]$. However, $\sin\theta = 1$, giving the solution $\theta = \frac{\pi}{2}$.

Analysis

Make sure to check all solutions on the given domain as some factors have no solution.

Exercise 3.6.6

Solve $\sin^2\theta = 2\cos\theta + 2$, $0 \le \theta \le 2\pi$. [Hint: Make a substitution to express the equation only in terms of cosine.]

Answer

$\cos\theta = -1$, $\theta = \pi$

Example 3.6.7A: Solving a Trigonometric Equation Using Algebra

Solve exactly: $2\sin^2\theta + \sin\theta = 0;\ 0 \le \theta < 2\pi$

Solution

This problem should appear familiar as it is similar to a quadratic. Let $\sin\theta = x$. The equation becomes $2x^2 + x = 0$. We begin by factoring:

$$\begin{aligned} 2x^2 + x &= 0 \\ x(2x + 1) &= 0 \qquad \text{Set each factor equal to zero.} \\ x &= 0 \\ 2x + 1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

Then, substitute back into the equation the original expression $\sin\theta$ for x. Thus,

$$\begin{aligned} \sin\theta &= 0 \\ \theta &= 0, \pi \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6}, \frac{11\pi}{6} \end{aligned}$$

The solutions within the domain $0 \le \theta < 2\pi$ are $\theta = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}$.

If we prefer not to substitute, we can solve the equation by following the same pattern of factoring and setting each factor equal to zero.

$$\begin{aligned} \sin^2\theta+\sin\theta &= 0 \\ \sin\theta(2\sin\theta+1) &= 0 \\ \sin\theta &= 0 \\ \theta &= 0,\pi \\ 2\sin\theta+1 &= 0 \\ 2\sin\theta &= -1 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \frac{7\pi}{6},\frac{11\pi}{6} \end{aligned}$$

Analysis

We can see the solutions on the graph in Figure 3.6.3. On the interval $0 \le \theta < 2\pi$,the graph crosses the x-axis four times, at the solutions noted. Notice that trigonometric equations that are in quadratic form can yield up to four solutions instead of the expected two that are found with quadratic equations. In this example, each solution (angle) corresponding to a positive sine value will yield two angles that would result in that value.

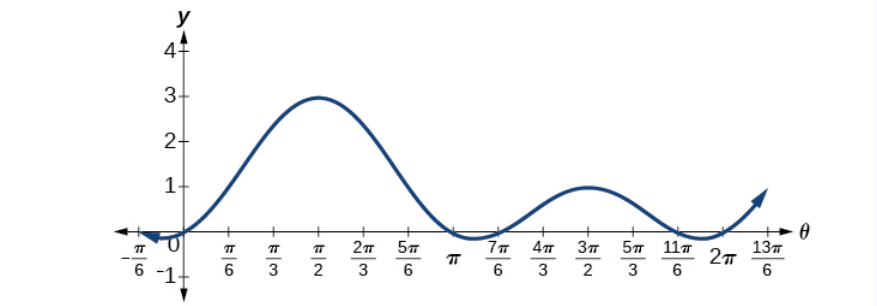

Figure 3.6.3

We can verify the solutions on the unit circle in via the result in the section on Sum and Difference Identities as well.

Example 3.6.7B: Solving a Trigonometric Equation Quadratic in Form

Solve the equation quadratic in form exactly: $2\sin^2\theta-3\sin\theta+1=0$, $0 \le \theta < 2\pi$.

Solution

We can factor using grouping. Solution values of θ can be found on the unit circle.

$$\begin{aligned} (2\sin\theta-1)(\sin\theta-1) &= 0 \\ 2\sin\theta-1 &= 0 \\ \sin\theta &= \frac{1}{2} \\ \theta &= \frac{\pi}{6},\frac{5\pi}{6} \\ \sin\theta &= 1 \\ \theta &= \frac{\pi}{2} \end{aligned}$$

Exercise 3.6.7

Solve the quadratic equation $2\cos^2\theta+\cos\theta=0$.

Answer

$\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}$

Solving Trigonometric Equations Using Fundamental Identities

While algebra can be used to solve a number of trigonometric equations, we can also use the fundamental identities because they make solving equations simpler. Remember that the techniques we use for solving are not the same as those for verifying identities. The basic rules of algebra apply here, as opposed to rewriting one side of the identity to match the other side. In the next example, we use two identities to simplify the equation.

Example 3.6.8A: Use Identities to Solve an Equation

Use identities to solve exactly the trigonometric equation over the interval $0 \le x < 2\pi$.

$$\cos x \cos(2x) + \sin x \sin(2x) = \frac{\sqrt{3}}{2}$$

Solution

Notice that the left side of the equation is the difference formula for cosine.

$$\begin{aligned} \cos x \cos(2x) + \sin x \sin(2x) &= \frac{\sqrt{3}}{2} \\ \cos(x - 2x) &= \frac{\sqrt{3}}{2} && \text{Difference formula for cosine} \\ \cos(-x) &= \frac{\sqrt{3}}{2} && \text{Use the negative angle identity.} \\ \cos x &= \frac{\sqrt{3}}{2} \end{aligned}$$

From the unit circle in the section on Sum and Difference Identities, we see that $\cos x = \frac{\sqrt{3}}{2}$ when $x = \frac{\pi}{6}, \frac{11\pi}{6}$.

Example 3.6.8B: Solving the Equation Using a Double-Angle Formula

Solve the equation exactly using a double-angle formula: $\cos(2\theta) = \cos\theta$.

Solution

We have three choices of expressions to substitute for the double-angle of cosine. As it is simpler to solve for one trigonometric function at a time, we will choose the double-angle identity involving only cosine:

$$\begin{aligned} \cos(2\theta) &= \cos\theta \\ 2\cos^2\theta - 1 &= \cos\theta \\ 2\cos^2\theta - \cos\theta - 1 &= 0 \\ (2\cos\theta + 1)(\cos\theta - 1) &= 0 \\ 2\cos\theta + 1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \cos\theta - 1 &= 0 \\ \cos\theta &= 1 \end{aligned}$$

So, if $\cos\theta = -\frac{1}{2}$,then $\theta = \frac{2\pi}{3} \pm 2\pi k$ and $\theta = \frac{4\pi}{3} \pm 2\pi k$; if $\cos\theta = 1$,then $\theta = 0 \pm 2\pi k$.

Example 3.6.8C: Solving an Equation Using an Identity

Solve the equation exactly using an identity: $3\cos\theta + 3 = 2\sin^2\theta$, $0 \le \theta < 2\pi$.

Solution

If we rewrite the right side, we can write the equation in terms of cosine:

$$\begin{aligned} 3\cos\theta+3 &= 2\sin^2\theta \\ 3\cos\theta+3 &= 2(1-\cos^2\theta) \\ 3\cos\theta+3 &= 2-2\cos^2\theta \\ 2\cos^2\theta+3\cos\theta+1 &= 0 \\ (2\cos\theta+1)(\cos\theta+1) &= 0 \\ 2\cos\theta+1 &= 0 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3},\ \frac{4\pi}{3} \\ \cos\theta+1 &= 0 \\ \cos\theta &= -1 \\ \theta &= \pi \end{aligned}$$

Our solutions are $\theta = \frac{2\pi}{3},\ \frac{4\pi}{3},\ \pi$.

Solving Trigonometric Equations with Multiple Angles

Sometimes it is not possible to solve a trigonometric equation with identities that have a multiple angle, such as $\sin(2x)$ or $\cos(3x)$. When confronted with these equations, recall that $y = \sin(2x)$ is a horizontal compression by a factor of 2 of the function $y = \sin x$. On an interval of 2π,we can graph two periods of $y = \sin(2x)$,as opposed to one cycle of $y = \sin x$. This compression of the graph leads us to believe there may be twice as many x-intercepts or solutions to $\sin(2x) = 0$ compared to $\sin x = 0$. This information will help us solve the equation.

Example 3.6.9: Solving a Multiple Angle Trigonometric Equation

Solve exactly: $\cos(2x) = \frac{1}{2}$ on $[0, 2\pi)$.

Solution

We can see that this equation is the standard equation with a multiple of an angle. If $\cos(\alpha) = \frac{1}{2}$,we know α is in quadrants I and IV. While $\theta = \cos^{-1}\frac{1}{2}$ will only yield solutions in quadrants I and II, we recognize that the solutions to the equation $\cos\theta = \frac{1}{2}$ will be in quadrants I and IV.

Therefore, the possible angles are $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$. So, $2x = \frac{\pi}{3}$ or $2x = \frac{5\pi}{3}$, which means that $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$. Does this make sense? Yes, because $\cos\left(2\left(\frac{\pi}{6}\right)\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

Are there any other possible answers? Let us return to our first step.

In quadrant I, $2x = \frac{\pi}{3}$, so $x = \frac{\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{\pi}{3} + 2\pi \\ &= \frac{\pi}{3} + \frac{6\pi}{3} \\ &= \frac{7\pi}{3} \\ x &= \frac{7\pi}{6} \end{aligned}$$

One more rotation yields

$$\begin{aligned} 2x &= \frac{\pi}{3} + 4\pi \\ &= \frac{\pi}{3} + \frac{12\pi}{3} \\ &= \frac{13\pi}{3} \end{aligned}$$

$x = \frac{13\pi}{6} > 2\pi$, so this value for x is larger than 2π, so it is not a solution on $[0, 2\pi)$.

In quadrant IV, $2x = \frac{5\pi}{3}$, so $x = \frac{5\pi}{6}$ as noted. Let us revolve around the circle again:

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 2\pi \\ &= \frac{5\pi}{3} + \frac{6\pi}{3} \\ &= \frac{11\pi}{3} \end{aligned}$$

so $x = \frac{11\pi}{6}$.

One more rotation yields

$$\begin{aligned} 2x &= \frac{5\pi}{3} + 4\pi \\ &= \frac{5\pi}{3} + \frac{12\pi}{3} \\ &= \frac{17\pi}{3} \end{aligned}$$

$x = \frac{17\pi}{6} > 2\pi$,so this value for x is larger than 2π,so it is not a solution on $[0, 2\pi)$.

Our solutions are $x = \frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Note that whenever we solve a problem in the form of $sin(nx) = c$, we must go around the unit circle n times.

Solving Right Triangle Problems

We can now use all of the methods we have learned to solve problems that involve applying the properties of right triangles and the Pythagorean Theorem. We begin with the familiar Pythagorean Theorem,

$$a^2 + b^2 = c^2 \tag{3.6.2}$$

and model an equation to fit a situation.

Example 3.6.10A: Using the Pythagorean Theorem to Model an Equation

One of the cables that anchors the center of the London Eye Ferris wheel to the ground must be replaced. The center of the Ferris wheel is 69.5 meters above the ground, and the second anchor on the ground is 23 meters from the base of the Ferris wheel. Approximately how long is the cable, and what is the angle of elevation (from ground up to the center of the Ferris wheel)? See Figure 3.6.4.

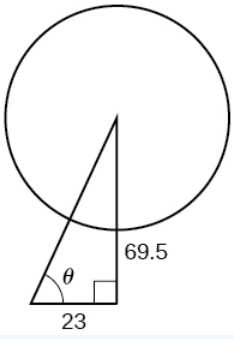

Figure 3.6.4

Solution

Use the Pythagorean Theorem (Equation 3.6.2) and the properties of right triangles to model an equation that fits the problem. Using the information given, we can draw a right triangle. We can find the length of the cable with the Pythagorean Theorem.

$$\begin{aligned} a^2+b^2 &= c^2 \\ (23)^2+(69.5)^2 &\approx 5359 \\ \sqrt{5359} &\approx 73.2\ m \end{aligned}$$

The angle of elevation is θ,formed by the second anchor on the ground and the cable reaching to the center of the wheel. We can use the tangent function to find its measure. Round to two decimal places.

$$\begin{aligned} \tan\theta &= 69.523 \\ \tan^{-1}(69.523) &\approx 1.2522 \\ &\approx 71.69^\circ \end{aligned}$$

The angle of elevation is approximately 71.7°, and the length of the cable is 73.2 meters.

Example 3.6.10B: Using the Pythagorean Theorem to Model an Abstract Problem

OSHA safety regulations require that the base of a ladder be placed 1 foot from the wall for every 4 feet of ladder length. Find the angle that a ladder of any length forms with the ground and the height at which the ladder touches the wall.

Solution

For any length of ladder, the base needs to be a distance from the wall equal to one fourth of the ladder's length. Equivalently, if the base of the ladder is "a" feet from the wall, the length of the ladder will be $4a$ feet. See Figure 3.6.5.

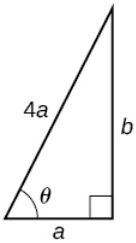

Figure 3.6.5

The side adjacent to θ is a and the hypotenuse is $4a$. Thus,

$$\begin{aligned} \cos\theta &= \frac{a}{4a} \\ &= \frac{1}{4} \\ \cos^{-1}\left(\frac{1}{4}\right) &\approx 75.5^\circ \end{aligned}$$

The elevation of the ladder forms an angle of 75.5° with the ground. The height at which the ladder touches the wall can be found using the Pythagorean Theorem:

$$\begin{aligned} a^2+b^2 &= (4a)^2 \\ b^2 &= (4a)^2-a^2 \\ b^2 &= 16a^2-a^2 \\ b^2 &= 15a^2 \\ b &= a\sqrt{15} \end{aligned}$$

Thus, the ladder touches the wall at $a\sqrt{15}$ feet from the ground.

Media

Access these online resources for additional instruction and practice with solving trigonometric equations.

- Solving Trigonometric Equations I
- Solving Trigonometric Equations II
- Solving Trigonometric Equations III
- Solving Trigonometric Equations IV
- Solving Trigonometric Equations V
- Solving Trigonometric Equations VI

Key Concepts

- When solving linear trigonometric equations, we can use algebraic techniques just as we do solving algebraic equations. Look for patterns, like the difference of squares, quadratic form, or an expression that lends itself well to substitution. See Example 3.6.1, Example 3.6.2, and Example 3.6.3.
- Equations involving a single trigonometric function can be solved or verified using the unit circle. See Example 3.6.4, Example 3.6.5, and Example 3.6.6, and Example 3.6.7.
- We can also solve trigonometric equations using a graphing calculator. See Example 3.6.8 and Example 3.6.9.
- Many equations appear quadratic in form. We can use substitution to make the equation appear simpler, and then use the same techniques we use solving an algebraic quadratic: factoring, the quadratic formula, etc. See Example 3.6.10, Example 3.6.11, Example 3.6.12, and Example 3.6.13.
- We can also use the identities to solve trigonometric equation. See Example 3.6.14, Example 3.6.15, and Example 3.6.16.
- We can use substitution to solve a multiple-angle trigonometric equation, which is a compression of a standard trigonometric function. We will need to take the compression into account and verify that we have found all solutions on the given interval. See Example 3.6.17.
- Real-world scenarios can be modeled and solved using the Pythagorean Theorem and trigonometric functions. See Example 3.6.18.

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3.7: Modeling with Trigonometric Equations

Suppose we charted the average daily temperatures in New York City over the course of one year. We would expect to find the lowest temperatures in January and February and highest in July and August. This familiar cycle repeats year after year, and if we were to extend the graph over multiple years, it would resemble a periodic function.

Figure 3.7.1: The hands on a clock are periodic: they repeat positions every twelve hours. (credit: “zoutedrop”/Flickr)

Many other natural phenomena are also periodic. For example, the phases of the moon have a period of approximately 28 days, and birds know to fly south at about the same time each year. So how can we model an equation to reflect periodic behavior? First, we must collect and record data. We then find a function that resembles an observed pattern. Finally, we make the necessary alterations to the function to get a model that is dependable. In this section, we will take a deeper look at specific types of periodic behavior and model equations to fit data.

Determining the Amplitude and Period of a Sinusoidal Function

Any motion that repeats itself in a fixed time period is considered **periodic motion** and can be modeled by a **sinusoidal function**. The amplitude of a sinusoidal function is the distance from the midline to the maximum value, or from the midline to the minimum value. The midline is the average value. Sinusoidal functions oscillate above and below the midline, are periodic, and repeat values in set cycles. Recall from Graphs of the Sine and Cosine Functions that the **period** of the sine function and the cosine function is 2π. In other words, for any value of x,

$$\sin(x \pm 2\pi k) = \sin x \text{ and } \cos(x \pm 2\pi k) = \cos x \tag{3.7.1}$$

where k is an integer.

STANDARD FORM OF SINUSOIDAL EQUATIONS

The general forms of a sinusoidal equation are given as

$$y = A\sin(Bt - C) + D \tag{3.7.2}$$

or

$$y = A\cos(Bt - C) + D \tag{3.7.3}$$

where $\text{amplitude} = |A|$, B is related to period such that the $\text{period} = \frac{2\pi}{B}$, C is the phase shift such that $\frac{C}{B}$ denotes the horizontal shift, and D represents the vertical shift from the graph’s parent graph.

Note that the models are sometimes written as

$$y = a\sin(\omega t \pm C) + D \tag{3.7.4}$$

or

$$y = a\cos(\omega t \pm C) + D, \tag{3.7.5}$$

with a period that is given as $\frac{2\pi}{\omega}$.

The difference between the sine and the cosine graphs is that the sine graph begins with the average value of the function and the cosine graph begins with the maximum or minimum value of the function.

Example 3.7.1: Showing How the Properties of a Trigonometric Function Can Transform a Graph

Show the transformation of the graph of $y = \sin x$ into the graph of $y = 2\sin(4x - \frac{\pi}{2}) + 2$.

Solution

Consider the series of graphs in Figure 3.7.1 and the way each change to the equation changes the image.

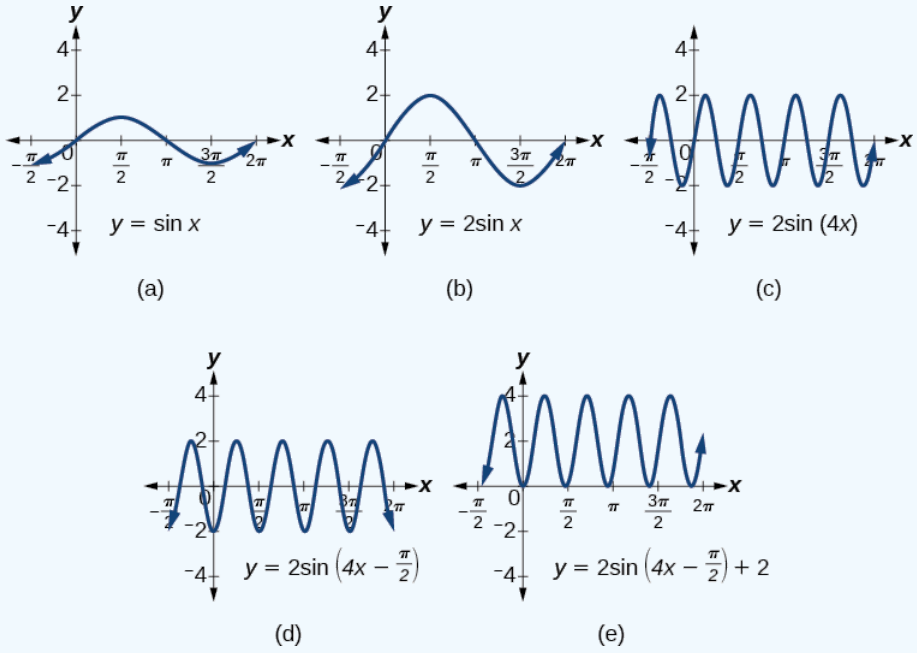

Figure 3.7.2:

a. The basic graph of $y = \sin x$
b. Changing the amplitude from 1 to 2 generates the graph of $y = 2\sin x$.
c. The period of the sine function changes with the value of B, such that period $= \frac{2\pi}{B}$. Here we have $B = 4$, which translates to a period of $\frac{\pi}{2}$. The graph completes one full cycle in $\frac{\pi}{2}$ units.
d. The graph displays a horizontal shift equal to $\frac{C}{B}$, or $\frac{\frac{\pi}{2}}{4} = \frac{\pi}{8}$.
e. Finally, the graph is shifted vertically by the value of D. In this case, the graph is shifted up by 2 units.

Example 3.7.2: Finding the Amplitude and Period of a Function

Find the amplitude and period of the following functions and graph one cycle.

a. $y = 2\sin(\frac{1}{4}x)$
b. $y = -3\sin(2x + \frac{\pi}{2})$
c. $y = \cos x + 3$

Solution

We will solve these problems according to the models.

a. $y = 2\sin(\frac{1}{4}x)$ involves sine, so we use the form

$$y = A\sin(Bt + C) + D$$

We know that $|A|$ is the amplitude, so the amplitude is 2. Period is $\frac{2\pi}{B}$ so the period is

$$\frac{2\pi}{B} = \frac{2\pi}{\frac{1}{4}}$$
$$= 8\pi$$

See the graph in Figure 3.7.3.

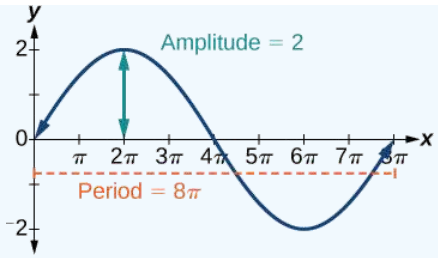

Figure 3.7.3:

b. $y = -3\sin(2x + \frac{\pi}{2})$ involves sine, so we use the form

$$y = A\sin(Bt - C) + D$$

Amplitude is $|A|$, so the amplitude is $|-3| = 3$. Since A is negative, the graph is reflected over the *x*-axis. Period is $\frac{2\pi}{B}$,so the period is

$$\frac{2\pi}{B} = \frac{2\pi}{2} = \pi$$

The graph is shifted to the left by $\frac{C}{B} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$ units. See Figure 3.7.4.

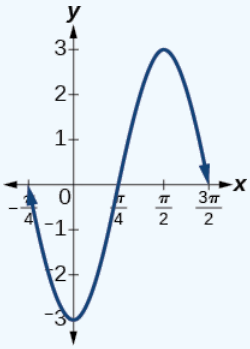

Figure 3.7.4:

c. $y = \cos x + 3$ involves cosine, so we use the form

d. $$y = A\cos(Bt \pm C) + D$$

Amplitude is $|A|$, so the amplitude is 1 and the period is 2π (Figure 3.7.5. This is the standard cosine function shifted up three units.

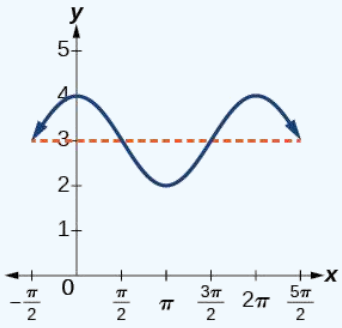

Figure 3.7.5

Exercise 3.7.1:

What are the amplitude and period of the function $y = 3\cos(3\pi x)$?

Answer

The amplitude is 3, and the period is $\frac{2}{3}$.

Finding Equations and Graphing Sinusoidal Functions

One method of graphing sinusoidal functions is to find five key points. These points will correspond to intervals of equal length representing $\frac{1}{4}$ of the period. The key points will indicate the location of maximum and minimum values. If there is no vertical shift, they will also indicate x-intercepts. For example, suppose we want to graph the function $y = cos\theta$. We know that the period is 2π, so we find the interval between key points as follows.

$$\frac{2\pi}{4} = \frac{\pi}{2}$$

Starting with $\theta = 0,$ we calculate the first y-value, add the length of the interval $\frac{\pi}{2}$ to 0, and calculate the second y-value. We then add $\frac{\pi}{2}$ repeatedly until the five key points are determined. The last value should equal the first value, as the calculations cover one full period. Making a table similar to Table 3.7.1, we can see these key points clearly on the graph shown in Figure 3.7.6.

Table 3.7.1

θ	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \cos\theta$	1	0	−1−1	0	1

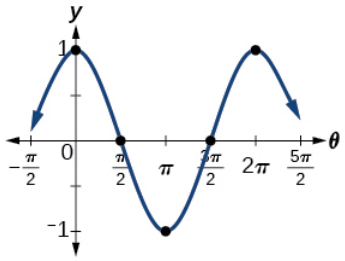

Figure 3.7.6

Example 3.7.3: Graphing Sinusoidal Functions Using Key Points

Graph the function $y = -4\cos(\pi x)$ using amplitude, period, and key points.

Solution

The amplitude is $|-4| = 4$. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2$. (Recall that we sometimes refer to B as ω.) One cycle of the graph can be drawn over the interval $[0, 2]$. To find the key points, we divide the period by 4. Make a table similar to Figure 3.7.1:, starting with $x = 0$ and then adding $\frac{1}{2}$ successively to x and calculate y. See the graph in Figure 3.7.7.

Table 3.7.1

x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$y = -4\cos(\pi x)$	−4	0	4	0	−4

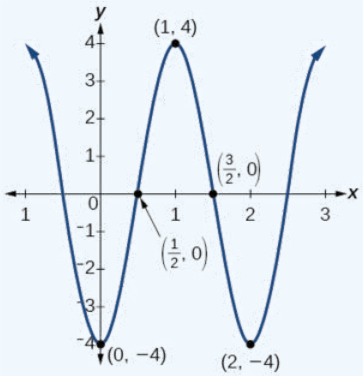

Figure 3.7.7

Exercise 3.7.2:

Graph the function $y = 3\sin(3x)$ using the amplitude, period, and five key points.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$3\sin(3x)$	0	3	0	-3	0

Answer

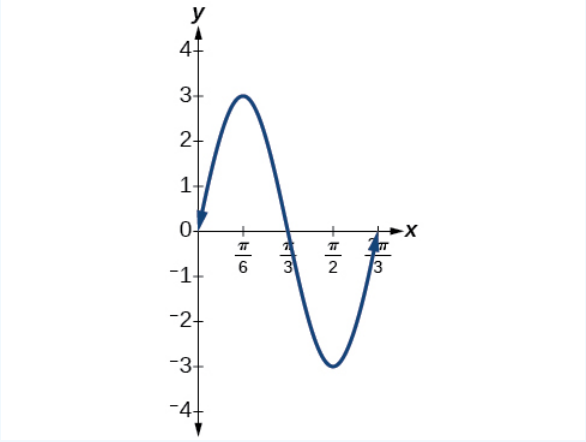

Figure 3.7.8

Modeling Periodic Behavior

We will now apply these ideas to problems involving periodic behavior.

Example 3.7.4: Modeling an Equation and Sketching a Sinusoidal Graph to Fit Criteria

The average monthly temperatures for a small town in Oregon are given in Table 3.7.1. Find a sinusoidal function of the form $y = A\sin(Bt - C) + D$ that fits the data (round to the nearest tenth) and sketch the graph.

Table 3.7.2

Month	Temperature, ^{o}F
January	42.5
February	44.5
March	48.5
April	52.5
May	58
June	63
July	68.5
August	69
September	64.5
October	55.5
November	46.5
December	43.5

Solution

Recall that amplitude is found using the formula

$$A = \frac{\text{largest value} - \text{smallest value}}{2} \tag{3.7.6}$$

Thus, the amplitude is

$$\begin{aligned} |A| &= \frac{69 - 42.5}{2} \\ &= 13.25 \end{aligned}$$

The data covers a period of 12 months, so $\frac{2\pi}{B} = 12$ which gives $B = \frac{2\pi}{12} = \frac{\pi}{6}$.

The vertical shift is found using the following equation.

$$D = \frac{\text{highest value} + \text{lowest value}}{2} \tag{3.7.7}$$

Thus, the vertical shift is

$$D = \frac{69 + 42.5}{2} \qquad = 55.8$$

So far, we have the equation $y = 13.3\sin(\frac{\pi}{6}x - C) + 55.8$.

To find the horizontal shift, we input the x and y values for the first month and solve for C.

$$\begin{aligned} 42.5 &= 13.3\sin(\frac{\pi}{6}(1) - C) + 55.8 \\ -13.3 &= 13.3\sin(\frac{\pi}{6} - C) \\ -1 &= \sin(\frac{\pi}{6} - C) \qquad \sin\theta = -1 \to \theta = -\frac{\pi}{2} \\ \frac{\pi}{6} - C &= -\frac{\pi}{2} \\ \frac{\pi}{6} + \frac{\pi}{2} &= C \\ \frac{2\pi}{3} &= C \end{aligned}$$

We have the equation $y = 13.3\sin(\frac{\pi}{6}x - \frac{2\pi}{3}) + 55.8$. See the graph in Figure 3.7.9.

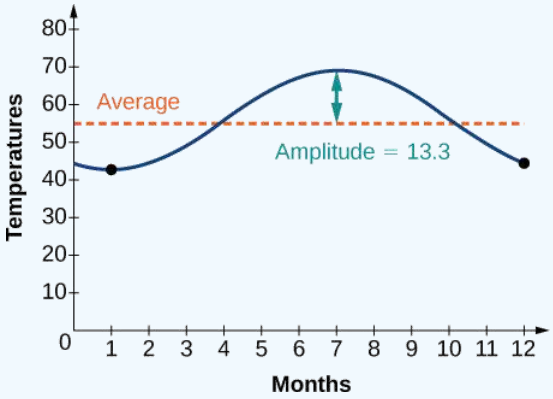

Figure 3.7.9

Example 3.7.5: Describing Periodic Motion

The hour hand of the large clock on the wall in Union Station measures 24 inches in length. At noon, the tip of the hour hand is 30 inches from the ceiling. At 3 PM, the tip is 54 inches from the ceiling, and at 6 PM, 78 inches. At 9 PM, it is again 54 inches from the ceiling, and at midnight, the tip of the hour hand returns to its original position 30 inches from the ceiling. Let

y equal the distance from the tip of the hour hand to the ceiling x hours after noon. Find the equation that models the motion of the clock and sketch the graph.

Solution

Begin by making a table of values as shown in Table 3.7.3.

Table 3.7.3

x	y	Points to plot
Noon	30 in	$(0, 30)$
3 PM	54 in	$(3, 54)$
6 PM	78 in	$(6, 78)$
9 PM	54 in	$(9, 54)$
Midnight	30 in	$(12, 30)$

To model an equation, we first need to find the amplitude.

$$\begin{aligned} |A| &= \left|\frac{78-30}{2}\right| \\ &= 24 \end{aligned}$$

The clock's cycle repeats every 12 hours. Thus,

$$\begin{aligned} B &= \frac{2\pi}{12} \\ &= \frac{\pi}{6} \end{aligned}$$

The vertical shift is

$$\begin{aligned} D &= \frac{78+30}{2} \\ &= 54 \end{aligned}$$

There is no horizontal shift, so $C = 0$. Since the function begins with the minimum value of y when $x = 0$ (as opposed to the maximum value), we will use the cosine function with the negative value for A. In the form $y = A\cos(Bx \pm C) + D$, the equation is

$$y = -24\cos\left(\frac{\pi}{6}x\right) + 54 \tag{3.7.8}$$

See Figure 3.7.10.

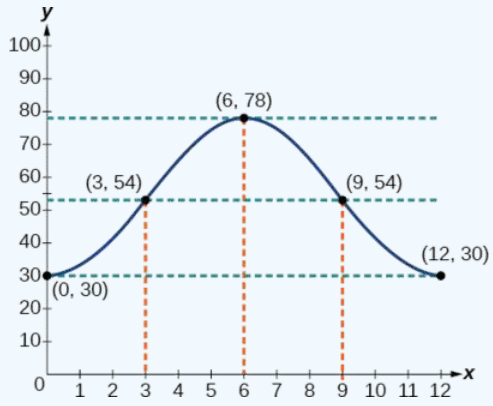

Figure 3.7.10

Example 3.7.6: Determining a Model for Tides

The height of the tide in a small beach town is measured along a seawall. Water levels oscillate between 7 feet at low tide and 15 feet at high tide. On a particular day, low tide occurred at 6 AM and high tide occurred at noon. Approximately every 12 hours, the cycle repeats. Find an equation to model the water levels.

Solution

As the water level varies from 7 ft to 15 ft, we can calculate the amplitude as

$$\begin{aligned} |A| &= |\frac{(15-7)}{2}| \\ &= 4 \end{aligned}$$

The cycle repeats every 12 hours; therefore, B is

$$\frac{2\pi}{12} = \frac{\pi}{6}$$

There is a vertical translation of $\frac{(15+8)}{2} = 11.5$. Since the value of the function is at a maximum at $t = 0$, we will use the cosine function, with the positive value for A.

$$y = 4\cos(\frac{\pi}{6})t + 11$$

See Figure 3.7.11.

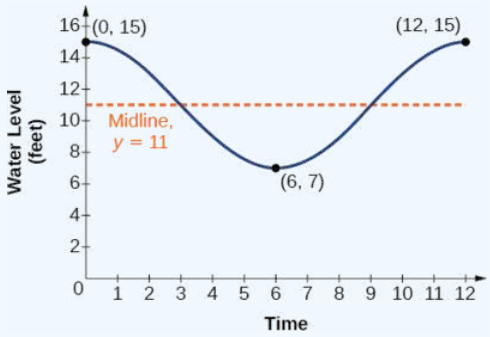

Figure 3.7.11

Exercise 3.7.3:

The daily temperature in the month of March in a certain city varies from a low of $24^\circ F$ to a high of $40^\circ F$. Find a sinusoidal function to model daily temperature and sketch the graph. Approximate the time when the temperature reaches the freezing point $32^\circ F$. Let $t = 0$ correspond to noon.

Answer

$$y = 8\sin(\frac{\pi}{12}t) + 32$$

The temperature reaches freezing at noon and at midnight.

Exercise 3.7.4: Interpreting the Periodic Behavior Equation

The average person's blood pressure is modeled by the function $f(t) = 20\sin(160\pi t) + 100$, where $f(t)$ represents the blood pressure at time t, measured in minutes. Interpret the function in terms of period and frequency. Sketch the graph and find the blood pressure reading.

Analysis

Blood pressure of $\frac{120}{80}$ is considered to be normal. The top number is the maximum or systolic reading, which measures the pressure in the arteries when the heart contracts. The bottom number is the minimum or diastolic reading, which measures the pressure in the arteries as the heart relaxes between beats, refilling with blood. Thus, normal blood pressure can be modeled by a periodic function with a maximum of 120 and a minimum of 80.

Example 3.7.7: Interpreting the Periodic Behavior Equation

The average person's blood pressure is modeled by the function $f(t) = 20\sin(160\pi t) + 100$, where $f(t)$ represents the blood pressure at time t, measured in minutes. Interpret the function in terms of period and frequency. Sketch the graph and find the blood pressure reading.

Solution

The period is given by

$$\begin{aligned} \frac{2\pi}{\omega} &= \frac{2\pi}{160\pi} \\ &= \frac{1}{80} \end{aligned}$$

In a blood pressure function, frequency represents the number of heart beats per minute. Frequency is the reciprocal of period and is given by

$$\begin{aligned} \frac{\omega}{2\pi} &= \frac{160\pi}{2\pi} \\ &= 80 \end{aligned}$$

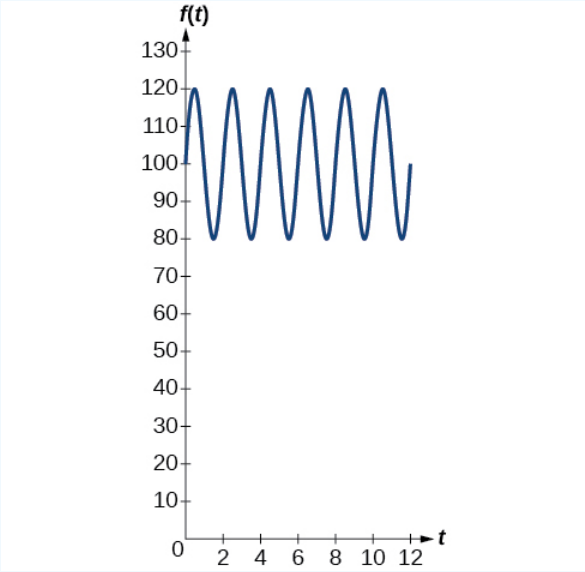

Figure 3.7.12: The blood pressure reading on the graph is $\frac{120}{80}\left(\frac{\text{maximum}}{\text{minimum}}\right)$.

Analysis

Blood pressure of $\frac{120}{80}$ is considered to be normal. The top number is the maximum or systolic reading, which measures the pressure in the arteries when the heart contracts. The bottom number is the minimum or diastolic reading, which measures the pressure in the arteries as the heart relaxes between beats, refilling with blood. Thus, normal blood pressure can be modeled by a periodic function with a maximum of 120 and a minimum of 80.

Modeling Harmonic Motion Functions

Harmonic motion is a form of periodic motion, but there are factors to consider that differentiate the two types. While general **periodic motion** applications cycle through their periods with no outside interference, **harmonic motion** requires a restoring force. Examples of harmonic motion include springs, gravitational force, and magnetic force.

Simple Harmonic Motion

A type of motion described as **simple harmonic motion** involves a restoring force but assumes that the motion will continue forever. Imagine a weighted object hanging on a spring, When that object is not disturbed, we say that the object is at rest, or in equilibrium. If the object is pulled down and then released, the force of the spring pulls the object back toward equilibrium and harmonic motion begins. The restoring force is directly proportional to the displacement of the object from its equilibrium point. When $t = 0, d = 0$.

SIMPLE HARMONIC MOTION

We see that **simple harmonic motion** equations are given in terms of displacement:

$$d = a\cos(\omega t) \text{ or } d = a\sin(\omega t) \tag{3.7.9}$$

where $|a|$ is the amplitude, $\frac{2\pi}{\omega}$ is the period, and $\frac{\omega}{2\pi}$ is the frequency, or the number of cycles per unit of time.

Exercise 3.7.5: Finding the Displacement, Period, and Frequency, and Graphing a Function

For each the given functions:

a. $y = 5\sin(3t)$
b. $y = 6\cos(\pi t)$
c. $y = 5\cos(\frac{\pi}{2})t$

address the following questions:

1. Find the maximum displacement of an object.
2. Find the period or the time required for one vibration.
3. Find the frequency.
4. Sketch the graph.

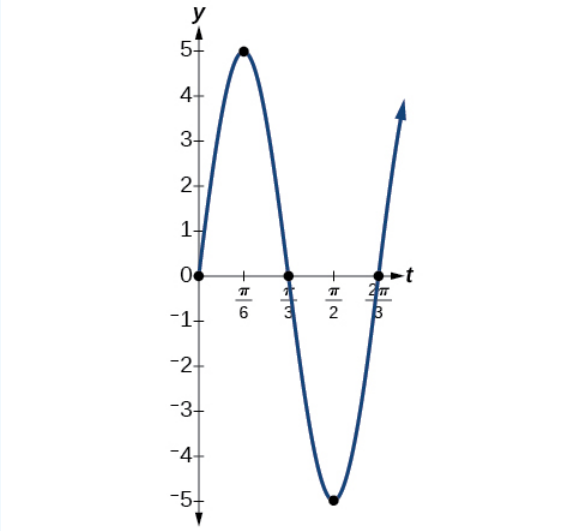

Figure 3.7.13

Answer a

1. The maximum displacement is equal to the amplitude,$|a|$, which is 5.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{3}$.
3. The frequency is given as $\frac{\omega}{2\pi} = \frac{3}{2\pi}$.
4. See Figure 3.7.13 The graph indicates the five key points.

Answer b

1. The maximum displacement is 6.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2$.

3. The frequency is $\frac{\omega}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}$.
4. See Figure 3.7.14

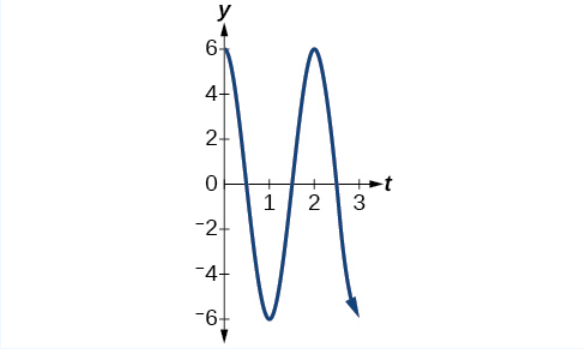

Figure 3.7.14

Answer c

1. The maximum displacement is 5.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4$.
3. The frequency is $\frac{1}{4}$.
4. See Figure 3.7.15

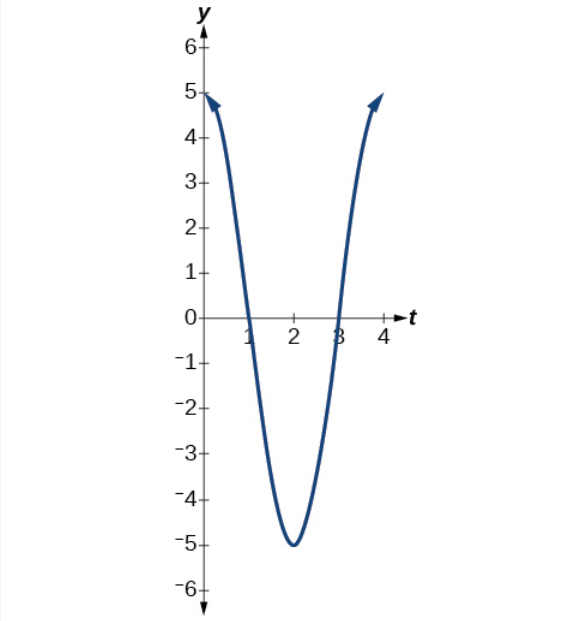

Figure 3.7.15

Damped Harmonic Motion

In reality, a pendulum does not swing back and forth forever, nor does an object on a spring bounce up and down forever. Eventually, the pendulum stops swinging and the object stops bouncing and both return to equilibrium. Periodic motion in which an energy-dissipating force, or damping factor, acts is known as **damped harmonic motion**. Friction is typically the damping factor.

In physics, various formulas are used to account for the damping factor on the moving object. Some of these are calculus-based formulas that involve derivatives. For our purposes, we will use formulas for basic damped harmonic motion models.

Definition: DAMPED HARMONIC MOTION

In *damped harmonic motion*, the displacement of an oscillating object from its rest position at time t is given as

$$f(t) = ae^{-ct}\sin(\omega t) \text{ or } f(t) = ae^{-ct}\cos(\omega t) \tag{3.7.10}$$

where c is a damping factor, $|a|$ is the initial displacement and $\frac{2\pi}{\omega}$ is the period.

Example 3.7.8: Modeling Damped Harmonic Motion

Model the equations that fit the two scenarios and use a graphing utility to graph the functions: Two mass-spring systems exhibit damped harmonic motion at a frequency of 0.5 cycles per second. Both have an initial displacement of 10 cm. The first has a damping factor of 0.5 and the second has a damping factor of 0.1.

Solution

At time $t = 0$, the displacement is the maximum of 10 cm, which calls for the cosine function. The cosine function will apply to both models.

We are given the frequency $f = \frac{\omega}{2\pi}$ of 0.5 cycles per second. Thus,

$$\begin{aligned} \frac{\omega}{2\pi} &= 0.5 \\ \omega &= (0.5)2\pi \\ &= \pi \end{aligned}$$

The first spring system has a damping factor of $c = 0.5$. Following the general model for damped harmonic motion, we have

$$f(t) = 10e^{-0.5t}\cos(\pi t)$$

Figure 3.7.16 models the motion of the first spring system.

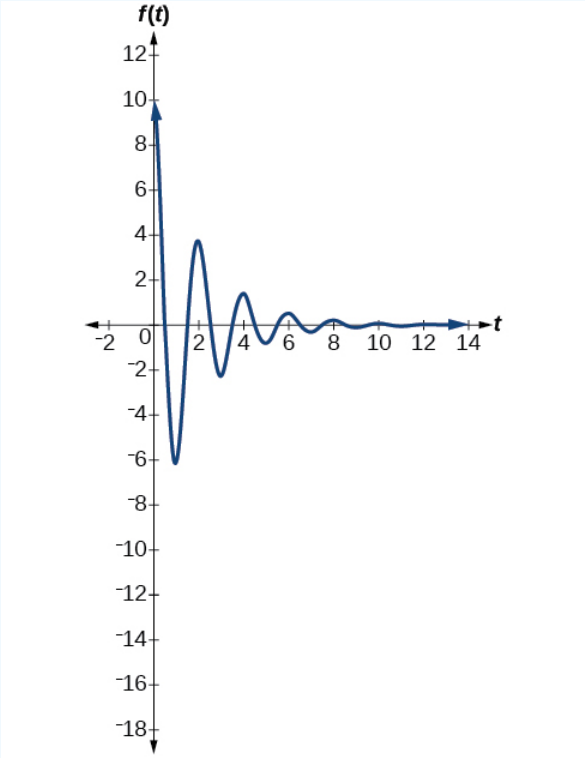

Figure 3.7.16

The second spring system has a damping factor of $c = 0.1$ and can be modeled as

$$f(t) = 10e^{-0.1t}\cos(\pi t) \tag{3.7.11}$$

Figure 3.7.17 models the motion of the second spring system.

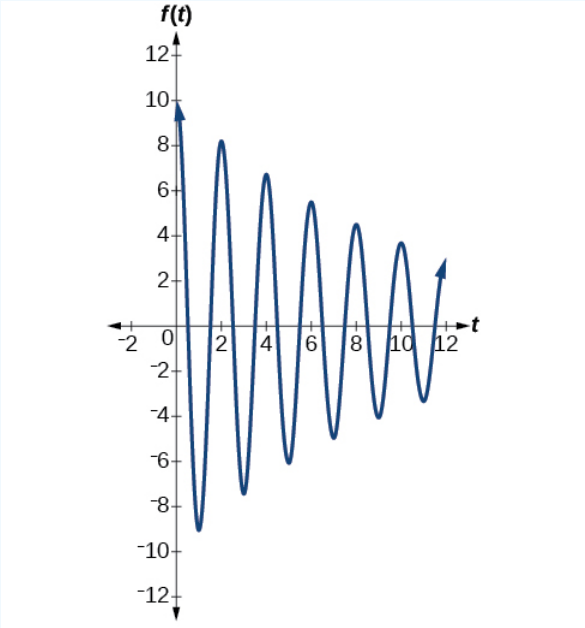

Figure 3.7.17

Analysis

Notice the differing effects of the damping constant. The local maximum and minimum values of the function with the damping factor $c = 0.5$ decreases much more rapidly than that of the function with $c = 0.1$.

Exercise 3.7.6: Finding a Cosine Function that Models Damped Harmonic Motion

Find and graph a function of the form $y = ae^{-ct}\cos(\omega t)$ that models the information given.

a. $a = 20, c = 0.05, p = 4$
b. $a = 2, c = 1.5, f = 3$

Solution

Substitute the given values into the model. Recall that period is $\frac{2\pi}{\omega}$ and frequency is $\frac{\omega}{2\pi}$.

a. $y = 20e^{-0.05t}\cos(\frac{\pi}{2}t)$. See Figure 3.7.18

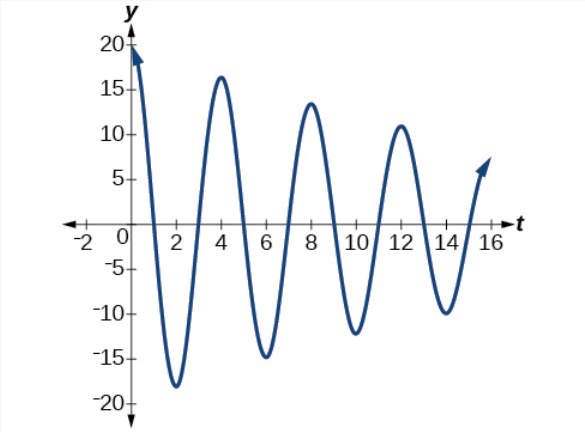

Figure 3.7.18

b. $y = 2e^{-1.5t}\cos(6\pi t)$. See Figure 3.7.19

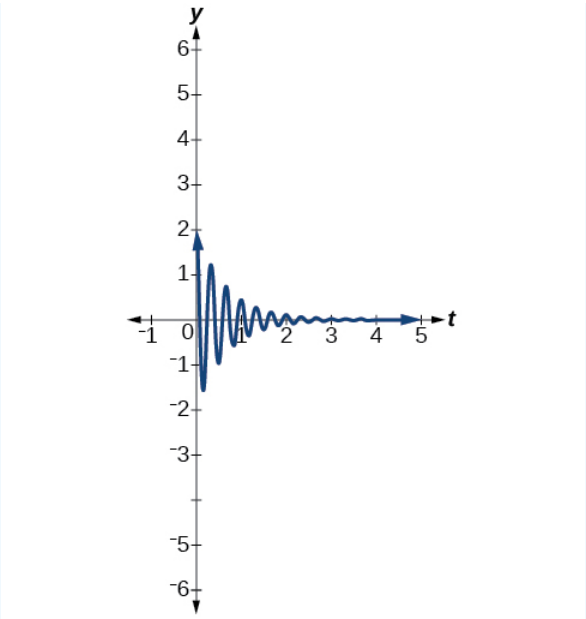

Figure 3.7.19

Exercise 3.7.7

The following equation represents a damped harmonic motion model: $f(t) = 5e^{-6t}\cos(4t)$ Find the initial displacement, the damping constant, and the frequency.

Answer

initial displacement =6, damping constant = -6, frequency = $\frac{2}{\pi}$

Example 3.7.9: Finding a Sine Function that Models Damped Harmonic Motion

Find and graph a function of the form $y = ae^{-ct}\sin(\omega t)$ that models the information given.

a. $a = 7, c = 10, p = \frac{\pi}{6}$
b. $a = 0.3, c = 0.2, f = 20$

Solution

Calculate the value of ω and substitute the known values into the model.

a. As period is $\frac{2\pi}{\omega}$, we have

$$\frac{\pi}{6} = \frac{2\pi}{\omega}$$
$$\omega\pi = 6(2\pi)$$
$$\omega = 12$$

The damping factor is given as 10 and the amplitude is 7. Thus, the model is $y = 7e^{-10t}\sin(12t)$. See Figure 3.7.20

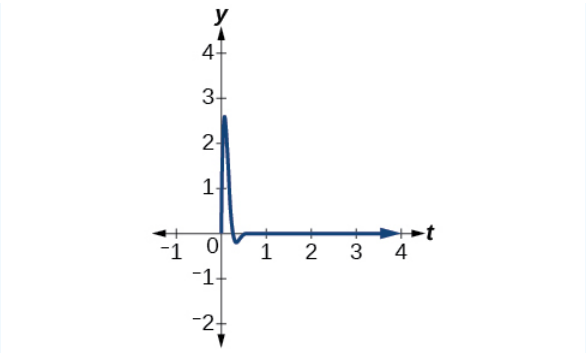

Figure 3.7.20

b. As frequency is $\frac{\omega}{2\pi}$, we have

$$\begin{aligned} 20 &= \frac{\omega}{2\pi} \\ 40\pi &= \omega \end{aligned}$$

The damping factor is given as 0.2 and the amplitude is 0.3. The model is $y = 0.3e^{-0.2t}\sin(40\pi t)$. See Figure 3.7.21.

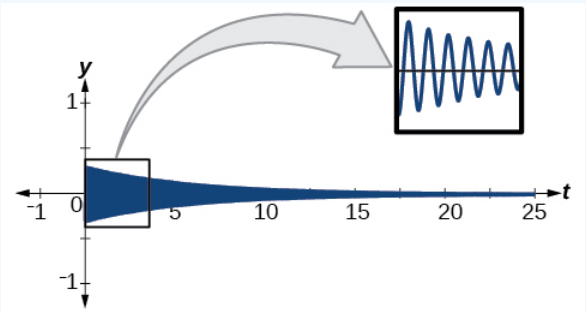

Figure 3.7.21

Analysis

A comparison of the last two examples illustrates how we choose between the sine or cosine functions to model sinusoidal criteria. We see that the cosine function is at the maximum displacement when $t = 0$, and the sine function is at the equilibrium point when $t = 0$. For example, consider the equation $y = 20e^{-0.05t}\cos(\frac{\pi}{2}t)$ from Example. We can see from the graph that when $t = 0, y = 20$, which is the initial amplitude. Check this by setting $t = 0$ in the cosine equation:

$$\begin{aligned} y &= 20e^{-0.05(0)}\cos(\frac{\pi}{2})(0) \\ &= 20(1)(1) \\ &= 20 \end{aligned}$$

Using the sine function yields

$$\begin{aligned} y &= 20e^{-0.05(0)}\sin(\frac{\pi}{2})(0) \\ &= 20(1)(0) \\ &= 0 \end{aligned}$$

Thus, cosine is the correct function.

Exercise 3.7.8:

Write the equation for damped harmonic motion given $a = 10, c = 0.5$, and $p = 2$.

Answer

$y = 10e^{-0.5t}\cos(\pi t)$

Example 3.7.10: Modeling the Oscillation of a Spring

A spring measuring 10 inches in natural length is compressed by 5 inches and released. It oscillates once every 3 seconds, and its amplitude decreases by 30% every second. Find an equation that models the position of the spring t seconds after being released.

Solution

The amplitude begins at 5 in. and deceases 30% each second. Because the spring is initially compressed, we will write A as a negative value. We can write the amplitude portion of the function as

$$A(t) = 5(1-0.30)^t$$

We put $(1-0.30)^t$ in the form e^{ct} as follows:

$$\begin{aligned} 0.7 &= e^c \\ c &= \ln .7 \\ c &= -0.357 \end{aligned}$$

Now let's address the period. The spring cycles through its positions every 3 seconds, this is the period, and we can use the formula to find omega.

$$\begin{aligned} 3 &= \frac{2\pi}{\omega} \\ \omega &= \frac{2\pi}{3} \end{aligned}$$

The natural length of 10 inches is the midline. We will use the cosine function, since the spring starts out at its maximum displacement. This portion of the equation is represented as

$$y = \cos(\frac{2\pi}{3}t) + 10$$

Finally, we put both functions together. Our the model for the position of the spring at t seconds is given as

$$y = -5e^{-0.357t}\cos(\frac{2\pi}{3}t) + 10$$

See the graph in Figure 3.7.22

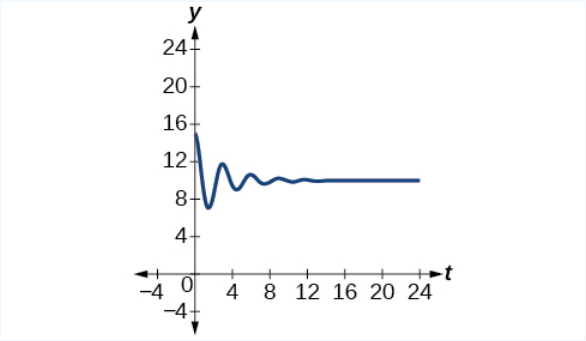

Figure 3.7.22

Exercise 3.7.9

A mass suspended from a spring is raised a distance of 5 cm above its resting position. The mass is released at time $t = 0$ and allowed to oscillate. After $\frac{1}{3}$ second, it is observed that the mass returns to its highest position. Find a function to model this motion relative to its initial resting position.

Answer

$y = 5\cos(6\pi t)$

According to the Given Criteria

A guitar string is plucked and vibrates in damped harmonic motion. The string is pulled and displaced 2 cm from its resting position. After 3 seconds, the displacement of the string measures 1 cm. Find the damping constant.

Solution

The displacement factor represents the amplitude and is determined by the coefficient ae^{-ct} in the model for damped harmonic motion. The damping constant is included in the term e^{-ct}. It is known that after 3 seconds, the local maximum measures one-half of its original value. Therefore, we have the equation

$$ae^{-c(t+3)} = \frac{1}{2}ae^{-ct} \tag{3.7.12}$$

Use algebra and the laws of exponents to solve for c.

$$\begin{aligned} ae^{-c(t+3)} &= \tfrac{1}{2}ae^{-ct} && \\ e^{-ct}\cdot e^{-3c} &= \tfrac{1}{2}e^{-ct} && \text{Divide out } a. \\ e^{-3c} &= \tfrac{1}{2} && \text{Divide out } e^{-ct}. \\ e^{3c} &= 2 && \text{Take reciprocals.} \end{aligned} \tag{3.7.13}$$

Then use the laws of logarithms.

$$\begin{aligned} e^{3c} &= 2 \\ 3c &= \ln 2 \\ c &= \frac{\ln 2}{3} \end{aligned}$$

The damping constant is $\frac{\ln 2}{3}$.

Bounding Curves in Harmonic Motion

Harmonic motion graphs may be enclosed by bounding curves. When a function has a varying **amplitude**, such that the amplitude rises and falls multiple times within a period, we can determine the bounding curves from part of the function.

Example 3.7.12: Graphing an Oscillating Cosine Curve

Graph the function $f(x) = \cos(2\pi x)\cos(16\pi x)$.

Solution

The graph produced by this function will be shown in two parts. The first graph will be the exact function $f(x)$ (see Figure $3.7.23; top$), and the second graph is the exact function $f(x)$ plus a bounding function (see Figure $3.7.23; bottom$). The graphs look quite different.

Figure_07_07_016.jpg" />

***Figure* 3.7.23**

Analysis

The curves $y = \cos(2\pi x)$ and $y = -\cos(2\pi x)$ are bounding curves: they bound the function from above and below, tracing out the high and low points. The harmonic motion graph sits inside the bounding curves. This is an example of a function whose amplitude not only decreases with time, but actually increases and decreases multiple times within a period.

Key Equations

Standard form of sinusoidal equation	$y = A\sin(Bt - C) + D$ or $y = A\cos(Bt - C) + D$
Simple harmonic motion	$d = a\cos(\omega t)$ or $d = a\sin(\omega t)$
Damped harmonic motion	$f(t) = ae^{-ct}\sin(\omega t)$ or $f(t) = ae^{-ct}\cos(\omega t)$

Key Concepts

- Sinusoidal functions are represented by the sine and cosine graphs. In standard form, we can find the amplitude, period, and horizontal and vertical shifts. See Example and Example.
- Use key points to graph a sinusoidal function. The five key points include the minimum and maximum values and the midline values. See Example.
- Periodic functions can model events that reoccur in set cycles, like the phases of the moon, the hands on a clock, and the seasons in a year. See Example, Example, Example and Example.
- Harmonic motion functions are modeled from given data. Similar to periodic motion applications, harmonic motion requires a restoring force. Examples include gravitational force and spring motion activated by weight. See Example.
- Damped harmonic motion is a form of periodic behavior affected by a damping factor. Energy dissipating factors, like friction, cause the displacement of the object to shrink. See Example, Example, Example, Example, and Example.
- Bounding curves delineate the graph of harmonic motion with variable maximum and minimum values. See Example.

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3.E: Trigonometric Identities and Equations (Exercises)

7.1: Solving Trigonometric Equations with Identities

In this section, we will begin an examination of the fundamental trigonometric identities, including how we can verify them and how we can use them to simplify trigonometric expressions.

Verbal

1) We know $g(x) = \cos x$ is an even function, and $f(x) = \sin x$ and $h(x) = \tan x$are odd functions. What about $G(x) = \cos^2 x$, $F(x) = \sin^2 x$ and $H(x) = \tan^2 x$? Are they even, odd, or neither? Why?

Answer

All three functions, F, G, and H, are even.

This is because

$F(-x) = \sin(-x)\sin(-x) = (-\sin x)(-\sin x) = \sin^2 x = F(x), G(-x) = \cos(-x)\cos(-x) = \cos x \cos x = cos^2 x = H(-x) = \tan(-x)\tan(-x) = (-\tan x)(-\tan x) = \tan 2x = H(x)$

2) Examine the graph of $f(x) = \sec x$ on the interval $[-\pi, \pi]$. How can we tell whether the function is even or odd by only observing the graph of $f(x) = \sec x$?

3) After examining the reciprocal identity for $\sec t$, explain why the function is undefined at certain points.

Answer

When $\cos t = 0$, then $\sec t = 10$, which is undefined.

4) All of the Pythagorean identities are related. Describe how to manipulate the equations to get from $\sin^2 t + \cos^2 t = 1$ to the other forms.

Algebraic

For the exercises 5-15, use the fundamental identities to fully simplify the expression.

5) $\sin x \cos x \sec x$

Answer

$\sin x$

6) $\sin(-x)\cos(-x)\csc(-x)$

7) $\tan x \sin x + \sec x \cos^2 x$

Answer

$\sec x$

8) $\csc x + \cos x \cot(-x)$

9) $\dfrac{\cot t + \tan t}{\sec(-t)}$

Answer

$\csc x$

10) $3\sin^3 t \csc t + \cos^2 t + 2\cos(-t)\cos t$

11) $-\tan(-x)\cot(-x)$

Answer

-1

12) $\dfrac{-\sin(-x)\cos x \sec x \csc x \tan x}{\cot x}$

13) $\dfrac{1+\tan^2\theta}{\csc^2\theta} + \sin^2\theta + \dfrac{1}{\sec^2\theta}$

Answer

$\sec^2 x$

14) $\left(\dfrac{\tan x}{\csc^2 x}+\dfrac{\tan x}{\sec^2 x}\right)\left(\dfrac{1+\tan x}{1+\cot x}\right)-\dfrac{1}{\cos^2 x}$

15) $\dfrac{1-\cos^2 x}{\tan^2 x}+2\sin^2 x$

Answer

$\sin^2 x+1$

For the exercises 16-28, simplify the first trigonometric expression by writing the simplified form in terms of the second expression.

16) $\dfrac{\tan x+\cot x}{\csc x};\cos x$

17) $\dfrac{\sec x+\csc x}{1+\tan x};\sin x$

Answer

$\dfrac{1}{\sin x}$

18) $\dfrac{\cos x}{1+\sin x}+\tan x;\cos x$

19) $\dfrac{1}{\sin x\cos x}-\cot x;\cot x$

Answer

$\dfrac{1}{\cot x}$

20) $\dfrac{1}{1-\cos x}-\dfrac{\cos x}{1+\cos x};\csc x$

21) $(\sec x+\csc x)(\sin x+\cos x)-2-\cot x;\tan x$

Answer

$\tan x$

22) $\dfrac{1}{\csc x-\sin x};\sec x \text{ and } \tan x$

23) $\dfrac{1-\sin x}{1+\sin x}-\dfrac{1+\sin x}{1-\sin x};\sec x \text{ and } \tan x$

Answer

$-4\sec x\tan x$

24) $\tan x;\sec x$

25) $\sec x;\cot x$

Answer

$\pm\sqrt{\dfrac{1}{\cot^2 x}+1}$

26) $\sec x;\sin x$

27) $\cot x;\sin x$

Answer

$\dfrac{\pm\sqrt{1-\sin^2 x}}{\sin x}$

28) $\cot x; \csc x$

For the exercises 29-33, verify the identity.

29) $\cos x - \cos^3 x = \cos x \sin^2 x$

Answer

Answers will vary. Sample proof:

$$\begin{aligned}\cos x - \cos^3 x &= \cos x(1 - \cos^2 x) \\ &= \cos x \sin^x\end{aligned}$$

30) $\cos x(\tan x - \sec(-x)) = \sin x - 1$

31) $\dfrac{1+\sin^2 x}{\cos^2 x} = \dfrac{1}{\cos^2 x} + \dfrac{\sin^2 x}{\cos^2 x} = 1 + 2\tan^2 x$

Answer

Answers will vary. Sample proof:

$$\begin{aligned}\frac{1+\sin^2 x}{\cos^2 x} &= \frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} \\ &= \sec^2 x + \tan^2 x \\ &= \tan^2 x + 1 + \tan^2 x \\ &= 1 + 2\tan^2 x\end{aligned}$$

32) $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

33) $\cos^2 x - \tan^2 x = 2 - \sin^2 x - \sec^2 x$

Answer

Answers will vary. Sample proof:

$$\begin{aligned}\cos^2 x - \tan^2 x &= 1 - \sin^2 x - \left(\sec^2 x - 1\right) \\ &= 1 - \sin^2 x - \sec^2 x + 1 \\ &= 2 - \sin^2 x - \sec^2 x\end{aligned}$$

Extensions

For the exercises 34-39, prove or disprove the identity.

34) $\dfrac{1}{1+\cos x} - \dfrac{1}{1-\cos(-x)} = -2\cot x \csc x$

35) $\csc^2 x(1 + \sin^2 x) = \cot^2 x$

Answer

False

36) $\left(\dfrac{\sec^2(-x) - \tan^2 x}{\tan x}\right)\left(\dfrac{2+2\tan x}{2+2\cot x}\right) - 2\sin^2 x = \cos(2x)$

37) $\dfrac{\tan x}{\sec x}\sin(-x) = \cos^2 x$

Answer

False

38) $\dfrac{\sec(-x)}{\tan x + \cot x} = -\sin(-x)$

39) $\dfrac{1+\sin x}{\cos x} = \dfrac{\cos x}{1+\sin(-x)}$

Answer

Proved with negative and Pythagorean identities

For the exercises 40-, determine whether the identity is true or false. If false, find an appropriate equivalent expression.

40) $\dfrac{\cos^2\theta-\sin^2\theta}{1-\tan^2\theta}=\sin^2\theta$

41) $3\sin^2\theta+4\cos^2\theta=3+\cos^2\theta$

Answer

True

$$\begin{aligned}3\sin^2\theta+4\cos^2\theta &= 3\sin^2\theta+3\cos^2\theta+\cos^2\theta\\ &= 3\left(\sin^2\theta+\cos^2\theta\right)+\cos^2\theta\\ &= 3+\cos^2\theta\end{aligned}$$

42) $\dfrac{\sec\theta+\tan\theta}{\cot\theta+\cos^{\theta}}=\sec^2\theta$

7.2: Sum and Difference Identities

In this section, we will learn techniques that will enable us to solve useful problems. The formulas that follow will simplify many trigonometric expressions and equations. Keep in mind that, throughout this section, the term *formula* is used synonymously with the word identity.

Verbal

1) Explain the basis for the cofunction identities and when they apply.

Answer

The cofunction identities apply to complementary angles. Viewing the two acute angles of a right triangle, if one of those angles measures x, the second angle measures $\dfrac{\pi}{2}-x$. Then $\sin x=\cos\left(\dfrac{\pi}{2}-x\right)$. The same holds for the other cofunction identities. The key is that the angles are complementary.

2) Is there only one way to evaluate $\cos\left(\dfrac{5\pi}{4}\right)$? Explain how to set up the solution in two different ways, and then compute to make sure they give the same answer.

3) Explain to someone who has forgotten the even-odd properties of sinusoidal functions how the addition and subtraction formulas can determine this characteristic for $f(x)=\sin(x)$ and $g(x)=\cos(x)$. (Hint: $0-x=-x$)

Answer

$\sin(-x)=-\sin x$, so $\sin x$ is odd. $\cos(-x)=\cos(0-x)=\cos x$, so $\cos x$ is even.

Algebraic

For the exercises 4-9, find the exact value.

4) $\cos\left(\dfrac{7\pi}{12}\right)$

5) $\cos\left(\dfrac{\pi}{12}\right)$

Answer

$\dfrac{\sqrt{2}+\sqrt{6}}{4}$

6) $\sin\left(\dfrac{5\pi}{12}\right)$

7) $\sin\left(\dfrac{11\pi}{12}\right)$

Answer

$\dfrac{\sqrt{6}-\sqrt{2}}{4}$

8) $\tan\left(-\dfrac{\pi}{12}\right)$

9) $\tan\left(\frac{19\pi}{12}\right)$

Answer

$-2-\sqrt{3}$

For the exercises 10-13, rewrite in terms of $\sin x$ and $\cos x$

10) $\sin\left(x+\frac{11\pi}{6}\right)$

11) $\sin\left(x-\frac{3\pi}{4}\right)$

Answer

$-\frac{\sqrt{2}}{2}\sin x-\frac{\sqrt{2}}{2}\cos x$

12) $\cos\left(x-\frac{5\pi}{6}\right)$

13) $\cos\left(x+\frac{2\pi}{3}\right)$

Answer

$-\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x$

For the exercises 14-19, simplify the given expression.

14) $\csc\left(\frac{\pi}{2}-t\right)$

15) $\sec\left(\frac{\pi}{2}-\theta\right)$

Answer

$\csc\theta$

16) $\cot\left(\frac{\pi}{2}-x\right)$

17) $\tan\left(\frac{\pi}{2}-x\right)$

Answer

$\cot x$

18) $\sin(2x)\cos(5x)-\sin(5x)\cos(2x)$

19) $\frac{\tan\left(\frac{3}{2}x\right)-\tan\left(\frac{7}{5}x\right)}{1+\tan\left(\frac{3}{2}x\right)\tan\left(\frac{7}{5}x\right)}$

Answer

$\tan\left(\frac{x}{10}\right)$

For the exercises 20-21, find the requested information.

20) Given that $\sin a=\frac{2}{3}$ and $\cos b=-\frac{1}{4}$, with a and b both in the interval $\left[\frac{\pi}{2},\pi\right)$, find $\sin(a+b)$ and $\cos(a-b)$.

21) Given that $\sin a=\frac{4}{5}$ and $\cos b=\frac{1}{3}$, with a and b both in the interval $\left[0,\frac{\pi}{2}\right)$, find $\sin(a-b)$ and $\cos(a+b)$.

Answer

$$\sin(a-b)=\left(\frac{4}{5}\right)\left(\frac{1}{3}\right)-\left(\frac{3}{5}\right)\left(\frac{2\sqrt{2}}{3}\right)=\frac{4-6\sqrt{2}}{15}$$

$$\cos(a+b)=\left(\frac{3}{5}\right)\left(\frac{1}{3}\right)-\left(\frac{4}{5}\right)\left(\frac{2\sqrt{2}}{3}\right)=\frac{3-8\sqrt{2}}{15}$$

For the exercises 22-24, find the exact value of each expression.

22) $\sin\left(\cos^{-1}(0)-\cos^{-1}\left(\frac{1}{2}\right)\right)$

23) $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$

Answer

$$\frac{\sqrt{2}-\sqrt{6}}{4}$$

24) $\tan\left(\sin^{-1}\left(\frac{1}{2}\right)-\cos^{-1}\left(\frac{1}{2}\right)\right)$

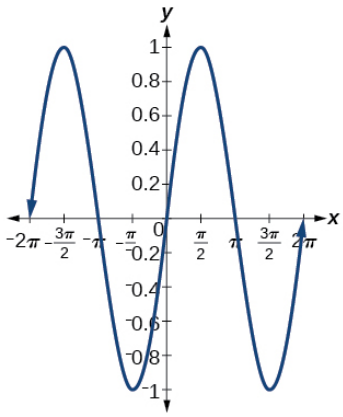
Graphical

For the exercises 25-32, simplify the expression, and then graph both expressions as functions to verify the graphs are identical.

25) $\cos\left(\frac{\pi}{2}-x\right)$

Answer

$\sin x$

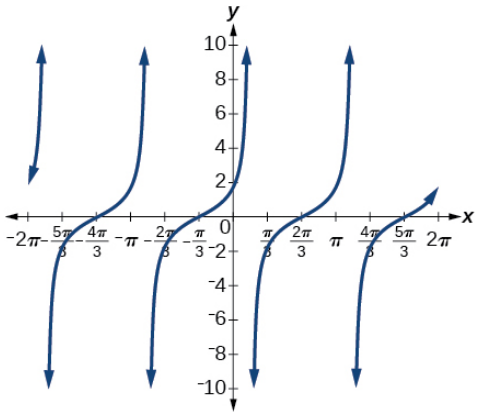


26) $\sin(\pi-x)$

27) $\tan\left(\frac{\pi}{3}+x\right)$

Answer

$\cot\left(\frac{\pi}{6}-x\right)$

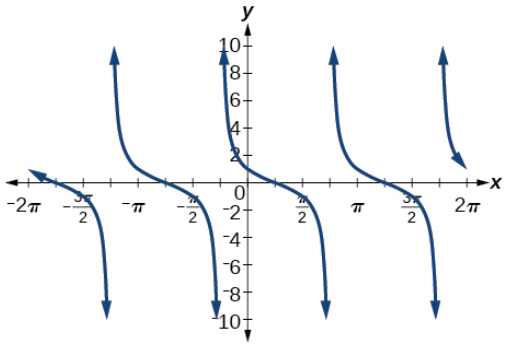


28) $\sin\left(\frac{\pi}{3}+x\right)$

29) $\tan\left(\frac{\pi}{4}-x\right)$

Answer

$\cot\left(\frac{\pi}{4}+x\right)$

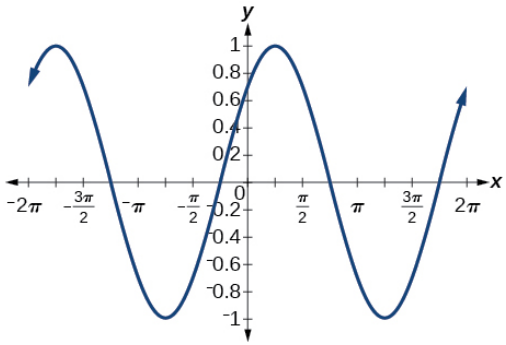


30) $\cos\left(\frac{7\pi}{6}+x\right)$

31) $\sin\left(\frac{\pi}{4}+x\right)$

Answer

$\frac{\sin x}{\sqrt{2}}+\frac{\cos x}{\sqrt{2}}$



32) $\cos\left(\frac{5\pi}{4}+x\right)$

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For the exercises 33-41, use a graph to determine whether the functions are the same or different. If they are the same, show why. If they are different, replace the second function with one that is identical to the first. (Hint: think $2x = x + x$)

33) $f(x) = \sin(4x) - \sin(3x)\cos x, g(x) = \sin x \cos(3x)$

Answer

They are the same.

34) $f(x) = \cos(4x) + \sin x \sin(3x), g(x) = -\cos x \cos(3x)$

35) $f(x) = \sin(3x)\cos(6x), g(x) = -\sin(3x)\cos(6x)$

Answer

They are different, try $g(x) = \sin(9x) - \cos(3x)\sin(6x)$

36) $f(x) = \sin(4x), g(x) = \sin(5x)\cos x - \cos(5x)\sin x$

37) $f(x) = \sin(2x), g(x) = 2\sin x \cos x$

Answer

They are the same.

38) $f(\theta) = \cos(2\theta), g(\theta) = \cos^2\theta - \sin^2\theta$

39) $f(\theta) = \tan(2\theta), g(\theta) = \dfrac{\tan\theta}{1 + \tan^2\theta}$

Answer

They are different, try $g(\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

40) $f(x) = \sin(3x)\sin x, g(x) = \sin^2(2x)\cos^2 x - \cos^2(2x)\sin 2x$

41) $f(x) = \tan(-x), g(x) = \dfrac{\tan x - \tan(2x)}{1 - \tan x \tan(2x)}$

Answer

They are different, try $g(x) = \dfrac{\tan x - \tan(2x)}{1 + \tan x \tan(2x)}$

Technology

For the exercises 42-46, find the exact value algebraically, and then confirm the answer with a calculator to the fourth decimal point.

42) $\sin(75^\circ)$

43) $\sin(195^\circ)$

Answer

$-\dfrac{\sqrt{3}-1}{2\sqrt{2}}$, or -0.2588

44) $\cos(165^\circ)$

45) $\cos(345^\circ)$

Answer

$\dfrac{1+\sqrt{3}}{2\sqrt{2}}$, or -0.9659

46) $\tan(-15^\circ)$

Extensions

For the exercises 47-51, prove the identities provided.

47) $\tan\left(x+\frac{\pi}{4}\right)=\frac{\tan x+1}{1-\tan x}$

Answer

$$\tan\left(x+\frac{\pi}{4}\right)=$$
$$\frac{\tan x+\tan\left(\frac{\pi}{4}\right)}{1-\tan x\tan\left(\frac{\pi}{4}\right)}=$$
$$\frac{\tan x+1}{1-\tan x(1)}=\frac{\tan x+1}{1-\tan x}$$

48) $\frac{\tan(a+b)}{\tan(a-b)}=\frac{\sin a\cos a+\sin b\cos b}{\sin a\cos a-\sin b\cos b}$

49) $\frac{\cos(a+b)}{\cos a\cos b}=1-\tan a\tan b$

Answer

$$\frac{\cos(a+b)}{\cos a\cos b}=$$
$$\frac{\cos a\cos b}{\cos a\cos b}-\frac{\sin a\sin b}{\cos a\cos b}=1-\tan a\tan b$$

50) $\cos(x+y)\cos(x-y)=\cos^2 x-\sin^2 y$

51) $\frac{\cos(x+h)-\cos(x)}{h}=\cos x\frac{\cos h-1}{h}-\sin x\frac{\sin h}{h}$

Answer

$$\frac{\cos(x+h)-\cos(x)}{h}=$$
$$\frac{\cos x\cosh-\sin x\sinh-\cos x}{h}=$$
$$\frac{\cos x(\cosh-1)-\sin x(\sinh-1)}{h}=\cos x\frac{\cos h-1}{h}-\sin x\frac{\sin h}{h}$$

For the exercises 52-, prove or disprove the statements.

52) $\tan(u+v)=\frac{\tan u+\tan v}{1-\tan u\tan v}$

53) $\tan(u-v)=\frac{\tan u-\tan v}{1+\tan u\tan v}$

Answer

True

54) $\frac{\tan(x+y)}{1+\tan x\tan x}=\frac{\tan x+\tan y}{1-\tan^2 x\tan^2 y}$

55) If α, β, and γ are angles in the same triangle, then prove or disprove $\sin(\alpha+\beta)=\sin\gamma$.

Answer

True. Note that $\sin(\alpha+\beta)=\sin(\pi-\gamma)$ and expand the right hand side.

56) If α, β, and γ are angles in the same triangle, then prove or disprove $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$.

7.3: Double-Angle, Half-Angle, and Reduction Formulas

In this section, we will investigate three additional categories of identities. Double-angle identities are derived from the sum formulas of the fundamental trigonometric functions: sine, cosine, and tangent. Reduction formulas are especially useful in calculus, as they allow us to reduce the power of the trigonometric term. Half-angle formulas allow us to find the value of trigonometric functions involving half-angles, whether the original angle is known or not.

Verbal

1) Explain how to determine the reduction identities from the double-angle identity $\cos(2x) = \cos^2 x - \sin^2 x$

Answer

Use the Pythagorean identities and isolate the squared term.

2) Explain how to determine the double-angle formula for $\tan(2x)$ using the double-angle formulas for $\cos(2x)$ and $\sin(2x)$.

3) We can determine the half-angle formula for $\tan\left(\frac{x}{2}\right) = \frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}$ by dividing the formula for $\sin\left(\frac{x}{2}\right)$ by $\cos\left(\frac{x}{2}\right)$. Explain how to determine two formulas for $\tan\left(\frac{x}{2}\right)$ that do not involve any square roots.

Answer

$\frac{1-\cos x}{\sin x}$, $\frac{\sin x}{1+\cos x}$, multiplying the top and bottom by $\sqrt{1-\cos x}$ and $\sqrt{1+\cos x}$, respectively.

4) For the half-angle formula given in the previous exercise for $\tan\left(\frac{x}{2}\right)$, explain why dividing by 0 is not a concern. (Hint: examine the values of $\cos x$ necessary for the denominator to be 0.)

Algebraic

For the exercises 5-8, find the exact values of a) $\sin(2x)$, b) $\cos(2x)$, and c) $\tan(2x)$ without solving for x.

5) If $\sin x = \frac{1}{8}$, and x is in quadrant I.

Answer

a. $\frac{3\sqrt{7}}{32}$
b. $\frac{31}{32}$
c. $\frac{3\sqrt{7}}{31}$

6) If $\cos x = \frac{2}{3}$, and x is in quadrant I.

7) If $\cos x = -\frac{1}{2}$, and x is in quadrant III.

Answer

a. $\frac{\sqrt{3}}{2}$
b. $-\frac{1}{2}$
c. $-\sqrt{3}$

8) If $\tan x = -8$, and x is in quadrant IV.

For the exercises 9-10, find the values of the six trigonometric functions if the conditions provided hold.

9) $\cos(2\theta) = \frac{3}{5}$ and $90^\circ \le \theta \le 180^\circ$

Answer

$\cos\theta = -\frac{2\sqrt{5}}{5}, \sin\theta = \frac{\sqrt{5}}{5}, \tan\theta = -\frac{1}{2}, \csc\theta = \sqrt{5}, \sec\theta = -\frac{\sqrt{5}}{2}, \cot\theta = -2$

10) $\cos(2\theta) = \frac{1}{\sqrt{2}}$ and $180^\circ \le \theta \le 270^\circ$

For the exercises 11-12, simplify to one trigonometric expression.

11) $2\sin\left(\frac{\pi}{4}\right)2\cos\left(\frac{\pi}{4}\right)$

Answer

$2\sin\left(\frac{\pi}{2}\right)$

12) $4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$

For the exercises 13-19, find the exact value using half-angle formulas.

13) $\sin\left(\frac{\pi}{8}\right)$

Answer

$\frac{\sqrt{2-\sqrt{2}}}{2}$

14) $\cos\left(-\frac{11\pi}{12}\right)$

15) $\sin\left(\frac{11\pi}{12}\right)$

Answer

$\frac{\sqrt{2-\sqrt{3}}}{2}$

16) $\cos\left(\frac{7\pi}{8}\right)$

17) $\tan\left(\frac{5\pi}{12}\right)$

Answer

$2+\sqrt{3}$

18) $\tan\left(-\frac{3\pi}{12}\right)$

19) $\tan\left(-\frac{3\pi}{8}\right)$

Answer

$-1-\sqrt{2}$

For the exercises 20-23, find the exact values of a) $\sin\left(\frac{x}{2}\right)$ b) $\cos\left(\frac{x}{2}\right)$, and c) $\tan\left(\frac{x}{2}\right)$ without solving for x, when $0^\circ \le \theta \le 360^\circ$

20) If $\tan x = -\frac{4}{3}$, and x is in quadrant IV.

21) If $\sin x = -\frac{12}{13}$, and x is in quadrant III.

Answer

a. $\frac{3\sqrt{13}}{13}$

b. $-\frac{2\sqrt{13}}{13}$

c. $-\frac{3}{2}$

22) If $\csc x = 7$, and x is in quadrant II.

23) If $\sec x = -4$, and x is in quadrant II.

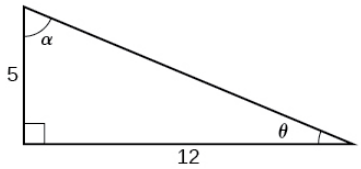
Answer

a. $\frac{\sqrt{10}}{4}$

b. $\frac{\sqrt{6}}{4}$

c. $\frac{\sqrt{15}}{3}$

For the exercises 24-27, use Figure below to find the requested half and double angles.



24) Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$.

25) Find $\sin(2\alpha)$, $\cos(2\alpha)$, and $\tan(2\alpha)$.

Answer

$\frac{120}{169}, -\frac{119}{169}, -\frac{120}{119}$

26) Find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$.

27) Find $\sin\left(\frac{\alpha}{2}\right)$, $\cos\left(\frac{\alpha}{2}\right)$, and $\tan\left(\frac{\alpha}{2}\right)$.

Answer

$\frac{2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}, \frac{2}{3}$

For the exercises 28-33, simplify each expression. Do not evaluate.

28) $\cos^2(28^\circ) - \sin^2(28^\circ)$

29) $2\cos^2(37^\circ) - 1$

Answer

$\cos(74^\circ)$

30) $1 - 2\sin^2(17^\circ)$

31) $\cos^2(9x) - \sin^2(9x)$

Answer

$\cos(18x)$

32) $4\sin(8x)\cos(8x)$

33) $6\sin(5x)\cos(5x)$

Answer

$3\sin(10x)$

For the exercises 34-37, prove the identity given.

34) $(\sin t - \cos t)^2 = 1 - \sin(2t)$

35) $\sin(2x) = -2\sin(-x)\cos(-x)$

Answer

$-2\sin(-x)\cos(-x) = -2(-\sin(x)\cos(x)) = \sin(2x)$

36) $\cot x - \tan x = 2\cot(2x)$

37) $\frac{1+\cos(2\theta)}{\sin(2\theta)}\tan^2\theta = \tan\theta$

Answer

$$\frac{\sin(2\theta)}{1+\cos(2\theta)}\tan^2\theta = \frac{2\sin(\theta)\cos(\theta)}{1+\cos^2\theta-\sin^2\theta}\tan^2\theta =$$

$$\frac{2\sin(\theta)\cos(\theta)}{2\cos^2\theta}\tan^2\theta = \frac{\sin(\theta)}{\cos(\theta)}\tan^2\theta =$$

$$\cot(\theta)\tan^2\theta = \tan\theta$$

For the exercises 38-44, rewrite the expression with an exponent no higher than 1.

38) $\cos^2(5x)$

39) $\cos^2(6x)$

Answer

$$\frac{1+\cos(12x)}{2}$$

40) $\sin^4(8x)$

41) $\sin^4(3x)$

Answer

$$\frac{3+\cos(12x)-4\cos(6x)}{8}$$

42) $\cos^2 x \sin^4 x$

43) $\cos^4 x \sin^2 x$

Answer

$$\frac{2+\cos(2x)-2\cos(4x)-\cos(6x)}{32}$$

44) $\tan^2 x \sin^2 x$

Technology

For the exercises 45-52, reduce the equations to powers of one, and then check the answer graphically.

45) $\tan^4 x$

Answer

$$\frac{3+\cos(4x)-4\cos(2x)}{3+\cos(4x)+4\cos(2x)}$$

46) $\sin^2(2x)$

47) $\sin^2 x \cos^2 x$

Answer

$$\frac{1-\cos(4x)}{8}$$

48) $\tan^2 x \sin x$

49) $\tan^4 x \cos^2 x$

Answer

$$\frac{3+\cos(4x)-4\cos(2x)}{4(\cos(2x)+1)}$$

50) $\cos^2 x \sin(2x)$

51) $\cos^2(2x)\sin x$

Answer

$$\frac{(1+\cos(4x))\sin x}{2}$$

52) $\tan^2\left(\frac{x}{2}\right)\sin x$

For the exercises 53-54, algebraically find an equivalent function, only in terms of $\sin x$ and/or $\cos x$, and then check the answer by graphing both equations.

53) $\sin(4x)$

Answer

$$4\sin x\cos x(\cos^2 x-\sin^2 x)$$

54) $\cos(4x)$

Extensions

For the exercises 55-63, prove the identities.

55) $\sin(2x)=\dfrac{2\tan x}{1+\tan^2 x}$

Answer

$$\frac{2\tan x}{1+\tan^2 x}=\frac{\frac{2\sin x}{\cos x}}{1+\frac{\sin^2 x}{\cos^2 x}}=\frac{\frac{2\sin x}{\cos x}}{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}=\frac{2\sin x}{\cos x}\cdot\frac{\cos^2 x}{1}=2\sin x\cos x=\sin(2x)$$

56) $\cos(2\alpha)=\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}$

57) $\tan(2x)=\dfrac{2\sin x\cos x}{2\cos^2 x-1}$

Answer

$$\frac{2\sin x\cos x}{2\cos^2 x-1}=\frac{\sin(2x)}{\cos(2x)}=\tan(2x)$$

58) $(\sin^2 x-1)^2=\cos(2x)+\sin^4 x$

59) $\sin(3x)=3\sin x\cos^2 x-\sin^3 x$

Answer

$$\begin{aligned}\sin(x+2x)&=\sin x\cos(2x)+\sin(2x)\cos x\\&=\sin x(\cos^2 x-\sin^2 x)+2\sin x\cos x\cos x\\&=\sin x\cos^2 x-\sin^3 x+2\sin x\cos^2 x\\&=3\sin x\cos^2 x-\sin^3 x\end{aligned}$$

60) $\cos(3x)=\cos^3 x-3\sin^2 x\cos x$

61) $\dfrac{1+\cos(2t)}{\sin(2t)-\cos t}=\dfrac{2\cos t}{2\sin t-1}$

Answer

$$\begin{aligned}\frac{1+\cos(2t)}{\sin(2t)-\cos t}&=\frac{1+2\cos^2 t-1}{2\sin t\cos t-\cos t}\\&=\frac{2\cos^2 t}{\cos t(2\sin t-1)}\\&=\frac{2\cos t}{2\sin t-1}\end{aligned}$$

62) $\sin(16x)=16\sin x\cos x\cos(2x)\cos(4x)\cos(8x)$

63) $\cos(16x)=(\cos^2(4x)-\sin^2(4x)-\sin(8x))(\cos^2(4x)-\sin^2(4x)+\sin(8x))$

Answer

$$\begin{aligned}(\cos^2(4x)-\sin^2(4x)-\sin(8x))(\cos^2(4x)-\sin^2(4x)+\sin(8x)) &= (\cos(8x)-\sin(8x))(\cos(8x)+\sin(8x)) \\ &= \cos^2(8x)-\sin^2(8x) \\ &= \cos(16x)\end{aligned}$$

7.4: Sum-to-Product and Product-to-Sum Formulas

From the sum and difference identities, we can derive the product-to-sum formulas and the sum-to-product formulas for sine and cosine. The product-to-sum formulas can rewrite products of sines, products of cosines, and products of sine and cosine as sums or differences of sines and cosines. We can also derive the sum-to-product identities from the product-to-sum identities using substitution. The sum-to-product formulas are used to rewrite sum or difference as products of sines and cosines.

Verbal

1) Starting with the product to sum formula $\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$, explain how to determine the formula for $\cos\alpha\sin\beta$.

Answer

Substitute α into cosine and β into sine and evaluate.

2) Explain two different methods of calculating $\cos(195^\circ)\cos(105^\circ)$, one of which uses the product to sum. Which method is easier?

3) Explain a situation where we would convert an equation from a sum to a product and give an example.

Answer

Answers will vary. There are some equations that involve a sum of two trig expressions where when converted to a product are easier to solve. For example: $\frac{\sin(3x)+\sin x}{\cos x}=1$. When converting the numerator to a product the equation becomes: $\frac{2\sin(2x)\cos x}{\cos x}=1$

4) Explain a situation where we would convert an equation from a product to a sum, and give an example.

Algebraic

For the exercises 5-10, rewrite the product as a sum or difference.

5) $16\sin(16x)\sin(11x)$

Answer

$8(\cos(5x)-\cos(27x))$

6) $2\cos(36t)\cos(6t)$

7) $2\sin(5x)\cos(3x)$

Answer

$\sin(2x)+\sin(8x)$

8) $10\cos(5x)\sin(10x)$

9) $\sin(-x)\sin(5x)$

Answer

$\frac{1}{2}(\cos(6x)-\cos(4x))$

10) $\sin(3x)\cos(5x)$

For the exercises 11-16, rewrite the sum or difference as a product.

11) $\cos(6t)+\cos(4t)$

Answer

$2\cos(5t)\cos t$

12) $\sin(3x)+\sin(7x)$

13) $\cos(7x)+\cos(-7x)$

Answer

$2\cos(7x)$

14) $\sin(3x)-\sin(-3x)$

15) $\cos(3x)+\cos(9x)$

Answer

$2\cos(6x)\cos(3x)$

16) $\sin h-\sin(3h)$

For the exercises 17-21, evaluate the product for the following using a sum or difference of two functions.

17) $\cos(45^\circ)\cos(15^\circ)$

Answer

$\frac{1}{4}(1+\sqrt{3})$

18) $\cos(45^\circ)\sin(15^\circ)$

19) $\sin(-345^\circ)\sin(-15^\circ)$

Answer

$\frac{1}{4}(\sqrt{3}-2)$

20) $\sin(195^\circ)\cos(15^\circ)$

21) $\sin(-45^\circ)\sin(-15^\circ)$

Answer

$\frac{1}{4}(\sqrt{3}-1)$

For the exercises 22-26, evaluate the product using a sum or difference of two functions. Leave in terms of sine and cosine.

22) $\cos(23^\circ)\sin(17^\circ)$

23) $2\sin(100^\circ)\sin(20^\circ)$

Answer

$\cos(80^\circ)-\cos(120^\circ)$

24) $2\sin(-100^\circ)\sin(-20^\circ)$

25) $\sin(213^\circ)\cos(8^\circ)$

Answer

$\frac{1}{2}(\sin(221^\circ)+\sin(205^\circ))$

26) $2\cos(56^\circ)\cos(47^\circ)$

For the exercises 27-31, rewrite the sum as a product of two functions. Leave in terms of sine and cosine.

27) $\sin(76^\circ)+\sin(14^\circ)$

Answer

$\sqrt{3}\cos(31^\circ)$

28) $\cos(58^\circ)-\cos(12^\circ)$

29) $\sin(101^\circ)-\sin(32^\circ)$

Answer

$2\cos(66.5^\circ)\sin(34.5^\circ)$

30) $\cos(100^\circ)+\cos(200^\circ)$

31) $\sin(-1^\circ)+\sin(-2^\circ)$

Answer

$2\sin(-1.5^\circ)\cos(0.5^\circ)$

For the exercises 32-38, prove the identity.

32) $\dfrac{\cos(a-b)}{\cos(a+b)}=\dfrac{1-\tan a\tan b}{1+\tan a\tan b}$

33) $4\sin(3x)\cos(4x)=2\sin(7x)-2\sin x$

Answer

$$\begin{aligned}2\sin(7x)-2\sin x &= 2\sin(4x+3x)-2\sin(4x-3x)\\ &= 2(\sin(4x)\cos(3x)+\sin(3x)\cos(4x))-2(\sin(4x)\cos(3x)-\sin(3x)\cos(4x))\\ &= 2\sin(4x)\cos(3x)+2\sin(3x)\cos(4x))-2\sin(4x)\cos(3x)+2\sin(3x)\cos(4x))\\ &= 4\sin(3x)\cos(4x)\end{aligned}$$

34) $\dfrac{6\cos(8x)\sin(2x)}{\sin(-6x)}=-3\sin(10x)\csc(6x)+3$

35) $\sin x+\sin(3x)=4\sin x\cos^2 x$

Answer

$$\begin{aligned}\sin x+\sin(3x) &= 2\sin\left(\frac{4x}{2}\right)\cos\left(-\frac{2x}{2}\right)\\ &= 2\sin(2x)\cos x\\ &= 2(2\sin x\cos x)\cos x\\ &= 4\sin x\cos^2 x\end{aligned}$$

36) $2(\cos^3 x-\cos x\sin^2 x)=\cos(3x)+\cos x$

37) $2\tan x\cos(3x)=\sec x(\sin(4x)-\sin(2x))$

Answer

$$\begin{aligned}2\tan x\cos(3x) &= \frac{2\sin x\cos(3x)}{\cos x}\\ &= \frac{2(.5(\sin(4x)-\sin(2x)))}{\cos x}\\ &= \frac{1}{\cos x}(\sin(4x)-\sin(2x))\\ &= \sec x(\sin(4x)-\sin(2x))\end{aligned}$$

38) $\cos(a+b)+\cos(a-b)=2\cos a\cos b$

Numeric

For the exercises 39-43, rewrite the sum as a product of two functions or the product as a sum of two functions. Give your answer in terms of sines and cosines. Then evaluate the final answer numerically, rounded to four decimal places.

39) $\cos(58^\circ)+\cos(12^\circ)$

Answer

$2\cos(35^\circ)\cos(23^\circ), 1.5081$

40) $\sin(2^\circ)-\sin(3^\circ)$

41) $\cos(44^\circ)-\cos(22^\circ)$

Answer

$-2\sin(33^\circ)\sin(11^\circ), -0.2078$

42) $\cos(176^\circ)\sin(9^\circ)$

43) $\sin(-14^\circ)\sin(85^\circ)$

Answer

$\frac{1}{2}(\cos(99^\circ)-\cos(71^\circ)), -0.2410$

Technology

For the exercises 44-48, algebraically determine whether each of the given expressions is a true identity. If it is not an identity, replace the right-hand side with an expression equivalent to the left side. Verify the results by graphing both expressions on a calculator.

44) $2\sin(2x)\sin(3x)=\cos x-\cos(5x)$

45) $\frac{\cos(10\theta)+\cos(6\theta)}{\cos(6\theta)-\cos(10\theta)}=\cot(2\theta)\cot(\theta)$

Answer

It is an identity.

46) $\frac{\sin(3x)-\sin(5x)}{\cos(3x)+\cos(5x)}=\tan x$

47) $2\cos(2x)\cos x+\sin(2x)\sin x=2\sin x$

Answer

It is not an identity, but $2\cos^3 x$ is.

48) $\frac{\sin(2x)+\sin(4x)}{\sin(2x)-\sin(4x)}=-\tan(3x)\cot x$

For the exercises 49-53, simplify the expression to one term, then graph the original function and your simplified version to verify they are identical.

49) $\frac{\sin(9t)-\sin(3t)}{\cos(9t)+\cos(3t)}$

Answer

$\tan(3t)$

50) $2\sin(8x)\cos(6x)-\sin(2x)$

51) $\frac{\sin(3x)-\sin(x)}{\sin x}$

Answer

$2\cos(2x)$

52) $\frac{\cos(5x)+\cos(3x)}{\sin(5x)+\sin(3x)}$

53) $\sin x\cos(15x)-\cos x\sin(15x)$

Answer

$-\sin(14x)$

Extensions

For the exercises 54-55, prove the following sum-to-product formulas.

54) $\sin x-\sin y=2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$

55) $\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

Answer

Start with $\cos x + \cos y$. Make a substitution and let $x = \alpha + \beta$ and let $y = \alpha - \beta$, so $\cos x + \cos y$ becomes:

$\cos(\alpha+\beta) + \cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta = 2\cos\alpha\cos\beta$

Since $x = \alpha + \beta$ and $y = \alpha - \beta$, we can solve for α and β in terms of x and y and substitute in for $2\cos\alpha\cos\beta$ and get

$2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

For the exercises 56-63, prove the identity.

56) $\frac{\sin(6x)+\sin(4x)}{\sin(6x)-\sin(4x)} = \tan(5x)\cot x$

57) $\frac{\cos(3x)+\cos x}{\cos(3x)-\cos(4x)} = -\cot(2x)\cot x$

Answer

$$\frac{\cos(3x)+\cos x}{\cos(3x)-\cos(4x)} = \frac{2\cos(2x)\cos x}{-2\sin(2x)\sin x} = -\cot(2x)\cot x$$

58) $\frac{\cos(6y)+\cos(8y)}{\sin(6y)-\sin(4y)} = \cot y\cos(7y)\sec(5y)$

59) $\frac{\cos(2y)-\cos(4y)}{\sin(2y)+\sin(4y)} = \tan y$

Answer

$$\begin{aligned}\frac{\cos(2y)-\cos(4y)}{\sin(2y)+\sin(4y)} &= \frac{-2\sin(3y)\sin(-y)}{2\sin(3y)\cos y}\\ &= \frac{2\sin(3y)\sin(y)}{2\sin(3y)\cos y}\\ &= \tan y\end{aligned}$$

60) $\frac{\sin(10x)-\sin(2x)}{\cos(10x)+\cos(2x)} = \tan(4x)$

61) $\cos x - \cos(3x) = 4\sin^2 x\cos x$

Answer

$$\begin{aligned}\cos x - \cos(3x) &= -2\sin(2x)\sin(-x)\\ &= 2(2\sin x\cos x)\sin x\\ &= 4\sin^2 x\cos x\end{aligned}$$

62) $(\cos(2x)-\cos(4x))^2 + (\sin(4x)+\sin(2x))^2 = 4\sin^2(3x)$

63) $\tan\left(\frac{\pi}{4}-t\right) = \frac{1-\tan t}{1+\tan t}$

Answer

$$\tan\left(\frac{\pi}{4}-t\right) = \frac{\tan\left(\frac{\pi}{4}\right)-\tan t}{1+\tan\left(\frac{\pi}{4}\right)\tan t} = \frac{1-\tan t}{1+\tan t}$$

7.5: Solving Trigonometric Equations

In earlier sections of this chapter, we looked at trigonometric identities. Identities are true for all values in the domain of the variable. In this section, we begin our study of trigonometric equations to study real-world scenarios such as the finding the dimensions of the pyramids.

Verbal

1) Will there always be solutions to trigonometric function equations? If not, describe an equation that would not have a solution. Explain why or why not.

Answer

There will not always be solutions to trigonometric function equations. For a basic example, $\cos(x) = -5$.

2) When solving a trigonometric equation involving more than one trig function, do we always want to try to rewrite the equation so it is expressed in terms of one trigonometric function? Why or why not?

3) When solving linear trig equations in terms of only sine or cosine, how do we know whether there will be solutions?

Answer

If the sine or cosine function has a coefficient of one, isolate the term on one side of the equals sign. If the number it is set equal to has an absolute value less than or equal to one, the equation has solutions, otherwise it does not. If the sine or cosine does not have a coefficient equal to one, still isolate the term but then divide both sides of the equation by the leading coefficient. Then, if the number it is set equal to has an absolute value greater than one, the equation has no solution.

Algebraic

For the exercises 4-12, find all solutions exactly on the interval $0 \le \theta < 2\pi$

4) $2\sin\theta = -\sqrt{2}$

5) $2\sin\theta = \sqrt{3}$

Answer

$\frac{\pi}{3}, \frac{2\pi}{3}$

6) $2\cos\theta = 1$

7) $2\cos\theta = -\sqrt{2}$

Answer

$\frac{3\pi}{4}, \frac{5\pi}{4}$

8) $\tan\theta = -1$

9) $\tan x = 1$

Answer

$\frac{\pi}{4}, \frac{5\pi}{4}$

10) $\cot x + 1 = 0$

11) $4\sin^2 x - 2 = 0$

Answer

$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

12) $\csc^2 x - 4 = 0$

For the exercises 13-22, solve exactly on $[0, 2\pi)$

13) $2\cos\theta = \sqrt{2}$

Answer

$\frac{\pi}{4}, \frac{7\pi}{4}$

14) $2\cos\theta = -1$

15) $2\sin\theta = -1$

Answer

$\frac{7\pi}{6}, \frac{11\pi}{6}$

16) $2\sin\theta = -\sqrt{3}$

17) $2\sin(3\theta) = 1$

Answer

$\frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$

18) $2\sin(2\theta) = \sqrt{3}$

19) $2\cos(3\theta) = -\sqrt{2}$

Answer

$\frac{3\pi}{12}, \frac{5\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}, \frac{21\pi}{12}$

20) $\cos(2\theta) = -\frac{\sqrt{3}}{2}$

21) $2\sin(\pi\theta) = 1$

Answer

$\frac{1}{6}, \frac{5}{6}, \frac{13}{6}, \frac{17}{6}, \frac{25}{6}, \frac{29}{6}, \frac{37}{6}$

22) $2\cos\left(\frac{\pi}{5}\theta\right) = \sqrt{3}$

For the exercises 23-32, find all exact solutions on $[0, 2\pi)$

23) $\sec(x)\sin(x) - 2\sin(x) = 0$

Answer

$0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

24) $\tan(x) - 2\sin(x)\tan(x) = 0$

25) $2\cos^2 t + \cos(t) = 1$

Answer

$\frac{\pi}{3}, \pi, \frac{5\pi}{3}$

26) $2\tan^2(t) = 3\sec(t)$

27) $2\sin(x)\cos(x) - \sin(x) + 2\cos(x) - 1 = 0$

Answer

$\frac{\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

28) $\cos^2\theta = \frac{1}{2}$

29) $\sec^2 x = 1$

Answer

$0, \pi$

30) $\tan^2(x) = -1 + 2\tan(-x)$

31) $8\sin^2(x) + 6\sin(x) + 1 = 0$

Answer

$$\pi - \sin^{-1}\left(-\frac{1}{4}\right), \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi + \sin^{-1}\left(-\frac{1}{4}\right)$$

32) $\tan^5(x) = \tan(x)$

For the exercises 33-40, solve with the methods shown in this section exactly on the interval $[0, 2\pi)$

33) $\sin(3x)\cos(6x) - \cos(3x)\sin(6x) = -0.9$

Answer

$$\frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{\pi}{3} - \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{2\pi}{3} + \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \pi - \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{4\pi}{3} + \frac{1}{3}\left(\sin^{-1}\left(\frac{9}{10}\right)\right), \frac{5\pi}{3} -$$

34) $\sin(6x)\cos(11x) - \cos(6x)\sin(11x) = -0.1$

35) $\cos(2x)\cos x + \sin(2x)\sin x = 1$

Answer

0

36) $6\sin(2t) + 9\sin t = 0$

37) $9\cos(2\theta) = 9\cos^2\theta - 4$

Answer

$$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

38) $\sin(2t) = \cos t$

39) $\cos(2t) = \sin t$

Answer

$$\frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

40) $\cos(6x) - \cos(3x) = 0$

For the exercises 41-49, solve exactly on the interval $[0, 2\pi)$. Use the quadratic formula if the equations do not factor.

41) $\tan^2 x - \sqrt{3}\tan x = 0$

Answer

$$0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}$$

42) $\sin^2 x + \sin x - 2 = 0$

43) $\sin^2 x - 2\sin x - 4 = 0$

Answer

There are no solutions.

44) $5\cos^2 x + 3\cos x - 1 = 0$

45) $3\cos^2 x - 3\cos x - 2 = 0$

Answer

$$\cos^{-1}\left(\frac{1}{3}\left(1 - \sqrt{7}\right)\right), 2\pi - \cos^{-1}\left(\frac{1}{3}\left(1 - \sqrt{7}\right)\right)$$

46) $5\sin^2 x + 2\sin x - 1 = 0$

47) $\tan^2 x + 5\tan x - 1 = 0$

Answer

$$\tan^{-1}\left(\frac{1}{2}\left(\sqrt{29}-5\right)\right), \pi+\tan^{-1}\left(\frac{1}{2}\left(-\sqrt{29}-5\right)\right), \pi+\tan^{-1}\left(\frac{1}{2}\left(\sqrt{29}-5\right)\right), 2\pi+\tan^{-1}\left(\frac{1}{2}\left(-\sqrt{29}-5\right)\right)$$

48) $\cot^2 x = -\cot x$

49) $-\tan^2 x - \tan x - 2 = 0$

Answer

There are no solutions.

For the exercises 50-65, find exact solutions on the interval $[0, 2\pi)$. Look for opportunities to use trigonometric identities.

50) $\sin^2 x - \cos^2 x - \sin x = 0$

51) $\sin^2 x + \cos^2 x = 0$

Answer

There are no solutions.

52) $\sin(2x) - \sin x = 0$

53) $\cos(2x) - \cos x = 0$

Answer

$$0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

54) $\dfrac{2\tan x}{2-\sec^2 x} - \sin^2 x = \cos^2 x$

55) $1 - \cos(2x) = 1 + \cos(2x)$

Answer

$$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

56) $\sec^2 x = 7$

57) $10 \sin x \cos x = 6 \cos x$

Answer

$$\sin^{-1}\left(\frac{3}{5}\right), \frac{\pi}{2}, \pi - \sin^{-1}\left(\frac{3}{5}\right), \frac{3\pi}{2}$$

58) $-3 \sin t = 15 \cos t \sin t$

59) $4 \cos^2 x - 4 = 15 \cos x$

Answer

$$\cos^{-1}\left(-\frac{1}{4}\right), 2\pi - \cos^{-1}\left(-\frac{1}{4}\right)$$

60) $8 \sin^2 x + 6 \sin x + 1 = 0$

61) $8 \cos^2 \theta = 3 - 2 \cos\theta$

Answer

$$\frac{\pi}{3}, \cos^{-1}\left(-\frac{3}{4}\right), 2\pi - \cos^{-1}\left(-\frac{3}{4}\right), \frac{5\pi}{3}$$

62) $6 \cos^2 x + 7 \sin x - 8 = 0$

63) $12 \sin^2 t + \cos t - 6 = 0$

Answer

$$\cos^{-1}\left(\frac{3}{4}\right), \cos^{-1}\left(-\frac{2}{3}\right), 2\pi - \cos^{-1}\left(-\frac{2}{3}\right), 2\pi - \cos^{-1}\left(\frac{3}{4}\right)$$

64) $\tan x = 3\sin x$

65) $\cos^3 t = \cos t$

Answer

$$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

Graphical

For the exercises 66-72, algebraically determine all solutions of the trigonometric equation exactly, then verify the results by graphing the equation and finding the zeros.

66) $6\sin^2 x - 5\sin x + 1 = 0$

67) $8\cos^2 x - 2\cos x - 1 = 0$

Answer

$$\frac{\pi}{3}, \cos^{-1}\left(-\frac{1}{4}\right), 2\pi - \cos^{-1}\left(-\frac{1}{4}\right), \frac{5\pi}{3}$$

68) $100\tan^2 x + 20\tan x - 3 = 0$

69) $2\cos^2 x - \cos x + 15 = 0$

Answer

There are no solutions.

70) $20\sin^2 x - 27\sin x + 7 = 0$

71) $2\tan^2 x + 7\tan x + 6 = 0$

Answer

$$\pi + \tan^{-1}(-2), \pi + \tan^{-1}\left(-\frac{3}{2}\right), 2\pi + \tan^{-1}(-2), 2\pi + \tan^{-1}\left(-\frac{3}{2}\right)$$

72) $130\tan^2 x + 69\tan x - 130 = 0$

Technology

For the exercises 73-76, use a calculator to find all solutions to four decimal places.

73) $\sin x = 0.27$

Answer

$$2\pi k + 0.2734, 2\pi k + 2.8682$$

74) $\sin x = -0.55$

75) $\tan x = -0.34$

Answer

$$\pi k - 0.3277$$

76) $\cos x = 0.71$

77) $\tan^2 x + 3\tan x - 3 = 0$

Answer

$$0.6694, 1.8287, 3.8110, 4.9703$$

78) $6\tan^2 x + 13\tan x = -6$

79) $\tan^2 x - \sec x = 1$

Answer

$1.0472, 3.1416, 5.2360$

80) $\sin^2 x - 2\cos^2 x = 0$

81) $2\tan^2 x + 9\tan x - 6 = 0$

Answer

$0.5326, 1.7648, 3.6742, 4.9064$

82) $4\sin^2 x + \sin(2x)\sec x - 3 = 0$

Extensions

For the exercises 83-92, find all solutions exactly to the equations on the interval $[0, 2\pi)$.

83) $\csc^2 x - 3\csc x - 4 = 0$

Answer

$\sin^{-1}\left(\frac{1}{4}\right), \pi - \sin^{-1}\left(\frac{1}{4}\right), \frac{3\pi}{2}$

84) $\sin^2 x - \cos^2 x - 1 = 0$

85) $\sin^2 x(1 - \sin^2 x) + \cos^2 x(1 - \sin^2 x) = 0$

Answer

$\frac{\pi}{2}, \frac{3\pi}{2}$

86) $3\sec^2 x + 2 + \sin^2 x - \tan^2 x + \cos^2 x = 0$

87) $\sin^2 x - 1 + 2\cos(2x) - \cos^2 x = 1$

Answer

There are no solutions.

88) $\tan^2 x - 1 - \sec^3 x \cos x = 0$

89) $\frac{\sin(2x)}{\sec^2 x} = 0$

Answer

$0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

90) $\frac{\sin(2x)}{2\csc^2 x} = 0$

91) $2\cos^2 x - \sin^2 x - \cos x - 5 = 0$

Answer

There are no solutions.

92) $\frac{1}{\sec^2 x} + 2 + \sin^2 x + 4\cos^2 x = 4$

Real-World Applications

93) An airplane has only enough gas to fly to a city 200 miles northeast of its current location. If the pilot knows that the city is 25 miles north, how many degrees north of east should the airplane fly?

Answer

7.2°

94) If a loading ramp is placed next to a truck, at a height of 4 feet, and the ramp is 15 feet long, what angle does the ramp make with the ground?

95) If a loading ramp is placed next to a truck, at a height of 2 feet, and the ramp is 20 feet long, what angle does the ramp make with the ground?

Answer

5.7°

96) A woman is watching a launched rocket currently 11 miles in altitude. If she is standing 4 miles from the launch pad, at what angle is she looking up from horizontal?

97) An astronaut is in a launched rocket currently 15 miles in altitude. If a man is standing 2 miles from the launch pad, at what angle is she looking down at him from horizontal? (Hint: this is called the angle of depression.)

Answer

82.4°

98) A woman is standing 8 meters away from a 10-meter tall building. At what angle is she looking to the top of the building?

99) A man is standing 10 meters away from a 6-meter tall building. Someone at the top of the building is looking down at him. At what angle is the person looking at him?

Answer

31.0°

100) A 20-foot tall building has a shadow that is 55 feet long. What is the angle of elevation of the sun?

101) A 90-foot tall building has a shadow that is 2 feet long. What is the angle of elevation of the sun?

Answer

88.7°

102) A spotlight on the ground 3 meters from a 2-meter tall man casts a 6 meter shadow on a wall 6 meters from the man. At what angle is the light?

103) A spotlight on the ground 3 feet from a 5-foot tall woman casts a 15-foot tall shadow on a wall 6 feet from the woman. At what angle is the light?

Answer

59.0°

For the exercises 104-106, find a solution to the word problem algebraically. Then use a calculator to verify the result. Round the answer to the nearest tenth of a degree.

104) A person does a handstand with his feet touching a wall and his hands 1.5 feet away from the wall. If the person is 6 feet tall, what angle do his feet make with the wall?

105) A person does a handstand with her feet touching a wall and her hands 3 feet away from the wall. If the person is 5 feet tall, what angle do her feet make with the wall?

Answer

36.9°

106) A 23-foot ladder is positioned next to a house. If the ladder slips at 7 feet from the house when there is not enough traction, what angle should the ladder make with the ground to avoid slipping?

7.6: Modeling with Trigonometric Equations

Many natural phenomena are also periodic. For example, the phases of the moon have a period of approximately 28 days, and birds know to fly south at about the same time each year. So how can we model an equation to reflect periodic behavior? First, we must collect and record data.

We then find a function that resembles an observed pattern and alter the function to get a dependable model. Here. we will take a deeper look at specific types of periodic behavior and model equations to fit data.

Verbal

1) Explain what types of physical phenomena are best modeled by sinusoidal functions. What are the characteristics necessary?

Answer

Physical behavior should be periodic, or cyclical.

2) What information is necessary to construct a trigonometric model of daily temperature? Give examples of two different sets of information that would enable modeling with an equation.

3) If we want to model cumulative rainfall over the course of a year, would a sinusoidal function be a good model? Why or why not?

Answer

Since cumulative rainfall is always increasing, a sinusoidal function would not be ideal here.

4) Explain the effect of a damping factor on the graphs of harmonic motion functions.

Algebraic

For the exercises 5-13, find a possible formula for the trigonometric function represented by the given table of values.

5)

x	y
0	-4
3	-1
6	2
9	-1
12	-4
15	-1
18	2

Answer

$y = -3\cos\left(\frac{\pi}{6}x\right) - 1$

6)

x	y
0	5
2	1
4	-3
6	1
8	5
10	1
12	-3

7)

x	y
0	2
$\frac{\pi}{4}$	7
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	-3
π	2
$\frac{5\pi}{4}$	7
$\frac{3\pi}{2}$	2

Answer

$5\sin(2x)+2$

8)

x	y
0	2
$\frac{\pi}{4}$	7
$\frac{\pi}{2}$	2
$\frac{3\pi}{4}$	-3
π	2
$\frac{5\pi}{4}$	7
$\frac{3\pi}{2}$	2

9)

x	y
0	1
1	-3
2	-7
3	-3
4	1
5	-3
6	-7

Answer

$4\cos\left(\frac{x\pi}{2}\right)-3$

10)

x	y
0	-2
1	4
2	10
3	4

x	y
4	-2
5	4
6	10

11)

x	y
0	5
1	-3
2	5
3	13
4	5
5	-3
6	5

Answer

$5-8\sin\left(\frac{x\pi}{2}\right)$

12)

x	y
-3	$-1-\sqrt{2}$
-2	-1
-1	$1-\sqrt{2}$
0	0
1	$\sqrt{2}-1$
2	1
3	$\sqrt{2}+1$

13)

x	y
-1	$\sqrt{3}-2$
0	0
1	$2-\sqrt{3}$
2	$\frac{\sqrt{3}}{3}$
3	1
4	$\sqrt{3}$
5	$2+\sqrt{3}$

Answer

$\tan\left(\frac{x\pi}{12}\right)$

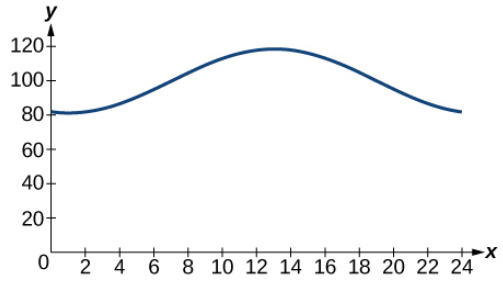
Graphical

For the exercises 14-16, graph the given function, and then find a possible physical process that the equation could model.

14) $f(x) = -30\cos\left(\frac{x\pi}{6}\right) - 20\cos^2\left(\frac{x\pi}{6}\right) + 80$ $[0, 12]$

15) $f(x) = -18\cos\left(\frac{x\pi}{12}\right) - 5\sin\left(\frac{x\pi}{12}\right) + 100$ on the interval $[0, 24]$

Answer



Answers will vary. Sample answer: This function could model temperature changes over the course of one very hot day in Phoenix, Arizona.

16) $f(x) = 10 - \sin\left(\frac{x\pi}{6}\right) + 24\tan\left(\frac{x\pi}{240}\right)$ on the interval $[0, 80]$

Technology

For the exercise 17, construct a function modeling behavior and use a calculator to find desired results.

17) A city's average yearly rainfall is currently 20 inches and varies seasonally by 5 inches. Due to unforeseen circumstances, rainfall appears to be decreasing by 15% each year. How many years from now would we expect rainfall to initially reach 0 inches? Note, the model is invalid once it predicts negative rainfall, so choose the first point at which it goes below 0.

Answer

9 years from now

Real-World Applications

For the exercises 18-29, construct a sinusoidal function with the provided information, and then solve the equation for the requested values.

18) Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the high temperature of $105^\circ F$ occurs at 5PM and the average temperature for the day is $85^\circ F$. Find the temperature, to the nearest degree, at 9AM.

19) Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the high temperature of $84^\circ F$ occurs at 6PM and the average temperature for the day is $70^\circ F$. Find the temperature, to the nearest degree, at 7AM.

Answer

$56^\circ F$

20) Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the temperature varies between $47^\circ F$ and $63^\circ F$ during the day and the average daily temperature first occurs at 10 AM. How many hours after midnight does the temperature first reach $51^\circ F$?

21) Outside temperatures over the course of a day can be modeled as a sinusoidal function. Suppose the temperature varies between $64^\circ F$ and $86^\circ F$ during the day and the average daily temperature first occurs at 12 AM. How many hours after midnight does the temperature first reach $70^\circ F$?

Answer

1.8024hours

22) A Ferris wheel is 20 meters in diameter and boarded from a platform that is 2 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 6 minutes. How much of the ride, in minutes and seconds, is spent higher than 13 meters above the ground?

23)A Ferris wheel is 45 meters in diameter and boarded from a platform that is 1 meter above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. How many minutes of the ride are spent higher than 27 meters above the ground? Round to the nearest second.

Answer

4 : 30

24) The sea ice area around the North Pole fluctuates between about 6 million square kilometers on September 1 to 14 million square kilometers on March 1. Assuming a sinusoidal fluctuation, when are there less than 9 million square kilometers of sea ice? Give your answer as a range of dates, to the nearest day.

25) The sea ice area around the South Pole fluctuates between about 18 million square kilometers in September to 3 million square kilometers in March. Assuming a sinusoidal fluctuation, when are there more than 15 million square kilometers of sea ice? Give your answer as a range of dates, to the nearest day.

Answer

From July 8 to October 23

26) During a 90-day monsoon season, daily rainfall can be modeled by sinusoidal functions. If the rainfall fluctuates between a low of 2 inches on day 10 and 12 inches on day 55, during what period is daily rainfall more than 10 inches?

27) During a 90-day monsoon season, daily rainfall can be modeled by sinusoidal functions. A low of 4 inches of rainfall was recorded on day 30, and overall the average daily rainfall was 8 inches. During what period was daily rainfall less than 5 inches?

Answer

From day 19 through day 40

28) In a certain region, monthly precipitation peaks at 8 inches on June 1 and falls to a low of 1 inch on December 1. Identify the periods when the region is under flood conditions (greater than 7 inches) and drought conditions (less than 2 inches). Give your answer in terms of the nearest day.

29) In a certain region, monthly precipitation peaks at 24 inches in September and falls to a low of 4 inches in March. Identify the periods when the region is under flood conditions (greater than 22 inches) and drought conditions (less than 5 inches). Give your answer in terms of the nearest day.

Answer

Floods: July 24 through October 7. Droughts: February 4 through March 27

For the exercises 30-32, find the amplitude, period, and frequency of the given function.

30) The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = 8\sin(6\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

31) The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = 11\sin(12\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

Answer

Amplitude: 11, period: $\frac{1}{6}$, frequency: 6 Hz

32) The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = 4\cos\left(\frac{\pi}{2}t\right)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

For the exercise 33, construct an equation that models the described behavior.

33) The displacement $h(t)$, in centimeters, of a mass suspended by a spring is modeled by the function $h(t) = -5\cos(60\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

Answer

Amplitude: 5, period: $\frac{1}{30}$, frequency: 30 Hz

For the exercises 34-41, construct an equation that models the described behavior.

34) A deer population oscillates 19 above and below average during the year, reaching the lowest value in January. The average population starts at 800 deer and increases by 160 each year. Find a function that models the population, P, in terms of months since January, t.

35) A rabbit population oscillates 15 above and below average during the year, reaching the lowest value in January. The average population starts at 650 rabbits and increases by 110 each year. Find a function that models the population, P, in terms of months since January, t.

Answer

$$P(t) = -15\cos\left(\frac{\pi}{6}t\right) + 650 + \frac{55}{6}t$$

36) A muskrat population oscillates 33 above and below average during the year, reaching the lowest value in January. The average population starts at 900 muskrats and increases by 7% each month. Find a function that models the population, P, in terms of months since January, t.

37) A fish population oscillates 40 above and below average during the year, reaching the lowest value in January. The average population starts at 800 fish and increases by 4% each month. Find a function that models the population, P, in terms of months since January, t.

Answer

$$P(t) = -40\cos\left(\frac{\pi}{6}t\right) + 800(1.04)^t$$

38) A spring attached to the ceiling is pulled 10 cm down from equilibrium and released. The amplitude decreases by 15% each second. The spring oscillates 18 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

39) A spring attached to the ceiling is pulled 7 cm down from equilibrium and released. The amplitude decreases by 11% each second. The spring oscillates 20 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

Answer

$$D(t) = 7(0.89)^t\cos(40\pi t)$$

40) A spring attached to the ceiling is pulled 17 cm down from equilibrium and released. After 3 seconds, the amplitude has decreased to 13 cm. The spring oscillates 14 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

41) A spring attached to the ceiling is pulled 19 cm down from equilibrium and released. After 4 seconds, the amplitude has decreased to 14 cm. The spring oscillates 13 times each second. Find a function that models the distance, D, the end of the spring is from equilibrium in terms of seconds, t, since the spring was released.

Answer

$$D(t) = 19(0.9265)^t\cos(26\pi t)$$

For the exercises 42-47, create a function modeling the described behavior. Then, calculate the desired result using a calculator.

42) A certain lake currently has an average trout population of 20,000. The population naturally oscillates above and below average by 2,000 every year. This year, the lake was opened to fishermen. If fishermen catch 3,000 fish every year, how long will it take for the lake to have no more trout?

43) Whitefish populations are currently at 500 in a lake. The population naturally oscillates above and below by 25 each year. If humans overfish, taking 4% of the population every year, in how many years will the lake first have fewer than 200 whitefish?

Answer

20.1 years

44) A spring attached to a ceiling is pulled down 11 cm from equilibrium and released. After 2 seconds, the amplitude has decreased to 6 cm. The spring oscillates 8 times each second. Find when the spring first comes between -0.1 and 0.1 cm, effectively at rest.

45) A spring attached to a ceiling is pulled down 21 cm from equilibrium and released. After 6 seconds, the amplitude has decreased to 4 cm. The spring oscillates 20 times each second. Find when the spring first comes between -0.1 and 0.1 cm, effectively at rest.

Answer

17.8 seconds

46) Two springs are pulled down from the ceiling and released at the same time. The first spring, which oscillates 8 times per second, was initially pulled down 32 cm from equilibrium, and the amplitude decreases by 50% each second. The second spring, oscillating 18 times per second, was initially pulled down 15 cm from equilibrium and after 4 seconds has an amplitude of 2 cm. Which spring comes to rest first, and at what time? Consider "rest" as an amplitude less than 0.1 cm.

47) Two springs are pulled down from the ceiling and released at the same time. The first spring, which oscillates 14 times per second, was initially pulled down 2 cm from equilibrium, and the amplitude decreases by 8% each second. The second spring, oscillating 22 times per second, was initially pulled down 10 cm from equilibrium and after 3 seconds has an amplitude of 2 cm. Which spring comes to rest first, and at what time? Consider "rest" as an amplitude less than 0.1 cm.

Answer

Spring 2 comes to rest first after 8.0 seconds.

Extensions

48) A plane flies 1 hour at 150 mph at 22° east of north, then continues to fly for 1.5 hours at 120 mph, this time at a bearing of 112° east of north. Find the total distance from the starting point and the direct angle flown north of east.

49) A plane flies 2 hours at 200 mph at a bearing of 60°,then continues to fly for 1.5 hours at the same speed, this time at a bearing of 150°. Find the distance from the starting point and the bearing from the starting point. (**Hint:** bearing is measured counterclockwise from north.)

Answer

500 miles, at 90°

For the exercises 50-52, find a function of the form $y = ab^x \sin\left(\frac{\pi}{2}x\right) + c$ that fits the given data.

50)

x	0	1	2	3
y	6	29	96	379

51)

x	0	1	2	3
y	6	34	150	746

Answer

$y = 6(5)^x + 4\sin\left(\frac{\pi}{2}x\right)$

52)

x	0	1	2	3
y	4	0	16	-40

For the exercises 53-54, find a function of the form $y = ab^x \cos\left(\frac{\pi}{2}x\right) + c$ that fits the given data.

53)

x	0	1	2	3
y	11	3	1	3

Answer

$y = 8\left(\frac{1}{2}\right)^x \cos\left(\frac{\pi}{2}x\right) + 3$

54)

x	0	1	2	3
y	4	1	−11	1

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3.R: Trigonometric Identities and Equations (Review)

7.1: Solving Trigonometric Equations with Identities

For the exercises 1-6, find all solutions exactly that exist on the interval $[0, 2\pi)$.

1) $\csc^2 t = 3$

Answer

$\sin^{-1}\left(\frac{\sqrt{3}}{3}\right), \pi - \sin^{-1}\left(\frac{\sqrt{3}}{3}\right), \pi + \sin^{-1}\left(\frac{\sqrt{3}}{3}\right), 2\pi - \sin^{-1}\left(\frac{\sqrt{3}}{3}\right)$

2) $\cos^2 x = \frac{1}{4}$

3) $2\sin\theta = -1$

Answer

$\frac{7\pi}{6}, \frac{11\pi}{6}$

4) $\tan x \sin x + \sin(-x) = 0$

5) $9\sin\omega - 2 = 4\sin^2\omega$

Answer

$\sin^{-1}\left(\frac{1}{4}\right), \pi - \sin^{-1}\left(\frac{1}{4}\right)$

6) $1 - 2\tan(\omega) = \tan^2(\omega)$

For the exercises 7-8, use basic identities to simplify the expression.

7) $\sec x \cos x + \cos x - \frac{1}{\sec x}$

Answer

1

8) $\sin^3 x + \cos^2 x \sin x$

For the exercises 9-10, determine if the given identities are equivalent.

9) $\sin^2 x + \sec^2 x - 1 = \frac{(1 - \cos^2 x)(1 + \cos^2 x)}{\cos^2 x}$

Answer

Yes

10) $\tan^3 x \csc^2 x \cot^2 x \cos x \sin x = 1$

7.2: Sum and Difference Identities

For the exercises 1-4, find the exact value.

1) $\tan\left(\frac{7\pi}{12}\right)$

Answer

$-2 - \sqrt{3}$

2) $\cos\left(\frac{25\pi}{12}\right)$

3) $\sin(70^\circ)\cos(25^\circ)-\cos(70^\circ)\sin(25^\circ)$

Answer

$\frac{\sqrt{2}}{2}$

4) $\cos(83^\circ)\cos(23^\circ)+\sin(83^\circ)\sin(23^\circ)$

For the exercises 5-6, prove the identity.

5) $\cos(4x)-\cos(3x)\cos x=\sin^2 x-4\cos^2 x\sin^2 x$

Answer

$$\begin{aligned}\cos(4x)-\cos(3x)\cos x &= \cos(2x+2x)-\cos(x+2x)\cos x\\ &= \cos(2x)\cos(2x)-\sin(2x)\sin(2x)-\cos x\cos(2x)\cos x+\sin x\sin(2x)\cos x\\ &= (\cos^2 x-\sin^2 x)^2-4\cos^2 x\sin^2 x-\cos^2 x(\cos^2 x-\sin^2 x)+\sin x(2)\sin x\cos x\cos x\\ &= (\cos^2 x-\sin^2 x)^2-4\cos^2 x\sin^2 x-\cos^2 x(\cos^2 x-\sin^2 x)+2\sin^2 x\cos^2 x\\ &= \cos^4 x-2\cos^2 x\sin^2 x+\sin^4-\cos^2 x\sin^2 x-\cos^4 x+\cos^2 x\sin^2 x+2\sin^2 x\cos^2 x\\ &= \sin^4 x-4\cos^2 x\sin^2 x+\cos^2 x\sin^2 x\\ &= \sin^2 x(\sin^2 x+\cos^2 x)-4\cos^2 x\sin^2 x\\ &= \sin^2 x-4\cos^2 x\sin^2 x\end{aligned}$$

6) $\cos(3x)-\cos^3 x=-\cos x\sin^2 x-\sin x\sin(2x)$

For exercise 7, simplify the expression.

7) $\frac{\tan\left(\frac{1}{2}x\right)+\tan\left(\frac{1}{8}x\right)}{1-\tan\left(\frac{1}{8}x\right)\tan\left(\frac{1}{2}x\right)}$

Answer

$\tan\left(\frac{5}{8}x\right)$

For the exercises 8-9, find the exact value.

8) $\cos\left(\sin^{-1}(0)-\cos^{-1}\left(\frac{1}{2}\right)\right)$

9) $\tan\left(\sin^{-1}(0)-\sin^{-1}\left(\frac{1}{2}\right)\right)$

Answer

$\frac{\sqrt{3}}{3}$

7.3: Double-Angle, Half-Angle, and Reduction Formulas

For the exercises 1-4, find the exact value.

1) Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\cos\theta=-\frac{1}{3}$ and θ is in the interval $\left[\frac{\pi}{2},\pi\right]$.

2) Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\sec\theta=-\frac{5}{3}$ and θ is in the interval $\left[\frac{\pi}{2},\pi\right]$.

Answer

$$-\frac{24}{25}, -\frac{7}{25}, \frac{24}{7}$$

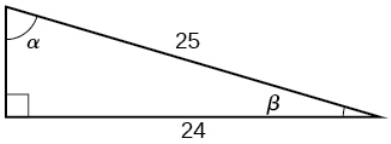
3) $\sin\left(\frac{7\pi}{8}\right)$

4) $\sec\left(\frac{3\pi}{8}\right)$

Answer

$$\sqrt{2(2+\sqrt{2})}$$

For the exercises 5-6, use Figure below to find the desired quantities.



5) $\sin(2\beta), \cos(2\beta), \tan(2\beta), \sin(2\alpha), \cos(2\alpha), \tan(2\alpha)$

6) $\sin\left(\frac{\beta}{2}\right), \cos\left(\frac{\beta}{2}\right), \tan\left(\frac{\beta}{2}\right), \sin\left(\frac{\alpha}{2}\right), \cos\left(\frac{\alpha}{2}\right), \tan\left(\frac{\alpha}{2}\right)$

Answer

$$\frac{\sqrt{2}}{10}, \frac{7\sqrt{2}}{10}, \frac{1}{7}, \frac{3}{5}, \frac{4}{5}, \frac{3}{4}$$

For the exercises 7-8, prove the identity.

7) $\frac{2\cos(2x)}{\sin(2x)} = \cot x - \tan x$

8) $\cot x \cos(2x) = -\sin(2x) + \cot x$

Answer

$$\begin{aligned}\cot x \cos(2x) &= \cot x(1 - 2\sin^2 x) \\ &= \cot x - \frac{\cos x}{\sin x}(2)\sin^2 x \\ &= -2\sin x \cos \\ &= -\sin(2x) + \cot x\end{aligned}$$

For the exercises 9-10, rewrite the expression with no powers.

9) $\cos^2 x \sin^4(2x)$

10) $\tan^2 x \sin^3 x$

Answer

$$\frac{10\sin x - 5\sin(3x) + \sin(5x)}{8(\cos(2x)+1)}$$

7.4: Sum-to-Product and Product-to-Sum Formulas

For the exercises 1-3, evaluate the product for the given expression using a sum or difference of two functions. Write the exact answer.

1) $\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

2) $2\sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{5\pi}{6}\right)$

Answer

$\frac{\sqrt{3}}{2}$

3) $2\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{3}\right)$

For the exercises 4-5, evaluate the sum by using a product formula. Write the exact answer.

4) $\sin\left(\frac{\pi}{12}\right)-\sin\left(\frac{7\pi}{12}\right)$

Answer

$-\frac{\sqrt{2}}{2}$

5) $\cos\left(\frac{5\pi}{12}\right)+\cos\left(\frac{7\pi}{12}\right)$

For the exercises 6-9, change the functions from a product to a sum or a sum to a product.

6) $\sin(9x)\cos(3x)$

Answer

$\frac{1}{2}(\sin(6x)+\sin(12x))$

7) $\cos(7x)\cos(12x)$

8) $\sin(11x)+\sin(2x)$

Answer

$2\sin\left(\frac{13}{2}x\right)\cos\left(\frac{9}{2}x\right)$

9) $\cos(6x)+\cos(5x)$

7.5: Solving Trigonometric Equations

For the exercises 1-2, find all exact solutions on the interval $[0,2\pi)$.

1) $\tan x+1=0$

Answer

$\frac{3\pi}{4},\frac{7\pi}{4}$

2) $2\sin(2x)+\sqrt{2}=0$

For the exercises 3-7, find all exact solutions on the interval $[0,2\pi)$.

3) $2\sin^2 x-\sin x=0$

Answer

$0,\frac{\pi}{6},\frac{5\pi}{6},\pi$

4) $\cos^2 x-\cos x-1=0$

5) $2\sin^2 x + 5\sin x + 3 = 0$

Answer

$$\frac{3\pi}{2}$$

6) $\cos x - 5\sin(2x) = 0$

7) $\dfrac{1}{\sec^2 x} + 2 + \sin^2 x + 4\cos^2 x = 0$

Answer

No solution.

For the exercises 8-9, simplify the equation algebraically as much as possible. Then use a calculator to find the solutions on the interval $[0, 2\pi)$. Round to four decimal places.

8) $\sqrt{3}\cot^2 x + \cot x = 1$

9) $\csc^2 x - 3\csc x - 4 = 0$

Answer

$0.2527, 2.8889, 4.7124$

For the exercises 10-11, graph each side of the equation to find the zeroes on the interval $[0, 2\pi)$.

10) $20\cos^2 x + 21\cos x + 1 = 0$

11) $\sec^2 x - 2\sec x = 15$

Answer

$1.3694, 1.9106, 4.3726, 4.9137$

7.6: Modeling with Trigonometric Equations

For the exercises 1-3, graph the points and find a possible formula for the trigonometric values in the given table.

1)

x	0	1	2	3	4	5
y	1	6	11	6	1	6

2)

x	y
0	-2
1	1
2	-2
3	-5
4	-2
5	1

Answer

$$3\sin\left(\frac{x\pi}{2}\right) - 2$$

3)

x	y
-3	$3+2\sqrt{2}$
-2	3
-1	$2\sqrt{2}-1$
0	1
1	$3-2\sqrt{2}$
2	-1
3	$-1-2\sqrt{2}$

4) A man with his eye level 6 feet above the ground is standing 3 feet away from the base of a 15-foot vertical ladder. If he looks to the top of the ladder, at what angle above horizontal is he looking?

Answer

71.6°

5) Using the ladder from the previous exercise, if a 6-foot-tall construction worker standing at the top of the ladder looks down at the feet of the man standing at the bottom, what angle from the horizontal is he looking?

For the exercises 6-7, construct functions that model the described behavior.

6) A population of lemmings varies with a yearly low of 500 in March. If the average yearly population of lemmings is 950, write a function that models the population with respect to t, the month.

Answer

$P(t)=950-450\sin\left(\frac{\pi}{6}t\right)$

7) Daily temperatures in the desert can be very extreme. If the temperature varies from 90°F to 30°F and the average daily temperature first occurs at 10 AM, write a function modeling this behavior.

For the exercises 8-9, find the amplitude, frequency, and period of the given equations.

8) $y=3\cos(x\pi)$

Answer

Amplitude: 3, period: 2, frequency: $\frac{1}{2}$Hz

9) $y=-2\sin(16x\pi)$

For the exercises 10-11, model the described behavior and find requested values.

10) An invasive species of carp is introduced to Lake Freshwater. Initially there are 100 carp in the lake and the population varies by 20 fish seasonally. If by year 5, there are 625 carp, find a function modeling the population of carp with respect to t, the number of years from now.

Answer

$C(t)=20\sin(2\pi t)+100(1.4427)^t$

11) The native fish population of Lake Freshwater averages 2500 fish, varying by 100 fish seasonally. Due to competition for resources from the invasive carp, the native fish population is expected to decrease by 5% each year. Find a function modeling the population of native fish with respect to t, the number of years from now. Also determine how many years it will take for the carp to overtake the native fish population.

Practice Test

For the exercises 1-2, simplify the given expression.

1) $\cos(-x)\sin x\cot x+\sin^2 x$

Answer

1

2) $\sin(-x)\cos(-2x)-\sin(-x)\cos(-2x)$

For the exercises 3-6, find the exact value.

3) $\cos\left(\frac{7\pi}{12}\right)$

Answer

$\frac{\sqrt{2}-\sqrt{6}}{4}$

4) $\tan\left(\frac{3\pi}{8}\right)$

5) $\tan\left(\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)+\tan^{-1}\sqrt{3}\right)$

Answer

$-\sqrt{2}-\sqrt{3}$

6) $2\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right)$

For the exercises 7-16, find all exact solutions to the equation on $[0,2\pi)$.

7) $\cos^2 x-\sin^2 x-1=0$

Answer

$0,\pi$

8) $\cos^2 x=\cos x$

Answer

$\sin^{-1}\left(\frac{1}{4}(\sqrt{13}-1)\right),\pi-\sin^{-1}\left(\frac{1}{4}(\sqrt{13}-1)\right)$

9) $\cos(2x)+\sin^2 x=0$

10) $2\sin^2 x-\sin x=0$

Answer

$0,\frac{\pi}{6},\frac{5\pi}{6},\pi$

11) Rewrite the expression as a product instead of a sum: $\cos(2x)+\cos(-8x)$

12) Find all solutions of $\tan(x)-\sqrt{3}=0$.

Answer

$\frac{\pi}{3}+k\pi$

13) Find the solutions of $\sec^2 x - 2\sec x = 15$ on the interval $[0, 2\pi)$ algebraically; then graph both sides of the equation to determine the answer.

14) Find $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given $\cot\theta = -\frac{3}{4}$ and θ is on the interval $\left[\frac{\pi}{2}, \pi\right]$.

Answer

$$-\frac{24}{25}, -\frac{7}{25}, \frac{24}{7}$$

15) Find $\sin\left(\frac{\theta}{2}\right)$, $\cos\left(\frac{\theta}{2}\right)$, and $\tan\left(\frac{\theta}{2}\right)$ given $\cos\theta = -\frac{7}{25}$ and θ is in quadrant IV.

16) Rewrite the expression $\sin^4 x$ with no powers greater than 1.

Answer

$$\frac{1}{8}(3 + \cos(4x) - 4\cos(2x))$$

For the exercises 17-19, prove the identity.

17) $\tan^3 x - \tan x \sec^2 x = \tan(-x)$

18) $\sin(3x) - \cos x \sin(2x) = \cos^2 x \sin x - \sin^3 x$

Answer

$$\begin{aligned} \sin(3x) - \cos x \sin(2x) &= \\ \sin(x + 2x) - \cos x(2\sin x \cos x) &= \\ \sin x \cos(2x) + \sin(2x)\cos x - 2\sin x \cos^2 x &= \\ \sin x(\cos^2 x - \sin^2 x) + 2\sin x \cos x \cos x - 2\sin x \cos^2 x &= \\ \sin x \cos^2 x - \sin^3 x + 0 &= \\ \cos^2 x \sin x - \sin^3 x &= \cos^2 x \sin x - \sin^3 x \end{aligned}$$

19) $\frac{\sin(2x)}{\sin x} - \frac{\cos(2x)}{\cos x} = \sec x$

20) Plot the points and find a function of the form $y = A\cos(Bx + C) + D$ that fits the given data.

x	0	1	2	3	4	5
y	-2	2	-2	2	-2	2

Answer

$$y = 2\cos(\pi x + \pi)$$

21) The displacement $h(t)$ in centimeters of a mass suspended by a spring is modeled by the function $h(t) = \frac{1}{4}\sin(120\pi t)$, where t is measured in seconds. Find the amplitude, period, and frequency of this displacement.

22) A woman is standing 300 feet away from a 2000-foot building. If she looks to the top of the building, at what angle above horizontal is she looking? A bored worker looks down at her from the 15th floor (1500 feet above her). At what angle is he looking down at her? Round to the nearest tenth of a degree.

Answer

$$81.5^\circ, 78.7^\circ$$

23) Two frequencies of sound are played on an instrument governed by the equation $n(t) = 8\cos(20\pi t)\cos(1000\pi t)$. What are the period and frequency of the "fast" and "slow" oscillations? What is the amplitude?

24) The average monthly snowfall in a small village in the Himalayas is 6 inches, with the low of 1 inch occurring in July. Construct a function that models this behavior. During what period is there more than 10 inches of snowfall?

Answer

$6+5\cos\left(\frac{\pi}{6}(1-x)\right)$. From November 23 to February 6.

25) A spring attached to a ceiling is pulled down 20 cm. After 3 seconds, wherein it completes 6 full periods, the amplitude is only 15 cm. Find the function modeling the position of the spring t seconds after being released. At what time will the spring come to rest? In this case, use 1 cm amplitude as rest.

26) Water levels near a glacier currently average 9 feet, varying seasonally by 2 inches above and below the average and reaching their highest point in January. Due to global warming, the glacier has begun melting faster than normal. Every year, the water levels rise by a steady 3 inches. Find a function modeling the depth of the water t months from now. If the docks are 2 feet above current water levels, at what point will the water first rise above the docks?

Answer

$D(t)=2\cos\left(\frac{\pi}{6}t\right)+108+\frac{1}{4}t$, 93.5855 months (or 7.8 years) from now

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CHAPTER OVERVIEW

4: Further Applications of Trigonometry

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in Trigonometric Functions and investigate applications more deeply and meaningfully.

- 4.1: Prelude to Further Applications of Trigonometry
- 4.2: Non-right Triangles - Law of Sines
- 4.3: Non-right Triangles - Law of Cosines
- 4.4: Polar Coordinates
- 4.5: Polar Coordinates - Graphs
- 4.6: Polar Form of Complex Numbers
- 4.7: Parametric Equations
- 4.8: Parametric Equations - Graphs
- 4.9: Vectors
- 4.E: Further Applications of Trigonometry (Exercises)
- 4.R: Further Applications of Trigonometry (Review)

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4.1: Prelude to Further Applications of Trigonometry

The world's largest tree by volume, named General Sherman, stands 274.9 feet tall and resides in Northern California. Just how do scientists know its true height? A common way to measure the height involves determining the angle of elevation, which is formed by the tree and the ground at a point some distance away from the base of the tree. This method is much more practical than climbing the tree and dropping a very long tape measure.

Figure 4.1.1: General Sherman, the world's largest living tree. (credit: Mike Baird, Flickr)

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in Trigonometric Functions and investigate applications more deeply and meaningfully.

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4.2: Non-right Triangles - Law of Sines

Learning Objectives

- Use the Law of Sines to solve oblique triangles.
- Find the area of an oblique triangle using the sine function.
- Solve applied problems using the Law of Sines.

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station is 35 degrees, whereas the **angle of elevation** measured by the second station is 15 degrees. How can we determine the altitude of the aircraft? We see in Figure 4.2.1 that the triangle formed by the aircraft and the two stations is not a right triangle, so we cannot use what we know about right triangles. In this section, we will find out how to solve problems involving **non-right triangles**.

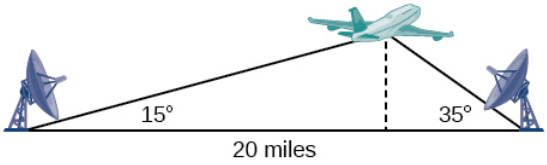

Figure 4.2.1

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

ASA (angle-side-angle) We know the measurements of two angles and the included side. See Figure 4.2.2.

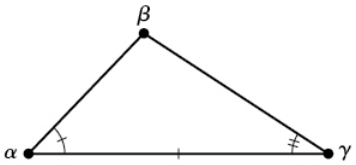

Figure 4.2.2

AAS (angle-angle-side) We know the measurements of two angles and a side that is not between the known angles. See Figure 4.2.3.

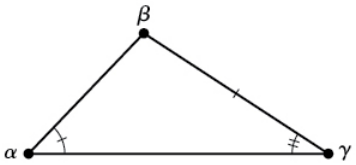

Figure 4.2.3

SSA (side-side-angle) We know the measurements of two sides and an angle that is not between the known sides. See Figure 4.2.4.

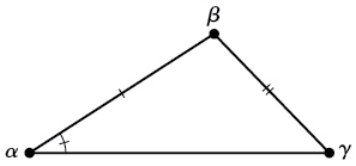

Figure 4.2.4

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in Figure 4.2.5.

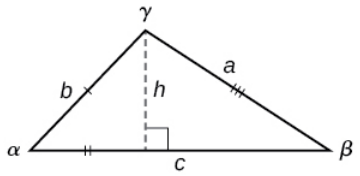

Figure 4.2.5

Using the right triangle relationships, we know that $\sin\alpha = \frac{h}{b}$ and $\sin\beta = \frac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b\sin\alpha \text{ and } h = a\sin\beta$$

We then set the expressions equal to each other.

$$\begin{aligned} b\sin\alpha &= a\sin\beta \\ \left(\frac{1}{ab}\right)(b\sin\alpha) &= (a\sin\beta)\left(\frac{1}{ab}\right) \qquad \text{Multiply both sides by } \frac{1}{ab} \\ \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \end{aligned}$$

Similarly, we can compare the other ratios.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \text{ and } \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Collectively, these relationships are called the **Law of Sines**.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See Figure 4.2.6.

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

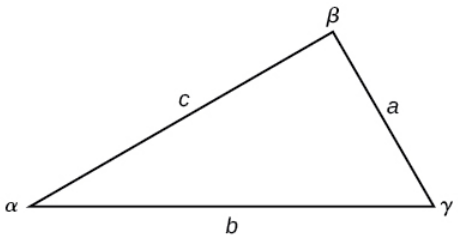

Figure 4.2.6

Note: LAW OF SINES

Given a triangle with angles and opposite sides labeled as in Figure 4.2.6, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \tag{4.2.1}$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \tag{4.2.2}$$

To solve an oblique triangle, use any pair of applicable ratios.

Example 4.2.1: Solving for Two Unknown Sides and Angle of an AAS Triangle

Solve the triangle shown in Figure 4.2.7 to the nearest tenth.

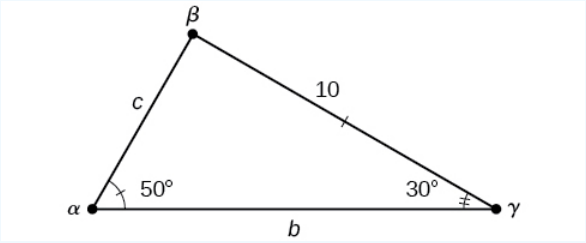

Figure 4.2.7

Solution

The three angles must add up to 180 degrees. From this, we can determine that

$$\begin{aligned}\beta &= 180^\circ - 50^\circ - 30^\circ \\ &= 100^\circ\end{aligned}$$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha = 50^\circ$ and its corresponding side $a = 10$. We can use the following proportion from the Law of Sines to find the length of c.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(30^\circ)}{c} \\ c\frac{\sin(50^\circ)}{10} &= \sin(30^\circ) \qquad \text{Multiply both sides by } c \\ c &= \sin(30^\circ)\frac{10}{\sin(50^\circ)} \qquad \text{Multiply by the reciprocal to isolate } c \\ c &\approx 6.5\end{aligned}$$

Similarly, to solve for b, we set up another proportion.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(100^\circ)}{b} \\ b\sin(50^\circ) &= 10\sin(100^\circ) \qquad \text{Multiply both sides by } b \\ b &= \frac{10\sin(100^\circ)}{\sin(50^\circ)} \qquad \text{Multiply by the reciprocal to isolate } b \\ b &\approx 12.9\end{aligned}$$

Therefore, the complete set of angles and sides is

$$\begin{aligned}\alpha &= 50^\circ \quad & a &= 10 \\ \beta &= 100^\circ \quad & b &\approx 12.9 \\ \gamma &= 30^\circ \quad & c &\approx 6.5\end{aligned}$$

Exercise 4.2.1

Solve the triangle shown in Figure 4.2.8 to the nearest tenth.

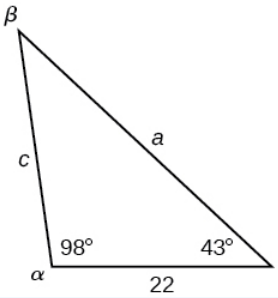

Figure 4.2.8

Answer

$\alpha = 98^\circ \quad a = 34.6$
$\beta = 39^\circ \quad b = 22$
$\gamma = 43^\circ \quad c = 23.8$

Using The Law of Sines to Solve SSA Triangles

We can use the Law of Sines to solve any oblique triangle, but some solutions may not be straightforward. In some cases, more than one triangle may satisfy the given criteria, which we describe as an ambiguous case. Triangles classified as SSA, those in which we know the lengths of two sides and the measurement of the angle opposite one of the given sides, may result in one or two solutions, or even no solution.

Note: POSSIBLE OUTCOMES FOR SSA TRIANGLES

Oblique triangles in the category SSA may have four different outcomes. Figure $4.2.9$ illustrates the solutions with the known sides a and b and known angle α.

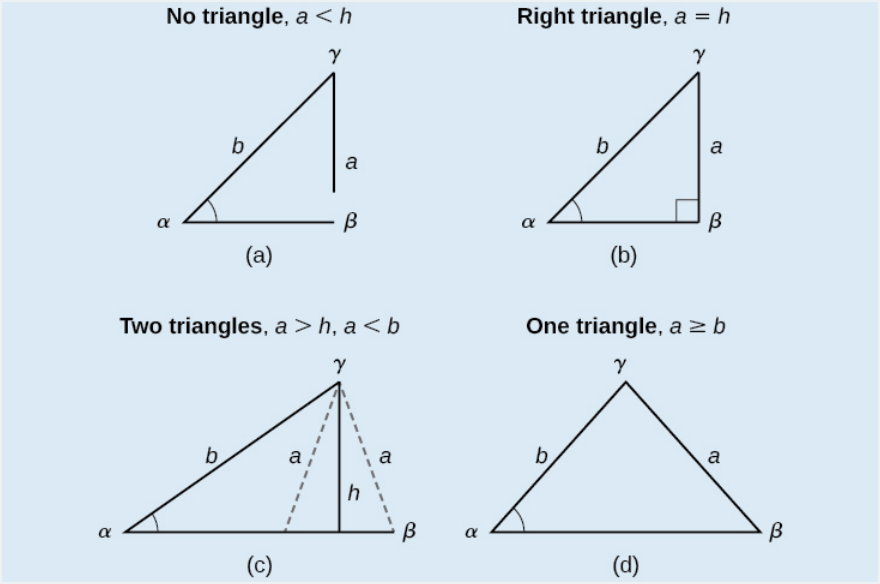

Figure 4.2.9

Example $4.2.2$: Solving an Oblique SSA Triangle

Solve the triangle in Figure $4.2.10$ for the missing side and find the missing angle measures to the nearest tenth.

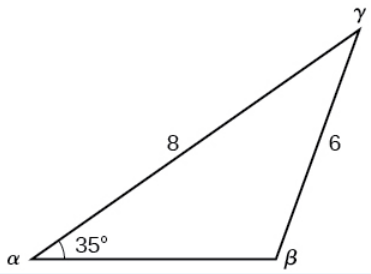

Figure 4.2.10

Solution

Use the Law of Sines to find angle β and angle γ, and then side c. Solving for β, we have the proportion

$$\begin{aligned} \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \\ \frac{\sin(35^\circ)}{6} &= \frac{\sin\beta}{8} \\ \frac{8\sin(35^\circ)}{6} &= \sin\beta \\ 0.7648 &\approx \sin\beta \\ \sin^{-1}(0.7648) &\approx 49.9^\circ \\ \beta &\approx 49.9^\circ \end{aligned}$$

However, in the diagram, angle β appears to be an obtuse angle and may be greater than 90°. How did we get an acute angle, and how do we find the measurement of β? Let's investigate further. Dropping a perpendicular from γ and viewing the triangle from a right angle perspective, we have Figure 4.2.11. It appears that there may be a second triangle that will fit the given criteria.

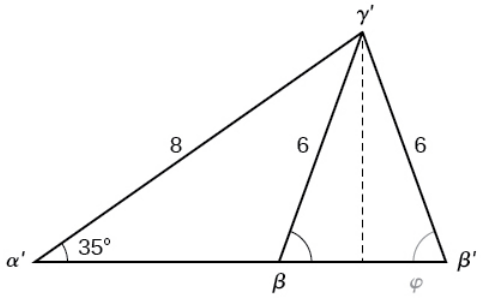

Figure 4.2.11

The angle supplementary to β is approximately equal to 49.9°, which means that $\beta = 180^\circ - 49.9^\circ = 130.1^\circ$. (Remember that the sine function is positive in both the first and second quadrants.) Solving for γ, we have

$$\begin{aligned} \gamma &= 180^\circ - 35^\circ - 130.1^\circ \\ &\approx 14.9^\circ \end{aligned}$$

We can then use these measurements to solve the other triangle. Since β' is supplementary to β, we have

$$\begin{aligned} \gamma' &= 180^\circ - 35^\circ - 49.5^\circ \\ &\approx 95.1^\circ \end{aligned}$$

Now we need to find c and c'.

We have

$$\begin{aligned} \frac{c}{\sin(14.9^\circ)} &= \frac{6}{\sin(35^\circ)} \\ c &= \frac{6\sin(14.9^\circ)}{\sin(35^\circ)} \\ &\approx 2.7 \end{aligned}$$

Finally,

$$\begin{aligned}\frac{c'}{\sin(95.1^\circ)} &= \frac{6}{\sin(35^\circ)}\\ c' &= \frac{6\sin(95.1^\circ)}{\sin(35^\circ)}\\ &\approx 10.4\end{aligned}$$

To summarize, there are two triangles with an angle of 35°, an adjacent side of 8, and an opposite side of 6, as shown in Figure 4.2.12

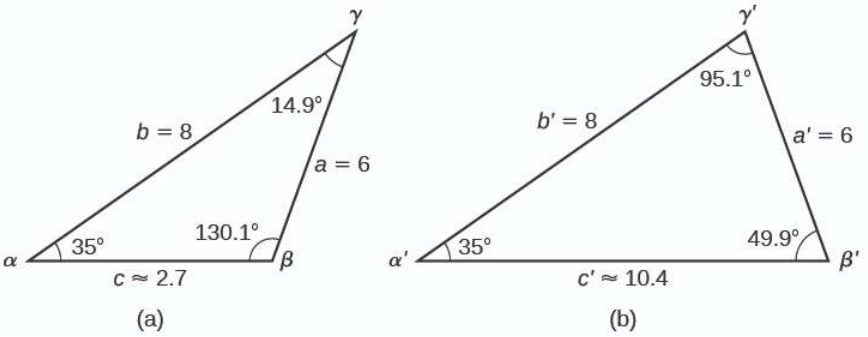

Figure 4.2.12

However, we were looking for the values for the triangle with an obtuse angle β. We can see them in the first triangle (a) in Figure 4.2.12

Exercise 4.2.2

Given $\alpha = 80^\circ$, $a = 120$, and $b = 121$, find the missing side and angles. If there is more than one possible solution, show both.

Answer

Solution 1

$\alpha = 80^\circ$ $\quad a = 120$
$\beta \approx 83.2^\circ$ $\quad b = 121$
$\gamma \approx 16.8^\circ$ $\quad c \approx 35.2$

Solution 2

$\alpha' = 80^\circ$ $\quad a' = 120$
$\beta' \approx 96.8^\circ$ $\quad b' = 121$
$\gamma' \approx 3.2^\circ$ $\quad c' \approx 6.8$

Example 4.2.3: Solving for the Unknown Sides and Angles of a SSA Triangle

In the triangle shown in Figure 4.2.13, solve for the unknown side and angles. Round your answers to the nearest tenth.

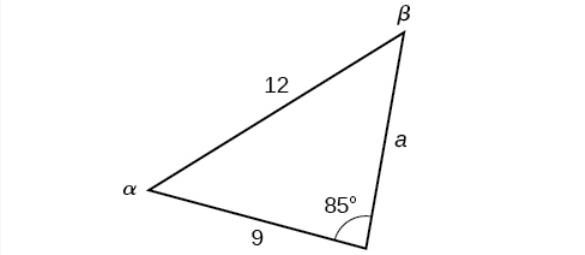

Figure 4.2.13

Solution

In choosing the pair of ratios from the Law of Sines to use, look at the information given. In this case, we know the angle, $\gamma = 85^\circ$, and its corresponding side $c = 12$, and we know side $b = 9$. We will use this proportion to solve for β.

$$\frac{\sin(85^\circ)}{12} = \frac{\sin\beta}{9} \qquad \text{Isolate the unknown.}$$
$$\frac{9\sin(85^\circ)}{12} = \sin\beta$$

To find β, apply the inverse sine function. The inverse sine will produce a single result, but keep in mind that there may be two values for β. It is important to verify the result, as there may be two viable solutions, only one solution (the usual case), or no solutions.

$$\beta = \sin^{-1}\left(\frac{9\sin(85^\circ)}{12}\right)$$
$$\beta \approx \sin^{-1}(0.7471)$$
$$\beta \approx 48.3^\circ$$

In this case, if we subtract β from 180°, we find that there may be a second possible solution. Thus, $\beta = 180^\circ - 48.3^\circ \approx 131.7^\circ$. To check the solution, subtract both angles, 131.7° and 85°, from 180°. This gives

$$\begin{aligned}\alpha &= 180^\circ - 85^\circ - 131.7^\circ \\ &\approx -36.7^\circ\end{aligned}$$

which is impossible, and so $\beta \approx 48.3^\circ$.

To find the remaining missing values, we calculate $\alpha = 180^\circ - 85^\circ - 48.3^\circ \approx 46.7^\circ$. Now, only side a is needed. Use the Law of Sines to solve for a by one of the proportions.

$$\frac{\sin(85^\circ)}{12} = \frac{\sin(46.7^\circ)}{a}$$
$$a\frac{\sin(85^\circ)}{12} = \sin(46.7^\circ)$$
$$a = \frac{12\sin(46.7^\circ)}{\sin(85^\circ)}$$
$$\approx 8.8$$

The complete set of solutions for the given triangle is

$$\begin{aligned}\alpha &\approx 46.7^\circ & a &\approx 8.8 \\ \beta &\approx 48.3^\circ & b &= 9 \\ \gamma &= 85^\circ & c &= 12\end{aligned}$$

? Exercise 4.2.3

Given $\alpha = 80^\circ$, $a = 100$, $b = 10$, find the missing side and angles. If there is more than one possible solution, show both. Round your answers to the nearest tenth.

Answer

$\beta \approx 5.7^\circ$, $\gamma \approx 94.3^\circ$, $c \approx 101.3$

✓ Example 4.2.4: Finding the Triangles That Meet the Given Criteria

Find all possible triangles if one side has length 4 opposite an angle of 50°, and a second side has length 10.

Solution

Using the given information, we can solve for the angle opposite the side of length 10. See Figure 4.2.14

$$\frac{\sin\alpha}{10} = \frac{\sin(50^\circ)}{4}$$
$$\sin\alpha = \frac{10\sin(50^\circ)}{4}$$
$$\sin\alpha \approx 1.915$$

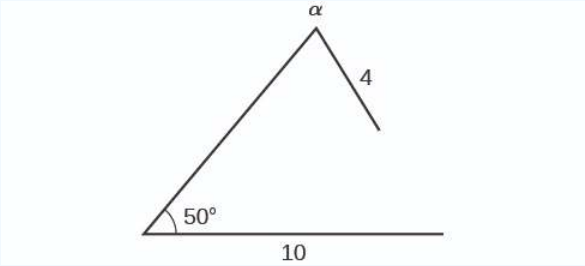

Figure 4.2.14

We can stop here without finding the value of α. Because the range of the sine function is $[-1, 1]$, it is impossible for the sine value to be 1.915. In fact, inputting $\sin^{-1}(1.915)$in a graphing calculator generates an ERROR DOMAIN. Therefore, no triangles can be drawn with the provided dimensions.

? Exercise 4.2.4

Determine the number of triangles possible given $a = 31$, $b = 26$, $\beta = 48^\circ$.

Answer

two

Finding the Area of an Oblique Triangle Using the Sine Function

Now that we can solve a triangle for missing values, we can use some of those values and the sine function to find the area of an oblique triangle. Recall that the area formula for a triangle is given as $Area = \frac{1}{2}bh$, where b is base and h is height. For oblique triangles, we must find h before we can use the area formula. Observing the two triangles in Figure 4.2.15, one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property $\sin\alpha = \frac{opposite}{hypotenuse}$ to write an equation for area in oblique triangles. In the acute triangle, we have $\sin\alpha = \frac{h}{c}$ or $c\sin\alpha = h$. However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base b to form a right triangle. The angle used in calculation is α', or $180 - \alpha$.

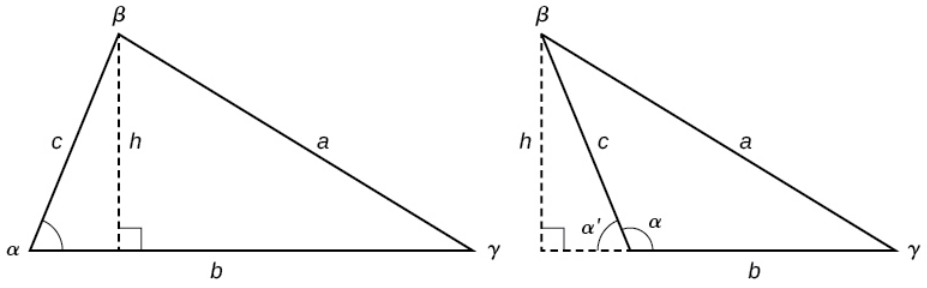

Figure 4.2.15

Thus,

$$Area = \frac{1}{2}(base)(height) = \frac{1}{2}b(c\sin\alpha)$$

Similarly,

$$Area = \frac{1}{2}a(b\sin\gamma) = \frac{1}{2}a(c\sin\beta)$$

Note: AREA OF AN OBLIQUE TRIANGLE

The formula for the area of an oblique triangle is given by

$$Area = \frac{1}{2}bc\sin\alpha \tag{4.2.3}$$

$$Area = \frac{1}{2}ac\sin\beta \tag{4.2.4}$$

$$Area = \frac{1}{2}ab\sin\gamma \tag{4.2.5}$$

This is equivalent to one-half of the product of two sides and the sine of their included angle.

Example 4.2.5: Finding the Area of an Oblique Triangle

Find the area of a triangle with sides $a = 90$, $b = 52$, and angle $\gamma = 102^\circ$. Round the area to the nearest integer.

Solution

Using the formula, we have

$$Area = \frac{1}{2}ab\sin\gamma$$
$$Area = \frac{1}{2}(90)(52)\sin(102^\circ)$$
$$Area \approx 2289 \text{ square units}$$

Exercise 4.2.5

Find the area of the triangle given $\beta = 42^\circ$, $a = 7.2\ ft$, $c = 3.4\ ft$. Round the area to the nearest tenth.

Answer

about 8.2 square feet

Solving Applied Problems Using the Law of Sines

The more we study trigonometric applications, the more we discover that the applications are countless. Some are flat, diagram-type situations, but many applications in calculus, engineering, and physics involve three dimensions and motion.

Example 4.2.6: Finding an Altitude

Find the altitude of the aircraft in the problem introduced at the beginning of this section, shown in Figure 4.2.16. Round the altitude to the nearest tenth of a mile.

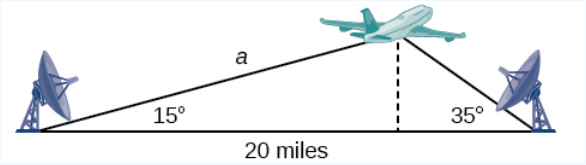

Figure 4.2.16

Solution

To find the elevation of the aircraft, we first find the distance from one station to the aircraft, such as the side a, and then use right triangle relationships to find the height of the aircraft, h.

Because the angles in the triangle add up to 180 degrees, the unknown angle must be $180^\circ - 15^\circ - 35^\circ = 130^\circ$. This angle is opposite the side of length 20, allowing us to set up a Law of Sines relationship.

$$\begin{aligned} \frac{\sin(130^\circ)}{20} &= \frac{\sin(35^\circ)}{a} \\ a\sin(130^\circ) &= 20\sin(35^\circ) \\ a &= \frac{20\sin(35^\circ)}{\sin(130^\circ)} \\ a &\approx 14.98 \end{aligned}$$

The distance from one station to the aircraft is about 14.98 miles.

Now that we know a, we can use right triangle relationships to solve for h.

$$\begin{aligned} \sin(15^\circ) &= \frac{opposite}{hypotenuse} \\ \sin(15^\circ) &= \frac{h}{a} \\ \sin(15^\circ) &= \frac{h}{14.98} \\ h &= 14.98\sin(15^\circ) \\ h &\approx 3.88 \end{aligned}$$

The aircraft is at an altitude of approximately 3.9 miles.

Exercise 4.2.6

The diagram shown in Figure $4.2.17$ represents the height of a blimp flying over a football stadium. Find the height of the blimp if the angle of elevation at the southern end zone, point A, is 70°, the angle of elevation from the northern end zone, point B, is 62°, and the distance between the viewing points of the two end zones is 145 yards.

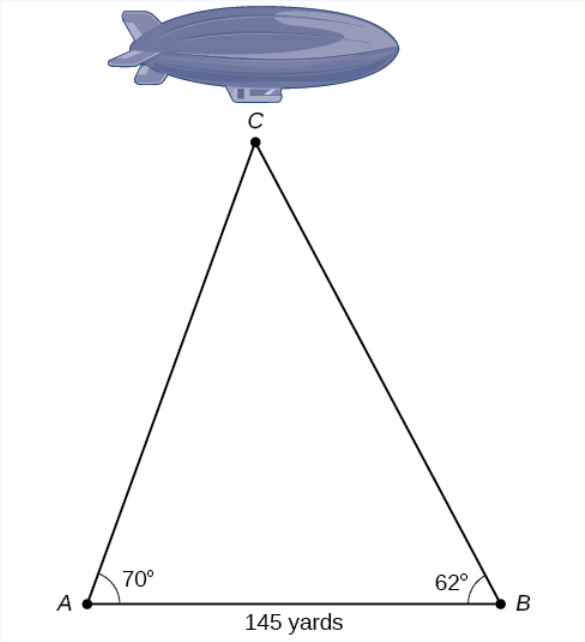

Figure 4.2.17

Answer

161.9yd.

Media

Access these online resources for additional instruction and practice with trigonometric applications.

- Law of Sines: The Basics
- Law of Sines: The Ambiguous Case

Key Equations

Law of Sines	$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$ $\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$
Area for oblique triangles	$Area=\frac{1}{2}bc\sin\alpha$ $=\frac{1}{2}ac\sin\beta$ $=\frac{1}{2}ab\sin\gamma$

Key Concepts

- The Law of Sines can be used to solve oblique triangles, which are non-right triangles.
- According to the Law of Sines, the ratio of the measurement of one of the angles to the length of its opposite side equals the other two ratios of angle measure to opposite side.
- There are three possible cases: ASA, AAS, SSA. Depending on the information given, we can choose the appropriate equation to find the requested solution. See Example $4.2.1$.
- The ambiguous case arises when an oblique triangle can have different outcomes.
- There are three possible cases that arise from SSA arrangement—a single solution, two possible solutions, and no solution. See Example $4.2.2$ and Example $4.2.3$.
- The Law of Sines can be used to solve triangles with given criteria. See Example $4.2.4$.
- The general area formula for triangles translates to oblique triangles by first finding the appropriate height value. See Example $4.2.5$.
- There are many trigonometric applications. They can often be solved by first drawing a diagram of the given information and then using the appropriate equation. See Example $4.2.6$.

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4.3: Non-right Triangles - Law of Cosines

Learning Objectives

In this section, you will:

- Use the Law of Cosines to solve oblique triangles.
- Solve applied problems using the Law of Cosines.
- Use Heron's formula to find the area of a triangle.

Suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles as shown in Figure 4.3.1 How far from port is the boat?

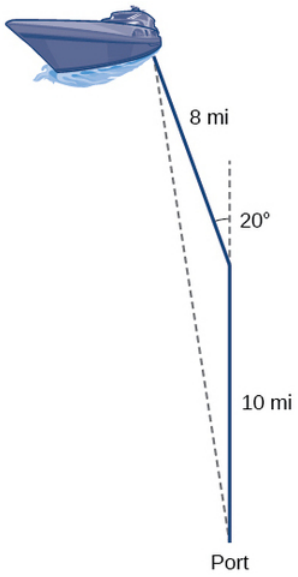

Figure 4.3.1

Unfortunately, while the Law of Sines enables us to address many non-right triangle cases, it does not help us with triangles where the known angle is between two known sides, a SAS (side-angle-side) triangle, or when all three sides are known, but no angles are known, a SSS (side-side-side) triangle. In this section, we will investigate another tool for solving oblique triangles described by these last two cases.

Using the Law of Cosines to Solve Oblique Triangles

The tool we need to solve the problem of the boat's distance from the port is the *Law of Cosines*, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the x-axis, and vertex C located at some point (x, y) in the plane, as illustrated in Figure 4.3.2. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.

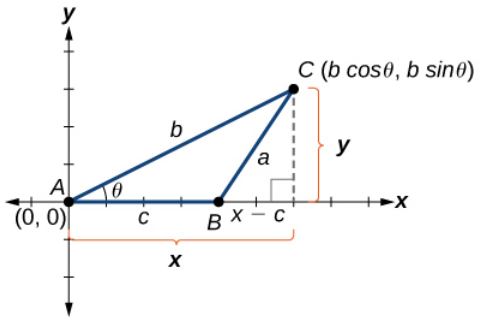

Figure 4.3.2

We can drop a perpendicular from C to the x-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos\theta = \frac{x(adjacent)}{b(hypotenuse)} \text{ and } \sin\theta = \frac{y(opposite)}{b(hypotenuse)}$$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x, y) point located at C has coordinates $(b\cos\theta, b\sin\theta)$. Using the side $(x - c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$\begin{aligned} a^2 &= (x-c)^2 + y^2 \\ &= (b\cos\theta - c)^2 + (b\sin\theta)^2 && \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y \\ &= (b^2\cos^2\theta - 2bc\cos\theta + c^2) + b^2\sin^2\theta && \text{Expand the perfect square.} \\ &= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta && \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1 \\ &= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta && \text{Factor out } b^2 \\ a^2 &= b^2 + c^2 - 2bc\cos\theta \end{aligned}$$

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

The LAW OF COSINES

The Law of Cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle.

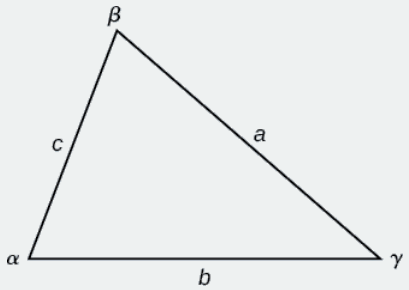

Figure 4.3.3

For triangles labeled as in Figure 4.3.3, with angles α, β and γ, and opposite corresponding sides a, b, and c, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \tag{4.3.1}$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \tag{4.3.2}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{4.3.3}$$

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \tag{4.3.4}$$

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac} \tag{4.3.5}$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} \tag{4.3.6}$$

How to: Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

Example 4.3.1: Finding the Unknown Side and Angles of a SAS Triangle

Find the unknown side and angles of the triangle in Figure 4.3.4.

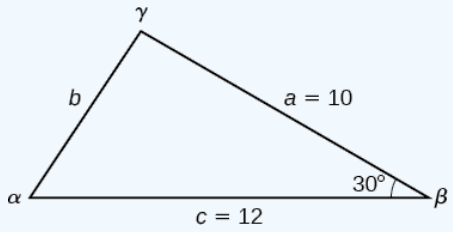

Figure 4.3.4

Solution

First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side b, as we know the measurement of the opposite angle β.

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$b^2 = 10^2 + 12^2 - 2(10)(12)\cos(30^\circ) \quad \text{Substitute the measurements for the known quantities.}$$

$$b^2 = 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right) \quad \text{Evaluate the cosine and begin to simplify.}$$

$$b^2 = 244 - 120\sqrt{3}$$

$$b = \sqrt{244 - 120\sqrt{3}} \quad \text{Use the square root property.}$$

$$b \approx 6.013$$

Because we are solving for a length, we use only the positive square root. Now that we know the length b, we can use the Law of Sines to fill in the remaining angles of the triangle. Solving for angle α, we have

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

$$\frac{\sin\alpha}{10} = \frac{\sin(30^\circ)}{6.013}$$

$$\sin\alpha = \frac{10\sin(30^\circ)}{6.013} \qquad \text{Multiply both sides of the equation by } 10.$$

$$\alpha = \sin^{-1}\left(\frac{10\sin(30^\circ)}{6.013}\right) \qquad \text{Find the inverse sine of } \frac{10\sin(30^\circ)}{6.013}.$$

$$\alpha \approx 56.3^\circ$$

The other possibility for α would be $\alpha = 180^\circ - 56.3^\circ \approx 123.7^\circ$. In the original diagram,α is adjacent to the longest side, so α is an acute angle and, therefore, 123.7° does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between 0° and 180°. Proceeding with $\alpha \approx 56.3^\circ$, we can then find the third angle of the triangle.

$$\begin{aligned}\gamma &= 180^\circ - 30^\circ - 56.3^\circ \\ &\approx 93.7^\circ\end{aligned}$$

The complete set of angles and sides is

$$\alpha \approx 56.3^\circ \; a = 10$$

$$\beta = 30^\circ \; b \approx 6.013$$

$$\gamma \approx 93.7^\circ \; c = 12$$

? Exercise 4.3.1

Find the missing side and angles of the given triangle: $\alpha = 30^\circ$, $b = 12$, $c = 24$.

Answer

$a \approx 14.9$, $\beta \approx 23.8^\circ$, $\gamma \approx 126.2^\circ$.

✓ Example 4.3.2: Solving for an Angle of a SSS Triangle

Find the angle α for the given triangle if side $a = 20$, side $b = 25$, and side $c = 18$.

Solution

For this example, we have no angles. We can solve for any angle using the Law of Cosines. To solve for angle α, we have

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

$$20^2 = 25^2 + 18^2 - 2(25)(18)\cos\alpha \qquad \text{Substitute the appropriate measurements.}$$

$$400 = 625 + 324 - 900\cos\alpha \qquad \text{Simplify in each step.}$$

$$400 = 949 - 900\cos\alpha$$

$$-549 = -900\cos\alpha \qquad \text{Isolate } \cos\alpha.$$

$$0.61 \approx \cos\alpha \qquad \text{Find the inverse cosine.}$$

$$\alpha \approx 52.4^\circ$$

See Figure 4.3.5.

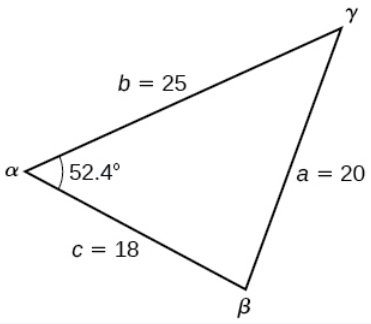

Figure 4.3.5

Analysis

Because the inverse cosine can return any angle between 0 and 180 degrees, there will not be any ambiguous cases using this method.

Exercise 4.3.2

Given $a = 5$, $b = 7$, and $c = 10$, find the missing angles.

Answer

$\alpha \approx 27.7^\circ, \beta \approx 40.5^\circ, \gamma \approx 111.8^\circ$

Solving Applied Problems Using the Law of Cosines

Just as the Law of Sines provided the appropriate equations to solve a number of applications, the Law of Cosines is applicable to situations in which the given data fits the cosine models. We may see these in the fields of navigation, surveying, astronomy, and geometry, just to name a few.

Example 4.3.3A: Using the Law of Cosines to Solve a Communication Problem

On many cell phones with GPS, an approximate location can be given before the GPS signal is received. This is accomplished through a process called triangulation, which works by using the distances from two known points. Suppose there are two cell phone towers within range of a cell phone. The two towers are located 6000 feet apart along a straight highway, running east to west, and the cell phone is north of the highway. Based on the signal delay, it can be determined that the signal is 5050 feet from the first tower and 2420 feet from the second tower. Determine the position of the cell phone north and east of the first tower, and determine how far it is from the highway.

Solution

For simplicity, we start by drawing a diagram similar to Figure $4.3.6$ and labeling our given information.

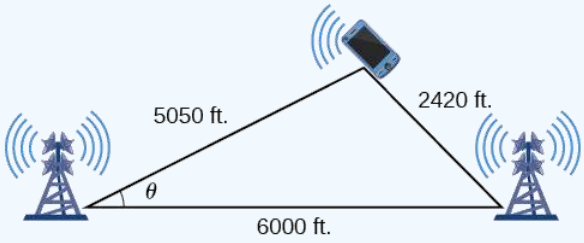

Figure 4.3.6

Using the Law of Cosines, we can solve for the angle θ. Remember that the Law of Cosines uses the square of one side to find the cosine of the opposite angle. For this example, let $a = 2420$, $b = 5050$, and $c = 6000$. Thus, θ corresponds to the opposite side $a = 2420$.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\theta \\ (2420)^2 &= (5050)^2 + (6000)^2 - 2(5050)(6000)\cos\theta \\ \cos\theta &\approx 0.9183 \\ \theta &\approx \cos^{-1}(0.9183) \\ \theta &\approx 23.3^\circ \end{aligned}$$

To answer the questions about the phone's position north and east of the tower, and the distance to the highway, drop a perpendicular from the position of the cell phone, as in Figure 4.3.7. This forms two right triangles, although we only need the right triangle that includes the first tower for this problem.

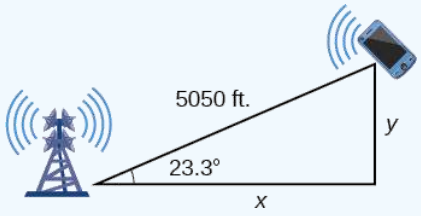

Figure 4.3.7

Using the angle $\theta = 23.3°$ and the basic trigonometric identities, we can find the solutions. Thus

$$\begin{aligned} \cos(23.3^\circ) &= \frac{x}{5050} \\ x &= 5050\cos(23.3^\circ) \\ x &\approx 4638.15\ feet \\ \sin(23.3^\circ) &= \frac{y}{5050} \\ y &= 5050\sin(23.3^\circ) \\ y &\approx 1997.5\ feet \end{aligned}$$

The cell phone is approximately 4638 feet east and 1998 feet north of the first tower, and 1998 feet from the highway.

Example 4.3.3B: Calculating Distance Traveled Using a SAS Triangle

Returning to our problem at the beginning of this section, suppose a boat leaves port, travels 10 miles, turns 20 degrees, and travels another 8 miles. How far from port is the boat? The diagram is repeated here in Figure 4.3.8.

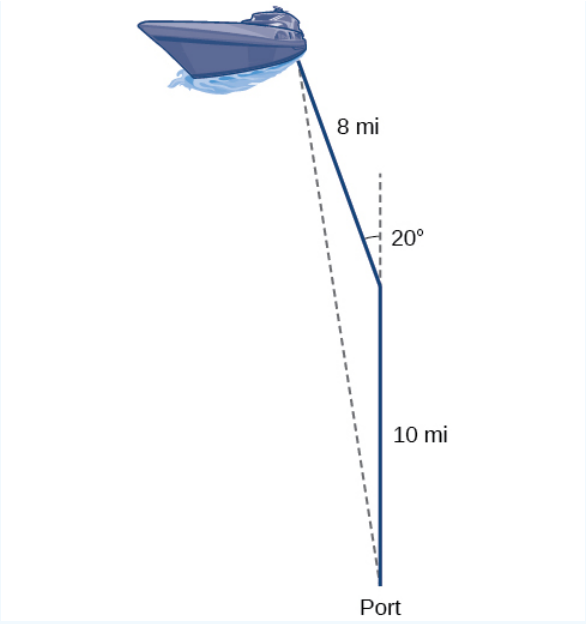

Figure 4.3.8

Solution

The boat turned 20 degrees, so the obtuse angle of the non-right triangle is the supplemental angle, $180^\circ - 20^\circ = 160^\circ$. With this, we can utilize the Law of Cosines to find the missing side of the obtuse triangle—the distance of the boat to the port.

$$\begin{aligned} x^2 &= 8^2 + 10^2 - 2(8)(10)\cos(160^\circ) \\ x^2 &= 314.35 \\ x &= \sqrt{314.35} \\ x &\approx 17.7\, miles \end{aligned}$$

The boat is about 17.7 miles from port.

Using Heron's Formula to Find the Area of a Triangle

We already learned how to find the area of an oblique triangle when we know two sides and an angle. We also know the formula to find the area of a triangle using the base and the height. When we know the three sides, however, we can use **Heron's formula** instead of finding the height. **Heron of Alexandria** was a geometer who lived during the first century A.D. He discovered a formula for finding the area of oblique triangles when three sides are known.

HERON'S FORMULA

Heron's formula finds the area of oblique triangles in which sides a, b,and c are known.

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \tag{4.3.7}$$

where $s = \dfrac{(a+b+c)}{2}$ is one half of the perimeter of the triangle, sometimes called the semi-perimeter.

Example 4.3.4: Using Heron's Formula to Find the Area of a Given Triangle

Find the area of the triangle in Figure $4.3.9$ using Heron's formula.

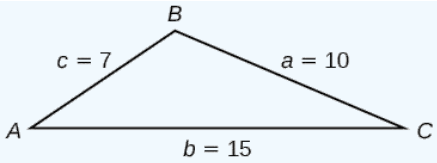

Figure 4.3.9

Solution

First, we calculate s.

$$s = \frac{(a+b+c)}{2}$$
$$s = \frac{(10+15+7)}{2}$$
$$= 16$$

Then we apply the formula.

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$
$$Area = \sqrt{16(16-10)(16-15)(16-7)}$$
$$Area \approx 29.4$$

The area is approximately 29.4 square units.

? Exercise 4.3.3

Use Heron's formula to find the area of a triangle with sides of lengths $a = 29.7$ ft, $b = 42.3$ ft, and $c = 38.4$ ft.

Answer

Area = 552 square feet

✔ Example 4.3.5: Applying Heron's Formula to a Real-World Problem

A Chicago city developer wants to construct a building consisting of artist's lofts on a triangular lot bordered by Rush Street, Wabash Avenue, and Pearson Street. The frontage along Rush Street is approximately 62.4 meters, along Wabash Avenue it is approximately 43.5 meters, and along Pearson Street it is approximately 34.1 meters. How many square meters are available to the developer? See Figure $4.3.10$ for a view of the city property.

Figure 4.3.10

Solution

Find the measurement for s, which is one-half of the perimeter.

$$s = \frac{(62.4 + 43.5 + 34.1)}{2}$$
$$s = 70\ m$$

Apply Heron's formula.

$$Area = \sqrt{70(70 - 62.4)(70 - 43.5)(70 - 34.1)}$$
$$Area = \sqrt{506,118.2}$$
$$Area \approx 711.4$$

The developer has about 711.4 square meters.

Exercise 4.3.4

Find the area of a triangle given $a = 4.38$ ft , $b = 3.79$ ft, and $c = 5.22$ ft.

Answer

about 8.15 square feet

Media

Access these online resources for additional instruction and practice with the Law of Cosines.

- Law of Cosines
- Law of Cosines: Applications
- Law of Cosines: Applications 2

Key Equations

Law of Cosines	$a^2 = b^2 + c^2 - 2bc\cos\alpha$ $b^2 = a^2 + c^2 - 2ac\cos\beta$ $c^2 = a^2 + b^2 - 2ab\cos\gamma$
Heron's formula	$Area = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{(a+b+c)}{2}$

Key Concepts

- The Law of Cosines defines the relationship among angle measurements and lengths of sides in oblique triangles.
- The Generalized Pythagorean Theorem is the Law of Cosines for two cases of oblique triangles: SAS and SSS. Dropping an imaginary perpendicular splits the oblique triangle into two right triangles or forms one right triangle, which allows sides to be related and measurements to be calculated. See Example $4.3.1$ and Example $4.3.2$.
- The Law of Cosines is useful for many types of applied problems. The first step in solving such problems is generally to draw a sketch of the problem presented. If the information given fits one of the three models (the three equations), then apply the Law of Cosines to find a solution. See Example $4.3.3$ and Example $4.3.4$.
- Heron's formula allows the calculation of area in oblique triangles. All three sides must be known to apply Heron's formula. See Example $4.3.5$ and See Example $4.3.6$.

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4.4: Polar Coordinates

Learning Objectives

- Plot points using polar coordinates.
- Convert from polar coordinates to rectangular coordinates.
- Convert from rectangular coordinates to polar coordinates.
- Transform equations between polar and rectangular forms.
- Identify and graph polar equations by converting to rectangular equations.

Over 12 kilometers from port, a sailboat encounters rough weather and is blown off course by a 16-knot wind (see Figure 4.4.1). How can the sailor indicate his location to the Coast Guard? In this section, we will investigate a method of representing location that is different from a standard coordinate grid.

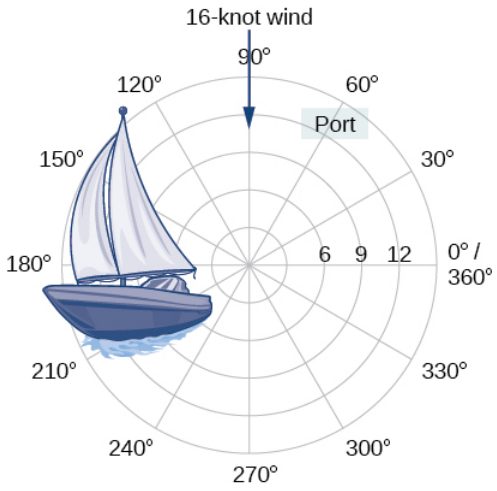
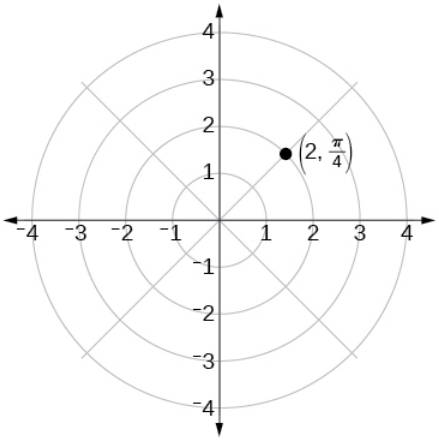


Figure 4.4.1

Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of *rectangular coordinates* (x, y) in the *Cartesian coordinate plane*. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to polar coordinates, which are points labeled (r, θ) and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the pole, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive x-axis now viewed as the polar axis and the origin as the pole. The first coordinate r is the radius or length of the directed line segment from the pole. The angle θ, measured in radians, indicates the direction of r. We move counterclockwise from the polar axis by an angle of θ,and measure a directed line segment the length of r in the direction of θ. Even though we measure θ first and then r, the polar point is written with the r-coordinate first. For example, to plot the point $\left(2, \frac{\pi}{4}\right)$,we would move $\frac{\pi}{4}$ units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in Figure 4.4.2.



Polar Grid

Figure 4.4.2

Example 4.4.1: Plotting a Point on the Polar Grid

Plot the point $\left(3, \frac{\pi}{2}\right)$ on the polar grid.

Solution

The angle $\frac{\pi}{2}$ is found by sweeping in a counterclockwise direction 90° from the polar axis. The point is located at a length of 3 units from the pole in the $\frac{\pi}{2}$ direction, as shown in Figure 4.4.3.

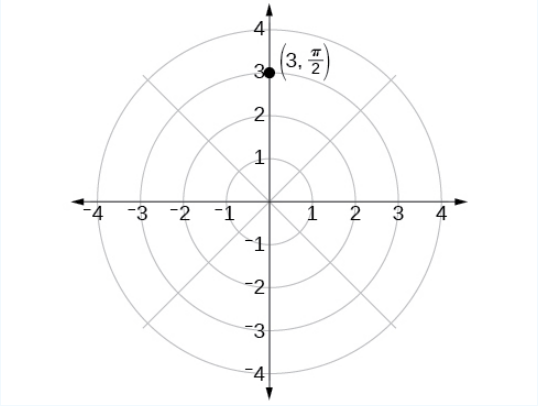


Figure 4.4.3

Exercise 4.4.1

Plot the point $\left(2, \frac{\pi}{3}\right)$ in the polar grid.

Answer

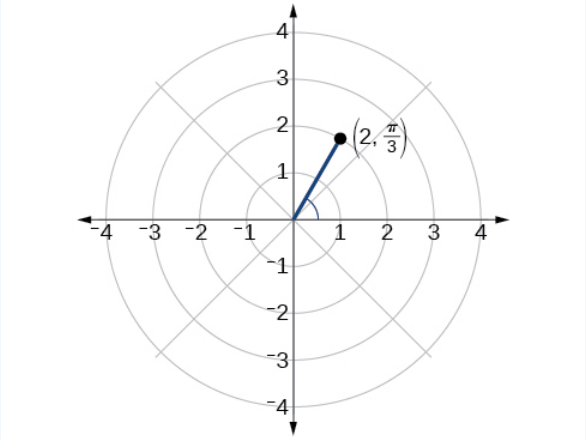


Figure 4.4.4

Example 4.4.2: Plotting a Point in the Polar Coordinate System with a Negative Component

Plot the point $\left(-2, \frac{\pi}{6}\right)$ on the polar grid.

Solution

We know that $\frac{\pi}{6}$ is located in the first quadrant. However, $r = -2$. We can approach plotting a point with a negative r in two ways:

1. Plot the point $\left(2, \frac{\pi}{6}\right)$ by moving $\frac{\pi}{6}$ in the counterclockwise direction and extending a directed line segment 2 units into the first quadrant. Then retrace the directed line segment back through the pole, and continue 2 units into the third quadrant;
2. Move $\frac{\pi}{6}$ in the counterclockwise direction, and draw the directed line segment from the pole 2 units in the negative direction, into the third quadrant.

See Figure $4.4.5a$. Compare this to the graph of the polar coordinate $(2, \pi 6)$ shown in Figure $4.4.5b$

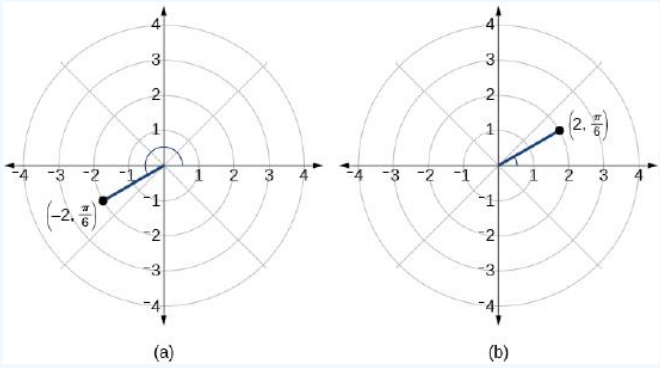


Figure 4.4.5

Exercise 4.4.2

Plot the points $\left(3, -\frac{\pi}{6}\right)$ and $\left(2, \frac{9\pi}{4}\right)$ on the same polar grid.

Answer

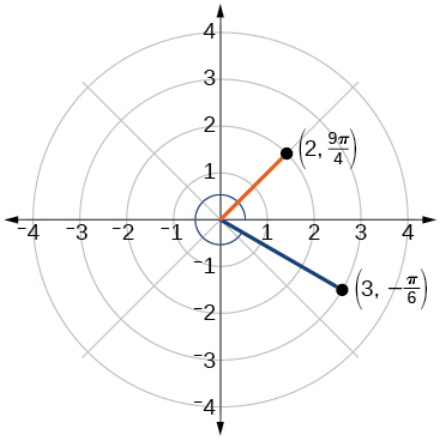


Figure 4.4.6

Converting from Polar Coordinates to Rectangular Coordinates

When given a set of **polar coordinates**, we may need to convert them to rectangular coordinates. To do so, we can recall the relationships that exist among the variables x, y, r, and θ.

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$$

Dropping a perpendicular from the point in the plane to the *x*-axis forms a right triangle, as illustrated in Figure 4.4.7. An easy way to remember the equations above is to think of $\cos\theta$ as the adjacent side over the hypotenuse and $\sin\theta$ as the opposite side over the hypotenuse.

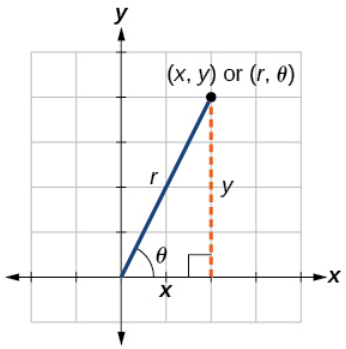


Figure 4.4.7

CONVERTING FROM POLAR COORDINATES TO RECTANGULAR COORDINATES

To convert polar coordinates (r, θ) to rectangular coordinates (x, y), let

$$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta \tag{4.4.1}$$

$$\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta \tag{4.4.2}$$

How to: Given polar coordinates, convert to rectangular coordinates.

1. Given the polar coordinate (r, θ), write $x = r\cos\theta$ and $y = r\sin\theta$.
2. Evaluate $\cos\theta$ and $\sin\theta$.
3. Multiply $\cos\theta$ by r to find the x-coordinate of the rectangular form.
4. Multiply $\sin\theta$ by r to find the y-coordinate of the rectangular form.

Example 4.4.3A: Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $\left(3, \frac{\pi}{2}\right)$ as rectangular coordinates.

Solution

Use the equivalent relationships.

$$\begin{aligned} x &= r\cos\theta \\ x &= 3\cos\frac{\pi}{2} \\ &= 0 \\ y &= r\sin\theta \\ y &= 3\sin\frac{\pi}{2} \\ &= 3 \end{aligned}$$

The rectangular coordinates are $(0, 3)$. See Figure 4.4.8.

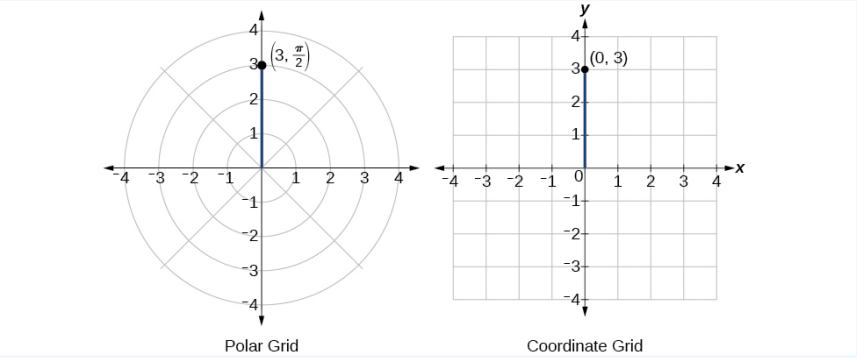


Figure 4.4.8

Example 4.4.3B: Writing Polar Coordinates as Rectangular Coordinates

Write the polar coordinates $(-2, 0)$ as rectangular coordinates.

Solution

See Figure 4.4.9. Writing the polar coordinates as rectangular, we have

$$\begin{aligned} x &= r\cos\theta \\ x &= -2\cos(0) \\ &= -2 \\ y &= r\sin\theta \\ y &= -2\sin(0) \\ &= 0 \end{aligned}$$

The rectangular coordinates are also $(-2, 0)$.

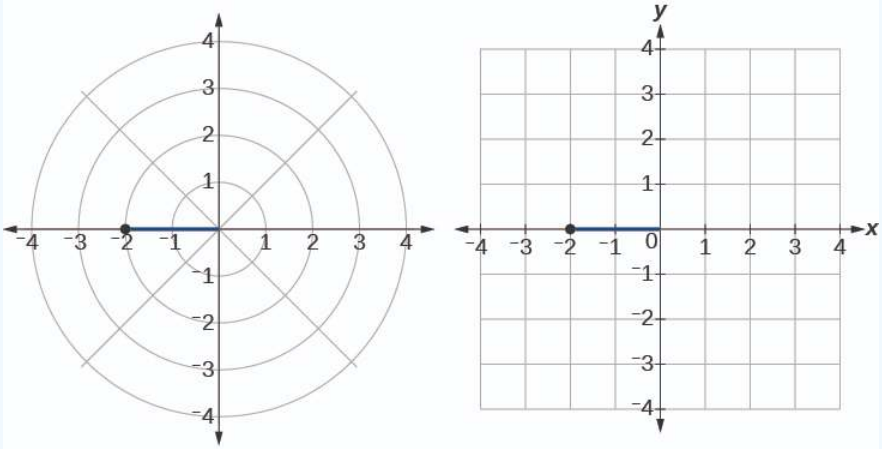


Figure 4.4.9

? Exercise 4.4.3

Write the polar coordinates $\left(-1, \frac{2\pi}{3}\right)$ as rectangular coordinates.

Answer

$$(x, y) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

CONVERTING FROM RECTANGULAR COORDINATES TO POLAR COORDINATES

Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in Figure 4.4.10.

$$\cos\theta = \frac{x}{r} \text{ or } x = r\cos\theta$$

$$\sin\theta = \frac{y}{r} \text{ or } y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$\tan\theta = \frac{y}{x}$$

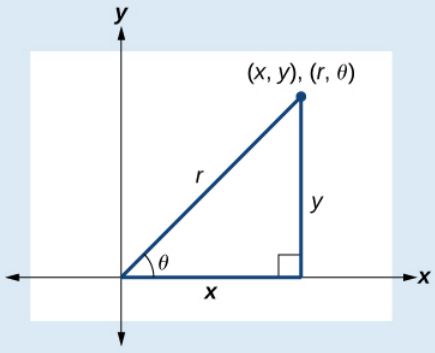


Figure 4.4.10

https://math.libretexts.org/@go/page/126596

Example 4.4.4: Writing Rectangular Coordinates as Polar Coordinates

Convert the rectangular coordinates $(3, 3)$ to polar coordinates.

Solution

We see that the original point $(3, 3)$ is in the first quadrant. To find θ, use the formula $\tan\theta = \dfrac{y}{x}$. This gives

$$\begin{aligned} \tan\theta &= \frac{3}{3} \\ \tan\theta &= 1 \\ \tan^{-1}(1) &= \frac{\pi}{4} \end{aligned}$$

To find r, we substitute the values for x and y into the formula $r = \sqrt{x^2 + y^2}$. We know that r must be positive, as $\dfrac{\pi}{4}$ is in the first quadrant. Thus

$$\begin{aligned} r &= \sqrt{3^2 + 3^2} \\ r &= \sqrt{9 + 9} \\ r &= \sqrt{18} \\ &= 3\sqrt{2} \end{aligned}$$

So, $r = 3\sqrt{2}$ and $\theta = \dfrac{\pi}{4}$, giving us the polar point $(3\sqrt{2}, \dfrac{\pi}{4})$. See Figure 4.4.11.

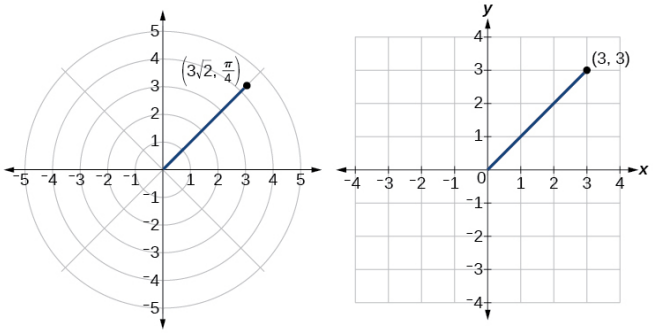


Figure 4.4.11

Analysis

There are other sets of polar coordinates that will be the same as our first solution. For example, the points $\left(-3\sqrt{2}, \dfrac{5\pi}{4}\right)$ and $\left(3\sqrt{2}, -\dfrac{7\pi}{4}\right)$ will coincide with the original solution of $\left(3\sqrt{2}, \dfrac{\pi}{4}\right)$. The point $\left(-3\sqrt{2}, \dfrac{5\pi}{4}\right)$ indicates a move further counterclockwise by π, which is directly opposite $\dfrac{\pi}{4}$. The radius is expressed as $-3\sqrt{2}$. However, the angle $\dfrac{5\pi}{4}$ is located in the third quadrant and, as r is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as $\left(3\sqrt{2}, \dfrac{\pi}{4}\right)$. The point $\left(3\sqrt{2}, -\dfrac{7\pi}{4}\right)$ is a move further clockwise by $-\dfrac{7\pi}{4}$, from $\dfrac{\pi}{4}$. The radius, $3\sqrt{2}$, is the same.

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

How to: Given an equation in polar form, graph it using a graphing calculator

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: r_1, r_2,..., r_6.
3. Enter the polar equation, set equal to r.
4. Press **GRAPH**.

Example 4.4.5A: Writing a Cartesian Equation in Polar Form

Write the Cartesian equation $x^2+y^2=9$ in polar form.

Solution

The goal is to eliminate x and y from the equation and introduce r and θ. Ideally, we would write the equation r as a function of θ. To obtain the polar form, we will use the relationships between (x,y) and (r,θ). Since $x=r\cos\theta$ and $y=r\sin\theta$, we can substitute and solve for r.

$$\begin{aligned}(r\cos\theta)^2+(r\sin\theta)^2 &= 9\\ r^2\cos^2\theta+r^2\sin^2\theta &= 9\\ r^2(\cos^2\theta+\sin^2\theta) &= 9\\ r^2(1) &= 9 \qquad \text{Substitute } \cos^2\theta+\sin^2\theta=1\\ r &= \pm 3 \qquad \text{Use the square root property.}\end{aligned}$$

Thus, $x^2+y^2=9$,$r=3$,and $r=-3$ should generate the same graph. See Figure 4.4.12

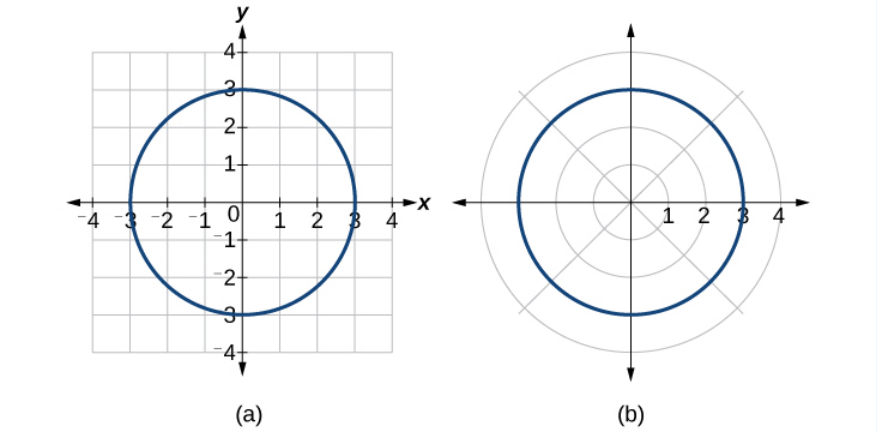


Figure 4.4.12: (a) Cartesian form $x^2+y^2=9$ (b) Polar form $r=3$

To graph a circle in rectangular form, we must first solve for y.

$$\begin{aligned}x^2+y^2 &= 9\\ y^2 &= 9-x^2\\ y &= \pm\sqrt{9-x^2}\end{aligned}$$

Note that this is two separate functions, since a circle fails the vertical line test. Therefore, we need to enter the positive and negative square roots into the calculator separately, as two equations in the form $Y_1=\sqrt{9-x^2}$ and $Y_2=-\sqrt{9-x^2}$. Press **GRAPH.**

Example 4.4.5B: Rewriting a Cartesian Equation as a Polar Equation

Rewrite the **Cartesian equation** $x^2+y^2=6y$ as a polar equation.

Solution

This equation appears similar to the previous example, but it requires different steps to convert the equation.

We can still follow the same procedures we have already learned and make the following substitutions:

$r^2 = 6y$ Use $x^2 + y^2 = r^2$.
$r^2 = 6r\sin\theta$ Substitute $y = r\sin\theta$.
$r^2 - 6r\sin\theta = 0$ Set equal to 0.
$r(r - 6\sin\theta) = 0$ Factor and solve.
$r = 0$ We reject $r = 0$, as it only represents one point, $(0, 0)$.
or $r = 6\sin\theta$

Therefore, the equations $x^2 + y^2 = 6y$ and $r = 6\sin\theta$ should give us the same graph. See Figure 4.4.13

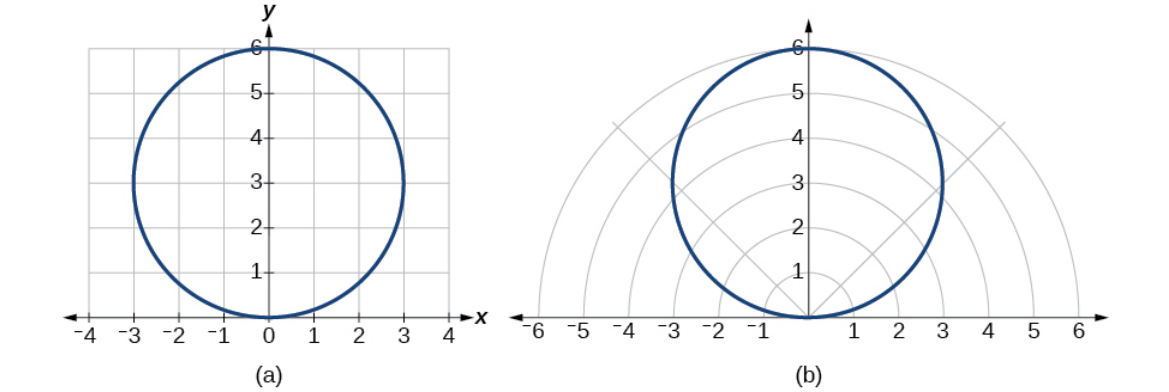


Figure 4.4.13: (a) Cartesian form $x^2 + y^2 = 6y$ (b) polar form $r = 6\sin\theta$

The Cartesian or rectangular equation is plotted on the rectangular grid, and the **polar equation** is plotted on the polar grid. Clearly, the graphs are identical.

? Exercise 4.4.4A:

Rewriting a Cartesian Equation in Polar Form

Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.

Answer

We will use the relationships $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} y &= 3x + 2 \\ r\sin\theta &= 3r\cos\theta + 2 \\ r\sin\theta - 3r\cos\theta &= 2 \\ r(\sin\theta - 3\cos\theta) &= 2 \quad \text{Isolate } r. \\ r &= \frac{2}{\sin\theta - 3\cos\theta} \quad \text{Solve for } r. \end{aligned}$$

? Exercise 4.4.4B:

Rewrite the Cartesian equation $y^2 = 3 - x^2$ in polar form.

Answer

$r = \sqrt{3}$

Identify and Graph Polar Equations by Converting to Rectangular Equations

We have learned how to convert rectangular coordinates to polar coordinates, and we have seen that the points are indeed the same. We have also transformed polar equations to rectangular equations and vice versa. Now we will demonstrate that their graphs, while drawn on different grids, are identical.

Example 4.4.6A: Graphing a Polar Equation by Converting to a Rectangular Equation

Covert the polar equation $r = 2\sec\theta$ to a rectangular equation, and draw its corresponding graph.

Solution

The conversion is

$$\begin{aligned} r &= 2\sec\theta \\ r &= \frac{2}{\cos\theta} \\ r\cos\theta &= 2 \\ x &= 2 \end{aligned}$$

Notice that the equation $r = 2\sec\theta$ drawn on the polar grid is clearly the same as the vertical line $x = 2$ drawn on the rectangular grid (see Figure 4.4.14). Just as $x = c$ is the standard form for a vertical line in rectangular form, $r = c\sec\theta$ is the standard form for a vertical line in polar form.

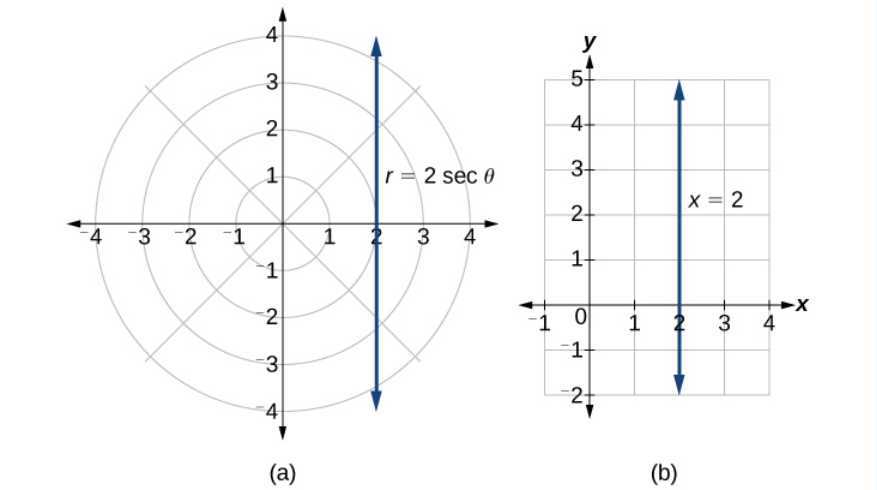


Figure 4.4.14: (a) Polar grid (b) Rectangular coordinate system

A similar discussion would demonstrate that the graph of the function $r = 2\csc\theta$ will be the horizontal line $y = 2$. In fact, $r = c\csc\theta$ is the standard form for a horizontal line in polar form, corresponding to the rectangular form $y = c$.

Example 4.4.6B: Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \dfrac{3}{1-2\cos\theta}$ as a Cartesian equation.

Solution

The goal is to eliminate θ and r,and introduce x and y. We clear the fraction, and then use substitution. In order to replace r with x and y, we must use the expression $x^2 + y^2 = r^2$.

$$\begin{aligned} &= \frac{3}{1-2\cos\theta} \\ r(1-2\cos\theta) &= 3 \\ r\left(1-2\left(\frac{x}{r}\right)\right) &= 3 \quad \text{Use } \cos\theta = \frac{x}{r} \text{ to eliminate } \theta. \\ r - 2x &= 3 \\ r &= 3+2x \quad \text{Isolate } r. \\ r^2 &= (3+2x)^2 \quad \text{Square both sides.} \\ x^2+y^2 &= (3+2x)^2 \quad \text{Use } x^2+y^2=r^2. \end{aligned}$$

The Cartesian equation is $x^2+y^2=(3+2x)^2$. However, to graph it, especially using a graphing calculator or computer program, we want to isolate y.

$$\begin{aligned} x^2+y^2 &= (3+2x)^2 \\ y^2 &= (3+2x)^2-x^2 \\ y &= \pm(3+2x)^2-x^2 \end{aligned}$$

When our entire equation has been changed from r and θ to x and y, we can stop, unless asked to solve for y or simplify. See Figure 4.4.15.

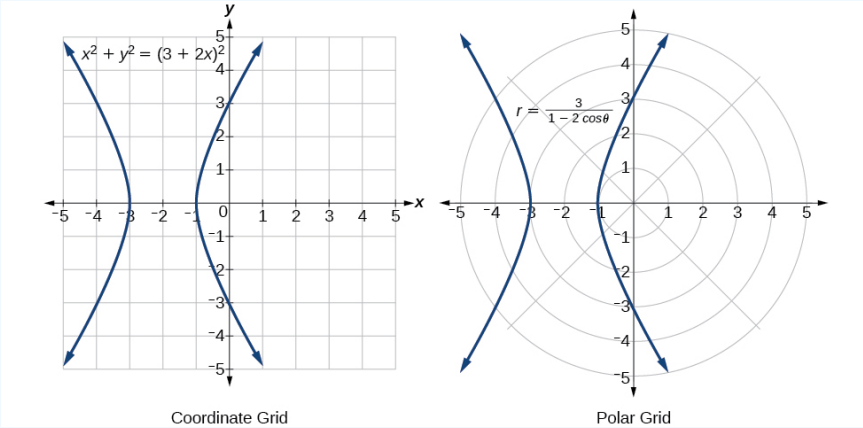


Figure 4.4.15

The "hour-glass" shape of the graph is called a *hyperbola*. Hyperbolas have many interesting geometric features and applications, which we will investigate further in Analytic Geometry.

Analysis

In this example, the right side of the equation can be expanded and the equation simplified further, as shown above. However, the equation cannot be written as a single function in Cartesian form. We may wish to write the rectangular equation in the hyperbola's standard form. To do this, we can start with the initial equation.

$$\begin{aligned} &x^2+y^2=(3+2x)^2 \\ &x^2+y^2-(3+2x)^2=0 \\ &x^2+y^2-(9+12x+4x^2)=0 \\ &x^2+y^2-9-12x-4x^2=0 \\ &-3x^2-12x+y^2=9 && \text{Multiply through by } -1. \\ &3x^2+12x-y^2=-9 \\ &3(x^2+4x)-y2=-9 && \text{Organize terms to complete the square for } x. \\ &3(x^2+4x+4)-y^2=-9+12 \\ &3(x+2)^2-y^2=3 \\ &(x+2)^2-\frac{y^2}{3}=1 \end{aligned}$$

? Exercise 4.4.5

Rewrite the polar equation $r=2\sin\theta$ in Cartesian form.

Answer

$x^2+y^2=2y$ or, in the standard form for a circle, $x^2+(y-1)^2=1$

Example 4.4.7: Rewriting a Polar Equation in Cartesian Form

Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

Solution

$$
\begin{aligned}
r &= \sin(2\theta) && \text{Use the double angle identity for sine.}\\
r &= 2\sin\theta\cos\theta && \text{Use } \cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}.\\
r &= 2\frac{x}{r})(\frac{y}{r}) && \text{Simplify.}\\
r &= \frac{2xy}{r^2} && \text{Multiply both sides by } r^2.\\
r^3 &= 2xy &&\\
(x^2+y^2)^{\,3} &= 2xy && \text{As } x^2+y^2 = r^2, r = \sqrt{x^2+y^2}.
\end{aligned}
$$

This equation can also be written as

$$(x^2+y^2)^{\frac{3}{2}} = 2xy \text{ or } x^2+y^2 = (2xy)^{\frac{2}{3}}$$

Media

Access these online resources for additional instruction and practice with polar coordinates.

- Introduction to Polar Coordinates
- Comparing Polar and Rectangular Coordinates

Key Equations

Conversion formulas	$\cos\theta = \frac{x}{r} \rightarrow x = r\cos\theta$ $\sin\theta = \frac{y}{r} \rightarrow y = r\sin\theta$ $r^2 = x^2 + y^2$ $\tan\theta = \frac{y}{x}$

Key Concepts

- The polar grid is represented as a series of concentric circles radiating out from the pole, or origin.
- To plot a point in the form (r, θ), $\theta > 0$, move in a counterclockwise direction from the polar axis by an angle of θ, and then extend a directed line segment from the pole the length of r in the direction of θ. If θ is negative, move in a clockwise direction, and extend a directed line segment the length of r in the direction of θ. See Example 4.4.1.
- If r is negative, extend the directed line segment in the opposite direction of θ. See Example 4.4.2.
- To convert from polar coordinates to rectangular coordinates, use the formulas $x = r\cos\theta$ and $y = r\sin\theta$. See Example 4.4.3 and Example 4.4.4.
- To convert from rectangular coordinates to polar coordinates, use one or more of the formulas: $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, $\tan\theta = \frac{y}{x}$, and $r = \sqrt{x^2+y^2}$. See Example 4.4.5.
- Transforming equations between polar and rectangular forms means making the appropriate substitutions based on the available formulas, together with algebraic manipulations. See Example 4.4.6, Example 4.4.7, and Example 4.4.8.
- Using the appropriate substitutions makes it possible to rewrite a polar equation as a rectangular equation, and then graph it in the rectangular plane. See Example 4.4.9, Example 4.4.10, and Example 4.4.11.

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4.5: Polar Coordinates - Graphs

Learning Objectives

- Test polar equations for symmetry.
- Graph polar equations by plotting points.

Kepler's First Law of Planetary Motion argues that the planets move through space in elliptical, periodic orbits about the sun, as shown in Figure 4.5.1. They are in constant motion, so fixing an exact position of any planet is valid only for a moment. In other words, we can fix only a planet's instantaneous position. This is one application of polar coordinates, represented as (r, θ). We interpret r as the distance from the sun and θ as the planet's angular bearing, or its direction from a fixed point on the sun. In this section, we will focus on the polar system and the graphs that are generated directly from polar coordinates.

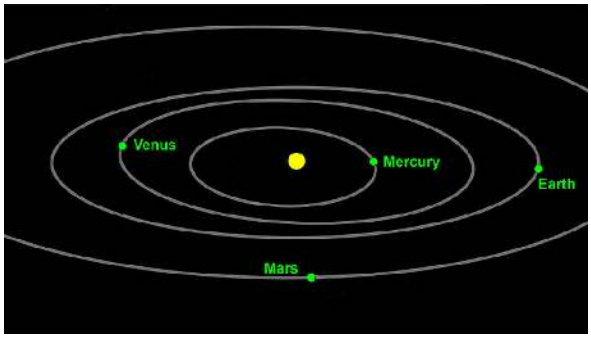

Figure 4.5.1: Planets follow elliptical paths as they orbit around the Sun. (credit: modification of work by NASA/JPL-Caltech)

Testing Polar Equations for Symmetry

Just as a rectangular equation such as $y = x^2$ describes the relationship between x and y on a Cartesian grid, a **polar equation** describes a relationship between r and θ on a polar grid. Recall that the coordinate pair (r, θ) indicates that we move counterclockwise from the polar axis (positive x-axis) by an angle of θ, and extend a ray from the pole (origin) r units in the direction of θ. All points that satisfy the polar equation are on the graph.

Symmetry is a property that helps us recognize and plot the graph of any equation. If an equation has a graph that is symmetric with respect to an axis, it means that if we folded the graph in half over that axis, the portion of the graph on one side would coincide with the portion on the other side. By performing three tests, we will see how to apply the properties of symmetry to polar equations. Further, we will use symmetry (in addition to plotting key points, zeros, and maximums of r) to determine the graph of a polar equation.

In the first test, we consider symmetry with respect to the line $\theta = \frac{\pi}{2}$ (y-axis). We replace (r, θ) with $(-r, -\theta)$ to determine if the new equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin\theta$;

$$\begin{aligned} r &= 2\sin\theta \\ -r &= 2\sin -\theta \quad \text{Replace } (r, \theta) \text{ with } (-r, -\theta). \\ -r &= -2\sin\theta \quad \text{Identity: } \sin(-\theta) = -\sin\theta. \\ r &= 2\sin\theta \quad \text{Multiply both sides by } -1 \end{aligned}$$

This equation exhibits symmetry with respect to the line $\theta = \frac{\pi}{2}$.

In the second test, we consider symmetry with respect to the polar axis (x-axis). We replace (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ to determine equivalency between the tested equation and the original. For example, suppose we are given the equation $r = 1 - 2\cos\theta$.

$$\begin{aligned} r &= 1 - 2\cos\theta \\ r &= 1 - 2\cos(-\theta) \quad \text{Replace } (r, \theta) \text{ with } (r, -\theta). \\ r &= 1 - 2\cos\theta \quad \text{Even/Odd identity} \end{aligned}$$

The graph of this equation exhibits symmetry with respect to the polar axis.

In the third test, we consider symmetry with respect to the pole (origin). We replace (r, θ) with $(-r, \theta)$ to determine if the tested equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin(3\theta)$.

$$\begin{aligned} r &= 2\sin(3\theta) \\ -r &= 2\sin(3\theta) \end{aligned}$$

The equation has failed the symmetry test, but that does not mean that it is not symmetric with respect to the pole. Passing one or more of the symmetry tests verifies that symmetry will be exhibited in a graph. However, failing the symmetry tests does not necessarily indicate that a graph will not be symmetric about the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole. In these instances, we can confirm that symmetry exists by plotting reflecting points across the apparent axis of symmetry or the pole. Testing for symmetry is a technique that simplifies the graphing of polar equations, but its application is not perfect.

Note: SYMMETRY TESTS

A polar equation describes a curve on the polar grid. The graph of a polar equation can be evaluated for three types of symmetry, as shown in Figure 4.5.2.

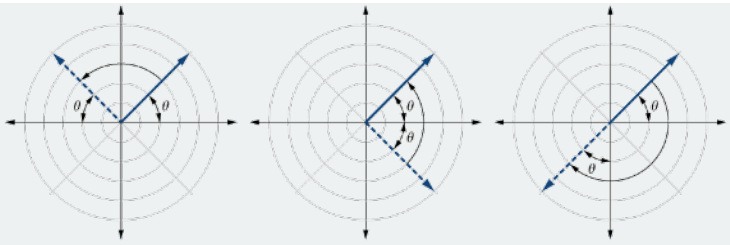

Figure 4.5.2: (a) A graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$ (y-axis) if replacing (r, θ) with $(-r, -\theta)$ yields an equivalent equation. (b) A graph is symmetric with respect to the polar axis (x-axis) if replacing (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ yields an equivalent equation. (c) A graph is symmetric with respect to the pole (origin) if replacing (r, θ) with $(-r, \theta)$ yields an equivalent equation.

How to: Given a polar equation, test for symmetry.

1. Substitute the appropriate combination of components for (r, θ): $(-r, -\theta)$ for $\theta = \frac{\pi}{2}$ symmetry; $(r, -\theta)$ for polar axis symmetry; and $(-r, \theta)$ for symmetry with respect to the pole.
2. If the resulting equations are equivalent in one or more of the tests, the graph produces the expected symmetry.

Example 4.5.1: Testing a Polar Equation for Symmetry

Test the equation $r = 2\sin\theta$ for symmetry.

Solution

Test for each of the three types of symmetry.

1) Replacing (r, θ) with $(-r, -\theta)$ yields the same result. Thus, the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.	$-r = 2\sin(-\theta)$ $-r = -2\sin\theta$ Even-odd identity $r = 2\sin\theta$ Multiply by -1 Passed
2) Replacing θ with $-\theta$ does not yield the same equation. Therefore, the graph fails the test and may or may not be symmetric with respect to the polar axis.	$r = 2\sin(-\theta)$ $r = -2\sin\theta$ Even-odd identity $r = -2\sin\theta \neq 2\sin\theta$ Failed
3) Replacing r with $-r$ changes the equation and fails the test. The graph may or may not be symmetric with respect to the pole.	$-r = 2\sin\theta$ $r = -2\sin\theta \neq 2\sin\theta$ Failed

Analysis

Using a graphing calculator, we can see that the equation $r = 2\sin\theta$ is a circle centered at $(0, 1)$ with radius $r = 1$ and is indeed symmetric to the line $\theta = \frac{\pi}{2}$. We can also see that the graph is not symmetric with the polar axis or the pole. See Figure 4.5.3.

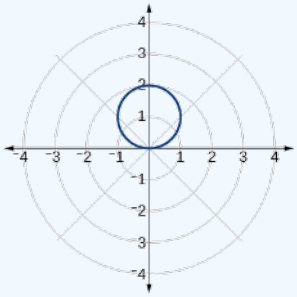

Figure 4.5.3

Exercise 4.5.1

Test the equation for symmetry: $r = -2\cos\theta$.

Answer

The equation fails the symmetry test with respect to the line $\theta = \frac{\pi}{2}$ and with respect to the pole. It passes the polar axis symmetry test.

Graphing Polar Equations by Plotting Points

To graph in the rectangular coordinate system we construct a table of x and y values. To graph in the polar coordinate system we construct a table of θ and r values. We enter values of θ into a polar equation and calculate r. However, using the properties of symmetry and finding key values of θ and r means fewer calculations will be needed.

Finding Zeros and Maxima

To find the zeros of a polar equation, we solve for the values of θ that result in $r = 0$. Recall that, to find the zeros of polynomial functions, we set the equation equal to zero and then solve for x. We use the same process for polar equations. Set $r = 0$, and solve for θ.

For many of the forms we will encounter, the maximum value of a polar equation is found by substituting those values of θ into the equation that result in the maximum value of the trigonometric functions. Consider $r = 5\cos\theta$; the maximum distance between the curve and the pole is 5 units. The maximum value of the cosine function is 1 when $\theta = 0$, so our polar equation is $5\cos\theta$, and the value $\theta = 0$ will yield the maximum $|r|$.

Similarly, the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, and if our polar equation is $r = 5\sin\theta$, the value $\theta = \frac{\pi}{2}$ will yield the maximum $|r|$. We may find additional information by calculating values of r when $\theta = 0$. These points would be polar axis intercepts, which may be helpful in drawing the graph and identifying the curve of a polar equation.

Example 4.5.2: Finding Zeros and Maximum Values for a Polar Equation

Using the equation in Example 4.5.1, find the zeros and maximum $|r|$ and, if necessary, the polar axis intercepts of $r = 2\sin\theta$.

Solution

To find the zeros, set r equal to zero and solve for θ.

$$\begin{aligned} 2\sin\theta &= 0 \\ \sin\theta &= 0 \\ \theta &= \sin^{-1} 0 \\ \theta &= n\pi \qquad \text{where n is an integer} \end{aligned}$$

Substitute any one of the θ values into the equation. We will use 0.

$$\begin{aligned} r &= 2\sin(0) \\ r &= 0 \end{aligned}$$

The points $(0, 0)$ and $(0, \pm n\pi)$ are the zeros of the equation. They all coincide, so only one point is visible on the graph. This point is also the only polar axis intercept.

To find the maximum value of the equation, look at the maximum value of the trigonometric function $\sin\theta$, which occurs when $\theta = \frac{\pi}{2} \pm 2k\pi$ resulting in $\sin\left(\frac{\pi}{2}\right) = 1$. Substitute $\frac{\pi}{2}$ for θ.

$$\begin{aligned} r &= 2\sin\left(\frac{\pi}{2}\right) \\ r &= 2(1) \\ r &= 2 \end{aligned}$$

Analysis

The point $\left(2, \frac{\pi}{2}\right)$ will be the maximum value on the graph. Let's plot a few more points to verify the graph of a circle. See Table 4.5.1 and Figure 4.5.4.

Table 4.5.1

θ	$r = 2\sin\theta$	r
0	$r = 2\sin(0) = 0$	0
$\frac{\pi}{6}$	$r = 2\sin\left(\frac{\pi}{6}\right) = 1$	1
$\frac{\pi}{3}$	$r = 2\sin\left(\frac{\pi}{3}\right) \approx 1.73$	1.73
$\frac{pi}{2}$	$r = 2\sin\left(\frac{\pi}{2}\right) = 2$	2
$\frac{2\pi}{3}$	$r = 2\sin\left(\frac{2\pi}{3}\right) \approx 1.73$	1.73
$\frac{5\pi}{6}$	$r = 2\sin\left(\frac{5\pi}{6}\right) = 1$	1
π	$r = 2\sin(\pi) = 0$	0

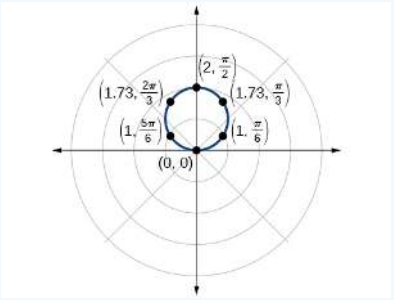

Figure 4.5.4

Exercise 4.5.2

Without converting to Cartesian coordinates, test the given equation for symmetry and find the zeros and maximum values of $|r|$: $r = 3\cos\theta$.

Answer

Tests will reveal symmetry about the polar axis. The zero is $\left(0, \frac{\pi}{2}\right)$, and the maximum value is $(3, 0)$.

Investigating Circles

Now we have seen the equation of a circle in the polar coordinate system. In the last two examples, the same equation was used to illustrate the properties of symmetry and demonstrate how to find the zeros, maximum values, and plotted points that produced the graphs. However, the circle is only one of many shapes in the set of polar curves.

There are five classic polar curves**: cardioids**, **limaçons, lemniscates, rose curves**, and **Archimedes' spirals**. We will briefly touch on the polar formulas for the circle before moving on to the classic curves and their variations.

FORMULAS FOR THE EQUATION OF A CIRCLE

Some of the formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$, whereа a is the diameter of the circle or the distance from the pole to the farthest point on the circumference. The radius is $\frac{|a|}{2}$, or one-half the diameter. For $r = a\cos\theta$, the center is $\left(\frac{a}{2}, 0\right)$. For $r = a\sin\theta$, the center is $\left(\frac{a}{2}, \pi\right)$. Figure 4.5.5 shows the graphs of

these four circles.

0. The center is at (a/2,0). Second is r=acos(theta), a<0. The center is at (a/2,0). Third is r=asin(theta), a>0. The center is at (a/2, pi). Fourth is r=asin(theta), a<0. The center is at (a/2, pi)." src="/api/deki/files/7429/CNX_Precalc_Figure_08_04_005new.jpg">

Figure 4.5.5

Example 4.5.3: Sketching the Graph of a Polar Equation for a Circle

Sketch the graph of $r = 4\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph is symmetric about the polar axis. Next, we find the zeros and maximum $|r|$ for $r = 4\cos\theta$. First, set $r = 0$, and solve for θ. Thus, a zero occurs at $\theta = \frac{\pi}{2} \pm k\pi$. A key point to plot is $\left(0, \frac{\pi}{2}\right)$.

To find the maximum value of r, note that the maximum value of the cosine function is 1 when $\theta = 0 \pm 2k\pi$. Substitute $\theta = 0$ into the equation:

$$\begin{aligned} r &= 4\cos\theta \\ r &= 4\cos(0) \\ r &= 4(1) \\ &= 4 \end{aligned}$$

The maximum value of the equation is 4. A key point to plot is $(4, 0)$.

As $r = 4\cos\theta$ is symmetric with respect to the polar axis, we only need to calculate r-values for θ over the interval $[0, \pi]$. Points in the upper quadrant can then be reflected to the lower quadrant. Make a table of values similar to Table 4.5.2. The graph is shown in Figure 4.5.6.

Table 4.5.2

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	3.46	2.83	2	0	-2	-2.83	-3.46	4

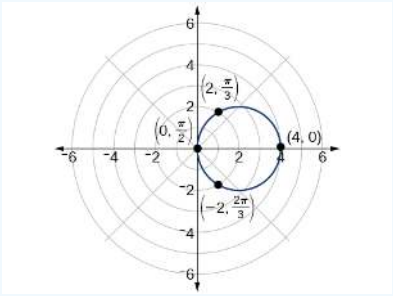

Figure 4.5.6

Investigating Cardioids

While translating from polar coordinates to Cartesian coordinates may seem simpler in some instances, graphing the classic curves is actually less complicated in the polar system. The next curve is called a cardioid, as it resembles a heart. This shape is often included with the family of curves called limaçons, but here we will discuss the cardioid on its own.

FORMULAS FOR A CARDIOID

The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $\frac{a}{b} = 1$. The cardioid graph passes through the pole, as we can see in Figure 4.5.7.

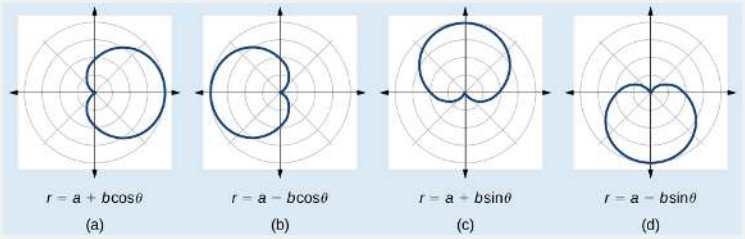

Figure 4.5.7

How to: Given the polar equation of a cardioid, sketch its graph

1. Check equation for the three types of symmetry.
2. Find the zeros. Set $r = 0$.
3. Find the maximum value of the equation according to the maximum value of the trigonometric expression.
4. Make a table of values for r and θ.
5. Plot the points and sketch the graph.

Example 4.5.4: Sketching the Graph of a Cardioid

Sketch the graph of $r = 2 + 2\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph of this equation will be symmetric about the polar axis. Next, we find the zeros and maximums. Setting $r = 0$, we have $\theta = \pi + 2k\pi$. The zero of the equation is located at $(0, \pi)$. The graph passes through this point.

The maximum value of $r = 2 + 2\cos\theta$ occurs when $\cos\theta$ is a maximum, which is when $\cos\theta = 1$ or when $\theta = 0$. Substitute $\theta = 0$ into the equation, and solve for r.

$$\begin{aligned} r &= 2 + 2\cos(0) \\ r &= 2 + 2(1) \\ &= 4 \end{aligned}$$

The point $(4, 0)$ is the maximum value on the graph.

We found that the polar equation is symmetric with respect to the polar axis, but as it extends to all four quadrants, we need to plot values over the interval $[0, \pi]$. The upper portion of the graph is then reflected over the polar axis. Next, we make a table of values, as in Table 4.5.3, and then we plot the points and draw the graph. See Figure 4.5.8.

Table 4.5.3

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	3.41	2	1	0

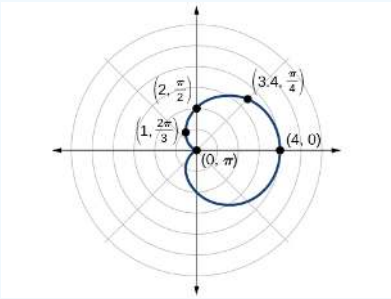

Figure 4.5.8

Investigating Limaçons

The word *limaçon* is Old French for "snail," a name that describes the shape of the graph. As mentioned earlier, the cardioid is a member of the limaçon family, and we can see the similarities in the graphs. The other images in this category include the one-loop limaçon and the two-loop (or inner-loop) limaçon. **One-loop limaçons** are sometimes referred to as dimpled limaçons when $1 < \frac{a}{b} < 2$ and convex limaçons when $\frac{a}{b} \geq 2$.

FORMULAS FOR ONE-LOOP LIMAÇONS

The formulas that produce the graph of a dimpled one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $1 < ab < 2$. All four graphs are shown in Figure 4.5.9.

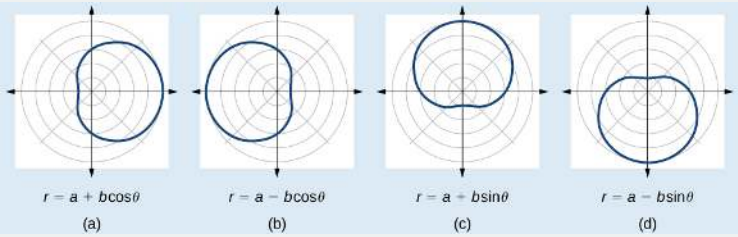

Figure 4.5.9

How to: Given a polar equation for a one-loop limaçon, sketch the graph

1. Test the equation for symmetry. Remember that failing a symmetry test does not mean that the shape will not exhibit symmetry. Often the symmetry may reveal itself when the points are plotted.
2. Find the zeros.
3. Find the maximum values according to the trigonometric expression.
4. Make a table.
5. Plot the points and sketch the graph.

Example 4.5.5: Sketching the Graph of a One-Loop Limaçon

Graph the equation $r = 4 - 3\sin\theta$.

Solution

First, testing the equation for symmetry, we find that it fails all three symmetry tests, meaning that the graph may or may not exhibit symmetry, so we cannot use the symmetry to help us graph it. However, this equation has a graph that clearly displays symmetry with respect to the line $\theta = \frac{\pi}{2}$, yet it fails all the three symmetry tests. A graphing calculator will immediately illustrate the graph's reflective quality.

Next, we find the zeros and maximum, and plot the reflecting points to verify any symmetry. Setting $r = 0$ results in θ being undefined. What does this mean? How could θ be undefined? The angle θ is undefined for any value of $\sin\theta > 1$. Therefore, θ is undefined because there is no value of θ for which $\sin\theta > 1$. Consequently, the graph does not pass through the pole. Perhaps the graph does cross the polar axis, but not at the pole. We can investigate other intercepts by calculating r when $\theta = 0$.

$$\begin{aligned} r(0) &= 4 - 3\sin(0) \\ r &= 4 - 3 \cdot 0 \\ &= 4 \end{aligned}$$

So, there is at least one polar axis intercept at $(4, 0)$.

Next, as the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, we will substitute $\theta = \frac{\pi}{2}$ into the equation and solve for r. Thus, $r = 1$.

Make a table of the coordinates similar to Table 4.5.4.

Table 4.5.4

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	4	2.5	1.4	1	1.4	2.5	4	5.5	6.6	7	6.6	5.5	4

The graph is shown in Figure 4.5.10.

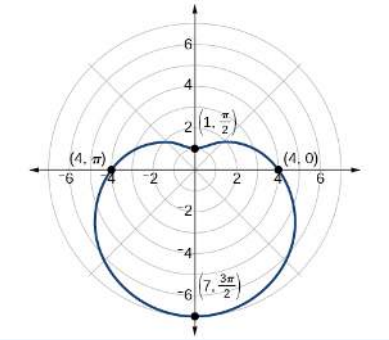

Figure 4.5.10: One-loop limaçon

Analysis

This is an example of a curve for which making a table of values is critical to producing an accurate graph. The symmetry tests fail; the zero is undefined. While it may be apparent that an equation involving $\sin\theta$ is likely symmetric with respect to the line $\theta = \frac{\pi}{2}$, evaluating more points helps to verify that the graph is correct.

Exercise 4.5.3

Sketch the graph of $r = 3 - 2\cos\theta$.

Answer

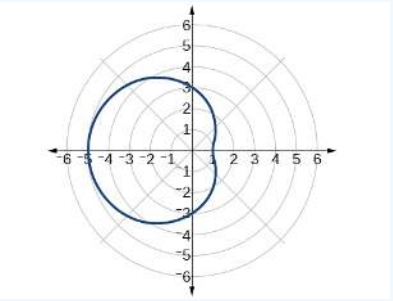

Figure 4.5.11

Another type of limaçon, the **inner-loop limaçon**, is named for the loop formed inside the general limaçon shape. It was discovered by the German artist Albrecht Dürer(1471-1528), who revealed a method for drawing the inner-loop limaçon in his 1525 book *Underweysung der Messing*. A century later, the father of mathematician Blaise Pascal, Étienne Pascal(1588-1651), rediscovered it.

FORMULAS FOR INNER-LOOP LIMAÇONS

The formulas that generate the inner-loop limaçons are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $a < b$. The graph of the inner-loop limaçon passes through the pole twice: once for the outer loop, and once for the inner loop. See Figure 10.5.12 for the graphs.

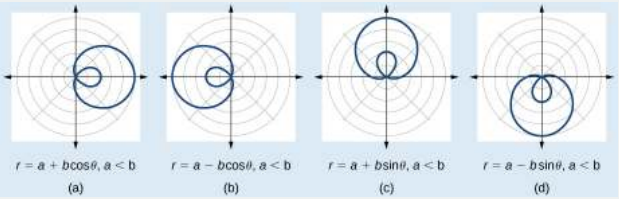

Figure 4.5.12

Example 4.5.6: Sketching the Graph of an Inner-Loop Limaçon

Sketch the graph of $r = 2 + 5\cos\theta$.

Solution

Testing for symmetry, we find that the graph of the equation is symmetric about the polar axis. Next, finding the zeros reveals that when $r = 0$, $\theta = 1.98$. The maximum $|r|$ is found when $\cos\theta = 1$ or when $\theta = 0$. Thus, the maximum is found at the point $(7, 0)$.

Even though we have found symmetry, the zero, and the maximum, plotting more points will help to define the shape, and then a pattern will emerge. See Table 4.5.5.

Table 4.5.5

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

r	7	6.3	4.5	2	-0.5	-2.3	-3	-2.3	-0.5	2	4.5	6.3	7

As expected, the values begin to repeat after $\theta = \pi$. The graph is shown in Figure 4.5.13.

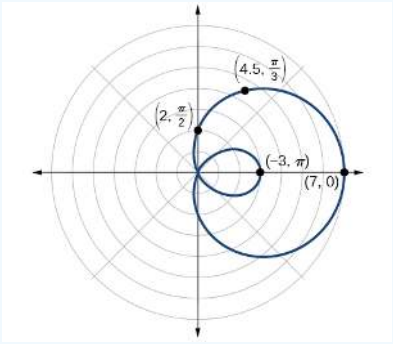

Figure 4.5.13: Inner-loop limaçon

Investigating Lemniscates

The lemniscate is a polar curve resembling the infinity symbol ∞ or a figure 8. Centered at the pole, a lemniscate is symmetrical by definition.

FORMULAS FOR LEMNISCATES

The formulas that generate the graph of a lemniscate are given by $r^2 = a^2 \cos 2\theta$ and $r^2 = a^2 \sin 2\theta$ where $a \neq 0$. The formula $r^2 = a^2 \sin 2\theta$ is symmetric with respect to the pole. The formula $r^2 = a^2 \cos 2\theta$ is symmetric with respect to the pole, the line $\theta = \frac{\pi}{2}$, and the polar axis. See Figure 4.5.14 for the graphs.

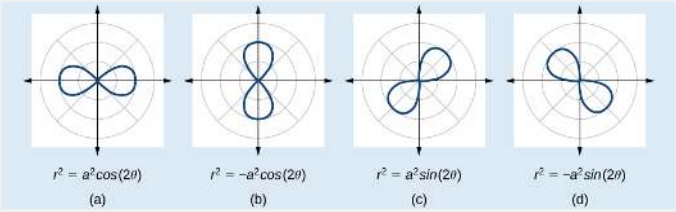

Figure 4.5.14

Example 4.5.7: Sketching the Graph of a Lemniscate

Sketch the graph of $r^2 = 4 \cos 2\theta$.

Solution

The equation exhibits symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, and the pole.

Let's find the zeros. It should be routine by now, but we will approach this equation a little differently by making the substitution $u = 2\theta$.

$$\begin{aligned} 0 &= 4\cos 2\theta \\ 0 &= 4\cos u \\ 0 &= \cos u \\ \cos^{-1} 0 &= \frac{\pi}{2} \\ u &= \frac{\pi}{2} \qquad \text{Substitute } 2\theta \text{ back in for } u. \\ 2\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{4} \end{aligned}$$

So, the point $\left(0, \frac{\pi}{4}\right)$ is a zero of the equation.

Now let's find the maximum value. Since the maximum of $\cos u = 1$ when $u = 0$, the maximum $\cos 2\theta = 1$ when $2\theta = 0$. Thus,

$$\begin{aligned} r^2 &= 4\cos(0) \\ r^2 &= 4(1) \\ r^2 &= 4 \\ r &= \pm 4 \\ &= 2 \end{aligned}$$

We have a maximum at $(2, 0)$. Since this graph is symmetric with respect to the pole, the line $\theta = \frac{\pi}{2}$, and the polar axis, we only need to plot points in the first quadrant.

Make a table similar to Table 4.5.6.

Table 4.5.6

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
r	2	$\sqrt{2}$	0	$\sqrt{2}$	0

Plot the points on the graph, such as the one shown in Figure 4.5.15.

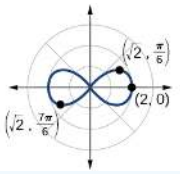

Figure 4.5.15: Lemniscate

Analysis

Making a substitution such as $u = 2\theta$ is a common practice in mathematics because it can make calculations simpler. However, we must not forget to replace the substitution term with the original term at the end, and then solve for the unknown.

Some of the points on this graph may not show up using the Trace function on the TI-84 graphing calculator, and the calculator table may show an error for these same points of r. This is because there are no real square roots for these values of θ. In other words, the corresponding r-values of $\sqrt{4\cos(2\theta)}$ are complex numbers because there is a negative number under the radical.

Investigating Rose Curves

The next type of polar equation produces a petal-like shape called a rose curve. Although the graphs look complex, a simple polar equation generates the pattern.

ROSE CURVES

The formulas that generate the graph of a **rose curve** are given by $r = a\cos n\theta$ and $r = a\sin n\theta$ where $a \neq 0$. If n is even, the curve has $2n$ petals. If n is odd, the curve has n petals. See Figure 4.5.16

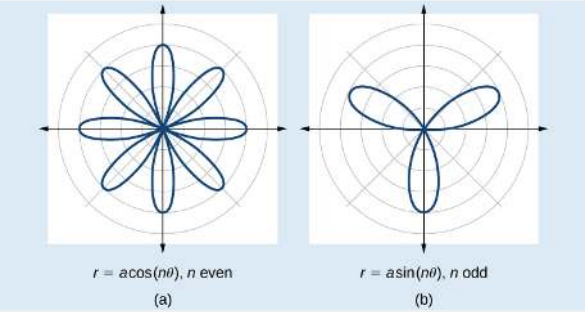

Figure 4.5.16

Example 4.5.8: Sketching the Graph of a Rose Curve (n Even)

Sketch the graph of $r = 2\cos 4\theta$.

Solution

Testing for symmetry, we find again that the symmetry tests do not tell the whole story. The graph is not only symmetric with respect to the polar axis, but also with respect to the line $\theta = \frac{\pi}{2}$ and the pole.

Now we will find the zeros. First make the substitution $u = 4\theta$.

$$\begin{aligned} 0 &= 2\cos 4\theta \\ 0 &= \cos 4\theta \\ 0 &= \cos u \\ \cos^{-1} 0 &= u \\ u &= \frac{\pi}{2} \\ 4\theta &= \frac{\pi}{2} \\ \theta &= \frac{\pi}{8} \end{aligned}$$

The zero is $\theta = \frac{\pi}{8}$. The point $\left(0, \frac{\pi}{8}\right)$ is on the curve.

Next, we find the maximum $|r|$. We know that the maximum value of $\cos u = 1$ when $\theta = 0$. Thus,

$$\begin{aligned} r &= 2\cos(4 \cdot 0) \\ r &= 2\cos(0) \\ r &= 2(1) \\ &= 2 \end{aligned}$$

The point $(2, 0)$ is on the curve.

The graph of the rose curve has unique properties, which are revealed in Table 4.5.7.

Table 4.5.7

θ	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$5\pi 8$	$3\pi 4$
r	2	0	-2	0	2	0	-2

As $r = 0$ when $\theta = \frac{\pi}{8}$, it makes sense to divide values in the table by $\frac{\pi}{8}$ units. A definite pattern emerges. Look at the range of r-values: $2, 0, -2, 0$ and so on. This represents the development of the curve one petal at a time. Starting at $r = 0$, each petal extends out a distance of $r = 2$, and then turns back to zero $2n$ times for a total of eight petals. See the graph in Figure 4.5.17.

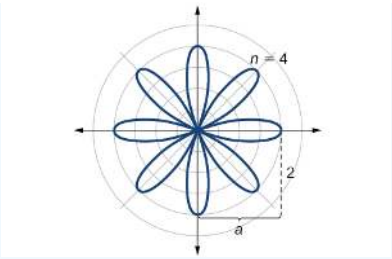

Figure 4.5.17: Rose curve, n even

Analysis

When these curves are drawn, it is best to plot the points in order, as in the Table 4.5.7. This allows us to see how the graph hits a maximum (the tip of a petal), loops back crossing the pole, hits the opposite maximum, and loops back to the pole. The action is continuous until all the petals are drawn.

? Exercise 4.5.4

Sketch the graph of $r = 4\sin(2\theta)$.

Answer

The graph is a rose curve, n even

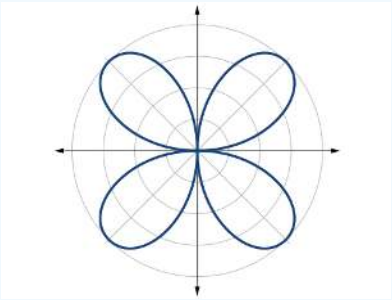
Figure 4.5.18

✓ Example 4.5.9: Sketching the Graph of a Rose Curve (n Odd)

Sketch the graph of $r = 2\sin(5\theta)$.

Solution

The graph of the equation shows symmetry with respect to the line $\theta = \frac{\pi}{2}$. Next, find the zeros and maximum. We will want to make the substitution $u = 5\theta$.

$$\begin{aligned} 0 &= 2\sin(5\theta) \\ 0 &= \sin u \\ \sin^{-1} 0 &= 0 \\ u &= 0 \\ 5\theta &= 0 \\ \theta &= 0 \end{aligned}$$

The maximum value is calculated at the angle where $\sin\theta$ is a maximum. Therefore,

$$\begin{aligned} r &= 2\sin\left(5 \cdot \frac{\pi}{2}\right) \\ r &= 2(1) \\ &= 2 \end{aligned}$$

Thus, the maximum value of the polar equation is 2. This is the length of each petal. As the curve for n odd yields the same number of petals as n, there will be five petals on the graph. See Figure 4.5.19.

Create a table of values similar to Table 4.5.8.

Table 4.5.8

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	\(0\0	1	**−1.73**	2	**−1.73**	1	0

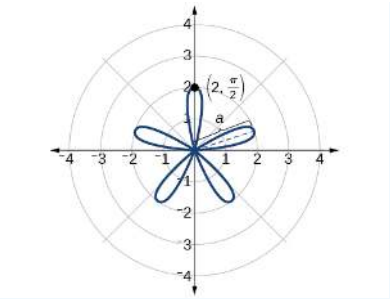

Figure 4.5.19: Rose curve, n odd

Exercise 4.5.5

Sketch the graph of $r = 3\cos(3\theta)$.

Answer

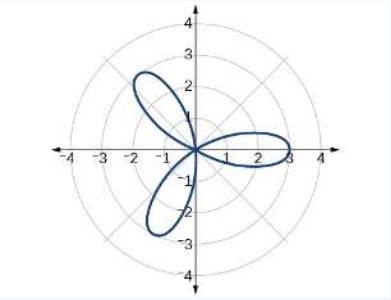

Figure 4.5.20

Rose curve, n odd

Investigating the Archimedes' Spiral

The final polar equation we will discuss is the Archimedes' spiral, named for its discoverer, the Greek mathematician Archimedes (c. 287 BCE - c. 212 BCE), who is credited with numerous discoveries in the fields of geometry and mechanics.

ARCHIMEDES' SPIRAL

The formula that generates the graph of the Archimedes' spiral is given by $r = \theta$ for $\theta \geq 0$. As θ increases, r increases at a constant rate in an ever-widening, never-ending, spiraling path. See Figure 4.5.21.

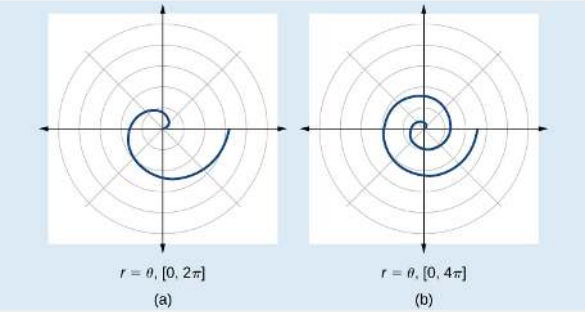

Figure 4.5.21

How to: Given an Archimedes' spiral over $[0, 2\pi]$, sketch the graph

1. Make a table of values for r and θ over the given domain.
2. Plot the points and sketch the graph.

Example 4.5.10: Sketching the Graph of an Archimedes' Spiral

Sketch the graph of $r = \theta$ over $[0, 2\pi]$.

Solution

As r is equal to θ, the plot of the Archimedes' spiral begins at the pole at the point $(0, 0)$. While the graph hints of symmetry, there is no formal symmetry with regard to passing the symmetry tests. Further, there is no maximum value, unless the domain is restricted.

Create a table such as Table 4.5.9.

Table 4.5.9

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0.785	1.57	3.14	4.71	5.50	6.28

Notice that the r-values are just the decimal form of the angle measured in radians. We can see them on a graph in Figure 4.5.22

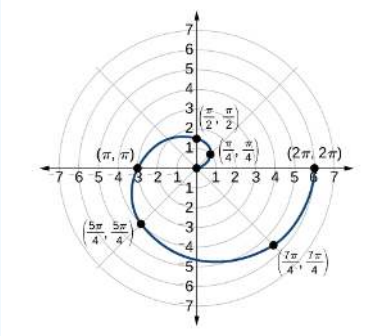

Figure 4.5.22: Archimedes' spiral

Analysis

The domain of this polar curve is $[0, 2\pi]$. In general, however, the domain of this function is $(-\infty, \infty)$. Graphing the equation of the Archimedes' spiral is rather simple, although the image makes it seem like it would be complex.

Exercise 4.5.6

Sketch the graph of $r = -\theta$ over the interval $[0, 4\pi]$.

Answer

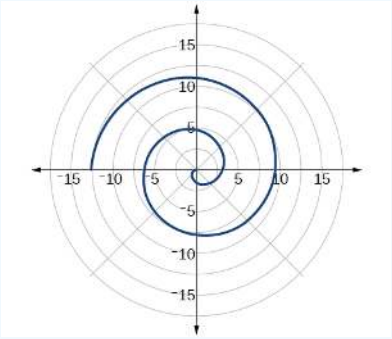

Figure 4.5.23

Summary of Curves

We have explored a number of seemingly complex polar curves in this section. Figure 4.5.24 and Figure 4.5.25 summarize the graphs and equations for each of these curves.

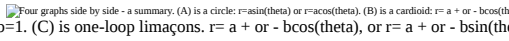

Figure 4.5.24

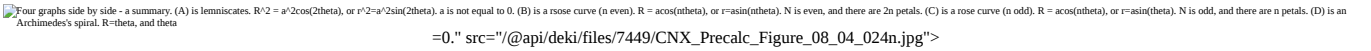

Figure 4.5.25

Media

Access these online resources for additional instruction and practice with graphs of polar coordinates.

- Graphing Polar Equations Part 1
- Graphing Polar Equations Part 2
- Animation: The Graphs of Polar Equations
- Graphing Polar Equations on the TI-84

Key Concepts

- It is easier to graph polar equations if we can test the equations for symmetry with respect to the line $\theta = \frac{\pi}{2}$, the polar axis, or the pole.
- There are three symmetry tests that indicate whether the graph of a polar equation will exhibit symmetry. If an equation fails a symmetry test, the graph may or may not exhibit symmetry. See Example 4.5.1.
- Polar equations may be graphed by making a table of values for θ and r.
- The maximum value of a polar equation is found by substituting the value θ that leads to the maximum value of the trigonometric expression.
- The zeros of a polar equation are found by setting $r = 0$ and solving for θ. See Example 4.5.2.
- Some formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$. See Example 4.5.3.
- The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$, for $a > 0$, $b > 0$, and $ab = 1$. See Example 4.5.4.
- The formulas that produce the graphs of a one-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $1 < ab < 2$. See Example 4.5.5.
- The formulas that produce the graphs of an inner-loop limaçon are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ for $a > 0$, $b > 0$, and $a < b$. See Example 4.5.6.
- The formulas that produce the graphs of a lemniscates are given by $r^2 = a^2\cos 2\theta$ and $r^2 = a^2\sin 2\theta$, where $a \neq 0$. See Example 4.5.7.
- The formulas that produce the graphs of rose curves are given by $r = a\cos n\theta$ and $r = a\sin n\theta$, where $a \neq 0$; if n is even, there are $2n$ petals, and if n is odd, there aren n petals. See Example 4.5.8 and Example 4.5.9.
- The formula that produces the graph of an Archimedes' spiral is given by $r = \theta$, $\theta \geq 0$. See Example 4.5.10.

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4.6: Polar Form of Complex Numbers

Learning Objectives

- Plot complex numbers in the complex plane.
- Find the absolute value of a complex number.
- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find quotients of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

"God made the integers; all else is the work of man." This rather famous quote by nineteenth-century German mathematician Leopold Kronecker sets the stage for this section on the polar form of a complex number. Complex numbers were invented by people and represent over a thousand years of continuous investigation and struggle by mathematicians such as Pythagoras, Descartes, De Moivre, Euler, Gauss, and others. Complex numbers answered questions that for centuries had puzzled the greatest minds in science.

We first encountered complex numbers in the section on Complex Numbers. In this section, we will focus on the mechanics of working with complex numbers: translation of complex numbers from polar form to rectangular form and vice versa, interpretation of complex numbers in the scheme of applications, and application of De Moivre's Theorem.

Plotting Complex Numbers in the Complex Plane

Plotting a complex number $a+bi$ is similar to plotting a real number, except that the horizontal axis represents the real part of the number, a, and the vertical axis represents the imaginary part of the number, bi.

How to: Given a complex number $a+bi$, plot it in the complex plane.

1. Label the horizontal axis as the *real* axis and the vertical axis as the *imaginary axis.*
2. Plot the point in the complex plane by moving a units in the horizontal direction and b units in the vertical direction.

Example 4.6.1: Plotting a Complex Number in the Complex Plane

Plot the complex number $2-3i$ in the complex plane.

Solution

From the origin, move two units in the positive horizontal direction and three units in the negative vertical direction. See Figure 4.6.1.

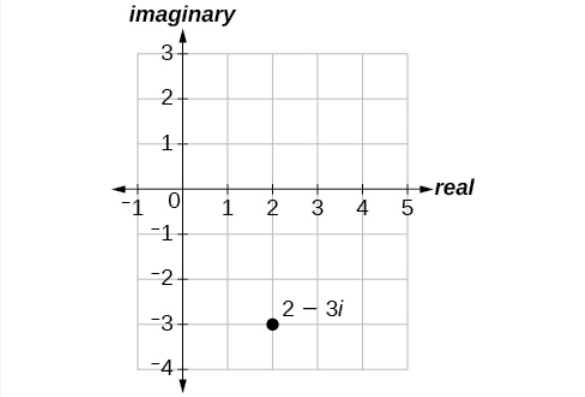

Figure 4.6.1

Exercise 4.6.1

Plot the point $1+5i$ in the complex plane.

Answer

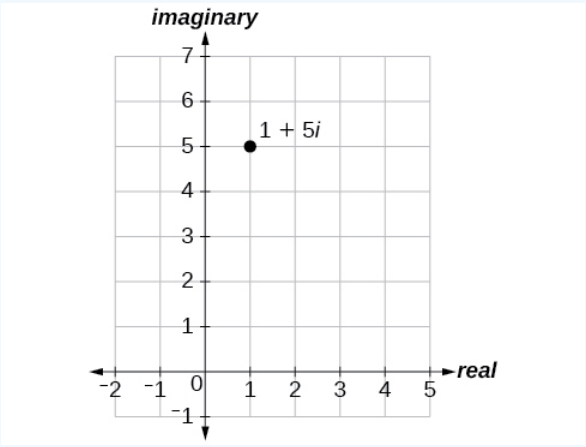

Figure 4.6.2

Finding the Absolute Value of a Complex Number

The first step toward working with a complex number in polar form is to find the absolute value. The absolute value of a complex number is the same as its magnitude, or $|z|$. It measures the distance from the origin to a point in the plane. For example, the graph of $z=2+4i$, in Figure 4.6.3, shows $|z|$.

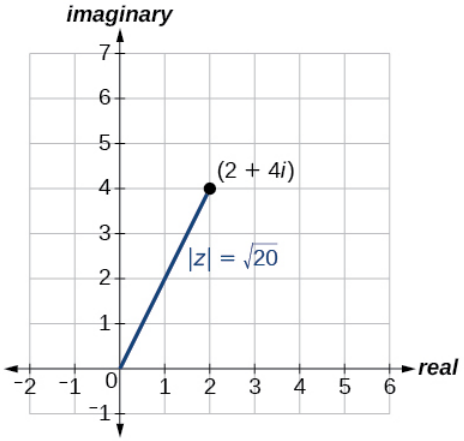

Figure 4.6.3

ABSOLUTE VALUE OF A COMPLEX NUMBER

Given $z=x+yi$, a complex number, the absolute value of z is defined as

$$|z|=\sqrt{x^2+y^2} \tag{4.6.1}$$

It is the distance from the origin to the point (x,y).

Notice that the absolute value of a real number gives the distance of the number from 0, while the absolute value of a complex number gives the distance of the number from the origin, $(0,0)$.

Example 4.6.2: Finding the Absolute Value of a Complex Number with a Radical

Find the absolute value of $z = \sqrt{5} - i$.

Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(\sqrt{5})^2 + (-1)^2}$$
$$|z| = \sqrt{5 + 1}$$
$$|z| = \sqrt{6}$$

See Figure 4.6.4.

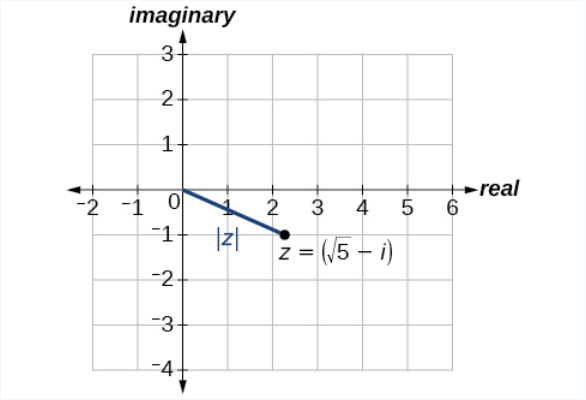

Figure 4.6.4

Exercise 4.6.2

Find the absolute value of the complex number $z = 12 - 5i$.

Answer

13

Example 4.6.3: Finding the Absolute Value of a Complex Number

Given $z = 3 - 4i$, find $|z|$.

Solution

Using the formula, we have

$$|z| = \sqrt{x^2 + y^2}$$
$$|z| = \sqrt{(3)^2 + (-4)^2}$$
$$|z| = \sqrt{9 + 16}$$
$$|z| = \sqrt{25}$$
$$|z| = 5$$

The absolute value z is 5. See Figure 4.6.5.

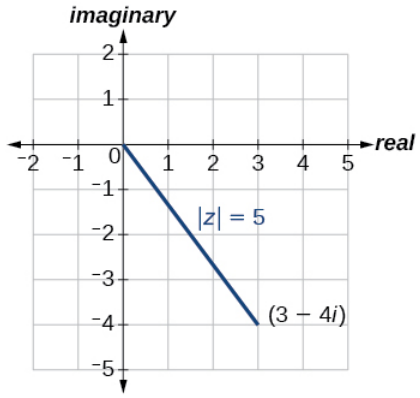

Figure 4.6.5

Exercise 4.6.3

Given $z = 1 - 7i$, find $|z|$.

Answer

$|z| = \sqrt{50} = 5\sqrt{2}$

Writing Complex Numbers in Polar Form

The polar form of a complex number expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z = x + yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

We review these relationships in Figure 4.6.6.

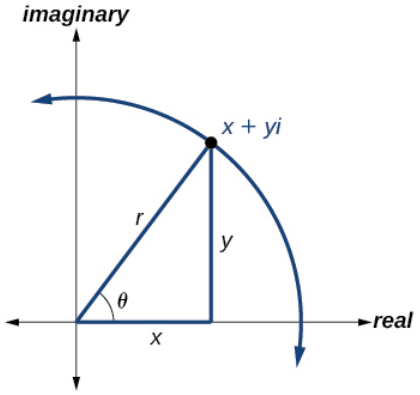

Figure 4.6.6

We use the term **modulus** to represent the absolute value of a complex number, or the distance from the origin to the point (x, y). The modulus, then, is the same as r, the radius in polar form. We use θ to indicate the angle of direction (just as with polar coordinates). Substituting, we have

$$\begin{aligned} z &= x + yi \\ z &= r\cos\theta + (r\sin\theta)i \\ z &= r(\cos\theta + i\sin\theta) \end{aligned}$$

POLAR FORM OF A COMPLEX NUMBER

Writing a complex number in polar form involves the following conversion formulas:

$$x = r\cos\theta \tag{4.6.2}$$
$$y = r\sin\theta \tag{4.6.3}$$
$$r = \sqrt{x^2+y^2} \tag{4.6.4}$$

Making a direct substitution, we have

$$z = x + yi \tag{4.6.5}$$
$$z = (r\cos\theta) + i(r\sin\theta) \tag{4.6.6}$$
$$z = r(\cos\theta + i\sin\theta) \tag{4.6.7}$$

where r is the modulus and θ is the argument. We often use the abbreviation $r\ cis\theta$ to represent $r(\cos\theta + i\sin\theta)$.

Example 4.6.4: Expressing a Complex Number Using Polar Coordinates

Express the complex number $4i$ using polar coordinates.

Solution

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. Writing it in polar form, we have to calculate r first.

$$r = \sqrt{x^2+y^2}$$
$$r = \sqrt{0^2+4^2}$$
$$r = \sqrt{16}$$
$$r = 4$$

Next, we look at x. If $x = r\cos\theta$, and $x = 0$, then $\theta = \dfrac{\pi}{2}$. In polar coordinates, the complex number $z = 0 + 4i$ can be written as $z = 4\left(\cos\left(\dfrac{\pi}{2}\right) + i\sin\left(\dfrac{\pi}{2}\right)\right)$ or $4\ cis\left(\dfrac{\pi}{2}\right)$. See Figure 4.6.7.

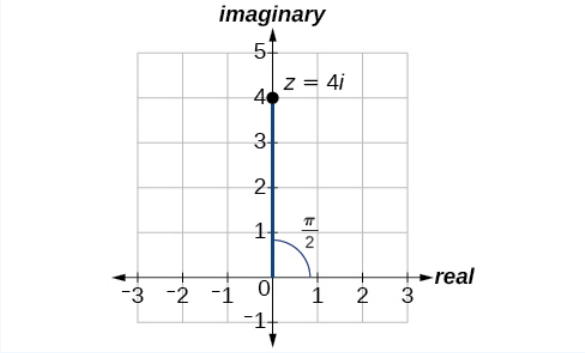

Figure 4.6.7

Exercise 4.6.4

Express $z = 3i$ as $r\ cis\theta$ in polar form.

Answer

$$z = 3\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

Example 4.6.5: Finding the Polar Form of a Complex Number

Find the polar form of $-4+4i$.

Solution

First, find the value of r.

$$r=\sqrt{x^2+y^2}$$
$$r=\sqrt{(-4)^2+(4^2)}$$
$$r=\sqrt{32}$$
$$r=4\sqrt{2}$$

Find the angle θ using the formula:

$$\cos\theta=\frac{x}{r}$$
$$\cos\theta=\frac{-4}{4\sqrt{2}}$$
$$\cos\theta=-\frac{1}{\sqrt{2}}$$
$$\theta=\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$$
$$=\frac{3\pi}{4}$$

Thus, the solution is $4\sqrt{2}\,cis\left(\frac{3\pi}{4}\right)$.

Exercise 4.6.5

Write $z=\sqrt{3}+i$ in polar form.

Answer

$$z=2\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$

Converting a Complex Number from Polar to Rectangular Form

Converting a complex number from polar form to rectangular form is a matter of evaluating what is given and using the distributive property. In other words, given $z=r(\cos\theta+i\sin\theta)$, first evaluate the trigonometric functions $\cos\theta$ and $\sin\theta$. Then, multiply through by r.

Example 4.6.6A: Converting from Polar to Rectangular Form

Convert the polar form of the given complex number to rectangular form:

$$z=12\left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)$$

Solution

We begin by evaluating the trigonometric expressions.

$$\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}\text{ and }\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

After substitution, the complex number is

$$z=12\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$

We apply the distributive property:

$$\begin{aligned} z &= 12\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right) \\ &= (12)\frac{\sqrt{3}}{2}+(12)\frac{1}{2}i \\ &= 6\sqrt{3}+6i \end{aligned}$$

The rectangular form of the given point in complex form is $6\sqrt{3}+6i$.

Example 4.6.6B: Finding the Rectangular Form of a Complex Number

Find the rectangular form of the complex number given $r=13$ and $\tan\theta=\frac{5}{12}$.

Solution

If $\tan\theta=\frac{5}{12}$, and $\tan\theta=\frac{y}{x}$, we first determine $r=\sqrt{x^2+y^2}=\sqrt{122+52}=13$. We then find $\cos\theta=\frac{x}{r}$ and $\sin\theta=\frac{y}{r}$.

$$\begin{aligned} z &= 13\left(\cos\theta+i\sin\theta\right) \\ &= 13\left(\frac{12}{13}+\frac{5}{13}i\right) \\ &= 12+5i \end{aligned}$$

The rectangular form of the given number in complex form is $12+5i$.

Exercise 4.6.6

Convert the complex number to rectangular form:

$$z=4\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$$

Answer

$z=2\sqrt{3}-2i$

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham **de Moivre** (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

PRODUCTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, then the product of these numbers is given as:

$$z_1z_2 = r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)] \quad (4.6.8)$$

$$z_1z_2 = r_1r_2\,cis(\theta_1+\theta_2) \quad (4.6.9)$$

Notice that the product calls for multiplying the moduli and adding the angles.

Example 4.6.7: Finding the Product of Two Complex Numbers in Polar Form

Find the product of $z_1 z_2$, given $z_1 = 4(\cos(80^\circ) + i\sin(80^\circ))$ and $z_2 = 2(\cos(145^\circ) + i\sin(145^\circ))$.

Solution

Follow the formula

$$\begin{aligned} z_1 z_2 &= 4 \cdot 2[\cos(80^\circ + 145^\circ) + i\sin(80^\circ + 145^\circ)] \\ z_1 z_2 &= 8[\cos(225^\circ) + i\sin(225^\circ)] \\ z_1 z_2 &= 8\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right] \\ z_1 z_2 &= 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right] \\ z_1 z_2 &= -4\sqrt{2} - 4i\sqrt{2} \end{aligned}$$

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form is the quotient of the two moduli and the difference of the two arguments.

QUOTIENTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the quotient of these numbers is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)],\ z_2 \neq 0 \tag{4.6.10}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,cis(\theta_1 - \theta_2),\ z_2 \neq 0 \tag{4.6.11}$$

Notice that the moduli are divided, and the angles are subtracted.

How to: Given two complex numbers in polar form, find the quotient

1. Divide $\frac{r_1}{r_2}$.
2. Find $\theta_1 - \theta_2$.
3. Substitute the results into the formula: $z = r(\cos\theta + i\sin\theta)$. Replace r with $\frac{r_1}{r_2}$, and replace θ with $\theta_1 - \theta_2$.
4. Calculate the new trigonometric expressions and multiply through by r.

Example 4.6.8: Finding the Quotient of Two Complex Numbers

Find the quotient of $z_1 = 2(\cos(213^\circ) + i\sin(213^\circ))$ and $z_2 = 4(\cos(33^\circ) + i\sin(33^\circ))$.

Solution

Using the formula, we have

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2}{4}[\cos(213^\circ - 33^\circ) + i\sin(213^\circ - 33^\circ)] \\ \frac{z_1}{z_2} &= \frac{1}{2}[\cos(180^\circ) + i\sin(180^\circ)] \\ \frac{z_1}{z_2} &= \frac{1}{2}[-1 + 0i] \\ \frac{z_1}{z_2} &= -\frac{1}{2} + 0i \\ \frac{z_1}{z_2} &= -\frac{1}{2} \end{aligned}$$

Exercise 4.6.8

Find the product and the quotient of $z_1 = 2\sqrt{3}(\cos(150^\circ) + i\sin(150^\circ))$ and $z_2 = 2(\cos(30^\circ) + i\sin(30^\circ))$.

Answer

$$z_1 z_2 = -4\sqrt{3}\,;\ \frac{z_1}{z_2} = -\frac{\sqrt{3}}{2} + \frac{3}{2}i$$

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is greatly simplified using **De Moivre's Theorem**. It states that, for a positive integer n, z^n is found by raising the modulus to the nth power and multiplying the argument by n. It is the standard method used in modern mathematics.

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] \tag{4.6.12}$$
$$z^n = r^n\, cis(n\theta) \tag{4.6.13}$$

where n is a positive integer.

Example 4.6.9: Evaluating an Expression Using De Moivre's Theorem

Evaluate the expression $(1+i)^5$ using De Moivre's Theorem.

Solution

Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $(1+i)$ in polar form. Let us find r.

$$r = \sqrt{x^2 + y^2}$$
$$r = \sqrt{(1)^2 + (1)^2}$$
$$r = \sqrt{2}$$

Then we find θ. Using the formula $\tan\theta = \frac{y}{x}$ gives

$$\tan\theta = \frac{1}{1}$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

Use De Moivre's Theorem to evaluate the expression.

$$(a+bi)^n = r^n[\cos(n\theta) + i\sin(n\theta)]$$
$$(1+i)^5 = (\sqrt{2})^5\left[\cos\left(5\cdot\frac{\pi}{4}\right) + i\sin\left(5\cdot\frac{\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right]$$
$$(1+i)^5 = 4\sqrt{2}\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right]$$
$$(1+i)^5 = -4 - 4i$$

Finding Roots of Complex Numbers in Polar Form

To find the nth root of a complex number in polar form, we use the nth Root Theorem or De Moivre's Theorem and raise the complex number to a power with a rational exponent. There are several ways to represent a formula for finding n^{th} roots of complex numbers in polar form.

THE N^{TH} ROOT THEOREM

To find the nth root of a complex number in polar form, use the formula given as

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right] \tag{4.6.14}$$

where $k = 0, 1, 2, 3, \ldots, n-1$. We add $\dfrac{2k\pi}{n}$ to $\dfrac{\theta}{n}$ in order to obtain the periodic roots.

th Root of a Complex Number

Evaluate the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)$.

Solution

We have

$$z^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)\right]$$
$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)\right]$$

There will be three roots: $k = 0, 1, 2$. When $k = 0$, we have

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)$$

When $k = 1$, we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)\right] \qquad \text{Add } \frac{2(1)\pi}{3} \text{ to each angle.}$$
$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{8\pi}{9}\right)+i\sin\left(\frac{8\pi}{9}\right)\right)$$

When $k = 2$, we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)\right] \qquad \text{Add } \frac{2(2)\pi}{3} \text{ to each angle.}$$
$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{14\pi}{9}\right)+i\sin\left(\frac{14\pi}{9}\right)\right)$$

Remember to find the common denominator to simplify fractions in situations like this one. For $k = 1$, the angle simplification is

$$\frac{\frac{2\pi}{3}}{3}+\frac{2(1)\pi}{3} = \frac{2\pi}{3}\left(\frac{1}{3}\right)+\frac{2(1)\pi}{3}\left(\frac{3}{3}\right)$$
$$= \frac{2\pi}{9}+\frac{6\pi}{9}$$
$$= \frac{8\pi}{9}$$

Exercise 4.6.10

Find the four fourth roots of $16(\cos(120^\circ)+i\sin(120^\circ))$.

Answer

$z_0 = 2(\cos(30^\circ)+i\sin(30^\circ))$

$z_1 = 2(\cos(120^\circ)+i\sin(120^\circ))$

$z_2 = 2(\cos(210^\circ)+i\sin(210^\circ))$

$z_3 = 2(\cos(300^\circ)+i\sin(300^\circ))$

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Access these online resources for additional instruction and practice with polar forms of complex numbers.

- The Product and Quotient of Complex Numbers in Trigonometric Form
- De Moivre's Theorem

Key Concepts

- Complex numbers in the form $a+bi$ are plotted in the complex plane similar to the way rectangular coordinates are plotted in the rectangular plane. Label the x-axis as the real axis and the y-axis as the *imaginary* axis. See Example 4.6.1.
- The absolute value of a complex number is the same as its magnitude. It is the distance from the origin to the point: $|z| = \sqrt{a^2+b^2}$. See Example 4.6.2 and Example 4.6.3.
- To write complex numbers in polar form, we use the formulas $x = r\cos\theta$, $y = r\sin\theta$, and $r = \sqrt{x^2+y^2}$. Then, $z = r(\cos\theta+i\sin\theta)$. See Example 4.6.4 and Example 4.6.5.
- To convert from polar form to rectangular form, first evaluate the trigonometric functions. Then, multiply through by r. See Example 4.6.6 and Example 4.6.7.
- To find the product of two complex numbers, multiply the two moduli and add the two angles. Evaluate the trigonometric functions, and multiply using the distributive property. See Example 4.6.8.
- To find the quotient of two complex numbers in polar form, find the quotient of the two moduli and the difference of the two angles. See Example 4.6.9.
- To find the power of a complex number z^n, raise r to the power n, and multiply θ by n. See Example 4.6.10.
- Finding the roots of a complex number is the same as raising a complex number to a power, but using a rational exponent. See Example 4.6.11.

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4.7: Parametric Equations

Learning Objectives

- Parameterize a curve.
- Eliminate the parameter.
- Find a rectangular equation for a curve defined parametrically.
- Find parametric equations for curves defined by rectangular equations.

Consider the path a moon follows as it orbits a planet, which simultaneously rotates around the sun, as seen in Figure 4.7.1. At any moment, the moon is located at a particular spot relative to the planet. But how do we write and solve the equation for the position of the moon when the distance from the planet, the speed of the moon's orbit around the planet, and the speed of rotation around the sun are all unknowns? We can solve only for one variable at a time.

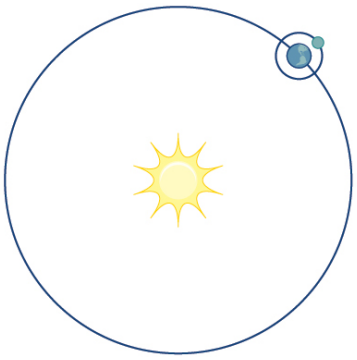


Figure 4.7.1

In this section, we will consider sets of equations given by $x(t)$ and $y(t)$ where t is the independent variable of time. We can use these parametric equations in a number of applications when we are looking for not only a particular position but also the direction of the movement. As we trace out successive values of t, the orientation of the curve becomes clear. This is one of the primary advantages of using parametric equations: we are able to trace the movement of an object along a path according to time. We begin this section with a look at the basic components of parametric equations and what it means to parameterize a curve. Then we will learn how to eliminate the parameter, translate the equations of a curve defined parametrically into rectangular equations, and find the parametric equations for curves defined by rectangular equations.

Parameterizing a Curve

When an object moves along a curve—or curvilinear path—in a given direction and in a given amount of time, the position of the object in the plane is given by the x-coordinate and the y-coordinate. However, both x and y vary over time and so are functions of time. For this reason, we add another variable, the parameter, upon which both x and y are dependent functions. In the example in the section opener, the parameter is time, t. The x position of the moon at time, t, is represented as the function $x(t)$, and the y position of the moon at time, t, is represented as the function $y(t)$. Together, $x(t)$ and $y(t)$ are called parametric equations, and generate an ordered pair $(x(t), y(t))$. Parametric equations primarily describe motion and direction.

When we parameterize a curve, we are translating a single equation in two variables, such as x and y,into an equivalent pair of equations in three variables, x, y, and t. One of the reasons we parameterize a curve is because the parametric equations yield more information: specifically, the direction of the object's motion over time.

When we graph parametric equations, we can observe the individual behaviors of x and of y. There are a number of shapes that cannot be represented in the form $y = f(x)$, meaning that they are not functions. For example, consider the graph of a circle, given as $r^2 = x^2 + y^2$. Solving for y gives $y = \pm\sqrt{r^2 - x^2}$, or two equations: $y_1 = \sqrt{r^2 - x^2}$ and $y_2 = -\sqrt{r^2 - x^2}$. If we graph y_1 and y_2 together, the graph will not pass the vertical line test, as shown in Figure 4.7.2. Thus, the equation for the graph of a circle is not a function.

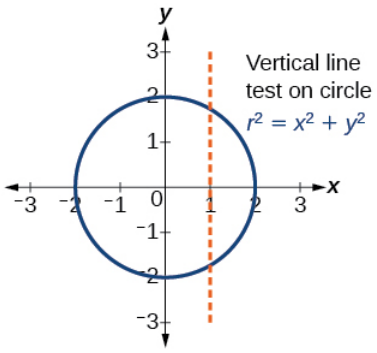


Figure 4.7.2

However, if we were to graph each equation on its own, each one would pass the vertical line test and therefore would represent a function. In some instances, the concept of breaking up the equation for a circle into two functions is similar to the concept of creating parametric equations, as we use two functions to produce a non-function. This will become clearer as we move forward.

PARAMETRIC EQUATIONS

Suppose t is a number on an interval, I. The set of ordered pairs, $(x(t), y(t))$, where $x = f(t)$ and $y = g(t)$,forms a plane curve based on the parameter t. The equations $x = f(t)$ and $y = g(t)$ are the parametric equations.

Example 4.7.1: Parameterizing a Curve

Parameterize the curve $y = x^2 - 1$ letting $x(t) = t$. Graph both equations.

Solution

If $x(t) = t$, then to find $y(t)$ we replace the variable x with the expression given in $x(t)$. In other words, $y(t) = t^2 - 1$ .Make a table of values similar to Table 4.7.1, and sketch the graph.

Table 4.7.1

t	$x(t)$	$y(t)$
-4	-4	$y(-4) = (-4)^2 - 1 = 15$
-3	-3	$y(-3) = (-3)^2 - 1 = 8$
-2	-2	$y(-2) = (-2)^2 - 1 = 3$
-1	-1	$y(-1) = (-1)^2 - 1 = 0$
0	0	$y(0) = (0)^2 - 1 = -1$
1	1	$y(1) = (1)^2 - 1 = 0$
2	2	$y(2) = (2)^2 - 1 = 3$
3	3	$y(3) = (3)^2 - 1 = 8$
4	4	$y(4) = (4)^2 - 1 = 15$

See the graphs in Figure $4.7.3$. It may be helpful to use the TRACE feature of a graphing calculator to see how the points are generated as t increases.

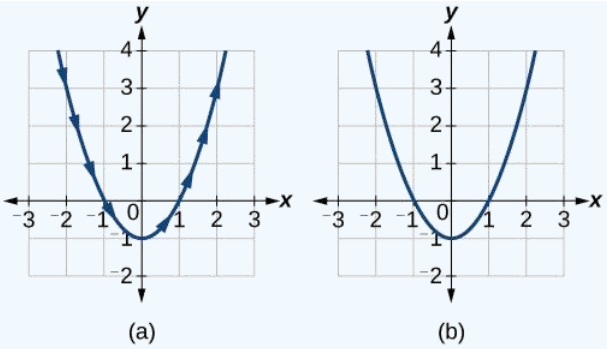


Figure 4.7.3: (a) Parametric $y(t) = t^2 - 1$ (b) Rectangular $y = x^2 - 1$

Analysis

The arrows indicate the direction in which the curve is generated. Notice the curve is identical to the curve of $y = x^2 - 1$.

Exercise 4.7.1

Construct a table of values and plot the parametric equations: $x(t) = t - 3$, $y(t) = 2t + 4$; $-1 \leq t \leq 2$.

Answer

t	$x(t)$	$y(t)$
-1	-4	2
0	-3	4
1	-2	6
2	-1	8

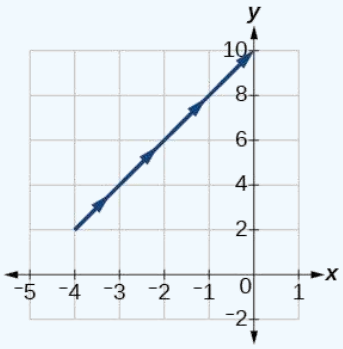


Figure 4.7.4

Example 4.7.2: Finding a Pair of Parametric Equations

Find a pair of parametric equations that models the graph of $y = 1 - x^2$, using the parameter $x(t) = t$. Plot some points and sketch the graph.

Solution

If $x(t) = t$ and we substitute t for x into the y equation, then $y(t) = 1 - t^2$. Our pair of parametric equations is

$$x(t) = t$$
$$y(t) = 1 - t^2$$

To graph the equations, first we construct a table of values like that in Table 4.7.2. We can choose values around $t = 0$, from $t = -3$ to $t = 3$. The values in the $x(t)$ column will be the same as those in the t column because $x(t) = t$. Calculate values

for the column $y(t)$.

Table 4.7.2

\(t)	$x(t)=t$	$y(t)=1-t^2$
-3	-3	$y(-3)=1-(-3)^2=-8$
-2	-2	$y(-2)=1-(-2)^2=-3$
-1	-1	$y(-1)=1-(-1)^2=0$
0	0	$y(0)=1-0=1$
1	1	$y(1)=1-(1)^2=0$
2	2	$y(2)=1-(2)^2=-3$
3	3	$y(3)=1-(3)^2=-8$

The graph of $y=1-t^2$ is a parabola facing downward, as shown in Figure 4.7.5. We have mapped the curve over the interval $[-3,3]$, shown as a solid line with arrows indicating the orientation of the curve according to t. Orientation refers to the path traced along the curve in terms of increasing values of t. As this parabola is symmetric with respect to the line $x=0$, the values of x are reflected across the *y*-axis.

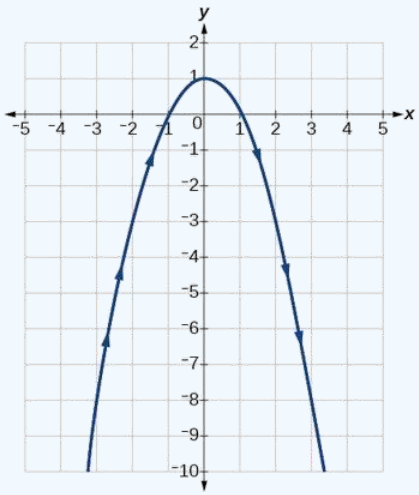


Figure 4.7.5

? Exercise 4.7.2

Parameterize the curve given by $x=y^3-2y$.

Answer

$x(t)=t^3-2t$

$y(t)=t$

✓ Example 4.7.3: Finding Parametric Equations That Model Given Criteria

An object travels at a steady rate along a straight path $(-5,3)$ to $(3,-1)$ in the same plane in four seconds. The coordinates are measured in meters. Find parametric equations for the position of the object.

Solution

The parametric equations are simple linear expressions, but we need to view this problem in a step-by-step fashion. The *x*-value of the object starts at -5 meters and goes to 3 meters. This means the distance x has changed by 8 meters in 4 seconds, which

is a rate of $\frac{8\ m}{4\ s}$, or $2\ m/s$. We can write the x-coordinate as a linear function with respect to time as $x(t) = 2t - 5$. In the linear function template $y = mx + b$, $2t = mx$ and $-5 = b$.

Similarly, the y-value of the object starts at 3 and goes to -1, which is a change in the distance y of -4 meters in 4 seconds, which is a rate of $\frac{-4\ m}{4\ s}$, or $-1\ m/s$. We can also write the y-coordinate as the linear function $y(t) = -t + 3$. Together, these are the parametric equations for the position of the object, where x and y are expressed in meters and t represents time:

$$\begin{aligned} x(t) &= 2t - 5 \\ y(t) &= -t + 3 \end{aligned}$$

Using these equations, we can build a table of values for t, x, and y (see Table 4.7.3). In this example, we limited values of t to non-negative numbers. In general, any value of t can be used.

Table 4.7.3

t	$x(t) = 2t - 5$	$y(t) = -t + 3$
0	$x = 2(0) - 5 = -5$	$y = -(0) + 3 = 3$
1	$x = 2(1) - 5 = -3$	$y = -(1) + 3 = 2$
2	$x = 2(2) - 5 = -1$	$y = -(2) + 3 = 1$
3	$x = 2(3) - 5 = 1$	$y = -(3) + 3 = 0$
4	$x = 2(4) - 5 = 3$	$y = -(4) + 3 = -1$

From this table, we can create three graphs, as shown in Figure 4.7.6.

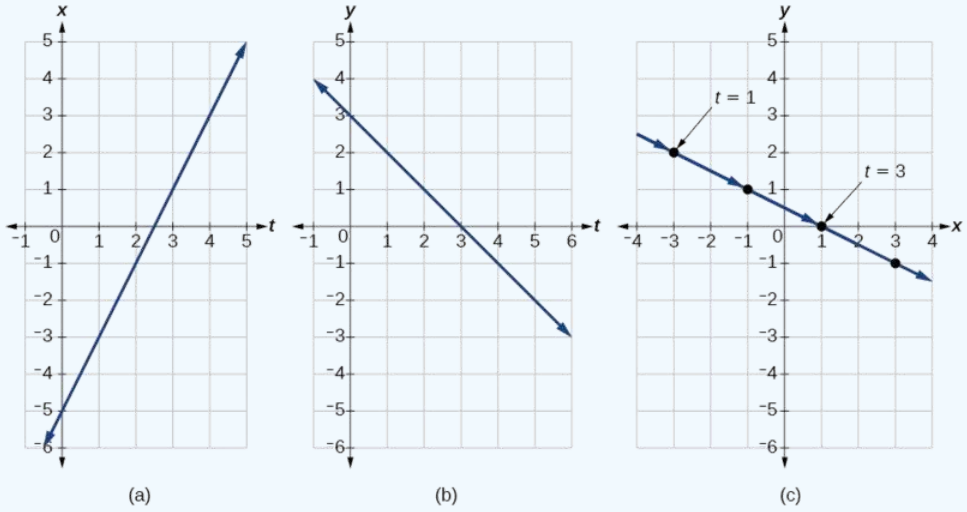


Figure 4.7.6 : (a) A graph of x vs. t, representing the horizontal position over time. (b) A graph of y vs. t, representing the vertical position over time. (c) A graph of y vs. x, representing the position of the object in the plane at time t.

Analysis

Again, we see that, in Figure 4.7.6 (c), when the parameter represents time, we can indicate the movement of the object along the path with arrows.

Eliminating the Parameter

In many cases, we may have a pair of parametric equations but find that it is simpler to draw a curve if the equation involves only two variables, such as x and y. Eliminating the parameter is a method that may make graphing some curves easier. However, if we are concerned with the mapping of the equation according to time, then it will be necessary to indicate the orientation of the curve as well. There are various methods for eliminating the parameter t from a set of parametric equations; not every method works for every type of equation. Here we will review the methods for the most common types of equations.

Eliminating the Parameter from Polynomial, Exponential, and Logarithmic Equations

For polynomial, exponential, or logarithmic equations expressed as two parametric equations, we choose the equation that is most easily manipulated and solve for t. We substitute the resulting expression for t into the second equation. This gives one equation in x and y.

Example 4.7.4: Eliminating the Parameter in Polynomials

Given $x(t)=t^2+1$ and $y(t)=2+t$, eliminate the parameter, and write the parametric equations as a Cartesian equation.

Solution

We will begin with the equation for y because the linear equation is easier to solve for t.

$$\begin{aligned} y &= 2+t \\ y-2 &= t \end{aligned}$$

Next, substitute $y-2$ for t in $x(t)$.

$$\begin{aligned} x &= t^2+1 \\ x &= (y-2)^2+1 \qquad \text{Substitute the expression for } t \text{ into } x. \\ x &= y^2-4y+4+1 \\ x &= y^2-4y+5 \\ x &= y^2-4y+5 \end{aligned}$$

The Cartesian form is $x=y^2-4y+5$.

Analysis

This is an equation for a parabola in which, in rectangular terms, x is dependent on y. From the curve's vertex at $(1,2)$, the graph sweeps out to the right. See Figure 4.7.7. In this section, we consider sets of equations given by the functions $x(t)$ and $y(t)$, where t is the independent variable of time. Notice, both x and y are functions of time; so in general y is not a function of x.

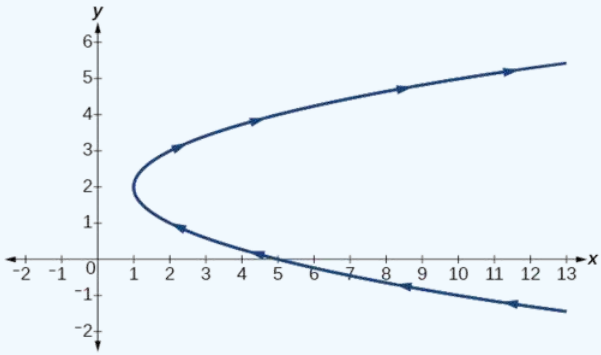


Figure 4.7.7

Exercise 4.7.3

Given the equations below, eliminate the parameter and write as a rectangular equation for y as a function of x.

$$\begin{aligned} x(t) &= 2t^2+6 \\ y(t) &= 5-t \end{aligned}$$

Answer

$$y=5-\sqrt{\tfrac{1}{2}x-3}$$

Example 4.7.5: Eliminating the Parameter in Exponential Equations

Eliminate the parameter and write as a Cartesian equation: $x(t) = e^{-t}$ and $y(t) = 3e^{t}, t > 0$.

Solution

Isolate e^{t}.

$$x = e^{-t}$$
$$e^{t} = \frac{1}{x}$$

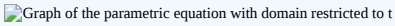
Substitute the expression into $y(t)$.

$$y = 3e^{t}$$
$$y = 3\left(\frac{1}{x}\right)$$
$$y = \frac{3}{x}$$

The Cartesian form is $y = \frac{3}{x}$.

Analysis

The graph of the parametric equation is shown in Figure $4.7.8a$. The domain is restricted to $t > 0$. The Cartesian equation, $y = \frac{3}{x}$ is shown in Figure $4.7.8b$ and has only one restriction on the domain, $x \neq 0$.



0, and a graph of that parametric equation in polar coordinates with domain only restricted to x not equal to 0. The Cartesian coordinate version has an extra reflection of the function across the origin in Q 3 (original was just in Q 1). " src="/@api/deki/files/12489/imageedit_21_3487926125.png">

Figure 4.7.8

Example 4.7.6: Eliminating the Parameter in Logarithmic Equations

Eliminate the parameter and write as a Cartesian equation: $x(t) = \sqrt{t} + 2$ and $y(t) = \log(t)$.

Solution

Solve the first equation for t.

$$\begin{aligned} x &= \sqrt{t} + 2 \\ x - 2 &= \sqrt{t} \\ (x-2)^2 &= t \qquad \text{Square both sides.} \end{aligned}$$

Then, substitute the expression for t into the y equation.

$$y = \log(t)$$
$$y = \log\left(x-2\right)^2$$

The Cartesian form is $y = \log\left(x-2\right)^2$.

Analysis

To be sure that the parametric equations are equivalent to the Cartesian equation, check the domains. The parametric equations restrict the domain on $x = \sqrt{t} + 2$ to $t > 0$; we restrict the domain on x to $x > 2$. The domain for the parametric equation $y = \log(t)$ is restricted to $t > 0$; we limit the domain on $y = \log\left(x-2\right)^2$ to $x > 2$.

Exercise 4.7.4

Eliminate the parameter and write as a **rectangular equation.**

$$x(t) = t^2$$
$$y(t) = \ln t,\ t > 0$$

Answer

$y = \ln \sqrt{x}$

Eliminating the Parameter from Trigonometric Equations

Eliminating the parameter from trigonometric equations is a straightforward substitution. We can use a few of the familiar trigonometric identities and the Pythagorean Theorem.

First, we use the identities:

$$x(t) = a\cos t$$
$$y(t) = b\sin t$$

Solving for $\cos t$ and $\sin t$, we have

$$\frac{x}{a} = \cos t$$
$$\frac{y}{b} = \sin t$$

Then, use the Pythagorean Theorem:

$$\cos^2 t + \sin^2 t = 1$$

Substituting gives

$$\cos^2 t + \sin^2 t = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

Example 4.7.7: Eliminating the Parameter from a Pair of Trigonometric Parametric Equations

Eliminate the parameter from the given pair of **trigonometric equations** where $0 \le t \le 2\pi$ and sketch the graph.

$$x(t) = 4\cos t$$
$$y(t) = 3\sin t$$

Solution

Solving for $\cos t$ and $\sin t$, we have

$$x = 4\cos t$$
$$\frac{x}{4} = \cos t$$
$$y = 3\sin t$$
$$\frac{y}{3} = \sin t$$

Next, use the Pythagorean identity and make the substitutions.

$$\cos^2 t + \sin^2 t = 1$$
$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The graph for the equation is shown in Figure 4.7.9.

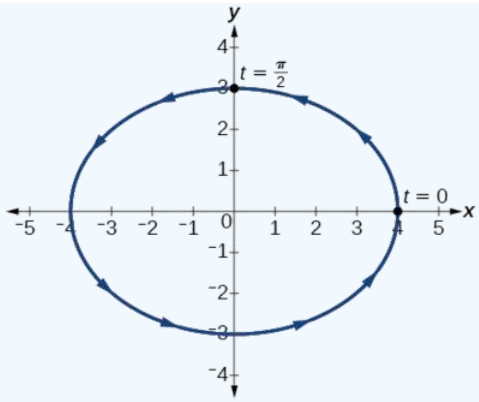


Figure 4.7.9

Analysis

Applying the general equations for conic sections (introduced in Analytic Geometry, we can identify $\frac{x^2}{16}+\frac{y^2}{9}=1$ as an ellipse centered at $(0,0)$. Notice that when $t=0$ the coordinates are $(4,0)$, and when $t=\frac{\pi}{2}$ the coordinates are $(0,3)$. This shows the orientation of the curve with increasing values of t.

? Exercise 4.7.5

Eliminate the parameter from the given pair of parametric equations and write as a Cartesian equation: $x(t)=2\cos t$ and $y(t)=3\sin t$.

Answer

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

Finding Cartesian Equations from Curves Defined Parametrically

When we are given a set of parametric equations and need to find an equivalent Cartesian equation, we are essentially "eliminating the parameter." However, there are various methods we can use to rewrite a set of parametric equations as a Cartesian equation. The simplest method is to set one equation equal to the parameter, such as $x(t)=t$. In this case, $y(t)$ can be any expression. For example, consider the following pair of equations.

$$\begin{aligned} x(t) &= t \\ y(t) &= t^2-3 \end{aligned}$$

Rewriting this set of parametric equations is a matter of substituting x for t. Thus, the Cartesian equation is $y=x^2-3$.

✓ Example 4.7.8: Finding a Cartesian Equation Using Alternate Methods

Use two different methods to find the Cartesian equation equivalent to the given set of parametric equations.

$$\begin{aligned} x(t) &= 3t-2 \\ y(t) &= t+1 \end{aligned}$$

Solution

Method 1. First, let's solve the x equation for t. Then we can substitute the result into the y equation.

$$\begin{aligned} x &= 3t-2 \\ x+2 &= 3t \\ \frac{x+2}{3} &= t \end{aligned}$$

Now substitute the expression for t into the y equation.

$$y = t+1$$
$$y = \left(\frac{x+2}{3}\right)+1$$
$$y = \frac{x}{3}+\frac{2}{3}+1$$
$$y = \frac{1}{3}x+\frac{5}{3}$$

Method 2. Solve the y equation for t and substitute this expression in the x equation.

$$y = t+1$$
$$y-1 = t$$

Make the substitution and then solve for y.

$$x = 3(y-1)-2$$
$$x = 3y-3-2$$
$$x = 3y-5$$
$$x+5 = 3y$$
$$\frac{x+5}{3} = y$$
$$y = \frac{1}{3}x+\frac{5}{3}$$

? Exercise 4.7.6

Write the given parametric equations as a Cartesian equation: $x(t) = t^3$ and $y(t) = t^6$.

Answer

$y = x^2$

Finding Parametric Equations for Curves Defined by Rectangular Equations

Although we have just shown that there is only one way to interpret a set of parametric equations as a rectangular equation, there are multiple ways to interpret a rectangular equation as a set of parametric equations. Any strategy we may use to find the parametric equations is valid if it produces equivalency. In other words, if we choose an expression to represent x, and then substitute it into the y equation, and it produces the same graph over the same domain as the rectangular equation, then the set of parametric equations is valid. If the domain becomes restricted in the set of parametric equations, and the function does not allow the same values for x as the domain of the rectangular equation, then the graphs will be different.

✓ Example 4.7.9: Finding a Set of Parametric Equations for Curves Defined by Rectangular Equations

Find a set of equivalent parametric equations for $y = (x+3)^2+1$.

Solution

An obvious choice would be to let $x(t) = t$. Then $y(t) = (t+3)^2+1$. But let's try something more interesting. What if we let $x = t+3$? Then we have

$$y = (x+3)^2+1$$
$$y = ((t+3)+3)^2+1$$
$$y = (t+6)^2+1$$

The set of parametric equations is

$$x(t) = t+3$$
$$y(t) = (t+6)^2+1$$

See Figure 4.7.10.

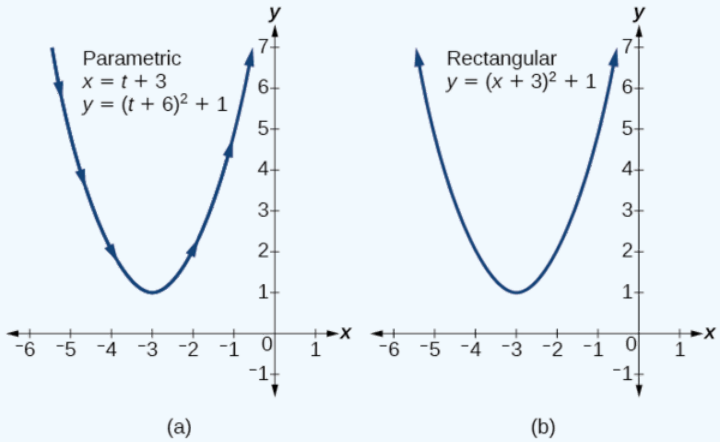


Figure 4.7.10

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Access these online resources for additional instruction and practice with parametric equations.

- Introduction to Parametric Equations
- Converting Parametric Equations to Rectangular Form

Key Concepts

- Parameterizing a curve involves translating a rectangular equation in two variables, x and y, into two equations in three variables, x, y, and t. Often, more information is obtained from a set of parametric equations. See Example 4.7.1, Example 4.7.2, and Example 4.7.3.
- Sometimes equations are simpler to graph when written in rectangular form. By eliminating t, an equation in x and y is the result.
- To eliminate t, solve one of the equations for t, and substitute the expression into the second equation. See Example 4.7.4, Example 4.7.5, Example 4.7.6, and Example 4.7.7.
- Finding the rectangular equation for a curve defined parametrically is basically the same as eliminating the parameter. Solve for t in one of the equations, and substitute the expression into the second equation. See Example 4.7.8.
- There are an infinite number of ways to choose a set of parametric equations for a curve defined as a rectangular equation.
- Find an expression for x such that the domain of the set of parametric equations remains the same as the original rectangular equation. See Example 4.7.9.

Contributors

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4.8: Parametric Equations - Graphs

Learning Objectives

- Graph plane curves described by parametric equations by plotting points.
- Graph parametric equations.

It is the bottom of the ninth inning, with two outs and two men on base. The home team is losing by two runs. The batter swings and hits the baseball at 140 feet per second and at an angle of approximately 45° to the horizontal. How far will the ball travel? Will it clear the fence for a game-winning home run? The outcome may depend partly on other factors (for example, the wind), but mathematicians can model the path of a projectile and predict approximately how far it will travel using parametric equations. In this section, we'll discuss parametric equations and some common applications, such as projectile motion problems.

Figure 4.8.1: Parametric equations can model the path of a projectile. (credit: Paul Kreher, Flickr)

Graphing Parametric Equations by Plotting Points

In lieu of a graphing calculator or a computer graphing program, plotting points to represent the graph of an equation is the standard method. As long as we are careful in calculating the values, point-plotting is highly dependable.

How to: Given a pair of parametric equations, sketch a graph by plotting points

1. Construct a table with three columns: t, $x(t)$,and $y(t)$.
2. Evaluate x and y for values of tt over the interval for which the functions are defined.
3. Plot the resulting pairs (x, y).

Example 4.8.1: Sketching the Graph of a Pair of Parametric Equations by Plotting Points

Sketch the graph of the parametric equations $x(t) = t^2 + 1$, $y(t) = 2 + t$.

Solution

Construct a table of values for t, $x(t)$, and $y(t)$, as in Table 4.8.1, and plot the points in a plane.

Table 4.8.1

t	$x(t) = t^2 + 1$	$y(t) = 2 + t$
-5	26	-3
-4	17	-2
-3	10	-1
-2	5	0
-1	2	1
0	1	2
1	2	3
2	5	4
3	10	5
4	17	6
5	26	7

The graph is a parabola with vertex at the point $(1, 2)$,opening to the right. See Figure 4.8.2.

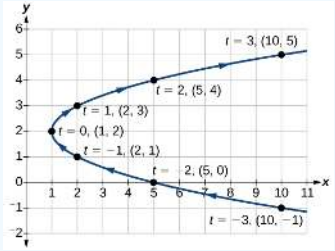

Figure 4.8.2

Analysis

As values for t progress in a positive direction from 0 to 5, the plotted points trace out the top half of the parabola. As values oft t become negative, they trace out the lower half of the parabola. There are no restrictions on the domain. The arrows indicate direction according to increasing values of t. The graph does not represent a function, as it will fail the vertical line test. The graph is drawn in two parts: the positive values for t, and the negative values for t.

? Exercise 4.8.1

Sketch the graph of the parametric equations $x = \sqrt{t}$, $y = 2t + 3$, $0 \le t \le 3$.

Answer

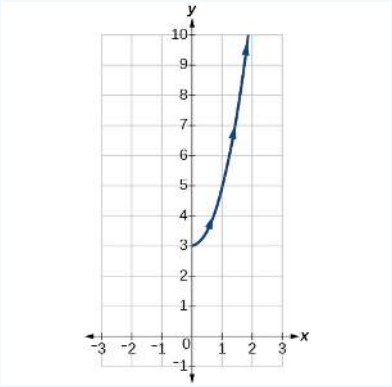

Figure 4.8.3

✓ Example 4.8.2: Sketching the Graph of Trigonometric Parametric Equations

Construct a table of values for the given parametric equations and sketch the graph:

$$x = 2\cos t$$
$$y = 4\sin t$$

Solution

Construct a table like that in Table 4.8.2 using angle measure in radians as inputs for t, and evaluating x and y. Using angles with known sine and cosine values for t makes calculations easier.

Table 4.8.2

t	$x = 2\cos t$	$y = 4\sin t$
0	$x = 2\cos(0) = 2$	$y = 4\sin(0) = 0$
$\frac{\pi}{6}$	$x = 2\cos(\frac{\pi}{6}) = \sqrt{3}$	$y = 4\sin(\frac{\pi}{6}) = 2$
$\frac{\pi}{3}$	$x = 2\cos(\frac{\pi}{3}) = 1$	$y = 4\sin(\frac{\pi}{3}) = 2\sqrt{3}$
$\frac{\pi}{2}$	$x = 2\cos(\frac{\pi}{2}) = 0$	$y = 4\sin(\frac{\pi}{2}) = 4$
$\frac{2\pi}{3}$	$x = 2\cos(\frac{2\pi}{3}) = -1$	$y = 4\sin(\frac{2\pi}{3}) = 2\sqrt{3}$
$\frac{5\pi}{6}$	$x = 2\cos(\frac{5\pi}{6}) = -\sqrt{3}$	$y = 4\sin(\frac{5\pi}{6}) = 2$
π	$x = 2\cos(\pi) = -2$	$y = 4\sin(\pi) = 0$
$\frac{7\pi}{6}$	$x = 2\cos(\frac{7\pi}{6}) = -\sqrt{3}$	$y = 4\sin(\frac{7\pi}{6}) = -2$
$\frac{4\pi}{3}$	$x = 2\cos(\frac{4\pi}{3}) = -1$	$y = 4\sin(\frac{4\pi}{3}) = -2\sqrt{3}$
$\frac{3\pi}{2}$	$x = 2\cos(\frac{3\pi}{2}) = 0$	$y = 4\sin(\frac{3\pi}{2}) = -4$
$\frac{5\pi}{3}$	$x = 2\cos(\frac{5\pi}{3}) = 1$	$y = 4\sin(\frac{5\pi}{3}) = -2\sqrt{3}$
$\frac{11\pi}{6}$	$x = 2\cos(\frac{11\pi}{6}) = \sqrt{3}$	$y = 4\sin(\frac{11\pi}{6}) = -2$
2π	$x = 2\cos(2\pi) = 2$	$y = 4\sin(2\pi) = 0$

Figure 4.8.4 shows the graph.

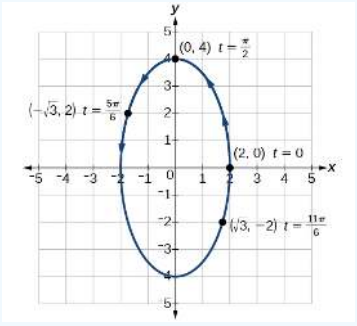

Figure 4.8.4

https://math.libretexts.org/@go/page/126600

By the symmetry shown in the values of x and y, we see that the parametric equations represent an ellipse. The ellipse is mapped in a counterclockwise direction as shown by the arrows indicating increasing t values.

Analysis

We have seen that parametric equations can be graphed by plotting points. However, a graphing calculator will save some time and reveal nuances in a graph that may be too tedious to discover using only hand calculations. Make sure to change the mode on the calculator to parametric (PAR). To confirm, the $Y=$ window should show

$$X_{1T} =$$
$$Y_{1T} =$$

instead of $Y_1 =$.

? Exercise 4.8.2

Graph the parametric equations: $x = 5\cos t$, $y = 3\sin t$.

Answer

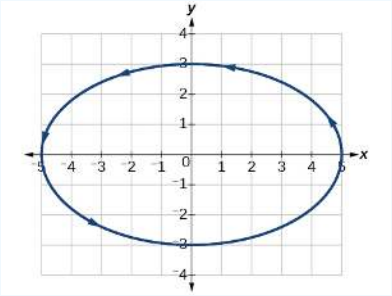

Figure 4.8.5

✓ Example 4.8.3: Graphing Parametric Equations and Rectangular Form Together

Graph the parametric equations $x = 5\cos t$ and $y = 2\sin t$. First, construct the graph using data points generated from the parametric form. Then graph the rectangular form of the equation. Compare the two graphs.

Solution

Construct a table of values like that in Table 4.8.3.

Table 4.8.3

t	$x = 5\cos t$	$y = 2\sin t$
0	$x = 5\cos(0) = 5$	$y = 2\sin(0) = 0$
1	$x = 5\cos(1) \approx 2.7$	$y = 2\sin(1) \approx 1.7$
2	$x = 5\cos(2) \approx -2.1$	$y = 2\sin(2) \approx 1.8$
3	$x = 5\cos(3) \approx -4.95$	$y = 2\sin(3) \approx 0.28$
4	$x = 5\cos(4) \approx -3.3$	$y = 2\sin(4) \approx -1.5$
5	$x = 5\cos(5) \approx 1.4$	$y = 2\sin(5) \approx -1.9$
-1	$x = 5\cos(-1) \approx 2.7$	$y = 2\sin(-1) \approx -1.$
-2	$x = 5\cos(-2) \approx -2.1$	$y = 2\sin(-2) \approx -1.$
-3	$x = 5\cos(-3) \approx -4.95$	$y = 2\sin(-3) \approx -0.$
-4	$x = 5\cos(-4) \approx -3.3$	$y = 2\sin(-4) \approx 1.5$
-5	$x = 5\cos(-5) \approx 1.4$	$y = 2\sin(-5) \approx 1.9$

Plot the (x, y) values from the table (Figure 4.8.6).

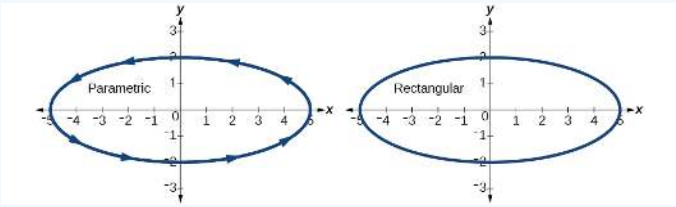

Figure 4.8.6

Next, translate the parametric equations to rectangular form. To do this, we solve for t in either $x(t)$ or $y(t)$, and then substitute the expression for t in the other equation. The result will be a function $y(x)$ if solving for t as a function of x, or $x(y)$ if solving for t as a function of y.

$$x = 5\cos t$$
$$\frac{x}{5} = \cos t$$

Solve for $\cos t$.

$$y = 2\sin t$$

Solve for $\sin t$.

$$\frac{y}{2} = \sin t$$

Then, use the Pythagorean Theorem.

$$\begin{aligned} \cos^2 t + \sin^2 t &= 1 \\ \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 &= 1 \\ \frac{x^2}{25} + \frac{y^2}{4} &= 1 \end{aligned}$$

Analysis

In Figure 4.8.7, the data from the parametric equations and the rectangular equation are plotted together. The parametric equations are plotted in blue; the graph for the rectangular equation is drawn on top of the parametric in a dashed style colored red. Clearly, both forms produce the same graph.

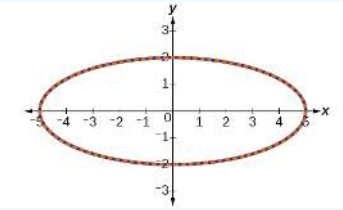

Figure 4.8.7

Example 4.8.4: Graphing Parametric Equations and Rectangular Equations on the Coordinate System

Graph the parametric equations $x = t + 1$ and $y = \sqrt{t}, t \geq 0$, and the rectangular equivalent $y = \sqrt{x-1}$ on the same coordinate system.

Solution

Construct a table of values for the parametric equations, as we did in the previous example, and graph $y = \sqrt{t}, t \geq 0$ on the same grid, as in Figure 4.8.8.

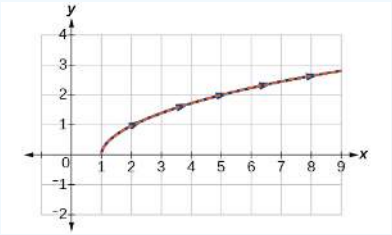

Figure 4.8.8

Analysis

With the domain on t restricted, we only plot positive values of t. The parametric data is graphed in blue and the graph of the rectangular equation is dashed in red. Once again, we see that the two forms overlap.

Exercise 4.8.3

Sketch the graph of the parametric equations $x = 2\cos\theta$ and $y = 4\sin\theta$, along with the rectangular equation on the same grid.

Answer

The graph of the parametric equations is in red and the graph of the rectangular equation is drawn in blue dots on top of the parametric equations.

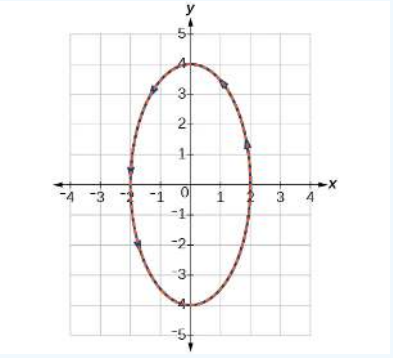

Figure 4.8.9

Applications of Parametric Equations

Many of the advantages of parametric equations become obvious when applied to solving real-world problems. Although rectangular equations in x and y give an overall picture of an object's path, they do not reveal the position of an object at a specific time. Parametric equations, however, illustrate how the values of x and y change depending on t, as the location of a moving object at a particular time.

A common application of parametric equations is solving problems involving projectile motion. In this type of motion, an object is propelled forward in an upward direction forming an angle of θ to the horizontal, with an initial speed of v_0, and at a height h above the horizontal.

The path of an object propelled at an inclination of θ to the horizontal, with initial speed v_0, and at a height h above the horizontal, is given by

$$x = (v_0 \cos\theta)t$$
$$y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + h$$

where g accounts for the effects of **gravity** and h is the initial height of the object. Depending on the units involved in the problem, use $g = 32ft/s^2$ or $g = 9.8m/s^2$. The equation for x gives horizontal distance, and the equation for y gives the vertical distance.

How to: Given a projectile motion problem, use parametric equations to solve.

1. The horizontal distance is given by $x = (v_0 \cos\theta)t$. Substitute the initial speed of the object for v_0.
2. The expression $\cos\theta$ indicates the angle at which the object is propelled. Substitute that angle in degrees for $\cos\theta$.
3. The vertical distance is given by the formula $y = -\frac{1}{2}gt^2 + (v_0 \sin\theta)t + h$. The term $-\frac{1}{2}gt^2$ represents the effect of gravity. Depending on units involved, use $g = 32ft/s^2$ or $g = 9.8m/s^2$. Again, substitute the initial speed for v_0, and the height at which the object was propelled for h.
4. Proceed by calculating each term to solve for t.

Example 4.8.5: Finding the Parametric Equations to Describe the Motion of a Baseball

Solve the problem presented at the beginning of this section. Does the batter hit the game-winning home run? Assume that the ball is hit with an initial velocity of 140 feet per second at an angle of 45° to the horizontal, making contact 3 feet above the ground.

1. Find the parametric equations to model the path of the baseball.
2. Where is the ball after 2 seconds?
3. How long is the ball in the air?
4. Is it a home run?

Solution

1. Use the formulas to set up the equations. The horizontal position is found using the parametric equation for x. Thus,

$$x = (v_0 \cos\theta)t$$
$$x = (140\cos(45^\circ))t$$

The vertical position is found using the parametric equation for y. Thus,

$$y = -16t^2 + (v_0 \sin\theta)t + h$$
$$y = -16t^2 + (140\sin(45^\circ))t + 3$$

2. Substitute 2 into the equations to find the horizontal and vertical positions of the ball.

$$x = (140\cos(45^\circ))(2)$$
$$x = 198\ feet$$
$$y = -16(2)^2 + (140\sin(45^\circ))(2) + 3$$
$$y = 137\ feet$$

After 2 seconds, the ball is 198 feet away from the batter's box and 137 feet above the ground.

3. To calculate how long the ball is in the air, we have to find out when it will hit ground, or when $y = 0$. Thus,

$$y = -16t^2 + (140\sin(45\circ))t + 3$$
$$y = 0 \text{ Set } y(t) = 0 \text{ and solve the quadratic.}$$
$$t = 6.2173$$

When $t = 6.2173$ seconds, the ball has hit the ground. (The quadratic equation can be solved in various ways, but this problem was solved using a computer math program.)

4. We cannot confirm that the hit was a home run without considering the size of the outfield, which varies from field to field. However, for simplicity's sake, let's assume that the outfield wall is 400 feet from home plate in the deepest part of the park. Let's also assume that the wall is 10 feet high. In order to determine whether the ball clears the wall, we need to calculate how high the ball is when \(x = 400\) feet. So we will set \(x = 400\), solve for t, and input tt into y.

$$x = (140\cos(45^\circ))t$$
$$400 = (140\cos(45^\circ))t$$
$$t = 4.04$$
$$y = -16(4.04)^2 + (140\sin(45^\circ))(4.04) + 3$$
$$y = 141.8$$

The ball is 141.8 feet in the air when it soars out of the ballpark. It was indeed a home run. See Figure 4.8.10.

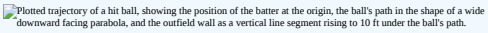

Figure 4.8.10

Media

Access the following online resource for additional instruction and practice with graphs of parametric equations.

- Graphing Parametric Equations on the TI-84

Key Concepts

- When there is a third variable, a third parameter on which x and y depend, parametric equations can be used.
- To graph parametric equations by plotting points, make a table with three columns labeled t, $x(t)$, and $y(t)$. Choose values fort t in increasing order. Plot the last two columns for x and y. See Example 4.8.1 and Example 4.8.2.
- When graphing a parametric curve by plotting points, note the associated t-values and show arrows on the graph indicating the orientation of the curve. See Example 4.8.3 and Example 4.8.4.

- Parametric equations allow the direction or the orientation of the curve to be shown on the graph. Equations that are not functions can be graphed and used in many applications involving motion. See Example 4.8.5.
- Projectile motion depends on two parametric equations: $x = (v_0 \cos\theta)t$ and $y = -16t^2 + (v_0 \sin\theta)t + h$. Initial velocity is symbolized as v_0. θ represents the initial angle of the object when thrown, and h represents the height at which the object is propelled.

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4.9: Vectors

Learning Objectives

- View vectors geometrically.
- Find magnitude and direction.
- Perform vector addition and scalar multiplication.
- Find the component form of a vector.
- Find the unit vector in the direction of v.
- Perform operations with vectors in terms of i and j.
- Find the dot product of two vectors.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour, as shown in Figure $4.9.1$. What are the ground speed and actual bearing of the plane?

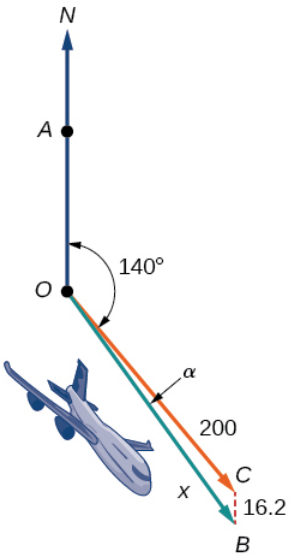

Figure 4.9.1

Ground speed refers to the speed of a plane relative to the ground. Airspeed refers to the speed a plane can travel relative to its surrounding air mass. These two quantities are not the same because of the effect of wind. In an earlier section, we used triangles to solve a similar problem involving the movement of boats. Later in this section, we will find the airplane's ground speed and bearing, while investigating another approach to problems of this type. First, however, let's examine the basics of vectors.

A Geometric View of Vectors

A vector is a specific quantity drawn as a line segment with an arrowhead at one end. It has an **initial point**, where it begins, and a **terminal point**, where it ends. A vector is defined by its magnitude, or the length of the line, and its direction, indicated by an arrowhead at the terminal point. Thus, a vector is a directed line segment. There are various symbols that distinguish vectors from other quantities:

- Lower case, boldfaced type, with or without an arrow on top such as $\boldsymbol{u}$, $\boldsymbol{w}$, $\vec{\boldsymbol{v}}$, $\vec{\boldsymbol{u}}$, $\vec{\boldsymbol{w}}$.
- Given initial point P and terminal point Q, a vector can be represented as $\overrightarrow{PQ}$. The arrowhead on top is what indicates that it is not just a line, but a directed line segment.
- Given an initial point of $(0, 0)$ and terminal point (a, b), a vector may be represented as $\langle a, b\rangle$.

This last symbol $\langle a, b\rangle$ has special significance. It is called the standard position. The position vector has an initial point $(0, 0)$ and a terminal point $\langle a, b\rangle$. To change any vector into the position vector, we think about the change in the *x*-coordinates and the change in the *y*-coordinates. Thus, if the initial point of a vector $\overrightarrow{CD}$ is $C(x_1, y_1)$ and the terminal point is $D(x_2, y_2)$, then the position vector is found by calculating

$$\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1 \rangle$$
$$= \langle a, b \rangle$$

In Figure 4.9.2, we see the original vector $\overrightarrow{CD}$ and the position vector $\overrightarrow{AB}$.

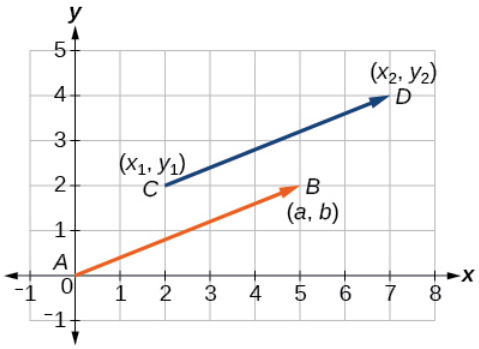

Figure 4.9.2

PROPERTIES OF VECTORS

A vector is a directed line segment with an initial point and a terminal point. Vectors are identified by magnitude, or the length of the line, and direction, represented by the arrowhead pointing toward the terminal point. The position vector has an initial point at $(0, 0)$ and is identified by its terminal point $\langle a, b \rangle$.

Example 4.9.1A: Find the Position Vector

Consider the vector whose initial point is $P(2, 3)$ and terminal point is $Q(6, 4)$. Find the position vector.

Solution

The position vector is found by subtracting one x-coordinate from the other x-coordinate, and one y-coordinate from the other y-coordinate. Thus

$$v = \langle 6 - 2, 4 - 3 \rangle$$
$$= \langle 4, 1 \rangle$$

The position vector begins at $(0, 0)$ and terminates at $(4, 1)$. The graphs of both vectors are shown in Figure 4.9.3.

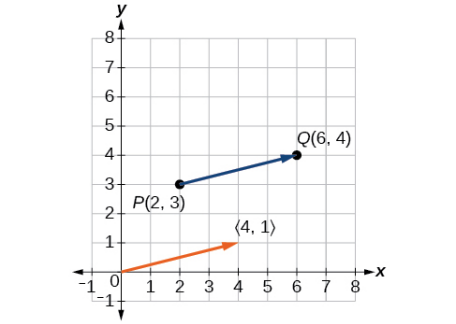

Figure 4.9.3

We see that the position vector is $\langle 4, 1 \rangle$.

Example 4.9.1B: Drawing a Vector with the Given Criteria and Its Equivalent Position Vector

Find the position vector given that vector v has an initial point at $(-3, 2)$ and a terminal point at $(4, 5)$, then graph both vectors in the same plane.

Solution

The position vector is found using the following calculation:

$$\begin{aligned} v &= \langle 4-(-3), 5-2\rangle \\ &= \langle 7, 3\rangle \end{aligned}$$

Thus, the position vector begins at $(0, 0)$ and terminates at $(7, 3)$. See Figure 4.9.4.

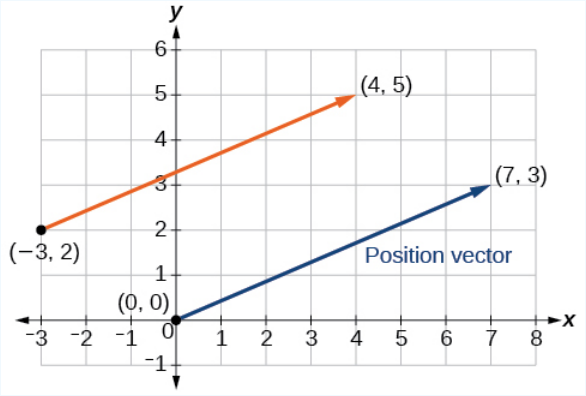

Figure 4.9.4

? Exercise 4.9.1

Draw a vector $\vec{v}$ that connects from the origin to the point $(3, 5)$.

Answer

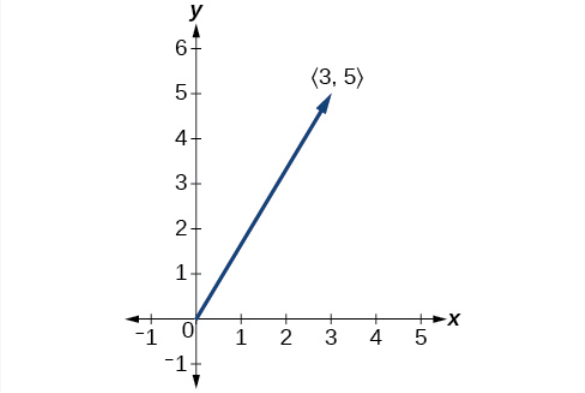

Figure 4.9.5

Finding Magnitude and Direction

To work with a vector, we need to be able to find its magnitude and its direction. We find its magnitude using the Pythagorean Theorem or the distance formula, and we find its direction using the inverse tangent function.

MAGNITUDE AND DIRECTION OF A VECTOR

Given a position vector $\vec{v} = \langle a, b\rangle$,the magnitude is found by $|v| = \sqrt{a^2 + b^2}$ .The direction is equal to the angle formed with the x-axis, or with the y-axis, depending on the application. For a position vector, the direction is found by $\tan\theta = \left(\frac{b}{a}\right) \Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$, as illustrated in Figure 4.9.6.

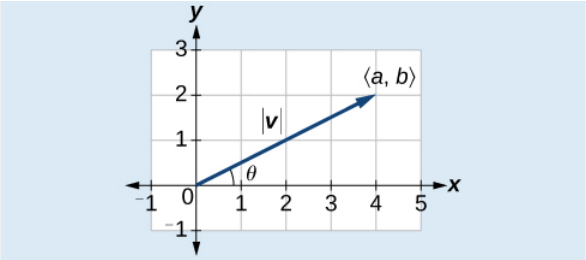

Figure 4.9.6

Two vectors $\vec{v}$ and $\vec{u}$ are considered equal if they have the same magnitude and the same direction. Additionally, if both vectors have the same position vector, they are equal.

Example $4.9.2A$: Finding the Magnitude and Direction of a Vector

Find the magnitude and direction of the vector with initial point $P(-8, 1)$ and terminal point $Q(-2, -5)$.Draw the vector.

Solution

First, find the position vector.

$$\begin{aligned} u &= \langle -2, -(-8), -5-1 \rangle \\ &= \langle 6, -6 \rangle \end{aligned}$$

We use the Pythagorean Theorem to find the magnitude.

$$\begin{aligned} |u| &= \sqrt{(6)^2 + (-6)^2} \\ &= \sqrt{72} \\ &= \sqrt{62} \end{aligned}$$

The direction is given as

$$\begin{aligned} \tan\theta &= \frac{-6}{6} = -1 \rightarrow \theta = \tan^{-1}(-1) \\ &= -45^\circ \end{aligned}$$

However, the angle terminates in the fourth quadrant, so we add 360° to obtain a positive angle. Thus, $-45^\circ + 360^\circ = 315^\circ$. See Figure $4.9.7$.

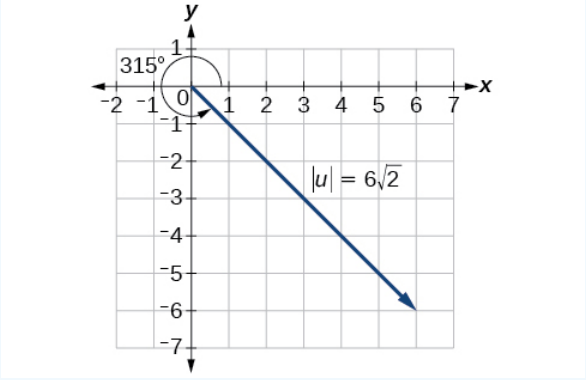

Figure 4.9.7

Example $4.9.2B$: Showing That Two Vectors Are Equal

Show that vector $\vec{v}$ with initial point at $(5, -3)$ and terminal point at $(-1, 2)$ is equal to vector $\vec{u}$ with initial point at $(-1, -3)$ and terminal point at $(-7, 2)$. Draw the position vector on the same grid as $\vec{v}$ and $\vec{u}$. Next, find the magnitude and direction of each vector.

Solution

As shown in Figure 4.9.8, draw the vector $\vec{v}$ starting at initial $(5,-3)$ and terminal point $(-1,2)$. Draw the vector $\vec{u}$ with initial point $(-1,-3)$ and terminal point $(-7,2)$. Find the standard position for each.

Next, find and sketch the position vector for $\vec{v}$ and $\vec{u}$. We have

$$\begin{aligned} v &= \langle -1-5, 2-(-3)\rangle \\ &= \langle -6, 5\rangle u \\ &= \langle -7-(-1), 2-(-3)\rangle \\ &= \langle -6, 5\rangle \end{aligned}$$

Since the position vectors are the same, $\vec{v}$ and $\vec{u}$ are the same.

An alternative way to check for vector equality is to show that the magnitude and direction are the same for both vectors. To show that the magnitudes are equal, use the Pythagorean Theorem.

$$\begin{aligned} |v| &= \sqrt{(-1-5)^2+(2-(-3))^2} \\ &= \sqrt{(-6)^2+(5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \\ |u| &= \sqrt{(-7-(-1))^2+(2-(-3))^2} \\ &= \sqrt{(-6)^2+(5)^2} \\ &= \sqrt{36+25} \\ &= \sqrt{61} \end{aligned}$$

As the magnitudes are equal, we now need to verify the direction. Using the tangent function with the position vector gives

$$\begin{aligned} \tan\theta = -\frac{5}{6} \Rightarrow \theta &= \tan^{-1}\left(-\frac{5}{6}\right) \\ &= -39.8^\circ \end{aligned}$$

However, we can see that the position vector terminates in the second quadrant, so we add 180°. Thus, the direction is $-39.8^\circ + 180^\circ = 140.2^\circ$.

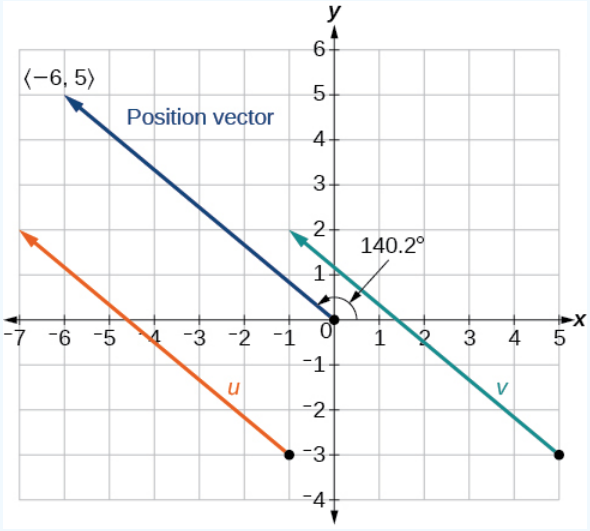

Figure 4.9.8

Performing Vector Addition and Scalar Multiplication

Now that we understand the properties of vectors, we can perform operations involving them. While it is convenient to think of the vector $u = \langle x, y \rangle$ as an arrow or directed line segment from the origin to the point (x, y), vectors can be situated anywhere in the plane. The sum of two vectors $\vec{u}$ and $\vec{v}$, or vector addition, produces a third vector $\overrightarrow{u+v}$, the resultant vector.

To find $\overrightarrow{u+v}$, we first draw the vector $\vec{u}$, and from the terminal end of $\vec{u}$, we drawn the vector $\vec{v}$. In other words, we have the initial point of $\vec{v}$ meet the terminal end of $\vec{u}$. This position corresponds to the notion that we move along the first vector and then, from its terminal point, we move along the second vector. The sum $\overrightarrow{u+v}$ is the resultant vector because it results from addition or subtraction of two vectors. The resultant vector travels directly from the beginning of $\vec{u}$ to the end of $\vec{v}$ in a straight path, as shown in Figure 4.9.9.

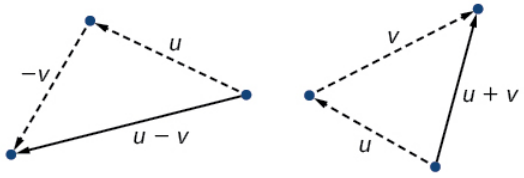

Figure 4.9.9

Vector subtraction is similar to vector addition. To find $\overrightarrow{u-v}$, view it as $\overrightarrow{u+(-v)}$. Adding $\overrightarrow{-v}$ is reversing direction of $\vec{v}$ and adding it to the end of $\vec{u}$. The new vector begins at the start of $\vec{u}$ and stops at the end point of $\overrightarrow{-v}$. See Figure 4.9.10 for a visual that compares vector addition and vector subtraction using parallelograms.

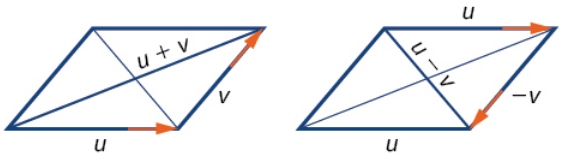

Figure 4.9.10

Example 4.9.3: Adding and Subtracting Vectors

Given $u = \langle 3, -2 \rangle$ and $v = \langle -1, 4 \rangle$, find two new vectors $\overrightarrow{u+v}$, and $\overrightarrow{u-v}$.

Solution

To find the sum of two vectors, we add the components. Thus,

$$\begin{aligned} u+v &= \langle 3, -2 \rangle + \langle -1, 4 \rangle \\ &= \langle 3+(-1), -2+4 \rangle \\ &= \langle 2, 2 \rangle \end{aligned}$$

See Figure 4.9.11a.

To find the difference of two vectors, add the negative components of $\vec{v}$ to $\vec{u}$. Thus,

$$\begin{aligned} u+(-v) &= \langle 3, -2 \rangle + \langle 1, -4 \rangle \\ &= \langle 3+1, -2+(-4) \rangle \\ &= \langle 4, -6 \rangle \end{aligned}$$

See Figure 4.9.11b

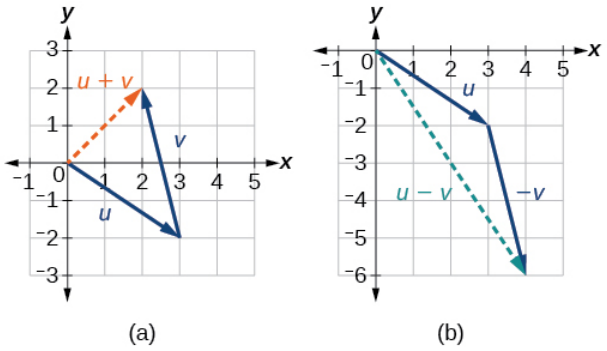

Figure 4.9.11: (a) Sum of two vectors (b) Difference of two vectors

Multiplying By a Scalar

While adding and subtracting vectors gives us a new vector with a different magnitude and direction, the process of multiplying a vector by a **scalar**, a constant, changes only the magnitude of the vector or the length of the line. Scalar multiplication has no effect on the direction unless the scalar is negative, in which case the direction of the resulting vector is opposite the direction of the original vector.

SCALAR MULTIPLICATION

Scalar multiplication involves the product of a vector and a scalar. Each component of the vector is multiplied by the scalar. Thus, to multiply $v = \langle a, b \rangle$ by k, we have

$kv = \langle ka, kb \rangle$

Only the magnitude changes, unless k is negative, and then the vector reverses direction.

Example 4.9.4: Performing Scalar Multiplication

Given vector $\vec{v} = \langle 3, 1 \rangle$, find $3\vec{v}$, $\frac{1}{2}$, and $\overrightarrow{-v}$.

Solution

See Figure 4.9.12 for a geometric interpretation. If $\vec{v} = \langle 3, 1 \rangle$, then

$$\begin{aligned} 3v &= \langle 3 \cdot 3, 3 \cdot 1 \rangle \\ &= \langle 9, 3 \rangle \\ \frac{1}{2}v &= \langle \frac{1}{2} \cdot 3, \frac{1}{2} \cdot 1 \rangle \\ &= \langle \frac{3}{2}, \frac{1}{2} \rangle \\ -v &= \langle -3, -1 \rangle \end{aligned}$$

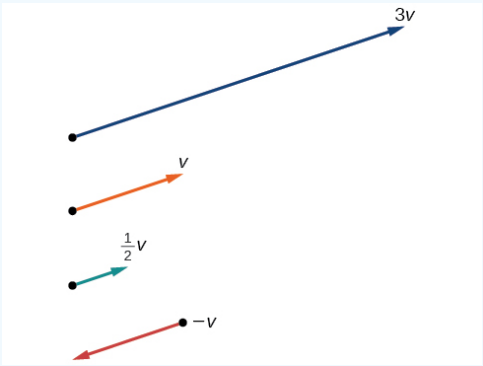

Figure 4.9.12

Analysis

Notice that the vector $3\vec{v}$ is three times the length of $\vec{v}$, $\frac{1}{2}\vec{v}$ is half the length of $\vec{v}$, and $\overrightarrow{-v}$ is the same length of $\vec{v}$, but in the opposite direction.

Exercise 4.9.2

Find the scalar multiple $3u$ given $\vec{u} = \langle 5, 4 \rangle$.

Answer

$3u = \langle 15, 12 \rangle$

Example 4.9.5

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31. Find the two numbers.

Solution

First, we must multiply each vector by the scalar.

$$\begin{aligned} 3u &= 3\langle 3, -2 \rangle \\ &= \langle 9, -6 \rangle \\ 2v &= 2\langle -1, 4 \rangle \\ &= \langle -2, 8 \rangle \end{aligned}$$

Then, add the two together.

$$\begin{aligned} w &= 3u + 2v \\ &= \langle 9, -6 \rangle + \langle -2, 8 \rangle \\ &= \langle 9 - 2, -6 + 8 \rangle \\ &= \langle 7, 2 \rangle \end{aligned}$$

So, $w = \langle 7, 2 \rangle$.

Finding Component Form

In some applications involving vectors, it is helpful for us to be able to break a vector down into its components. Vectors are comprised of two components: the horizontal component is the x direction, and the vertical component is the y direction. For example, we can see in the graph in Figure 4.9.13 that the position vector $\langle 2, 3 \rangle$ comes from adding the vectors v_1 and v_2. We have v_2 with initial point $(0, 0)$ and terminal point $(2, 0)$.

$$\begin{aligned} v_1 &= \langle 2 - 0, 0 - 0 \rangle \\ &= \langle 2, 0 \rangle \end{aligned}$$

We also have v_2 with initial point $(0, 0)$ and terminal point $(0, 3)$.

$$\begin{aligned} v_2 &= \langle 0 - 0, 3 - 0 \rangle \\ &= \langle 0, 3 \rangle \end{aligned}$$

Therefore, the position vector is

$$\begin{aligned} v &= \langle 2 + 0, 3 + 0 \rangle \\ &= \langle 2, 3 \rangle \end{aligned}$$

Using the Pythagorean Theorem, the magnitude of v_1 is 2, and the magnitude of v_2 is 3. To find the magnitude of v, use the formula with the position vector.

$$\begin{aligned} |v| &= \sqrt{|v_1|^2 + |v_2|^2} \\ &= \sqrt{2^2 + 3^2} \\ &= \sqrt{13} \end{aligned}$$

The magnitude of v is $\sqrt{13}$. To find the direction, we use the tangent function $\tan\theta = \dfrac{y}{x}$.

$$\tan\theta = \frac{v_2}{v_1}$$

$$\tan\theta = \frac{3}{2}$$

$$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.3^\circ$$

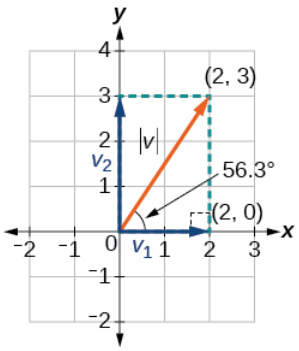

Figure 4.9.13

Thus, the magnitude of $\vec{v}$ is $\sqrt{13}$ and the direction is 56.3° off the horizontal.

Example 4.9.6: Finding the Components of the Vector

Find the components of the vector $\vec{v}$ with initial point $(3, 2)$ and terminal point $(7, 4)$.

Solution

First find the standard position.

$$\begin{aligned} v &= \langle 7-3, 4-2 \rangle \\ &= \langle 4, 2 \rangle \end{aligned}$$

See the illustration in Figure 4.9.14

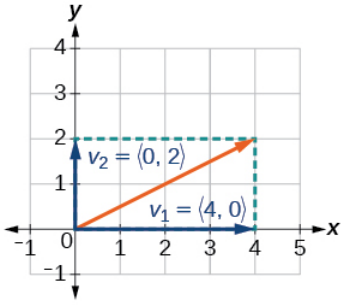

Figure 4.9.14

The horizontal component is $\overrightarrow{v_1} = \langle 4, 0 \rangle$ and the vertical component is $\overrightarrow{v_2} = \langle 0, 2 \rangle$.

Finding the Unit Vector in the Direction of v

In addition to finding a vector's components, it is also useful in solving problems to find a vector in the same direction as the given vector, but of magnitude 1. We call a vector with a magnitude of 1 a unit vector. We can then preserve the direction of the original vector while simplifying calculations.

Unit vectors are defined in terms of components. The horizontal unit vector is written as $\vec{i} = \langle 1, 0 \rangle$ and is directed along the positive horizontal axis. The vertical unit vector is written as $\vec{j} = \langle 0, 1 \rangle$ and is directed along the positive vertical axis. See Figure 4.9.15.

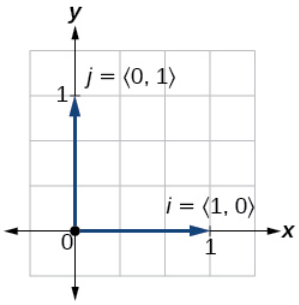

Figure 4.9.15

THE UNIT VECTORS

If $\vec{v}$ is a nonzero vector, then $\dfrac{v}{|v|}$ is a unit vector in the direction of $\boldsymbol{v}$**.** Any vector divided by its magnitude is a unit vector. Notice that magnitude is always a scalar, and dividing by a scalar is the same as multiplying by the reciprocal of the scalar.

Example 4.9.7: Finding the Unit Vector in the Direction of v

Find a unit vector in the same direction as $v = \langle -5, 12 \rangle$.

Solution

First, we will find the magnitude.

$$\begin{aligned} |v| &= \sqrt{(-5)^2 + (12)^2} \\ &= \sqrt{25 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Then we divide each component by $|v|$, which gives a unit vector in the same direction as $\vec{v}$:

$$\frac{v}{|v|} = -\frac{5}{13}i + \frac{12}{13}j$$

or, in component form

$$\frac{v}{|v|} = \left\langle -\frac{5}{13}, \frac{12}{13} \right\rangle$$

See Figure 4.9.16.

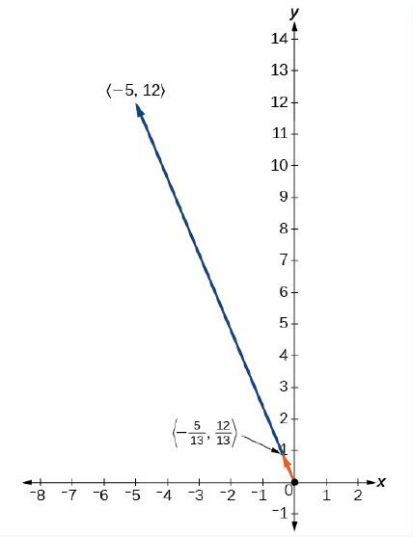

Figure 4.9.16

Verify that the magnitude of the unit vector equals 1. The magnitude of $-\frac{5}{13}i+\frac{12}{13}j$ is given as

$$\begin{aligned}\sqrt{\left(-\frac{5}{13}\right)^2+\left(\frac{12}{13}\right)^2} &= \sqrt{\frac{25}{169}+\frac{144}{169}} \\ &= \sqrt{\frac{169}{169}} \\ &= 1\end{aligned}$$

The vector $u=\frac{5}{13}i+\frac{12}{13}j$ is the unit vector in the same direction as $v=\langle -5,12\rangle$.

Performing Operations with Vectors in Terms of i and j

So far, we have investigated the basics of vectors: magnitude and direction, vector addition and subtraction, scalar multiplication, the components of vectors, and the representation of vectors geometrically. Now that we are familiar with the general strategies used in working with vectors, we will represent vectors in rectangular coordinates in terms of i and j.

VECTORS IN THE RECTANGULAR PLANE

Given a vector $\vec{v}$ with initial point $P=(x_1,y_1)$ and terminal point $Q=(x_2,y_2)$, $\vec{v}$ is written as

$$v=(x_2-x_1)i+(y_1-y_2)j \tag{4.9.1}$$

The position vector from $(0,0)$ to (a,b), where $(x_2-x_1)=a$ and $(y_2-y_1)=b$, is written as $\vec{v}=\overrightarrow{ai}+\overrightarrow{bj}$. This vector sum is called a linear combination of the vectors $\vec{i}$ and $\vec{j}$.

The magnitude of $\vec{v}=\overrightarrow{ai}+\overrightarrow{bj}$ is given as $|v|=\sqrt{a^2+b^2}$. See Figure 4.9.17.

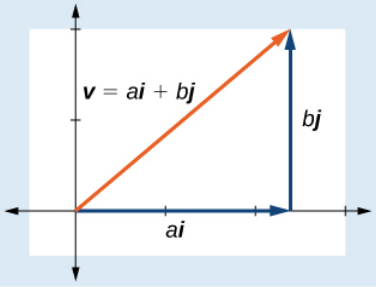

Figure 4.9.17

Example 4.9.8A: Writing a Vector in Terms of i and j

Given a vector $\vec{v}$ with initial point $P=(2,-6)$ and terminal point $Q=(-6,6)$, write the vector in terms of $\vec{i}$ and $\vec{j}$.

Solution

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2-x_1)i+(y_2-y_1)j \\ &= (-6-2)i+(6-(-6))j \\ &= -8i+12j \end{aligned}$$

Example 4.9.8B: Writing a Vector in Terms of i and j Using Initial and Terminal Points

Given initial point $P_1=(-1,3)$ and terminal point $P_2=(2,7)$, write the vector $\vec{v}$ in terms of $\vec{i}$ and $\vec{j}$.

Solution

Begin by writing the general form of the vector. Then replace the coordinates with the given values.

$$\begin{aligned} v &= (x_2-x_1)i+(y_2-y_1)j \\ v &= (2-(-1))i+(7-3)j \\ &= 3i+4j \end{aligned}$$

Exercise 4.9.3

Write the vector $\vec{u}$ with initial point $P=(-1,6)$ and terminal point $Q=(7,-5)$ in terms of $\vec{i}$ and $\vec{j}$.

Answer

$u=8i-11j$

Performing Operations on Vectors in Terms of i and j

When vectors are written in terms of i and j, we can carry out addition, subtraction, and scalar multiplication by performing operations on corresponding components.

ADDING AND SUBTRACTING VECTORS IN RECTANGULAR COORDINATES

Given $v=ai+bj$ and $u=ci+dj$, then

$$\begin{aligned} v+u &= (a+c)i+(b+d)j \\ v-u &= (a-c)i+(b-d)j \end{aligned}$$

Example 4.9.9: Finding the Sum of the Vectors

Find the sum of $v_1=2i-3j$ and $v_2=4i+5j$.

Solution

$$\begin{aligned} v_1+v_2 &= (2+4)i+(-3+5)j \\ &= 6i+2j \end{aligned}$$

Calculating the Component Form of a Vector: Direction

We have seen how to draw vectors according to their initial and terminal points and how to find the position vector. We have also examined notation for vectors drawn specifically in the Cartesian coordinate plane using i and j. For any of these vectors, we can calculate the magnitude. Now, we want to combine the key points, and look further at the ideas of magnitude and direction.

Calculating direction follows the same straightforward process we used for polar coordinates. We find the direction of the vector by finding the angle to the horizontal. We do this by using the basic trigonometric identities, but with $|v|$ replacing r.

VECTOR COMPONENTS IN TERMS OF MAGNITUDE AND DIRECTION

Given a position vector $v = \langle x, y \rangle$ and a direction angle θ,

$$\cos\theta = \frac{x}{|v|} \text{ and } \sin\theta = y|v|$$

$$x = |v|\cos\theta$$

$$y = |v|\sin\theta$$

Thus, $v = xi + yj = |v|\cos\theta i + |v|\sin\theta j$, and magnitude is expressed as $|v| = \sqrt{x^2 + y^2}$.

Example 4.9.10: Writing a Vector in Terms of Magnitude and Direction

Write a vector with length 7 at an angle of 135° to the positive *x*-axis in terms of magnitude and direction.

Solution

Using the conversion formulas $x = |v|\cos\theta i$ and $y = |v|\sin\theta j$, we find that

$$x = 7\cos(135^\circ)i$$

$$= -\frac{7\sqrt{2}}{2}$$

$$y = 7\sin(135^\circ)j$$

$$= \frac{7\sqrt{2}}{2}$$

This vector can be written as $v = 7\cos(135^\circ)i + 7\sin(135^\circ)j$ or simplified as

$$v = -\frac{7\sqrt{2}}{2}i + \frac{7\sqrt{2}}{2}j$$

Exercise 4.9.4

A vector travels from the origin to the point $(3, 5)$. Write the vector in terms of magnitude and direction.

Answer

$v = \sqrt{34}\cos(59^\circ)i + \sqrt{34}\sin(59^\circ)j$

Magnitude = 34

$\theta = \tan^{-1}\left(\frac{5}{3}\right) = 59.04^\circ$

Finding the Dot Product of Two Vectors

As we discussed earlier in the section, scalar multiplication involves multiplying a vector by a scalar, and the result is a vector. As we have seen, multiplying a vector by a number is called scalar multiplication. If we multiply a vector by a vector, there are two possibilities: the *dot product* and the *cross product*. We will only examine the dot product here; you may encounter the cross product in more advanced mathematics courses.

The dot product of two vectors involves multiplying two vectors together, and the result is a scalar.

DOT PRODUCT

The dot product of two vectors $v = \langle a, b \rangle$ and $u = \langle c, d \rangle$ is the sum of the product of the horizontal components and the product of the vertical components.

$$v \cdot u = ac + bd \tag{4.9.2}$$

To find the angle between the two vectors, use the formula below.

$$\cos\theta = \frac{v}{|v|} \cdot \frac{u}{|u|} \tag{4.9.3}$$

Example 4.9.11A: Finding the Dot Product of Two Vectors

Find the dot product of $v = \langle 5, 12 \rangle$ and $u = \langle -3, 4 \rangle$.

Solution

Using the formula, we have

$$\begin{aligned} v \cdot u &= \langle 5, 12 \rangle \cdot \langle -3, 4 \rangle \\ &= 5 \cdot (-3) + 12 \cdot 4 \\ &= -15 + 48 \\ &= 33 \end{aligned}$$

Example 4.9.11B: Finding the Dot Product of Two Vectors and the Angle between Them

Find the dot product of $v_1 = 5i + 2j$ and $v_2 = 3i + 7j$. Then, find the angle between the two vectors.

Solution

Finding the dot product, we multiply corresponding components.

$$\begin{aligned} v_1 \cdot v_2 &= \langle 5, 2 \rangle \cdot \langle 3, 7 \rangle \\ &= 5 \cdot 3 + 2 \cdot 7 \\ &= 15 + 14 \\ &= 29 \end{aligned}$$

To find the angle between them, we use the formula $\cos\theta = \frac{v}{|v|} \cdot \frac{u}{|u|}$.

$$\begin{aligned} \frac{v}{|v|} \cdot \frac{u}{|u|} &= \left\langle \frac{5}{\sqrt{29}} + \frac{2}{\sqrt{29}} \right\rangle \cdot \left\langle \frac{3}{\sqrt{58}} + \frac{7}{\sqrt{58}} \right\rangle \\ &= \frac{5}{\sqrt{29}} \cdot \frac{3}{\sqrt{58}} + \frac{2}{\sqrt{29}} \cdot \frac{7}{\sqrt{58}} \\ &= \frac{15}{\sqrt{1682}} + \frac{14}{\sqrt{1682}} \\ &= \frac{29}{\sqrt{1682}} \\ &= 0.707107 \end{aligned}$$

$$\cos^{-1}(0.707107) = 45^\circ$$

See Figure 4.9.18

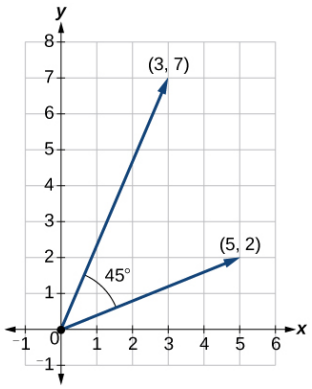

Figure 4.9.18

Example $4.9.11C$: Finding the Angle between Two Vectors

Find the angle between $u = \langle -3, 4 \rangle$ and $v = \langle 5, 12 \rangle$.

Solution

Using the formula, we have

$$\theta = \cos^{-1}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right)$$

$$\begin{aligned}\left(\frac{u}{|u|} \cdot \frac{v}{|v|}\right) &= \frac{-3i+4j}{5} \cdot \frac{5i+12j}{13} \\ &= \left(-\frac{3}{5} \cdot \frac{5}{13}\right) + \left(\frac{4}{5} \cdot \frac{12}{13}\right) \\ &= -\frac{15}{65} + \frac{48}{65} \\ &= \frac{33}{65}\end{aligned}$$

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{33}{65}\right) \\ &= 59.5^\circ\end{aligned}$$

See Figure $4.9.19$

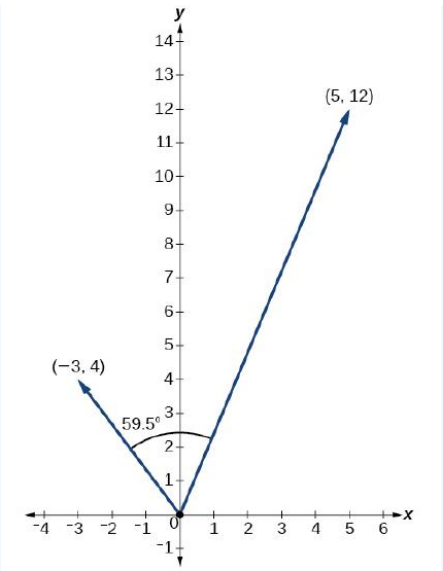

Figure 4.9.19

Example 4.9.11D: Finding Ground Speed and Bearing Using Vectors

We now have the tools to solve the problem we introduced in the opening of the section.

An airplane is flying at an airspeed of 200 miles per hour headed on a SE bearing of 140°. A north wind (from north to south) is blowing at 16.2 miles per hour. What are the ground speed and actual bearing of the plane? See Figure 4.9.20.

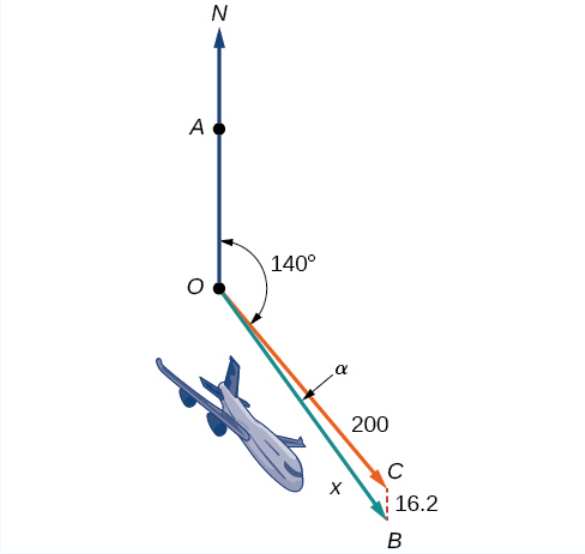

Figure 4.9.20

Solution

The ground speed is represented by x in the diagram, and we need to find the angle α in order to calculate the adjusted bearing, which will be $140^\circ + \alpha$.

Notice in Figure 4.9.20, that angle $\angle BCO$ must be equal to angle $\angle AOC$ by the rule of alternating interior angles, so angle $\angle BCO$ is 140°. We can find x by the Law of Cosines:

$$\begin{aligned} x^2 &= (16.2)^2 + (200)^2 - 2(16.2)(200)\cos(140^\circ) \\ x^2 &= 45,226.41 \\ x &= \sqrt{45,226.41} \\ x &= 212.7 \end{aligned}$$

The ground speed is approximately 213 miles per hour. Now we can calculate the bearing using the Law of Sines.

$$\begin{aligned} \frac{\sin\alpha}{16.2} &= \frac{\sin(140^\circ)}{212.7} \\ \sin\alpha &= \frac{16.2\sin(140^\circ)}{212.7} \\ &= 0.04896 \\ \sin^{-1}(0.04896) &= 2.8^\circ \end{aligned}$$

Therefore, the plane has a SE bearing of $140^\circ + 2.8^\circ = 142.8^\circ$. The ground speed is 212.7 miles per hour.

Media: Access these online resources for additional instruction and practice with vectors.

- Introduction to Vectors
- Vector Operations
- The Unit Vector

Key Concepts

- The position vector has its initial point at the origin. See Example $4.9.1$.
- If the position vector is the same for two vectors, they are equal. See Example $4.9.2$.
- Vectors are defined by their magnitude and direction. See Example $4.9.3$.
- If two vectors have the same magnitude and direction, they are equal. See Example $4.9.4$.
- Vector addition and subtraction result in a new vector found by adding or subtracting corresponding elements. See Example $4.9.5$.
- Scalar multiplication is multiplying a vector by a constant. Only the magnitude changes; the direction stays the same. See Example $4.9.6$ and Example $4.9.7$.
- Vectors are comprised of two components: the horizontal component along the positive x-axis, and the vertical component along the positive y-axis. See Example $4.9.8$.
- The unit vector in the same direction of any nonzero vector is found by dividing the vector by its magnitude.
- The magnitude of a vector in the rectangular coordinate system is $|v| = \sqrt{a^2 + b^2}$. See Example $4.9.9$.
- In the rectangular coordinate system, unit vectors may be represented in terms of ii and jj where i represents the horizontal component and j represents the vertical component. Then, $v = ai + bj$ is a scalar multiple of v by real numbers a and b. See Example $4.9.10$ and Example $4.9.11$.
- Adding and subtracting vectors in terms of i and j consists of adding or subtracting corresponding coefficients of i and corresponding coefficients of j. See Example $4.9.12$
- A vector $v = ai + bj$ is written in terms of magnitude and direction as $v = |v|\cos\theta i + |v|\sin\theta j$. See Example $4.9.13$
- The dot product of two vectors is the product of the i terms plus the product of the j terms. See Example $4.9.14$
- We can use the dot product to find the angle between two vectors. Example $4.9.15$ and Example $4.9.16$.
- Dot products are useful for many types of physics applications. See Example $4.9.17$.

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4.E: Further Applications of Trigonometry (Exercises)

8.1: Non-right Triangles: Law of Sines

Verbal

1) Describe the altitude of a triangle.

Answer

The altitude extends from any vertex to the opposite side or to the line containing the opposite side at a 90° angle.

2) Compare right triangles and oblique triangles.

3) When can you use the Law of Sines to find a missing angle?

Answer

When the known values are the side opposite the missing angle and another side and its opposite angle.

4) In the Law of Sines, what is the relationship between the angle in the numerator and the side in the denominator?

5) What type of triangle results in an ambiguous case?

Answer

A triangle with two given sides and a non-included angle.

Algebraic

For the exercises 6-10, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve each triangle, if possible. Round each answer to the nearest tenth.

6) $\alpha = 43^\circ, \gamma = 69^\circ, a = 20$

7) $\alpha = 35^\circ, \gamma = 73^\circ, c = 20$

Answer

$\beta = 72^\circ, a \approx 12.0, b \approx 19.9$

8) $\alpha = 60^\circ, \beta = 60^\circ, \gamma = 60^\circ$

9) $a = 4, \alpha = 60^\circ, \beta = 100^\circ$

Answer

$\gamma = 20^\circ, b \approx 4.5, c \approx 1.6$

10) $b = 10, \beta = 95^\circ, \gamma = 30^\circ$

11) Find side b when $A = 37^\circ, B = 49^\circ, c = 5$

Answer

$b \approx 3.78$

12) Find side a when $A = 132^\circ, C = 23^\circ, b = 10$

13) Find side c when $B = 37^\circ, C = 21^\circ, b = 23$

Answer

$c \approx 13.70$

For the exercises 14-23, assume α is opposite side a, β is opposite side b, and γ is opposite side c. Determine whether there is no triangle, one triangle, or two triangles. Then solve each triangle, if possible. Round each answer to the nearest tenth.

14) $\alpha = 119^\circ, a = 14, b = 26$

15) $\gamma = 113^\circ, b = 10, c = 32$

Answer

one triangle, $\alpha \approx 50.3^\circ, \beta \approx 16.7^\circ, a \approx 26.7$

16) $b = 3.5, c = 5.3, \gamma = 80^\circ$

17) $a = 12, c = 17, \alpha = 35^\circ$

Answer

two triangles, $\gamma \approx 54.3^\circ, \beta \approx 90.7^\circ, b \approx 20.9$ or $\gamma' \approx 125.7^\circ, \beta' \approx 19.3^\circ, b' \approx 6.9$

18) $a = 20.5, b = 35.0, \beta = 25^\circ$

19) $a = 7, c = 9, \alpha = 43^\circ$

Answer

two triangles, $\beta \approx 75.7^\circ, \gamma \approx 61.3^\circ, b \approx 9.9$ or $\beta' \approx 18.3^\circ, \gamma' \approx 118.7^\circ, b' \approx 3.2$

20) $a = 7, b = 3, \beta = 24^\circ$

21) $b = 13, c = 5, \gamma = 10^\circ$

Answer

two triangles, $\alpha \approx 143.2^\circ, \beta \approx 26.8^\circ, a \approx 17.3$ or $\alpha' \approx 16.8^\circ, \beta' \approx 153.2^\circ, a' \approx 8.3$

22) $a = 2.3, c = 1.8, \gamma = 28^\circ$

23) $\beta = 119^\circ, b = 8.2, a = 11.3$

Answer

no triangle possible

For the exercises 24-26, use the Law of Sines to solve, if possible, the missing side or angle for each triangle or triangles in the ambiguous case. Round each answer to the nearest tenth.

24) Find angle A when $a = 24, b = 5, B = 22^\circ$

25) Find angle A when $a = 13, b = 6, B = 20^\circ$

Answer

$A \approx 47.8^\circ$ or $A' \approx 132.2^\circ$

26) Find angle B when $A = 12^\circ, a = 2, b = 9$

For the exercises 27-30, find the area of the triangle with the given measurements. Round each answer to the nearest tenth.

27) $a = 5, c = 6, \beta = 35^\circ$

Answer

8.6

28) $b = 11, c = 8, \alpha = 28^\circ$

29) $a = 32, b = 24, \gamma = 75^\circ$

Answer

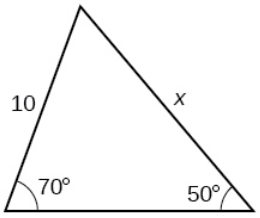
370.9

30) $a = 7.2, b = 4.5, \gamma = 43^\circ$

Graphical

For the exercises 31-36, find the length of side x. Round to the nearest tenth.

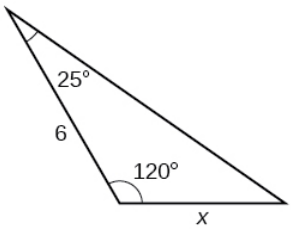
31)



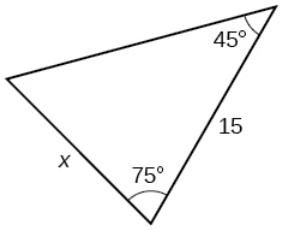
Answer

12.3

32)



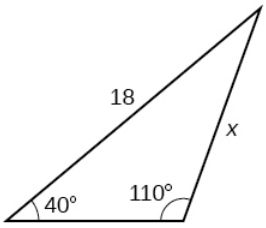
33)



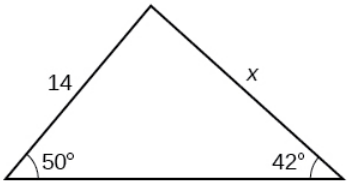
Answer

12.2

34)



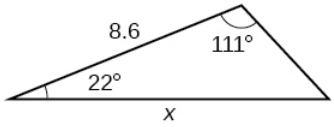
35)



Answer

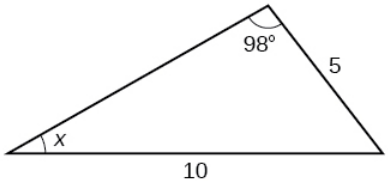
16.0

36)



For the exercises 37-,42 find the measure of angle x, if possible. Round to the nearest tenth.

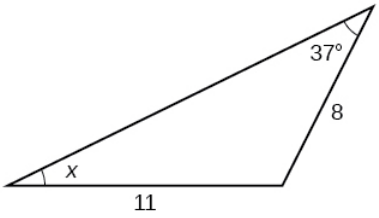
37)



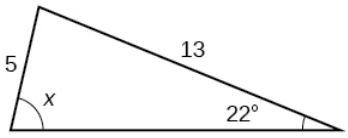
Answer

29.7°

38)



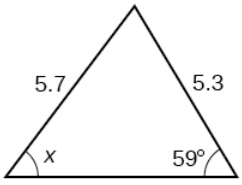
39)



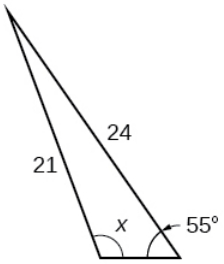
Answer

$x = 76.9^\circ$ or $x = 103.1^\circ$

40)



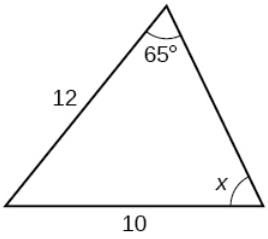
41) Notice that x is an obtuse angle.



Answer

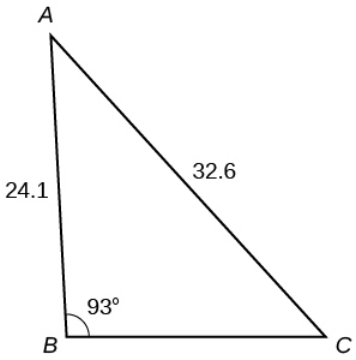
110.6°

42)



For the exercises 43-49, find the area of each triangle. Round each answer to the nearest tenth.

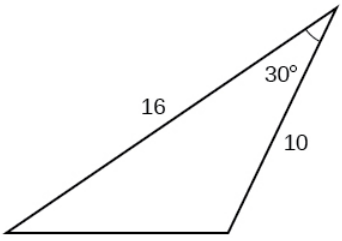
43)



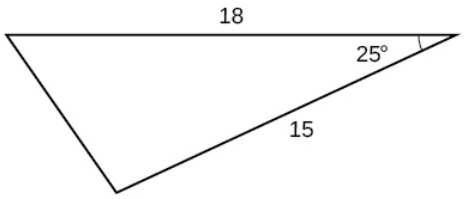
Answer

$A \approx 39.4, C \approx 47.6, BC \approx 20.7$

44)



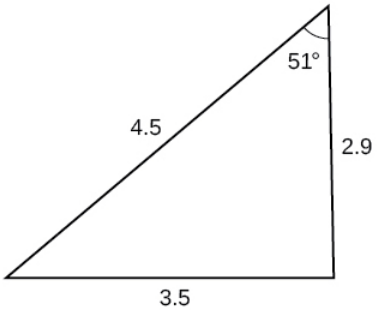
45)



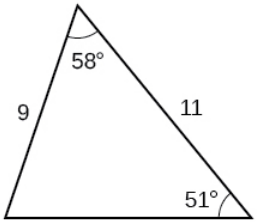
Answer

57.1

46)



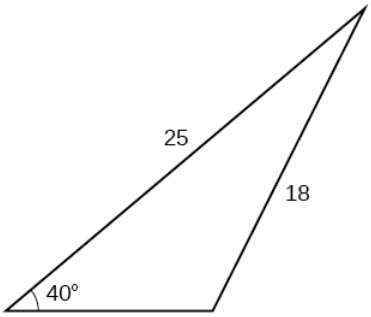
47)



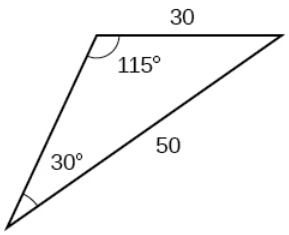
Answer

42.0

48)



49)



Answer

430.2

Extensions

50) Find the radius of the circle in the Figure below. Round to the nearest tenth.

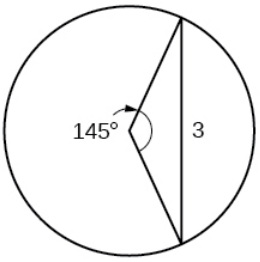
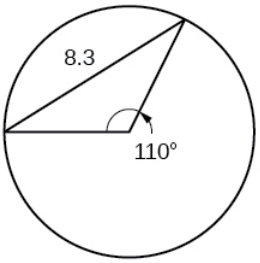


Figure 50.

51) Find the diameter of the circle in the Figure below. Round to the nearest tenth.



Answer

10.1

52) Find $m\angle ADC$ in the Figure below. Round to the nearest tenth.

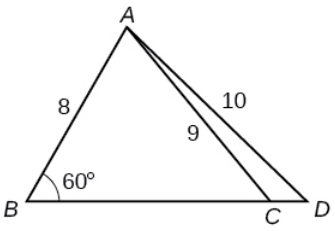
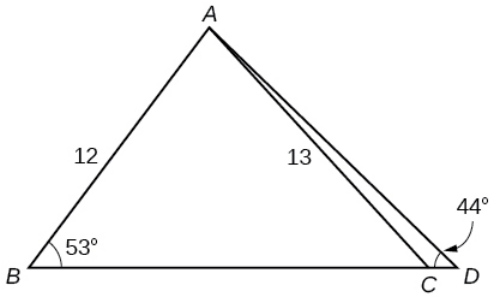


Figure 52.

53) Find side AD in the Figure below. Round to the nearest tenth.



Answer

$AD \approx 13.8$

54) Solve both triangles in the Figure below. Round each answer to the nearest tenth.

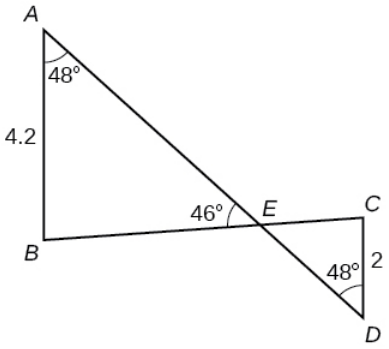
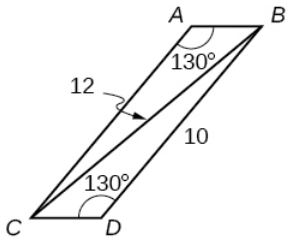


Figure 54.

55) Find side AB in the parallelogram shown below. Round to the nearest tenth.



Answer

$AB \approx 2.8$

56) Solve the triangle in the Figure below. (**Hint:** Draw a perpendicular from H to JK. Round each answer to the nearest tenth.

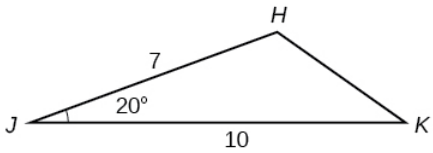
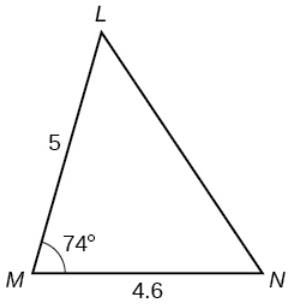


Figure 56.

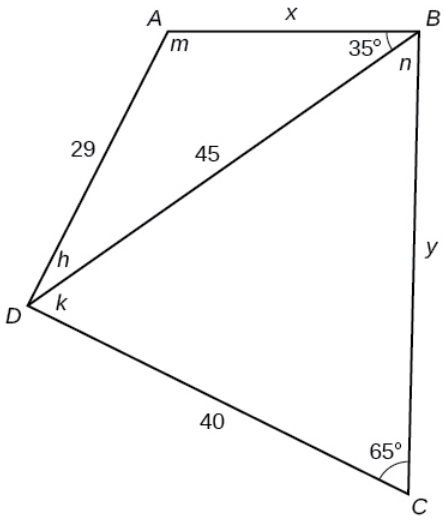
57) Solve the triangle in the Figure below. (Hint: Draw a perpendicular from N to LM. Round each answer to the nearest tenth.



Answer

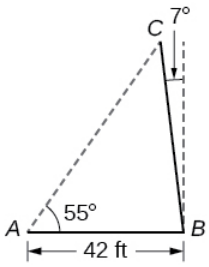
$L \approx 49.7^\circ, N \approx 56.3^\circ, LN \approx 5.8$

58) In the Figure below, $ABCD$ is not a parallelogram. $\angle m$ is obtuse. Solve both triangles. Round each answer to the nearest tenth.



Real-World Applications

59) A pole leans away from the sun at an angle of 7° to the vertical, as shown in the Figure below. When the elevation of the sun is 55°, the pole casts a shadow 42 feet long on the level ground. How long is the pole? Round the answer to the nearest tenth.



Answer

51.4ft

60) To determine how far a boat is from shore, two radar stations 500 feet apart find the angles out to the boat, as shown in the Figure below. Determine the distance of the boat from station A and the distance of the boat from shore. Round your answers to the nearest whole foot.

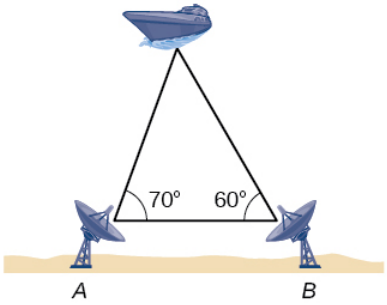
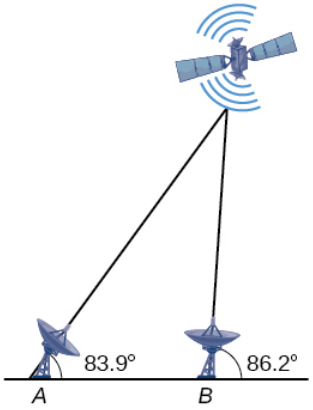


Figure 60.

61) The Figure below shows a satellite orbiting Earth. The satellite passes directly over two tracking stations A and B, which are 69 miles apart. When the satellite is on one side of the two stations, the angles of elevation at A and B are measured to be $86.2°$ and $83.9°$ respectively. How far is the satellite from station A and how high is the satellite above the ground? Round answers to the nearest whole mile.



Answer

The distance from the satellite to station A is approximately 1716 miles. The satellite is approximately 1706 miles above the ground.

62) A communications tower is located at the top of a steep hill, as shown in the Figure below. The angle of inclination of the hill is $67°$. A guy wire is to be attached to the top of the tower and to the ground, 165 meters downhill from the base of the tower. The angle formed by the guy wire and the hill is $16°$. Find the length of the cable required for the guy wire to the nearest whole meter.

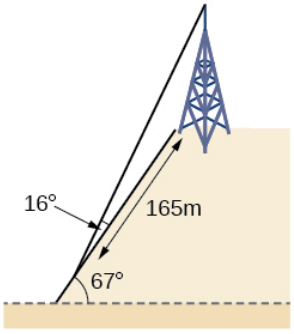
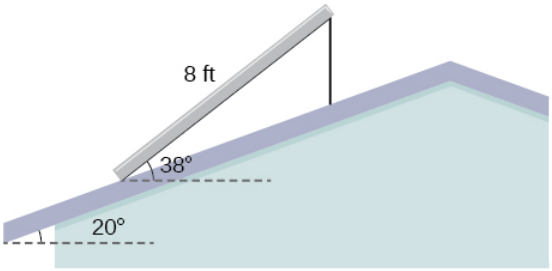


Figure 62.

63) The roof of a house is at a $20°$ angle. An 8-foot solar panel is to be mounted on the roof and should be angled $38°$ relative to the horizontal for optimal results. (See the Figure below). How long does the vertical support holding up the back of the panel need to be? Round to the nearest tenth.



Answer

2.6 ft

64) Similar to an angle of elevation, an *angle of depression* is the acute angle formed by a horizontal line and an observer's line of sight to an object below the horizontal. A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 6.6 km apart, to be 37° and 44° as shown in the Figure below. Find the distance of the plane from point A to the nearest tenth of a kilometer.

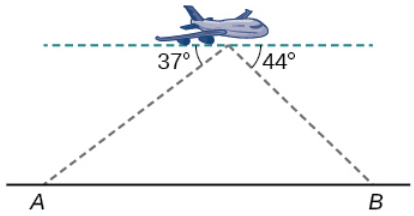
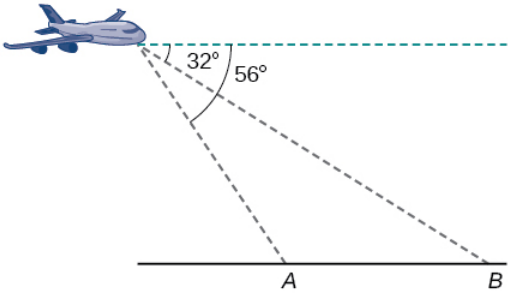


Figure 64.

65) A pilot is flying over a straight highway. He determines the angles of depression to two mileposts, 4.3 km apart, to be 32° and 56°, as shown in the Figure below. Find the distance of the plane from point A to the nearest tenth of a kilometer.



Answer

5.6 km

66) In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 39°. They then move 300 feet closer to the building and find the angle of elevation to be 50°. Assuming that the street is level, estimate the height of the building to the nearest foot.

67) In order to estimate the height of a building, two students stand at a certain distance from the building at street level. From this point, they find the angle of elevation from the street to the top of the building to be 35°. They then move 250 feet closer to the building and find the angle of elevation to be 53°. Assuming that the street is level, estimate the height of the building to the nearest foot.

Answer

371 ft

68) Points A and B are on opposite sides of a lake. Point C is 97 meters from A. The measure of $\angle BAC$ is determined to be 101°, and the measure of $\angle ACB$ is determined to be 53°. What is the distance from A to B, rounded to the nearest whole meter?

69) A man and a woman standing $3\frac{1}{2}$ miles apart spot a hot air balloon at the same time. If the angle of elevation from the man to the balloon is 27°, and the angle of elevation from the woman to the balloon is 41°, find the altitude of the balloon to the nearest foot.

Answer

5936ft

70) Two search teams spot a stranded climber on a mountain. The first search team is 0.5 miles from the second search team, and both teams are at an altitude of 1 mile. The angle of elevation from the first search team to the stranded climber is 15°. The angle of elevation from the second search team to the climber is 22°. What is the altitude of the climber? Round to the nearest tenth of a mile.

71) A street light is mounted on a pole. A 6-foot-tall man is standing on the street a short distance from the pole, casting a shadow. The angle of elevation from the tip of the man's shadow to the top of his head of 28°. A 6-foot-tall woman is standing on the same street on the opposite side of the pole from the man. The angle of elevation from the tip of her shadow to the top of her head is 28°. If the man and woman are 20 feet apart, how far is the street light from the tip of the shadow of each person? Round the distance to the nearest tenth of a foot.

Answer

24.1ft

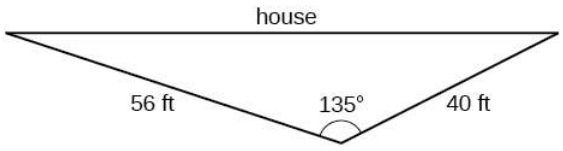
72) Three cities, A, B, and C, are located so that city A is due east of city B. If city C is located 35° west of north from city B and is 100 miles from city A and 70 miles from city B, how far is city A from city B? Round the distance to the nearest tenth of a mile.

73) Two streets meet at an 80° angle. At the corner, a park is being built in the shape of a triangle. Find the area of the park if, along one road, the park measures 180 feet, and along the other road, the park measures 215 feet.

Answer

$19,056$ft^2

74) Brian's house is on a corner lot. Find the area of the front yard if the edges measure 40 and 56 feet, as shown in the Figure below.



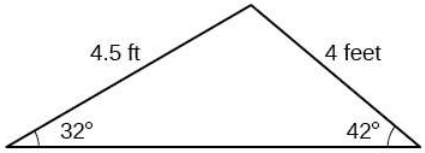
75) The Bermuda triangle is a region of the Atlantic Ocean that connects Bermuda, Florida, and Puerto Rico. Find the area of the Bermuda triangle if the distance from Florida to Bermuda is 1030 miles, the distance from Puerto Rico to Bermuda is 980 miles, and the angle created by the two distances is 62°.

Answer

$445,624$square miles

76) A yield sign measures 30 inches on all three sides. What is the area of the sign?

77) Naomi bought a modern dining table whose top is in the shape of a triangle. Find the area of the table top if two of the sides measure 4 feet and 4.5 feet, and the smaller angles measure 32° and 42°, as shown in the Figure below.



Answer

8.65ft^2

8.2: Non-right Triangles - Law of Cosines

Verbal

1) If you are looking for a missing side of a triangle, what do you need to know when using the Law of Cosines?

Answer

two sides and the angle opposite the missing side.

2) If you are looking for a missing angle of a triangle, what do you need to know when using the Law of Cosines?

3) Explain what s represents in Heron's formula.

Answer

s is the semi-perimeter, which is half the perimeter of the triangle.

4) Explain the relationship between the Pythagorean Theorem and the Law of Cosines.

5) When must you use the Law of Cosines instead of the Pythagorean Theorem?

Answer

The Law of Cosines must be used for any oblique (non-right) triangle.

Algebraic

For the exercises 6-15, assume α is opposite side a, β is opposite side b, and γ is opposite side c. If possible, solve each triangle for the unknown side. Round to the nearest tenth.

6) $\gamma = 41.2^\circ, a = 2.49, b = 3.13$

7) $\alpha = 120^\circ, b = 6, c = 7$

Answer

11.3

8) $\beta = 58.7^\circ, a = 10.6, c = 15.7$

9) $\alpha = 115^\circ, a = 18, b = 23$

Answer

34.7

10) $\alpha = 119^\circ, a = 26, b = 14$

11) $\gamma = 113^\circ, b = 10, c = 32$

Answer

26.7

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12) $\beta = 67^\circ, a = 49, b = 38$

13) $\alpha = 43.1^\circ, a = 184.2, b = 242.8$

Answer

257.4

14) $\alpha = 36.6^\circ, a = 186.2, b = 242.2$

15) $\beta = 50^\circ, a = 105, b = 45$

Answer

not possible

For the exercises 16-20, use the Law of Cosines to solve for the missing angle of the oblique triangle. Round to the nearest tenth.

16) $a = 42, b = 19, c = 30$; find angle A.

17) $a = 14, b = 13, c = 20$; find angle C.

Answer

95.5°

18) $a = 16, b = 31, c = 20$; find angle B.

19) $a = 13, b = 22, c = 28$; find angle A.

Answer

26.9°

20) $a = 108, b = 132, c = 160$; find angle C.

For the exercises 21-26, solve the triangle. Round to the nearest tenth.

21) $A = 35^\circ, b = 8, c = 11$

Answer

$B \approx 45.9^\circ, C \approx 99.1^\circ, a \approx 6.4$

22) $B = 88^\circ, a = 4.4, c = 5.2$

23) $C = 121^\circ, a = 21, b = 37$

Answer

$A \approx 20.6^\circ, B \approx 38.4^\circ, c \approx 51.1$

24) $a = 13, b = 11, c = 15$

25) $a = 3.1, b = 3.5, c = 5$

Answer

$A \approx 37.8^\circ, B \approx 43.8^\circ, C \approx 98.4$

26) $a = 51, b = 25, c = 29$

For the exercises 27-,31 use Heron's formula to find the area of the triangle. Round to the nearest hundredth.

27) Find the area of a triangle with sides of length 18 in, 21 in, and 32 in. Round to the nearest tenth.

Answer

177.56in^2

28) Find the area of a triangle with sides of length 20 cm, 26 cm, and 37 cm. Round to the nearest tenth.

29) $a = \frac{1}{2}$ m, $b = \frac{1}{3}$ m, $c = \frac{1}{4}$ m

Answer

0.04m^2

30) $a = 12.4$ ft, $b = 13.7$ ft, $c = 20.2$ ft

31) $a = 1.6$ yd, $b = 2.6$ yd, $c = 4.1$ yd

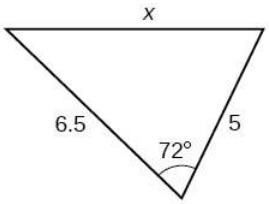
Answer

0.91yd^2

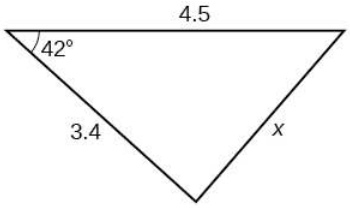
Graphical

For the exercises 32-37, find the length of side x. Round to the nearest tenth.

32)



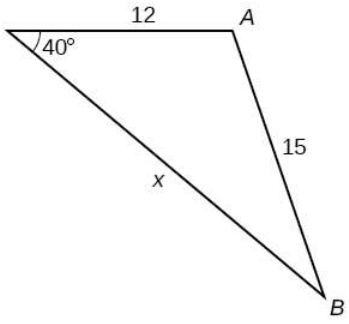
33)



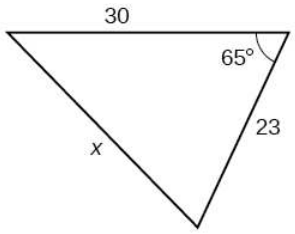
Answer

3.0

34)



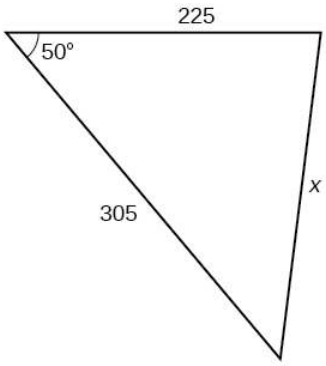
35)



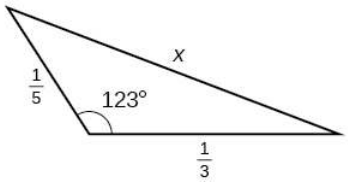
Answer

29.1

36)



37)

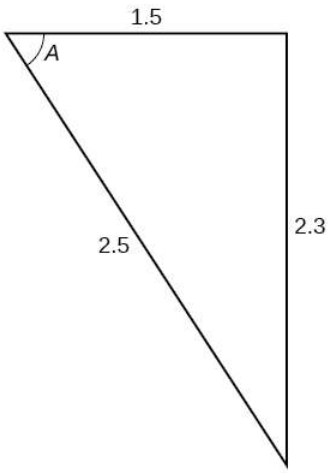


Answer

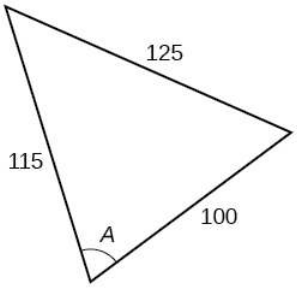
0.5

For the exercises 38-41, find the measurement of angle A

38)



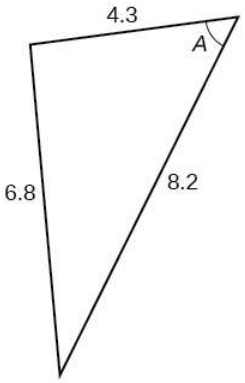
39)



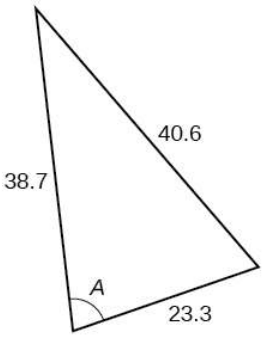
Answer

70.7°

40)



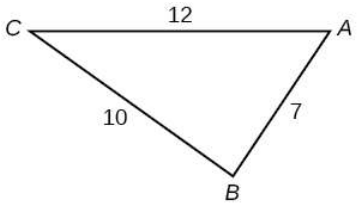
41)



Answer

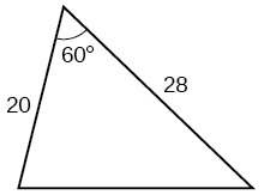
77.4°

42) Find the measure of each angle in the triangle shown in the Figure below. Round to the nearest tenth.



For the exercises 43-46, solve for the unknown side. Round to the nearest tenth.

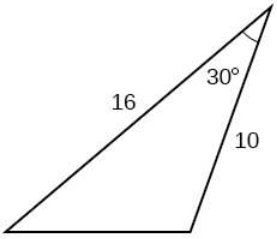
43)



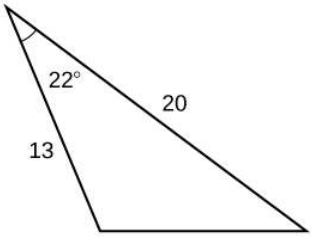
Answer

25.0

44)



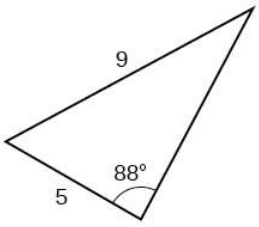
45)



Answer

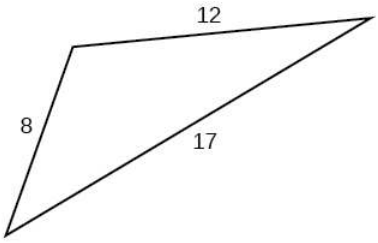
9.3

46)



For the exercises 47-51, find the area of the triangle. Round to the nearest hundredth.

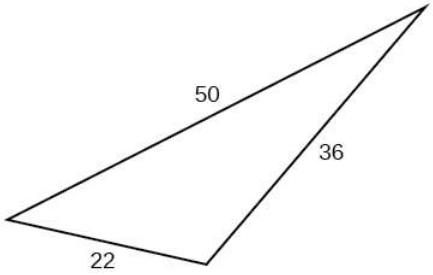
47)



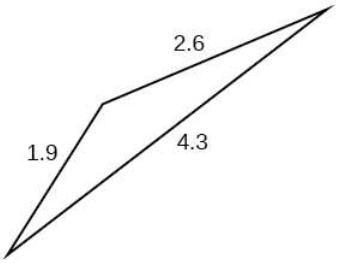
Answer

43.52

48)



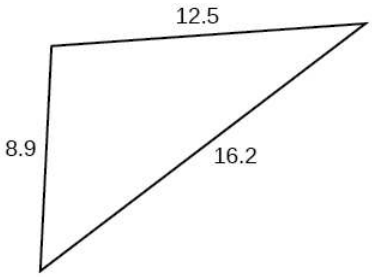
49)



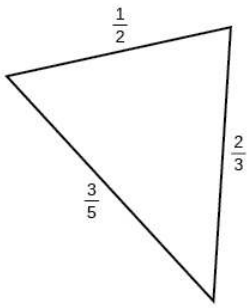
Answer

1.41

50)



51)



Answer

0.14

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Extensions

52) A parallelogram has sides of length 16 units and 10 units. The shorter diagonal is 12 units. Find the measure of the longer diagonal.

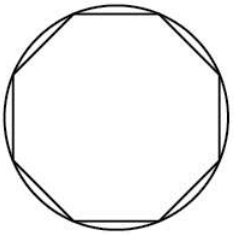
53) The sides of a parallelogram are 11 feet and 17 feet. The longer diagonal is 22 feet. Find the length of the shorter diagonal.

Answer

18.3

54) The sides of a parallelogram are 28 centimeters and 40 centimeters. The measure of the larger angle is 100°. Find the length of the shorter diagonal.

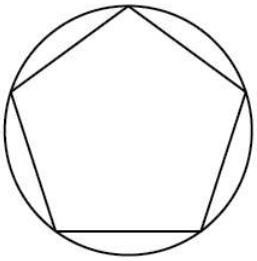
55) A regular octagon is inscribed in a circle with a radius of 8 inches. (See Figure below.) Find the perimeter of the octagon.



Answer

48.98

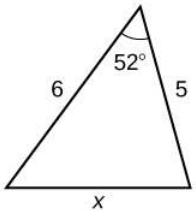
56) A regular pentagon is inscribed in a circle of radius 12 cm. (See Figure below.) Find the perimeter of the pentagon. Round to the nearest tenth of a centimeter.



For the exercises 57-58, suppose that $x^2 = 25 + 36 - 60\cos(52)$ represents the relationship of three sides of a triangle and the cosine of an angle.

57) Draw the triangle.

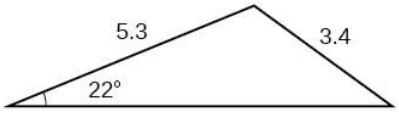
Answer



58) Find the length of the third side.

For the exercises 59-61, find the area of the triangle.

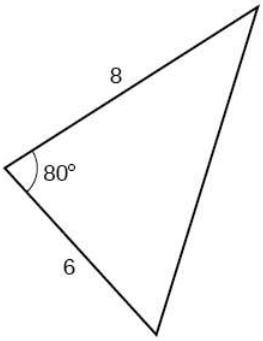
59)



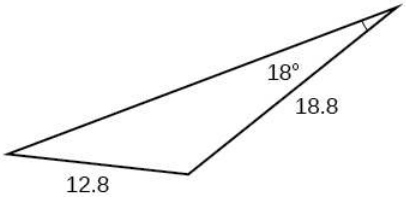
Answer

7.62

60)



61)



Answer

85.1

Real-World Applications

62) A surveyor has taken the measurements shown in the Figure below. Find the distance across the lake. Round answers to the nearest tenth.

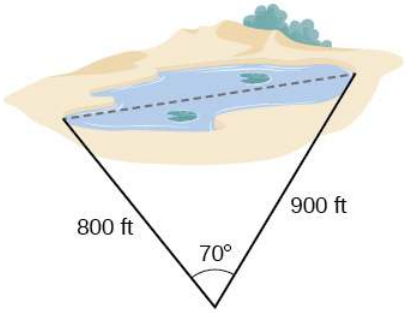
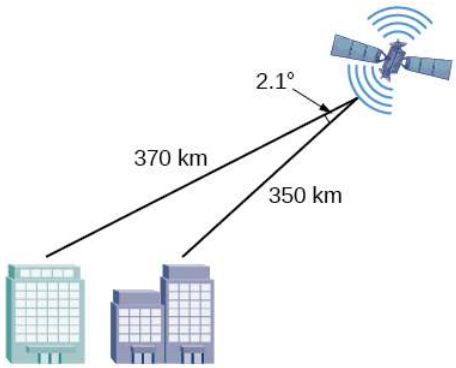


Figure below (not to scale). Find the distance between the two cities. Round answers to the nearest tenth.

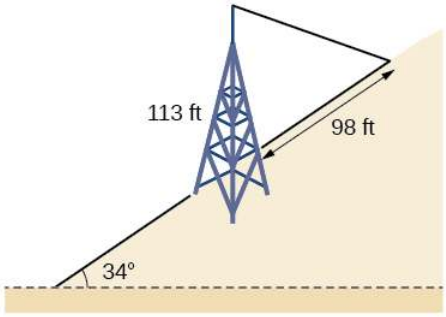


Answer

24.0km

64) An airplane flies 220 miles with a heading of 40°, and then flies 180 miles with a heading of 170°. How far is the plane from its starting point, and at what heading? Round answers to the nearest tenth.

65) A 113-foot tower is located on a hill that is inclined 34° to the horizontal, as shown in the Figure below. A guy-wire is to be attached to the top of the tower and anchored at a point 98 feet uphill from the base of the tower. Find the length of wire needed.

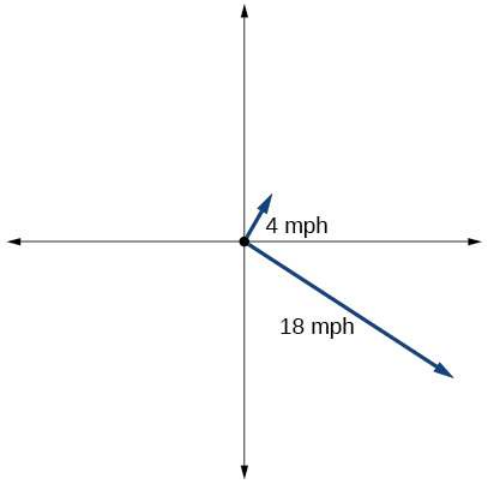


Answer

99.9ft

66) Two ships left a port at the same time. One ship traveled at a speed of 18 miles per hour at a heading of 320°. The other ship traveled at a speed of 22 miles per hour at a heading of 194°. Find the distance between the two ships after 10 hours of travel.

67) The graph in the Figure below represents two boats departing at the same time from the same dock. The first boat is traveling at 18 miles per hour at a heading of 327° and the second boat is traveling at 4 miles per hour at a heading of 60°. Find the distance between the two boats after 2 hours.



Answer

37.3miles

68) A triangular swimming pool measures 40 feet on one side and 65 feet on another side. These sides form an angle that measures 50°. How long is the third side (to the nearest tenth)?

69) A pilot flies in a straight path for 1 hour 30 min. She then makes a course correction, heading 10° to the right of her original course, and flies 2 hours in the new direction. If she maintains a constant speed of 680 miles per hour, how far is she from her starting position?

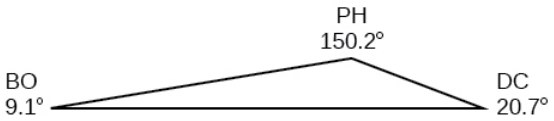
Answer

2371miles

70) Los Angeles is $1,744$ miles from Chicago, Chicago is 714 miles from New York, and New York is $2,451$ miles from Los Angeles. Draw a triangle connecting these three cities, and find the angles in the triangle.

71) Philadelphia is 140 miles from Washington, D.C., Washington, D.C. is 442 miles from Boston, and Boston is 315 miles from Philadelphia. Draw a triangle connecting these three cities and find the angles in the triangle.

Answer



72) Two planes leave the same airport at the same time. One flies at 20° east of north at 500 miles per hour. The second flies at 30° east of south at 600 miles per hour. How far apart are the planes after 2 hours?

73) Two airplanes take off in different directions. One travels 300 mph due west and the other travels 25° north of west at 420 mph. After 90 minutes, how far apart are they, assuming they are flying at the same altitude?

Answer

599.8miles

74) A parallelogram has sides of length 15.4 units and 9.8 units. Its area is 72.9 square units. Find the measure of the longer diagonal.

75) The four sequential sides of a quadrilateral have lengths 4.5 cm, 7.9 cm, 9.4 cm, and 12.9 cm. The angle between the two smallest sides is 117°. What is the area of this quadrilateral?

Answer

65.4cm^2

76) The four sequential sides of a quadrilateral have lengths 5.7 cm, 7.2 cm, 9.4 cm, and 12.8 cm. The angle between the two smallest sides is 106°. What is the area of this quadrilateral?

77) Find the area of a triangular piece of land that measures 30 feet on one side and 42 feet on another; the included angle measures 132°. Round to the nearest whole square foot.

Answer

$468\,\text{ft}^2$

78) Find the area of a triangular piece of land that measures 110 feet on one side and 250 feet on another; the included angle measures 85°. Round to the nearest whole square foot.

8.3: Polar Coordinates

Verbal

1) How are polar coordinates different from rectangular coordinates?

Answer

For polar coordinates, the point in the plane depends on the angle from the positive x-axis and distance from the origin, while in Cartesian coordinates, the point represents the horizontal and vertical distances from the origin. For each point in the coordinate plane, there is one representation, but for each point in the polar plane, there are infinite representations

2) How are the polar axes different from the x- and y-axes of the Cartesian plane?

3) Explain how polar coordinates are graphed.

Answer

Determine θ for the point, then move r units from the pole to plot the point. If r is negative, move r units from the pole in the opposite direction but along the same angle. The point is a distance of r away from the origin at an angle of θ from the polar axis.

4) How are the points $\left(3, \frac{\pi}{2}\right)$ and $\left(-3, \frac{\pi}{2}\right)$ related?

5) Explain why the points $\left(-3, \frac{\pi}{2}\right)$ and $\left(3, -\frac{\pi}{2}\right)$ are the same.

Answer

The point $\left(-3, \frac{\pi}{2}\right)$ has a positive angle but a negative radius and is plotted by moving to an angle of $\frac{\pi}{2}$ and then moving 3 units in the negative direction. This places the point 3 units down the negative y-axis. The point $\left(3, -\frac{\pi}{2}\right)$ has a negative angle and a positive radius and is plotted by first moving to an angle of $-\frac{\pi}{2}$ and then moving 3 units down, which is the positive direction for a negative angle. The point is also 3 units down the negative y-axis.

Algebraic

6) $\left(7, \frac{7\pi}{6}\right)$

7) $(5, \pi)$

Answer

$(-5, 0)$

8) $\left(6, -\frac{\pi}{4}\right)$

9) $\left(-3, \frac{\pi}{6}\right)$

Answer

$\left(-\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

10) $\left(4, \frac{7\pi}{4}\right)$

For the exercises 11-15, convert the given Cartesian coordinates to polar coordinates with $r > 0$ and $0 \leq \theta \leq 2\pi$. Remember to consider the quadrant in which the given point is located.

11) $(4, 2)$

Answer

$(2\sqrt{5}, 0.464)$

12) $(-4, 6)$

13) $(3, -5)$

Answer

$(\sqrt{34}, 5.253)$

14) $(-10, -13)$

15) $(8, 8)$

Answer

$\left(8\sqrt{2}, \frac{\pi}{4}\right)$

For the exercises 16-27, convert the given Cartesian equation to a polar equation.

16) $x = 3$

17) $y = 4$

Answer

$r = 4 \csc \theta$

18) $y = 4x^2$

19) $y = 2x^4$

Answer

$r = \sqrt[3]{\frac{\sin \theta}{2 \cos^4 \theta}}$

20) $x^2 + y^2 = 4y$

21) $x^2 + y^2 = 3x$

Answer

$r = 3 \cos \theta$

22) $x^2 - y^2 = x$

23) $x^2 - y^2 = 3y$

Answer

$$r = \frac{3\sin\theta}{\cos(2\theta)}$$

24) $x^2 + y^2 = 9$

25) $x^2 = 9y$

Answer

$$r = \frac{9\sin\theta}{\cos^2\theta}$$

26) $y^2 = 9x$

27) $9xy = 1$

Answer

$$r = \sqrt{\frac{1}{9\cos\theta\sin\theta}}$$

For the exercises 28-39, convert the given polar equation to a Cartesian equation. Write in the standard form of a conic if possible, and identify the conic section represented.

28) $r = 3\sin\theta$

29) $r = 4\cos\theta$

Answer

$x^2 + y^2 = 4x$ or $\frac{(x-2)^2}{4} + \frac{y^2}{4} = 1$; circle

30) $r = \frac{4}{\sin\theta + 7\cos\theta}$

31) $r = \frac{6}{\cos\theta + 3\sin\theta}$

Answer

$3y + x = 6$; line

32) $r = 2\sec\theta$

33) $r = 3\csc\theta$

Answer

$y = 3$; line

34) $r = \sqrt{r\cos\theta + 2}$

35) $r^2 = 4\sec\theta\csc\theta$

Answer

$xy = 4$; hyperbola

36) $r = 4$

37) $r^2 = 4$

Answer

$x^2 + y^2 = 4$; circle

38) $r = \dfrac{1}{4\cos\theta - 3\sin\theta}$

39) $r = \dfrac{3}{\cos\theta - 5\sin\theta}$

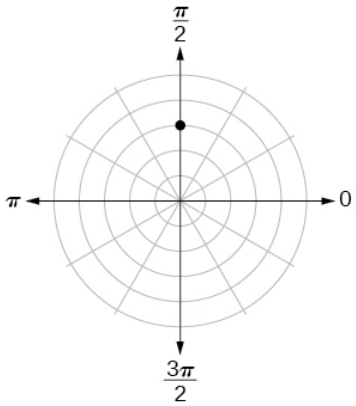
Answer

$x - 5y = 3$; line

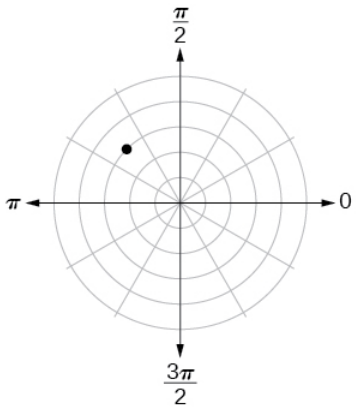
Graphical

For the exercises 40-44, find the polar coordinates of the point.

40)



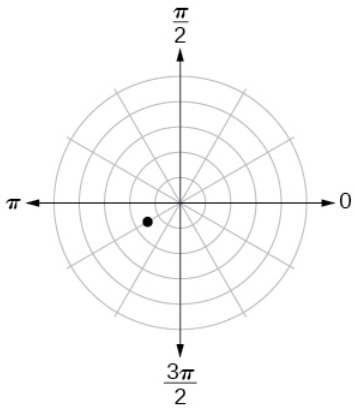
41)



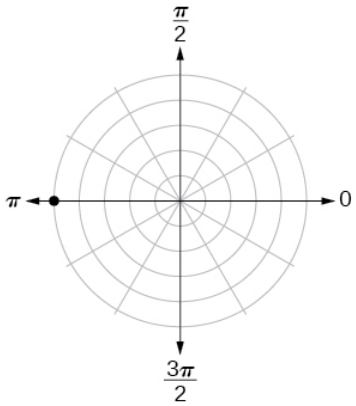
Answer

$\left(3, \dfrac{3\pi}{4}\right)$

42)



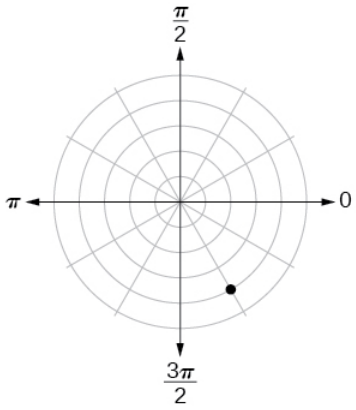
43)



Answer

$(5, \pi)$

44)



For the exercises 45-54, plot the points.

45) $\left(-2, \frac{\pi}{3}\right)$

Answer

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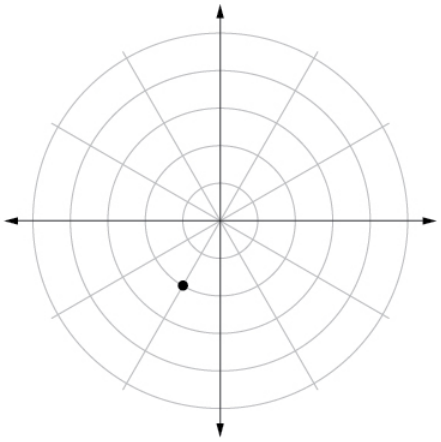
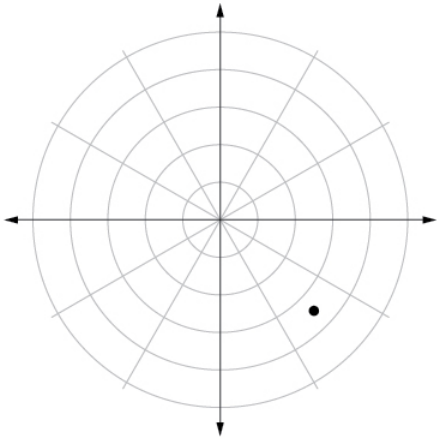


46) $\left(-1, -\frac{\pi}{2}\right)$

47) $\left(3.5, \frac{7\pi}{4}\right)$

Answer



48) $\left(-4, \frac{\pi}{3}\right)$

49) $\left(5, \frac{\pi}{2}\right)$

Answer

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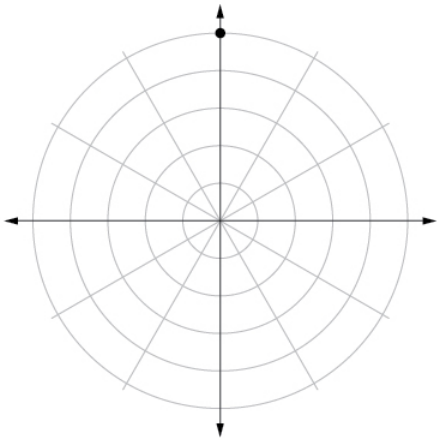
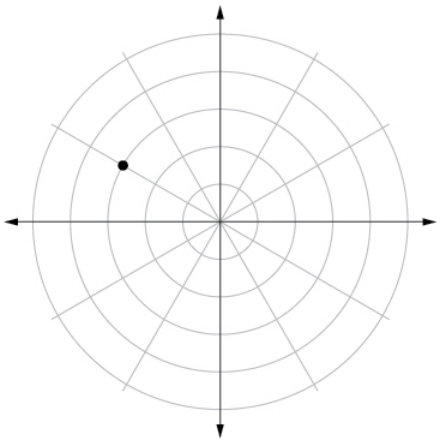


50) $\left(4, \frac{-5\pi}{4}\right)$

51) $\left(3, \frac{5\pi}{6}\right)$

Answer



52) $\left(-1.5, \frac{7\pi}{6}\right)$

53) $\left(-2, \frac{\pi}{4}\right)$

Answer

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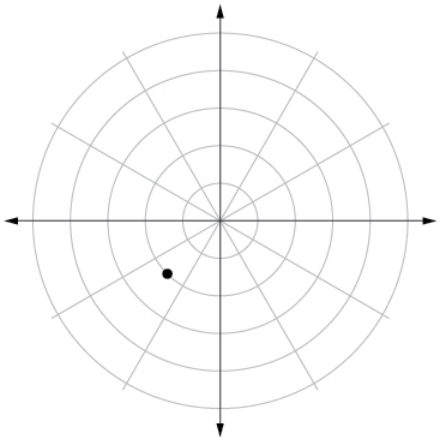


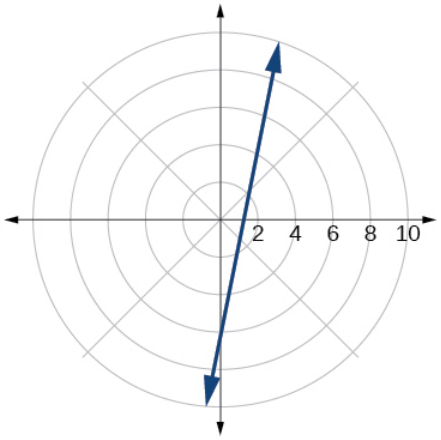
54) $\left(1, \frac{3\pi}{2}\right)$

For the exercises 55-61, convert the equation from rectangular to polar form and graph on the polar axis.

55) $5x - y = 6$

Answer

$$r = \frac{6}{5\cos\theta - \sin\theta}$$



56) $2x + 7y = -3$

57) $x^2 + (y-1)^2 = 1$

Answer

$$r = 2\sin\theta$$

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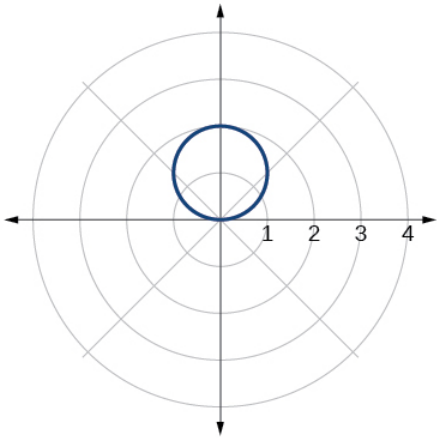
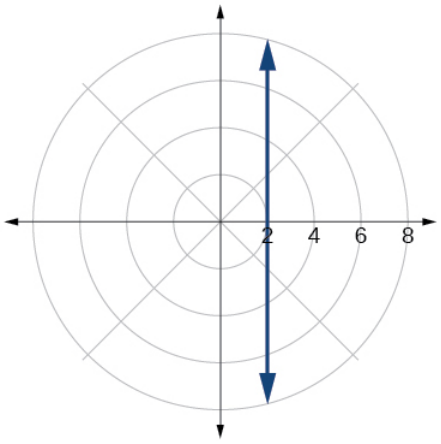


58) $(x+2)^2+(y+3)^2=13$

59) $x=2$

Answer

$r=\dfrac{2}{\cos\theta}$



60) $x^2+y^2=5y$

61) $x^2+y^2=3x$

Answer

$r=3\cos\theta$

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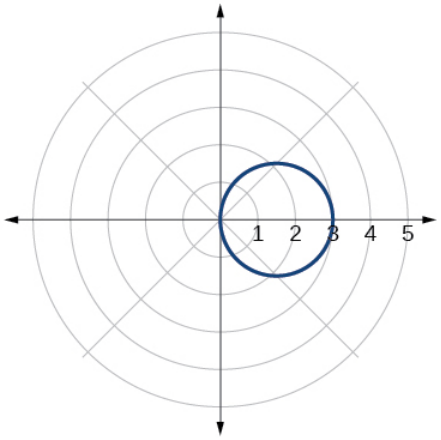


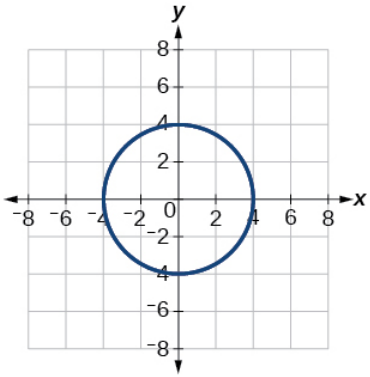
For the exercises 62-68, convert the equation from polar to rectangular form and graph on the rectangular plane.

62) $r = 6$

63) $r = -4$

Answer

$x^2 + y^2 = 16$

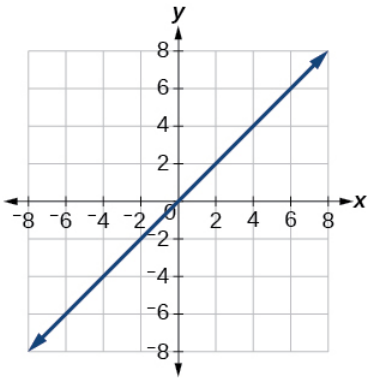


64) $\theta = -\frac{2\pi}{3}$

65) $\theta = \frac{\pi}{4}$

Answer

$y = x$



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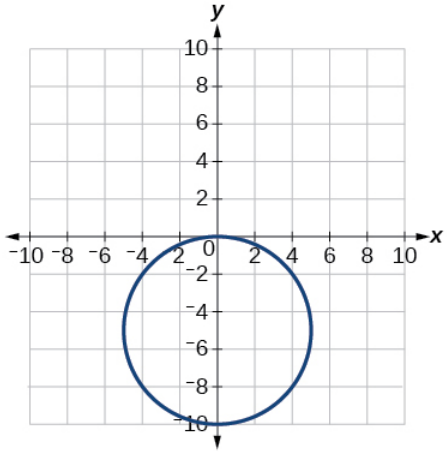
66) $r = \sec\theta$

67) $r = -10\sin\theta$

Answer

$x^2 + (y+5)^2 = 25$



68) $r = 3\cos\theta$

Technology

69) Use a graphing calculator to find the rectangular coordinates of $\left(2, -\frac{\pi}{5}\right)$. Round to the nearest thousandth.

Answer

$(1.618, -1.176)$

70) Use a graphing calculator to find the rectangular coordinates of $\left(-3, \frac{3\pi}{7}\right)$. Round to the nearest thousandth.

71) Use a graphing calculator to find the polar coordinates of $(-7, 8)$ in degrees. Round to the nearest thousandth.

Answer

$(10.630, 131.186^\circ)$

72) Use a graphing calculator to find the polar coordinates of $(3, -4)$ in degrees. Round to the nearest hundredth.

73) Use a graphing calculator to find the polar coordinates of $(-2, 0)$ in degrees. Round to the nearest hundredth.

Answer

$(2, 3.14)$or $(2, \pi)$

Extensions

74) Describe the graph of $r = a\sec\theta$; $a > 0$.

75) Describe the graph of $r = a\sec\theta$; $a < 0$.

Answer

A vertical line with a units left of the y-axis.

76) Describe the graph of $r = a\csc\theta$; $a > 0$.

77) Describe the graph of $r = a\csc\theta$; $a < 0$.

Answer

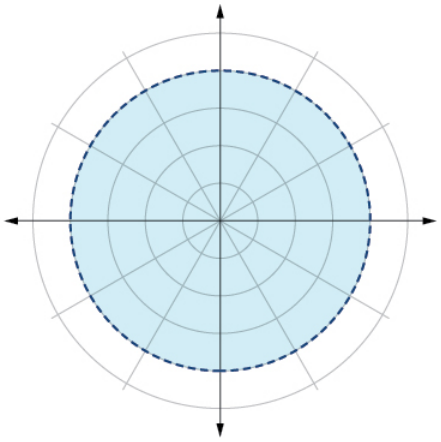
A horizontal line with a units below the x-axis.

78) What polar equations will give an oblique line?

For the exercises 79-84, graph the polar inequality.

79) $r < 4$

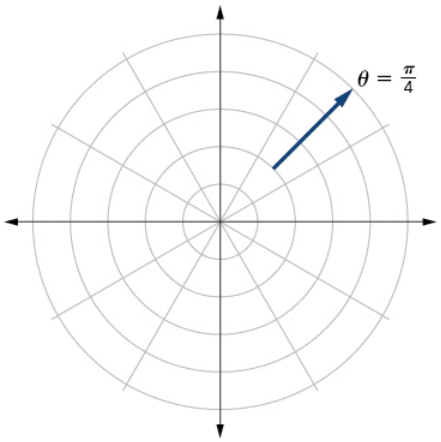
Answer



80) $0 \leq \theta \leq \frac{\pi}{4}$

81) $\theta = \frac{\pi}{4}, r \geq 2$

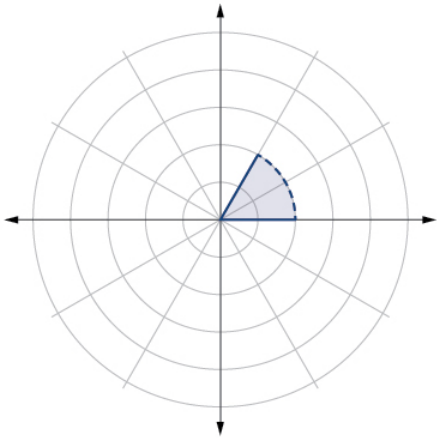
Answer



82) $\theta = \frac{\pi}{4}, r \geq -3$

83) $0 \leq \theta \leq \frac{\pi}{3}, r < 2$

Answer



84) $\frac{-\pi}{6} < \theta \leq \frac{\pi}{3}, -3 < r < 2$

8.4: Polar Coordinates - Graphs

Verbal

1) Describe the three types of symmetry in polar graphs, and compare them to the symmetry of the Cartesian plane.

Answer

Symmetry with respect to the polar axis is similar to symmetry about the x-axis, symmetry with respect to the pole is similar to symmetry about the origin, and symmetric with respect to the line $\theta = \frac{\pi}{2}$ is similar to symmetry about the y-axis.

2) Which of the three types of symmetries for polar graphs correspond to the symmetries with respect to the x-axis, y-axis, and origin?

3) What are the steps to follow when graphing polar equations?

Answer

Test for symmetry; find zeros, intercepts, and maxima; make a table of values. Decide the general type of graph, cardioid, limaçon, lemniscate, etc., then plot points at $\theta = 0$, $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ and sketch the graph.

4) Describe the shapes of the graphs of cardioids, limaçons, and lemniscates.

5) What part of the equation determines the shape of the graph of a polar equation?

Answer

The shape of the polar graph is determined by whether or not it includes a sine, a cosine, and constants in the equation.

Graphical

For the exercises 6-15, test the equation for symmetry.

6) $r = 5\cos 3\theta$

7) $r = 3 - 3\cos\theta$

Answer

symmetric with respect to the polar axis

8) $r = 3 + 2\sin\theta$

9) $r = 3\sin 2\theta$

Answer

symmetric with respect to the polar axis, symmetric with respect to the line $\theta = \frac{\pi}{2}$, symmetric with respect to the pole

10) $r = 4$

11) $r = 2\theta$

Answer

no symmetry

12) $r = 4\cos\frac{\theta}{2}$

13) $r = \frac{2}{\theta}$

Answer

no symmetry

14) $r = 3\sqrt{1-\cos^2\theta}$

15) $r = \sqrt{5\sin 2\theta}$

Answer

symmetric with respect to the pole

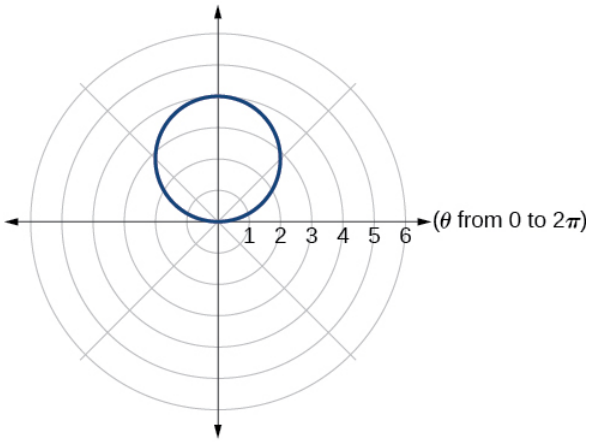
For the exercises 16-43, graph the polar equation. Identify the name of the shape.

16) $r = 3\cos\theta$

17) $r = 4\sin\theta$

Answer

circle

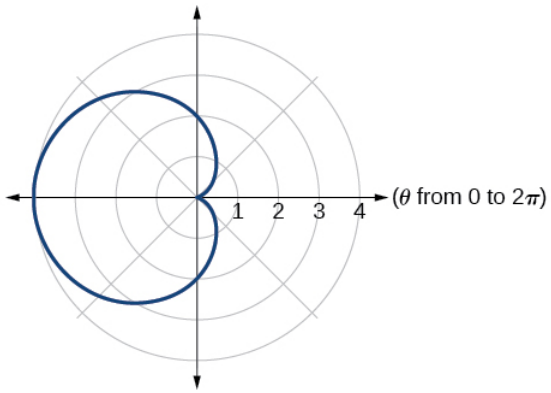


18) $r = 2 + 2\cos\theta$

19) $r = 2 - 2\cos\theta$

Answer

cardioid

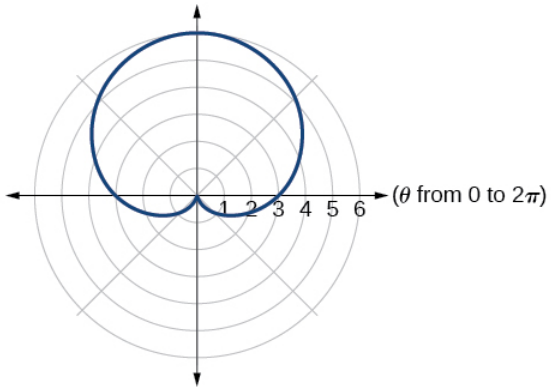


20) $r = 5 - 5\sin\theta$

21) $r = 3 + 3\sin\theta$

Answer

cardioid

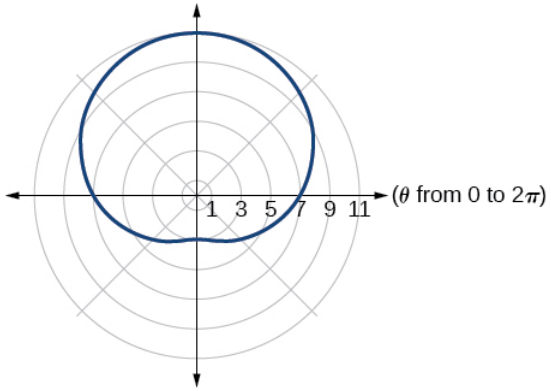


22) $r = 3 + 2\sin\theta$

23) $r = 7 + 4\sin\theta$

Answer

one-loop/dimpled limaçon



24) $r = 4 + 3\cos\theta$

25) $r = 5 + 4cos\theta$

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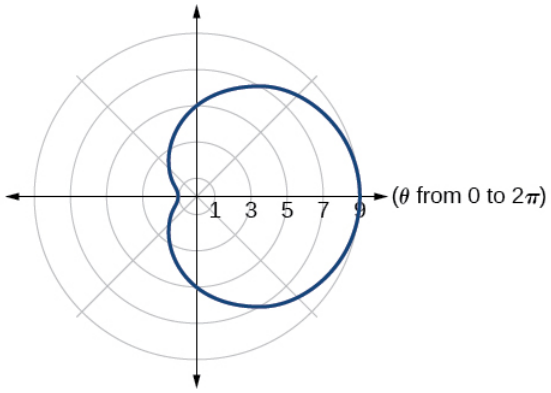
Answer

one-loop/dimpled limaçon

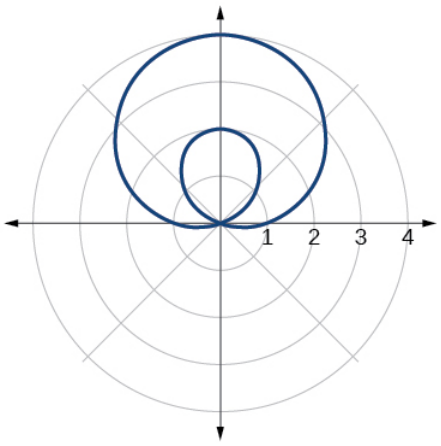


26) $r = 10 + 9\cos\theta$

27) $r = 1 + 3\sin\theta$

Answer

inner loop/ two-loop limaçon

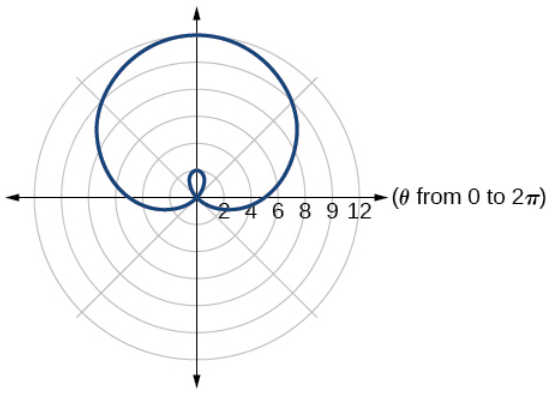


28) $r = 2 + 5\sin\theta$

29) $r = 5 + 7\sin\theta$

Answer

inner loop/ two-loop limaçon

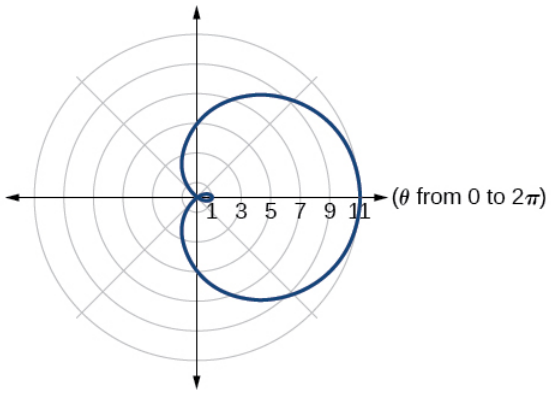


30) $r = 2 + 4\cos\theta$

31) $r = 5 + 6\cos\theta$

Answer

inner loop/ two-loop limaçon

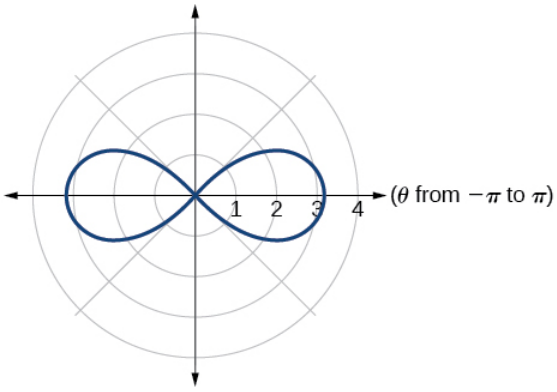


32) $r^2 = 36\cos(2\theta)$

33) $r^2 = 10\cos(2\theta)$

Answer

lemniscate



34) $r^2 = 4\sin(2\theta)$

35) $r^2 = 10\sin(2\theta)$

Answer

lemniscate

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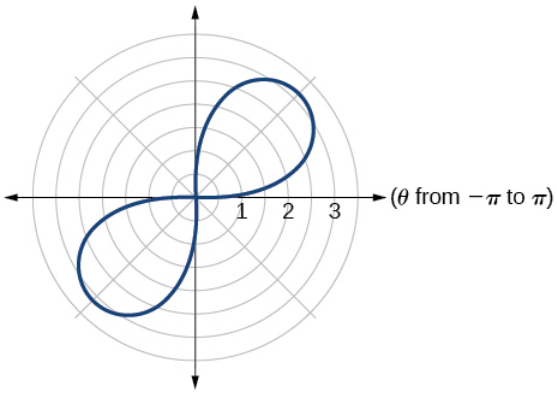
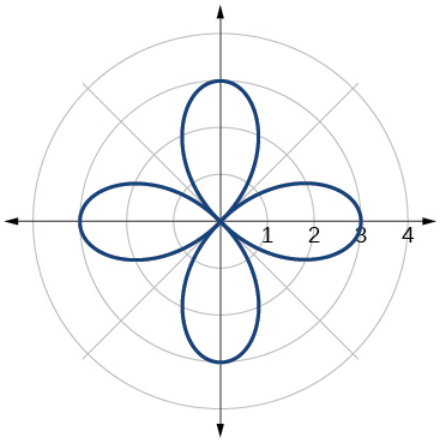


36) $r = 3\sin(2\theta)$

37) $r = 3\cos(2\theta)$

Answer

rose curve

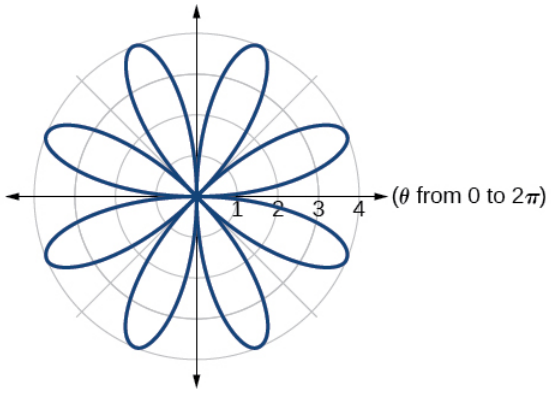


38) $r = 5\sin(3\theta)$

39) $r = 4\sin(4\theta)$

Answer

rose curve



40) $r = 4\sin(5\theta)$

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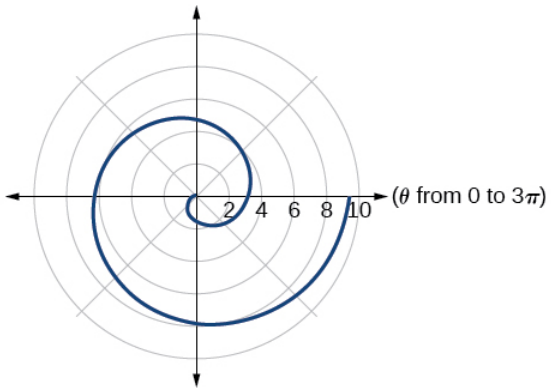
41) $r = -\theta$

Answer

Archimedes' spiral

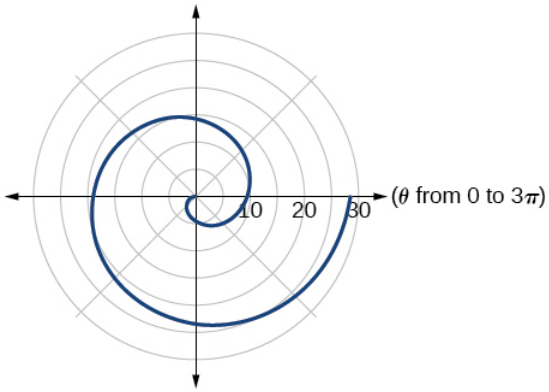


42) $r = 2\theta$

43) $r = -3\theta$

Answer

Archimedes' spiral



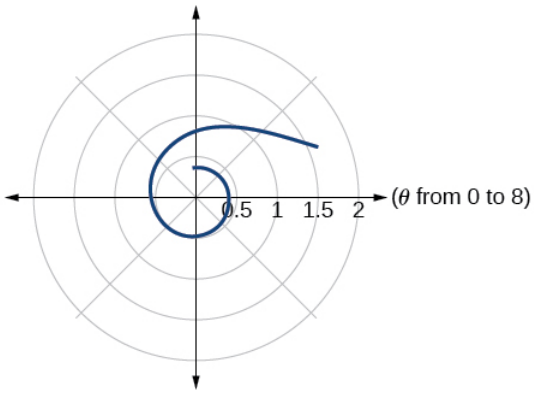
Technology

For the exercises 44-53, use a graphing calculator to sketch the graph of the polar equation.

44) $r = \dfrac{1}{\theta}$

45) $r = \dfrac{1}{\sqrt{\theta}}$

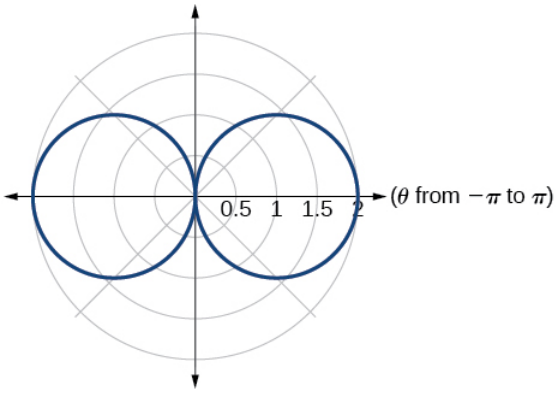
Answer



46) $r = 2\sin\theta\tan\theta$, a cissoid

47) $r = 2\sqrt{1-\sin^2\theta}$, a hippopede

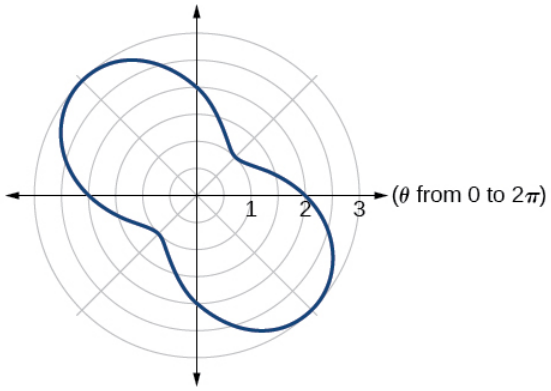
Answer



48) $r = 5 + \cos(4\theta)$

49) $r = 2 - \sin(2\theta)$

Answer



50) $r = \theta^2$

51) $r = \theta + 1$

Answer

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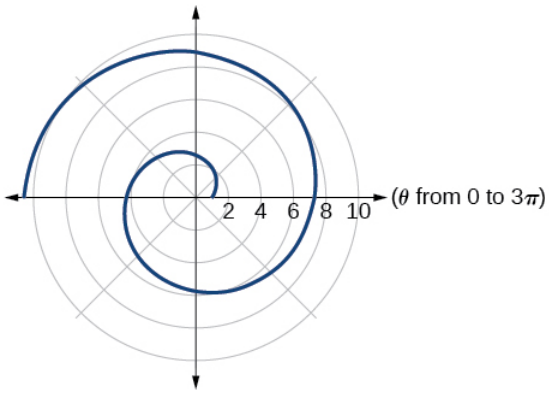
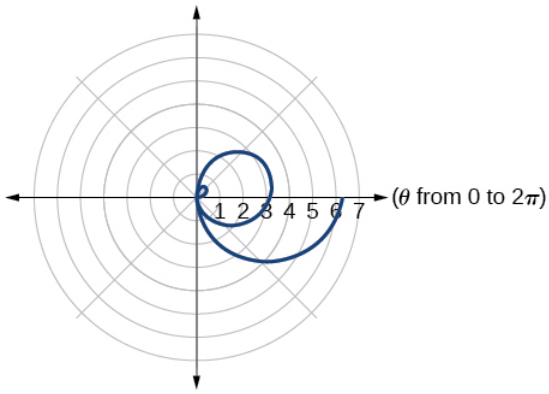


52) $r = \theta \sin \theta$

53) $r = \theta \cos \theta$

Answer



For the exercises 54-63, use a graphing utility to graph each pair of polar equations on a domain of $[0, 4\pi]$ and then explain the differences shown in the graphs.

54) $r = \theta, r = -\theta$

55) $r = \theta, r = \theta + \sin \theta$

Answer

They are both spirals, but not quite the same.

56) $r = \sin \theta + \theta, r = \sin \theta - \theta$

57) $r = 2 \sin\left(\frac{\theta}{2}\right), r = \theta \sin\left(\frac{\theta}{2}\right)$

Answer

Both graphs are curves with 2 loops. The equation with a coefficient of θ has two loops on the left, the equation with a coefficient of 2 has two loops side by side. Graph these from 0 to 4π to get a better picture.

58) $r = \sin(\cos(3\theta)), r = \sin(3\theta)$

59) On a graphing utility, graph $r = \sin\left(\frac{16}{5}\theta\right)$ on $[0, 4\pi]$, $[0, 8\pi]$, $[0, 12\pi]$ and $[0, 16\pi]$ Describe the effect of increasing the width of the domain.

Answer

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When the width of the domain is increased, more petals of the flower are visible.

60) On a graphing utility, graph and sketch $r = \sin\theta + \left(\sin\left(\frac{5}{2}\theta\right)\right)^3$ on $[0, 4\pi]$.

61) On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$$r_1 = 3\sin(3\theta)$$
$$r_2 = 2\sin(3\theta)$$
$$r_3 = \sin(3\theta)$$

Answer

The graphs are three-petal, rose curves. The larger the coefficient, the greater the curve's distance from the pole.

62) On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$$r_1 = 3 + 3\cos(\theta)$$
$$r_2 = 2 + 2\cos(\theta)$$
$$r_3 = 1 + \cos(\theta)$$

63) On a graphing utility, graph each polar equation. Explain the similarities and differences you observe in the graphs.

$$r_1 = 3\theta$$
$$r_2 = 2\theta$$
$$r_3 = \theta$$

Answer

The graphs are spirals. The smaller the coefficient, the tighter the spiral.

Extensions

For the exercises 64-72, draw each polar equation on the same set of polar axes, and find the points of intersection.

64) $r_1 = 3 + 2\sin\theta, r_2 = 2$

65) $r_1 = 6 - 4\cos\theta, r_2 = 4$

Answer

$$\left(4, \frac{\pi}{3}\right), \left(4, \frac{5\pi}{3}\right)$$

66) $r_1 = 1 + \sin\theta, r_2 = 3\sin\theta$

67) $r_1 = 1 + \cos\theta, r_2 = 3\cos\theta$

Answer

$$\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right)$$

68) $r_1 = \cos(2\theta), r_2 = \sin(2\theta)$

69) $r_1 = \sin^2(2\theta), r_2 = 1 - \cos(4\theta)$

Answer

$$\left(0, \frac{\pi}{2}\right), (0, \pi), \left(0, \frac{3\pi}{2}\right), (0, 2\pi)$$

70) $r_1 = \sqrt{3}, r_2 = 2\sin(\theta)$

71) $r_1^2 = \sin\theta, r_2^2 = \cos\theta$

Answer

$\left(\frac{\sqrt[4]{8}}{2}, \frac{\pi}{4}\right), \left(\frac{\sqrt[4]{8}}{2}, \frac{5\pi}{4}\right)$, and at $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ since r is squared

72) $r_1 = 1 + \cos\theta, r_2 = 1 - \sin\theta$

8.5: Polar Form of Complex Numbers

Verbal

1) A complex number is $a + bi$. Explain each part.

Answer

a is the real part, b is the imaginary part, and $i = \sqrt{-1}$

2) What does the absolute value of a complex number represent?

3) How is a complex number converted to polar form?

Answer

Polar form converts the real and imaginary part of the complex number in polar form using $x = r\cos\theta$ and $y = r\sin\theta$.

4) How do we find the product of two complex numbers?

5) What is De Moivre's Theorem and what is it used for?

Answer

$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$

It is used to simplify polar form when a number has been raised to a power.

Algebraic

For the exercises 6-11, find the absolute value of the given complex number.

6) $5 + 3i$

7) $-7 + i$

Answer

$5\sqrt{2}$

8) $-3 - 3i$

9) $\sqrt{2} - 6i$

Answer

$\sqrt{38}$

10) $2i$

11) $2.2 - 3.1i$

Answer

$\sqrt{14.45}$

For the exercises 12-16, write the complex number in polar form.

12) $2+2i$

13) $8-4i$

Answer

$4\sqrt{5}\textbf{cis}(333.4^\circ)$

14) $-\frac{1}{2}-\frac{1}{2}i$

15) $\sqrt{3}+i$

Answer

$2\textbf{cis}\left(\frac{\pi}{6}\right)$

16) $3i$

For the exercises 17-22, convert the complex number from polar to rectangular form.

17) $z=7\textbf{cis}\left(\frac{\pi}{6}\right)$

Answer

$\frac{7\sqrt{3}}{2}+i\frac{7}{2}$

18) $z=2\textbf{cis}\left(\frac{\pi}{3}\right)$

19) $z=4\textbf{cis}\left(\frac{7\pi}{6}\right)$

Answer

$-2\sqrt{3}-2i$

20) $z=7\textbf{cis}(25^\circ)$

21) $z=3\textbf{cis}(240^\circ)$

Answer

$-1.5-i\frac{3\sqrt{3}}{2}$

22) $z=\sqrt{2}\textbf{cis}(100^\circ)$

For the exercises 23-28, find $z_1 z_2$ in polar form.

23) $z_1=2\sqrt{3}\textbf{cis}(116^\circ);\ z_2=2\textbf{cis}(82^\circ)$

Answer

$4\sqrt{3}\textbf{cis}(198^\circ)$

24) $z_1=\sqrt{2}\textbf{cis}(205^\circ);\ z_2=2\sqrt{2}\textbf{cis}(118^\circ)$

25) $z_1=3\textbf{cis}(120^\circ);\ z_2=\frac{1}{4}\textbf{cis}(60^\circ)$

Answer

$\frac{3}{4}\textbf{cis}(180^\circ)$

26) $z_1 = 3\mathbf{cis}\left(\frac{\pi}{4}\right); z_2 = 5\mathbf{cis}\left(\frac{\pi}{6}\right)$

27) $z_1 = \sqrt{5}\mathbf{cis}\left(\frac{5\pi}{8}\right); z_2 = \sqrt{15}\mathbf{cis}\left(\frac{\pi}{12}\right)$

Answer

$5\sqrt{3}\mathbf{cis}\left(\frac{17\pi}{24}\right)$

28) $z_1 = 4\mathbf{cis}\left(\frac{\pi}{2}\right); z_2 = 2\mathbf{cis}\left(\frac{\pi}{4}\right)$

For the exercises 29-,34 find $\frac{z_1}{z_2}$ in polar form.

29) $z_1 = 21\mathbf{cis}(135^\circ); z_2 = 3\mathbf{cis}(65^\circ)$

Answer

$7\mathbf{cis}(70^\circ)$

30) $z_1 = \sqrt{2}\mathbf{cis}(90^\circ); z_2 = 2\mathbf{cis}(60^\circ)$

31) $z_1 = 15\mathbf{cis}(120^\circ); z_2 = 3\mathbf{cis}(40^\circ)$

Answer

$5\mathbf{cis}(80^\circ)$

32) $z_1 = 6\mathbf{cis}\left(\frac{\pi}{3}\right); z_2 = 2\mathbf{cis}\left(\frac{\pi}{4}\right)$

33) $z_1 = 5\sqrt{2}\mathbf{cis}(\pi); z_2 = \sqrt{2}\mathbf{cis}\left(\frac{2\pi}{3}\right)$

Answer

$5\mathbf{cis}\left(\frac{\pi}{3}\right)$

34) $z_1 = 2\mathbf{cis}\left(\frac{3\pi}{5}\right); z_2 = 3\mathbf{cis}\left(\frac{\pi}{4}\right)$

For the exercises 35-40, find the powers of each complex number in polar form.

35) Find z^3 when $z = 5\mathbf{cis}(45^\circ)$.

Answer

$125\mathbf{cis}(135^\circ)$

36) Find z^4 when $z = 2\mathbf{cis}(70^\circ)$.

37) Find z^2 when $z = 3\mathbf{cis}(120^\circ)$.

Answer

$9\mathbf{cis}(240^\circ)$

38) Find z^2 when $z = 4\mathbf{cis}\left(\frac{\pi}{4}\right)$.

39) Find z^4 when $z = \mathbf{cis}\left(\frac{3\pi}{16}\right)$.

Answer

$\mathbf{cis}\left(\frac{3\pi}{4}\right)$

40) Find z^3 when $z = 3\mathbf{cis}\left(\frac{5\pi}{3}\right)$.

For the exercises 41-45, evaluate each root.

41) Evaluate the cube root of z when $z = 27\mathbf{cis}(240^\circ)$.

Answer

$3\mathbf{cis}(80^\circ), 3\mathbf{cis}(200^\circ), 3\mathbf{cis}(320^\circ)$

42) Evaluate the square root of z when $z = 16\mathbf{cis}(100^\circ)$.

43) Evaluate the cube root of z when $z = 32\mathbf{cis}\left(\frac{2\pi}{3}\right)$.

Answer

$2\sqrt[3]{4}\mathbf{cis}\left(\frac{2\pi}{9}\right), 2\sqrt[3]{4}\mathbf{cis}\left(\frac{8\pi}{9}\right), 2\sqrt[3]{4}\mathbf{cis}\left(\frac{14\pi}{9}\right)$

44) Evaluate the square root of z when $z = 32\mathbf{cis}(\pi)$.

45) Evaluate the cube root of z when $z = 8\mathbf{cis}\left(\frac{7\pi}{4}\right)$.

Answer

$2\sqrt{2}\mathbf{cis}\left(\frac{7\pi}{8}\right), 2\sqrt{2}\mathbf{cis}\left(\frac{15\pi}{8}\right)$

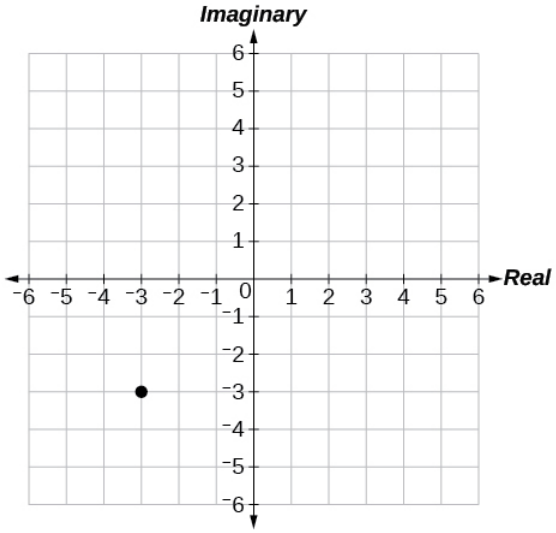
Graphical

For the exercises 46-55, plot the complex number in the complex plane.

46) $2 + 4i$

47) $-3 - 3i$

Answer



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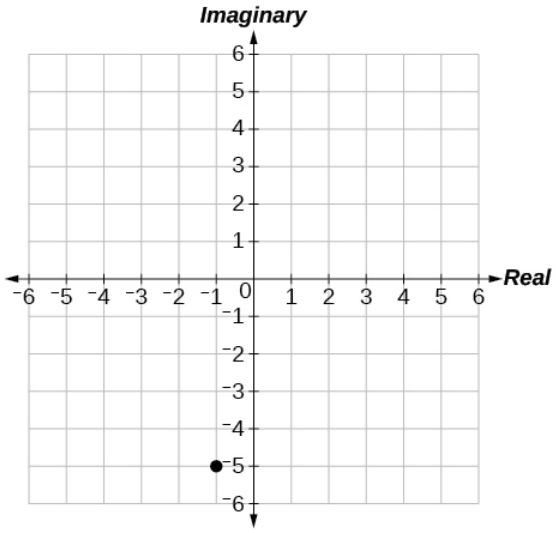
48) $5-4i$

49) $-1-5i$

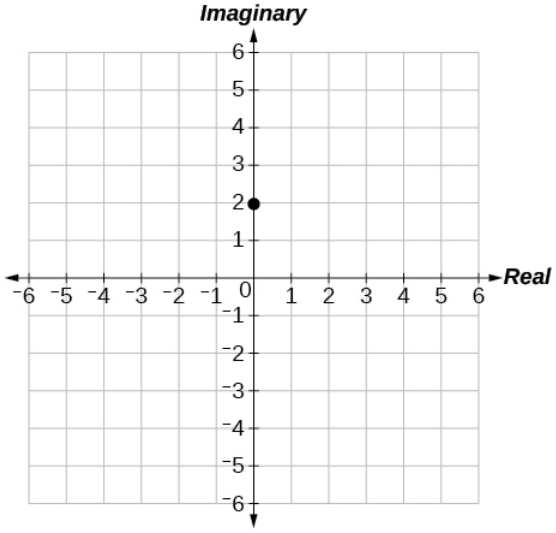
Answer



50) $3+2i$

51) $2i$

Answer



52) -4

53) $6-2i$

Answer

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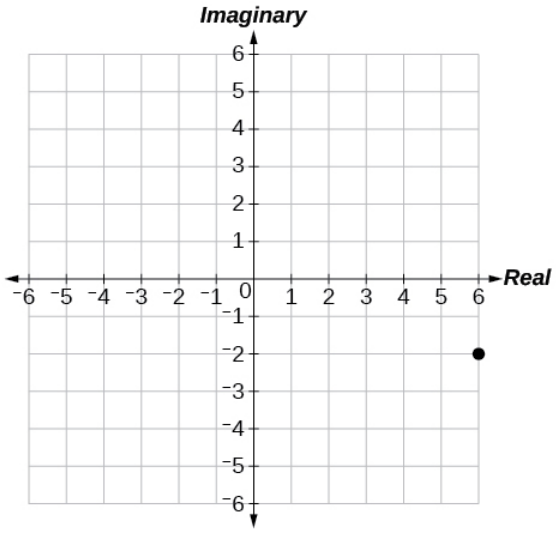
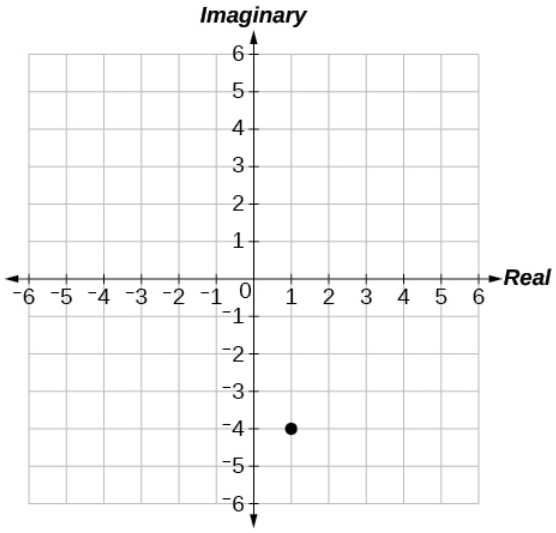


54) $-2+i$

55) $1-4i$

Answer



Technology

For the exercises 56-, find all answers rounded to the nearest hundredth.

56) Use the rectangular to polar feature on the graphing calculator to change $5+5i$ to polar form.

57) Use the rectangular to polar feature on the graphing calculator to change $3-2i$ to polar form.

Answer

$3.61e^{-0.59i}$

58) Use the rectangular to polar feature on the graphing calculator to change $-3-8i$ to polar form.

59) Use the polar to rectangular feature on the graphing calculator to change $4\mathbf{cis}(120^\circ)$ to rectangular form.

Answer

$-2+3.46i$

60) Use the polar to rectangular feature on the graphing calculator to change $2\mathbf{cis}(45^\circ)$ to rectangular form.

61) Use the polar to rectangular feature on the graphing calculator to change $5\mathbf{cis}(210^\circ)$ to rectangular form.

Answer

$-4.33-2.50i$

8.6: Parametric Equations

Verbal

1) What is a system of parametric equations?

Answer

A pair of functions that is dependent on an external factor. The two functions are written in terms of the same parameter. For example, $x=f(t)$ and $y=f(t)$.

2) Some examples of a third parameter are time, length, speed, and scale. Explain when time is used as a parameter.

3) Explain how to eliminate a parameter given a set of parametric equations.

Answer

Choose one equation to solve for t, substitute into the other equation and simplify.

4) What is a benefit of writing a system of parametric equations as a Cartesian equation?

5) What is a benefit of using parametric equations?

Answer

Some equations cannot be written as functions, like a circle. However, when written as two parametric equations, separately the equations are functions.

6) Why are there many sets of parametric equations to represent on Cartesian function?

Algebraic

For the exercises 7-25, eliminate the parameter t to rewrite the parametric equation as a Cartesian equation.

7) $\begin{cases} x(t)=5-t \\ y(t)=8-2t \end{cases}$

Answer

$y=-2+2x$

8) $\begin{cases} x(t)=6-3t \\ y(t)=10-t \end{cases}$

9) $\begin{cases} x(t)=2t+1 \\ y(t)=3\sqrt{t} \end{cases}$

Answer

$y=3\sqrt{\dfrac{x-1}{2}}$

10) $\begin{cases} x(t)=3t-1 \\ y(t)=2t^2 \end{cases}$

11) $\begin{cases} x(t)=2e^t \\ y(t)=1-5t \end{cases}$

Answer

$$x = 2e^{\frac{1-y}{5}} \text{ or } y = 1 - 5\ln\left(\frac{x}{2}\right)$$

12) $\begin{cases} x(t) = e^{-2t} \\ y(t) = 2e^{-t} \end{cases}$

13) $\begin{cases} x(t) = 4\log(t) \\ y(t) = 3 + 2t \end{cases}$

Answer

$$x = 4\log\left(\frac{y-3}{2}\right)$$

14) $\begin{cases} x(t) = \log(2t) \\ y(t) = \sqrt{t-1} \end{cases}$

15) $\begin{cases} x(t) = t^3 - t \\ y(t) = 2t \end{cases}$

Answer

$$x = \left(\frac{y}{2}\right)^3 - \frac{y}{2}$$

16) $\begin{cases} x(t) = t - t^4 \\ y(t) = t + 2 \end{cases}$

17) $\begin{cases} x(t) = e^{2t} \\ y(t) = e^{6t} \end{cases}$

Answer

$$y = x^3$$

18) $\begin{cases} x(t) = t^5 \\ y(t) = t^{10} \end{cases}$

19) $\begin{cases} x(t) = 4\cos t \\ y(t) = 5\sin t \end{cases}$

Answer

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{5}\right)^2 = 1$$

20) $\begin{cases} x(t) = 3\sin t \\ y(t) = 6\cos t \end{cases}$

21) $\begin{cases} x(t) = 2\cos^2 t \\ y(t) = -\sin t \end{cases}$

Answer

$$y^2 = 1 - \frac{1}{2}x$$

22) $\begin{cases} x(t) = \cos t + 4 \\ y(t) = 2\sin^2 t \end{cases}$

23) $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$

Answer

$y = x^2 + 2x + 1$

24) $\begin{cases} x(t) = -t \\ y(t) = t^3 + 1 \end{cases}$

25) $\begin{cases} x(t) = 2t - 1 \\ y(t) = t^3 - 2 \end{cases}$

Answer

$y = \left(\frac{x+1}{2}\right)^3 - 2$

For the exercises 26-29, rewrite the parametric equation as a Cartesian equation by building an $x - y$ table.

26) $\begin{cases} x(t) = 2t - 1 \\ y(t) = t + 4 \end{cases}$

27) $\begin{cases} x(t) = 4 - t \\ y(t) = 3t + 2 \end{cases}$

Answer

$y = -3x + 14$

28) $\begin{cases} x(t) = 2t - 1 \\ y(t) = 5t \end{cases}$

29) $\begin{cases} x(t) = 4t - 1 \\ y(t) = 4t + 2 \end{cases}$

Answer

$y = x + 3$

For the exercises 30-33, parameterize (write parametric equations for) each Cartesian equation by setting $x(t) = t$ or by setting $y(t) = t$.

30) $y(x) = 3x^2 + 3$

31) $y(x) = 2\sin x + 1$

Answer

$\begin{cases} x(t) = t \\ y(t) = 2\sin t + 1 \end{cases}$

32) $x(y) = 3\log(y) + y$

33) $x(y) = \sqrt{y} + 2y$

Answer

$\begin{cases} x(t) = \sqrt{t} + 2t \\ y(t) = t \end{cases}$

For the exercises 34-41, parameterize (write parametric equations for) each Cartesian equation by using $x(t) = a\cos t$ and $y(t) = b\sin t$. Identify the curve.

34) $\frac{x^2}{4} + \frac{x^2}{9} = 1$

35) $\frac{x^2}{16} + \frac{x^2}{36} = 1$

Answer

$\begin{cases} x(t) = 4\cos t \\ y(t) = 6\sin t \end{cases}$; Ellipse

36) $x^2 + y^2 = 16$

37) $x^2 + y^2 = 10$

Answer

$\begin{cases} x(t) = \sqrt{10}\cos t \\ y(t) = \sqrt{10}\sin t \end{cases}$; Circle

38) Parameterize the line from $(3, 0)$ to $(-2, -5)$ so that the line is at $(3, 0)$ at $t = 0$, and at $(-2, -5)$ at $t = 1$.

39) Parameterize the line from $(-1, 0)$ to $(3, -2)$ so that the line is at $(-1, 0)$ at $t = 0$, and at $(3, -2)$ at $t = 1$.

Answer

$\begin{cases} x(t) = -1 + 4t \\ y(t) = -2t \end{cases}$

40) Parameterize the line from $(-1, 5)$ to $(2, 3)$ so that the line is at $(-1, 5)$ at $t = 0$, and at $(2, 3)$ at $t = 1$.

41) Parameterize the line from $(4, 1)$ to $(6, -2)$ so that the line is at $(4, 1)$ at $t = 0$, and at $(6, -2)$ at $t = 1$.

Answer

$\begin{cases} x(t) = 4 + 2t \\ y(t) = 1 - 3t \end{cases}$

Technology

For the exercises 42-43, use the table feature in the graphing calculator to determine whether the graphs intersect.

42) $\begin{cases} x_1(t) = 3t \\ y_1(t) = 2t - 1 \end{cases}$ and $\begin{cases} x_2(t) = t + 3 \\ y_2(t) = 4t - 4 \end{cases}$

43) $\begin{cases} x_1(t) = t^2 \\ y_1(t) = 2t - 1 \end{cases}$ and $\begin{cases} x_2(t) = -t + 6 \\ y_2(t) = t + 1 \end{cases}$

Answer

yes, at $t = 2$

For the exercises 44-46, use a graphing calculator to complete the table of values for each set of parametric equations.

44) $\begin{cases} x_1(t) = 3t^2 - 3t + 7 \\ y_1(t) = 2t + 3 \end{cases}$

t	x	y
-1		
0		
1		

45) $\begin{cases} x_1(t) = t^2 - 4 \\ y_1(t) = 2t^2 - 1 \end{cases}$

t	x	y

t	x	y
1		
2		
3		

Answer

t	x	y
1	-3	1
2	0	7
3	5	17

46) $\begin{cases} x_1(t) = t^4 \\ y_1(t) = t^3 + 4 \end{cases}$

t	x	y
-1		
0		
1		
2		

Extensions

47) Find two different sets of parametric equations for $y = (x+1)^2$.

Answer

answers may vary: $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$ and $\begin{cases} x(t) = t + 1 \\ y(t) = (t+2)^2 \end{cases}$

48) Find two different sets of parametric equations for $y = 3x - 2$.

49) Find two different sets of parametric equations for $y = x^2 - 4x + 4$.

Answer

answers may vary: $\begin{cases} x(t) = t \\ y(t) = t^2 - 4t + 4 \end{cases}$ and $\begin{cases} x(t) = t + 2 \\ y(t) = t^2 \end{cases}$

8.7: Parametric Equations - Graphs

Verbal

1) What are two methods used to graph parametric equations?

Answer

plotting points with the orientation arrow and a graphing calculator

2) What is one difference in point-plotting parametric equations compared to Cartesian equations?

3) Why are some graphs drawn with arrows?

Answer

The arrows show the orientation, the direction of motion according to increasing values of t.

4) Name a few common types of graphs of parametric equations.

5) Why are parametric graphs important in understanding projectile motion?

Answer

The parametric equations show the different vertical and horizontal motions over time.

Graphical

For the exercises 6-11, graph each set of parametric equations by making a table of values. Include the orientation on the graph.

6) $\begin{cases} x(t) = t \\ y(t) = t^2 - 1 \end{cases}$

t	x	y
-3		
-2		
-1		
0		
1		
2		
3		

7) $\begin{cases} x(t) = t - 1 \\ y(t) = t^2 \end{cases}$

t	-3	-2	-1	0	1	2
x						
y						

Answer

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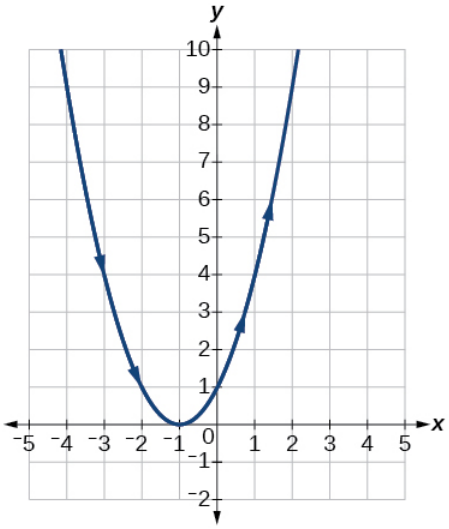


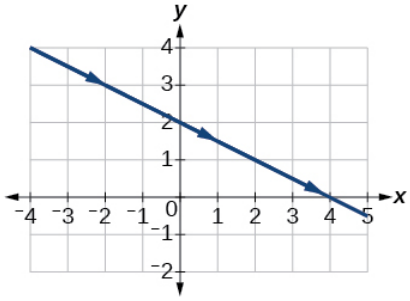
8) $\begin{cases} x(t) = 2 + t \\ y(t) = 3 - 2t \end{cases}$

t	-2	-1	0	1	2	3
x						
y						

9) $\begin{cases} x(t) = -2 - 2t \\ y(t) = 3 + t \end{cases}$

t	-3	-2	-1	0	1
x					
y					

Answer



10) $\begin{cases} x(t) = t^3 \\ y(t) = t + 2 \end{cases}$

t	-2	-1	0	1	2

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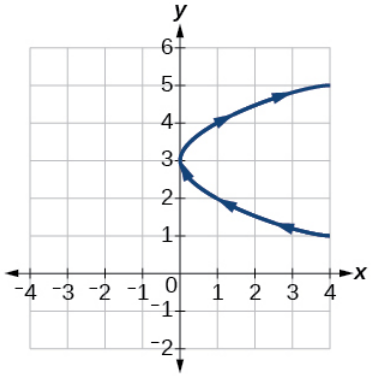
x					
y					

11) $\begin{cases} x(t) = t^2 \\ y(t) = t + 3 \end{cases}$

t	-2	-1	0	1	2
x					
y					

Answer

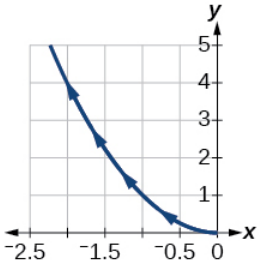


For the exercises 12-22, sketch the curve and include the orientation.

12) $\begin{cases} x(t) = t \\ y(t) = \sqrt{t} \end{cases}$

13) $\begin{cases} x(t) = -\sqrt{t} \\ y(t) = t \end{cases}$

Answer



(t from -1 to 5)

14) $\begin{cases} x(t) = 5 - |t| \\ y(t) = t + 2 \end{cases}$

15) $\begin{cases} x(t) = -t + 2 \\ y(t) = 5 - |t| \end{cases}$

Answer

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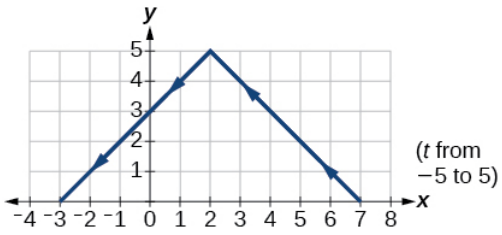
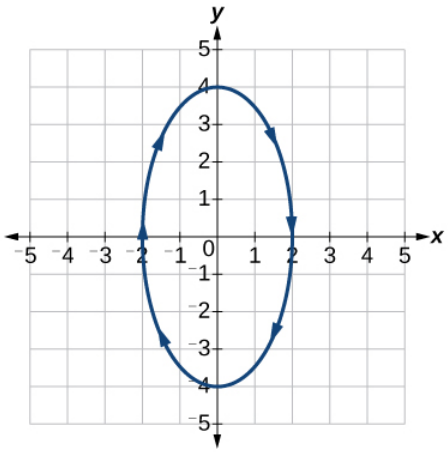


16) $\begin{cases} x(t) = 4\sin t \\ y(t) = 2\cos t \end{cases}$

17) $\begin{cases} x(t) = 2\sin t \\ y(t) = 4\cos t \end{cases}$

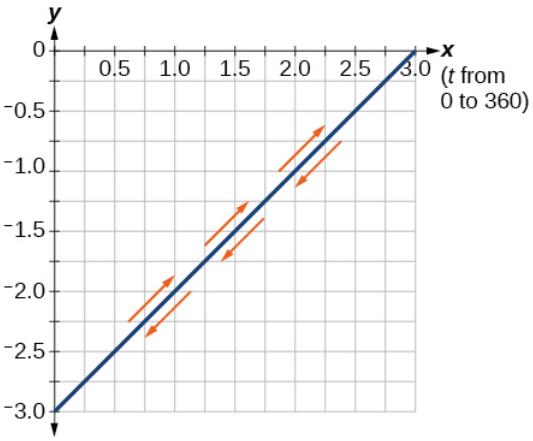
Answer



18) $\begin{cases} x(t) = 3\cos^2 t \\ y(t) = -3\sin t \end{cases}$

19) $\begin{cases} x(t) = 3\cos^2 t \\ y(t) = -3\sin^2 t \end{cases}$

Answer



20) $\begin{cases} x(t) = \sec t \\ y(t) = \tan t \end{cases}$

21) $\begin{cases} x(t) = \sec t \\ y(t) = \tan^2 t \end{cases}$

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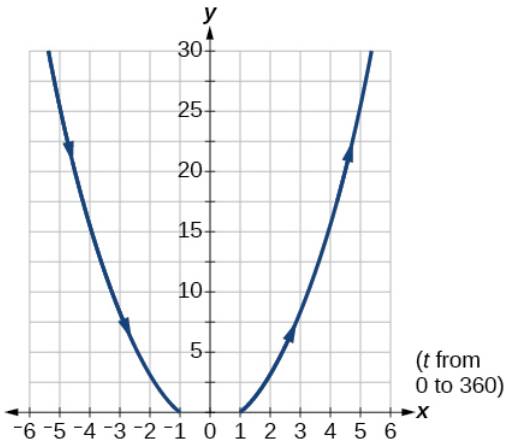
Answer

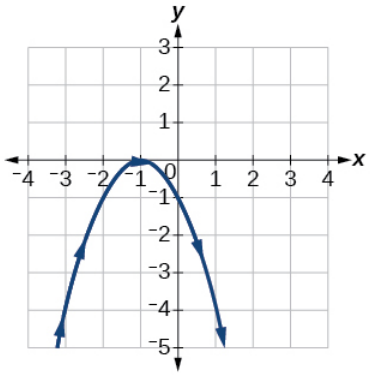


22) $\begin{cases} x(t) = \dfrac{1}{e^{2t}} \\ y(t) = e^{-t} \end{cases}$

For the exercises 23-27, graph the equation and include the orientation. Then, write the Cartesian equation.

23) $\begin{cases} x(t) = t - 1 \\ y(t) = -t^2 \end{cases}$

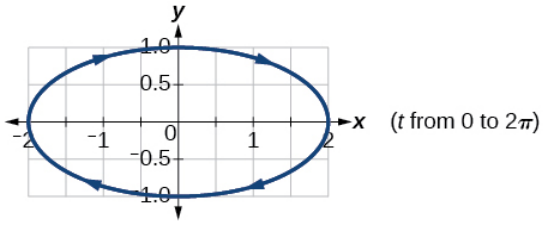
Answer



24) $\begin{cases} x(t) = t^3 \\ y(t) = t + 3 \end{cases}$

25) $\begin{cases} x(t) = 2\cos t \\ y(t) = -\sin t \end{cases}$

Answer



26) $\begin{cases} x(t) = 7\cos t \\ y(t) = 7\sin t \end{cases}$

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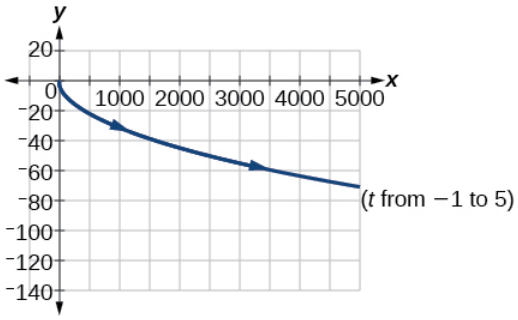
27) $\begin{cases} x(t) = e^{2t} \\ y(t) = -e^{t} \end{cases}$

Answer

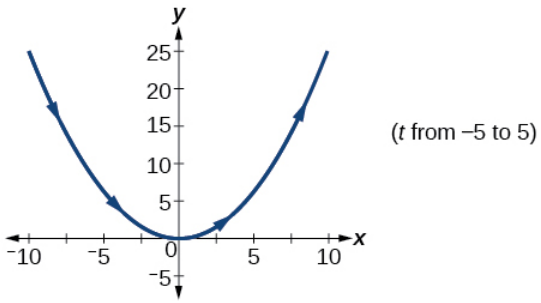


For the exercises 28-33, graph the equation and include the orientation.

28) $x = t^2, y = 3t, 0 \le t \le 5$

29) $x = 2t, y = t^2, -5 \le t \le 5$

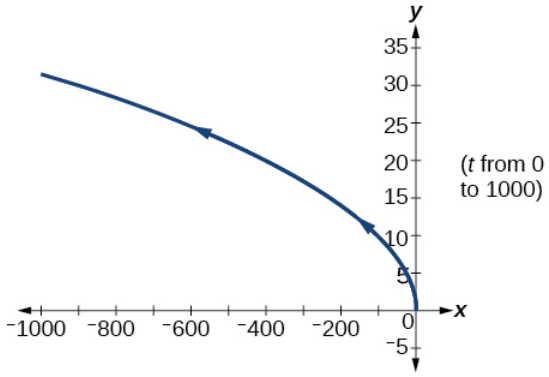
Answer



30) $x = t, y = \sqrt{25 - t^2}, 0 < t \le 5$

31) $x(t) = -t, y(t) = \sqrt{t}, t \ge 0$

Answer



32) $x = -2\cos t, y = 6\sin t, 0 \le t \le \pi$

33) $x = -\sec t, y = \tan t, -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}$

Answer

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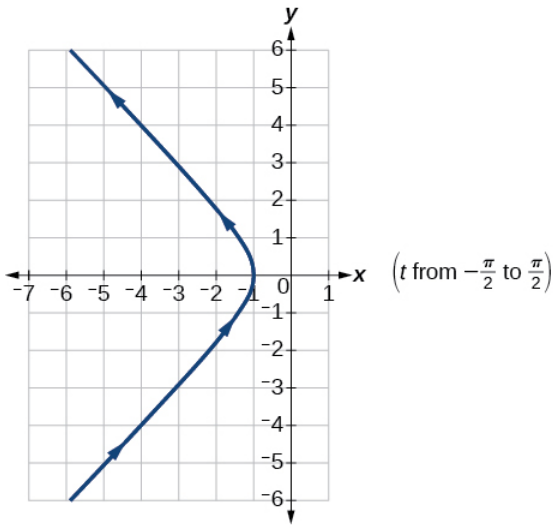


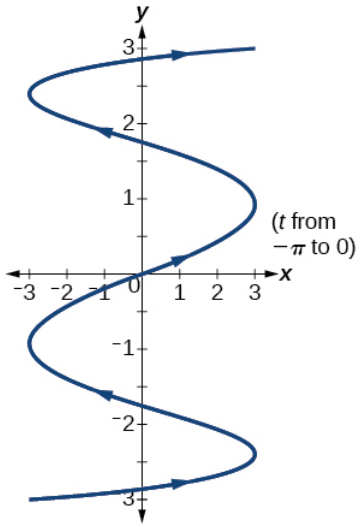
For the exercises 34-41, use the parametric equations for integers a and b:

$$x(t) = a\cos((a+b)t)$$
$$y(t) = a\cos((a-b)t)$$

34) Graph on the domain $[-\pi, 0]$, where $a = 2$ and $b = 1$, and include the orientation.

35) Graph on the domain $[-\pi, 0]$, where $a = 3$ and $b = 2$, and include the orientation.

Answer



36) Graph on the domain $[-\pi, 0]$, where $a = 4$ and $b = 3$, and include the orientation.

37) Graph on the domain $[-\pi, 0]$, where $a = 5$ and $b = 4$, and include the orientation.

Answer

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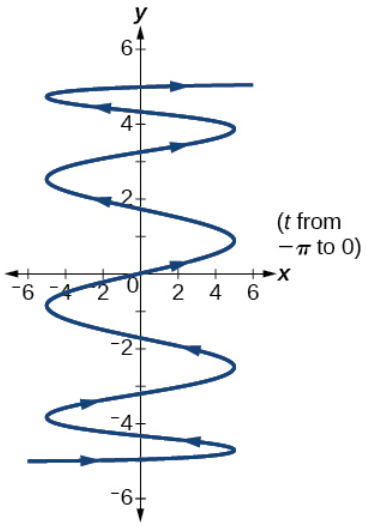

38) If a is 1 more than b, describe the effect the values of a and b have on the graph of the parametric equations.

39) Describe the graph if $a = 100$ and $b = 99$.

Answer

There will be 100 back-and-forth motions.

40) What happens if b is 1 more than a? Describe the graph.

41) If the parametric equations $x(t) = t^2$ and $y(t) = 6 - 3t$ have the graph of a horizontal parabola opening to the right, what would change the direction of the curve?

Answer

Take the opposite of the $x(t)$ equation.

For the exercises 42-46, describe the graph of the set of parametric equations.

42) $x(t) = -t^2$ and $y(t)$ is linear

43) $y(t) = t^2$ and $x(t)$ is linear

Answer

The parabola opens up.

44) $y(t) = -t^2$ and $x(t)$ is linear

45) Write the parametric equations of a circle with center $(0, 0)$, radius 5, and a counterclockwise orientation.

Answer

$$\begin{cases} x(t) = 5\cos t \\ y(t) = 5\sin t \end{cases}$$

46) Write the parametric equations of an ellipse with center $(0, 0)$,major axis of length 10, minor axis of length 6, and a counterclockwise orientation.

For the exercises 47-52, use a graphing utility to graph on the window $[-3, 3]$ by $[-3, 3]$ on the domain $[0, 2\pi)$ for the following values of a and b, and include the orientation.

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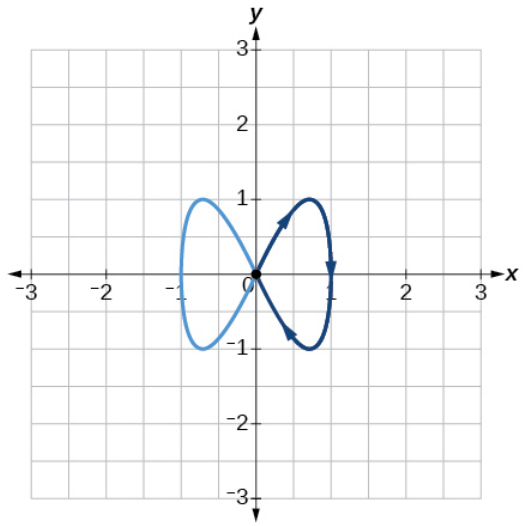
$$\begin{cases} x(t) = \sin(at) \\ y(t) = \sin(bt) \end{cases}$$

47) $a = 1, b = 2$

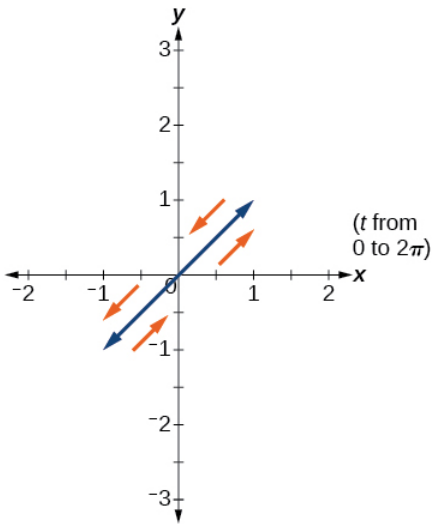
Answer



48) $a = 2, b = 1$

49) $a = 3, b = 3$

Answer



50) $a = 5, b = 5$

51) $a = 2, b = 5$

Answer

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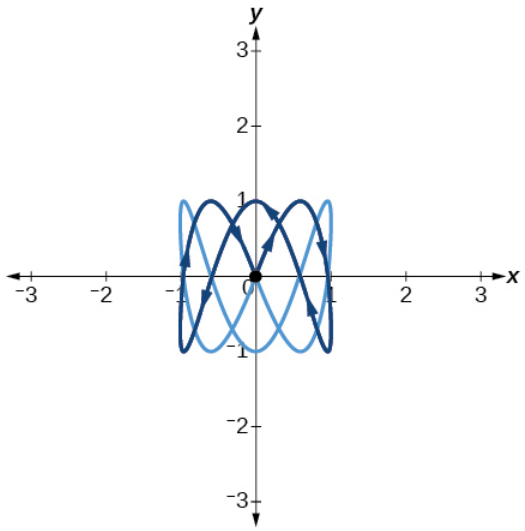


52) $a = 5, b = 2$

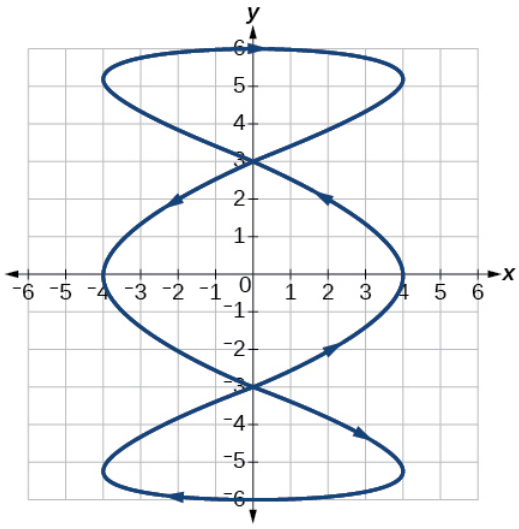
Technology

For the exercises 53-56, look at the graphs that were created by parametric equations of the form

$$\begin{cases} x(t) = a\cos(bt) \\ y(t) = c\sin(dt) \end{cases}$$

Use the parametric mode on the graphing calculator to find the values of a, b, c, and d to achieve each graph.

53)



Answer

$a = 4, b = 3, c = 6, d = 1$

54)

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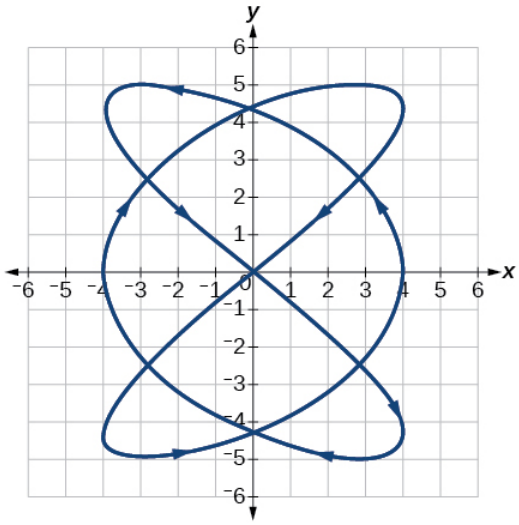
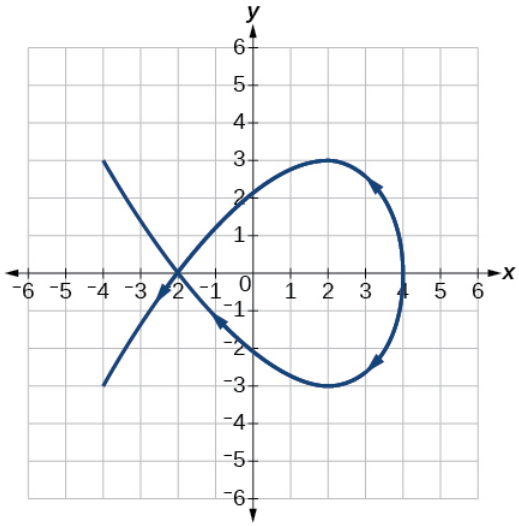


55)



Answer

$a = 4, b = 2, c = 3, d = 3$

56)

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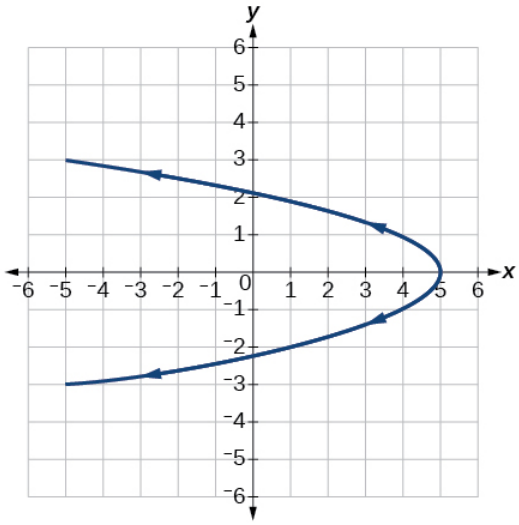


For the exercises 57-62, use a graphing utility to graph the given parametric equations.

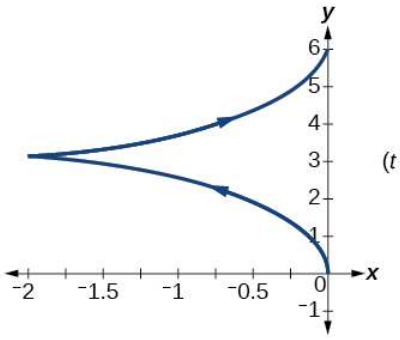
$$\begin{cases} x(t) = \cos t - 1 \\ y(t) = \sin t + t \end{cases}$$

$$\begin{cases} x(t) = \cos t + t \\ y(t) = \sin t - 1 \end{cases}$$

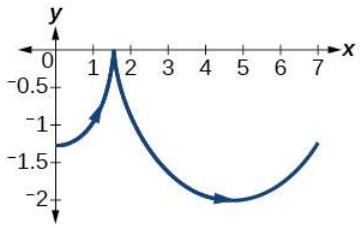
$$\begin{cases} x(t) = t - \sin t \\ y(t) = \cos t - 1 \end{cases}$$

57) Graph all three sets of parametric equations on the domain $[0, 2\pi]$.

Answer



(t from 0 to 2π)



(t from 0 to 2π)

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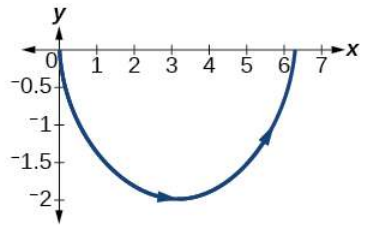
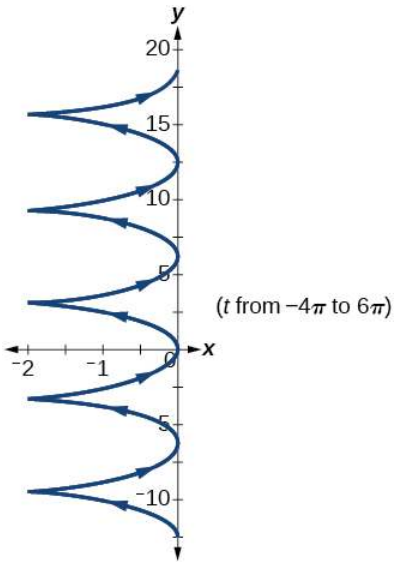


(t from 0 to 2π)

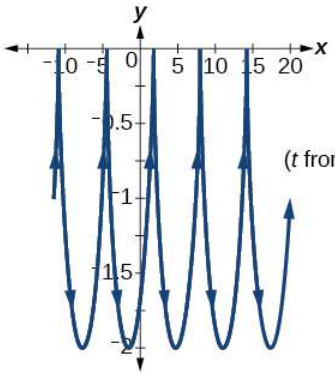
58) Graph all three sets of parametric equations on the domain $[0, 4\pi]$.

59) Graph all three sets of parametric equations on the domain $[-4\pi, 6\pi]$.

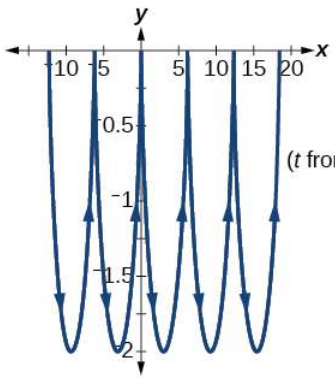
Answer



(t from -4π to 6π)



(t from -4π to 6π)



(t from -4π to 6π)

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60) The graph of each set of parametric equations appears to “creep” along one of the axes. What controls which axis the graph creeps along?

61) Explain the effect on the graph of the parametric equation when we switched $\sin t$ and $\cos t$.

Answer

The y-intercept changes.

62) Explain the effect on the graph of the parametric equation when we changed the domain.

Extensions

63) An object is thrown in the air with vertical velocity of 20 ft/s and horizontal velocity of 15 ft/s. The object’s height can be described by the equation $y(t) = -16t^2 + 20t$, while the object moves horizontally with constant velocity 15 ft/s. Write parametric equations for the object’s position, and then eliminate time to write height as a function of horizontal position.

Answer

$$y(x) = -16\left(\frac{x}{15}\right)^2 + 20\left(\frac{x}{15}\right)$$

64) A skateboarder riding on a level surface at a constant speed of 9 ft/s throws a ball in the air, the height of which can be described by the equation $y(t) = -16t^2 + 10t + 5$. Write parametric equations for the ball’s position, and then eliminate time to write height as a function of horizontal position.

For the exercises 65-69, use this scenario: A dart is thrown upward with an initial velocity of 65 ft/s at an angle of elevation of 52°. Consider the position of the dart at any time t. Neglect air resistance.

65) Find parametric equations that model the problem situation.

Answer

$$\begin{cases} x(t) = 64t\cos(52^\circ) \\ y(t) = -16t^2 + 64t\sin(52^\circ) \end{cases}$$

66) Find all possible values of x that represent the situation.

67) When will the dart hit the ground?

Answer

approximately 3.2 seconds

68) Find the maximum height of the dart.

69) At what time will the dart reach maximum height?

Answer

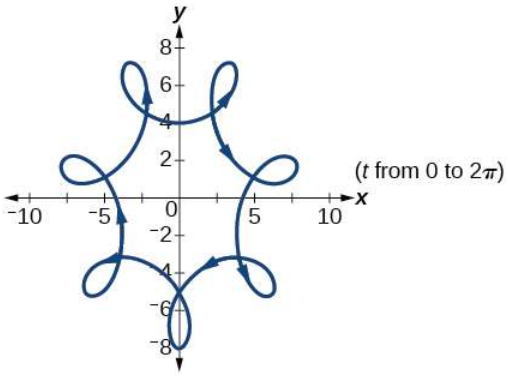
1.6 seconds

For the exercises 70-73, look at the graphs of each of the four parametric equations. Although they look unusual and beautiful, they are so common that they have names, as indicated in each exercise. Use a graphing utility to graph each on the indicated domain.

70) An epicycloid $\begin{cases} x(t) = 14\cos t - \cos(14t) \\ y(t) = 14\sin t + \sin(14t) \end{cases}$ on the domain $[0, 2\pi]$

71) An hypocycloid $\begin{cases} x(t) = 6\sin t + 2\sin(6t) \\ y(t) = 6\cos t - 2\cos(6t) \end{cases}$ on the domain $[0, 2\pi]$

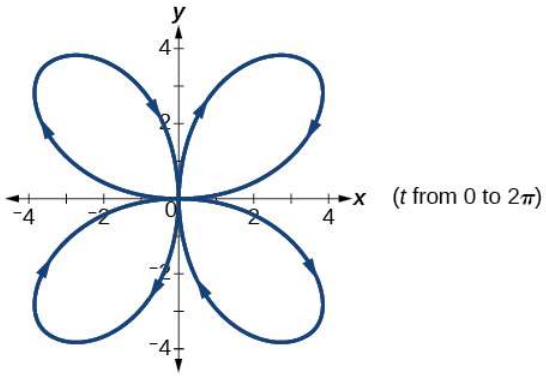
Answer



72) An hypotrochoid $\begin{cases} x(t) = 2\sin t + 5\cos(6t) \\ y(t) = 5\cos t - 2\sin(6t) \end{cases}$ on the domain $[0, 2\pi]$

73) A rose $\begin{cases} x(t) = 5\sin(2t)\sin t \\ y(t) = 5\sin(2t)\cos t \end{cases}$ on the domain $[0, 2\pi]$

Answer



8.8: Vectors

Verbal

1) What are the characteristics of the letters that are commonly used to represent vectors?

Answer

lowercase, bold letter, usually u, v, w

2) How is a vector more specific than a line segment?

3) What are i and j, and what do they represent?

Answer

They are unit vectors. They are used to represent the horizontal and vertical components of a vector. They each have a magnitude of 1.

4) What is component form?

5) When a unit vector is expressed as $\langle a, b \rangle$ which letter is the coefficient of the i and which the j?

Answer

The first number always represents the coefficient of the i and the second represents the j.

6) Given a vector with initial point $(5,2)$ and terminal point $(-1,-3)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b\rangle$.

7) Given a vector with initial point $(-4,2)$ and terminal point $(3,-3)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b\rangle$.

Answer

$\langle 7,-5\rangle$

8) Given a vector with initial point $(7,-1)$ and terminal point $(-1,-7)$, find an equivalent vector whose initial point is $(0,0)$. Write the vector in component form $\langle a,b\rangle$.

For the exercises 9-15, determine whether the two vectors u and v are equal, where u has an initial point P_1 and a terminal point P_2 and v has an initial point P_3 and a terminal point P_4.

9) $P_1=(5,1), P_2=(3,-2), P_3=(-1,3), P4=(9,-4)$

Answer

not equal

10) $P_1=(2,-3), P_2=(5,1), P_3=(6,-1), P_4=(9,3)$

11) $P_1=(-1,-1), P_2=(-4,5), P_3=(-10,6), P_4=(-13,12)$

Answer

equal

12) $P_1=(3,7), P_2=(2,1), P_3=(1,2), P_4=(-1,-4)$

13) $P_1=(8,3), P_2=(6,5), P_3=(11,8), P4=(9,10)$

Answer

equal

14) Given initial point $P_1=(-3,1)$ and terminal point $P_2=(5,2)$, write the vector v in terms of i and j.

15) Given initial point $P_1=(6,0)$ and terminal point $P_2=(-1,-3)$, write the vector v in terms of i and j.

Answer

$7i-3j$

For the exercises 16-17, use the vectors $u=i+5j, v=-2i-3j, w=4i-j$

16) Find $u+(v-w)$

17) Find $4v+2u$

Answer

$-6i-2j$

For the exercises 18-21, use the given vectors to compute $u+v, u-v, 2u-3v$

18) $u=\langle 2,-3\rangle, v=\langle 1,5\rangle$

19) $u=\langle -3,4\rangle, v=\langle -2,1\rangle$

Answer

$u+v=\langle -5,5\rangle, u-v=\langle -1,3\rangle, 2u-3v=\langle 0,5\rangle$

20) Let $v=-4i+3j$. Find a vector that is half the length and points in the same direction as v.

21) Let $v=5i+2j$. Find a vector that is twice the length and points in the opposite direction as v.

Answer

$-10i-4j$

For the exercises 22-27, find a unit vector in the same direction as the given vector.

22) $a=3i+4j$

23) $b=-2i+5j$

Answer

$-\frac{2\sqrt{29}}{29}i+\frac{5\sqrt{29}}{29}j$

24) $c=10i-j$

25) $d=-\frac{1}{3}i+\frac{5}{2}j$

Answer

$-\frac{2\sqrt{229}}{229}i+\frac{15\sqrt{229}}{229}j$

26) $u=100i+200j$

27) $u=-14i+2j$

Answer

$-\frac{7\sqrt{2}}{10}i+\frac{\sqrt{2}}{10}j$

For the exercises 28-35, find the magnitude and direction of the vector, $0\le\theta<2\pi$.

28) $\langle 0,4\rangle$

29) $\langle 6,5\rangle$

Answer

$|v|=7.810, \theta=39.806^{\circ}$

30) $\langle 2,-5\rangle$

31) $\langle -4,-6\rangle$

Answer

$|v|=7.211, \theta=236.310^{\circ}$

32) Given $u=3i-4j$ and $v=-2i+3j$, calculate $u\cdot v$.

33) Given $u=-i-j$ and $v=i+5j$, calculate $u\cdot v$.

Answer

-6

34) Given $u = \langle -2, 4 \rangle$ and $v = \langle -3, 1 \rangle$, calculate $u \cdot v$.

35) Given $u = \langle -1, 6 \rangle$ and $v = \langle 6, -1 \rangle$,calculate $u \cdot v$.

Answer

-12

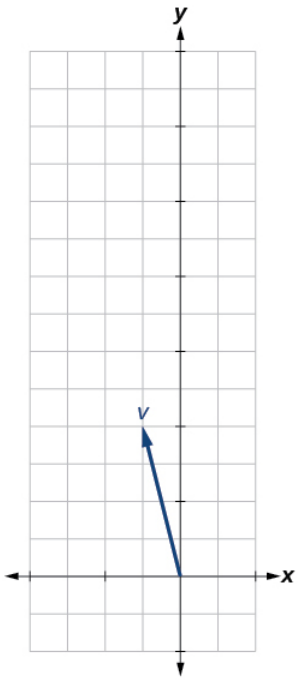
Graphical

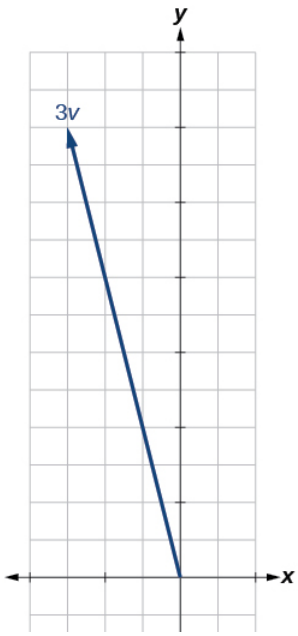
For the exercises 36-,38 given v, draw v, $3v$, and $\frac{1}{2}v$.

36) $\langle 2, -1 \rangle$

37) $\langle -1, 4 \rangle$

Answer





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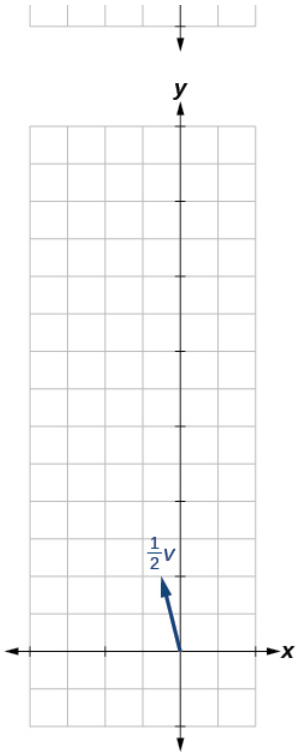
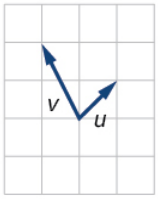


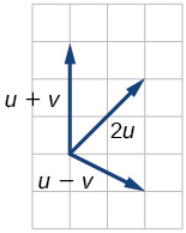
38) $\langle -3, -2 \rangle$

For the exercises 39-41, use the vectors shown to sketch $u + v$, $u - v$, and $2u$.

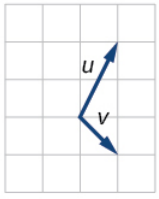
39)



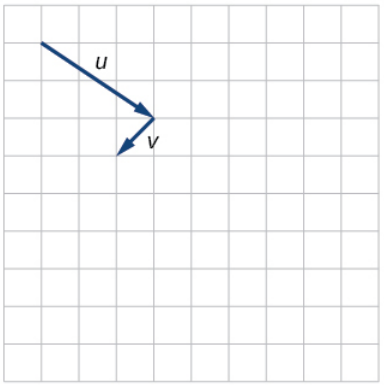
Answer



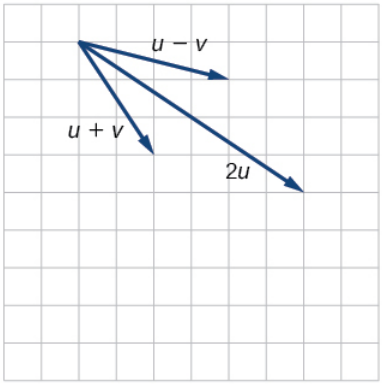
40)



41)

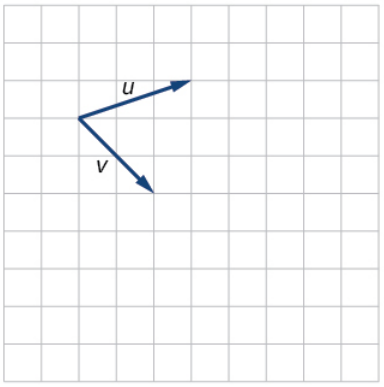


Answer



For the exercises 42-43, use the vectors shown to sketch $2u + v$.

42)



43)

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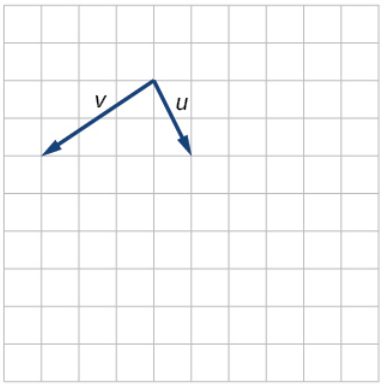
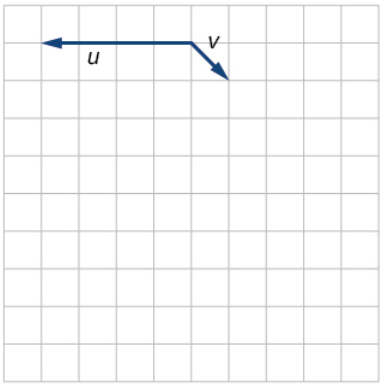


Answer

For the exercises 44-45, use the vectors shown to sketch $u - 3v$.

44)



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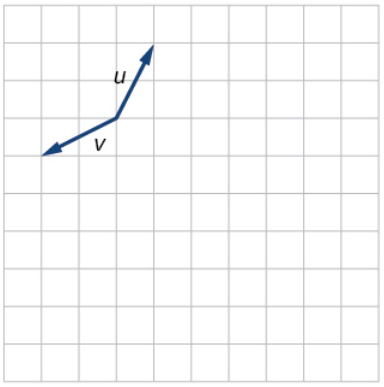
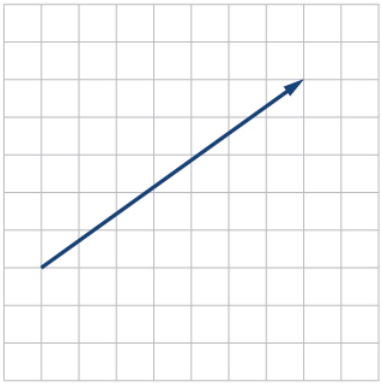


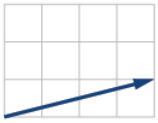
Answer



For the exercises 46-47, write the vector shown in component form.

46)

47)



Answer

$\langle 4, 1 \rangle$

48) Given initial point $P_1 = (2, 1$ and terminal point $P_2 = (-1, 2)$, write the vector v in terms of i and j, then draw the vector on the graph.

49) Given initial point $P_1 = (4, -1$ and terminal point $P_2 = (-3, 2)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

Answer

$v = -7i + 3j$

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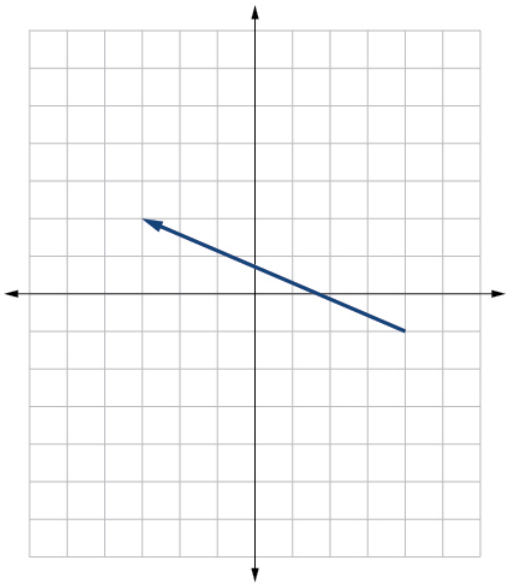

50) Given initial point $P_1 = (3, 3$ and terminal point $P_2 = (-3, 3)$, write the vector v in terms of i and j. Draw the points and the vector on the graph.

Extensions

For the exercises 51-54, use the given magnitude and direction in standard position, write the vector in component form.

51) $|v| = 6, \theta = 45^\circ$

Answer

$3\sqrt{2}i + 3\sqrt{2}j$

52) $|v| = 8, \theta = 220^\circ$

53) $|v| = 2, \theta = 300^\circ$

Answer

$i - \sqrt{3}j$

54) $|v| = 5, \theta = 135^\circ$

55) A 60-pound box is resting on a ramp that is inclined 12°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.
b. Find the magnitude of the component of the force that is parallel to the ramp.

Answer

a. 58.7
b. 12.5

56) A 25-pound box is resting on a ramp that is inclined 8°. Rounding to the nearest tenth,

a. Find the magnitude of the normal (perpendicular) component of the force.
b. Find the magnitude of the component of the force that is parallel to the ramp.

57) Find the magnitude of the horizontal and vertical components of a vector with magnitude 8 pounds pointed in a direction of 27° above the horizontal. Round to the nearest hundredth.

Answer

$x = 7.13$ pounds, $y = 3.63$ pounds

58) Find the magnitude of the horizontal and vertical components of the vector with magnitude 4 pounds pointed in a direction of 127° above the horizontal. Round to the nearest hundredth.

59) Find the magnitude of the horizontal and vertical components of a vector with magnitude 5 pounds pointed in a direction of 55° above the horizontal. Round to the nearest hundredth.

Answer

$x = 2.87$ pounds, $y = 4.10$ pounds

60) Find the magnitude of the horizontal and vertical components of the vector with magnitude 1 pound pointed in a direction of 8° above the horizontal. Round to the nearest hundredth.

Real-World Applications

61) A woman leaves home and walks 3 miles west, then 2 miles southwest. How far from home is she, and in what direction must she walk to head directly home?

Answer

4.635miles, 17.764° N of E

62) A boat leaves the marina and sails 6 miles north, then 2 miles northeast. How far from the marina is the boat, and in what direction must it sail to head directly back to the marina?

63) A man starts walking from home and walks 4 miles east, 2 miles southeast, 5 miles south, 4 miles southwest, and 2 miles east. How far has he walked? If he walked straight home, how far would he have to walk?

Answer

17 miles, 10.318miles

64) A woman starts walking from home and walks 4 miles east, 7 miles southeast, 6 miles south, 5 miles southwest, and 3 miles east. How far has she walked? If she walked straight home, how far would she have to walk?

65) A man starts walking from home and walks 3 miles at 20° north of west, then 5 miles at 10° west of south, then 4 miles at 15° north of east. If he walked straight home, how far would he have to the walk, and in what direction?

Answer

Distance: 2.868, Direction: 86.474° North of West, or 3.526° West of North

66) A woman starts walking from home and walks 6 miles at 40° north of east, then 2 miles at 15° east of south, then 5 miles at 30° south of west. If she walked straight home, how far would she have to walk, and in what direction?

67) An airplane is heading north at an airspeed of 600 km/hr, but there is a wind blowing from the southwest at 80 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

Answer

4.924°, 659 km/hr

68) An airplane is heading north at an airspeed of 500 km/hr, but there is a wind blowing from the northwest at 50 km/hr. How many degrees off course will the plane end up flying, and what is the plane's speed relative to the ground?

69) An airplane needs to head due north, but there is a wind blowing from the southwest at 60 km/hr. The plane flies with an airspeed of 550 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

Answer

4.424°

70) An airplane needs to head due north, but there is a wind blowing from the northwest at 80 km/hr. The plane flies with an airspeed of 500 km/hr. To end up flying due north, how many degrees west of north will the pilot need to fly the plane?

71) As part of a video game, the point $(5, 7)$ is rotated counterclockwise about the origin through an angle of \(35^{\circ}\). Find the new coordinates of this point.

Answer

$(0.081, 8.602)$

72) As part of a video game, the point $(7, 3)$ is rotated counterclockwise about the origin through an angle of 40°. Find the new coordinates of this point.

73) Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 10 mph relative to the car, and the car is traveling 25 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

Answer

21.801°, relative to the car's forward direction

74) Two children are throwing a ball back and forth straight across the back seat of a car. The ball is being thrown 8 mph relative to the car, and the car is traveling 45 mph down the road. If one child doesn't catch the ball, and it flies out the window, in what direction does the ball fly (ignoring wind resistance)?

75) A 50-pound object rests on a ramp that is inclined 19°. Find the magnitude of the components of the force parallel to and perpendicular to (normal) the ramp to the nearest tenth of a pound.

Answer

parallel: 16.28, perpendicular: 47.28pounds

76) Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it upward, and 5 pounds acting on it 45° from the horizontal. What single force is the resultant force acting on the body?

77) Suppose a body has a force of 10 pounds acting on it to the right, 25 pounds acting on it -135° from the horizontal, and 5 pounds acting on it directed 150° from the horizontal. What single force is the resultant force acting on the body?

Answer

19.35pounds, 231.54° from the horizontal

78) The condition of equilibrium is when the sum of the forces acting on a body is the zero vector. Suppose a body has a force of 2 pounds acting on it to the right, 5 pounds acting on it upward, and 3 pounds acting on it 45° from the horizontal. What single force is needed to produce a state of equilibrium on the body?

79) Suppose a body has a force of 3 pounds acting on it to the left, 4 pounds acting on it upward, and 2 pounds acting on it \(30^{\circ}\) from the horizontal. What single force is needed to produce a state of equilibrium on the body? Draw the vector.

Answer

5.1583pounds, 75.8° from the horizontal

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4.R: Further Applications of Trigonometry (Review)

8.1: Non-right Triangles: Law of Sines

For the exercises 1-5 assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve each triangle, if possible. Round each answer to the nearest tenth.

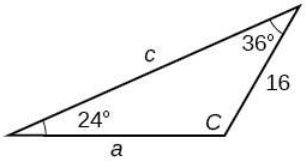
1) $\beta = 50^\circ, a = 105, b = 45$

Answer

Not possible

2) $\alpha = 43.1^\circ, a = 184.2, b = 242.8$

3) Solve the triangle.



Answer

$C = 120^\circ, a = 23.1, c = 34.1$

4) Find the area of the triangle.

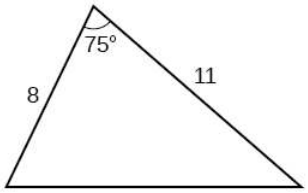
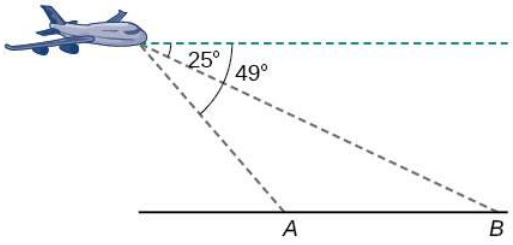


Figure 4.

5) A pilot is flying over a straight Highway. He determines the angles of depression to two mile posts 2.1 km apart to be 25° and 49°, as shown in the figure below. Find the distance of the plane from point A and the elevation of the plane.



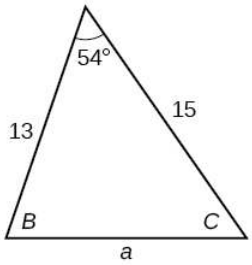
Answer

distance of the plane from point A : 2.2 km, elevation of the plane: 1.6 km

8.2: Non-right Triangles - Law of Cosines

1) Solve the triangle, rounding to the nearest tenth, assuming α is opposite side a, β is opposite side b, and γ s opposite side c : $a = 4, b = 6, c = 8$.

2) Solve the triangle in the Figure below, rounding to the nearest tenth.

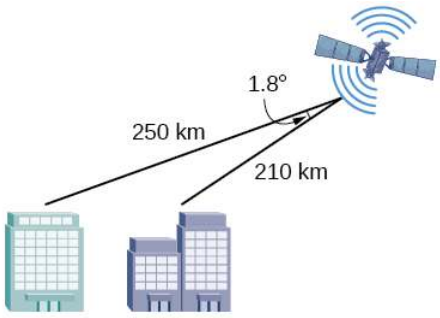


Answer

$B = 71.0^\circ, C = 55.0^\circ, a = 12.8$

3) Find the area of a triangle with sides of length 8.3, 6.6, and 9.1.

4) To find the distance between two cities, a satellite calculates the distances and angle shown in the Figure below (not to scale). Find the distance between the cities. Round answers to the nearest tenth.



Answer

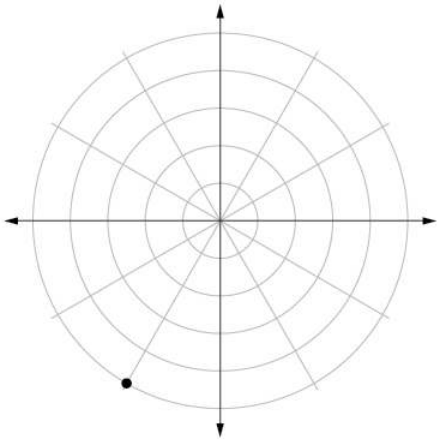
40.6km

8.3: Polar Coordinates

1) Plot the point with polar coordinates $\left(3, \frac{\pi}{6}\right)$.

2) Plot the point with polar coordinates $\left(5, \frac{-2\pi}{3}\right)$.

Answer



3) Convert $\left(6, \frac{-3\pi}{4}\right)$ to rectangular coordinates.

4) Convert $\left(-2, \frac{3\pi}{2}\right)$ to rectangular coordinates.

Answer

$(0, 2)$

5) Convert $(7, -2)$ to polar coordinates.

6) Convert $(-9, -4)$ to polar coordinates.

Answer

$(9.8489, 203.96^\circ)$

For the exercises 7-9, convert the given Cartesian equation to a polar equation.

7) $x = -2$

8) $x^2 + y^2 = 64$

Answer

$r = 8$

9) $x^2 + y^2 = -2y$

For the exercises 10-11, convert the given polar equation to a Cartesian equation.

10) $r = 7\cos\theta$

Answer

$x^2 + y^2 = 7x$

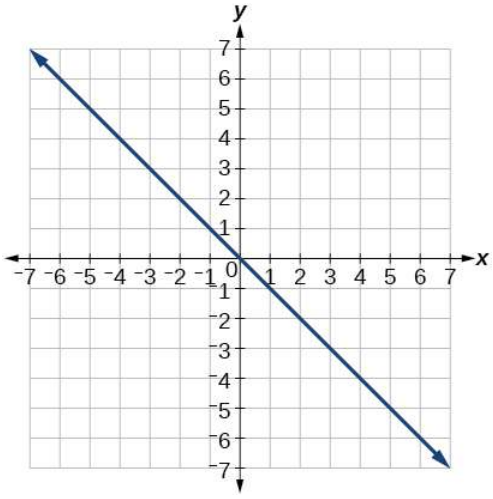
11) $r = \dfrac{-2}{4\cos\theta + \sin\theta}$

For the exercises 12-13, convert to rectangular form and graph.

12) $\theta = \dfrac{3\pi}{4}$

Answer

$y = -x$



13) $r = 5\sec\theta$

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8.4: Polar Coordinates - Graphs

For the exercises 1-5, test each equation for symmetry.

1) $r = 4 + 4\sin\theta$

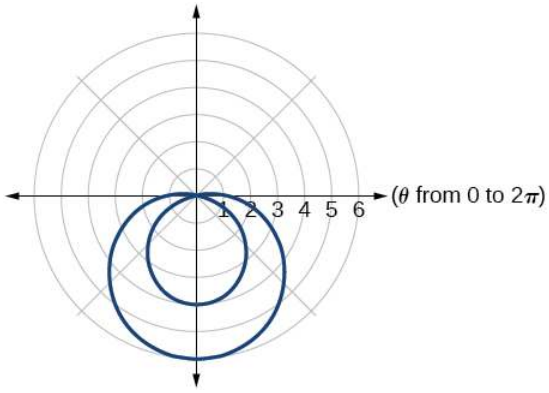
Answer

symmetric with respect to the line $\theta = \dfrac{\pi}{2}$

2) $r = 7$

3) Sketch a graph of the polar equation $r = 1 - 5\sin\theta$. Label the axis intercepts.

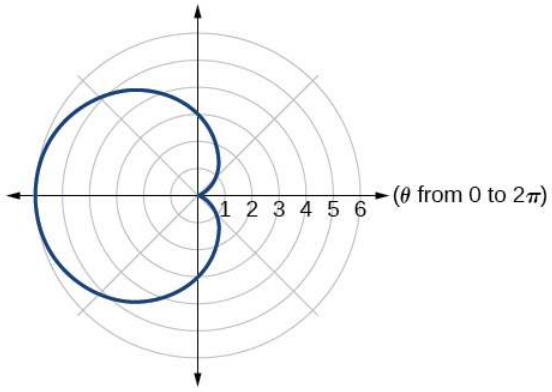
Answer



4) Sketch a graph of the polar equation $r = 5\sin(7\theta)$.

5) Sketch a graph of the polar equation $r = 3 - 3\cos\theta$

Answer



8.5: Polar Form of Complex Numbers

For the exercises 1-2, find the absolute value of each complex number.

1) $-2 + 6i$

2) $4 - 3i$

Answer

5

Write the complex number in polar form.

3) $5+9i$

4) $\frac{1}{2}-\frac{\sqrt{3}}{2}i$

Answer

$\text{cis}\left(-\frac{\pi}{3}\right)$

For the exercises 5-6, convert the complex number from polar to rectangular form.

5) $z=5\text{cis}\left(\frac{5\pi}{6}\right)$

6) $z=3\text{cis}(40^\circ)$

Answer

$2.3+1.9i$

For the exercises 7-8, find the product $z_1 z_2$ in polar form.

7) $z_1=2\text{cis}(89^\circ)$
$z_2=5\text{cis}(23^\circ)$

8) $z_1=10\text{cis}\left(\frac{\pi}{6}\right)$
$z_2=6\text{cis}\left(\frac{\pi}{3}\right)$

Answer

$60\text{cis}\left(\frac{\pi}{2}\right)$

For the exercises 9-10, find the quotient $\frac{z_1}{z_2}$ in polar form.

9) $z_1=12\text{cis}(55^\circ)$
$z_2=3\text{cis}(18^\circ)$

10) $z_1=27\text{cis}\left(\frac{5\pi}{3}\right)$
$z_2=9\text{cis}\left(\frac{\pi}{3}\right)$

Answer

$3\text{cis}\left(\frac{4\pi}{3}\right)$

For the exercises 11-12, find the powers of each complex number in polar form.

11) Find z^4 when $z=2\text{cis}(70^\circ)$

12) Find z^2 when $z=5\text{cis}\left(\frac{3\pi}{4}\right)$

Answer

$25\text{cis}\left(\frac{3\pi}{2}\right)$

For the exercises 13-14, evaluate each root.

13) Evaluate the cube root of z when $z=64\text{cis}(210^\circ)$.

14) Evaluate the square root of z when $z = 25\text{cis}\left(\frac{3\pi}{2}\right)$.

Answer

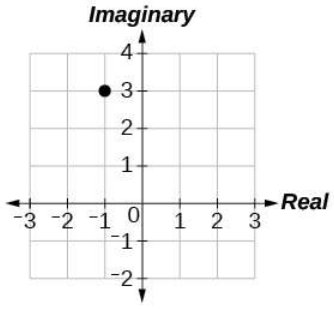
$5\text{cis}\left(\frac{3\pi}{4}\right), 5\text{cis}\left(\frac{7\pi}{4}\right)$

For the exercises 15-16, plot the complex number in the complex plane.

15) $6 - 2i$

16) $-1 + 3i$

Answer



8.6: Parametric Equations

1) $\begin{cases} x(t) = 3t - 1 \\ y(t) = \sqrt{t} \end{cases}$

2) $\begin{cases} x(t) = -\cos t \\ y(t) = 2\sin^2 t \end{cases}$

Answer

$x^2 + \frac{1}{2}y = 1$

3) Parameterize (write a parametric equation for) each Cartesian equation by using $x(t) = a\cos t$ and $y(t) = b\sin t$ for $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

4) Parameterize the line from $(-2, 3)$ to $(4, 7)$ so that the line is at $(-2, 3)$ at $t = 0$ and $(4, 7)$ at $t = 1$.

Answer

$\begin{cases} x(t) = -2 + 6t \\ y(t) = 3 + 4t \end{cases}$

8.7: Parametric Equations - Graphs

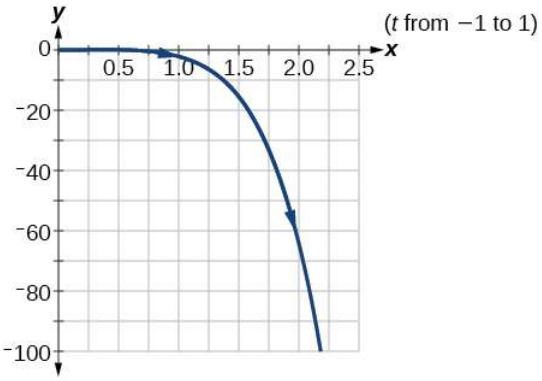
For the exercises 1-, make a table of values for each set of parametric equations, graph the equations, and include an orientation; then write the Cartesian equation.

1) $\begin{cases} x(t) = 3t^2 \\ y(t) = 2t - 1 \end{cases}$

2) $\begin{cases} x(t) = e^t \\ y(t) = -2e^{5t} \end{cases}$

Answer

$y = -2x^5$



3) $\begin{cases} x(t) = 3\cos t \\ y(t) = 2\sin t \end{cases}$

4) A ball is launched with an initial velocity of 80 feet per second at an angle of 40° to the horizontal. The ball is released at a height of 4 feet above the ground.

a. Where is the ball after 3 seconds?

b. How long is the ball in the air?

Answer

a. $\begin{cases} x(t) = (80\cos(40^\circ))t \\ y(t) = -16t^2 + (80\sin(40^\circ))t + 4 \end{cases}$

b. The ball is 14 feet high and 184 feet from where it was launched.

c. 3.3 seconds

8.8: Vectors

For the exercises 1-2, determine whether the two vectors, $\vec{u}$ and $\vec{v}$, are equal, where $\vec{u}$ has an initial point P_1 and a terminal point P_2, and $\vec{v}$ has an initial point P_3 and a terminal point P_4.

1) $P_1 = (-1, 4), P_2 = (3, 1), P_3 = (5, 5), P_4 = (9, 2)$

2) $P_1 = (6, 11), P_2 = (-2, 8), P_3 = (0, -1), P_4 = (-8, 2)$

Answer

not equal

For the exercises 3-4, use the vectors $\vec{u} = 2\hat{i} - \hat{j}$, $\vec{v} = 4\hat{i} - 3\hat{j}$, and $\vec{w} = -2\hat{i} + 5\hat{j}$ to evaluate the expression.

3) $\vec{u} - \vec{v}$

4) $2\vec{v} - \vec{u} + \vec{w}$

Answer

$4\hat{i}$

For the exercises 5-6, find a unit vector in the same direction as the given vector.

5) $\vec{a} = 8\hat{i} - 6\hat{j}$

6) $\vec{b} = -3\hat{i} - \hat{j}$

Answer

$$-\frac{3\sqrt{10}}{10}\hat{\mathbf{i}}-\frac{\sqrt{10}}{10}\hat{\mathbf{j}}$$

For the exercises 7-11, calculate $\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}$

7) $\vec{\mathbf{u}}=-2\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\vec{\mathbf{v}}=3\hat{\mathbf{i}}+7\hat{\mathbf{j}}$

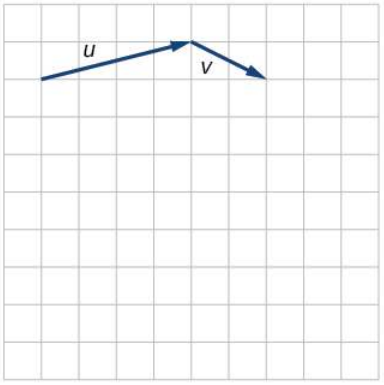
8) $\vec{\mathbf{u}}=\hat{\mathbf{i}}+4\hat{\mathbf{j}}$ and $\vec{\mathbf{v}}=4\hat{\mathbf{i}}+3\hat{\mathbf{j}}$

Answer

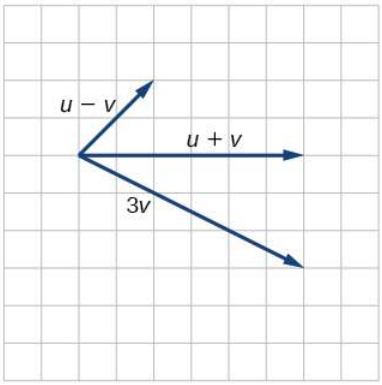
16

9) Given $\vec{\mathbf{v}}=\langle -3,4\rangle$ draw $\vec{\mathbf{v}}$, $2\vec{\mathbf{v}}$, and $\frac{1}{2}\vec{\mathbf{v}}$.

10) Given the vectors shown in the Figure below, sketch $\vec{\mathbf{u}}+\vec{\mathbf{v}}$, $\vec{\mathbf{u}}-\vec{\mathbf{v}}$ and $3\vec{\mathbf{v}}$.



Answer



11) Given initial point $P_1=(3,2)$ and terminal point $P_2=(-5,-1)$, write the vector $\vec{\mathbf{v}}$ in terms of $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$. Draw the points and the vector on the graph.

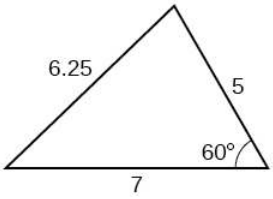
Practice Test

1) Assume α is opposite side a, β is opposite side b, and γ is opposite side c. Solve the triangle, if possible, and round each answer to the nearest tenth, given $\beta=68^\circ, b=21, c=16$.

Answer

$\alpha=67.1^\circ, \gamma=44.9^\circ, a=20.9$

2) Find the area of the triangle in the Figure below. Round each answer to the nearest tenth.



3) A pilot flies in a straight path for 2 hours. He then makes a course correction, heading 15° to the right of his original course, and flies 1 hour in the new direction. If he maintains a constant speed of 575 miles per hour, how far is he from his starting position?

Answer

1712miles

4) Convert $(2, 2)$ to polar coordinates, and then plot the point.

5) Convert $\left(2, \frac{\pi}{3}\right)$ to rectangular coordinates.

Answer

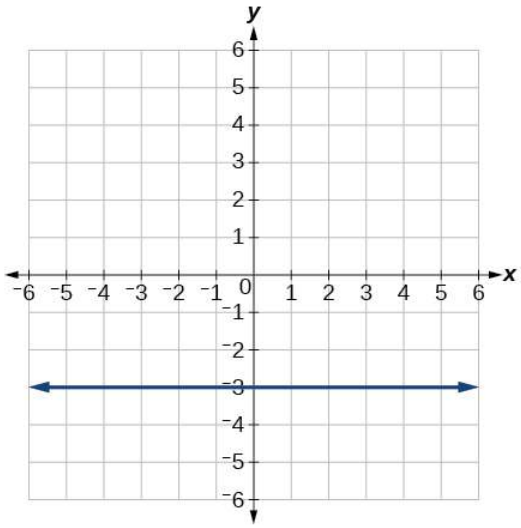
$(1, \sqrt{3})$

6) Convert the polar equation to a Cartesian equation: $x^2 + y^2 = 5y$.

7) Convert to rectangular form and graph:$r = -3 \csc \theta$.

Answer

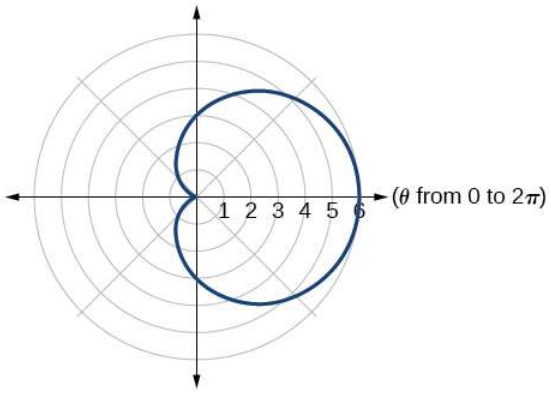
$y = -3$



8) Test the equation for symmetry: $r = -4 \sin(2\theta)$.

9) Graph $r = 3 + 3 \cos \theta$.

Answer



10) Graph $r = 3 - 5\sin\theta$.

11) Find the absolute value of the complex number $5 - 9i$.

Answer

$\sqrt{106}$

12) Write the complex number in polar form: $4 + i$.

13) Convert the complex number from polar to rectangular form: $z = 5\text{cis}\left(\frac{2\pi}{3}\right)$

Answer

$\frac{-5}{2} + i\frac{5\sqrt{3}}{2}$

14) $z_1 z_2$

15) $\frac{z_1}{z_2}$

Answer

$4\text{cis}(21^\circ)$

16) $(z_2)^3$

17) $\sqrt{z_1}$

Answer

$2\sqrt{2}\text{cis}(18^\circ), 2\sqrt{2}\text{cis}(198^\circ)$

18) Plot the complex number $-5 - i$ in the complex plane.

19) Eliminate the parameter t to rewrite the following parametric equations as a Cartesian equation: $\begin{cases} x(t) = t + 1 \\ y(t) = 2t^2 \end{cases}$

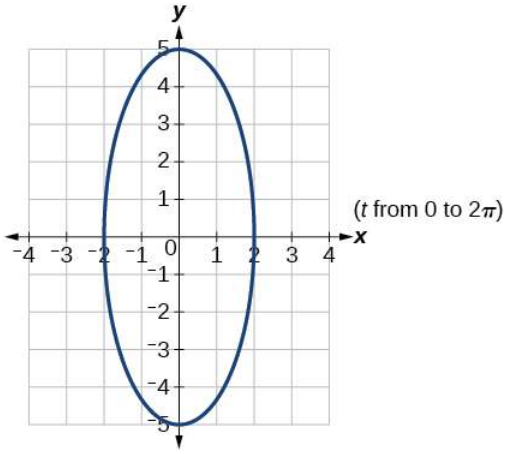
Answer

$y = 2(x-1)^2$

20) Parameterize (write a parametric equation for) the following Cartesian equation by using $x(t) = a\cos t$ and $y(t) = b\sin t$: $\frac{x^2}{36} + \frac{y^2}{100} = 1$

21) Graph the set of parametric equations and find the Cartesian equation: $\begin{cases} x(t) = -2\sin t \\ y(t) = 5\cos t \end{cases}$

Answer



22) A ball is launched with an initial velocity of 95 feet per second at an angle of 52° to the horizontal. The ball is released at a height of 3.5 feet above the ground.

a. Where is the ball after 2 seconds?

b. How long is the ball in the air?

For the exercises 23-26, use the vectors $\vec{\mathbf{u}} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}}$ and $\vec{\mathbf{v}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$.

23) Find $2\vec{\mathbf{u}} - 3\vec{\mathbf{v}}$.

Answer

$-4\hat{\mathbf{i}} - 15\hat{\mathbf{j}}$

24) Calculate $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$.

25) Find a unit vector in the same direction as $\vec{\mathbf{v}}$.

Answer

$\frac{2\sqrt{3}}{13}\hat{\mathbf{i}} + \frac{3\sqrt{3}}{13}\hat{\mathbf{j}}$

26) Given vector $\vec{\mathbf{v}}$ has an initial point $P_1 = (2, 2)$ and terminal point $P_2 = (-1, 0)$, write the vector $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$.

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Index

A

adjacent side
- 1.5: Right Triangle Trigonometry
- 1.R: Trigonometric Functions (Review)
- 2.R: Periodic Functions (Review)

amplitude
- 2.2: Graphs of the Sine and Cosine Functions

angle
- 1.2: Angles

angle of depression
- 1.5: Right Triangle Trigonometry
- 1.R: Trigonometric Functions (Review)
- 2.R: Periodic Functions (Review)

angle of elevation
- 1.5: Right Triangle Trigonometry
- 1.R: Trigonometric Functions (Review)
- 2.R: Periodic Functions (Review)

angular speed
- 1.2: Angles

arc length
- 1.2: Angles

ARC LENGTH ON A CIRCLE
- 1.2: Angles

Archimedean spiral
- 4.5: Polar Coordinates - Graphs

area of a sector
- 1.2: Angles

Area of oblique triangles
- 4.2: Non-right Triangles - Law of Sines

C

cardioid
- 4.5: Polar Coordinates - Graphs

circles
- 4.5: Polar Coordinates - Graphs

circumference
- 1.2: Angles

Cofunction Identities
- 1.5: Right Triangle Trigonometry
- 1.R: Trigonometric Functions (Review)
- 2.R: Periodic Functions (Review)
- 3.3: Sum and Difference Identities

complex number
- 4.6: Polar Form of Complex Numbers

compressed cosecant function
- 2.3: Graphs of the Other Trigonometric Functions

compressed cotangent function
- 2.3: Graphs of the Other Trigonometric Functions

compressed secant function
- 2.3: Graphs of the Other Trigonometric Functions

compressed tangent function
- 2.3: Graphs of the Other Trigonometric Functions

conversion formula
- 4.4: Polar Coordinates

cosecant
- 1.4: The Other Trigonometric Functions

Cosine Function
- 1.3: Unit Circle - Sine and Cosine Functions

cotangent
- 1.4: The Other Trigonometric Functions

coterminal angles
- 1.2: Angles

D

damped harmonic motion
- 3.7: Modeling with Trigonometric Equations

DeMoivre's Theorem
- 4.6: Polar Form of Complex Numbers

Difference formula for cosine
- 3.3: Sum and Difference Identities

Difference Formula for Sine
- 3.3: Sum and Difference Identities

Difference Formula for Tangent
- 3.3: Sum and Difference Identities

E

Endpoint
- 1.2: Angles

G

graphing
- 4.8: Parametric Equations - Graphs

H

Heron's Formula
- 4.3: Non-right Triangles - Law of Cosines

Hypotenuse
- 1.5: Right Triangle Trigonometry
- 1.R: Trigonometric Functions (Review)
- 2.R: Periodic Functions (Review)

I

identities
- 1.4: The Other Trigonometric Functions

inverse function
- 2.4: Inverse Trigonometric Functions

L

law of cosines
- 4.3: Non-right Triangles - Law of Cosines

Law of Sines
- 4.2: Non-right Triangles - Law of Sines

lemniscate
- 4.5: Polar Coordinates - Graphs

limacon
- 4.5: Polar Coordinates - Graphs

linear speed
- 1.2: Angles

linear speed related to angular speed
- 1.2: Angles

O

opposite side
- 1.5: Right Triangle Trigonometry
- 1.R: Trigonometric Functions (Review)
- 2.R: Periodic Functions (Review)

P

parameterization of a curve
- 4.7: Parametric Equations

period
- 1.4: The Other Trigonometric Functions
- 2.2: Graphs of the Sine and Cosine Functions

polar coordinates
- 4.4: Polar Coordinates
- 4.5: Polar Coordinates - Graphs

polar form
- 4.6: Polar Form of Complex Numbers

Product to sum formulas
- 3.5: Sum-to-Product and Product-to-Sum Formulas

projectiles
- 4.8: Parametric Equations - Graphs

Pythagorean Identity
- 1.3: Unit Circle - Sine and Cosine Functions
- 3.2: Solving Trigonometric Equations with Identities

Q

quotient identities
- 3.2: Solving Trigonometric Equations with Identities

R

radians
- 1.2: Angles

ray
- 1.2: Angles

reciprocal
- 3.2: Solving Trigonometric Equations with Identities

reduction formulas
- 3.4: Double-Angle, Half-Angle, and Reduction Formulas

REFERENCE ANGLES
- 1.2: Angles

right triangles
- 1.5: Right Triangle Trigonometry
- 1.R: Trigonometric Functions (Review)
- 2.R: Periodic Functions (Review)

rose curves
- 4.5: Polar Coordinates - Graphs

S

scalar
- 4.9: Vectors

secant
- 1.4: The Other Trigonometric Functions

shifted cosecant function
- 2.3: Graphs of the Other Trigonometric Functions

shifted cotangent function
- 2.3: Graphs of the Other Trigonometric Functions

shifted secant function
- 2.3: Graphs of the Other Trigonometric Functions

shifted tangent function
- 2.3: Graphs of the Other Trigonometric Functions

simple harmonic motion
- 3.7: Modeling with Trigonometric Equations

sine function
- 1.3: Unit Circle - Sine and Cosine Functions

sinusoidal functions
- 2.2: Graphs of the Sine and Cosine Functions

standard form of sinusoidal equation
- 3.7: Modeling with Trigonometric Equations

standard position
- 1.2: Angles

stretched cotangent function
- 2.3: Graphs of the Other Trigonometric Functions

stretched secant function
- 2.3: Graphs of the Other Trigonometric Functions

stretched tangent function
- 2.3: Graphs of the Other Trigonometric Functions

Sum formula for cosine
- 3.3: Sum and Difference Identities

Sum Formula for Sine
3.3: Sum and Difference Identities
Sum Formula for Tangent
3.3: Sum and Difference Identities
Sum to product formulas
3.5: Sum-to-Product and Product-to-Sum Formulas

T

tangent
1.4: The Other Trigonometric Functions
terminal side
1.2: Angles
trigonometric functions
1.5: Right Triangle Trigonometry
1.R: Trigonometric Functions (Review)
2.4: Inverse Trigonometric Functions
2.R: Periodic Functions (Review)
3.6: Solving Trigonometric Equations

U

unit circle
1.3: Unit Circle - Sine and Cosine Functions

V

vector magnitude
4.9: Vectors
vectors
4.9: Vectors
vertical asymptote
2.4: Inverse Trigonometric Functions
vetex
1.2: Angles

Glossary

Sample Word 1 | Sample Definition 1

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    - InfoPage - *Undeclared*
    - Table of Contents - *Undeclared*
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    - 1.1: Prelude to Trigonometric Functions - *CC BY 4.0*
    - 1.2: Angles - *CC BY 4.0*
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    - 2.2: Graphs of the Sine and Cosine Functions - *CC BY 4.0*
    - 2.3: Graphs of the Other Trigonometric Functions - *CC BY 4.0*
    - 2.4: Inverse Trigonometric Functions - *CC BY 4.0*
    - 2.E: Periodic Functions (Exercises) - *CC BY 4.0*
    - 2.R: Periodic Functions (Review) - *CC BY 4.0*
  - 3: Trigonometric Identities and Equations - *CC BY 4.0*
    - 3.1: Prelude to Trigonometric Identities and Equations - *CC BY 4.0*
    - 3.2: Solving Trigonometric Equations with Identities - *CC BY 4.0*
    - 3.3: Sum and Difference Identities - *CC BY 4.0*
    - 3.4: Double-Angle, Half-Angle, and Reduction Formulas - *CC BY 4.0*
    - 3.5: Sum-to-Product and Product-to-Sum Formulas - *CC BY 4.0*
    - 3.6: Solving Trigonometric Equations - *CC BY 4.0*
    - 3.7: Modeling with Trigonometric Equations - *CC BY 4.0*
    - 3.E: Trigonometric Identities and Equations (Exercises) - *CC BY 4.0*
    - 3.R: Trigonometric Identities and Equations (Review) - *CC BY 4.0*
  - 4: Further Applications of Trigonometry - *CC BY 4.0*
    - 4.1: Prelude to Further Applications of Trigonometry - *CC BY 4.0*
    - 4.2: Non-right Triangles - Law of Sines - *CC BY 4.0*
    - 4.3: Non-right Triangles - Law of Cosines - *CC BY 4.0*
    - 4.4: Polar Coordinates - *CC BY 4.0*
    - 4.5: Polar Coordinates - Graphs - *CC BY 4.0*
    - 4.6: Polar Form of Complex Numbers - *CC BY 4.0*
    - 4.7: Parametric Equations - *CC BY 4.0*
    - 4.8: Parametric Equations - Graphs - *CC BY 4.0*
    - 4.9: Vectors - *CC BY 4.0*
    - 4.E: Further Applications of Trigonometry (Exercises) - *CC BY 4.0*
    - 4.R: Further Applications of Trigonometry (Review) - *CC BY 4.0*
  - Back Matter - *Undeclared*
    - Index - *Undeclared*
    - Glossary - *Undeclared*
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    - 1.2: Angles - *CC BY 4.0*
    - 1.3: Unit Circle - Sine and Cosine Functions - *CC BY 4.0*
    - 1.4: The Other Trigonometric Functions - *CC BY 4.0*
    - 1.5: Right Triangle Trigonometry - *CC BY 4.0*
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    - 1.R: Trigonometric Functions (Review) - *CC BY 4.0*
  - 2: Periodic Functions - *CC BY 4.0*
    - 2.1: Prelude to Periodic Functions - *CC BY 4.0*
    - 2.2: Graphs of the Sine and Cosine Functions - *CC BY 4.0*
    - 2.3: Graphs of the Other Trigonometric Functions - *CC BY 4.0*
    - 2.4: Inverse Trigonometric Functions - *CC BY 4.0*
    - 2.E: Periodic Functions (Exercises) - *CC BY 4.0*
    - 2.R: Periodic Functions (Review) - *CC BY 4.0*
  - 3: Trigonometric Identities and Equations - *CC BY 4.0*
    - 3.1: Prelude to Trigonometric Identities and Equations - *CC BY 4.0*
    - 3.2: Solving Trigonometric Equations with Identities - *CC BY 4.0*
    - 3.3: Sum and Difference Identities - *CC BY 4.0*
    - 3.4: Double-Angle, Half-Angle, and Reduction Formulas - *CC BY 4.0*
    - 3.5: Sum-to-Product and Product-to-Sum Formulas - *CC BY 4.0*
    - 3.6: Solving Trigonometric Equations - *CC BY 4.0*
    - 3.7: Modeling with Trigonometric Equations - *CC BY 4.0*
    - 3.E: Trigonometric Identities and Equations (Exercises) - *CC BY 4.0*
    - 3.R: Trigonometric Identities and Equations (Review) - *CC BY 4.0*
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    - 4.1: Prelude to Further Applications of Trigonometry - *CC BY 4.0*
    - 4.2: Non-right Triangles - Law of Sines - *CC BY 4.0*
    - 4.3: Non-right Triangles - Law of Cosines - *CC BY 4.0*
    - 4.4: Polar Coordinates - *CC BY 4.0*
    - 4.5: Polar Coordinates - Graphs - *CC BY 4.0*
    - 4.6: Polar Form of Complex Numbers - *CC BY 4.0*
    - 4.7: Parametric Equations - *CC BY 4.0*
    - 4.8: Parametric Equations - Graphs - *CC BY 4.0*
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    - 4.E: Further Applications of Trigonometry (Exercises) - *CC BY 4.0*
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    - Index - *Undeclared*
    - Glossary - *Undeclared*
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