COREQUISITE PRECALCULUS



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Highline College



Acknowledgements

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Highline College
Corequisite Precalculus

Katherine Skelton

Course Audience and Printing Information

This textbook aligns with the common course numbering outcomes in the state of Washington, USA for Precalculus I. It is designed for approximately 10 weeks of instruction in a quarter system pacing. There is also an academic success skills chapter that can be referenced throughout the course as part of a first-year experience cohort or as a way to integrate good learning practices into a new student's college life.

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1.1: The Power to Learn

Questions to consider

- What actually happens to me when I learn something?
- Am I aware of different types of learning?
- Do I approach studying or practicing differently depending on the desired outcome?

Welcome to one of the most empowering chapters in this book! While each chapter focuses on showing you clear paths to success as a student, this one deals specifically with what is at the core of being a student: the act of learning.

We humans have been obsessed with how we learn and understand things since ancient times. Because of this, some of our earliest recorded philosophies have tried to explain how we take in information about the world around us, how we acquire new knowledge, and even how we can be certain what we learn is correct. This obsession has produced a large number of theories, ideas, and research into how we learn. There is a great deal of information out there on the subject—some of it is very good, and some of it, while well intentioned, has been a bit misguided.

Because of this obsession with learning, over the centuries, people have continually come up with new ideas about how we acquire knowledge. The result has been that commonly held “facts” about education have been known to change frequently. Often, what was once thought to be the newest, greatest discovery about learning was debunked later on. One well-known example of this is that of corporal punishment. For most of the time formal education has existed in our society, educators truly believed that beating students when they made a mistake actually helped them learn faster. Thankfully, *birching* (striking someone with a rod made from a birch tree) has fallen out of favor in education circles, and our institutions of learning have adopted different approaches. In this chapter, not only will you learn about current learning theories that are backed by neuroscience (something we did not have back in the days of birching), but you will also learn other learning theories that did not turn out to be as effective or as thoroughly researched as once thought. That does not mean those ideas about learning are useless. Instead, in these cases you find ways to separate the valuable parts from the myths to make good learning choices.

> ***"Research has shown that one of the most influential aids in learning is an understanding about learning itself."***

What Is the Nature of Learning?

To begin with, it is important to recognize that learning is work. Sometimes it is easy and sometimes it is difficult, but there is always work involved. For many years people made the error of assuming that learning was a passive activity that involved little more than just absorbing information. Learning was thought to be a lot like copying and pasting words in a document; the student’s mind was blank and ready for an instructor to teach them facts that they could quickly take in. As it turns out, learning is much more than that. In fact, at its most rudimentary level, it is an actual process that physically changes our brains. Even something as simple as learning the meaning of a new word requires the physical alteration of neurons and the creation of new paths to receptors. These new electrochemical pathways are formed and strengthened as we utilize, practice, or remember what we have learned. If the new skill or knowledge is used in conjunction with other things we have already learned, completely different sections of the brain, our nerves, or our muscles may be tied in as a part of the process. A good example of this would be studying a painting or drawing that depicts a scene from a story or play you are already familiar with. Adding additional connections, memories, and mental associations to things you already know something about expands your knowledge and understanding in a way that cannot be reversed. In essence, it can be said that every time we learn something new we are no longer the same.

In addition to the physical transformation that takes place during learning, there are also a number of other factors that can influence how easy or how difficult learning something can be. While most people would assume that the ease or difficulty would really depend on what is being learned, there are actually several other factors that play a greater role.

In fact, research has shown that one of the most influential factors in learning is a clear understanding about learning itself. This is not to say that you need to become neuroscientists in order to do well in school, but instead, knowing a thing or two about learning and how we learn in general can have strong, positive results for your own learning. This is called *metacognition* (i.e., thinking about thinking).

Some of the benefits to how we learn can be broken down into different areas such as

- attitude and motivation toward learning,
- types of learning,
- methods of learning, and
- your own preferences for learning.

In this chapter you will explore these different areas to better understand how they may influence your own learning, as well as how to make conscious decisions about your own learning process to maximize positive outcomes.

All Learning Is Not the Same

The first, fundamental point to understand about learning is that there are several types of learning. Different kinds of knowledge are learned in different ways. Each of these different types of learning can require different processes that may take place in completely different parts of our brain.

For example, simple memorization is a form of learning that does not always require deeper understanding. Children often learn this way when they memorize poems or verses they recite. An interesting example of this can be found in the music industry, where there have been several hit songs sung in English by vocalists who do not speak English. In these cases, the singers did not truly understand what they were singing, but instead they were taught to memorize the sounds of the words in the proper order.



Figure 1.1.2 Learning has many levels and forms. For example, collaborative learning and showing your work require different skills and produce different results than reading or notetaking on your own. (Credit: StartUpStockPhotos / Pexels)

Memorizing sounds is a very different type of learning than, say, acquiring a deep understanding of Einstein's general theory of relativity.

Notice in the comparative examples of music and physics that the different levels of learning are being defined by what they allow you to know or do. When classifying learning in this way, people usually agree on six different levels of learning. In this next section we will take a detailed look at each of these.

In the table below, the cells in the left column each contain one of the main levels of learning, categorized by what the learning allows you to do. To the right of each category are the "skill acquired" and a set of real-world examples of what those skills might be as applied to a specific topic. This set of categories is called Bloom's Taxonomy, and it is often used as a guide for educators when they are determining what students should learn within a course.

Table 2.1

Category of Learning	Skill acquired	Example 1: Musical ability	Example 2: Historical information on Charles the Bald

Category of Learning	Skill acquired	Example 1: Musical ability	Example 2: Historical information on Charles the Bald
Create	Produce new or original work	Compose a piece of music	Write a paper on Charles that draws a new conclusion about his reign
Evaluate	Justify or support an idea or decision	Make critical decisions about the notes that make up a melody—what works, what doesn't, and why	Make arguments that support the idea that Charles was a good ruler
Analyze	Draw connections	Play the specific notes that are found in the key of A	Compare and contrast the historical differences between the reign of Charles and his grandfather, Charlemagne
Apply	Use information in new ways	Use knowledge to play several notes that sound good together	Use the information to write a historical account on the reign of Charles
Understand/ Comprehend	Explain ideas or concepts	Understand the relationship between the musical notes and how to play each on a musical instrument	Explain the historical events that enabled Charles to become Emperor
Remember	Recall facts and basic concepts	Memorize notes on a musical scale	Recall that Charles the Bald was Holy Roman Emperor from 875–877 CE

A review of the above table shows that actions in the left column (or what you will be able to do with the new knowledge) has a direct influence over what needs to be learned and can even dictate the type of learning approach that is best. For example, *remembering* requires a type of learning that allows the person basic memorization. In the case of Charles the Bald and his reign, it is simply a matter of committing the dates to memory. When it comes to *understanding and comprehension*, being able to explain how Charles came to power requires not only the ability to recall several events, but also for the learner to be able to understand the cause and effect of those events and how they worked together to make Charles emperor. Another example would be the ability to *analyze*. In this particular instance the information learned would not only be about Charles, but also about other rulers, such as Charlemagne. The information would have to be of such a depth that the learner could compare the events and facts about each ruler.

When you engage in any learning activity, take the time to understand what you will do with the knowledge once you have attained it. This can help a great deal when it comes to making decisions on how to go about it. Using flashcards to help memorize angles does not really help you solve problems using geometry formulas. Instead, practicing problem-solving with the actual formulas is a much better approach. The key is to make certain the learning activity fits your needs.

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1.2: It's All in the Mindset

Questions to consider

- What is a growth mindset, and how does it affect my learning?
- What are performance goals versus learning goals?

In the previous sections of this chapter you have focused on a number of concepts and models about learning. One of the things they all have in common is that they utilize different approaches to education by presenting new ways to think about learning. In each of these, the common element has been a better understanding of yourself as a learner and how to apply what you know about yourself to your own learning experience. If you were to distill all that you have learned in this chapter so far down to a single factor, it would be about using your mindset to your best advantage. In this next section, you will examine how all of this works in a broader sense by learning about the significance of certain mindsets and how they can hinder or promote your own learning efforts.



Figure 1.2.7: Many fields of study and work create intersections of growth and fixed mindset. People may feel great ability to grow and learn in some areas, like art and communication, but feel more limited in others, such as planning and financials. Recognizing these intersections will help you approach new topics and tasks. (Credit: mentatdgt / Pexels)

Performance vs. Learning Goals

As you have discovered in this chapter, much of our ability to learn is governed by our motivations and goals. What has not yet been covered in detail has been how sometimes hidden goals or mindsets can impact the learning process. In truth, we all have goals that we might not be fully aware of, or if we are aware of them, we might not understand how they help or restrict our ability to learn. An illustration of this can be seen in a comparison of a student that has *performance*-based goals with a student that has *learning*-based goals.

If you are a student with strict performance goals, your primary psychological concern might be to appear intelligent to others. At first, this might not seem to be a bad thing for college, but it can truly limit your ability to move forward in your own learning. Instead, you would tend to play it safe without even realizing it. For example, a student who is strictly performance-goal-oriented will often only says things in a classroom discussion when they think it will make them look knowledgeable to the instructor or their classmates. For example, a performance-oriented student might ask a question that she knows is beyond the topic being covered (e.g., asking about the economics of Japanese whaling while discussing the book *Moby Dick* in an American literature course). Rarely will they ask a question in class because they actually do not understand a concept. Instead they will ask questions that make them look intelligent to others or in an effort to “stump the teacher.” When they do finally ask an honest question, it may be because they are more afraid that their lack of understanding will result in a poor performance on an exam rather than simply wanting to learn.

If you are a student who is driven by learning goals, your interactions in classroom discussions are usually quite different. You see the opportunity to share ideas and ask questions as a way to gain knowledge quickly. In a classroom discussion you can ask for

clarification immediately if you don't quite understand what is being discussed. If you are a person guided by learning goals, you are less worried about what others think since you are there to learn and you see that as the most important goal.

Another example where the difference between the two mindsets is clear can be found in assignments and other coursework. If you are a student who is more concerned about performance, you may avoid work that is challenging. You will take the "easy A" route by relying on what you already know. You will not step out of your comfort zone because your psychological goals are based on approval of your performance instead of being motivated by learning.

This is very different from a student with a learning-based psychology. If you are a student who is motivated by learning goals, you may actively seek challenging assignments, and you will put a great deal of effort into using the assignment to expand on what you already know. While getting a good grade is important to you, what is even more important is the learning itself.

If you find that you sometimes lean toward performance-based goals, do not feel discouraged. Many of the best students tend to initially focus on performance until they begin to see the ways it can restrict their learning. The key to switching to learning-based goals is often simply a matter of first recognizing the difference and seeing how making a change can positively impact your own learning.

What follows in this section is a more in-depth look at the difference between performance- and learning-based goals. This is followed by an exercise that will give you the opportunity to identify, analyze, and determine a positive course of action in a situation where you believe you could improve in this area.

WHAT STUDENTS SAY

1. In the past, did you feel like you had control over your own learning?
    a. No. Someone has always dictated how and what I learned.
    b. Yes. I always look for ways to take control of what and how I learned.
    c. I am uncertain. I never thought about it before.
2. Have you ever heard of learning styles or do you know your own learning style?
    a. No. I have never heard of learning styles.
    b. Yes. I have heard of learning styles and know my own.
    c. Yes. I have heard of learning styles, but I don't think they're accurate or relate to me.
3. Which factors other than intelligence do you think have the greatest influence on learning?
    a. Motivation
    b. Perseverance
    c. Understanding how I learn
    d. Good teachers and support

You can also take the anonymous What Students Say surveys to add your voice to this textbook. Your responses will be included in updates.

Students offered their views on these questions, and the results are displayed in the graphs below.

In the past, did you feel like you had control over your own learning?

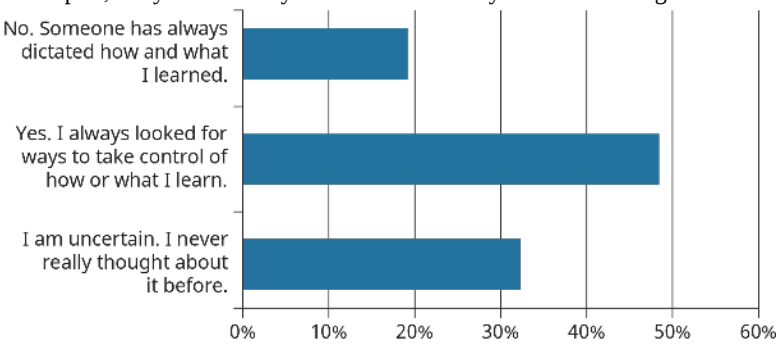


Figure 1.2.8

Have you ever heard of learning styles or do you know your own learning style?

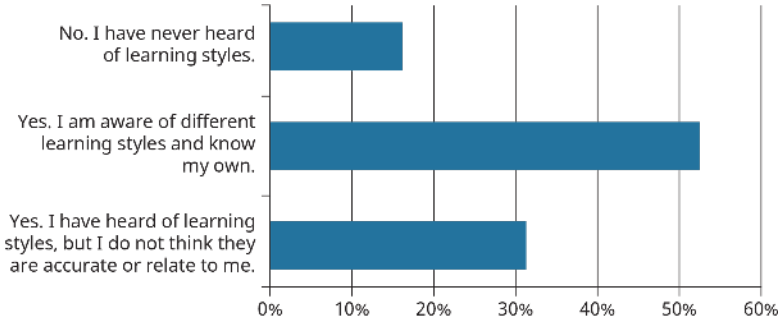


Figure 1.2.9

Which factors other than intelligence do you think have the greatest influence on learning?

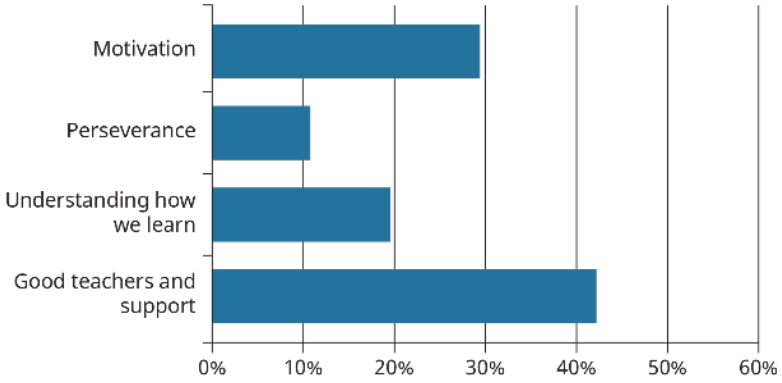


Figure 1.2.10

Fixed vs. Growth Mindset

The research-based model of these two mindsets and their influence on learning was presented in 1988 by Carol Dweck.[7] In Dr. Dweck's work, she determined that a student's perception about their own learning accompanied by a broader goal of learning had a significant influence on their ability to overcome challenges and grow in knowledge and ability. This has become known as the Fixed vs. Growth Mindset model. In this model, the *performance*-goal-oriented student is represented by the *fixed* mindset, while the *learning*-goal-oriented student is represented by the *growth* mindset.

In the following graphic, based on Dr. Dweck's research, you can see how many of the components associated with learning are impacted by these two mindsets.

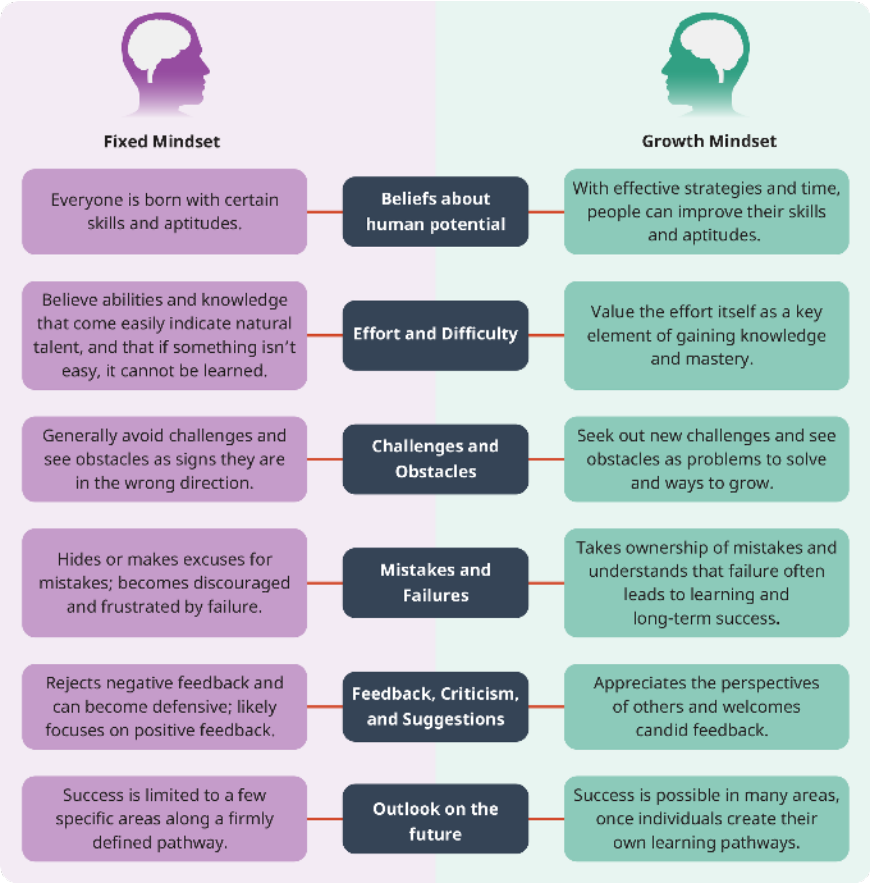


Figure 1.2.11: The differences between fixed and growth mindset are clear when aligned to key elements of learning and personality. (Credit: Based on work by Dr. Carol Dweck)

The Growth Mindset and Lessons About Failing

Something you may have noticed is that a growth mindset would tend to give a learner grit and persistence. If you had learning as your major goal, you would normally keep trying to attain that goal even if it took you multiple attempts. Not only that, but if you learned a little bit more with each try you would see each attempt as a success, even if you had not achieved complete mastery of whatever it was you were working to learn.

With that in mind, it should come as no surprise that Dr. Dweck found that those people who believed their abilities could change through learning (growth vs. a fixed mindset) readily accepted learning challenges and persisted despite early failures.

Improving Your Ability to Learn

As strange as it may seem, research into fixed vs. growth mindsets has shown that if you believe you can learn something new, you greatly improve your ability to learn. At first, this may seem like the sort of feel-good advice we often encounter in social media posts or quotes that are intended to inspire or motivate us (e.g., *believe in yourself!*), but in looking at the differences outlined between a fixed and a growth mindset, you can see how each part of the growth mindset path would increase your probability of success when it came to learning.

ACTIVITY

Very few people have a strict fixed or growth mindset all of the time. Often we tend to lean one way or another in certain situations. For example, a person trying to improve their ability in a sport they enjoy may exhibit all of the growth mindset traits and characteristics, but they find themselves blocked in a fixed mindset when they try to learn something in another area like computer programming or arithmetic.

In this exercise, do a little self-analysis and think of some areas where you may find yourself hindered by a fixed mindset. Using the outline presented below, in the far right column, write down how you can change your own behavior for each of the parts of the learning process. What will you do to move from a fixed to a growth mindset? For example, say you were trying to

learn to play a musical instrument. In the *Challenges* row, you might pursue a growth path by trying to play increasingly more difficult songs rather than sticking to the easy ones you have already mastered. In the *Criticism* row, you might take someone's comment about a weakness in timing as a motivation for you to practice with a metronome. For *Success of others* you could take inspiration from a famous musician that is considered a master and study their techniques.

Whatever it is that you decide you want to use for your analysis, apply each of the Growth characteristics to determine a course of action to improve.

Table 2.4

Parts of the learning process	Growth characteristic	What will you do to adopt a growth mindset?
Challenges	Embraces challenges	
Obstacles	Persists despite setbacks	
Effort	Sees effort as a path to success	
Criticism	Learns from criticism	
Success of Others	Finds learning and inspiration in the success of others	

Footnotes

- Dweck, C.S. & Leggett, E.L. (1988). A Social-Cognitive Approach to Motivation and Personality

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1.3: Failure Is Not an Option

Failure Is Not an Option

Nathan Wallace

In the movie Apollo 13, Ed Harris portrays NASA flight director Gene Kranz as he successfully guides the crew of a damaged spacecraft to safety. In a famous scene during which Kranz and his staff are attempting to overcome some extremely daunting challenges, Harris shouts, “Failure is not an option!” This singular statement perfectly articulated the determination of Kranz to bring the Apollo astronauts back to Earth.

This “failure is not an option” credo was perfect for the life and death situation that NASA was facing. Failure meant that the astronauts on Apollo 13 would never come home, and that outcome was unacceptable. Attending college, on the other hand, shouldn’t be a life or death experience, though it sometimes might feel like one. Failure, though never the intended outcome, can and sometimes does happen. Sometimes failure manifests itself in election results for a student government post, in a test score, or even in a final grade.

Throughout my life I have had many failures. In high school I drove my parents and teachers crazy because of my lack of academic achievement. I even managed to get an F- in Spanish on my report card. When I told my mom that it was a typo she responded, “So you didn’t get an F?” “No,” I said, “I definitely earned the F, but there’s no such thing as an F-.” To this day I’m not so sure that my reply was accurate. I might have earned that minus after all.

My failures in high school led to only one acceptance from of all the colleges I applied to attend. Furthermore, I was not accepted to the school’s main campus, but to their branch campus. During my first semester there my effort wasn’t much better than in high school, but since my parents were now paying for my education I did enough work to avoid academic probation. It wasn’t until my second semester that I found my niche as a Religious Studies major and started getting good grades, moved to the main campus, and eventually graduated with honors.

Since graduating from college, my career path has taken me into higher education as a Student Affairs administrator. This career has exposed me to many great theories regarding student success, and many of them gave me insight into my own college experience. But it was Stanford psychologist Carol Dweck who appeared to be thinking of me when she wrote the following about fixed mindsets in the introduction to her book titled *Mindset: The New Psychology of Success*:

Believing that your qualities are carved in stone—the fixed mindset—creates an urgency to prove yourself over and over. If you have only a certain amount of intelligence, a certain personality, and a certain moral character—well, then you’d better prove that you have a healthy dose of them. It simply wouldn’t do to look or feel deficient in these most basic characteristics. (Dweck, 2006)

This statement was a revelation to me. I finally understood my problem throughout high school and even in college. I earned good grades because I liked Religious Studies but never really challenged myself inside or outside of the classroom. My problem was that I had a fixed mindset about academic success. I believed that a person is either smart or they’re not, and nothing could be done to significantly change that. I also believed that I was one of the fortunate ones to be “gifted” with an abundance of intelligence.

One might think that having confidence in your intelligence is a whole lot better than thinking that you’re stupid, but the result was the same. My fixed mindset was holding me back because it led to a paralyzing fear of failure. Since as far back as I could remember, my family, friends, and teachers were always telling me how smart I was, and I believed them. But that belief was a double-edged sword. High school and college offered many occasions when self-confidence in my inherent intelligence could be threatened. If I fail on this test or in this course it means that I’m not the smart person I thought I was. If I fail, my family and friends will find out that they were wrong about me.

However, there was a way to avoid all of the risks of academic rigor. I could just not try. If I don’t try I’ll get bad marks on my report card, but those won’t be true indicators of my intelligence. By not putting forth any effort, my intelligence would never be disproven. I would always be able to say to myself and others that, “I could do the work and be a straight A student, but I’m just not interested.” Looking back on this time in my life, it is clear to me that this wasn’t a conscious decision to save face. It was fear, not logic, which was guiding my behavior.

After reading *Mindset* I have made a conscious effort to identify and thwart any remaining fixed mindset thoughts that I continue to hold. Dweck’s book acts as a manual for rooting out fixed mindset thoughts, because she explains that the idea of fixed mindsets is only half of her mindset theory. There is another kind of mindset, and she calls it growth mindset. Dweck writes that, “This growth

mindset is based on the belief that your basic qualities are things you can cultivate through your efforts" (Dweck, 2006). Dweck goes on to explain that we can choose to have a growth mindset about any type of ability, whether it's math, art, athletics, or any other skill that one wishes to cultivate.

I put this theory to the test not long after reading the book. A few years ago I attended a meeting only to find out that it wasn't any ordinary meeting. During this meeting we would be brainstorming solutions to a specific problem. This was going to be a true brainstorming session, led by a facilitator trained in the science of soliciting uninhibited ideas from an audience. As soon as I heard the word *brainstorming* I froze. I have always hated brainstorming. I'm the type of person that likes to think things through two or three times before expressing an opinion. My fear of failing at this task in front of my coworkers paralyzed my mind. I couldn't think.

That's when it hit me. This was fixed mindset thinking. My belief in my brainstorming inadequacies was preventing me from even trying. So I flipped this thinking on its head and decided the best way to improve my brainstorming abilities was to clear my mind and start firing out ideas. I gave it a shot, and though the ideas didn't come out at the prolific rate of some of my colleagues, I had never before had such a positive outcome and experience while brainstorming. Through this experience I found that I really could choose to have a growth mindset, and that this choice produces a greater chance of success. With a greater chance of success comes a smaller chance of failure.

Nevertheless, when it comes to academic success and success in all phases of life, failure is always an option. Though it can be painful, failure can lead to great learning and progress when a specific failure is analyzed through the lens of a growth mindset. By focusing more on effort than on outcomes anyone can learn and grow, regardless of their skill level. Therefore, to make the most of their time in college, students must seek out challenges that will stretch their abilities. These challenges can take many forms and they can occur in a variety of settings, both inside and outside of the classroom. When seeking out challenges there is always the possibility of agonizing defeat, but out of that defeat can be the seeds of great success in the future.

Reference

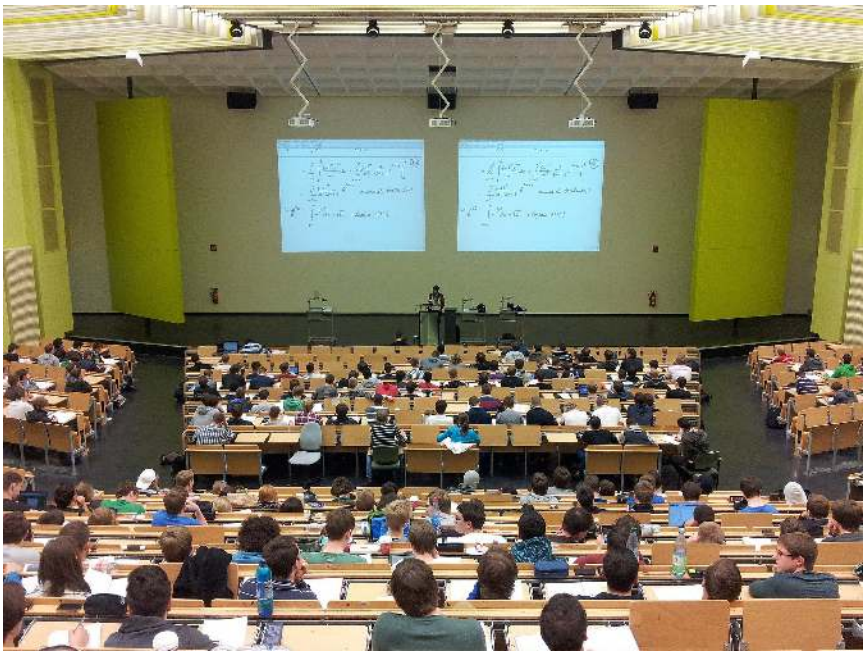
Dweck, C. (2006). *Mindset: The new psychology of success.* New York: Ballantine Books.

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1.4: Active Learning from Lectures

The learning of information and skills presented in class lectures or discussions should be viewed as a process: preparation to take in the new information, the act of taking in the new information, and then reviewing the information so that it is later accessible (recalled from memory) to use for a project, paper, or test.



The activities suggested for “Before Class” in Table 5-2 help students develop a mental framework into which this new information can be “filed” or organized which aids information recall (Bower, Clark, Lesgold, & Winzenz, 1969). It also makes note taking easier because you are familiar with the material and feel less panicked about getting everything noted in full detail.

Attending class and taking the notes is important primarily because the professor may be presenting information that is not presented in the text. It should also be seen as a structured opportunity to engage with the material that will further aid recall when needed. Oftentimes, students can also get clues as to what a professor perceives as important material from attending class that will aid in test preparation.

Table 5-2: Effective Techniques for Learning from Class

Before Class	During Class	After Class
Complete assigned readings to prepare for class. Take notes or mark your text for easy reference.	Take notes. Active engagement with the material (like writing down what you are hearing) helps improve memory and concentration and material to later review. Have a positive mindset about taking notes you are writing down answers to questions on the test.”	Review your notes and handouts within 24 hours of taking them. Fill in details that you remember but did not have time to write down. Write questions for the notes that you can use for self-testing later. Write a summary of the day’s notes.
Prepare questions that you may have about the reading.	Put a date on your note paper. It helps to also include the text chapter or general topic that might be related to these notes as well.	Compare your notes with a reliable classmate who may help give you ideas on what you missed that might be important or how to better organize the information. Discuss the notes to add another-layer of processing.

Before Class	During Class	After Class
Review notes from the previous class period to get a sense of where you may have left off.	Use active listening techniques. Refer to the informational box called “Ten Tips for Active Listening” for more information.	Follow up on questions that you have written down with your professor, the teaching assistant, or another reliable classmate.
Be sure you have your materials ready for class: text, three-ring notebook with paper, writing utensils, highlighters, charged laptop, etc.	Have your textbook accessible in case your faculty member refers to specific information. Write down that reference in your notes as it is probably important information.	Integrate your class notes with your text notes or markings. If you highlighted in your book, add some of that information to your notes for the “whole” picture.
Get a good night’s sleep. It is easier to concentrate on the lecture when you are alert.	Write down the main points a speaker is making along with any supple-mental details like examples or experiments. Do not expect to be able to write down every word the speaker says.	Develop tools that swill serve to help you review the material now, as well as later. Create visual diagrams* of concepts, develop ram cards* of concepts, reorganize your notes, etc. *Refer to informational boxes on “Visualize Your Information” and “Cram Cards for Long-Term Review.”
Make it a habit to attend every class. Put your classes on your schedule and treat them as a limited opportunity to engage with your professor and this material.	Use abbreviations for commonly used words to increase your speed in note taking. Create some abbreviations of your own.	Self-test your comprehension of the information on an ongoing basis. Do not just “look over” or read what you have written. Use the questions you have created. Cover up the “answer” and see if you can explain what is in the notes. Use the cram cards or visual diagrams to test yourself.
If you cannot attend class due to illness or an-other extraordinary reason, contact a classmate to arrange for notes. It is always a good idea to send an e-mail or call your professor as well.	Develop a system of organization for your notes that works for you. Common methods include the Cornell, Outlining, Mapping, Text-Class, and Sentence Methods. Refer to informational box called “Note Taking Methods: What Is Right for You?”	
	Leave space between main points so that you can add in new information that you may have missed or from the text.	
	If the professor writes it down or puts it on a PowerPoint slide, it is probably important, so write it down.	
	If the professor repeats it, it is important, so write it down.	
	Listen for cues like “very important,” “in summary,” or sequence words like “first,” “second,” etc. This information provides cues for what is important and possible note organization.	
	Ask questions you may have. If you are not comfortable talking in class, write them down and talk with your professor later.	

Adapted from Dembo Sell (2008) and Downing (2008).

Ten Tips for Active Listening

Effective note taking in class requires the use of active listening techniques for optimal results.

1. Accept responsibility as a listener.
2. Adopt a positive attitude toward listening. Listening is a choice.
3. Sit somewhere in the classroom that will minimize distractions and allow you to focus on the professor.
4. Maintain eye contact with your professor.
5. Focus on the content being presented rather than the delivery.
6. Ask questions in class.
7. Ask mental questions and search for the answer in the lecture or discussion.
8. Avoid emotional involvement that may impair your ability to concentrate or filter important information.
9. Clue in on non-verbal communication, such as exaggerated movements, excitement, etc., as it may serve as a cue to important information.
10. Monitor your concentration throughout the class period and continually refocus.

Adapted from Kline (2002) and Treuer (2006).



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1.5: Active Learning from Reading

As with learning from lectures and discussions, the learning of information and skills presented in readings, texts, and written text from a Web site should be viewed as a process: preparation to take in the new information, the act of taking in the new information, and then reviewing the information so that it is later accessible (recalled from memory) to use for a project, paper, or test.



Students who simply start and finish a reading with no further actions taken are really wasting their time. Reading a text needs to be approached properly in order to ensure comprehension and retention of the information. Planning for an active reading session by engaging in the "before reading" activities facilitates an active mindset to the reading session and discourages "auto-pilot reading" that results in scanning of the words but no real careful thought about what is being said.

Preparing for class by reading assigned texts related to the lecture material is extremely important to optimize the lecture experience. If you have a professor who tends to lecture at a fast pace, you have the foundational information and only need to focus on writing down unfamiliar information. If your professor expects class participation in discussions or utilizes active classroom techniques, such as case studies, small group activities and discussions, simulations, or demonstrations, you will understand what is happening and be able to contribute to the learning experience. Marking or highlighting your text for main ideas while reading not only keeps your mind focused on the reading but also serves to prepare you for easier exam review.

Table 5-3. Effective Techniques for Reading

Before Reading	During Reading	After Reading
Preview the reading headings to get a "big picture" of the outline of the reading. Look at the pictures and figures. Check out the bold and italicized words.	As you are reading, seek out the answers to the questions that the reading or you generated rather than passively scanning the words.	Take reading notes from the highlighted material from the text. Utilize visual organizers and summaries to capture information. Revisit the informational box on "Note-Taking Methods: What Is Right for You?" and visit "Visualize Your information and "Cram Cards for Long-term Review."
Preview any questions that the chapter may offer so that you can actively seek answers to those questions.	Monitor your concentration and comprehension. After each paragraph, ask yourself, "What was the main point of that paragraph?" After each section, summarize what you have read.	Answer the questions that a the reading may have provided or that you developed. Reread sections in which you remember little information.
Turn text headings into questions so that you can actively seek answers to those questions. Jot those questions down in the margin of the text or in your notes if you plan to take reading notes.	Mark your text. Highlight the main idea of a paragraph or write down the main idea of a paragraph in your notes. If you cannot mark your text, take reading notes on the main ideas. Refer to the informational box called "In Search of Main Ideas" for more information	Integrate your reading notes and lecture/discussion notes into one location for easier review for an exam.
Plan for a high energy time of the day to read. Consider breaking up a long reading assignment into multiple, shorter reading sessions.	Look up the definitions to words that you do not understand to help build your vocabulary and facilitate understanding of the topic.	Discuss readings with classmates on a regular basis. Take turns explaining to each other sections of the reading. Ask and answer reading questions. Jot down questions that are unclear to the study group to ask the professor in class or during office hours.
Be sure you have your materials ready for reading: text, notebook paper, writing utensils, high-lighters, charged laptop, etc.		

Adapted from Dembo and Seli (2008) and Downing (2008).

In Search of Main Ideas

Some students do not actively take notes or mark their readings and texts because they determine it is a lot of extra work to write down so much information. However, the problem is that students are usually writing down too much information and not really cluing in to the main ideas. Here are some tips for identifying the main ideas when reading:

1. The Table of Contents and chapter outlines provide a broad view of the main points that will be covered in a reading. Flesh out the outlines that are already provided for you.
2. Oftentimes in a textbook, the main idea is the first or last sentence of a paragraph. If it is not the first or the last sentence, then look back at the entire paragraph to see what the overall issue seems to be. Look for the overall pat- terns of your textbooks.
3. Titles, headings, and subheadings announce the major subject. Make these headings into questions, and the answers to the questions will likely be the main ideas.
4. Bold and italic words point to a main idea or key concept that you need to understand.
5. Repetition of key words or phrases throughout the text point to a main idea.
6. Chapter questions at the end of the chapter are about the main ideas of the text. Answer those questions and you will identify your main ideas.
7. Summaries presented at the end of the chapter also tend to restate the main ideas briefly. Flesh these ideas out with some supporting ideas, and you would have a good view of the entire chapter.
8. Stop and look at the visuals—pictures, diagrams, tables, etc. Oftentimes, the message depicted in the graph or picture is typically a main idea.
9. Detailed statistics, several examples in a row, and other details often signal that a main idea is being clarified, proven, supported, etc. Track back or ahead to find the main idea they are trying to illustrate.
10. Text that includes bullet points, numbering, or sequences is often a main idea.
11. Look for organizational patterns in the reading that might highlight the main ideas. For instance, are two issues being compared or contrasted? What was the effect of a certain event? Are problems and various solutions being presented? Is there a timeline of events that is important?
12. Be intentional about searching for the main ideas. Ask yourself at the end of each section or paragraph, "What is the point?" or "What is it that the author wants me to know?"
13. If you are reading a narrative, ask yourself questions like, "Who are the main characters?" "Why is this character important to this story?" "Why did the author chose to tell this part of the story?"

Finally, taking the time to think critically about the reading you have done further solidifies that information in your memory and can serve to prepare you for exam review. Summarizing material using visual organizers (refer to informational box "Visualize Your Information" and Figure 5-1) moves your thinking from low levels of thinking to high levels of thinking on Bloom's Taxonomy. Integrating text and lecture notes and discussing readings with your classmates and professor promotes memory storage of that information and thinking at higher levels as well.

Visualize Your Information

Summarizing main ideas of the readings or lectures can be difficult without the right tools. Visual organizers can help students condense important information into visual diagrams that can be easily reviewed for exams or help organize a writing assignment. Muskingum College's Center for Advancement and Learning (n.d.) hosts a useful Web site that summarizes and provides examples of various visual organizers (a.k.a. Information Organization Strategies) at

muskingum.edu/~cal/database/g...anization.html (Web site wants username and password now.)

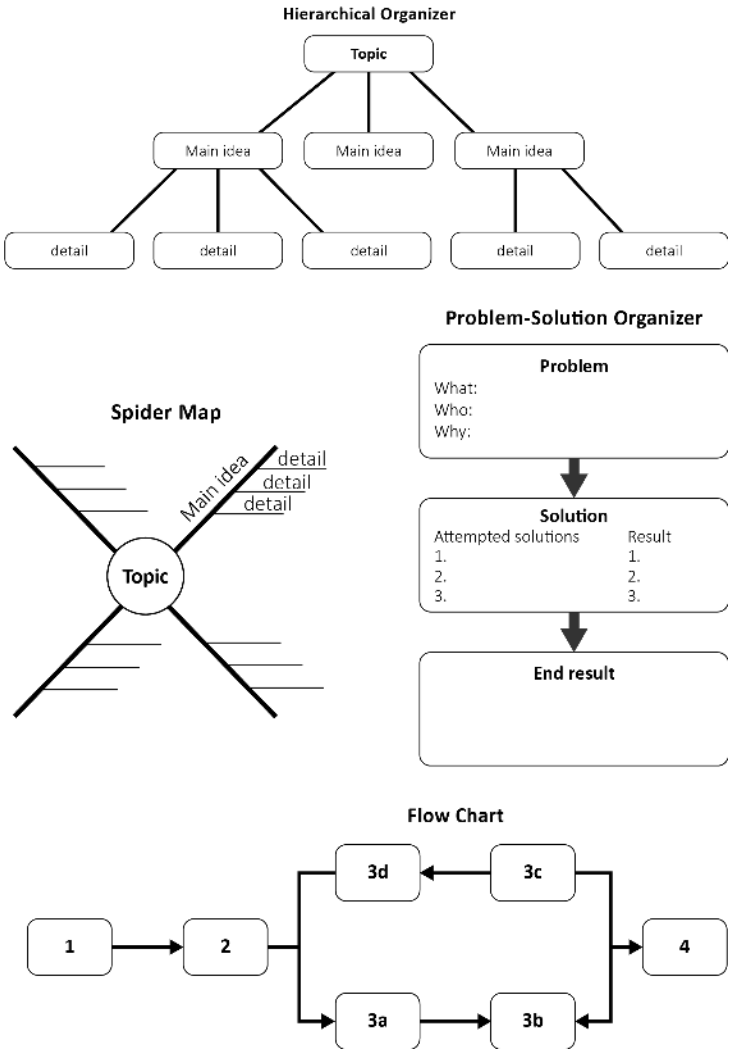
Consider how you might be able to apply several pages of text to a simple flowchart, hierarchical organizer, comparison-contrast organizer, spider map, etc. (See Figure 5-1.)

An additional benefit of using visual organizers is that you are thinking about the material at a deeper level as you identify the proper visual tool and apply the information. While at first it may take more time to determine the proper tool, it will get easier with practice and become second nature.

Activity 5-2

Utilizing what you have learned from the informational boxes "In Search of Main Ideas" and "Visualize Your Information," select a text chapter or a week of class notes and create some visual organizers that summarize the main ideas of that material. Ask a classmate to share in the same exercise and compare your work in a critical manner. What have you learned from this exercise and each other?

Figure 5-1. Examples of visual organizers



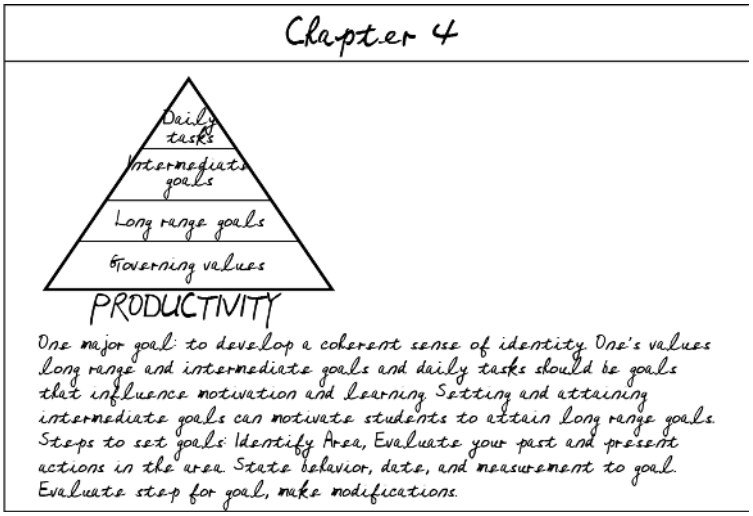
Source: Muskingum College's Center for Advancement and Learning (n.d)

Cram Cards for Long-term Review

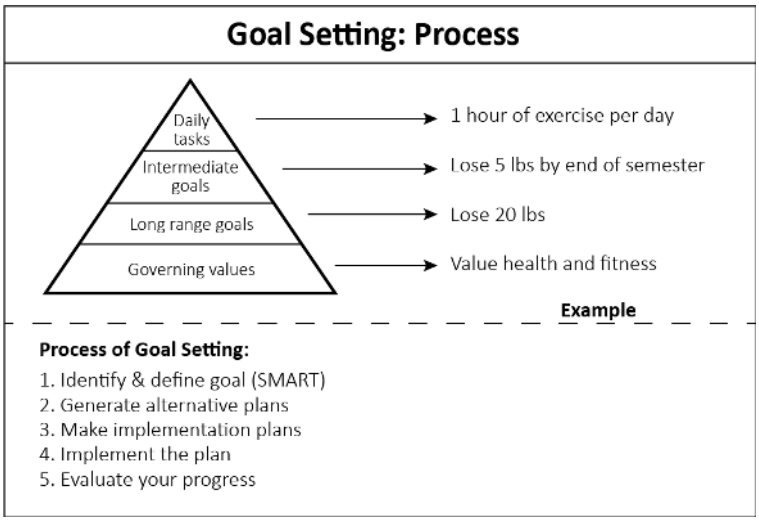
The use of "cram" in cram cards is deceiving. Cram cards are a study tool that promote the active learning of reading or lecture notes and allows for the easy and portable review of material over a longer period of time, therefore promoting later recall. It is a technique that will take some practice to hone your skills in selecting main ideas, but students who use cram cards indicate it is well worth their time.

Tips for Writing Cram Cards:

- Use 4 x 6 index cards (3 x 5 size tends to be too small)—some enthusiasts will even use the colored index cards for different sources of information (i.e., notes versus text reading) or topics (e.g., blue for English and green for Biology).
- Make one cram card per major concept or main idea.
- Do not simply take notes on the card; capture a summary of the main ideas maintaining the proper relationships of the concepts.
- Write the information in your own words.
- Include book examples or provide real-life examples for difficult concepts.
- Include key words.
- Be sure to write legibly.



Example of a Bad Cram Card



Example of a Good Cram Card

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1.6: Taking Notes

Questions to consider

- How can you prepare to take notes to maximize the effectiveness of the experience?
- What are some specific strategies you can employ for better notetaking?
- Why is annotating your notes after the notetaking session a critical step to follow?

Beyond providing a record of the information you are reading or hearing, notes help you organize the ideas and help you make meaning out of something about which you may not be familiar, so notetaking and reading are two compatible skill sets. Taking notes also helps you stay focused on the question at hand. Nanami often takes notes during presentations or class lectures so she can follow the speaker's main points and condense the material into a more readily usable format. Strong notes build on your prior knowledge of a subject, help you discuss trends or patterns present in the information, and direct you toward areas needing further research or reading.



Figure 1.6.9 Strong notes build on your prior knowledge of a subject, help you discuss trends or patterns present in the information, and direct you toward areas needing further research or reading.

It is not a good habit to transcribe every single word a speaker utters—even if you have an amazing ability to do that. Most of us don't have that court-reporter-esque skill level anyway, and if we try, we would end up missing valuable information. Learn to listen for main ideas and distinguish between these main ideas and details that typically support the ideas. Include examples that explain the main ideas, but do so using understandable abbreviations.

Think of all notes as potential study guides. In fact, if you only take notes without actively working on them after the initial notetaking session, the likelihood of the notes helping you is slim. Research on this topic concludes that without active engagement after taking notes, most students forget 60–75 percent of material over which they took the notes—within two days! That sort of defeats the purpose, don't you think? This information about memory loss was first brought to light by 19th-century German psychologist Hermann Ebbinghaus. Fortunately, you do have the power to thwart what is sometimes called the Ebbinghaus Forgetting Curve by reinforcing what you learned through review at intervals shortly after you take in the material and frequently thereafter.

If you are a musician, you'll understand this phenomenon well. When you first attempt a difficult piece of music, you may not remember the chords and notes well at all, but after frequent practice and review, you generate a certain muscle memory and cognitive recall that allows you to play the music more easily.

Notetaking may not be the most glamorous aspect of your higher-education journey, but it is a study practice you will carry throughout college and into your professional life. Setting yourself up for successful notetaking is almost as important as the actual taking of notes, and what you do after your notetaking session is equally significant. Well-written notes help you organize your thoughts, enhance your memory, and participate in class discussion, and they prepare you to respond successfully on exams. With all that riding on your notes, it would behoove you to learn how to take notes properly and continue to improve your notetaking skills.

Analysis Question

Do you currently have a preferred way to take notes? When did you start using it? Has it been effective? What other strategy might work for you?

Preparing to Take Notes

Preparing to take notes means more than just getting out your laptop or making sure you bring pen and paper to class. You'll do a much better job with your notes if you understand why we take notes, have a strong grasp on your preferred notetaking system, determine your specific priorities depending on your situation, and engage in some version of efficient shorthand.

Like handwriting and fingerprints, we all have unique and fiercely independent notetaking habits. These understandably and reasonably vary from one situation to the next, but you can only improve your skills by learning more about ways to take effective notes and trying different methods to find a good fit.

The very best notes are the ones you take in an organized manner that encourages frequent review and use as you progress through a topic or course of study. For this reason, you need to develop a way to organize all your notes for each class so they remain together and organized. As old-fashioned as it sounds, a clunky three-ring binder is an excellent organizational container for class notes. You can easily add to previous notes, insert handouts you may receive in class, and maintain a running collection of materials for each separate course. If the idea of carrying around a heavy binder has you rolling your eyes, then transfer that same structure into your computer files. If you don't organize your many documents into some semblance of order on your computer, you will waste significant time searching for improperly named or saved files.

You may be interested in relatively new research on what is the more effective notetaking strategy: handwriting versus typing directly into a computer. While individuals have strong personal opinions on this subject, most researchers agree that the format of student notes is less important than what students do with the notes they take afterwards. Both handwriting notes and using a computer for notetaking have pros and cons.

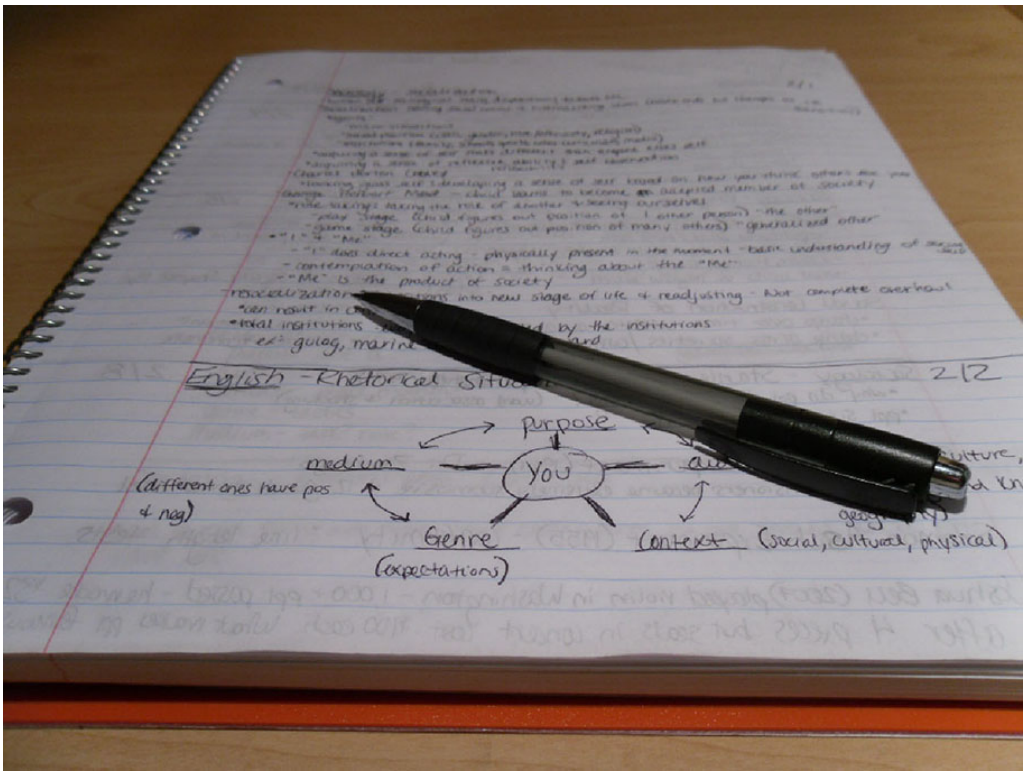


Figure 1.6.10 The best notes are the ones you take in an organized manner. Frequent review and further annotation are important to build a deep and useful understanding of the material. (Credit: English106 / Flickr / Attribution 2.0 Generic (CC-BY 2.0))

Managing Notetaking Systems (Computer, Paper/Pen, Note Cards, Textbook)

Whichever of the many notetaking systems you choose (and new ones seem to come out almost daily), the very best one is the one that you will use consistently. The skill and art of notetaking is not automatic for anyone; it takes a great deal of practice, patience, and continuous attention to detail. Add to that the fact that you may need to master multiple notetaking techniques for different classes, and you have some work to do. Unless you are specifically directed by your instructor, you are free to combine the best parts of different systems if you are most comfortable with that hybrid system.

Just to keep yourself organized, all your notes should start off with an identifier, including at the very least the date, the course name, the topic of the lecture/presentation, and any other information you think will help you when you return to use the notes for further study, test preparation, or assignment completion. Additional, optional information may be the number of notetaking sessions about this topic or reminders to cross-reference class handouts, textbook pages, or other course materials. It's also always a good idea to leave some blank space in your notes so you can insert additions and questions you may have as you review the material later.

Notetaking Strategies

You may have a standard way you take all your notes for all your classes. When you were in high school, this one-size-fits-all approach may have worked. Now that you're in college, reading and studying more advanced topics, your general method may still work some of the time, but you should have some different strategies in place if you find that your method isn't working as well with college content. You probably will need to adopt different notetaking strategies for different subjects. The strategies in this section represent various ways to take notes in such a way that you are able to study after the initial notetaking session.

Cornell Method

One of the most recognizable notetaking systems is called the *Cornell Method*, a relatively simple way to take effective notes devised by Cornell University education professor Dr. Walter Pauk in the 1940s. In this system, you take a standard piece of note paper and divide it into three sections by drawing a horizontal line across your paper about one to two inches from the bottom of the page (the summary area) and then drawing a vertical line to separate the rest of the page above this bottom area, making the left side about two inches (the recall column) and leaving the biggest area to the right of your vertical line (the notes column). You may want to make one page and then copy as many pages as you think you'll need for any particular class, but one advantage of this

system is that you can generate the sections quickly. Because you have divided up your page, you may end up using more paper than you would if you were writing on the entire page, but the point is not to keep your notes to as few pages as possible. The Cornell Method provides you with a well-organized set of notes that will help you study and review your notes as you move through the course. If you are taking notes on your computer, you can still use the Cornell Method in Word or Excel on your own or by using a template someone else created.

Topic/Objective:	Name:
	Class/Period:
	Date:
Essential Question:	

Questions:	Notes:

Summary:

Figure 1.6.11: The Cornell Method provides a straightforward, organized, and flexible approach

Now that you have the notetaking format generated, the beauty of the Cornell Method is its organized simplicity. Just write on one side of the page (the right-hand notes column)—this will help later when you are reviewing and revising your notes. During your notetaking session, use the notes column to record information over the main points and concepts of the lecture; try to put the ideas into your own words, which will help you not transcribe the speaker's words verbatim. Skip lines between each idea in this column. Practice the shortcut abbreviations covered in the next section and avoid writing in complete sentences. Don't make your notes too cryptic, but you can use bullet points or phrases equally well to convey meaning—we do it all the time in conversation. If you know you will need to expand the notes you are taking in class but don't have time, you can put reminders directly in the notes by adding and underlining the word *expand* by the ideas you need to develop more fully.

As soon as possible after your notetaking session, preferably within eight hours but no more than twenty-four hours, read over your notes column and fill in any details you missed in class, including the places where you indicated you wanted to expand your notes. Then in the recall column, write any key ideas from the corresponding notes column—you can't stuff this smaller recall column as if you're explaining or defining key ideas. Just add the one- or two-word main ideas; these words in the recall column serve as cues to help you remember the detailed information you recorded in the notes column.

Once you are satisfied with your notes and recall columns, summarize this page of notes in two or three sentences using the summary area at the bottom of the sheet. This is an excellent time to get with another classmate or a group of students who all heard the same lecture to make sure you all understood the key points. Now, before you move onto something else, cover the large notes column, and quiz yourself over the key ideas you recorded in the recall column. Repeat this step often as you go along, not just immediately before an exam, and you will help your memory make the connections between your notes, your textbook reading, your in-class work, and assignments that you need to succeed on any quizzes and exams.

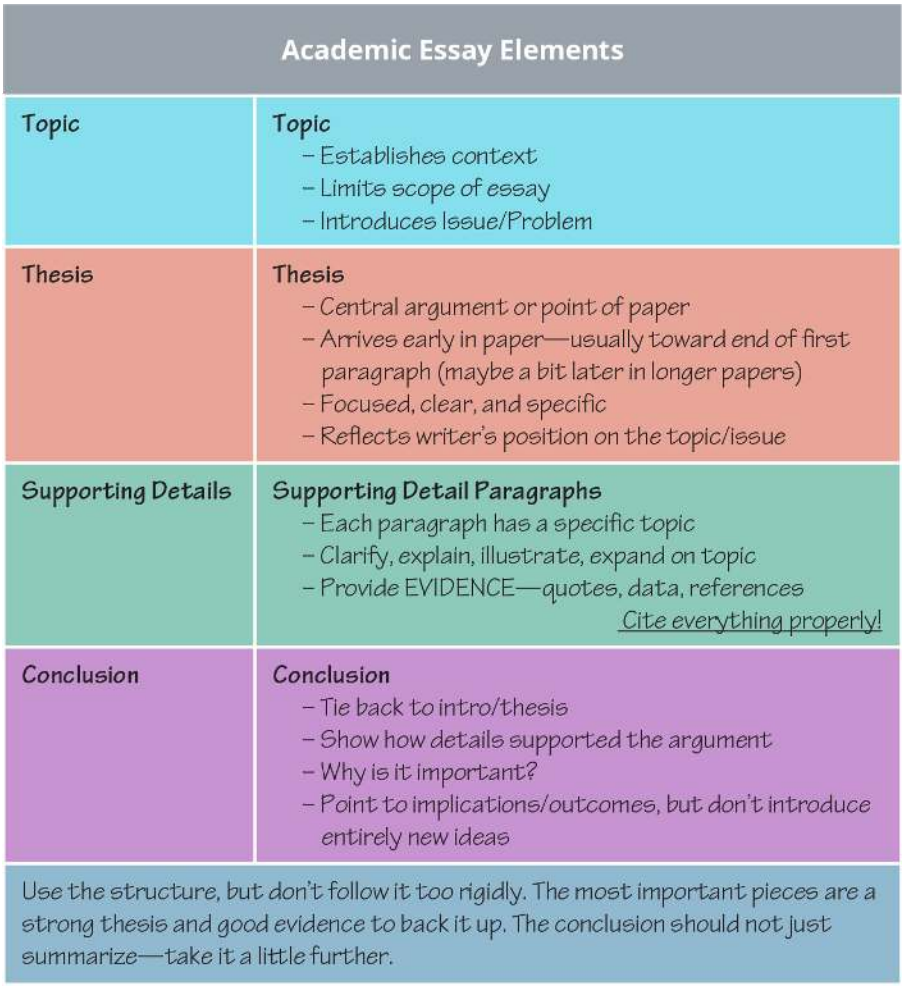


Figure 1.6.12 This sample set of notes in the Cornell Method is designed to make sense of a large amount of information. The process of organizing the notes can help you retain the information more effectively than less consistent methods.

The main advantage of the Cornell Method is that you are setting yourself up to have organized, workable notes. The neat format helps you move into study-mode without needing to re-copy less organized notes or making sense of a large mass of information you aren't sure how to process because you can't remember key ideas or what you meant. If you write notes in your classes without any sort of system and later come across something like "Napoleon—short" in the middle of a glob of notes, what can you do at this point? Is that important? Did it connect with something relevant from the lecture? How would you possibly know? You are your best advocate for setting yourself up for success in college.

Outlining

Other note organizing systems may help you in different disciplines. You can take notes in a formal outline if you prefer, using Roman numerals for each new topic, moving down a line to capital letters indented a few spaces to the right for concepts related to the previous topic, then adding details to support the concepts indented a few more spaces over and denoted by an Arabic numeral. You can continue to add to a formal outline by following these rules.

You don't absolutely have to use the formal numerals and letter, but you have to then be careful to indent so you can tell when you move from a higher level topic to the related concepts and then to the supporting information. The main benefit of an outline is how

organized it is. You have to be on your toes when you are taking notes in class to ensure you keep up the organizational format of the outline, which can be tricky if the lecture or presentation is moving quickly or covering many diverse topics.

The following formal outline example shows the basic pattern:

I. Dogs *(main topic–usually general)*
   A. German Shepherd *(concept related to main topic)*
      1. Protection *(supporting info about the concept)*
      2. Assertive
      3. Loyal
   B. Weimaraner *(concept related to main topic)*
      1. Family-friendly *(supporting info about the concept)*
      2. Active
      3. Healthy
II. Siamese

You would just continue on with this sort of numbering and indenting format to show the connections between main ideas, concepts, and supporting details. Whatever details you do not capture in your notetaking session, you can add after the lecture as you review your outline.

Chart or table

Similar to creating an outline, you can develop a chart to compare and contrast main ideas in a notetaking session. Divide your paper into four or five columns with headings that include either the main topics covered in the lecture or categories such as How?, What?, When used?, Advantages/Pros, Disadvantages/Cons, or other divisions of the information. You write your notes into the appropriate columns as that information comes to light in the presentation.

Example of a Chart to Organize Ideas and Categories

	Structure	Types	Functions in Body	Additional Notes
Carbohydrates				
Lipids				
Proteins				
Nucleic Acid				

This format helps you pull out the salient ideas and establishes an organized set of notes to study later. (If you haven't noticed that this *reviewing later* idea is a constant across all notetaking systems, you should…take note of that.) Notes by themselves that you never reference again are little more than scribblings. That would be a bit like compiling an extensive grocery list so you stay on budget when you shop, work all week on it, and then just throw it away before you get to the store. You may be able to recall a few items, but likely won't be as efficient as you could be if you had the notes to reference. Just as you cannot read all the many books, articles, and documents you need to peruse for your college classes, you cannot remember the most important ideas of all the notes you will take as part of your courses, so you must review.

Concept Mapping and Visual Notetaking

One final notetaking method that appeals to learners who prefer a visual representation of notes is called *mapping* or sometimes *mind mapping* or *concept mapping*, although each of these names can have slightly different uses. Variations of this method abound, so you may want to look for more versions online, but the basic principles are that you are making connections between main ideas through a graphic depiction; some can get rather elaborate with colors and shapes, but a simple version may be more useful at least to begin. Main ideas can be circled or placed in a box with supporting concepts radiating off these ideas shown with a connecting line and possibly details of the support further radiating off the concepts. You can present your main ideas vertically or horizontally, but turning your paper long-ways, or in landscape mode, may prove helpful as you add more main ideas.

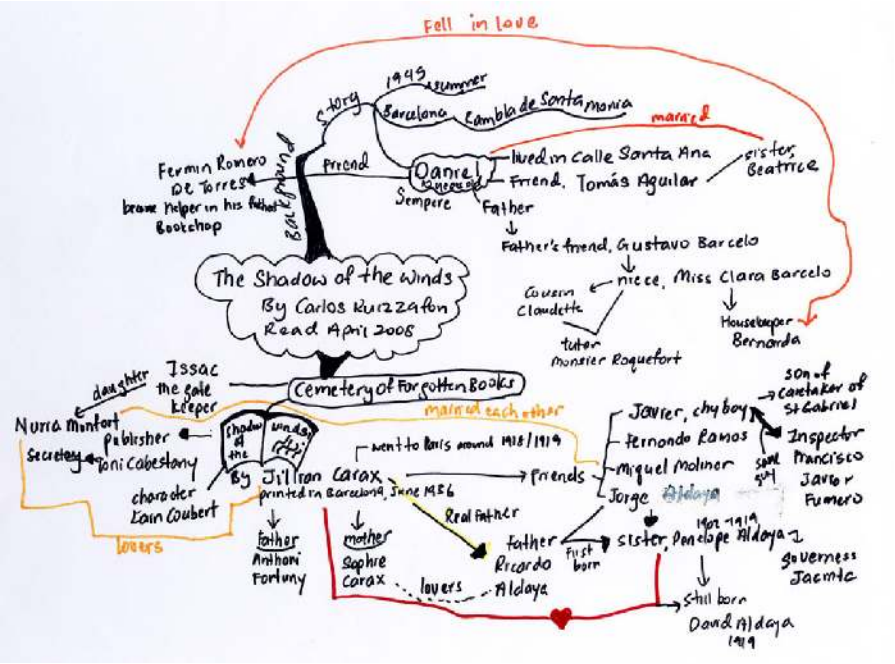


Figure 1.6.13 Concept mapping, sometimes referred to as mind mapping, can be an effective and very personalized approach to capturing information. (Credit: ArtistIvanChew / Flickr / Attribution 2.0 Generic (CC-BY 2.0))

You may be interested in trying visual notetaking or adding pictures to your notes for clarity. Sometimes when you can't come up with the exact wording to explain something or you're trying to add information for complex ideas in your notes, sketching a rough image of the idea can help you remember. According to educator Sherrill Knezel in an article entitled "The Power of Visual Notetaking," this strategy is effective because "When students use images and text in notetaking, it gives them two different ways to pull up the information, doubling their chances of recall." Don't shy away from this creative approach to notetaking just because you believe you aren't an artist; the images don't need to be perfect. You may want to watch Rachel Smith's TEDx Talk called "Drawing in Class" to learn more about visual notetaking.

You can play with different types of notetaking suggestions and find the method(s) you like best, but once you find what works for you, stick with it. You will become more efficient with the method the more you use it, and your notetaking, review, and test prep will become, if not easier, certainly more organized, which can delete decrease your anxiety.

Practicing Decipherable Shorthand

Most college students don't take a class in shorthand, once the domain of secretaries and executive assistants, but maybe they should. That almost-lost art in the age of computers could come in very handy during intense notetaking sessions. Elaborate shorthand systems do exist, but you would be better served in your college notetaking adventures to hone a more familiar, personalized form of shorthand to help you write more in a shorter amount of time. Seemingly insignificant shortcuts can add up to ease the stress notetaking can induce—especially if you ever encounter an "I'm not going to repeat this" kind of presenter! Become familiar with these useful abbreviations:

Shortcut symbol	Meaning
w/, w/o, w/in	with, without, within
&	and
#	number
b/c	because
X, √	incorrect, correct
Diff	different, difference
etc.	and so on

ASAP	as soon as possible
US, UK	United States, United Kingdom
info	information
Measurements: ft, in, k, m	foot, inch, thousand, million
¶	paragraph or new paragraph
Math symbols: =, +, >, <, ÷	equal, plus, greater, less, divided by
WWI, WWII	World Wars I and II
impt	important
?, !, **	denote something is very significant; don't over use

Do you have any other shortcuts or symbols that you use in your notes? Ask your parents if they remember any that you may be able to learn.

Annotating Notes After Initial Notetaking Session

Annotating notes after the initial notetaking session may be one of the most valuable study skills you can master. Whether you are highlighting, underlining, or adding additional notes, you are reinforcing the material in your mind and memory.

Admit it—who can resist highlighting markers? Gone are the days when yellow was the star of the show, and you had to be very careful not to press too firmly for fear of obliterating the words you were attempting to emphasize. Students now have a veritable rainbow of highlighting options and can color-code notes and text passages to their hearts' content. Technological advances may be important, but highlighter color choice is monumental! Maybe.

The only reason to highlight anything is to draw attention to it, so you can easily pick out that ever-so-important information later for further study or reflection. One problem many students have is not knowing when to stop. If what you need to recall from the passage is a particularly apt and succinct definition of the term important to your discipline, highlighting the entire paragraph is less effective than highlighting just the actual term. And if you don't rein in this tendency to color long passages (possibly in multiple colors) you can end up with a whole page of highlighted text. Ironically, that is no different from a page that is not highlighted at all, so you have wasted your time. Your mantra for highlighting text should be *less is more*. Always read your text selection first before you start highlighting anything. You need to know what the overall message is before you start placing emphasis in the text with highlighting.

Another way to annotate notes after initial notetaking is underlying significant words or passages. Albeit not quite as much fun as its colorful cousin highlighting, underlining provides precision to your emphasis.

Some people think of annotations as only using a colored highlighter to mark certain words or phrases for emphasis. Actually, annotations can refer to anything you do with a text to enhance it for your particular use (either a printed text, handwritten notes, or other sort of document you are using to learn concepts). The annotations may include highlighting passages or vocabulary, defining those unfamiliar terms once you look them up, writing questions in the margin of a book, underlining or circling key terms, or otherwise marking a text for future reference. You can also annotate some electronic texts.

Realistically, you may end up doing all of these types of annotations at different times. We know that repetition in studying and reviewing is critical to learning, so you may come back to the same passage and annotate it separately. These various markings can be invaluable to you as a study guide and as a way to see the evolution of your learning about a topic. If you regularly begin a reading session writing down any questions you may have about the topic of that chapter or section and also write out answers to those questions at the end of the reading selection, you will have a good start to what that chapter covered when you eventually need to study for an exam. At that point, you likely will not have time to reread the entire selection especially if it is a long reading selection, but with strong annotations in conjunction with your class notes, you won't need to do that. With experience in reading discipline-specific texts and writing essays or taking exams in that field, you will know better what sort of questions to ask in your annotations

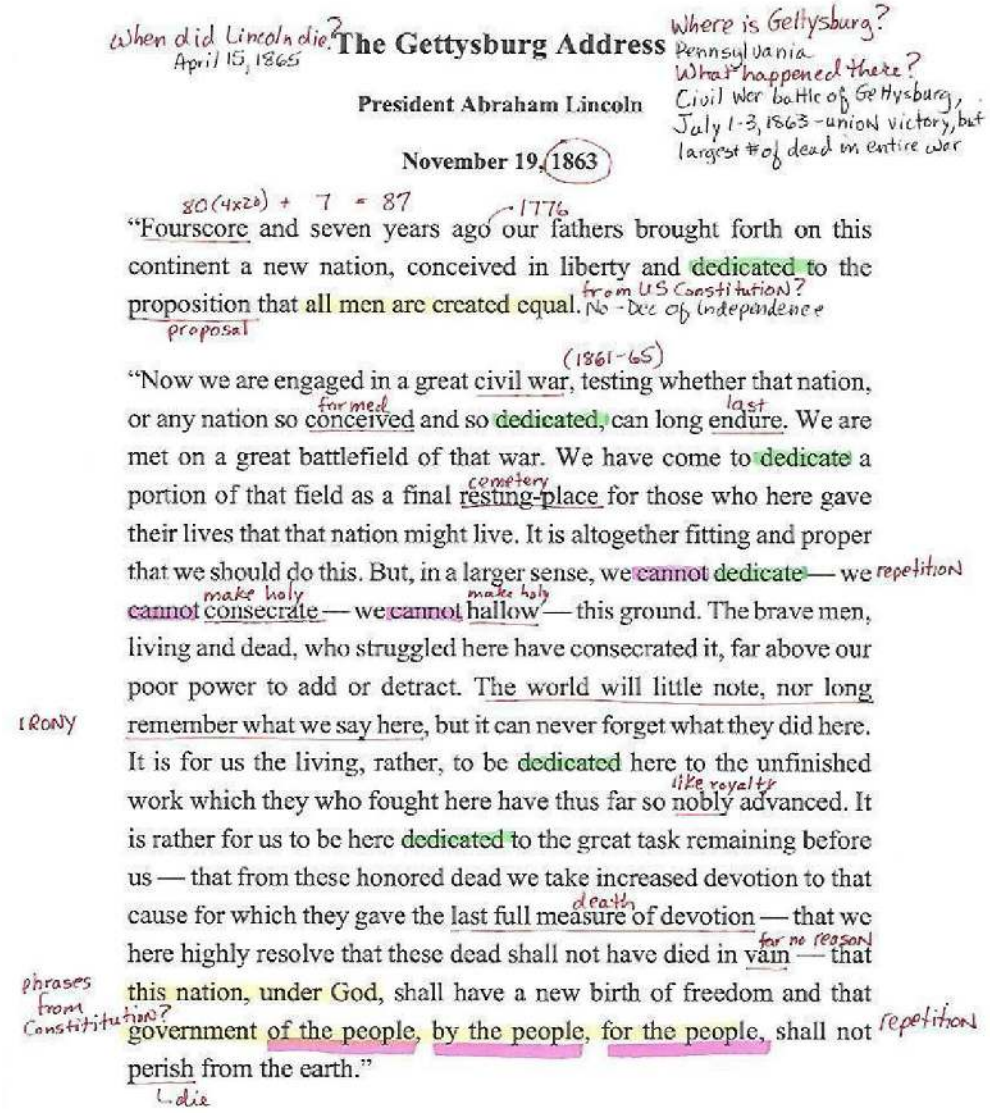


Figure 1.6.14 Annotations may include highlighting important topics, defining unfamiliar terms, writing questions in, underlining or circling key terms, or otherwise marking a text for future reference. Whichever approach you choose, try not to overdo it; neat, organized, and efficient notes are more effective than crowded or overdone notes.

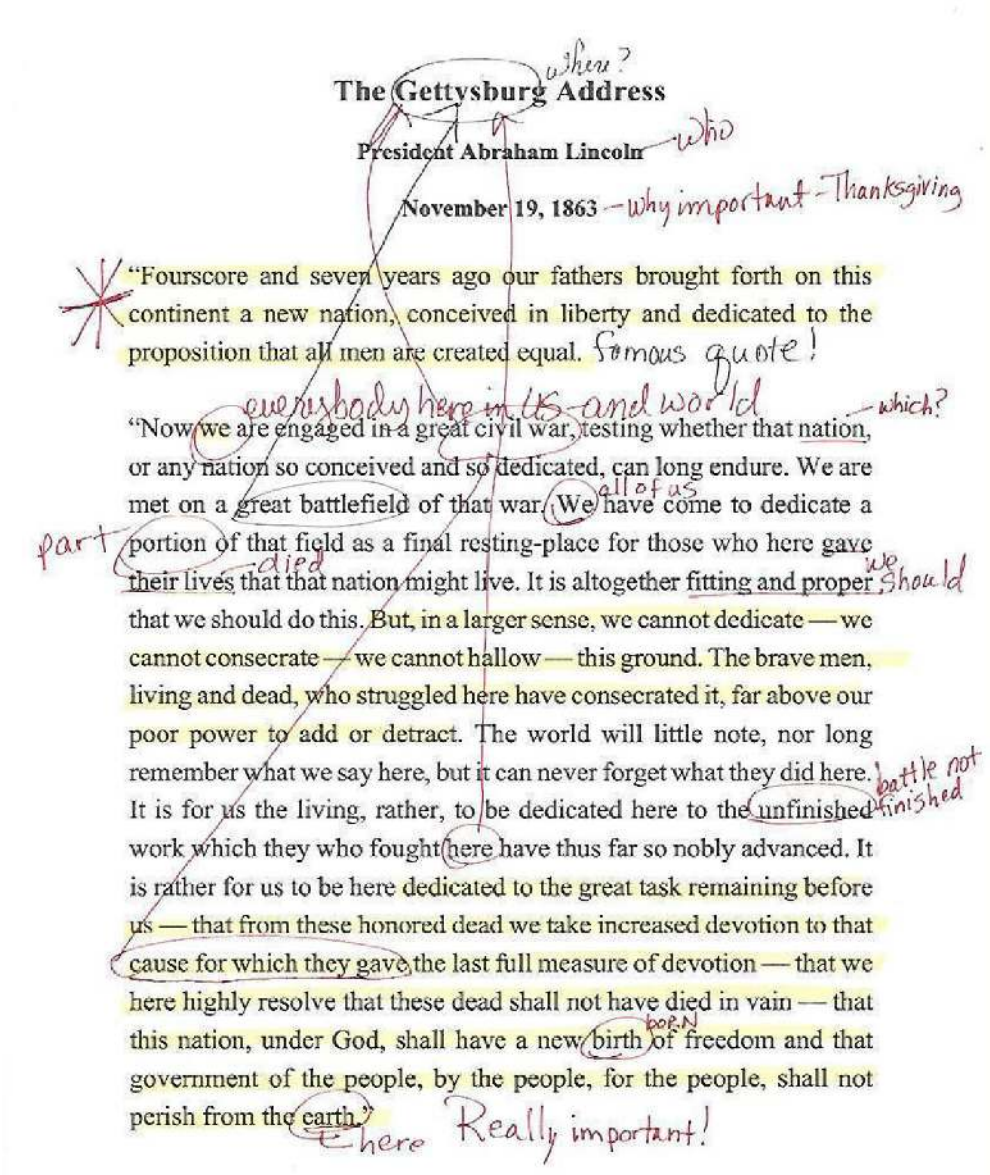


Figure 1.6.15 While these notes may be meaningful to the person who took them, they are neither organized nor consistent. For example, note that some of the more commonly used terms, like "we" and "unfinished," are defined, but less common ones -- "consecrate" and "hallow" -- are not.

What you have to keep in the front of your mind while you are annotating, especially if you are going to conduct multiple annotation sessions, is to not overdo whatever method you use. Be judicious about what you annotate and how you do it on the page, which means you must be neat about it. Otherwise, you end up with a mess of either color or symbols combined with some cryptic notes that probably took you quite a long time to create, but won't be worth as much to you as a study aid as they could be. This is simply a waste of time and effort.

You cannot eat up every smidgen of white space on the page writing out questions or summaries and still have a way to read the original text. If you are lucky enough to have a blank page next to the beginning of the chapter or section you are annotating, use this, but keep in mind that when you start writing notes, you aren't exactly sure how much space you'll need. Use a decipherable shorthand and write only what you need to convey the meaning in very small print. If you are annotating your own notes, you can make a habit of using only one side of the paper in class, so that if you need to add more notes later, you could use the other side. You can also add a blank page to your notes before beginning the next class date in your notebook so you'll end up with extra paper for annotations when you study.

Professional resources may come with annotations that can be helpful to you as you work through the various documentation requirements you'll encounter in college as well. Purdue University's Online Writing Lab (OWL) provides an annotated sample for

how to format a college paper according to guidelines in the Modern Language Association (MLA) manual that you can see, along with other annotations.

Adding Needed Additional Explanations to Notes

Marlon was totally organized and ready to take notes in a designated course notebook at the beginning of every philosophy class session. He always dated his page and indicated what the topic of discussion was. He had various colored highlighters ready to denote the different note purposes he had defined: vocabulary in pink, confusing concepts in green, and note sections that would need additional explanations later in yellow. He also used his own shorthand and an impressive array of symbols to indicate questions (red question mark), highly probable test material (he used a tiny bomb exploding here), additional reading suggestions, and specific topics he would ask his instructor before the next class. Doing everything so precisely, Marlon's methods seemed like a perfect example of how to take notes for success. Inevitably though, by the end of the hour-and-a-half class session, Marlon was frantically switching between writing tools, near to tears, and scouring his notes as waves of yellow teased him with uncertainty. What went wrong?

As with many of us who try diligently to do everything we know how to do for success or what we think we know because we read books and articles on success in between our course work, Marlon is suffering from trying to do too much simultaneously. It's an honest mistake we can make when we are trying to save a little time or think we can multitask and kill two birds with one stone.

Unfortunately, this particular error in judgement can add to your stress level exponentially if you don't step back and see it for what it is. Marlon attempted to take notes in class as well as annotate his notes to get them ready for his test preparation. It was too much to do at one time, but even if he could have done all those things during class, he's missing one critical point about notetaking.

As much as we may want to hurry and get it over with, notetaking in class is just the beginning. Your instructor likely gave you a pre-class assignment to read or complete before coming to that session. The intention of that preparatory lesson is for you to come in with some level of familiarity for the topic under consideration and questions of your own. Once you're in class, you may also need to participate in a group discussion, work with your classmates, or perform some other sort of lesson-directed activity that would necessarily take you away from taking notes. Does that mean you should ignore taking notes for that day? Most likely not. You may just need to indicate in your notes that you worked on a project or whatever other in-class event you experienced that date.

Very rarely in a college classroom will you engage in an activity that is not directly related to what you are studying in that course. Even if you enjoyed every minute of the class session and it was an unusual format for that course, you still need to take some notes. Maybe your first note could be to ask yourself why you think the instructor used that unique teaching strategy for the class that day. Was it effective? Was it worth using the whole class time? How will that experience enhance what you are learning in that course?

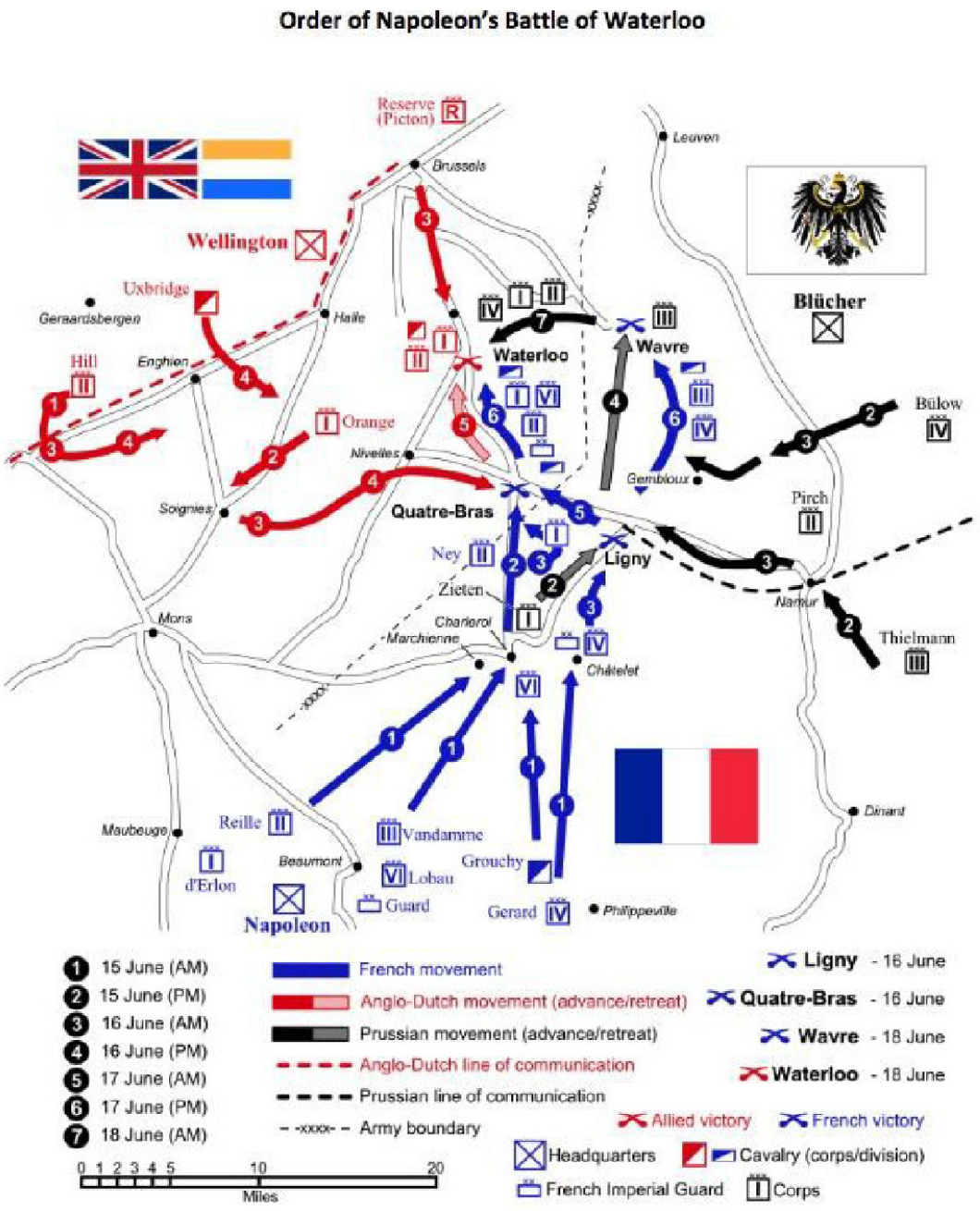
If you use an ereader or ebooks to read texts for class or read articles from the Internet on your laptop or tablet, you can still take effective notes. Depending on the features of your device, you have many choices. Almost all electronic reading platforms allow readers to highlight and underline text. Some devices allow you to add a written text in addition to marking a word or passage that you can collect at the end of your notetaking session. Look into the specific tools for your device and learn how to use the features that allow you to take notes electronically. You can also find apps on devices to help with taking notes, some of which you may automatically have installed when you buy the product. Microsoft's OneNote, Google Keep, and the Notes feature on phones are relatively easy to use, and you may already have free access to those.

Taking Notes on Non-Text Items (i.e., Tables, Maps, Figures, etc.)

You may also encounter situations as you study and read textbooks, primary sources, and other resources for your classes that are not actually texts. You can still take notes on maps, charts, graphs, images, and tables, and your approach to these non-text features is similar to when you prepare to take notes over a passage of text. For example, if you are looking at the following map, you may immediately come up with several questions. Or it may initially appear overwhelming. Start by asking yourself these questions:

What is the main point of this map?

- Who is the intended audience?
- Where is it?
- What time period does it depict?
- What does the map's legend (the explanation of symbols) include?
- What other information do I need to make sense of this map?



Source: Wikipedia Creative Commons, https://en.wikipedia.org/wiki/Order_of_battle_of_the_Waterloo_campaign

Figure 1.6.16 Graphics, charts, graphs, and other visual items are also important to annotate. Not only do they often convey important information, but they may appear on exams or in other situations where you'll need to use or demonstrate knowledge. Credit: "Lpankonin" / Wikipedia Commons / Attribution 3.0 Generic (CC BY 3.0)

You may want to make an extra copy of a graphic or table before you add annotations if you are dealing with a lot of information. Making sense of all the elements will take time, and you don't want to add to the confusion.

Returning to Your Notes

Later, as soon as possible after the class, you can go back to your notes and add in missing parts. Just as you may generate questions as you're reading new material, you may leave a class session or lecture or activities with many questions. Write those

down in a place where they won't get lost in all your other notes.

The exact timing of when you get back to the notes you take in class or while you are reading an assignment will vary depending on how many other classes you have or what other obligations you have in your daily schedule. A good starting place that is also easy to remember is to make every effort to review your notes within 24 hours of first taking them. Longer than that and you are likely to have forgotten some key features you need to include; must less time than that, and you may not think you need to review the information you so recently wrote down, and you may postpone the task too long.

Use your phone or computer to set reminders for all your note review sessions so that it becomes a habit and you keep on top of the schedule.

Your personal notes play a significant role in your test preparation. They should enhance how you understand the lessons, textbooks, lab sessions, and assignments. All the time and effort you put into first taking the notes and then annotating and organizing the notes will be for naught if you do not formulate an effective and efficient way to use them before sectional exams or comprehensive tests.

The whole cycle of reading, notetaking in class, reviewing and enhancing your notes, and preparing for exams is part of a continuum you ideally will carry into your professional life. Don't try to take short cuts; recognize each step in the cycle as a building block. Learning doesn't end, which shouldn't fill you with dread; it should help you recognize that all this work you're doing in the classroom and during your own study and review sessions is ongoing and cumulative. Practicing effective strategies now will help you be a stronger professional.

Activity

What resources can you find about reading and notetaking that will actually help you with these crucial skills? How do you go about deciding what resources are valuable for improving your reading and notetaking skills?

The selection and relative value of study guides and books about notetaking vary dramatically. Ask your instructors for recommendations and see what the library has available on this topic. The following list is not comprehensive, but will give you a starting point for books and articles on notetaking in college.

- *College Rules!: How to Study, Survive, and Succeed in College*, by Sherri Nist-Olejnik and Jodi Patrick Holschuh. More than just notetaking, this book covers many aspects of transitioning into the rigors of college life and studying.
- *Effective Notetaking*, by Fiona McPherson. This small volume has suggestions for using your limited time wisely before, during, and after notetaking sessions.
- *How to Study in College*, by Walter Pauk. This is the book that introduced Pauk's notetaking suggestions we now call the Cornell Method. It is a bit dated (from the 1940s), but still contains some valuable information.
- *Learn to Listen, Listen to Learn 2: Academic Listening and Note-taking*, by Roni S. Lebauer. The main point of this book is to help students get the most from college lectures by watching for clues to lecture organization and adapting this information into strong notes.
- *Study Skills: Do I Really Need this Stuff?*, by Steve Piscitelli. Written in a consistently down-to-earth manner, this book will help you with the foundations of strong study skills, including time management, effective notetaking, and seeing the big picture.
- "What Reading Does for the Mind," by Anne Cunningham and Keith Stanovich, 1998, https://www.aft.org/sites/default/files/periodicals/cunningham.pdf
- Adler, Mortimer J. and Charles Van Doren. *How to Read a Book: The Classic Guide to Intelligent Reading. NY: Simon & Schuster, 1940*.
- Berns, Gregory S., Kristina Blaine, Michael J. Prietula, and Brandon E. Pye. *Brain Connectivity.* Dec 2013.ahead of print http://doi.org/10.1089/brain.2013.0166

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1.7: How to Manage Time

Questions to consider:

- How can I use time-on-task estimates to improve time management?
- What behaviors can help or hinder when it comes to managing time?

In this next section you will learn about managing time and prioritizing tasks. This is not only a valuable skill for pursuing an education, but it can become an ability that follows you through the rest of your life, especially if your career takes you into a leadership role.



Figure 1.7.1 An online calendar is a very useful tool for keeping track of classes, meetings, and other events. Most learning management systems contain these features, or you can use a calendar application.

ANALYSIS QUESTION

Read each statement in the brief self-evaluation tool below, and check the answer that best applies to you. There are no right or wrong answers.

Table 3.3

	Always	Usually	Sometimes	Rarely	Never
I like to be given strict deadlines for each task. It helps me stay organized and on track.					
I would rather be 15 minutes early than 1 minute late.					
I like to improvise instead of planning everything out ahead of time.					

	Always	Usually	Sometimes	Rarely	Never
I prefer to be able to manage when and how I do each task.					
I have a difficult time estimating how long a task will take.					
I have more motivation when there is an upcoming deadline. It helps me focus.					
I have difficulty keeping priorities in the most beneficial order.					

This exercise is intended to help you recognize some things about your own time management style. The important part is for you to identify any areas where you might be able to improve and to find solutions for them. This chapter will provide some solutions, but there are many others that can be found by researching time management strategies.

After you have decided your best response to each statement, think about what they may mean in regard to potential strengths and/or challenges for you when it comes to time management in college. If you are a person that likes strict deadlines, what would you do if you took a course that only had one large paper due at the end? Would you set yourself a series of mini deadlines that made you more comfortable and that kept things moving along for you? Or, if you have difficulty prioritizing tasks, would it help you to make a list of the tasks to do and order them, so you know which ones must be finished first?

How to Manage Time

The simplest way to manage your time is to accurately plan for how much time it will take to do each task, and then set aside that amount of time. How you divide the time is up to you. If it is going to take you five hours to study for a final exam, you can plan to spread it over five days, with an hour each night, or you can plan on two hours one night and three hours the next. What you would not want to do is plan on studying only a few hours the night before the exam and find that you fell very short on the time you estimated you would need. If that were to happen, you would have run out of time before finishing, with no way to go back and change your decision. In this kind of situation, you might even be tempted to “pull an all-nighter,” which is a phrase that has been used among college students for decades. In essence it means going without sleep for the entire night and using that time to finish an assignment. While this method of trying to make up for poor planning is common enough to have a name, rarely does it produce the best work.

Table 3.4 Sample Time Estimate Table

Activity	Estimated Time	Actual Time
Practice Quiz	5 minutes	15 minutes
Lab Conclusions	20 minutes	35 minutes
Food shopping	45 minutes	30 minutes
Drive to work	20 minutes	20 minutes
Physical Therapy	1 hour	50 minutes

Of all the parts of time management, accurately predicting how long a task will take is usually the most difficult—and the most elusive. Part of the problem comes from the fact that most of us are not very accurate timekeepers, especially when we are busy applying ourselves to a task. The other issue that makes it so difficult to accurately estimate time on task is that our estimations must also account for things like interruptions or unforeseen problems that cause delays.?

When it comes to academic activities, many tasks can be dependent upon the completion of other things first, or the time a task takes can vary from one instance to another, both of which add to the complexity and difficulty of estimating how much time and effort are required.

For example, if an instructor assigned three chapters of reading, you would not really have any idea how long each chapter might take to read until you looked at them. The first chapter might be 30 pages long while the second is 45. The third chapter could be only 20 pages but made up mostly of charts and graphs for you to compare. By page count, it might seem that the third chapter would take the least amount of time, but actually studying charts and graphs to gather information can take longer than regular reading.?

To make matters even more difficult, when it comes to estimating time on task for something as common as reading, not all reading takes the same amount of time. Fiction, for example, is usually a faster read than a technical manual. But something like the novel *Finnegan's Wake* by James Joyce is considered so difficult that most readers never finish it.

ACTIVITY

To better understand how much time different kinds of material can take to read, try this experiment. You will use two examples of famous texts that are very close to being the same number of words: *The Gettysburg Address* and the opening paragraphs from *A Christmas Carol*. Before you begin, estimate how long it will take you to read each, and predict which you think will take longer. When you do the reading, use a stopwatch function on a device such as a phone or some other timer to see how long it actually takes.

Make certain that you are reading for understanding, not just skimming over words. If you must reread a section to better comprehend what is being said, that is appropriate. The goal here is to compare reading of different texts, not to see how fast you can sight-read the words on a page.

After you have finished *The Gettysburg Address*, read and time *A Christmas Carol* and compare both of your times.

The Gettysburg Address
Abraham Lincoln
Gettysburg, Pennsylvania November 19, 1863
Word count: 278

Four score and seven years ago our fathers brought forth on this continent, a new nation, conceived in Liberty, and dedicated to the proposition that all men are created equal.

Now we are engaged in a great civil war, testing whether that nation, or any nation so conceived and so dedicated, can long endure. We are met on a great battle-field of that war. We have come to dedicate a portion of that field, as a final resting place for those who here gave their lives that that nation might live. It is altogether fitting and proper that we should do this.

But, in a larger sense, we can not dedicate -- we can not consecrate -- we can not hallow -- this ground. The brave men, living and dead, who struggled here, have consecrated it, far above our poor power to add or detract. The world will little note, nor long remember what we say here, but it can never forget what they did here. It is for us the living, rather, to be dedicated here to the unfinished work which they who fought here have thus far so nobly advanced. It is rather for us to be here dedicated to the great task remaining before us -- that from these honored dead we take increased devotion to that cause for which they gave the last full measure of devotion -- that we here highly resolve that these dead shall not have died in vain -- that this nation, under God, shall have a new birth of freedom -- and that government of the people, by the people, for the people, shall not perish from the earth.

A Christmas Carol
Charles Dickens
Chapman & Hall, 1843
Word count: 260

Marley was dead: to begin with. There is no doubt whatever about that. The register of his burial was signed by the clergyman, the clerk, the undertaker, and the chief mourner. Scrooge signed it: and Scrooge's name was good upon 'Change, for anything he chose to put his hand to. Old Marley was as dead as a door-nail.

Mind! I don't mean to say that I know, of my own knowledge, what there is particularly dead about a door-nail. I might have been inclined, myself, to regard a coffin-nail as the deadest piece of ironmongery in the trade. But the wisdom of our ancestors is in the simile; and my unhallowed hands shall not disturb it, or the Country's done for. You will therefore permit me to repeat, emphatically, that Marley was as dead as a door-nail.

Scrooge knew he was dead? Of course he did. How could it be otherwise? Scrooge and he were partners for I don't know how many years. Scrooge was his sole executor, his sole administrator, his sole assign, his sole residuary legatee, his sole friend, and sole mourner. And even Scrooge was not so dreadfully cut up by the sad event, but that he was an excellent man of business on the very day of the funeral, and solemnised it with an undoubted bargain.

The mention of Marley's funeral brings me back to the point I started from. There is no doubt that Marley was dead. This must be distinctly understood, or nothing wonderful can come of the story I am going to relate.

In comparing the two, was one or the other easier to understand or faster to read? Was it the piece you predicted you would read faster?

It is important to note that in this case both readings were only three paragraphs long. While there may have only been half a minute or so between the reading of each, that amount of time would multiply greatly over an entire chapter.

Knowing Yourself

While you can find all sorts of estimates online as to how long a certain task may take, it is important to know these are only averages. People read at different speeds, people write at different speeds, and those numbers even change for each individual depending on the environment.

If you are trying to read in surroundings that have distractions (e.g., conversations, phone calls, etc.), reading 10 pages can take you a lot longer than if you are reading in a quiet area. By the same token, you may be reading in a quiet environment (e.g., in bed after everyone in the house has gone to sleep), but if you are tired, your attention and retention may not be what it would be if you were refreshed.

In essence, the only way you are going to be able to manage your time accurately is to know yourself and to know how long it takes you to do each task. But where to begin?

Below, you will find a table of common college academic activities. This list has been compiled from a large number of different sources, including colleges, publishers, and professional educators, to help students estimate their own time on tasks. The purpose of this table is to both give you a place to begin in your estimates and to illustrate how different factors can impact the actual time spent.

You will notice that beside each task there is a column for the *unit*, followed by the average *time on task*, and a column for notes. The *unit* is whatever is being measured (e.g., pages read, pages written, etc.), and the *time on task* is an average time it takes students to do these tasks. It is important to pay attention to the notes column, because there you will find factors that influence the time on task. These factors can dramatically change the amount of time the activity takes.

Table 3.5 Time on task for common college activities.

Time on Task			
Activity	**Unit**	**Time on task**	**Notes**
General academic reading (textbook, professional journals)	1 page	5–7 minutes	Be aware that your personal reading speed may differ and may change over time.
Technical reading (math, charts and data)	1 page	10–15 minutes	Be aware that your personal reading speed may differ and may change over time.

Time on Task			
Simple Quiz or homework question: short answer—oriented toward recall or identification type answers	Per question	1–2 minutes	Complexity of question will greatly influence the time required.
Complex Quiz or homework question: short answer—oriented toward application, evaluation, or synthesis of knowledge	Per question	2–3 minutes	Complexity of question will greatly influence the time required.
Math problem sets, complex	Per question	15 minutes	For example, algebra, complex equations, financial calculations
Writing: short, no research	Per page	60 minutes	Short essays, single-topic writing assignments, summaries, freewriting assignments, journaling—includes drafting, writing, proofing, and finalizing
Writing: research paper	Per page	105 minutes	Includes research time, drafting, editing, proofing, and finalizing (built into per-page calculation)
Study for quiz	Per chapter	60 minutes	45–90 minutes per chapter, depending upon complexity of material
Study for exam	Per exam	90 minutes	1–2 hours, depending upon complexity of material

Again, these are averages, and it does not mean anything if your times are a little slower or a little faster. There is no "right amount of time," only the time that it takes you to do something so you can accurately plan and manage your time.

There is also another element to look for in the table. These are differentiations in the similar activities that will also affect the time you spend. A good example of this can be found in the first four rows. Each of these activities involves reading, but you can see that depending on the material being read and its complexity, the time spent can vary greatly. Not only do these differences in time account for the different types of materials you might read (as you found in the comparative reading exercise earlier in this chapter), but also they also take into consideration the time needed to think about what you are reading to truly understand and comprehend what it is saying.

GET CONNECTED

Which apps help you best prepare for success when managing your time?

Do you have trouble keeping track of multiple tasks over the course of a term?

Trello lets you organize all your obligations in helpful boards. You can share them with others (project collaborators), set alerts as reminders, and mark tasks off as you complete them.

Do you use a particular app to help you manage your time?

Sticky note apps are available for PC, Mac, and mobile devices. They let you post quick reminders, reorganize them as needed, and view them separately or as a full to-do list.

What do you wish you could improve about your time management skills?

> Toggl helps you keep track of how and where you are spending your time so you can budget better and make time management changes that free you up for the really important stuff.

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1.8: Enhanced Strategies for Time and Task Management

Questions to consider:

- What strategy helps me prioritize my top tasks?
- How do I make the best use of my time when prioritizing?
- How do I make sure I tackle unpleasant tasks instead of putting them off?
- What's the best way to plan for long-term tasks?
- How do I find time in a busy schedule?

Over the years, people have developed a number of different strategies to manage time and tasks. Some of the strategies have proven to be effective and helpful, while others have been deemed not as useful.

The good news is that the approaches that do not work very well or do not really help in managing time do not get passed along very often. But others, those which people find of value, do. What follows here are three unique strategies that have become staples of time management. While not everyone will find that all three work for them in every situation, enough people have found them beneficial to pass them along with high recommendations.

Daily Top Three

The idea behind the *daily top three* approach is that you determine which three things are the most important to finish that day, and these become the tasks that you complete. It is a very simple technique that is effective because each day you are finishing tasks and removing them from your list. Even if you took one day off a week and completed no tasks on that particular day, a *daily top three* strategy would have you finishing 18 tasks in the course of a single week. That is a good amount of things crossed off your list.

analysis question

Analysis: Think about what would be your top three tasks for today? What would you have on the list tomorrow?

Pomodoro Technique



Figure 1.8.14 The Pomodoro Technique is named after a type of kitchen timer, but you can use any clock or countdown timer. (Marco Verch /Flickr / Attribution 2.0 Generic (CC BY 2.0))

The Pomodoro Technique was developed by Francesco Cirillo. The basic concept is to use a timer to set work intervals that are followed by a short break. The intervals are usually about 25 minutes long and are called *pomodoros*, which comes from the Italian word for tomato because Cirillo used a tomato-shaped kitchen timer to keep track of the intervals.

In the original technique there are six steps:

1. Decide on the task to be done.
2. Set the timer to the desired interval.
3. Work on the task.
4. When the timer goes off, put a check mark on a piece of paper.
5. If you have fewer than four check marks, take a short break (3–5 minutes), then go to Step 1 or 2 (whichever is appropriate).
6. After four pomodoros, take a longer break (15–30 minutes), reset your check mark count to zero, and then go to Step 1 or 2.

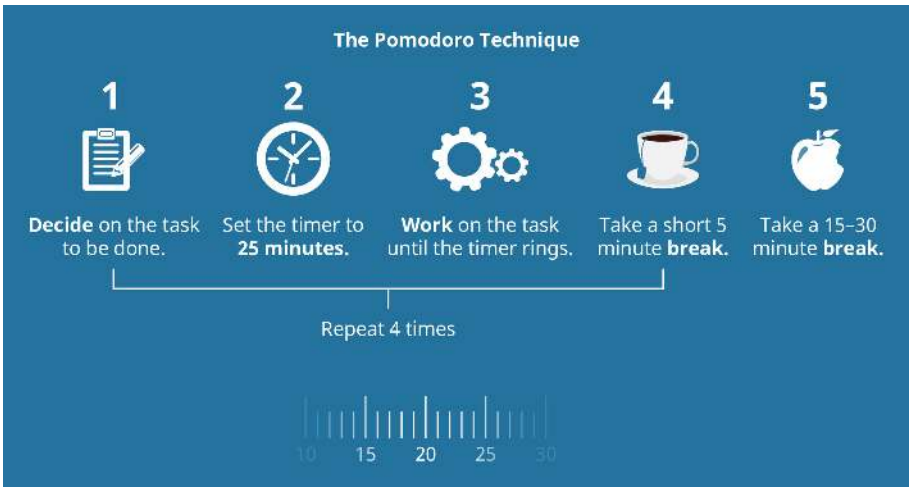


Figure 1.8.15 The Pomodoro Technique contains five defined steps.

There are several reasons this technique is deemed effective for many people. One is the benefit that is derived from quick cycles of work and short breaks. This helps reduce mental fatigue and the lack of productivity caused by it. Another is that it tends to encourage practitioners to break tasks down to things that can be completed in about 25 minutes, which is something that is usually manageable from the perspective of time available. It is much easier to squeeze in three 25-minute sessions of work time during the day than it is to set aside a 75- minute block of time.

Eat the Frog

Of our three quick strategies, *eat the frog* probably has the strangest name and may not sound the most inviting. The name comes from a famous quote, attributed to Mark Twain: “Eat a live frog first thing in the morning and nothing worse will happen to you the rest of the day.” *Eat the Frog* is also the title of a best-selling book by Brian Tracy that deals with time management and avoiding procrastination.

How this applies to time and task management is based on the concept that if a person takes care of the biggest or most unpleasant task first, everything else will be easier after that.

Although stated in a humorous way, there is a good deal of truth in this. First, we greatly underestimate how much worry can impact our performance. If you are continually distracted by anxiety over a task you are dreading, it can affect the task you are working on at the time. Second, not only will you have a sense of accomplishment and relief when the task you are concerned with is finished and out of the way, but other tasks will seem lighter and not as difficult.

Application: Try Three Time Management Strategies

Over the next two weeks, try each of these three methods to see which ones might work for you. Is there one you favor over the others? Might each of these three approaches serve you better in different situations or with different tasks? Do you have a creative alternative or possibly a way to use some combination of these techniques?

In addition to these three strategies, you could also develop whole new approaches from suggestions found earlier in this chapter. For example, you could apply some of the strategies for avoiding procrastination or for setting appropriate priorities and see how they work in combination with these techniques or on their own.

The key is to find which system works best for you.

Breaking Down the Steps and Spreading Them over Shorter Work Periods

Above, you read about several different tried-and-tested strategies for effective time management—approaches that have become staples in the professional world. In this section you will read about two more creative techniques that combine elements from these other methods to handle tasks when time is scarce and long periods of time are a luxury you just do not have.

The concept behind this strategy is to break tasks into smaller, more manageable units that do not require as much time to complete. As an illustration of how this might work, imagine that you are assigned a two-page paper that is to include references. You estimate that to complete the paper—start to finish—would take you between four and a half and five hours. You look at your

calendar over the next week and see that there simply are no open five-hour blocks (unless you decided to only get three hours of sleep one night). Rightly so, you decide that going without sleep is not a good option. While looking at your calendar, you do see that you can squeeze in an hour or so every night. Instead of trying to write the entire paper in one sitting, you break it up into much smaller components as shown in the table below:

Table 3.8: Breaking Down Projects into Manageable-Sized Tasks

Day/Time	Task	Time
Monday, 6:00 p.m.	Write outline; look for references.	60 minutes
Tuesday, 6:00 p.m.	Research references to support outline; look for good quotes.	60 minutes
Wednesday, 7:00 p.m.	Write paper introduction and first page draft.	60 minutes
Thursday, 6:00 p.m.	Write second page and closing draft.	60 minutes
Friday, 5:00 p.m.	Rewrite and polish final draft.	60 minutes
Saturday, 10:00 a.m.	*Only if needed—finish or polish final draft.*	60 minutes?

While this is a simple example, you can see how it would redistribute tasks to fit your available time in a way that would make completing the paper possible. In fact, if your time constraints were even more rigid, it would be possible to break these divided tasks down even further. You could use a variation of the Pomodoro Technique and write for three 20-minute segments each day at different times. The key is to look for ways to break down the entire task into smaller steps and spread them out to fit your schedule.

Table 3.9

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
8:00–10:00		Work		Work			
10:00–12:00	Algebra	Work	Algebra	Work	Algebra	10 a.m.–11 a.m. *Only if needed*	Work
12:00–2:00	Lunch/study	1 p.m. English Comp	Lunch/study	1 p.m. English Comp	Lunch/study	Family picnic	Work
2:00–4:00	History	English Comp	History	English Comp	History	Family picnic	
4:00–6:00	Study for Algebra quiz.	Grocery	Study for History exam.	Study for History exam.	5 p.m.–6 p.m. Rewrite and polish final draft.	Family picnic	Laundry
6:00–7:00	Write outline; look for references.	Research references to support outline; look for good quotes.	Research presentation project.	Write second page and closing draft	Create presentation.	Meet with Darcy.	Prepare school stuff for next week.
7:00–8:00	Free time	Free time	Write paper introduction and first page draft.	Research presentation project.	Create presentation.		Free time

STUDENT PROFILE

"Time management is probably one of the hardest things I had to pick up when I got to college. For starters, I didn't have anyone to come wake me up if I forgot to set an alarm or to tell me to get out of bed so that I wouldn't be late. I had to start placing my phone far away from my bed; so that way, I would have to get out of bed in order to turn the alarm off. Accomplishing work on time can also be difficult. It's tough to find the fine balance between when you have to stay in and work on assignments and when is acceptable to go out and do leisure activities.

"I learned the 8-8-8 rule. Every day you spend eight hours working on school work or going to class, eight hours of free time to do what you want, and then eight hours to sleep at night so that you will get enough rest. Sleep is crucial for time management. I learned very quickly that you cannot focus or be productive if you are struggling to keep your head from falling over because you are so tired. Basically, I've learned that if you want to be successful in college, then you have to be on top of your game when it comes to time. It's something thing you cannot make up once it's gone."

—Preston Allen, University of Central Arkansas

Analyzing Your Schedule and Creating Time to Work

Of all the strategies covered in this chapter, this one may require the most discipline, but it can also be the most beneficial in time management. The fact is most of us waste time throughout the day. Some of it is due to a lack of awareness, but it can also be caused by the constraints of our current schedules. An example of this is when we have 15 to 20 minutes before we must leave to go somewhere. We don't do anything with that time because we are focused on leaving or where we are going, and we might not be organized enough to accomplish something in that short of a time period. In fact, a good deal of our 24- hour days are spent a few minutes at a time waiting for the next thing scheduled to occur. These small units of time add up to a fair amount each day.

The intent of this strategy is to recapture those lost moments and use them to your advantage. This may take careful observation and consideration on your part, but the results of using this as a method of time management are more than worth it.

The first step is to look for those periods of time that are wasted or that can be repurposed. In order to identify them, you will need to pay attention to what you do throughout the day and how much time you spend doing it. The example of waiting for the next thing in your schedule has already been given, but there are many others. How much time do you spend in activities after you have really finished doing them but are still lingering because you have not begun to do something else (e.g., letting the next episode play while binge-watching, reading social media posts or waiting for someone to reply, surfing the Internet, etc.)? You might be surprised to learn how much time you use up each day by just adding a few unproductive minutes here and there.

If you set a limit on how much time you spend on each activity, you might find that you can recapture time to do other things. An example of this would be limiting yourself to reading news for 30 minutes. Instead of reading the main things that interest you and then spending an additional amount of time just looking at things that you are only casually interested in because that is what you are doing at the moment, you could stop after a certain allotted period and use the extra time you have gained on something else.

After you identify periods of lost time, the next step will be to envision how you might restructure your activities to bring those extra minutes together into useful blocks of time. Using the following scenario as an illustration, we will see how this could be accomplished.



Figure 1.8.16 Sarah has to balance a lot of obligations.

On Tuesday nights, Sarah has a routine: After work, she does her shopping for the week (2 hours driving and shopping) and then prepares and eats dinner (1 hour). After dinner, she spends time on homework (1 hour) and catching up with friends, reading the news, and other Internet activities (1 hour), and then she watches television or reads before going to bed (1 hour). While it may seem that there is very little room for improvement in her schedule without cutting out something she enjoys, limiting the amount of time she spends on each activity and rethinking how she goes about each task can make a significant difference.

In this story, Sarah's Tuesday-night routine includes coming home from work, taking stock of which items in her home she might need to purchase, and then driving to the store. While at the store, she spends time picking out and selecting groceries as she plans for meals she will eat during the rest of the week. Then, after making her purchases, she drives home. Instead, if she took the time to make a list and plan for what she needed at the store before she arrived, she would not spend as much time looking for inspiration in each aisle. Also, if she had a prepared list, not only could she quickly pick up each item, but she could stop at the store on the way home from work, thus cutting out the extra travel time. If purchasing what she needed took 30 minutes less because she was more organized and she cut out an additional 20 minutes of travel time by saving the extra trip to the store from her house, she could recapture a significant amount of her Tuesday evening. If she then limited the time she spent catching up with friends and such to 30 minutes or maybe did some of that while she prepared dinner, she would find that she had added almost an extra hour and a half to the time available to her on that evening, without cutting out anything she needed to do or enjoys. If she decided to spend her time on study or homework, this would more than double the time she previously had available in her schedule for homework.

Analysis Question: Reflection

Analysis: Identify areas in the way you spend your day where you may be able to recapture and repurpose time. Are there things you can move around to gain more time? Are there ways you can combine tasks or reduce travel time?

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1.9: Goal Setting and Motivation

Questions to consider:

- How do I set motivational goals?
- What are SMART goals?
- What's the importance of an action plan?
- How do I keep to my plan?

Motivation often means the difference between success and failure. That applies to school, to specific tasks, and to life in general. One of the most effective ways to keep motivated is to set goals.

Goals can be big or small. A goal can range from *I am going to write one extra page tonight*, to *I am going to work to get an A in this course*, all the way to *I am going to graduate in the top of my class so I can start my career with a really good position*. The great thing about goals is that they can include and influence a number of other things that all work toward a much bigger picture. For example, if your goal is to get an A in a certain course, all the reading, studying, and every assignment you do for that course contributes to the larger goal. You have motivation to do each of those things and to do them well.

Setting goals is something that is frequently talked about, but it is often treated as something abstract. Like time management, goal setting is best done with careful thought and planning. This next section will explain how you can apply tested techniques to goal setting and what the benefits of each can be.

Set Goals That Motivate You

The first thing to know about goal setting is that a goal is a specific end result you desire. If the goal is not something you are really interested in, there is little motivational drive to achieve it. Think back to when you were much younger and some well-meaning adult set a goal for you—something that didn't really appeal to you at all. How motivated were you to achieve the goal? More than likely, if you were successful at all in meeting the goal, it was because you were motivated by earning the approval of someone or receiving a possible reward, or you were concerned with avoiding something adverse that might happen if you did not do what you were told. From an honest perspective in that situation, your real goal was based on something else, not the meeting of the goal set for you. To get the most from the goals you set, make sure they are things that you are interested in achieving.

That is not to say you shouldn't set goals that are supported by other motivations (e.g., If I finish studying by Friday, I can go out on Saturday), but the idea is to be intellectually honest with your goals.

Set SMART Goals

Goals should also be SMART. In this case, the word *smart* is not only a clever description of the type of goal, but it is also an acronym that stands for Specific, Measurable, Attainable, Relevant, and Time-bound. The reason these are all desirable traits for your goals is because they not only help you plan how to meet the goal, but they can also contribute to your decision-making processes during the planning stage.

What does it mean to create SMART goals?

- Specific—For a goal to be specific, it must be defined enough to actually determine the goal. A goal of *get a good job when I graduate* is too general. It doesn't define what a good job is. In fact, it doesn't even necessarily include a job in your chosen profession. A more specific goal would be something like *be hired as a nurse in a place of employment where it is enjoyable to work and that has room for promotion.*
- Measurable—The concept of *measurable* is one that is often overlooked when setting goals. What this means is that the goal should have clearly defined outcomes that are detailed enough to measure and can be used for planning of how you will achieve the goal. For example, setting a goal of *doing well in school* is a bit undefined, but making a goal of *graduating with a GPA above 3.0* is measurable and something you can work with. If your goal is measurable, you can know ahead of time how many points you will have to earn on a specific assignment to stay in that range or how many points you will need to make up in the next assignment if you do not do as well as you planned.
- Attainable—*Attainable* or *achievable* goals means they are reasonable and within your ability to accomplish. While a goal of *make an extra one million dollars by the end of the week* is something that would be nice to achieve, the odds that you could make that happen in a single week are not very realistic.

- Relevant—For goal setting, *relevant* means it applies to the situation. In relation to college, a goal of *getting a horse to ride* is not very relevant, but *getting dependable transportation* is something that would contribute to your success in school.
- Time-bound—Time-bound means you set a specific time frame to achieve the goal. *I will get my paper written by Wednesday* is time-bound. You know when you have to meet the goal. *I will get my paper written sometime soon* does not help you plan how and when you will accomplish the goal.

In the following table you can see some examples of goals that do and do not follow the SMART system. As you read each one, think about what elements make them SMART or how you might change those that are not.

Table 3.6

Goal	Is it SMART?	
I am going to be rich someday.	No	There is nothing really specific, measurable, or time-bound in this goal.
I will graduate with my degree, on time.	Yes	The statement calls out specific, measureable, and time-bound details. The other attributes of attainable and relevant are implied.
I am going to save enough money to buy a newer car by June.	Yes	All SMART attributes are covered in this goal.
I would like to do well in all my courses next semester.	No	While this is clearly time-bound and meets most of the SMART goal attributes, it is not specific or measurable without defining what “do well” means.
I am going to start being a nicer person.	No	While most of the SMART attributes are implied, there is nothing really measurable in this goal.
I will earn at least a 3.0 GPA in all my courses next semester.	Yes	All of the SMART attributes are present in this goal.
I am going to start being more organized.	No	While most of the SMART attributes are implied, there is nothing really measurable in this goal.

APPLICATION

Try writing two SMART goals—something with a one-week time frame and something that you will accomplish over the next year. Make certain that you include all the appropriate elements—Specific, Measurable, Attainable, Relevant, and Time-bound.

Make an Action Plan

Like anything else, making a step-by-step action plan of how you will attain your goals is the best way to make certain you achieve them. It doesn’t matter if it is a smaller goal with immediate results (e.g., finish all your homework due by Friday) or something bigger that takes several years to accomplish (graduate with my degree in the proper amount of time).

The planning techniques you use for time management and achieving goals can be similar. In fact, accurate goal setting is very much a part of time management if you treat the completion of each task as a goal.

What follows is an example of a simple action plan that lists the steps for writing a short paper. You can use something like this or modify it in a way that would better suit your own preferences.

Table 3.7

Action Plan		
Task	**Objective**	**When**

Action Plan		
Choose topic.	Select something interesting.	Needs to be done by Monday!
Write outline, look for references.	Create structure of paper and outline each part.	Monday, 6:00 p.m.
Research references to support outline, look for good quotes.	Strengthen paper and resources.	Tuesday, 6:00 p.m.
Write paper introduction and first page draft.	Get main ideas and thesis statement down.	Wednesday, 7:00 p.m.
Write second page and closing draft.	Finish main content and tie it all together.	Thursday, 6:00 p.m.
Rewrite and polish final draft.	Clean up for grammar, writing style, and effective communication.	Friday, 5:00 p.m.

Another useful approach to goal setting is to create SMART goals and then write them down. For most people there is a higher level of commitment when we write something down. If you have your goals written out, you can refer to each component of the SMART acronym and make certain you are on track to achieve it.

Stick with It!

As with anything else, the key to reaching goals is to keep at it, keep yourself motivated, and overcome any obstacles along the way. In the following graphic you will find seven methods that highly successful people use to accomplish this.



Figure 1.9.13 These seven ways to stay motivated are good suggestions from highly successful people. What other strategies would you suggest?

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1.10: Test Taking and Preparation

Students need to develop a mindset that every time they attend class or complete a reading assignment they are preparing for the next exam, because oftentimes so much of their grade rides on those exams. In college, it is not uncommon for your entire course grade to be based on two or three exams with no options for a retake or extra credit. Instead of an exam being over one or two chapters, exams are more likely to be over 5+ chapters of material, and in the case of a comprehensive exam, an entire semester's worth of material. Intentionally working through the process of preparing for, taking, and learning from the testing process will facilitate successful outcomes.



Table 5-4. Effective Technique for Test Preparation and Taking

Before the Test	During the Test	After the Test
Go to all of your classes and take lecture notes and read all of your assigned reading using techniques described in this chapter.	Mark your exam with the necessary identification. Read all instructions on the exam. Scan the exam for question types, point values, etc.	Conduct an error analysis for items missed on the test. Look up the correct answers. Determine the nature of the errors you made.
Develop a study plan and schedule the time to review for an exam over several days. Refer to "Create a Five-Day Study Plan for Exams."	Develop a plan for how you will use your time to complete all questions on the exam.	Talk with your professor about items you missed on the exam. Discuss adjustments in your test preparation methods if necessary.

Schedule study sessions with classmates in advance of the night before the exam.	Read each item carefully and fully before marking an answer. You might find clues to the answer and will be less likely to make a preventable mistake.	Analyze the test questions to see if patterns arise, such as,"Were most of the questions from the lecture, the book, or both?" "What types of questions were asked?" "What levels of thinking were being tested based on Bloom's Taxonomy?" etc.
Be sure you have materials you need for the exam ready at least the night before the exam, such as multiple writing utensils, scantrons, blue books, notebook paper, etc.	Skip difficult questions and come back to them if you are pressed for time. Caution: if using a scantron to record answers be very careful that you do not incorrectly record your answers on the wrong line and get off track.	Review the methods that you used to study for the exam and determine what adjustments you need to make in preparing for the next exam.
Get a good night's rest the night before the exam.	If time is available at the end of the exam, review questions you marked as "difficult" in order to check for errors.	Determine your overall grade in the course based on the new information that this grade has provided you. Talk with your professor and academic advisor if you have concerns about your overall grade.
Eat a good breakfast prior to the exam. Avoid caffeine prior to the exam.	If you have an essay question to write, take the time to briefly outline an answer to ensure that you are answering all aspects of the question and that you provide a well-organized answer for your faculty member to read.	Archive your exam for future reference.
To ensure that you are on time to your exam, set two alarm clocks or arrange to have a friend call you to make sure you are awake.	Try not to leave any answers blank. You might be awarded partial credit or even make a correct guess.	
	Take a moment to take some deep cleansing breaths (or any other quick relaxation techniques) if you feel stressed by a question or the exam as a whole.	

Preparation for an exam should begin the first day of class. Repeated exposure to material over a longer period of time (Baddeley, 1982; Bahrick & Hall, 1991) optimizes the recall of information when it is needed. Every assigned reading, every opportunity to take notes and engage in class activities, and a routine of reviewing notes should be considered activities toward exam preparation. The volume of information that students are responsible for on each exam is much larger than experienced in high school. Careful review of the material prior to each exam is required for optimal performance, and student reports tell us that the habit of "cramming" the night before the exam is not going to lead to positive, long-term results. Research of the brain supports this typical finding. Physical and neurochemical changes in the synapses of the brain occur when information is learned, a process called long-term potentiation (Kandel & Schwartz, 1982). Without proper time and repeated stimulation of the neural circuits, long-term learning of this information will not occur efficiently and may not be accessible when needed for a test. Planning for a review period over five days prior to the exam and healthy eating and sleep habits cannot be overemphasized.

Activity 5-4

Identify when your next substantial exam will be given in a class. Create a five-day study plan using the formula described in "Create a Five-Day Study Plan for Exams" and the "Five-Day Study Plan Worksheet" at the end of this chapter. Once you have created this plan, incorporate the schedule into your time management calendar for implementation.

Oftentimes, students jump right in to taking an exam feeling stressed and rushed with a need to regurgitate information immediately. However, approaching a test in a careful and methodical way can help ensure the best results. You can best utilize your exam time by taking the time to survey the exam directions and point values and planning how to approach sections of the

test. Strategy is particularly important for tests with mixed types of questions (i.e., multiple choice and essay) or tests with multiple essay questions. For ex- ample, if you were to spend too much time on the multiple choice questions and not allocate enough time to complete an essay question, you may have answered the majority of the total number of questions but still do poorly on the exam if the essay question was worth a larger portion of the exam's total points. Plan your time carefully and manage your stress.

Create a Five-Day Study Plan for Exams

Making a plan that will allow for the careful review of all of the assigned and presented material leads to less stress the day of an exam and the best conditions for recall and performance on an exam. Here is a simple method for creating a study plan:

1. Break the material for the exam into four manageable "chunks." If material can be logically divided by chapters, use that method. If not, make up your own chunks based on the content of the material. Note: It is not a good idea to break up information by the method of delivery (i.e., one chunk equals notes, one chunk equals readings, etc.). You want to include all information on specific topics together in one chunk (e.g., information about the topic of osmosis from the lecture notes and the textbook in a chunk).
2. Plan to spend about two hours studying on each of the five days.
3. You should work with the material on the exam in two ways, by preparing that material for review and by reviewing that material. Preparation of material incorporates study techniques discussed earlier in this chapter such as combining and clarifying lecture and textbook notes, creating visual organizers for topics, developing cram cards, solving practice problems, creating self-testing questions using multiple levels of Bloom's Taxonomy, etc. Once you get the hang of preparing material for review, you could do some of that along the way (for instance creating cram cards for readings and notes) to reduce the amount of time needed for preparation at this time, leaving more time for review.

Example of the Schedule for a Five-Day Study Plan

Day One
Prepare 1st Chunk – 2 hours

Day Two
Prepare 2nd Chunk – 2 hours
Review 1st Chunk – 30 minutes

Day Three
Prepare 3rd Chunk – 1.5 hours
Review 2nd Chunk – 30 minutes
Review 1st Chunk – 15 minutes

Day Four
Prepare 4th Chunk – 1 hour
Review 3rd Chunk – 30 minutes
Review 2nd Chunk – 15 minutes
Review 1st Chunk – 10 minutes

Day Five
Review 4th Chunk – 1 hour
Review 3rd Chunk – 30 minutes
Review 2nd Chunk – 30 minutes
Review 1st Chunk – 30 minutes

Taking some time to plan out what material will go into each chunk will ensure that you do not miss anything. Use the "Five-Day Study Plan Worksheet" at the end of this chapter as a guide for your planning.

This method typically works very well for courses that offer two or three tests throughout the semester; however, a similar method could be modified by the number of days or the amount of time spent each day if a course offers more exams with less material on each exam.

Tips for Better Test Taking

The types of test items on a test are as different as the professors who created the tests. "Objective" test questions refer to items in which students have to recognize the correct answer from a list of provided options (e.g., multiple choice, true-false, matching). In

most cases there is one best answer, though it is important to note that a professor could indicate in the directions that you can select more than one answer. “Subjective” test items are the fill-in-the-blank, short answer, or essay questions in which students must recall and produce the answer.

Multiple Choice Questions

- Read the question and all of the answer options first.
- Mark out any answers that you know are not correct.
- Once you have selected an answer, do not change it unless you misread the question and know the new answer is correct. Usually, your first answer is the correct one if you are making an educated guess.
- Many times, the correct answer has more information in it.
- If there is no penalty for guessing, do not leave any items blank—make an educated guess.
- If there is an “All of the Above” option and there are two options that are correct, select “All of the Above.”
- In a question with “All of the Above” and “None of the Above,” if you are certain that one of the answers is true, do not select “None of the Above” and, likewise, if one of the answers is false, then do not select “All of the Above.”

True-False Questions

- Read the statement carefully, but do not read too much into the statement. Base your answer on the information provided.
- With statements that have multiple facts, if any one of the facts is false the entire statement is false.
- Qualifying words like “always,” “never,” and “every” indicate that this would have to be true all of the time. If it is not true all of the time, then you should answer false.
- Qualifying words like “usually,” “sometimes,” and “generally” indicate that it could be true or false depending on the situation. Oftentimes the answer is true.

Essay Questions

- Most Important Rule: Make sure you understand what the question is asking you to do. An essay question is not an opportunity to simply regurgitate everything you know about a particular topic. Revisit Question Cues listed in the “Table 5-1: “Bloom’s Taxonomy: Cognitive Domain” and be sure you know what you are supposed to accomplish when you see those cues. Underline key words and let those words help guide you. If you are not sure, ask your professor for clarification on what he or she wants you to accomplish in your answer.
- Take a minute or so to plan out your answer and jot down a brief outline of key points to guide your writing before you start. A well-organized answer tends to score more points when graded.
- Formula for Essay Questions:
  - Introductory statements that define terms and describe what you will accomplish in your essay. This section should be brief.
  - State your first main idea, and then give supporting facts, examples, statistics, or details. Follow with your next main idea and continue the pattern until complete. The bulk of your time should be spent developing this section of the essay as it would likely include the details your professor wants to see in grading. Typically, each main idea would have its own paragraph. Use transitional works like “first, second, third…,” “next,” “also,” “however,” etc., to aid your reader.
  - End with a summary or final conclusion. This section is also likely to be brief.
- Budget time to proofread and revise if needed.
- Write legibly. Neater papers tend to receive higher marks.

Adapted from Dembo and Seli (2008) and TestTakingTips.com (2003–2009).

Once the exam has been completed, the learning experience is not over. Look over the test after you get it back from your professor and continue your review of the material that may be important for a comprehensive final or information that serves as a foundation for material to be learned in other courses. Even if only a scantron answer sheet is returned to you, professors are willing to give students access to review the test results during their office hours. Comprehensive post-test review not only includes reviewing your correct and incorrect answers to the question but also looking at where questions came from (i.e., lecture notes, readings, or both). Also, consider the study strategies you used and how they worked for you in order to help you prepare for future exams. Overall, it is important to emphasize the benefit of talking with your professor after each exam to gain insights into the information presented in the class as well as your learning and test strategies.

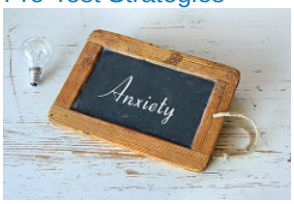
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1.11: Pre- Mid- and Post-Test-Taking Strategies

Phyllis Nissila

Pre-Test Strategies



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Q: When should you start preparing for the first test? Circle…

1. The night before.
2. The week prior.
3. The first day of classes.

If you answered "3. The first day of classes," you are correct. If you circled all three, you are also correct. Preparing to pass tests is something that begins when learning begins and continues all the way through to the final exam.

Many students, however, don't start thinking about test taking, whether weekly exams, mid-terms, or finals, until the day before when they engage in an all-nighter, or cramming. From the previous unit on memory, you might recall that the brain can only process an average of 5-7 new pieces of information at a time. Additionally, unless memory devices are used to aid memory and to cement information into long term memory (or at least until the test is over tomorrow!) chances are slim that students who cram will effectively learn and remember the information.

Additionally, a lot of students are unaware of the many strategies available to help with the test-taking experience before, during, and after. For starters, take a look at what has helped you so far.

EXERCISE 30-1

Pre-Test Taking Strategies

PART A:

Put a check mark next to the pre-test strategies you already employ.

____ Organize your notebook and other class materials the first week of classes.

____ Maintain your organized materials throughout the term.

____ Take notes on key points from lectures and other materials.

____ Make sure you understand the information as you go along.

____ Access your instructor's help and the help of a study group, as needed.

____ Organize a study group, if desired.

____ Create study tools such as flashcards, graphic organizers, etc. as study aids.

____ Complete all homework assignments on time.

____ Review likely test items several times beforehand.

____ Ask your instructor what items are likely to be covered on the test.

____ Ask your instructor if she or he can provide a study guide or practice test.

____ Ask your instructor if he/she gives partial credit for test items such as essays.

____ Maintain an active learner attitude.

____ Schedule extra study time in the days just prior to the test.

____ Gather all notes, handouts, and other materials needed before studying.

____ Review all notes, handouts, and other materials.

____ Organize your study area for maximum concentration and efficiency.

____ Create and use mnemonic devices to aid memory.

____ Put key terms, formulas, etc., on a single study sheet that can be quickly reviewed.

____ Schedule study times short enough (1-2 hours) so you do not get burned out.

____ Get plenty of sleep the night before.

____ Set a back-up alarm in case the first alarm doesn't sound or you sleep through it.

____ Have a good breakfast with complex carbs and protein to see you through.

____ Show up 5-10 minutes early to get completely settled before the test begins.

____ Use the restroom beforehand to minimize distractions.

PART B

By reviewing the pre-test strategies, above, you have likely discovered new ideas to add to what you already use. Make a list of them.

Mid-Test Strategies

Here is a list of the most common–and useful–strategies to survive this ubiquitous college experience.

- Scan the test, first, to get the big picture of how many test items there are, what types there are (multiple choice, matching, essay, etc.), and the point values of each item or group of items.
- Determine which way you want to approach the test: Some students start with the easy questions first, that is, the ones they immediately know the answers to, saving the difficult ones for later, knowing they can spend the remaining time on them. Some students begin with the biggest-point items first, to make sure they get the most points.
- Determine a schedule that takes into consideration how long you have to test, and the types of questions on the test. Essay questions, for example, will require more time than multiple choice, or matching questions.
- Keep your eye on the clock.
- If you can mark on the test, put a check mark next to items you are not sure of just yet. It is easy to go back and find them to answer later on. You might just find help in other test questions covering similar information.
- Sit where you are most comfortable. That said, sitting near the front has a couple of advantages: You may be less distracted by other students. If a classmate comes up with a question for the instructor and there is an important clarification given, you will be better able to hear it and apply it, if needed.
- Wear ear plugs, if noise distracts you.
- You do NOT have to start with #1! It you are unsure of it, mark it to come back to later on.
- Bring water…this helps calm the nerves, for one, and water is also needed for optimum brain function.
- If permitted, get up and stretch (or stretch in your chair) from time to time to relieve tension and assist the blood to the brain!
- Remember to employ strategies to reduce test-taking anxiety (covered in the next lesson)
- If despite all of your best efforts to prepare for a test you just cannot remember the answer to a given item for multiple choice, matching, and/or true/false questions, employ one or more of the following educated guessing (also known as "educated selection") techniques. By using these techniques, you have a *better* chance of selecting the correct answer. It is usually best to avoid selecting an extreme or all-inclusive answer (also known as 100% modifiers) such as "always," and "never". Choose, instead, words such as "usually," "sometimes," etc. (also known as in-between modifiers). If the answers are numbers, choose one of the middle numbers. If you have options such as "all of the above," or "both A and B," make sure each item is true before selecting those options. Choose the longest, or most inclusive, answer. Make sure to match the grammar of question and answer. For example, if the question indicates a plural answer, look for the plural answer. Regarding matching tests: count both sides to be matched. If there are more questions than answers, ask if you can use an answer more than once. Pay close attention to items that ask you to choose the "best" answer. This means one answer is better or more inclusive than a similar answer. Read all of the response options.

Post-Test Strategies

In addition to sighing that big sigh of relief, here are a few suggestions to help with future tests.

- If you don't understand why you did not get an item right, ask the instructor. This is especially useful for quizzes that contain information that may be incorporated into more inclusive exams such as mid-terms and finals.
- Analyze your results to help you in the future. For example, see if most of your incorrect answers were small things such as failing to include the last step in a math item, or neglecting to double-check for simple errors in a short-answer or essay item. See where in the test you made the most errors: beginning, middle, or end. Also analyze which type of questions, true/false, multiple choice, essay, etc. And which topics were missed. This will help you pay closer attention to those sections in the future.

EXERCISE 30-2

Write a letter of advice to Chen incorporating 10 test-taking tips and strategies you think will help him.

Chen believes he is good at organization, and he usually is–for about the first two weeks of classes. He then becomes overwhelmed with all of the handouts and materials and tends to start slipping in the organization department. When it comes to tests, he worries that his notes might not cover all of the right topics and that he will not be able to remember all of the key terms and points–especially for his math class. During tests, he sometimes gets stuck on an item and tends to spend too much time there. He also sometimes changes answers but finds out later that his original selection was correct. Chen is also easily distracted by other students and noises which makes it hard for him to concentrate sometimes, and, unfortunately, he does admit to occasionally "cramming" the night before.

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Adaptions: Changed formatting, removed one exercise.

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1.12: Test-Taking Strategy Specifics

Dave Dillon

"By failing to prepare, you are preparing to fail."

– Benjamin Franklin

Discipline, Preparation and Execution

From what I have seen in my own educational experiences, along with coaching and teaching at the college and university level for over 20 years, test taking (with few exceptions) comes down to discipline, preparation and execution. Students wanting to be successful have to have the self-discipline to schedule time to study well in advance of the exam. They have to actually do the work: the preparation needed in order to have the best opportunity for success on the exam. Then they must execute: they have to be able to apply their preparation accordingly and perform well on the exam.

Preparation for an exam is not glamorous. It's easy to find other things to do that are more interesting and fun. Students need to keep themselves motivated with their "eyes on the prize." Think of it like this: if the most important event of your life was coming up and you wanted to perform to the best of your ability in that event, you would likely spend some time preparing for it, rehearsing for it, practicing it, etc. A student may argue that an exam they will be taking would not be the most important event of their life, but think back to the chapter on Passion. If you're already spending the time, effort, energy and money to attend college, why not do it to the best of your ability?

It would be beneficial to spread this preparation and practice out over time and prepare periodically rather than to wait until the last minute and binge study or cram. Your preparation would not be the same and this may affect your test score. Binge studying and cramming also are not healthy. Staying up late puts stress on our brain and body, and not getting adequate sleep places our bodies at risk for getting sick.

"One of the most important keys to success is having the discipline to do what you know you should do, even when you don't feel like doing it."

– Unknown

"The will to succeed is important, but what's more important is the will to prepare."

– Bobby Knight

Everyone wants to be successful. When the exam is passed out, everyone wants to perform well. But what often separates successful students and less successful students is the preparation time put in.

Studying the right thing is a process and a skill. As you gain more experience, you will learn how to become better at knowing what to study. It can be very frustrating to spend a lot of time preparing and studying and then finding out that what you studied was not on the exam. You will see a lot of variance with exams due to different instructors, classes and types of tests. The better you become at predicting what will be on the exam and study accordingly, the better you will perform on your exams. Try placing yourself in your instructor's shoes and design questions you think your instructor would ask. It's often an eye-opening experience for students and a great study strategy.

Preparation for Exam Strategies

Find out as much about the exam in advance as you can.

Some professors will tell you how many questions there will be, what format the exam will be in, how much time you will have, etc., and others will not. I encourage students to ask questions about the exam if there is not information given. I also encourage students to ask those questions before class, after class, in professors' office hours or via e-mail rather than during class.

Know the test

If you know how many questions, what the format is, and/or how much time you will have, you can start to mentally prepare for the exam much more so than if you are coming in with no information. There are two more important aspects that you may or may not know: a) what will be covered or asked on the exam; b) how the exam will be scored. Obviously, the more you know about what will be covered, the easier it is for you to be able to prepare for the exam. Most exam scoring is standardized, but not always.

Look for opportunities where some areas of the exam are worth more points than others. For example: An exam consists of 21 questions, with 10 being True/False, 10 being multiple choice, and one essay question. The T/F questions are worth 1 point each (10 points), the multiple-choice questions are worth 2 points each (20 points), and the essay question is worth 30 points. We know that the essay question is the most valuable (it is worth half of the value of the exam). And we should allocate our time for it accordingly. I would advise starting with the essay question. Do a quick analysis of time to be able to spend your time on the exam wisely. You want to spend some time with the exam question since it is so valuable, without sacrificing adequate time to ensure the T/F and multiple-choice questions are answered.

Often, the order of the exam in this scenario will be: T/F first, multiple choice second and essay third. Most students will go in the chronological order of the exam, but a savvy student would start with the essay. If an exam were to last for 30 minutes with this format of questions, I might recommend a student spent 15 minutes on the essay question, ten minutes on the multiple choice, three minutes on the T/F and two minutes reviewing their answer.

Also, look for situations where exams penalize students for incorrectly answering a question. This does not occur very often, but is the case with some exams. With the SAT for example, students are awarded one point for a correct answer and ¼ of a point subtracted for an incorrect answer. Points are not awarded nor subtracted for leaving a question blank. Thus, the strategy for a multiple-choice question is: if you can narrow down the potentially correct answer to two rather than four or five, it is statistically advantageous to answer the question and guess between the two answers; however, if a student had no idea if any of the answers were correct or incorrect, it would be best to leave the answer blank. Remember, this is rare, but it is important to understand the strategy when students take these exams.

In conclusion, the more information you have about the exam, the better you can prepare for content, allocation of time spent on aspects of the exam, and the more confident you will be in knowing how and when to attempt to answer questions.

Take care of your body

Before the exam, it is important to prepare your brain and body for optimal performance for your exam. Do not cram the night before. Get a good night's sleep. Make sure you eat (nutritiously) before the exam. I recommend exercising the day before and if possible a few hours before the exam.

Strategies for Specific Exam Formats

True or False Questions

Look for qualifiers. A qualifier is a word that is absolute. Examples are: all, never, no, always, none, every, only, entirely. They are often seen in false statements. This is because it is more difficult to create a true statement using a qualifier like never, no, always, etc. For example, "All cats chase mice." Cats may be known for chasing mice, but not all of them do so. The answer here is false and the qualifier "all" gave us a tip. Qualifiers such as: sometimes, many, some, most, often, and usually are commonly found in true statements. For example: "Most cats chase mice." This is true and the qualifier "most" gave us a tip.

Make sure to read the entire statement. All parts of a sentence must be true if the whole statement is to be true. If one part of it is false, the whole sentence is false. Long sentences are often false for this reason.

Students should guess on True or False questions they do not know the answer to unless there is a penalty for an incorrect answer.

Multiple Choice Questions

Think of multiple choice questions as four (or five) true or false statements in one. One of the statements is true (the correct answer) and the others will be false. Apply the same strategy toward qualifiers. If you see an absolute qualifier in one of the answer choices, it is probably false and not the correct answer. Try to identify the true statement. If you can do this, you have the answer as there is only one. If you cannot do this at first, try eliminating answers you know to be false.

If there is no penalty for incorrect answers, my suggestion is to guess if you are not certain of the answer. If there is a penalty for incorrect answers, common logic is to guess if you can eliminate two of the answers as incorrect (pending what the penalty is). If there's a penalty and you cannot narrow down the answers, it's best to leave it blank. You may wish to ask your instructor for clarification.

Answers that are strange and unrelated to the question are usually false. If two answers have a word that looks or sounds similar, one of those is usually correct. For example: abductor/ adductor. If you see these as two of the four or five choices, one of them is usually correct. Also look for answers that are grammatically incorrect. These are usually incorrect answers. If you have to completely guess, choose B or C. It is statistically proven to be correct more than 25 percent of the time. If there are four answers

for each question, and an exam had standardized the answers, each answer on the exam A, B, C and D would be equal. But most instructors do not standardize their answers, and more correct answers are found in the middle (B and C then the extremes A and D or E). “People writing isolated four-choice questions hide the correct answer in the two middle positions about 70% of the time.”[1] This is 20 percent more correct answers found in B or C than a standardized exam with equal correct answers for each letter.

Matching Questions

Although less common than the other types of exams, you will likely see some matching exams during your time in college. First, read the instructions and take a look at both lists to determine what the items are and their relationship. It is especially important to determine if both lists have the same number of items and if all items are to be used, and used only once.

Matching exams become much more difficult if one list has more items than the other or if items either might not be used or could be used more than once. If your exam instructions do not discern this, you may wish to ask your instructor for further clarification. I advise students to take a look at the whole list before selecting an answer because a more correct answer may be found further into the list. Mark items when you are sure you have a match (pending the number of items in the list this may eliminate answers for the future). Guessing (if needed) should take place once you have selected answers you are certain about.

Short-Answer Questions

Read all of the instructions first. Budget your time and then read all of the questions. Answer the ones you know best or feel the most confident with. Then go back to the other ones. If you do not know the answer and there is no penalty for incorrect answers, guess. Use common sense. Sometimes instructors will award partial credit for a logical answer that is related even if it is not the correct answer.

Essay Questions

Keep in mind that knowing the format of the exam can help you determine how to study. If I know that I am taking a True-False exam, I know that I will need to discern whether a statement is True or False. I will need to know subject content for the course. But if I am studying for short answer and especially for essay questions, I must know a lot more. For essay questions, I must have much greater content knowledge and be able to make a coherent argument that answers the question using information from textbooks, lectures or other course materials. I have to place a lot more time and thought into studying for an essay exam than for True-False or Multiple-Choice exams.

Read the essay question(s) and the instructions first. Plan your time wisely and organize your answer before you start to write. Address the answer to the question in your first or second sentence. It may help to restate the original question. Write clearly and legibly. Instructors have difficulty grading essays that they cannot read. Save some time for review when you have finished writing to check spelling, grammar and coherent thought in your answer. Make sure you have addressed all parts of the essay question.

During the Exam:

Always read the directions first. Read them thoroughly.

Preview the exam to help you allocate proper time for each area.

Skip questions if you do not know the answer but make a mark somewhere to ensure you are able to go back to those questions (you may need to reallocate your prepared time for this depending on how many there are).

Allocate some time to review your answers before submitting your exam or the exam time expiring.

One of the biggest mistakes that students make after they take an exam in a course is that they do not use the exam for the future. The exam contains a lot of information that can be helpful in studying for future exams. Students that perform well on an exam often put it away thinking they do not need it anymore. Students who do poorly on an exam often put it away, not wanting to think about it any further.

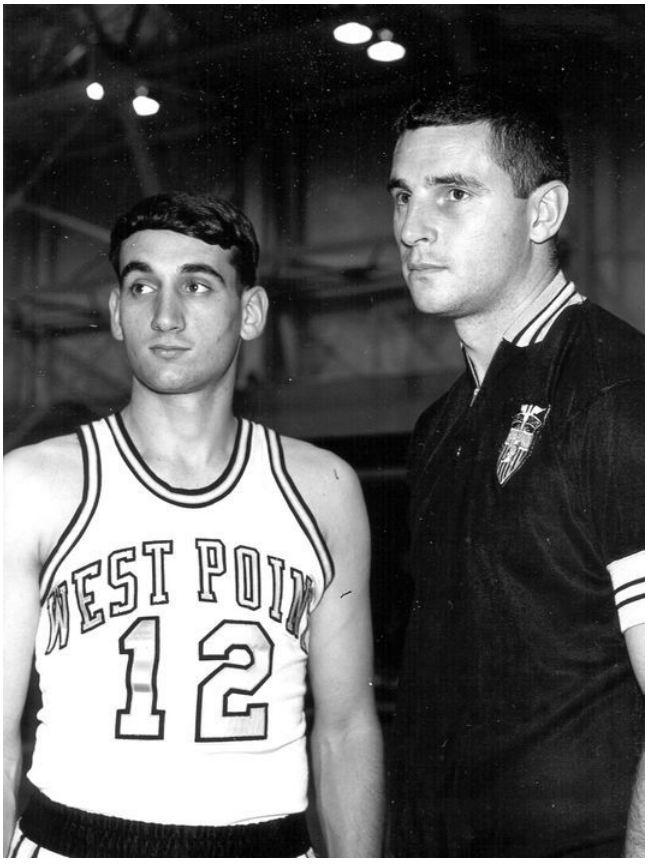
In both cases, students are missing out on the value of reviewing their exams. It is wise to review exams for three reasons: 1) students should review the answers that were correct because they may see those questions on future exams and it is important to reinforce learning; 2) students should review the answers that were incorrect in order to learn what the correct answer was and why. These questions also may appear on a future exam. In addition, occasionally an answer is marked incorrect, when it should have been marked correct. The student would never know this if they didn’t review their exam; and 3) there is value in reviewing the exam to try to predict what questions or what format will be used by a professor for a future exam in the same course.

Author's Story

What does it mean to be a "poor test taker?" Think about that. Does it mean that the student has put effort into studying but has difficulty under pressure? Does it mean that a student studies the wrong material? Is the student prepared but does not execute well? Could the student have a learning disability? Are they missing key strategies for taking tests? It could mean any of those things. And while I believe that it may be true that a student may be a "poor test taker," it does not by any means mean it is permanent. Students willing to work hard and learn can improve their test taking skills and raise their confidence.

In high school chemistry, I earned a D grade. In microeconomics at UC Santa Cruz, I struggled and barely passed. Both of these classes had grading systems that were heavily weighted by exams. I was an above average student in my other classes in both high school and college, and so I was told (and believed for a long time) that I was not a "good test taker." I no longer believe this. I will not make excuses. The responsibility for the grade I earned rests solely with me. I mention this because many students come to my office and complain that their professor grades in a way that is unfair to them because so much emphasis is on exams and that they understand the material but are "not good test takers." It may be the case that my grade or another students' grade would have been better had the course grade been determined with less weight levied to exams. That to me is irrelevant. It does not absolve me from being responsible for knowing what the grading metrics were at the beginning of the course and choosing to continue in the course after learning this knowledge. In the end, I realized I was not a poor test taker and that I needed to spend more time studying and preparing to perform well on exams. I changed my attitude from being afraid of exams to one of believing I was going to do well regardless of the class or the format of the exam. Once I had confidence that I had the necessary tools and was willing to work hard, it changed my entire perspective and experience with exams.

Note: if you think you may have a learning disability and would like to get assessed, contact your college to see what steps to follow to begin the assessment process.



Mike Krzyzewski, point guard at Army with Bob Knight, coach at Army.

Thinking about excuses reminds me of a story about Mike Krzyzewski, the successful Duke University men's basketball coach. Before Krzyzewski coached, he was a student at the United States Military Academy at West Point. One day in his first year, Krzyzweski and his roommate were walking and his roommate stepped in a puddle. Mud splashed onto Krzyzewski's uniform. Immediately, an upperclassman screamed at Krzyzewski about not knowing the rules of wearing a clean uniform. Plebes (first-year

students at military academies) are allowed three answers when asked a question: “Yes sir!” “No sir!” And “No excuse sir!” Krzyzewski repeated himself, “No excuse sir!” Despite wanting to explain what had happened and that it was not his fault, he realized that there was no excuse. It was his responsibility to keep a clean uniform.

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1. Yigal Attali and Maya Bar-Hillel, “Guess Where: The Position of Correct Answers in Multiple-Choice Test Items as a Psychometric Variable,” *Journal of Educational Measurement* 40 no. 2 (2003):109-128. ↵

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CHAPTER OVERVIEW

2: Inequalities and Functions

2.1: Prelude to Inequalities and Functions
2.2: Solving Nonlinear Inequalities
2.3: Functions and Function Notation
2.4: Graphing Functions and Common Function Behavior
2.5: Finding Information About Functions From Graphs
2.6: Average Rate of Change
2.7: Transformations of Functions
2.8: Algebra and Composition of Functions
2.9: Inverse Functions
2.E: Functions (Exercises)
2.R: Functions (Review)

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2.1: Prelude to Inequalities and Functions

Toward the end of the twentieth century, the values of stocks of internet and technology companies rose dramatically. As a result, the Standard and Poor's stock market average rose as well. The *Standard and Poor's Index* tracks the value of that initial investment of just under $100 over the 40 years. It shows that an investment that was worth less than $500 until about 1995 skyrocketed up to about $1100 by the beginning of 2000. That five-year period became known as the "dot-com bubble" because so many internet startups were formed. As bubbles tend to do, though, the dot-com bubble eventually burst. Many companies grew too fast and then suddenly went out of business. The result caused the sharp decline represented on the graph beginning at the end of 2000.

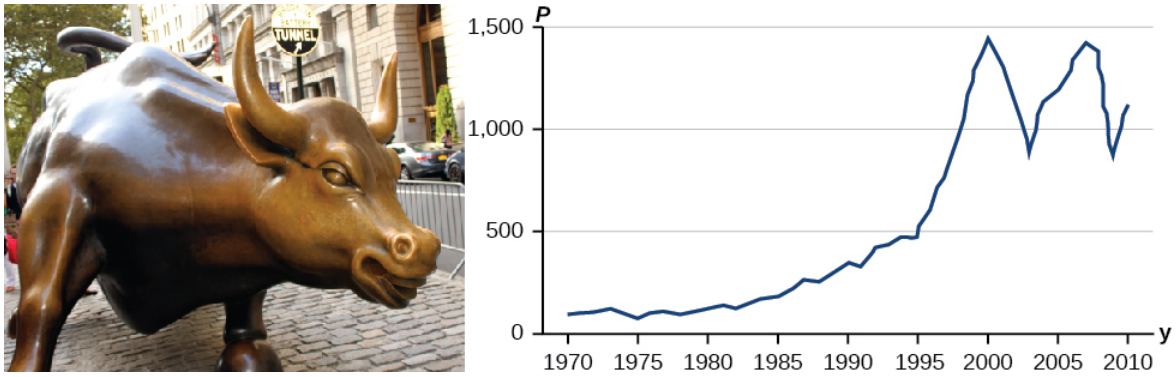


Figure 2.1.1: Figure of a Bull and a graph of market prices. Standard and Poor's Index with dividends reinvested (credit "bull": modification of work by Prayitno Hadinata; credit "graph": modification of work by MeasuringWorth)

Notice, as we consider this example, that there is a definite relationship between the year and stock market average. For any year we choose, we can determine the corresponding value of the stock market average. In this chapter, we will explore these kinds of relationships and their properties.

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2.2: Solving Nonlinear Inequalities

Learning Objectives

- Determine if a number is a solution to a nonlinear inequality with one variable.
- Understand the geometric relationship between solutions to nonlinear inequalities and their graphs and use this relationship to solve nonlinear inequalities.
- Solve non-linear inequalities algebraically using sign charts.

Precalculus is a course that focuses on the concept of functions. Functions are the basis of programming languages that are used in the programs that run the technology we interact with in our daily lives. We use functions to help describe the world around us. They help us describe how we expect things to behave and they help us to make predictions. As we start our journey into studying of functions, we will be begin with nonlinear inequalities.

Inequalities are very useful as they give us a way to define constraints in real-life situations. Consider a container that is being designed to collect rain water for future drinking use. The container is designed to hold a positive quantity of water which would be described by the volume the container can hold. It would not make sense for the volume to be less than zero since the minimum amount of water the container can hold is no water. Since the volume cannot be negative, we can describe this with an inequality, $\text{volume} \geq 0$. The size and shape of the container would describe the exact inequality for this situation.



Figure 2.2.2: Photo of a Large Water Container. (This Photo by Unknown Author is licensed under CC BY-SA)

Nonlinear inequalities contain expressions that are nonlinear. Based on things you have had experience with in algebra courses, variables raised to powers and variables in denominators within an inequality would indicate that an inequality is nonlinear.

Some examples of nonlinear inequalities:

$$m^2 - 8m > 9\,, \qquad \frac{x-4}{(3x-2)(x+5)} \leq 0$$

A solution to an inequality is any real number that will produce a true statement when substituted in for the variable.

Example 2.2.1

Are -3, -2, and -1 solutions to $x^2 - x - 6 \leq 0$?

Solution

Substitute the given value in for x and simplify.

$x^2 - x - 6 \leq 0$	$x^2 - x - 6 \leq 0$	$x^2 - x - 6 \leq 0$
$(-3)^2 - (-3) - 6 \leq 0$	$(-2)^2 - (-2) - 6 \leq 0$	$(-1)^2 - (-1) - 6 \leq 0$
$9 + 3 - 6 \leq 0$	$4 + 2 - 6 \leq 0$	$1 + 1 - 6 \leq 0$
$6 \leq 0$ ✗	$0 \leq 0$ ✓	$-4 \leq 0$ ✓

Answer:

-2 and -1 are solutions and -3 is not.

Solving Nonlinear Inequalities Using Graphs

Graphing the function defined by $y = x^2 - x - 6$ found in the previous example we have,

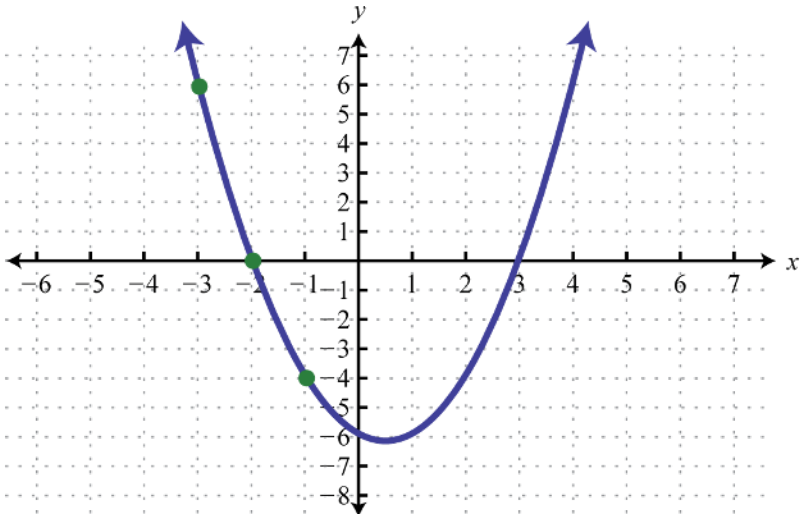


Figure 2.2.3: Graph of y=x^2-x+6

Notice that the x-intercepts are at $(-2, 0)$ and $(3, 0)$. Remember that to find the x-intercepts of a function, set $y = 0$. Finding them algebraically,

$$\begin{aligned} 0 &= x^2 - x - 6 \\ 0 &= (x+2)(x-3) \\ x+2 &= 0 \quad \text{or} \quad x-3 = 0 \\ x &= -2 \quad \text{or} \quad x = 3 \end{aligned}$$

The values -2 and 3 are called **critical numbers**. These values bound the regions where the function is positive (above the x-axis) or negative (below the x-axis).

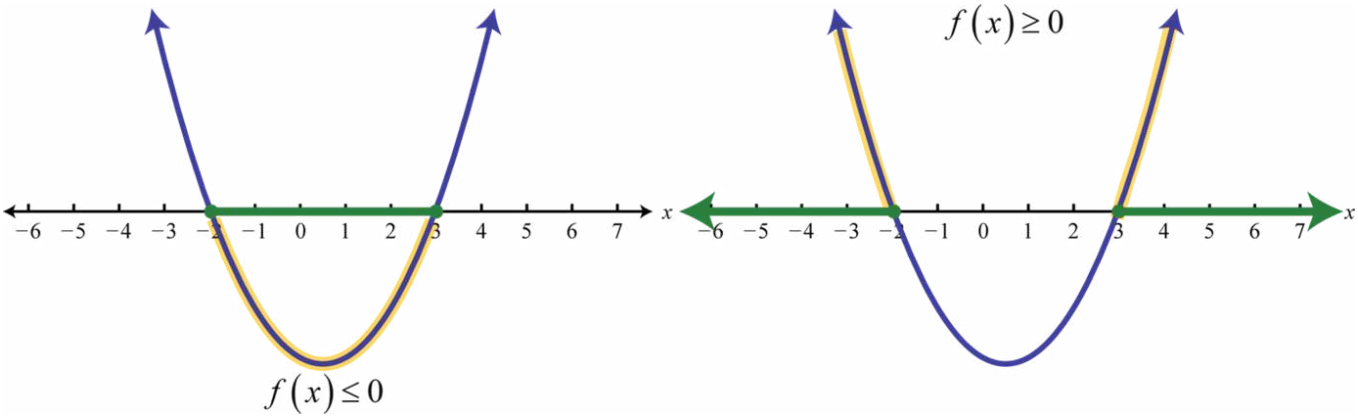


Figure 2.2.4: Graphs showing intervals where y=x^2-x+6 is positive and negative

We can see that the function is below the x-axis between -2 and 3. Therefore $x^2 - x - 6 \leq 0$ has solutions where $-2 \leq x \leq 3$, using interval notation $[-2, 3]$. Furthermore, $x^2 - x - 6 \geq 0$ has solutions where $x \leq -2$ or $x \geq 3$, using interval notation $(-\infty, -2] \cup [-3, \infty)$.

Example 2.2.2

Given the graph of $f(x)$ determine the solutions to $f(x) > 0$:

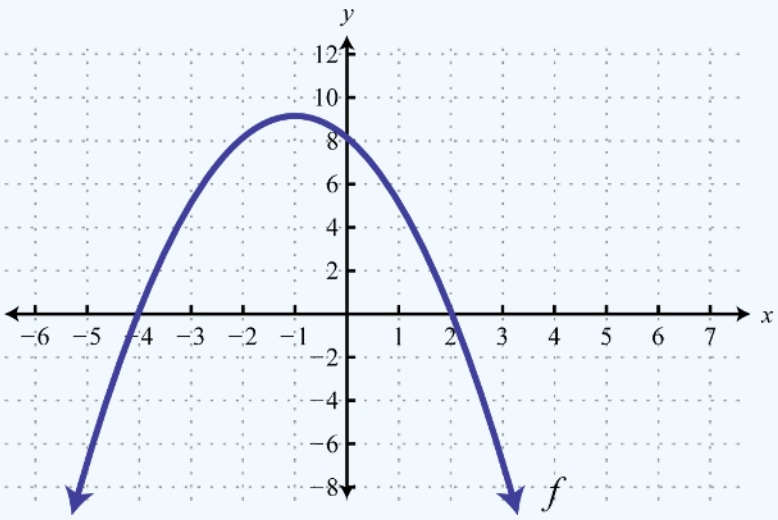


Figure 2.2.5: Graph of function f(x)

Because of the strict inequality, the solution set is shaded with an open dot on each of the boundaries. This indicates that these critical numbers are not actually included in the solution set. This solution set can be expressed two ways,

$$\{x | -4 < x < 2\} \qquad \textit{Set Notation}$$
$$(-4,2) \qquad \textit{Interval Notation}$$

In this textbook, we will continue to present answers in interval notation.

Answer:

$(-4, 2)$

You Try 2.2.1

Use the graph of f to determine the solutions to $f(x) \leq 0$. State answer in interval notation.

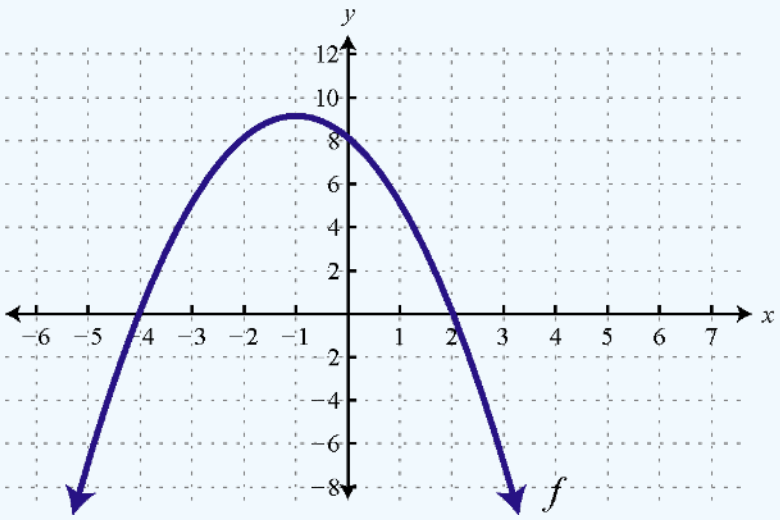


Figure 2.2.6: Graph of function f(x)

Answer

$(-\infty, -4] \cup [2, \infty)$

Solving Polynomial Inequalities Using Sign Charts

A **polynomial inequality** is a mathematical statement that relates a polynomial expression as either less than or greater than another.

Some examples of polynomial inequalities:

$$(3x+1)(x-2) < 0, \qquad t^3 + 30 \geq 7t^2 + 7t$$

We have seen how to solve an inequality graphically. To solve inequalities without graphing, we make use of a **sign chart** which models a function using a number line that represents the x-axis and signs $(+ \text{ or } -)$ to indicate where the function is positive or negative. For example,

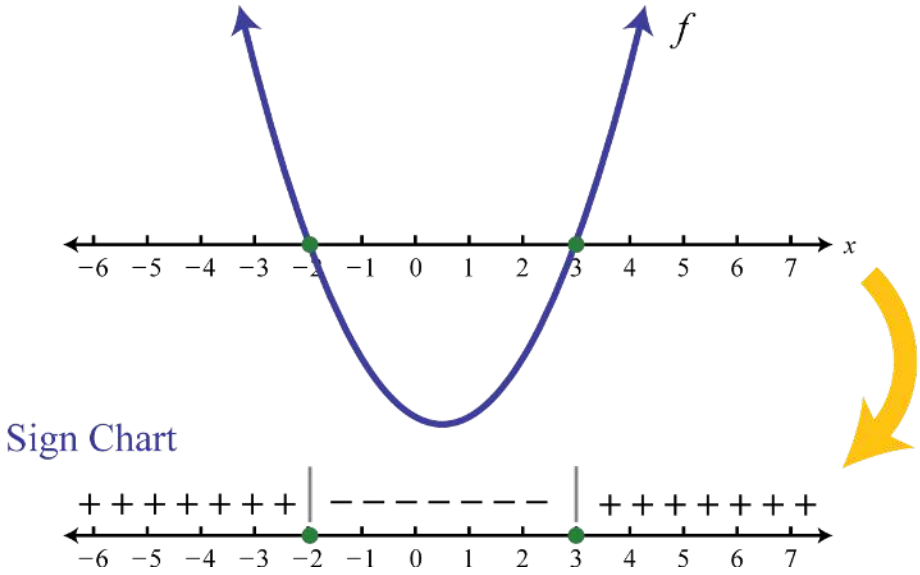


Figure 2.2.7

The plus signs indicate that the function is positive on the region. The negative signs indicate that the function is negative on the region. The boundaries are the critical numbers, -2 and 3 in this case. Sign charts are useful when a detailed picture of the graph is

not needed and are used extensively in higher level mathematics. The steps for solving a polynomial inequality with one variable are outlined in the following example.

Example 2.2.3

Solve: $-x^2+6x+7\geq 0$.

Solution

It is important to note that this quadratic polynomial inequality is in standard form, with zero on one side of the inequality. Zero must be on one side of the inequality to be able to find the critical numbers in the next step.

Step 1: Determine the critical numbers. To find them, set the function equal to zero and solve.

$$\begin{aligned} -x^2+6x+7&=0\\ -\left(x^2-6x-7\right)&=0\\ -(x+1)(x-7)&=0 \end{aligned}$$

$$\begin{aligned} x+1=0 \quad &\text{or} \quad x-7=0\\ x=-1 \quad &\qquad x=7 \end{aligned}$$

The critical numbers are -1 and 7.

Step 2: Create a sign chart by putting the critical numbers on a number line. Since the critical numbers bound the regions where the function is positive or negative, we need to test a single value in each region. In this case, the critical numbers partition the number line into three regions. We choose test values $x=-3, x=0$, and $x=10$.

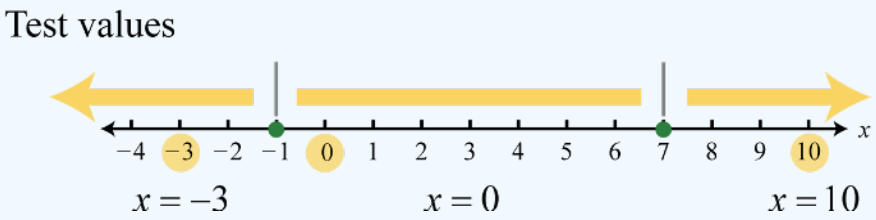


Figure 2.2.8

Test values may vary. In fact, we need only determine the sign $(+$ or $-)$ of the result when evaluating $f(x)=-x^2+6x+7=-(x+1)(x-7)$. Here we evaluate using the factored form.

$$\begin{aligned} f(-3)&=-(-3+1)(-3-7)=-(-2)(-10) &&=-Negative\\ f(0)&=-(0+1)(0-7)=-(1)(-7) &&=+Positive\\ f(10)&=-(10+1)(10-7)=-(11)(3) &&=-Negative \end{aligned}$$

Since the result of evaluating for -3 was negative, we place negative signs above the first region. The result of evaluating for 0 was positive, so we place positive signs above the middle region. Finally, the result of evaluating for 10 was negative, so we place negative signs above the last region, and the sign chart is complete.

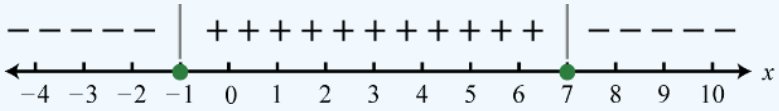


Figure 2.2.9

Step 3: Use the sign chart to answer the question. In this case, we are asked to determine where $f(x)\geq 0$, or where the function is positive or zero. From the sign chart we see this occurs when x-values are inclusively between -1 and 7.

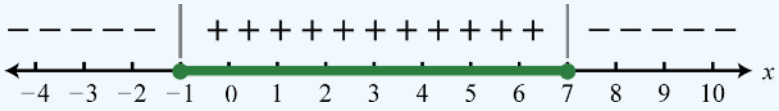


Figure 2.2.9

Using interval notation, the answer is expressed as $[-1,7]$.

The graph is not required; however, for the sake of completeness it is provided below.

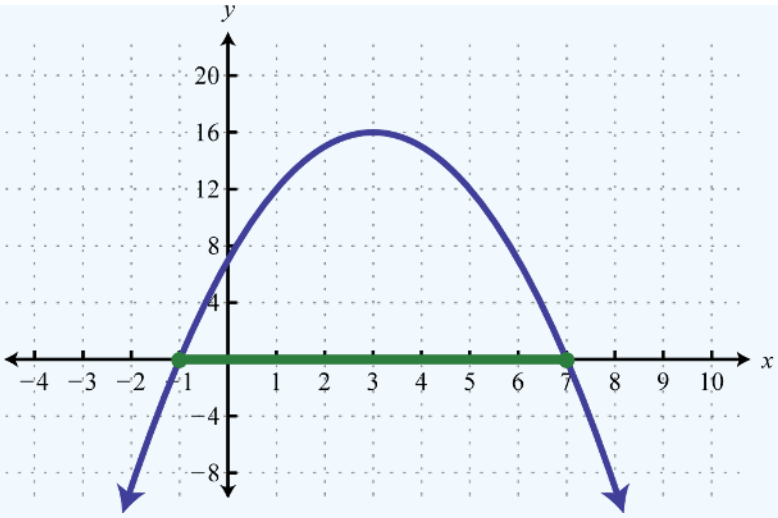


Figure 2.2.10

Indeed the function is greater than or equal to zero, above or on the x-axis, for x-values in the specified interval.

Answer:

$[-1, 7]$

Before looking at another example it is important to note that there are multiple ways to set up sign charts. Some like to set up sign charts and evaluate each factor individually. Below is an example of a sign chart set up this way:

	(- ∞, -1)	(-1,7)	(7, ∞)
-1	-	-	-
(x+1)	-	+	+
(x-7)	-	-	+
-(x+1)(x-7)	-	+	-

Figure 2.2.11: Sign chart for -(x+1)(x-7)

Since -1 is a constant factor that is negative, we fill in negative signs everywhere for this factor. Notice that the factor x+1 changes sign, from negative to positive, at the critical number -1. This factor gave this critical value. Similarly, this is also true for the factor x-7 and critical value of 7.

Example 2.2.4

Solve: $2x^2 - 7x + 3 > 0$.

Solution

Begin by finding the critical numbers, in this case, the roots of $f(x) = 2x^2 - 7x + 3$.

$$\begin{aligned} 2x^2 - 7x + 3 &= 0 \\ (2x-1)(x-3) &= 0 \end{aligned}$$

$$\begin{aligned} 2x - 1 = 0 \quad &\text{or} \quad x - 3 = 0 \\ 2x = 1 \quad &\qquad\quad x = 3 \\ x = \frac{1}{2} \quad &\end{aligned}$$

The critical numbers are $\frac{1}{2}$ and 3. Because of the strict inequality > we will use open dots.

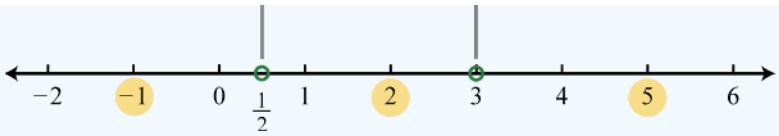


Figure 2.2.12

Next choose a test value in each region and determine the sign after evaluating $f(x)=2x^2-7x+3=(2x-1)(x-3)$. Here we choose test values $-1, 2$, and 5.

$$\begin{aligned} f(-1)&=(2(-1)-1)(-1-3) &&=(-)(-)=+\\ f(2)&=(2(2)-1)(2-3) &&=(+)(-)=-\\ f(5)&=(2(5)-1)(5-3) &&=(+)(+)=+ \end{aligned}$$

And we can complete the sign chart.

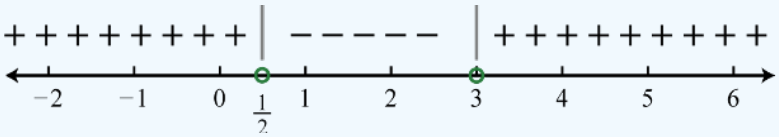


Figure 2.2.13

The question asks us to find the x-values that produce positive results (greater than zero). Therefore, shade in the regions with a $+$ over them. This is the solution set.

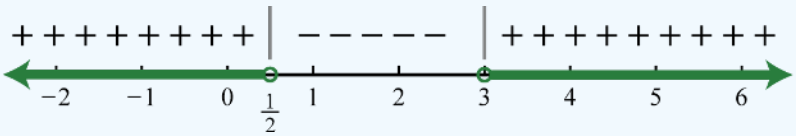


Figure 2.2.14

Answer:

$\left(-\infty, \frac{1}{2}\right) \cup (3, \infty)$

Sometimes the quadratic function does not factor. In this case we can make use of the quadratic formula.

Example 2.2.5

Solve: $x^2-2x-11 \leq 0$

Solution

Find the critical numbers.

$$x^2-2x-11=0$$

Identify a, b, and c for use in the quadratic formula. Here $a=1, b=-2$, and $c=-11$. Substitute the appropriate values into the quadratic formula and then simplify.

$$\begin{aligned} x&=\frac{-b \pm \sqrt{b^2-4ac}}{2a}\\ &=\frac{-(-2) \pm \sqrt{(-2)^2-4(1)(-11)}}{2(1)}\\ &=\frac{2 \pm \sqrt{48}}{2}\\ &=\frac{2 \pm 4\sqrt{3}}{2}\\ &=1 \pm 2\sqrt{3} \end{aligned}$$

Therefore the critical numbers are $1-2\sqrt{3} \approx -2.5$ and $1+2\sqrt{3} \approx 4.5$. Use a closed dot on the number to indicate that these values will be included in the solution set.

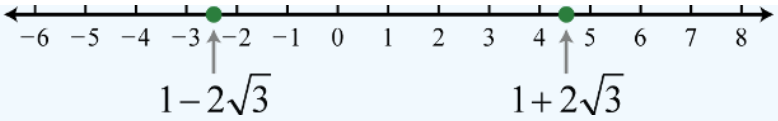


Figure 2.2.15

Here we will use test values $-5, 0$, and 7.

$$\begin{aligned} f(-5) &= (-5)^2 - 2(-5) - 11 = 25 + 10 - 11 \quad &= + \\ f(0) &= (0)^2 - 2(0) - 11 = 0 + 0 - 11 \quad &= - \\ f(7) &= (7)^2 - 2(7) - 11 = 49 - 14 - 11 \quad &= + \end{aligned}$$

After completing the sign chart shade in the values where the function is negative as indicated by the question $(f(x) \leq 0)$.

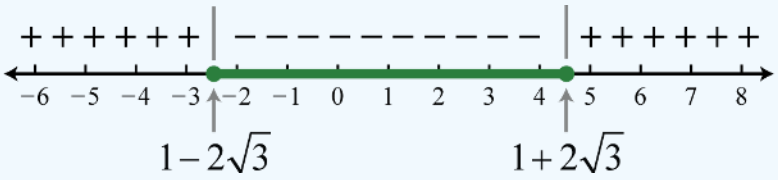


Figure 2.2.16

Answer:

$[1-2\sqrt{3}, 1+2\sqrt{3}]$

You Try 2.2.2

Solve: $9 - x^2 > 0$.

Answer

$(-3, 3)$

It may be the case that there are no critical numbers.

Example 2.2.6

Solve: $x^2 - 2x + 3 > 0$.

Solution

To find the critical numbers solve,

Substitute $a = 1, b = -2$, and $c = 3$ into the quadratic formula and then simplify.

Because the solutions are not real, we conclude there are no real roots; hence there are no critical numbers. When this is the case, the graph has no x-intercepts and is completely above or below the x-axis. We can test any value to create a sign chart. Here we choose $x = 0$.

$$f(0) = (0)^2 - 2(0) + 3 = +$$

Because the test value produced a positive result the sign chart looks as follows:

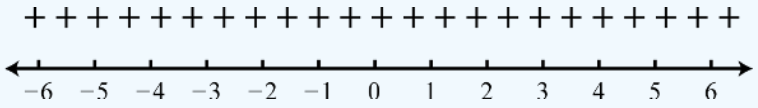


Figure 2.2.17

We are looking for the values where $f(x) > 0$; the sign chart implies that any real number for x will satisfy this condition.

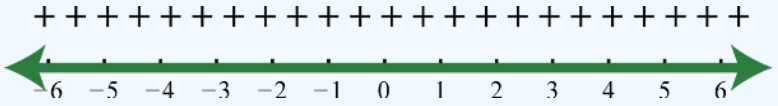


Figure 2.2.18

Answer:

$(-\infty, \infty)$

The function in the previous example is graphed below.

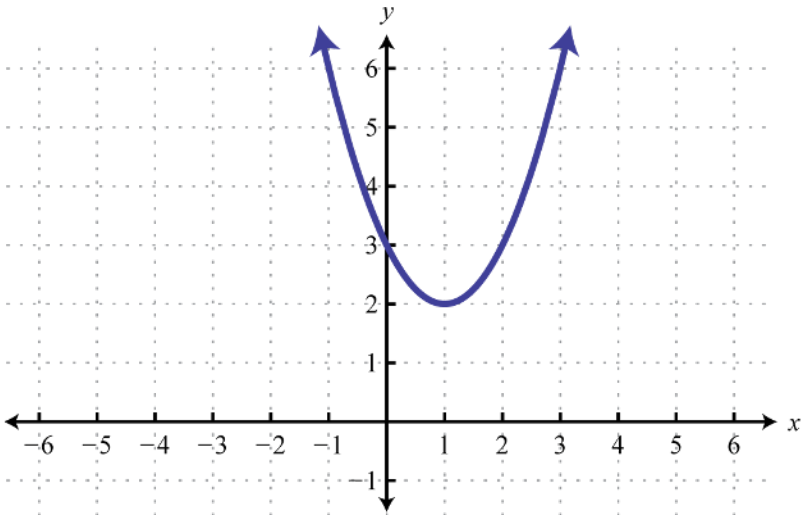


Figure 2.2.19

We can see that it has no x-intercepts and is always above the x-axis (positive). If the question was to solve $x^2 - 2x + 3 < 0$, then the answer would have been no solution. The function is never negative.

? You Try 2.2.3

Solve: $9x^2 - 12x + 4 \leq 0$

Answer

One solution, $\frac{2}{3}$.

✓ Example 2.2.7

Solve $x(x+3)^2(x-4) < 0$.

Solution

Begin by finding the critical numbers. Because $f(x) = x(x+3)^2(x-4)$ is given in its factored form and zero is on one side of the inequality, the critcal numbers are apparent. Here the critical values are: $0, -3$, and 4. Because of the strict inequality, plot them using open dots on a number line.

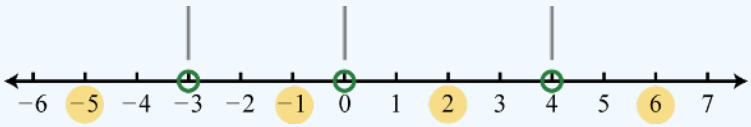


Figure 2.2.20

In this case, the critical numbers partition the number line into four regions. Test values in each region to determine if f is positive or negative. Here we choose test values $-5, -1, 2$, and 6. Remember that we are only concerned with the sign $(+$ or $-)$ of the result.

$$\begin{aligned} f(-5) &= (-5)(-5+3)^2(-5-4) = (-)(-)^2(-) && = +Positive \\ f(-1) &= (-1)(-1+3)^2(-1-4) = (-)(+)^2(-) && = +Positive \\ f(2) &= (2)(2+3)^2(2-4) = (+)(+)^2(-) && = -Negative \\ f(6) &= (6)(6+3)^2(6-4) = (+)(+)^2(+) && = +Positive \end{aligned}$$

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After testing values we can complete a sign chart.

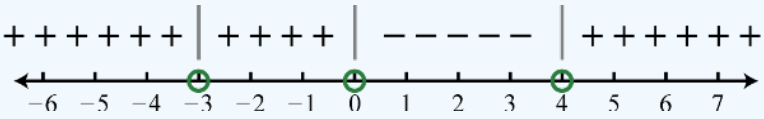


Figure 2.2.21

The question asks us to find the values where $f(x) < 0$, or where the function is negative. From the sign chart we can see that the function is negative for x-values in between 0 and 4.

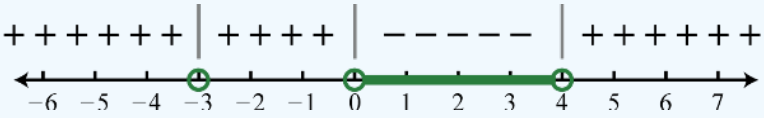


Figure 2.2.22

Answer:

$(0, 4)$

Although we have not discussed the graphs of polynomial functions yet, the graph of this function is provided below. Compare the graph to its corresponding sign chart.

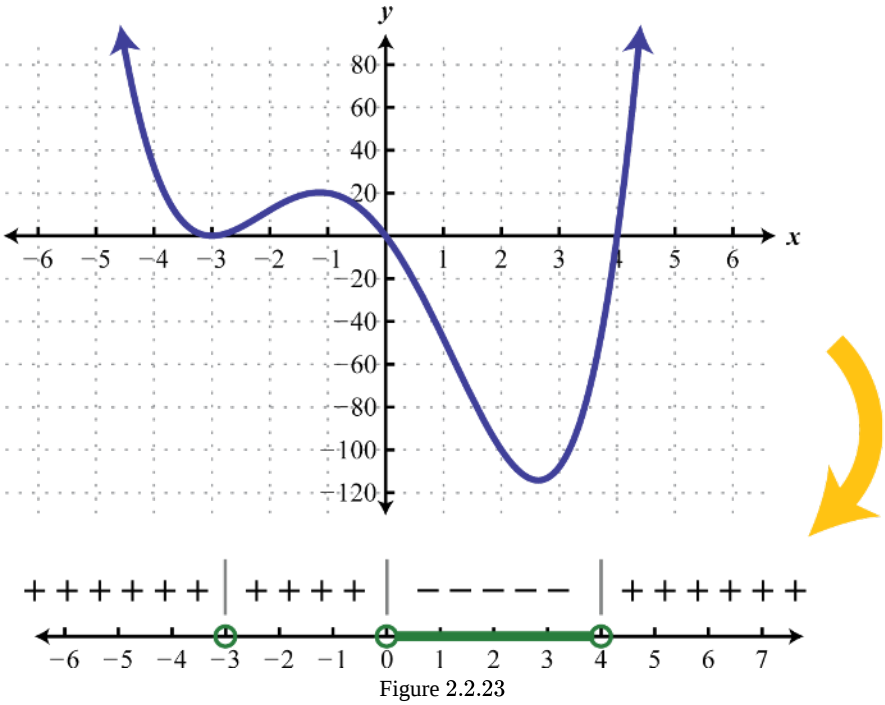


Figure 2.2.23

Certainly it may not be the case that the polynomial is factored nor that it has zero on one side of the inequality. To model a function using a sign chart, all of the terms should be on one side and zero on the other. The general steps for solving a polynomial inequality are listed in the following example.

Example 2.2.8

Solve: $2x^4 > 3x^3 + 9x^2$.

Solution

Step 1: Obtain zero on one side of the inequality. In this case, subtract to obtain a polynomial on the left side in standard from.

$$2x^4 > 3x^3 + 9x^2$$
$$2x^4 - 3x^3 - 9x^2 > 0$$

Step 2: Find the critical numbers. Here we can find the zeros by factoring.

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$$2x^4 - 3x^3 - 9x^2 = 0$$
$$x^2\left(2x^2 - 3x - 9\right) = 0$$
$$x^2(2x+3)(x-3) = 0$$

There are three solutions, hence, three critical numbers $-\frac{3}{2}, 0$, and 3. The strict inequality indicates that we should use open dots.

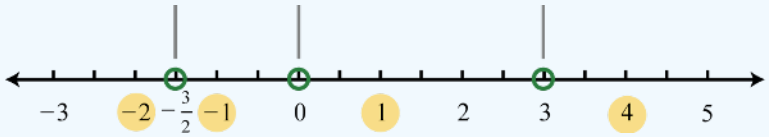


Figure 2.2.24

Step 3: Create a sign chart. In this case use $f(x) = x^2(2x+3)(x-3)$ and test values $-2, -1, 1$, and 4 to determine the sign of the function in each interval.

$$\begin{aligned} f(-2) &= (-2)^2(2(-2)+3)(-2-3) &&= (-)^2(-)(-) = + \\ f(-1) &= (-1)^2(2(-1)+3)(-1-3) &&= (-)^2(+)(-) = - \\ f(1) &= (1)^2(2(1)+3)(1-3) &&= (+)^2(+)(-) = - \\ f(4) &= (4)^2(2(4)+3)(4-3) &&= (+)^2(+)(+) = + \end{aligned}$$

With this information we can complete the sign chart.

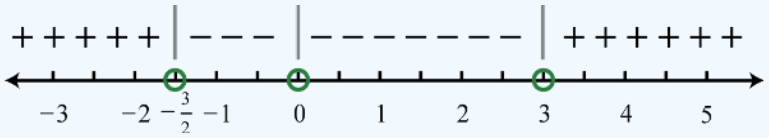


Figure 2.2.25

Step 4: Use the sign chart to answer the question. Here the solution consists of all values for which $f(x) > 0$. Shade in the values that produce positive results and then express this set in interval notation.



Figure 2.2.26

Answer:

$\left(-\infty, -\frac{3}{2}\right) \cup (3, \infty)$

Example 2.2.9

Solve: $x^3 + x^2 \leq 4(x+1)$.

Solution

Begin by rewriting the inequality in standard form, with zero on one side.

$$\begin{aligned} x^3 + x^2 &\leq 4(x+1) \\ x^3 + x^2 &\leq 4x + 4 \\ x^3 + x^2 - 4x - 4 &\leq 0 \end{aligned}$$

Next find the critical numbers of $f(x) = x^3 + x^2 - 4x - 4$:

$$\begin{aligned} x^3 + x^2 - 4x - 4 &= 0 \quad \textit{Factor by grouping.} \\ x^2(x+1) - 4(x+1) &= 0 \\ (x+1)\left(x^2 - 4\right) &= 0 \\ (x+1)(x+2)(x-2) &= 0 \end{aligned}$$

The critical numbers are -2, -1, and 2. Because of the inclusive inequality $(\leq)$ we will plot them using closed dots.

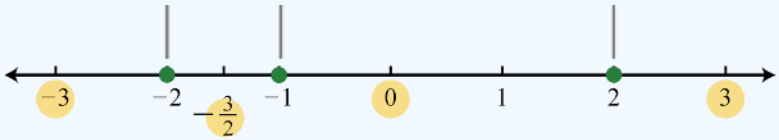


Figure 2.2.27

Use test values, -3, $-\frac{3}{2}$, 0, and 3 to create a sign chart.

$$\begin{aligned} f(-3) &= (-3+1)(-3+2)(-3-2) &&= (-)(-)(-) = - \\ f(-\tfrac{3}{2}) &= (-\tfrac{3}{2}+1)(-\tfrac{3}{2}+2)(-\tfrac{3}{2}-2) &&= (-)(+)(-) = + \\ f(0) &= (0+1)(0+2)(0-2) &&= (+)(+)(-) = - \\ f(3) &= (3+1)(3+2)(3-2) &&= (+)(+)(+) = + \end{aligned}$$

And we have

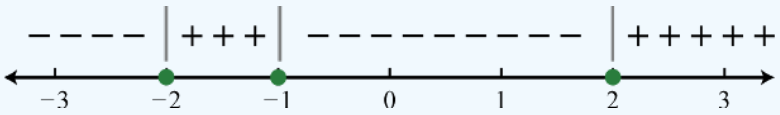


Figure 2.2.28

Use the sign chart to shade in the values that have negative results $(f(x) \leq 0)$.

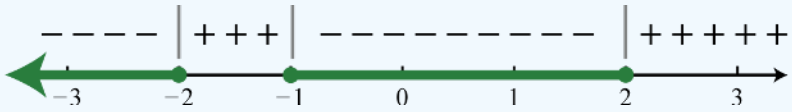


Figure 2.2.29

Answer

$(-\infty, -2] \cup [-1, 2]$

? You Try 2.2.4

Solve $-3x^4 + 12x^3 - 9x^2 > 0$.

Answer

$(1, 3)$

Solving Rational Inequalities Using Sign Charts

A **rational inequality** is a mathematical statement that relates a rational expression as either less than or greater than another. Because rational functions have restrictions to the domain, which we will discuss more in the next section, we must take care when solving rational inequalities. In addition to the zeros, we will include the restrictions to the domain of the function which are values that make the denominator zero in the set of critical numbers.

✔ Example 2.2.10

Solve: $\dfrac{(x-4)(x+2)}{(x-1)} \geq 0$

Solution

The zeros of a rational function occur when the numerator is zero, provided they are unique to the numerator, and the values that produce zero in the denominator are the restrictions. In this case,

Roots (Numerator)	Restriction(Denominator)
$x-4=0$ or $x+2=0$	$x-1=0$
$x=4 \quad x=-2$	$x=1$

Therefore the critical numbers are $-2, 1$, and 4. Because of the inclusive inequality $(\geq)$ use a closed dot for the roots $-2, 4$ and always use an open dot for restrictions 1. Restrictions are never included in the solution set.

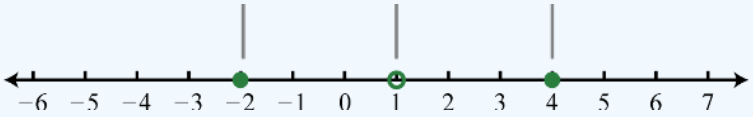


Figure 2.2.30

Use test values $x=-4, 0, 2, 6$.

$$f(-4)=\frac{(-4-4)(-4+2)}{(-4-1)} \quad =\frac{(-)(-)}{(-)}=-$$
$$f(0)=\frac{(0-4)(0+2)}{(0-1)} \quad =\frac{(-)(+)}{(-)}=+$$
$$f(2)=\frac{(2-4)(2+2)}{(2-1)} \quad =\frac{(-)(+)}{(+)}=-$$
$$f(6)=\frac{(6-4)(6+2)}{(6-1)} \quad =\frac{(+)(+)}{(+)}=+$$

And then complete the sign chart.

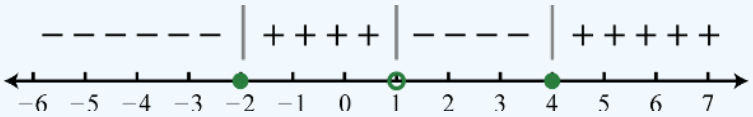


Figure 2.2.31

The question asks us to find the values for which $f(x)\geq 0$, in other words, positive or zero. Shade in the appropriate regions and present the solution set in interval notation.

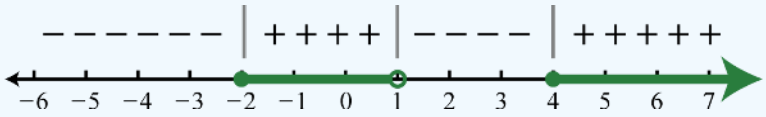


Figure 2.2.32

Answer

$[-2, 1)\cup[4, \infty)$

Although we have not learned about the graphs of rational functions yet, the graph of this function is provided below. Compare the graph to its corresponding sign chart above.

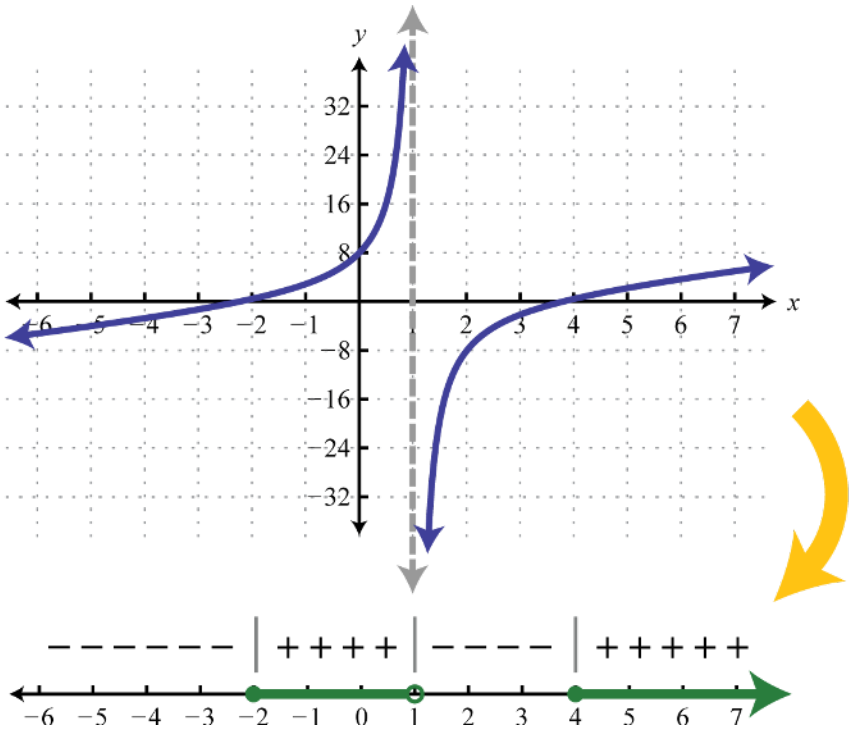


Figure 2.2.33

Notice that the restriction $x = 1$ corresponds to a place where the function values get very large and also very small. This is called a vertical asymptote which bounds regions where the function changes from positive to negative. While not included in the solution set because it makes the denominator zero, this value is a critical number. Before creating a sign chart we must ensure the inequality has a zero on one side. The general steps for solving a rational inequality are outlined in the following example.

Example 2.2.11

Solve $\dfrac{7}{x+3} < 2$.

Solution

Step 1: Begin by obtaining zero on the right side.

$$\frac{7}{x+3} < 2$$

$$\frac{7}{x+3} - 2 < 0$$

Step 2: Determine the critical numbers. The critical numbers are the zeros and restrictions. Begin by simplifying to a single algebraic fraction.

$$\frac{7}{x+3} - \frac{2}{1} < 0$$

$$\frac{7-2(x+3)}{x+3} < 0$$

$$\frac{7-2x-6}{x+3} < 0$$

$$\frac{-2x+1}{x+3} < 0$$

Next find the critical numbers. Set the numerator and denominator equal to zero and solve.

Root	Restriction
$-2x+1=0$	
$-2x=-1$	$x+3=0$
$x=\frac{1}{2}$	$x=-3$

In this case, the strict inequality indicates that we should use an open dot for the root.

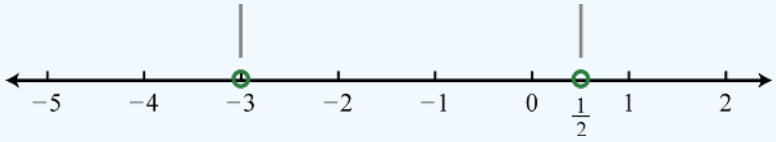


Figure 2.2.34

Step 3: Create a sign chart. Choose test values -4, 0, and 1.

$$f(-4)=\frac{-2(-4)+1}{-4+3} \quad =\frac{+}{-}=-$$

$$f(0)=\frac{-2(0)+1}{0+3} \quad =\frac{+}{+}=+$$

$$f(1)=\frac{-2(1)+1}{1+3} \quad =\frac{-}{+}=-$$

And we have

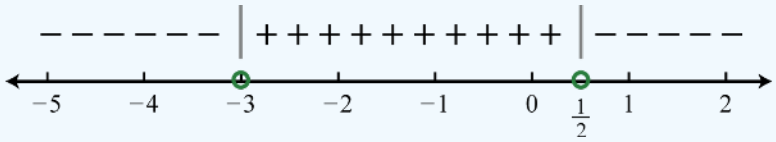


Figure 2.2.35

Step 4: Use the sign chart to answer the question. In this example we are looking for the values for which the function is negative, $f(x)<0$. Shade the appropriate values and then present your answer using interval notation.

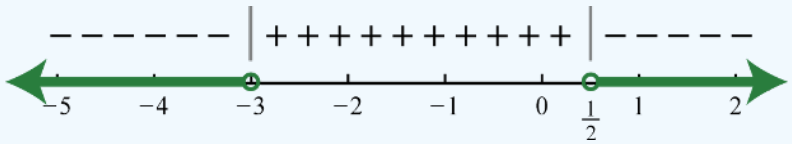


Figure 2.2.36

Answer:

$(-\infty,-3)\cup\left(\frac{1}{2},\infty\right)$

Example 2.2.12

Solve: $\frac{1}{x^2-4}\leq\frac{1}{2-x}$.

Solution

Begin by obtaining zero on the right side.

$$\frac{1}{x^2-4}\leq\frac{1}{2-x}$$

$$\frac{1}{x^2-4}-\frac{1}{2-x}\leq 0$$

Next simplify the left side to a single algebraic fraction.

$$\frac{1}{x^2-4}-\frac{1}{2-x}\le 0$$

$$\frac{1}{(x+2)(x-2)}-\frac{1}{-(x-2)}\le 0$$

$$\frac{1}{(x+2)(x-2)}+\frac{1(x+2)}{(x-2)(x+2)}\le 0$$

$$\frac{1+x+2}{(x+2)(x-2)}\le 0$$

$$\frac{x+3}{(x+2)(x-2)}\le 0$$

The critical numbers are -3, -2, and 2. Note that ± 2 are restrictions and thus we will use open dots when plotting them on a number line. Because of the inclusive inequality we will use a closed dot at the root -3.

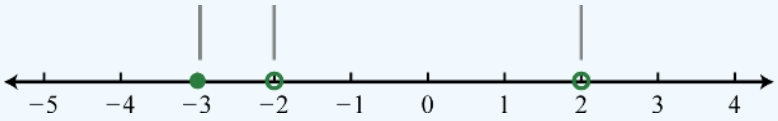


Figure 2.2.37

Choose test values -4, $-2\frac{1}{2}=-\frac{5}{2}$, 0, and 3.

$$f(-4)=\frac{-4+3}{(-4+2)(-4-2)} \qquad =\frac{(-)}{(-)(-)}=-$$

$$f(-\tfrac{5}{2})=\frac{-\frac{5}{2}+3}{(-\frac{5}{2}+2)(-\frac{5}{2}-2)} \qquad =\frac{(+)}{(-)(-)}=+$$

$$f(0)=\frac{0+3}{(0+2)(0-2)} \qquad =\frac{(+)}{(+)(-)}=-$$

$$f(3)=\frac{3+3}{(3+2)(3-2)} \qquad =\frac{(+)}{(+)(+)}=+$$

Construct a sign chart.

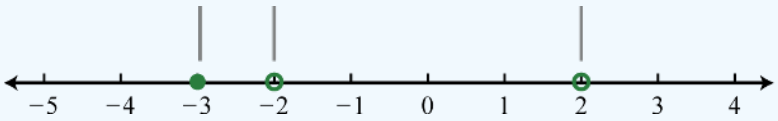


Figure 2.2.38

Answer the question; in this case, find x where $f(x)\le 0$.

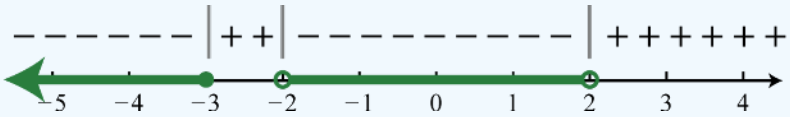


Figure 2.2.39

Answer

$$(-\infty,-3]\cup(-2,2)$$

? You Try 2.2.5

Solve: $\dfrac{2x^2}{2x^2+7x-4}\ge\dfrac{x}{x+4}$.

Answer

$$(-4, 0] \cup \left(\frac{1}{2}, \infty\right)$$

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2.3: Functions and Function Notation

Learning Objectives

- Determine whether a relation represents a function.
- Evaluate functions.
- Find domain of functions.

Imagine visiting a faraway city or even outer space from the comfort of your living room. It could be possible using virtual reality. This technology creates realistic images that make you feel as if you are truly immersed in the scene and even enable you to interact with them. It is being developed for fun applications, such as video games, but also for architects to plan buildings, car companies to design prototypes, the military to train, and medical students to learn. Developing virtual reality devices requires modeling the environment using functions.



Figure 2.3.1: This odd-looking headgear provides the user with a virtual world. (credit: fill/Pixabay)

What is a Function?

The natural world is full of relationships between quantities that change. When we see these relationships, it is natural for us to ask "If I know one quantity, can I then determine the other?" This establishes the idea of an input quantity, or independent variable, and a corresponding output quantity, or dependent variable. From this we get the notion of a functional relationship in which the output can be determined from the input.

For some quantities, like height and age, there are certainly relationships between these quantities. Given a specific person and any age, it is easy enough to determine their height, but if we tried to reverse that relationship and determine age from a given height, that would be problematic, since most people maintain the same height for many years.

Determining Whether a Relation Represents a Function

A **relation** is a set of ordered pairs. The set of the first components of the ordered pairs is called the **domain** and the set of the second components of the ordered pairs is called the **range**.

Consider the following set of ordered pairs: $\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$

The domain is $\{1,2,3,4,5\}$ The range is $\{2,4,6,8,10\}$

Each value in the domain is also known as an **input** value. Each value in the range is also known as an **output** value.

A **function** is a relation that assigns each value in the domain to a single value in the range.

To determine if a relation represents a function we can use a **mapping diagram**.

Figure 2.3.1 shows mapping diagrams for relations that are functions and are not functions.

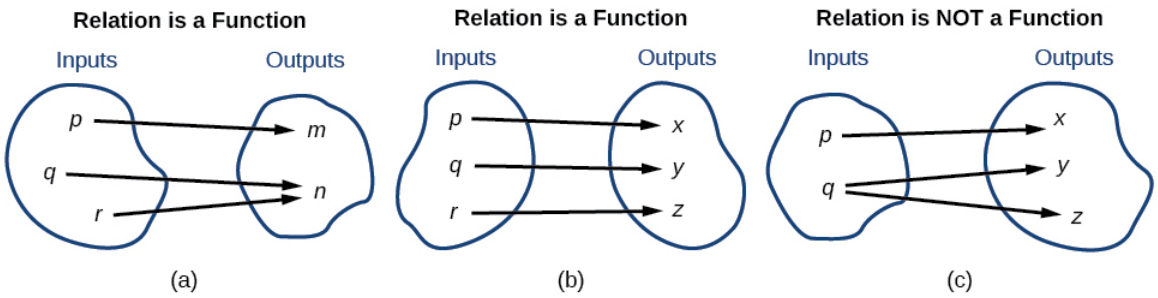


Figure 2.3.1: (a) This relation is a function because each input is associated with a single output. Note that input q and r both give output n. (b) This relation is also a function. In this case, each input is associated with a single output. (c) This relation is not a function because input q is associated with two different outputs.

Function

A **function** is a relation in which each input value in the domain is assigned to exactly one output value in the range. We say "the output is a function of the input."

The set of all input values are called the domain values, and the set of all output values are called the range values.

Example 2.3.1

The coffee shop menu, shown in Figure 2.3.2 consists of items and their prices.

a. Is price a function of the item?
b. Is the item a function of the price?



Figure 2.3.2: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Solution

a. Let's begin by considering the input as the items on the menu. The output values are then the prices. See Figure 2.3.3.



Figure 2.3.3: A menu of donut prices from a coffee shop where a plain donut is $1.49 and a jelly donut and chocolate donut are $1.99.

Each item on the menu has only one price, so the price is a function of the item.

a. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it. See Figure 2.3.4.

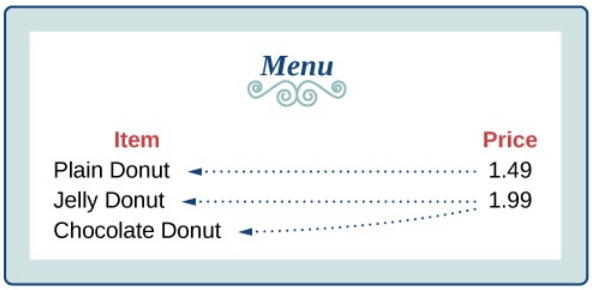


Figure 2.3.4: Association of the prices to the donuts.

Therefore, the item is a not a function of price.

Example 2.3.2

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? Table 2.3.1 shows a possible rule for assigning grade points.

Table 2.3.1: Class grade points.

Percent grade	0–56	57–61	62–66	67–71	72–77	78–86	87–91	92–100
Grade point average	0.0	1.0	1.5	2.0	2.5	3.0	3.5	4.0

Solution

For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.

In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86. Thus, percent grade is not a function of grade point average.

You Try 2.3.1

Table 2.3.2 lists the five greatest baseball players of all time in order of rank.

Table 2.3.2: Five greatest baseball players.

Player	Rank
Babe Ruth	1
Willie Mays	2
Ty Cobb	3
Walter Johnson	4
Hank Aaron	5

a. Is the rank a function of the player name?
b. Is the player name a function of the rank?

Answer a

Yes

Answer b

Yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

Function Notation

Once we determine that a relation is a function of the input, we can define a name to describe this relationship. This is especially useful in programming. We typically use a letter or symbol when naming a function. The letters f, g,and h are often used to represent functions. Sometimes we refer to the function relationship as a rule.

For example, to represent "height is a function of age," we start by identifying variables for the domain and range. Let a represent age and h represent height. Next, define a name for the function. Let f be the function name.

h is f of a	We name the function f; height is a function of age.
$h = f(a)$	We use parentheses to indicate the function input.
$f(a)$	We name the function f ; the expression is read as " f of a."

The notation $f(a)$ shows us that h depends on a. The value a must be put into the function f to get an output h. The parentheses indicate that age is the input for the function; they do not indicate multiplication.

Note that the input and output of a function do not have to be numbers. They can be names of people, labels of geometric objects, etc. Most of the functions we will work with in this book will have numbers as inputs and outputs.

Function Notation

For the function $y = f(x)$

f is the name of the function
x is the domain value
$f(x)$ is the range value y corresponding to the value x

We read $f(x)$ as f of x or the value of f at x.

We call *x* the **independent** variable as it can be any value in the domain. We call *y* the **dependent** variable as its value depends on *x*.

Example 2.3.3

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

Solution

The number of days in a month is a function of the name of the month, so if we name the function f, we write $\text{days} = f(\text{month})$ or $d = f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.

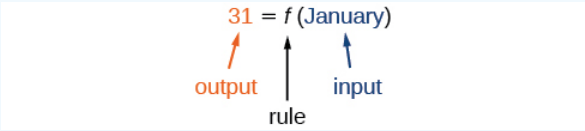


Figure 2.3.5: The function $31 = f(January)$ where 31 is the output, f is the rule, and January is the input.

For example, $f(\text{March}) = 31$, because March has 31 days. The notation $d = f(m)$ reminds us that the number of days, d (the output), is dependent on the name of the month, m (the input).

Analysis

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

Example 2.3.4

A function $N = f(y)$ gives the number of police officers, N, in a town in year y. What does $f(2005) = 300$ represent?

Solution

When we read $f(2005) = 300$, we see that the input year is 2005. The value for the output, the number of police officers (N), is 300. Remember, $N = f(y)$. The statement $f(2005) = 300$ tells us that in the year 2005 there were 300 police officers in the town.

Q&A

Instead of a notation such as $y = f(x)$, could we use the same symbol for the output as for the function, such as $y = y(x)$, meaning "y is a function of x?"

Yes, this is often done with formulas, especially in applied subjects that use higher level math, such as physics and engineering. While exploring math itself we like to maintain a distinction between a function such as f, which is a rule or procedure, and the output y we get by applying f to a particular input x. This is why we usually use notation such as $y = f(x)$, $P = W(d)$, and so on.

Evaluating a Function

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

Let's look at the function $y = 4x - 5$. To find the value of y when $x = 2$, we know to substitute $x = 2$ into the function and then simplify.

	$y = 4x - 5$
Let x=2.	$y = 4 \cdot 2 - 5$
	$y = 3$

The value of the function at $x = 2$ is 3.

We do the same thing using function notation, the equation $y = 4x - 5$ can be written as $f(x) = 4x - 5$. To find the value when $x = 2$, we write:

	$f(x) = 4x - 5$
Let x=2.	$f(2) = 4 \cdot 2 - 5$
	$f(2) = 3$

The value of the function at $x = 2$ is 3.

This process of finding the value of $f(x)$ for a given value of x is called ***evaluating a function***.

Example 2.3.5

For the function $f(x) = 2x^2 + 3x - 1$, evaluate the function.

a. $f(3)$
b. $f(-2)$
c. $f(a)$

Solution

a.

	$f(x) = 2x^2 + 3x - 1$

To evaluate $f(3)$, substitute 3 for x.	$f(3) = 2(3)^2 + 3 \cdot 3 - 1$
Simplify.	$f(3) = 2 \cdot 9 + 3 \cdot 3 - 1$
	$f(3) = 18 + 9 - 1$
	$f(3) = 26$

b.

		$f(x) = 2$
To evaluate $f(-2)$, substitute -2 for x.		$f(-2) = 2(-2)^2 + 3(-2) - 1$
Simplify.		$f(-2) = 2 \cdot 4 + (-6) - 1$
		$f(-2) = 8 + (-6) - 1$
		$f(-2) = 1$

c.

	$f(x) = 2x^2 + 3x - 1$
To evaluate f(a),f(a), substitute a for x.	$f(x) = 2a^2 + 3a - 1$
Simplify.	

? You Try 2.3.2

For the function $f(x) = 3x^2 - 2x + 1$, find the following:

a. $f(3)$
b. $f(-1)$
c. $f(t)$

Answer

a. $f(3) = 22$

b. $f(-1) = 6$

c. $f(t) = 3t^2 - 2t + 1$

✓ Example 2.3.6

For the function $g(x) = 3x - 5$, evaluate the function.

a. $g(h^2)$
b. $g(x + 2)$
c. $g(x) + g(2)$

Solution

a.

	$g(x) = 3x - 5$
To evaluate $g(h^2)$, substitute h^2 for x.	$g\left(h^2\right) = 3h^2 - 5$
	$g\left(h^2\right) = 3h^2 - 5$

b.

	$g(x)=3x-5$
To evaluate $g(x+2)$, substitute $x+2$ for x.	$g(x+2)=3(x+2)-5$
Simplify.	$g(x+2)=3x+6-5$
	$g(x+2)=3x+1$

c.

	$g(x)=3x-5$
To evaluate $g(x)+g(2)$, first find $g(2)$.	$g(2)=3\cdot 2-5$
	$g(2)=1$
	$g(x)+g(2)=\underbrace{3x-5}_{g(x)}\quad\underbrace{}_{g(2)}+1$
Simplify.	$g(x)+g(2)=3x-5+1$
	$g(x)+g(2)=3x-4$

Notice the difference between part b and c. We get $g(x+2)=3x+1$ and $g(x)+g(2)=3x-4$. So we see that $g(x+2)\neq g(x)+g(2)$.

You Try 2.3.3

For the function $g(x)=4x-7$, evaluate the function.

a. $g(m^2)$
b. $g(x-3)$
c. $g(x)-g(3)$

Answer

a. $4m^2-7$

b. $4x-19$

c. $4x-12$

You Try 2.3.4

Given the function $g(m)=\sqrt{m-4}$, evaluate

1. $g(5)$
2. $g(7)$

Answer

1. $g(5)=1$
2. $g(7)=\sqrt{3}$

Note: $\sqrt{3}$ is an irrational number. Since the directions did not indicate to approximate, we leave the answer in this form, called ***exact form***, since irrational numbers are non-terminating non-repeating decimals. If the directions indicated to round to one decimal place, then we could state the answer as an approximatimation as, $g(7)\approx 1.7$. Do remember to round appropriately.

Example 2.3.7

Evaluate $f(x) = x^2 + 3x - 4$ at

a. 2
b. a
c. $a + h$
d. Evaluate $\frac{f(a+h)-f(a)}{h}$

Solution

Replace the x in the function with each specified value.

a. Because the input value is a number, 2, we can use simple algebra to simplify.

$$\begin{aligned} f(2) &= 2^2 + 3(2) - 4 \\ &= 4 + 6 - 4 \\ &= 6 \end{aligned}$$

b. In this case, the input value is a letter so we cannot simplify the answer any further.

$$f(a) = a^2 + 3a - 4$$

c. With an input value of $a + h$, we must use the distributive property.

$$\begin{aligned} f(a+h) &= (a+h)^2 + 3(a+h) - 4 \\ &= a^2 + 2ah + h^2 + 3a + 3h - 4 \end{aligned}$$

d. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that

$$f(a+h) = a^2 + 2ah + h^2 + 3a + 3h - 4$$

and we know that

$$f(a) = a^2 + 3a - 4$$

Now we combine the results and simplify.

$$\begin{aligned} \frac{f(a+h) - f(a)}{h} &= \frac{(a^2 + 2ah + h^2 + 3a + 3h - 4) - (a^2 + 3a - 4)}{h} \\ &= \frac{(2ah + h^2 + 3h)}{h} \\ &= \frac{h(2a + h + 3)}{h} && \text{Factor out h.} \\ &= 2a + h + 3 && \text{Simplify.} \end{aligned}$$

In the previous example, the formula in part d is called a **difference quotient**. In calculus, the difference quotient provides us with a way to describe instantaneous rate of change which is very useful.

Many everyday situations can also be modeled using functions. Let's look at an example.

You Try 2.3.5

The number of unread emails in Sylvia's account is 75. This number grows by 10 unread emails a day. The function $N(t) = 75 + 10t$ represents the relation between the number of emails, *N*, and the time, *t*, measured in days.

a. Determine the independent and dependent variable.
b. Find $N(5)$. Explain what this result means.

Answer

a. The number of unread emails is a function of the number of days. The number of unread emails, N, depends on the number of days, t. Therefore, the variable N, is the dependent variable and the variable tt is the independent variable.

b. Find $N(5)$. Explain what this result means.

	$N(t) = 75 + 10t$
Substitute in t=5.t=5.	$N(5) = 75 + 10 \cdot 5$
Simplify.	$N(5) = 75 + 50$
	$N(5) = 125$

Since 5 is the number of days, $N(5)$, is the number of unread emails after 5 days. After 5 days, there are 125 unread emails in the account.

Finding Input Value(s) When Given an Output Value

When we know an output value and want to determine the input values that would produce the given output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

Example 2.3.8

Given the function $h(p) = p^2 + 2p$, solve for $h(p) = 3$.

Solution

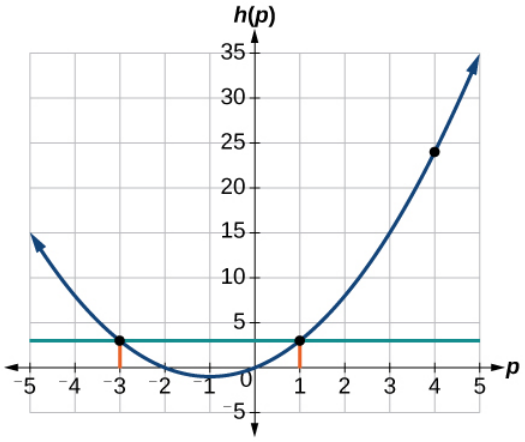
$$\begin{aligned} h(p) &= 3 \\ p^2 + 2p &= 3 \quad \text{Substitute the original function} \\ p^2 + 2p - 3 &= 0 \quad \text{Subtract 3 from each side.} \\ (p+3)(p-1) &= 0 \quad \text{Factor.} \end{aligned}$$

If $(p+3)(p-1) = 0$, either $(p+3) = 0$ or $(p-1) = 0$ (or both of them equal 0). We will set each factor equal to 0 and solve for p in each case.

$$(p+3) = 0,\ p = -3$$

$$(p-1) = 0,\ p = 1$$

This gives us two solutions. The output $h(p) = 3$ when the input is either $p = 1$ or $p = -3$. We can also verify by graphing as in Figure 2.3.6. The graph verifies that $h(1) = h(-3) = 3$ and $h(4) = 24$.



p	−3	−2	0	1	4
$h(p)$	3	0	0	3	24

Figure 2.3.6: Graph of $h(p) = p^2 + 2p$

? You Try 2.3.6

Given the function $g(m) = \sqrt{m-4}$, solve $g(m) = 2$.

Answer

$m = 8$

Writing Equations as Functions

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can write the equation using function notation. For example, the equation $2n + 6p = 12$ represents a functional relationship between n and p. We can rewrite the equation as p is a function of n by solving the equation for p.

✓ Example 2.3.9

Express the relationship $2n + 6p = 12$ as a function $p = f(n)$

Solution

To express the relationship in this form, we need to write the relationship where p is a function of n, which means writing it as $p = [\text{expression involving } n]$.

$$\begin{aligned} 2n + 6p &= 12 \\ 6p &= 12 - 2n && \text{Subtract 2n from both sides.} \\ p &= \frac{12 - 2n}{6} && \text{Divide both sides by 6 and simplify.} \\ p &= \frac{12}{6} - \frac{2n}{6} \\ p &= 2 - \frac{1}{3}n \end{aligned}$$

Therefore, p as a function of n is written as

$$p = f(n) = 2 - \frac{1}{3}n$$

Note: Not every relationship can be expressed using function notation. This can only be done when the equation represents a function.

Piecewise Defined Functions

Piecewise defined functions are used in many real-world phenomena (e.g. postal rates and income tax formulas) are modeled by such functions. It is important that we are familiar with them and know how to evaluate them.

Consider the absolute value function $f(x) = |x|$.

The domain is real numbers and the range is values greater than or equal to 0.

If we input 0 or any positive value, the output is the same value as the input ($f(0) = |0| = 0$, $f(3) = |3| = 3$, $f(7.2) = |7.2| = 7.2$, ...).

$f(x) = x$ if $x \geq 0$

If we input a negative value the sign of the output must change from negative to positive ($f(-1) = |-1| = 1$, $f(-4.5) = |-4.5| = 4.5$, ...).

$f(x) = -x$ if $x < 0$, since multiplying a negative value by -1 makes it positive.

Since this requires two different processes or pieces, the absolute value function is often called the most basic piecewise defined function.

Piecewise Function

A **piecewise function** is a function in which the formula used depends upon the domain the input lies in. We notate this idea like:

$$f(x) = \begin{cases} \text{formula 1, if domain value satisfies given criteria 1} \\ \text{formula 2, if domain value satisfies given criteria 2} \\ \text{formula 3, if domain value satisfies given criteria 3} \end{cases}$$

Note that piecewise defined functions are not rescricted to 3 formulas, they may contain 2 or more.

Example 2.3.10

A museum charges $5 per person for a guided tour with a group of 1 to 9 people, or a fixed $50 fee for 10 or more people in the group. Set up a function relating the number of people, n, to the cost, C.

Solution

To set up this function, two different formulas would be needed. $C = 5n$ would work for n values under 10, and $C = 50$ would work for values of n ten or greater. Notating this:

$$C(n) = \begin{cases} 5n & \text{if } 0 < \text{n} < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

Example 2.3.11

For the function $f(x) = \begin{cases} x+6, & \text{if } x < 1 \\ 3 - x^2, & \text{if } x \geq 1 \end{cases}$, find the following:

1. $f(0)$
2. $f(1)$
3. $f(3)$

Solution

1. Since the input is 0, we input into the top formula, $x + 6$. So, $f(0) = 0 + 6 = 6$
2. Since the input is 1, we input into the top formula, $x + 6$. So, $f(1) = 1 + 6 = 7$
3. Since the input is 3 we input into the bottom formula, $3 - x^2$. So, $f(0) = 0 + 6 = 6$

Example 2.3.12

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if} \quad 0 < g < 2 \\ 25 + 10(g-2) & \text{if} \quad g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data, and the cost of using 4 gigabytes of data.

Solution

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which piece of domain our input falls in. Since 1.5 is less than 2, we use the first formula, giving $C(1.5)$ = $25.

The find the cost of using 4 gigabytes of data, $C(4)$, we see that our input of 4 is greater than 2, so we'll use the second formula. $C(4)$ = 25 + 10(4 - 2) = $45.

Example 2.3.13

The height h in feet of a model rocket above the ground t seconds after lift-off is given by

$$h(t) = \begin{cases} -5t^2 + 100t, & \text{if} \quad 0 \leq t \leq 20 \\ 0, & \text{if} \quad t > 20 \end{cases}$$

1. Find and interpret $h(10)$ and $h(60)$.
2. Solve $h(t) = 375$ and interpret your answers.

Solution

1. We first note that the independent variable here is t, chosen because it represents time. Secondly, the function is broken up into two rules: one formula for values of t between 0 and 20 inclusive, and another for values of t greater than 20. Since $t = 10$ satisfies the inequality $0 \leq t \leq 20$, we use the first formula listed, $h(t) = -5t^2 + 100t$, to find $h(10)$. We get $h(10) = -5(10)^2 + 100(10) = 500$. Since t represents the number of seconds since lift-off and $h(t)$ is the height above the ground in feet, the equation $h(10) = 500$ means that 10 seconds after lift-off, the model rocket is 500 feet above the ground. To find $h(60)$, we note that $t = 60$ satisfies $t > 20$, so we use the rule $h(t) = 0$. This function returns a value of 0 regardless of what value is substituted in for t, so $h(60) = 0$. This means that 60 seconds after lift-off, the rocket is 0 feet above the ground; in other words, a minute after lift-off, the rocket has already returned to Earth.
2. Since the function h is defined in pieces, we need to solve $h(t) = 375$ in pieces. For $0 \leq t \leq 20$, $h(t) = -5t^2 + 100t$, so for these values of t, we solve $-5t^2 + 100t = 375$. Rearranging terms, we get $5t^2 - 100t + 375 = 0$, and factoring gives $5(t-5)(t-15) = 0$. Our answers are $t = 5$ and $t = 15$, and since both of these values of t lie between 0 and 20, we keep both solutions. For $t > 20$, $h(t) = 0$, and in this case, there are no solutions to $0 = 375$. In terms of the model rocket, solving $h(t) = 375$ corresponds to finding when, if ever, the rocket reaches 375 feet above the ground. Our two answers, $t = 5$ and $t = 15$ correspond to the rocket reaching this altitude twice - once 5 seconds after launch, and again 15 seconds after launch.

Domain of Functions

We have learned that the domain of a function is the set of input values. How do find the domain when we are given a formula for a function?

First, let's recall some things from algebra.

If $f(x) = \frac{x}{x-5}$, find $f(5)$. Evaluating this,

$$f(5) = \frac{(5)}{(5)-5} = \frac{5}{0},$$

which is undefined. Since the number 5 makes the function undefined, this value is excluded from the domain. Since 5 is the only real number that would make the denominator zero, the domain is $(-\infty, 5) \cup (5, \infty)$.

Next, If $g(x) = \sqrt{x+3}$, find $g(-4)$. Evaluating this,

$$g(4)=\sqrt{(-4)+3}=\sqrt{-1}$$

which is not a real number.

Any real number that we input into this function that makes $x+3$ negative must be excluded from the domain. So, the domain of this function is all real numbers such that $x+3\geq 0$ which yields $x\geq -3$. The domain in interval notation is $[-3,\infty)$.

When finding the domain of a function, we start with the set of all real numbers. Then, we exclude any values that make a denominator zero or result in an even root of a negative number.

How To Find the Domain of a Function

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain. If there is a denominator in the function's formula, set the denominator equal to zero and solve for x . If the function's formula contains an even root, set the radicand greater than or equal to 0, and then solve.
3. Write the domain in interval notation.

Example 2.3.14

Find the domain of the function $f(x)=x^2-1$.

Solution

The input value, shown by the variable x in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of f is $(-\infty,\infty)$.

You Try 2.3.7

Find the domain of the function:

$$f(x)=5-x+x^3$$

Answer

$(-\infty,\infty)$

Example 2.3.15

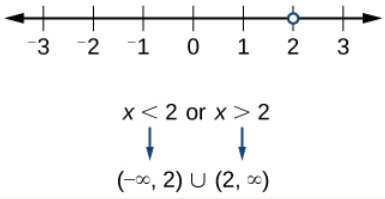
Find the domain of the function $f(x)=\dfrac{x+1}{2-x}$.

Solution

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for x.

$$\begin{aligned}2-x&=0\\-x&=-2\\x&=2\end{aligned}$$

Now, we will exclude 2 from the domain. The answers are all real numbers where $x<2$ or $x>2$. We can use a symbol known as the union, $\cup$,to combine the two sets. In interval notation, we write the solution:$(-\infty,2)\cup(2,\infty)$.



In interval notation, the domain of f is $(-\infty, 2) \cup (2, \infty)$.

? You Try 2.3.8

Find the domain of the function:

$$f(x) = \frac{1+4x}{2x-1}$$

Answer

$$(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$$

✔ Example 2.3.16

Find the domain of the function:

$$f(x) = \sqrt{7-x}$$

Solution

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for x.

$$\begin{aligned} 7-x &\geq 0 \\ -x &\geq -7 \\ x &\leq 7 \end{aligned}$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or $(-\infty, 7]$.

? You Try 2.3.9

Find the domain of the function

$$f(x) = \sqrt{5+2x}.$$

Answer

$$[-2.5, \infty)$$

✔ Example 2.3.17

Find the domain of the following functions. State answer in interval notation.

1. $g(x) = \sqrt{4-3x}$
2. $h(x) = \sqrt[5]{4-3x}$
3. $R(x) = \sqrt{x^2+5}$
4. $F(x) = \dfrac{\sqrt[4]{2x+1}}{x^2-1}$

5. $r(t) = \dfrac{4}{6 - \sqrt{t+3}}$
6. $I(p) = \dfrac{3p^2}{p}$

Solution

1. Set $4 - 3x \geq 0$. Solving this, we get $x \leq \frac{4}{3}$. This shows that as long as $x \leq \frac{4}{3}$, the radicand is greater than or equal to 0, $4 - 3x \geq 0$, and the formula $g(x)$ outputs a real number. The domain is $\left(-\infty, \frac{4}{3}\right]$.
2. Although the formula for $h(x)$ is very similar to that of $g(x)$ there is one key difference: $g(x)$ is an even indexed root (namely a square root) and $h(x)$ is an odd indexed root (the fifth root). Since odd roots of real numbers (even negative real numbers) are real numbers, there is no restriction on the inputs to h. Hence, the domain is $(-\infty, \infty)$.
3. Since $x^2 + 5 > 0$ for all x, there is no restrictions on the inputs. Hence, the domain is $(-\infty, \infty)$.
4. There are two issues here: not only do we have a denominator, we have a fourth (even-indexed) root. We must determine the restrictions on each part and select the real numbers which satisfy both conditions. To satisfy the fourth root, $2x + 1 \geq 0$. From this we get $x \geq -\frac{1}{2}$. Next, we find the values of x which make the denominator equal to 0, $x^2 - 1 = 0$. Solving this, $x = \pm 1$. Hence, in order for a real number x to be in the domain of F, $x \geq -\frac{1}{2}$ but $x \neq \pm 1$. In interval notation, this set is $\left[-\frac{1}{2}, 1\right) \cup (1, \infty)$.
5. Don't be put off by the 't' here. It is an independent variable representing a real number, just like x does, and is subject to the same restrictions. As in the previous problem, we have two issues here: we have a square root and a denominator. To satisfy the square root, we need a non-negative radicand so we set $t + 3 \geq 0$ to get $t \geq -3$. Setting the denominator equal to zero gives $6 - \sqrt{t+3} = 0$, or $\sqrt{t+3} = 6$. Squaring both sides gives $t + 3 = 36$, or $t = 33$. Remember that since we squared both sides in the course of solving this equation, we need to check our answer. Sure enough, when $t = 33$, $6 - \sqrt{t+3} = 6 - \sqrt{36} = 0$, so $t = 33$ will cause problems in the denominator. For the domain of r: we need $t \geq -3$, but $t \neq 33$. Hence, the domain is $[-3, 33) \cup (33, \infty)$.
6. It's tempting to simplify $I(x) = \frac{3x^2}{x} = 3x$. Since there are no longer any denominators it appears that the domain is $(-\infty, \infty)$ which is not the case. When simplifying $I(x)$, we are assuming $x \neq 0$, since $\frac{0}{0}$ is undefined. Proceeding as before, we find the domain of I to be all real numbers except 0: $(-\infty, 0) \cup (0, \infty)$.

It is worth reiterating the importance of finding the domain of a function before simplifying, as evidenced by the function I in the previous example. Even though the formula $I(x)$ simplifies to $3x$, it would be inaccurate to write $I(x) = 3x$ without adding the stipulation that $x \neq 0$.

Example 2.3.18

Find the domain of the function $v(t) = \sqrt{6 - 5t - t^2}$.

Solution

A square root is only defined when the quantity inside the square root, is zero or greater. Thus, the domain of this function will be when $6 - 5t - t^2 \geq 0$. Solving this non-linear inequality yields $[-6, 1]$ which is the domain of the $v(t)$ function.

Example 2.3.19

Given $g(x) = \dfrac{4 + x}{x - 3}$,

1. Find and simplify $g\left(\dfrac{1}{t}\right)$.
2. What is the domain of the function you found in part 1? State your answer in interval notation.

Solution

1.

https://math.libretexts.org/@go/page/126508

$$g\left(\frac{1}{t}\right) = \frac{4+\frac{1}{t}}{\frac{1}{t}-3}$$
$$= \left(\frac{t}{t}\right)\left(\frac{4+\frac{1}{t}}{\frac{1}{t}-3}\right)$$
$$= \frac{4t+1}{1-3t}$$

2. For $g\left(\frac{1}{t}\right)$ to be defined, $t \neq 0$. In addition, for the function to be defined, $1-3t \neq 0$. So, $t \neq \frac{1}{3}$.

The domain is $(-\infty, 0) \cup \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

? You Try 2.3.10

Given $f(a) = \frac{3a}{a^2-7a+10}$ and $g(a) = \frac{2a}{a^2-8a+15}$,

1. Find and simplify $f(a) - g(a)$.
2. What is the domain of the function you found in part 1? State your answer in interval notation.

Answer

1. $\frac{a}{(a-3)(a-2)}$
2. $(-\infty, 2) \cup (2, 3) \cup (3, 5) \cup (5, \infty)$

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2.4: Graphing Functions and Common Function Behavior

Learning Objectives

- Use points to graph functions.
- Graph functions using a graphing calculator.
- Determine if a graph represents a function (Vertical Line Test).
- Evaluating functions using graphs.
- Identify basic functions.
- Graph piecewise-defined functions.

You have experience plotting points and graph linear equations. In this section, we will extend on this knowledge for graphing functions and we will become familiar with some functions that we commonly work with so that we can later develop some ways for graphing non-linear functions more efficiently.

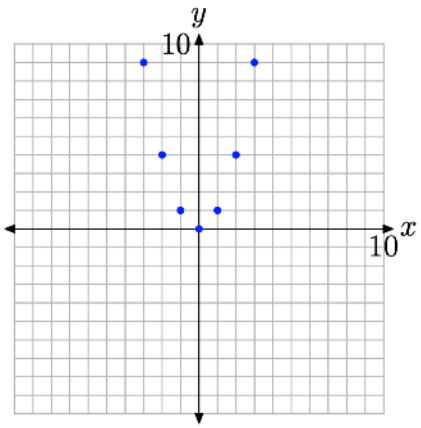
Graphing Functions Using Points

The **graph of a function** is the set of all points plotted in the Cartesian coordinate system that satisfy the function.

Consider the function defined by $f(x) = x^2$. To graph a function using points, we begin by creating a table of points $(x, f(x))$, where x is in the domain of the function f . Pick some values for x. Then evaluate the function at these values. Plot the points.

x	$f(x) = x^2$	$(x, f(x))$
-3	9	$(-3, 9)$
-2	4	$(-2, 4)$
-1	1	$(-1, 1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	4	$(2, 4)$
3	9	$(3, 9)$

(a)



(b)

Figure 2.4.1. Plotting pairs satisfying the functional relationship defined by the equation $f(x) = x^2$.

The points we found appear to establish a pattern for the function behavior so we draw the graph by connecting the points. Since the domain is all real numbers, we draw arrows at the edge of the coordinate system to indicate the graphical pattern continues upward.

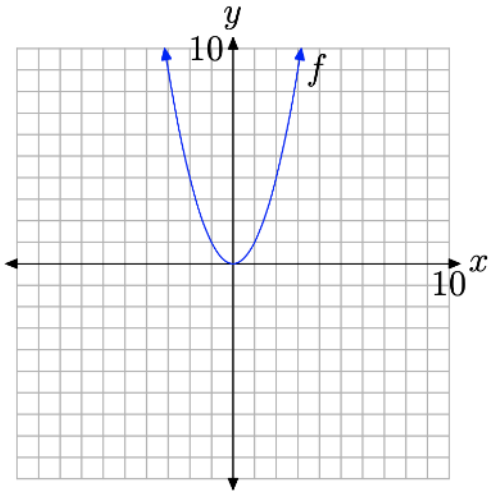


Figure 2.4.2. Plotting all pairs $(x, f(x))$ so that x is in the domain of f.

Strategy for Graphing by Hand Using Points

If a function is defined by an equation, you can create the graph of the function as follows.

1. Select several values of x in the domain of the function f.
2. Use the selected values of x to create a table of pairs (x, f(x)) that satisfy the equation that defines the function f.
3. Create a Cartesian coordinate system on a sheet of graph paper. Label and scale each axis, then plot the pairs (x, f(x)) from your table on your coordinate system.
4. If the plotted pairs (x, f(x)) provide enough of a pattern for you, make the "leap of faith" and plot all pairs that satisfy the equation defining f by drawing a graph on your coordinate system.
5. If your plotted pairs do not provide enough of a pattern to determine the final shape of the graph of f, then find more add more pairs and plot them. Continue in this manner until you are confident in the shape of the graph of f.

Let's look at some more examples of graphing by hand.

Example 2.4.1

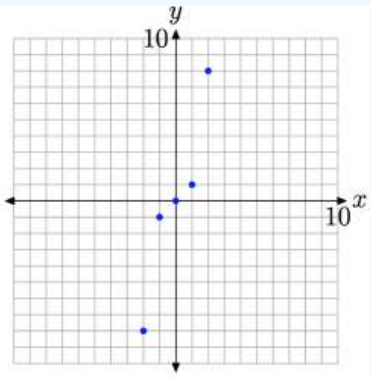
Sketch the graph of the function defined by the equation $f(x) = x^3$.

Solution

We'll start with x-values $-2, -1, 0, 1$, and 2, then use the equation $f(x) = x^3$ to determine pairs (x, f(x)) (e.g., $f(-2) = (-2)^3 = -8$). These are listed in the table in Figure 2.4.3(a). We then plot the points from the table on a Cartesian coordinate system, as shown in Figure 2.4.3(b).

x	$f(x) = x^3$	$(x, f(x))$
-2	-8	$(-2, -8)$
-1	-1	$(-1, -1)$
0	0	$(0, 0)$
1	1	$(1, 1)$
2	8	$(2, 8)$

(a)



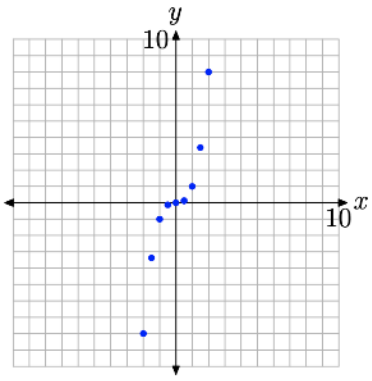
(b)

Figure 2.4.3. Plotting pairs (x, f(x)) defined by the equation $f(x) = x^3$.

We're a bit unsure of the shape of the graph of f, so we'll add a few more points to our table and plot them. This is shown in Figures 2.4.4(a) and (b).

x	$f(x) = x^3$	$(x, f(x))$
$-3/2$	$-27/8$	$(-3/2, -27/8)$
$-1/2$	$-1/8$	$(-1/2, -1/8)$
$1/2$	$1/8$	$(1/2, 1/8)$
$3/2$	$27/8$	$(3/2, 27/8)$

(a)



(b)

Figure 2.4.4. Plotting additional pairs (x, f(x)) defined by the equation $f(x) = x^3$.

The additional pairs fill in the shape of f in Figure 2.4.4(b) a bit better than those in Figure 2.4.3(b), enough so that we're confident enough to draw the final shape of the graph of $f(x) = x^3$ in Figure 2.4.5.

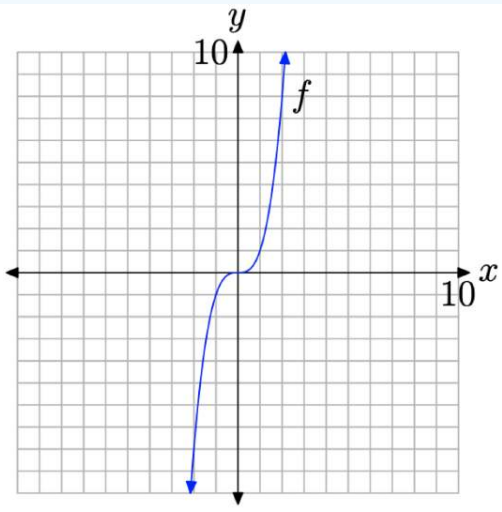


Figure 2.4.5. The final graph of $f(x) = x^3$.

Let's look at another example.

Example 2.4.2

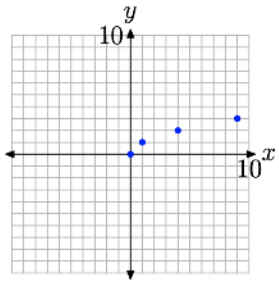
Sketch the graph of $f(x) = \sqrt{x}$

Solution

Again, we'll start by selecting several values of x in the domain of f. In this case, $f(x) = \sqrt{x}$, and it's not possible to take the square root of a negative number. Also, if we're creating a table of pairs by hand, it's good strategy to select known squares. Thus, we'll use x = 0, 1, 4, and 9 for starters.

x	$f(x)=\sqrt{x}$	$(x, f(x))$
0	0	$(0,0)$
1	1	$(1,1)$
4	2	$(4,2)$
9	3	$(9,3)$

(a)



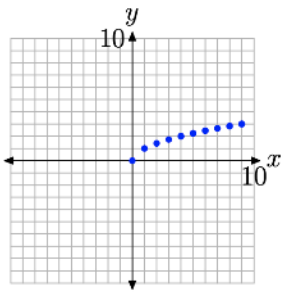
(b)

Figure 2.4.6. Plotting pairs (x, f(x)) defined by the equation $f(x)=\sqrt{x}$.

Some might be ready to make a “leap of faith” based on these initial results. Others might want to use a calculator to compute decimal approximations for additional square roots. The resulting pairs are shown in the table in Figure 2.4.7(a) and the additional pairs are plotted in Figure 2.4.7(b).

x	$f(x)=\sqrt{x}$	$(x, f(x))$
2	1.4	$(2,1.4)$
3	1.7	$(3,1.7)$
5	2.2	$(5,2.2)$
6	2.4	$(6,2.4)$
7	2.6	$(7,2.6)$
8	2.8	$(8,2.8)$

(a)



(b)

Figure 2.4.7 Plotting additional pairs (x, f(x)) defined by the equation $f(x)=\sqrt{x}$

The pattern in Figure 2.4.7(b) is clear enough to complete the graph as shown in Figure 2.4.8.

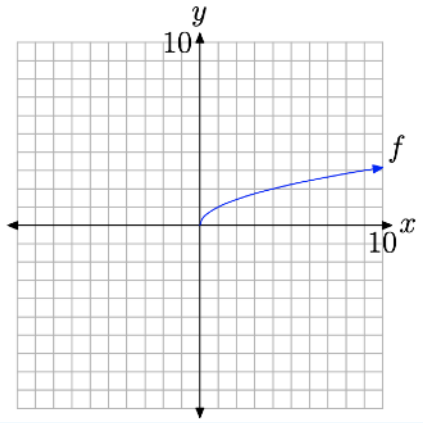


Figure 2.4.8. The graph of f defined by the equation $f(x)=\sqrt{x}$.

Using a Graphing Calculator to Find Points and Graph Functions

The TABLE feature on your graphing calculator can be of immense help when creating tables of points that satisfy the equation defining the function f. Let’s look at an example

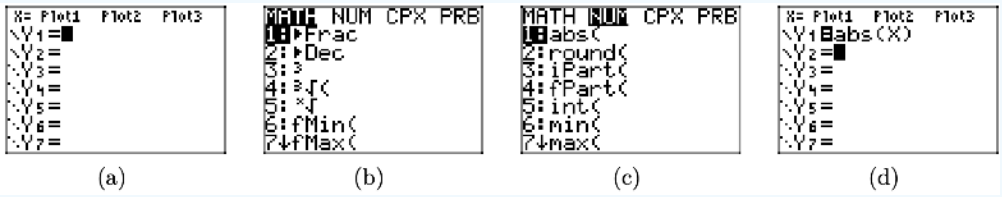
Example 2.4.3

Sketch the graph of $f(x)=|x|$

Solution

Enter the function $f(x) = |x|$ in the Y = menu as follows.

1. Press the Y= button on your calculator. This will open the Y= menu as shown in Figure 2.4.9(a). Use the arrow keys and the CLEAR button on your calculator to delete any existing functions.
2. Press the MATH button to open the menu shown in Figure 2.4.9(b).
3. Press the right-arrow on your calculator to select the NUM submenu as shown in Figure 2.4.9(c).
4. Select 1:abs(, then enter X and close the parentheses, as shown in Figure 2.4.9(d).



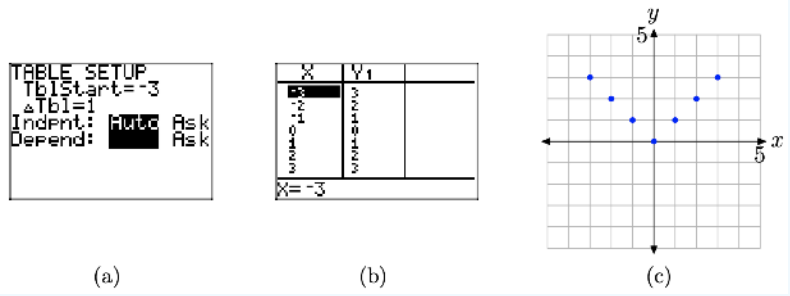
(a) (b) (c) (d)

Figure 2.4.9. Entering $f(x) = |x|$ in the Y = menu.

We will now use the TABLE feature of the graphing calculator to help create a table of pairs (x, f(x)) satisfying the equation $f(x) = |x|$. Proceed as follows.

1. Select 2nd TBLSET (i.e., push the 2nd button followed by TBLSET), which is located over the WINDOW button. Enter TblStart=-3, ΔTbl = 1, and set the independent and dependent variables to Auto (this is done by highlighting Auto and pressing the Enter button), as shown in Figure 2.4.10(a).
2. Press 2nd TABLE, which is located above the GRAPH button, to produce the table of pairs (x, f(x)) shown in Figure 2.4.10(b).

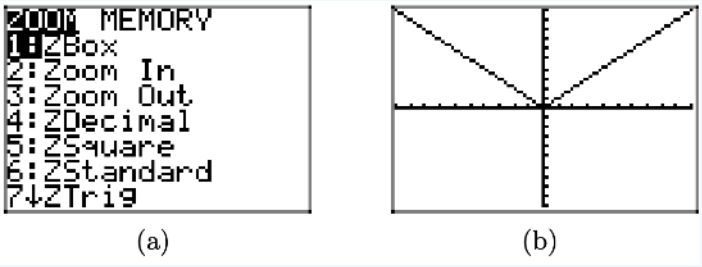
We've plotted the pairs directly from the calculator onto a Cartesian coordinate system on graph paper in Figure 2.4.10(c).



(a) (b) (c)

Figure 2.4.10. Creating a table with the TABLE feature of the graphing calculator.

We can have the graphing calculator draw the graph for us. Push the ZOOM button, then select 6:ZStandard (shown in Figure 2.4.11(a)) to produce the graph shown in Figure 2.4.11(b).



(a) (b)

Figure 2.4.11. Creating the graph of $f(x) = |x|$ with the graphing calculator.

Note: The standard graphing calculator window, 6:ZStandard, is:

$$-10 \le x \le -10 \text{ and } -10 \le y \le -10$$

The functions discussed so far are sometimes referred to as *parent functions*, *basic functions*, or *tool box functions*. They will be important later when we look at more efficient ways to graph functions.

Adjusting the Calculator Viewing Window

In Example 2.4.3, we used the graphing calculator to draw the graph of the function defined by the equation $f(x) = |x|$. For the functions we've encountered thus far, drawing their graphs using the graphing calculator is pretty trivial. Simply enter the equation in the Y= menu, then press the ZOOM button and select 6:ZStandard. However, if the graph of a function doesn't fit (or even appear) in the "standard" viewing window, it can be quite challenging to find optimal view settings so that the important features of the graph are visible.

Indeed, as one might not even know what "important" features to look for, setting the viewing window is usually highly subjective and experimental by nature. Let's look at some examples.

Example 2.4.4

Use a graphing calculator to sketch the graph of $f(x) = 56 - x - x^2$ using the window: Xmin = -15, Xmax = 15, Ymin = -100, Ymax = 100

Solution

First, start by entering the function in the *Y=* menu, as shown in Figure 2.4.12(a). The caret ^ on the keyboard is used for exponents. Press the *ZOOM* button and then select *6:ZStandard* to produce the graph shown in Figure 2.4.12(b).

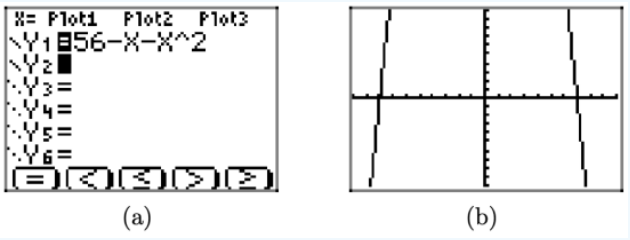


Figure 2.4.12 The graph of $f(x) = 56 - x - x^2$ in the "standard" viewing window.

To change the window, push the *WINDOW* button. Now, set Xmin = -15 and Xmax = 15. Xscl changes the units on the x-axis. If we want the tick marks on the scale to be 5 units, set Xscl = 5. Next, set Ymax = 100 and Ymin = -100. Since this is a wide interval, let's set tick marks on the y-axis every 10 units with Yscl = 10.

Push the GRAPH button to view the effects of these changes to the WINDOW parameters in Figure 2.4.13(b).

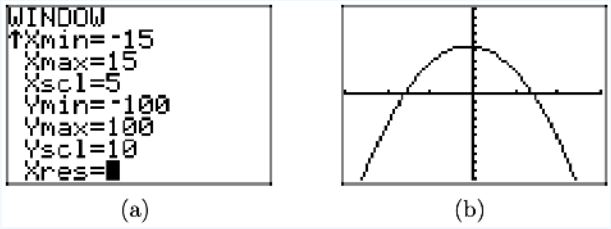


Figure 2.4.13 Improving the WINDOW settings.

In the previous example, the window setting were provided. We can see in Figure 2.4.12(b) and 2.4.13(b) that different windows provide different visuals of the graph. We usually prefer to use window settings that show the "important features" of function. If one is just beginning to learn about the graphs of functions, how is one to determine what are good windows for viewing the "important features" of the graph? Unfortunately, the answer to this question is, "through experience." Undoubtedly, this is a very frustrating phrase for readers to hear, but at least it's truthful. The more graphs that you draw, the more you will learn how to look for "turning points," "end-behavior," "x- and y-intercepts," and the like.

Legend for Graphing Calculator Window Parameters

The following explains each of the WINDOW parameters in Figure 2.4.13(a).

Xmin is the x-value of left edge of viewing window

Xmax is the x-value of right edge of viewing window

Xscl is the x-axis tick increment

Ymin is the y-value of bottom edge of viewing window

Ymax is the y-value of top edge of viewing window

Yscl is the y-axis tick increment

Xres is a pixel setting for viewing. Most people leave this setting at 1

Note: If the Xmin $\geq$ Xmax, the calculator will not graph and it will give you an error message. Similiarly, this will also occur if Ymin $\geq$ Ymax

Determining If a Graph Represents a Function (Vertical Line Test)

As we have seen in some examples above, we can represent a function using a graph. Graphs display a great many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. By convention, graphs are typically constructed with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value x and the output y, and we say y is a function of x, or $y = f(x)$ when the function is named f. The graph of the function is the set of all points (x, y) in the plane that satisfies the equation $y = f(x)$. If the function is defined for only a few input values, then the graph of the function is only a few points, where the x-coordinate of each point is an input value and the y-coordinate of each point is the corresponding output value. For example, the black dots on the graph in Figure 2.4.10 tell us that $f(0) = 2$ and $f(6) = 1$. However, the set of all points (x, y) satisfying $y = f(x)$ is a curve. The curve shown includes $(0, 2)$ and $(6, 1)$ because the curve passes through those points

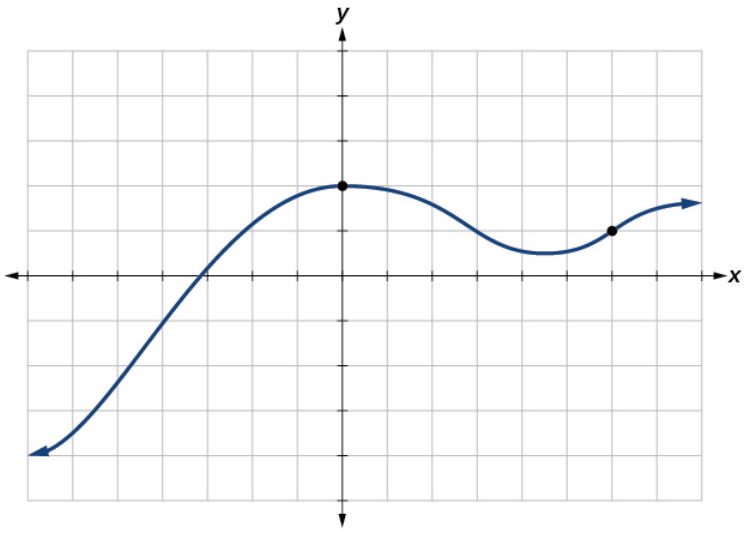


Figure 2.4.14: Graph of a polynomial.

The vertical line test can be used to determine whether a graph represents a function of x. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function of x because a function has only one output value for each input value. See Figure 2.4.15.

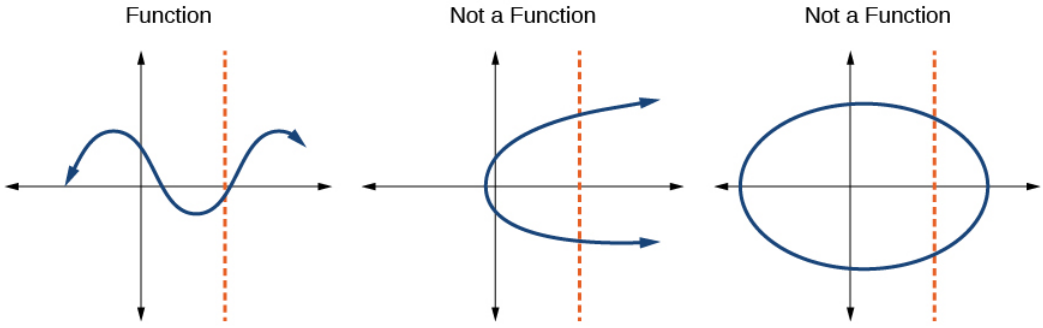


Figure 2.4.15: Three graphs visually showing what is and is not a function.

Example 2.4.5

Which of the graphs in Figure 2.4.16 represent(s) a function $y = f(x)$?

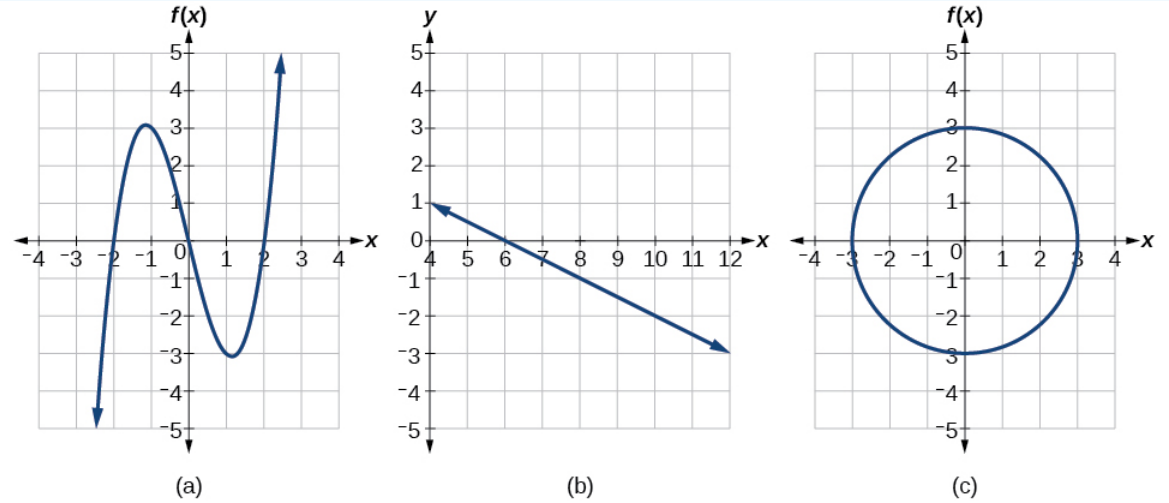


Figure 2.4.16: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

Solution

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of Figure 2.4.16. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most x-values, a vertical line would intersect the graph at more than one point, as shown in Figure 2.4.17.

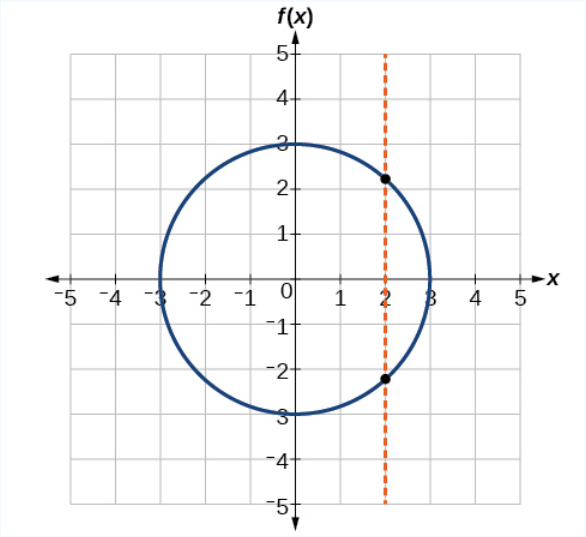


Figure 2.4.17: Graph of a circle.

You Try 2.4.3

Does the graph in Figure 2.4.18 represent a function?

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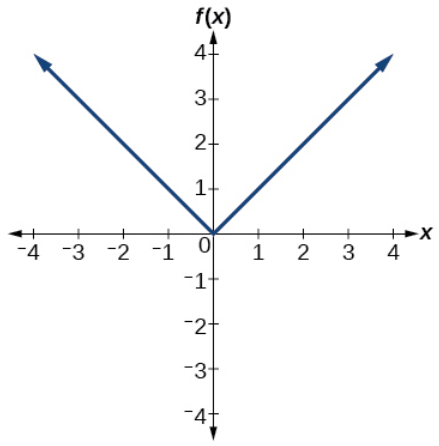


Figure 2.4.18: Graph of absolute value function.

Answer

yes

Determining if an Equation Represents a Function

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula. How do we then determine if an equation represents a function?

Example 2.4.6

Does the equation $x^2+y^2=1$ represent a function of x? If so, express the relationship as a function $y=f(x)$.

Solution

First we subtract x^2 from both sides.

$$y^2=1-x^2$$

We now try to solve for y in this equation.

$$y=\pm\sqrt{1-x^2}$$

$$\text{so, } y=\sqrt{1-x^2} \text{ and } y=-\sqrt{1-x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y=f(x)$.

Note: Most graphing calculators are designed to only graph functions. To graph this equation in a graphing calculator properly, you must input both formulas into the calculator

Example: Y1=$\sqrt{1-x^2}$ and Y2=$-\sqrt{1-x^2}$

You Try 2.4.2

Does the equation $x^3-8y=1$ represent a function of x? If so, express the relationship as a function $y=f(x)$.

Answer

$$y=f(x)=\frac{x^3}{8}-\frac{1}{8} \text{ or } y=f(x)=\frac{x^3-1}{8}$$

Q & A

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?

Yes, this can happen. For example, given the equation $x = y + 2^y$, if we want to express y as a function of x, there is no simple algebraic formula involving only x that equals y. However, each x does determine a unique value for y, and there are mathematical procedures by which y can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for y as a function of x, even though the formula cannot be written explicitly.

Evaluating Functions Using Graphs

We know that the graph of f pictured in Figure 2.4.19 is the graph of a function of x. We know this because no vertical line will cut the graph of f more than once.

We earlier defined the graph of f as the set of all ordered pairs $(x, f(x))$, so that x is in the domain of f. Consequently, if we select a point P on the graph of f, as in Figure 2.4.19(a), we label the point P(x, f(x)). However, we can also label this point as $P(x, y)$ since y = f(x), as shown in Figure 2.4.19(b).

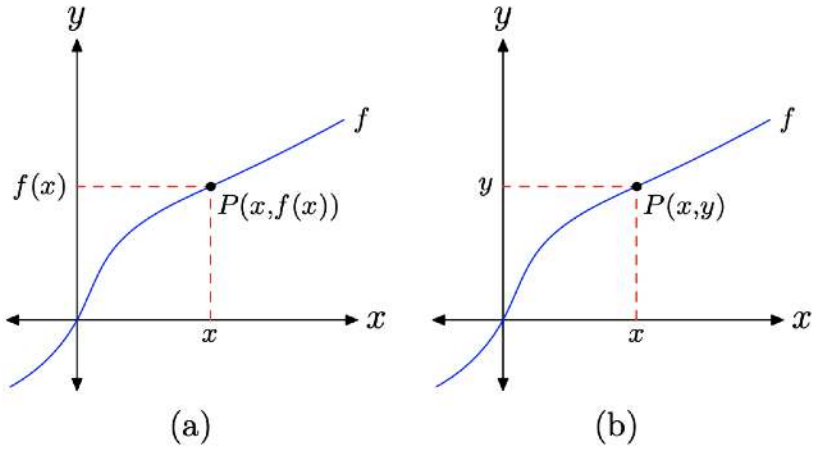


Figure 2.4.19. Reading the graph of a function.

Example 2.4.7

Given the graph of a function f in Figure 2.4.20(a), find f(4).

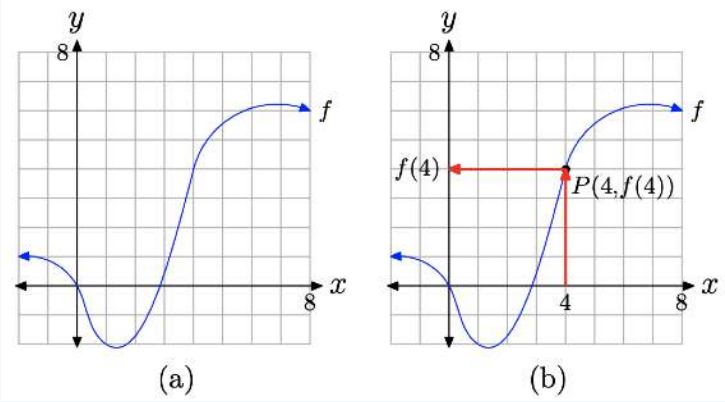


Figure 2.4.20. Finding the value of f(4).

Solution

Because f(4) represents the y-value that is paired with the x-value of 4, we first locate 4 on the x-axis, as shown in Figure 2.4.20(b). We then draw a vertical arrow until we intercept the graph of f at the point P(4, f(4)). Finally, we draw a horizontal

arrow from the point P until we intercept the y-axis. The projection of the point P onto the y-axis is the value of f(4).

Because we have a grid that shows a scale on each axis, we can approximate the value of f(4). It would appear that the y-value of point P is approximately 4. Thus, $f(4) \approx 4$.

Let's reverse this process in another example.

Example 2.4.8

Given the graph of a function, f, in Figure 2.4.21(a), for what value of x does f(x) = −4?

Solution

This time, in the equation $f(x) = -4$, we're given a y-value equal to −4. We must reverse the process used in the previous example. We first locate the y-value −4 on the y-axis, then draw a horizontal arrow until we intercept

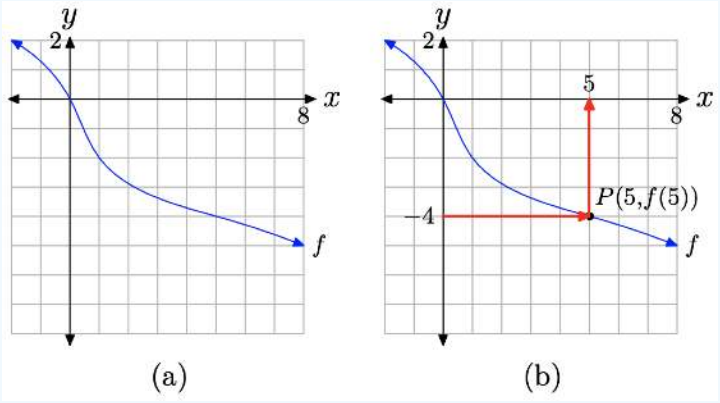


Figure 2.4.21. Finding x so that $f(x) = -4$.

the graph of f at P, as shown in Figure 2.4.21(b). Finally, we draw a vertical arrow from the point P until we intercept the x-axis. The projection of the point P onto the x-axis is the solution of $f(x) = -4$.

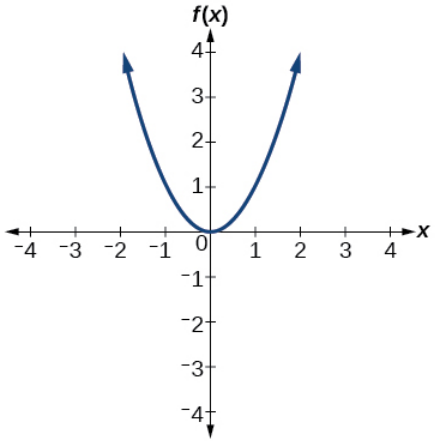
Because we have a grid that shows a scale on each axis, we can approximate the x-value of the point P. It seems that $x \approx 5$. Thus, we label the point P(5, f(5)), and the solution of $f(x) = -4$ is approximately $x \approx 5$.

This solution can easily be checked by computing f(5). Simply start with 5 on the x-axis, then reverse the order of the arrows shown in Figure (\PageIndex{21}\)(b). You should wind up at −4 on the y-axis, demonstrating that $f(5) \approx -4$.

Basic Functions

Earlier it was mentioned that there are some functions we commonly work with. These functions are commonly referred to as *parent functions*, *basic functions*, or *tool box functions*. We want to be familiar with these functions so let's take a moment to define what they are.

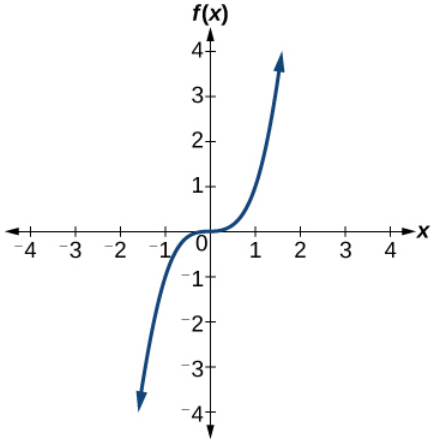
For the **basic quadratic function** $f(x) = x^2$, the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 2.4.22: Quadratic function $f(x) = x^2$.

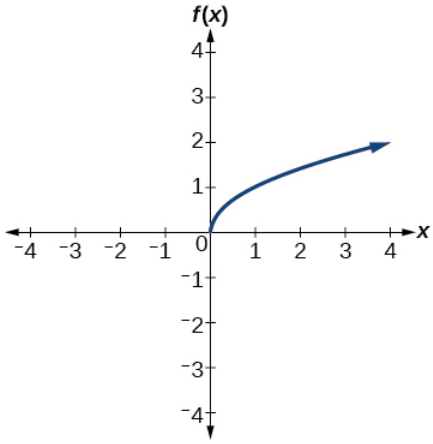
For the **basic cubic function** $f(x) = x^3$, the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.



Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$

Figure 2.4.23: Cubic function $f(x) - x^3$.

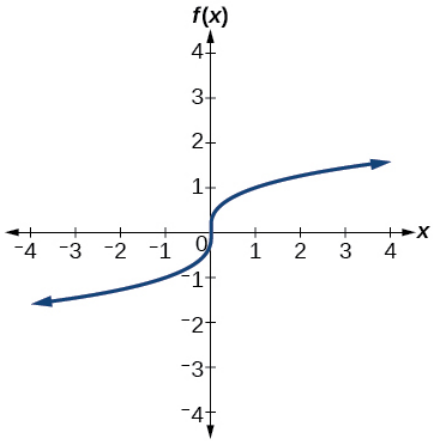
For the **basic square root function** $f(x) = \sqrt{x}$, we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number x is defined to be positive, even though the square of the negative number $-\sqrt{x}$ also gives us x.



Domain: [0, ∞)
Range: [0, ∞)

Figure 2.4.24: Square root function $f(x) = \sqrt{x}$.

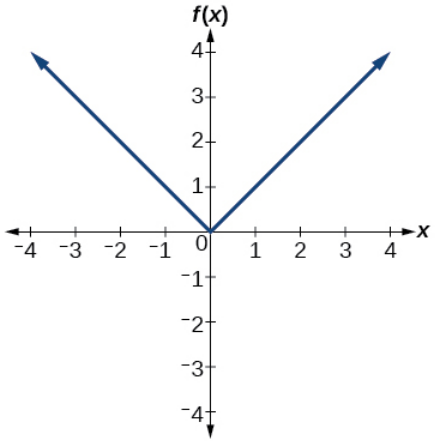
For the **basic cube root function** $f(x) = \sqrt[3]{x}$, the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).



Domain: (−∞, ∞)
Range: (−∞, ∞)

Figure 2.4.25: Cube root function $f(x) = \sqrt[3]{x}$.

For the **basic absolute value function** $f(x) = |x|$, there is no restriction on x. However, because absolute value is defined as a distance from 0, the output can only be greater than or equal to 0.



Domain: $(-\infty, \infty)$
Range: $[0, \infty)$

Figure 2.4.26: Absolute function $f(x) = |x|$.

All the basic functions listed above have a point at (0,0) and (1,1). This will be useful for later.

Graphing Piecewise Functions

Previously, we learned how to evaluate piecewise defined functions. Next, let's focus on how to graph them.

Given a piecewise function, sketch a graph.

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Note: It is important to not just plot some points when graphing piecewise functions.

Example 2.4.9

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 3 & \text{if } 1 < x \leq 2 \\ x & \text{if } x > 2 \end{cases}$$

Solution

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the piece we need for the indicated domain of the piecewise function we are graphing. At the endpoints of the domain interval, draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 2.4.27 shows the three components of the piecewise function graphed on separate coordinate systems.

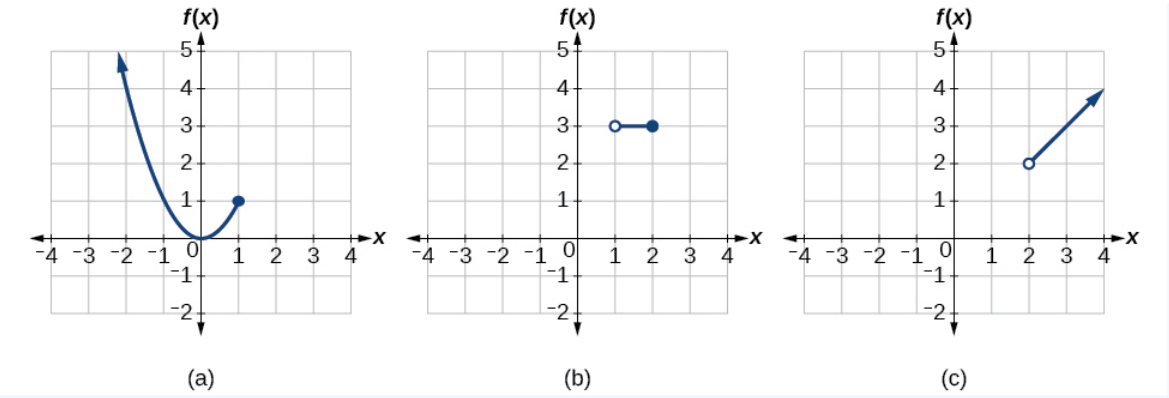


Figure 2.4.27: Graph of each part of the piece-wise function f(x)

(a) $f(x) = x^2$ if $x \leq 1$; (b) $f(x) = 3$ if $1 < x \leq 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we then combine them in the same coordinate plane to form the graph of the piecewise function. See Figure 2.4.28

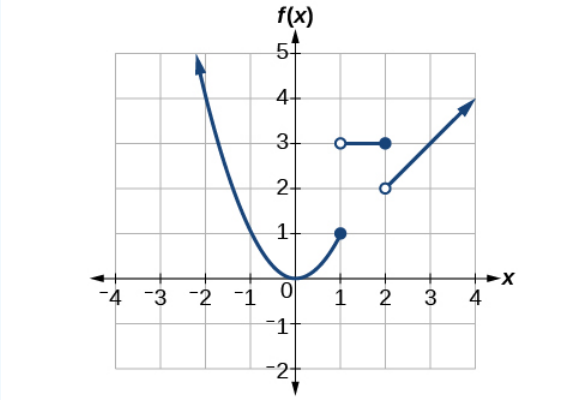


Figure 2.4.28: Graph of the entire function.

Analysis

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

? You Try 2.4.10

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Answer

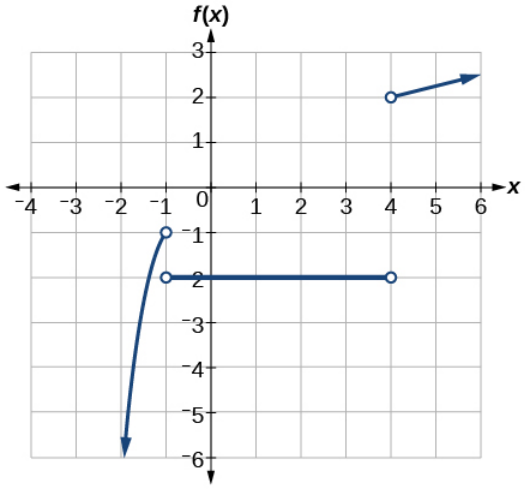


Figure 2.4.29

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2.5: Finding Information About Functions From Graphs

Learning Objectives

- Use a graph to find domain and range.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graphing calculator to describe behavior of functions.

We have been learning a lot about functions in this course. Understanding their unique features and how they behave is helpful when determining which functions will be the best models when describing real-life situations. In this section, we will focus on what information we can gain about functions from a graph.

Finding Domain and Range from Graphs

We have learned how to find the domain of functions algebraically. In many cases, find range algebraically is not so simple as there are no systematic approaches since the output behavior of functions can vary a lot depending on the equation of the functions. Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the x-axis. The range is the set of possible output values, which are shown on the y-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see (as noted by arrows on the ends of the graph), the domain and range may be greater than the visible values shown in the coordinate system. See Figure 2.5.1.

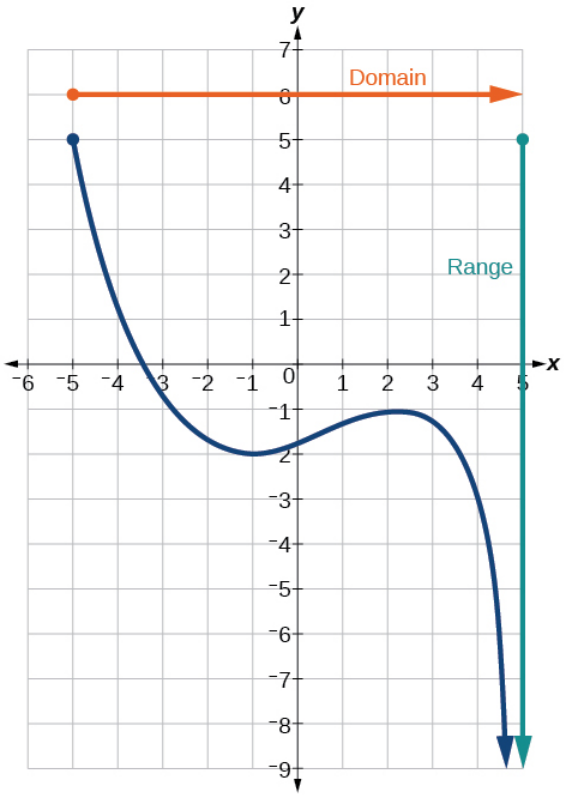


Figure 2.5.1: Graph of a polynomial that shows the x-axis is the domain and the y-axis is the range

We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is $[-5, \infty)$. The vertical extent of the graph is all range values 5 and below, so the range is $(-\infty, 5]$. Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

Example 2.5.1

Find the domain and range of the function f whose graph is shown in Figure 2.5.2.

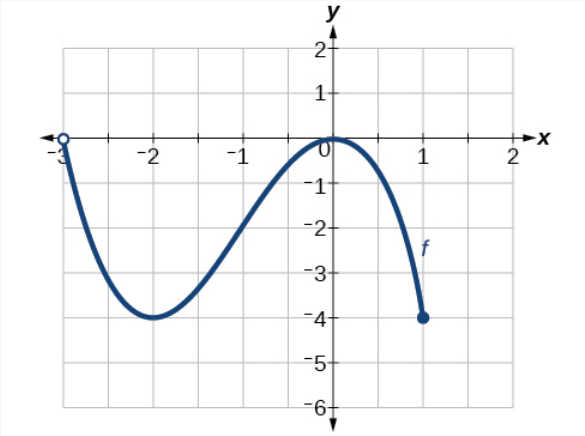


Figure 2.5.2: Graph of a function from (-3, 1].

Solution

We can observe that the horizontal extent of the graph is –3 to 1, so the domain of f is $(-3, 1]$.

The vertical extent of the graph is 0 to –4 and it appears there is a point at $(0, 0)$ so the range is $[-4, 0]$.

Example 2.5.2

Find the domain and range of the function f whose graph is shown in Figure 2.5.3.

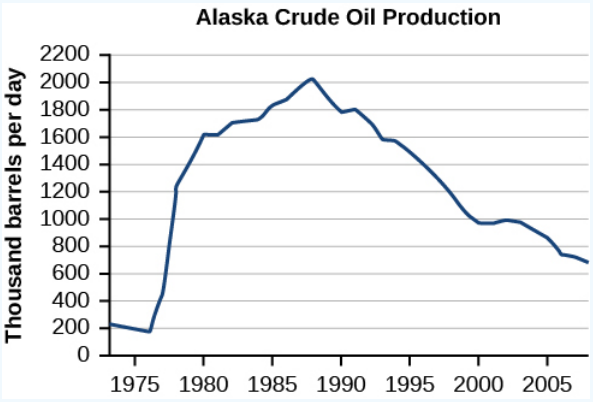


Figure 2.5.3: Graph of the Alaska Crude Oil Production where the y-axis is thousand barrels per day and the -axis is the years(credit: modification of work by the U.S. Energy Information Administration)

Solution

The input quantity along the horizontal axis is “years,” which we represent with the variable t for time. The output quantity is “thousands of barrels of oil per day,” which we represent with the variable b for barrels. The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as $1973 \le t \le 2008$ and the range as approximately $180 \le b \le 2010$.

In interval notation, the domain is $[1973, 2008]$, and the range is about $[180, 2010]$. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

Using a Graphing Calculator to Find Domain and Range

We’ve learned how to find the domain and range of a function by looking at a graph. If we define a function by means of an equation, such as $f(x) = \sqrt{4 - x}$, then we should be able to capture the domain and range of f from its graph, provided, of course, that we can draw the graph of f. If we don't know what a graph looks like, we can use graphing technology.

Example 2.5.3

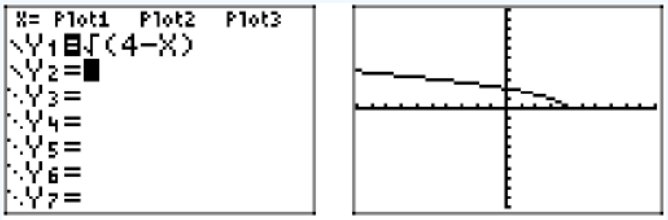
Use a graphing calculator to estimate the domain and range using interval notation for the function defined by

$$f(x) = \sqrt{4 - x}$$

.

Solution

Load the expression defining f into the Y= menu, as shown in Figure 2.5.4(a). Select 6:ZStandard from the ZOOM menu to produce the graph of f shown in Figure 2.5.5(b).



(a) (b)

Figure 2.5.4. Sketching the graph of $f(x) = \sqrt{4 - x}$.

Next, project each point on the graph of f onto the x-axis, as shown in Figure 2.5.4(b) to estimate the domain. Note that we've made two assumptions about the graph of f.

1. At the left end of the graph in Figure 2.5.4(b), we assume that the graph of f continues upward and to the left indefinitely. Hence, the "shadow" or projection onto the x-axis will move indefinitely to the left. This is indicated by attaching an arrowhead to the left-hand end of the region that "lies in shadow" on the x-axis, as shown in Figure 2.5.4(b).
2. We also assume that the right end of the graph ends at the point $(4, 0)$. This accounts for the "filled dot" when this point on the graph of f is projected onto the x-axis.

Note that the "shadow" or projection onto the x axis in Figure 2.5.4(b) includes all values of x less than or equal to 4. Thus, the domain of f is

$$\text{Domain of } f = (-\infty, 4] = \{x : x \leq 4\}$$

We can also see this result by considering how f is defined.

$$f(x) = \sqrt{4 - x}$$

Recall that we earlier defined the domain of f as the set of "permissible" x-values. In this case, it is impossible to take the square root of a negative number, so we must be careful selecting the x-values we use in this rule. Note that $x = 4$ is allowable, as

$$f(0) = \sqrt{4 - 4} = \sqrt{0} = 0$$

However, numbers larger than 4 cannot be used in this rule. For example, consider what happens when we attempt to use $x = 5$.

$$f(x) = \sqrt{4 - 5} = \sqrt{-1}$$

We'll leave it to our readers to test other values of x that are less than 4. They will also produce real answers when they are input into the rule $f(x) = \sqrt{4 - x}$. Note that this also verifies our earlier conjecture that the "shadow" or projection shown in Figure 2.5.4(b) continues indefinitely to the left.

Instead of "guessing and checking," we can speed up the analysis of the domain of $f(x) = \sqrt{4 - x}$ by noting that the expression under the radical must not be a negative number. Hence, $4 - x$ must either be greater than or equal to zero. This argument produces an inequality that is easily solved for x.

$$4 - x \geq 0$$
$$-x \geq -4$$
$$x \leq 4$$

This last result verifies that the domain of f is all values of x that are less than or equal to 4, which is in complete agreement with the "shadow" or projection onto the x-axis shown in Figure 2.5.4(b).

Simliary, to determine the range of f, we project each point on the graph of f onto the y-axis. We estimate the range as

$$\text{Range of } f = [0, \infty)$$

Using a Graph to Determine Where a Function is Increasing, Decreasing, or Constant

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. Figure 2.5.5 shows examples of increasing and decreasing intervals on a function.

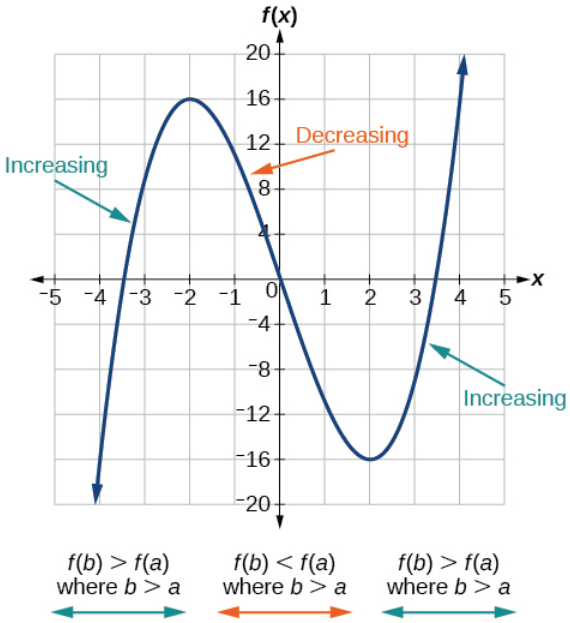


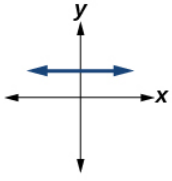
Figure 2.5.5: The function $f(x) = x^3 - 12x$ is increasing on $(-\infty, -2) \cup (2, \infty)$ and is decreasing on $(-2, 2)$.

Increasing and Decreasing Function Behavior

- A function f is **increasing** on an open interval if $f(b) > f(a)$ for every a, b within the interval where $b > a$.
- A function f is **decreasing** on an open interval if $f(b) < f(a)$ for every a, b within interval where $b > a$.
- A function f is **constant** on an open interval otherwise.

Increasing and Decreasing Behavior for Basic Functions

We will now return to our basic toolkit functions and discuss their graphical behavior in Figure 2.5.6. Note that this list includes some other functions that were not discussed in the previous section.

Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function	Increasing	

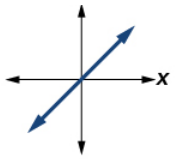
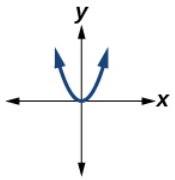
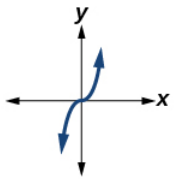
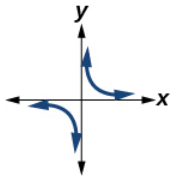
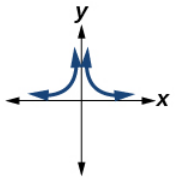
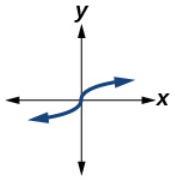
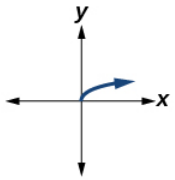
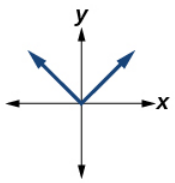
$f(x) = x$		
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \|x\|$	Increasing on $(0, \infty)$ Decreasing on $(\infty, 0)$	

Figure 2.5.6

Local Extrema

While some functions are increasing (or decreasing) over their entire domain, many are not. A **continuous function** is a function whose graph can be drawn in one movement without having to lift your pencil. If a function is continuous, a value of the input where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a **local maximum**. If a function has more than one, we say it has local maxima. Similarly, a value of the input where a function

changes from decreasing to increasing as the input variable increases is called a **local minimum**. The plural form is "local minima." Together, local maxima and minima are called **local extrema** of the function. Local extrema may are be referred to as **relative extrema**.

Clearly, a function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of local extrema, because any given local extrema as defined here is not necessarily the highest maximum or lowest minimum in the function's entire domain.

For the function whose graph is shown in Figure 2.5.7, the local maximum is 16, and it occurs at $x = -2$. The local minimum is -16 and it occurs at $x = 2$.

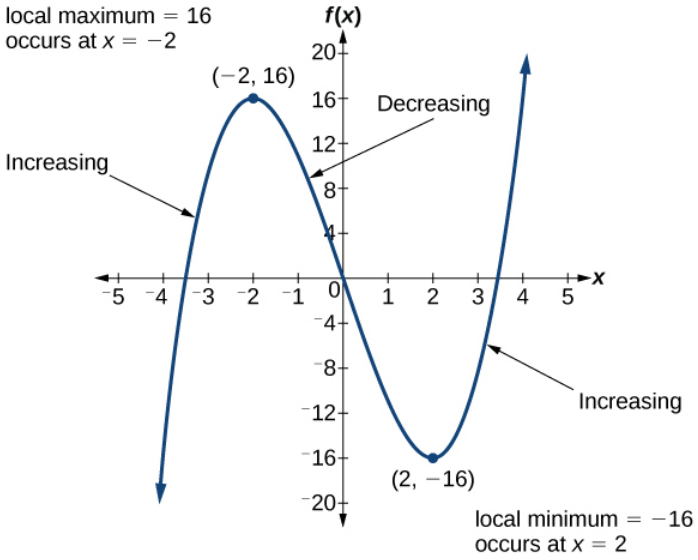


Figure 2.5.7: Graph of a polynomial that shows the increasing and decreasing intervals and local maximum.maximum

To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. Figure 2.5.8 illustrates these ideas for a local maximum.

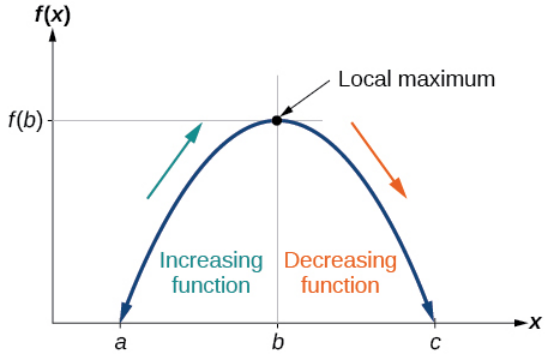


Figure 2.5.8: Definition of a local maximum

These observations lead us to a formal definition of local extrema.

Local Extrema

- A function f has a local maximum at a point b in an open interval (a, c) if $f(b)$ is greater than or equal to $f(x)$ for every point x (x does not equal b) in the interval.
- A function f has a local minimum at a point b in (a, c) if $f(b)$ is less than or equal to $f(x)$ for every x (x does not equal b) in the interval.

Example 2.5.4

Given the function $p(t)$ in Figure 2.5.9, identify the intervals on which the function appears to be increasing.

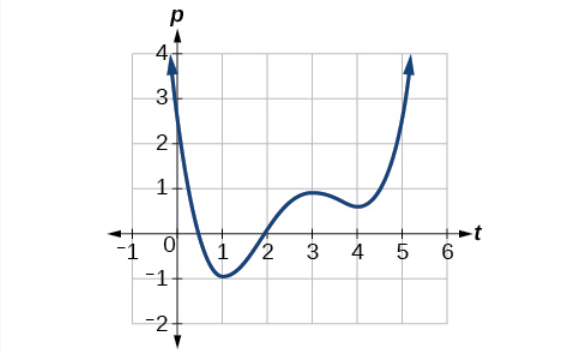


Figure 2.5.9: Graph of a polynomial.

Solution

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, we would say the function appears to be increasing on the interval $(1, 3)$ and the interval $(4, \infty)$.

Analysis

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t = 1$, $t = 3$, and $t = 4$. These points are the local extrema (two minima and a maximum).

Example 2.5.5

Graph the function $f(x) = \frac{2}{x} + \frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

Solution

Using technology, we find that the graph of the function looks like that in Figure 2.5.10. It appears there is a low point, or local minimum, between $x = 2$ and $x = 3$, and a mirror-image high point, or local maximum, somewhere between $x = -3$ and $x = -2$

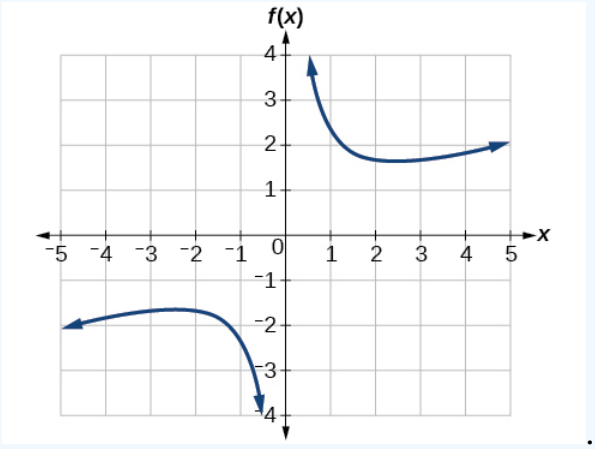


Figure 2.5.10: Graph of a rational function.

Analysis

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. Figure 2.5.11 provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

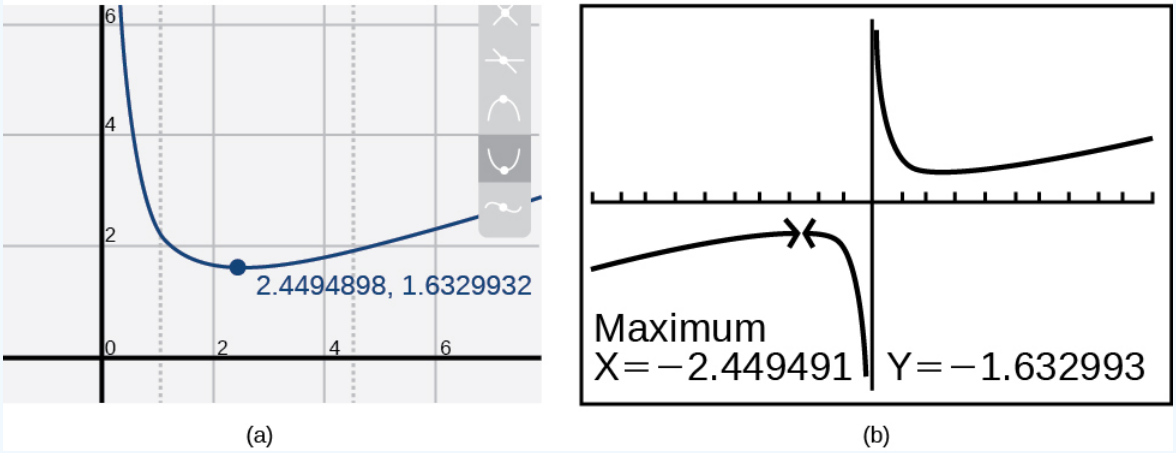


Figure 2.5.11: Graph of the reciprocal function on a graphing calculator.

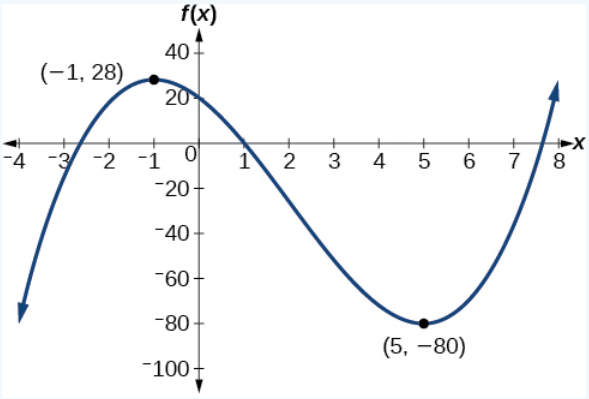
Based on these estimates, the function is increasing on the interval $(-\infty, -2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact location of the extrema is at $\pm\sqrt{6}$, but determining this requires calculus.)

You Try 2.5.1

Graph the function $f(x) = x^3 - 6x^2 - 15x + 20$ in a graphing calculator to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

Solution

The local maximum appears to occur at $(-1, 28)$, and the local minimum occurs at $(5, -80)$. The function is increasing on $(-\infty, -1) \cup (5, \infty)$ and decreasing on $(-1, 5)$.



Graph of a polynomial with a local maximum at (-1, 28) and local minimum at (5, -80).

Example 2.5.6

For the function f whose graph is shown in Figure 2.5.12, find all local maxima and minima.

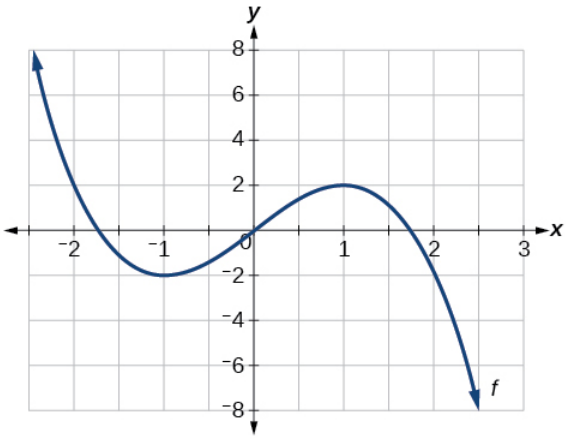


Figure 2.5.12: Graph of a polynomial.

Solution

Observe the graph of f. The graph attains a local maximum at $x = 1$ because it is the highest point in an open interval around $x = 1$.The local maximum is the y-coordinate at $x = 1$, which is 2.

The graph attains a local minimum at $x = -1$ because it is the lowest point in an open interval around $x = -1$. The local minimum is the y-coordinate at $x = -1$, which is -2.

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2.6: Average Rate of Change

Learning Objectives

- Find the average rate of change of a function.

Since functions represent how an output quantity varies with an input quantity, it is natural to ask about the rate at which the values of the function are changing. For example, the function $C(t)$ below gives the average cost, in dollars, of a gallon of gasoline t years after 2000.

t	2	3	4	5	6	7	8	9
$C(t)$	1.47	1.69	1.94	2.30	2.51	2.64	3.01	2.14

If we were interested in how the gas prices had changed between 2002 and 2009, we could compute that the cost per gallon had increased from \$1.47 to \$2.14, an increase of \$0.67. While this is interesting, it might be more useful to look at how much the price changed *per year*. You are probably noticing that the price didn't change the same amount each year, so we would be finding the **average rate of change** over a specified amount of time.

The gas price increased by \$0.67 from 2002 to 2009, over 7 years, for an average of $\frac{\$0.67}{7 years} \approx 0.096$ dollars per year. On average, the price of gas increased by about 9.6 cents each year.

Rate of Change

A **rate of change** describes how the output quantity changes in relation to the input quantity. The units on a rate of change are output units per input units.

Some other examples of rates of change would be quantities like:

- A population of rats increases by 40 rats per week
- A barista earns \$9 per hour (dollars per hour)
- A farmer plants 60,000 onions per acre
- A car can drive 27 miles per gallon
- A population of grey whales decreases by 8 whales per year
- The amount of money in your college account decreases by \$4,000 per quarter

Average Rate of Change

The **average rate of change** between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\text{Average rate of change} = \frac{\text{Change in Output}}{\text{Change in Input}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 2.6.1

Using the cost-of-gas function from earlier, find the average rate of change between 2007 and 2009.

Solution

From the table, in 2007 the cost of gas was \$2.64. In 2009 the cost was \$2.14.

The input (years) has changed by 2. The output has changed by \$2.14 - \$2.64 = -0.50.

The average rate of change is then $\frac{-\$0.50}{2 years}$ = -0.25 dollars per year

You Try 2.6.1

Using the same cost-of-gas function, find the average rate of change between 2003 and 2008

Answer

$\frac{\$3.01 - \$1.69}{5\ years} = \frac{\$1.32}{5\ years} = 0.264$ dollars per year.

Notice that in the last example the change of output was *negative* since the output value of the function had decreased. Correspondingly, the average rate of change is negative.

Example 2.6.2

Given the function $g(t)$ shown below in Figure 2.6.1, find the average rate of change on the interval [0, 3].

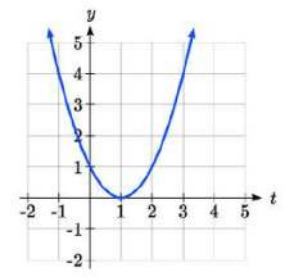


Figure 2.6.1: Graph of $g(t)$

Solution

At $t = 0$, the graph shows $g(0) = 1$

At $t = 3$, the graph shows $g(3) = 4$

The output has changed by 3 while the input has changed by 3, giving an average rate of change of:

$$\frac{4-1}{3-0} = \frac{3}{3} = 1$$

Example 2.6.3

On a road trip, after picking up your friend who lives 10 miles away, you decide to record your distance from home over time. Find your average speed over the first 6 hours.

t (hours)	0	1	2	3	4	5	6	7
$D(t)$ (miles)	10	55	90	153	214	240	292	300

Solution

Here, your average speed is the average rate of change.

You traveled 282 miles in 6 hours, for an average speed of

$$\frac{292-10}{6-0} = \frac{282}{6} = 47 \text{ miles per hour}$$

We can more formally state the average rate of change calculation using function notation.

Average Rate of Change Using Function Notation

Given a function $f(x)$, the average rate of change on the interval [a, b] is

$$\text{Average rate of change} = \frac{\text{Change of Output}}{\text{Change of Input}} = \frac{f(b)-f(a)}{b-a}$$

Example 2.6.4

Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval [2, 4].

Solution

We can start by computing the function values at each endpoint of the interval

$$f(2) = 2^2 - \frac{1}{2} = 4 - \frac{1}{2} = \frac{7}{2}$$

$$f(4) = 4^2 - \frac{1}{4} = 16 - \frac{1}{4} = \frac{63}{4}$$

Now computing the average rate of change

$$\text{Average rate of change} = \frac{f(4)-f(2)}{4-2} = \frac{\frac{63}{4}-\frac{7}{2}}{4-2} = \frac{\frac{49}{4}}{2} = \frac{49}{8}$$

You Try 2.6.2

Find the average rate of change of $f(x) = x - 2\sqrt{x}$ on the interval [1, 9].

Answer

$$\text{Average rate of change} = \frac{f(9)-f(1)}{9-1} = \frac{(9-2\sqrt{9})-(1-2\sqrt{1})}{9-1} = \frac{(3)-(-1)}{9-1} = \frac{4}{8} = \frac{1}{2}$$

Example 2.6.5

The magnetic force F, measured in Newtons, between two magnets is related to the distance between the magnets d, in centimeters, by the formula $F(d) = \frac{2}{d^2}$. Find the average rate of change of force if the distance between the magnets is increased from 2 cm to 6 cm.

Solution

We are computing the average rate of change of $F(d) = \frac{2}{d^2}$ on the interval [2, 6].

Average rate of change = $\frac{F(6)-F(2)}{6-2}$

Evaluating the function

$$\frac{F(6)-F(2)}{6-2} =$$

$$= \frac{\frac{2}{6^2}-\frac{2}{2^2}}{6-2}$$

Simplifying

$$= \frac{\frac{2}{36}-\frac{2}{4}}{4}$$

Combining the numerator terms

$$= \frac{\frac{-16}{36}}{4}$$

Simplifying further

$$= \frac{-1}{9}$$

Newtons per centimeter

This tells us the magnetic force decreases, on average, by 1/9 Newtons per centimeter over this interval.

Example 2.6.6

Find the average rate of change of $g(t) = t^2 + 3t + 1$ on the interval $[0, a]$. Your answer will be an expression involving a.

Solution

Using the average rate of change formula

$$\frac{g(a)-g(0)}{a-0}$$

Evaluating the function

$$\frac{(a^2+3a+1)-(0^2+3(0)+1)}{a-0}$$

Simplifying

$$\frac{a^2+3a+1-1}{a}$$

Simplifying further, and factoring

$$\frac{a(a+3)}{a}$$

Cancelling the common factor a

$$a+3$$

This result tells us the average rate of change between t = 0 and any other point t = a. For example, on the interval [0, 5], the average rate of change would be 5+3 = 8.

? You Try 2.6.3

Find the average rate of change of $f(x)=x^3+2$ on the interval $[a, a+h]$.

Answer

$$\frac{f(a+h)-f(a)}{(a+h)-a}=\frac{((a+h)^3+2)-(a^3+2)}{h}=\frac{a^3+3a^2h+3ah^2+h^3+2-a^3-2}{h}=\frac{3a^2h+3ah^2+h^3}{h}=\frac{h(3a^2+3ah+h^2)}{h}=3a^2+3ah+h^2$$

Caution!

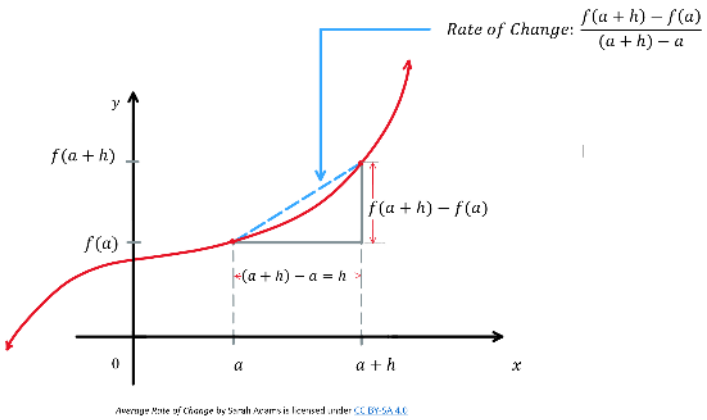
Do remember to watch out for common arithmetic errors: $(a+h)^3 \neq a^3+h^3$.

Instead, expand the product to help ensure the multiplication is done correctly: $(a+h)^3=(a+h)(a+h)(a+h)$

You may be wondering why we would want to find the average rate of change on an interval such as $[a, a+h]$. The value "a" represents an arbitrary x-coordinate for a point. The value "a+h" represents another arbitrary x-coordinate of another point. The value "h" represents the horizontal distance between these two points. The average rate of change on this interval, $[a, a+h]$, is

$$\frac{f(a+h)-f(a)}{(a+h)-a}$$

which is visually the slope of a **secant line** passing through these two points on the graph of the function, as shown in Figure 2.6.2 below. In calculus, you will investigate what happens to the slope of this secant line as you change the distance "h" between the two points. By shrinking this distance we can describe instantaneous rate of change which is a very useful topic in many real-life situations. Stay tuned for more about this in your future calculus course!



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Figure 2.6.2: Rate of Change Image with a and h (Sarah Adams: CC-BY-SA 4.0)

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2.7: Transformations of Functions

Learning Objectives

- Identify transformations of functions.
- Graph functions using transformations.
- Find functions that model transformed graphs.
- Determine whether a function is even, odd, or neither from its graph.

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.



Figure 2.7.1: (credit: "Misko"/Flickr)

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the basic functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + c$, the function $f(x)$ is shifted vertically c units. See Figure 2.7.2 for an example.

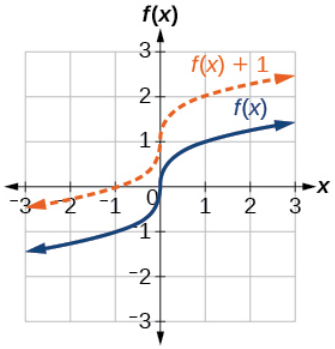


Figure 2.7.2: Vertical shift by $c = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + c$ is equivalent to $y + c$. Every unit of y is replaced by $y + c$, so the y-value increases or decreases depending on the value of c. The result is a shift upward or downward.

Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + c$, where c is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by c units. If c is positive, the graph will shift up. If c is negative, the graph will shift down.

Example 2.7.1

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 2.7.3 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

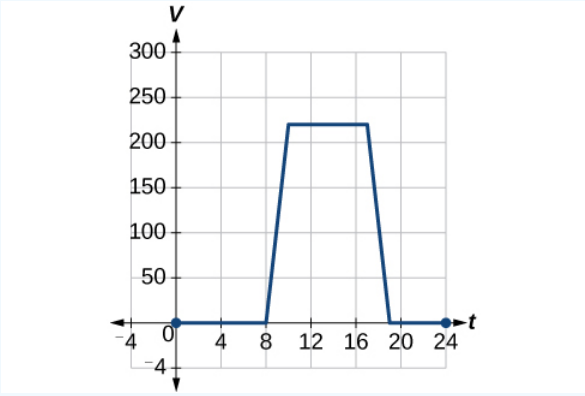


Figure 2.7.3

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 2.7.4.

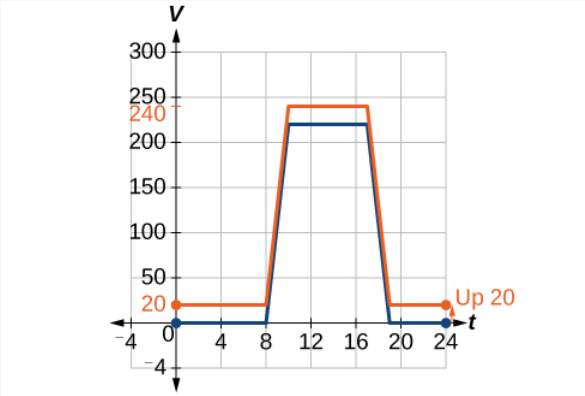


Figure 2.7.4

Notice that in Figure 2.7.4, for each input value, the output value has increased by 20, so if we call the new function $S(t)$,we could write

$$S(t) = V(t) + 20$$

This notation tells us that, for any value of t,$S(t)$ can be found by evaluating the function V at the same input and then adding 20 to the result. This defines S as a transformation of the function V, in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 2.7.1.

Table 2.7.1

t	0	8	10	17	19	24
$V(t)$	0	0	220	220	0	0

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$S(t)$	20	20	240	240	20	20

Identifying Horizontal Shifts

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a **horizontal shift**, shown in Figure 2.7.5.

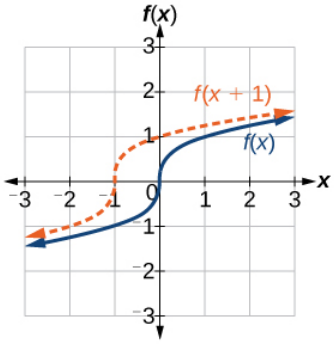


Figure 2.7.5: Horizontal shift of the function $f(x) = \sqrt[3]{x}$. Note that $h = +1$ shifts the graph to the left, that is, towards negative values of x.

For example, if $f(x) = x^2$, then $g(x) = (x-2)^2$ is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in f.

Horizontal Shift

Given a function f, a new function $g(x) = f(x-c)$, where c is a constant, is a **horizontal shift** of the function f. If c is positive, the graph will shift right. If c is negative, the graph will shift left.

Example 2.7.2

Returning to our building airflow example from Figure 2.7.3, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

Solution

We can set $V(t)$ to be the original program and $F(t)$ to be the revised program.

$$V(t) = \text{ the original venting plan}$$

$$F(t) = \text{ starting 2 hrs sooner}$$

In the new graph, at each time, the airflow is the same as the original function V was 2 hours later. For example, in the original function V, the airflow starts to change at 8 a.m., whereas for the function F, the airflow starts to change at 6 a.m. The comparable function values are $V(8) = F(6)$. See Figure 2.7.6. Notice also that the vents first opened to 220ft^2 at 10 a.m. under the original plan, while under the new plan the vents reach 220ft^2 at 8 a.m., so $V(10) = F(8)$.

In both cases, we see that, because $F(t)$ starts 2 hours sooner, $c = -2$. That means that the same output values are reached when $F(t) = V(t-(-2)) = V(t+2)$.

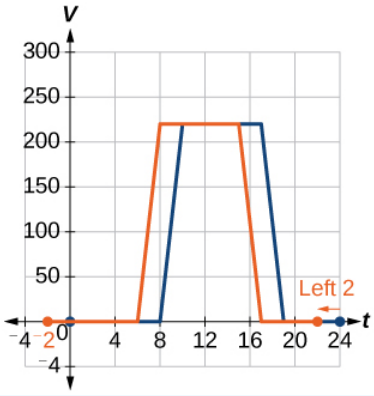


Figure 2.7.6

Analysis

Note that $V(t+2)$ has the effect of shifting the graph to the left.

Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function $F(t)$ uses the same outputs as $V(t)$, but matches those outputs to inputs 2 hours earlier than those of $V(t)$. Said another way, we must add 2 hours to the input of V to find the corresponding output for $F: F(t)=V(t+2)$.

Notice that, with a vertical shift, the input values stay the same and only the output values change. See Table 2.7.2.

Table 2.7.2

t	0	8	10	17	19	24
$t+2$	-2	6	8	15	17	22
$V(t)$	0	0	220	220	0	0
$S(t)$	0	0	220	220	0	0

Example 2.7.3

A graph of $f(x)$ is shown in Figure 2.7.7 which represents a transformation of the basic function $y=x^2$. Find an equation for $f(x)$.

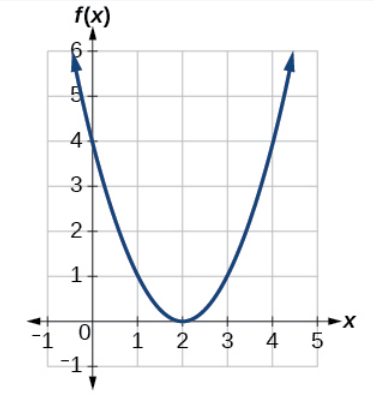


Figure 2.7.7: Graph of a parabola

Solution

Notice that the graph is identical in shape to the $f(x)=x^2$ function, but the x-values are shifted to the right 2 units. The vertex used to be at $(0,0)$, but now the vertex is at $(2,0)$. The graph is the basic quadratic function shifted 2 units to the right, so

$$g(x) = f(x-2)$$

Notice how we must input the value $x = 2$ to get the output value $y = 0$; the x-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the $f(x)$ function to write a formula for $g(x)$ by evaluating $f(x-2)$.

$$\begin{aligned} f(x) &= x^2 \\ g(x) &= f(x-2) \\ g(x) &= f(x-2) = (x-2)^2 \end{aligned}$$

Analysis

To determine whether the shift is $+2$ or -2, consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, $f(0) = 0$. In our shifted function, $g(2) = 0$. To obtain the output value of 0 from the function f, we need to decide whether a plus or a minus sign will work to satisfy $g(2) = f(x-2) = f(0) = 0$. For this to work, we will need to subtract 2 units from our input values.

Example 2.7.4

The function $G(m)$ gives the number of gallons of gas required to drive m miles. Interpret $G(m)+10$ and $G(m+10)$

Solution

$G(m)+10$ can be interpreted as adding 10 to the output, gallons. This is the gas required to drive m miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.

$G(m+10)$ can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than m miles. The graph would indicate a horizontal shift.

You Try 2.7.1

Given the function $f(x) = \sqrt{x}$, graph the original function $f(x)$ and the transformation $g(x) = f(x+2)$ on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?

Answer

The graphs of $f(x)$ and $g(x)$ are shown below. The transformation is a horizontal shift. The function is shifted to the left by 2 units.

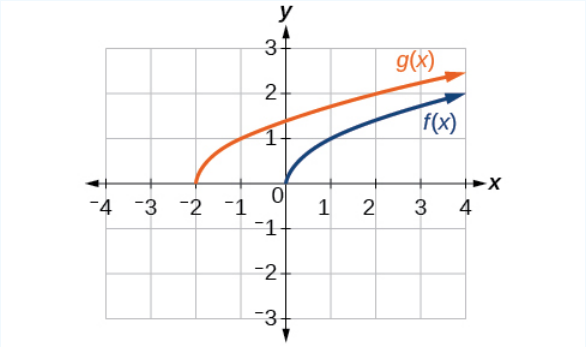


Figure 2.7.8

Combining Vertical and Horizontal Shifts

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the y-coordinates of points and shift the function up or down. Horizontal shifts are inside changes that affect the x-coordinates of points and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

Example 2.7.5

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x+1) - 3$.

Solution

The function f is our basic absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of h has transformed f in two ways: $f(x+1)$ is a change on the inside of the function, giving a horizontal shift left by 1, and the subtraction by 3 in $f(x+1)-3$ is a change to the outside of the function, giving a vertical shift down by 3. The transformation of the graph is illustrated in Figure 2.7.9.

Let us follow one point of the graph of $f(x) = |x|$.

- The point $(0, 0)$ is transformed first by shifting left 1 unit:$(0, 0) \to (-1, 0)$
- The point $(-1, 0)$ is transformed next by shifting down 3 units:$(-1, 0) \to (-1, -3)$

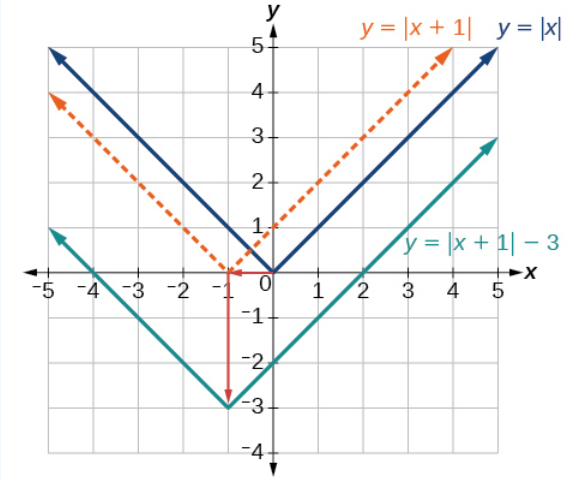


Figure 2.7.9: Graph of an absolute function, $y = |x|$, and how it was transformed to $y = |x+1| - 3$

Figure 2.7.10shows the graph of h.

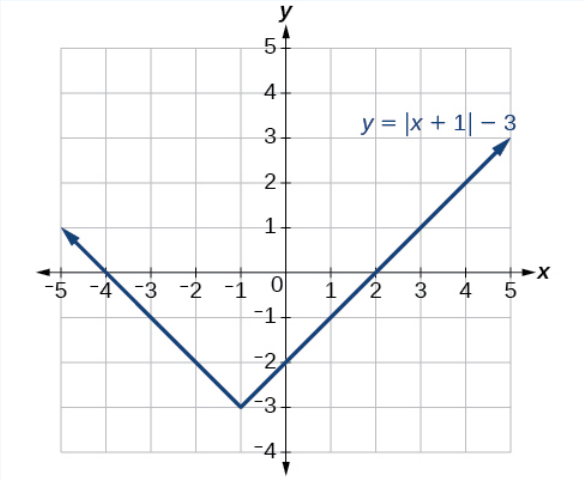


Figure 2.7.10: The final function $y = |x+1| - 3$.

You Try 2.7.2

Given $f(x) = |x|$, sketch a graph of $h(x) = f(x-2) + 4$.

Answer

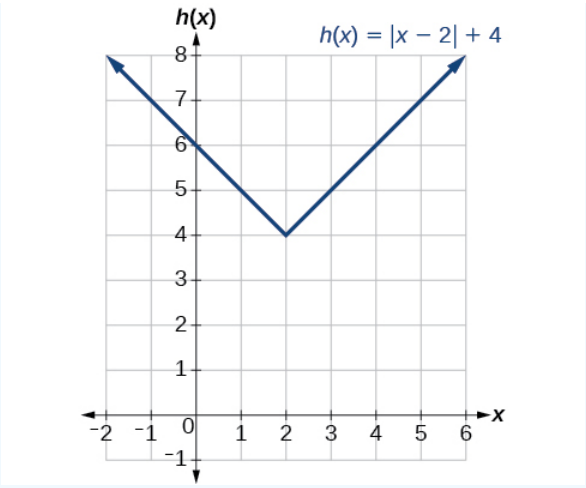


Figure 2.7.11

Example 2.7.6

Write an equation for the function, $h(x)$ that models the graph shown in Figure 2.7.12, which is a transformation of the basic square root function.

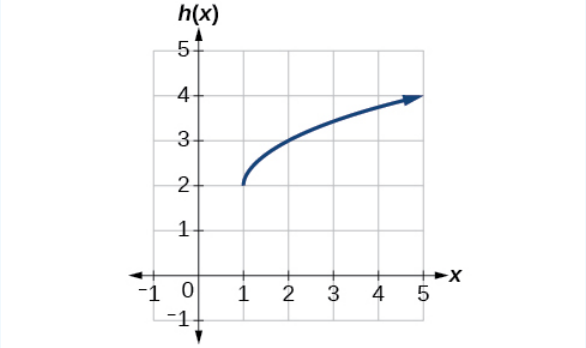


Figure 2.7.12: Graph of a square root function transposed right one unit and up 2.

Solution

The graph of the basic function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as

$$h(x) = f(x-1)+2$$

Using the formula for the square root function, we can write

$$h(x) = \sqrt{x-1}+2$$

Analysis

Note that this transformation has changed the domain and range of the function. This new graph has domain $[1, \infty)$ and range $[2, \infty)$.

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 2.7.13.

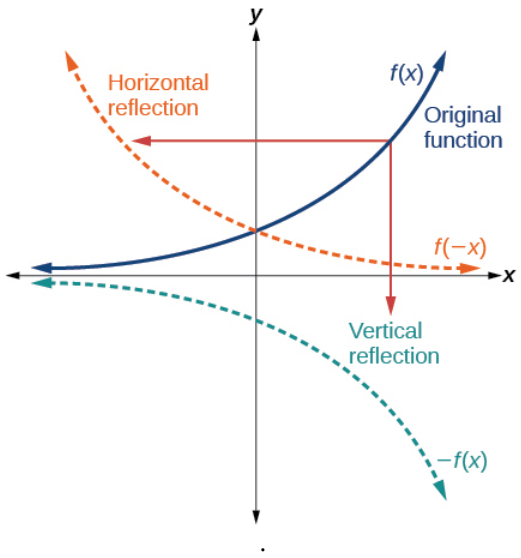


Figure 2.7.13: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. As a result, vertical reflections are also called **x-axis reflections**. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis. As a result, horizontal reflections are also called **y-axis reflections**.

Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

Example 2.7.7

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 2.7.14

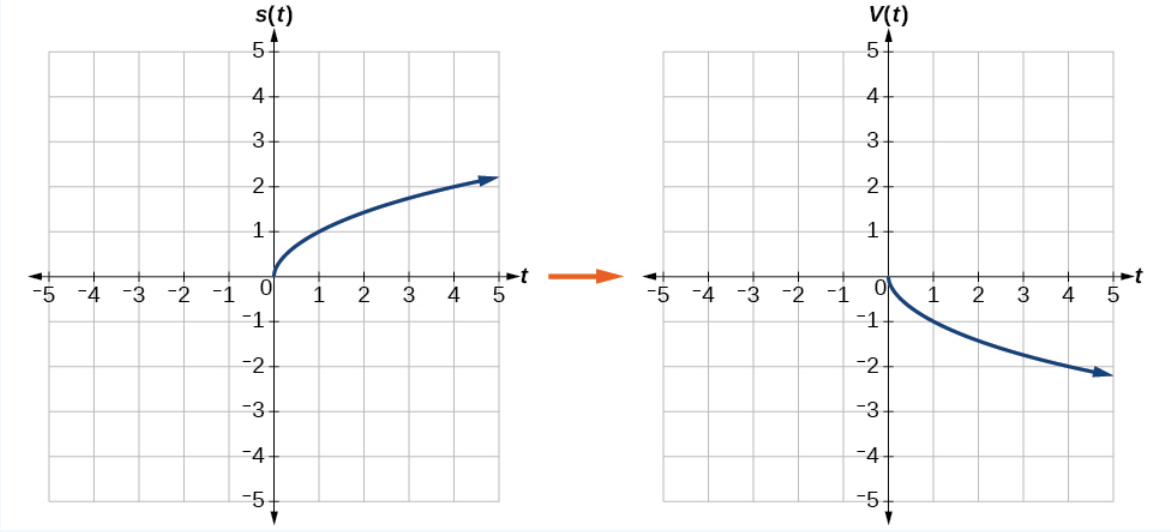


Figure 2.7.14: Graph of the vertical reflection of the square root function.

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Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 2.7.15.

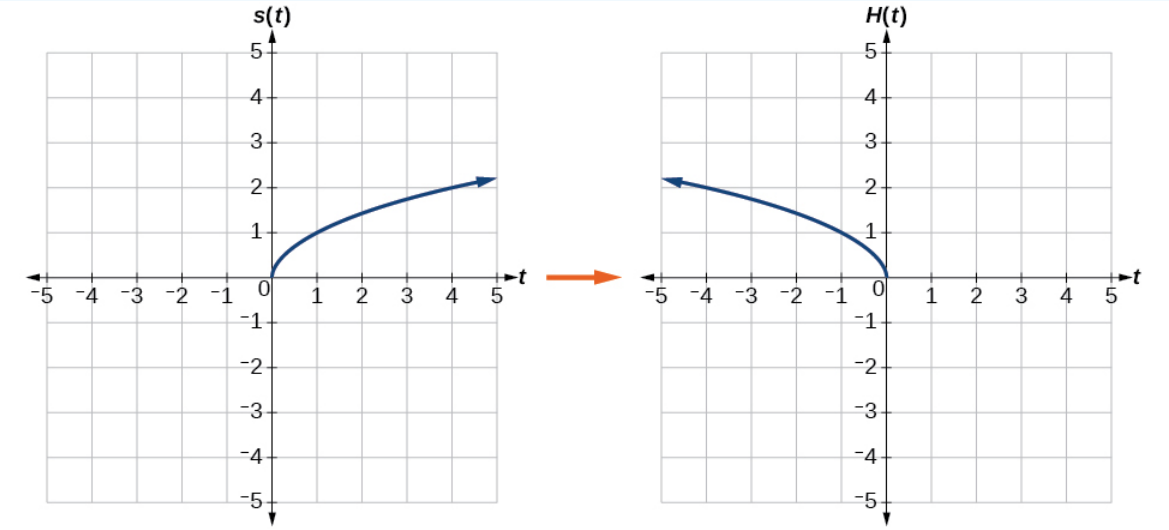


Figure 2.7.15: Horizontal reflection of the square root function

Because each input value is the opposite of the original input value, we can write

$$H(t) = s(-t) \text{ or } H(t) = \sqrt{-t}$$

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.

Note that these transformations can affect the domain and range of the functions. While the original square root function has domain $[0, \infty)$ and range $[0, \infty)$, the vertical reflection gives the $V(t)$ function the range $(-\infty, 0]$ and the horizontal reflection gives the $H(t)$ function the domain $(-\infty, 0]$.

You Try 2.7.3

Reflect the graph of $f(x) = |x - 1|$ (a) vertically and (b) horizontally.

Answer

a.

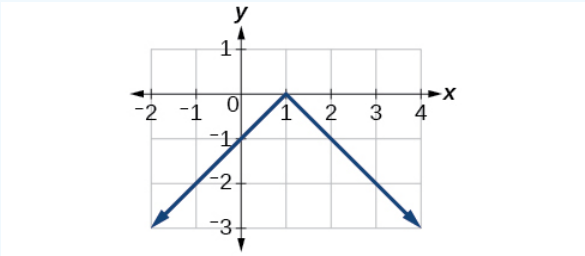


Figure 2.7.16: Graph of a vertically reflected absolute function.

The function that describes this graph is $g(x) = -|x - 1|$

b.

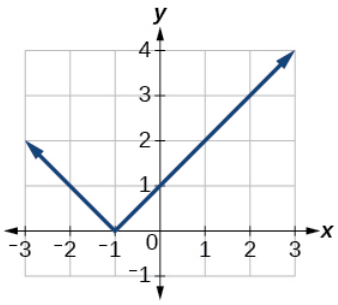


Figure 2.7.17: Graph of an absolute function translated one unit left.

The function that describes this graph is $h(x) = |-(x+1)|$ or $h(x) = |-x-1|$

Example 2.7.8

A function $f(x)$ is given as Table 2.7.3. Create a table for the functions below.

a. $g(x) = -f(x)$
b. $h(x) = f(-x)$

Table 2.7.3

x	2	4	6	8
$f(x)$	1	3	7	11

a. For $g(x)$, the negative sign outside the function indicates a vertical reflection, so the x-values stay the same and each output value will be the opposite of the original output value. See Table 2.7.4.

Table 2.7.4

x	2	4	6	8
$g(x)$	-1	-3	-7	-11

b. For $h(x)$, the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the $h(x)$ values stay the same as the $f(x)$ values. See Table 2.7.5.

Table 2.7.5

x	-2	-4	-6	-8
$h(x)$	1	3	7	11

You Try 2.7.4

Given the basic function $f(x) = x^2$, graph $g(x) = -f(x)$ and $h(x) = f(-x)$.

Answer

https://math.libretexts.org/@go/page/126512

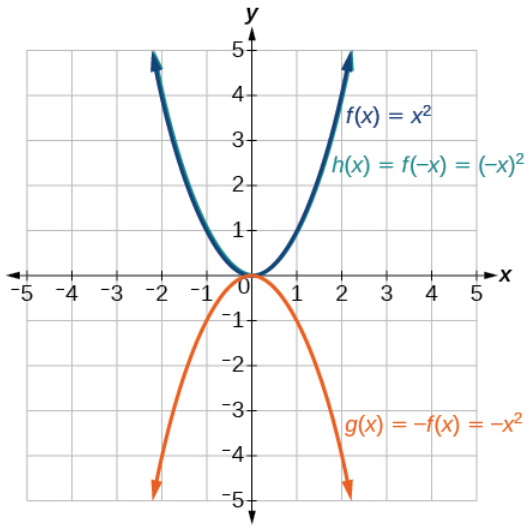


Figure 2.7.18: Graph of x^2 and its reflections.

Notice: $g(x) = f(-x)$ looks the same as $f(x)$.

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 2.7.19 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

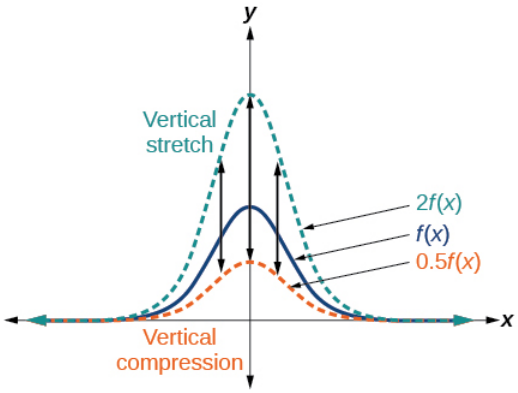


Figure 2.7.19: Vertical stretch and compression

Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched vertically.
- If $0 < a < 1$, then the graph will be compressed vertically.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

Example 2.7.9

A function $P(t)$ models the population of fruit flies. The graph is shown in Figure 2.7.20.

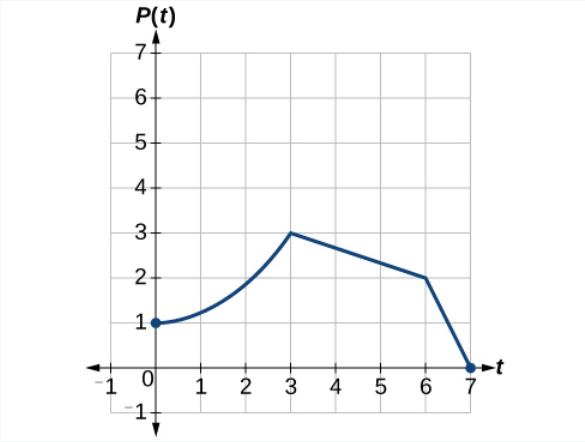


Figure 2.7.20: Graph to represent the growth of the population of fruit flies.

A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

Solution

Because the population is always twice as large, the new population's output values are always twice the original function's output values. Graphically, this is shown in Figure 2.7.21.

If we choose four reference points, $(0, 1)$, $(3, 3)$, $(6, 2)$ and $(7, 0)$ we will multiply all of the outputs by 2.

The following shows where the new points for the new graph will be located.

$$(0, 1) \to (0, 2)$$

$$(3, 3) \to (3, 6)$$

$$(6, 2) \to (6, 4)$$

$$(7, 0) \to (7, 0)$$

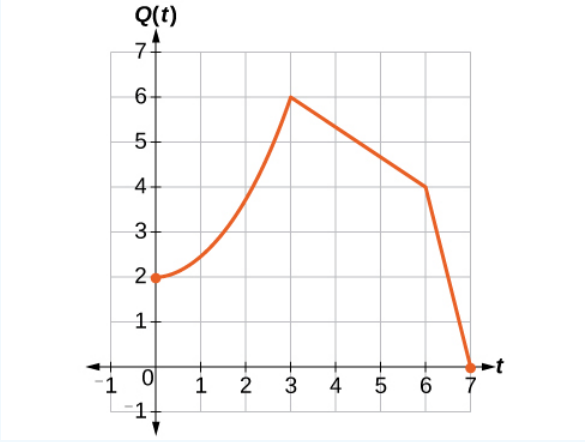


Figure 2.7.21: Graph of the population function doubled.

Symbolically, the relationship is written as

$$Q(t) = 2P(t)$$

This means that for any input t, the value of the function Q is twice the value of the function P. Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, t,

stay the same while the output values are twice as large as before.

Example 2.7.10

Write an equation for the function, $g(x)$ that models the graph shown in Figure 2.7.22, which is a transformation of the basic cubic function.

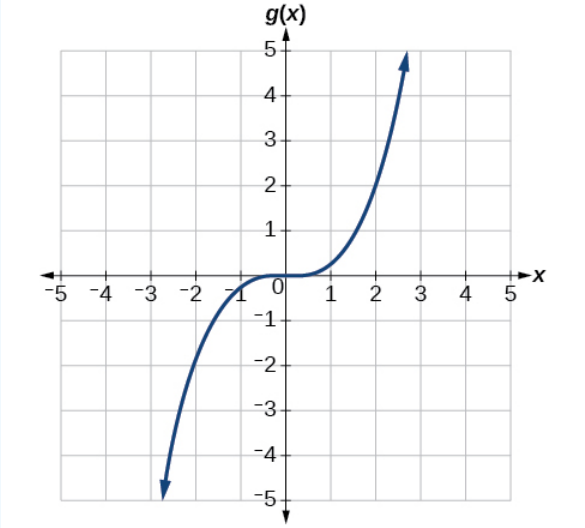


Figure 2.7.22: Graph of a transformation of $f(x) = x^3$

When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that $g(2) = 2$. With the basic cubic function at the same input, $f(2) = 2^3 = 8$. Based on that, it appears that the outputs of g are $\frac{1}{4}$ the outputs of the function f because $g(2) = \frac{1}{4}f(2)$. From this we can fairly safely conclude that $g(x) = \frac{1}{4}f(x)$.

We can write a formula for g by using the definition of the function f.

$$g(x) = \frac{1}{4}f(x) = \frac{1}{4}x^3.$$

You Try 2.7.5

Write an equation for a function that is a stretch of the basic cubic function vertically by a factor of 4, shifted right 5 units, and then shifted down by 2 units.

Answer

$g(x) = 4(x-5)^3 - 2$

Horizontal Stretches and Compressions

Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1, we get a **horizontal stretch**; if the constant is greater than 1, we get a **horizontal compression** of the function.

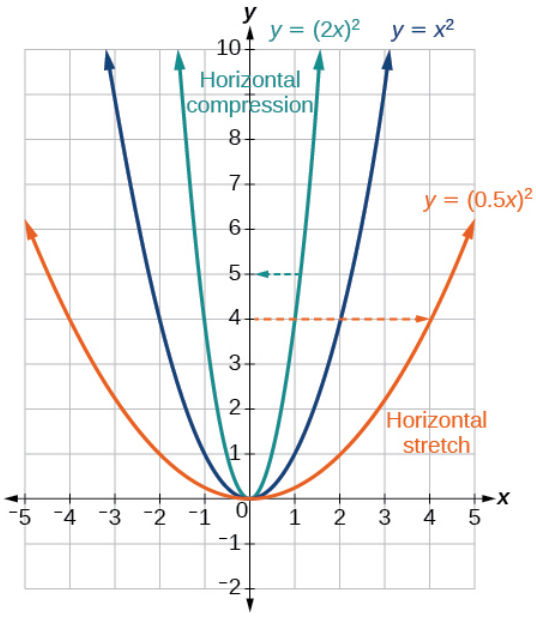


Figure 2.7.23: Graph of the vertical stretch and compression of x^2.

Given a function $y = f(x)$, the form $y = f(bx)$ results in a horizontal stretch or compression. Consider the function $y = x^2$. Observe Figure 2.7.23 The graph of $y = (0.5x)^2$ is a horizontal stretch of the graph of the function $y = x^2$ by a factor of 2. The graph of $y = (2x)^2$ is a horizontal compression of the graph of the function $y = x^2$ by a factor of 2.

Horizontal Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a **horizontal stretch** or **horizontal compression** of the function $f(x)$.

- If $b > 1$, then the graph will be compressed horizontally by $\frac{1}{b}$.
- If $0 < b < 1$, then the graph will be stretched horizontally by $\frac{1}{b}$.
- If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.

Example 2.7.11

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, R, will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

Solution

Symbolically, we could write

$$\begin{aligned} R(1) &= P(2), \\ R(2) &= P(4), \qquad \text{and in general,} \\ R(t) &= P(2t). \end{aligned}$$

See Figure 2.7.24for a graphical comparison of the original population and the compressed population.

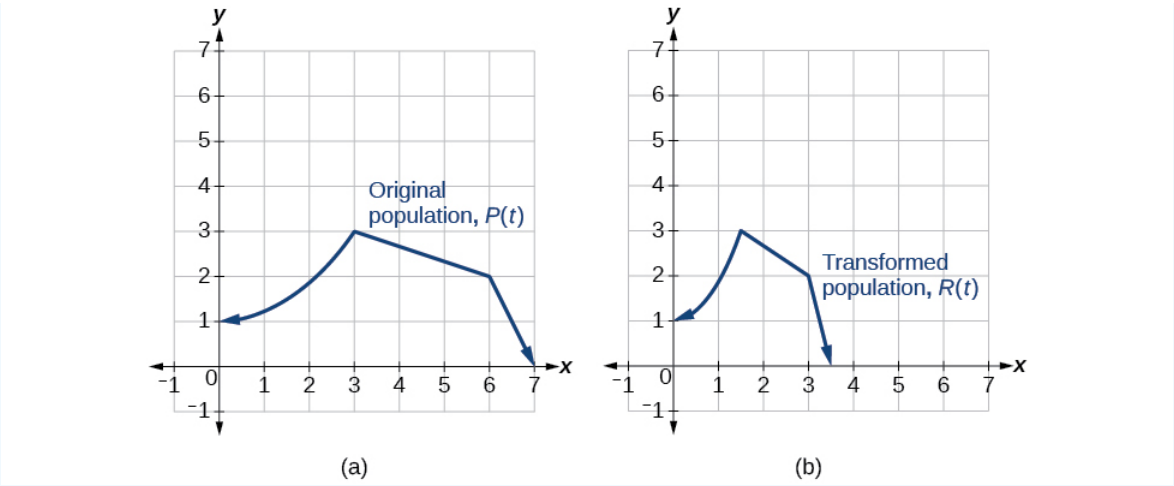


Figure 2.7.24: (a) Original population graph (b) Compressed population graph

Analysis

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

Example 2.7.12

Use transformations to describe the function $g(x)$ in terms of $f(x)$ in Figure 2.7.25.

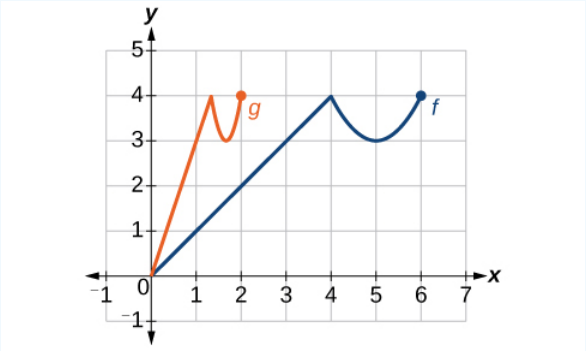


Figure 2.7.25: Graph of $f(x)$ being vertically compressed to $g(x)$.

Solution

The graph of $g(x)$ looks like the graph of $f(x)$ horizontally compressed. Because $f(x)$ ends at (6,4) and $g(x)$ ends at (2,4), we can see that the x-values have been compressed by $\frac{1}{3}$, because $6(\frac{1}{3}) = 2$. We might also notice that $g(2) = f(6)$ and $g(1) = f(3)$. Either way, we can describe this relationship as $g(x) = f(3x)$. This is a horizontal compression by $\frac{1}{3}$.

Analysis

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4, we need a coefficient of $\frac{1}{4}$ in our function: $f(\frac{1}{4}x)$. This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

You Try 2.7.6

Write a formula for the basic square root function horizontally stretched by a factor of 3.

Answer

$g(x) = f(\frac{1}{3}x)$, so using the square root function we get $g(x) = \sqrt{\frac{1}{3}x}$

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about sometimes. When we write $g(x)=f(2x+6)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. A way to identify the transformations is to factor inside the functionfirst to rewrite the function in a form that we can identify all the transformations, $g(x)=f(2x+6)=f(2(x+3)$. Function $g(x)$ is a horizontal compression of $f(x)$ by 2 and a horizontal shifting of $f(x)$ to the left by 3.

Let's work through an example.

$$f(x)=(3x-12)^2$$

We can factor out a 3.

$$f(x)=(3(x-4))^2$$

Now we can more clearly observe a horizontal shift to the right 4 units and a horizontal compression by 3. Factoring in this way allows us to horizontally compress first and then shift horizontally.

Combining Transformations

- When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h. If $b<0$ there is also a horizontal reflection over the y-axis.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h. If $b<0$ there is also a horizontal reflection over the y-axis.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical reflections are performed first.

Example 2.7.13

Use the graph of $f(x)$ in Figure $2.7.26$to sketch a graph of $k(x)=f\left(\frac{1}{2}x+1\right)-3$.

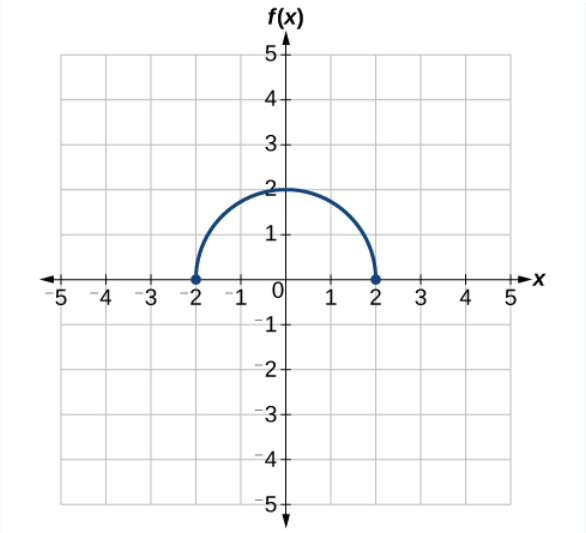


Figure 2.7.26: Graph of a half-circle.

To simplify, let's start by factoring out the inside of the function.

$$f\left(\frac{1}{2}x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3$$

By factoring the inside, we can first horizontally stretch by 2, as indicated by the $\frac{1}{2}$ on the inside of the function. Remember that twice the size of 0 is still 0, so the point $(0,2)$ remains at $(0,2)$ while the point $(2,0)$ will stretch to $(4,0)$. See Figure 2.7.27.

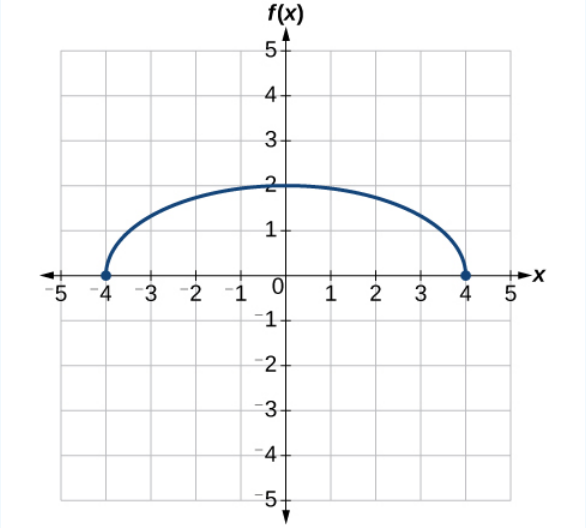


Figure 2.7.27: Graph of a vertically stretch half-circle.

Next, we horizontally shift left by 2 units, as indicated by $x+2$. See Figure 2.7.28

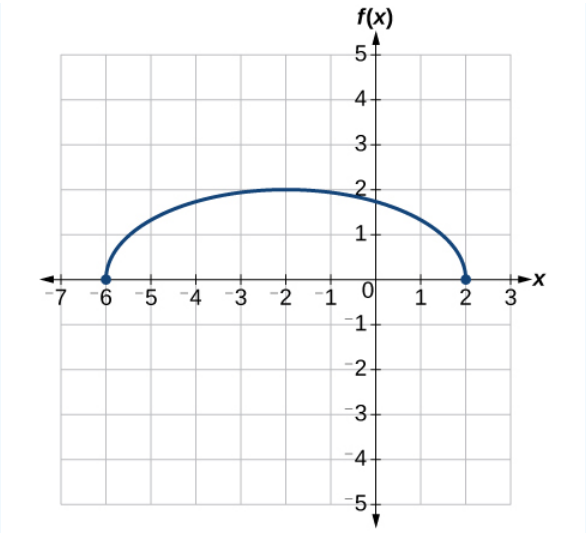


Figure 2.7.28: Graph of a vertically stretch and translated half-circle.

Last, we vertically shift down by 3 to complete our sketch, as indicated by the −3 on the outside of the function. See Figure 2.7.29

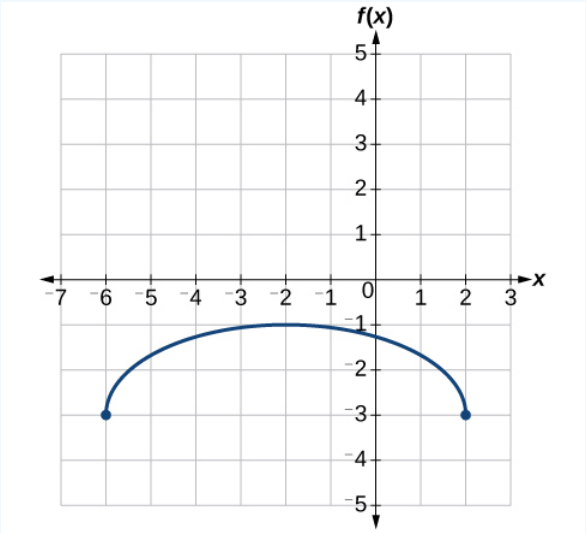


Figure 2.7.29: Graph of a vertically stretch and translated half-circle.

Example 2.7.14

Given the function $r(x) = \sqrt{x-3} - 5$,

1. Identify the transformations and use them to sketch the graph.
2. Based on what you found in part 1, how many x-intercepts do you expect there to be for this function? Explain.
3. Find the x-intercept(s) of $r(x)$.

Solution

1. The basic function shifts right 3 units and down 5 units.

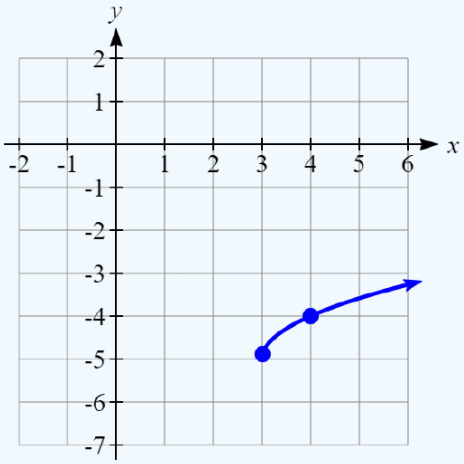


Figure 2.7.30: Graph of $r(x)$

2. It is expected that there is one x-intercept based on the transformations and the shape of the square root function. The x-intercept should have a fairly large x-coordinate, likely larger than 10, since this square root function increases fairly slowly.
3. Solving for the x-intercept:

$$\begin{aligned} 0 &= \sqrt{x-3}-5 \\ 5 &= \sqrt{x-3} \\ (5)^2 &= (\sqrt{x-3})^2 \\ 25 &= x-3 \\ 28 &= x \end{aligned}$$

The x-intercept is $(28, 0)$.

Example 2.7.15

Given the function $p(x) = -5(x-4)^2 + 2$,

1. Identify the transformations and use them to sketch the graph.
2. Based on what you found in part 1, how many x-intercepts do you expect there to be for this function? Explain.
3. Now, use your knowledge of solving equations to find the x-intercept(s) of $p(x)$ algebraically, if they exist.

Solution

1. The basic function shifts right 4 units and up 2 units. There is an x-axis reflection and a vertical stretch by 5.

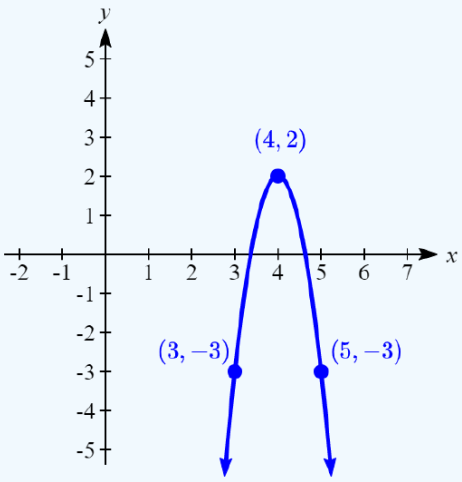


Figure 2.7.31: Graph of $p(x)$

1. Based on the transformations and shape of the basic graph, it is expected that there should be two x-intercepts with x-coordinates within the intervals $(3,4)$ and $(4,5)$.
2. Solving for the x-intercepts:

$$\begin{aligned} 0 &= -5(x-4)^2+2 \\ -2 &= -5(x-4)^2 \\ \frac{2}{5} &= (x-4)^2 \\ \pm\sqrt{\frac{2}{5}} &= \sqrt{(x-4)^2} \\ \pm\sqrt{\frac{2}{5}} &= x-4 \\ 4\pm\sqrt{\frac{2}{5}} &= x \end{aligned}$$

The x-intercepts are $\left(4-\sqrt{\frac{2}{5}},0\right)$ and $\left(4+\sqrt{\frac{2}{5}},0\right)$.

Note: $\left(4-\sqrt{\frac{2}{5}},0\right)\approx(3.37,0)$ and $\left(4+\sqrt{\frac{2}{5}},0\right)\approx(4.63,0)$ which follows what was expected visually.

? You Try 2.7.7

Given the function $k(x)=4(x+2)^3-3$:

1. Based on your knowledge of transformations, how many x-intercepts do you expect to have and where do you anticipate the x-intercept(s) to be within the coordinate system? Explain.
2. Now, use your knowledge of solving equations to find the x-intercept(s) of $k(x)$algebraically, if they exist.

Answer

1. This is a cubic function with a vertical stretch by 4 with a horizontal shift left 2 units and vertical shift down 3 units. Based on this knowledge, there should be one x-intercept with a x-coordinate within the interval $(-2,-1)$.
2. The x-intercept is $\left(-2+\sqrt[3]{\frac{3}{4}},0\right)$.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the basic functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph, as shown in Figure 2.7.32

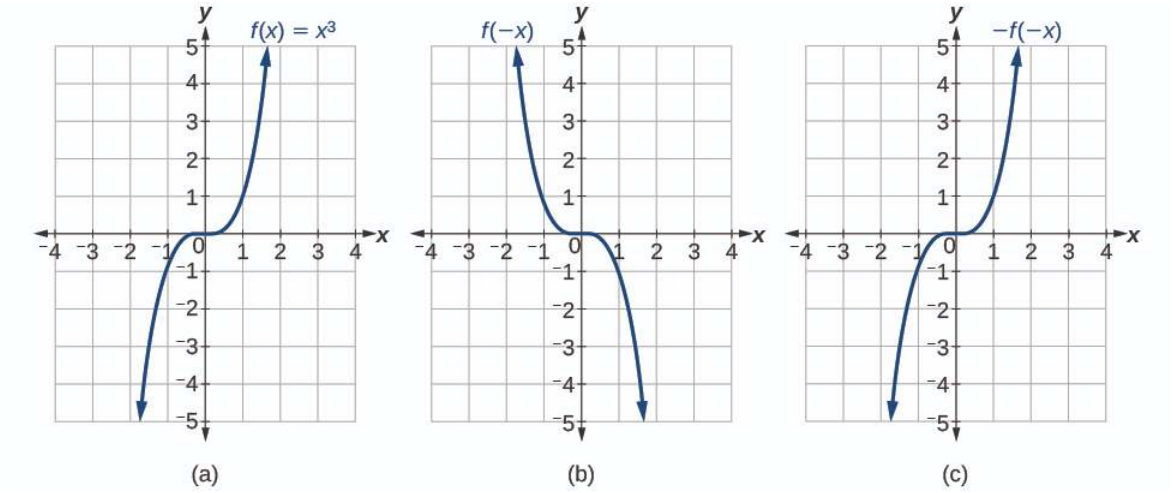


Figure 2.7.32: (a) The cubic basic function (b) Horizontal reflection of the cubic basic function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Even and Odd Functions

A function is called an **even function** if for every input x

$f(x) = f(-x)$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input x

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin.

Example 2.7.16

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis

Consider the graph of f in Figure 2.7.33. Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

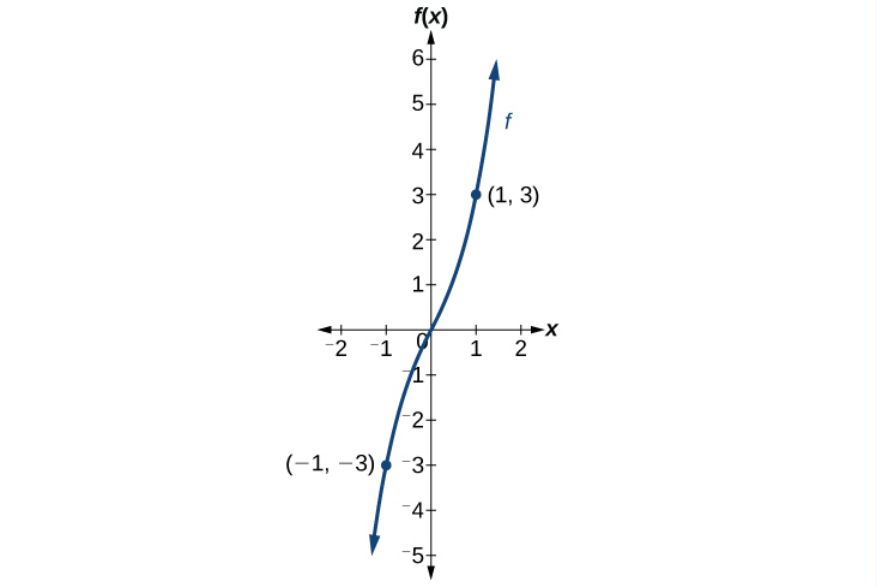


Figure 2.7.33: Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.

You Try 2.7.8

Is the function $f(s) = s^4 + 3s^2 + 7$ even, odd, or neither?

Answer

even

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2.8: Algebra and Composition of Functions

Learning Objectives

- Create new functions using algebraic operations.
- Evaluate combined algebraic functions.
- Find the domain of combined algebraic functions.
- Create new functions by composition of functions.
- Evaluate composite functions.
- Describe what composite functions represent in real-life situations.
- Find the domain of composite functions.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

Using descriptive variables, we can notate these two functions. The function $C(T)$ gives the cost C of heating a house for a given average daily temperature in T degrees Celsius. The function $T(d)$ gives the average daily temperature on day d of the year. For any given day, $Cost = C(T(d))$ means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature $T(d)$. For example, we could evaluate $T(5)$ to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the **cost function** at that temperature. We would write $C(T(5))$.

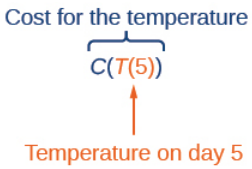


Figure 2.8.1: Explanation of $C(T(5))$, which is the cost for the temperature and $T(5)$ is the temperature on day 5.

By combining these two relationships into one function, we have performed function composition, which is one way to combine functions together to form new functions.

Combining Functions Using Algebraic Operations

Function composition is only one way to combine existing functions. Another way is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If $w(y)$ is the wife's income and $h(y)$ is the husband's income in year y, and we want T to represent the total income, then we can define a new function.

$$T(y) = h(y) + w(y)$$

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write

$$T = h + w$$

We can define other algebraic operations with functions in a similar way.

Algebra of Functions

For two functions $f(x)$ and $g(x)$ where x is in the domain of both f and g, we define new functions $f+g$, $f-g$, fg, and $\frac{f}{g}$ by the relations:

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x) \\
(f-g)(x) &= f(x)-g(x) \\
(fg)(x) &= f(x)g(x) \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, g(x) \neq 0
\end{aligned}
$$

Example 2.8.1

Given $f(x)=x-1$ and $g(x)=x^2-1$. find and simplify the functions $(g-f)(x)$ and $\left(\frac{g}{f}\right)(x)$.

Solution

Begin by writing the general form, and then substitute in the given functions.

$$
\begin{aligned}
(g-f)(x) &= g(x)-f(x) \\
&= x^2-1-(x-1) \\
&= x^2-x \\
&= x(x-1) \\
\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\
&= \frac{x^2-1}{x-1} \\
&= \frac{(x+1)(x-1)}{x-1} \\
&= x+1, x \neq 1
\end{aligned}
$$

Note: For $\left(\frac{g}{f}\right)(x)$, the condition $x \neq 1$ must be stated after simplifying. Remember that function operations can only be performed for all x-values in the domain of both f and g where $g(x) \neq 0$ which was assumed when we simplified.

Caution!

Remember to state domain restrictions for functions formed using algebra of functions. If the domain restriction is not visually apparent after simplifying, as in the previous example, it must be stated for the combined function to be correct.

You Try 2.8.1

Given $f(x)=x-1$ and $g(x)=x^2-1$, find and simplify the functions $(fg)(x)$ and $(f-g)(x)$.

Answer

$(fg)(x)=f(x)g(x)=(x-1)(x^2-1)=x^3-x^2-x+1$

$(f-g)(x)=f(x)-g(x)=(x-1)-(x^2-1)=x-x^2$

When performing function operations, make sure to use correct notation by including parentheses around each function so that operations are performed correctly.

In the previous example, it would be incorrect to write

$$(fg)(x)=f(x)g(x)=x-1*x^2-1 \quad \text{since this simplifies to } x-x^2-1$$

Similarly, it would be incorrect to write

$$(f-g)(x) = f(x) - g(x) = x - 1 - x^2 - 1 \quad \text{since this simplifies to } -x^2 + x - 2$$

Caution!

Using correct mathematical notation is important. When performing operations with multiple terms, use parentheses. Provided below are some examples of notations that are not equivalent.

$$(x+2)(x-5) \neq x + 2 * x - 5$$

$$(x+2) - (x-5) \neq x + 2 - x - 5$$

Evaluating Functions Formed Algebraically From Two Functions

Just like with any function we can evaluate functions that were combined algebraically. Let's look at an example.

Example 2.8.2

Let $f(x) = 6x^2 - 2x$ and $g(x) = 3 - \dfrac{1}{x}$.

1. Find $(f+g)(-1)$
2. Find $(fg)(2)$

Solution

1. To find $(f+g)(-1)$ we first find $f(-1) = 8$ and $g(-1) = 4$. By definition, we have that

$$\begin{aligned}(f+g)(-1) &= f(-1) + g(-1) \\ &= 8 + 4 \\ &= 12\end{aligned}$$

2. To find $(fg)(2)$, we first need $f(2)$ and $g(2)$. Since $f(2) = 20$ and $g(2) = \frac{5}{2}$, our formula yields

$$\begin{aligned}((fg)(2) &= f(2)g(2) \\ &= (20)\left(\frac{5}{2}\right) \\ &= 50\end{aligned}$$

Note that you can also find the formula for the combined function first, then evaluate. You will get the same result as above.

Finding Domain of a Function Formed Algebraically From Two Functions

Remember that to combine two functions algebraically, f and g, that the input values must be in the domain of both functions. Let's look at an example of finding domain.

Example 2.8.3

Let $f(x) = 6x^2 - 2x$ and $g(x) = 3 - \dfrac{1}{x}$.

1. Find the domain of $g - f$ then find and simplify $(g-f)(x)$.
2. Find the domain of $\left(\frac{g}{f}\right)$ then find and simplify $\left(\frac{g}{f}\right)(x)$.

Solution

1. To find the domain of $g - f$, find the domain of g and of f separately, then find the intersection of these two sets. The domain of g is $(-\infty, 0) \cup (0, \infty)$. The domain of f is $(-\infty, \infty)$. Thus the domain of $g - f$ is $(-\infty, 0) \cup (0, \infty)$.

   Moving along, we need to simplify $(g-f)(x)$.

$$\begin{aligned}(g-f)(x) &= g(x)-f(x)\\ &= \left(3-\frac{1}{x}\right)-\left(6x^2-2x\right)\\ &= 3-\frac{1}{x}-6x^2+2x\\ &= \frac{3x}{x}-\frac{1}{x}-\frac{6x^3}{x}+\frac{2x^2}{x}\\ &= \frac{3x-1-6x^3+2x^2}{x}\\ &= \frac{-6x^3+2x^2+3x-1}{x},\ x\neq 0\end{aligned}$$

2. To find the domain of $\frac{g}{f}$, first find the domain of g and f separately and then find the intersection of these two sets. Also, since $\left(\frac{g}{f}\right)(x)=\frac{g(x)}{f(x)}$, we are introducing a new denominator, namely $f(x)$ in this case, we need to guard against this being 0. Our previous work tells us that the domain of g is $(-\infty,0)\cup(0,\infty)$ and the domain of f is $(-\infty,\infty)$. Setting $f(x)=0$ gives $6x^2-2x=0$ or $x=0,\frac{1}{3}$. As a result, the domain of $\frac{g}{f}$ is all real numbers except $x=0$ and $x=\frac{1}{3}$, or $(-\infty,0)\cup\left(0,\frac{1}{3}\right)\cup\left(\frac{1}{3},\infty\right)$.

Next, we find and simplify $\left(\frac{g}{f}\right)(x)$.

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)}\\ &= \frac{3-\frac{1}{x}}{6x^2-2x}\\ &= \frac{x}{x}\cdot\frac{3-\frac{1}{x}}{6x^2-2x}\\ &= \frac{x\cdot\left(3-\frac{1}{x}\right)}{x\cdot(6x^2-2x)}\\ &= \frac{3x-1}{2x^2\cdot(3x-1)}\\ &= \frac{1}{2x^2},\ x\neq 0,\frac{1}{3}\end{aligned}$$

Hence, $\left(\frac{g}{f}\right)(x)=\frac{1}{2x^2}, x\neq 0,\frac{1}{3}$

Note the importance of finding the domain of a function *before* simplifying its expression. In part 2 in the previous example, had we waited to find the domain of $\frac{g}{f}$ until after simplifying, we'd just have the formula $\frac{1}{2x^2}$ to go by, and we would (incorrectly!) state the domain as $(-\infty,0)\cup(0,\infty)$, since the other troublesome number, $x=\frac{1}{3}$, was canceled away.

Example 2.8.4

Given $h(x)=\dfrac{x}{3x-2}$, $l(x)=\dfrac{x}{2x+3}$, and $m(x)=\dfrac{x+4}{6x^2+5x-6}$,

1. Find and simplify $(h-l)(x)$.
2. What is the domain of $h-l$? State your answer in interval notation.
3. For what x-values does $(h-l)(x)=m(x)$?

Solution

1.
$$\begin{aligned}(h-l)(x) &= \frac{x}{3x-2} - \frac{x}{2x+3} \\ &= \frac{x(2x+3)-x(3x-2)}{(3x-2)(2x+3)} \\ &= \frac{2x^2+3x-3x^2+2x}{(3x-2)(2x+3)} \\ &= \frac{-x^2+5x}{(3x-2)(2x+3)}\end{aligned}$$

2. The domain of $h-l$ is $\left(-\infty,-\frac{3}{2}\right)\cup\left(-\frac{3}{2},\frac{2}{3}\right)\cup\left(\frac{2}{3},\infty\right)$.
3. You may choose to work with simplified version of $h-l$ that was found in part 1 or the original functions, h and l, when solving. The simplified function found in part 1 was used for solving in the following steps.

$$\begin{aligned}(h-l)(x) &= m(x) \\ \frac{-x^2+5x}{(3x-2)(2x+3)} &= \frac{x+4}{6x^2+5x-6} \\ \frac{-x^2+5x}{(3x-2)(2x+3)} &= \frac{x+4}{(3x-2)(2x+3)}\end{aligned}$$

Multiplying both sides by the common denominator,$(3x-2)(2x+3)$, we get

$$\begin{aligned}-x^2+5x &= x+4 \\ -x^2+4x-4 &= 0 \\ x^2-4x+4 &= 0 \\ (x-2)^2 &= 0 \\ x &= 2\end{aligned}$$

Verifying that this is not an extraneous solution, we can conclude that $(h+l)(x)=m(x)$ at $x=2$.

Remember that solutions to equations relate to where graphs of functions intersect. If we want to know the point where the graph of $h+l$ intersects m, we can evaluate both functions at $x=2$:

$$(h-l)(2)=\frac{-(2)^2+5(2)}{(3(2)-2)(2(2)+3)} \qquad \text{and} \qquad m(2)=\frac{(2)+4}{6(2)^2+5(2)-6}$$

$$(h-l)(2)=\frac{3}{14} \qquad\qquad m(2)=\frac{3}{14}$$

The point where $(h-l)(x)$ intersects $m(x)$ is at $\left(2,\frac{3}{14}\right)$.

Looking at the graph of $h-l$ and m, denoted by blue and red graphs respectively, it is difficult to see this point where the graphs intersect.

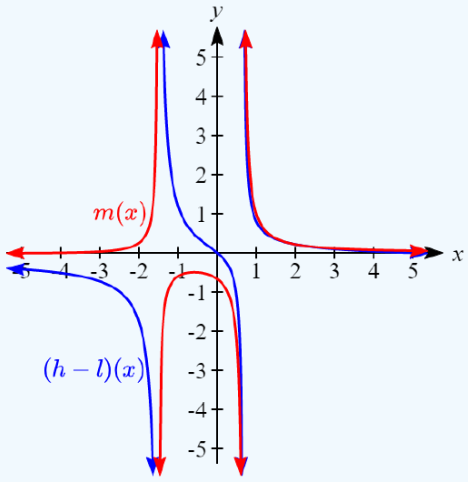


Figure 2.8.2: Graphs of (h-l)(x) and m(x)

If you try different windows for viewing the graphs of these two functions you will find that in almost all windows it will be very difficult to clearly see the point of intersection. This is just one of many examples that illustrates the importance of why we need to have good algebra skills so we can apply them in situations such as this. In addition, understanding the behavior of functions helps us to better have an intuitive sense of the types of solutions we might expect when solving equations.

? You Try 2.8.2

Given $A(t)=\sqrt{7t+4}$ and $B(t)=\sqrt{t+1}$, where does $(A-B)(t)=3$?

Answer

$(A-B)(t)=3$ at $t=3$.

Note that $t=0$ is extraneous. Upon viewing the graph of $A-B$ with the line $y=3$ we can see that the graphs intersect at $t=3$, as seen below in Figure 2.8.3

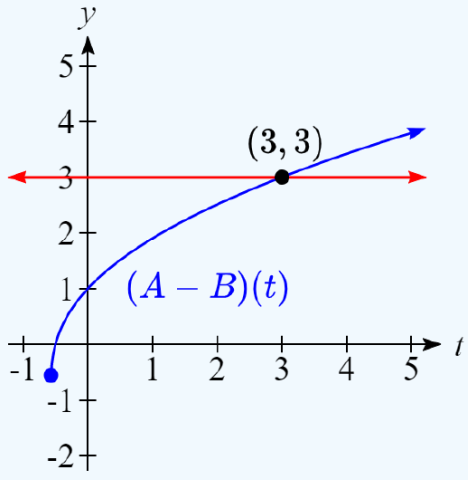


Figure 2.8.3: Graphs of (A-B)(x) and y=3 showing the point (3,3) where they intersect

Composition of Functions

Earlier, we discussed wanting to compute a heating cost from a day of the year. To do this we created a new function that takes a day as input and yields a cost as output. This process of combining functions so that the output of one function becomes the input

of another is known as **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$(f \circ g)(x) = f(g(x))$$

We read the left-hand side as"f composed with g at x," and the right-hand side as"f of g of x." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. It is important not to confuse function composition with multiplication because they are different operations.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual order of operations convention with parentheses by starting with the innermost parentheses first, and then working to the outside. To perform composition, $(f \circ g)(x) = f(g(x))$, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

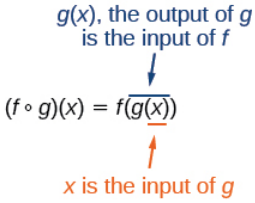


Figure 2.8.4: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. The composition of functions f and g, defined as $f \circ g$, is called composite function such that

$$(f \circ g)(x) = f(g(x))$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

Remember: It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Example 2.8.5

Using the functions provided, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$f(x) = 2x + 1 \quad g(x) = 3 - x$$

Solution

Let's begin by substituting $g(x)$ into $f(x)$.

$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= 2(3-x)+1\\ &= 6-2x+1\\ &= 7-2x\end{aligned}$$

Now we can substitute $f(x)$ into $g(x)$.

$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= 3-(2x+1)\\ &= 3-2x-1\\ &= 2-2x\end{aligned}$$

Note that $(f\circ g)(x)\neq(g\circ f)(x)$.

Evaluating Composite Functions

Next, let's look at an example of evaluating composite functions.

Example 2.8.6

Given $f(t)=t^2-t$ and $h(x)=3x+2$, evaluate $(f\circ h)(1)$.

Solution

Because the inside expression is $h(1)$, we start by evaluating $h(x)$ at 1.

$$\begin{aligned}h(1) &= 3(1)+2\\ h(1) &= 5\end{aligned}$$

Then $f(h(1))=f(5)$, so we evaluate $f(t)$ at an input of 5.

$$\begin{aligned}(f\circ h)(1) &= f(h(1))\\ &= f(5)\\ &= 5^2-5\\ &= 20\end{aligned}$$

Note that it makes no difference what the input variables t and x were called in this problem because we evaluated for specific numerical values.

You could also find the formula for the composite function, $f\circ h$, then evaluate. You will get the same result as above.

You Try 2.8.3

Given $f(t)=t^2-t$ and $h(x)=3x+2$, evaluate

a. $(h\circ f)(2)$
b. $(h\circ f)(-2)$

Answer a

8

Answer b

20

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

Example 2.8.7

Using the tables below, evaluate $(f \circ g)(3)$ and $(g \circ f)(4)$

x	$f(x)$
1	6
2	8
3	3
4	1

x	$g(x)$
1	3
2	5
3	2
4	7

Solution

To evaluate $(f \circ g)(3)$, we start from the inside with the value 3. We then evaluate the inside expression $g(3)$ using the table that defines the function g : $g(3) = 2$.

We can then use that result as the input to the f function, so $g(3)$ is replaced by the equivalent value 2 and we can evaluate $f(2)$. Then using the table that defines the function f, we find that $f(2) = 8$.

$$(f \circ g)(3) = f(2) = 8.$$

To evaluate $(g \circ f)(4)$, we first evaluate the inside expression $f(4)$using the first table: $f(4) = 1$. Then using the table for g we can evaluate:

$$(g \circ f)(4) = g(1) = 3.$$

You Try 2.8.4

Using the tables from the example above, evaluate $(f \circ g)(1)$ and $(g \circ f)(3)$.

Answer

$(f \circ g)(1) = f(3) = 3$ and $(g \circ f)(3) = g(3) = 2$

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the x- and y-axes of the graphs.

Example 2.8.8

Using Figure 2.8.5, evaluate $(f \circ g)(1)$.

Solution

To evaluate $(f \circ g)(1)$, we start with the inside evaluation. See Figure 2.8.5.

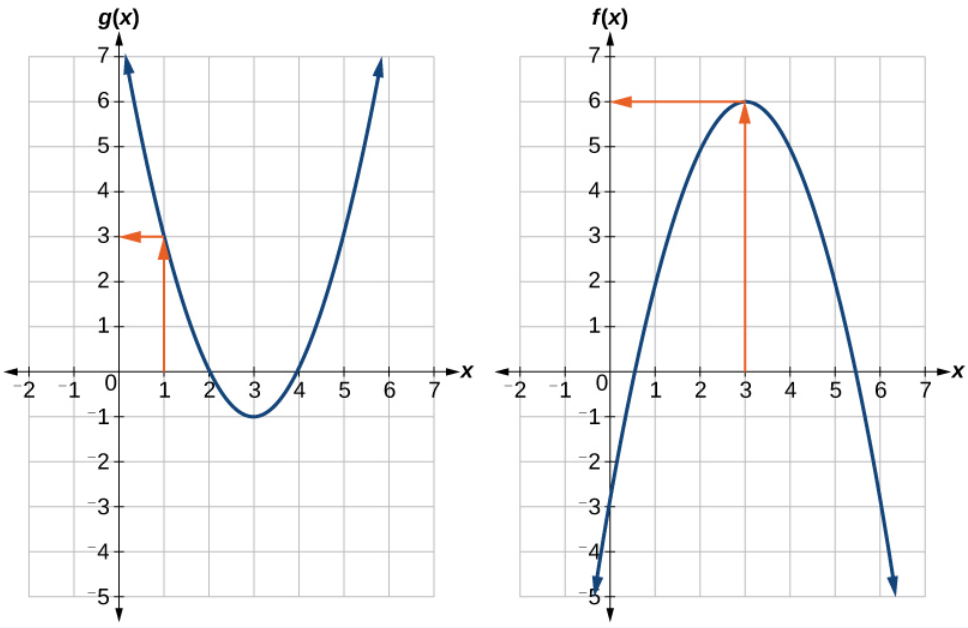


Figure 2.8.5: Two graphs of parabolas, one that opens upward, $g(x)$, and one that opens downward, $f(x)$. The following points are plotted: $g(1) = 3$ and $f(3) = 6$.

We evaluate $g(1)$ using the graph of $g(x)$, finding the input of 1 on the x-axis and finding the output value of the graph at that input. Here, $g(1) = 3$. We use this value as the input to the function f.

$$(f \circ g)(1) = f(3)$$

We can then evaluate the composite function by looking to the graph of $f(x)$, finding the input of 3 on the x-axis and reading the output value of the graph at this input. Here, $f(3) = 6$, so $(f \circ g)(1) = 6$.

? You Try 2.8.5

Using Figure 2.8.5 in the previous example, evaluate $(g \circ f)(5)$.

Answer

$(g \circ f)(5) = g(2) = 0$

Interpretation of Composite Functions

There are many real-life situations that can be modeled using composite functions. It is important that we understand how to combine functions appropriately so that the resulting composite function makes sense. Let's take a look.

✓ Example 2.8.9

The function $c(s)$ gives the number of calories burned completing s sit-ups, and $s(t)$ gives the number of sit-ups a person can complete in t minutes. Interpret $(c \circ s)(3)$.

Solution

The inside expression in the composition is $s(3)$. Because the input to the s-function is time, $t = 3$ represents 3 minutes, and $s(3)$ is the number of sit-ups completed in 3 minutes.

Using $s(3)$ as the input to the function $c(s)$ gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

Example 2.8.10

Suppose $f(x)$ gives miles that can be driven in x hours and $g(y)$ gives the gallons of gas used in driving y miles. Which of these expressions is meaningful: $f(g(y))$ or $g(f(x))$?

Solution

The function $y = f(x)$ is a function whose output is the number of miles driven corresponding to the number of hours driven.

$$\text{number of miles } = f(\text{number of hours})$$

The function $g(y)$ is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:

$$\text{number of gallons } = g(\text{number of miles})$$

The expression $g(y)$ takes miles as the input and a number of gallons as the output. The function $f(x)$ requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression $f(g(y))$ is meaningless.

The expression $f(x)$ takes hours as input and a number of miles driven as the output. The function $g(y)$ requires a number of miles as the input. Using $f(x)$ (miles driven) as an input value for $g(y)$, where gallons of gas depends on miles driven, does make sense. The expression $g(f(x))$ makes sense, and will yield the number of gallons of gas used, g, driving a certain number of miles, $f(x)$, in x hours.

You Try 2.8.6

The gravitational force on a planet a distance r from the sun is given by the function $G(r)$. The acceleration of a planet subjected to any force F is given by the function $a(F)$. Form a meaningful composition of these two functions, and explain what it means.

Answer

A gravitational force is still a force, so $a(G(r))$ makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but $G(a(F))$ does not make sense.

Finding the Domain of a Composite Function

As we discussed previously, the **domain of a composite function** such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and g(x)\) is in the domain of f.

Domain of a Composite Function

The **domain of a composite function** $f \circ g$ is the set of all inputs x in the domain of g for which $g(x)$ is in the domain of f.

Example 2.8.11

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \frac{5}{x-1} \text{ and } g(x) = \frac{4}{3x-2}$$

Solution

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x) = 1$.

$$\begin{aligned} \frac{4}{3x-2} &= 1 \\ 4 &= 3x-2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

Example 2.8.12

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \sqrt{x+2} \text{ and } g(x) = \sqrt{3-x}$$

Solution

Because we cannot take the square root of a negative number, the domain of g is $(-\infty, 3]$. Now we check the domain of the composite function

$$(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$$

For $(f \circ g)(x) = \sqrt{\sqrt{3-x}+2}$, $\sqrt{3-x}+2 \geq 0$, since the radicand of a square root must be non-negative. Since square roots are positive, $\sqrt{3-x} \geq 0$, or, $3-x \geq 0$, which gives a domain of $(-\infty, 3]$.

Analysis

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of $f \circ g$ can contain values that are not in the domain of f, though they must be in the domain of g.

You Try 2.8.7

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \frac{1}{x-2} \text{ and } g(x) = \sqrt{x+4}$$

Answer

$[-4, 0) \cup (0, \infty)$

Decomposing a Composite Function into its Component Functions

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There may be more than one way to **decompose a composite function**, so we may choose the decomposition that appears to be most expedient.

Example 2.8.13

Write $f(x) = \sqrt{5 - x^2}$ as the composition of two functions.

Solution

We are looking for two functions, g and h, so $f(x) = g(h(x))$. To do this, we look for a function inside a function in the formula for $f(x)$. As one possibility, we might notice that the expression $5 - x^2$ is the inside of the square root. We could then decompose the function as

$$h(x) = 5 - x^2 \text{ and } g(x) = \sqrt{x}$$

We can check our answer by recomposing the functions.

$$g(h(x)) = g(5 - x^2) = \sqrt{5 - x^2}$$

You Try 2.8.8

Write $f(x) = \dfrac{4}{3 - \sqrt{4 + x^2}}$ as the composition of two functions.

Answer

There are multiple possible answers. Here's one:

$g(x) = \sqrt{4 + x^2}$

$h(x) = \dfrac{4}{3 - x}$

$f = h \circ g$

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2.9: Inverse Functions

Learning Objectives

- Determine if a function is 1-1.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Verify inverse functions.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure 2.9.1 provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

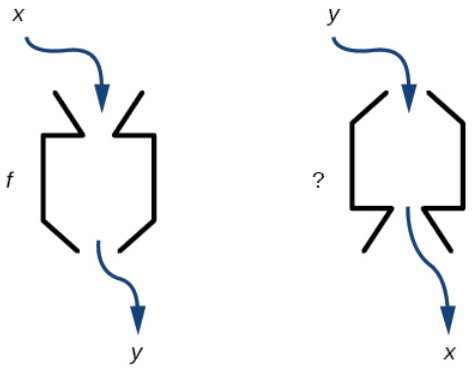


Figure 2.9.1: Can a function "machine" operate in reverse?

This idea of operating a function in reverse is known as an **inverse** operation. It is important when doing so that we know if the inverse is itself a function also. Let's take a moment to review what a function is and how we may determine if an inverse of a function will itself be a function.

Recall: A function of (\x\) is a relation in which each input, (\x\), value in the domain is assigned to exactly one output, value in the range.

One-to-One Functions

Some functions have a given output value that corresponds to two or more input values. However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions **one-to-one functions**.

To visualize this concept, let's look again at some simple relations in Figures 2.9.2 below. The function in part (a) shows a relationship that is not a one-to-one function because inputs q and r both give output n. The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

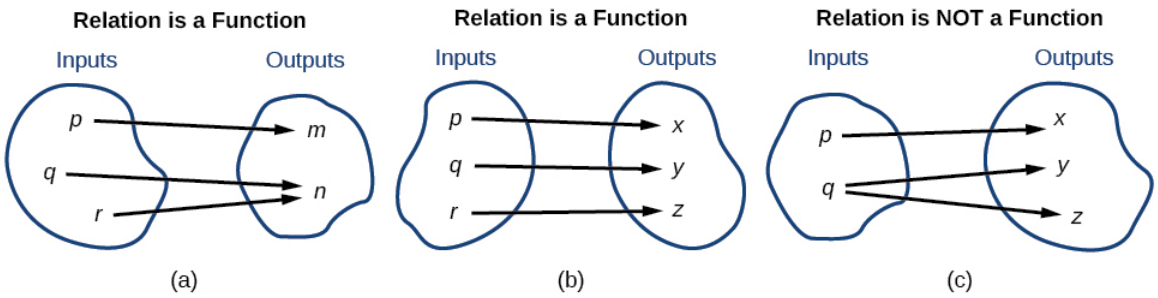


Figure 2.9.2: (a) This function is a not a one-to-one function because there are two inputs, q and r, associated with a single output, n. (b) This function is a one-to-one function since it is a function where each output has only one input associated with it. (c) This relation is not a function so it can not be a one-to-one function.

One-to-One Function

A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

Example 2.9.1

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

Solution

A circle of radius r has a unique area measure given by $A = \pi r^2$, so for any input, r, there is only one output, A. The area is a function of radius$r$.

If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure A is given by the formula $A = \pi r^2$. Because areas and radii are positive numbers, there is exactly one solution: $\sqrt{\frac{A}{\pi}}$. So the area of a circle is a one-to-one function of the circle's radius.

You Try 2.9.1

1. Is a balance in a bank account a function of the bank account number?
2. Is a bank account number a function of the balance?
3. Is a balance a one-to-one function of the bank account number?

Answer

1. yes, because each bank account has a single balance at any given time;
2. no, because several bank account numbers may have the same balance;
3. no, because the same output may correspond to more than one input.

Horizontal Line Test

An easy way to determine if a function is a one-to-one function is to use the **horizontal line test**. Draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, as seen in Figure 2.9.3, then the graph does not represent a one-to-one function.

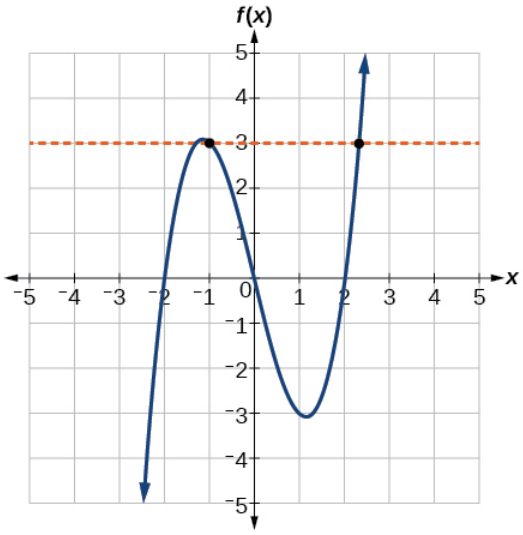


Figure 2.9.3: Graph of a polynomial with a horizontal line crossing through 2 points.

Example 2.9.2

Consider the relations shown below in Figure 2.9.4. Are any of these one-to-one functions?

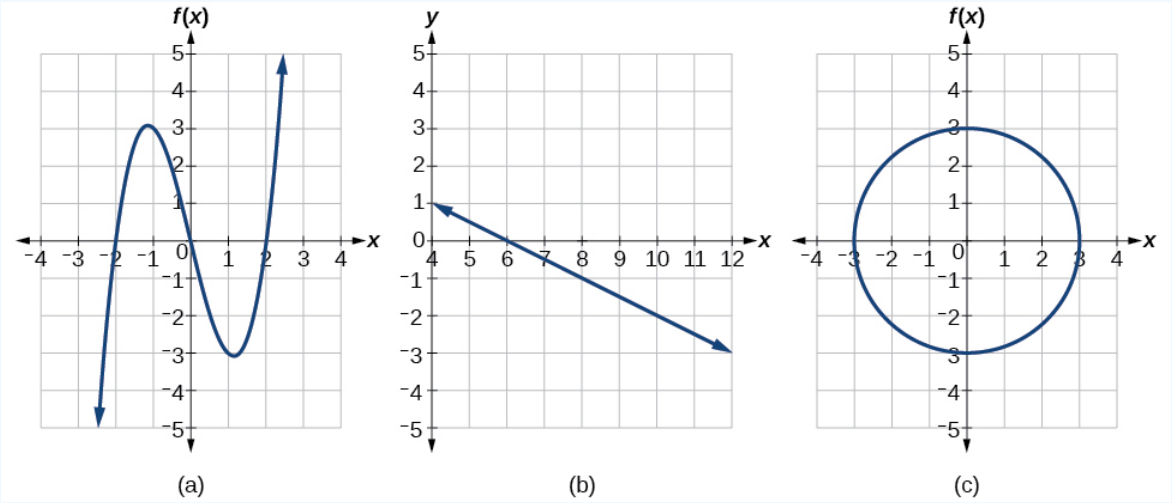


Figure 2.9.4: Graph of a polynomial (a), a downward-sloping line (b), and a circle (c).

Solution

The graphs in (a) and (b) are functions since they satisfy the vertical line test. The graph in (a) is not a one-to-one function since a horizontal line will intersect the graph more than once. The graph in (b) is a one-to-one function since there is not any horizontal line that will intersect the graph more than once. The graph is part (c) is not a function so it not a one-to-one function.

Inverse Functions

Suppose a fashion designer will be traveling to Milan for a fashion show and wants to know what the temperature will be. She finds the weather forecast online, as shown in Figure 2.9.5. She is not familiar with the Celsius scale and her internet connection went down right after she found the forecast. She decided to convert the temperature from degrees Celsius to degrees Fahrenheit because she remembers the formula

$$C = \frac{5}{9}(F - 32)$$

75 degrees Fahrenheit is a comfortable temperature for his. So, she substitutes 75 for F into the formula

$$\frac{5}{9}(75-32) \approx 24^{\circ}$$

and finds that 75 degrees Fahrenheit is about 24 degrees Celsius.

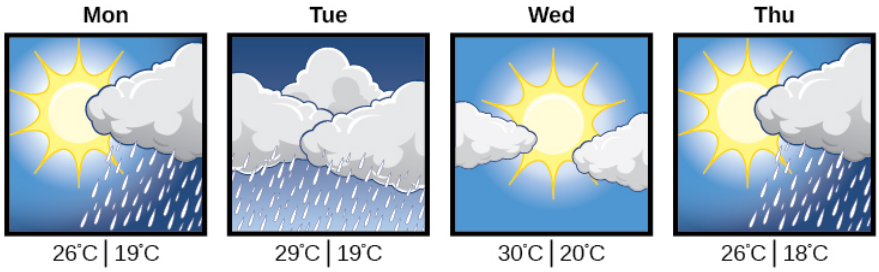


Figure 2.9.5: A forecast of Monday's through Thursday's weather.

She could do this for other temperatures but quickly realized that the most efficient thing to do is solve the equation for F. Then she can just plug in the Celsius degrees into the equation and evaluate to get the temperatures in Fahrenheit degrees. The equation,

$$F = \frac{9}{5}C + 32$$

is known as an **inverse function**.

Inverse Function

If $f(x) = y$, then a function g is an **inverse** of f if $g(y) = x$.

The inverse function of $f(x)$ is typically notated as $f^{-1}(x)$, which is read "f inverse of x". , so equivalently, if $f(x) = y$ then $f^{-1}(y) = x$.

The notation $f^{-1}(x)$ may only be used when the inverse of f is itself a a function. There are some functions that do not have inverse functions.

Caution!

The -1 used in the notation $f^{-1}(x)$ is simply a function notation where the name of the function is "f inverse", and does not designate an exponent or power of -1.

Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

Example 2.9.3

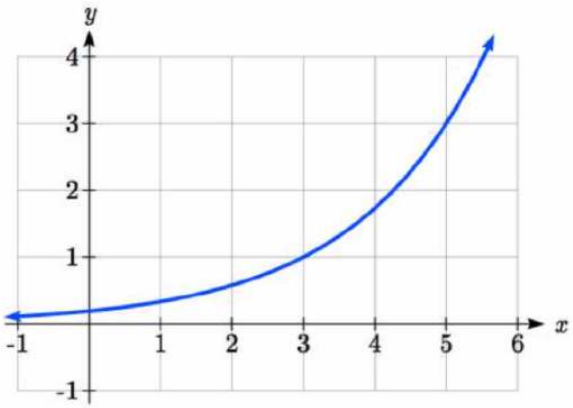
Given that $h^{-1}(6) = 2$, what are the corresponding input and output values of the original function h?

Solution

$h(2) = 6$

Example 2.9.4

A function $g(x)$ is given as a graph below. Find $g(3)$ and $g^{-1}(3)$



Solution

To evaluate $g(3)$, we find 3 on the horizontal axis and find the corresponding output value on the vertical axis. The point (3, 1) tells us that $g(3) = 1$

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means $g(x) = 3$. By looking for the output value 3 on the vertical axis we find the point (5, 3) on the graph, which means $g(5) = 3$, so by definition $g^{-1}(3) = 5$.

Example 2.9.5

A function $f(t)$ is given as a table below, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

Solution

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, the 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. Interpreting this, it means that to drive 70 miles, it took 90 minutes.

Alternatively, recall the definition of the inverse was that if $f(a) = b$ then $f^{-1}(b) = a$. By this definition, if you are given $f^{-1}(70) = a$ then you are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Notice that original function and the inverse function *undo* each other. If $f(a) = b$, then $f^{-1}(b) = a$, returning us to the original input. More simply put, if you compose a function and its inverse function together, you get the original input as your answer. Composition may be used to verify this.

Determining if Two Functions are Inverses of Each Other

How To Determine if Two Functions are Inverses of Each Other:

For two one-to-one functions, f and g,

1. Determine whether $f(g(x)) = x$ and $g(f(x)) = x$.
2. If both statements in part (1) are true, then the functions are inverses of each other ($g = f^{-1}$ and $f = g^{-1}$). If one or both statements in part (1) are false, then the functions are not inverses of each other ($g \neq f^{-1}$ and $f \neq g^{-1}$).

Example 2.9.6

If $f(x)=\dfrac{1}{x+2}$ and $g(x)=\dfrac{1}{x}-2$, is $g=f^{-1}$?

Solution

$$\begin{aligned} g(f(x)) &= \frac{1}{(\frac{1}{x+2})}-2 \\ &= x+2-2 \\ &= x \end{aligned}$$

so

$$g=f^{-1} \text{ and } f=g^{-1}$$

This is enough to answer yes to the question, but we can also verify the other formula.

$$\begin{aligned} f(g(x)) &= \frac{1}{(\frac{1}{x}-2)+2} \\ &= \frac{1}{\frac{1}{x}} \\ &= x \end{aligned}$$

You Try 2.9.2

If $f(x)=x^3-4$ and $g(x)=\sqrt[3]{x+4}$, is $g=f^{-1}$?

Answer

Yes

Example 2.9.7

If $f(x)=x^3$ (the basic cubic function) and $g(x)=\frac{1}{3}x$, is $g=f^{-1}$?

Solution

$$f(g(x))=\frac{x^3}{27}\neq x$$

No, the functions are not inverses.

Note that the correct inverse to the basic cubic function is, of course, the cube root $\sqrt[3]{x}=x^{\frac{1}{3}}$, that is, the one-third is an exponent, not a multiplier.

You Try 2.9.3

If $f(x)=(x-1)^3$ and $g(x)=\sqrt[3]{x}+1$, is $g=f^{-1}$?

Answer

Yes

Domain and Range of Inverse Functions

We have seen that the outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 2.9.6.

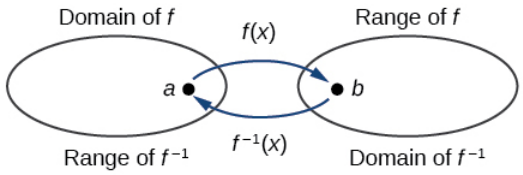


Figure 2.9.6: Domain and range of a function and its inverse.

This is important to remember when finding inverse functions. For example, the inverse of $f(x)=\sqrt{x}$ is $f^{-1}(x)=x^2, x\geq 0$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0,\infty)$, since that is the range of $f(x)=\sqrt{x}$.

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, y as a function of x— we can often find the inverse function by solving to obtain x as a function of y.

How To Find an Inverse Function:

1. Make sure f is a one-to-one function.
2. Solve for x
3. Interchange x and y to write the inverse as a function of x. Do remember to include any restrictions for the domain if it is not clearly implied in the equation you found.

Example 2.9.8

Returning to our designer traveling to Milan, find a formula for the inverse function that gives Fahrenheit temperature given a Celsius temperature.

Solution

A quick Google search would find the inverse function, but alternatively, the designer might look back at how she solved for the Fahrenheit temperature for a specific Celsius value, and repeat the process in general

$$C=\frac{5}{9}(F-32)$$
$$C\cdot\frac{9}{5}=F-32$$
$$F=\frac{9}{5}C+32$$

By solving in general, we have uncovered the inverse function. If

$$C=h(F)=\frac{5}{9}(F-32)$$

Then

$$F=h^{-1}(C)=\frac{9}{5}C+32$$

In this case, we introduced a notation for a function h to represent the conversion since the input and output variables are descriptive and writing C^{-1} could get confusing.

Example 2.9.9

Find the inverse of the function $f(x) = 2 + \sqrt{x-4}$.

Solution

$$\begin{aligned} y &= 2 + \sqrt{x-4} \\ y-2 &= \sqrt{x-4} \\ (y-2)^2 &= x-4 \\ x &= (y-2)^2 + 4 \end{aligned}$$

interchanging x and y,

$$y = (x-2)^2 + 4$$

Notice that the range of f is $[2, \infty)$, so this means that the domain of the inverse function f^{-1} is also $[2, \infty)$. The formula we found for $f^{-1}(x)$ above looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to state the restricted domain of f^{-1} to $[2, \infty)$ in order to make f^{-1} a one-to-one function.

So, $f^{-1}(x) = (x-2)^2 + 4$ where $x \geq 2$.

You Try 2.9.4

What is the inverse of the function $f(x) = 2 - \sqrt{x}$?

Answer

$f^{-1}(x) = (2-x)^2$ where $x \leq 2$

Inverse Functions and Their Graphs

Now that we know how to find the inverse of a function, let's explore the graphical relationship of functions and their inverses. Let's look at the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$. This is a one-to-one function, and it's graph is shown in Figure 2.9.7.

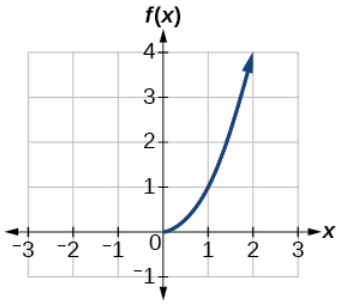


Figure 2.9.7: Quadratic function with domain restricted to $[0, \infty)$.

The inverse of this function is $f^{-1}(x) = \sqrt{x}$. The graph of both f and f^{-1} are shown in Figure 2.9.8 on the same set of axes.

Notice that the graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 2.9.8.

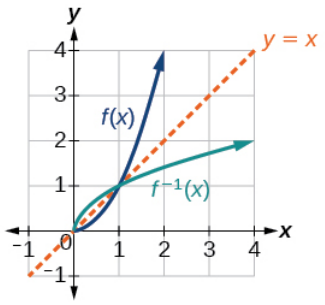


Figure 2.9.8: Square and square-root functions on the non-negative domain

This relationship will be observed for all one-to-one functions and their inverses, because it is a result of the function and its inverse swapping inputs and outputs. This is useful to know for checking for errors when finding an inverse of a one-to-one function since the inverse equation found should visually be a reflection over the line (y=x\).

Example 2.9.10

Given the graph of $f(x)$ in Figure 2.9.9, sketch a graph of $f^{-1}(x)$.

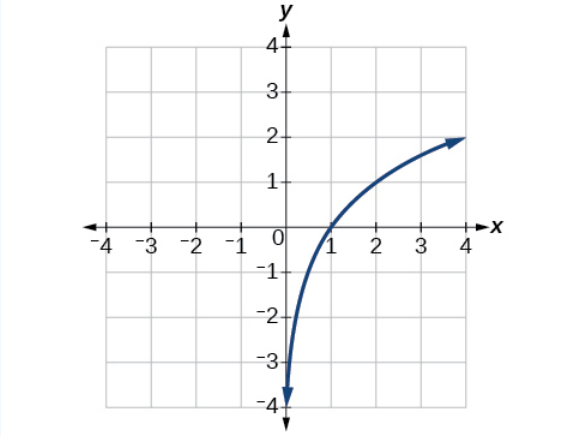


Figure 2.9.9: Graph of $f^{-1}(x)$.

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$, the point $(4, 2)$ reflects to $(2, 4)$, and so on. Sketching the inverse on the same axes as the original graph gives Figure 2.9.10

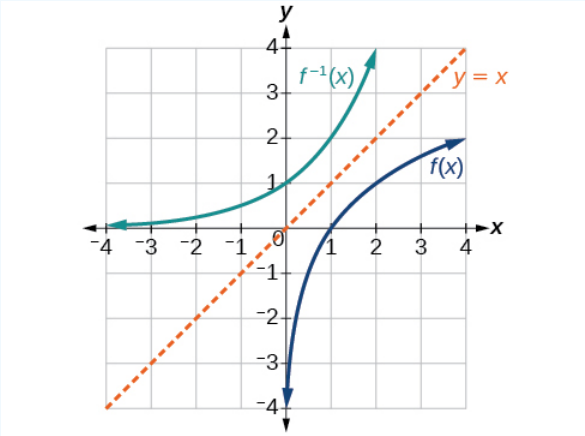


Figure 2.9.10: The function and its inverse, showing reflection about the identity line

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the basic reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

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2.E: Functions (Exercises)

This page is a draft and is under active development.

Functions and Function Notation

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

Verbal

1) What is the difference between a relation and a function?

Answer

A relation is a set of ordered pairs. A function is a special kind of relation in which no two ordered pairs have the same first coordinate.

2) What is the difference between the input and the output of a function?

3) Why does the vertical line test tell us whether the graph of a relation represents a function?

Answer

When a vertical line intersects the graph of a relation more than once, that indicates that for that input there is more than one output. At any particular input value, there can be only one output if the relation is to be a function.

4) How can you determine if a relation is a one-to-one function?

5) Why does the horizontal line test tell us whether the graph of a function is one-to-one?

Answer

When a horizontal line intersects the graph of a function more than once, that indicates that for that output there is more than one input. A function is one-to-one if each output corresponds to only one input.

Algebraic

For the exercises 6-7, determine whether the relation represents a function.

6) $\{(a,b),(c,d),(a,c)\}$

7) $\{(a,b),(b,c),(c,c)\}$

Answer

function

For the exercises 8-26, determine whether the relation represents y as a function of x.

8) $5x+2y=10$

9) $y=x^2$

Answer

function

10) $x=y^2$

11) $3x^2+y=14$

Answer

function

12) $2x+y^2=6$

13) $y=-2x^2+40x$

Answer

function

14) $y=\dfrac{1}{x}$

15) $x=\dfrac{3y+5}{7y-1}$

Answer

function

16) $x=\sqrt{1-y^2}$

17) $y=\dfrac{3x+5}{7x-1}$

Answer

function

18) $x^2+y^2=9$

19) $2xy=1$

Answer

function

20) $x=y^3$

21) $y=x^3$

Answer

function

22) $y=\sqrt{1-x^2}$

23) $x=\pm\sqrt{1-y}$

Answer

function

24) $y=\pm\sqrt{1-x}$

25) $y^2=x^2$

Answer

not a function

26) $y^3=x^2$

For the exercises 27-39, evaluate the function f at the indicated values $f(-3), f(2), f(-a), -f(a), f(a+h)$.

27) $f(x)=2x-5$

Answer

$f(-3)=-11$; $f(2)=-1$; $f(-a)=-2a-5$; $-f(a)=-2a+5$; $f(a+h)=2a+2h-5$

28) $f(x)=-5x^2+2x-1$

29) $f(x)=\sqrt{2-x}+5$

Answer

$f(-3)=\sqrt{5+5}$; $f(2)=5$; $f(-a)=\sqrt{2+a+5}$; $-f(a)=-\sqrt{2-a}-5$; $f(a+h)=\sqrt{2-a-h}+5$

30) $f(x)=\dfrac{6x-1}{5x+2}$

31) $f(x)=|x-1|-|x+1|$

Answer

$f(-3)=2$; $f(2)=1-3=-2$; $f(-a)=|-a-1|-|-a+1|$; $-f(a)=-|a-1|+|a+1|$;
$f(a+h)=|a+h-1|-|a+h+1|$

32) Given the function $g(x)=5-x^2$, evaluate $g(x+h)-g(x)h$, $h\neq 0$.

33) Given the function $g(x)=x^2+2x$, evaluate $\dfrac{g(x)-g(a)}{x-a}$, $x\neq a$.

Answer

$\dfrac{g(x)-g(a)}{x-a}=x+a+2$, $x\neq a$

34) Given the function $k(t)=2t-1$:

a. Evaluate $k(2)$.
b. Solve $k(t)=7$.

35) Given the function $f(x)=8-3x$:

a. Evaluate $f(-2)$.
b. Solve $f(x)=-1$.

Answer

a. $f(-2)=14$; b. $x=3$

36) Given the function $p(c)=c^2+c$:

a. Evaluate $p(-3)$.
b. Solve $p(c)=2$.

37) Given the function $f(x)=x^2-3x$:

a. Evaluate $f(5)$.
b. Solve $f(x)=4$.

Answer

a. $f(5)=10$; b. $x=-1$ or $x=4$

38) Given the function $f(x)=\sqrt{x+2}$:

a. Evaluate $f(7)$.
b. Solve $f(x)=4$.

39) Consider the relationship $3r+2t=18$.

a. Write the relationship as a function $r=f(t)$.
b. Evaluate $f(-3)$.

c. Solve $f(t) = 2$.

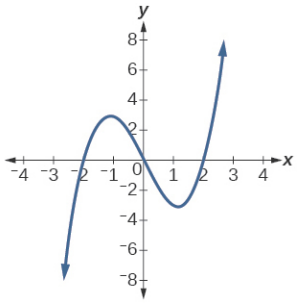
Answer

a. $f(t) = 6 - \frac{2}{3}t$; b. $f(-3) = 8$; c. $t = 6$

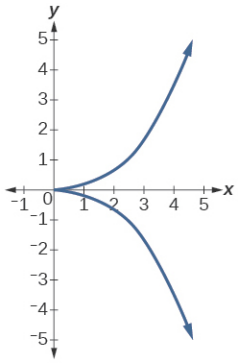
Graphical

For the exercises 40-54, use the vertical line test to determine which graphs show relations that are functions.

40)



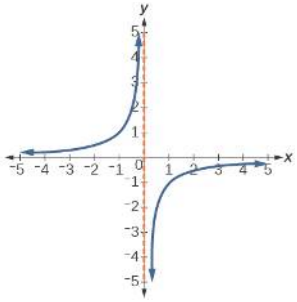
41)



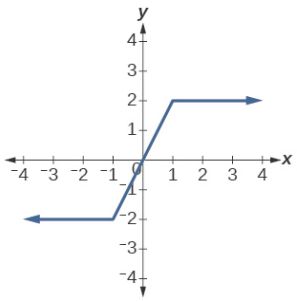
Answer

not a function

42)



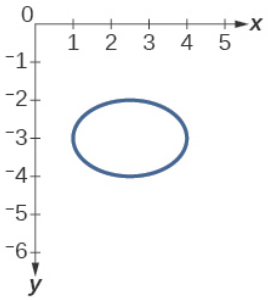
43)



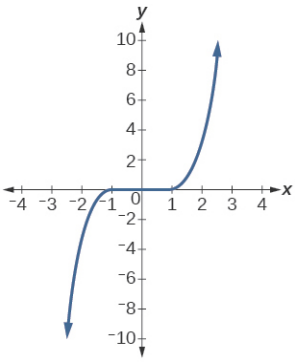
Answer

function

44)



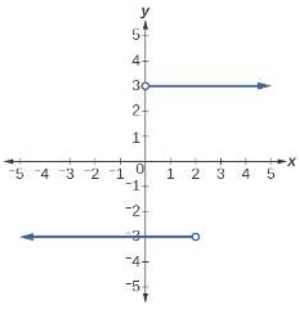
45)



Answer

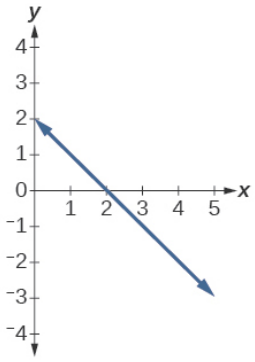
function

46)



47)

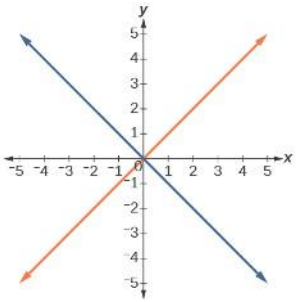
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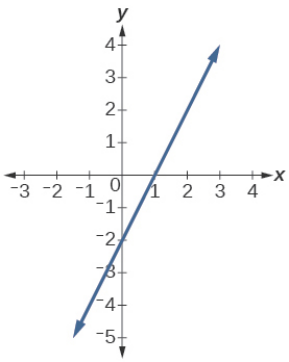
Answer

function

48)



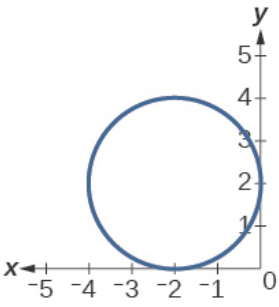
49)



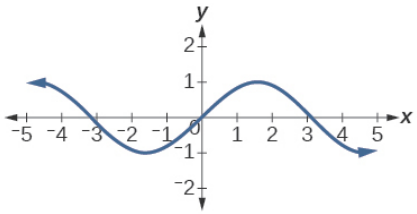
Answer

function

50)



51)

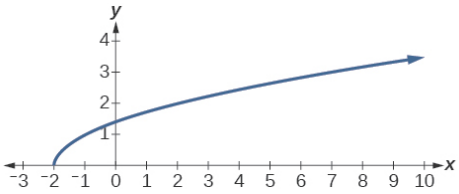


Answer

function

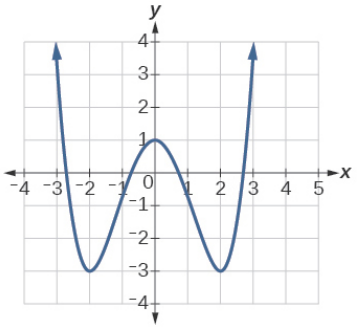
52) Given the following graph,

- Evaluate $f(-1)$.
- Solve for $f(x) = 3$.



53) Given the following graph,

- Evaluate $f(0)$.
- Solve for $f(x) = -3$.

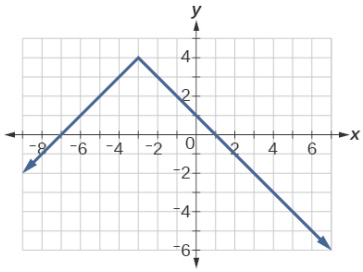


Answer

a.$f(0) = 1$; b. $f(x) = -3$, $x = -2$ or $x = 2$

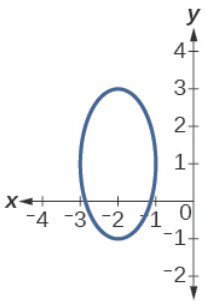
54) Given the following graph,

- Evaluate $f(4)$.
- Solve for $f(x) = 1$.



For the exercises 55-59, determine if the given graph is a one-to-one function.

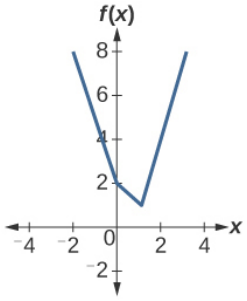
55)



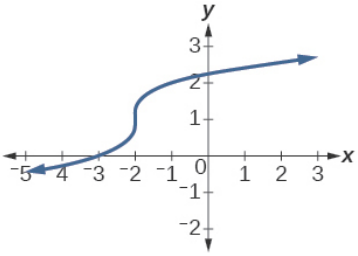
Answer

not a function so it is also not a one-to-one function

56)



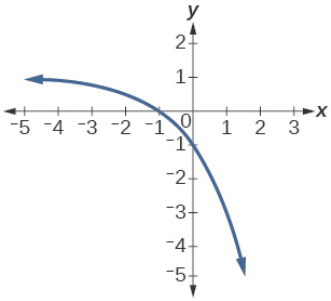
57)



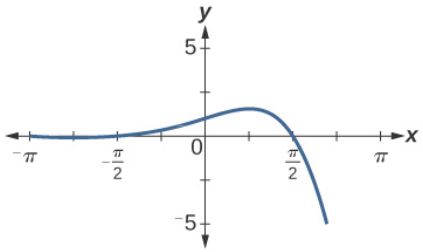
Answer

one-to-one function

58)



59)



Answer

function, but not one-to-one

Numeric

For the exercises 60-62, determine whether the relation represents a function.

60) $\{(-1,-1),(-2,-2),(-3,-3)\}$

61) $\{(3,4),(4,5),(5,6)\}$

Answer

function

62) $(2,5),(7,11),(15,8),(7,9)\}$

For the exercises 63-65, determine if the relation represented in table form represents y as a function of x.

63)

x	5	10	15
y	3	8	14

Answer

function

64)

x	5	10	15
y	3	8	8

65)

x	5	10	10
y	3	8	14

Answer

not a function

For the exercises 66-67, use the function f represented in the Table below

x	$f(x)$
0	74
1	28
2	1

x	$f(x)$
3	53
4	56
5	3
6	36
7	45
8	14
9	47

66) Evaluate $f(3)$.

67) Solve $f(x)=1$.

Answer

$f(x)=1$, $x=2$

For the exercises 68-73, evaluate the function f at the values $f(-2)$, $f(-1)$, $f(0)$, $f(1)$,and $f(2)$.

68) $f(x)=4-2x$

69) $f(x)=8-3x$

Answer

$f(-2)=14; f(-1)=11; f(0)=8; f(1)=5; f(2)=2$

70) style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:normal;font-variant:normal;font-weight:400;letter-spacing:normal;line-height:22.4px;orphans:2;text-align:justify;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">$f(x)=8x^2-7x+3$

71) $f(x)=3+\sqrt{x+3}$

Answer

$f(-2)=4$; $f(-1)=4.414$; $f(0)=4.732$; $f(1)=4.5$; $f(2)=5.236$

72) $f(x)=\dfrac{x-2}{x+3}$

73) $f(x)=3^x$

Answer

$f(-2)=\frac{1}{9}$; $f(-1)=\frac{1}{3}$; $f(0)=1$; $f(1)=3$; $f(2)=9$

For the exercises 74-75, evaluate the expressions, given functions f, g, and h:

- $f(x)=3x-2$
- $g(x)=5-x^2$
- $h(x)=-2x^2+3x-1$

74) $3f(1)-4g(-2)$

75) $f\left(\dfrac{7}{3}\right)-h(-2)$

Answer

20

Technology

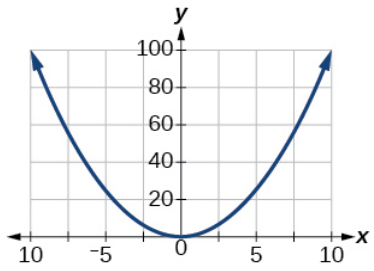
For the exercises 76-78, graph $y = x^2$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

76) $[-0.1, 0.1]$

77) $[-10, 10]$

Answer

$[0, 100]$



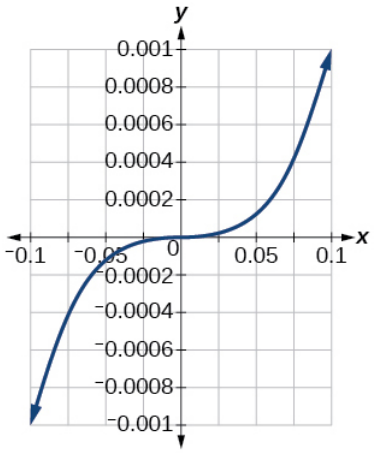
78) $[-100, 100]$

For the exercises 79-81, graph $y = x^3$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

79) $[-0.1, 0.1]$

Answer

$[-0.001, 0.001]$

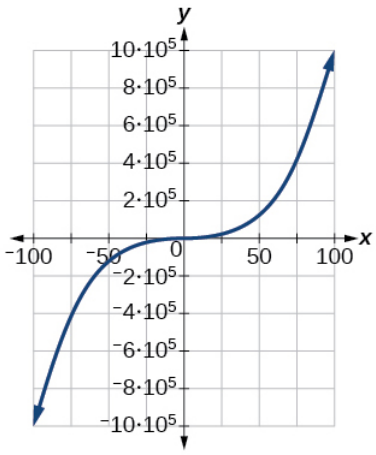


80) $[-10, 10]$

81) $[-100, 100]$

Answer

$[-1,000,000, 1,000,000]$



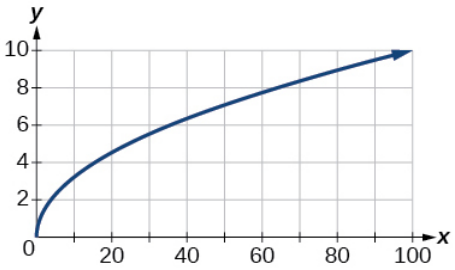
For the exercises 82-84, graph $y = \sqrt{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

82) $[0, 0.01]$

83) $[0, 100]$

Answer

$[0, 10]$



84) $[0, 10,000]$

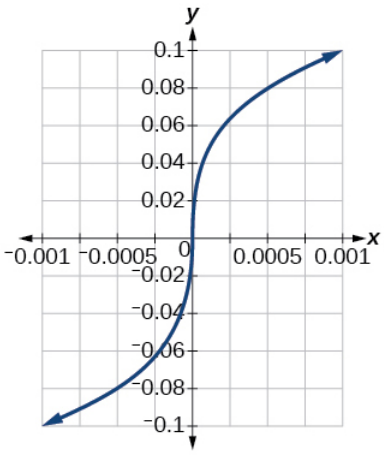
For the exercises 85-87, graph $y = \sqrt[3]{x}$ on the given viewing window. Determine the corresponding range for each viewing window. Show each graph.

85) $[-0.001, 0.001]$

Answer

$[-0.1, 0.1]$

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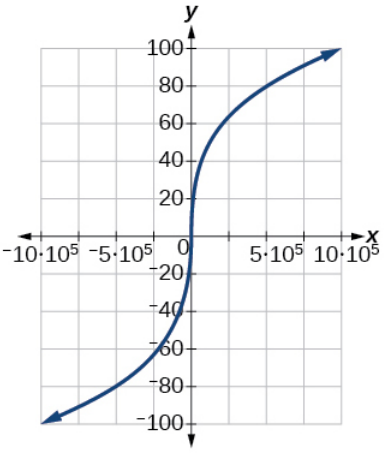


86) $[-1000, 1000]$

87) $[-1,000,000, 1,000,000]$

Answer

$[-100, 100]$



Real-World Applications

88) The amount of garbage, G, produced by a city with population p is given by $G = f(p)$. G is measured in tons per week, and p is measured in thousands of people.

The town of Tola has a population of $40,000$ and produces 13 tons of garbage each week. Express this information in terms of the function f.Tola has a population of $40,000$ and produces 13 tons of garbage each week. Express this information in terms of the function f.
Explain the meaning of the statement $f(5) = 2$.

89) The number of cubic yards of dirt, D, needed to cover a garden with area a square feet is given by $D = g(a)$.

a. A garden with area $5000\ ft^2$ requires $50\ yd^3$ of dirt. Express this information in terms of the function g.
b. Explain the meaning of the statement $g(100) = 1$.

Answer

a. $g(5000) = 50$; b. The number of cubic yards of dirt required for a garden of 100 square feet is 1.

90) Let $f(t)$ be the number of ducks in a lake t years after 1990. Explain the meaning of each statement:

a. f(5)=30
b. f(10)=40

91) Let $h(t)$ be the height above ground, in feet, of a rocket t seconds after launching. Explain the meaning of each statement:

a. $h(1) = 200$
b. $h(2) = 350$

Answer

a. The height of a rocket above ground after 1 second is 200 ft. b. the height of a rocket above ground after 2 seconds is 350 ft.

92) Show that the function $f(x) = 3(x-5)2+7$ is not one-to-one.

Domain and Range

In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section, we will investigate methods for determining the domain and range of functions.

Verbal

1) Why does the domain differ for different functions?

Answer

The domain of a function depends upon what values of the independent variable make the function undefined or imaginary.

2) How do we determine the domain of a function defined by an equation?

3) Explain why the domain of $f(x) = \sqrt[3]{x}$ is different from the domain of $f(x) = \sqrt{x}$.

Answer

There is no restriction on x for $f(x) = \sqrt[3]{x}$ because you can take the cube root of any real number. So the domain is all real numbers, $(-\infty, \infty)$. When dealing with the set of real numbers, you cannot take the square root of negative numbers. So x-values are restricted for $f(x) = \sqrt{x}$ to nonnegative numbers and the domain is $[0, \infty)$.

4) When describing sets of numbers using interval notation, when do you use a parenthesis and when do you use a bracket?

5) How do you graph a piecewise function?

Answer

Graph each formula of the piecewise function over its corresponding domain. Use the same scale for the x-axis and y-axis for each graph. Indicate inclusive endpoints with a solid circle and exclusive endpoints with an open circle. Use an arrow to indicate $-\infty$ or ∞. Combine the graphs to find the graph of the piecewise function.

Algebraic

For the exercises 6-25, find the domain of each function using interval notation.

6) style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:normal;font-variant:normal;font-weight:400;letter-spacing:normal;line-height:22.4px;orphans:2;text-align:justify;text-decoration:none;text-indent:0px;text-transform:none;visibility:visible;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">$f(x) = -2x(x-1)(x-2)$

style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:normal;font-variant:normal;font-weight:400;letter-spacing:normal;line-height:22.4px;orphans:2;text-align:justify;text-decoration:none;text-indent:0px;text-transform:none;visibility:visible;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">7) style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:normal;font-variant:normal;font-weight:400;letter-spacing:normal;line-height:22.4px;orphans:2;text-align:justify;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">$f(x) = 5 - 2x^2$

Answer

$(-\infty, \infty)$

8) $f(x) = 3\sqrt{x-2}$

9) $f(x) = 3 - \sqrt{6-2x}$

Answer

$(-\infty, 3]$

10) $f(x) = \sqrt{4-3x}$

11) $f(x) = \sqrt{x^2+4}$

Answer

$(-\infty, \infty)$

12) $f(x) = \sqrt[3]{1-2x}$

13) $f(x) = \sqrt[3]{x-1}$

Answer

$(-\infty, \infty)$

14) $f(x) = \dfrac{9}{x-6}$

15) $f(x) = \dfrac{3x+1}{4x+2}$

Answer

$\left(-\infty, -\dfrac{1}{2}\right) \cup \left(-\dfrac{1}{2}, \infty\right)$

16) $f(x) = \dfrac{\sqrt{x+4}}{x-4}$

17) $f(x) = \dfrac{x-3}{x^2+9x-22}$

Answer

$(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$

18) $f(x) = \dfrac{1}{x^2-x-6}$

19) $f(x) = \dfrac{2x^3-250}{x^2-2x-15}$

Answer

$(-\infty, -3) \cup (-3, 5) \cup (5, \infty)$

20) $\dfrac{5}{\sqrt{x-3}}$

21) $\dfrac{2x+1}{\sqrt{5-x}}$

Answer

$(-\infty, 5)$

22) $\dfrac{\sqrt{x-4}}{\sqrt{x-6}}$

23) $\dfrac{\sqrt{x-6}}{\sqrt{x-4}}$

Answer

$[6, \infty)$

24) $f(x) = \dfrac{x}{x}$

25) $f(x) = \dfrac{x^2 - 9x}{x^2 - 81}$

Answer

$(-\infty, -9) \cup (-9, 9) \cup (9, \infty)$

26) Find the domain of the function $f(x) = \sqrt{2x^3 - 50x}$ by:

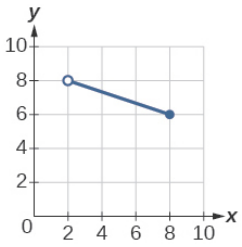
a. using algebra

b. graphing the function in the radicand and determining intervals on the x-axis for which the radicand is nonnegative.

Graphical

For the exercises 27-37, write the domain and range of each function using interval notation.

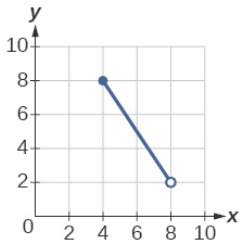
27)



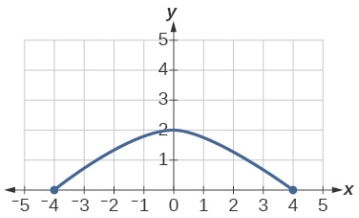
Answer

domain: $(2, 8]$, range $[6, 8)$

28)



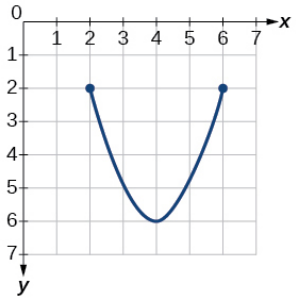
29)



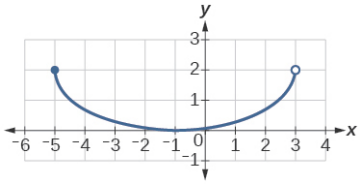
Answer

domain: $[-4, 4]$, range: $[0, 2]$

30)



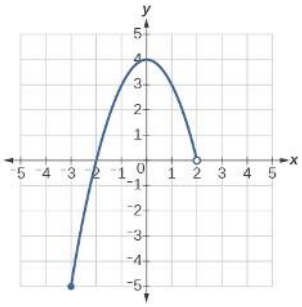
31)



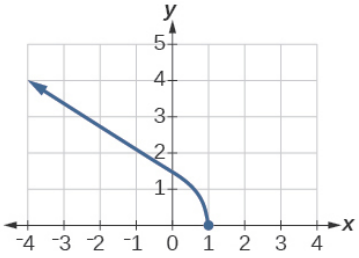
Answer

domain: $[-5, 3)$, range: $[0, 2]$

32)



33)

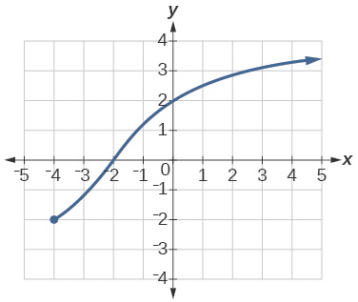


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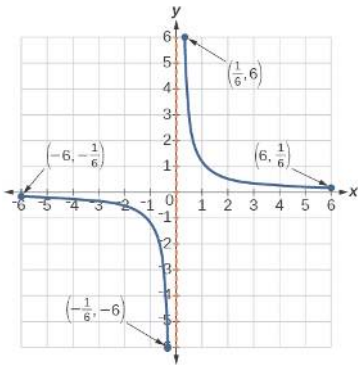
Answer

domain: $(-\infty, 1]$, range: $[0, \infty)$

34)



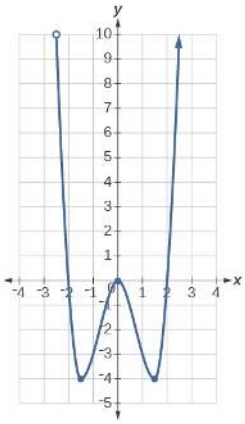
35)



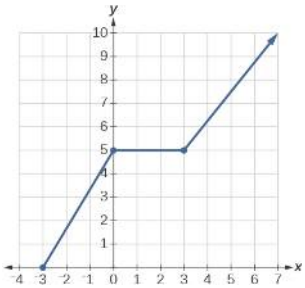
Answer

domain: $[-6, -16] \cup [16, 6]$; range: $[-6, -16] \cup [16, 6]$

36)



37)



Answer

domain: $[-3, \infty)$; range: $[0, \infty)$

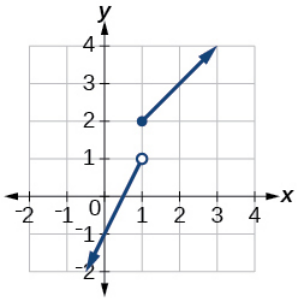
For the exercises 38-45, sketch a graph of the piecewise function. Write the domain in interval notation.

38) $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

39) $f(x) = \begin{cases} 2x-1 & \text{if } x < 1 \\ 1+x & \text{if } x \geq 1 \end{cases}$

Answer

domain: $(-\infty, \infty)$

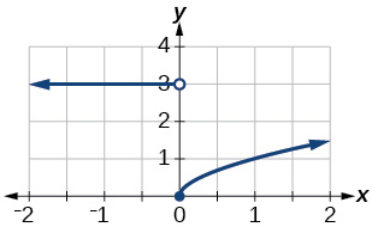


40) $f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x-1 & \text{if } x > 0 \end{cases}$

41) $f(x) = \begin{cases} 3 & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$

Answer

domain: $(-\infty, \infty)$

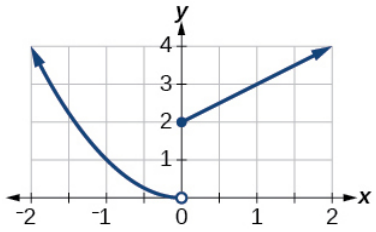


42) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1-x & \text{if } x > 0 \end{cases}$

43) $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ x+2 & \text{if } x \geq 0 \end{cases}$

Answer

domain: $(-\infty, \infty)$

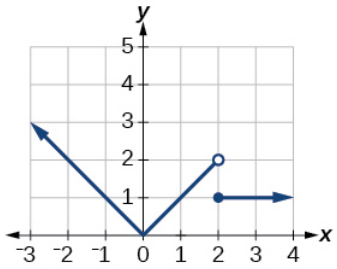


44) $f(x) = \begin{cases} x+1 & \text{if } x < 1 \\ x^3 & \text{if } x \geq 1 \end{cases}$

45) $f(x) = \begin{cases} |x| & \text{if } x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$

Answer

domain: $(-\infty, \infty)$



Numeric

For the exercises 46-48, given each function f,evaluate $f(-3)$, $f(-2)$, $f(-1)$, and $f(0)$.

46) $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

47) $f(x) = \begin{cases} 1 & \text{if } x \leq -3 \\ 0 & \text{if } x > -3 \end{cases}$

Answer

$f(-3) = 1$; $f(-2) = 0$; $f(-1) = 0$; $f(0) = 0$

48) $f(x) = \begin{cases} -2x^2+3 & \text{if } x \leq -1 \\ 5x-7 & \text{if } x > -1 \end{cases}$

For the exercises 49-51, given each function f, evaluate $f(-1)$, $f(0)$, $f(2)$, and $f(4)$.

49) $f(x) = \begin{cases} 7x+3 & \text{if } x < 0 \\ 7x+6 & \text{if } x \geq 0 \end{cases}$

Answer

$f(-1) = -4$; $f(0) = 6$; $f(2) = 20$; $f(4) = 34$

50) $f(x) = \begin{cases} x^2-2 & \text{if } x < 2 \\ 4+|x-5| & \text{if } x \geq 2 \end{cases}$

51) $f(x) = \begin{cases} 5x & \text{if } x < 0 \\ 3 & \text{if } 0 \geq x \leq 2 \\ x^2 & \text{if } x > 3 \end{cases}$

Answer

$f(-1) = -5$; $f(0) = 3$; $f(2) = 3$; $f(4) = 16$

For the exercises 52-54, write the domain for the piecewise function in interval notation.

52) $f(x) = \begin{cases} x+1 & \text{if } x < -2 \\ -2x-3 & \text{if } x \geq -2 \end{cases}$

53) $f(x) = \begin{cases} x^2-2 & \text{if } x < 1 \\ -x^2+2 & \text{if } x > 1 \end{cases}$

Answer

domain: $(-\infty, 1) \cup (1, \infty)$

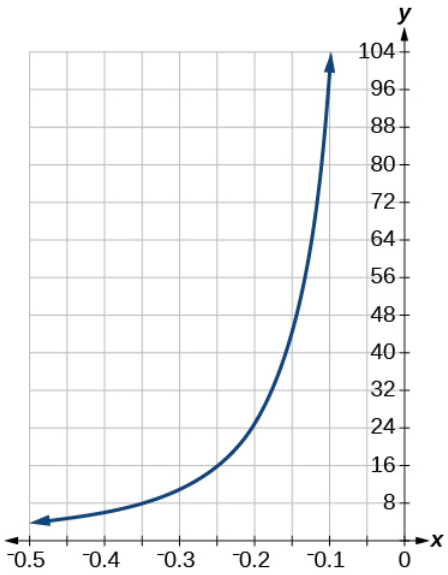
54) $f(x) = \begin{cases} x^2-3 & \text{if } x < 0 \\ -3x^2 & \text{if } x \geq 2 \end{cases}$

Technology

55) Graph $y = \dfrac{1}{x^2}$ on the viewing window $[-0.5, -0.1]$ and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

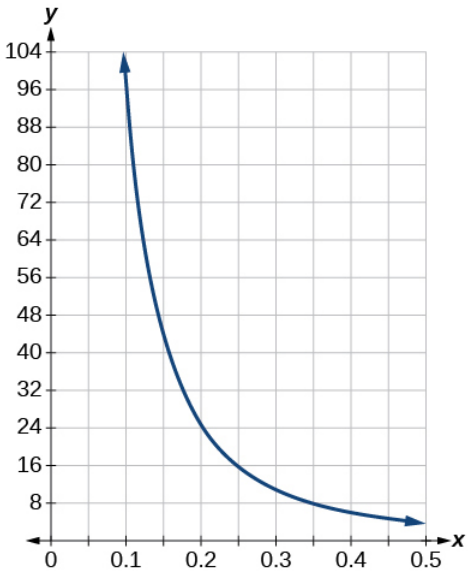
Answer

window: $[-0.5, -0.1]$; range: $[4, 100]$



window: $[0.1, 0.5]$; range: $[4, 100]$

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56) Graph $y = \frac{1}{x}$ on the viewing window $[-0.5, -0.1]$and $[0.1, 0.5]$. Determine the corresponding range for the viewing window. Show the graphs.

Extension

57) Suppose the range of a function f is $[-5, 8]$. What is the range of $|f(x)|$?

Answer

$[0, 8]$

58) Create a function in which the range is all nonnegative real numbers.

59) Create a function in which the domain is $x > 2$.

Answer

Many answers. One function is $f(x) = \frac{1}{\sqrt{x-2}}$.

Real-World Applications

60) The height h of a projectile is a function of the time t it is in the air. The height in feet for t seconds is given by the function $h(t) = -16t^2 + 96t$. What is the domain of the function? What does the domain mean in the context of the problem?

Answer

The domain is $[0, 6]$; it takes 6 seconds for the projectile to leave the ground and return to the ground

61) The cost in dollars of making x items is given by the function $C(x) = 10x + 500$.

a. The fixed cost is determined when zero items are produced. Find the fixed cost for this item.

b. What is the cost of making 25 items?

c. Suppose the maximum cost allowed is $\$1500$. What are the domain and range of the cost function, $C(x)$?

Rates of Change and Behavior of Graphs

In this section, we will investigate changes in functions. For example, a rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data. Identifying points that mark the interval on a graph can be used to find the average rate of change. Comparing pairs of input and output values in a table can also be used to find the average rate of change.

Verbal

1) Can the average rate of change of a function be constant?

Answer

Yes, the average rate of change of all linear functions is constant.

2) If a function f is increasing on (a, b) and decreasing on (b, c), then what can be said about the local extremum of f on (a, c)?

3) How are the absolute maximum and minimum similar to and different from the local extrema?

Answer

The absolute maximum and minimum relate to the entire graph, whereas the local extrema relate only to a specific region around an open interval.

4) How does the graph of the absolute value function compare to the graph of the quadratic function, $y = x^2$, in terms of increasing and decreasing intervals?

Algebraic

For the exercises 5-15, find the average rate of change of each function on the interval specified for real numbers b or h.

5) $f(x) = 4x^2 - 7$ on $[1, b]$

Answer

$4(b+1)$

6) $g(x) = 2x^2 - 9$ on $[4, b]$

7) $p(x) = 3x + 4$ on $[2, 2+h]$

Answer

3

8) $k(x) = 4x - 2$ on $[3, 3+h]$

9) $f(x) = 2x^2 + 1$ on $[x, x+h]$

Answer

$4x + 2h$

10) $g(x) = 3x^2 - 2$ on $[x, x+h]$

11) $a(t) = \dfrac{1}{t+4}$ on $[9, 9+h]$

Answer

$\dfrac{-1}{13(13+h)}$

12) $b(x) = 1x + 3$ on $[1, 1+h]$

13) $j(x) = 3x^3$ on $[1, 1+h]$

Answer

$3h^2 + 9h + 9$

14) $r(t) = 4t^3$ on $[2, 2+h]$

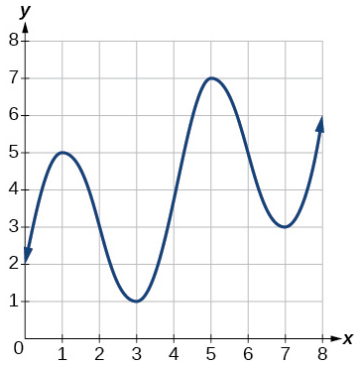
15) $\dfrac{f(x+h) - f(x)}{h}$ given $f(x) = 2x^2 - 3x$ on $[x, x+h]$

Answer

$4x+2h-3$

Graphical

For the exercises 16-21, consider the graph of f shown in the Figure below



Graph of a polynomial.

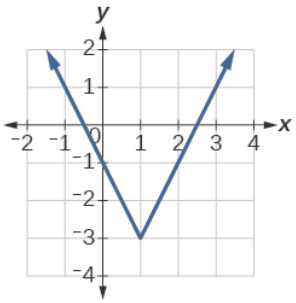
16) Estimate the average rate of change from $x=1$ to $x=4$.

17) Estimate the average rate of change from $x=2$ to $x=5$.

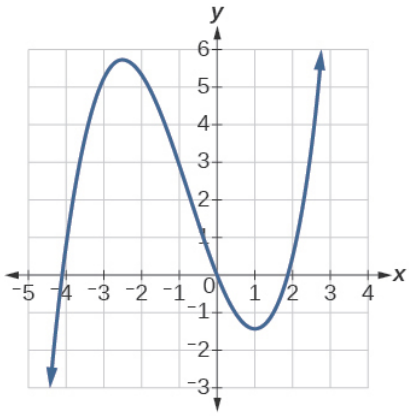
Answer

$\frac{4}{3}$

18)



19)

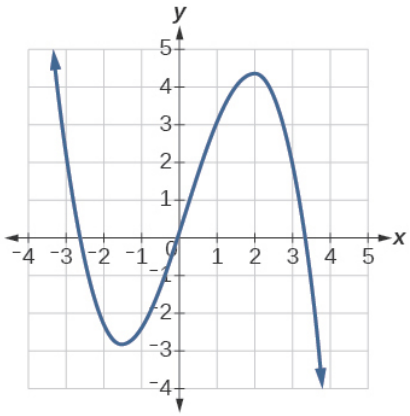


Answer

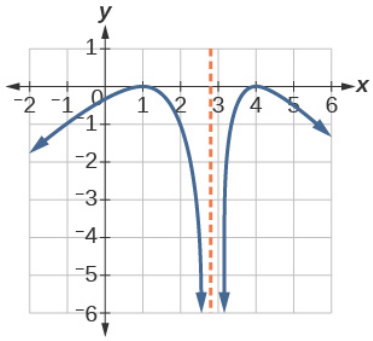
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increasing on $(-\infty, -2.5) \cup (1, \infty)$, decreasing on $(-2.5, 1)$

20)



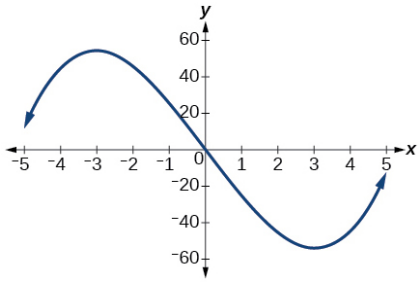
21)



Answer

increasing on $(-\infty, 1) \cup (3, 4)$, decreasing on $(1, 3) \cup (4, \infty)$

For the exercises 22-23, consider the graph shown in the Figure below.



Graph of a cubic function.

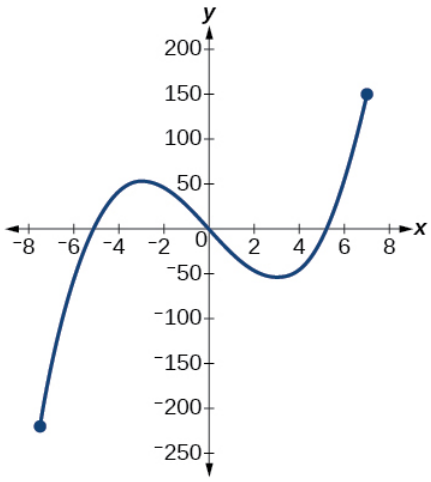
22) style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:normal;font-variant:normal;font-weight:400;letter-spacing:normal;line-height:22.4px;orphans:2;text-align:justify;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Estimate the intervals where the function is increasing or decreasing.

style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:normal;font-variant:normal;font-weight:400;letter-spacing:normal;line-height:22.4px;orphans:2;text-align:justify;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">23) Estimate the point(s) at which the graph of f has a local maximum or a local minimum.

Answer

local maximum: $(-3, 60)$, local minimum: $(3, -60)$

For the exercises 24-25, consider the graph in the Figure below.



Graph of a cubic function.

24) If the complete graph of the function is shown, estimate the intervals where the function is increasing or decreasing.

25) If the complete graph of the function is shown, estimate the absolute maximum and absolute minimum.

Answer

absolute maximum at approximately $(7, 150)$, absolute minimum at approximately $(-7.5, -220)$

Numeric

26) The Table below gives the annual sales (in millions of dollars) of a product from 1998 to 2006. What was the average rate of change of annual sales (a) between 2001 and 2002, and (b) between 2001 and 2004?

Year	Sales (millions of dollars)
1998	201
1999	219
2000	233
2001	243
2002	249
2003	251
2004	249
2005	243
2006	233

27) The Table below gives the population of a town (in thousands) from 2000 to 2008. What was the average rate of change of population (a) between 2002 and 2004, and (b) between 2002 and 2006?

Year	Population (thousands)
2000	87
2001	84

Year	Population (thousands)
2002	83
2003	80
2004	77
2005	76
2006	78
2007	81
2008	85

Answer

a. -3000; b. -1250

For the exercises 28-34, find the average rate of change of each function on the interval specified.

28) $f(x) = x^2$ on $[1, 5]$

29) $h(x) = 5 - 2x^2$ on $[-2, 4]$

Answer

-4

30) $q(x) = x^3$ on $[-4, 2]$

31) $g(x) = 3x^3 - 1$ on $[-3, 3]$

Answer

27

32) $y = \dfrac{1}{x}$ on $[1, 3]$

33) $p(t) = \dfrac{(t^2 - 4)(t + 1)}{t^2 + 3}$ on $[-3, 1]$

Answer

-0.167

34) $k(t) = 6t^2 + \dfrac{4}{t^3}$ on $[-1, 3]$

Technology

For the exercises 35-40, use a graphing utility to estimate the local extrema of each function and to estimate the intervals on which the function is increasing and decreasing.

35) $f(x) = x^4 - 4x^3 + 5$

Answer

Local minimum at $(3, -22)$, decreasing on $(-\infty, 3)$, increasing on $(3, \infty)$

36) $h(x) = x^5 + 5x^4 + 10x^3 + 10x^2 - 1$

37) $g(t) = t\sqrt{t + 3}$

Answer

Local minimum at $(-2,-2)$, decreasing on $(-3,-2)$, increasing on $(-2,\infty)$

38) $k(t)=3t^{\frac{2}{3}}-t$

39) $m(x)=x^4+2x^3-12x^2-10x+4$

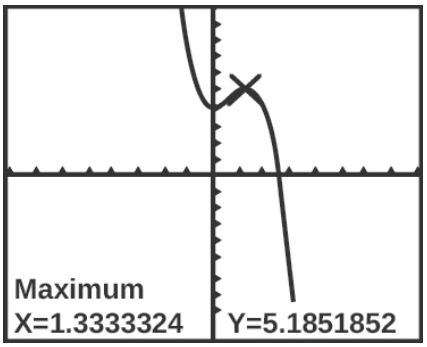
Answer

Local maximum at $(-0.5,6)$, local minima at $(-3.25,-47)$ and $(2.1,-32)$, decreasing on $(-\infty,-3.25)$ and $(-0.5,2.1)$, increasing on $(-3.25,-0.5)$and $(2.1,\infty)$

40) $n(x)=x^4-8x^3+18x^2-6x+2$

Extension

41) The graph of the function f is shown in the Figure below:



style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of f(x) on a graphing calculator.

Based on the calculator screen shot, the point $(1.333,5.185)$is which of the following?

a. a relative (local) maximum of the function
b. the vertex of the function
c. the absolute maximum of the function
d. a zero of the function

Answer

a

42) Let $f(x)=\dfrac{1}{x}$. Find a number c such that the average rate of change of the function f on the interval $(1,c)$ is $-\dfrac{1}{4}$.

43) Let $f(x)=\dfrac{1}{x}$. Find the number b such that the average rate of change of f on the interval $(2,b)$ is $-\dfrac{1}{10}$.

Answer

$b=5$

Real-World Applications

44) At the start of a trip, the odometer on a car read $21,395$. At the end of the trip, 13.5 hours later, the odometer read $22,125$. Assume the scale on the odometer is in miles. What is the average speed the car traveled during this trip?

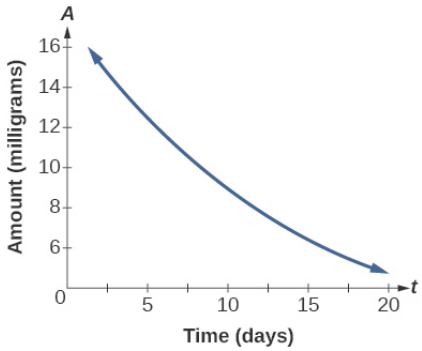
45) A driver of a car stopped at a gas station to fill up his gas tank. He looked at his watch, and the time read exactly 3:40 p.m. At this time, he started pumping gas into the tank. At exactly 3:44, the tank was full and he noticed that he had pumped 10.7 gallons. What is the average rate of flow of the gasoline into the gas tank?

Answer

2.7 gallons per minute

46) Near the surface of the moon, the distance that an object falls is a function of time. It is given by $d(t) = 2.6667t^2$, where t is in seconds and $d(t)$ is in feet. If an object is dropped from a certain height, find the average velocity of the object from $t = 1$ to $t = 2$.

47) The graph in the Figure below illustrates the decay of a radioactive substance over t days.



style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of an exponential function.

Use the graph to estimate the average decay rate from $t = 5$ to $t = 15$.

Answer

approximately -0.6 milligrams per day

Composition of Functions

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. The cost depends on the temperature, and the temperature depends on the day. By combining these two relationships into one function, we have performed function composition, which is the focus of this section.

Verbal

1) How does one find the domain of the quotient of two functions, $\frac{f}{g}$?

Answer

Find the numbers that make the function in the denominator g equal to zero, and check for any other domain restrictions on f and g, such as an even-indexed root or zeros in the denominator.

2) What is the composition of two functions, $f \circ g$?

3) If the order is reversed when composing two functions, can the result ever be the same as the answer in the original order of the composition? If yes, give an example. If no, explain why not.

Answer

Yes. Sample answer: Let $f(x) = x + 1$ and $g(x) = x - 1$. Then $f(g(x)) = f(x-1) = (x-1)+1 = x$ and $g(f(x)) = g(x+1) = (x+1)-1 = x$. So $f \circ g = g \circ f$.

4) How do you find the domain for the composition of two functions, $f \circ g$?

5) Given $f(x)=x^2+2x$ and $g(x)=6-x^2$,find $f+g$, $f-g$, fg, and $\frac{f}{g}$.Determine the domain for each function in interval notation.

Answer

$(f+g)(x)=2x+6$, domain: $(-\infty,\infty)$

$(f-g)(x)=2x^2+2x-6$, domain: $(-\infty,\infty)$

$(fg)(x)=-x^4-2x^3+6x^2+12x$, domain: $(-\infty,\infty)$

$\left(\frac{f}{g}\right)(x)=\frac{x^2+2x}{6-x^2}$, domain: $(-\infty,-\sqrt{6})\cup(\sqrt{6},\sqrt{6})\cup(\sqrt{6},\infty)$

6) Given $f(x)=-3x^2+x$ and $g(x)=5$, find $f+g$, $f-g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

7) Given $f(x)=2x^2+4x$ and $g(x)=\frac{1}{2x}$,find $f+g$, $f-g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

Answer

$(f+g)(x)=\frac{4x^3+8x^2+1}{2x}$, domain: $(-\infty,0)\cup(0,\infty)$

$(f-g)(x)=\frac{4x3+8x2-1}{2x}$, domain: $(-\infty,0)\cup(0,\infty)$

$(fg)(x)=x+2$, domain: $(-\infty,0)\cup(0,\infty)$

$(fg)(x)=4x^3+8x^2$, domain: $(-\infty,0)\cup(0,\infty)$

8) Given $f(x)=\frac{1}{x-4}$ and $g(x)=\frac{1}{6-x}$, find$f+g$, $f-g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

9) Given $f(x)=3x^2$ and $g(x)=\sqrt{x-5}$, find $f+g$, $f-g$, fg, and $\frac{f}{g}$. Determine the domain for each function in interval notation.

Answer

$(f+g)(x)=3x^2+\sqrt{x-5}$, domain: $[5,\infty)$

$(f-g)(x)=3x^2-\sqrt{x-5}$, domain: $[5,\infty)$

$(fg)(x)=3x^2\sqrt{x-5}$, domain: $[5,\infty)$

$\left(\frac{f}{g}\right)(x)=\frac{3x^2}{\sqrt{x-5}}$, domain: $(5,\infty)$

10) Given $f(x)=\sqrt{x}$ and $g(x)=|x-3|$, find $\frac{g}{f}$. Determine the domain of the function in interval notation.

11) Given $f(x)=2x^2+1$ and $g(x)=3x-5$, find the following:

a. $f(g(2))$
b. $f(g(x))$
c. $g(f(x))$
d. $(g\circ g)(x)$
e. $(f\circ f)(-2)$

Answer

a. 3
b. $f(g(x)) = 2(3x-5)^2+1$
c. $f(g(x)) = 6x^2-2$
d. $(g\circ g)(x) = 3(3x-5)-5 = 9x-20$
e. $(f\circ f)(-2) = 163$

For the exercises 12-17, use each pair of functions to find $f(g(x))$ and $g(f(x))$. Simplify your answers.

12) $f(x) = x^2+1$, $g(x) = \sqrt{x+2}$

13) $f(x) = \sqrt{x}+2$, $g(x) = x^2+3$

Answer

$f(g(x)) = \sqrt{x+3}$, $g(f(x)) = x+4\sqrt{x}+7$

14) $f(x) = |x|$, $g(x) = 5x+1$

15) $f(x) = \sqrt[3]{x}$, $g(x) = \dfrac{x+1}{x^3}$

Answer

$f(g(x)) = \sqrt[3]{\dfrac{x+1}{x^3}} = \dfrac{\sqrt[3]{x+1}}{x}$, $g(f(x)) = \dfrac{\sqrt[3]{x}+1}{x}$

16) $f(x) = \dfrac{1}{x-6}$, $g(x) = \dfrac{7}{x}+6$

17) $f(x) = \dfrac{1}{x-4}$, $g(x) = \dfrac{2}{x}+4$

Answer

$(f\circ g)(x) = \dfrac{1}{\frac{2}{x}+4-4} = \dfrac{x}{2}$, $(g\circ f)(x) = 2x-4$

For the exercises 18-25, use each set of functions to find f(g(h(x))). Simplify your answers.

18) $f(x) = x^4+6$, $g(x) = x-6$, and $h(x) = \sqrt{x}$

19) $f(x) = x^2+1$, $g(x) = \dfrac{1}{x}$, and $h(x) = x+3$

Answer

$f(g(h(x))) = \left(\dfrac{1}{x+3}\right)^2+1$

20) Given $f(x) = \dfrac{1}{x}$ and $g(x) = x-3$, find the following:

a. $(f\circ g)(x)$
b. the domain of $(f\circ g)(x)$ in interval notation
c. $(g\circ f)(x)$
d. the domain of $(g\circ f)(x)$
e. $\left(\dfrac{f}{g}\right)x$

21) Given $f(x) = \sqrt{2-4x}$ and $g(x) = -\dfrac{3}{x}$, find the following:

a. $(g\circ f)(x)$
b. the domain of $(g\circ f)(x)$ in interval notation

Answer

a. $(g\circ f)(x)=-\dfrac{3}{\sqrt{2-4x}}$

b. $\left(-\infty,\dfrac{1}{2}\right)$

22) Given the functions $f(x)=\dfrac{1-x}{x}$ and $g(x)=\dfrac{1}{1+x^2}$, find the following:

a. $(g\circ f)(x)$

b. $(g\circ f)(2)$

23) Given functions $p(x)=\dfrac{1}{\sqrt{x}}$ and $m(x)=x^2-4$, state the domain of each of the following functions using interval notation:

a. $\dfrac{p(x)}{m(x)}$

b. $p(m(x))$

c. $m(p(x))$

Answer

a. $(0,2)\cup(2,\infty)$

b. $(-\infty,-2)\cup(2,\infty)$

c. $(0,\infty)$

24) Given functions $q(x)=\dfrac{1}{\sqrt{x}}$ and $h(x)=x^2-9$, state the domain of each of the following functions using interval notation.

a. $\dfrac{q(x)}{h(x)}$

b. $q(h(x))$

c. $h(q(x))$

25) For $f(x)=\dfrac{1}{x}$ and $g(x)=\sqrt{x-1}$, write the domain of $(f\circ g)(x)$ in interval notation.

Answer

$(1,\infty)$

For the exercises 26-41, find functions $f(x)$ and $g(x)$ so the given function can be expressed as $h(x)=f(g(x))$.

26) $h(x)=(x+2)^2$

27) $h(x)=(x-5)^3$

Answer

sample: $f(x)=x^3$

$g(x)=x-5$

28) $h(x)=\dfrac{3}{x-5}$

29) $h(x)=\dfrac{4}{(x+2)^2}$

Answer

sample: $f(x)=\dfrac{4}{x}$

$g(x)=(x+2)^2$

30) $h(x) = 4 + \sqrt[3]{x}$

31) $h(x) = \sqrt[3]{\dfrac{1}{2x-3}}$

Answer

sample: $f(x) = \sqrt[3]{x}$

$g(x) = \dfrac{1}{2x-3}$

32) $h(x) = \dfrac{1}{(3x^2-4)^{-3}}$

33) $h(x) = \sqrt[4]{\dfrac{3x-2}{x+5}}$

Answer

sample: $f(x) = \sqrt[4]{x}$

$g(x) = \dfrac{3x-2}{x+5}$

34) $h(x) = (\dfrac{8+x^3}{8-x^3})^4$

35) $h(x) = \sqrt{2x+6}$

Answer

sample: $f(x) = \sqrt{x}$

$g(x) = 2x+6$

36) $h(x) = (5x-1)^3$

37) $h(x) = \sqrt[3]{x-1}$

Answer

sample: $f(x) = \sqrt[3]{x}$
$g(x) = (x-1)$

38) $h(x) = |x^2+7|$

39) $h(x) = \dfrac{1}{(x-2)^3}$

Answer

sample: $f(x) = x^3$

$g(x) = \dfrac{1}{x-2}$

40) $h(x) = \left(\dfrac{1}{2x-3}\right)^2$

41) $h(x) = \sqrt{\dfrac{2x-1}{3x+4}}$

Answer

sample: $f(x) = \sqrt{x}$
$g(x) = \dfrac{2x-1}{3x+4}$

Graphical

For the exercises 42-49, use the graphs of f,shown in Figure 1, and g,shown in Figure 2, to evaluate the expressions.

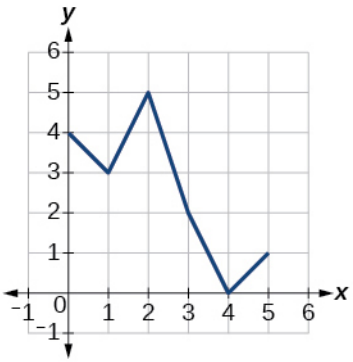


Figure 1: Graph of a function.

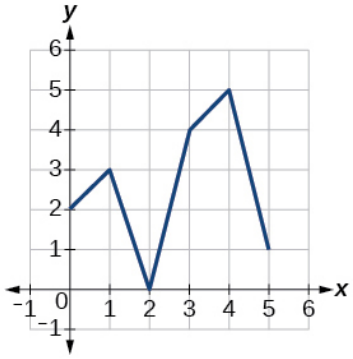


Figure 2: *Graph of a function.*

42) $f(g(3))$

43) $f(g(1))$

Answer

2

44) $g(f(1))$

45) $g(f(0))$

Answer

5

46) $f(f(5))$

47) $f(f(4))$

Answer

4

48) $g(g(2))$

49) $g(g(0))$

Answer

0

For the exercises 50-57, use graphs of $f(x)$,shown in Figure 3, $g(x)$,shown in Figure 4, and $h(x)$,shown in Figure 5, to evaluate the expressions.

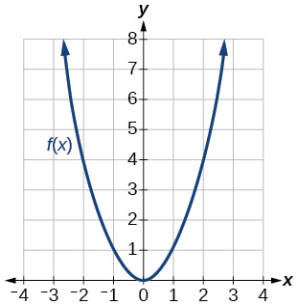


Figure 3: Graph of a parabola.

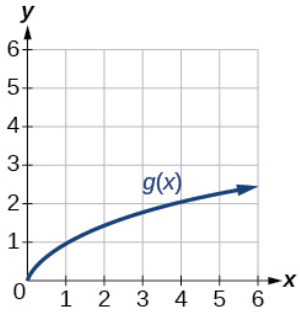


Figure 4: *Graph of a square root function.*

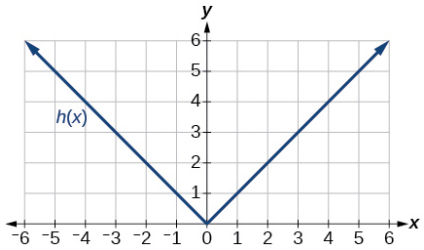


Figure 5: Graph of an absolute value function.

50) $g(f(1))$

51) $g(f(2))$

Answer

2

52) $f(g(4))$

53) $f(g(1))$

Answer

1

54) $f(h(2))$

55) $h(f(2))$

Answer

4

56) $f(g(h(4)))$

57) $f(g(f(-2)))$

Answer

4

Numeric

For the exercises 58-65, use the function values for f and g shown in the Table below to evaluate each expression.

x	$f(x)$	$g(x)$
0	7	9
1	6	5
2	5	6
3	8	2
4	4	1
5	0	8
6	2	7
7	1	3
8	9	4
9	3	0

58) $f(g(8))$

59) $f(g(5))$

Answer

9

60) $g(f(5))$

61) $g(f(3))$

Answer

4

62) $f(f(4))$

63) $f(f(1))$

Answer

2

64) $g(g(2))$

65) $g(g(6))$

Answer

3

For the exercises 66-71, use the function values for f and g shown in the Table below to evaluate the expressions.

https://math.libretexts.org/@go/page/126515

x	$f(x)$	$g(x)$
-3	11	-8
-2	9	-3
-1	7	0
0	5	1
1	3	0
2	1	-3
3	-1	-8

66) $(f \circ g)(1)$

67) $(f \circ g)(2)$

Answer

11

68) $(g \circ f)(2)$

69) $(g \circ f)(3)$

Answer

0

70) $(g \circ g)(1)$

71) $(f \circ f)(3)$

Answer

7

For the exercises 72-75, use each pair of functions to find $f(g(0))$ and $g(f(0))$.

72) $f(x) = 4x + 8$, $g(x) = 7 - x^2$

73) $f(x) = 5x + 7$, $g(x) = 4 - 2x^2$

Answer

$f(g(0)) = 27$, $g(f(0)) = -94$

74) $f(x) = \sqrt{x + 4}$, $g(x) = 12 - x^3$

75) $f(x) = \frac{1}{x+2}$, $g(x) = 4x + 3$

Answer

$f(g(0)) = \frac{1}{5}$, $g(f(0)) = 5$

For the exercises 76-79, use the functions $f(x) = 2x^2 + 1$ and $g(x) = 3x + 5$ to evaluate or find the composite function as indicated.

76) $f(g(2))$

77) $f(g(x))$

Answer

$18x^2+60x+51$

78) $g(f(-3))$

79) $(g\circ g)(x)$

Answer

$g\circ g(x)=9x+20$

Extensions

For the exercises 80-84, use $f(x)=x^3+1$ and $g(x)=\sqrt[3]{x-1}$.

80) Find $(f\circ g)(x)$ and $(g\circ f)(x)$. Compare the two answers.

81) Find $(f\circ g)(2)$ and $(g\circ f)(2)$.

Answer

2

82) What is the domain of $(g\circ f)(x)$?

83) What is the domain of $(f\circ g)(x)$?

Answer

$(-\infty,\infty)$

84) Let $f(x)=\frac{1}{x}$.

a. Find $(f\circ f)(x)$.
b. Is $(f\circ f)(x)$ for any function f the same result as the answer to part (a) for any function? Explain.

For the exercises 85-87, let $F(x)=(x+1)^5$, $f(x)=x^5$, and $g(x)=x+1$.

85) True or False: $(g\circ f)(x)=F(x)$.

Answer

False

86) True or False: $(f\circ g)(x)=F(x)$.

For the exercises 87-89, find the composition when $f(x)=x^2+2$ for all $x\geq 0$ and $g(x)=\sqrt{x-2}$.

87) $(f\circ g)(6)$; $(g\circ f)(6)$

Answer

$(f\circ g)(6)=6$; $(g\circ f)(6)=6$

88) $(g\circ f)(a)$; $(f\circ g)(a)$

89) $(f\circ g)(11)$; $(g\circ f)(11)$

Answer

$(f\circ g)(11)=11$, $(g\circ f)(11)=11$

Real-World Applications

90) The function $D(p)$ gives the number of items that will be demanded when the price is p. The production cost $C(x)$ is the cost of producing x items. To determine the cost of production when the price is $\$6$, you would do which of the following?

a. Evaluate $D(C(6))$
b. Evaluate $C(D(6))$

c. Solve $D(C(x)) = 6$
d. Solve $C(D(p)) = 6$

91) The function $A(d)$ gives the pain level on a scale of 0 to 10 experienced by a patient with d milligrams of a pain-reducing drug in her system. The milligrams of the drug in the patient's system after t minutes is modeled by $m(t)$. Which of the following would you do in order to determine when the patient will be at a pain level of 4?

a. Evaluate $A(m(4))$
b. Evaluate $m(A(4))$
c. Solve $A(m(t)) = 4$
d. Solve $m(A(d)) = 4$

Answer

c

92) A store offers customers a 30% discount on the price x of selected items. Then, the store takes off an additional 15% at the cash register. Write a price function $P(x)$ that computes the final price of the item in terms of the original price x. (Hint: Use function composition to find your answer.)

93) A rain drop hitting a lake makes a circular ripple. If the radius, in inches, grows as a function of time in minutes according to $r(t) = 25\sqrt{t+2}$, find the area of the ripple as a function of time. Find the area of the ripple at $t = 2$.

Answer

$A(t) = \pi(25\sqrt{t+2})^2$ and $A(2) = \pi(25\sqrt{4})^2 = 2500\pi$ square inches

94) A forest fire leaves behind an area of grass burned in an expanding circular pattern. If the radius of the circle of burning grass is increasing with time according to the formula $r(t) = 2t + 1$, express the area burned as a function of time, t (minutes).

95) Use the function you found in the previous exercise to find the total area burned after 5 minutes.

Answer

$A(5) = \pi(2(5)+1)2 = 121\pi$ square units

96) The radius r, in inches, of a spherical balloon is related to the volume, V, by $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$. Air is pumped into the balloon, so the volume after t seconds is given by $V(t) = 10 + 20t$.

a. Find the composite function $r(V(t))$.
b. Find the exact time when the radius reaches 10 inches.

97) The number of bacteria in a refrigerated food product is given by $N(T) = 23T^2 - 56T + 1$, $3 < T < 33$, where T is the temperature of the food. When the food is removed from the refrigerator, the temperature is given by $T(t) = 5t + 1.5$, where t is the time in hours.

a. Find the composite function $N(T(t))$.
b. Find the time (round to two decimal places) when the bacteria count reaches 6752.

Answer

a. $N(T(t)) = 23(5t+1.5)^2 - 56(5t+1.5) + 1$
b. 3.38hours

Transformation of Functions

Often when given a problem, we try to model the scenario using mathematics in the form of words, tables, graphs, and equations. One method we can employ is to adapt the basic graphs of the toolkit functions to build new models for a given scenario. There are systematic ways to alter functions to construct appropriate models for the problems we are trying to solve.

Verbal

1) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal shift from a vertical shift?

Answer

A horizontal shift results when a constant is added to or subtracted from the input. A vertical shifts results when a constant is added to or subtracted from the output.

2) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal stretch from a vertical stretch?

3) When examining the formula of a function that is the result of multiple transformations, how can you tell a horizontal compression from a vertical compression?

Answer

A horizontal compression results when a constant greater than 1 is multiplied by the input. A vertical compression results when a constant between 0 and 1 is multiplied by the output.

4) When examining the formula of a function that is the result of multiple transformations, how can you tell a reflection with respect to the x-axis from a reflection with respect to the y-axis?

5) How can you determine whether a function is odd or even from the formula of the function?

Answer

For a function f, substitute (−x) for (x) in $f(x)$. Simplify. If the resulting function is the same as the original function, $f(-x)=f(x)$, then the function is even. If the resulting function is the opposite of the original function, $f(-x)=-f(x)$, then the original function is odd. If the function is not the same or the opposite, then the function is neither odd nor even.

Algebraic

6) Write a formula for the function obtained when the graph of $f(x)=\sqrt{x}$ is shifted up 1 unit and to the left 2 units.

7) Write a formula for the function obtained when the graph of $f(x)=|x|$ is shifted down 3 units and to the right 1 unit.

Answer

$g(x)=|x-1|-3$

8) Write a formula for the function obtained when the graph of $f(x)=\dfrac{1}{x}$ is shifted down 4 units and to the right 3 units.

9) Write a formula for the function obtained when the graph of $f(x)=\dfrac{1}{x^2}$ is shifted up 2 units and to the left 4 units.

Answer

$g(x)=\dfrac{1}{(x+4)^2}+2$

For the exercises 10-19, describe how the graph of the function is a transformation of the graph of the original function f.

10) $y=f(x-49)$

11) $y=f(x+43)$

Answer

The graph of $f(x+43)$ is a horizontal shift to the left 43 units of the graph of f.

12) $y=f(x+3)$

13) $y=f(x-4)$

Answer

The graph of $f(x-4)$ is a horizontal shift to the right 4 units of the graph of f.

14) $y=f(x)+5$

15) $y=f(x)+8$

Answer

The graph of $f(x)+8$ is a vertical shift up 8 units of the graph of f.

16) $y=f(x)-2$

17) $y=f(x)-7$

Answer

The graph of $f(x)-7$ is a vertical shift down 7 units of the graph of f.

18) $y=f(x-2)+3$

19) style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:normal;font-variant:normal;font-weight:400;letter-spacing:normal;line-height:22.4px;orphans:2;text-align:justify;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">$y=f(x+4)-1$

Answer

The graph of $f(x+4)-1$ is a horizontal shift to the left 4 units and a vertical shift down 1 unit of the graph of f.

For the exercises 20-23, determine the interval(s) on which the function is increasing and decreasing.

20) $f(x)=4(x+1)^2-5$

21) $g(x)=5(x+3)^2-2$

Answer

decreasing on $(-\infty,-3)$ and increasing on $(-3,\infty)$

22) $a(x)=\sqrt{-x+4}$

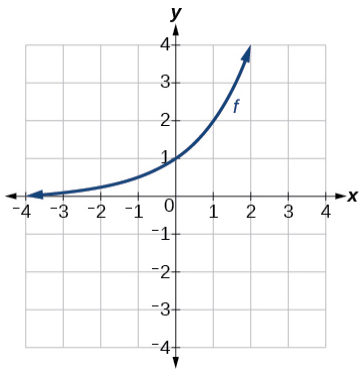
23) $k(x)=-3\sqrt{x}-1$

Answer

decreasing on $(0,\infty)$

Graphical

For the exercises 24-26, use the graph of $f(x)=2^x$ shown in the Figure below to sketch a graph of each transformation of $f(x)$.

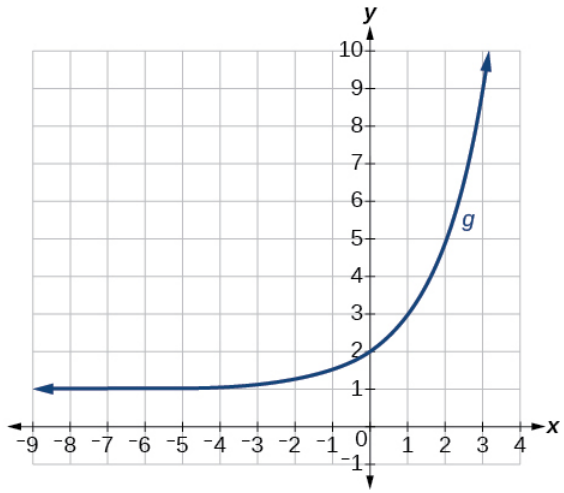


Graph of $f(x)$.

25) $h(x) = 2^x - 3$

24) $g(x) = 2^x + 1$

Answer



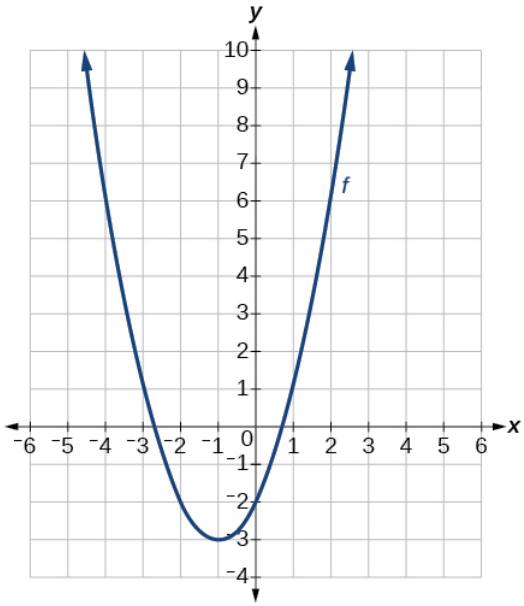
Graph of $k(x)$

26) $w(x) = 2^x - 1$

For the exercises 27-30, sketch a graph of the function as a transformation of the graph of one of the toolkit functions.

27) $f(t) = (t+1)^2 - 3$

Answer

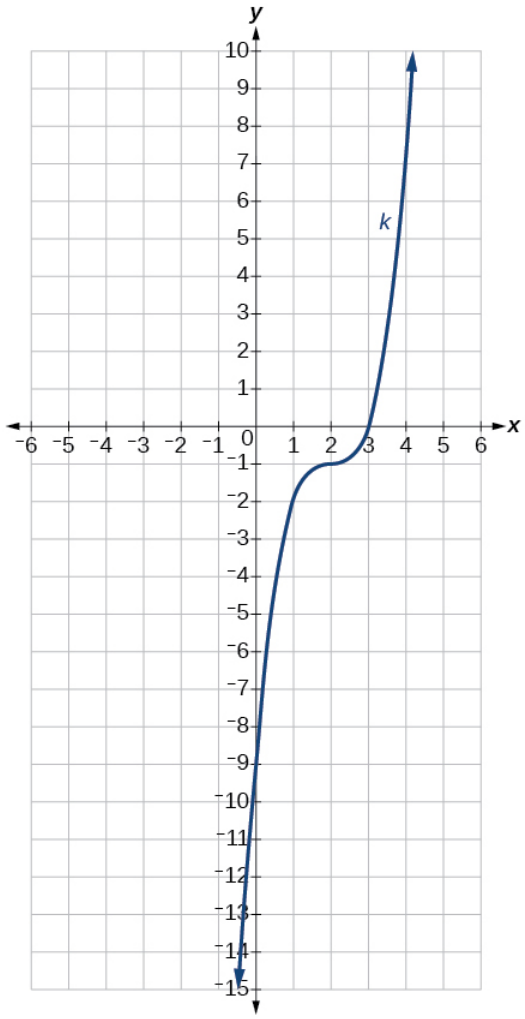


Graph of $f(t)$.

28) $h(x) = |x-1| + 4$

29) $k(x) = (x-2)^3 - 1$

Answer



Graph of $k(x)$.

30) $m(t) = 3 + \sqrt{t+2}$

Numeric

31) Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

x	-2	-1	0	1	2
$f(x)$	-2	-1	-3	1	2

x	-1	0	1	2	3
$g(x)$	-2	-1	-3	1	2

x	-2	-1	0	1	2
$h(x)$	-1	0	-2	2	3

Answer

$g(x) = f(x-1)$, $h(x) = f(x)+1$

32) Tabular representations for the functions f, g, and h are given below. Write $g(x)$ and $h(x)$ as transformations of $f(x)$.

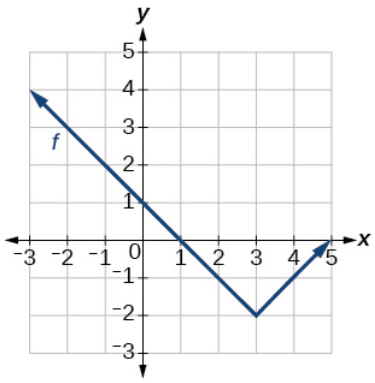
x	-2	-1	0	1	2
$f(x)$	-1	-3	4	2	1

x	-3	-2	-1	0	1
$g(x)$	-1	-3	4	2	1

x	-2	-1	0	1	2
$h(x)$	-2	-4	3	1	0

For the exercises 33-40, write an equation for each graphed function by using transformations of the graphs of one of the toolkit functions.

33)

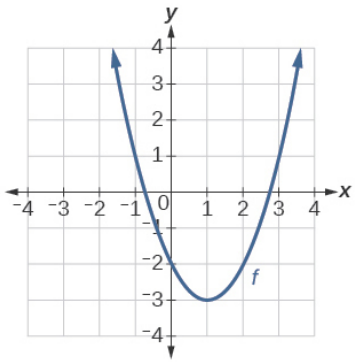


style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of an absolute function.

Answer

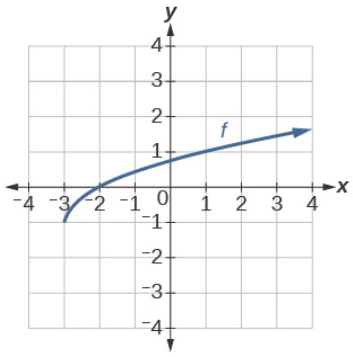
$f(x) = |x - 3| - 2$

34)



style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of a parabola.

35)

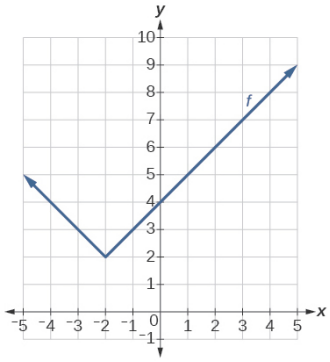


style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of a square root function.

Answer

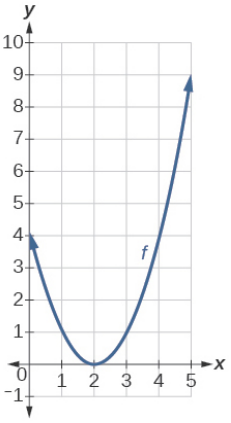
$f(x) = \sqrt{x+3} - 1$

36)



style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of an absolute function.

37)



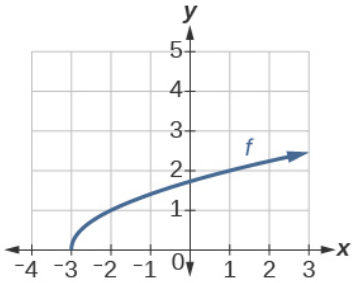
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style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of a parabola

Answer

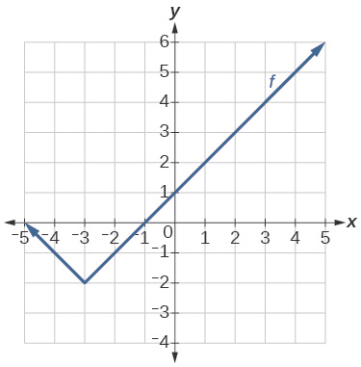
$f(x) = (x-2)^2$

38)



style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of a square root function.

39)

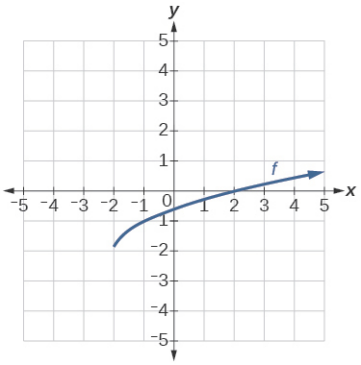


Graph of an absolute function.

Answer

$f(x) = |x+3| - 2$

40)

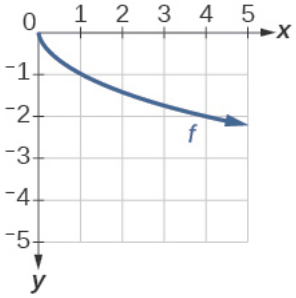


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style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of a square root function.

For the exercises 41-42, use the graphs of transformations of the square root function to find a formula for each of the functions.

41)

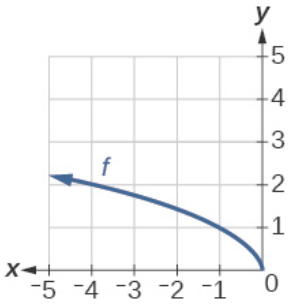


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Answer

$f(x) = -\sqrt{x}$

42)

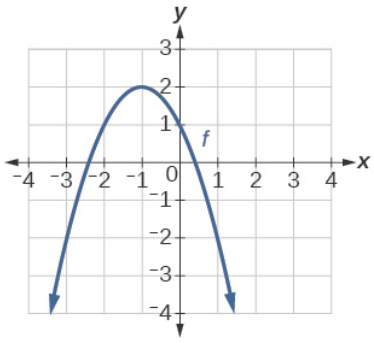


style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of a square root function.

For the exercises 43-46, use the graphs of the transformed toolkit functions to write a formula for each of the resulting functions.

43)

https://math.libretexts.org/@go/page/126515

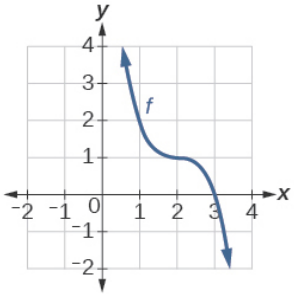


style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of a parabola.

Answer

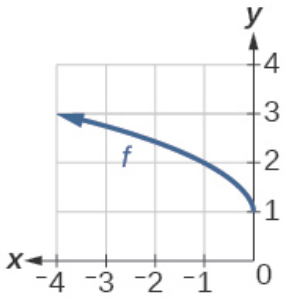
$f(x) = -(x+1)^2 + 2$

44)



style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of a cubic function.

45)

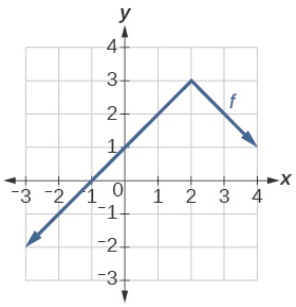


style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of a square root function.

Answer

$f(x) = \sqrt{-x} + 1$

46)



style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:italic;font-variant:normal;font-weight:400;letter-spacing:normal;orphans:2;text-align:center;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">Graph of an absolute function.

For the exercises 47-52, determine whether the function is odd, even, or neither.

47) $f(x) = 3x^4$

Answer

even

48) $g(x) = \sqrt{x}$

49) $h(x) = \dfrac{1}{x} + 3x$

Answer

odd

50) $f(x) = (x-2)^2$

51) $g(x) = 2x^4$

Answer

even

52) $h(x) = 2x - x^3$

For the exercises 53-62, describe how the graph of each function is a transformation of the graph of the original function f.

53) $g(x) = -f(x)$

Answer

The graph of g is a vertical reflection (across the x-axis) of the graph of f.

54) $g(x) = f(-x)$

55) $g(x) = 4f(x)$

Answer

The graph of g is a vertical stretch by a factor of 4 of the graph of f.

56) $g(x) = 6f(x)$

57) $g(x) = f(5x)$

Answer

The graph of g is a horizontal compression by a factor of 15 of the graph of f.

58) $g(x) = f(2x)$

59) $g(x) = f\left(\frac{1}{3}x\right)$

Answer

The graph of g is a horizontal stretch by a factor of 3 of the graph of f.

60) $g(x) = f\left(\frac{1}{5}x\right)$

61) $g(x) = 3f(-x)$

Answer

The graph of g is a horizontal reflection across the y-axis and a vertical stretch by a factor of 3 of the graph of f.

62) $g(x) = -f(3x)$

For the exercises 63-68, write a formula for the function g that results when the graph of a given toolkit function is transformed as described.

63) The graph of $f(x) = |x|$ is reflected over the y-axis and horizontally compressed by a factor of $\frac{1}{4}$.

Answer

$g(x) = |-4x|$

64) The graph of $f(x) = \sqrt{x}$ is reflected over the x-axis and horizontally stretched by a factor of 2.

65) The graph of $f(x) = \frac{1}{x^2}$ is vertically compressed by a factor of $\frac{1}{3}$, then shifted to the left 2 units and down 3 units.

Answer

$g(x) = \frac{1}{3(x+2)^2} - 3$

66) The graph of $f(x) = \frac{1}{x}$ is vertically stretched by a factor of 8, then shifted to the right 4 units and up 2 units.

67) The graph of $f(x) = x^2$ is vertically compressed by a factor of $\frac{1}{2}$, then shifted to the right 5 units and up 1 unit.

Answer

$g(x) = \frac{1}{2}(x-5)^2 + 1$

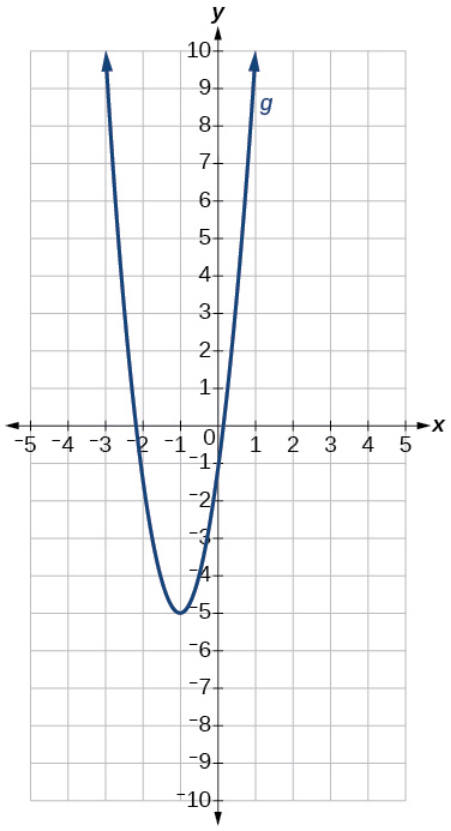
68) The graph of $f(x) = x^2$ is horizontally stretched by a factor of 3, then shifted to the left 4 units and down 3 units.

For the exercises 69-77, describe how the formula is a transformation of a toolkit function. Then sketch a graph of the transformation.

69) $g(x) = 4(x+1)^2 - 5$

Answer

The graph of the function $f(x) = x^2$ is shifted to the left 1 unit, stretched vertically by a factor of 4, and shifted down 5 units.



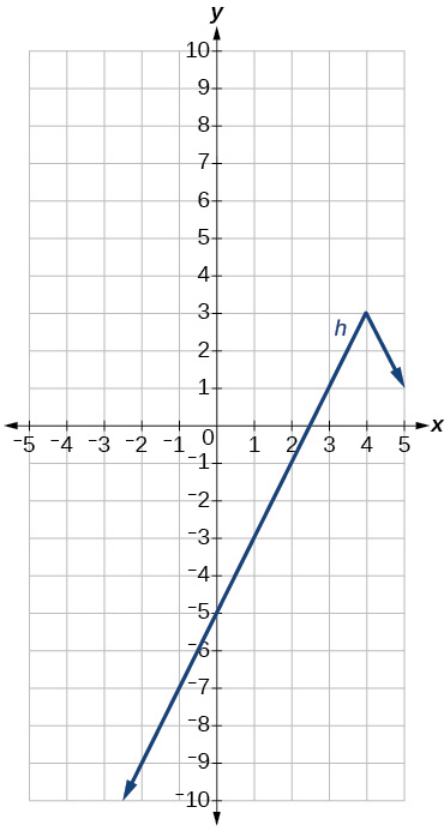
Graph of a parabola.

70) $g(x) = 5(x+3)^2 - 2$

71) $h(x) = -2|x-4| + 3$

Answer

The graph of $f(x) = |x|$ is stretched vertically by a factor of 2, shifted horizontally 4 units to the right, reflected across the horizontal axis, and then shifted vertically 3 units up.



Graph of an absolute value function.

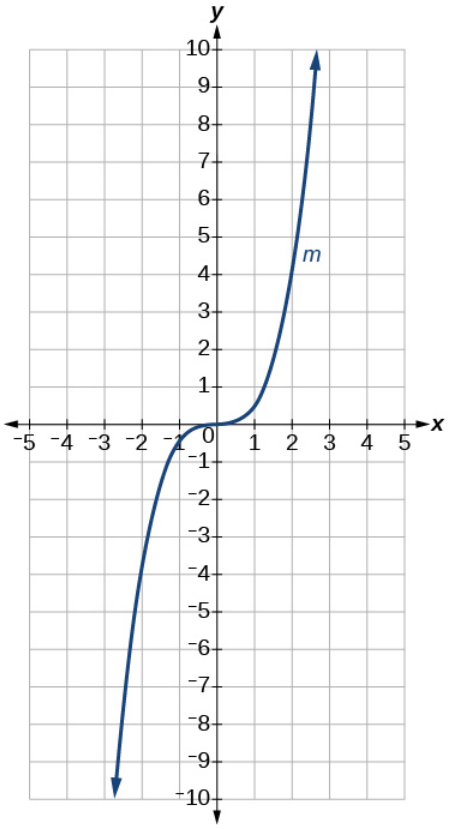
72) $k(x) = -3\sqrt{x} - 1$

73) $m(x) = \frac{1}{2}x^3$

Answer

The graph of the function $f(x) = x^3$ is compressed vertically by a factor of $\frac{1}{2}$.

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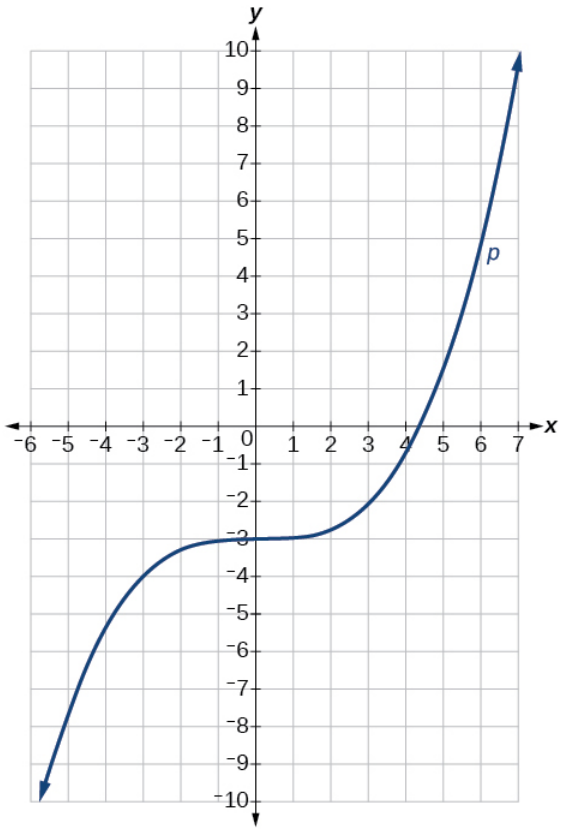
Graph of a cubic function.

74) $n(x) = \frac{1}{3}|x-2|$

75) $p(x) = (\frac{1}{3}x)^3 - 3$

Answer

The graph of the function is stretched horizontally by a factor of 3 and then shifted vertically downward by 3 units.



Graph of a cubic function.

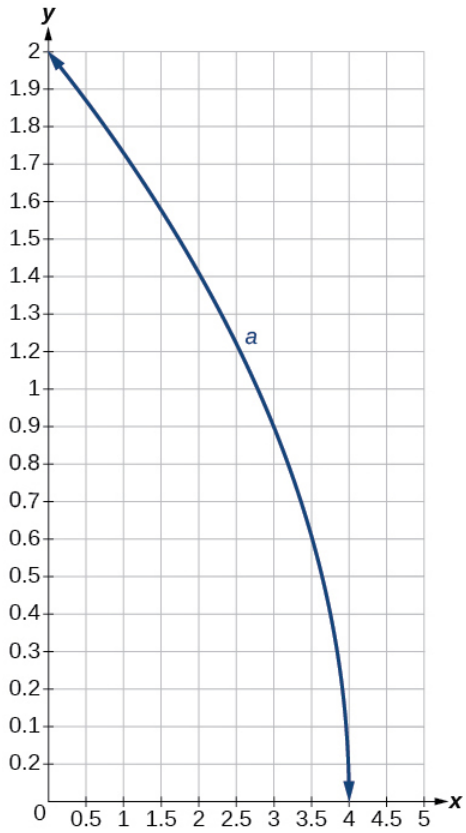
76) $q(x) = (\frac{1}{4}x)^3 + 1$

77) $a(x) = \sqrt{-x+4}$

Answer

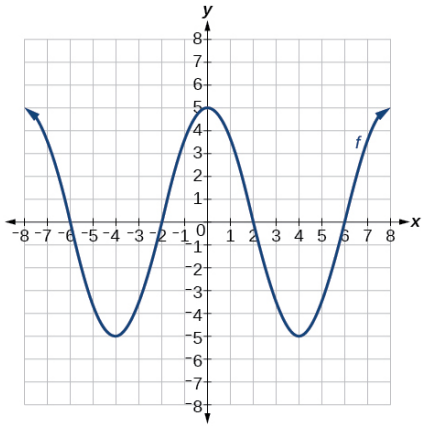
The graph of $f(x) = \sqrt{x}$ is shifted right 4 units and then reflected across the vertical line $x = 4$.

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Graph of a square root function.

For the exercises 78-81, use the graph in the Figure below to sketch the given transformations.



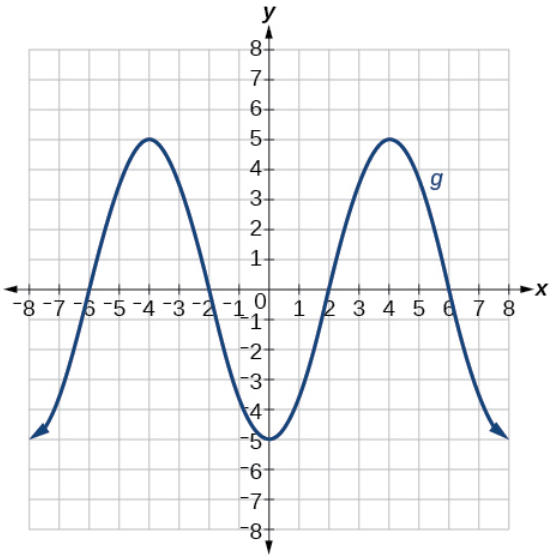
Graph of a polynomial.

78) $g(x) = f(x) - 2$

79) $g(x) = -f(x)$

Answer

https://math.libretexts.org/@go/page/126515

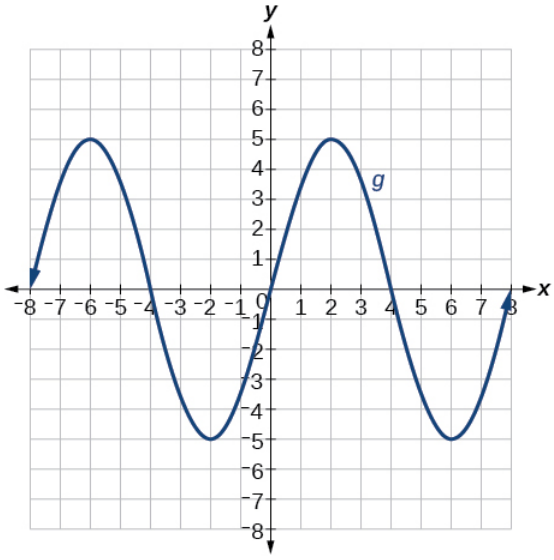


Graph of a polynomial.

80) $g(x) = f(x+1)$

81) $g(x) = f(x-2)$

Answer



Graph of a polynomial.

Absolute Value Functions

Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate absolute value functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

Verbal

1) How do you solve an absolute value equation?

Answer

Isolate the absolute value term so that the equation is of the form $|A| = B$. Form one equation by setting the expression inside the absolute value symbol, A, equal to the expression on the other side of the equation, B. Form a second equation by setting A equal to the opposite of the expression on the other side of the equation, $-B$. Solve each equation for the variable.

2) How can you tell whether an absolute value function has two x-intercepts without graphing the function?

3) When solving an absolute value function, the isolated absolute value term is equal to a negative number. What does that tell you about the graph of the absolute value function?

Answer

The graph of the absolute value function does not cross the x-axis, so the graph is either completely above or completely below the x-axis.

4) How can you use the graph of an absolute value function to determine the x-values for which the function values are negative?

5) How do you solve an absolute value inequality algebraically?

Answer

First determine the boundary points by finding the solution(s) of the equation. Use the boundary points to form possible solution intervals. Choose a test value in each interval to determine which values satisfy the inequality.

Algebraic

6) Describe all numbers x that are at a distance of 4 from the number 8. Express this using absolute value notation.

7) Describe all numbers x that are at a distance of $\frac{1}{2}$ from the number -4. Express this using absolute value notation.

Answer

$|x+4| = \frac{1}{2}$

8) Describe the situation in which the distance that point x is from 10 is at least 15 units. Express this using absolute value notation.

9) Find all function values $f(x)$ such that the distance from $f(x)$ to the value 8 is less than 0.03 units. Express this using absolute value notation.

Answer

$|f(x) - 8| < 0.03$

For the exercises 10-24, solve the equations below and express the answer using set notation.

10) $|x+3| = 9$

11) $|6-x| = 5$

Answer

$1, 11$

12) $|5x-2| = 11$

13) $|4x-2| = 11$

Answer

$\left\{\frac{9}{4}, \frac{13}{4}\right\}$

14) $2|4-x| = 7$

15) $3|5-x|=5$

Answer

$\left\{\frac{10}{3},\frac{20}{3}\right\}$

16) $3|x+1|-4=5$

17) $5|x-4|-7=2$

Answer

$\left\{\frac{11}{5},\frac{29}{5}\right\}$

18) $0=-|x-3|+2$

19) $2|x-3|+1=2$

Answer

$\left\{\frac{5}{2},\frac{7}{2}\right\}$

20) $|3x-2|=7$

21) $|3x-2|=-7$

Answer

No solution

22) $\left|\frac{1}{2}x-5\right|=11$

23) $\left|\frac{1}{3}x+5\right|=14$

Answer

$\{-57,27\}$

24) $-\left|\frac{1}{3}x+5\right|+14=0$

For the exercises 25-28, find the x- and y-intercepts of the graphs of each function.

25) style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:normal;font-variant:normal;font-weight:400;letter-spacing:normal;line-height:22.4px;orphans:2;text-align:justify;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">$f(x)=2|x+1|-10$

Answer

$(0,-8)$; $(-6,0)$, $(4,0)$

26) style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:normal;font-variant:normal;font-weight:400;letter-spacing:normal;line-height:22.4px;orphans:2;text-align:justify;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">$f(x)=4|x-3|+4$

style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:normal;font-variant:normal;font-weight:400;letter-spacing:normal;line-height:22.4px;orphans:2;text-

align:justify;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">27) $f(x) = -3|x-2| - 1$

Answer

$(0, -7)$; no x-intercepts

28) style="display:inline !important;float:none;background-color:transparent;font-family:lato,arial,helvetica,sans-serif,arial unicode ms;font-style:normal;font-variant:normal;font-weight:400;letter-spacing:normal;line-height:22.4px;orphans:2;text-align:justify;text-decoration:none;text-indent:0px;text-transform:none;-webkit-text-stroke-width:0px;white-space:normal;word-spacing:0px;">$f(x) = -2|x+1| + 6$

For the exercises 29-36, solve each inequality and write the solution in interval notation.

29) $|x-2| > 10$

Answer

$(-\infty, -8) \cup (12, \infty)$

30) $2|v-7| - 4 \geq 42$

31) $|3x-4| \geq 8$

Answer

$-\frac{4}{3} \leq x \leq 4$

32) $|x-4| \geq 8$

33) $|3x-5| \geq -13$

Answer

$\left(-\infty, -\frac{8}{3}\right] \cup [6, \infty)$

34) $|3x-5| \geq -13$

35) $|\frac{3}{4}x - 5| \geq 7$

Answer

$\left(-\infty, -\frac{8}{3}\right] \cup [16, \infty)$

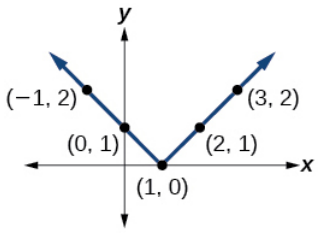
36) $\left|\frac{3}{4}x - 5\right| + 1 \leq 16$

Graphical

For the exercises 37-39, graph the absolute value function. Plot at least five points by hand for each graph.

37) $y = |x-1|$

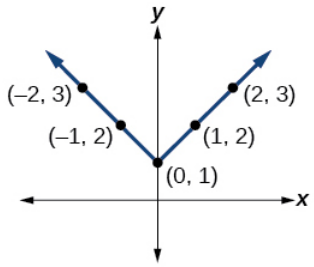
Answer



38) $y = |x + 1|$

39) $y = |x| + 1$

Answer

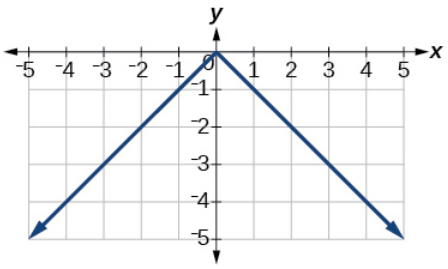


For the exercises 40-52, graph the given functions by hand.

40) $y = |x| - 2$

41) $y = -|x|$

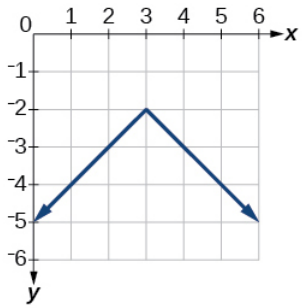
Answer



42) $y = -|x| - 2$

43) $y = -|x - 3| - 2$

Answer

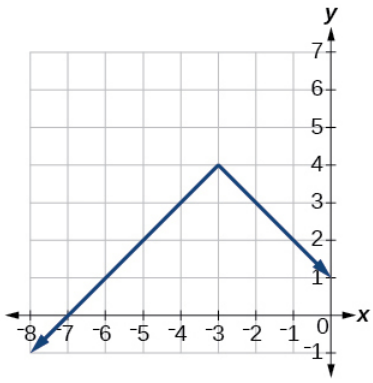


44) $f(x) = -|x - 1| - 2$

45) $f(x) = -|x + 3| + 4$

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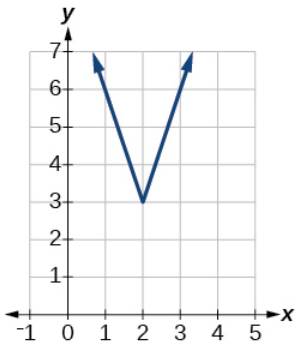
Answer



46) $f(x) = 2|x+3|+1$

47) $f(x) = 3|x-2|+3$

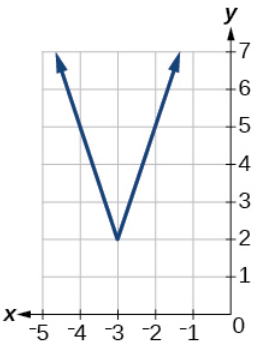
Answer



48) $f(x) = |2x-4|-3$

49) $f(x) = |3x+9|+2$

Answer

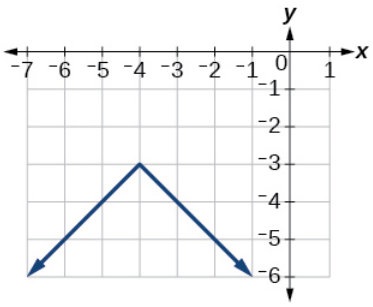


50) $f(x) = -|x-1|-3$

51) $f(x) = -|x+4|-3$

Answer

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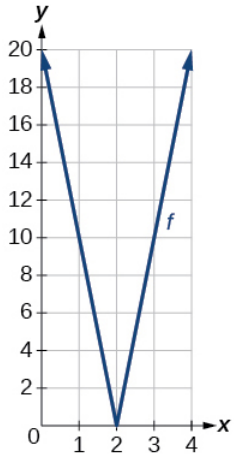
52) $f(x) = \frac{1}{2}|x+4| - 3$

Technology

53) Use a graphing utility to graph $f(x) = 10|x-2|$ on the viewing window $[0, 4]$. Identify the corresponding range. Show the graph.

Answer

range: $[0, 20]$



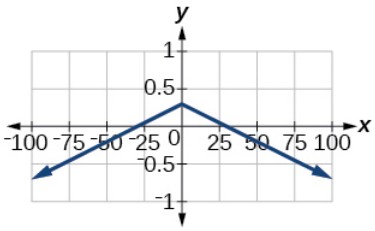
54) Use a graphing utility to graph $f(x) = -100|x| + 100$ on the viewing window $[-5, 5]$. Identify the corresponding range. Show the graph.

For the exercises 55-56, graph each function using a graphing utility. Specify the viewing window.

55) $f(x) = -0.1|0.1(0.2 - x)| + 0.3$

Answer

x-intercepts:



56) $f(x) = 4 \times 10^9|x - (5 \times 10^9)| + 2 \times 10^9$

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Extensions

For the exercises 57-59, solve the inequality.

57) $\left|-2x-\frac{2}{3}(x+1)\right|+3>-1$

Answer

$(-\infty,\infty)$

58) If possible, find all values of a such that there are no x-intercepts for $f(x)=2|x+1|+a$.

59) If possible, find all values of a such that there are no y-intercepts for $f(x)=2|x+1|+a$.

Answer

There is no solution for a that will keep the function from having a y-intercept. The absolute value function always crosses the y-intercept when $x=0$.

Real-World Applications

60) Cities A and B are on the same east-west line. Assume that city A is located at the origin. If the distance from city A to city B is at least 100 miles and x represents the distance from city B to city A, express this using absolute value notation.

61) The true proportion p of people who give a favorable rating to Congress is 8% with a margin of error of 1.5%. Describe this statement using an absolute value equation.

Answer

$|p-0.08|\leq 0.015$

62) Students who score within 18 points of the number 82 will pass a particular test. Write this statement using absolute value notation and use the variable x for the score.

63) A machinist must produce a bearing that is within 0.01 inches of the correct diameter of 5.0 inches. Using x as the diameter of the bearing, write this statement using absolute value notation.

Answer

$|x-5.0|\leq 0.01$

64) The tolerance for a ball bearing is 0.01. If the true diameter of the bearing is to be 2.0 inches and the measured value of the diameter is x inches, express the tolerance using absolute value notation.

Inverse Functions

If some physical machines can run in two directions, we might ask whether some of the function “machines” we have been studying can also run backwards. In this section, we will consider the reverse nature of functions.

Verbal

1) Describe why the horizontal line test is an effective way to determine whether a function is one-to-one?

Answer

Each output of a function must have exactly one output for the function to be one-to-one. If any horizontal line crosses the graph of a function more than once, that means that y-values repeat and the function is not one-to-one. If no horizontal line crosses the graph of the function more than once, then no y-values repeat and the function is one-to-one.

2) Why do we restrict the domain of the function $f(x)=x^2$ to find the function’s inverse?

3) Can a function be its own inverse? Explain.

Answer

Yes. For example, $f(x)=\dfrac{1}{x}$ is its own inverse.

4) Are one-to-one functions either always increasing or always decreasing? Why or why not?

5) How do you find the inverse of a function algebraically?

Answer

Given a function $y=f(x)$, solve for x in terms of y. Interchange the x and y. Solve the new equation for y. The expression for y is the inverse, $y=f^{-1}(x)$.

Algebraic

6) Show that the function $f(x)=a-x$ is its own inverse for all real numbers a.

For the exercises 7-12, find $f^{-1}(x)$ for each function.

7) $f(x)=x+3$

Answer

$f^{-1}(x)=x-3$

8) $f(x)=x+5$

9) $f(x)=2-x$

Answer

$f^{-1}(x)=2-x$

10) $f(x)=3-x$

11) $f(x)=\dfrac{x}{x+2}$

Answer

$f^{-1}(x)=\dfrac{-2x}{x-1}$

12) $f(x)=\dfrac{2x+3}{5x+4}$

For the exercises 13-16, find a domain on which each function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

13) $f(x)=(x+7)^2$

Answer

domain of $f(x)$: $[-7,\infty)$; $f^{-1}(x)=\sqrt{x}-7$

14) $f(x)=(x-6)^2$

15) $f(x)=x^2-5$

Answer

domain of $f(x)$: $[0,\infty)$; $f^{-1}(x)=\sqrt{x+5}$

16) Given $f(x)=\dfrac{x}{2+x}$ and $g(x)=\dfrac{2x}{1-x}$:

a. Find $f(g(x))$ and $g(f(x))$.
b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

Answer

a. $f(g(x)) = x$ and $g(f(x)) = x$

b. This tells us that f and g are inverse functions

For the exercises 17-18, use function composition to verify that $f(x)$ and $g(x)$ are inverse functions.

17) $f(x) = \sqrt[3]{x-1}$ and $g(x) = x^3 + 1$

Answer

$f(g(x)) = x$, $g(f(x)) = x$

18) $f(x) = -3x + 5$ and $g(x) = \dfrac{x-5}{-3}$

Graphical

For the exercises 19-22, use a graphing utility to determine whether each function is one-to-one.

19) $f(x) = \sqrt{x}$

Answer

one-to-one

20) $f(x) = \sqrt[3]{3x+1}$

21) $f(x) = -5x + 1$

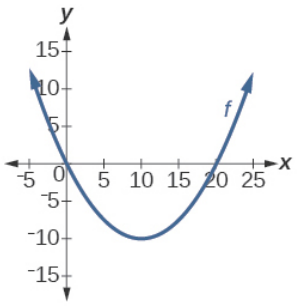
Answer

one-to-one

22) $f(x) = x^3 - 27$

For the exercises 23-24, determine whether the graph represents a one-to-one function.

23)



Answer

not one-to-one

24)

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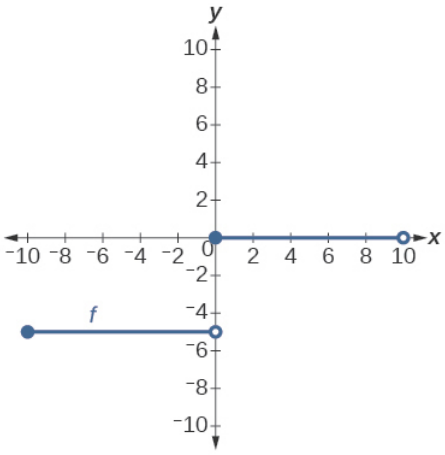
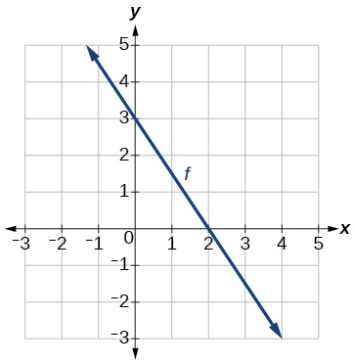


Figure below.



Graph of a line

25) Find $f(0)$.

Answer

3

26) Solve $f(x) = 0$.

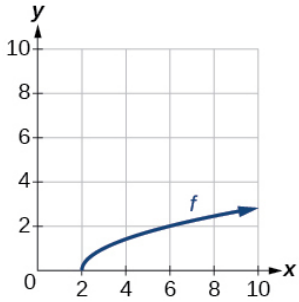
27) Find $f^{-1}(0)$.

Answer

2

28) Solve $f^{-1}(x) = 0$.

For the exercises 29-32, use the graph of the one-to-one function shown in the Figure below

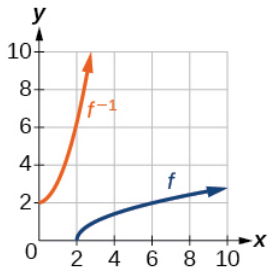


Graph of a square root function.

29) Sketch the graph of f^{-1}.

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Answer



30) Find $f(6)$ and $f^{-1}(2)$.

31) If the complete graph of f is shown, find the domain of f.

Answer

$[2, 10]$

32) If the complete graph of f is shown, find the range of f

Numeric

For the exercises 33-36, evaluate or solve, assuming that the function f is one-to-one.

33) If $f(6) = 7$, find $f^{-1}(7)$.

Answer

6

34) If $f(3) = 2$, find $f^{-1}(2)$.

35) If $f^{-1}(-4) = -8$, find $f(-8)$.

Answer

-4

36) If $f^{-1}(-2)$=-1, find $f(-1)$.

For the exercises 37-40, use the values listed in the Table below to evaluate or solve.

x	$f(x)$
0	8
1	0
2	7
3	4
4	2
5	6
6	5
7	3
8	9
9	1

37) Find $f(1)$.

Answer

0

38) Solve $f(x) = 3$.

39) Find $f^{-1}(0)$.

Answer

1

40) Solve $f^{-1}(x) = 7$.

41) Use the tabular representation of f in the Table below to create a table for $f^{-1}(x)$.

x	3	6	9	13	14
$f(x)$	1	4	7	12	16

Answer

x	1	4	7	12	16
$f^{-1}(x)$	3	6	9	13	14

Technology

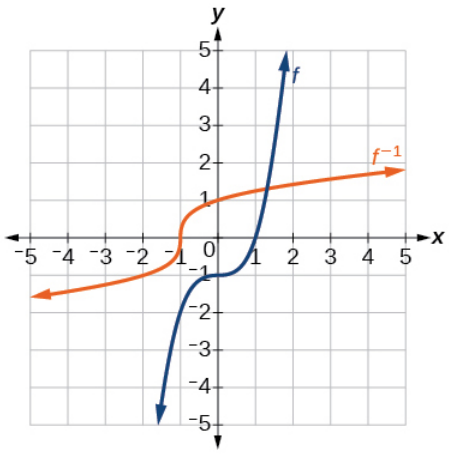
For the exercises 42-44, find the inverse function. Then, graph the function and its inverse.

42) $f(x) = \dfrac{3}{x-2}$

43) $f(x) = x^3 - 1$

Answer

$f^{-1}(x) = (1+x)^{1/3}$



44) Find the inverse function of $f(x) = \dfrac{1}{x-1}$. Use a graphing utility to find its domain and range. Write the domain and range in interval notation.

Real-World Applications

45) To convert from x degrees Celsius to y degrees Fahrenheit, we use the formula $f(x) = \dfrac{9}{5}x + 32$. Find the inverse function, if it exists, and explain its meaning.

Answer

$f^{-1}(x) = \frac{5}{9}(x - 32)$. Given the Fahrenheit temperature, x, this formula allows you to calculate the Celsius temperature.

46) The circumference C of a circle is a function of its radius given by $C(r) = 2\pi r$. Express the radius of a circle as a function of its circumference. Call this function $r(C)$. Find $r(36\pi)$ and interpret its meaning.

47) A car travels at a constant speed of 50 miles per hour. The distance the car travels in miles is a function of time, t, in hours given by $d(t) = 50t$. Find the inverse function by expressing the time of travel in terms of the distance traveled. Call this function $t(d)$. Find $t(180)$ and interpret its meaning.

Answer

$t(d) = \frac{d}{50}$, $t(180) = \frac{180}{50}$. The time for the car to travel 180 miles is 3.6 hours.

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2.R: Functions (Review)

This page is a draft and is under active development.

Functions and Function Notation

For the exercises 1-4, determine whether the relation is a function.

1) $\{(a,b),(c,d),(e,d)\}$

Answer

function

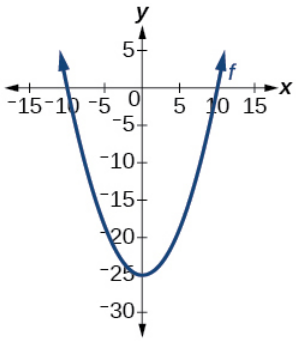
2) $\{(5,2),(6,1),(6,2),(4,8)\}$

3) $y^2+4=x$ for x the independent variable and y the dependent variable

Answer

not a function

4) Is the graph in the Figure below a function?



For the exercises 5-6, evaluate the function at the indicated values: $f(-3); f(2); f(-a); -f(a); f(a+h)$

5) $f(x)=-2x^2+3x$

Answer

$f(-3)=-27; f(2)=-2; f(-a)=-2a^2-3a; -f(a)=2a^2-3a; f(a+h)=-2a^2+3a-4ah+3h-2h^2$

6) $f(x)=2|3x-1|$

For the exercises 7-8, determine whether the functions are one-to-one.

7) $f(x)=-3x+5$

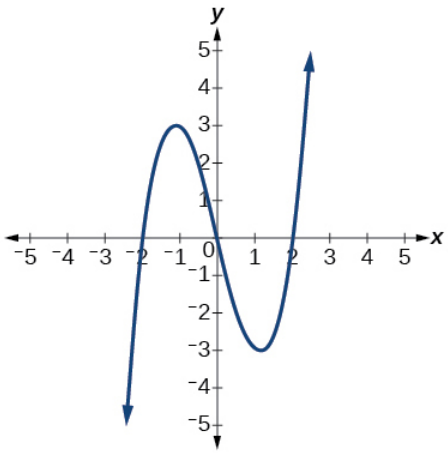
Answer

one-to-one

8) $f(x)=|x-3|$

For the exercises 9-11, use the vertical line test to determine if the relation whose graph is provided is a function.

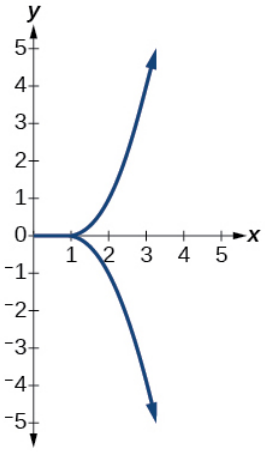
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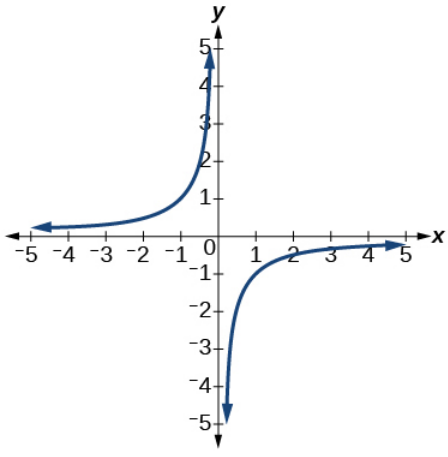
Answer

function

10)



11)



Answer

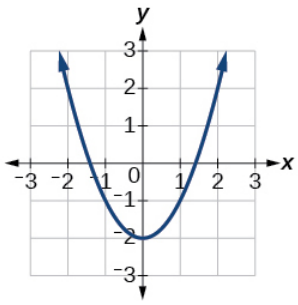
function

For the exercises 12-13, graph the functions.

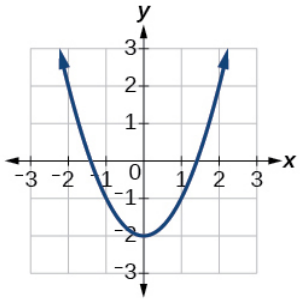
12) $f(x) = |x+1|$

13) $f(x) = x^2 - 2$

Answer



For the exercises 14-17, use the Figure below to approximate the values.



14) $f(2)$

15) $f(-2)$

Answer

2

16) If $f(x) = -2$, then solve for x

17) If $f(x) = 1$, then solve for x

Answer

$x = -1.8$ or $x = 1.8$

For the exercises 18-19, use the function $h(t) = -16t^2 + 80t$ to find the values.

18) $\dfrac{h(2) - h(1)}{2 - 1}$

19) $\dfrac{h(a) - h(1)}{a - 1}$

Answer

$$\frac{-64 + 80a - 16a^2}{-1 + a} = -16a + 64$$

Domain and Range

For the exercises 1-4, find the domain of each function, expressing answers using interval notation.

1) $f(x) = \dfrac{2}{3x + 2}$

2) $f(x) = \frac{x-3}{x^2-4x-12}$

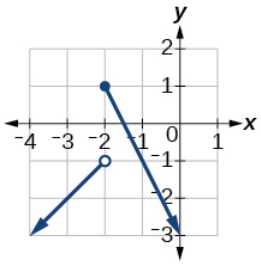
Answer

$(-\infty, -2) \cup (-2, 6) \cup (6, \infty)$

3)

4) Graph this piecewise function: $f(x) = \begin{cases} x+1 & x < -2 \\ -2x-3 & x \geq -2 \end{cases}$

Answer



Rates of Change and Behavior of Graphs

For the exercises 1-3, find the average rate of change of the functions from $x = 1$ to $x = 2$

1) $f(x) = 4x - 3$

2) $f(x) = 10x^2 + x$

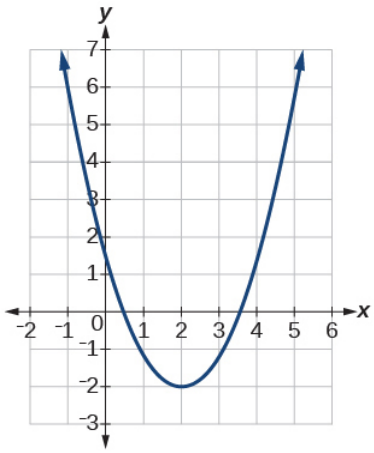
Answer

31

3) $f(x) = -\frac{2}{x^2}$

For the exercises 4-6, use the graphs to determine the intervals on which the functions are increasing, decreasing, or constant.

4)

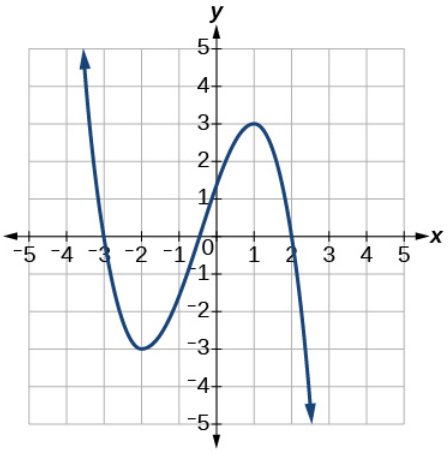


Answer

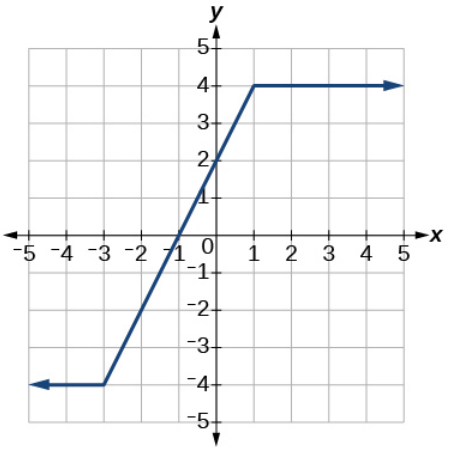
increasing $(2, \infty)$; decreasing $(-\infty, 2)$

5)

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6)



Answer

increasing $(-3, 1)$; constant $(-\infty, -3) \cup (1, \infty)$

7) Find the local minimum of the function graphed in Exercise 4.

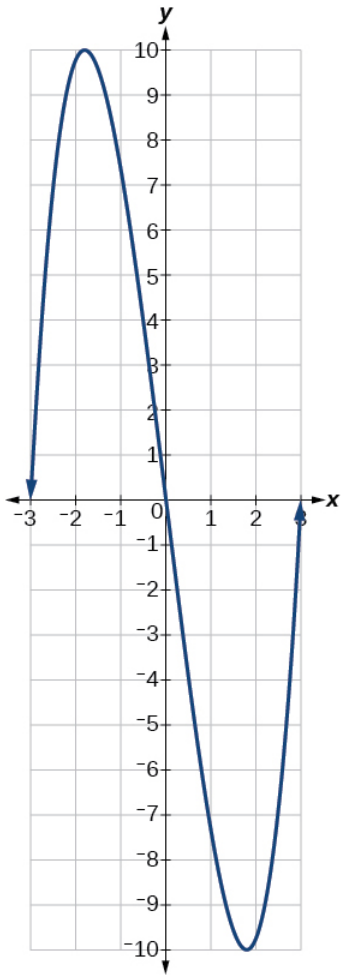
8) Find the local extrema for the function graphed in Exercise 5.

Answer

local minimum $(-2, -3)$; local maximum $(1, 3)$

9) For the graph in the Figure in Exercise 10, the domain of the function is $[-3, 3]$. The range is $[-10, 10]$. Find the absolute minimum of the function on this interval.

10) Find the absolute maximum of the function graphed in the Figure below.



Answer

$(-1.8, 10)$

Composition of Functions

For the exercises 1-5, find $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions.

1) $f(x) = 4 - x, g(x) = -4x$

2) $f(x) = 3x + 2, g(x) = 5 - 6x$

Answer

$(f \circ g)(x) = 17 - 18x; (g \circ f)(x) = -7 - 18x$

3) $f(x) = x^2 + 2x, g(x) = 5x + 1$

4) $f(x) = \sqrt{x+2}, g(x) = \dfrac{1}{x}$

Answer

$(f \circ g)(x) = \sqrt{\dfrac{1}{x} + 2}; (g \circ f)(x) = \dfrac{1}{\sqrt{x+2}}$

5) $f(x) = \dfrac{x+3}{2}, g(x) = \sqrt{1-x}$

For the exercises 6-9, find $(f \circ g)$ and the domain for $(f \circ g)(x)$ for each pair of functions.

6) $f(x)=\frac{x+1}{x+4}, g(x)=\frac{1}{x}$

Answer

$(f\circ g)(x)=\dfrac{1+x}{1+4x}, x\neq 0, x\neq -\dfrac{1}{4}$

7) $f(x)=\dfrac{1}{x+3}, g(x)=\dfrac{1}{x-9}$

8) $f(x)=\dfrac{1}{x}, g(x)=\sqrt{x}$

Answer

$(f\circ g)(x)=\frac{1}{\sqrt{x}}, x>0$

9) $f(x)=\frac{1}{x^2-1}, g(x)=\sqrt{x+1}$

For the exercises 10-11, express each function H as a composition of two functions f and g where $H(x)=(f\circ g)(x)$

10) $H(x)=\sqrt{\frac{2x-1}{3x+4}}$

Answer

sample: $g(x)=\dfrac{2x-1}{3x+4}; f(x)=\sqrt{x}$

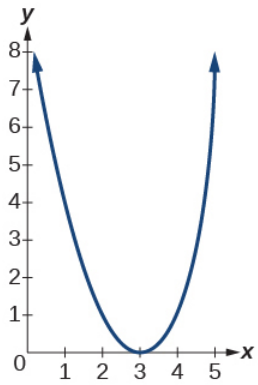
11) $H(x)=\dfrac{1}{(3x^2-4)^{-3}}$

Transformation of Functions

For the exercises 1-8, sketch a graph of the given function.

1) $f(x)=(x-3)^2$

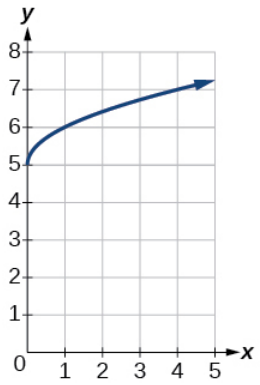
Answer



2) $f(x)=(x+4)^3$

3) $f(x)=\sqrt{x}+5$

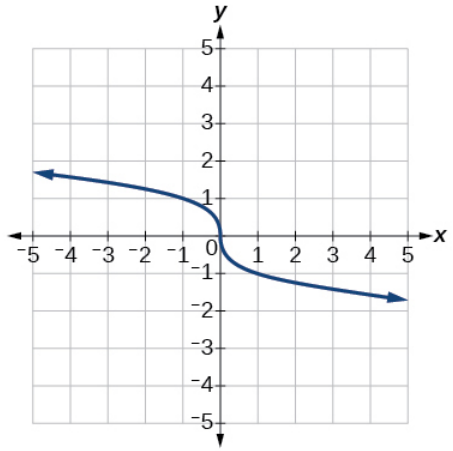
Answer



4) $f(x) = -x^3$

5) $f(x) = \sqrt[3]{-x}$

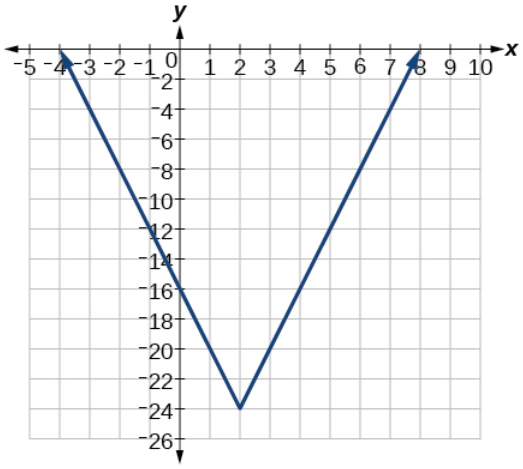
Answer



6) $f(x) = 5\sqrt{-x} - 4$

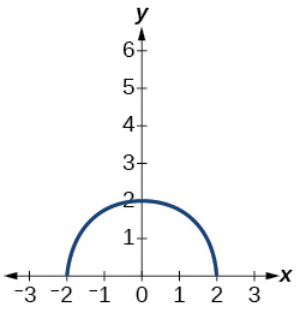
7) $f(x) = 4[|x-2| - 6]$

Answer



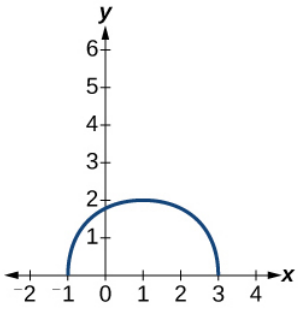
8) $f(x) = -(x+2)^2 - 1$

For the exercises 9-10, sketch the graph of the function g if the graph of the function f is shown in the Figure below.



9) $g(x) = f(x-1)$

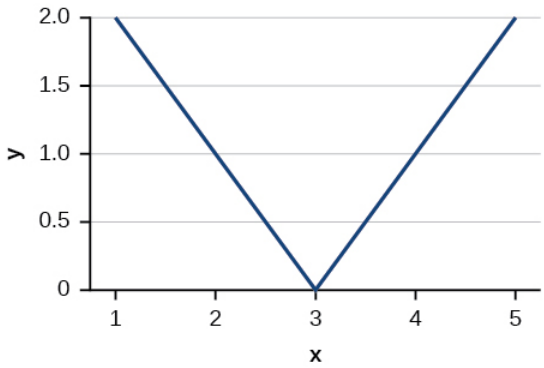
Answer



10) $g(x) = 3f(x)$

For the exercises 11-12, write the equation for the standard function represented by each of the graphs below.

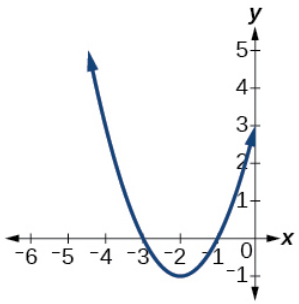
11)



Answer

$f(x) = |x-3|$

12)



For the exercises 13-15, determine whether each function below is even, odd, or neither.

13) $f(x) = 3x^4$

Answer

even

14) $g(x) = \sqrt{x}$

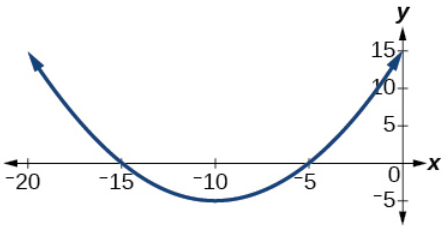
15) $h(x) = \frac{1}{x} + 3x$

Answer

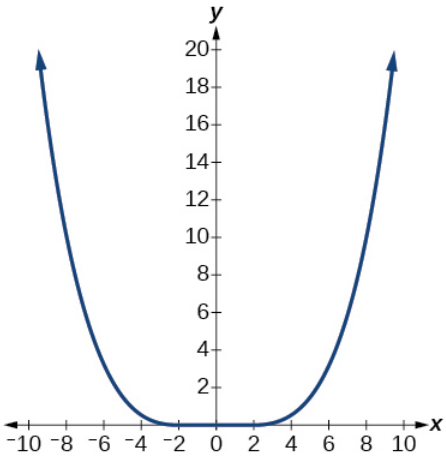
odd

For the exercises 16-18, analyze the graph and determine whether the graphed function is even, odd, or neither.

16)



17)

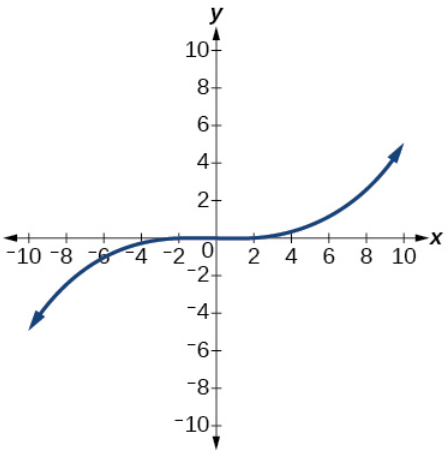


Answer

even

18)

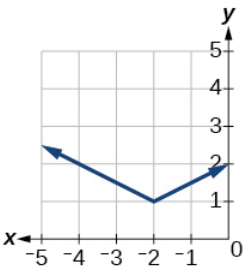
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Absolute Value Functions

For the exercises 1-3, write an equation for the transformation of $f(x) = |x|$.

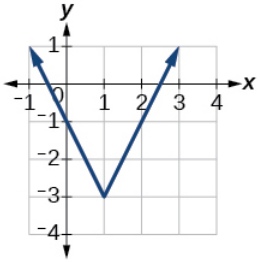
1)



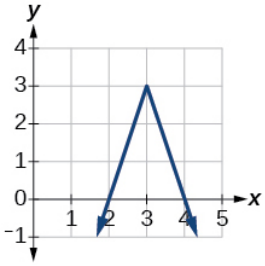
Answer

$$f(x) = \frac{1}{2}|x+2|+1$$

2)



3)



Answer

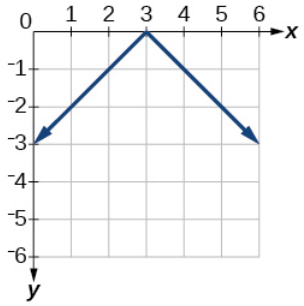
$f(x) = -3|x-3|+3$

For the exercises 4-6, graph the absolute value function.

4) $f(x) = |x-5|$

5) $f(x) = -|x-3|$

Answer



6) $f(x) = |2x-4|$

For the exercises 7-8, solve the absolute value equation.

7) $|x+4| = 18$

Answer

$x = -22, x = 14$

8) $\left|\frac{1}{3}x+5\right| = \left|\frac{3}{4}x-2\right|$

For the exercises 9-10, solve the inequality and express the solution using interval notation.

9) $|3x-2| < 7$

Answer

$\left(-\frac{5}{3}, 3\right)$

10) $\left|\frac{1}{3}x-2\right| \leq 7$

Inverse Functions

For the exercises 1-2, find $f^{-1}(x)$ for each function.

1) $f(x) = 9+10x$

2) $f(x) = \frac{x}{x+2}$

Answer

$f^{-1}(x) = \frac{-2x}{x-1}$

3) For the following exercise, find a domain on which the function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

$$f(x) = x^2+1 \tag{2.R.1}$$

4) Given $f(x) = x^3-5$ and $g(x) = \sqrt[3]{x+5}$:

a. Find $f(g(x))$ and $g(f(x))$.
b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

Answer

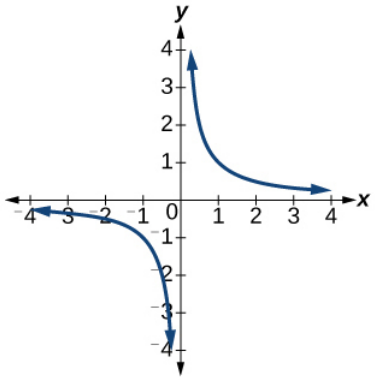
a. $f(g(x)) = x$ and $g(f(x)) = x$
b. This tells us that f and g are inverse functions

For the exercises 5-8, use a graphing utility to determine whether each function is one-to-one.

5) $f(x) = \dfrac{1}{x}$

Answer

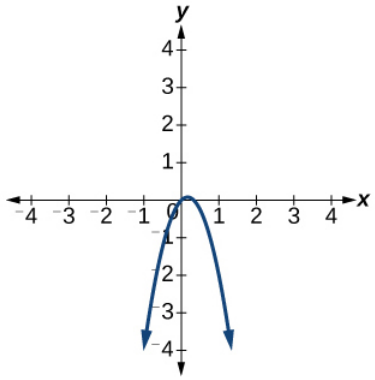
The function is one-to-one.



6) $f(x) = -3x^2 + x$

Answer

The function is not one-to-one.



7) If $f(5) = 2$, find $f^{-1}(2)$

Answer

5

8) If $f(1) = 4$, find $f^{-1}(4)$

Practice Test

For the exercises 1-2, determine whether each of the following relations is a function.

1) $y = 2x + 8$

Answer

The relation is a function.

2) $\{(2,1),(3,2),(-1,1),(0,-2)\}$

For the exercises 3-4, evaluate the function $f(x)=-3x^2+2x$ at the given input.

3) $f(-2)$

Answer

-16

4) $f(a)$

5) Show that the function $f(x)=-2(x-1)^2+3$ is not one-to-one.

Answer

The graph is a parabola and the graph fails the horizontal line test.

6) Write the domain of the function $f(x)=\sqrt{3-x}$ in interval notation.

7) Given $f(x)=2x^2-5x,$ find $f(a+1)-f(1)$

Answer

$2a^2-a$

8) Graph the function $f(x)=\begin{cases} x+1 & \text{if} \quad -2<x<3 \\ -x & \text{if} \quad x\geq 3 \end{cases}$

9) Find the average rate of change of the function $f(x)=3-2x^2+x$ by finding $\dfrac{f(b)-f(a)}{b-a}$

Answer

$-2(a+b)+1$

For the exercises 10-11, use the functions $f(x)=3-2x^2+x$ and $g(x)=\sqrt{x}$ to find the composite functions.

10) $(g\circ f)(x)$

11) $(g\circ f)(1)$

Answer

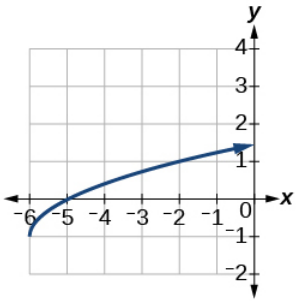
$\sqrt{2}$

12) Express $H(x)=\sqrt[3]{5x^2-3x}$ a composition of two functions, f and g, where $(f\circ g)(x)=H(x)$

For the exercises 13-14, graph the functions by translating, stretching, and/or compressing a toolkit function.

13) $f(x)=\sqrt{x+6}-1$

Answer



14) $f(x)=\dfrac{1}{x+2}-1$

For the exercises 15-17, determine whether the functions are even, odd, or neither.

15) $f(x)=-\dfrac{5}{x^2}+9x^6$

Answer

even

16) $f(x)=-\dfrac{5}{x^3}+9x^5$

17) $f(x)=\dfrac{1}{x}$

Answer

odd

18) Graph the absolute value function $f(x)=-2|x-1|+3$.

19) Solve $|2x-3|=17$.

Answer

$x=-7$ and $x=10$

20) Solve $-\left|\dfrac{1}{3}x-3\right|\geq 17$. Express the solution in interval notation.

For the exercises 21-22, find the inverse of the function.

21) $f(x)=3x-5$

Answer

$f^{-1}(x)=\dfrac{x+5}{3}$

22) $f(x)=\dfrac{4}{x+7}$

For the exercises 23-26, use the graph of *g* shown in the Figure below.

23) On what intervals is the function increasing?

Answer

$(-\infty,-1.1)$ and $(1.1,\infty)$

24) On what intervals is the function decreasing?

25) Approximate the local minimum of the function. Express the answer as an ordered pair.

Answer

$(1.1, -0.9)$

26) Approximate the local maximum of the function. Express the answer as an ordered pair.

For the exercises 27-29, use the graph of the piecewise function shown in the Figure below.

27) Find $f(2)$.

Answer

$f(2) = 2$

28) Find $f(-2)$.

29) Write an equation for the piecewise function.

Answer

$$f(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$$

For the exercises 30-35, use the values listed in the Table below.

x	$F(x)$
0	1
1	3
2	5
3	7
4	9
5	11
6	13
7	15
8	17

30) Find $F(6)$.

31) Solve the equation $F(x) = 5$

Answer

$x = 2$

32) Is the graph increasing or decreasing on its domain?

33) Is the function represented by the graph one-to-one?

Answer

yes

34) Find $F^{-1}(15)$.

35) Given $f(x) = -2x + 11$, find $f^{-1}(x)$.

Answer

$$f^{-1}(x) = -\frac{x-11}{2}$$

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CHAPTER OVERVIEW

3: Polynomial and Rational Functions

3.1: Prelude to Polynomial and Rational Functions
3.2: Quadratic Functions
3.3: Graphs of Polynomial Functions
3.4: Polynomial Division
3.5: Zeros of Polynomials
3.6: Introduction to Rational Functions
3.7: Graphing Rational Functions
3.E: Polynomial and Rational Functions (Exercises)
3.R: Polynomial and Rational Functions(Review)

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3.1: Prelude to Polynomial and Rational Functions

Digital photography has dramatically changed the nature of photography. No longer is an image etched in the emulsion on a roll of film. Instead, nearly every aspect of recording and manipulating images is now governed by mathematics. An image becomes a series of numbers, representing the characteristics of light striking an image sensor. When we open an image file, software on a camera or computer interprets the numbers and converts them to a visual image. Photo editing software uses complex polynomials to transform images, allowing us to manipulate the image in order to crop details, change the color palette, and add special effects. Inverse functions make it possible to convert from one file format to another. In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.

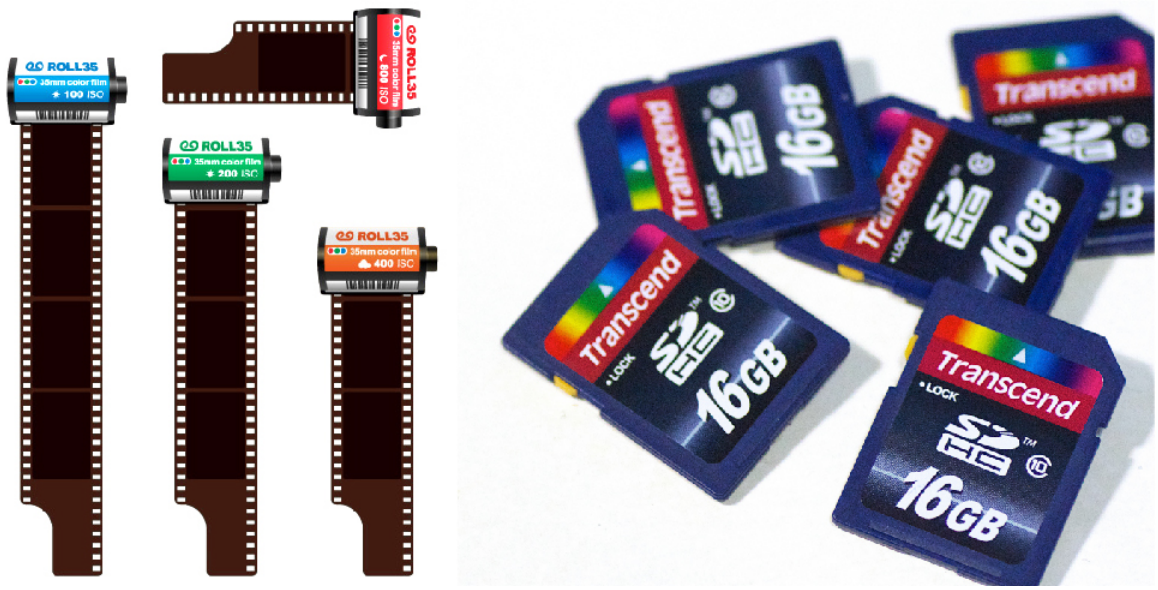


Figure* 3.1.1*: *35-mm film, once the standard for capturing photographic images, has been made largely obsolete by digital photography. (credit "film": modification of work by Horia Varlan; credit "memory cards": modification of work by Paul Hudson)*

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3.2: Quadratic Functions

Learning Objectives

- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.

Curved antennas, such as the ones shown in Figure 3.2.1, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The shape used in many antenna and satellite designs are called paraboloids which are studied in multivariable calculus. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

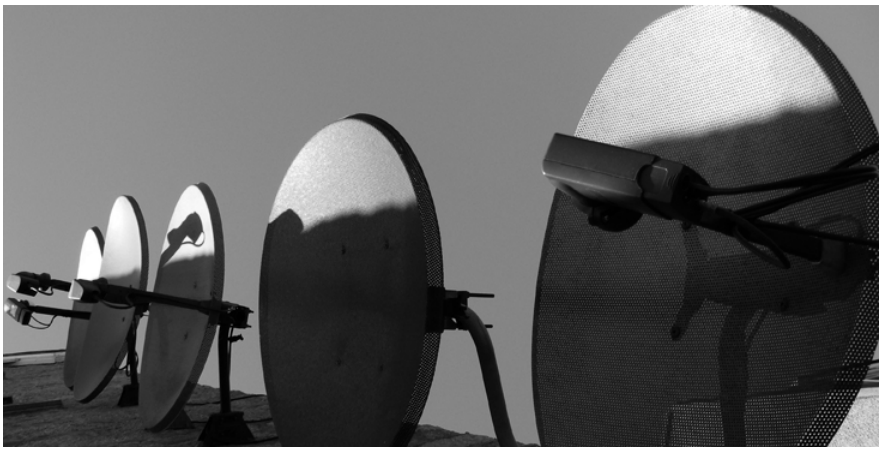


Figure 3.2.1: An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has a local extrema point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. These features are illustrated in Figure 3.2.2.

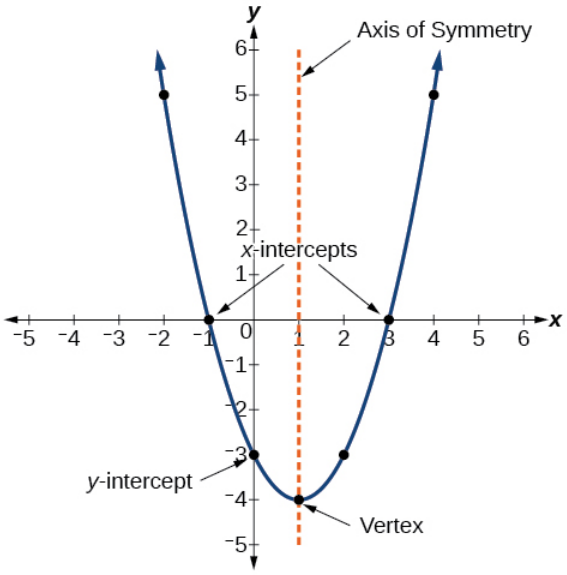


Figure 3.2.2: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are.

Example 3.2.1

Determine the vertex, axis of symmetry, x-intercepts, and y-intercept of the parabola shown in Figure 3.2.3.

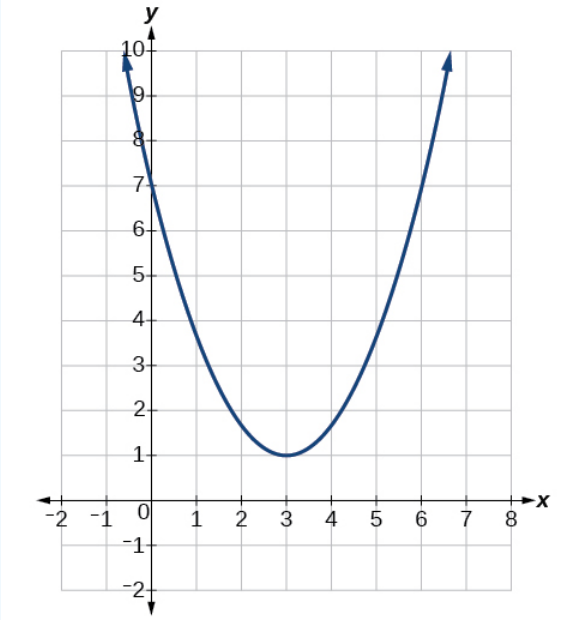


Figure 3.2.3.

Solution

The vertex is the lowest point of the graph. We can see that the vertex is at $(3, 1)$. The axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no x-intercepts. It crosses the y-axis at $(0, 7)$ so this is the y-intercept.

We have discussed transformations of functions. Given the function $f(x) = 3(x + 2)^2 - 13$, we know that the parabola open upward and that the vertex is at $(-2, -13)$. With quadratics we sometimes call this transformation form of the function **vertex form**.

Now, consider what happens when we expand this function out:

$$f(x) = 3(x+2)^2 - 13$$
$$f(x) = 3(x+2)(x+2) - 13$$
$$f(x) = 3(x^2+4x+4) - 13$$
$$f(x) = 3x^2+12x+12) - 13$$
$$f(x) = 3x^2+12x-1$$

This expanded form, $f(x) = 3x^2+12x-1$, is commonly called **general form** of a quadratic function. *(Note that this terminology is not universal but we will refer to this as general form throughout this text).*

Also, notice that the leading coefficient of 3 was the same value in the vertex and general form of the function. This value tells us whether the graph has a vertical stretch or compression as well as an x-axis reflection.

General Form of a Quadratic Function

The **general form** of a quadratic function is $f(x) = ax^2+bx+c$ where a, b, and c are real numbers and $a \neq 0$.

- If $a > 0$, then the parabola opens upward.
- If $a < 0$, then the parabola opens downward.

Next, how do we determine what the vertex of a quadratic function is when the function is given in general form? To do so, we rewrite the function in vertex form by completing the square. Let's look at an example of this:

Given the function $f(x) = x^2+2x+4$, begin by grouping the variable terms together,

$$f(x) = (x^2+2x)+4$$

Then complete the square by taking half of the linear term coefficient and the squaring it, $(\frac{1}{2}(2))^2 = 1$. Next, add and subtract this value to the function,

$$f(x) = (x^2+2x+1) - 1 + 4$$

Now factor and simplify to get the vertex form,

$$f(x) = (x+1)^2+3$$

Completing the square is a very useful skill for writing quadratic functions in vertex form. It is also useful for writing many other equations in more useful forms in calculus. Let's look at another example using this important skill.

Example 3.2.2

Rewrite $f(x) = 2x^2-12x+14$ into vertex form by completing the square and identify the vertex.

Solution

We start by factoring the leading coefficient from the quadratic and linear terms.

$$2(x^2-6x)+14$$

Take half of the linear coefficient, -6, which is -3, then square to get $(-3)^2 = 9$.

$$(x^2-6x+9) = (x-3)^2$$

Now, we can't just add 9 into the expression – that would change the value of the expression. In fact, adding 9 inside the parentheses actually adds 18 to the expression, since the 2 outside the parentheses will distribute. To keep the expression balanced, we can subtract 18.

$$2(x^2-6x+9)+14-18$$

Simplifying, we are left with vertex form.

$$2(x-3)^2-4$$

The vertex is $(3, -4)$.

Vertex Form of a Quadratic Function

The **vertex form** of a quadratic function is

$$f(x) = a(x-h)^2 + k$$

The vertex of the quadratic function is located at (h, k), where h and k are the numbers in the transformation form of the function.

In the previous example, we saw that it is possible to rewrite a quadratic function in vertex form by completing the square. It would be useful to reverse this process since the vertex form reveals the vertex.

Expanding out the vertex form of a quadratic gives:

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ f(x) &= a(x-h)(x-h) + k \\ f(x) &= a(x^2 - 2xh + h^2 + k \\ f(x) &= ax^2 - 2ahx + ah^2 + k \end{aligned}$$

This is equivalent to the general form of the quadratic:

$$ax^2 - 2ahx + ah^2 + k = ax^2 + bx + c$$

The second degree terms are already equal and have the same labels. For the linear terms to be equal, the coefficients must be equal. So,

$$-2ah = b$$

then solving for h we get:

$$h = -\frac{b}{2a}$$

Likewise, set the constant terms equal,

$$ah^2 + k = c$$

Solving for k we get:

$$k = c - ah^2 = c - a\left(-\frac{b}{2a}\right)^2 = c - a\frac{b^2}{4a^2} = c - \frac{b^2}{4a}$$

In practice, though, it is usually easier to remember that k is the output value of the function when the input is h, so $k = f(h)$.

Vertex Formula

Given a quadratic function in general form, $f(x) = ax^2 + bx + c$, the vertex is (h, k) where

$$h = -\frac{b}{2a} \quad \text{and } k = f(h)$$

Example 3.2.3

Find the vertex of the quadratic $f(x) = 2x^2 - 6x + 7$.

Solution

The x-coordinate of the vertex will be at

$$h = -\frac{b}{2a} = -\frac{-6}{2(2)} = \frac{6}{4} = \frac{3}{2}$$

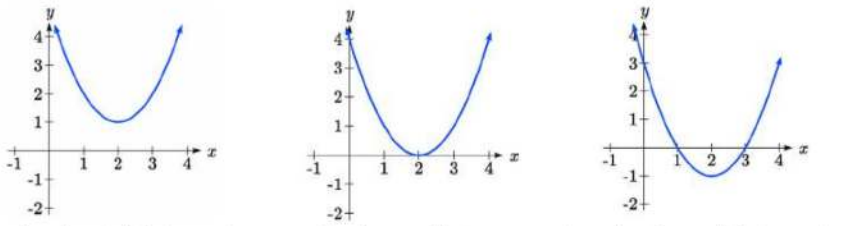
The y-coordinate of the vertex will be at

$$f(\frac{3}{2}) = 2(\frac{3}{2})^2 - 6(\frac{3}{2}) + 7 = \frac{5}{2}$$

The vertex is $\left(\frac{3}{2}, \frac{5}{2}\right)$

Intercept Behavior

As with any function, we can find the y-intercepts of a quadratic by evaluating the function at an input of zero. All quadratics functions have one y-intercept since the domain is all real numbers. We can find the x-intercepts by solving for when the output will be zero. Notice that depending upon the location of the graph, we might have zero, one, or two horizontal intercepts.



zero horizontal intercepts | one horizontal intercept | two horizontal intercepts

Example 3.2.4

Find intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$

Solution

We can find the y-intercept by evaluating the function at an input of zero:

$$f(0) = 3(0)^2 + 5(0) - 2 = -2$$

Vertical intercept at (0,-2)

For the x-intercepts, we solve for when the output will be zero

$$0 = 3x^2 + 5x - 2$$

In this case, the quadratic can be factored easily, providing the simplest method for solution

$$0 = (3x - 1)(x + 2)$$

$$0 = 3x - 1 \qquad \text{or} \qquad 0 = x + 2$$
$$x = \frac{1}{3} \qquad\qquad x = -2$$

x-intercepts at $\left(\frac{1}{3}, 0\right)$ and (-2,0)

Notice that in the standard form of a quadratic, the constant term *c* reveals the vertical intercept of the graph.

Example 3.2.5

Find the x-intercepts of the quadratic $f(x) = 2x^2 + 4x - 4$

Solution

Again we will solve for when the output will be zero

$$0 = 2x^2 + 4x - 4$$

Since the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic into transformation form.

$$h = -\frac{b}{2a} = -\frac{4}{2(2)} = -1 \quad k = f(-1) = 2(-1)^2 + 4(-1) - 4 = -6$$

$$f(x) = 2(x+1)^2 - 6$$

Now we can solve for when the output will be zero

$$\begin{aligned} 0 &= 2(x+1)^2 - 6 \\ 6 &= 2(x+1)^2 \\ 3 &= (x+1)^2 \\ x+1 &= \pm\sqrt{3} \\ x &= -1 \pm \sqrt{3} \end{aligned}$$

The graph has x-intercepts at $(-1-\sqrt{3}, 0)$ and $(-1+\sqrt{3}, 0)$

? You Try 3.2.1

For the function $g(x) = 13 + x^2 - 6x$, find the y and x-intercepts.

Answer

The y-intercept at (0, 13), There are no x-intercepts since the vertex is above the x-axis and the graph opens upwards.

Note that if there are no x-intercepts, it is important to explain how you determined that.

The process for finding the x-intercepts is done commonly enough that sometimes people find it easier to solve the problem once in general and remember the formula for the result, rather than repeating the process each time. Based on our previous work we showed that any quadratic in general form can be written into vertex form as:

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Solving for the horizontal intercepts using this general equation gives:

$$0 = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

start to solve for x by moving the constants to the other side

$$\frac{b^2}{4a} - c = a\left(x + \frac{b}{2a}\right)^2$$

divide both sides by a

$$\frac{b^2}{4a^2} - \frac{c}{a} = \left(x + \frac{b}{2a}\right)^2$$

find a common denominator to combine fractions

$$\frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

combine the fractions on the left side of the equation

$$\frac{b^2 - 4ac}{4a^2} = \left(x + \frac{b}{2a}\right)^2$$

take the square root of both sides

$$\pm\sqrt{\frac{b^2-4ac}{4a^2}}=x+\frac{b}{2a}$$

subtract $b/2a$ from both sides

$$-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=x$$

combining the fractions

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Notice that this can yield two different answers for x

Quadratic Formula

For a quadratic function given in standard form $f(x)=ax^2+bx+c$, the **quadratic formula** gives the horizontal intercepts of the graph of this function.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Let's practice a few more examples of applying all the concepts we've learned about quadratics.

Example 3.2.6

Find the domain and range of $f(x)=-5x^2+9x-1$.

Solution

As with any quadratic function, the domain is all real numbers.

Because a is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the x-value of the vertex.

$$\begin{aligned}h&=-\frac{b}{2a}\\&=-\frac{9}{2(-5)}\\&=\frac{9}{10}\end{aligned}$$

The maximum value is given by $f(h)$.

$$\begin{aligned}f\left(\frac{9}{10}\right)&=5\left(\frac{9}{10}\right)^2+9\left(\frac{9}{10}\right)-1\\&=\frac{61}{20}\end{aligned}$$

Since we know the parabola opens down, we know the range is $f(x)\leq\frac{61}{20}$, or $\left(-\infty,\frac{61}{20}\right]$.

Example 3.2.7

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.

a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
b. What dimensions should she make her garden to maximize the enclosed area?

Solution

Let's use a diagram such as Figure 3.2.4 to record the given information. It is also helpful to introduce a temporary variable, W, to represent the width of the garden and the length of the fence section parallel to the backyard fence.

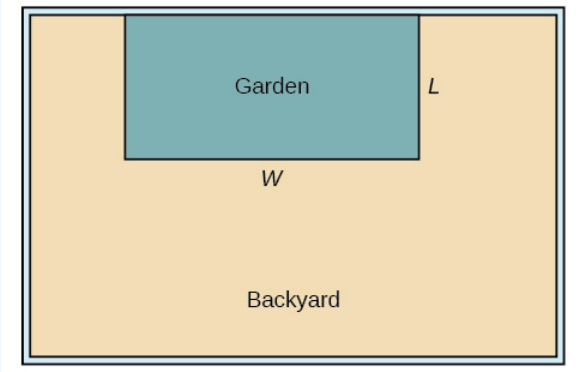


Figure 3.2.4: Diagram of the garden and the backyard.

a. We know we have only 80 feet of fence available, and $L+W+L=80$, or more simply, $2L+W=80$. This allows us to represent the width, W, in terms of L.

$$W=80-2L$$

Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so

$$\begin{aligned} A &= LW = L(80-2L) \\ A(L) &= 80L-2L^2 \end{aligned}$$

This formula represents the area of the fence in terms of the variable length L. The function, written in general form, is

$$A(L)=-2L^2+80L$$

The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since a is the coefficient of the squared term, $a=-2$, $b=80$, and $c=0$.

To find the vertex:

$$\begin{aligned} h &= -\frac{80}{2(-2)} \\ &= 20 \end{aligned}$$

and

$$\begin{aligned} k &= A(20) \\ &= 80(20)-2(20)^2 \\ &= 800 \end{aligned}$$

The maximum value of the function is an area of 800 square feet, which occurs when $L=20$ feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

Analysis

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function in Figure 3.2.5.

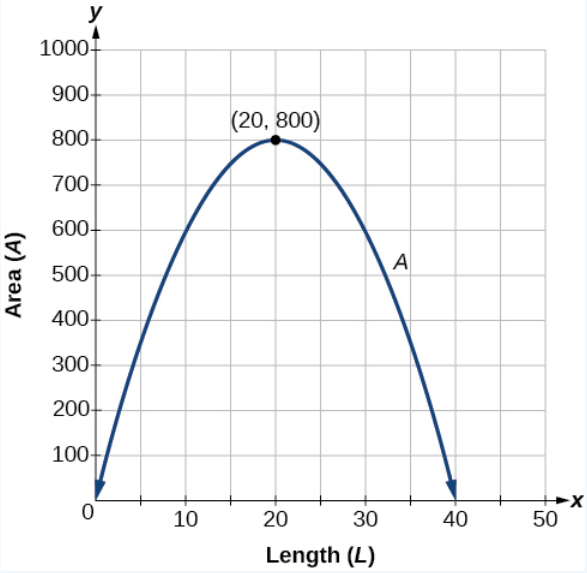


Figure 3.2.5: Graph of the parabolic function $A(L) = -2L^2 + 80L$

Example 3.2.8

A ball is thrown upwards from the top of a 40-foot-tall building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.

What is the maximum height of the ball?

When does the ball hit the ground? Round to the nearest thousandth.

Solution

To find the maximum height of the ball, we would need to know the vertex of the quadratic.

$$h = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2}, k = H\left(\frac{5}{2}\right) = -16\left(\frac{5}{2}\right)^2 + 80\left(\frac{5}{2}\right) + 40 = 140$$

The ball reaches a maximum height of 140 feet after 2.5 seconds.

To find when the ball hits the ground, we need to determine when the height is zero – when $H(t) = 0$. While we could do this using the transformation form of the quadratic, we can also use the quadratic formula:

$$t = \frac{-80 \pm \sqrt{80^2 - 4(-16)(40)}}{2(-16)} = \frac{-80 \pm \sqrt{8960}}{-32}$$

Since the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions:

$$t = \frac{-80 - \sqrt{8960}}{-32} \approx 5.458 \text{ or } t = \frac{-80 + \sqrt{8960}}{-32} \approx -0.458$$

The second answer is outside the reasonable domain of our model, so we conclude the ball will hit the ground after about 5.458 seconds.

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3.3: Graphs of Polynomial Functions

Learning Objectives

- Identify polynomial functions.
- Identify end behavior of polynomial functions.
- Find x-intercepts of polynomial functions.

Polynomial functions are used widely for modeling many real-life situations. In this section, we will investigate the common traits of this important family of functions.

Polynomial Function

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

where a_0, a_1, ... , a_n are real numbers and $n \geq 1$ is a natural number. The domain of a polynomial function is $(-\infty, \infty)$.

- The natural number n is called the **degree** of the polynomial f.
- The term $a_n x^n$ is called the **leading term** of the polynomial f.
- The real number a_n is called the **leading coefficient** of the polynomial f.
- The real number a_0 is called the **constant term** of the polynomial f.

Example 3.3.1

Determine if the following functions are polynomials. If they are, identify the degree, leading term, leading coefficient and constant term.

1. $f(x) = 4x^5 - 3x^2 + 2x - 5$
2. $g(x) = 12x - x^3$
3. $p(x) = \dfrac{4x + x^3}{x}$
4. $h(x) = \dfrac{4 - x}{5}$
5. $A(x) = (2x - 1)^3 (x - 2)(3x + 2)$

Solution

1. This is a polynomial function by definition, and we see that the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is -5.
2. The definition of a polynomial function has the highest power of x first. To that end, we re-write $g(x) = 12x - x^3 = -x^3 + 12x$, and see that the degree of g is 3, the leading term is $-x^3$, the leading coefficient is -1 and the constant term is 0.
3. Even though $p(x) = \dfrac{x^3 + 4x}{x}$ simplifies to $p(x) = x^2 + 4$, which certainly looks like a polynomial function, the domain of p excludes 0. Alas, p is not a polynomial function.
4. h can be rewritten as follows: $h(x) = \dfrac{4 - x}{5} = \dfrac{4}{5} - \dfrac{x}{5} = -\dfrac{1}{5}x + \dfrac{4}{5}$. The degree of h is 1, the leading term is $-\dfrac{1}{5}x$, the leading coefficient is $-\dfrac{1}{5}$ and the constant term is $\dfrac{4}{5}$.
5. It may seem that we have some work ahead of us to get A in the form provided in the definition of a polynomial function. However, it is possible to gain the information requested about A without multiplying out the entire expression $(2x - 1)^3 (x - 2)(3x + 2)$. The leading term of A will be the term which has the highest power of x. The way to get this term is to multiply the terms with the highest power of x from each factor together - in other words, the leading term of $A(x)$ is the product of the leading terms of the factors of $A(x)$. Hence, the leading term of A is $(2x)^3 (x)(3x) = 24x^5$.

This means that the degree of A is 5 and the leading coefficient is 24. As for the constant term, we can perform a similar trick. The constant term is obtained by multiplying the constant terms from each of the factors $(-1)^3(-2)(2) = 4$.

Now that we know how to identify polynomial functions, we can start to investigate their graphical behavior. Polynomial functions share two properties which help distinguish them from other animals (functions) in the algebra zoo: they are continuous and smooth. While these concepts are formally defined in Calculus, informally, polynomial graphs have no 'breaks', 'holes', or 'sharp turns' (also known as a 'cusp'). Below, in Figure 3.3.1, on the left we a graph of a function which is neither smooth nor continuous, and to its right we have a graph of a polynomial, for comparison.

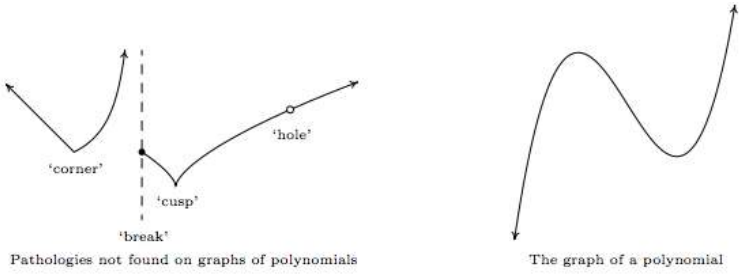


Figure 3.3.1: Graph of a function that is not a polynomial and a graph of a polynomial.

Next, let's look at monomials (single term polynomials) to help us establish some graphical pattern that is similar with all polynomials.

End Behavior of Polynomial Functions

Figure 3.3.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and and $h(x) = x^6$, which are monomial functions with even degrees. Notice that these graphs have similar shapes, very much like that of the basic quadratic function. They are commonly called *basic even-power functions*. However, as the power increases, the graphs flattens near the origin within the interval $(-1, 1)$ and becomes steeper must faster outside of this interval.

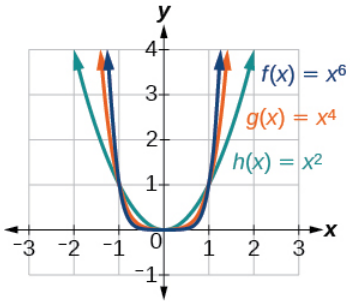


Figure 3.3.2: Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the even-degree monomial functions, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we write this as

$$\text{as } x \to \pm\infty,\ f(x) \to \infty$$

Figure 3.3.3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all odd degree monomial functions, commonly called basic *odd-degree power functions*. Notice that these graphs look similar to the basic cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

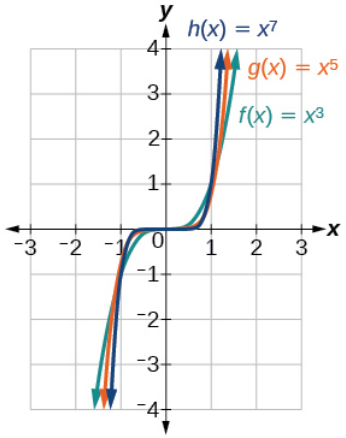


Figure 3.3.3: Odd-power function

These examples illustrate that functions of the form $f(x) = x^n$ where n is a positive integer reveal symmetry of one kind or another. First, in Figure 3.3.2 we see that functions of the form $f(x) = x^n$, n even, are symmetric about the y-axis. In Figure 3.3.3 we see that functions of the form $f(x) = x^n$, n odd, are symmetric about the origin.

For these odd powered functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$
$$\text{as } x \to \infty,\ f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small $(x \to -\infty)$ and get very large $x \to \infty$ is referred to as the **end behavior** of the function. We can use words or symbols to describe end behavior.

Figure 3.3.4 shows the end behavior of various monomial functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power and the constant.

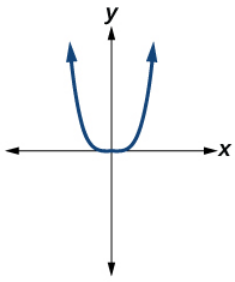
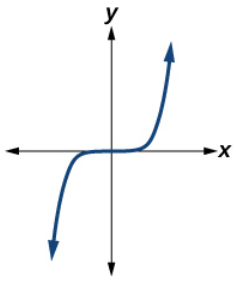
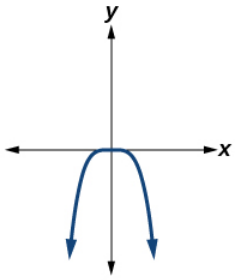
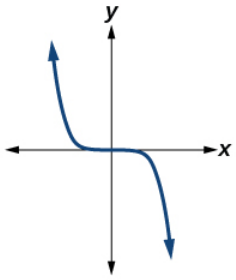
	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to \infty$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty$, $f(x) \to -\infty$ and $x \to \infty$, $f(x) \to -\infty$	 $x \to -\infty$, $f(x) \to \infty$ and $x \to \infty$, $f(x) \to -\infty$

Figure 3.3.4

The end behavior describes above can be generalized for all polynomials as follows:

End Behavior of Even Degree Polynomial Functions

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and n is an even natural number. The end behavior of the graph of $y = f(x)$ matches one of the following:

- for $a > 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$
- for $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$

Graphically:

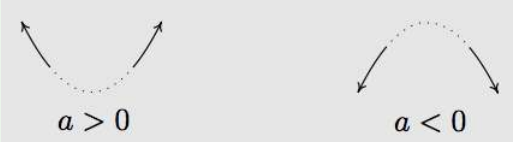


Figure 3.3.5

End Behavior of Odd Degree Polynomial Functions

Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \geq 3$ is an odd natural number. The end behavior of the graph of $y = f(x)$ matches one of the following:

1. for $a > 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$
2. for $a < 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$

Graphically:

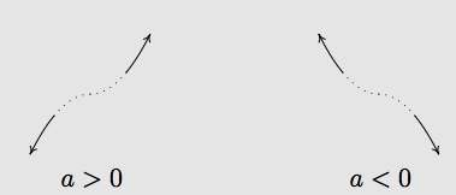


Figure 3.3.6

Example 3.3.2

Describe the end behavior and determine a possible degree of the polynomial function in Figure 3.3.7.

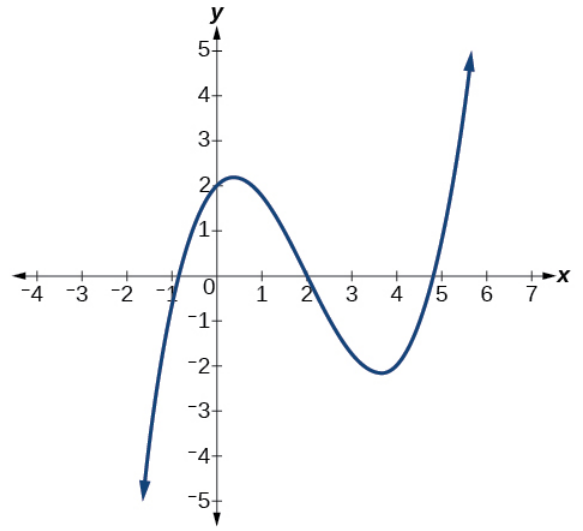


Figure 3.3.7.

Solution

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to \infty,\ f(x) \to \infty$$

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree polynomial that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

You Try 3.3.1

Describe the end behavior, and determine a possible degree of the polynomial function in Figure 3.3.8.

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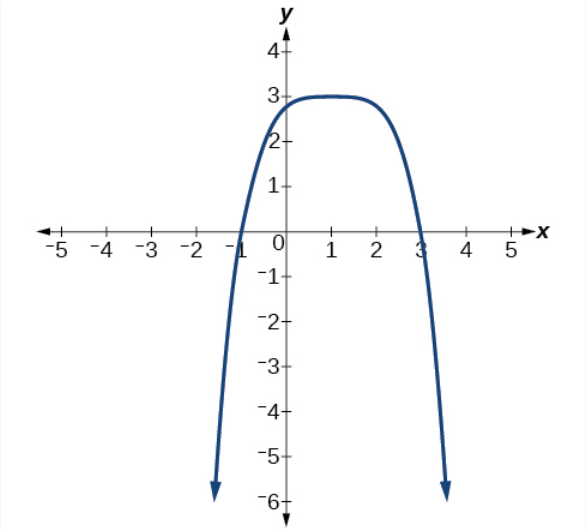


Figure 3.3.8

Answer

As $x\to\infty$, $f(x)\to-\infty$; as $x\to-\infty$, $f(x)\to-\infty$. It has the shape of an even degree polynomial function with a negative coefficient.

Example 3.3.3

Given the function $f(x)=-3x^2(x-1)(x+4)$, determine the leading term, degree, and end behavior of the function.

Solution

First, expand the function,

$$\begin{aligned} f(x) &= -3x^2(x-1)(x+4) \\ &= -3x^2(x^2+3x-4) \\ &= -3x^4-9x^3+12x^2 \end{aligned}$$

The leading term is $-3x^4$; therefore, the degree of the polynomial is 4. The degree is even (4) and the leading coefficient is negative (–3), so the end behavior is

$$\text{as } x\to-\infty,\ f(x)\to-\infty$$

$$\text{as } x\to\infty,\ f(x)\to-\infty$$

You Try 3.3.2

Given the function $f(x)=0.2(x-2)(x+1)(x-5)$, determine the leading term, degree, and end behavior of the function.

Answer

The leading term is $0.2x^3$, so it is a degree 3 polynomial.

$$\text{as } x\to-\infty,\ f(x)\to-\infty$$

$$\text{as } x\to\infty,\ f(x)\to\infty$$

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in the intercepts and turning points. As with all functions, the y-intercept is the point at which the

graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one y-intercept $(0, a_0)$. The x-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See Figure 3.3.9. Turning points are local extrema values. We may approximate them using technology but it will take calculus to find them exactly. Hence, we will focus our time in this course are finding intercepts only.

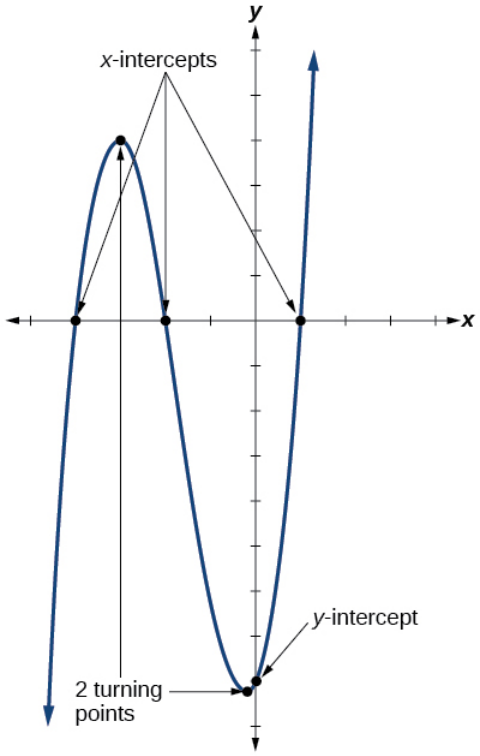


Figure 3.3.9

Before we look at finding intercepts, let's note some important theory about graphical behavior.

The Intermediate Value Theorem

Suppose f is a continuous function on an interval containing $x = a$ and $x = b$ with $a < b$. If $f(a)$ and $f(b)$ have different signs, then f has at least one zero between $x = a$ and $x = b$; that is, for at least one real number c such that $a < c < b$, we have $f(c) = 0$.

The Intermediate Value Theorem is extremely profound; it gets to the heart of what it means to be a real number, and is one of the most often used and under appreciated theorems in Mathematics. With that being said, most students see the result as common sense since it says, geometrically, that the graph of a polynomial function cannot be above the x-axis at one point and below the x-axis at another point without crossing the x-axis somewhere in between.

Example 3.3.4

Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, determine the y- and x-intercepts.

Solution

The y-intercept occurs when the input is zero, so substitute 0 for x.

$$\begin{aligned} f(0) &= (0-2)(0+1)(0-4) \\ &= (-2)(1)(-4) \\ &= 8 \end{aligned}$$

The y-intercept is $(0, 8)$.

The x-intercepts occur when the output is zero. The function is already factored which makes this equation very nice to solve:

$$0 = (x-2)(x+1)(x-4)$$

$$\begin{array}{lll} x-2=0 & \text{or} & x+1=0 \quad \text{or} \quad x-4=0 \\ x=2 & & x=-1 \qquad\quad x=4 \end{array}$$

The x-intercepts are $(2,0)$,$(-1,0)$, and $(4,0)$.

We can see these intercepts on the graph of the function shown in Figure 3.3.10

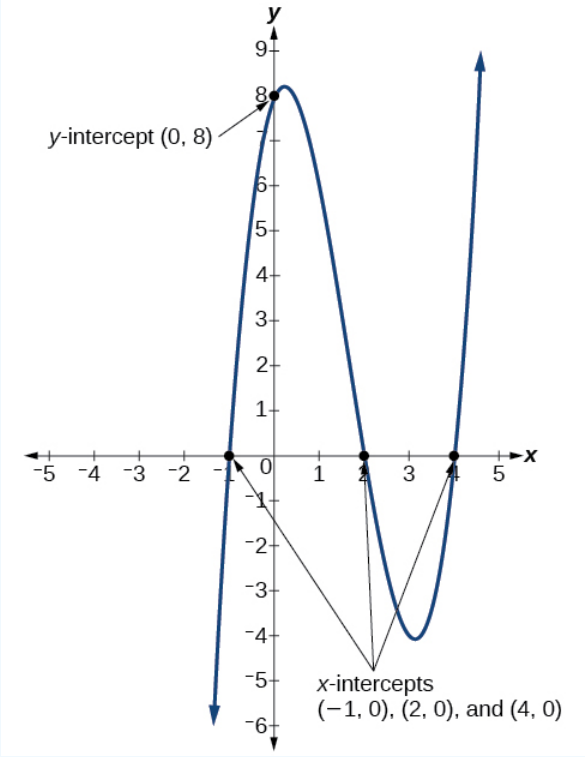


Figure 3.3.10: Graph of $f(x) = (x-2)(x+1)(x-4)$.

You Try 3.3.3

Given the polynomial function $f(x) = 2x^3 - 6x^2 - 20x$, determine the intercepts.

Answer

x-intercepts $(0,0)$,$(-2,0)$, and $(5,0)$, y-intercept $(0,0)$;

Example 3.3.5

Find the x-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

Solution

We can attempt to factor this polynomial to find solutions for $f(x) = 0$.

$$x^6 - 3x^4 + 2x^2 = 0$$

Factoring out the greatest common factor

$$x^2(x^4 - 3x^2 + 2) = 0$$

Factoring the inside as a quadratic in x^2

$$x^2(x^2-1)(x^2-2)=0$$

Then break apart to find solutions

$$\begin{array}{lll} x^2=0 & x^2-1=0 & x^2-2=0 \\ x=0 \quad \text{or} & x^2=1 \quad \text{or} & x^2=2 \\ & x=\pm 1 & x=\pm\sqrt{2} \end{array}$$

This gives us 5 x-intercepts: $(0,0)$, $(-\sqrt{2},0)$, $(-1,0)$, $(1,0)$, and $(\sqrt{2},0)$.

Example 3.3.6

Given the polynomial function $f(x)=x^4-4x^2-45$, determine the y- and x-intercepts.

Solution

The y-intercept occurs when the input is zero.

$$\begin{aligned} f(0) &= (0)^4-4(0)^2-45 \\ &= -45 \end{aligned}$$

The y-intercept is $(0,-45)$.

The x-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.

$$\begin{aligned} f(x) &= x^4-4x^2-45 \\ &= (x^2-9)(x^2+5) \\ &= (x-3)(x+3)(x^2+5) \end{aligned}$$

$$0=(x-3)(x+3)(x^2+5)$$

$$\begin{array}{lll} x-3=0 & x+3=0 & x^2+5=0 \\ x=3 \quad \text{or} & x=-3 \quad \text{or} & \text{no real solution} \end{array}$$

The x-intercepts are $(3,0)$ and $(-3,0)$.

We can see these intercepts on the graph of the function shown in Figure 3.3.11. We can see that the function is even because $f(x)=f(-x)$.

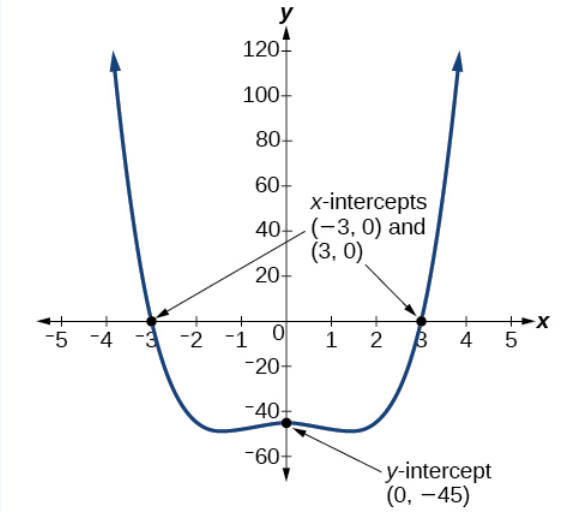


Figure 3.3.11: Graph of $f(x)=x^4-4x^2-45$.

https://math.libretexts.org/@go/page/126520

Example 3.3.7

Find the intercepts of $D(x) = 3(x-2)^6 - 8$.

Solution

This is a basic even-power function that has has a horizontal shift right 2 units, vertical shift down 8 units, and a vertical stretch by 3. Since the basic shape is similar to a quadratic function, we know the graph opens up so that indicates that there are two x-intercepts. If you take a moment to sketch the function on graph paper using your knowledge of transformations, you will also notice that the x-intercepts should have x-coordinates close to 1 and 3. See Figure 3.3.12

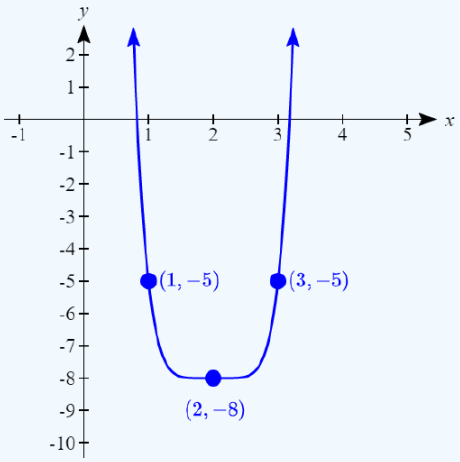


Figure 3.3.12: Graph of D(x)

Finding the x-intercepts,

$$\begin{aligned} 0 &= 3(x-2)^6 - 8 \\ 8 &= 3(x-2)^6 \\ \frac{8}{3} &= (x-2)^6 \\ \pm\sqrt[6]{\frac{8}{3}} &= \sqrt[6]{(x-2)^6} \\ \pm\sqrt[6]{\frac{8}{3}} &= x-2 \\ 2 \pm \sqrt[6]{\frac{8}{3}} &= x \end{aligned}$$

The x-intercepts are $\left(2-\sqrt[6]{\frac{8}{3}}, 0\right)$ and $\left(2+\sqrt[6]{\frac{8}{3}}, 0\right)$.

Note that if you approximate the x-intercepts that they should be about where we anticipated them to be when we sketched the graph.

Finding the y-intercept,

$$\begin{aligned} D(0) &= 3(x-2)^6 - 8 \\ &= 3(0-2)^6 - 8 \\ &= 3(-2)^6 - 8 \\ &= 3(64) - 8 \\ &= 184 \end{aligned}$$

The y-intercept is $(0, 184)$.

Local Extrema of Polynomials

Recall that local extrema are function occur where a function changes from increasing to decreasing or from decreasing to increasing. Turning points for a graph of a polynomial function occur at extrema values. Although calculus is required to prove the following, this is an important theorem to know about the number of local extrema values we can expect for a polynomial function.

Number of Local Extrema for Polynomials

A polynomial of degree n will have at most $n-1$ local extrema.

Note that a polynomial of degree n can have less than $n-1$ local extrema. For example, $P(x) = 2x^5$ is a polynomial of degree 5 but it has no local extrema.

Graphical Behavior Near X-Intercepts

If we graph the function $f(x) = (x+3)(x-2)^2(x+1)^3$, as shown in Figure 3.3.13, notice that the behavior at each of the x-intercepts is different.

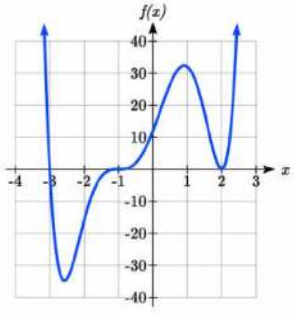


Figure 3.3.13: Graph of $f(x) = (x+3)(x-2)^2(x+1)^3$

At the x-intercept at $(-3, 0)$, which came from the $(x+3)$ factor of the polynomial, the graph passes directly through this horizontal intercept.

The factor $(x+3)$ is linear (has a power of 1), so the behavior near the intercept is like that of a line - it passes directly through the intercept. We call this a single zero, since the zero corresponds to a single factor of the function.

At the x-intercept at $(2, 0)$, which came from the $(x-2)^2$ factor of the polynomial, the graph touches the axis at the intercept and then the graph bounces. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic – it bounces off the horizontal axis at the intercept. Since $(x-2)^2 = (x-2)(x-2)$, the factor is repeated twice, so we call this a double zero. We could also say the zero has multiplicity 2.

At the x-intercept at $(-1, 0)$, which came from the $(x+1)^3$ factor of the polynomial, the graph passes through the axis at the intercept, but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic, with the same "S" type shape near the intercept that the cubic parent function, x^3, has. We call this a triple zero. We could also say the zero has multiplicity 3.

By utilizing these behaviors, we can sketch a reasonable graph of a factored polynomial function without needing technology.

Graphical Behavior of Polynomials at X-intercepts

If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the horizontal intercept h is determined by the power on the factor.

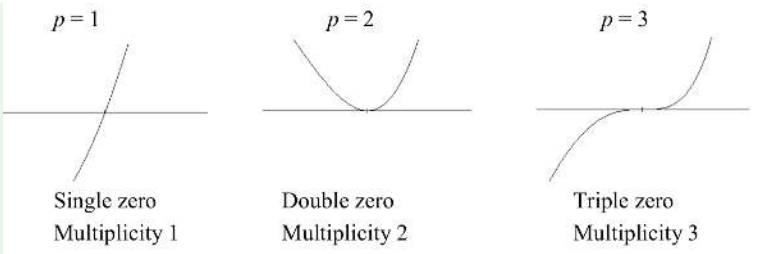


Figure 3.3.14: Graphical behavior of roots with various roots of multiplicities

For higher even powers 4,6,8 etc.… the graph will bounce off the horizontal axis but the graph will appear flatter with each increasing even power as it approaches and leaves the axis.

For higher odd powers, 5,7,9 etc… the graph will pass through the horizontal axis but the graph will appear flatter with each increasing odd power as it approaches and leaves the axis.

Example 3.3.8

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution

This graph has two horizontal intercepts. At -3, the factor it came from is squared, indicating the graph will bounce at this horizontal intercept. At 5, the factor t came from is linear, indicating the graph will pass through the axis at this intercept.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x^3$, so the long-run behavior the left side of the graph will go upward and the right side will go downward.

To sketch this we consider the following:

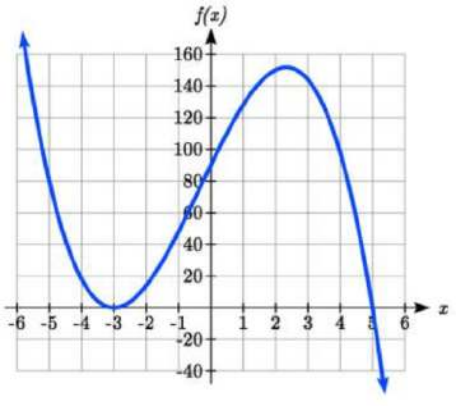


Figure 3.3.15: Copy and Paste Caption here. (Copyright; author via source)

As $x \to -\infty$ the function $f(x) \to \infty$ so we know the graph starts in the 2nd quadrant and is decreasing toward the horizontal axis.

At (-3, 0) the graph bounces off the horizontal axis and so the function must start increasing.

At (0, 90) the graph crosses the vertical axis at the vertical intercept.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis since the graph passes through the next intercept at (5,0).

As $x \to \infty$ the function $f(x) \to -\infty$ so we know the graph continues to decrease and we can stop drawing the graph in the 4 th quadrant.

Using technology and an appropriate viewing window we can verify the shape of the graph.

Example 3.3.9

Write a formula for the polynomial function graphed in Figure $PageIndex16$.

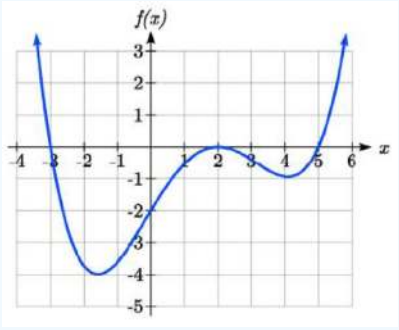


Figure 3.3.16: Graph of a polynomial with x-intercepts at (-3,0), (2,0), and (5,0)

Solution

This graph has three horizontal intercepts: x = -3, 2, and 5. At x = -3 and 5 the graph passes through the axis, suggesting the corresponding factors of the polynomial will be linear. At $x = 2$ the graph bounces at the intercept, suggesting the corresponding factor of the polynomial will be 2nd degree (quadratic).

Together, this gives us:

$$f(x) = a(x+3)(x-2)^2(x-5)$$

To determine the stretch factor, we can utilize another point on the graph. Here, the vertical intercept appears to be (0,-2), so we can plug in those values to solve for a:

$$-2 = a(0+3)(0-2)^2(0-5)$$
$$-2 = -60a$$
$$a = \frac{1}{30}$$

The graphed polynomial appears to represent the function

$$f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$$

You Try 3.3.4

Given the graph, write a formula for the function shown in Figure 3.3.17.

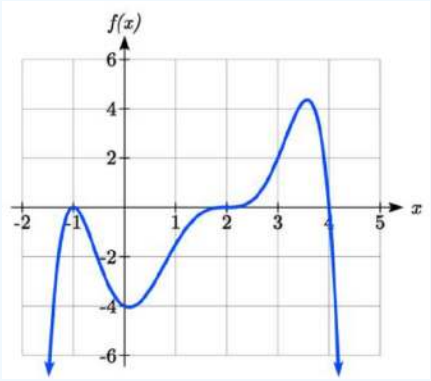


Figure 3.3.17: Graph of a polynomial with x-intercepts at (-1,0), (2,0), and (4,0)

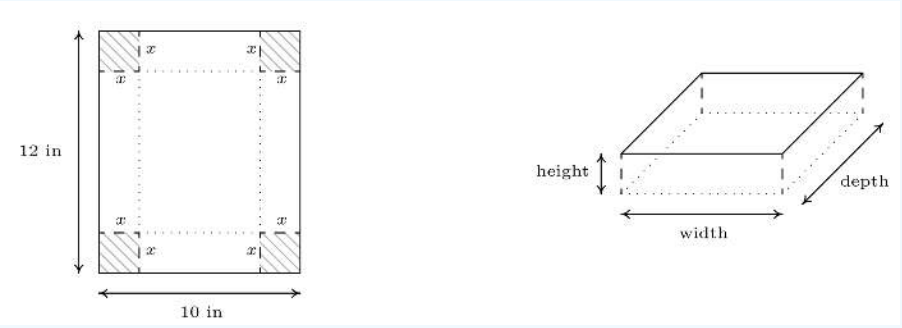
Answer

Double zero at $x=-1$, triple zero at $x=2$. Single zero at $x=4$. $f(x)=a(x-2)^3(x+1)^2(x-4)$. Substituting (0,-4) and solving for a, $f(x)=-\frac{1}{8}(x-2)^3(x+1)^2(x-4)$

Let's look at a couple more examples to apply all that we have been learning.

Example 3.3.10

A box with no top, also called a tray, is to be fashioned from a 10 inch $\times$ 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. Let x denote the length of the side of the square which is removed from each corner.



Find the volume V of the box as a function of x. Include an appropriate applied domain.

Use a graphing calculator to graph $y=V(x)$ on the domain you found in part 1 and approximate the dimensions of the box with maximum volume to two decimal places. What is the maximum volume?

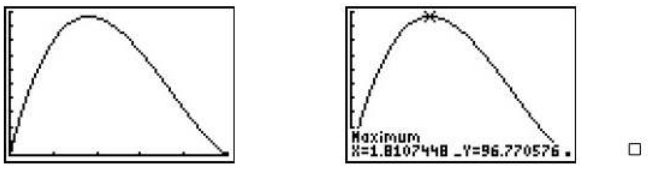
Solution

1. From Geometry, we know that $\text{Volume}=\text{width}\times\text{height}\times\text{depth}$. The key is to find each of these quantities in terms of x. From the figure, we see that the height of the box is x itself. The cardboard piece is initially 10 inches wide. Removing squares with a side length of x inches from each corner leaves $10-2x$ inches for the width.\footnote{There's no harm in taking an extra step here and making sure this makes sense. If we chopped out a 1 inch square from each side, then the width would be 8 inches, so chopping out x inches would leave $10-2x$ inches.} As for the depth, the cardboard is initially 12 inches long, so after cutting out x inches from each side, we would have $12-2x$ inches remaining. As a function\footnote{When we write $V(x)$, it is in the context of function notation, not the volume V times the quantity x.} of x, the volume is

$$V(x)=x(10-2x)(12-2x)=4x^3-44x^2+120x$$

To find a suitable applied domain, we note that to make a box at all we need $x>0$. Also the shorter of the two dimensions of the cardboard is 10 inches, and since we are removing $2x$ inches from this dimension, we also require $10-2x>0$ or $x<5$. Hence, our applied domain is $0<x<5$.

2. Using a graphing calculator, we see that the graph of $y=V(x)$ has a relative maximum. For $0<x<5$, this is also the absolute maximum. Using the 'Maximum' feature of the calculator, we get $x\approx1.81$, $y\approx96.77$. This yields a height of $x\approx1.81$ inches, a width of $10-2x\approx6.38$ inches, and a depth of $12-2x\approx8.38$ inches. The y-coordinate is the maximum volume, which is approximately 96.77 cubic inches (also written in^3).



In order to solve, we made good use of the graph of the polynomial $y=V(x)$, so we ought to turn our attention to graphs of polynomials in general. Below are the graphs of $y=x^2$, $y=x^4$ and $y=x^6$, side-by-side. We have omitted the axes to allow

you to see that as the exponent increases, the 'bottom' becomes 'flatter' and the 'sides' become 'steeper.' If you take the the time to graph these functions by hand,\footnote{Make sure you choose some x-values between -1 and 1.} you will see why.

We learned earlier in this course how to solve non-linear inequalities. Inequalities also tell us when a function will be positive and negative. Let's look at an example to ensure we understand the relationship between inequalities and graphs of functions.

Example 3.3.11

Solve $(x+3)(x+1)^2(x-4)>0$

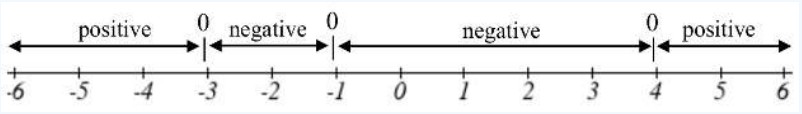
Solution

As with all inequalities, we start by solving the equality $(x+3)(x+1)^2(x-4)=0$, which has solutions at *x* = -3, -1, and 4. We know the function can only change from positive to negative at these values, so these divide the inputs into 4 intervals.

We could choose a test value in each interval and evaluate the function $f(x)=(x+3)(x+1)^2(x-4)$ at each test value to determine if the function is positive or negative in that interval

Interval	Test x in interval	f (test value)	> 0 or < 0?
$x<-3$	-4	72	> 0
$-3<x<-1$	-2	-6	< 0
$-1<x<4$	0	-12	< 0
$x>4$	5	288	> 0

On a number line this would look like:



From our test values, we can determine this function is positive when $x<-3$ or $x>4$, or in interval notation, $(-\infty,-3)\bigcup(4,\infty)$

You Try 3.3.5

Given the function $g(x)=x^3-x^2-6x$ use the methods that we have learned so far to find the vertical & horizontal intercepts, determine where the function is negative and positive, describe the end behavior, and sketch the graph without technology.

Answer

Vertical intercept (0, 0), Horizontal intercepts (-2, 0), (0, 0), (3, 0)

The function is negative on $(-\infty$, -2) and (0, 3)

The function is positive on (-2, 0) and (3,∞)

The leading term is x^3 so as $x\to-\infty$, $g(x)\to-\infty$ and as$x\to\infty$, $g(x)\to\infty$

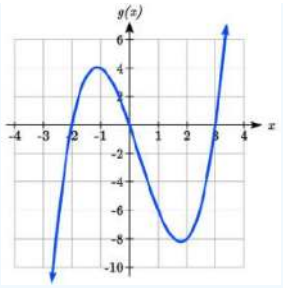


Figure 3.3.18: Graph of g(x)=x^3-x^2-6x

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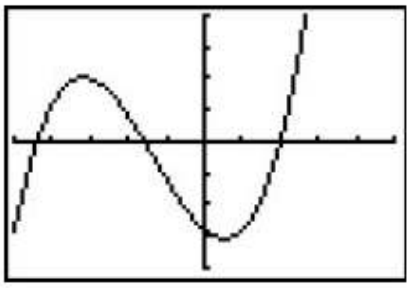
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3.4: Polynomial Division

Learning Objectives

- Divide polynomials using long division
- Divide polynomials using synthetic division
- Apply the factor and remainder theorems to polynomials

Suppose we wish to find the x-intercepts of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all of the factoring techniques we learned in Intermediate Algebra courses, this equation foils us at every turn since these factoring techniques do no work. If we graph f using a graphing calculator window [-5,5]x[-16,16], we get



The graph suggests that the function has three x-intercepts, one of which is at $x = 2$. It's easy to show that $f(2) = 0$, but the other two x-intercepts seem to be less friendly. Even though we could use the 'Zero' command to find decimal approximations for these x-intercepts, we seek a method to find the exact values of the other x-intercepts. Based on our experience in the previous section, if there is an x-intercept at $x = 2$, it seems that there should be a factor of $(x - 2)$ for $f(x)$. In other words, we should expect that $x^3 + 4x^2 - 5x - 14 = (x - 2)\,q(x)$, where $q(x)$ is some other polynomial. How could we find such a $q(x)$, if it even exists? The answer comes from our old friend, division.

We are familiar with long division from ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

Long Division

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ -15\downarrow \\ \hline 28 \\ -27 \\ \hline 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$
Answer: $59R1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{dividend} &= (\text{divisor}\cdot\text{quotient}) + \text{remainder} \\ 178 &= (3\cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We can also state that $\frac{178}{3} = 59 + \frac{1}{3}$ which is the algebraic form of the mixed number $59\frac{1}{3}$

We call this the **Division Algorithm**.

Long Division of Polynomials

Now, let's revisit the the polynomial $f(x) = x^3 + 4x^2 - 5x - 14$. In the last section we saw that we could write a polynomial as a product of factors, each corresponding to a x-intercept. If we know that there is an x-intercept at $x = 2$ for $f(x)$, then we might guess that the polynomial could be factored as $x^3 + 4x^2 - 5x - 14 = (x - 2)$ (something). To find that "something," we can use polynomial division.

Example 3.4.1

Divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$.

Solution

Start by writing the problem out in long division form

$$x - 2 \longdiv{x^3 + 4x^2 - 5x - 14}$$

Now we divide the leading terms: $x^3 \div x = x^2$. It is best to align it above the same-powered term in the dividend. Now, multiply that x^2 by $x - 2$ and write the result below the dividend.

$$\begin{array}{r} x^2 \\ x-2\overline{)\,x^3+4x^2-5x-14} \\ x^3-2x^2 \end{array}$$

Now subtract that expression from the dividend.

$$\begin{array}{r} x^2 \\ x-2\overline{)\,x^3+4x^2-5x-14} \\ \underline{-\left(x^3-2x^2\right)} \\ 6x^2-5x-14 \end{array}$$

Again, divide the leading term of the remainder by the leading term of the divisor. $6x^2 \div x = 6x$. We add this to the result, multiply 6*x* by $x - 2$, and subtract.

$$\begin{array}{r} x^2+6x \\ x-2\overline{)\,x^3+4x^2-5x-14} \\ \underline{-\left(x^3-2x^2\right)} \\ 6x^2-5x-14 \\ \underline{-\left(6x^2-12x\right)} \\ 7x-14 \end{array}$$

Repeat the process one last time.

$$\begin{array}{r} x^2+6x+7 \\ x-2\overline{)\,x^3+4x^2-5x-14} \\ \underline{-\left(x^3-2x^2\right)} \\ 6x^2-5x-14 \\ \underline{-\left(6x^2-12x\right)} \\ 7x-14 \\ \underline{-\left(7x-14\right)} \\ 0 \end{array}$$

This tells us $x^3 + 4x^2 - 5x - 14$ divided by $x - 2$ is $x^2 + 6x + 7$, with a remainder of zero. This also means that we can factor $x^3 + 4x^2 - 5x - 14$ as $(x - 2)\left(x^2 + 6x + 7\right)$.

Now, if we want to find the other x-intercepts, we now solve $x^2 + 6x + 7 = 0$. Since this doesn't factor nicely we can complete the square or use the Quadratic Formula to get $x = -3 \pm \sqrt{2}$. The point of this section is to focus on how to perform division. Before looking at more examples, let's note what we can expect when we divide polynomials.

The Division Algorithm for Polynomials

Suppose $d(x)$ and $p(x)$ are nonzero polynomials where the degree of p is greater than or equal to the degree of d. There exist two unique polynomials, $q(x)$ and $r(x)$, such that

$$p(x) = d(x)\,q(x) + r(x),$$

or

$$\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

The polynomial p is called the **dividend**; d is the **divisor**; q is the **quotient**; r is the **remainder**. If $r(x) = 0$ then d is called a **factor** of p.

Because of the division, the remainder will either be zero, or a polynomial of lower degree than d(x). Because of this, if we divide a polynomial by a term of the form $x - c$, then the remainder will be zero or a constant.

If $p(x) = (x-c)q(x) + r$, then $p(c) = (c-c)q(c) + r = 0 + r = r$, which establishes the Remainder Theorem.

The Remainder Theorem

If $p(x)$ is a polynomial of degree 1 or greater and c is a real number, then when p(x) is divided by $x - c$, the remainder is $p(c)$.

If $x - c$ is a factor of the polynomial p, then $p(x) = (x-c)q(x)$ for some polynomial q. Then $p(c) = (c-c)q(c) = 0$, showing c is a zero of the polynomial. This shouldn't surprise us - we already knew that if the polynomial factors it reveals the roots.

If $p(c) = 0$, then the remainder theorem tells us that if p is divided by $x - c$, then the remainder will be zero, which means $x - c$ is a factor of p.

The solutions to an equation $p(x) = 0$ are called zeros of $p(x)$. Zeros can be real numbers or complex numbers. As we have seen with quadratics, real zeros corresponding to x-coordinates of x-intercepts and complex zeros do not.

The Factor Theorem

If $p(x)$ is a nonzero polynomial, then the real number c is a zero of $p(x)$ if and only if $x - c$ is a factor of $p(x)$.

Example 3.4.2

Divide $5x^2 + 3x - 2$ by $x + 1$.

Solution

$x+1\ \overline{)\,5x^2+3x-2}$	Set up division problem.
$\phantom{x+1\ \overline{)\,}5x$	
$x+1\ \overline{)\,5x^2+3x-2}$	$5x^2$ divided by x is $5x$.
$\phantom{x+1\ \overline{)\,}5x$	
$x+1\ \overline{)\,5x^2+3x-2}$	Multiply $x+1$ by $5x$.
$\phantom{x+1\ \overline{)\,}-(5x^2+5x)$	
$\phantom{x+1\ \overline{)\,}-2x-2$	Subtract.
$\phantom{x+1\ \overline{)\,}5x-2$	Bring down the next term.
$x+1\ \overline{)\,5x^2+3x-2}$	$-2x$ divided by x is -2.
$\phantom{x+1\ \overline{)\,}-(5x^2+5x)$	
$\phantom{x+1\ \overline{)\,}-2x-2$	
$\phantom{x+1\ \overline{)\,}-(-2x-2)$	Multiply $x+1$ by -2.
$\phantom{x+1\ \overline{)\,}0$	Subtract.

The quotient is $5x-2$. The remainder is 0. We write the result as

$$\frac{5x^2+3x-2}{x+1}=5x-2$$

Analysis

This division problem had a remainder of 0. This tells us that $(x+1)$ is a factor of $5x^2+3x-2$, meaning $5x^2+3x-2=(x+1)(5x-2)$, and that $x=-1$ is a zero of $p(x)=5x^2+3x-2$.

Example 3.4.3

Divide $6x^3+11x^2-31x+15$ by $3x-2$.

Solution

$$\begin{array}{r} 2x^2+5x-7 \\ 3x-2\overline{)\,6x^3+11x^2-31x+1} \\ \underline{-(6x^3-4x^2)} \\ 15x^2-31x \\ \underline{-(15x^2+10x)} \\ 21x+15 \\ \underline{-(-21x+14)} \\ 1 \end{array}$$

$6x^3$ divided by $3x$ is $2x^2$.

Multiply $x+2$ by $2x^2$.
Subtract. Bring down the next term. $15x^2$ divided by $3x$ is $5x$.
Multiply $3x-2$ by $5x$.
Subtract. Bring down the next term. $-21x$ divided by $3x$ is -7.
Multiply $3x-2$ by -7.
Subtract. The remainder is 1.

There is a remainder of 1. We can express the result as:

$$\frac{6x^3+11x^2-31x+15}{3x-2}=2x^2+5x-7+\frac{1}{3x-2}$$

Analysis

We can check our work by using the Division Algorithm to rewrite the solution, then multiply.

$$(3x-2)(2x^2+5x-7)+1=6x^3+11x^2-31x+15$$

Notice, as we write our result,

- the dividend is $6x^3+11x^2-31x+15$
- the divisor is $3x-2$
- the quotient is $2x^2+5x-7$
- the remainder is 1. Since the remainder is 1 we know that $3x-2$ is not a factor of $6x^3+11x^2-31x+15$.

Synthetic Division

Since dividing by $x-c$ is a way to check if a number, c, is a zero of the polynomial, it would be nice to have a faster way to divide by $x-c$ than having to use long division every time. Happily, quicker ways have been discovered.

Let's look back at the long division we did in Example 1 and try to streamline it. First, let's distribute the negatives in the subtraction steps:

$$
\begin{array}{r}
x^2+6x+7 \\
x-2\overline{)\ x^3+4x^2-5x-14} \\
\underline{-x^3+2x^2} \\
6x^2-5x-14 \\
\underline{-6x^2+12x} \\
7x-14 \\
\underline{-7x+14} \\
0
\end{array}
$$

Next, observe that the terms $-x^3$, $-6x^2$, and $-7x$ are the exact opposite of the terms above them. Also note that the terms we 'bring down' (namely the -5x and -14) aren't really necessary to recopy since we know they will cancel out when we subtract, so let's omit them, too:

$$
\begin{array}{r}
x^2+6x+7 \\
x-2\overline{)\ x^3+4x^2-5x-14} \\
\underline{2x^2} \\
6x^2 \\
\underline{12x} \\
7x \\
\underline{14} \\
0
\end{array}
$$

Now, let's move things up a bit and, for reasons which will become clear in a moment, copy the x^3 into the last row.

$$
\begin{array}{r}
x^2+6x+7 \\
x-2\overline{)x^3+4x^2-5x-14} \\
2x^2\ 12x\ \ 14 \\
\hline
x^3\ \ 6x^2\ \ 7x\ \ \ 0
\end{array}
$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial.

This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$
\begin{array}{r}
x-2\overline{)\ x^3+4x^2-5x-14} \\
2x^2\ 12x\ \ 14 \\
\hline
x^3\ \ 6x^2\ \ 7x\ \ \ 0
\end{array}
$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, 6x and 7, respectively, by the -2 in $x-2$, then by -1 when we changed the subtraction to addition. Multiplying by -2 then by -1 is the same

as multiplying by 2, so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

We have constructed a **synthetic division** tableau for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3+4x^2-5x-14$ by $x-2$, we write 2 in the place of the divisor and the coefficients of $x^3+4x^2-5x-14$ in for the dividend. Then "bring down" the first coefficient of the dividend.

$$\begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ \hline \end{array} \qquad \begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was "brought down" to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|cccc} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient is a second degree polynomial. Hence the quotient is x^2+6x+7. The number in the box is the remainder. Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x-c$. It is important to note that it works only for these kinds of divisors. Also take note that when a polynomial (of degree at least 1) is divided by $x-c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division.

Example 3.4.4

Use synthetic division to divide $5x^3-2x^2+1$ by $x-3$.

Solution

When setting up the synthetic division tableau, we need to enter 0 for the coefficient of x in the dividend. Doing so gives

$$\begin{array}{r|cccc} 3 & 5 & -2 & 0 & 1 \\ & \downarrow & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \boxed{118} \end{array}$$

Since the dividend was a third degree polynomial, the quotient is a quadratic polynomial with coefficients 5, 13 and 39. Our quotient is $q(x) = 5x^2 + 13x + 39$ and the remainder is $r(x) = 118$. We can express this as

$$\frac{5x^3 - 2x^2 + 1}{x - 3} = 5x^2 + 13x + 39 + \frac{118}{x - 3}$$

We also know that $x - 3$ is not a factor of $5x^3 - 2x^2 + 1$ since the remainder is not zero.

Example 3.4.5

Divide $x^3 + 8$ by $x + 2$.

Solution

For this division, we rewrite $x + 2$ as $x - (-2)$ and proceed as before.

-2	1	0	0	8
	↓	-2	4	-8
	1	-2	4	0

The quotient is $x^2 - 2x + 4$ and the remainder is zero. Since the remainder is zero, $x + 2$ is a factor of $x^3 + 8$.

$$x^3 + 8 = (x + 2)\left(x^2 - 2x + 4\right)$$

We can write our answer as

$$\frac{x^3 + 8}{x + 2} = x^2 - 2x + 4$$

Note in the last example that there are placeholders for the missing x^2 and x terms. When using synthetic division you must put place holders for the missing terms for the division to work properly.

You Try 3.4.1

Divide $4x^4 - 8x^2 - 5x$ by $x - 3$ using synthetic division.

Answer

3	4	0	-8	-5	0
	↓	12	36	84	237
	4	12	28	79	237

$4x^4 - 8x^2 - 5x$ divided by $x - 3$ is $4x^3 + 12x^2 + 28x + 79$ with remainder 237

Using this process allows us to find the real zeros of polynomials, presuming we can figure out at least one root. We'll explore how to do that in the next section.

Example 3.4.6

The polynomial $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$ has a horizontal intercept at $x = \frac{1}{2}$ with multiplicity 2. Find the other intercepts of $p(x)$.

Solution

Since $x = \frac{1}{2}$ is an intercept with multiplicity 2, then $x - \frac{1}{2}$ is a factor twice. Use synthetic division to divide by $x - \frac{1}{2}$ twice.

1/2	4	-4	-11	12	-3
	↓	2	-1	-6	3
	4	-2	-1	-6	0
1/2	4	-2	-1	-6	
	↓	2	0	-6	
	4	0	-12	0	

From the first division, we get $4x^4-4x^3-11x^2+12x-3=\left(x-\frac{1}{2}\right)\left(4x^3-2x^2-x-6\right)$ The second division tells us

$$4x^4-4x^3-11x^2+12x-3=\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2}\right)\left(4x^2-12\right)$$

To find the remaining intercepts, we set $4x^2-12=0$ and get $x=\pm\sqrt{3}$.

Note this also means $4x^4-4x^3-11x^2+12x-3=4\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2}\right)\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)$.

Let's summarize what we've learned about zeros and polynomial functions:

Connections Between Zeros, Factors, and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n\geq 1$. The following statements are equivalent:

- The real number c is a zero of p
- $p(c)=0$
- $x=c$ is a solution to the polynomial equation $p(x)=0$
- $(x-c)$ is a factor of $p(x)$
- The point $(c,0)$ is an x-intercept of the graph of $y=p(x)$

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3.5: Zeros of Polynomials

Learning Objectives

- Find intervals that contain all real zeros.
- Use the Rational Zero Theorem to find rational zeros.
- Find zeros of a polynomial function.
- Write polynomial functions as a product of linear factors.
- Use Descartes' Rule of Signs.
- Find polynomial functions that model given criteria.

In the previous section, we looked at the function $f(x)=x^3+4x^2-5x-14$. We discussed how this function does not factor nicely using factoring techniques from intermediate algebra courses. We found from using technology that there was a "nice" x-intercept at $(2,0)$, "nice" meaning that the x-coordinate of the intercept was an integer value. We knew that this x-intercept came from a factor of $(x-2)$. We were then able to divide f by this factor to get the factored form of f, $f(x)=x^3+4x^2-5x-14=(x-2)(x^2+6x+7)$. Since the factor x^2+6x+7 was quadratic, we are able to solve $x^2+6x+7=0$ to find the other x-intercepts. If we did not have access to graphing technology or were unable to identify a "nice" x-intercept from graphing technology, how would be begin the process of finding zeros and factors of polynomials that do not factor nicely using techniques from intermediate algebra? This poses a real-life dilemma as we use polynomials when modeling many real-life situations.

In this section we will discuss some very important theory and techniques that will help us tackle this important question of finding zeros, also called roots, when polynomial functions do not factor "nicely." Let's begin with some useful theory that will help in determining intervals where we expect real zeros to be at as well as where me might start looking for possible "nice" zeros?

Bounds for Real Zeros

The theory below, which is due to the famous mathematician Augustin Cauchy, gives us an interval on which all of the real zeros of a polynomial can be found. This theory is useful since it guarantees us an interval where we can expect all our real zeros to be. Remember that real zeros correspond to x-intercepts on a graph.

Cauchy's Bound Theorem

Suppose

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$$

is a polynomial of degree n with $n\geq 1$. Let M be the largest of the numbers:

$$\frac{|a_0|}{|a_n|},\frac{|a_1|}{|a_n|},\ldots,\frac{|a_{n-1}|}{|a_n|}$$

Then all the real zeros of f lie in in the interval $[-(M+1),M+1]$.

The proof of this theorem is not easily explained within the confines of this text. Like many of the things discussed in this section, Cauchy's Bound Theorem is best understood with an example.

Example 3.5.1

Let $f(x)=2x^4+4x^3-x^2-6x-3$. Determine an interval which contains all of the real zeros of f.

Solution

To find the M stated in Cauchy's Bound, we take the absolute value of leading coefficient, in this case $|2|=2$ and divide it into the largest (in absolute value) of the remaining coefficients, in this case $|-6|=6$. We find $M=3$, so it is guaranteed that the real zeros of f all lie in $[-4,4]$.

Cauchy's Bound is useful but it can provide us a very large interval when a polynomial has large coefficients in absolute value. The next theorem can be useful in finding a smaller interval where real zeros are within.

Upper and Lower Bounds Theorem

Suppose f is a polynomial of degree $n \geq 1$.

- If $c > 0$ is synthetically divided into f and all of the coefficients of the quotient and remainder are all non-negative values, then c is an upper bound for the real zeros of f. That is, there are no real zeros greater than c.
- If $c < 0$ is synthetically divided into f and the coefficients of the quotient and remainder alternate signs, then c is a lower bound for the real zeros of f. That is, there are no real zeros less than c.

If the number 0 occurs as a coefficient of the quotient or remainder in the final line of the division table in either of the above cases, it can be treated as $(+)$ or $(-)$ as needed.

You may be wondering how to determine what value to use for c in the previous theorem. There is no systematic method for finding a c value to try, just pick a number and divide to see if it is an upper or lower bound for the real zeros.

Example 3.5.2

Given $P(x) = 2x^4 + 27x^3 - 68x^2 - 333x + 180$, use the Upper and Lower Bounds Theorem to find an interval which contains all of the real zeros of P that is smaller than the interval found using Cauchy's Bound Theorem.

Solution

From Cauchy's Bound, $|a_n| = |2| = 2$. The largest coefficient in absolute value is $|-333| = 333$. Then $M = \dfrac{333}{2} = 166.5$. All the real zeros lie in $[-167.5, 167.5]$. We can see that this is a pretty large interval. This is where finding upper and lower bounds using the Upper and Lower Bounds Theorem can be helpful as it could give us a much smaller interval.

Let's begin with finding an upper bound. Let's try $c = 10$.

10	2	27	−68	−333	180
		20	470	4020	36870
	2	47	402	3687	37050

We can see that 10 is an upper bound for the real zeros of P since all the numbers on the last row of the table are all non-negative (which are the coefficients of the quotient and remainder). This is a significant improvement from Cauchy's Bound. Note that if 10 was not an upper bound, then we would pick another c value larger than 10 to test. We'd continue this process until we find a value that is an upper bound.

Next, let's find a lower bound. Let's pick $c = -10$.

−10	2	27	−68	−333	−180
		−20	−70	1380	−1047
	2	7	−138	1047	−10290

-10 is not a lower bound since the numbers on the last row of the table do not alternate signs between + and -. This indicates we must pick a number smaller to test. Let's try $c = -20$.

−20	2	27	−68	−333	−180
		−40	260	−3840	83460
	2	−13	192	−4173	83640

Since the numbers on the last row of the table alternate signs between + and -, we can conclude that the real zeros will be in $[-20, 10]$.

Do note that this interval is not unique as there are other values of c that will work.

Next, let's look at some theory that will help in providing a list of values that could be potential zeros for a polynomial.

Rational Zeros

Recall that rational numbers are numbers that can be written in the form $\frac{a}{b}$ where a and b are integers.

The Rational Zeros Theorem

If a polynomial $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ has integer coefficients, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

Example 3.5.3

List all possible rational zeros of $f(x)=2x^4-5x^3+x^2-4$.

Solution

The only possible rational zeros of $f(x)$ are the quotients of the factors of the last term, –4, and the factors of the leading coefficient, 2.

The constant term is –4; the factors of –4 are $p=\pm1,\pm2,\pm4$.

The leading coefficient is 2; the factors of 2 are $q=\pm1,\pm2$.

If any of the four real zeros are rational zeros, then they will be of one of the following factors of –4 divided by one of the factors of 2.

$$\frac{p}{q}=\pm\frac{1}{1},\pm\frac{1}{2}\qquad\frac{p}{q}=\pm\frac{2}{1},\pm\frac{2}{2}\qquad\frac{p}{q}=\pm\frac{4}{1},\pm\frac{4}{2}$$

Note that $\frac{2}{2}=1$ and $\frac{4}{2}=2$, which have already been listed. So we can shorten our list.

$$\frac{p}{q}=\frac{\text{Factors of the constant term}}{\text{Factors of the leading coefficient}}=\pm1,\pm2,\pm4,\pm\frac{1}{2}$$

Example 3.5.4

Use the Rational Zero Theorem to find the rational zeros of $f(x)=2x^3+x^2-4x+1$.

Solution

The factors of the constant term are ±1 and the factors of the leading coefficient are ±1 and ±2. The possible values for $\frac{p}{q}$ are ±1 and $\pm\frac{1}{2}$. These are all the possible rational zeros for the function. We can determine which of these are actual zeros by substituting these values for x in $f(x)$.

$$f(-1)=2(-1)^3+(-1)^2-4(-1)+1=4$$

$$f(1)=1(1)^3+(1)^2-4(1)+1=0$$

$$f\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^3+\left(-\frac{1}{2}\right)^2-4\left(-\frac{1}{2}\right)+1=3$$

$$f\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^2-4\left(\frac{1}{2}\right)+1=-\frac{1}{2}$$

1 is the only rational zero of $f(x)$.

Example 3.5.5

Find the rational zeros of $f(x)=4x^3-3x-1$.

Solution

The factors of the constant term are ±1 and the factors of the leading coefficient are ±1, ±2, and ±4. The possible values for $\frac{p}{q}$ are $\pm 1, \pm\frac{1}{2}$, and $\pm\frac{1}{4}$. These are the possible rational zeros for the function. Let's begin with 1. $f(1) = 4(1)^3 - 3(1) - 1 = 0$ so 1 is a zero of f. Dividing by $(x-1)$ gives us the following factor for the polynomial:

$$(x-1)(4x^2+4x+1)$$

The quadratic is a perfect square. $f(x)$ can be written as

$$(x-1)(2x+1)^2$$

We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0.

$$\begin{aligned} 2x+1 &= 0 \\ x &= -\frac{1}{2} \end{aligned}$$

The zeros of the function are 1 and $-\frac{1}{2}$ with multiplicity 2.

Analysis

Look at the graph of the function f in Figure 3.5.1. Notice, at $x = -0.5$, the graph bounces off the x-axis, indicating the even multiplicity (2,4,6…) for the zero −0.5. At $x = 1$, the graph crosses the x-axis, indicating the odd multiplicity (1,3,5…) for the zero $x = 1$.

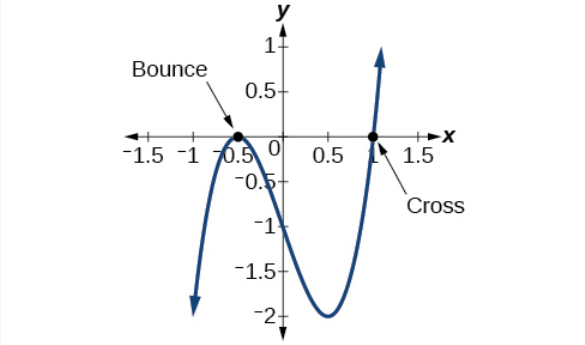


Figure 3.5.1: Graph of $f(x) = 4x^3 - 3x - 1$

? You Try 3.5.1

Use the Rational Zero Theorem to find the rational zeros of $f(x) = x^3 - 5x^2 + 2x + 1$.

Answer

There are no rational zeros for this function.

The Fundamental Theorem of Algebra

The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose f is a polynomial function of degree four, and $f(x) = 0$. The Fundamental Theorem of Algebra states that there is at least one complex solution, call it c_1. By the Factor Theorem, we can write $f(x)$ as a product of $x - c_1$ and a polynomial quotient. Since $x - c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it c_2. So we can write the polynomial quotient as a product of $x - c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

The Fundamental Theorem Of Algebra

If $f(x)$ is a polynomial of degree $n > 0$, then $f(x)$ has at least one complex zero.

Example 3.5.6

Find the zeros of $f(x) = 3x^3 + 9x^2 + x + 3$.

Solution

The Rational Zero Theorem tells us that if $\frac{p}{q}$ is a zero of $f(x)$, then p is a factor of 3 and q is a factor of 3.

$$\frac{p}{q} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$
$$= \frac{\text{factors of } 3}{\text{factors of } 3}$$

The factors of 3 are ±1 and ±3. The possible values for $\frac{p}{q}$, and therefore the possible rational zeros for the function, are ±3,±1, and $\pm\frac{1}{3}$. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0. Let's begin with –3.

Dividing by $(x+3)$ gives a remainder of 0, so –3 is a zero of the function. The polynomial can be written as

$$(x+3)(3x^2+1)$$

We can then set the quadratic equal to 0 and solve to find the other zeros of the function.

$$3x^2 + 1 = 0$$
$$x^2 = -\frac{1}{3}$$
$$x = \pm -\sqrt{\frac{1}{3}}$$
$$= \pm\frac{i\sqrt{3}}{3}$$

The zeros of $f(x)$ are -3 and $\pm\frac{i\sqrt{3}}{3}$.

Analysis

Look at the graph of the function f in Figure 3.5.2. Notice that, at $x = -3$, the graph crosses the x-axis, indicating an odd multiplicity (1) for the zero $x = -3$. Also note the presence of the two turning points. This means that, since there is a 3rd degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the x-intercepts for the function are shown. So either the multiplicity of $x = -3$ is 1 and there are two complex solutions, which is what we found, or the multiplicity at $x = -3$ is three. Either way, our result is correct.

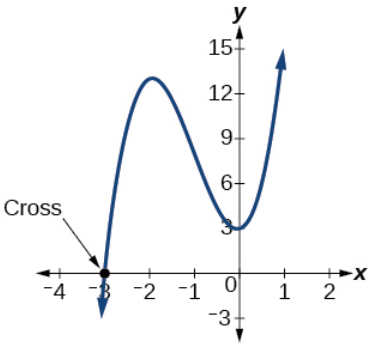


Figure 3.5.2: Graph of $f(x) = 3x^3 + 9x^2 + x + 3$

A vital implication of the Fundamental Theorem of Algebra, as we stated above, is that a polynomial function of degree n will have n zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into n factors. As a result of this, the **Linear Factorization Theorem** tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x - c)$, where c is a complex number.

Let f be a polynomial function with real coefficients, and suppose $a + bi$, $b \neq 0$, is a zero of $f(x)$. Then, by the Factor Theorem, $x - (a + bi)$ is a factor of $f(x)$. For f to have real coefficients, $x - (a - bi)$ must also be a factor of $f(x)$. This is true because any factor other than $x - (a - bi)$, when multiplied by $x - (a + bi)$, will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function f with real coefficients has a complex zero $a + bi$, then the complex conjugate $a - bi$ must also be a zero of $f(x)$. This is called the **Complex Conjugate Theorem**.

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree $n > 0$, and a is a non-zero real number, then $f(x)$ has exactly n linear factors

$$f(x) = a(x - c_1)(x - c_2)\ldots(x - c_n)$$

where $c_1, c_2, ..., c_n$ are complex numbers.

$f(x)$ has n zeros total when we allow for multiplicities.

Complex Conjugate Theorem

If the polynomial function f has real coefficients and a complex zero in the form $a + bi$, then the complex conjugate of the zero, $a - bi$, is also a zero.

Q&A

If $2 + 3i$ were given as a zero of a polynomial with real coefficients, would $2 - 3i$ also need to be a zero?

Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

Example 3.5.7

Find a fourth degree polynomial with real coefficients that has zeros of -3, 2, i, such that $f(-2) = 100$.

Solution

Because $x = i$ is a zero, by the Complex Conjugate Theorem $x = -i$ is also a zero. The polynomial must have factors of $(x + 3), (x - 2), (x - i)$, and $(x + i)$. Since we are looking for a degree 4 polynomial, and now have four zeros, we have all four factors. Let's begin by multiplying these factors.

$$\begin{aligned} f(x) &= a(x+3)(x-2)(x-i)(x+i) \\ f(x) &= a(x^2+x-6)(x^2+1) \\ f(x) &= a(x^4+x^3-5x^2+x-6) \end{aligned}$$

We need to find a to ensure $f(-2)=100$. Substitute $x=-2$ and $f(-2)=100$ into $f(x)$.

$$\begin{aligned} 100 &= a((-2)^4+(-2)^3-5(-2)^2+(-2)-6) \\ 100 &= a(-20) \\ -5 &= a \end{aligned}$$

So the polynomial function is

$$f(x) = -5(x^4+x^3-5x^2+x-6)$$

or

$$f(x) = -5x^4-5x^3+25x^2-5x+30$$

Analysis

We found that both i and $-i$ were zeros, but only one of these zeros needed to be given. If i is a zero of a polynomial with real coefficients, then $-i$ must also be a zero of the polynomial because $-i$ is the complex conjugate of i.

Example 3.5.8

Let $f(x) = 12x^5-20x^4+19x^3-6x^2-2x+1$.

1. Find all of the complex zeros of f and state their multiplicities.
2. Write $f(x)$ as a product of linear factors.

Solution

1.Since f is a fifth degree polynomial, we know that we need to perform at least three successful divisions to get the quotient down to a quadratic function. At that point, we can find the remaining zeros using the Quadratic Formula, if necessary. Using the techniques developed in Section 3.3, we get

$\frac{1}{2}$ |	12	−20	19	−6	−2	1
	↓	6	−7	6	0	−1
$\frac{1}{2}$ |	12	−14	12	0	−2	[0]
	↓	6	−4	4	2	
$-\frac{1}{3}$ |	12	−8	8	4	[0]	
	↓	−4	4	−4		
	12	−12	12	[0]		

Our quotient is $12x^2-12x+12$, whose zeros we find to be $\frac{1\pm i\sqrt{3}}{2}$. From Theorem 3.14, we know f has exactly 5 zeros, counting multiplicities, and as such we have the zero $\frac{1}{2}$ with multiplicity 2, and the zeros $-\frac{1}{3}$, $\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$, each of multiplicity 1.

2. Applying th eLinear Factorization Theorem, we are guaranteed that f factors as

$$f(x) = 12\left(x-\frac{1}{2}\right)^2\left(x+\frac{1}{3}\right)\left(x-\left[\frac{1+i\sqrt{3}}{2}\right]\right)\left(x-\left[\frac{1-i\sqrt{3}}{2}\right]\right)$$

A true test of the Linear Factorization Theorem (and a student's mettle!) would be to take the factored form of $f(x)$ in the previous example and multiply it out {You really should do this once in your life to convince yourself that all of the theory actually does work!} to see that it really is the original function

$$f(x) = 12x^5-20x^4+19x^3-6x^2-2x+1.$$

When factoring a polynomial using the Linear Factorization Theorem, we say that it is **factored completely over the complex numbers**, meaning that it is impossible to factor the polynomial any further using complex numbers. If we wanted to completely factor $f(x)$ over the real numbers then we would have stopped short of finding the nonreal zeros of f and factored f using our work from the synthetic division to write

$$f(x) = \left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right)\left(12x^2 - 12x + 12\right)$$

or

$$(f(x) = 12\left(x - \frac{1}{2}\right)^2 \left(x + \frac{1}{3}\right)\left(x^2 - x + 1\right).$$

Since the zeros of $x^2 - x + 1$ are nonreal, we call $x^2 - x + 1$ an **irreducible quadratic** meaning it is impossible to break it down any further using *real* numbers.

The last two results of the section show us that, at least in theory, if we have a polynomial function with real coefficients, we can always factor it down enough so that any nonreal zeros come from irreducible quadratics.

Descartes' Rule of Signs

The process of testing possible rational zeros and dividing can be tedious for some polynomial functions. There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.

$$f(x) = x^4 + x^3 + x^2 + x - 1$$

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.

$$f(-x) = (-x)^4 + (-x)^3 + (-x)^2 + (-x) - 1$$
$$f(-x) = +x^4 - x^3 + x^2 - x - 1$$

In this case, $f(-x)$ has 3 sign changes. This tells us that $f(x)$ could have 3 or 1 negative real zeros.

Theorem: Descartes' Rule of Signs

Suppose $f(x)$ is the formula for a polynomial function written with descending powers of x.

1. If P denotes the number of variations of sign in the formula for $f(x)$, then the number of positive real zeros (counting multiplicity) is one of the numbers $P, P-2, P-4, \ldots$.
2. If N denotes the number of variations of sign in the formula for $f(-x)$, then the number of negative real zeros (counting multiplicity) is one of the numbers \{N\), $N-2$, $N-4$, \dots\}.
3. All other zeros are complex zeros.

Example 3.5.9

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$.

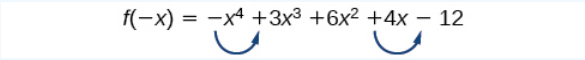
Solution

Begin by determining the number of sign changes.

There are two sign changes for $f(x)$, so there are either 2 or 0 positive real roots.

Next, we examine $f(-x)$ to determine the number of negative real roots.

$$f(-x) = -(-x)^4 - 3(-x)^3 + 6(-x)^2 - 4(-x) - 12$$
$$f(-x) = -x^4 + 3x^3 + 6x^2 + 4x - 12$$



Again, there are two sign changes for $f(-x)$, so there are either 2 or 0 negative real roots.

There are four possibilities, as we can see in Table 3.5.1.

Table 3.5.1

Positive Real Zeros	Negative Real Zeros	Complex Zeros	Total Zeros
2	2	0	4
2	0	2	4
0	2	2	4
0	0	4	4

Analysis

We can confirm the numbers of positive and negative real roots by examining a graph of the function. See Figure 3.5.3. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.

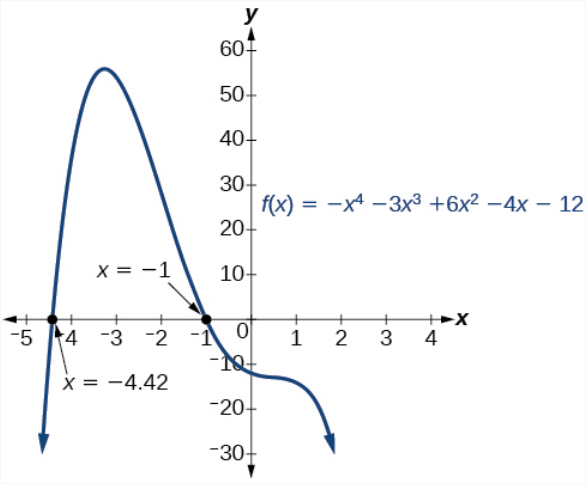


Figure 3.5.3: Graph of $f(x) = -x^4 - 3x^3 + 6x^2 - 4x - 12$ with x-intercepts at $(-4.42, 0)$ and $(-1, 0)$

? You Try 3.5.2

Use Descartes' Rule of Signs to determine the maximum possible numbers of positive and negative real zeros for $f(x) = 2x^4 - 10x^3 + 11x^2 - 15x + 12$.

Solution

There must be 4, 2, or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.

✓ Example 3.5.10

Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

1. Find all of the zeros of f and their multiplicities.
2. Sketch the graph of $y = f(x)$.

Solution

1. We know from Cauchy's Bound that all of the real zeros lie in the interval $[-4, 4]$ and that our possible rational zeros are $\pm\frac{1}{2}$, ± 1, $\pm\frac{3}{2}$ and ± 3. Descartes' Rule of Signs guarantees us at least one negative real zero and exactly one positive real zero, counting multiplicity. We try our positive rational zeros, starting with the smallest, $\frac{1}{2}$. Since the remainder isn't zero, we know $\frac{1}{2}$ isn't a zero. Sadly, the final line in the division tableau has both positive and negative numbers, so $\frac{1}{2}$ is not an upper bound. The only information we get from this division is courtesy of the Remainder Theorem which tells us $f\left(\frac{1}{2}\right) = -\frac{45}{8}$ so the point $\left(\frac{1}{2}, -\frac{45}{8}\right)$ is on the graph of f. We continue to our next possible zero, 1. As before, the only information we can glean from this is that $(1, -4)$ is on the graph of f. When we try our next possible zero, $\frac{3}{2}$, we get that it is not a zero, and we also see that it is an upper bound on the zeros of f, since all of the numbers in the final line of the division tableau are positive. This means there is no point trying our last possible rational zero, 3. Descartes' Rule of Signs guaranteed us a positive real zero, and at this point we have shown this zero is irrational. Furthermore, the Intermediate Value Theorem, Theorem 3.1, tells us the zero lies between 1 and $\frac{3}{2}$, since $f(1) < 0$ and $f\left(\frac{3}{2}\right) > 0$.

$\frac{1}{2}$	2	4	-1	-6	-3
	$\downarrow$	1	$\frac{5}{2}$	$\frac{3}{4}$	$-\frac{21}{8}$
	2	5	$\frac{3}{2}$	$-\frac{21}{4}$	$-\frac{45}{8}$

1	2	4	-1	-6	-3
	$\downarrow$	2	6	5	-1
	2	6	5	-1	-4

$\frac{3}{2}$	2	4	-1	-6	-3
	$\downarrow$	3	$\frac{21}{2}$	$\frac{57}{4}$	$\frac{99}{8}$
	2	7	$\frac{19}{2}$	$\frac{33}{4}$	$\frac{75}{8}$

2. We now turn our attention to negative real zeros. We try the largest possible zero, $-\frac{1}{2}$. Synthetic division shows us it is not a zero, nor is it a lower bound (since the numbers in the final line of the division tableau do not alternate), so we proceed to -1. This division shows -1 is a zero. Descartes' Rule of Signs told us that we may have up to three negative real zeros, counting multiplicity, so we try -1 again, and it works once more. At this point, we have taken f, a fourth degree polynomial, and performed two successful divisions. Our quotient polynomial is quadratic, so we look at it to find the remaining zeros.

$-\frac{1}{2}$	2	4	-1	-6	-3
	$\downarrow$	-1	$-\frac{3}{2}$	$\frac{5}{4}$	$\frac{19}{8}$
	2	3	$-\frac{5}{2}$	$-\frac{19}{4}$	$-\frac{5}{8}$

-1	2	4	-1	-6	-3
	$\downarrow$	-2	-2	3	3
-1	2	2	-3	-3	
	$\downarrow$	-2	0	3	
	2	0	-3	0	

Setting the quotient polynomial equal to zero yields $2x^2 - 3 = 0$, so that $x^2 = \frac{3}{2}$, or $x = \pm\frac{\sqrt{6}}{2}$. Descartes' Rule of Signs tells us that the positive real zero we found, $\frac{\sqrt{6}}{2}$, has multiplicity 1. Descartes also tells us the total multiplicity of negative real zeros is 3, which forces -1 to be a zero of multiplicity 2 and $-\frac{\sqrt{6}}{2}$ to have multiplicity 1.

We know the end behavior of $y = f(x)$ resembles that of its leading term $y = 2x^4$. This means that the graph enters the scene in Quadrant II and exits in Quadrant I. Since $\pm\frac{\sqrt{6}}{2}$ are zeros of odd multiplicity, we have that the graph crosses through the x-axis at the points $\left(-\frac{\sqrt{6}}{2}, 0\right)$ and $\left(\frac{\sqrt{6}}{2}, 0\right)$. Since -1 is a zero of multiplicity 2, the graph of $y = f(x)$ touches and rebounds off the x-axis at $(-1, 0)$. Putting this together, we get the graph of f below in Figure 3.5.4

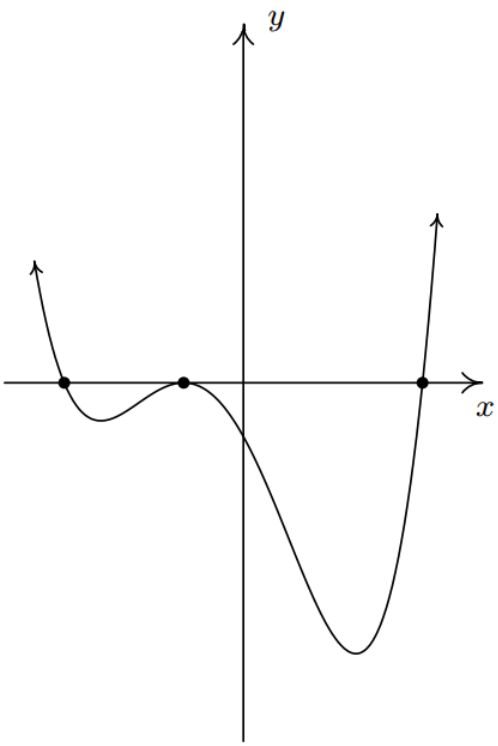


Figure 3.5.4: Graph of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$

Our last example turns the tables and asks us to manufacture a polynomial with certain properties of its graph and zeros.

Example 3.5.11

Find a polynomial p of lowest degree that has integer coefficients and satisfies all of the following criteria:

1. the graph of $y = p(x)$ touches (but does not cross) the x-axis at $\left(\frac{1}{3}, 0\right)$
2. $x = 3i$ is a zero of p.
3. as $x \to -\infty$, $p(x) \to -\infty$
4. as $x \to \infty$, $p(x) \to -\infty$

Solution

To solve this problem, we will need a good understanding of the relationship between the x-intercepts of the graph of a function and the zeros of a function, the Factor Theorem, the role of multiplicity, complex conjugates, the Complex Factorization Theorem, and end behavior of polynomial functions. (In short, you'll need most of the major concepts of this chapter.) Since the graph of p touches the x-axis at $\left(\frac{1}{3}, 0\right)$, we know $x = \frac{1}{3}$ is a zero of even multiplicity. Since we are after a polynomial of lowest degree, we need $x = \frac{1}{3}$ to have multiplicity exactly 2.

The Factor Theorem now tells us $\left(x - \frac{1}{3}\right)^2$ is a factor of $p(x)$. Since $x = 3i$ is a zero and our final answer is to have integer (real) coefficients, $x = -3i$ is also a zero. The *Factor Theorem* kicks in again to give us $(x - 3i)$ and $(x + 3i)$ as factors of $p(x)$. We are given no further information about zeros or intercepts so we conclude, by the *Complex Factorization Theorem* that $p(x) = a\left(x - \frac{1}{3}\right)^2 (x - 3i)(x + 3i)$ for some real number a. Expanding this, we get

$$p(x) = ax^4 - \frac{2a}{3}x^3 + \frac{82a}{9}x^2 - 6ax + a$$

In order to obtain integer coefficients, we know a must be an integer multiple of 9. Our last concern is end behavior. Since the leading term of $p(x)$ is ax^4, we need $a < 0$ to get $p(x) \to -\infty$ as $x \to \pm\infty$. Hence, if we choose $x = -9$, we get

$$p(x) = -9x^4 + 6x^3 - 82x^2 + 54x - 9$$

> We can verify our handiwork using the techniques developed in this chapter.

This example concludes our study of polynomial functions. The last few sections have contained what is considered by many to be 'heavy' Mathematics. Like a heavy meal, heavy Mathematics takes time to digest. Don't be overly concerned if it does not seem to sink in all at once and pace yourself as you practice. With time and dedication you've got this!

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3.6: Introduction to Rational Functions

Learning Objectives

- Find asymptotes of rational functions.
- Find holes of rational functions.
- Describe behavior of graph near asymptotes.
- Describe end behavior of rational functions.

If we add, subtract, or multiply polynomial functions, we will produce another polynomial function. If, on the other hand, we divide two polynomial functions, the result may not be a polynomial. Division of polynomials gives us another important family of functions, called **rational functions**.

Rational Function

A **rational function** is a function which is the ratio of polynomial functions of the form

$$r(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial function and $q(x) \neq 0$.

Recall that we have domain issues any time the denominator of a fraction is zero. In the example below, we review this concept as well as some of the algebra of rational expressions.

Example 3.6.1

Find the domain of the following rational functions. Write them in the form $\frac{p(x)}{q(x)}$ for polynomial functions p and q and simplify.

1. $f(x) = \dfrac{2x-1}{x+1}$
2. $g(x) = 2 - \dfrac{3}{x+1}$
3. $h(x) = \dfrac{2x^2-1}{x^2-1} - \dfrac{3x-2}{x^2-1}$
4. $r(x) = \dfrac{2x^2-1}{x^2-1} \div \dfrac{3x-2}{x^2-1}$

Solution

1. We find the zeros of the denominator and exclude them from the domain. Setting $x+1=0$ results in $x=-1$. The domain is $(-\infty,-1)\cup(-1,\infty)$. The function $f(x)$ is already in simplest form since there are no common factors among the numerator and denominator.

2. We determine the domain of g by solving $x+1=0$. The domain of g is $(-\infty,-1)\cup(-1,\infty)$. To simplify $g(x)$, we need to get a common denominator

$$\begin{aligned} g(x) &= 2 - \frac{3}{x+1} \\ &= \frac{2}{1} - \frac{3}{x+1} \\ &= \frac{(2)(x+1)}{(1)(x+1)} - \frac{3}{x+1} \\ &= \frac{(2x+2)-3}{x+1} \\ &= \frac{2x-1}{x+1} \end{aligned}$$

3. The denominators for $h(x)$ are both x^2-1 whose zeros are $x=\pm 1$. The domain of h is $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$. Since we have the same denominator in both terms, we subtract the numerators. We then factor the resulting numerator and denominator, and cancel out the common factor.

$$\begin{aligned} h(x) &= \frac{2x^2-1}{x^2-1} - \frac{3x-2}{x^2-1} \\ &= \frac{\left(2x^2-1\right)-(3x-2)}{x^2-1} \\ &= \frac{2x^2-1-3x+2}{x^2-1} \\ &= \frac{2x^2-3x+1}{x^2-1} \\ &= \frac{(2x-1)(x-1)}{(x+1)(x-1)} \\ &= \frac{2x-1}{x+1} \end{aligned}$$

4. To find the domain of r, it may help to temporarily rewrite $r(x)$ as

$$r(x) = \frac{\dfrac{2x^2-1}{x^2-1}}{\dfrac{3x-2}{x^2-1}}$$

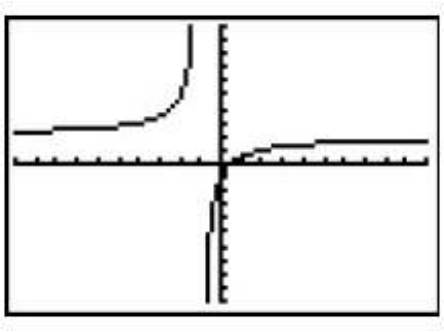
We need to set all of the denominators equal to zero which means we need to solve not only $x^2-1=0$, but also $\dfrac{3x-2}{x^2-1}=0$. We find $x=\pm 1$ for the former and $x=\dfrac{2}{3}$ for the latter. The domain is $(-\infty,-1)\cup\left(-1,\dfrac{2}{3}\right)\cup\left(\dfrac{2}{3},1\right)\cup(1,\infty)$. We simplify $r(x)$ by rewriting the division as multiplication by the reciprocal and then by canceling the common factor

$$\begin{aligned} r(x) &= \frac{2x^2-1}{x^2-1} \div \frac{3x-2}{x^2-1} \\ &= \frac{2x^2-1}{x^2-1} \cdot \frac{x^2-1}{3x-2} \\ &= \frac{\left(2x^2-1\right)\left(x^2-1\right)}{\left(x^2-1\right)(3x-2)} \\ &= \frac{2x^2-1}{3x-2} \end{aligned}$$

A few remarks about the previous example are in order. Note that the the functions $f(x)$, $g(x)$ and $h(x)$ all simplify to the same thing. However, only two of these functions are actually equal. For two functions to be equal, they need, among other things, to have the same domain. Since $f(x) = g(x)$ when simplified and f and g have the same domain, they are equal functions. Even though $h(x)$ simplifies to be the same as f and g, the domain of h is different than the domain of f and g, and thus h is a different function.

Vertical Asymptotes

We now turn our attention to the graphs of rational functions. Let's first focus on how these functions behave near values that are not in the domain. Consider the function $f(x) = \dfrac{2x-1}{x+1}$ from the previous example. Using a graphing calculator, we obtain the following graph:



(*note that older graphing calculator technologies will show a slightly different graph due to graphing limitations)*

Note that the graph appears to 'break' at $x = -1$. We know from our last example that $x = -1$ is not in the domain of f which means $f(-1)$ is undefined. When we make a table of values to study the behavior of f *near* $x = -1$ we see that we can get 'near' $x = -1$ from two directions. We can choose values a little less than -1, for example $x = -1.1$, $x = -1.01$, $x = -1.001$, and so on. These values are said to 'approach -1 from the *left*.' Similarly, the values $x = -0.9$, $x = -0.99$, $x = -0.999$, etc., are said to 'approach -1 from the *right*.' If we make two tables, we find that the numerical results confirm what we see graphically.

x	f(x)	(x, f(x))
-1.1	32	(-1.1,32)
-1.01	302	(-1.01,302)
-1.001	3002	(-1.001,3002)
-1.0001	30002	(-1.001,30002)

x	f(x)	(x, f(x))
-0.9	-28	(-0.9,-28)
-0.99	-298	(-0.99,-298)
-0.999	-2998	(-0.999,-2998)
-0.9999	-29998	(-0.9999,-29998)

As the x values approach -1 from the left, the function values become larger and larger. We express this symbolically as $x \to -1^-$, $f(x) \to \infty$. Similarly, we conclude from the table that as $x \to -1^+$, $f(x) \to -\infty$. For this type of unbounded behavior, we say the graph of $y = f(x)$ has a **vertical asymptote** of $x = -1$. Roughly speaking, this means that near $x = -1$, the graph looks very much like the vertical line $x = -1$.

Vertical Asymptote

The line $x = c$, where c is a real number, is called a **vertical asymptote** of the graph of a function $y = f(x)$ if as $x \to c^-$ or as $x \to c^+$, either $f(x) \to \infty$ or $f(x) \to -\infty$.

Let's revisit $h(x) = \dfrac{2x^2-1}{x^2-1} - \dfrac{3x-2}{x^2-1}$. We discussed that although $h(x)$ simplified to $\dfrac{2x-1}{x+1}$, the functions f and h are different since $x = 1$ is in the domain of f but $x = 1$ is not in the domain of h. If we graph $h(x)$ using a graphing calculator, we will get a graph that looks identical to $y = f(x)$. There is a vertical asymptote at $x = -1$, but near $x = 1$, everything seem fine. Tables of values provide numerical evidence which supports the graphical observation.

x	h(x)	(x, h(x))
0.9	≈ 0.4210	≈(0.9,0.4210)
0.99	≈ 0.4925	≈(0.99,0.4925)
0.999	≈ 0.4992	≈(0.999,0.4992)
0.9999	≈ 0.4999	≈(0.9999,0.4999)

x	h(x)	(x, h(x))
1.1	≈ 0.5714	≈(1.1,0.5714)
1.01	≈ 0.5075	≈(1.01,0.5075)
1.001	≈ 0.5007	≈(1.001,0.5007)
1.0001	≈ 0.5001	≈(1.0001,0.5001)

We see that as $x \to 1^-$, $h(x) \to 0.5^-$ and as $x \to 1^+$, $h(x) \to 0.5^+$. In other words, the points on the graph of $y = h(x)$ are approaching $(1, 0.5)$, which we can summarize as $x \to 1$, $h(x) \to 0.5$. Since $x = 1$ is not in the domain of h, we put an open circle, also called a **hole** or **removable discontinuity**, at $(1, 0.5)$. Below is a graph of $h(x)$ with the hole drawn in. Note that most graphing technologies do not sketch holes.

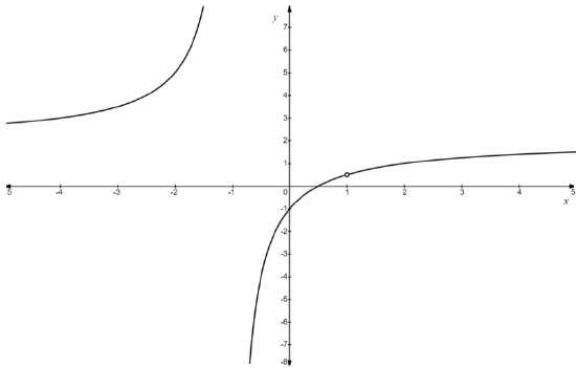


Figure 3.6.1: Copy and Paste Caption here. (Copyright; author via source)

We have seen that values excluded from the domain will yield a vertical asymptote or a hole for the graph. How do we know which graphical behavior will occur at a value not in the domain? Notice that when we simplified h, we had $h(x) = \dfrac{(2x-1)(x-1)}{(x+1)(x-1)}$. The factor $(x-1)$ cancels whereas $(x+1)$ does not. Loosely speaking, the trouble caused by $(x-1)$ in the denominator is canceled away while the factor $(x+1)$ remains to cause mischief. This is why the graph of $y = h(x)$ has a vertical asymptote at $x = -1$ but only a hole at $x = 1$. These observations are generalized and summarized in the theorem below, whose proof is found in Calculus.

How to Find Vertical Asymptotes and Holes

Suppose r is a rational function which can be written as $r(x) = \dfrac{p(x)}{q(x)}$ where p and q have no common zeros {in other words, $r(x)$ is in lowest terms.}. Let c be a real number which is not in the domain of r.

- If $q(c) \neq 0$, then the graph of $y = r(x)$ has a hole at $\left(c, \dfrac{p(c)}{q(c)}\right)$.
- If $q(c) = 0$, then the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$.

This theorem says that if $x = c$ is not in the domain of r but, when we simplify $r(x)$, it no longer makes the denominator 0, then we have a hole at $x = c$. Otherwise, the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$.

Example 3.6.2

Find the vertical asymptotes and/or holes in the graphs of the following rational functions. Describe the behavior of the graph near them using proper notation.

1. $f(x) = \dfrac{2x}{x^2 - 3}$

2. $g(x) = \dfrac{x^2 - x - 6}{x^2 - 9}$
3. $h(x) = \dfrac{x^2 - x - 6}{x^2 + 9}$
4. $r(x) = \dfrac{x^2 - x - 6}{x^2 + 4x + 4}$

Solution

1. To find the domain, we solve $x^2 - 3 = 0$ and get $x = \pm\sqrt{3}$. Since the expression $f(x)$ is in lowest terms, there is no cancellation possible, and we conclude that the lines $x = -\sqrt{3}$ and $x = \sqrt{3}$ are vertical asymptotes to the graph of $y = f(x)$. As $x \to -\sqrt{3}^{\,-}$, $f(x) \to -\infty$, as $x \to -\sqrt{3}^{\,+}$, $f(x) \to \infty$, as $x \to \sqrt{3}^{\,-}$, $f(x) \to -\infty$, and finally as $x \to \sqrt{3}^{\,+}$, $f(x) \to \infty$.
2. Solving $x^2 - 9 = 0$ gives $x = \pm 3$. In lowest terms $g(x) = \dfrac{x^2 - x - 6}{x^2 - 9} = \dfrac{(x-3)(x+2)}{(x-3)(x+3)} = \dfrac{x+2}{x+3}$. Since $x + 3$ continues to make trouble in the denominator, the line $x = -3$ is a vertical asymptote of the graph of $y = g(x)$. Since $x - 3$ cancelled out, we have a hole at $x = 3$. To find the y-coordinate of the hole, we substitute $x = 3$ into the simplified expression, $\dfrac{x+2}{x+3}$, and find the hole is at $\left(3, \dfrac{5}{6}\right)$. As $x \to -3^-$, $g(x) \to \infty$, as $x \to -3^+$, $g(x) \to -\infty$, as $x \to 3^-$, $g(x) \to \dfrac{5}{6}^{\,-}$, and as $x \to 3^+$, $g(x) \to \dfrac{5}{6}^{\,+}$.
3. The domain of h is all real numbers, since $x^2 + 9 = 0$ has no real solutions. Hence, the graph of $y = h(x)$ has no vertical asymptotes or holes.
4. Setting $x^2 + 4x + 4 = 0$ gives us $x = -2$. Simplifying, we see

$$r(x) = \frac{x^2 - x - 6}{x^2 + 4x + 4} = \frac{(x-3)(x+2)}{(x+2)^2} = \frac{x-3}{x+2}$$

Since $x = -2$ continues to produce a 0 in the denominator of the reduced function, we know $x = -2$ is a vertical asymptote to the graph. As $x \to -2^-$, $r(x) \to \infty$ and as $x \to -2^+$, $r(x) \to -\infty$.

Our next example gives us a physical interpretation of a vertical asymptote. This type of model arises from a family of equations cheerily named **'doomsday' equations**. These functions arise in Differential Equations. The unfortunate name will make sense shortly.

Example 3.6.3

A mathematical model for the population P, in thousands, of a certain species of bacteria, t days after it is introduced to an environment is given by $P(t) = \dfrac{100}{(5-t)^2}$, $0 \le t < 5$.

1. Find and interpret $P(0)$.
2. When will the population reach 100,000?
3. Determine the behavior of P as $t \to 5^-$. Interpret this result graphically and within the context of the problem.

Solution

1. Substituting $t = 0$ gives $P(0) = \dfrac{100}{(5-0)^2} = 4$, which means 4000 bacteria are initially introduced into the environment.
2. To find when the population reaches 100,000, we first need to remember that $P(t)$ is measured in *thousands*. In other words, 100,000 bacteria corresponds to $P(t) = 100$. Substituting for $P(t)$ gives the equation $\dfrac{100}{(5-t)^2} = 100$. Clearing denominators and dividing by 100 gives $(5-t)^2 = 1$, which, after extracting square roots, produces $t = 4$ or $t = 6$. Of these two solutions, only $t = 4$ in in our domain, so this is the solution we keep. Hence, it takes 4 days for the population of bacteria to reach 100,000.
3. To determine the behavior of P as $t \to 5^-$, we can make a table

t	P(t)
4.9	10000
4.99	1000000
4.999	100000000
4.9999	10000000000

In other words, as $t \to 5^-$, $P(t) \to \infty$. Graphically, the line $t = 5$ is a vertical asymptote of the graph of $y = P(t)$. Physically, this means that the population of bacteria is increasing without bound as we near 5 days, which cannot actually happen. For this reason, $t = 5$ is called the 'doomsday' for this population. There is no way any environment can support infinitely many bacteria, so shortly before $t = 5$ the environment would collapse.

Horizontal Asymptotes

Now that we have thoroughly investigated behavior or rational functions around values not in the domain, let's turn our attention to end behavior of these functions. Recall the graph of the function $y = f(x) = \dfrac{2x-1}{x+1}$ that we were discussing earlier. Notice that the graph of f seems to 'level off' on the left and right hand sides of the screen. This is a statement about the end behavior of the function. Looking at a tables of values to investigate what happens as x gets larger and larger and also more and more negative without bound, we find

-10	≈ 2.3333	≈(-10,2.3333)
-100	≈ 2.0303	≈(-100,2.0303)
-1000	≈ 2.0030	≈(-1000,2.0030)
-10000	≈2.0003	≈(-10000,2.0003)
-10	≈ 2.3333	≈(-10,2.3333)

x	f(x)	(x, f(x))
10	≈ 1.7273	≈(10,1.7273)
100	≈ 1.9703	≈(100,1.9703)
-1000	≈ 1.9970	≈(-10000,2.0003)
-10	≈ 2.3333	≈(-10,2.3333)

From the tables, we see that as $x \to -\infty$, $f(x) \to 2$ and as $x \to \infty$, $f(x) \to 2$. In this case, we say the graph of $y = f(x)$ has a **horizontal asymptote** of $y = 2$. This means that the end behavior of f resembles the horizontal line $y = 2$, which explains the 'leveling off' behavior we see in the calculator's graph.

Horizontal Asymptote

The line $y = c$, where c is a real number, is called a horizontal asymptote of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to c$.

Note that in in this definition, we write $f(x) \to c$ (not $f(x) \to c^+$ or $f(x) \to c^-$) because we are unconcerned from which direction the values $f(x)$ approach the value c, just as long as they do so.

How to Find Horizontal Asymptotes

Suppose r is a rational function and $r(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomial functions with leading coefficients a and b, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y = \frac{a}{b}$ is the horizontal asymptote of the graph of $r(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the horizontal asymptote of the graph of $r(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $r(x)$ has no horizontal asymptotes.

You may be wondering why the existence of horizontal asymptotes for rational functions is dependent upon the degree of the numerator and denominator. Remember that rational functions are division of polynomials. Let's revisit division for a moment to discover why. Using our example $f(x) = \frac{2x-1}{x+1}$ and division, we can write f as $\frac{2x-1}{x+1} = 2 - \frac{3}{x+1}$. As x becomes unbounded in either direction, the quantity $\frac{3}{x+1}$ gets closer and closer to 0 so that the values of $f(x)$ become closer and closer to 2 (you can verify this using tables). In symbols, as $x \to \pm\infty$, $f(x) \to 2$. By definition, this indicates f has a horizontal asymptote at $y = 2$.

Alternatively, we can use what we know about end behavior of polynomials to help us understand this theorem. We know the end behavior of a polynomial is determined by its leading term. Applying this to the numerator and denominator of $f(x)$, we get that as $x \to \pm\infty$, $f(x) = \frac{2x-1}{x+1} \approx \frac{2x}{x} = 2$.

Applying this reasoning to the general case, suppose $r(x) = \frac{p(x)}{q(x)}$ where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$. As $x \to \pm\infty$, $r(x) \approx \frac{ax^n}{bx^m}$, where n and m are the degrees of $p(x)$ and $q(x)$, respectively. If the degree of $p(x)$ and the degree of $q(x)$ are the same, then $n = m$ so that $r(x) \approx \frac{a}{b}$, which means $y = \frac{a}{b}$ is the horizontal asymptote in this case. If the degree of $p(x)$ is less than the degree of $q(x)$, then $n < m$, so $m - n$ is a positive number, and hence, $r(x) \approx \frac{a}{bx^{m-n}} \to 0$ as $x \to \pm\infty$. If the degree of $p(x)$ is greater than the degree of $q(x)$, then $n > m$, and hence $n - m$ is a positive number and $r(x) \approx \frac{ax^{n-m}}{b}$, which becomes unbounded as $x \to \pm\infty$.

Example 3.6.4

List the horizontal asymptotes, if any, of the graphs of the following functions. Describe the behavior of the graph near them using proper notation.

1. $f(x) = \frac{5x}{x^2+1}$
2. $g(x) = \frac{x^2-4}{x+1}$
3. $h(x) = \frac{6x^3-3x+1}{5-2x^3}$

Solution

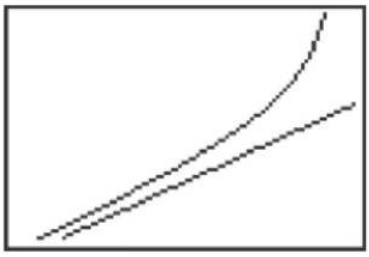
1. The numerator of $f(x)$ is $5x$, which has degree 1. The denominator of $f(x)$ is x^2+1, which has degree 2. Since the degree of the numerator is less than the denominator, $y = 0$ is the horizontal asymptote. As $x \to -\infty$, $f(x) \to 0^-$ and as $x \to \infty$, $f(x) \to 0^+$.
2. The numerator of $g(x)$, x^2-4, has degree 2, but the degree of the denominator, $x+1$, has degree 1. Since the degree of the numerator is larger thant he denominator, here is no horizontal asymptote. {Sit tight! We'll revisit this function and its end behavior shortly.}
3. The degrees of the numerator and denominator of $h(x)$ are both three, so $y = \frac{6}{-2} = -3$ is the horizontal asymptote. As $x \to -\infty$, $h(x) \to -3^+$, and as $x \to \infty$, $h(x) \to -3^-$.

Slant Asymptotes

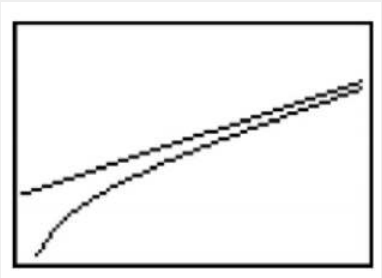
In the previous example, $g(x)=\dfrac{x^2-4}{x+1}$, did not have a horizontal asymptote. What is the end behavior of this function then? Performing long division, we get $g(x)=\dfrac{x^2-4}{x+1}=x-1-\dfrac{3}{x+1}$. Since the term $\dfrac{3}{x+1}\to 0$ as $x\to\pm\infty$, it stands to reason that as x becomes unbounded, the function values $g(x)=x-1-\dfrac{3}{x+1}\approx x-1$. Geometrically, this means that the graph of $y=g(x)$ should resemble the line $y=x-1$ as $x\to\pm\infty$. We see this play out both numerically and graphically below.

x	g(x)	x-1
-10	≈-10.6667	-11
-100	≈-100.9697	-101
-1000	≈-1000.9970	-1001
-10000	≈-10000.9997	-10001

x	g(x)	x-1
10	≈ 8.7273	9
100	≈ 98.9703	99
1000	≈ 998.9970	999
10000	≈ 9998.9997	9999



$y=g(x)$ and $y=x-1$ as $x\to-\infty$



$y=g(x)$ and $y=x-1$ as $x\to\infty$

The way we symbolize the relationship between the end behavior of $y=g(x)$ with that of the line $y=x-1$ is to write 'as $x\to\pm\infty$, $g(x)\to x-1$.' In this case, we say the line $y=x-1$ is a **slant asymptote** of $y=g(x)$. Informally, the graph of a rational function has a slant asymptote if, as $x\to\infty$ or as $x\to-\infty$, the graph resembles a non-horizontal, or 'slanted' line. Formally, we define a slant asymptote as follows.

Slant Asymptote

The line $y=mx+b$ where $m\neq 0$ is called a **slant asymptote** of the graph of a function $y=f(x)$ if as $x\to-\infty$ or as $x\to\infty$, $f(x)\to mx+b$.

Note that the stipulation $m\neq 0$ in this definition is what makes the 'slant' asymptote 'slanted' as opposed to the case when $m=0$ in which case we'd have a horizontal asymptote.

Also, since $y=x-1$ has the end behavior that $y\to-\infty$ as $x\to-\infty$ and $y\to\infty$ as $x\to\infty$ we can describe the end of behavior of g(x) as follows:

$g(x)\to-\infty$ as $x\to-\infty$ and $g(x)\to\infty$ as $x\to\infty$

Some mathematicians prefer to describe the end behavior with the actual asymptote and some prefer to describe the end behavior directly like was just done above.

Our next task is to determine the conditions under which the graph of a rational function has a slant asymptote, and if it does, how to find it. In the case of $g(x)=\dfrac{x^2-4}{x+1}$, the degree of the numerator x^2-4 is 2, which is *exactly one more* than the degree if its denominator $x+1$ which is 1. This results in a *linear* quotient polynomial, and it is this quotient polynomial which is the slant asymptote.

How to Find a Slant Asymptote

Suppose r is a rational function and $r(x) = \dfrac{p(x)}{q(x)}$, where the degree of p is **exactly** one more than the degree of q. Then the graph of $y = r(x)$ has the **slant asymptote** $y = L(x)$ where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$.

Example 3.6.5

Find the slant asymptotes of the graphs of the following functions if they exist. Describe the end behavior of the graph near them using proper notation.

1. $f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$
2. $g(x) = \dfrac{x^2 - 4}{x - 2}$
3. $h(x) = \dfrac{x^3 + 1}{x^2 - 4}$

Solution

1. The degree of the numerator is 2 and the degree of the denominator is 1, so f has a slant asymptote. To find it, we divide $1 - x = -x + 1$ into $x^2 - 4x + 2$ and get a quotient of $-x + 3$, so our slant asymptote is $y = -x + 3$. As $x \to -\infty$ or as as $x \to \infty$, $f(x) \to (-x + 3)$.
2. The degree of the numerator $g(x) = \dfrac{x^2 - 4}{x - 2}$ is 2 and the degree of the denominator is 1. However,
$$g(x) = \frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{(x - 2)} = x + 2, x \neq 2$$
so we have that the slant asymptote $y = x + 2$ is identical to the graph of $y = g(x)$ except at $x = 2$ (where the latter has a 'hole' at $(2, 4)$.) As $x \to -\infty$ and as $x \to \infty$, $g(x) \to (x + 2)$.
3. For $h(x) = \dfrac{x^3 + 1}{x^2 - 4}$, the degree of the numerator is 3 and the degree of the denominator is 2 so again, we are guaranteed the existence of a slant asymptote. The long division $\left(x^3 + 1\right) \div \left(x^2 - 4\right)$ gives a quotient of just x, so our slant asymptote is the line $y = x$. As $x \to -\infty$ and as $x \to \infty$, $g(x) \to x$

Oblique Asymptotes

We have discussed the end behavior and asymptotes of rational functions where the degree of the numerator is less than, equal, or one degree higher than the degree of the denominator. You may be wondering about the end behavior of rational functions where the degree of the numerator is a degree of at least two or larger than the degree of the denominator. We know from polynomial division that the quotient will be a polynomial. Since rational functions are division of polynomials that means that the quotient will describe the end behavior of the rational function in a similar way as slant asymptotes do.

The function $f(x) = \dfrac{x^4 + x^3 - 2x^2 + 3}{x^2 + x - 2}$ can be rewritten as $f(x) = x^2 + \dfrac{3}{x^2 + x - 2}$. As $x \to \pm\infty$, $f(x) \to x^2$. So, the end behavior for f is as $x \to \pm\infty$, $f(x) \to \infty$. We call $y = x^2$ an **oblique asymptote**. When graphing f and the oblique asymptote together, visually can see that, in fact, $f(x) \to x^2$ as $x \to \pm\infty$.

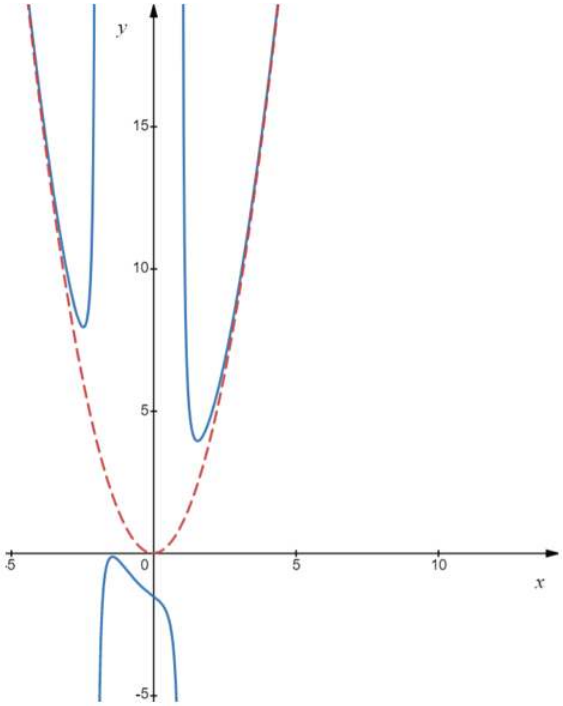


Figure 3.6.2: Graph of rational function with quadratic oblique asymptote

Some mathematicians will say that horizontal and/or slant asymptotes are types of oblique asymptotes. Others will argue that oblique asymptotes are strictly polynomials of degree two or higher. We are not in a position to argue that one description is better than another. This main thing is to be aware that the end behavior of rational functions is polynomial in nature. For this text, we will summarize the end behavior into three categories of asymptotes as follows:

For a rational function, $r(x)$, as $x \to \pm\infty$

1. If $r(x) \to c$ where c is a real number, then $y = c$ is a horizontal asymptote for $r(x)$.
2. If $r(x) \to mx + b$ where m and b are real numbers and $m \neq 0$, then $y = mx + b$ is a slant asymptote for $r(x)$.
3. If $r(x) \to P(x)$ where $P(x)$ is a polynomial of degree 2 or higher, then $y = P(x)$ is a slant asymptote for $r(x)$.

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3.7: Graphing Rational Functions

Learning Objectives

- Identify important behavior of rational functions and use this information to sketch their graphs.
- Find equations of rational functions that model graphical information.

In this section we are going look at sketching graphs of rational functions. Although technology can graph functions for us, it is important that we understand the details of how these functions behave. Window settings for viewing a graph in a graphing calculator can significantly affect what graphical behavior we see. What if you owned a company that produced and sold cookies. Your company would have some costs to produce cookies. In business, it is known that average cost models are rational functions. As a business owner, you'd want to reduce your costs as much as possible. This means you'd want to find the smallest of the local minimums for the functions. If you didn't know how the function model for your costs behaved, how would you be certain you had reduced your costs as much as possible? This is just one of many examples of why it is important to understand how functions behave.

Graphing Rational Functions

Let's take a moment to review what we know about rational function behavior. Rational functions have vertical asymptotes or holes for values not in the domain of the function. We know the function values increase or decrease without bound as we near vertical asymptotes. Rational functions have horizontal, slant, or oblique asymptotes which are polynomials that we can use to describe their end behavior. On the ends of the graph, the function will approach one of these asymptotes. To figure out how the function behaves elsewhere, we can use points such as intercepts and also sign charts.

Strategy for Graphing Rational Functions

Let $r(x)$ be a rational function.

1. Write r as a single fraction in factored form.
2. Determine the domain of r.
3. Find the vertical asymptotes of r, if any.
4. Find any holes in the graph, if they exist.
5. Find the horizontal, slant, or oblique asymptote for r.
6. Find all intercepts for r, if any.
7. Find all points where the function intersects the horizontal or slant or oblique asymptote, if any.
8. Use this information to sketch the graph of $r(x)$. If needed, construct a sign chart and find other points to help with sketching the graph.

A note is in order about checking to see if $r(x)$ intersects its horizontal or slant or oblique asymptote. Unlike vertical asymptotes that occur at values not in the domain of $r(x)$, these asymptotes describe end behavior of the function only. This means that it is possible that $r(x)$ can have the same function value as the horizontal or slant or oblique asymptote somewhere in between the ends. That is why we must check this.

Example 3.7.1

For the given function, $r(x) = \dfrac{x^2+2x-3}{x^2+2x-8}$,

1. Find the domain and state answer in interval notation.
2. Identify all the asymptotes, if any.
3. Identify any holes in the graph of r, if any.
4. Describe the end behavior of r using proper notation.
5. Find the x and y-intercepts.
6. Set up a sign chart and use it to describe where $r(x) > 0$ and $r(x) < 0$.
7. Use the information your found above to sketch a graph of r.

8. Find the range of $r(x)$ and state answer in interval notation. Use graphing technology and round to the first decimal place, if applicable.

Solution

1. To find domain, we first must factor:

$$r(x) = \frac{x^2+2x-3}{x^2+2x-8} = \frac{(x+3)(x-1)}{(x-2)(x+4)}$$

From here we can see that $x = 2$ and $x = -4$ will make r undefined. Hence, the domain is $(-\infty, -4) \cup (-4, 2) \cup (2, \infty)$.

2. Since there are no common factors between the numerator and denominator we can conclude that there are two vertical asymptotes at $x = 2$ and $x = -4$. The degree of the numerator and denominator are the same, degree 2, so there is a horizontal asymptote at $y = \frac{1}{1} = 1$.

Remember that we do need to check if the function intersects the horizontal asymptote so let's check this. To do so, we need to find out if there are any points for $r(x)$ that have a y-coordinate equal to 1. To find out, set $r(x) = 1$

$$\frac{x^2+2x-3}{x^2+2x-8} = 1$$

Multiplying both sides by the common denominator,

$$(x^2+2x-8)\left(\frac{x^2+2x-3}{x^2+2x-8}\right) = (x^2+2x-8)(1)$$

which simplifies to

$$x^2+2x-3 = x^2+2x-8$$

Subtracting x^2 and then $2x$ from both sides yields

$$-3 = -8$$

which is a false statement. We can conclude that r does not intersect it's horizontal asymptote.

3. Since there are no common factors between the numerator and denominator, we can conclude there are no holes in the graph of $r(x)$.
4. As $x \to \pm\infty$, $r(x) \to 1$.
5. To find the x-intercepts set $y = 0$,

$$\frac{x^2+2x-3}{x^2+2x-8} = 0$$

Multiplying both sides by the common denominator,

$$(x^2+2x-8)(\frac{x^2+2x-3}{x^2+2x-8}) = (x^2+2x-8)(0)$$

which simplifies to

$$x^2+2x-3 = 0$$

Notice that this is just the numerator or $r(x)$. Now factoring,

$$(x+3)(x-1) = 0$$

which gives us $x = -3$ and $x = 1$. The x-intercepts are $(-3, 0)$ and $(1, 0)$.

To find the y-intercept, evaluate the function at $x = 0$

$$r(0) = \frac{(0)^2+2(0)-3}{(0)^2+2(0)-8} = \frac{-3}{-8} = \frac{3}{8}$$

The y-intercept is $\left(0, \frac{3}{8}\right)$.

Note that for the x-intercepts we can just look at the factored numerator of $r(x)$ without having to actually go through the steps of solving $r(x) = 0$. Keep this in mind as this can save you some time! Also notice that for the y-intercept, the y-coordinate is just the constant terms of the numerator and denominator of $r(x)$ in unfactored form.

6. The critical values for $r(x)$ are -4, -3, 1, and 2. Setting up a sign chart:

		- 4	- 3	1	2
	(-∞,-4)	(-4,-3)	(-3,1)	(1,2)	(2,∞)
(x+3)	-	-	+	+	+
(x-1)	-	-	-	+	+
(x-2)	-	-	-	-	+
(x+4)	-	+	+	+	+
r(x)	+	-	+	-	+

Figure 3.7.1: Sign Chart for r(x)

$r(x) > 0$ on the intervals $(-\infty, -4)$, $(-3, 1)$, and $(2, \infty)$

$r(x) < 0$ on the intervals $(-4, -3)$ and $(1, 2)$

7. We begin by sketching dashed lines for the vertical asymptotes of $r(x)$. We use dashed lines since asymptotes describe behavior of the function but they are not actually part of the graph of the function. We know that $r(x)$ will increase or decrease without bound as the graph nears the vertical asymptotes so we can indicate this behavior on our graph, as shown in Figure 3.7.2.

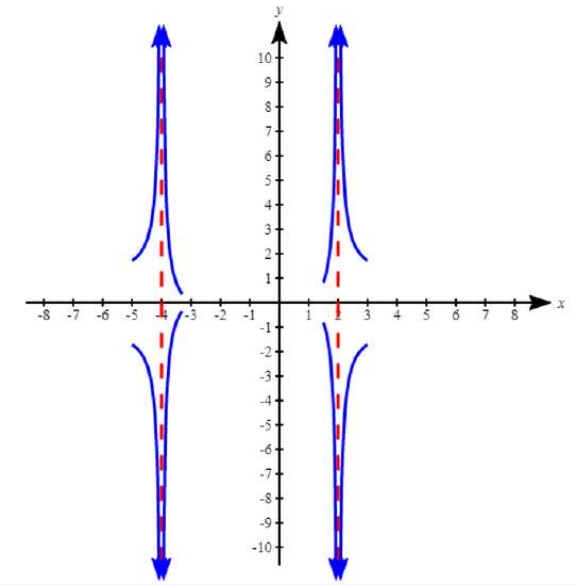


Figure 3.7.2: Vertical Asymptotes and Behavior of r(x) near them

Next, we sketch a dashed line for the horizontal asymptote. We know that as $x \to \pm\infty$, $r(x) \to 1$, so we can indicate this behavior on our graph, as shown in Figure 3.7.3.

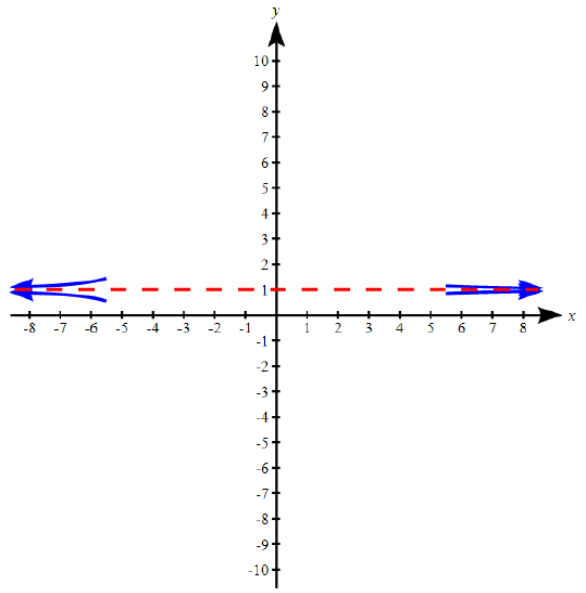


Figure 3.7.3: Horizontal asymptotes of r(x) and end behavior

Now add the intercepts to the graph, as shown in Figure 3.7.4

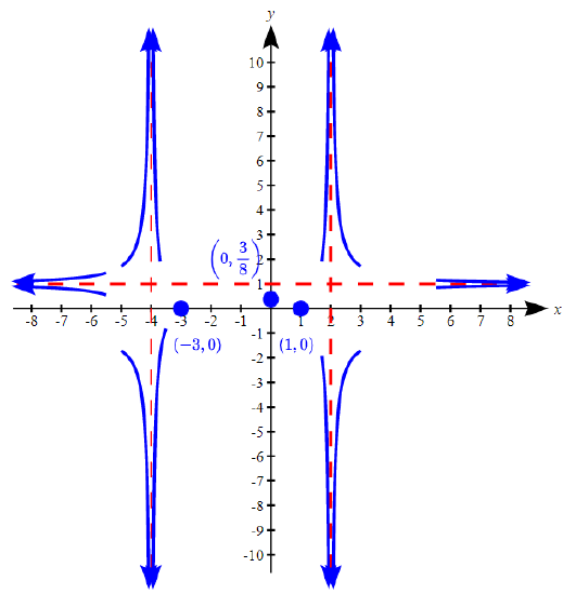


Figure 3.7.4: Asymptotic behavior and points of r(x)

Since the domain is $(-\infty, -4) \cup (-4, 2) \cup (2, \infty)$ there will be three parts to the graph, one part on each interval. On the interval $(-\infty, -4)$, we know that there are no x-intercepts and that the graph does not intersect the horizontal asymptote. This tells us that the graph must be above the horizontal asymptote. We can also see from the sign chart that $r(x) > 0$ on this interval which indicates the graph can not go below the x-axis. On the interval $(-4, 2)$, we know the graph does not intersect the horizontal asymptote. This tells us that the graph must come from below the x-axis to be able to pass through the intercepts. On the interval $(2, \infty)$, we know that there are no x-intercepts and that the graph does not intersect the horizontal asymptote. This means that the graph must be above the horizontal asymptote here. We can also see from the sign chart that $r(x) > 0$ on this interval which indicates the graph can not go below the x-axis. The graph of $r(x)$ is shown in Figure 3.7.5

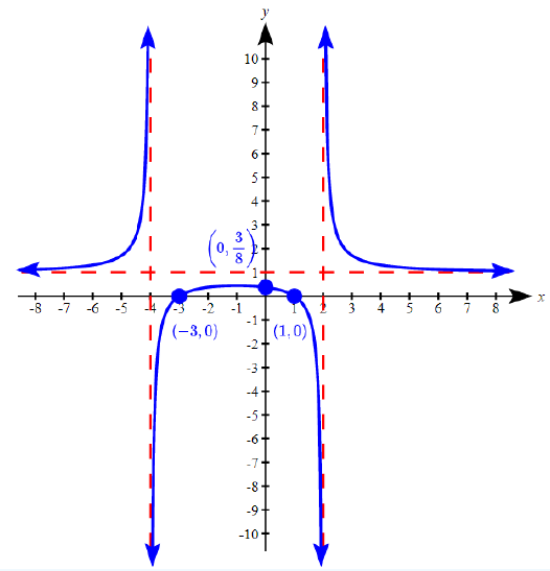


Figure 3.7.5: Graph of r(x)

8. We can see that there is a local maximum within the interval $(-4, 2)$ which we will need to approximate using a graphing calculator. The maximum occurs at approximately $(-1, 0.4)$. Rounding to the nearest first decimal place, the range is $(-\infty, 0.4] \cup (1, \infty)$.

In the previous example, we learned that we can find the x-intercepts quickly by looking at factors of the numerator. Do keep in mind that you must simplify the function first before determining the x-intercepts since there could be common factors between the numerator and denominator which would indicate that a hole exists.

Example 3.7.2

Find the asymptotes, intercepts, domain, range, and end behavior of $f(x) = \dfrac{3x^2 + 14x - 5}{x + 3}$. Round to the first decimal place for the range, if applicable. Then, use this information to sketch $f(x)$.

Solution

Let's begin with factoring to determine the domain. $f(x) = \dfrac{3x^2 + 14x - 5}{x + 3} = \dfrac{(x + 5)(3x - 1)}{x + 3}$.

From the factored form of $f(x)$, we find that the domain is $(-\infty, -3) \cup (-3, \infty)$. We can also tell from the factored that there is no hole and that there is a vertical asymptote at $x = -3$.

Since there are no common factors between the numerator and denominator, we can set the numerator factors equal to 0 to find the x-intercepts, $(-5, 0)$ and $(\frac{1}{3}, 0)$.

Evaluating at $x = 0$, $f(0) = \dfrac{-5}{3}$ so the y-intercept is $\left(-\dfrac{5}{3}, 0\right)$.

The degree of the numerator is one degree larger than the degree of the denominator so there is a slant asymptote. Using division,

$$f(x) = \frac{3x^2 + 14x - 5}{x + 3} = 3x + 5 - \frac{20}{x + 3}$$

The slant asymptote is $y = 3x + 5$.

Checking to see if f intersects its slant asymptote,

$$\frac{3x^2+14x-5}{x+3} = 3x+5$$

$$3x^2+14x-5 = (x+3)(3x+5) \qquad \text{multiply both sides by LCD}$$

$$3x^2+14x-5 = 3x^2+14x-15 \qquad \text{multiply}$$

$$-5 = -15$$

This is a false statement so we can conclude that f does not intersect its slant asymptote.

Since the slant asymptote is a line with a positive slope, the end behavior is: as $x \to -\infty$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to \infty$.

Next, sketch the the asymptotes as dashed lines. Fill in the behavior of $f(x)$ near the asymptotes and plot the intercepts, as shown in Figure 3.7.6.

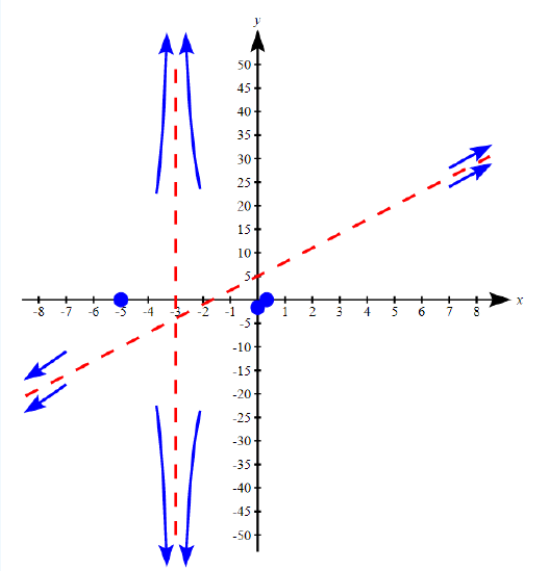


Figure 3.7.6: Asymptotic behavior and points of f(x)

We know there is an x-intercept on the interval $(-\infty, -3)$ and that the graph does not intercept the slant asymptote. This tells us that the graph must be above the slant asymptote and that as $x \to -3$, $f(x) \to \infty$. We also know there there is an x and y-intercept on the interval $(-3, \infty)$ and that the graph does not intercept the slant asymptote on this interval. This tells us that the graph must be below the slant asymptote and that as $x \to -3$, $f(x) \to \infty$. The graph of $f(x)$ is shown below in Figure 3.7.7.

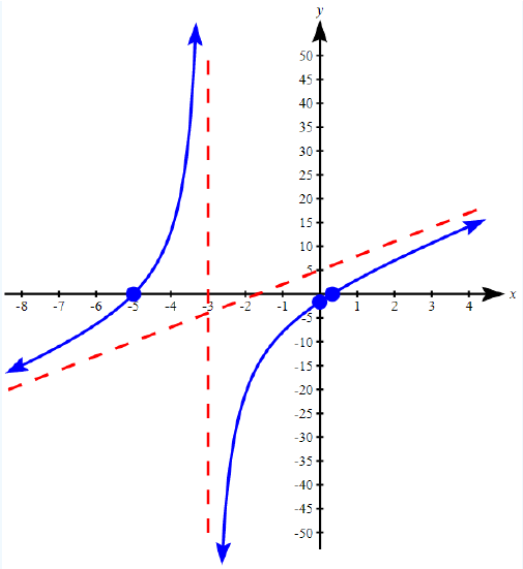


Figure 3.7.7: Graph of f(x)

Vertical asymptote at $x=-3$, slant asymptote at $y=3x+5$, intercepts at $(-5,0)$, $\left(\frac{1}{3},0\right)$, and $\left(0,-\frac{5}{3}\right)$, domain is $(-\infty,\infty)$, range is $(-\infty,\infty)$, and end behavior is as $x\to-\infty$, $f(x)\to-\infty$, and as $x\to\infty$, $f(x)\to\infty$.

Note that a sign chart and other points were not needed to help us determine the general shape of the graph of $f(x)$ in the previous example. This is not always the case when sketching graphs of rational functions.

You Try 3.7.1

Find the asymptotes, intercepts, domain, range, and end behavior of $B(x)=\dfrac{3x^2-7x-6}{x^2+3x-4}$. Round to the first decimal place for the range, if applicable. Then, use this information to sketch $B(x)$.

Answer

Vertical asymptotes at $x=-4$ and $x=1$, horizontal asymptote at $y=3$, graph intersects horizontal asymptote at $\left(\frac{3}{8},3\right)$,

intercepts at $\left(-\frac{2}{3},0\right)$, $(3,0)$ and $\left(0,\frac{3}{2}\right)$,

domain is $(-\infty,-4)\cup(-4,1)\cup(4,\infty)$, range is $(-\infty,\infty)$, as $x\to\pm\infty$, $B(x)\to3$.

The graph of $B(x)$ is shown in Figure 3.7.8

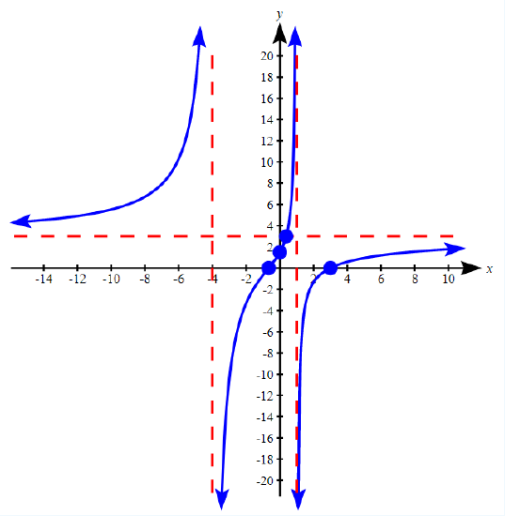


Figure 3.7.8: Graph of B(x)

Example 3.7.3

Find the asymptotes, intercepts, domain, range, and end behavior of $g(t)=\dfrac{9t+9-t^3-t^2}{t^3+t^2+3t+3}$. Round to the first decimal place for the range, if applicable. Then, use this information to sketch $g(t)$.

Solution

To begin, we need factor. Write terms in descending order and factoring by grouping,

$$g(t)=\frac{9t+9-t^3-t^2}{t^3+t^2+3t+3}=\frac{-(t+3)(t-3)(t+1)}{(t^2+3)(t+1)}$$

Note that t^2+3 does factor into $(t+\sqrt{3}i)(t-\sqrt{3}i)$ but we are graphing in a real coordinate system so we need not be concerned about these factors.

From the factored form of $g(t)$, we find that the domain is $(-\infty,-1)\cup(-1,\infty)$. There is a hole at $(-1,2)$ (*remember that the y-coordinate is found by simplifying the function first, then evaluating at -1*). The *t*-intercepts are at $(-3,0)$ and $(3,0)$.

The y-intercept is $(0,3)$ since $g(0)=\dfrac{9}{3}=3$.

There are no vertical asymptotes since there are no real numbers that make the denominator zero after simplifying.

There is a horizontal asymptote at $y=-1$. Verify on your own that g does not intersect the horizontal asymptote.

Sketch the the asymptote, points, and hole, as shown in Figure 3.7.9.

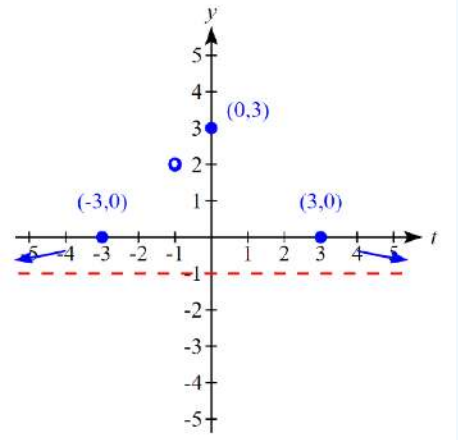


Figure 3.7.9: Asymptotic behavior and points of g(t)

We are able to connect this information to sketch the curve, as shown in Figure 3.7.10, without further information since the points found are below the asymptote and we know that the graph does not intersect the asymptote.

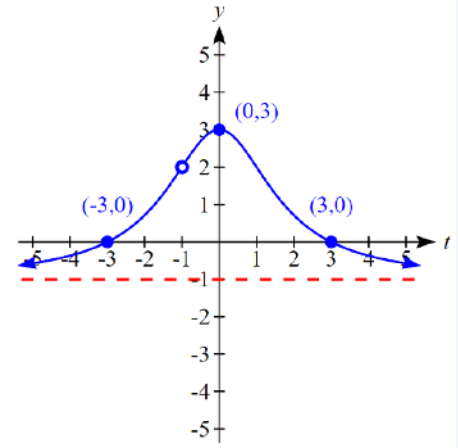


Figure 3.7.10: Graph of g(t)

No vertical asymptotes, horizontal asymptote at $y=-1$, intercepts at $(-3,0)$, $(3,0)$, and $(0,3)$, hole at $(-1,2)$, domain is $(-\infty,\infty)$, range is $(1,3]$, as $x\to\pm\infty$, $g(t)\to -1$.

Note that we know there is a local maximum from sketching the graph but we are unable to just assume that it is at the y-intercept. The graphing calculator helped us verify that there is a maximum at the y-intercept in this example. In calculus, you will learn ways to find local extrema without aid of graphing technology.

Example 3.7.4

Find the asymptotes, intercepts, domain, and range of $A(x)=\dfrac{x^2+6x-40}{2x^3-11x^2+10x+8}$. Round to the third decimal place for the range, if applicable. Then, use this information to sketch $A(x)$.

Solution

First, factor $A(x)$ (*you will need to use some factoring techniques discussed recently to factor*),

$$A(x)=\frac{x^2+6x-40}{2x^3-11x^2+10x+8}=\frac{(x-4)(x+10)}{(x-4)(x-2)(2x+1)}$$

From the factored form of $A(x)$, we find the domain is $(-\infty,-\frac{1}{2})\cup(-\frac{1}{2},2)\cup(2,4)\cup(4,\infty)$. There is a hole at $\left(4,\frac{7}{9}\right)$.

There are vertical asymptotes at $x = 2$ and $x = -\frac{1}{2}$.

There is a horizontal asymptote at $y = 0$. We do not need to check if $A(x)$ intersects the asymptote as we gain this information when finding the x-intercepts since the horizontal asymptote is the x-axis.

The x-intercept is $(-10, 0)$. The y-intercept is $(0, -5)$.

Sketch the the asymptote, points, and hole as shown in Figure 3.7.11.

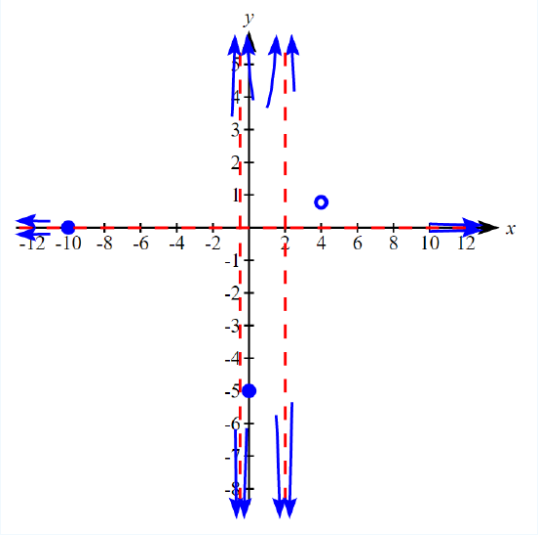


Figure 3.7.11: Asymptotic behavior and points of A(x)

On the interval $\left(-\infty, -\frac{1}{2}\right)$, we know that as $x \to -\infty$, $A \to 0$. Also, as $x \to -\frac{1}{2}^{-}$, $A \to -\infty$ or $A \to \infty$. There is also an x-intercept at $(-10, 0)$ within this interval. Applying what we have learned about polynomials, the graph must pass through the x-intercept since it comes from the factor $(x + 10)$ which is a single root of multiplicity.

Currently, this is not enough information to sketch the graph on this interval since we do not know if the graph will be above or below the x-axis on the interval $(-\infty, -10)$. This is also the case for the interval $\left(-10, -\frac{1}{2}\right)$. We can either set up a sign chart of find a point within one of these intervals to find out.

Using the latter, let's evaluate at $x = -6$.

$$A(-6) = \frac{1}{22}$$

so we can conclude that the graph is above the x-axis on $\left(-10, -\frac{1}{2}\right)$. The graph must be below the x-axis on $(-\infty, -10)$ because we know the graph passed through the x-intercept at $(-10, 0)$.

On the interval $\left(-\frac{1}{2}, 2\right)$, we know there is a y-intercept at $(0, -5)$ and no x-intercept. This indicates that the graph will be below the x-axis. Since we know that $A \to -\infty$ as we approach the vertical asymptotes, we also know that there will be a local maximum within this interval.

Currently, we do not have enough information to sketch the graph on the interval $(2, \infty)$. We know that A has to approach the asymptotes, but we do not know if the graph is above or below the x-axis. Again, we may use a sign chart or find a point within this interval.

Let's evaluate at $x = 5$ which is within this interval.

$$A(5) = \frac{5}{11}$$

so we can conclude that the graph is above the x-axis.

The graph of A is shown in Figure 3.7.12

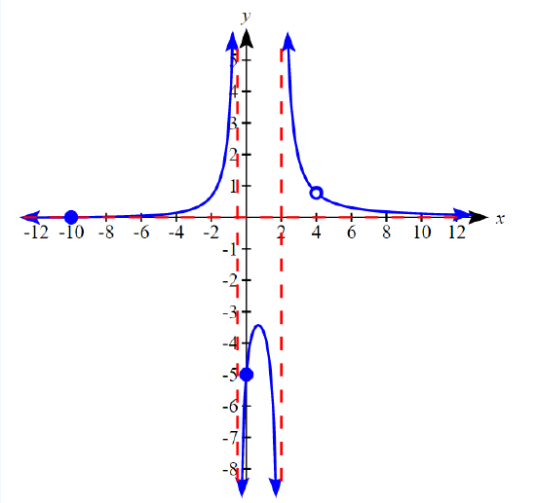


Figure 3.7.12: Graph of A(x)

We still need to find the range. Upon viewing the graph, we can see the range is not all real numbers so we will need to approximate the range using graphing technology. We can see there is a maximum within the interval $\left(-\frac{1}{2}, 2\right)$. Using our graphing calculator, we find the maximum is ≈ -3.428.

Remember that there is is an x-intercept at $(-10, 0)$. Although not easily viewable in the window used to sketch the graph in Figure 3.7.12, we know that they must be a local minimum within the interval $\left(-\frac{1}{2}, 2\right)$. To approximate it using graphing technology you will need to find an appropriate window to view the local minimum so that you can then find it. The local minimum is ≈ -0.012. We can conclude from this that the range is $(-\infty, -3.428] \cup [-0.012, \infty)$

One important thing to note about the graph of $A(x)$. Most graphing viewing windows will not show the local minimum. It is important for hand drawn sketches to include this information. Also, this example also illustrates the importance of understanding how functions behave as we would have not known there was a local minimum within the interval $\left(-\infty, -\frac{1}{2}\right)$ had we not investigated the behavior of the function.

? You Try 3.7.2

Find the asymptotes, intercepts, domain, and range of $w(t) = \frac{3}{t} - \frac{3t+2}{t^2+1}$. Round to the third decimal place for the range, if applicable. Then, use this information to sketch $w(t)$.

Answer

Note that the function must be simplified before finding the graphical information we are asked to find.

$$w(t) = \frac{3}{t} - \frac{3t+2}{t^2+1} = \frac{3-2t}{t(t^2+1)}$$

The vertical asymptote is $x = 0$. The horizontal asymptote is $y = 0$.

There is a *t*-intercept at $(\frac{3}{2}, 0)$. This is the only intercept.

The domain is $(-\infty, 0) \cup (0, \infty)$. The range is $(-\infty, \infty)$. (*note that there is a local minimum at* $\approx (2.382, -0.111)$

Graph is shown in Figure 3.7.13

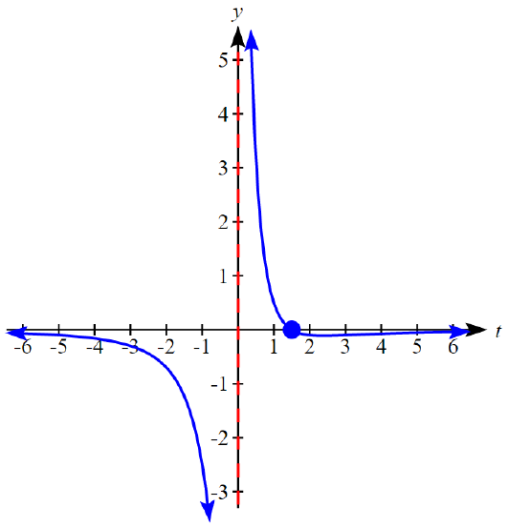


Figure 3.7.13: Graph of w(t)

Finding Equations of Rational Functions that Model Graphical Information

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an *x*-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of *x*-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

Example 3.7.5

Write an equation for the rational function shown in Figure 3.7.14

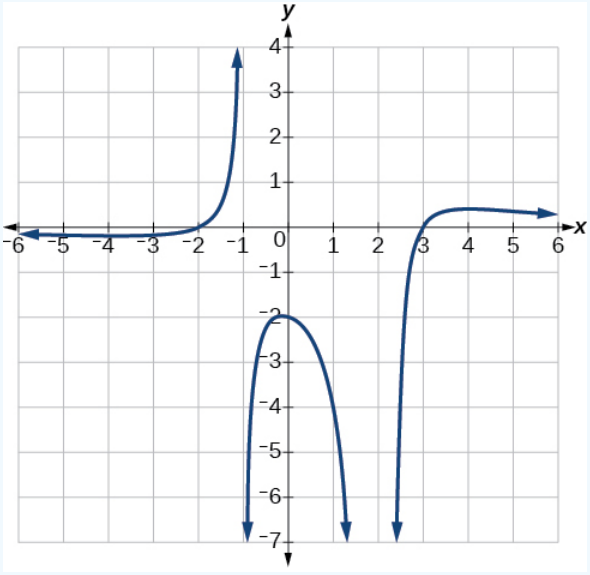


Figure 3.7.14.

Solution

The graph appears to have x-intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors for the numerator. The graph has two vertical asymptotes at $x = -1$ and $x = 2$ indicating denominator factors. Notice

that at $x = 2$ that the graph approaches the asymptote from the same side. This indicates the factor $x - 2$ would be squared (*think about what we know about sign charts and polynomial behavior*). See Figure 3.7.15

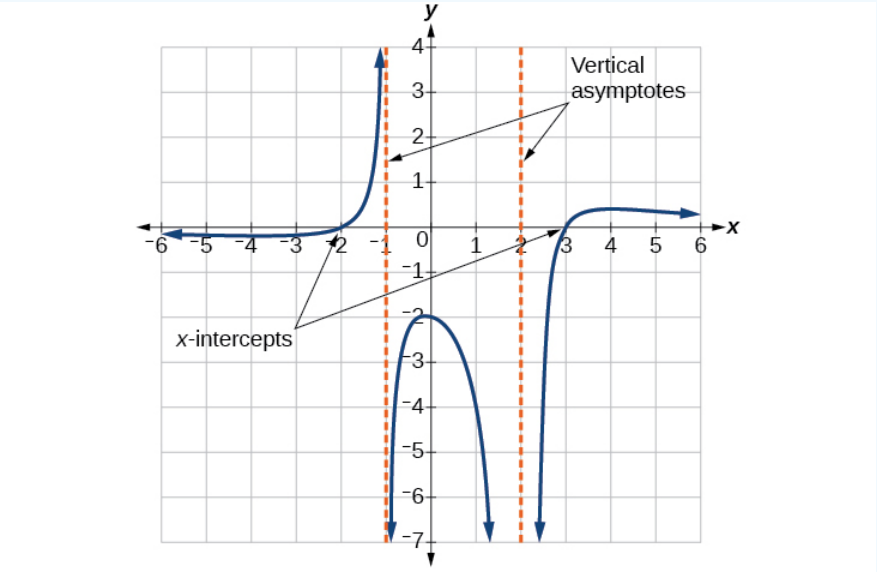


Figure 3.7.15.

We can use this information to write a function of the form

$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, a, we can evaluate at a clear point on the graph, such as the y-intercept $(0, -2)$.

$$-2 = a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$

$$-2 = a\left(\frac{-6}{4}\right)$$

$$a = \frac{-8}{-6} = \frac{4}{3}$$

This gives us a final function of $f(x) = \dfrac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.

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3.E: Polynomial and Rational Functions (Exercises)

Complex Numbers

Verbal

1) Explain how to add complex numbers.

Answer

Add the real parts together and the imaginary parts together.

2) What is the basic principle in multiplication of complex numbers?

3) Give an example to show the product of two imaginary numbers is not always imaginary.

Answer

i times i equals -1, which is not imaginary (answers vary)

4) What is a characteristic of the plot of a real number in the complex plane?

Algebraic

For the exercises 5-10, evaluate the algebraic expressions.

5) If $f(x)=x^2+x-4$, evaluate $f(2i)$.

Answer

$-8+2i$

6) If $f(x)=x^3-2$, evaluate $f(i)$.

7) If $f(x)=x^2+3x+5$,evaluate $f(2+i)$.

Answer

$14+7i$

8) If $f(x)=2x^2+x-3$, evaluate $f(2-3i)$.

9) If $f(x)=\dfrac{x+1}{2-x}$, evaluate $f(5i)$.

Answer

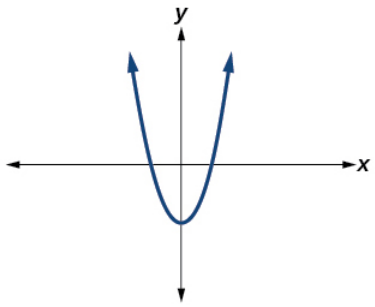
$-\dfrac{23}{29}+\dfrac{15}{29}i$

10) If $f(x)=\dfrac{1+2x}{x+3}$, evaluate $f(4i)$.

Graphical

For the exercises 11-12, determine the number of real and nonreal solutions for each quadratic function shown.

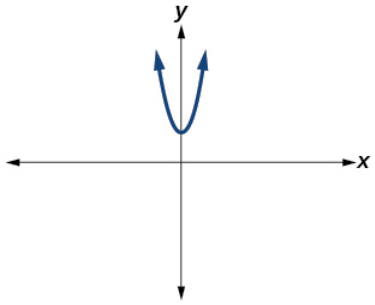
11)



Answer

2 real and 0 nonreal

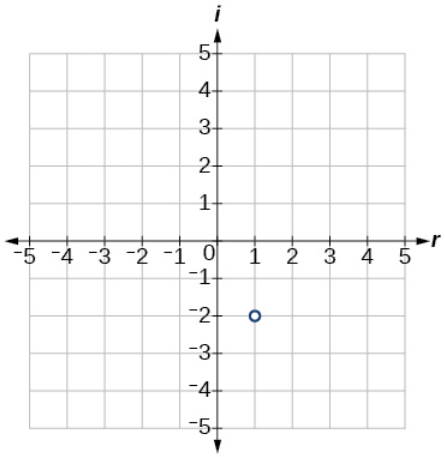
12)



For the exercises 13-16, plot the complex numbers on the complex plane.

13) $1-2i$

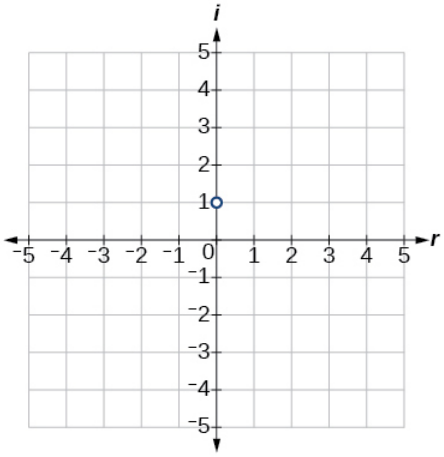
Answer



14) $-2+3i$

15) i

Answer



16) $-3-4i$

Numeric

For the exercises 17-43, perform the indicated operation and express the result as a simplified complex number.

17) $(3+2i)+(5-3i)$

Answer

$8-i$

18) $(-2-4i)+(1+6i)$

19) $(-5+3i)-(6-i)$

Answer

$-11+4i$

20) $(2-3i)-(3+2i)$

21) $(-4+4i)-(-6+9i)$

Answer

$2-5i$

22) $(2+3i)(4i)$

23) $(5-2i)(3i)$

Answer

$6+15i$

24) $(6-2i)(5)$

25) $(-2+4i)(8)$

Answer

$-16+32i$

26) $(2+3i)(4-i)$

27) $(-1+2i)(-2+3i)$

Answer

$-4-7i$

28) $(4-2i)(4+2i)$

29) $(3+4i)(3-4i)$

Answer

25

30) $\dfrac{3+4i}{2}$

31) $\dfrac{6-2i}{3}$

Answer

$2-\dfrac{2}{3}i$

32) $\dfrac{-5+3i}{2i}$

33) $\dfrac{6+4i}{i}$

Answer

$4-6i$

34) $\dfrac{2-3i}{4+3i}$

35) $\dfrac{3+4i}{2-i}$

Answer

$\dfrac{2}{5}+\dfrac{11}{5}i$

36) $\dfrac{2+3i}{2-3i}$

37) $\sqrt{-9}+3\sqrt{-16}$

Answer

$15i$

38) $-\sqrt{-4}-4\sqrt{-25}$

39) $\dfrac{2+\sqrt{-12}}{2}$

Answer

$1+i\sqrt{3}$

40) $\dfrac{4+\sqrt{-20}}{2}$

41) i^8

Answer

1

42) i^{15}

43) i^{22}

Answer

-1

Technology

For the exercises 44-48, use a calculator to help answer the questions.

44) Evaluate $(1+i)^k$ for $k=4, 8,$ and 12.Predict the value if $k=16$.

45) Evaluate $(1-i)^k$ for $k=2, 6,$ and 10.Predict the value if $k=14$.

Answer

$128i$

46) Evaluate $(1+i)^k-(1-i)^k$ for $k=4$, 8, and 12. Predict the value for $k=16$.

47) Show that a solution of $x^6+1=0$ is $\frac{\sqrt{3}}{2}+\frac{1}{2}i$.

Answer

$(\frac{\sqrt{3}}{2}+\frac{1}{2}i)^6=-1$

48) Show that a solution of $x^8-1=0$ is $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i$.

Extensions

For the exercises 49-58, evaluate the expressions, writing the result as a simplified complex number.

49) $\frac{1}{i}+\frac{4}{i^3}$

Answer

$3i$

50) $\frac{1}{i^{11}}-\frac{1}{i^{21}}$

51) $i^7(1+i^2)$

Answer

0

52) $i^{-3}+5i^7$

53) $\frac{(2+i)(4-2i)}{(1+i)}$

Answer

$5-5i$

54) $\frac{(1+3i)(2-4i)}{(1+2i)}$

55) $\frac{(3+i)^2}{(1+2i)^2}$

Answer

$-2i$

56) $\frac{3+2i}{2+i}+(4+3i)$

57) $\frac{4+i}{i}+\frac{3-4i}{1-i}$

Answer

$\frac{9}{2}-\frac{9}{2}i$

58) $\frac{3+2i}{1+2i}-\frac{2-3i}{3+i}$

Quadratic Functions

Verbal

1) Explain the advantage of writing a quadratic function in standard form.

Answer

When written in that form, the vertex can be easily identified.

2) How can the vertex of a parabola be used in solving real world problems?

3) Explain why the condition of $a \neq 0$ is imposed in the definition of the quadratic function.

Answer

If $a = 0$ then the function becomes a linear function.

4) What is another name for the standard form of a quadratic function?

5) What two algebraic methods can be used to find the horizontal intercepts of a quadratic function?

Answer

If possible, we can use factoring. Otherwise, we can use the quadratic formula.

Algebraic

For the exercises 6-13, rewrite the quadratic functions in standard form and give the vertex.

6) $f(x) = x^2 - 12x + 32$

7) $g(x) = x^2 + 2x - 3$

Answer

$g(x) = (x+1)^2 - 4$, Vertex $(-1, -4)$

8) $f(x) = x^2 - x$

9) $f(x) = x^2 + 5x - 2$

Answer

$f(x) = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$, Vertex $\left(-\frac{5}{2}, -\frac{33}{4}\right)$

10) $h(x) = 2x^2 + 8x - 10$

11) $k(x) = 3x^2 - 6x - 9$

Answer

$k(x) = 3(x-1)^2 - 12$, Vertex $(1, -12)$

12) $f(x) = 2x^2 - 6x$

13) $f(x) = 3x^2 - 5x - 1$

Answer

$f(x) = 3\left(x - \frac{5}{6}\right)^2 - \frac{37}{12}$, Vertex $\left(\frac{5}{6}, -\frac{37}{12}\right)$

For the exercises 14-20, determine whether there is a minimum or maximum value to each quadratic function. Find the value and the axis of symmetry.

14) $y(x) = 2x^2 + 10x + 12$

15) $f(x) = 2x^2 - 10x + 4$

Answer

Minimum is $-\frac{17}{2}$ and occurs at $\frac{5}{2}$. Axis of symmetry is $x = \frac{5}{2}$.

16) $f(x) = -x^2 + 4x + 3$

17) $f(x) = 4x^2 + x - 1$

Answer

Minimum is $-\frac{17}{16}$ and occurs at $-\frac{1}{8}$. Axis of symmetry is $x = -\frac{1}{8}$.

18) $h(t) = -4t^2 + 6t - 1$

19) $f(x) = \frac{1}{2}x^2 + 3x + 1$

Answer

Minimum is $-\frac{7}{2}$ and occurs at -3. Axis of symmetry is $x = -3$.

20) $f(x) = -\frac{1}{3}x^2 - 2x + 3$

For the exercises 21-25, determine the domain and range of the quadratic function.

21) $f(x) = (x-3)^2 + 2$

Answer

Domain is $(-\infty, \infty)$. Range is $[2, \infty)$.

22) $f(x) = -2(x+3)^2 - 6$

23) $f(x) = x^2 + 6x + 4$

Answer

Domain is $(-\infty, \infty)$. Range is $[-5, \infty)$.

24) $f(x) = 2x^2 - 4x + 2$

25) $k(x) = 3x^2 - 6x - 9$

Answer

Domain is $(-\infty, \infty)$. Range is $[-12, \infty)$.

For the exercises 26-44, solve the equations over the complex numbers.

26) $x^2 = -25$

27) $x^2 = -8$

Answer

$2i\sqrt{2}, -2i\sqrt{2}$

28) $x^2 + 36 = 0$

29) $x^2 + 27 = 0$

Answer

$3i\sqrt{3}, -3i\sqrt{3}$

30) $x^2 + 2x + 5 = 0$

31) $x^2 - 4x + 5 = 0$

Answer

$2 + i, 2 - i$

32) $x^2 + 8x + 25 = 0$

33) $x^2 - 4x + 13 = 0$

Answer

$2 + 3i, 2 - 3i$

34) $x^2 + 6x + 25 = 0$

35) $x^2 - 10x + 26 = 0$

Answer

$5 + i, 5 - i$

36) $x^2 - 6x + 10 = 0$

37) $x(x - 4) = 20$

Answer

$2 + 2\sqrt{6}, 2 - 2\sqrt{6}$

38) $x(x - 2) = 10$

39) $2x^2 + 2x + 5 = 0$

Answer

$\left\{-\frac{1}{2} + \frac{3}{2}i, -\frac{1}{2} - \frac{3}{2}i\right\}$

40) $5x^2 - 8x + 5 = 0$

41) $5x^2 + 6x + 2 = 0$

Answer

$$\left\{-\frac{3}{5}+\frac{1}{5}i,-\frac{3}{5}-\frac{1}{5}i\right\}$$

42) $2x^2-6x+5=0$

43) $x^2+x+2=0$

Answer

$$\left\{-\frac{1}{2}+\frac{1}{2}i\sqrt{7},-\frac{1}{2}-\frac{1}{2}i\sqrt{7}\right\}$$

44) $x^2-2x+4=0$

For the exercises 45-52, use the vertex (h,k) and a point on the graph (x,y) to find the general form of the equation of the quadratic function.

45) $(h,k)=(2,0),(x,y)=(4,4)$

Answer

$f(x)=x^2-4x+4$

46) $(h,k)=(-2,-1),(x,y)=(-4,3)$

47) $(h,k)=(0,1),(x,y)=(2,5)$

Answer

$f(x)=x^2+1$

48) $(h,k)=(2,3),(x,y)=(5,12)$

49) $(h,k)=(-5,3),(x,y)=(2,9)$

Answer

$$f(x)=\frac{6}{49}x^2+\frac{60}{49}x+\frac{297}{49}$$

50) $(h,k)=(3,2),(x,y)=(10,1)$

51) $(h,k)=(0,1),(x,y)=(1,0)$

Answer

$f(x)=-x^2+1$

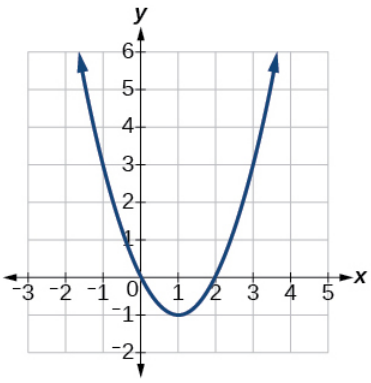
52) $(h,k)=(1,0),(x,y)=(0,1)$

Graphical

For the exercises 53-58, sketch a graph of the quadratic function and give the vertex, axis of symmetry, and intercepts.

53) $f(x)=x^2-2x$

Answer

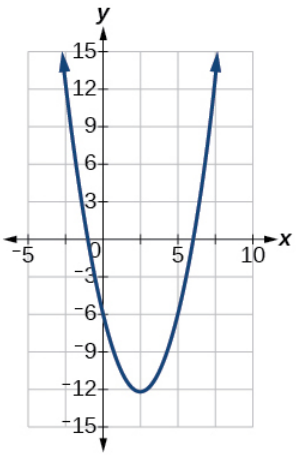


Vertex $(1, -1)$, Axis of symmetry is $x = 1$. Intercepts are $(0, 0), (2, 0)$.

54) $f(x) = x^2 - 6x - 1$

55) $f(x) = x^2 - 5x - 6$

Answer

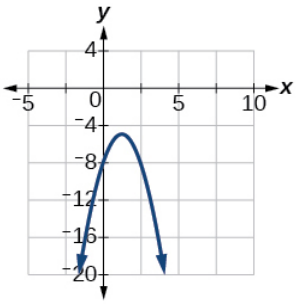


Vertex $\left(\frac{5}{2}, \frac{-49}{4}\right)$, Axis of symmetry is $x = \frac{5}{2}$. Intercepts are $(0, -6), (-1, 0), (6, 0)$.

56) $f(x) = x^2 - 7x + 3$

57) $f(x) = -2x^2 + 5x - 8$

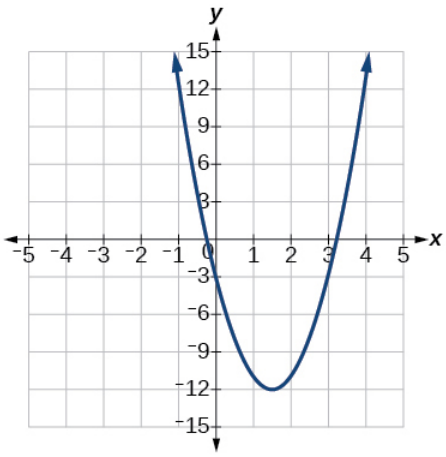
Answer



Vertex $\left(\frac{5}{4}, -\frac{39}{8}\right)$, Axis of symmetry is $x = \frac{5}{4}$. Intercepts are $(0, -8)$.

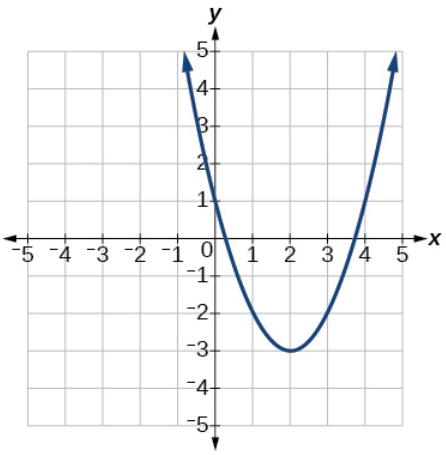
58) $f(x) = 4x^2 - 12x - 3$

Answer



For the exercises 59-64, write the equation for the graphed function.

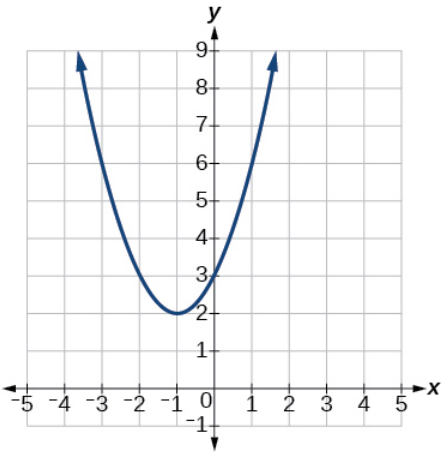
59)



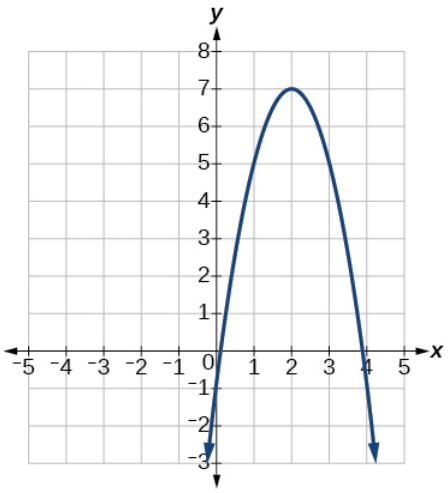
Answer

$f(x) = x^2 - 4x + 1$

60)



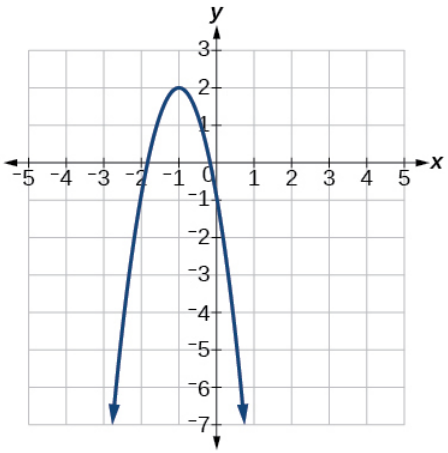
61)



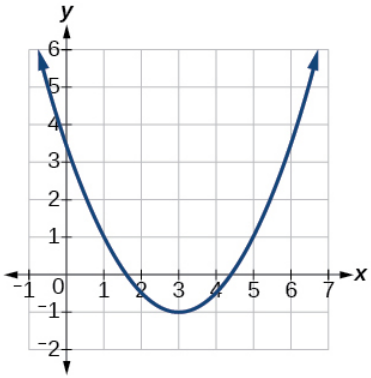
Answer

$f(x) = -2x^2 + 8x - 1$

62)



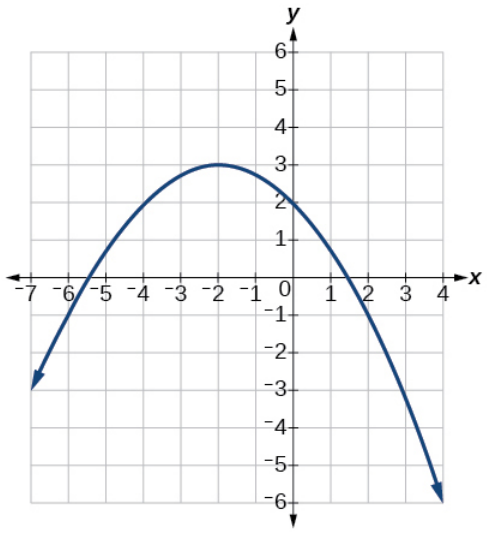
63)



Answer

$f(x) = \frac{1}{2}x^2 - 3x + \frac{7}{2}$

64)



Numeric

For the exercises 65-69, use the table of values that represent points on the graph of a quadratic function. By determining the vertex and axis of symmetry, find the general form of the equation of the quadratic function.

65)

x	–2	–1	0	1	2
y	5	2	1	2	5

Answer

$f(x) = x^2 + 1$

66)

x	–2	–1	0	1	2
y	1	0	1	4	9

67)

x	–2	–1	0	1	2
y	–2	1	2	1	–2

Answer

$f(x) = 2 - x^2$

68)

x	–2	–1	0	1	2
y	–8	–3	0	1	0

69)

x	–2	–1	0	1	2
y	8	2	0	2	8

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Answer

$f(x) = 2x^2$

Technology

For the exercises 70-74, use a calculator to find the answer.

70) Graph on the same set of axes the functions $f(x) = x^2, f(x) = 2x^2$, and $f(x) = \frac{1}{3}x^2$. What appears to be the effect of changing the coefficient?

71) Graph on the same set of axes $f(x) = x^2, f(x) = x^2 + 2$ and $f(x) = x^2, f(x) = x^2 + 5$ and $f(x) = x^2 - 3$. What appears to be the effect of adding a constant?

Answer

The graph is shifted up or down (a vertical shift).

72) Graph on the same set of axes $f(x) = x^2, f(x) = (x-2)^2, f(x-3)2,$ and $f(x) = (x+4)^2$. What appears to be the effect of adding or subtracting those numbers?

73) The path of an object projected at a 45 degree angle with initial velocity of 80 feet per second is given by the function $h(x) = -\frac{32}{(80)^2}x^2 + x$ where x is the horizontal distance traveled and $h(x)$ is the height in feet. Use the TRACE feature of your calculator to determine the height of the object when it has traveled 100 feet away horizontally.

Answer

50 feet

74) A suspension bridge can be modeled by the quadratic function $h(x) = .0001x^2$ with $-2000 \leq x \leq 2000$ where $|x|$ is the number of feet from the center and $h(x)$ is height in feet. Use the TRACE feature of your calculator to estimate how far from the center does the bridge have a height of 100 feet.

Extensions

For the exercises 75-78, use the vertex of the graph of the quadratic function and the direction the graph opens to find the domain and range of the function.

75) Vertex $(1, -2)$, opens up.

Answer

Domain is $(-\infty, \infty)$. Range is $[-2, \infty)$.

76) Vertex $(-1, 2)$ opens down.

77) Vertex $(-5, 11)$, opens down.

Answer

Domain is $(-\infty, \infty)$ Range is $(-\infty, 11]$.

78) Vertex $(-100, 100)$, opens up.

For the exercises 79-84, write the equation of the quadratic function that contains the given point and has the same shape as the given function.

79) Contains $(1, 1)$ and has shape of $f(x) = 2x^2$. Vertex is on the y-axis.

Answer

$f(x) = 2x^2 - 1$

80) Contains $(-1, 4)$ and has the shape of $f(x) = 2x^2$. Vertex is on the y-axis.

81) Contains $(2, 3)$ and has the shape of $f(x) = 3x^2$. Vertex is on the y-axis.

Answer

$f(x) = 3x^2 - 9$

82) Contains $(1, -3)$ and has the shape of $f(x) = -x^2$. Vertex is on the y-axis.

83) Contains $(4, 3)$ and has the shape of $f(x) = 5x^2$. Vertex is on the y-axis.

Answer

$f(x) = 5x^2 - 77$

84) Contains $(1, -6)$ has the shape of $f(x) = 3x^2$. Vertex has x-coordinate of -1.

Real-World Applications

85) Find the dimensions of the rectangular corral producing the greatest enclosed area given 200 feet of fencing.

Answer

50 feet by 50 feet. Maximize $f(x) = -x^2 + 100x$.

86) Find the dimensions of the rectangular corral split into 2 pens of the same size producing the greatest possible enclosed area given 300 feet of fencing.

87) Find the dimensions of the rectangular corral producing the greatest enclosed area split into 3 pens of the same size given 500 feet of fencing.

Answer

125 feet by 62.5 feet. Maximize $f(x) = -2x^2 + 250x$.

88) Among all of the pairs of numbers whose sum is 6, find the pair with the largest product. What is the product?

89) Among all of the pairs of numbers whose difference is 12, find the pair with the smallest product. What is the product?

Answer

6 and -6; product is -36; maximize $f(x) = x^2 + 12x$.

90) Suppose that the price per unit in dollars of a cell phone production is modeled by $p = \$45 - 0.0125x$, where x is in thousands of phones produced, and the revenue represented by thousands of dollars is $R = x \cdot p$. Find the production level that will maximize revenue.

91) A rocket is launched in the air. Its height, in meters above sea level, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 229t + 234$. Find the maximum height the rocket attains.

Answer

2909.56 meters

92) A ball is thrown in the air from the top of a building. Its height, in meters above ground, as a function of time, in seconds, is given by $h(t) = -4.9t^2 + 24t + 8$. How long does it take to reach maximum height?

93) A soccer stadium holds $62,000$ spectators. With a ticket price of $\$11$, the average attendance has been $26,000$. When the price dropped to $\$9$, the average attendance rose to $31,000$. Assuming that attendance is linearly related to ticket price, what ticket price would maximize revenue?

Answer

$\$10.70$

94) A farmer finds that if she plants 75 trees per acre, each tree will yield 20 bushels of fruit. She estimates that for each additional tree planted per acre, the yield of each tree will decrease by 3 bushels. How many trees should she plant per acre to maximize her harvest?

Polynomial Functions

Verbal

1) Explain the difference between the coefficient of a power function and its degree.

Answer

The coefficient of the power function is the real number that is multiplied by the variable raised to a power. The degree is the highest power appearing in the function.

2) If a polynomial function is in factored form, what would be a good first step in order to determine the degree of the function?

3) In general, explain the end behavior of a power function with odd degree if the leading coefficient is positive.

Answer

As x decreases without bound, so does $f(x)$. As x increases without bound, so does $f(x)$.

4) What is the relationship between the degree of a polynomial function and the maximum number of turning points in its graph?

5) What can we conclude if, in general, the graph of a polynomial function exhibits the following end behavior? As $x \to -\infty, f(x) \to -\infty$ and as $x \to \infty, f(x) \to -\infty$.

Answer

The polynomial function is of even degree and leading coefficient is negative.

Algebraic

For the exercises 6-11, identify the function as a power function, a polynomial function, or neither.

6) $f(x) = x^5$

7) $f(x) = (x^2)^3$

Answer

Power function

8) $f(x) = x - x^4$

9) $f(x) = \dfrac{x^2}{x^2 - 1}$

Answer

Neither

10) $f(x) = 2x(x+2)(x-1)^2$

11) $f(x) = 3^{x+1}$

Answer

Neither

For the exercises 12-16, find the degree and leading coefficient for the given polynomial.

12) $-3x^4$

13) $7 - 2x^2$

Answer

Degree = 2, Coefficient = -2

14) $-2x^2 - 3x^5 + x - 6$

15) $x(4 - x^2)(2x + 1)$

Answer

Degree =4, Coefficient = -2

16) $x^2(2x - 3)^2$

For the exercises 17-24, determine the end behavior of the functions.

17) $f(x) = x^4$

Answer

As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to \infty$

18) $f(x) = x^3$

19) $f(x) = -x^4$

Answer

As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$

20) $f(x) = -x^9$

21) $f(x) = -2x^4 - 3x^2 + x - 1$

Answer

As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to -\infty$

22) $f(x) = 3x^2 + x - 2$

23) $f(x) = x^2(2x^3 - x + 1)$

Answer

As $x \to \infty$, $f(x) \to \infty$, as $x \to -\infty$, $f(x) \to -\infty$

24) $f(x) = (2 - x)^7$

For the exercises 25-30, find the intercepts of the functions.

25) $f(t) = 2(t-1)(t+2)(t-3)$

Answer

y-intercept is $(0, 12)$, t-intercepts are $(1, 0)$; $(-2, 0)$;and $(3, 0)$.

26) $g(n) = -2(3n-1)(2n+1)$

27) $f(x) = x^4 - 16$

Answer

y-intercept is $(0, -16)$, x-intercepts are $(2, 0)$ and $(-2, 0)$.

28) $f(x) = x^3 + 27$

29) $f(x) = x(x^2 - 2x - 8)$

Answer

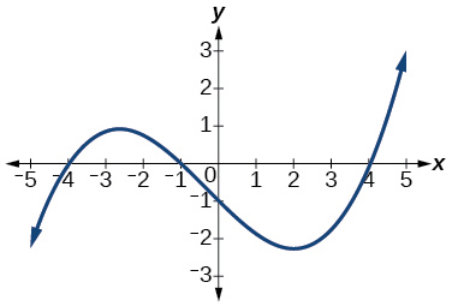
y-intercept is $(0, 0)$, x-intercepts are $(0, 0)$, $(4, 0)$,and $(-2, 0)$.

30) $f(x) = (x+3)(4x^2 - 1)$

Graphical

For the exercises 31-38, determine the least possible degree of the polynomial function shown.

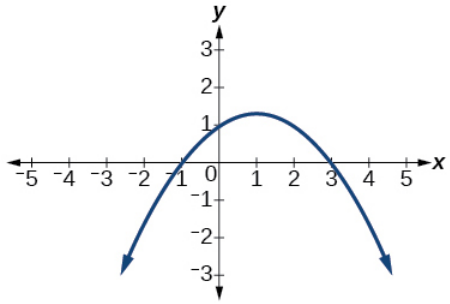
31)



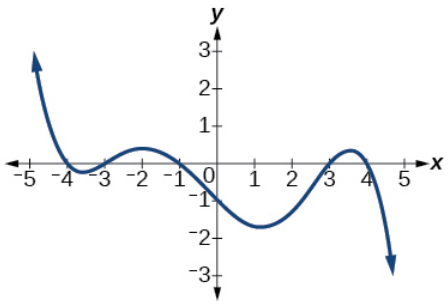
Answer

3

32)



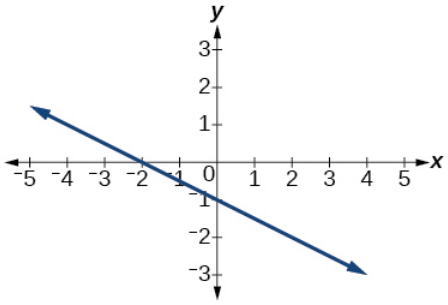
33)



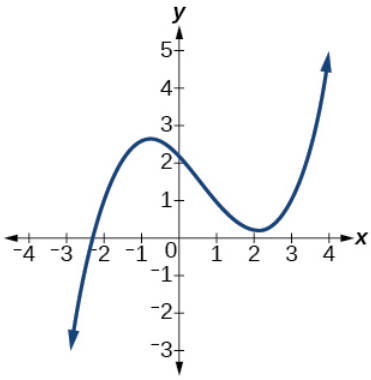
Answer

5

34)



35)

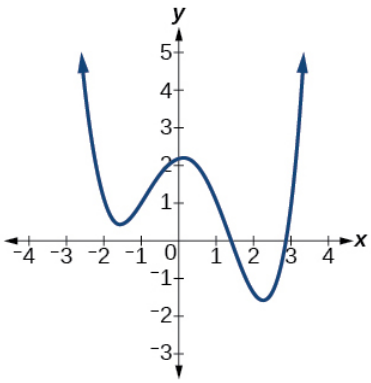


Answer

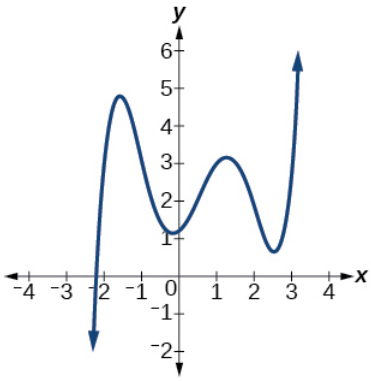
3

36)

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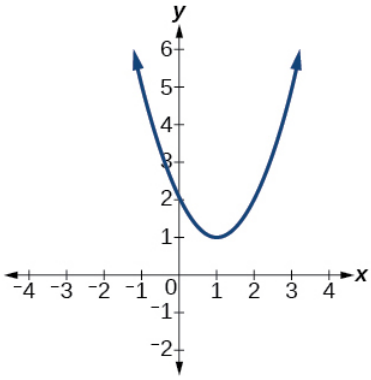
37)



Answer

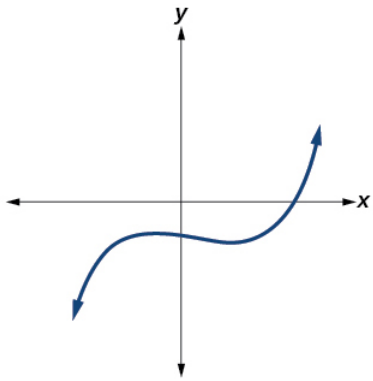
5

38)



For the exercises 39-45, determine whether the graph of the function provided is a graph of a polynomial function. If so, determine the number of turning points and the least possible degree for the function.

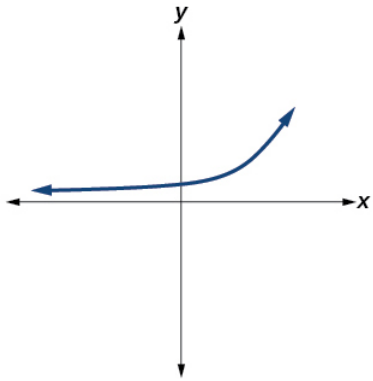
39)



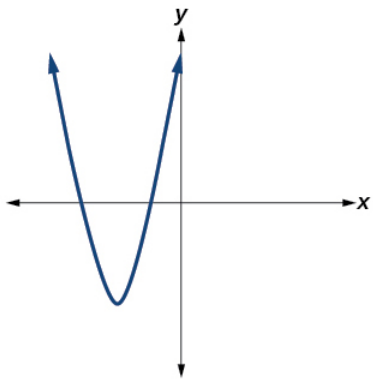
Answer

Yes. Number of turning points is 2. Least possible degree is 3.

40)



41)

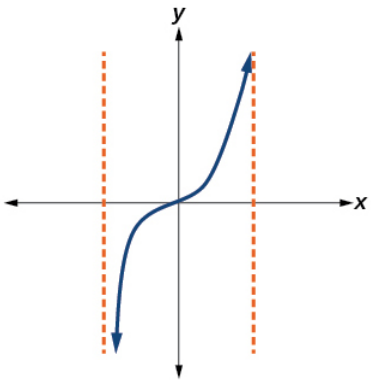


Answer

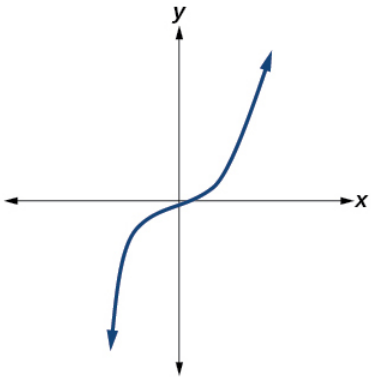
Yes. Number of turning points is 1. Least possible degree is 2.

42)

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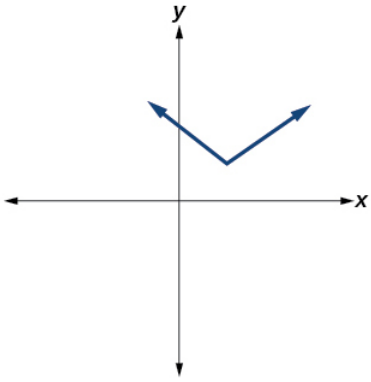
43)



Answer

Yes. Number of turning points is 0. Least possible degree is 1.

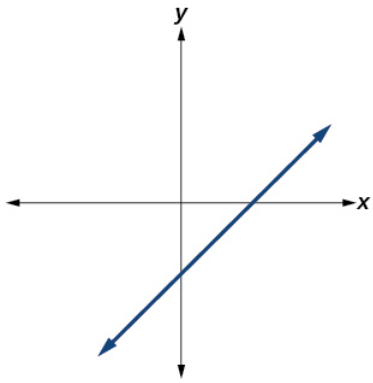
44)



Answer

No.

45)



Answer

Yes. Number of turning points is 0. Least possible degree is 1.

Numeric

For the exercises 46-50, make a table to confirm the end behavior of the function.

46) $f(x) = -x^3$

47) $f(x) = x^4 - 5x^2$

Answer

x	$f(x)$
10	9,500
100	99,950,000
–10	9,500
–100	99,950,000

as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$

48) $f(x) = x^2(1-x)^2$

49) $f(x) = (x-1)(x-2)(3-x)$

Answer

x	$f(x)$
10	–504
100	–941,094
–10	1,716
–100	1,061,106

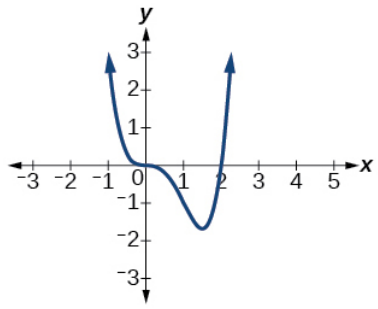
as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to -\infty$

50) $f(x) = \dfrac{x^5}{10} - x^4$

For the exercises 51-60, graph the polynomial functions using a calculator. Based on the graph, determine the intercepts and the end behavior.

51) $f(x) = x^3(x-2)$

Answer

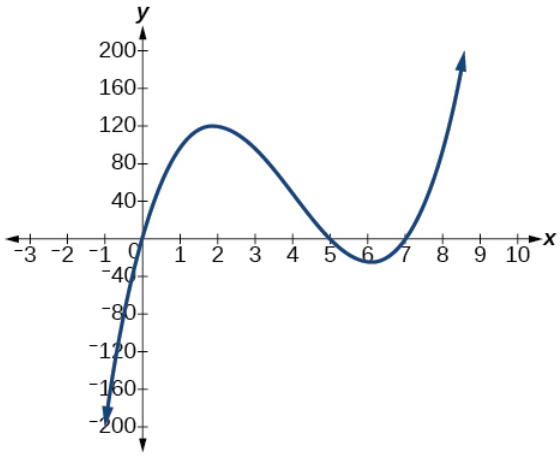


The y-intercept is $(0,0)$. The x-intercepts are $(0,0),(2,0)$. As $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$

52) $f(x) = x(x-3)(x+3)$

53) $f(x) = x(14-2x)(10-2x)$

Answer

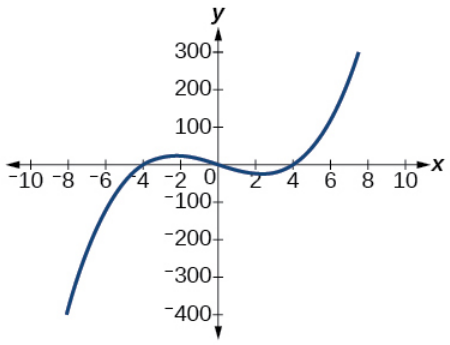


The y-intercept is $(0,0)$. The x-intercepts are $(0,0),(5,0),(7,0)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$

54) $f(x) = x(14-2x)(10-2x)^2$

55) $f(x) = x^3 - 16x$

Answer

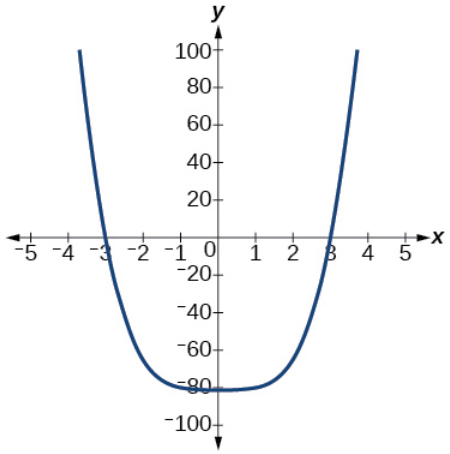


The y-intercept is $(0,0)$. The x-intercept is $(-4,0),(0,0),(4,0)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$

56) $f(x) = x^3 - 27$

57) $f(x) = x^4 - 81$

Answer

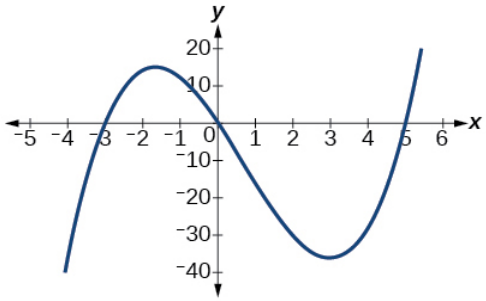


The y-intercept is $(0,-81)$. The x-intercept are $(3,0),(-3,0)$. As $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$

58) $f(x) = -x^3 + x^2 + 2x$

59) $f(x) = x^3 - 2x^2 - 15x$

Answer



The y-intercept is $(0,0)$. The x-intercepts are $(-3,0),(0,0),(5,0)$.As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$

60) $f(x) = x^3 - 0.01x$

Extensions

For the exercises 61-65, use the information about the graph of a polynomial function to determine the function. Assume the leading coefficient is 1 or -1. There may be more than one correct answer.

61) The y-intercept is $(0,-4)$. The x-intercepts are $(-2,0),(2,0)$. Degree is 2.

End behavior: as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$.

Answer

$f(x) = x^2 - 4$

62) The y-intercept is $(0,9)$. The x-intercepts are $(-3,0),(3,0)$. Degree is 2.

End behavior: as $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to -\infty$.

63) The y-intercept is $(0,0)$. The x-intercepts are $(0,0),(2,0)$. Degree is 3.

End behavior: as $x \to -\infty,\ f(x) \to -\infty$, as $x \to \infty,\ f(x) \to \infty$.

Answer

$f(x) = x^3 - 4x^2 + 4x$

64) The y-intercept is $(0,1)$. The x-intercept is $(1,0)$. Degree is 3.

End behavior: as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to -\infty$.

65) The y-intercept is $(0,1)$. There is no x-intercept. Degree is 4.

End behavior: as $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to \infty$.

Answer

$f(x) = x^4 + 1$

Real-World Applications

For the exercises 66-70, use the written statements to construct a polynomial function that represents the required information.

66) An oil slick is expanding as a circle. The radius of the circle is increasing at the rate of 20 meters per day. Express the area of the circle as a function of d, the number of days elapsed.

67) A cube has an edge of 3 feet. The edge is increasing at the rate of 2 feet per minute. Express the volume of the cube as a function of m, the number of minutes elapsed.

Answer

$V(m) = 8m^3 + 36m^2 + 54m + 27$

68) A rectangle has a length of 10 inches and a width of 6 inches. If the length is increased by x inches and the width increased by twice that amount, express the area of the rectangle as a function of x.

69) An open box is to be constructed by cutting out square corners of x-inch sides from a piece of cardboard 8 inches by 8 inches and then folding up the sides. Express the volume of the box as a function of x.

Answer

$V(x) = 4x^3 - 32x^2 + 64x$

70) A rectangle is twice as long as it is wide. Squares of side 2 feet are cut out from each corner. Then the sides are folded up to make an open box. Express the volume of the box as a function of the width (x).

Graphs of Polynomial Functions

Verbal

1) What is the difference between an x-intercept and a zero of a polynomial function f?

Answer

The x-intercept is where the graph of the function crosses the x-axis, and the zero of the function is the input value for which $f(x) = 0$.

2) If a polynomial function of degree n has n distinct zeros, what do you know about the graph of the function?

3) Explain how the Intermediate Value Theorem can assist us in finding a zero of a function.

Answer

If we evaluate the function at a and at b and the sign of the function value changes, then we know a zero exists between a and b.

4) Explain how the factored form of the polynomial helps us in graphing it.

5) If the graph of a polynomial just touches the x-axis and then changes direction, what can we conclude about the factored form of the polynomial?

Answer

There will be a factor raised to an even power.

Algebraic

For the exercises 6-23, find the x-or t-intercepts of the polynomial functions.

6) $C(t) = 2(t-4)(t+1)(t-6)$

7) $C(t) = 3(t+2)(t-3)(t+5)$

Answer

$(-2, 0), (3, 0), (-5, 0)$

8) $C(t) = 4t(t-2)^2(t+1)$

9) $C(t) = 2t(t-3)(t+1)^2$

Answer

$(3,0), (-1,0), (0,0)$

10) $C(t) = 2t^4 - 8t^3 + 6t^2$

11) $C(t) = 4t^4 + 12t^3 - 40t^2$

Answer

$(0,0), (-5,0), (2,0)$

12) $f(x) = x^4 - x^2$

13) $f(x) = x^3 + x^2 - 20x$

Answer

$(0,0), (-5,0), (4,0)$

14) $f(x) = x^3 + 6x^2 - 7x$

15) $f(x) = x^3 + x^2 - 4x - 4$

Answer

$(2,0), (-2,0), (-1,0)$

16) $f(x) = x^3 + 2x^2 - 9x - 18$

17) $f(x) = 2x^3 - x^2 - 8x + 4$

Answer

$(-2,0), (2,0), \left(\frac{1}{2}, 0\right)$

18) $f(x) = x^6 - 7x^3 - 8$

19) $f(x) = 2x^4 + 6x^2 - 8$

Answer

$(1,0), (-1,0)$

20) $f(x) = x^3 - 3x^2 - x + 3$

21) $f(x) = x^6 - 2x^4 - 3x^2$

Answer

$(0,0), (\sqrt{3},0), (-\sqrt{3},0)$

22) $f(x) = x^6 - 3x^4 - 4x^2$

23) $f(x) = x^5 - 5x^3 + 4x$

Answer

$(0,0),(1,0),(-1,0),(2,0),(-2,0)$

For the exercises 24-29, use the Intermediate Value Theorem to confirm that the given polynomial has at least one zero within the given interval.

24) $f(x)=x^3-9x$, between $x=-4$ and $x=-2$.

25) $f(x)=x^3-9x$, between $x=2$ and $x=4$.

Answer

$f(2)=-10$ and $f(4)=28$. Sign change confirms.

26) $f(x)=x^5-2x$, between $x=1$ and $x=2$.

27) $f(x)=-x^4+4$, between $x=1$ and $x=3$.

Answer

$f(1)=3$ and $f(3)=-77$. Sign change confirms.

28) $f(x)=-2x^3-x$, between $x=-1$ and $x=1$.

29) $f(x)=x^3-100x+2$, between $x=0.01$ and $x=0.1$

Answer

$f(0.01)=1.000001$ and $f(0.1)=-7.999$ Sign change confirms.

For the exercises 30-41, find the zeros and give the multiplicity of each.

30) $f(x)=(x+2)^3(x-3)^2$

31) $f(x)=x^2(2x+3)^5(x-4)^2$

Answer

0 with multiplicity 2, $-\frac{3}{2}$ with multiplicity 5, 4 with multiplicity 2

32) $f(x)=x^3(x-1)^3(x+2)$

33) $f(x)=x^2(x^2+4x+4)$

Answer

0 with multiplicity 2, -2 with multiplicity 2

34) $f(x)=(2x+1)^3(9x^2-6x+1)$

35) $f(x)=(3x+2)^5(x^2-10x+25)$

Answer

$-\frac{2}{3}$ with multiplicity 5, 5 with multiplicity 2

36) $f(x)=x(4x^2-12x+9)(x^2+8x+16)$

37) $f(x)=x^6-x^5-2x^4$

Answer

0 with multiplicity 4, 2 with multiplicity 1, -1 with multiplicity 1

38) $f(x) = 3x^4 + 6x^3 + 3x^2$

39) $f(x) = 4x^5 - 12x^4 + 9x^3$

Answer

$\frac{3}{2}$ with multiplicity 2, 0 with multiplicity 3

40) $f(x) = 2x^4(x^3 - 4x^2 + 4x)$

41) $f(x) = 4x^4(9x^4 - 12x^3 + 4x^2)$

Answer

0 with multiplicity 6, $\frac{2}{3}$ with multiplicity 2

Graphical

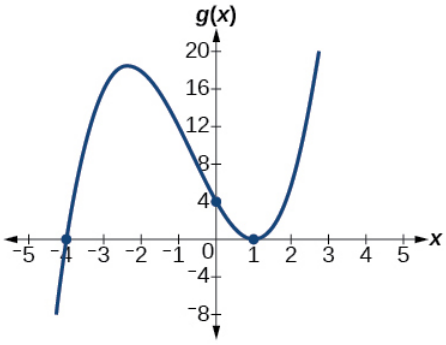
For the exercises 42-47, graph the polynomial functions. Note x- and y- intercepts, multiplicity, and end behavior.

42) $f(x) = (x + 3)^2(x - 2)$

43) $g(x) = (x + 4)(x - 1)^2$

Answer

x-intercepts, $(1, 0)$ with multiplicity 2, $(-4, 0)$ with multiplicity 1, y- intercept $(0, 4)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.



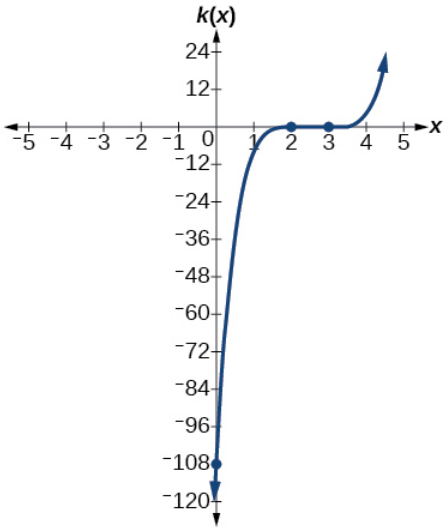
44) $h(x) = (x - 1)^3(x + 3)^2$

45) $k(x) = (x - 3)^3(x - 2)^2$

Answer

x-intercepts $(3, 0)$ with multiplicity 3, $(2, 0)$ with multiplicity 2, y- intercept $(0, -108)$. As $x \to -\infty$, $f(x) \to -\infty$, as $x \to \infty$, $f(x) \to \infty$.

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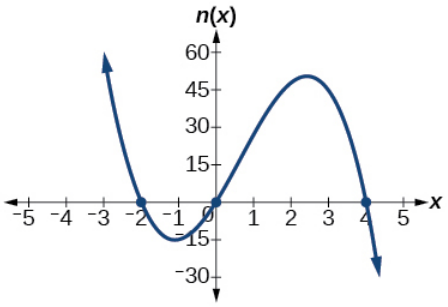


46) $m(x) = -2x(x-1)(x+3)$

47) $n(x) = -3x(x+2)(x-4)$

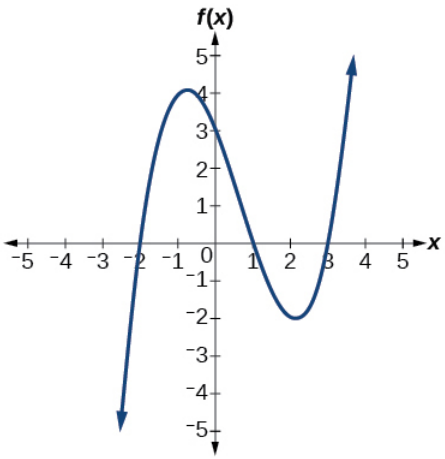
Answer

x-intercepts $(0,0), (-2,0), (4,0)$with multiplicity 1, y-intercept (0, 0). As $x \to -\infty,\ f(x) \to \infty$, as $x \to \infty,\ f(x) \to -\infty$.

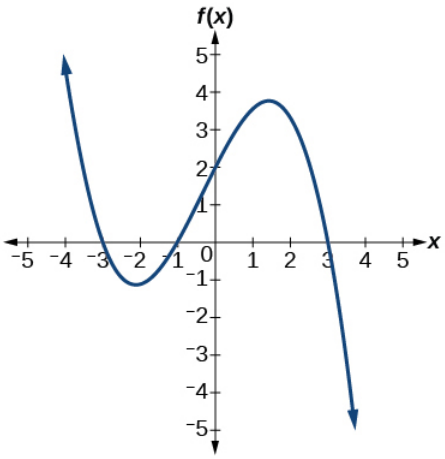


For the exercises 48-52, use the graphs to write the formula for a polynomial function of least degree.

48)



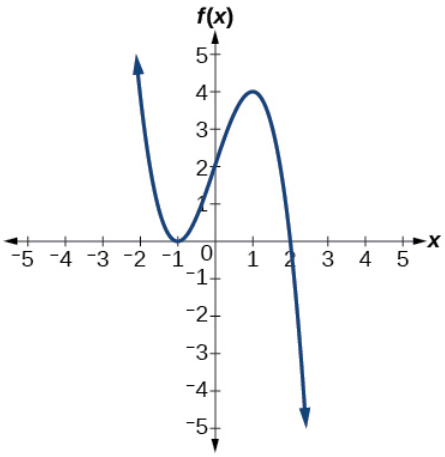
49)



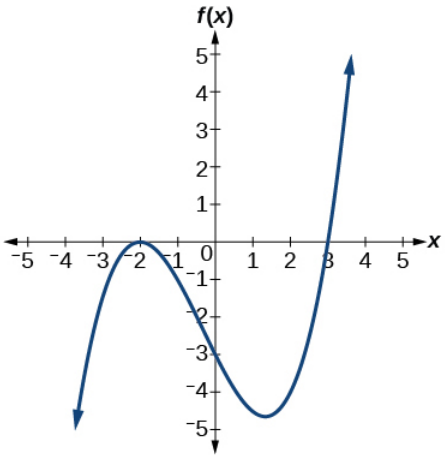
Answer

$f(x) = -\frac{2}{9}(x-3)(x+1)(x+3)$

50)



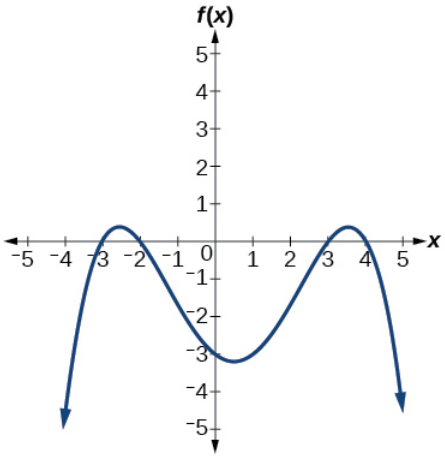
51)



Answer

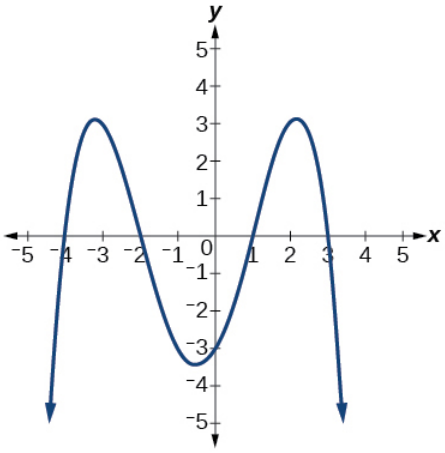
$$f(x) = \frac{1}{4}(x+2)^2(x-3)$$

52)



For the exercises 53-56, use the graph to identify zeros and multiplicity.

53)

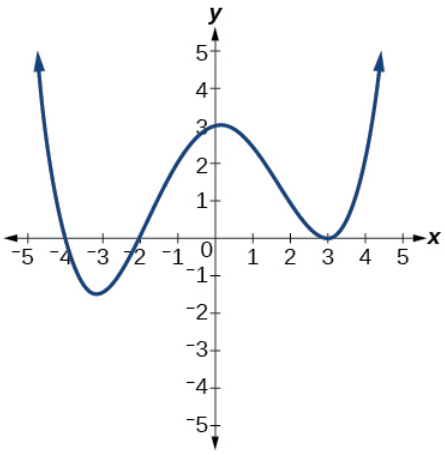


Answer

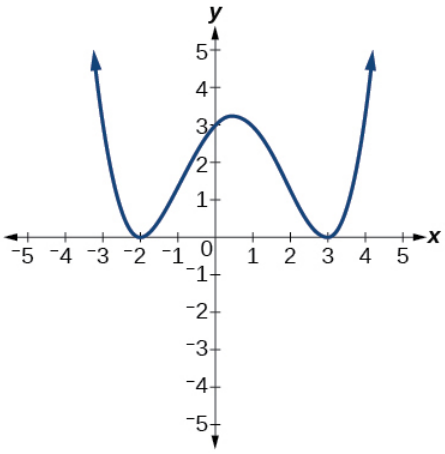
$-4, -2, 1, 3$with multiplicity 1

54)

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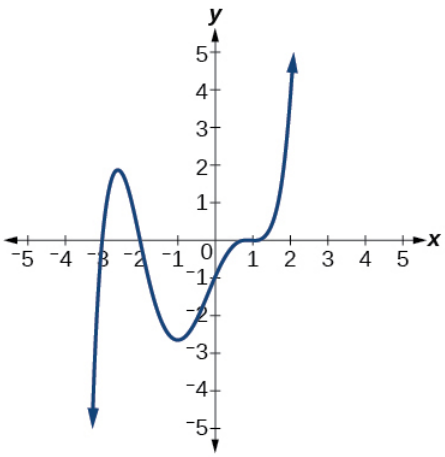
55)



Answer

$-2, 3$ each with multiplicity 2

56)



For the exercises 57-66, use the given information about the polynomial graph to write the equation.

57) Degree 3. Zeros at $x = -2, x = 1$, and $x = 3$. y-intercept at $(0, -4)$

Answer

$f(x) = -\frac{2}{3}(x+2)(x-1)(x-3)$

58) Degree 3. Zeros at $x = -5$, $x = -2$, and $x = 1$. y-intercept at $(0, 6)$

59) Degree 5. Roots of multiplicity 2 at $x = 3$ and $x = 1$, and a root of multiplicity 1 at $x = -3$. y-intercept at $(0, 9)$

Answer

$f(x) = \frac{1}{3}(x-3)^2(x-1)^2(x+3)$

60) Degree 4. Root of multiplicity 2 at $x = 4$, and a roots of multiplicity 1 at $x = 1$ and $x = -2$. y-intercept at $(0, -3)$

61) Degree 5. Double zero at $x = 1$, and triple zero at $x = 3$. Passes through the point $(2, 15)$

Answer

$f(x) = -15(x-1)^2(x-3)^3$

62) Degree 3. Zeros at $x = 4$, $x = 3$, and $x = 2$. y-intercept at $(0, -24)$

63) Degree 3. Zeros at $x = -3$, $x = -2$ and $x = 1$. y-intercept at $(0, 12)$

Answer

$f(x) = -2(x+3)(x+2)(x-1)$

64) Degree 5. Roots of multiplicity 2 at $x = -3$ and $x = 2$ and a root of multiplicity 1 at $x = -2$. y-intercept at $(0, 4)$.

65) Degree 4. Roots of multiplicity 2 at $x = \frac{1}{2}$ and roots of multiplicity 1 at $x = 6$ and $x = -2$. y-intercept at $(0, 18)$

Answer

$f(x) = -\frac{3}{2}(2x-1)^2(x-6)(x+2)$

66) Double zero at $x = -3$ and triple zero at $x = 0$. Passes through the point $(1, 32)$.

Technology

For the exercises 67-71, use a calculator to approximate local minima and maxima or the global minimum and maximum.

67) $f(x) = x^3 - x - 1$

Answer

local max $(-.58, -.62)$ local min $(.58, -1.38)$

68) $f(x) = 2x^3 - 3x - 1$

69) $f(x) = x^4 + x$

Answer

global min $(-.63, -.47)$

70) $f(x) = -x^4 + 3x - 2$

71) $f(x) = x^4 - x^3 + 1$

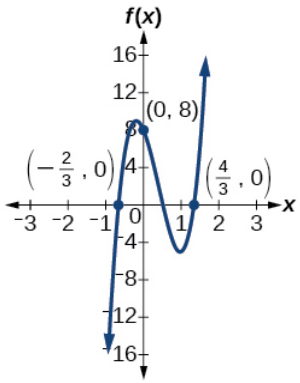
Answer

global min $(.75, .89)$

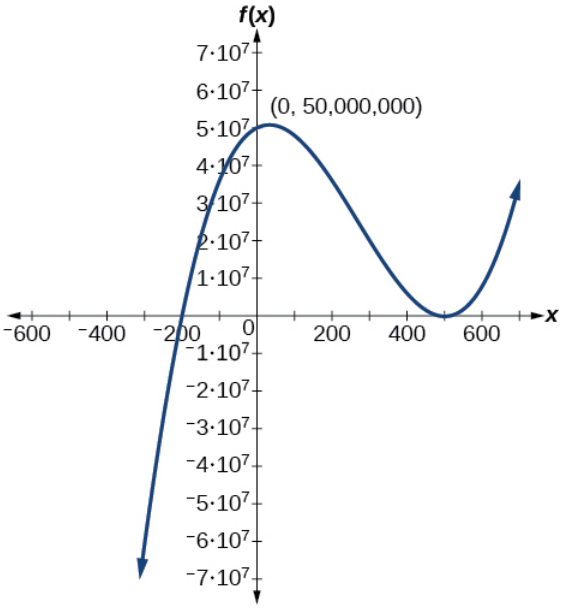
Extensions

For the exercises 72-74, use the graphs to write a polynomial function of least degree.

72)



73)

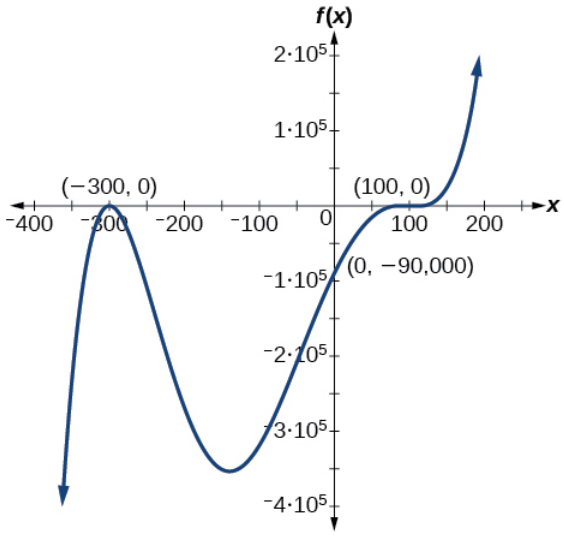


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Answer

$f(x) = (x - 500)^2(x + 200)$

74)



Real-World Applications

For the exercises 75-, write the polynomial function that models the given situation.

75) A rectangle has a length of 10 units and a width of 8 units. Squares of x by x units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a polynomial function in terms of x.

Answer

$f(x) = 4x^3 - 36x^2 + 80x$

76) Consider the same rectangle of the preceding problem. Squares of $2x$ by $2x$ units are cut out of each corner. Express the volume of the box as a polynomial in terms of x.

77) A square has sides of 12 units. Squares $x + 1$ by $x + 1$ units are cut out of each corner, and then the sides are folded up to create an open box. Express the volume of the box as a function in terms of x.

Answer

$f(x) = 4x^3 - 36x^2 + 60x + 100$

78) A cylinder has a radius of $x + 2$ units and a height of 3 units greater. Express the volume of the cylinder as a polynomial function.

79) A right circular cone has a radius of $3x + 6$ and a height 3 units less. Express the volume of the cone as a polynomial function. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$ for radius r and height h.

Answer

$f(x) = \pi(9x^3 + 45x^2 + 72x + 36)$

Dividing Polynomials

Verbal

1) If division of a polynomial by a binomial results in a remainder of zero, what can be conclude?

Answer

The binomial is a factor of the polynomial.

2) If a polynomial of degree n is divided by a binomial of degree 1, what is the degree of the quotient?

Algebraic

For the exercises 3-13, use long division to divide. Specify the quotient and the remainder.

3) $(x^2+5x-1)\div(x-1)$

Answer

$\mathrm{x}+6+\frac{5}{\mathrm{x}-1}$, quotient : $\mathrm{x}+6$, remainder : 5

4) $(2x^2-9x-5)\div(x-5)$

5) $(3x^2+23x+14)\div(x+7)$

Answer

$3\mathrm{x}+2$, quotient : $3\mathrm{x}+2$, remainder : 0

6) $(4x^2-10x+6)\div(4x+2)$

7) $(6x^2-25x-25)\div(6x+5)$

Answer

$\mathrm{x}-5$, quotient : $\mathrm{x}-5$, remainder : 0

8) $(-x^2-1)\div(x+1)$

9) $(2x^2-3x+2)\div(x+2)$

Answer

$2\mathrm{x}-7+\frac{16}{\mathrm{x}+2}$, quotient : $2\mathrm{x}-7$, remainder : 16

10) $(x^3-126)\div(x-5)$

11) $(3x^2-5x+4)\div(3x+1)$

Answer

$\mathrm{x}-2+\frac{6}{3\mathrm{x}+1}$, quotient : $\mathrm{x}-2$, remainder : 6

12) $(x^3-3x^2+5x-6)\div(x-2)$

13) $(2x^3 + 3x^2 - 4x + 15) \div (x + 3)$

Answer

$2x^2 - 3x + 5$, quotient : $2x^2 - 3x + 5$, remainder : 0

For the exercises 14-37, use synthetic division to find the quotient.

14) $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

15) $(2x^3 - 6x^2 - 7x + 6) \div (x - 4)$

Answer

$2x^2 + 2x + 1 + \dfrac{10}{x - 4}$

16) $(6x^3 - 10x^2 - 7x - 15) \div (x + 1)$

17) $(4x^3 - 12x^2 - 5x - 1) \div (2x + 1)$

Answer

$2x^2 - 7x + 1 - \dfrac{2}{2x + 1}$

18) $(9x^3 - 9x^2 + 18x + 5) \div (3x - 1)$

19) $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

Answer

$3x^2 - 11x + 34 - \dfrac{106}{x + 3}$

20) $(-6x^3 + x^2 - 4) \div (2x - 3)$

21) $(2x^3 + 7x^2 - 13x - 3) \div (2x - 3)$

Answer

$x^2 + 5x + 1$

22) $(3x^3 - 5x^2 + 2x + 3) \div (x + 2)$

23) $(4x^3 - 5x^2 + 13) \div (x + 4)$

Answer

$4x^2 - 21x + 84 - \dfrac{323}{x + 4}$

24) $(x^3 - 3x + 2) \div (x + 2)$

25) $(x^3 - 21x^2 + 147x - 343) \div (x - 7)$

Answer

$x^2 - 14x + 49$

26) $(x^3 - 15x^2 + 75x - 125) \div (x - 5)$

27) $(9x^3 - x + 2) \div (3x - 1)$

Answer

$3x^2 + x + \dfrac{2}{3x-1}$

28) $(6x^3 - x^2 + 5x + 2) \div (3x + 1)$

29) $(x^4 + x^3 - 3x^2 - 2x + 1) \div (x + 1)$

Answer

$x^3 - 3x + 1$

30) $(x^4 - 3x^2 + 1) \div (x - 1)$

31) $(x^4 + 2x^3 - 3x^2 + 2x + 6) \div (x + 3)$

Answer

$x^3 - x^2 + 2$

32) $(x^4 - 10x^3 + 37x^2 - 60x + 36) \div (x - 2)$

33) $(x^4 - 8x^3 + 24x^2 - 32x + 16) \div (x - 2)$

Answer

$x^3 - 6x^2 + 12x - 8$

34) $(x^4 + 5x^3 - 3x^2 - 13x + 10) \div (x + 5)$

35) $(x^4 - 12x^3 + 54x^2 - 108x + 81) \div (x - 3)$

Answer

$x^3 - 9x^2 + 27x - 27$

36) $(4x^4 - 2x^3 - 4x + 2) \div (2x - 1)$

37) $(4x^4 + 2x^3 - 4x^2 + 2x + 2) \div (2x + 1)$

Answer

$2x^3 - 2x + 2$

For the exercises 38-43, use synthetic division to determine whether the first expression is a factor of the second. If it is, indicate the factorization.

38) $x - 2, 4x^3 - 3x^2 - 8x + 4$

39) $x - 2, 3x^4 - 6x^3 - 5x + 10$

Answer

Yes $(x - 2)(3x^3 - 5)$

40) $x + 3, -4x^3 + 5x^2 + 8$

41) $x-2, 4x^4-15x^2-4$

Answer

Yes $(x-2)(4x^3+8x^2+x+2)$

42) $x-\frac{1}{2}, 2x^4-x^3+2x-1$

43) $x+\frac{1}{3}, 3x^4+x^3-3x+1$

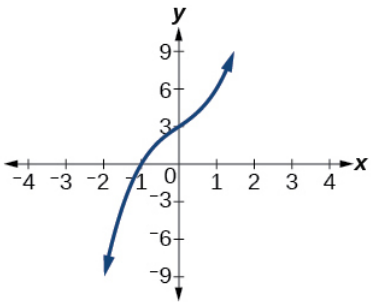
Answer

No

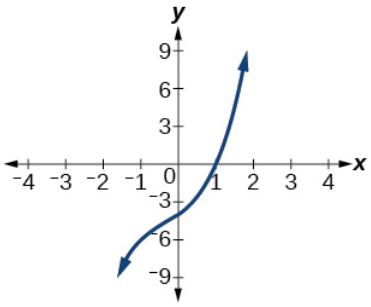
Graphical

For the exercises 44-48, use the graph of the third-degree polynomial and one factor to write the factored form of the polynomial suggested by the graph. The leading coefficient is one.

44) Factor is x^2-x+3



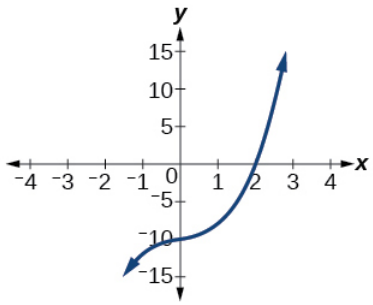
45) Factor is (x^2+2x+4)



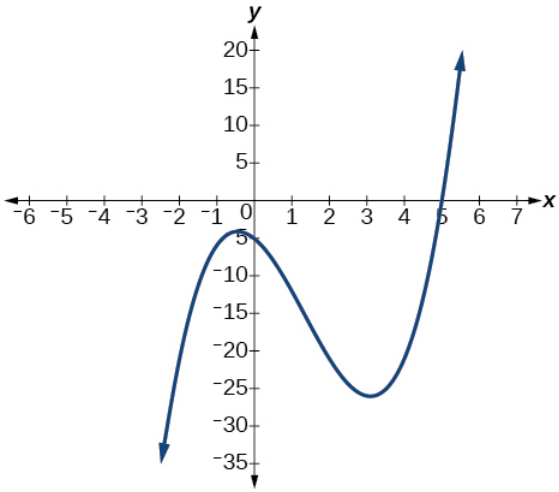
Answer

$(x-1)(x^2+2x+4)$

46) Factor is x^2+2x+5



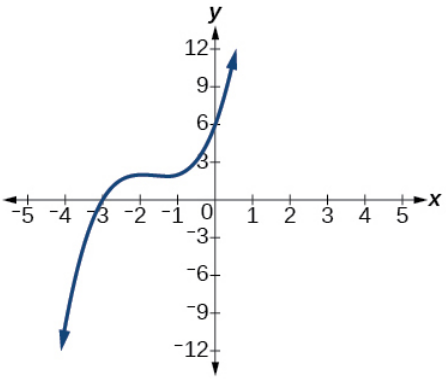
47) Factor is $x^2 + x + 1$



Answer

$(x-5)(x^2+x+1)$

48) Factor is $x^2 + 2x + 2$



For the exercises 49-53, use synthetic division to find the quotient and remainder.

49) $\dfrac{4x^3 - 33}{x - 2}$

Answer

Quotient : $4x^2 + 8x + 16$, remainder : -1

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50) $\dfrac{2x^3+25}{x+3}$

51) $\dfrac{3x^3+2x-5}{x-1}$

Answer

Quotient : $3x^2+3x+5$, remainder : 0

52) $\dfrac{-4x^3-x^2-12}{x+4}$

53) $\dfrac{x^4-22}{x+2}$

Answer

Quotient : x^3-2x^2+4x-8, remainder : -6

Technology

For the exercises 54-58, use a calculator with CAS to answer the questions.

54) Consider $\dfrac{x^k-1}{x-1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

55) Consider $\dfrac{x^k+1}{x+1}$ for $k=1,3,5$. What do you expect the result to be if $k=7$?

Answer

$x^6-x^5+x^4-x^3+x^2-x+1$

56) Consider $\dfrac{x^4-k^4}{x-k}$ for $k=1,2,3$. What do you expect the result to be if $k=4$?

57) Consider $\dfrac{x^k}{x+1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

Answer

$x^3-x^2+x-1+\dfrac{1}{x+1}$

58) Consider $\dfrac{x^k}{x-1}$ with $k=1,2,3$. What do you expect the result to be if $k=4$?

Extensions

For the exercises 59-63, use synthetic division to determine the quotient involving a complex number.

59) $\dfrac{x+1}{x-i}$

Answer

$1+\dfrac{1+i}{x-i}$

60) $\dfrac{x^2+1}{x-i}$

61) $\dfrac{x+1}{x+i}$

Answer

$1+\dfrac{1-i}{x+i}$

62) $\dfrac{x^2+1}{x+i}$

63) $\dfrac{x^3+1}{x-i}$

Answer

$x^2-ix-1+\dfrac{1-i}{x-i}$

Real-World Applications

For the exercises 64-66, use the given length and area of a rectangle to express the width algebraically.

64) Length is $x+5$, area is $2x^2+9x-5$.

65) Length is $2x+5$, area is $4x^3+10x^2+6x+15$.

Answer

$2x^2+3$

66) Length is $3x-4$, area is $6x^4-8x^3+9x^2-9x-4$.

For the exercises 67-70, use the given volume of a box and its length and width to express the height of the box algebraically.

67) Volume is $12x^3+20x^2-21x-36$, length is $2x+3$, width is $3x-4$.

Answer

$2x+3$

68) Volume is $18x^3-21x^2-40x+48$, length is $3x-4$, width is $3x-4$.

69) Volume is $10x^3+27x^2+2x-24$, length is $5x-4$, width is $2x+3$.

Answer

$x+2$

70) Volume is $10x^3+30x^2-8x-24$, length is 2, width is $x+3$.

For the exercises 71-73, use the given volume and radius of a cylinder to express the height of the cylinder algebraically.

71) Volume is $\pi(25x^3 - 65x^2 - 29x - 3)$, radius is $5x + 1$.

Answer

$x - 3$

72) Volume is $\pi(4x^3 + 12x^2 - 15x - 50)$, radius is $2x + 5$.

73) Volume is $\pi(3x^4 + 24x^3 + 46x^2 - 16x - 32)$, radius is $x + 4$.

Answer

$3x^2 - 2$

Zeros of Polynomial Functions

Verbal

1) Describe a use for the Remainder Theorem.

Answer

The theorem can be used to evaluate a polynomial.

2) Explain why the Rational Zero Theorem does not guarantee finding zeros of a polynomial function.

3) What is the difference between rational and real zeros?

Answer

Rational zeros can be expressed as fractions whereas real zeros include irrational numbers.

4) If Descartes' Rule of Signs reveals a no change of signs or one sign of changes, what specific conclusion can be drawn?

5) If synthetic division reveals a zero, why should we try that value again as a possible solution?

Answer

Polynomial functions can have repeated zeros, so the fact that number is a zero doesn't preclude it being a zero again.

Algebraic

For the exercises 6-13, use the Remainder Theorem to find the remainder.

6) $(x^4 - 9x^2 + 14) \div (x - 2)$

7) $(3x^3 - 2x^2 + x - 4) \div (x + 3)$

Answer

-106

8) $(x^4 + 5x^3 - 4x - 17) \div (x + 1)$

9) $(-3x^2+6x+24) \div (x-4)$

Answer

0

10) $(5x^5-4x^4+3x^3-2x^2+x-1) \div (x+6)$

11) $(x^4-1) \div (x-4)$

Answer

255

12) $(3x^3+4x^2-8x+2) \div (x-3)$

13) $(4x^3+5x^2-2x+7) \div (x+2)$

Answer

-1

For the exercises 14-21, use the Factor Theorem to find all real zeros for the given polynomial function and one factor.

14) $f(x)=2x^3-9x^2+13x-6; x-1$

15) $f(x)=2x^3+x^2-5x+2; x+2$

Answer

$-2, 1, \frac{1}{2}$

16) $f(x)=3x^3+x^2-20x+12; x+3$

17) $f(x)=2x^3+3x^2+x+6; x+2$

Answer

-2

18) $f(x)=-5x^3+16x^2-9; x-3$

19) $x^3+3x^2+4x+12; x+3$

Answer

-3

20) $4x^3-7x+3; x-1$

21) $2x^3+5x^2-12x-30, 2x+5$

Answer

$-\frac{5}{2}, \sqrt{6}, -\sqrt{6}$

For the exercises 22-39, use the Rational Zero Theorem to find all real zeros.

22) $x^3 - 3x^2 - 10x + 24 = 0$

23) $2x^3 + 7x^2 - 10x - 24 = 0$

Answer

$2, -4, -\frac{3}{2}$

24) $x^3 + 2x^2 - 9x - 18 = 0$

25) $x^3 + 5x^2 - 16x - 80 = 0$

Answer

$4, -4, -5$

26) $x^3 - 3x^2 - 25x + 75 = 0$

27) $2x^3 - 3x^2 - 32x - 15 = 0$

Answer

$5, -3, -\frac{1}{2}$

28) $2x^3 + x^2 - 7x - 6 = 0$

29) $2x^3 - 3x^2 - x + 1 = 0$

Answer

$\frac{1}{2}, \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

30) $3x^3 - x^2 - 11x - 6 = 0$

31) $2x^3 - 5x^2 + 9x - 9 = 0$

Answer

$\frac{3}{2}$

32) $2x^3 - 3x^2 + 4x + 3 = 0$

33) $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$

Answer

$2, 3, -1, -2$

34) $x^4 + 2x^3 - 9x^2 - 2x + 8 = 0$

35) $4x^4 + 4x^3 - 25x^2 - x + 6 = 0$

Answer

$\frac{1}{2}, -\frac{1}{2}, 2, -3$

36) $2x^4 - 3x^3 - 15x^2 + 32x - 12 = 0$

37) $x^4 + 2x^3 - 4x^2 - 10x - 5 = 0$

Answer

$-1, -1, \sqrt{5}, -\sqrt{5}$

38) $4x^3 - 3x + 1 = 0$

39) $8x^4 + 26x^3 + 39x^2 + 26x + 6$

Answer

$-\frac{3}{4}, -\frac{1}{2}$

For the exercises 40-45, find all complex solutions (real and non-real).

40) $x^3 + x^2 + x + 1 = 0$

41) $x^3 - 8x^2 + 25x - 26 = 0$

Answer

$2, 3 + 2i, 3 - 2i$

42) $x^3 + 13x^2 + 57x + 85 = 0$

43) $3x^3 - 4x^2 + 11x + 10 = 0$

Answer

$-\frac{2}{3}, 1 + 2i, 1 - 2i$

44) $x^4 + 2x^3 + 22x^2 + 50x - 75 = 0$

45) $2x^3 - 3x^2 + 32x + 17 = 0$

Answer

$-\frac{1}{2}, 1 + 4i, 1 - 4i$

Graphical

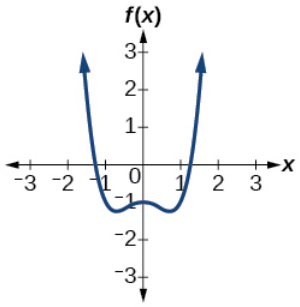
For the exercises 46-55, use Descartes' Rule to determine the possible number of positive and negative solutions. Confirm with the given graph.

46) $f(x) = x^3 - 1$

47) $f(x) = x^4 - x^2 - 1$

Answer

1 positive, 1 negative

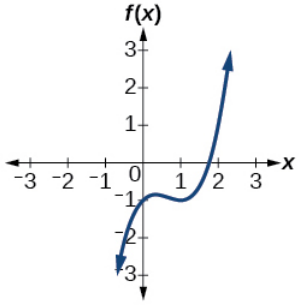


48) $f(x) = x^3 - 2x^2 - 5x + 6$

49) $f(x) = x^3 - 2x^2 + x - 1$

Answer

3 or 1 positive, 0 negative

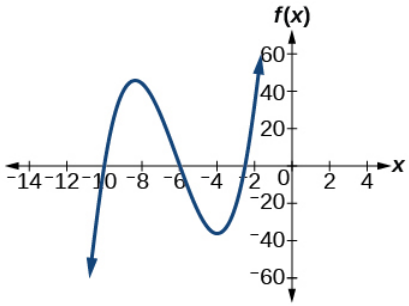


50) $f(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$

51) $f(x) = 2x^3 + 37x^2 + 200x + 300$

Answer

0 positive, 3 or 1 negative

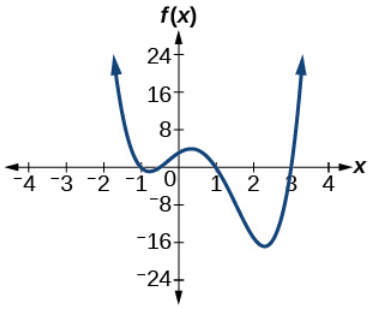


52) $f(x) = x^3 - 2x^2 - 16x + 32$

53) $f(x) = 2x^4 - 5x^3 - 5x^2 + 5x + 3$

Answer

2 or 0 positive, 2 or 0 negative

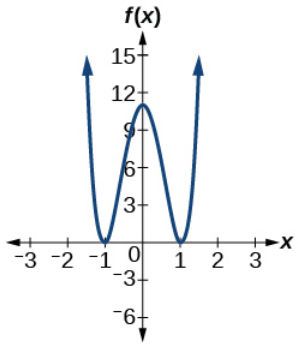


54) $f(x) = 2x^4 - 5x^3 - 14x^2 + 20x + 8$

55) $f(x) = 10x^4 - 21x^2 + 11$

Answer

2 or 0 positive, 2 or 0 negative



Numeric

For the exercises 56-60, list all possible rational zeros for the functions.

56) $f(x) = x^4 + 3x^3 - 4x + 4$

57) $f(x) = 2x^3 + 3x^2 - 8x + 5$

Answer

$\pm 5, \pm 1, \pm \frac{5}{2}$

58) $f(x) = 3x^3 + 5x^2 - 5x + 4$

59) $f(x) = 6x^4 - 10x^2 + 13x + 1$

Answer

$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}$

60) $f(x) = 4x^5 - 10x^4 + 8x^3 + x^2 - 8$

Technology

For the exercises 61-65, use your calculator to graph the polynomial function. Based on the graph, find the rational zeros. All real solutions are rational.

61) $f(x) = 6x^3 - 7x^2 + 1$

Answer

$1, \frac{1}{2}, -\frac{1}{3}$

62) $f(x) = 4x^3 - 4x^2 - 13x - 5$

63) $f(x) = 8x^3 - 6x^2 - 23x + 6$

Answer

$2, \frac{1}{4}, -\frac{3}{2}$

64) $f(x) = 12x^4 + 55x^3 + 12x^2 - 117x + 54$

65) $f(x) = 16x^4 - 24x^3 + x^2 - 15x + 25$

Answer

$\frac{5}{4}$

Extensions

For the exercises 66-70, construct a polynomial function of least degree possible using the given information.

66) Real roots: $-1, 1, 3$ and $(2, f(2)) = (2, 4)$

67) Real roots: -1 (with multiplicity 2 and 1) and $(2, f(2)) = (2, 4)$

Answer

$f(x) = \frac{4}{9}(x^3 + x^2 - x - 1)$

68) Real roots: $-2, \frac{1}{2}$ (with multiplicity 2) and $(-3, f(-3)) = (-3, 5)$

69) Real roots: $-\frac{1}{2}, 0, \frac{1}{2}$ and $(-2, f(-2)) = (-2, 6)$

Answer

$f(x) = -\frac{1}{5}(4x^3 - x)$

70) Real roots: $-4, -1, 1, 4$ and $(-2, f(-2)) = (-2, 10)$

Real-World Applications

For the exercises 71-75, find the dimensions of the box described.

71) The length is twice as long as the width. The height is 2 inches greater than the width. The volume is 192 cubic inches.

Answer

8 by 4 by 6 inches

72) The length, width, and height are consecutive whole numbers. The volume is 120cubic inches.

73) The length is one inch more than the width, which is one inch more than the height. The volume is 86.625 cubic inches.

Answer

5.5 by 4.5 by 3.5 inches

74) The length is three times the height and the height is one inch less than the width. The volume is 108 cubic inches.

75) The length is 3 inches more than the width. The width is 2 inches more than the height. The volume is 120 cubic inches.

Answer

8 by 5 by 3 inches

For the exercises 76-80, find the dimensions of the right circular cylinder described.

76) The radius is 3 inches more than the height. The volume is 16π cubic meters.

77) The height is one less than one half the radius. The volume is 72π cubic meters.

Answer

Radius = 6 meters, Height = 2 meters

78) The radius and height differ by one meter. The radius is larger and the volume is 48π cubic meters.

79) The radius and height differ by two meters. The height is greater and the volume is 28.125π cubic meters.

Answer

Radius = 2.5 meters, Height = 4.5 meters

80) The radius is $\frac{1}{3}$ meter greater than the height. The volume is $\frac{98}{9}\pi$ cubic meters.

Rational Functions

Verbal

1) What is the fundamental difference in the algebraic representation of a polynomial function and a rational function?

Answer

The rational function will be represented by a quotient of polynomial functions.

2) What is the fundamental difference in the graphs of polynomial functions and rational functions?

3) If the graph of a rational function has a removable discontinuity, what must be true of the functional rule?

Answer

The numerator and denominator must have a common factor.

4) Can a graph of a rational function have no vertical asymptote? If so, how?

5) Can a graph of a rational function have no x-intercepts? If so, how?

Answer

Yes. The numerator of the formula of the functions would have only complex roots and/or factors common to both the numerator and denominator.

Algebraic

For the exercises 6-9, find the domain of the rational functions.

6) $f(x) = \dfrac{x-1}{x+2}$

7) $f(x) = \dfrac{x+1}{x^2-1}$

Answer

All reals $x \neq -1, 1$

8) $f(x) = \dfrac{x^2+4}{x^2-2x-8}$

9) $f(x) = \dfrac{x^2+4x-3}{x^4-5x^2+4}$

Answer

All reals $x \neq -1, -2, 1, 2$

For the exercises 10-19, find the domain, vertical asymptotes, and horizontal asymptotes of the functions.

10) $f(x) = \dfrac{4}{x-1}$

11) $f(x) = \dfrac{2}{5x+2}$

Answer

V.A. at $x = -\dfrac{2}{5}$; H.A. at $y = 0$; Domain is all reals $x \neq -\dfrac{2}{5}$

12) $f(x) = \dfrac{x}{x^2-9}$

13) $f(x) = \dfrac{x}{x^2+5x-36}$

Answer

V.A. at $x = 4, -9$; H.A. at $y = 0$; Domain is all reals $x \neq 4, -9$

14) $f(x)=\dfrac{3+x}{x^3-27}$

15) $f(x)=\dfrac{3x-4}{x^3-16x}$

Answer

V.A. at $x=0,4,-4$; H.A. at $y=0$; Domain is all reals $x\neq 0,4,-4$

16) $f(x)=\dfrac{x^2-1}{x^3+9x^2+14x}$

17) $f(x)=\dfrac{x+5}{x^2-25}$

Answer

V.A. at $x=-5$; H.A. at $y=0$; Domain is all reals $x\neq 5,-5$

18) $f(x)=\dfrac{x-4}{x-6}$

19) $f(x)=\dfrac{4-2x}{3x-1}$

Answer

V.A. at $x=\dfrac{1}{3}$; H.A. at $y=-\dfrac{2}{3}$; Domain is all reals $x\neq\dfrac{1}{3}$

For the exercises 20-24, find the x- and y-intercepts for the functions.

20) $f(x)=\dfrac{x+5}{x^2+4}$

21) $f(x)=\dfrac{x}{x^2-x}$

Answer

none

22) $f(x)=\dfrac{x^2+8x+7}{x^2+11x+30}$

23) $f(x)=\dfrac{x^2+x+6}{x^2-10x+24}$

Answer

x -intercepts none, y -intercept $\left(0,\dfrac{1}{4}\right)$

24) $f(x)=\dfrac{94-2x^2}{3x^2-12}$

For the exercises 25-29, describe the local and end behavior of the functions.

25) $f(x)=\dfrac{x}{2x+1}$

Answer

Local behavior: $x\to-\dfrac{1}{2}^{+}, f(x)\to-\infty, x\to-\dfrac{1}{2}^{-}, f(x)\to\infty$

End behavior: $x\to\pm\infty, f(x)\to\dfrac{1}{2}$

26) $f(x)=\dfrac{2x}{x-6}$

27) $f(x)=\dfrac{-2x}{x-6}$

Answer

Local behavior: $x\to 6^{+}, f(x)\to -\infty, x\to 6^{-}, f(x)\to \infty$

End behavior: $x\to \pm\infty, f(x)\to -2$

28) $f(x)=\dfrac{x^2-4x+3}{x^2-4x-5}$

29) $f(x)=\dfrac{2x^2-32}{6x^2+13x-5}$

Answer

Local behavior: $x\to -\dfrac{1}{3}, f(x)\to \infty, x\to -\dfrac{1}{3}, f(x)\to -\infty, x\to \dfrac{5}{2}, f(x)\to \infty, x\to \dfrac{5}{2}+f(x)\to -\infty$

End behavior: $x\to \pm\infty, f(x)\to \dfrac{1}{3}$

For the exercises 30-34, find the slant asymptote of the functions.

30) $f(x)=\dfrac{24x^2+6x}{2x+1}$

31) $f(x)=\dfrac{4x^2-10}{2x-4}$

Answer

$y=2x+4$

32) $f(x)=\dfrac{81x^2-18}{3x-2}$

33) $f(x)=\dfrac{6x^3-5x}{3x^2+4}$

Answer

$y=2x$

34) $f(x)=\dfrac{x^2+5x+4}{x-1}$

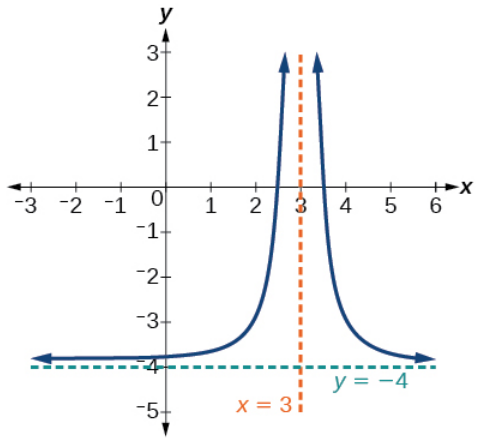
Graphical

For the exercises 35-38, use the given transformation to graph the function. Note the vertical and horizontal asymptotes.

35) The reciprocal function shifted up two units.

Answer

V.A. $x=0$, H.A. $y=2$

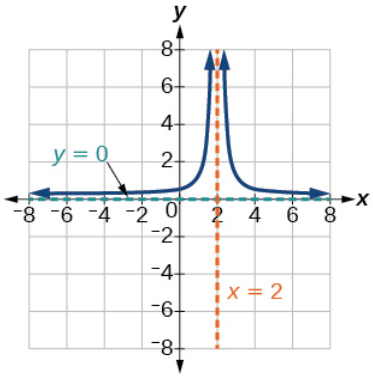


36) The reciprocal function shifted down one unit and left three units.

37) The reciprocal squared function shifted to the right 2 units.

Answer

V.A. $x = 2$, H.A. $y = 0$



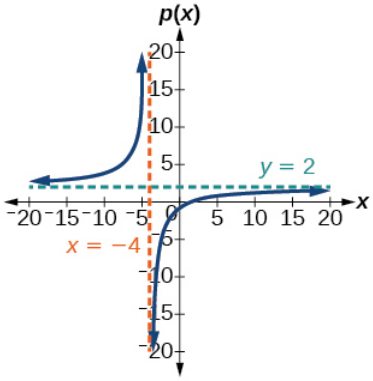
38) The reciprocal squared function shifted down 2 units and right 1 unit.

For the exercises 39-50, find the horizontal intercepts, the vertical intercept, the vertical asymptotes, and the horizontal or slant asymptote of the functions. Use that information to sketch a graph.

39) $p(x) = \dfrac{2x-3}{x+4}$

Answer

V.A. $x = -4$, H.A. $y = 2$; $\left(\dfrac{3}{2}, 0\right)$; $\left(0, -\dfrac{3}{4}\right)$

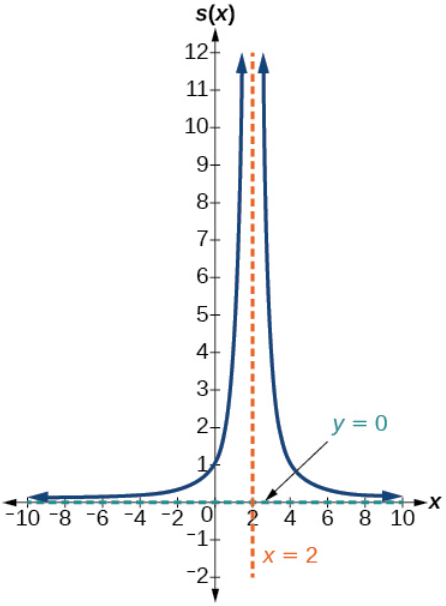


40) $q(x) = \dfrac{x-5}{3x-1}$

41) $s(x) = \dfrac{4}{(x-2)^2}$

Answer

V.A. $x = 2$, H.A. $y = 0$, $(0, 1)$



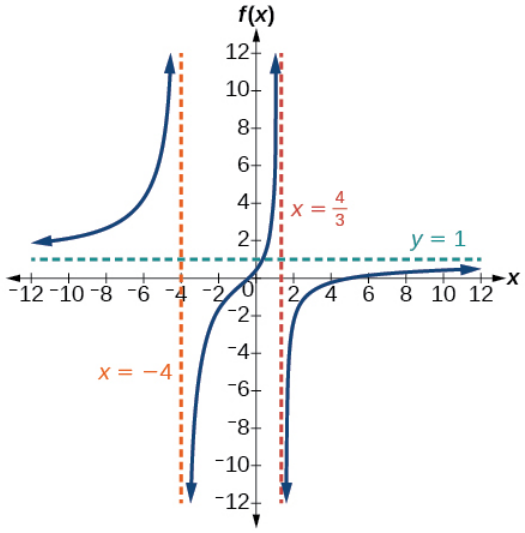
42) $r(x) = \dfrac{5}{(x+1)^2}$

43) $f(x) = \dfrac{3x^2-14x-5}{3x^2+8x-16}$

Answer

V.A. $x = -4$, $x = \dfrac{4}{3}$, H.A. $y = 1$; $(5, 0)$; $\left(-\dfrac{1}{3}, 0\right)$; $\left(0, \dfrac{5}{16}\right)$

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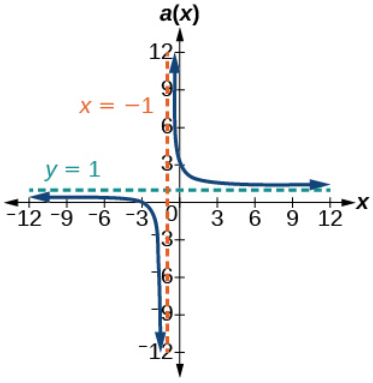


44) $g(x) = \dfrac{2x^2+7x-15}{3x^2-14+15}$

45) $a(x) = \dfrac{x^2+2x-3}{x^2-1}$

Answer

V.A. $x=-1$, H.A. $y=1$; $(-3,0)$; $(0,3)$



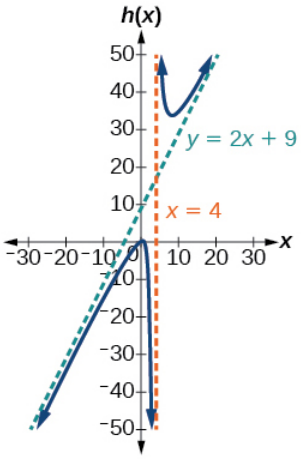
46) $b(x) = \dfrac{x^2-x-6}{x^2-4}$

47) $h(x) = \dfrac{2x^2+x-1}{x-4}$

Answer

V.A. $x=4$, S.A. $y=2x+9$; $(-1,0)$; $\left(\dfrac{1}{2},0\right)$; $\left(0,\dfrac{1}{4}\right)$

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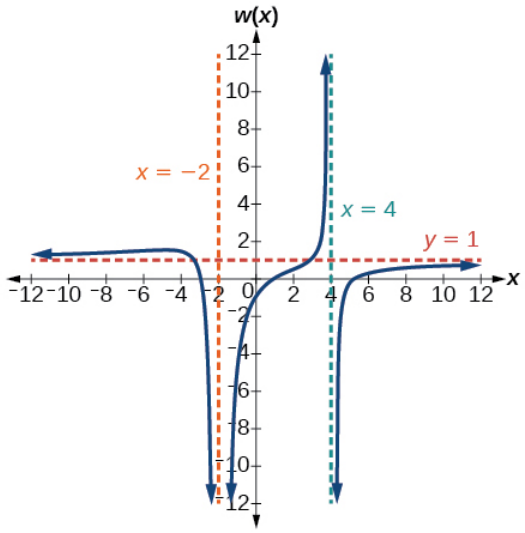


48) $k(x) = \dfrac{2x^2 - 3x - 20}{x - 5}$

49) $w(x) = \dfrac{(x-1)(x+3)(x-5)}{(x+2)^2(x-4)}$

Answer

V.A. $x = -2$, $x = 4$, H.A. $y = 1$, $(1, 0)$; $(5, 0)$; $(-3, 0)$; $\left(0, -\dfrac{15}{16}\right)$



50) $z(x) = \dfrac{(x+2)^2(x-5)}{(x-3)(x+1)(x+4)}$

For the exercises 51-56, write an equation for a rational function with the given characteristics.

51) Vertical asymptotes at $x = 5$, x -intercepts at $(2, 0)$ and $(-1, 0)$, y-intercept at $(0, 4)$

Answer

$y = 50\dfrac{x^2 - x - 2}{x^2 - 25}$

52) Vertical asymptotes at $x = -4$ and $x = -1$, x -intercepts at $(1, 0)$ and $(5, 0)$, y-intercept at $(0, 7)$

53) Vertical asymptotes at $x = -4$ and $x = -5$, x -intercepts at $(4, 0)$ and $(-6, 0)$, Horizontal asymptote at $y = 7$

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Answer

$$y = 7\frac{x^2+2x-24}{x^2+9x+20}$$

54) Vertical asymptotes at $x=-3$ and $x=6$, x -intercepts at $(-2,0)$ and $(1,0)$, Horizontal asymptote at $y=-2$

55) Vertical asymptote at $x=-1$, Double zero at $x=2$, y -intercept at $(0,2)$

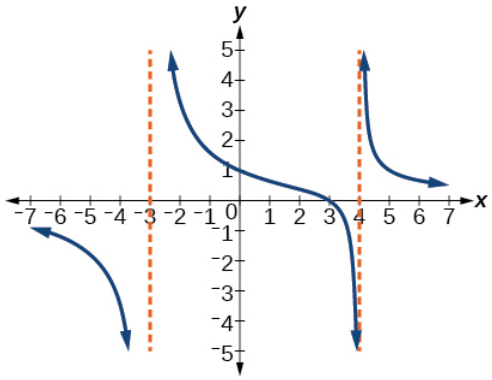
Answer

$$y = \frac{1}{2}\frac{x^2-4x+4}{x+1}$$

56) Vertical asymptote at $x=3$, Double zero at $x=1$, y -intercept at $(0,4)$

For the exercises 57-,64 use the graphs to write an equation for the function.

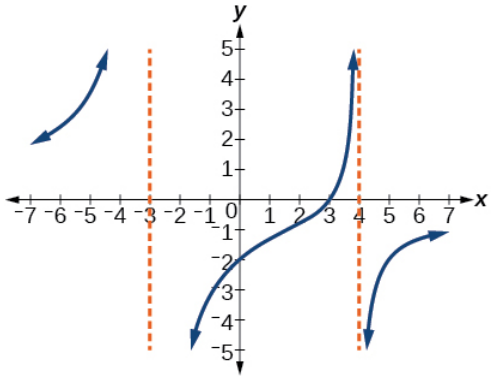
57)



Answer

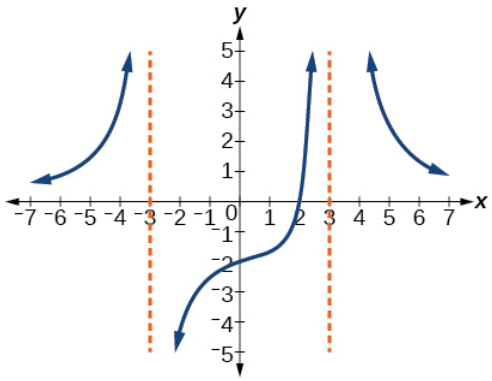
$$y = 4\frac{x-3}{x^2-x-12}$$

58)



59)

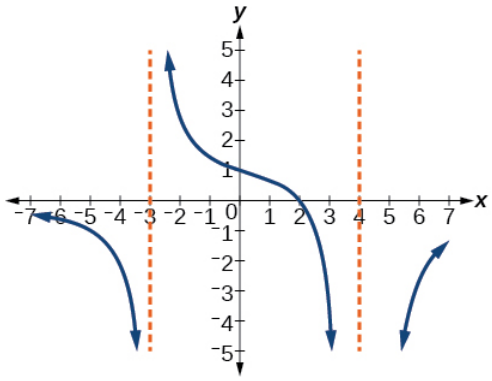
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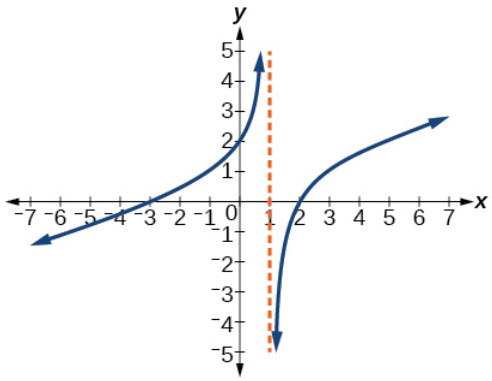
Answer

$$y = -9\frac{x-2}{x^2-9}$$

60)



61)

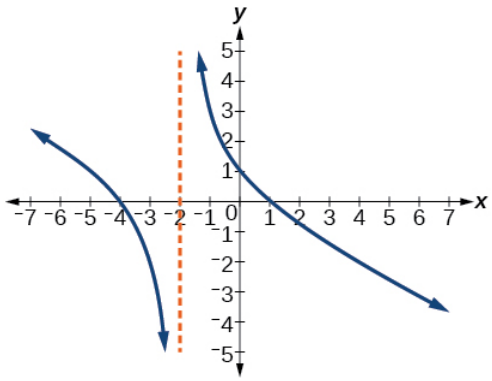


Answer

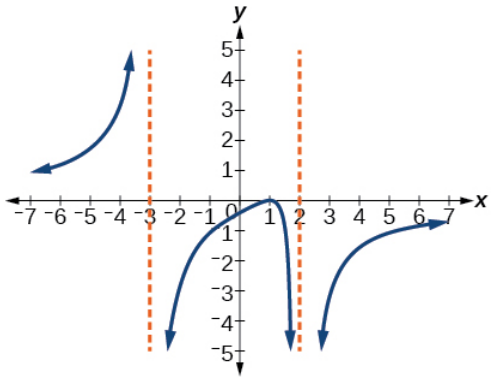
$$y = \frac{1}{3}\frac{x^2+x-6}{x-1}$$

62)

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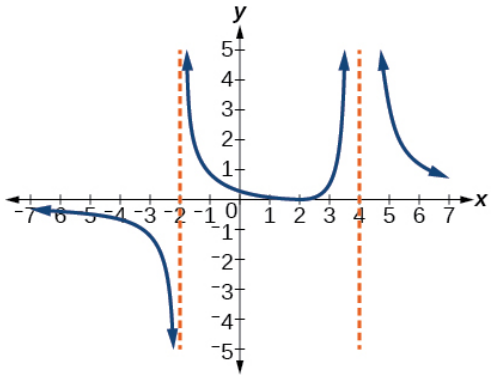
63)



Answer

$$y = -6\frac{(x-1)^2}{(x+3)(x-2)^2}$$

64)



Numeric

For the exercises 65-69, make tables to show the behavior of the function near the vertical asymptote and reflecting the horizontal asymptote

65) $f(x) = \dfrac{1}{x-2}$

Answer

x	2.01	2.001	2.0001	1.99	1.999
y	100	1,000	10,000	–100	–1,000

x	10	100	1,000	10,000	100,000
y	.125	.0102	.001	.0001	.00001

Vertical asymptote $x = 2$, Horizontal asymptote $y = 0$

66) $f(x) = \dfrac{x}{x-3}$

67) $f(x) = \dfrac{2x}{x+4}$

Answer

x	–4.1	–4.01	–4.001	–3.99	–3.999
y	82	802	8,002	–798	–7998

x	10	100	1,000	10,000	100,000
y	1.4286	1.9331	1.992	1.9992	1.999992

Vertical asymptote $x = -4$, Horizontal asymptote $y = 2$

68) $f(x) = \dfrac{2x}{(x-3)^2}$

69) $f(x) = \dfrac{x^2}{x^2+2x+1}$

Answer

x	–.9	–.99	–.999	–1.1	–1.01
y	81	9,801	998,001	121	10,201

x	10	100	1,000	10,000	100,000
y	.82645	.9803	.998	.9998	

Vertical asymptote $x = -1$ Horizontal asymptote $y = 1$

Technology

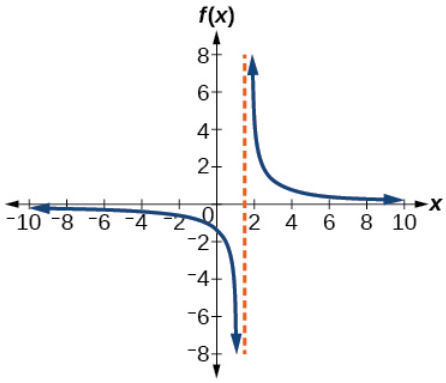
For the exercises 70-74, use a calculator to graph $f(x)$. Use the graph to solve $f(x) > 0$.

70) $f(x) = \dfrac{2}{x+1}$

71) $f(x) = \dfrac{4}{2x-3}$

Answer

$\left(\dfrac{3}{2}, \infty\right)$

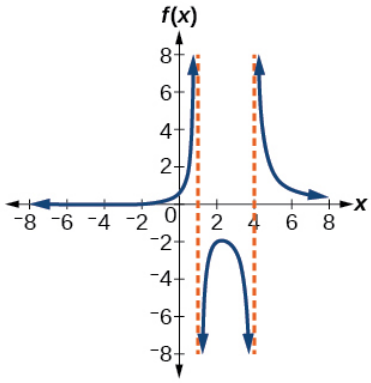


72) $f(x) = \dfrac{2}{(x-1)(x+2)}$

73) $f(x) = \dfrac{x+2}{(x-1)(x-4)}$

Answer

$(-2, 1) \cup (4, \infty)$



74) $f(x) = \dfrac{(x+3)^2}{(x-1)^2(x+1)}$

Extensions

For the exercises 75-79, identify the removable discontinuity.

75) $f(x) = \dfrac{x^2-4}{x-2}$

Answer

$(2, 4)$

76) $f(x) = \dfrac{x^3+1}{x+1}$

77) $f(x) = \dfrac{x^2+x-6}{x-2}$

Answer

$(2, 5)$

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78) $f(x)=\dfrac{2x^2+5x-3}{x+3}$

79) $f(x)=\dfrac{x^3+x^2}{x+1}$

Answer

$(-1,1)$

Real-World Applications

For the exercises 80-81, express a rational function that describes the situation.

80) A large mixing tank currently contains 200 gallons of water, into which 10 pounds of sugar have been mixed. A tap will open, pouring 10 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 3 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

81) A large mixing tank currently contains 300 gallons of water, into which 8 pounds of sugar have been mixed. A tap will open, pouring 20 gallons of water per minute into the tank at the same time sugar is poured into the tank at a rate of 2 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after t minutes.

Answer

$C(t)=\dfrac{8+2t}{300+20t}$

For the exercises 82-83, use the given rational function to answer the question.

82) The concentration C of a drug in a patient's bloodstream t hours after injection in given by $C(t)=\dfrac{2t}{3+t^2}$. What happens to the concentration of the drug as t increases?

83) The concentration C of a drug in a patients bloodstream, hours after injection is given by $C(t)=\dfrac{100t}{2t^2+75}$. Use a calculator to approximate the time when the concentration is highest.

Answer

After about 6.12hours.

For the exercises 84-88, construct a rational function that will help solve the problem. Then, use a calculator to answer the question.

84) An open box with a square base is to have a volume of 108 cubic inches. Find the dimensions of the box that will have minimum surface area. Let x = length of the side of the base.

85) A rectangular box with a square base is to have a volume of 20 cubic feet. The material for the base costs 30 cents/ square foot. The material for the sides costs 10 cents/square foot. The material for the top costs 20 cents/square foot. Determine the dimensions that will yield minimum cost. Let x = length of the side of the base.

Answer

$A(x)=50x^2+\dfrac{800}{x}\cdot 2$ by 2 by 5 feet.

86) A right circular cylinder has volume of 100 cubic inches. Find the radius and height that will yield minimum surface area. Let x = radius.

87) A right circular cylinder with no top has a volume of 50 cubic meters. Find the radius that will yield minimum surface area. Let x = radius.

Answer

$A(x)=\pi x^2+\dfrac{100}{x}$. Radius = 2.52meters.

88) A right circular cylinder is to have a volume of 40 cubic inches. It costs 4 cents/square inch to construct the top and bottom and 1 cent/square inch to construct the rest of the cylinder. Find the radius to yield minimum cost. Let x = radius.

Inverses and Radical Functions

Verbal

1) Explain why we cannot find inverse functions for all polynomial functions.

Answer

It can be too difficult or impossible to solve forin terms of y.

2) Why must we restrict the domain of a quadratic function when finding its inverse?

3) When finding the inverse of a radical function, what restriction will we need to make?

Answer

We will need a restriction on the domain of the answer.

4) The inverse of a quadratic function will always take what form?

Algebraic

For the exercises 5-12, find the inverse of the function on the given domain.

5) $f(x)=(x-4)^2, [4,\infty)$

Answer

$f^{-1}(x)=\sqrt{x}+4$

6) $f(x)=(x+2)^2, [-2,\infty)$

7) $f(x)=(x+1)^2-3, [-1,\infty)$

Answer

$f^{-1}(x)=\sqrt{x+3}-1$

8) $f(x)=2-\sqrt{3+x}$

9) $f(x)=3x^2+5, (-\infty,0]$

Answer

$f^{-1}(x)=-\sqrt{\frac{x-5}{3}}$

10) $f(x)=12-x^2, [0,\infty)$

11) $f(x)=9-x^2, [0,\infty)$

Answer

$f(x)=\sqrt{9-x}$

12) $f(x)=2x^2+4, [0,\infty)$

For the exercises 13-16, find the inverse of the functions.

13) $f(x)=x^3+5$

Answer

$f^{-1}(x)=\sqrt[3]{x-5}$

14) $f(x) = 3x^3 + 1$

15) $f(x) = 4 - x^3$

Answer

$f^{-1}(x) = \sqrt[3]{4 - x}$

16) $f(x) = 4 - 2x^3$

For the exercises 17-31, find the inverse of the functions.

17) $f(x) = \sqrt{2x + 1}$

Answer

$f^{-1}(x) = \dfrac{x^2 - 1}{2}, [0, \infty)$

18) $f(x) = \sqrt{3 - 4x}$

19) $f(x) = 9 + \sqrt{4x - 4}$

Answer

$f^{-1}(x) = \dfrac{(x - 9)^2 + 4}{4}, [9, \infty)$

20) $f(x) = \sqrt{6x - 8} + 5$

21) $f(x) = 9 + 2\sqrt[3]{x}$

Answer

$f^{-1}(x) = \left(\dfrac{x - 9}{2}\right)^3$

22) $f(x) = 3 - \sqrt[3]{x}$

23) $f(x) = \dfrac{2}{x + 8}$

Answer

$f^{-1}(x) = \dfrac{2 - 8x}{x}$

24) $f(x) = \dfrac{3}{x - 4}$

25) $f(x) = \dfrac{x + 3}{x + 7}$

Answer

$f^{-1}(x) = \dfrac{7x - 3}{1 - x}$

26) $f(x) = \dfrac{x - 2}{x + 7}$

27) $f(x) = \dfrac{3x + 4}{5}$

Answer

$f^{-1}(x) = \dfrac{5x - 4}{4x + 3}$

28) $f(x)=\dfrac{5x+1}{2-5x}$

29) $f(x)=x^2+2x,[-1,\infty)$

Answer

$f^{-1}(x)=\sqrt{x+1}-1$

30) $f(x)=x^2+4x+1,[-2,\infty)$

31) $f(x)=x^2-6x+3,[3,\infty)$

Answer

$f^{-1}(x)=\sqrt{x+6}+3$

Graphical

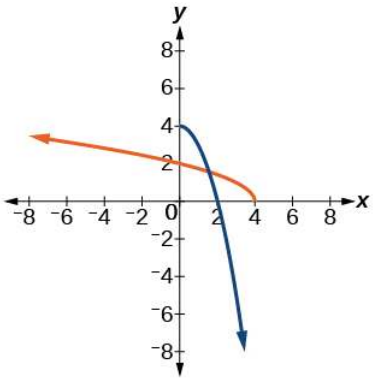
For the exercises 32-41, find the inverse of the function and graph both the function and its inverse.

32) $f(x)=x^2+2, x\geq 0$

33) $f(x)=4-x^2, x\geq 0$

Answer

$f^{-1}(x)=\sqrt{4-x}$

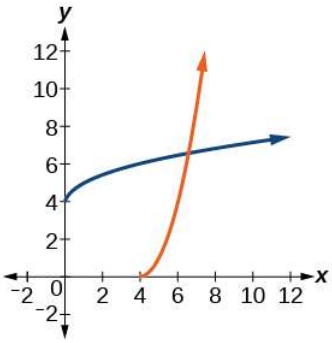


34) $f(x)=(x+3)^2, x\geq -3$

35) $f(x)=(x-4)^2, x\geq 4$

Answer

$f^{-1}(x)=\sqrt{x}+4$



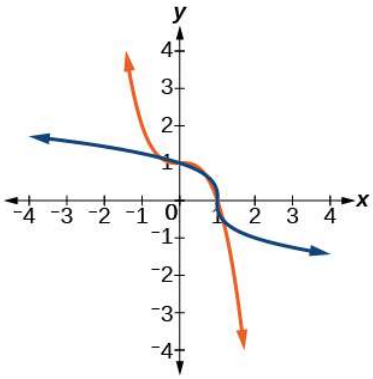
36) $f(x)=x^3+3$

37) $f(x)=1-x^3$

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Answer

$f^{-1}(x) = \sqrt[3]{1-x}$

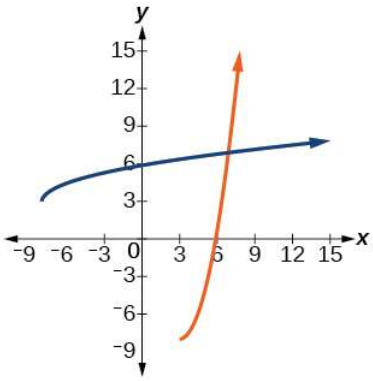


38) $f(x) = x^2 + 4x, x \geq -2$

39) $f(x) = x^2 - 6x + 1, x \geq 3$

Answer

$f^{-1}(x) = \sqrt{x+8} + 3$

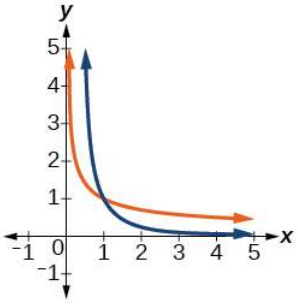


40) $f(x) = \dfrac{2}{x}$

41) $f(x) = \dfrac{1}{x^2}, x \geq 0$

Answer

$f^{-1}(x) = \sqrt{\dfrac{1}{x}}$



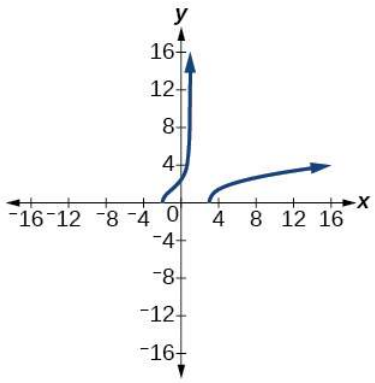
For the exercises 42-46, use a graph to help determine the domain of the functions.

42) $f(x)=\sqrt{\dfrac{(x+1)(x-1)}{x}}$

43) $f(x)=\sqrt{\dfrac{(x+2)(x-3)}{x-1}}$

Answer

$[-2,1)\cup[3,\infty)$

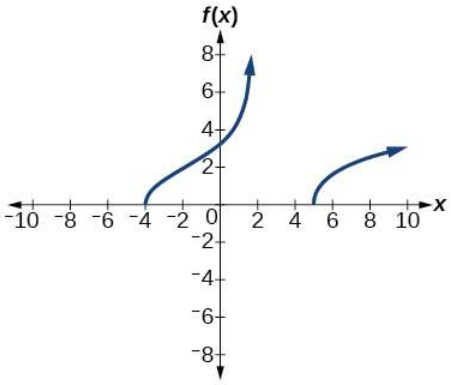


44) $f(x)=\sqrt{\dfrac{x(x+3)}{x-4}}$

45) $f(x)=\sqrt{\dfrac{x^2-x-20}{x-2}}$

Answer

$[-4,2)\cup[5,\infty)$



46) $f(x)=\sqrt{\dfrac{9x^2}{x+4}}$

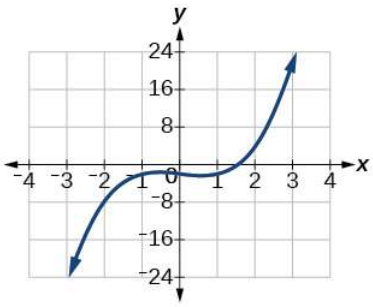
Technology

For the exercises 47-51, use a calculator to graph the function. Then, using the graph, give three points on the graph of the inverse with y-coordinates given.

47) $f(x)=x^3-x-2, y=1,2,3$

Answer

$(-2,0);(4,2);(22,3)$

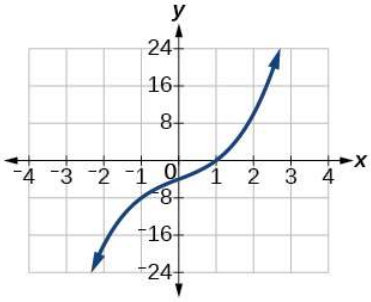


48) $f(x) = x^3 + x - 2, y = 0, 1, 2$

49) $f(x) = x^3 + 3x - 4, y = 0, 1, 2$

Answer

$(-4, 0); (0, 1); (10, 2)$



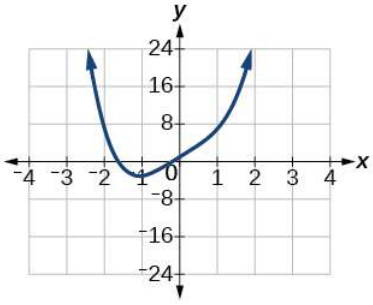
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50) $f(x) = x^3 + 8x - 4, y = -1, 0, 1$

51) $f(x) = x^4 + 5x + 1, y = -1, 0, 1$

Answer

$(-3, -1); (1, 0); (7, 1)$



Extensions

For the exercises 52-56, find the inverse of the functions with a, b, c positive real numbers.

52) $f(x) = ax^3 + b$

53) $f(x) = x^2 + bx$

Answer

$$f^{-1}(x) = \sqrt{x + \frac{b^2}{4}} - \frac{b}{2}$$

54) $f(x) = \sqrt{ax^2 + b}$

55) $f(x)=\sqrt[3]{ax+b}$

Answer

$$f^{-1}(x)=\frac{x^2-b}{a}$$

56) $f(x)=\dfrac{ax+b}{x+c}$

Real-World Applications

For the exercises 57-66, determine the function described and then use it to answer the question.

57) An object dropped from a height of 200 meters has a height, $h(t)$, in meters after t seconds have lapsed, such that $h(t)=200-4.9t^2$. Express t as a function of height, h, and find the time to reach a height of 50 meters.

Answer

$$t(h)=\sqrt{\frac{200-h}{4.9}},\ 5.53\text{seconds}$$

58) An object dropped from a height of 600 feet has a height, $h(t)$, in feet after t seconds have elapsed, such that $h(t)=600-16t^2$. Express t as a function of height h, and find the time to reach a height of 400 feet.

59) The volume, V, of a sphere in terms of its radius, r, is given by $V(r)=\frac{4}{3}\pi r^3$. Express r as a function of V, and find the radius of a sphere with volume of 200 cubic feet.

Answer

$$r(V)=\sqrt[3]{\frac{3V}{4\pi}},\ 3.63\text{feet}$$

60) The surface area, A, of a sphere in terms of its radius, r, is given by $A(r)=4\pi r^2$. Express r as a function of V, and find the radius of a sphere with a surface area of 1000 square inches.

61) A container holds 100 ml of a solution that is 25 ml acid. If n ml of a solution that is 60% acid is added, the function $C(n)=\dfrac{25+.6n}{100+n}$ gives the concentration, C, as a function of the number of ml added, n. Express n as a function of C and determine the number of mL that need to be added to have a solution that is 50% acid.

Answer

$$n(C)=\frac{100C-25}{6-C},250\text{mL}$$

62) The period T, in seconds, of a simple pendulum as a function of its length l, in feet, is given by $T(l)=2\pi\sqrt{\dfrac{l}{322}}$. Express l as a function of T and determine the length of a pendulum with period of 2 seconds.

63) The volume of a cylinder, V, in terms of radius, r, and height, h, is given by $V=\pi r^2 h$. If a cylinder has a height of 6 meters, express the radius as a function of V and find the radius of a cylinder with volume of 300 cubic meters.

Answer

$$r(V)=\sqrt{\frac{V}{6\pi}},3.99\text{ meters}$$

64) The surface area, A, of a cylinder in terms of its radius, r, and height, h, is given by $A=2\pi r^2+2\pi rh$. If the height of the cylinder is 4 feet, express the radius as a function of V and find the radius if the surface area is 200 square feet.

65) The volume of a right circular cone, V, in terms of its radius, r, and its height, h, is given by $V=\frac{1}{3}\pi r^2 h$. Express r in terms of V if the height of the cone is 12 feet and find the radius of a cone with volume of 50 cubic inches.

Answer

$r(V) = \sqrt{\frac{V}{4\pi}}, 1.99$ inches

66) Consider a cone with height of 30 feet. Express the radius, r, in terms of the volume, V, and find the radius of a cone with volume of 1000 cubic feet.

Modeling Using Variation

Verbal

1) What is true of the appearance of graphs that reflect a direct variation between two variables?

Answer

The graph will have the appearance of a power function.

2) If two variables vary inversely, what will an equation representing their relationship look like?

3) Is there a limit to the number of variables that can jointly vary? Explain.

Answer

No. Multiple variables may jointly vary.

Algebraic

For the exercises 4-23, write an equation describing the relationship of the given variables.

4) y varies directly as x and when $x = 6, y = 12$

5) y varies directly as the square of x and when $x = 4, y = 80$

Answer

$y = 5x^2$

6) y varies directly as the square root of x and when $x = 36, y = 24$

7) y varies directly as the cube of x and when $x = 36, y = 24$

Answer

$y = 10x^3$

8) y varies directly as the cube root of x and when $x = 27, y = 15$

9) y varies directly as the fourth power of x and when $x = 1, y = 6$

Answer

$y = 6x^4$

10) y varies inversely as x and when $x = 4, y = 2$

11) y varies inversely as the square of x and when $x = 3, y = 2$

Answer

$y = \frac{18}{x^2}$

12) y varies inversely as the cube of x and when $x = 2, y = 5$

13) y varies inversely as the fourth power of x and when $x = 3, y = 1$

Answer

$$y = \frac{81}{x^4}$$

14) y varies inversely as the square root of x and when $x = 25, y = 3$

15) y varies inversely as the cube root of x and when $x = 64, y = 5$

Answer

$$y = \frac{20}{\sqrt[3]{x}}$$

16) y varies jointly with x and z and when $x = 2$ and $z = 3, y = 36$

17) y varies jointly as x, z, and w and when $x = 1, z = 2, w = 5$, then $y = 100$

Answer

$$y = 10xzw$$

18) y varies jointly as the square of x and the square of z and when $x = 3$ and $z = 4$, then $y = 72$

19) y varies jointly as x and the square root of z and when $x = 2$ and $z = 25$, then $y = 100$

Answer

$$y = 10x\sqrt{z}$$

20) y varies jointly as the square of x the cube of z and the square root of w. When $x = 1, z = 2,$ and $w = 36$, then $y = 48$

21) y varies jointly as x and z and inversely as w. When $x = 3, z = 5,$ and $w = 6$, then $y = 10$

Answer

$$y = 4\frac{xz}{w}$$

22) y varies jointly as the square of x and the square root of z and inversely as the cube of w. When $x = 3, z = 4$, and $w = 3$, then $y = 6$

23) y varies jointly as x and z and inversely as the square root of wand the square of t. When $x = 3, z = 1, w = 25$, and $t = 2$, then $y = 6$

Answer

$$y = 40\frac{xz}{\sqrt{w}t^2}$$

Numeric

For the exercises 24-40, use the given information to find the unknown value.

24) y varies directly as x. When $x = 3$, then $y = 12$. Find y when $x = 20$.

25) y varies directly as the square of x. When $x = 2$, then $y = 16$. Find y when $x = 8$.

Answer

$$y = 256$$

26) y varies directly as the cube of x. When $x = 3$, then $y = 5$. Find y when $x = 4$.

27) y varies directly as the square root of x. When $x = 16$, then $y = 4$. Find y when $x = 36$.

Answer

$$y = 6$$

28) y varies directly as the cube root of x. When $x = 125$, then $y = 15$. Find y when $x = 1,000$.

29) y varies inversely with x. When $x = 3$, then $y = 2$. Find y when $x = 1$.

Answer

$y = 6$

30) y varies inversely with the square of x. When $x = 4$, then $y = 3$. Find y when $x = 2$.

31) y varies inversely with the cube of x. When $x = 3$, then $y = 1$. Find y when $x = 1$.

Answer

$y = 27$

32) y varies inversely with the square root of x. When $x = 64$, then $y = 12$. Find y when $x = 36$.

33) y varies inversely with the cube root of x. When $x = 27$, then $y = 5$. Find y when $x = 125$.

Answer

$y = 3$

34) y varies jointly as x and z. When $x = 4$ and $z = 2$, then $y = 16$. Find y when $x = 3$ and $z = 3$.

35) y varies jointly as x, z, and w. When $x = 2$, $z = 1$, and $w = 12$, then $y = 72$. Find y when $x = 1$, $z = 2$, and $w = 3$.

Answer

$y = 18$

36) y varies jointly as x and the square of z. When $x = 2$ and $z = 4$, then $y = 144$. Find y when $x = 4$ and $z = 5$.

37) y varies jointly as the square of x and the square root of z. When $x = 9$, then $y = 24$. Find y when $x = 3$ and $z = 25$.

Answer

$y = 90$

38) y varies jointly as x and inversely as w. When $x = 5$, $z = 2$, and $w = 20$, then $y = 4$. Find y when $x = 3$ and $z = 8$ and $w = 48$.

39) y varies jointly as the square of x and the cube of z and inversely as the square root of w. When $x = 2$, $z = 2$, and $w = 64$ then $y = 12$. Find y when $x = 1$, $z = 3$, and $w = 4$.

Answer

$y = \frac{81}{2}$

40) y varies jointly as the square of x and of z and inversely as the square root of wand of t. When $x = 2, z = 3, w = 16$, and $t = 3$, then $y = 1$. Find y when $x = 3, z = 2, w = 36$, and $t = 5$.

Technology

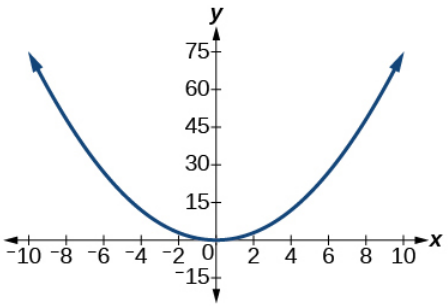
For the exercises 41-45, use a calculator to graph the equation implied by the given variation.

41) y varies directly with the square of x and when $x = 2, y = 3$

Answer

$y = \frac{3}{4}x^2$

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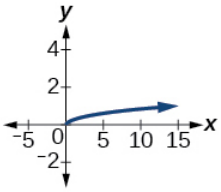


42) y varies directly as the cube of x and when $x = 2, y = 4$.

43) y varies directly as the square root of x and when $x = 36, y = 2$.

Answer

$y = \frac{1}{3}\sqrt{x}$

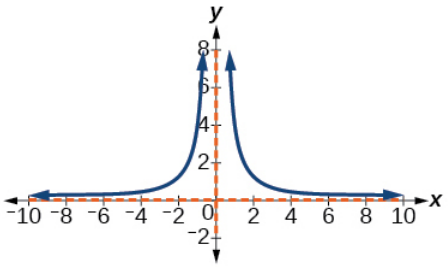


44) y varies inversely with x and when $x = 6, y = 2$.

45) y varies inversely as the square of x and when $x = 1, y = 4$.

Answer

$y = \frac{4}{x^2}$



Extensions

For the exercises 46-50, use Kepler's Law, which states that the square of the time, T, required for a planet to orbit the Sun varies directly with the cube of the mean distance, a, that the planet is from the Sun.

46) Using the Earth's time of 1 year and mean distance of 93 million miles, find the equation relating T and a.

47) Use the result from the previous exercise to determine the time required for Mars to orbit the Sun if its mean distance is 142 million miles.

Answer

1.89 years

48) Using Earth's distance of 150 million kilometers, find the equation relating T and a.

49) Use the result from the previous exercise to determine the time required for Venus to orbit the Sun if its mean distance is 108 million kilometers.

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Answer

0.61 years

50) Using Earth's distance of 1 astronomical unit (A.U.), determine the time for Saturn to orbit the Sun if its mean distance is 9.54 A.U.

Real-World Applications

For the exercises 51-60, use the given information to answer the questions.

51) The distance s that an object falls varies directly with the square of the time, t, of the fall. If an object falls 16 feet in one second, how long for it to fall 144 feet?

Answer

3 seconds

52) The velocity v of a falling object varies directly to the time, t, of the fall. If after 2 seconds, the velocity of the object is 64 feet per second, what is the velocity after 5 seconds?

53) The rate of vibration of a string under constant tension varies inversely with the length of the string. If a string is 24 inches long and vibrates 128 times per second, what is the length of a string that vibrates 64 times per second?

Answer

48 inches

54) The volume of a gas held at constant temperature varies indirectly as the pressure of the gas. If the volume of a gas is 1200 cubic centimeters when the pressure is 200 millimeters of mercury, what is the volume when the pressure is 300 millimeters of mercury?

55) The weight of an object above the surface of the Earth varies inversely with the square of the distance from the center of the Earth. If a body weighs 50 pounds when it is 3960 miles from Earth's center, what would it weigh it were 3970 miles from Earth's center?

Answer

49.75 pounds

56) The intensity of light measured in foot-candles varies inversely with the square of the distance from the light source. Suppose the intensity of a light bulb is 0.08 foot-candles at a distance of 3 meters. Find the intensity level at 8 meters.

57) The current in a circuit varies inversely with its resistance measured in ohms. When the current in a circuit is 40 amperes, the resistance is 10 ohms. Find the current if the resistance is 12 ohms.

Answer

33.33 amperes

58) The force exerted by the wind on a plane surface varies jointly with the square of the velocity of the wind and with the area of the plane surface. If the area of the surface is 40 square feet surface and the wind velocity is 20 miles per hour, the resulting force is 15 pounds. Find the force on a surface of 65 square feet with a velocity of 30 miles per hour.

59) The horsepower (hp) that a shaft can safely transmit varies jointly with its speed (in revolutions per minute (rpm)) and the cube of the diameter. If the shaft of a certain material 3 inches in diameter can transmit 45 hp at 100 rpm, what must the diameter be in order to transmit 60 hp at 150 rpm?

Answer

2.88 inches

60) The kinetic energy K of a moving object varies jointly with its mass m and the square of its velocity v. If an object weighing 40 kilograms with a velocity of 15 meters per second has a kinetic energy of 1000 joules, find the kinetic energy if the velocity is

increased to 20 meters per second.

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3.R: Polynomial and Rational Functions(Review)

Complex Numbers

Perform the indicated operation with complex numbers.

1) $(4+3i)+(-2-5i)$

Answer

$2-2i$

2) $(6-5i)-(10+3i)$

3) $(2-3i)(3+6i)$

Answer

$24+3i$

4) $\dfrac{2-i}{2+i}$

Solve the following equations over the complex number system.

5) $x^2-4x+5=0$

Answer

$\{2+i, 2-i\}$

6) $x^2+2x+10=0$

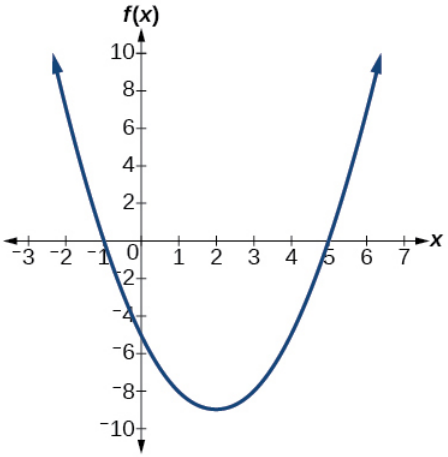
Quadratic Functions

For the exercises 1-2, write the quadratic function in standard form. Then, give the vertex and axes intercepts. Finally, graph the function.

1) $f(x)=x^2-4x-5$

Answer

$f(x)=(x-2)^2-9$ vertex $(2,-9)$, intercepts $(5,0);(-1,0);(0,-5)$



2) $f(x)=-2x^2-4x$

For the problems 3-4, find the equation of the quadratic function using the given information.

3) The vertex is $(-2,3)$ and a point on the graph is $(3,6)$.

Answer

$$f(x) = \frac{3}{25}(x+2)^2 + 3$$

4) The vertex is $(-3, 6.5)$ and a point on the graph is $(2, 6)$.

Answer the following questions.

5) A rectangular plot of land is to be enclosed by fencing. One side is along a river and so needs no fence. If the total fencing available is 600 meters, find the dimensions of the plot to have maximum area.

Answer

300 meters by 150 meters, the longer side parallel to river.

6) An object projected from the ground at a 45 degree angle with initial velocity of 120 feet per second has height, h, in terms of horizontal distance traveled, x, given by $h(x) = \frac{-32}{(120)^2}x^2 + x$. Find the maximum height the object attains.

Power Functions and Polynomial Functions

For the exercises 1-3, determine if the function is a polynomial function and, if so, give the degree and leading coefficient.

1) $f(x) = 4x^5 - 3x^3 + 2x - 1$

Answer

Yes, degree $= 5$, leading coefficient $= 4$

2) $f(x) = 5^{x+1} - x^2$

3) $f(x) = x^2\left(3 - 6x + x^2\right)$

Answer

Yes, degree $= 4$, leading coefficient $= 1$

For the exercises 4-6, determine end behavior of the polynomial function.

4) $f(x) = 2x^4 + 3x^3 - 5x^2 + 7$

5) $f(x) = 4x^3 - 6x^2 + 2$

Answer

As $x \to -\infty, f(x) \to -\infty$, as $x \to \infty, f(x) \to \infty$

6) $f(x) = 2x^2\left(1 + 3x - x^2\right)$

Graphs of Polynomial Functions

For the exercises 1-3, find all zeros of the polynomial function, noting multiplicities.

1) $f(x) = (x+3)^2(2x-1)(x+1)^3$

Answer

-3 with multiplicity 2, $-\frac{1}{2}$ with multiplicity 1, -1 with multiplicity 3

2) $f(x) = x^5 + 4x^4 + 4x^3$

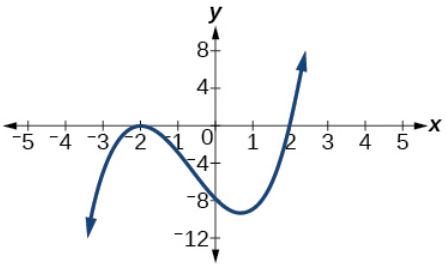
3) $f(x) = x^3 - 4x^2 + x - 4$

Answer

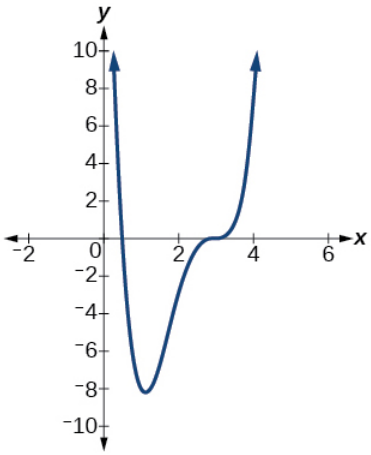
4 with multiplicity 1

For the exercises 4-5, based on the given graph, determine the zeros of the function and note multiplicity.

4)



5)



Answer

$\frac{1}{2}$ with multiplicity 1, 3 with multiplicity 3

6) Use the Intermediate Value Theorem to show that at least one zero lies between 2 and 3 for the function $f(x) = x^3 - 5x + 1$

Dividing Polynomials

For the exercises 1-2, use long division to find the quotient and remainder.

1) $\frac{x^3 - 2x^2 + 4x + 4}{x - 2}$

Answer

$x^2 + 4$ with remainder 12

2) $\frac{3x^4 - 4x^2 + 4x + 8}{x + 1}$

For the exercises 3-6, use synthetic division to find the quotient. If the divisor is a factor, then write the factored form.

3) $\frac{x^2 - 2x^2 + 5x - 1}{x + 3}$

Answer

$x^2 - 5x + 20 - \frac{61}{x + 3}$

4) $\frac{x^2 + 4x + 10}{x - 3}$

5) $\dfrac{2x^3+6x^2-11x-12}{x+4}$

Answer

$2x^2-2x-3$, so factored form is $(x+4)\left(2x^2-2x-3\right)$

6) $\dfrac{3x^4+3x^3+2x+2}{x+1}$

Zeros of Polynomial Functions

For the exercises 1-4, use the Rational Zero Theorem to help you solve the polynomial equation.

1) $2x^3-3x^2-18x-8=0$

Answer

$\left\{-2,4,-\dfrac{1}{2}\right\}$

2) $3x^3+11x^2+8x-4=0$

3) $2x^4-17x^3+46x^2-43x+12=0$

Answer

$\left\{1,3,4,\dfrac{1}{2}\right\}$

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For the exercises 5-6, use Descartes' Rule of Signs to find the possible number of positive and negative solutions.

5) $x^3-3x^2-2x+4=0$

Answer

0 or 2 positive, 1 negative

6) $2x^4-x^3+4x^2-5x+1=0$

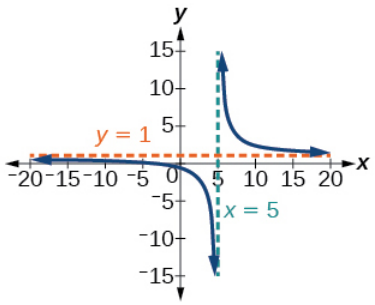
Rational Functions

For the following rational functions 1-4, find the intercepts and the vertical and horizontal asymptotes, and then use them to sketch a graph.

1) $f(x)=\dfrac{x+2}{x-5}$

Answer

Intercepts $(-2,0)$ and $\left(0,-\dfrac{2}{5}\right)$, Asymptotes $x=5$ and $y=1$

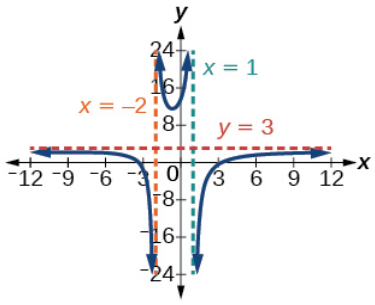


2) $f(x) = \dfrac{x^2+1}{x^2-4}$

3) $f(x) = \dfrac{3x^2-27}{x^2-9}$

Answer

Intercepts $(3,0)$, $(-3,0)$, and $\left(0, \dfrac{27}{2}\right)$, Asymptotes $x=1, x=-2, y=3$



4) $f(x) = \dfrac{x+2}{x^2-9}$

For the exercises 5-6, find the slant asymptote.

5) $f(x) = \dfrac{x^2-1}{x+2}$

Answer

$y = x-2$

6) $f(x) = \dfrac{2x^3-x^2+4}{x^2+1}$

Inverses and Radical Functions

For the exercises 1-6, find the inverse of the function with the domain given.

1) $f(x) = (x-2)^2, x \geq 2$

Answer

$f^{-1}(x) = \sqrt{x}+2$

2) $f(x) = (x+4)^2 - 3, x \geq -4$

3) $f(x) = x^2+6x-2, x \geq -3$

Answer

$f^{-1}(x) = \sqrt{x+11} - 3$

4) $f(x) = 2x^3 - 3$

5) $f(x) = \sqrt{4x+5} - 3$

Answer

$f^{-1}(x) = \dfrac{(x+3)^2 - 5}{4}, x \geq -3$

6) $f(x) = \dfrac{x-3}{2x+1}$

Modeling Using Variation

For the exercises 1-4, find the unknown value.

1) y varies directly as the square of x. If when $x = 3, y = 36$, find y if $x = 4$.

Answer

$y = 64$

2) y varies inversely as the square root of x. If when $x = 25, y = 2$, find y if $x = 4$.

3) y varies jointly as the cube of x and as z. If when $x = 1$ and $z = 2, y = 6$, find y if $x = 2$ and $z = 3$.

Answer

$y = 72$

4) y varies jointly as x and the square of z and inversely as the cube of w. If when $x = 3, z = 4$, and $w = 2, y = 48$, ind y if $x = 4, z = 5$, and $w = 3$.

For the exercises 5-6, solve the application problem.

5) The weight of an object above the surface of the earth varies inversely with the distance from the center of the earth. If a person weighs 150 pounds when he is on the surface of the earth $(3,960$ miles from center), find the weight of the person if he is 20 miles above the surface.

Answer

148.5pounds

6) The volume V of an ideal gas varies directly with the temperature T and inversely with the pressure$P$. A cylinder contains oxygen at a temperature of 310 degrees K and a pressure of 18 atmospheres in a volume of 120 liters. Find the pressure if the volume is decreased to 100 liters and the temperature is increased to 320 degrees K.

Practice Test

Perform the indicated operation or solve the equation.

1) $(3-4i)(4+2i)$

Answer

$20 - 10i$

2) $\dfrac{1-4i}{3+4i}$

3) $x^2 - 4x + 13 = 0$

Answer

$\{2+3i, 2-3i\}$

4) Give the degree and leading coefficient of the following polynomial function.

$$f(x) = x^3\left(3-6x^2-2x^2\right)$$

Determine the end behavior of the polynomial function.

5) $f(x) = 8x^3 - 3x^2 + 2x - 4$

Answer

As $x \to -\infty, f(x) \to -\infty$, as $x \to \infty, f(x) \to \infty$

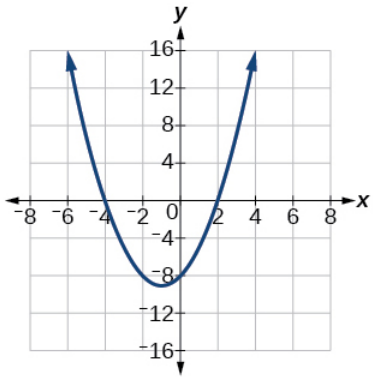
6) $f(x) = -2x^2\left(4-3x-5x^2\right)$

7) Write the quadratic function in standard form. Determine the vertex and axes intercepts and graph the function.

$$f(x) = x^2 + 2x - 8$$

Answer

$f(x) = (x+1)^2 - 9$, vertex $(-1, -9)$, intercepts $(2, 0); (-4, 0); (0, -8)$



8) Given information about the graph of a quadratic function, find its equation: Vertex $(2, 0)$ and point on graph $(4, 12)$

Solve the following application problem.

9) A rectangular field is to be enclosed by fencing. In addition to the enclosing fence, another fence is to divide the field into two parts, running parallel to two sides. If $1,200$ feet of fencing is available, find the maximum area that can be enclosed.

Answer

$60,000$square feet

Find all zeros of the following polynomial functions, noting multiplicities.

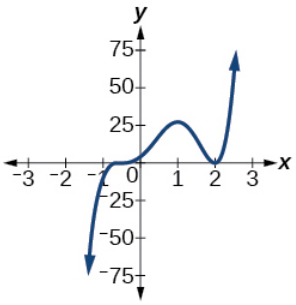
10) $f(x) = (x-3)^3(3x-1)(x-1)^2$

11) $f(x) = 2x^6 - 12x^5 + 18x^4$

Answer

0 with multiplicity 4, 3 with multiplicity 2

12) Based on the graph, determine the zeros of the function and multiplicities.



13) Use long division to find the quotient:

$$\frac{2x^2+3x-4}{x+2}$$

Answer

$2x^2-4x+11-\frac{26}{x+2}$

Use synthetic division to find the quotient. If the divisor is a factor, write the factored form.

14) $\frac{x^4+3x^2-4}{x-2}$

15) $\frac{2x^3+5x^2-7x-12}{x+3}$

Answer

$2x^2-x-4$. So factored form is $(x+3)\left(2x^2-x-4\right)$

Use the Rational Zero Theorem to help you find the zeros of the polynomial functions.

16) $f(x)=2x^3+5x^2-6x-9$

17) $f(x)=4x^4+8x^3+21x^2+17x+4$

Answer

$-\frac{1}{2}$ (has multiplicity 2), $\frac{-1+i\sqrt{15}}{2}$

18) $f(x)=4x^4+16x^3+13x^2-15x-18$

19) $f(x)=x^5+6x^4+13x^3+14x^2+12x+8$

Answer

-2 (has multiplicity 3), $\pm i$

Given the following information about a polynomial function, find the function.

20) It has a double zero at $x=3$ and zeroes at $x=1$ and $x=-2$. Its y -intercept is $(0,12)$.

21) It has a zero of multiplicity 3 at $x=\frac{1}{2}$ and another zero at $x=-3$. It contains the point $(1,8)$.

Answer

$f(x)=2(2x-1)^3(x+3)$

22) Use Descartes' Rule of Signs to determine the possible number of positive and negative solutions.

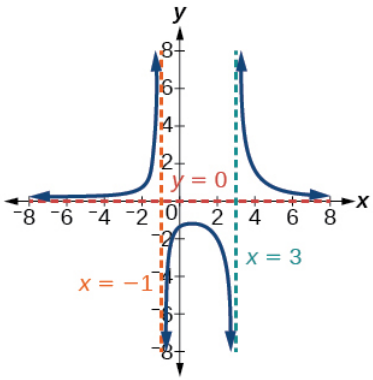
$$8x^3-21x^2+6=0$$

For the following rational functions, find the intercepts and horizontal and vertical asymptotes, and sketch a graph.

23) $f(x) = \dfrac{x+4}{x^2-2x-3}$

Answer

Intercepts $(-4, 0)$, $\left(0, -\dfrac{4}{3}\right)$, Asymptotes $x = 3, x = -1, y = 0$



24) $f(x) = \dfrac{x^2+2x-3}{x^2-4}$

25) Find the slant asymptote of the rational function.

$$f(x) = \frac{x^2+3x-3}{x-1}$$

Answer

$y = x + 4$

Find the inverse of the function.

26) $f(x) = \sqrt{x-2} + 4$

27) $f(x) = 3x^3 - 4$

Answer

$f^{-1}(x) = \sqrt[3]{\dfrac{x+4}{3}}$

28) $f(x) = \dfrac{2x+3}{3x-1}$

Find the unknown value.

29) y varies inversely as the square of x and when $x = 3, y = 2$. Find y if $x = 1$.

Answer

$y = 18$

30) y varies jointly with x and the cube root of z. If when $x = 27, y = 12$, find y if $x = 5$ and $z = 8$.

Solve the following application problem.

31) The distance a body falls varies directly as the square of the time it falls. If an object falls 64 feet in 2 seconds, how long will it take to fall 256 feet?

Answer

4 seconds

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CHAPTER OVERVIEW

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4.1: Prelude to Exponential and Logarithmic Functions

Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.

Bacteria commonly reproduce through a process called binary fission, during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years. For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. Table 4.1.1 shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million!

Table 4.1.1

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

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4.2: Exponential Functions

Learning Objectives

- Evaluate exponential functions.
- Use compound interest formulas.
- Evaluate exponential functions with base e.

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about $.2\%$ each year. If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say that the growth is "exponential," meaning that something is growing very rapidly. To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions*, which model this kind of rapid growth.

Exponential Growth and Decay

What exactly does it mean to *grow exponentially*? For us to gain a clear understanding of exponential growth, let us compare exponential growth with linear growth. We will construct two functions. The first function, $f(x) = 2^x$, is exponential. The second function, $g(x) = 2x$, is linear. We will start with an input of 0, and increase each input by 1. The corresponding outputs are shown in Table $4.2.1$. From Table $4.2.1$, we can see that for the exponential function, $f(x)$ that as we increase the input by 1 that the output doubles. The linear function, $g(x)$, is a constant increase by 2. From Table $4.2.1$ we can conclude that exponential growth dwarfs linear growth.

- *Exponential growth* refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- *Linear growth* refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

Table 4.2.1

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

For exponential growth, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

The general form of an **exponential function** is $f(x) = ab^x$, where a is any nonzero number, b is a positive real number not equal to 1.

- If $b > 1$,the function grows at a rate proportional to its size. This is called exponential growth.
- If $0 < b < 1$, the function decays at a rate proportional to its size. This is called exponential decay.

Exponential growth refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.

Exponential decay refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time .

Let's look at the function $f(x) = 2^x$ from our example. We will create a table (Table 4.2.2) to determine the corresponding outputs over an interval in the domain from -3 to 3.

Table 4.2.2

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$2^{-3} = \frac{1}{8}$	$2^{-2} = \frac{1}{4}$	$2^{-1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^2 = 4$	$2^3 = 8$

Let us examine the graph of f by plotting the ordered pairs from Table 4.2.2 and then make a few observations in Figure 4.2.1.

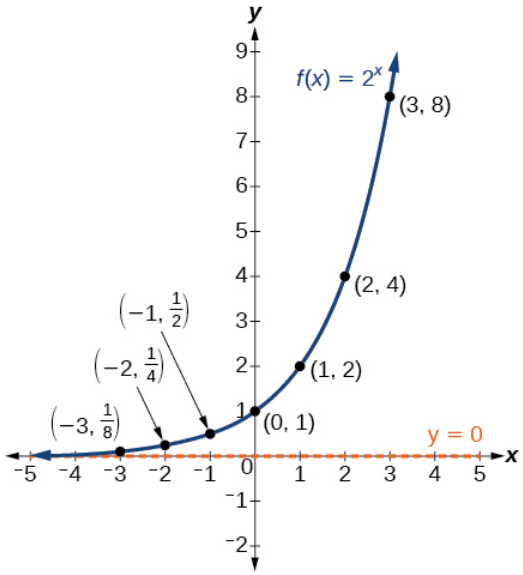


Figure 4.2.1

Let's define the behavior of the graph of the exponential function $f(x) = 2^x$ and highlight some its key characteristics.

- the domain is $(-\infty, \infty)$,
- the range is $(0, \infty)$,
- as $x \to \infty$, $f(x) \to \infty$,
- as $x \to -\infty$, $f(x) \to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.
- $y = 0$ is the horizontal asymptote.
- the *y*-intercept is $(0, 1)$.

Exponential Functions

For any real number x, an **exponential function** is a function with the form

$$f(x) = ab^x$$

where

- a is a non-zero real number called the initial value and
- b is any positive real number such that $b \neq 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a > 0$.
- The range of f is all negative real numbers if $a < 0$.
- The y-intercept is $(0, a)$,and the horizontal asymptote is $y = 0$.

b is called the base of the function and x is called the exponent.

Note that an exponential function does not intersect its horizontal asymptote. This is because there is no value for x that we can input into the function that would yield an output of 0, $ab^x \neq 0$.

Example 4.2.1

Which of the following equations are *not* exponential functions?

- $f(x) = 4^{3(x-2)}$
- $g(x) = x^3$
- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution

By definition, an exponential function has a constant base and a variable exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is variable and the exponent is constant.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

$f(x)$ and $h(x)$ are exponential functions.

You Try 4.2.1

Which of the following equations represent exponential functions?

- $f(x) = 2x^2 - 3x + 1$
- $g(x) = 0.875^x$
- $h(x) = 1.75x + 2$
- $j(x) = 1095.6^{-2x}$

Answer

$g(x) = 0.875^x$ and $j(x) = 1095.6^{-2x}$ represent exponential functions. The others are not exponential functions.

Recall that the base of an exponential function must be a positive real number other than 1. Why do we limit the base, b ,to positive values? To ensure that the outputs will be real numbers. Observe what happens if the base is not positive:

Let $b = -9$ and $x = \frac{1}{2}$. Then $f(x) = f\left(\frac{1}{2}\right) = (-9)^{\frac{1}{2}} = \sqrt{-9}$, which is not a real number.

Why do we limit the base to positive values other than 1? Because base 1 results in the constant function. Observe what happens if the base is 1:

Let $b = 1$. Then $f(x) = 1^x = 1$ for any value of x.

Evaluating Exponential Functions

To evaluate an exponential function with the form $f(x) = b^x$, we simply substitute x with the given value, and calculate the resulting power. For example:

Let $f(x) = 2^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 2^x \\ f(3) &= 2^3 \qquad \text{Substitute } x = 3 \\ &= 8 \qquad \text{Evaluate the power} \end{aligned}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let $f(x) = 30(2)^x$. What is $f(3)$?

$$\begin{aligned} f(x) &= 30(2)^x \\ f(3) &= 30(2)^3 \qquad \text{Substitute } x = 3 \\ &= 30(8) \qquad \text{Simplify the power first} \\ &= 240 \qquad \text{Multiply} \end{aligned}$$

Caution!

Remember to follow order of operations correctly. Exponent operations must be performed before multiplication.

$$f(3) = 30(2)^3 \neq 60^3 = 216,000$$

Example 4.2.2

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function $P(t) = 1.25(1.012)^t$, where t is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

Solution

To estimate the population in 2031, we evaluate the models for $t = 18$, because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549$$

There will be about 1.549 billion people in India in the year 2031.

Example 4.2.3

Bismuth-210 is an isotope that radioactively decays by about 13% each day, meaning 13% of the remaining Bismuth-210 transforms into another atom (polonium-210 in this case) each day. If you begin with 100 mg of Bismuth-210, how much remains after one week?

Solution

With radioactive decay, instead of the quantity increasing at a percent rate, the quantity is decreasing at a percent rate. Our initial quantity is a = 100 mg, and our growth rate will be negative 13%, since we are decreasing: r = -0.13. This gives the equation:

$$Q(d) = 100(1 - 0.13)^d = 100(0.87)^d$$

This can also be explained by recognizing that if 13% decays, then 87 % remains.

After one week, 7 days, the quantity remaining would be

$$Q(7) = 100(0.87)^7 = 37.73 \text{ mg of Bismuth-210 remains.}$$

Example 4.2.4

$T(q)$ represents the total number of Android smart phone contracts, in thousands, held by a certain Verizon store region measured quarterly since January 1, 2016, interpret all the parts of the equation $T(2) = 86(1.64)^2 = 231.3056$

Solution

Interpreting this from the basic exponential form, we know that 86 is our initial value. This means that on Jan. 1, 2016 this region had 86,000 Android smart phone contracts. Since $b = 1 + r = 1.64$, we know that every quarter the number of smart phone contracts grows by 64%. $T(2) = 231.3056$means that in the 2nd quarter (or at the end of the second quarter) there were approximately 231,306 Android smart phone contracts.

Compound Interest

A certificate of deposit (CD) is a type of savings account offered by banks, typically offering a higher interest rate in return for a fixed length of time you will leave your money invested. Typically bank accounts and other savings instruments in which earnings are reinvested, such as mutual funds and retirement accounts, utilize compound interest. The term *compounding* comes from the behavior that interest is earned not on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, annual percentage rate r, and number of compounding periods in a year n:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe Table $4.2.3$, which shows the result of investing $\$1,000$ at 10% APR for one year. Notice how the value of the account increases as the compounding frequency increases.

Table 4.2.3

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71
Daily	$1105.16

Compound Interest

Compound interest can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where

- $A(t)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Note that sometime rates are given without stating that they are annual percentage rates. In most textbooks, including this one, it is implied that the rate is an annual percentage rate unless explicitly stated otherwise.

Example 4.2.5

If you invest $3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution

Since we are starting with $3000, $a = 3000$

Our interest rate is 3%, so $r = 0.03$

Since we are compounding quarterly, we are compounding 4 times per year, so $k = 4$

We want to know the value of the account in 10 years, so we are looking for $A(10)$, the value when $t = 10$.

$$A(10) = 3000\left(1 + \frac{0.03}{4}\right)^{4(10)} = \$4045.05$$

The account will be worth $4045.05 in 10 years.

(*note that we round to the nearest cent when working with money)*

Example 4.2.6

A 529 plan is a college savings plan in which a relative can invest money to pay for a child's later college tuition, and the account grows tax free. If Lily wants to set up a 529 account for her new granddaughter, wants the account to grow to $40,000 over 18 years, and she believes the account will earn 6% compounded semi-annually (twice a year), how much will Lily need to invest in the account now?

Solution

Since the account is earning 6%, $r = 0.06$

Since interest is compounded twice a year, $k = 2$

In this problem, we don't know how much we are starting with, so we will be solving for a, the initial amount needed. We do know we want the end amount to be $40,000, so we will be looking for the value of a so that $A(18) = 40,000$.

$$40,000 = A(18) = a\left(1 + \frac{0.06}{2}\right)^{2(18)}$$
$$40,000 = a(2.8983)$$
$$a = \frac{40,000}{2.8983} \approx \$13,801$$

Lily will need to invest $13,801 to have $40,000 in 18 years.

Euler's Number and The Natural Exponential Function

As we saw earlier, the amount we earn increases as we increase the compounding frequency. The table, though, shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to believe that although increasing the frequency of compounding will increase our result, there is an upper limit to this process.

To see this, let us examine the value of $1 invested at 100% interest for 1 year.

Frequency	Value
Annual	$2
Quarterly	$2.441406
Monthly	$2.613035
Daily	$2.714567
Hourly	$2.718127
Once per minute	$2.718279
Once per second	$2.718282

These values do indeed appear to be approaching an upper limit. This value ends up being so important that it gets represented by its own letter, much like how π represents a number. This number is denoted by $\boldsymbol{e}$. The famous mathematician, Leonard Euler, was the first person to use this symbol to describe this number in 1731.

Euler's number: e

e is the letter used to represent the value that $\left(1+\frac{1}{k}\right)^k$ approaches as k gets big.

$$e \approx 2.718282$$

Because e is often used as the base of an exponential, most scientific and graphing calculators have a button that can calculate powers of e, usually labeled e^x. Some computer software instead defines a function $exp(x)$, where $exp(x) = e^x$.

Because e arises when the time between compounds becomes very small, e allows us to define **continuous growth** and allows us to define a new toolkit function, $f(x) = e^x$.

Continuous Growth Formula

Continuous Growth can be calculated using the formula

$$f(x) = ae^{rx}$$

where

a is the starting amount

r is the continuous growth rate

Euler's number is commonly used when describing quantities that change more or less continuously in nature, like chemical reactions, growth of large populations, and radioactive decay. The **natural exponential function** is an exponential function with a base e.

The Natural Exponential Function

The **natural exponential function** is the exponential function defined as

$$f(x) = e^x$$

Example 4.2.7

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100mg of Radon-222 decay to in 3 days?

Solution

Since we are given a continuous decay rate, we use the continuous growth formula. Since the substance is decaying, we know the growth rate will be negative: r = -0.173

$f(3) = 100e^{-0.173(3)} \approx 59.512$ mg of Radon-222 will remain.

You Try 4.2.2

Interpret the following: $S(t) = 20e^{0.12t}$ if $S(t)$ represents the growth of a substance in grams, and time is measured in days.

Answer

An initial substance weighing 20g is growing at a continuous rate of 12% per day.

Example 4.2.8

If $1000 is invested in an account earning 10% compounded continuously, find the value after 1 year.

Solution

Here, the continuous growth rate is 10%, so r = 0.10. We start with $1000, so a = 1000.

To find the value after 1 year,

$$f(1) = 1000e^{0.10(1)} \approx \$1105.17$$

Notice this is a \$105.17 increase for the year. As a percent increase, this is $\frac{105.17}{1000} = 0.10517 = 10.517\%$ increase over the original \$1000.

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4.3: Graphing Exponential Functions

Learning Objectives

- Identify basic graphs of exponential functions and sketch their graphs.
- Identify transformations of exponential functions.
- Graph exponential functions.
- Find equations of exponential functions that model graphical information.

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table $4.3.1$ change as the input increases by 1.

Table 4.3.1

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x) = ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

Figure $4.3.1$ shows the exponential growth function $f(x) = 2^x$.

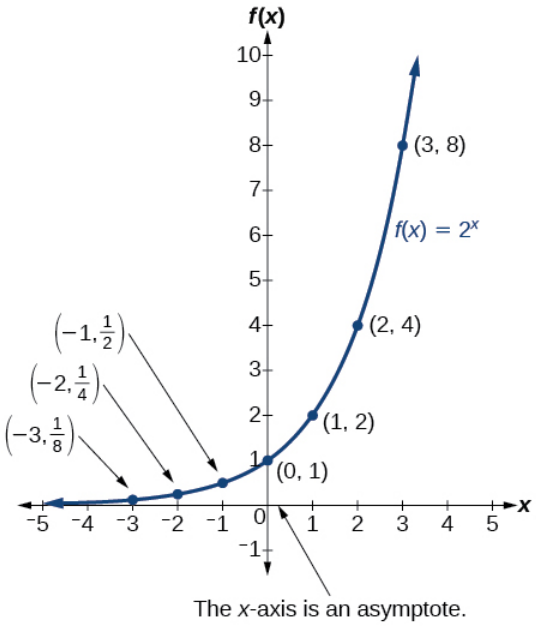


Figure 4.3.1: Notice that the graph gets close to the x-axis, but never touches it.

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table $4.3.2$ change as the input increases by 1.

Table 4.3.2

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

Figure $4.3.2$ shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

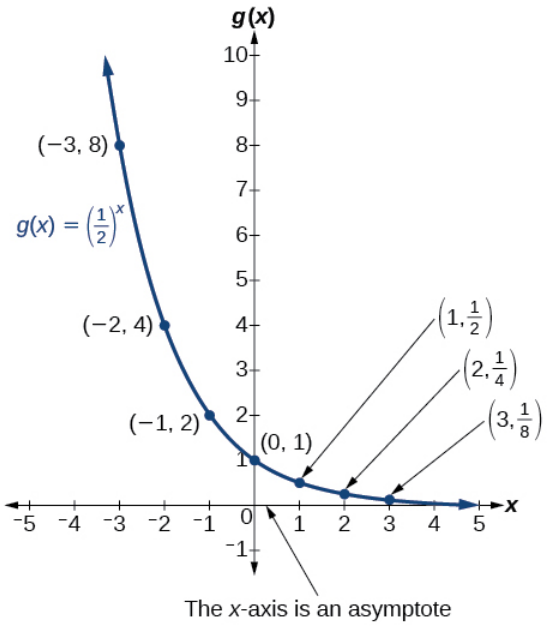


Figure 4.3.2

The domain of $g(x) = \left(\frac{1}{2}\right)^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

Characteristics of Basic Exponential Functions

An exponential function of the form $f(x) = b^x$, $b > 0$, $b \neq 1$, has the following characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x-intercept: none
- y-intercept: $(0, 1)$

- point at $(1, b)$
- increasing if $b > 1$
- decreasing if $b < 1$

Figure $4.3.3$ compares the graphs of exponential growth and decay functions.

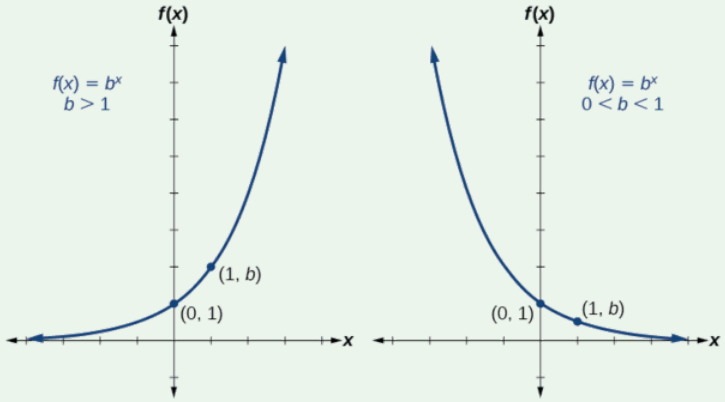


Figure 4.3.3

Example 4.3.1

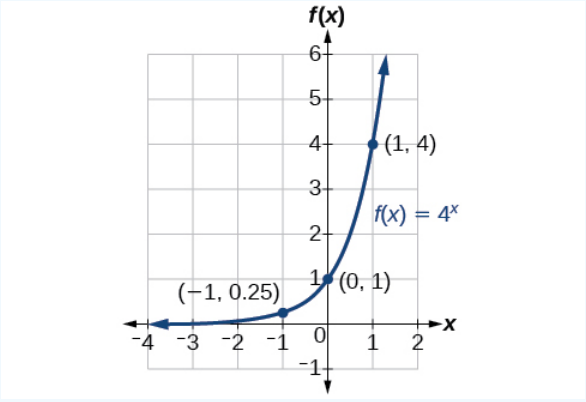
Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Solution

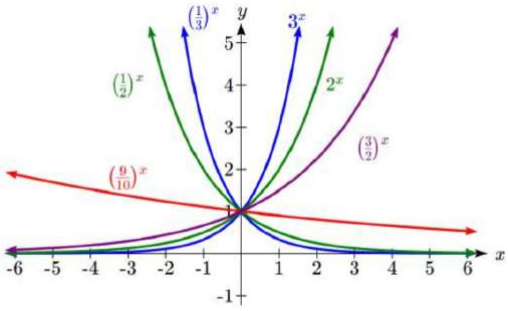
Before graphing, identify the behavior and key points for the graph.

- Since $b = 4$ is greater than one, we know the function is increasing. The left tail of the graph will approach the horizontal asymptote, $y = 0$, and the right tail will increase without bound.
- The y-intercept is $(0, 1)$.
- The point $(1, 4)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

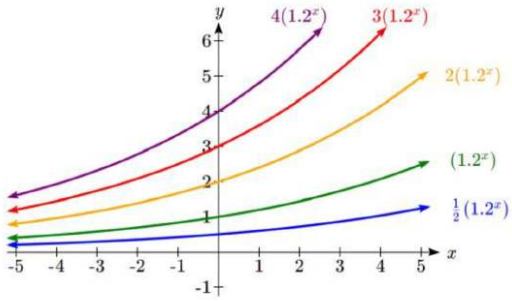


To get a better feeling for the effect of a and b on the graph of $f(x) = ab^x$, examine the sets of graphs below. The first set shows various graphs, where a remains the same and we only change the value for b.



Notice that the closer the value of b is to 1, the less steep the graph will be. The growth/decay is faster the further b is from 1.

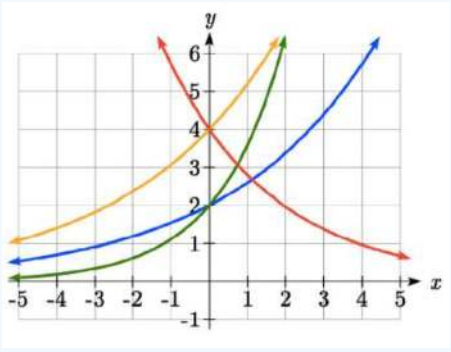
In the next set of graphs, a is altered and our value for b remains the same.



Notice that changing the value for a changes the y-intercept. Since a is multiplying the b^x term, a acts as a vertical stretch factor, not as a shift. Notice also that the end behavior for all of these functions is the same because the growth factor did not change and none of these a values introduced a vertical flip.

Example 4.3.2

Match each equation with its graph.



$$f(x) = 2(1.3)^x$$

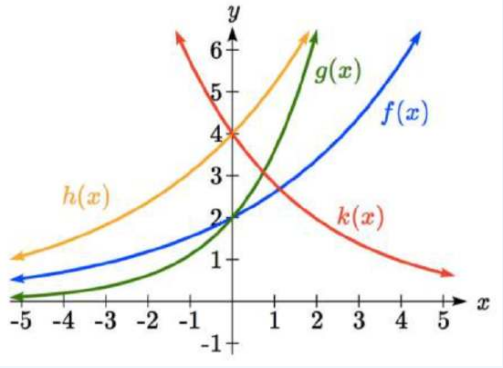
$$g(x) = 2(1.8)^x$$

$$h(x) = 4(1.3)^x$$

$$k(x) = 4(0.7)^x$$

Solution

The graph of $k(x)$ is the easiest to identify, since it is the only equation with a growth factor less than one, which will produce a decreasing graph. The graph of $h(x)$ can be identified as the only growing exponential function with a vertical intercept at (0,4). The graphs of $f(x)$ and $g(x)$ both have a vertical intercept at (0,2), but since $g(x)$ has a larger growth factor, we can identify it as the graph increasing faster.



Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other basic functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the basic function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the basic function $f(x) = b^x$, giving us a vertical shift d units in the same direction as the sign. For example, if we begin by graphing a basic function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure 4.3.4.

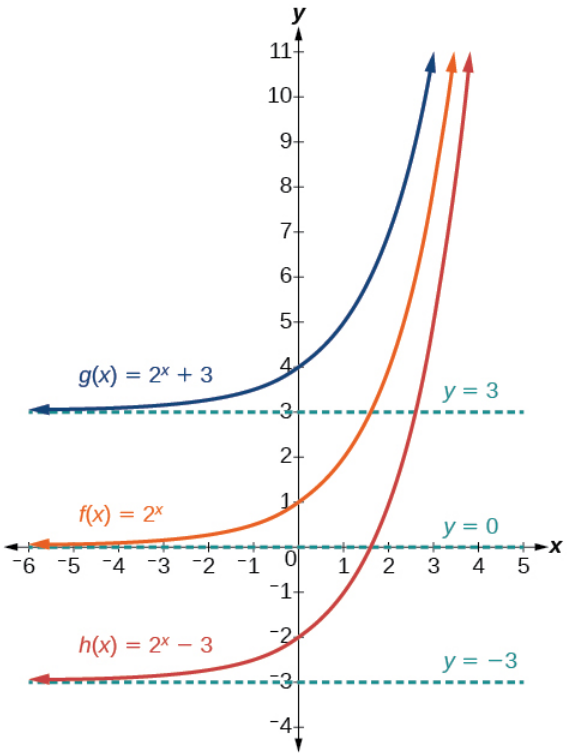


Figure 4.3.4

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The y-intercept shifts up 3 units to $(0, 4)$.

- The asymptote shifts up 3 units to $y = 3$.
- The range becomes $(3, \infty)$.

- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The *y*-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the basic function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the basic function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure 4.3.5.

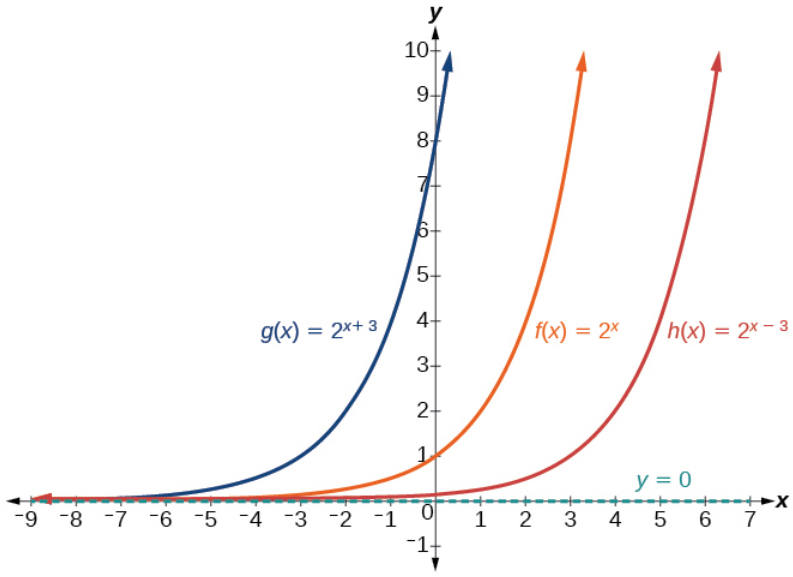


Figure 4.3.5

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$,remains unchanged.
- The asymptote, $y = 0$,remains unchanged.
- The *y*-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$, the y-intercept of the basic function shifts to (-3,1). The y-intercept of $g(x)$ is $(0, 8)$ which can be found by evaluating the function at $x = 0$.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$, the y-intercept of the basic function shifts to (3,1). The y-intercept of $h(x)$ is $(0, \frac{1}{8})$.

Example 4.3.3

Identify the transformations and use them to sketch the graph of $f(x) = 2^{x+1} - 3$. State the domain, range, and asymptote.

Solution

We have an exponential equation of the form $f(x) = b^{x+c} + d$, with $b = 2$, $c = 1$, and $d = -3$.

The basic function is $y = 2^x$. The graph will shift left 1 unit and down 3 units.

Shifting left 1 unit and down 3 units results in the y-intercept of the basic graph shifting to $(-1, -2)$. The point $(1, 2)$ on the basic graph shifts to $(0, -1)$. The horizontal asymptote $y = -3$.

Using these key points, asymptote, and the shape of $f(x) = 2^x$, shift the whole graph left 1 units and down 3 units.

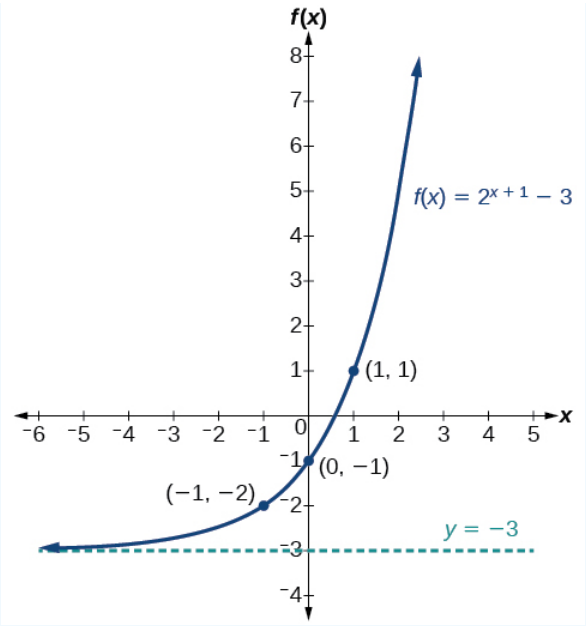


Figure 4.3.6

The domain is $(-\infty, \infty)$; the range is $(-3, \infty)$; the horizontal asymptote is $y = -3$.

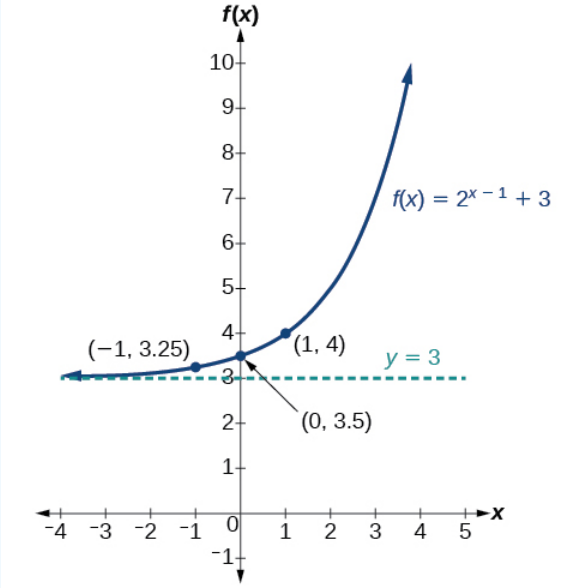
You Try 4.3.1

Identify the transformations and use them to sketch the graph of $f(x) = 2^{x-1} + 3$. State domain, range, and asymptote.

Answer

The basic function is $y = 2^x$. The graph will shift right 1 unit and up 3 units.

The domain is $(-\infty, \infty)$; the range is $(3, \infty)$; the horizontal asymptote is $y = 3$.



Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the x-axis or the y-axis. When we multiply the basic function $f(x) = b^x$ by -1, we get a reflection about the x-axis. When we multiply the input by -1, we get a reflection

about the y-axis. The reflection about the *x*-axis, $g(x) = -2^x$, is shown on the left side of Figure 4.3.7, and the reflection about the y-axis $h(x) = 2^{-x}$, is shown on the right side of Figure 4.3.7.

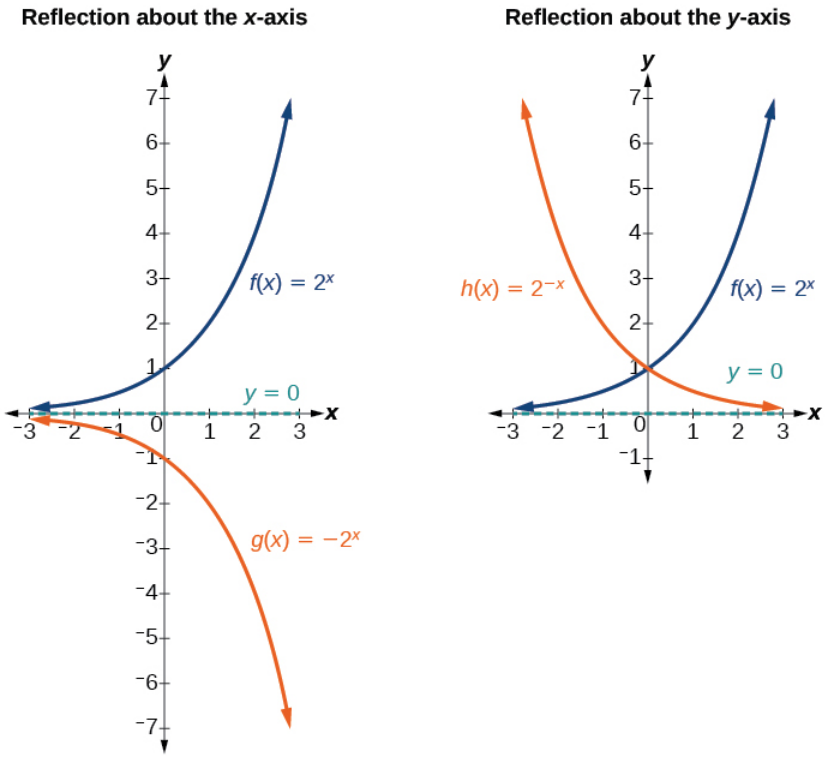


Figure 4.3.7: (a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the y-axis.

Example 4.3.4

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = \left(\frac{1}{4}\right)^x$ about the x-axis. State its domain, range, and asymptote.

Solution

Since we want to reflect the basic function $f(x) = \left(\frac{1}{4}\right)^x$ about the x-axis, we multiply $f(x)$ by -1 to get, $g(x) = -\left(\frac{1}{4}\right)^x$.

Next we create a table of points as in Table 4.3.3.

Table 4.3.3

x	-3	-2	-1	0	1	2	3
$g(x) = -(\frac{1}{4})^x$	-64	-16	-4	-1	-0.25	$\mathbf{-0.0625}$	$\mathbf{-0.0156}$

Plot the y-intercept, $(0, -1)$, along with two other points. We can use $(-1, -4)$ and $(1, -0.25)$.

Draw a smooth curve connecting the points:

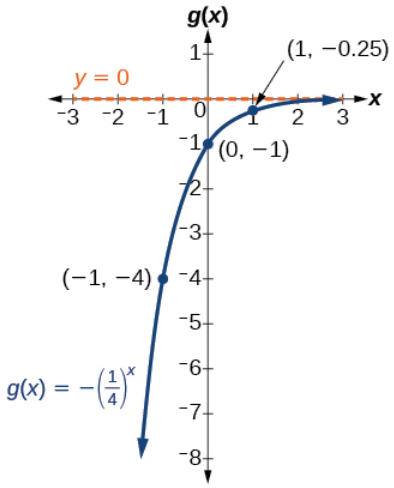


Figure 4.3.8

The domain is $(-\infty, \infty)$; the range is $(-\infty, 0)$; the horizontal asymptote is $y = 0$.

Example 4.3.5

Identify the transformations and use them to sketch the graph of $h(x) = 3^{-x+2} - 4$. State domain, range, and asymptote.

Solution

The basic function is $y = 3^x$. Since x is negative, we must first factor so we can identify the transformations.

$$\begin{aligned} f(x) &= 3^{-x+2} - 4 \\ &= 3^{-(x-2)} - 4 \end{aligned}$$

There is a y-axis reflection, a horizontal shift to the right 2 units, and a vertical shift down 4 units. The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = -4$.

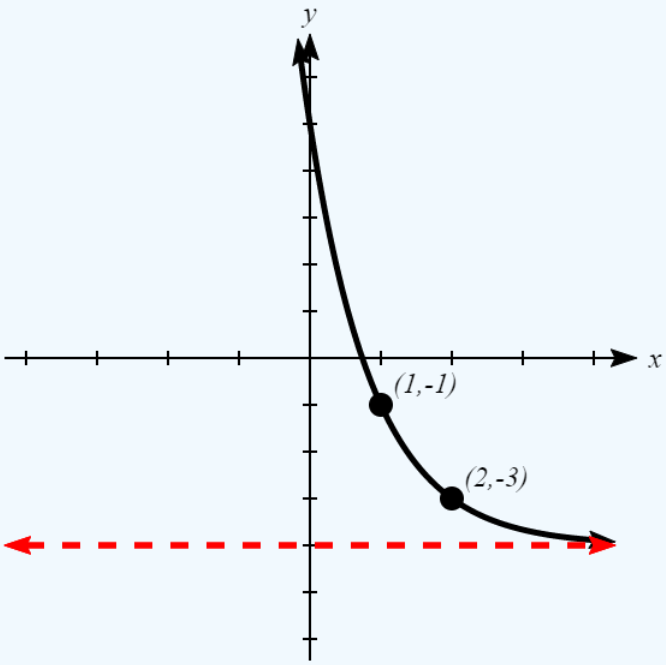


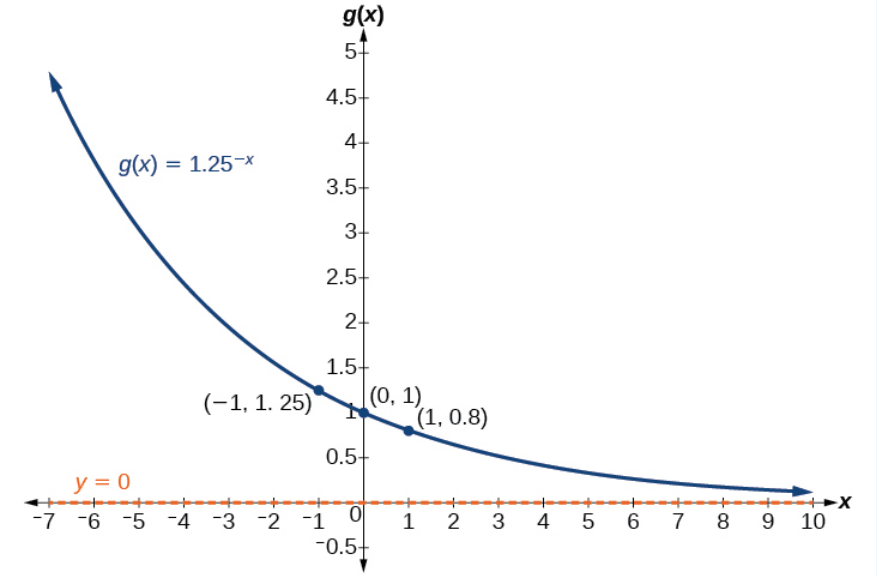
Figure 4.3.9: Graph of Transformed Exponential Function h(x).

? You Try 4.3.2

Find and graph the equation for a function, $g(x)$, that reflects $f(x) = 1.25^x$ about the y-axis. State its domain, range, and asymptote.

Answer

The equation of the reflected graph is $y == 1.25^{-}x$; the domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a vertical stretch or compression occurs when we multiply the basic function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the basic function $f(x) = 2^x$,we can then graph the stretch, using $a = 3$,to get $g(x) = 3(2)^x$ as shown on the left in Figure 4.3.10, and the compression, using $a = \frac{1}{3}$,to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure 4.3.10.

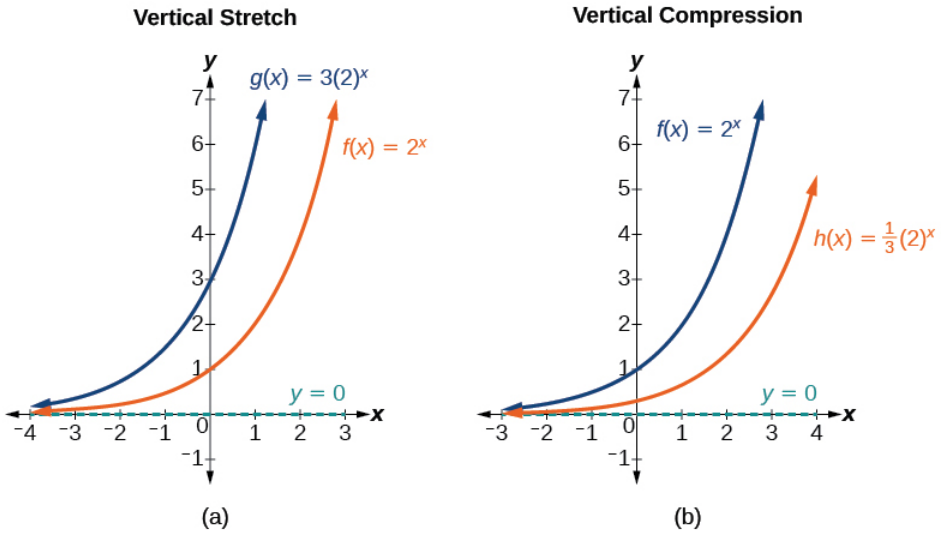


Figure 4.3.10: (a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

Example 4.3.6

Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ this is a decreasing exponential function. The left end of the graph will increase without bound as x decreases, and the right tail will approach the x-axis as x increases.
- Since $a = 4$, the graph of $f(x) = 4\left(\frac{1}{2}\right)^x$ will be stretched by a factor of 4.
- Create a table of points as shown in Table 4.3.4.

Table 4.3.4

x	-3	-2	-1	0	1	2	3
$f(x) = 4(\frac{1}{2})^x$	32	16	8	4	2	1	0.5

- Plot the y-intercept, $(0, 4)$, along with two other points. We can use $(-1, 8)$ and $(1, 2)$. You may plot other points so your graph is more accurate also.

Draw a smooth curve connecting the points, as shown in Figure 4.3.11.

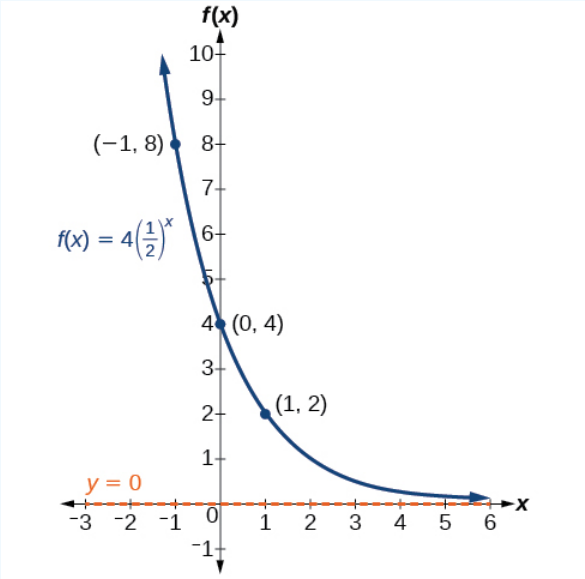


Figure 4.3.11

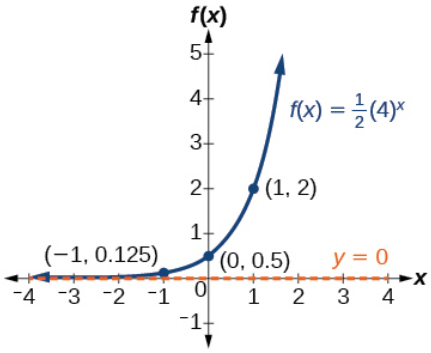
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

You Try 4.3.3

Sketch the graph of $f(x) = \frac{1}{2}(4)^x$. State the domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Summary of Transformations of Exponential Functions

Now that we have worked with each type of transformation for the exponential function, we can summarize them to arrive at the general equation for translating exponential functions.

Transformations of Exponential Functions

A transformation of an exponential function has the form

$$f(x) = ab^{x+c} + d$$

where the basic function, $y = b^x$, $b > 1$, is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- shifted vertically d units.
- reflected about the x-axis when $a < 0$.

For $f(x) = b^{-x}$, the graph of the basic function is reflected about the y-axis.

Remember that the order for transformations follow the order of operations. This means that you start with stretch/compressions and reflections first, then shifts.

You Try 4.3.4

What is the domain, range, asymptote, and end behavior of $y = 8 - e^{-x+7}$?

Answer

The domain is $(-\infty, \infty)$, the range is $(-\infty, 8)$, the vertical asymptote is at $x = 8$, as $x \to -\infty$, $y \to -\infty$, and as $x \to \infty$, $y \to 8$.

Finding Equations of Exponential Functions that Model Graphical Information

Example 4.3.7

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the y-axis, and then shifted up 4 units.

Solution

We want to find an equation of the general form $f(x) = ab^{x+c} + d$. We use the description provided to find a, b, c, and d.

- We are given the basic function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of 2, so $a = 2$.
- There is no horizontal shift so $c = 0$.

- The function is reflected about the *y*-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$$\begin{aligned} f(x) &= ab^{x+c} + d \\ &= 2e^{-x+0} + 4 \\ &= 2e^{-x} + 4 \end{aligned}$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

You Try 4.3.5

Write the equation for function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is compressed vertically by a factor of $\frac{1}{3}$, reflected across the *x*-axis and then shifted down 2 units.

Answer

$f(x) = -\frac{1}{3}e^x - 2$; the domain is $(-\infty, \infty)$; the range is $(-\infty, 2)$; the horizontal asymptote is $y = 2$.

Example 4.3.8

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$. Round to the nearest fourth decimal place.

Solution

Because we are not given the base for the function, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = ab^{-2}$
- Substituting $(2, 1)$ gives $1 = ab^2$

Use the first equation to solve for a in terms of b:

$$\begin{aligned} 6 &= ab^{-2} \\ 6 &= \frac{a}{b^2} && \text{Use properties of exponents to rewrite with positive expoents} \\ 6b^2 &= a && \text{Multiply by common denominator} \end{aligned}$$

Substitute a in the second equation, and solve for b:

$$\begin{aligned} 1 &= ab^2 \\ 1 &= 6b^2b^2 && \text{Substitute a} \\ 1 &= 6b^4 && \text{Now, solve for b} \\ \frac{1}{6} &= b^4 \\ \left(\frac{1}{6}\right)^{\frac{1}{4}} &= \left(b^4\right)^{\frac{1}{4}} \\ b &= \left(\frac{1}{6}\right)^{\frac{1}{4}} && \text{Round 4 decimal places rewrite the denominator} \\ b &\approx 0.6389 \end{aligned}$$

Use the value of b in the first equation to solve for the value of a:

$$\begin{aligned} a &= 6b^2 \\ &\approx 6(0.6389)^2 \\ &\approx 2.4492 \end{aligned}$$

Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in Figure $4.3.12$ passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

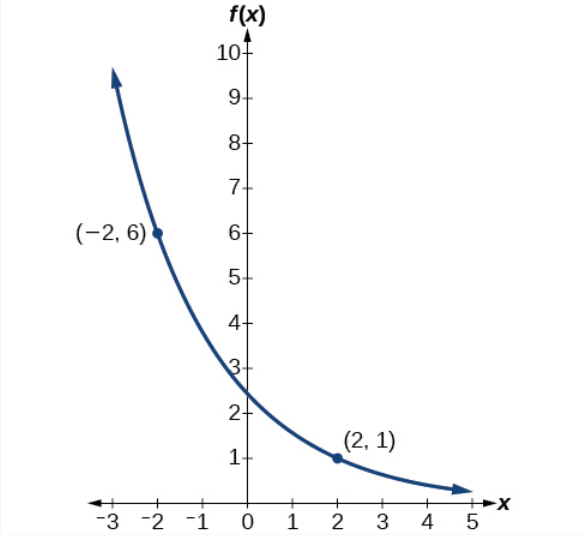


Figure 4.3.12: The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.

You Try 4.3.6

Given the two points $(1, 3)$ and $(2, 4.5)$, find the equation of the exponential function that passes through these two points.

Answer

$f(x) = 2(1.5)^x$

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4.4: Logarithmic Functions

Learning Objectives

- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms.
- Use common logarithms.
- Use natural logarithms.
- Finding inverses of exponential and logarithmic functions.

In 2010, a major earthquake struck Haiti, destroying or damaging over 285,000 homes. One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332,000 buildings, like those shown in Figure 4.4.1. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale whereas the Japanese earthquake registered a 9.0.



Figure 4.4.1: Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce).

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is

$$10^{8-4} = 10^4 = 10,000$$

times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in Figure 4.4.2, to better estimate the solution.

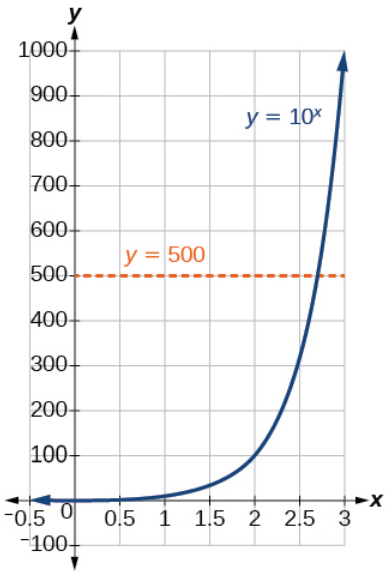


Figure 4.4.2

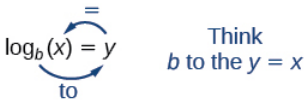
Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure 4.4.2 passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, "The logarithm with base b of x is equal to y," or, simplified, "log base b of x is y." We can also say, "b raised to the power of y is x," because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1$$

Note that the base b is always positive.



Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:

Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log^b(x)$ and $y = b^x$ are inverse functions.

Logarithmic Function

A logarithm base b of a positive number x satisfies the following definition.

For $x > 0, b > 0, b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x$$

where,

- we read $\log_b(x)$ as, “the logarithm with base b of x” or the “log base b of x."
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

We call the input, x, the *argument* of the function.

Example 4.4.1

Write the following logarithmic equations in exponential form.

a. $\log_3(9) = 2$
b. $\log_6(\sqrt{6}) = \frac{1}{2}$
c. $\log_5(1) = 0$
d. $\log_{10}(1,000,000) = 6$

Solution

First, identify the values of b, y,and x. Then, write the equation in the form $b^y = x$.

a. $\log_3(9) = 2$

Here, $b = 3$, $y = 2$,and $x = 9$. Therefore, the equation $\log_3(9) = 2$ is equivalent to

$$3^2 = 9$$

b. $\log_6(\sqrt{6}) = \frac{1}{2}$

Here, $b = 6$, $y = \frac{1}{2}$,and $x = \sqrt{6}$. Therefore, the equation $\log_6(\sqrt{6}) = \frac{1}{2}$ is equivalent to

$$6^{\frac{1}{2}} = \sqrt{6}$$

c. $\log_5(1) = 0$

Here, $b = 5$, $y = 0$,and $x = 1$. Therefore, the equation $\log_5(1) = 0$ is equivalent to

$$5^0 = 1$$

You Try 4.4.1

Write the following logarithmic equations in exponential form.

a. $\log_{10}(1,000,000) = 6$
b. $\log_5(25) = 2$

Answer a

$\log_{10}(1,000,000) = 6$is equivalent to $10^6 = 1,000,000$

Answer b

$\log_5(25) = 2$ is equivalent to $5^2 = 25$

By establishing the relationship between exponential and logarithmic functions, we can now solve basic logarithmic equations by rewriting them in exponential form.

Example 4.4.2

Solve $\log_4(x) = 2$ for x.

Solution

By rewriting this expression as an exponential, $4^2 = x$, so $x = 16$.

You Try 4.4.2

Solve $\log_2(t) = 5$ for t.

Answer

$t = 32$

Converting from Exponential to Logarithmic Form

To convert from exponents to logarithms, we follow the same steps in reverse. We identify the base b,exponent x,and output y. Then we write $x = \log_b(y)$.

Example 4.4.3

Write the following exponential equations in logarithmic form.

a. $2^3 = 8$
b. $4^2 = 16$
c. $10^{-4} = \dfrac{1}{10,000}$

Solution

First, identify the values of b, y,and x. Then, write the equation in the form $x = \log_b(y)$.

a. $2^3 = 8$

Here, $b = 2$, $x = 3$,and $y = 8$. Therefore, the equation $2^3 = 8$ is equivalent to $\log_2(8) = 3$.

b. $4^2 = 16$

Here, $b = 4$, $x = 2$,and $y = 16$. Therefore, the equation $4^2 = 16$ is equivalent to $\log_4(16) = 2$.

c. $10^{-4} = \dfrac{1}{10,000}$

Here, $b = 10$, $x = -4$,and $y = \dfrac{1}{10,000}$. Therefore, the equation $10^{-4} = \dfrac{1}{10,000}$ is equivalent to $\log_{10}\left(\dfrac{1}{10,000}\right) = -4$.

You Try 4.4.3

Write the following exponential equations in logarithmic form.

a. $3^2 = 9$
b. $5^3 = 125$
c. $2^{-1} = \dfrac{1}{2}$

Answer a

$3^2 = 9$ is equivalent to $\log_3(9) = 2$

Answer b

$5^3 = 125$ is equivalent to $\log_5(125) = 3$

Answer c

$2^{-1} = \frac{1}{2}$ is equivalent to $\log_2\left(\frac{1}{2}\right) = -1$

By establishing the relationship between exponential and logarithmic functions, we can now solve basic exponential equations also.

Example 4.4.4

Solve $2^x = 10$ for x.

Solution

By rewriting this expression as a logarithm, we get $x = \log_2(10)$.

You Try 4.4.4

Solve $3^m = 11$ for m.

Answer

$m = \log_3(11)$

Evaluating Logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider $\log_2 8$. We ask, "To what exponent must 2 be raised in order to get 8?" Because we already know $2^3 = 8$, it follows that $\log_2 8 = 3$.

Now consider solving $\log_7 49$ and $\log_3 27$ mentally.

- We ask, "To what exponent must 7 be raised in order to get 49?" We know $7^2 = 49$. Therefore, $\log_7 49 = 2$
- We ask, "To what exponent must 3 be raised in order to get 27?" We know $3^3 = 27$. Therefore, $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate $\log_{\frac{2}{3}} \frac{4}{9}$ mentally.

- We ask, "To what exponent must $\frac{2}{3}$ be raised in order to get $\frac{4}{9}$?" We know $2^2 = 4$ and $3^2 = 9$, so

$$\left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Therefore,

$$\log_{\frac{2}{3}}\left(\frac{4}{9}\right) = 2.$$

Example 4.4.5

Evaluate the following without using a calculator:

1. $\log_4(64)$
2. $\log_3\left(\frac{1}{27}\right)$
3. $\log_5(-3)$

Solution

1. First let's set the expression equal to y, $y = \log_4(64)$, then we can rewrite the logarithm in exponential form: $4^y = 64$. Next, we ask, "To what exponent must 4 be raised in order to get 64?"

We know

$$4^3 = 64$$

So

$$4^y = 4^3$$

Therefore,

$$\log_4(64) = 3$$

2. First, set the expression equal to a variable, $y = \log_3\left(\frac{1}{27}\right)$, Then rewrite the logarithm in exponential form: $3^y = \frac{1}{27}$.

   Next, we ask, "To what exponent must 3 be raised in order to get $\frac{1}{27}$?"

   We know $3^3 = 27$,but what must we do to get the reciprocal, $\frac{1}{27}$? Recall from working with exponents that $b^{-a} = \frac{1}{b^a}$. We use this information to write

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

   Therefore, $\log_3\left(\frac{1}{27}\right) = -3$.

3. Logarithms are only defined for positive values so $y = \log_5(-3)$ is undefined.

? You Try 4.4.5

Evaluate the following without using a calculator:

1. $\log_{121}(11)$
2. $\log_2\left(\frac{1}{32}\right)$

Answer

1. $\log_{121}(11) = \frac{1}{2}$
2. $\log_2\left(\frac{1}{32}\right) = -5$

Properties of Logarithms

We saw above that it was convenient to convert to exponential form to evaluate logarithms. Recall that $\log_4(64) = 3$. We know that $4^3 = 64$, so we substitute this in, we get

$$\log_4(64) = \log_4(4^3) = 3$$

This gives us our first property for logarithms:

$$\log_b(b^x) = x$$

Next, consider $x = \log_3 1$. Converting to exponential form, $3^x = 1$. From properties of exponents we know $x = 0$. We can conclude that \({\log}_31\=0). This bring sus to our second property for logarithms:

$$\log_b 1 = 0$$

Similarly, if we convert $x = \log_b b$ to exponetial form we will find that $x = \log_b b = 1$. This is our third property for logarithms:

$$\log_b b = 1$$

To evaluate $x = 4_4^{\log}(9)$, we can rewrite in logarithmic form as $\log_4 9 = \log_4 x$. This means $x = 9$. This give us the following property:

$b^{\log_b x} = x$ where $x > 0$

Properties of Logarithms

- $log_b(b^x) = x$
- $\log_b 1 = 0$
- $\log_b b = 1$
- $b^{\log_b x} = x$ where $x > 0$

Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base 10 logarithm a **common logarithm**. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

Common Logarithm

A common logarithm is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x$$

We read $\log(x)$ as, "the logarithm with base 10 of x " or "log base 10 of x."

The logarithm y is the exponent to which 10 must be raised to get x.

Example 4.4.6

Evaluate $y = \log(1000)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know

$$10^3 = 1000$$

Therefore, $\log(1000) = 3$.

You Try 4.4.6

Evaluate $y = \log(1,000,000)$ without using a calculator.

Answer

$\log(1,000,000) = 6$

Example 4.4.7

Evaluate $y = log(321)$ and round to four decimal places using a calculator.

Solution

Rounding to four decimal places, $\log(321) \approx 2.5065$

Analysis

Note that $10^2 = 100$ and that $10^3 = 1000$. Since 321 is between 100 and 1000, we know that $\log(321)$must be between $\log(100)$and $\log(1000)$. This gives us the following:

$$100 < 321 < 1000$$

$$2 < 2.5065 < 3$$

When using a calculator when approximating logarithms, it is important to make sure you input everything properly including having parentheses in the corect place.

Example 4.4.8

Use your calculator to approximate $log3 - 4$ to four decimal places.

Solution

Remember that sometimes parentheses are not written, $log3 - 4$ is equivalent to $log(3) - 4$.

$log3 - 4 \approx -3.5229$

You Try 4.4.7

Evaluate $y = \log(123)$and round to four decimal places using a calculator.

Answer

$\log(123) \approx 2.0899$

Example 4.4.9

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation $10^x = 500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. Rounding to three decimal places, what was the difference in magnitudes?

Solution

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$

$$\log(500) = x \text{ Use the definition of the common log.}$$

Next we evaluate the logarithm using a calculator:

To the nearest thousandth, $\log(500) \approx 2.699$

The difference in magnitudes was about 2.699.

You Try 4.4.8

The amount of energy released from one earthquake was $8,500$times greater than the amount of energy released from another. The equation $10^x = 8500$ represents this situation, where x is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

Answer

The difference in magnitudes was about 3.929

Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation,$\ln(x)$. Most values of $\ln(x)$ can be found

only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the ln key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

Natural Logarithm

A natural logarithm is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

Example 4.4.10

Approximate $y = \ln(500)$ to four decimal places using a calculator.

Solution

Rounding to four decimal places, $\ln(500) \approx 6.2146$

You Try 4.4.9

Approximate $\ln(-500)$ to four decimal places using a calculator.

Answer

It is not possible to take the logarithm of a negative number so $\ln(-500)$ is undefined.

You Try 4.4.10

Solve $\ln(a) = 7$ for a.

Answer

$a = e^7$

Note that this answer is exact as e is an irrational number. To represent as a decimal approximation when asked, use your calculator to approximate.

Note

There a couple of things worth mentioning from what we have learned about logarithms in this section.

- Logarithms are functions, $f(x) = \log_b(x)$. We call the input, x, the *argument* of the function.
- When evaluating, sometimes we do not write parentheses around the argument. Example:

$$\log_7(4) \text{ is equivalent to } \log_7 4$$

- Common logarithms are written as $y = \log(x)$. All other logarithms, excluding the natural logarithm must have the base written or else it is assumed to be a common logarithm
- The natural logarithm is written as $y = \ln(x)$. This is the only logarithm with a function name, ln, that is different than all other logarithms.

Let's end with some examples of using the relationship between exponential and logarithmic functions to find inverse functions.

Example 4.4.11

Given $J(x) = \log_3(x-2)+4$, find J^{-1}

Solution

$$\begin{aligned} J(x) &= \log_3(x-2)+4 \\ y &= \log_3(x-2)+4 \\ x &= \log_3(y-2)+4 \\ x-4 &= \log_3(y-2) \\ 3^{x-4} &= y-2 \\ 3^{x-4}+2 &= y \end{aligned}$$

Hence $J^{-1} = 3^{x-4}+2$

Notice that $J(x)$ is a basic logarithm function of base 3 that has a horizontal shift to the right 2 units and a vertical shift up 4 units. J^{-1} has a horizontal shift to the right 4 units and a vertical shift up 2 units. When working with transformed functions, the inverse will have its transformations reversed between the horizontal and vertical directions. Remember that the domain and range of functions and their inverse are also reversed.

You Try 4.4.11

Find the inverse of $V(x) = 5 - e^{x+3}$

Answer

$y = \ln(5-x)-3$

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4.5: Graphing Logarithmic Functions

Learning Objectives

- Identify the domain of logarithmic functions.
- Identify transformations of logarithmic functions.
- Graph logarithmic functions.
- Find equations of logarithmic functions that model graphical information.

Earlier in this chapter, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest $\$2500$ in an account that offers an annual interest rate of 5%, compounded continuously. We already know that the balance in our account for any year t can be found with the equation $A = 2500e^{0.05t}$.

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? Figure $4.5.1$ shows this point on the logarithmic graph.

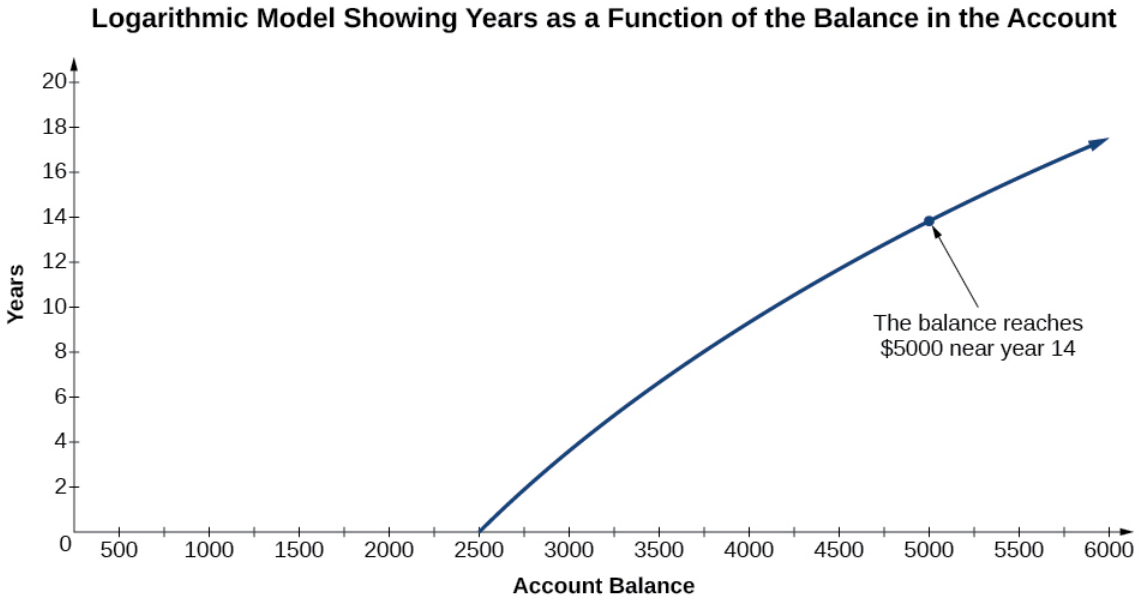


Figure 4.5.1

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.
- The range of y is $(0, \infty)$.

In the last section, we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the basic function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other basic functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the basic function without loss of shape.

We've seen that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the basic function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$, is greater than zero. To find the domain, we set up an inequality and solve for x:

$$\begin{aligned} 2x - 3 &> 0 && \text{Show the argument greater than zero} \\ 2x &> 3 && \text{Add 3} \\ x &> \frac{3}{2} && \text{Divide by 2} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $\left(\frac{3}{2}, \infty\right)$.

Example 4.5.1

Find the domain of the following functions:

1. $f(x) = \log_2(x + 3)$
2. $g(t) = 4 - \log(5 - 2t)$

Solution

1. The logarithmic function is defined only when the input is positive, so this function is defined when $x + 3 > 0$. Solving this inequality,

$$\begin{aligned} x + 3 &> 0 && \text{The input must be positive} \\ x &> -3 && \text{Subtract 3} \end{aligned}$$

   The domain of $f(x) = \log_2(x + 3)$ is $(-3, \infty)$.

2. This function is defined when $5 - 2t > 0$. Solving this inequality,

$$\begin{aligned} 5 - 2t &> 0 && \text{The input must be positive} \\ -2t &> -5 && \text{Subtract 5} \\ t &< \frac{5}{2} && \text{Divide by -2 and switch the inequality} \end{aligned}$$

   The domain of $g(t) = 4 - \log(5 - 2t)$ is $\left(-\infty, \frac{5}{2}\right)$.

Example 4.5.2

What is the domain of $h(x) = \log_4(x^2 - 4x + 3) + 2$?

Solution

This function is defined when $x^2 - 4x + 3 > 0$. Solving this inequality using a sign chart,

$$x^2 - 4x + 3 > 0$$

$$(x - 3)(x - 1) > 0$$

	(-∞,1)	(1,3)	(3,∞)
(x-3)	-	-	+
(x-1)	-	+	+
h(x)	+	-	+

Figure 4.5.2: Sign Chart for h(x)

The domain is $(-3, \infty)$

? You Try 4.5.1

What is the domain of $f(x) = \log(3x + 7) + 2$?

Answer

$(-/dfrac73, \infty)$

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the basic function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the basic function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table 4.5.1.

Table 4.5.1

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from Table 4.5.1, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See Table 4.5.2.

Table 4.5.2

$f(x) = 2^x$	$\left(-3, \frac{1}{8}\right)$	$\left(-2, \frac{1}{4}\right)$	$\left(-1, \frac{1}{2}\right)$	$(0, 1)$	$(1, 2)$	$(2, 4)$	$(3, 8)$
$g(x) = \log_2(x)$	$\left(\frac{1}{8}, -3\right)$	$\left(\frac{1}{4}, -2\right)$	$\left(\frac{1}{2}, -1\right)$	$(1, 0)$	$(2, 1)$	$(4, 2)$	$(8, 3)$

As we'd expect, the x- and y-coordinates are reversed for the inverse functions. Figure 4.5.3 shows the graph of f and g.

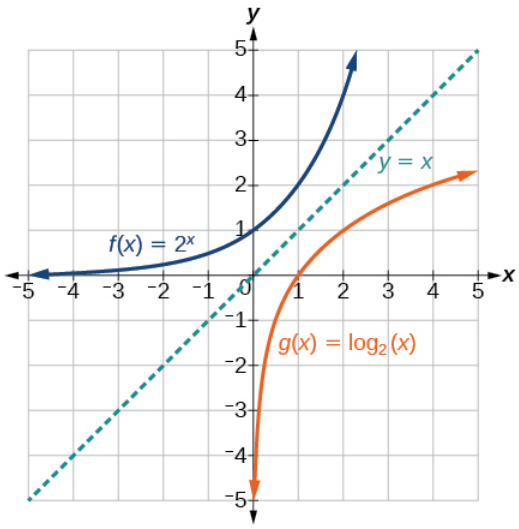


Figure 4.5.3: Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a y-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an x- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

Characteristics of the Graph of the Basic Function, $f(x) = log_b(x)$

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- x-intercept: $(1, 0)$
- y-intercept: none
- point at $(b, 1)$
- increasing if $b > 1$
- decreasing if $0 < b < 1$

Figure 4.5.4 compares the graphs of logarithmic functions with $b > 1$ and $0 < b < 1$

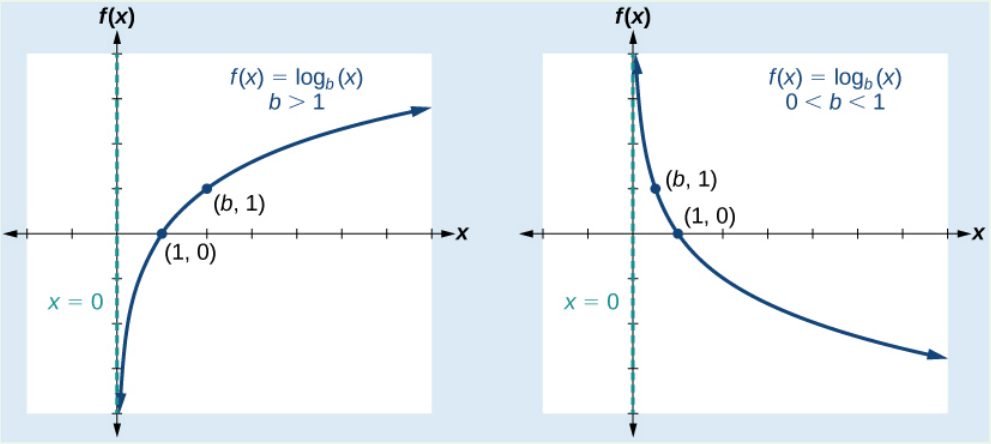


Figure 4.5.4

Figure 4.5.5 shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (Note: recall that the function $\ln(x)$ has base $e \approx 2.718$.)

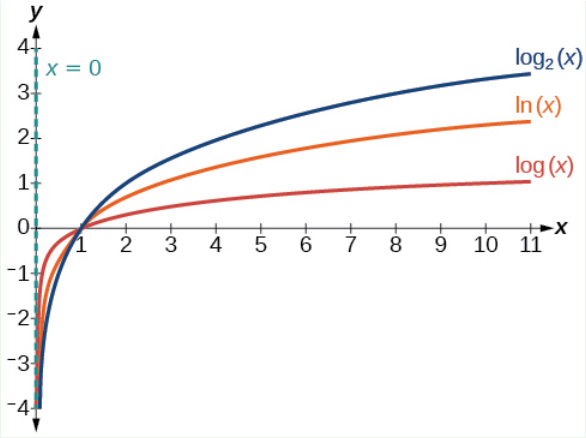


Figure 4.5.5: The graphs of three logarithmic functions with different bases, all greater than 1.

Example 4.5.3

Sketch the graph of $f(x) = \log_5(x)$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points for the graph.

- Since $b = 5$ is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote, $x = 0$, and the right tail will increase slowly without bound.
- The x-intercept is $(1, 0)$.
- The point $(5, 1)$ is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points (see Figure 4.5.6).

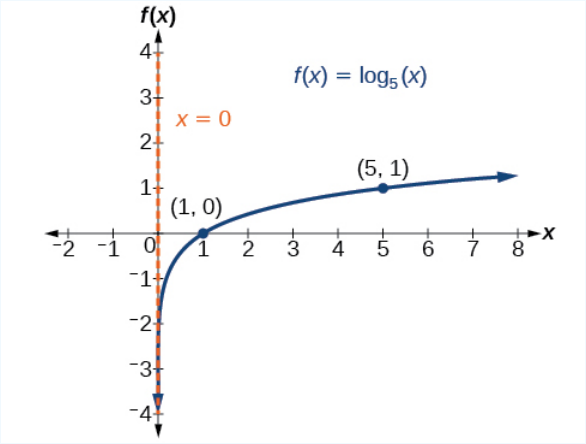


Figure 4.5.6

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other basic functions. We can shift, stretch, compress, and reflect the basic function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift

When a constant c is added to the input of the basic function $f(x) = \log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the basic function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 4.5.7.

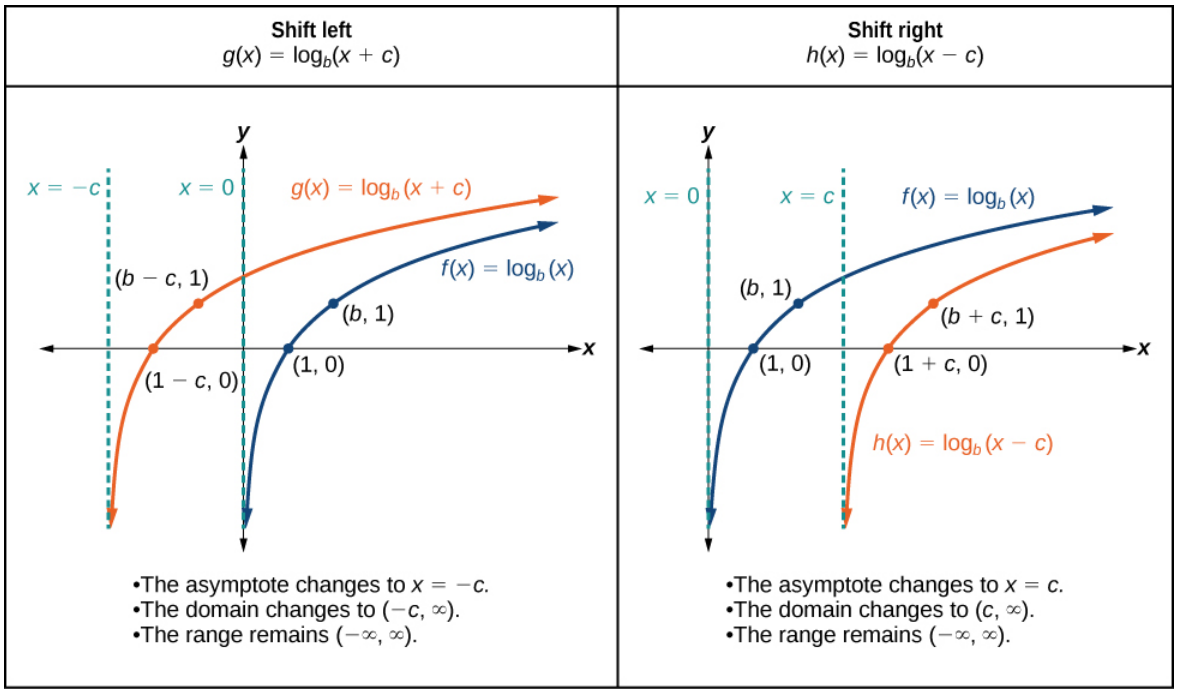


Figure 4.5.7

Example 4.5.4

Sketch the horizontal shift $f(x) = \log_3(x-2)$ alongside its basic function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = \log_3(x-2)$, we notice $x + (-2) = x - 2$.

Thus $c = -2$, so $c < 0$. This means we will shift the function $f(x) = \log_3(x)$ right 2 units.

The vertical asymptote is $x = -(-2)$ or $x = 2$.

Consider the three key points from the basic function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by adding 2 to the x coordinates.

Label the points $\left(\frac{7}{3}, -1\right)$, $(3, 0)$,and $(5, 1)$.

The domain is $(2, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 2$.

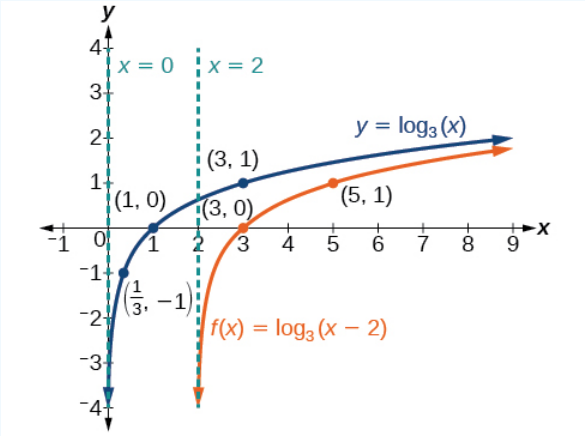


Figure 4.5.8

? You Try 4.5.2

Sketch a graph of $f(x) = \log_3(x+4)$ alongside its basic function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

Answer

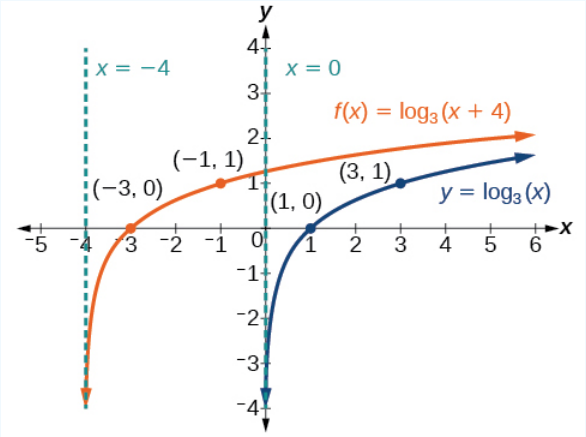


Figure 4.5.9

The domain is $(-4, \infty)$,the range $(-\infty, \infty)$,and the asymptote $x = -4$.

Graphing a Vertical Shift

When a constant d is added to the basic function $f(x) = \log_b(x)$,the result is a vertical shift d units in the direction of the sign on d. To visualize vertical shifts, we can observe the general graph of the basic function $f(x) = \log_b(x)$ alongside the shift up, $g(x) = \log_b(x) + d$ and the shift down, $h(x) = \log_b(x) - d$ .See Figure 4.5.10.

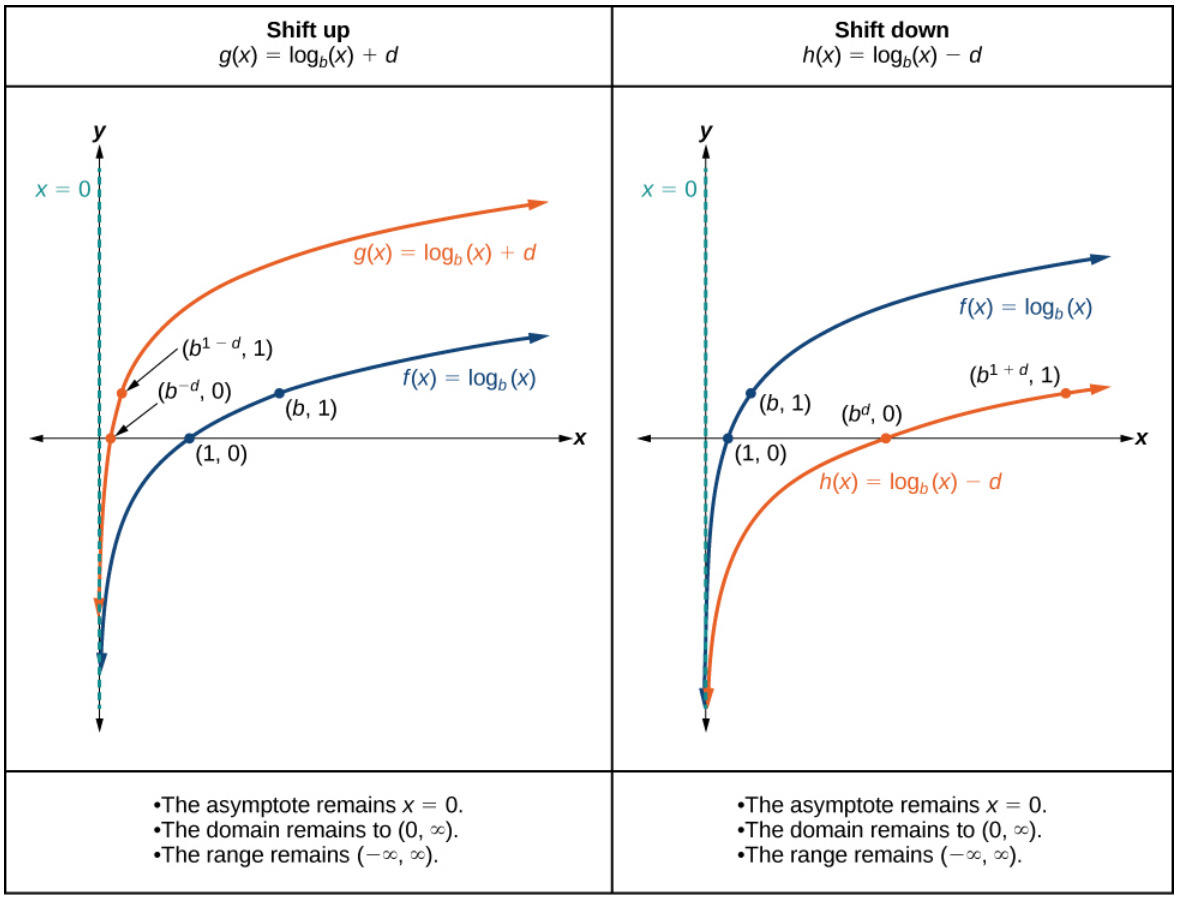


Figure 4.5.10

Example 4.5.5

Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its basic function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = \log_3(x) - 2$,we will notice $d = -2$. Thus $d < 0$.

This means we will shift the function $f(x) = \log_3(x)$ down 2 units.

The vertical asymptote is $x = 0$.

Consider the three key points from the basic function, $\left(\frac{1}{3}, -1\right)$, $(1, 0)$, and $(3, 1)$.

The new coordinates are found by subtracting 2 from the y coordinates.

Label the points $\left(\frac{1}{3}, -3\right)$, $(1, -2)$, and $(3, -1)$.

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

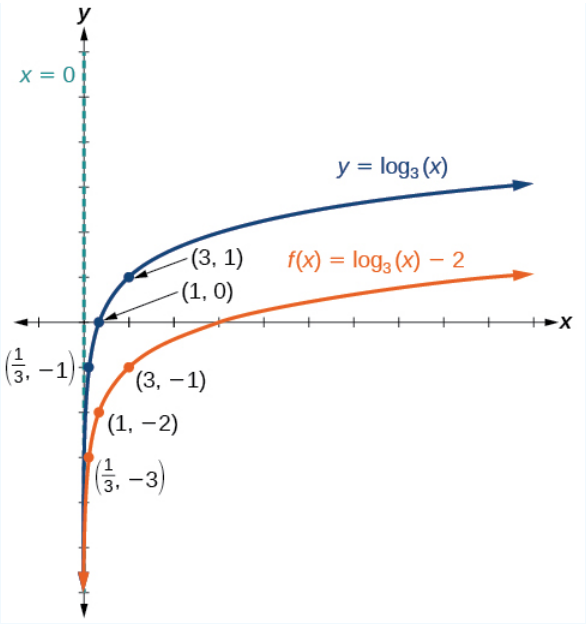


Figure 4.5.11

The domain is $(0, \infty)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Example 4.5.6

Identify the transformations and use them to sketch the graph of $L(x) = 2 + \log_3(x+1)$. State the domain, range, and asymptote.

Solution

We can rewrite $L(x) = 2 + \log_3(x+1)$ as $L(x) = \log_3(x+1) + 2$ which can be helpful for identifying transformations.

The basic function is $y = \log_3(x)$. The graph will shift left 1 unit and up 2 units.

Shifting left 1 unit and up 2 units results in the x-intercept of the basic graph shifting to $(0, 2)$. The point $(3, 1)$ on the basic graph shifts to $(2, 3)$. The horizontal asymptote $x = -1$.

Using these key points, asymptote, and the shape of $y = \log_3(x)$, shift the whole graph left 1 units and down 3 units.

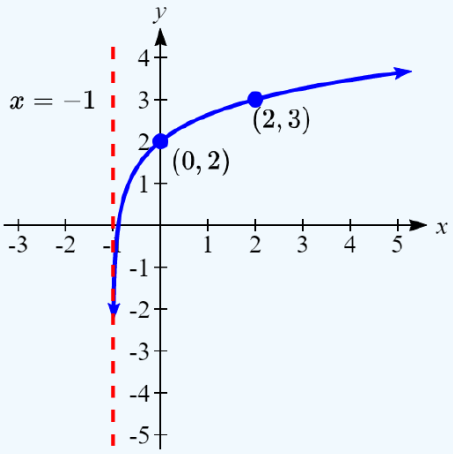


Figure 4.5.12: Graph of Transformed Logarithmic Function L(x)

The domain is $(-1, \infty)$, the range is $(-\infty, \infty)$, and the horizontal asymptote is $x = -1$

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You Try 4.5.3

Identify the transformations and use them to sketch the graph of $m(x) = \log_5(x-4) - 3$. State the domain, range, and asymptote.

Answer

The transformations are shift right 4 units and shift down 3 units.

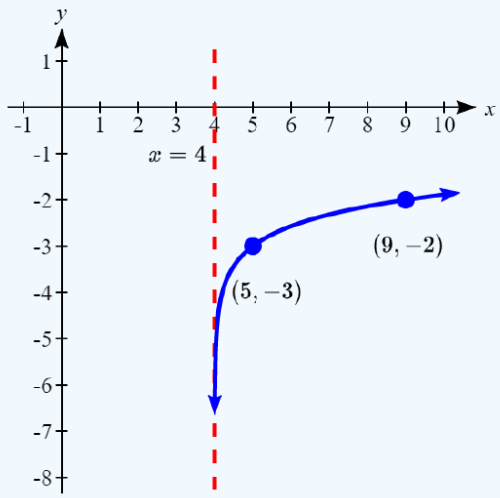


Figure 4.5.13: Graph of Transformed Logarithmic Function m(x)

The domain is $(4, \infty)$, the range is $(-\infty, \infty)$, and the horizontal asymptote is $x = 4$

Caution!

Remember that the input for a logarithm is called the argument. When the argument is a single term, it is a common practice to not write parentheses. Example:

$$f(x) = \log_4(x) \text{ is equivalent to } f(x) = \log_4 x$$

However, when the argument is more than one term, one must put parentheses around it. Example:

$$y = \ln(x+3) \text{ is not equivalent to } y = \ln x + 3$$

The function on the left is a natural logarithmic function shifted left 3 units, whereas, the function on the right is a natural logarithmic function shifted up 3 units.

In graphing calculators, one needs to make sure functions are inputted properly also.

Graphing Reflections of $f(x) = log_b(x)$

When the basic function $f(x) = \log_b(x)$ is multiplied by -1,the result is a reflection about the x-axis. When the *input* is multiplied by -1,the result is a reflection about the y-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the basic function $f(x) = \log_b(x)$ alongside the reflection about the x-axis, $g(x) = -\log_b(x)$ and the reflection about the y-axis, $h(x) = \log_b(-x)$.

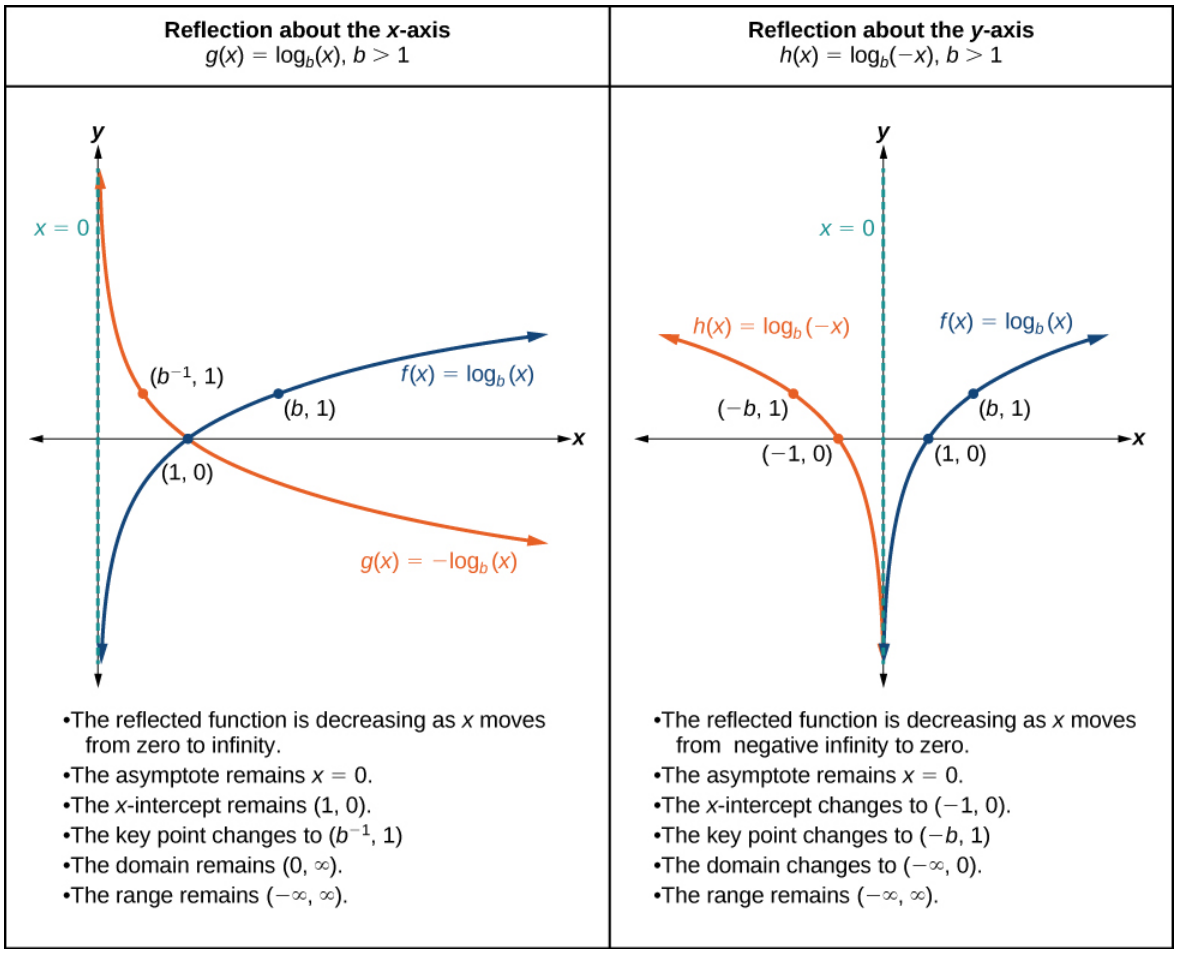


Figure 4.5.14

Example 4.5.7

Sketch a graph of $f(x) = \log(-x)$ alongside its basic function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Before graphing $f(x) = log(-x)$, $f(x) = log(-x)$,identify the behavior and key points for the graph.

- Since $b = 10$ is greater than one, we know that the basic function is increasing. Since the *input* value is multiplied by -1, $f(x)$ is a reflection of the basic graph about the y-axis. Thus, $f(x) = \log(-x)$ will be decreasing as x moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote $x = 0$.
- The x-intercept is $(-1, 0)$.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.

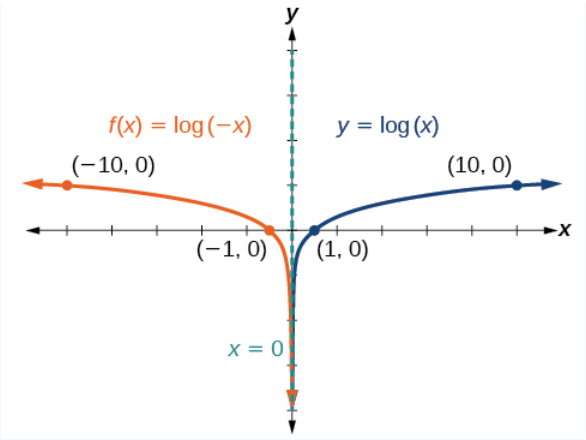


Figure 4.5.15

The domain is $(-\infty, 0)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

? You Try 4.5.4

Graph $f(x) = -\log(-x)$. State the domain, range, and asymptote.

Answer

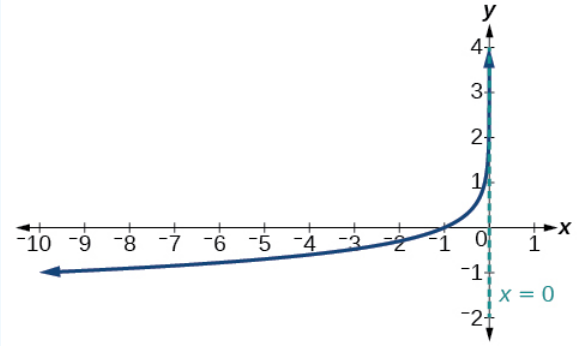


Figure 4.5.16

The domain is $(-\infty, 0)$,the range is $(-\infty, \infty)$,and the vertical asymptote is $x = 0$.

Graphing Stretches and Compressions of $y = log_b(x)$

When the basic function $f(x) = \log_b(x)$ is multiplied by a constant $a > 0$, the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set $a > 1$ and observe the general graph of the basic function $f(x) = \log_b(x)$ alongside the vertical stretch, $g(x) = a\log_b(x)$ and the vertical compression, $h(x) = \frac{1}{a}\log_b(x)$. See Figure 4.5.17.

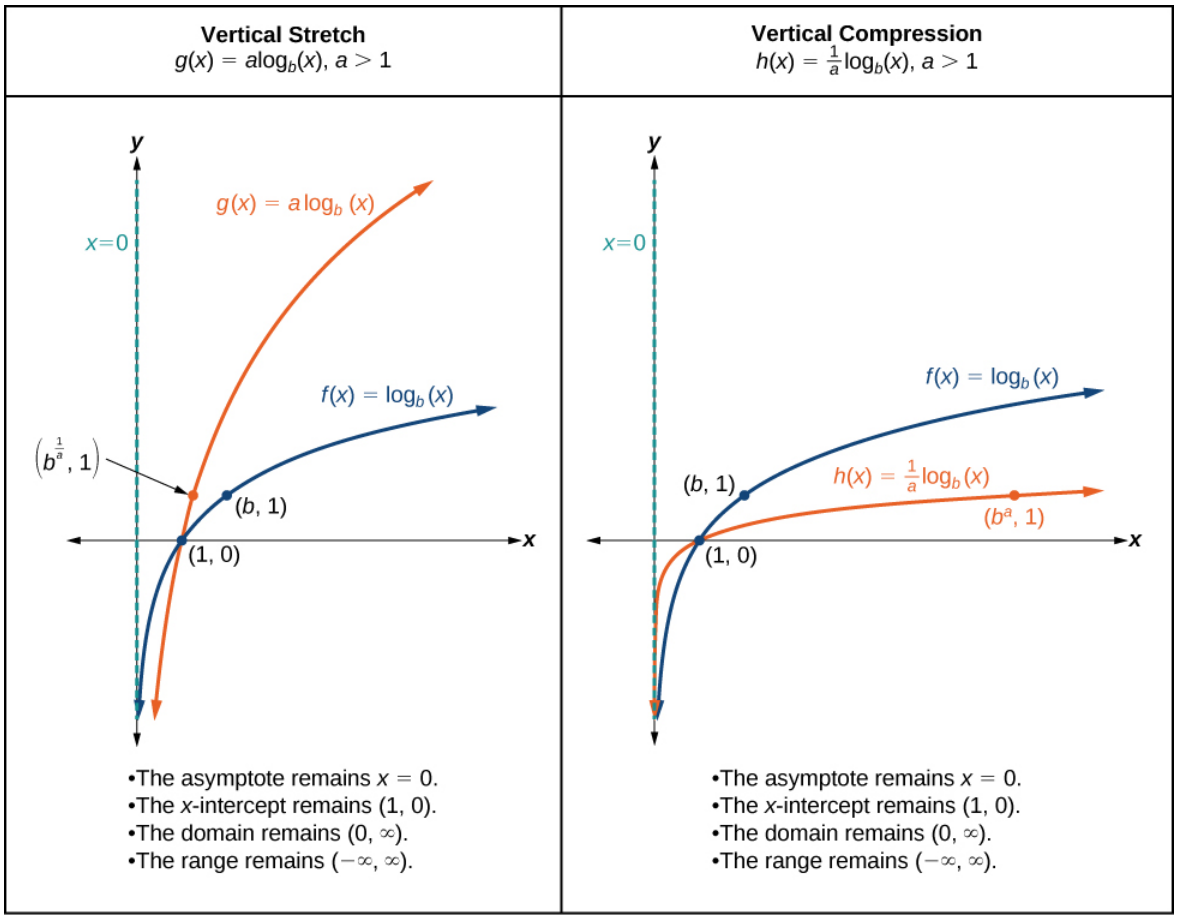


Figure 4.5.17

Example 4.5.8

Sketch a graph of $f(x) = 2\log_4(x)$ alongside its basic function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

Solution

Since the function is $f(x) = 2\log_4(x)$, we will notice $a = 2$.

This means we will stretch the function $f(x) = \log_4(x)$ by a factor of 2.

The vertical asymptote is $x = 0$.

Consider the three key points from the basic function, $\left(\frac{1}{4}, -1\right)$, $(1, 0)$, and $(4, 1)$.

The new coordinates are found by multiplying the y coordinates by 2.

Label the points $\left(\frac{1}{4}, -2\right)$, $(1, 0)$, and $(4, 2)$.

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$. See Figure 4.5.18

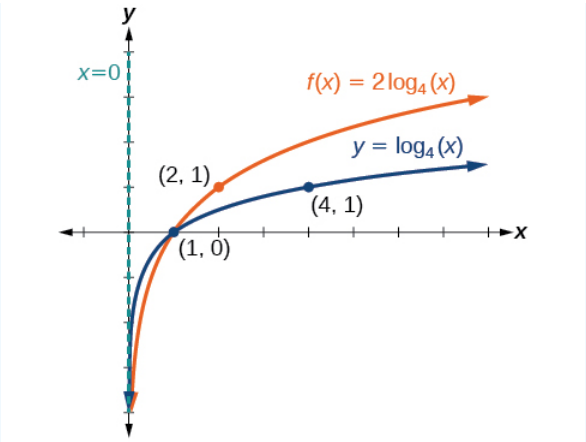


Figure 4.5.18

The domain is $(0, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 0$.

Example 4.5.9

Sketch a graph of $f(x) = 5\log(x+2)$. State the domain, range, and asymptote.

Solution

Remember: what happens inside basicheses happens first. First, we move the graph left 2 units, then stretch the function vertically by a factor of 5, as in Figure 4.5.19 The vertical asymptote will be shifted to $x = -2$. The x-intercept will be $(-1, 0)$. The domain will be $(-2, \infty)$. Two points will help give the shape of the graph: $(-1, 0)$ and $(8, 5)$. We chose $x = 8$ as the x-coordinate of one point to graph because when $x = 8$, $x + 2 = 10$, the base of the common logarithm.

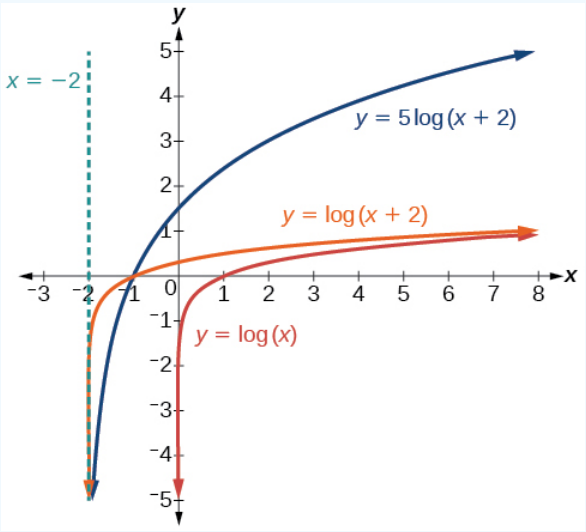


Figure 4.5.19

The domain is $(-2, \infty)$, the range is $(-\infty, \infty)$,and the vertical asymptote is $x = -2$.

You Try 4.5.5

Sketch a graph of the function $f(x) = 3\log(x-2)+1$. State the domain, range, and asymptote.

Answer

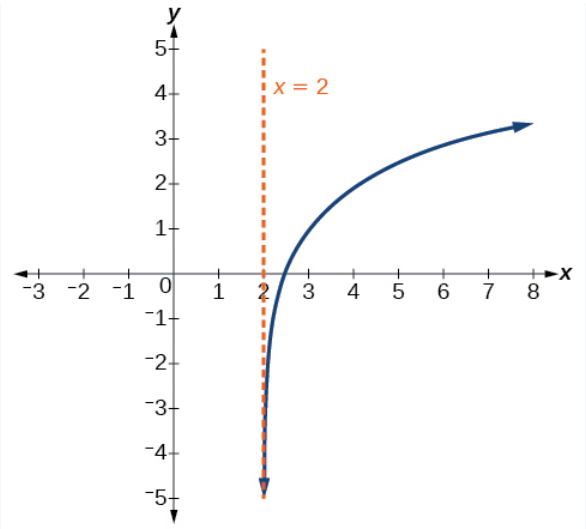


Figure 4.5.20

The domain is $(2, \infty)$, the range is $(-\infty, \infty)$, and the vertical asymptote is $x = 2$.

Summarizing Transformations of the Logarithmic Function

Now that we have worked with each type of transformation for the logarithmic function, we can summarize each to arrive at the general equation for translating exponential functions.

Transformations of Logarithmic Functions

All transformations of the basic logarithmic function, $y = \log_b(x)$, have the form

$$f(x) = a\log_b(x + c) + d$$

where the basic function, $y = \log_b(x)$, $b > 1$,is

- shifted vertically up d units.
- shifted horizontally to the left c units.
- stretched vertically by a factor of $|a|$ if $|a| > 0$.
- compressed vertically by a factor of $|a|$ if $0 < |a| < 1$.
- reflected about the x-axis when $a < 0$.

For $f(x) = \log(-x)$, the graph of the basic function is reflected about the y-axis.

Example 4.5.10

What is the vertical asymptote and domain of $f(x) = -2\log_3(x + 4) + 5$?

Solution

The vertical asymptote is at $x = -4$.

Analysis

The coefficient, the base, and the upward transformation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to$x = -4$. The domain is $(-4, \infty)$

You Try 4.5.6

What is the vertical asymptote and domain of of $f(x) = 3 + \ln(-x + 1)$?

Answer

The vertical asymptote is $x = 1$. The domain is $(-\infty, 1)$

Finding Equations of Logarithmic Functions that Model Graphical Information

Example 4.5.11

Write an equation of the natural logarithmic function that has no vertical stretch or compression that is graphed in Figure 4.5.21.

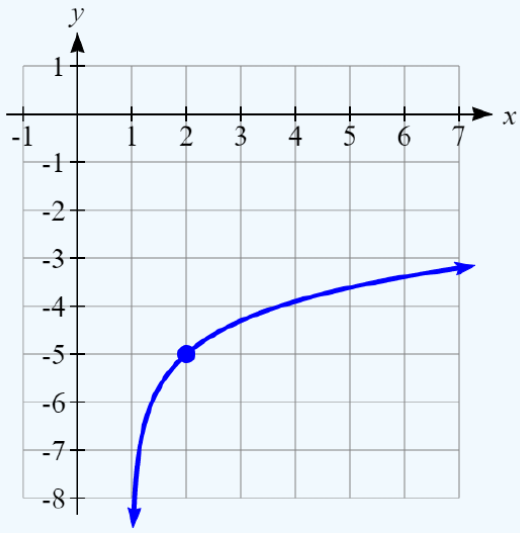


Figure 4.5.21: Graph of a Logarithmic Function

Solution

The vertical asymptote is at $x = 1$. There is a point at $(2, -5)$ that is one unit to the right of the asymptote which indicates this is the transformed x-intercept (*remember that for basic functions that the vertical asymptote is at* $x = 0$ *and the x-intercept is at* $(1, 0)$). This tells us the basic graph was shifted right 1 units and down 5 units. The equation for the graph is $y = \ln(x - 1) - 5$

You Try 4.5.7

Write an equation of the base 3 logarithmic function that is vertically stretched by 4, is reflected across the y-axis, and then shifted up 6 units.

Answer

$y = 4\log_3(-x) + 6$

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4.6: Laws of Logarithms

Learning Objectives

- Use the product rule for logarithms.
- Use the quotient rule for logarithms.
- Use the power rule for logarithms.
- Expand logarithmic expressions.
- Condense logarithmic expressions.
- Use the change-of-base formula for logarithms.

In chemistry, the pH scale is used as a measure of the acidity or alkalinity of a substance. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:

- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 at $25^{\circ}C$
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH, which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula, where $[\mathrm{H}^+]$ is the concentration of hydrogen ions in the solution

$$\begin{aligned} pH &= -\log([\mathrm{H}^+]) \\ &= \log\left(\frac{1}{[\mathrm{H}^+]}\right) \end{aligned}$$

These two equations forpH look different but, in fact, are equivalent. This is one of the logarithm properties we will examine in this section.



Figure 4.6.1: The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan).

Recall that the logarithmic and exponential functions “undo” each other. This means that logarithms have similar properties to exponents. First, let's recall the properties that we have learned thus far:

- $log_b(b^x) = x$
- $b^{\log_b x} = x$ where $x > 0$
- $\log_b 1 = 0$

- $\log_b b = 1$

The first two properties listed above are called *inverse properties*.

Recall that we use the product rule of exponents to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N) .$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for M and N} \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents} \\ &= m + n && \text{Apply the inverse property of logs} \\ &= \log_b(M) + \log_b(N) && \text{Substitute for m and n} \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

The Product Rule for Logarithms

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0$$

We can use this rule as well as others we will discuss shortly, to expand a single logarithm into sums and differences of simpler logarithms or we can use this rule to combine logarithms into a single logarithm. This is a useful things to be able to do as it will allow us a way to solve certain kings of logarithmic equations as well as determine behavior of logarithmic functions. As you will see in calculus, these rules are also very useful for making the process of taking derivatives and integrating much easier. Let's now look at some examples of applying this rule.

Example 4.6.1

Expand $\log_5(6a)$.

Solution

Notice that the argument, $6a$, is a product. Using the product rule, we can expand to write this as a sum of logarithms,

$$\log_5(6a) = \log_5(6) + \log_5(a)$$

Remember that is is a common practice to write single term arguments without parentheses. For the last example, it would be acceptable to also write the answer as $log_5 6 + \log_5 a$. However, it would not be correct to write $\log_5(6a)$ as $\log_5 6a$ since the argument is a product.

$$\begin{aligned} \log_5 6a &= \log_5(6)(a) && \text{a product of log base 5 of 6 which is multiplied by a} \\ &= a \log_5 6 \end{aligned}$$

When we have a product of a logarithm with a single term, it is common practice to write the logarithm last.

Caution!

A common mistake is to not put parentheses around arguments of products.

$$\log_b(mn) \neq \log_b mn$$

Example 4.6.2

Expand $\log_3(30x(3x+4))$.

Solution

We begin by factoring the argument completely, expressing 30 as a product of primes.

$$\log_3(30x(3x+4)) = \log_3(2 \cdot 3 \cdot 5 \cdot x \cdot (3x+4))$$

Next we write the equivalent equation by summing the logarithms of each factor.

$$\log_3(30x(3x+4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x+4)$$

We discussed earlier that it is acceptable to write a logarithm of a single term argument without parentheses. This is not the case with an argument that is a product or an argument that is a sum or difference.

You Try 4.6.1

Expand $\log_b(8k)$.

Answer

3{\log}_b2+{\log}_bk\)

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the quotient rule of exponents to combine the quotient of exponents by subtracting: $x^{\frac{a}{b}} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms.

The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number x and positive real numbers M, N, and b, b, where $b \neq 1$, we will show

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)\ .$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned}
\log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{Substitute for M and N} \\
&= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents} \\
&= m - n && \text{Apply the inverse property of logs} \\
&= \log_b(M) - \log_b(N) && \text{Substitute for m and n}
\end{aligned}$$

Example 4.6.3

Expand $\log_7\left(\frac{3}{x}\right)$.

Solution

Notice that the argument is a quotient. Using the quotient rule are are able to write this is as a difference of logarithms,

$$\log_7\left(\frac{3}{x}\right) = \log_7(3) - \log_7(x)$$

A note about notation here. In the previous example, the argument is a quotient. Notice that the logarithm is aligned with the fraction bar in the argument. Unlike with arguments that are products, sometimes you will see arguments that are quotients written without parentheses, $\log_7\left(\frac{3}{x}\right) = \log_7 \frac{3}{x}$. Logarithms are functions and functions have inputs. For the statement, $\log_7 \frac{3}{x}$, to make sense the argument must be the single term $\frac{3}{x}$.

Example 4.6.4

Expand $log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

Solution

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x))$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5.

$$\begin{aligned}\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3)+\log_2(5)+\log_2(x)+\log_2(x-1)] - [\log_2(3x+4)+\log_2(2-x)] \\ &= \log_2(3)+\log_2(5)+\log_2(x)+\log_2(x-1)-\log_2(3x+4)-\log_2(2-x)\end{aligned}$$

Analysis

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for $x = -\frac{4}{3}$ and $x = 2$. Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that $x > 0$, $x > 1$, $x > -\frac{4}{3}$, and $x < 2$. Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

You Try 4.6.2

Expand $\log_3\left(\frac{7x^2+21x}{7x(x-1)(x-2)}\right)$.

Answer

$\log_3(x+3) - \log_3(x-1) - \log_3(x-2)$

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\begin{aligned}\log_b(x^2) &= \log_b(x \cdot x) \\ &= \log_b x + \log_b x \\ &= 2\log_b x\end{aligned}$$

Notice that we used the product rule for logarithms to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \quad \sqrt{3} = 3^{\frac{1}{2}} \quad \frac{1}{e} = e^{-1}$$

The Power Rule for Logarithms

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b M$$

Note that since M^n is a single term that $\log_b(M^n) = \log_b M^n$.

Example 4.6.5

Expand $\log_2 x^5$.

Solution

The argument is already written as a power, so we identify the exponent, 5, and the base, x, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2 x$$

Notice in the previous example there were no parentheses around the argument. The argument is a single term so it is ok to do this.

You Try 4.6.3

Expand $\ln x^2$.

Answer

$2\ln x$

Example 4.6.6

Expand $\log_3(25)$ using the power rule for logs.

Solution

Expressing the argument as a power, we get $\log_3(25) = \log_3(5^2)$.

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(25) = 2\log_3(5)$$

You Try 4.6.4

Expand $\ln\left(\frac{1}{x^2}\right)$.

Answer

$-2\ln(x)$

(remember to simplify the answer, $\ln 1 = 0$)

Now that we are getting good with expanding logarithms, let's look at using this rule in the reverse order.

Example 4.6.7

Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Solution

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression $4\ln(x)$, we identify the factor, 4, as the exponent and the argument, x, as the base, and rewrite the product as a logarithm of a power: $4\ln(x) = \ln(x^4)$.

You Try 4.6.5

Rewrite $2\log_3 4$ using the power rule for logs to a single logarithm with a leading coefficient of 1.

Answer

$\log_3 16$

Expanding Logarithmic Expressions Using Multiple Rules

Taken together, the product rule, quotient rule, and power rule are often called **Laws of Logarithms**. Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b\left(\frac{6x}{y}\right) &= \log_b(6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\begin{aligned}\log_b\left(\frac{A}{C}\right) &= \log_b(AC^{-1}) \\ &= \log_b(A) + \log_b(C^{-1}) \\ &= \log_b A + (-1)\log_b C \\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Example 4.6.8

Expand $\ln\left(\frac{x^4 y}{7}\right)$.

Solution

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4 y}{7}\right) = \ln(x^4 y) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

So, $ln\left(\frac{x^4y}{7}\right) = 4\ln(x) + \ln(y) - \ln(7)$

? You Try 4.6.6

Expand $\log\left(\frac{x^2y^3}{z^4}\right)$.

Answer

$2\log x + 3\log y - 4\log z$

✓ Example 4.6.9

Expand $\log\sqrt{x}$.

Solution

$$\log\sqrt{x} = \log\left(x^{\frac{1}{2}}\right)$$
$$= \frac{1}{2}\log x$$

? You Try 4.6.7

Expand $\ln(\sqrt[3]{x^2})$.

Answer

$\frac{2}{3}\ln x$

✓ Example 4.6.10

Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

Solution

We can expand by applying the Product and Quotient Rules.

$$\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) = \log_6 64 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \quad \text{Apply the Quotient Rule}$$
$$= \log_6 2^6 + \log_6 x^3 + \log_6(4x+1) - \log_6(2x-1) \quad \text{Simplify by writing } 64 \text{ as } 2^6$$
$$= 6\log_6 2 + 3\log_6 x + \log_6(4x+1) - \log_6(2x-1) \quad \text{Apply the Power Rule}$$

? You Try 4.6.8

Expand $\ln\left(\dfrac{\sqrt{(x-1)(2x+1)^2}}{(x^2-9)}\right)$.

Answer

$\dfrac{1}{2}\ln(x-1)+\ln(2x+1)-\ln(x+3)-\ln(x-3)$

Condensing Logarithmic Expressions Using Multiple Rules

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

Example 4.6.11

Write $\log_3(5)+\log_3(8)-\log_3(2)$ as a single logarithm.

Solution

Using the product and quotient rules

$$\log_3(5)+\log_3(8)=\log_3(5\cdot 8)=\log_3(40)$$

This reduces our original expression to

$$\log_3(40)-\log_3(2)$$

Then, using the quotient rule

$$\log_3(40)-\log_3(2)=\log_3\left(\frac{40}{2}\right)=\log_3(20)$$

You Try 4.6.9

Write $\log 3-\log 4+\log 5-\log 6$ as a single logarithm.

Answer

$\log\left(\dfrac{5}{8}\right)$ after reducing the fraction to lowest terms.

Example 4.6.12

Condense $\log_2(x^2)+\dfrac{1}{2}\log_2(x-1)-3\log_2(x+3)^2$ into a single logarithm.

Solution

We apply the power rule first:

$$\log_2(x^2)+\frac{1}{2}\log_2(x-1)-3\log_2(x+3)^2=\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2(x+3)^6$$

Next we apply the product rule to the sum:

$$\log_2(x^2)+\log_2(\sqrt{x-1})-\log_2(x+3)^6=\log_2(x^2\sqrt{x-1})-\log_2(x+3)^6$$

Finally, we apply the quotient rule to the difference:

$$\log_2(x^2\sqrt{x-1})-\log_2(x+3)^6=\log_2\frac{x^2\sqrt{x-1}}{(x+3)^6}$$

Example 4.6.13

Rewrite $2\log x - 4\log(x+5) + \log(3x+5)$ as a single logarithm.

Solution

We apply the power rule first:

$$2\log x - 4\log(x+5) +$$
$$log(3x+5) = \log(x^2) - \log\left(x+5\right)^4 + \log((3x+5)^{x^{-1}})$$

Next we rearrange and apply the product rule to the sum:

$$\begin{aligned}\log(x^2) - \log\left(x+5\right)^4 + \log((3x+5)) &= \log(x^2) + \log((3x+5) - \log\left(x+5\right)^4 \\ &= \log(x^2(3x+5)) - \log\left(x+5\right)^4 \qquad \text{Apply the product rule} \\ &= \log\frac{x^2(3x+5)}{(x+5)^4} \qquad \text{Apply the quotient rule}\end{aligned}$$

You Try 4.6.10

Rewrite $\log(5) + 0.5\log(x) - \log(7x-1) + 3\log(x-1)$ as a single logarithm.

Answer

$$\log\frac{5\sqrt{x}(x-1)^3}{(7x-1)}$$

You Try 4.6.11

Condense $4(3\log(x) + \log(x+5) - \log(2x+3))$.

Answer

$\log\dfrac{x^{12}(x+5)^4}{(2x+3)^4}$; this answer could also be written $\log\left(\dfrac{x^3(x+5)}{(2x+3)}\right)^4$

Example 4.6.14

In chemistry, pH is defined as $pH = -\log[H+]$ where $H+$ is the concentration of hydrogen ions in a liquid. If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH = -\log(2C)$$

Using the product rule of logs

$$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$,the new pH is

$$pH = P - \log(2) \approx P - 0.301$$

You Try 4.6.12

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

Answer

The pH increases by about 0.301.

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or e, we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base;. When using a calculator to approximate a logarithm, we would change them to common or natural logs.

To derive the change-of-base formula, we use the one-to-one property and power rule for logarithms.

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M$. By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\begin{aligned} \log_n(b^y) &= \log_n M && \text{Apply the one-to-one property} \\ y\log_n b &= \log_n M && \text{Apply the power rule for logarithms} \\ y &= \frac{\log_n M}{\log_n b} && \text{Isolate y} \\ \log_b M &= \frac{\log_n M}{\log_n b} && \text{Substitute for y} \end{aligned}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\begin{aligned} \log_5 36 &= \frac{\log(36)}{\log(5)} && \text{Apply the change of base formula using base 10} \\ &\approx 2.2266 && \text{Use a calculator to evaluate to 4 decimal places} \end{aligned}$$

The Change-of-Base Formula

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b}$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b}$$

and

$$\log_b M = \frac{\log M}{\log b}$$

Example 4.6.15

Change $\log_5 3$ to a quotient of natural logarithms.

Solution

Because we will be expressing $\log_5 3$ as a quotient of natural logarithms, the new base, $n = e$.

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_b M = \frac{\ln M}{\ln b}$$

$$\log_5 3 = \frac{\ln 3}{\ln 5}$$

Example 4.6.16

Evaluate $\log_2(10)$ using the change-of-base formula with a calculator and round to four decimal places.

Solution

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base e.

$$\log_2 10 = \frac{\ln 10}{\ln 2} \qquad \text{Apply the change of base formula using base } e$$

$$\approx 3.3219 \qquad \text{Use a calculator to evaluate to 4 decimal places}$$

You Try 4.6.13

Evaluate $\log_5(100)$ using the change-of-base formula. and round to three decimal places.

Answer

$$\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094} = 2.861$$

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4.7: Exponential and Logarithmic Equations

Learning Objectives

- Use like bases to solve exponential equations.
- Use logarithms to solve exponential equations.
- Use the definition of a logarithm to solve logarithmic equations.
- Use the one-to-one property of logarithms to solve logarithmic equations.
- Solve applied problems involving exponential and logarithmic equations.

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.



Figure 4.7.1: Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the "rabbit plague." (credit: Richard Taylor, Flickr)

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential and logarithmic equations.

Exponential Equations

Exponential equations are equations that contain variables in the exponent.

Some examples of exponential equations:

$$2^x = 2^{3x-7} \qquad\qquad 8^{t-3} + 4 = 7$$

There are two main techniques to solving exponential equations. The first technique is for solving equations that involve like bases. The second technique is for solving equations that we are unable to write with the same base.

Using Like Bases to Solve Exponential Equations

Consider the equation $3^{4x-7} = 3^{2x}$. Notice that the bases are the same. To have equivalency, the exponents must be equal: $4x - 7 = 2x$. Solving this, we get $x = \dfrac{7}{2}$.

The **one-to-one property of exponential functions** tells us that, for any real numbers b, S, and T, where $b > 0$, $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T.$$

Example 4.7.1

Solve $2^{x-1} = 2^{2x-4}$

Solution

$2^{x-1} = 2^{2x-4}$	The common base is 2
$x - 1 = 2x - 4$	By the one-to-one property the exponents must be equal
$x = 3$	Solve for x

? You Try 4.7.1

Solve $5^{2x} = 5^{3x+2}$

Answer

$x = -2$

Sometimes the *common base* for an exponential equation is not explicitly shown. In these cases, we simply rewrite the terms in the equation as powers with a common base, and solve using the one-to-one property.

For example, consider the equation $256 = 4^{x-5}$. We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for x:

$16 = 2^{x-5}$	
$2^4 = 2^{x-5}$	Rewrite 16 as a power with base 2
$4 = x - 5$	Apply the one-to-one property of exponents
$9 = x$	Add 5 to both sides

✓ Example 4.7.2

Solve $8^{x+2} = 16^{x+1}$

Solution

$8^{x+2} = 16^{x+1}$	
$(2^3)^{x+2} = (2^4)^{x+1}$	Write 8 and 16 as powers of 2
$2^{3x+6} = 2^{4x+4}$	To take a power of a power, multiply exponents
$3x + 6 = 4x + 4$	Use the one-to-one property to set the exponents equal
$x = 2$	Solve for x

? You Try 4.7.2

Solve $5^{2x} = 25^{3x+2}$

Answer

$x = -1$

✓ Example 4.7.3

Solve $2^{5x} = \sqrt{2}$

Solution

$2^{5x} = 2^{\frac{1}{2}}$	Write the square root of 2 as a power of 2
$5x = \frac{1}{2}$	Use the one-to-one property
$x = \frac{1}{10}$	Solve for x

? You Try 4.7.3

Solve $5^x = \sqrt{5}$

Answer

$$x = \frac{1}{2}$$

Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. The **one-to-one property for logarithms** tells us that, for real numbers $a > 0$ and $b > 0$, $\log(a) = \log(b)$ is equivalent to $a = b$. This means that we may apply logarithms with the same base on both sides of an exponential equation. It is most common to apply the natural logarithm on both sides but any logarithms with the same base applied to both sides may be used.

✔ Example 4.7.4

Solve $5^{x+2} = 4^x$

Solution

$5^{x+2} = 4^x$	There is no easy way to get the powers to have the same base
$\ln 5^{x+2} = \ln 4^x$	Take ln of both sides
$(x+2)\ln 5 = x \ln 4$	Use laws of logs
$x \ln 5 + 2\ln 5 = x \ln 4$	Use the distributive law
$x \ln 5 - x \ln 4 = -2 \ln 5$	Get terms containing x on one side, terms without x on the other
$x(\ln 5 - \ln 4) = -2 \ln 5$	On the left hand side, factor out an x
$x \ln\left(\frac{5}{4}\right) = \ln\left(\frac{1}{25}\right)$	Use the laws of logs
$x = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}$	Divide by the coefficient of x
$x = \frac{-\ln 25}{\ln 5 - \ln 4}$	simplify using Laws of Logarithms

✔ Example 4.7.5

Solve $2^x = 3^x$

Solution

$\ln 2^x = \ln 3^x$	Take ln of both sides
$x \ln 2 = x \ln 3$	move exponents to the front
$x \ln 2 - x \ln 3 = 0$	get like terms on one side
$x(\ln 2 - \ln 3) = 0$	factor out x
$x = \frac{0}{\ln 2 - \ln 3}$	divide both sides by $\ln 2 - \ln 3$
$x = 0$	

? You Try 4.7.4

Solve $2^x = 3^{x+1}$

Answer

$$x = \frac{\ln 3}{\ln 2 - \ln 3\left(\frac{2}{3}\right)} \text{ or } x = \frac{\ln 3}{\ln 2 - \ln 3}$$

✓ Example 4.7.6

Solve $5^x = 11$. Find the exact answer and then approximate it to three decimal places.

Solution:

$$5^x = 11$$

Since the exponential is isolated, take the logarithm of both sides.

$$\log 5^x = \log 11$$

Use the Power Property to get the x as a factor, not an exponent.

$$x \log 5 = \log 11$$

Solve for x. Find the exact answer.

$$x = \frac{\log 11}{\log 5}$$

Approximate the answer.

$$x \approx 1.490$$

Since $5^1 = 5$ and $5^2 = 25$, it makes sense that $5^{1.490} \approx 11$.

Note in the previous example that $y = 5^x$ is a basic exponential function. We know what the graph looks like and how the function values will start to become large quite quickly for larger x-values since this is an increasing exponential function of base 5, it seems reasonable that the function value of 11 would occur at a smaller x-value. Also, knowing that there is are points on the graph at $(0, 1)$ and at $(1, 5)$, this would indicate that the function would equal 11 for an x-value that is is just a bit bigger than 1.

? You Try 4.7.5

Solve $7^x = 43$. Find the exact answer and then approximate it to three decimal places.

Answer

$$x = \frac{\log 43}{\log 7} \approx 1.933$$

One common type of exponential equations are those with base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

✓ Example 4.7.7

Solve $100 = 20e^{2t}$

Solution

$$
\begin{aligned}
100 &= 20e^{2t} \\
5 &= e^{2t} && \text{Divide by the coefficient of the power} \\
\ln 5 &= 2t && \text{Take ln of both sides. Use the fact that } ln(x) \text{ and } e^x \text{ are inverse functions} \\
t &= \frac{\ln 5}{2} && \text{Divide by the coefficient of t}
\end{aligned}
$$

Analysis

Using laws of logs, we can also write this answer in the form $t = \ln \sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

? You Try 4.7.6

Solve $3e^{0.5t} = 11$

Answer

$t = 2\ln\left(\frac{11}{3}\right)$ or $\ln\left(\frac{11}{3}\right)^2$

✓ Example 4.7.8

Solve $4e^{2x} + 5 = 12$

Solution

$$
\begin{aligned}
4e^{2x} + 5 &= 12 \\
4e^{2x} &= 7 && \text{Combine like terms} \\
e^{2x} &= \frac{7}{4} && \text{Divide by the coefficient of the power} \\
2x &= \ln\left(\frac{7}{4}\right) && \text{Take ln of both sides} \\
x &= \frac{1}{2}\ln\left(\frac{7}{4}\right) && \text{Solve for x}
\end{aligned}
$$

? You Try 4.7.7

Solve $3 + e^{2t} = 7e^{2t}$

Answer

$t = \ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}\ln(2)$

✓ Example 4.7.9

Solve $3e^{x+2} = 24$. Find the exact answer and then approximate it to three decimal places.

Solution:

$$3e^{x+2} = 24$$

Isolate the exponential by dividing both sides by 3.

$$e^{x+2} = 8$$

Take the natural logarithm of both sides.

$$\ln e^{x+2} = \ln 8$$

Use the Power Property to get the x as a factor, not an exponent.

$$(x+2)\ln e = \ln 8$$

Use the property $\ln e = 1$ to simplify.

$$x+2 = \ln 8$$

Solve the equation. Find the exact answer.

$$x = \ln 8 - 2$$

Approximate the answer.

$$x \approx 0.079$$

? You Try 4.7.8

Solve $5e^{2x} = 25$. Find the exact answer and then approximate it to three decimal places.

Answer

$x = \frac{\ln 5}{2} \approx 0.805$

You may be wondering if all exponential equations have solutions. The answer is not all exponential equations have solutions.

The equation $3^{x+1} = -2$ has no solution. There is no real value of x that will make the equation a true statement because any power of a positive number is positive.

Figure 4.7.2 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

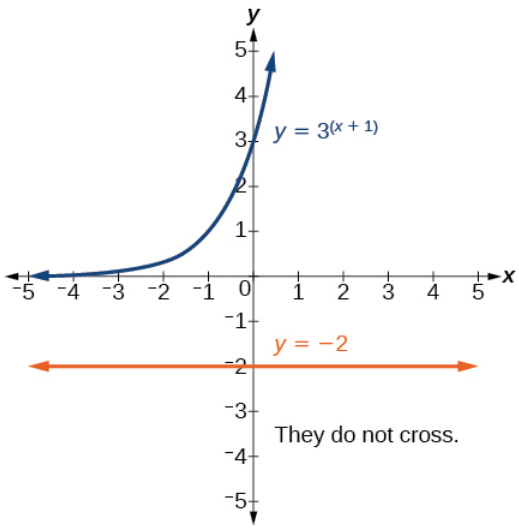


Figure 4.7.2

Another example of an equation that has no solution is $2 = -3e^t$.

Figure 4.7.3 shows that the two graphs do not cross so the left side is never equal to the right side. Thus the equation has no solution.

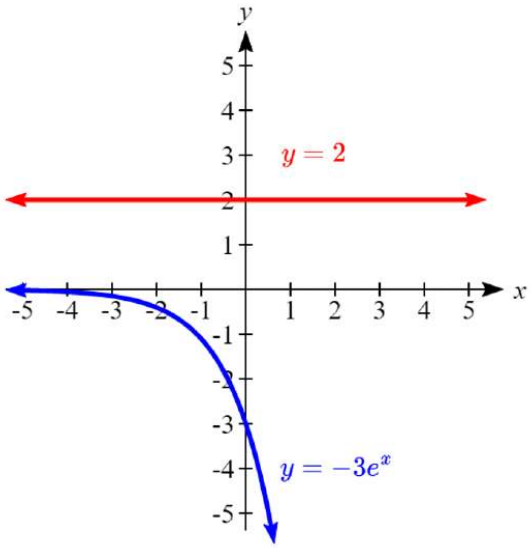


Figure 4.7.3: Graph shows that the functions do not intersect

Checking for Extraneous Solutions When Solving Exponential Equations

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

Example 4.7.10

Solve $e^{2x} - e^x = 56$

Solution

$$\begin{aligned} e^{2x} - e^x &= 56 \\ e^{2x} - e^x - 56 &= 0 \qquad \text{Get one side of the equation equal to zero since this equation is of quadratic form} \\ (e^x + 7)(e^x - 8) &= 0 \qquad \text{Factor by the FOIL method} \end{aligned}$$

Solving for each factor,

$$\begin{aligned} e^x + 7 = 0 \quad &\text{or} \quad e^x - 8 = 0 \\ e^x = -7 \quad &\text{or} \quad e^x = 8 \end{aligned}$$

$e^x = -7$ has no solution since $x = \ln(-7)$ is undefined. Solving $e^x = 8$,

$$x = \ln 8$$

You Try 4.7.9

Solve $e^{2x} = e^x + 2$

Answer

$x = \ln 2$

Logarithmic Equations

Logarithmic equations are equations that contain logarithms with variables.

Some examples of logarithmic equations:

$$\log_5 x = 8 \qquad \ln(t) - \ln(t-2) = \ln 11$$

There are two main techniques to solving logarithmic equations. The first technique is for solving equations that contain only logarithmic terms with the same base. The second technique is for solving equations that contain logarithmic terms and constants.

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x-1) = \log_2(8), \text{ then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for x,and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9-1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2) - \log(2) = \log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x:

$$\begin{aligned}
\log(3x-2) - \log(2) &= \log(x+4) && \\
\log\left(\frac{3x-2}{2}\right) &= \log(x+4) && \text{Apply the quotient rule of logarithms} \\
\frac{3x-2}{2} &= x+4 && \text{Apply the one to one property of a logarithm} \\
3x-2 &= 2x+8 && \text{Multiply both sides of the equation by 2} \\
x &= 10 && \text{Subtract 2x and add 2}
\end{aligned}$$

To check the result, substitute $x = 10$ into $\log(3x-2) - \log(2) = \log(x+4)$.

$$\begin{aligned}
\log(3(10)-2) - \log(2) &= \log((10)+4) \\
\log(28) - \log(2) &= \log(14) \\
\log\left(\frac{28}{2}\right) &= \log(14) && \text{The solution checks}
\end{aligned}$$

Example 4.7.11

Solve $\ln(x^2) = \ln(2x+3)$

Solution

$$\begin{aligned}
\ln(x^2) &= \ln(2x+3) && \\
x^2 &= 2x+3 && \text{Use the one-to-one property of the logarithm} \\
x^2 - 2x - 3 &= 0 && \text{Get zero on one side before factoring} \\
(x-3)(x+1) &= 0 && \text{Factor}
\end{aligned}$$

Solving for each factor,

$$x - 3 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = 3 \quad \text{or} \quad x = -1$$

Analysis

There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

? You Try 4.7.10

Solve $\ln(x^2) = \ln 1$

Answer

$x = 1$ or $x = -1$

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x:

$$\begin{aligned} \log_2(2) + \log_2(3x-5) &= 3 && \\ \log_2(2(3x-5)) &= 3 && \text{Apply the product rule of logarithms} \\ \log_2(6x-10) &= 3 && \text{Distribute} \\ 2^3 &= 6x-10 && \text{Convert to exponential form} \\ 8 &= 6x-10 && \text{Calculate } 2^3 \\ 18 &= 6x && \text{Add 10 to both sides} \\ x &= 3 && \text{Divide by 6} \end{aligned}$$

✓ Example 4.7.12

Solve $2\ln x + 3 = 7$

Solution

$$\begin{aligned} 2\ln x + 3 &= 7 && \\ 2\ln x &= 4 && \text{Subtract 3} \\ \ln x &= 2 && \text{Divide by 2} \\ x &= e^2 && \text{Rewrite in exponential form} \end{aligned}$$

? You Try 4.7.11

Solve $6 + \ln x = 10$

Answer

$x = e^4$

✓ Example 4.7.13

Solve $2\ln(6x) = 7$

Solution

$$
\begin{aligned}
2\ln(6x) &= 7 \\
\ln(6x) &= \frac{7}{2} && \text{Divide by 2} \\
6x &= e^{\left(\frac{7}{2}\right)} && \text{convert to exponential form ln} \\
x &= \frac{1}{6}e^{\left(\frac{7}{2}\right)} && \text{Divide by 6}
\end{aligned}
$$

? You Try 4.7.12

Solve $2\ln(x+1) = 10$

Answer

$x = e^5 - 1$

✓ Example 4.7.14

Solve $\ln x = 3$

Solution

$$
\begin{aligned}
\ln x &= 3 \\
x &= e^3
\end{aligned}
$$

Figure 4.7.4 represents the graph of the equation. On the graph, the x-coordinate of the point at which the two graphs intersect is close to 20. In other words $e^3 \approx 20$. A calculator gives a better approximation: $e^3 \approx 20.0855$.

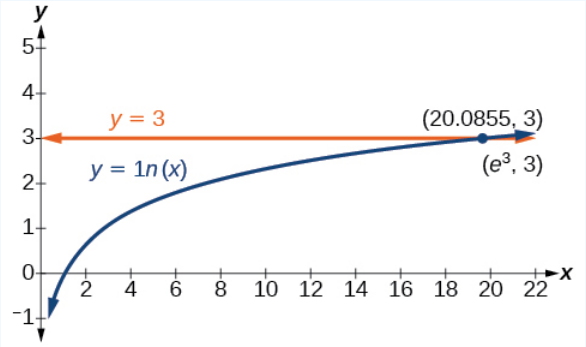


Figure 4.7.4: The graphs of $y = \ln x$ and $y = 3$ cross at the point $(e^3, 3)$, which is approximately $(20.0855, 3)$.

? You Try 4.7.13

Use a graphing calculator to estimate the approximate solution to the logarithmic equation $2^x = 1000$ to 2 decimal places.

Answer

$x \approx 9.97$

Checking for Extraneous Solutions When Solving Logarithmic Equations

It is possible to have extraneous solutions when solving logarithmic equations since the domain of a logarithm is $(0, \infty)$. Check your answers to see if they are extraneous or not.

Example 4.7.15

Solve $\ln x + \ln(2x-5) = \ln 3$

Solution

$$\begin{aligned} \ln x + \ln(2x-5) &= \ln 3 \\ \ln(x(2x-5)) &= \ln 3 \\ \ln(2x^2-5x) &= \ln 3 \\ 2x^2-5x &= 3 \\ 2x^2-5x-3 &= 0 \\ (2x+1)(x-3) &= 0 \\ 2x+1=0 \qquad x-3 &= 0 \\ x = \frac{1}{2} \qquad x &= 3 \end{aligned}$$

Example 4.7.16

Solve: $\log_4(x+6) - \log_4(2x+5) = -\log_4 x$

Solution:

$$\log_4(x+6) - \log_4(2x+5) = -\log_4 x$$

Use the Quotient Property on the left side and the Power Property on the right.

$$\log_4\left(\frac{x+6}{2x+5}\right) = \log_4 x^{-1}$$

Rewrite $x^{-1} = \frac{1}{x}$.

$$\log_4\left(\frac{x+6}{2x+5}\right) = \log_4 \frac{1}{x}$$

Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.

$$\frac{x+6}{2x+5} = \frac{1}{x}$$

Solve the rational equation.

$$x(x+6) = 2x+5$$

Distribute.

$$x^2+6x = 2x+5$$

Write in standard form.

$$x^2+4x-5 = 0$$

Factor.

$$(x+5)(x-1) = 0$$

Use the Zero-Product-Property.

$$x+5=0, \quad x-1=0$$

Solve each equation.

$$x = \cancel{-5}, \quad x = 1$$

Check.

We leave the check for you.

Example 4.7.17

Solve: $2\log_5 x = \log_5 81$

Solution:

$$2\log_5 x = \log_5 81$$

Use the Power Property.

$$\log_5 x^2 = \log_5 81$$

Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.

$$x^2 = 81$$

Solve using the Square Root Property.

$$x = \pm 9$$

We eliminate $x = -9$ as we cannot take the logarithm of a negative number.

$$x = 9,\ \cancel{x = -9}$$

Check. $x = 9$

$$2\log_5 x = \log_5 81$$
$$2\log_5 9 \stackrel{?}{=} \log_5 81$$
$$\log_5 9^2 \stackrel{?}{=} \log_5 81$$
$$\log_5 81 = \log_5 81$$

You Try 4.7.14

Solve: $2\log_3 x = \log_3 36$

Answer

$x = 6$

You Try 4.7.15

Solve: $\log(x+2) - \log(4x+3) = -\log x$

Answer

$x = 3$

You Try 4.7.16

Solve: $\log(x-2) - \log(4x+16) = \log\dfrac{1}{x}$

Answer

$x = 8$

Solving Applied Problems

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic

equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

Example 4.7.18

Suppose $\$2000$ is invested in an account which offers 7.125% compounded monthly.

1. Express the amount A in the account as a function of the term of the investment t in years.
2. How much is in the account after 5 years?
3. How long will it take for the initial investment to double?
4. Find and interpret the average rate of change of the amount in the account from the end of the fourth year to the end of the fifth year, and from the end of the thirty-fourth year to the end of the thirty-fifth year.

Solution

1. Substituting $P = 2000$, $r = 0.07125$, and $n = 12$ (since interest is compounded *monthly* into the compound interest formula yields
$$A(t) = 2000\left(1 + \frac{0.07125}{12}\right)^{12t} = 2000(1.0059375)^{12t}$$
2. Since t represents the length of the investment in years, we substitute $t = 5$ into $A(t)$ to find
$$A(5) = 2000(1.0059375)^{12(5)} \approx 2852.92$$
After 5 years, we have approximately $\$2852.92$
3. Our initial investment is $\$2000$, so to find the time it takes this to double, we need to find t when $A(t) = 4000$. We get $2000(1.0059375)^{12t} = 4000$, or $(1.0059375)^{12t} = 2$ Taking natural logs of both sides, we get
$$t = \frac{\ln(2)}{12\ln(1.0059375)} \approx 9.75.$$
Hence, it takes approximately 9 years 9 months for the investment to double.
4. To find the average rate of change of A from the end of the fourth year to the end of the fifth year, we compute
$$\frac{A(5) - A(4)}{5 - 4} \approx 195.63$$
Similarly, the average rate of change of A from the end of the thirty-fourth year to the end of the thirty-fifth year is
$$\frac{A(35) - A(34)}{35 - 34} \approx 1648.21$$
This means that the value of the investment is increasing at a rate of approximately $\$195.63$ per year between the end of the fourth and fifth years, while that rate jumps to $\$1648.21$ per year between the end of the thirty-fourth and thirty-fifth years. So, not only is it true that the longer you wait, the more money you have, but also the longer you wait, the faster the money increases. (in fact, the rate of increase of the amount in the account is exponential as well. This is the quality that really defines exponential functions and we refer the reader to a course in Calculus.

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4.8: Applications of Exponential and Logarithmic Functions

Learning Objectives

- Apply common exponential models to real-life situations.
- Apply common logarithmic models to real-life situations.

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and Newton's Law of Cooling.

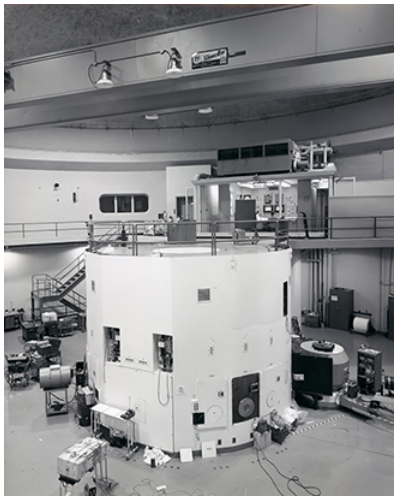


Figure 4.8.1: A nuclear research reactor inside the Neely Nuclear Research Center on the Georgia Institute of Technology campus (credit: Georgia Tech Research Institute)

Modeling Exponential Growth and Decay

In real-world applications, we need to model the behavior with functions. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of population growth, such as a bacteria culture that a biologist is studying for a new medical treatment, we may choose the exponential growth function:

$$N(t) = N_0 e^{rt}$$

where N_0 is equal to the value at time zero, e is Euler's constant, and k is a positive constant that determines the rate (percentage) of growth.

Exponential Growth

The model $N(t) = N_0 \cdot e^{kt}$ represents the population size after a given amount of time t.

N_0 is the initial population size and r is the rate of growth for the population.

Example 4.8.1

A population of bacteria doubles in size every hour. If the culture started with 10 bacteria, find a model that describes the population size after of the bacteria after t hours. Use this model to determine how many bacteria there will be after 10 hours.

Solution

Bacteria growth is exponential. To find N_0 we use the fact that N_0 is the amount at time zero, so $N_0 = 10$. To find r, use the fact that after one hour $(t = 1)$ the population doubles from 10 to 20. The formula is derived as follows

$$\begin{aligned} 20 &= 10e^{r\cdot 1} \\ 2 &= e^r \qquad \text{Divide by 10} \\ \ln 2 &= r \qquad \text{Convert to exponential form} \end{aligned}$$

so $r = \ln(2)$. Thus the equation that models the bacteria growth is $N(t) = 10e^{(\ln 2)t} = 10(e^{\ln 2})^t = 10 \cdot 2^t$. The graph is shown in Figure 4.8.2.

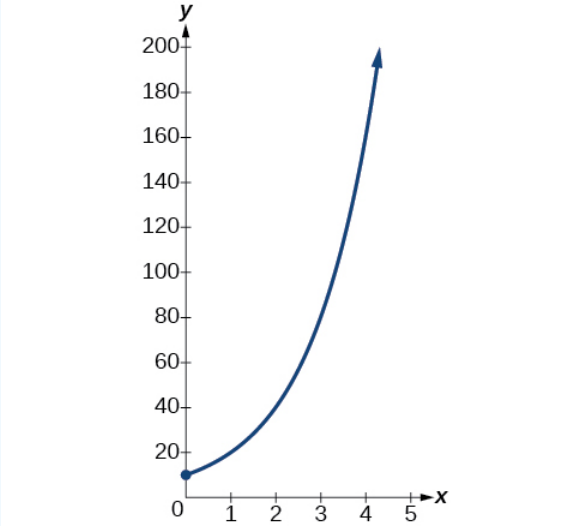


Figure 4.8.2: The graph of $y = 10e^{(\ln 2)t}$

The population of bacteria after ten hours is $N(10) = 10 \cdot 2^10 = 10,240$.

Analysis

We could describe this amount is being of the order of magnitude 10^4. The population of bacteria after twenty hours is $10,485,760$ which is of the order of magnitude 10^7, so we could say that the population has increased by three orders of magnitude in ten hours.

In the previous example, we were provided information about the time it took for the population to double in size. This time is called **doubling time**. Notice that the model simplified to an exponential function of base 2. We can describe a modified growth model for doubling time as follows:

Doubling Time

The model $N(t) = N_0 \cdot 2^{\frac{t}{a}}$ represents the population size after a given amount of time t.

N_0 is the initial population size and a is the time it takes for the population to double in size.

Example 4.8.2

Cancer cells sometimes increase exponentially. If a cancerous growth contained 300 cells last month and 360 cells this month, how long will it take for the number of cancer cells to double? Round your answer to the nearest tenth.

Solution

Defining t to be time in months, with $t = 0$ corresponding to the initial size of the population, we are given two pieces of data: last month, (0, 300) and this month, (1, 360).

From this data, we can find an equation for the growth. Using the form $N(t) = N_0 \cdot 2^{\frac{t}{a}}$, we know immediately $N_0 = 300$, giving $N(t) = 300 \cdot 2^t$. Substituting in $(1, 360)$,

$$
\begin{aligned}
360 &= 300 \cdot 2^{\frac{1}{d}} \\
\frac{360}{300} &= 2^{\frac{1}{d}} \\
1.2 &= 2^{\frac{1}{d}} \\
\ln 1.2 &= \ln\left(2^{\frac{1}{d}}\right) \\
\ln 1.2 &= \frac{1}{d}\ln 2 \\
d \cdot \ln 1.2 &= \ln 2 \\
d &= \frac{\ln 2}{\ln 1.2} \\
d &\approx 3.8 \text{ months}
\end{aligned}
$$

Radioactive Decay and Half-Life

We now turn to exponential decay. One of the common terms associated with exponential decay is **half-life**, the length of time it takes an exponentially decaying quantity to decrease to half its original amount. Every radioactive isotope has a half-life, and the process describing the exponential decay of an isotope is called radioactive decay.

Radioactive Decay

The model $M(t) = M_0 e^{-\frac{ln2}{h}t}$ represents the amount of material (mass) left after a given amount of time t.

M_0 is the initial mass size and h is the half-life of the material.

The formula for radioactive decay is important in **radiocarbon dating**, which is used to calculate the approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby, who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about 1% error for plants or animals that died within the last $60,000$ years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of $5,730$ years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12, which has an atomic weight of 12 and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last $60,000$ years, using tree rings and other organic samples of known dates—although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Example 4.8.3

The half-life of carbon-14 is $5,730$ years. Express the amount of carbon-14 remaining as a function of time, t.

Solution

The function that describes this continuous decay is $M(t) = M_0 e^{\left(-\frac{\ln(2)}{5730}\right)t}$.

Example 4.8.4

A bone fragment is found that contains 20% of its original carbon-14. To the nearest year, how old is the bone?

Solution

The percentage of carbon-14 in all living things is 100%. The percentage of carbon-14 after some amount of time, t, (which we are trying to find) is 20%. Using decimals for the percent, we can substitute in $M_0 = 1$ and $M(t) = 0.2$ into the function we found in the previous example,

$$\begin{aligned} M(t) &= M_0 e^{\left(-\frac{\ln(2)}{5730}\right)t} \\ 0.2 &= 1 \cdot e^{\left(-\frac{\ln(2)}{5730}\right)t} \\ 0.2 &= e^{\left(-\frac{\ln(2)}{5730}\right)t} \\ \ln 0.2 &= \frac{-\ln(2)}{5730} t \\ -\frac{5730}{\ln 2}(\ln 0.2) &= t \\ t &\approx 13,305 \text{ years} \end{aligned}$$

The bone fragment is about $13,305$ years old.

Analysis

The instruments that measure the percentage of carbon-14 are extremely sensitive and, as we mention above, a scientist will need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about 1%, so this age should be given as $13,305$ years $\pm 1\%$ or $13,305$ years ± 133 years.

Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless the room temperature is zero, this will correspond to a vertical shift of the generic exponential decay function. This translation leads to **Newton's Law of Cooling**, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature

Newton's Law of Cooling

The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$$T(t) = D_0 e^{-kt} + T_s$$

where

- t is time
- D_0 is the difference between the initial temperature of the object and the surroundings
- k is a constant, the continuous rate of cooling of the object

Example 4.8.5

A cheesecake is taken out of the oven with an ideal internal temperature of $165^\circ F$, and is placed into a $35^\circ F$ refrigerator. After 10 minutes, the cheesecake has cooled to $150^\circ F$. If we must wait until the cheesecake has cooled to $70^\circ F$ before we eat it, how long will we have to wait?

Solution

Because the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially toward 35, following the equation

$$T(t) = D_0 e^{-kt} + 35$$

We know the initial temperature was 165, so $T(0) = 165$.

$$165 = Ae^{-k0} + 35 \qquad \text{Substitute } (0, 165)$$
$$D_0 = 130 \qquad \text{Solve for } D_0$$

We were given another data point, $T(10) = 150$, which we can use to solve for k.

$$150 = 130e^{-k10} + 35 \qquad \text{Substitute } (10, 150)$$
$$115 = 130e^{-k10} \qquad \text{Subtract 35}$$
$$\frac{115}{130} = e^{-10k} \qquad \text{Divide by 130}$$
$$\ln\left(\frac{115}{130}\right) = -10k \qquad \text{Take the natural log of both sides}$$
$$k = \frac{\ln\left(\frac{115}{130}\right)}{-10} \qquad \text{Divide by the coefficient of k}$$

This gives us the equation for the cooling of the cheesecake: $T(t) = 130e^{\left(-\frac{1}{10}\ln\frac{23}{26}t\right)} + 35$.
Now we can solve for the time it will take for the temperature to cool to 70 degrees.

$$70 = 130e^{\left(-\frac{1}{10}\ln\frac{23}{26}\right)} + 35 \qquad \text{Substitute in 70 for } T(t)$$
$$35 = 130e^{\left(-\frac{1}{10}\ln\frac{23}{26}\right)t} \qquad \text{Subtract 35}$$
$$\frac{35}{130} = e^{\left(-\frac{1}{10}\ln\frac{23}{26}\right)t} \qquad \text{Divide by 130}$$
$$\ln\left(\frac{35}{130}\right) = \left(-\tfrac{1}{10}\ln\tfrac{23}{26}\right)t \qquad \text{Take the natural log of both sides}$$
$$t = \frac{\ln\left(\frac{35}{130}\right)}{\left(-\frac{1}{10}\ln\frac{23}{26}\right)t}$$
$$t \approx 107 \qquad \text{Divide by the coefficient of t}$$

It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool to $70^\circ F$.

? You Try 4.8.1

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees to three decimal places?

Answer

6.026hours

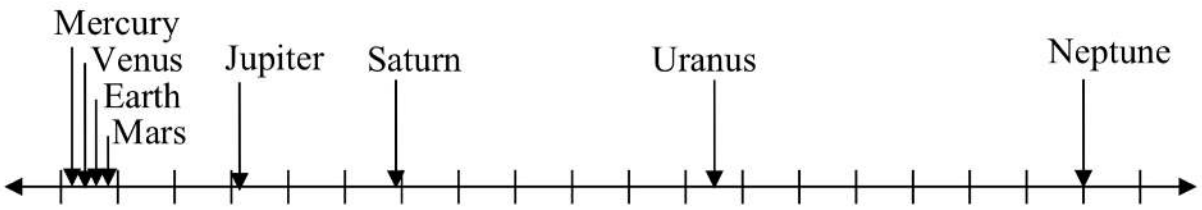
Modeling with Logarithms

For quantities that vary greatly in magnitude, a standard scale of measurement is not always effective, and utilizing logarithms can make the values more manageable. For example, if the average distances from the sun to the major bodies in our solar system are listed, you see they vary greatly.

Planet	Distance (millions of km)
Mercury	58
Venus	108
Earth	150

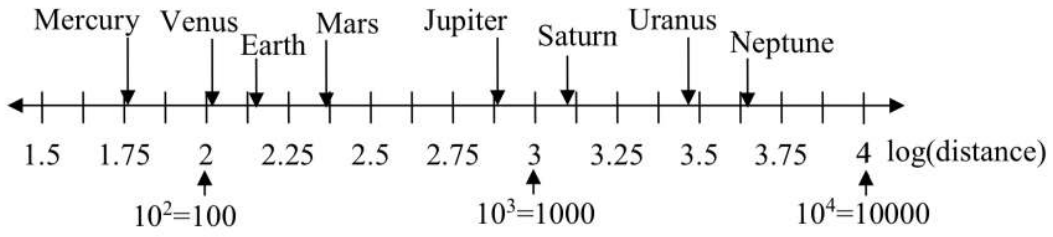
Mars	228
Jupiter	779
Saturn	1430
Uranus	2880
Neptune	4500

Placed on a linear scale – one with equally spaced values – these values get bunched up, as seen below.



However, computing the logarithm of each value and plotting these new values on a number line results in a more manageable graph, and makes the relative distances more apparent.

Planet	**Distance (millions of km)**	**log(distance)**
Mercury	58	1.76
Venus	108	2.03
Earth	150	2.18
Mars	228	2.36
Jupiter	779	2.89
Saturn	1430	3.16
Uranus	2880	3.46
Neptune	4500	3.65

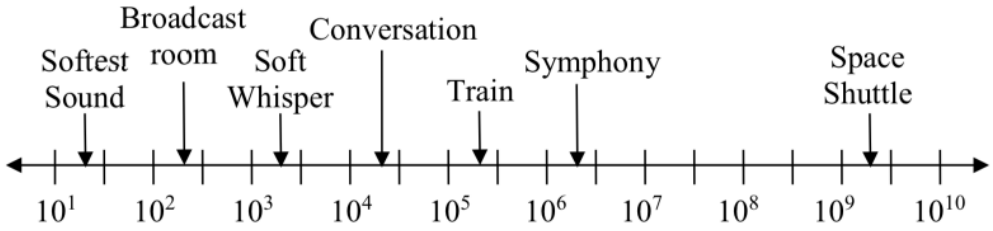


(It is interesting to note the large gap between Mars and Jupiter on the log number line. There is an asteroid belt located there, which some scientists believe is a planet that never formed because of the effects of the gravity of Jupiter.)

Logarithms are also used to measure sound intensity levels.

Source of Sound/Noise	Approximate Sound Pressure in μ Pa (micro Pascals)
Launching of the Space Shuttle	2000,000,000
Full Symphony Orchestra	2000,000

Diesel Freight Train at High Speed at 25 m	200,000
Normal Conversation	20,000
Soft Whispering at 2 m in Library	2,000
Unoccupied Broadcast Studio Room	200
Softest Sound a human can hear	20



Logarithms are useful for showing relative changes. For example, comparing the sound of a diesel freight train to a soft whisper, the diesel train is 100 times larger than a soft whisper.

$$\frac{200,000}{2,000} = 100 = 10^2$$

When one quantity is roughly ten times larger than another, we say it is one **order of magnitude** larger. The order of magnitude can be found as the common logarithm of the ratio of the quantities.

Orders of Magnitude

Given two values A and B, to determine how many **orders of magnitude** A is greater than B,

Difference in orders of magnitude = $\log(\frac{A}{B})$

One example of a logarithmic scale is the Moment Magnitude Scale (MMS) used for earthquakes. This scale is commonly and mistakenly called the Richter Scale, which was a very similar scale succeeded by the MMS.

Moment Magnitude Scale for Earthquakes

For an earthquake with seismic moment S, a measurement of earth movement, the MMS value, or magnitude M of an earthquake, is

$$M = \frac{2}{3} log(\frac{S}{S_0})$$

Where $S_0 = 10^{16}$ is a baseline measure for the seismic moment.

Example 4.8.6

If one earthquake has a MMS magnitude of 6.0, and another has a magnitude of 8.0, how much more powerful (in terms of earth movement) is the second earthquake?

Solution

Since the first earthquake has magnitude 6.0, we can find the amount of earth movement for that quake, which we'll denote S_1. The value of S_0 is not particularity relevant, so we will not replace it with its value.

$$6.0 = \frac{2}{3} log\left(\frac{S_1}{S_0}\right)$$

$$6.0\left(\frac{3}{2}\right) = log\left(\frac{S_1}{S_0}\right)$$

$$9 = log\left(\frac{S_1}{S_0}\right)$$

$$\frac{S_1}{S_0} = 10^9$$

$$S_1 = 10^9 S_0$$

This tells us the first earthquake has about 10^9 times more earth movement than the baseline measure, S_0.

Doing the same with the second earthquake, S_2, with a magnitude of 8.0,

$$8.0 = \frac{2}{3} log\left(\frac{S_2}{S_0}\right)$$

$$S_2 = 10^{12} S_0$$

Comparing the earth movement of the second earthquake to the first,

$$\frac{S_2}{S_1} = \frac{10^{12} S_0}{10^9 S_0} = 10^3 = 1000$$

The second value's earth movement is 1000 times as large as the first earthquake.

Example 4.8.7

One earthquake has magnitude of 3.0. If a second earthquake has twice as much earth movement as the first earthquake, find the magnitude of the second quake.

Solution

Since the first quake has magnitude 3.0,

$$3.0 = \frac{2}{3} log(\frac{S}{S_0})$$

Solving for S,

$$3.0\frac{3}{2} = log(\frac{S}{S_0})$$

$$4.5 = log(\frac{S}{S_0})$$

$$10^{4.5} = \frac{S}{S_0}$$

$$S = 10^{4.5} S_0$$

Since the second earthquake has twice as much earth movement, for the second quake,

$$S = 2 \cdot 10^{4.5} S_0$$

Finding the magnitude,

$$M = \frac{2}{3} log(\frac{2 \cdot 10^{4.5} S_0}{S_0})$$

$$M = \frac{2}{3} log(2 \cdot 10^{4.5}) \approx 3.201$$

The second earthquake with twice as much earth movement will have a magnitude of about 3.2.

As we saw in an earlier section, logarithms are also used to describe pH levels.

pH Level

The pH of a solutions is defined by the following formula, where $[\mathrm{H}^+]$ is the concentration of hydrogen ions in the solution

$$\begin{aligned} pH &= -\log([\mathrm{H}^+]) \\ &= \log\left(\frac{1}{[\mathrm{H}^+]}\right) \end{aligned}$$

Example 4.8.8

If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

Solution

Suppose C is the original concentration of hydrogen ions, and P is the original pH of the liquid. Then $P = -\log(C)$. If the concentration is doubled, the new concentration is $2C$. Then the pH of the new liquid is

$$pH = -\log(2C)$$

Using the product rule of logs

$$pH = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since $P = -\log(C)$,the new pH is

$$pH = P - \log(2) \approx P - 0.301$$

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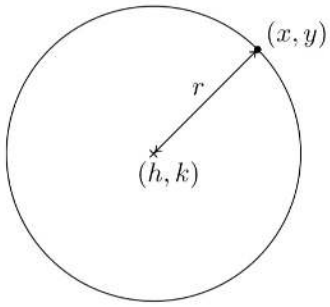
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4.9: Trigonometry Preview - Circles

Recall from Geometry that a circle can be determined by fixing a point (called the center) and a positive number (called the radius) as follows.

Circle

A **circle** with center (h,k) and radius $r>0$ is the set of all points (x,y) in the plane whose distance to (h,k) is r.



From the picture, we see that a point (x,y) is on the circle if and only if its distance to (h,k) is r. We express this relationship algebraically using the Distance Formula, as

$$r=\sqrt{(x-h)^2+(y-k)^2}$$

By squaring both sides of this equation, we get an equivalent equation (since $r>0$) which gives us the standard equation of a circle.

The Standard Form of a Circle

The standard form of the equation of a circle with center (h,k) and radius $r>0$ is

$$(x-h)^2+(y-k)^2=r^2.$$

Example 4.9.1

Write the standard equation of the circle with center (−2,3) and radius 5.

Solution.

Here, $(h,k)=(-2,3)$ and $r=5$, so we get

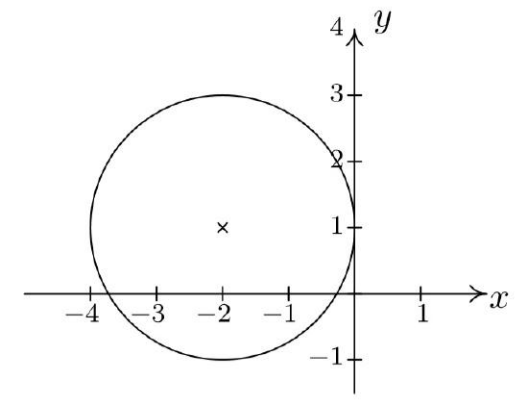
$$\begin{aligned}(x-(-2))^2+(y-3)^2 &= (5)^2\\ (x+2)^2+(y-3)^2 &= 25\end{aligned}$$

Example 4.9.2

Graph $(x+2)^2+(y-1)^2=4$. Find the center and radius.

Solution

From the standard form of a circle, we have that $x+2$ is $x-h$, so $h=-2$ and $y-1$ is $y-k$ so $k=1$. This tells us that our center is $(-2,1)$. Furthermore, $r^2=4$, so $r=2$. Thus we have a circle centered at $(-2,1)$ with a radius of 2. Graphing gives us



If we were to expand the equation in the previous example and gather up like terms, instead of the easily recognizable $(x+2)^2+(y-1)^2=4$, we'd be contending with $x^2+4x+y^2-2y+1=0$. If we're given such an equation, we can complete the square to rewrite the equation in standard form for a circle.

Example 4.9.3

Complete the square to find the center and radius of $3x^2-6x+3y^2+4y-4=0$.

Solution

$$\begin{aligned} 3x^2-6x+3y^2+4y-4 &= 0 \\ 3x^2-6x+3y^2+4y &= 4 && \text{add 4 to both sides} \\ 3\left(x^2-2x\right)+3\left(y^2+\frac{4}{3}y\right) &= 4 && \text{factor out leading coefficients} \\ 3\left(x^2-2x+\underline{1}\right)+3\left(y^2+\frac{4}{3}y+\underline{\underline{\frac{4}{9}}}\right) &= 4+3\underline{(1)}+3\underline{\underline{\left(\frac{4}{9}\right)}} && \text{complete the square in } x, y \\ 3(x-1)^2+3\left(y+\frac{2}{3}\right)^2 &= \frac{25}{3} && \text{factor} \\ (x-1)^2+\left(y+\frac{2}{3}\right)^2 &= \frac{25}{9} && \text{divide both sides by 3} \end{aligned}$$

From the standard form of a circle, we identify $x-1$ as $x-h$, so $h=1$, and $y+\frac{2}{3}$ as $y-k$, so $k=-\frac{2}{3}$. Hence, the center is $(h,k)=\left(1,-\frac{2}{3}\right)$. Furthermore, we see that $r^2=\frac{25}{9}$ so the radius is $r=\frac{5}{3}$.

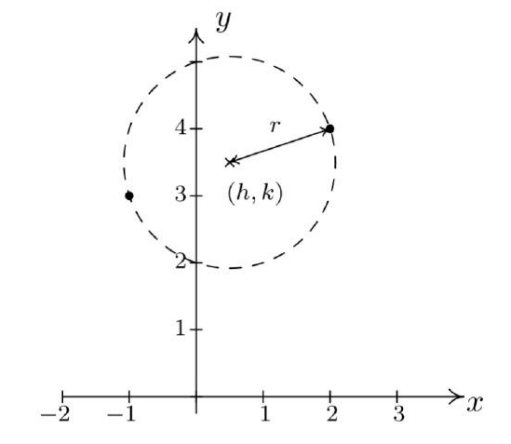
It is possible to obtain equations like $(x-3)^2+(y+1)^2=0$ or $(x-3)^2+(y+1)^2=-1$, neither of which describes a circle. (Do you see why not?) The reader is encouraged to think about what, if any, points lie on the graphs of these two equations.

Example 4.9.4

Write the standard equation of the circle which has $(-1,3)$ and $(2,4)$ as the endpoints of a diameter.

Solution

We recall that a diameter of a circle is a line segment containing the center and two points on the circle. Plotting the given data yields



Since the given points are endpoints of a diameter, we know their midpoint (h,k) is the center of the circle. Equation ??? gives us

$$\begin{aligned}(h,k) &= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)\\ &= \left(\frac{-1+2}{2}, \frac{3+4}{2}\right)\\ &= \left(\frac{1}{2}, \frac{7}{2}\right)\end{aligned}$$

The diameter of the circle is the distance between the given points, so we know that half of the distance is the radius. Thus,

$$\begin{aligned}r &= \frac{1}{2}\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2}\\ &= \frac{1}{2}\sqrt{(2-(-1))^2+(4-3)^2}\\ &= \frac{1}{2}\sqrt{3^2+1^2}\\ &= \frac{\sqrt{10}}{2}\end{aligned}$$

Finally, since $\left(\frac{\sqrt{10}}{2}\right)^2 = \frac{10}{4}$, our answer becomes $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{7}{2}\right)^2 = \frac{10}{4}$

We close this section with a very important circle that you will learn more about in trigonometry: the **Unit Circle**.

The Unit Circle

The **Unit Circle** is the circle centered at $(0,0)$ with a radius of 1. The standard equation of the Unit Circle is $x^2+y^2=1$.

Example 4.9.5

Find the points on the unit circle with y-coordinate $\frac{\sqrt{3}}{2}$.

Solution

We replace y with $\frac{\sqrt{3}}{2}$ in the equation $x^2+y^2=1$ to get

$$\begin{aligned} x^2+y^2 &= 1 \\ x^2+\left(\frac{\sqrt{3}}{2}\right)^2 &= 1 \\ \frac{3}{4}+x^2 &= 1 \\ x^2 &= \frac{1}{4} \\ x &= \pm\sqrt{\frac{1}{4}} \\ x &= \pm\frac{1}{2} \end{aligned}$$

Our final answers are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$.

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4.E: Exponential and Logarithmic Functions (Exercises)

Exponential Functions

When populations grow rapidly, we often say that the growth is “exponential,” meaning that something is growing very rapidly. To a mathematician, however, the term exponential growth has a very specific meaning. In this section, we will take a look at exponential functions, which model this kind of rapid growth.

Verbal

1) Explain why the values of an increasing exponential function will eventually overtake the values of an increasing linear function.

Answer:

Linear functions have a constant rate of change. Exponential functions increase based on a percent of the original.

2) Given a formula for an exponential function, is it possible to determine whether the function grows or decays exponentially just by looking at the formula? Explain.

3)The Oxford Dictionary defines the word nominal as a value that is “stated or expressed but not necessarily corresponding exactly to the real value.” Develop a reasonable argument for why the term nominal rate is used to describe the annual percentage rate of an investment account that compounds interest.

Answer:

When interest is compounded, the percentage of interest earned to principal ends up being greater than the annual percentage rate for the investment account. Thus, the annual percentage rate does not necessarily correspond to the real interest earned, which is the very definition of nominal.

Algebraic

For the following exercises, identify whether the statement represents an exponential function. Explain.

4) The average annual population increase of a pack of wolves is 25.

5) A population of bacteria decreases by a factor of $\frac{1}{8}$ every 24 hours.

Answer:

exponential; the population decreases by a proportional rate.

6) The value of a coin collection has increased by 3.25%annually over the last 20 years.

7) For each training session, a personal trainer charges his clients $\$5$ less than the previous training session.

Answer:

not exponential; the charge decreases by a constant amount each visit, so the statement represents a linear function.

8) The height of a projectile at time t is represented by the function $h(t) = -4.9t^2 + 18t + 40$

For the following exercises, consider this scenario: For each year t, the population of a forest of trees is represented by the function $A(t) = 115(1.025)^t$. In a neighboring forest, the population of the same type of tree is represented by the function $B(t) = 82(1.029)^t$. (Round answers to the nearest whole number.)

9) Which forest’s population is growing at a faster rate?

Answer:

The forest represented by the function $B(t) = 82(1.029)^t$.

10) Which forest had a greater number of trees initially? By how many?

11) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 20 years? By how many?

Answer:

After $t = 20$ years, forest A will have 43more trees than forest B.

12) Assuming the population growth models continue to represent the growth of the forests, which forest will have a greater number of trees after 100years? By how many?

13) Discuss the above results from the previous four exercises. Assuming the population growth models continue to represent the growth of the forests, which forest will have the greater number of trees in the long run? Why? What are some factors that might influence the long-term validity of the exponential growth model?

Answer:

Answers will vary. Sample response: For a number of years, the population of forest A will increasingly exceed forest B, but because forest B actually grows at a faster rate, the population will eventually become larger than forest A and will remain that way as long as the population growth models hold. Some factors that might influence the long-term validity of the exponential growth model are drought, an epidemic that culls the population, and other environmental and biological factors.

For the following exercises, determine whether the equation represents exponential growth, exponential decay, or neither. Explain.

14) $y = 300(1 - t)^5$

15) $y = 220(1.06)^x$

Answer:

exponential growth; The growth factor, $1.06$is greater than 1.

16) $y = 16.5(1.025)^{\frac{1}{x}}$

17) $y = 11,701(0.97)^t$

Answer:

exponential decay; The decay factor, $0.97$is between 0 and 1

For the following exercises, find the formula for an exponential function that passes through the two points given.

18) $(0, 6)$ and $(3, 750)$

19) $(0, 2000)$and $(2, 20)$

Answer:

$f(x) = 2000(0.1)^x$

20) $\left(-1, \frac{3}{2}\right)$ and $(3, 24)$

21) $(-2, 6)$and $(3, 1)$

Answer:

$f(x) = \left(\frac{1}{6}\right)^{-\frac{3}{5}} \left(\frac{1}{6}\right)^{\frac{x}{5}} \approx 2.93(0.699)^x$

22) $(3, 1)$ and $(5, 4)$

For the following exercises, determine whether the table could represent a function that is linear, exponential, or neither. If it appears to be exponential, find a function that passes through the points.

23)

x	1	2	3	4
$f(x)$	70	40	10	-20

Answer:

Linear

24)

x	1	2	3	4
$h(x)$	70	49	34.3	24.01

25)

x	1	2	3	4
$m(x)$	80	61	42.9	25.61

Answer:

Neither

26)

x	1	2	3	4
$f(x)$	10	20	40	80

27)

x	1	2	3	4
$g(x)$	-3.25	2	7.25	12.5

Answer:

Linear

For the following exercises, use the compound interest formula, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

28) After a certain number of years, the value of an investment account is represented by the equation $10,250\left(1 + \frac{0.04}{12}\right)^{120}$ What is the value of the account?

29) What was the initial deposit made to the account in the previous exercise?

Answer:

$\$10,250$

30) How many years had the account from the previous exercise been accumulating interest?

31) An account is opened with an initial deposit of $\$6,500$ and earns 3.6% interest compounded semi-annually. What will the account be worth in 20 years?

Answer:

$\$13,268.58$

32) How much more would the account in the previous exercise have been worth if the interest were compounding weekly?

33) Solve the compound interest formula for the principal, P.

Answer:

$P = A(t) \cdot \left(1 + \frac{r}{n}\right)^{-nt}$

34) Use the formula found in the previous exercise to calculate the initial deposit of an account that is worth $\$14,472.74$ after earning 5.5% interest compounded monthly for 5 years. (Round to the nearest dollar.)

35) How much more would the account in the previous two exercises be worth if it were earning interest for 5 more years?

Answer:

$\$4,572.56$

36) Use properties of rational exponents to solve the compound interest formula for the interest rate, r.

37) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded semi-annually, had an initial deposit of $\$9,000$ and was worth $\$13,373.53$ after 10 years.

Answer:

4%

38) Use the formula found in the previous exercise to calculate the interest rate for an account that was compounded monthly, had an initial deposit of $\$5,500$ and was worth $\$38,455$ after 30 years.

For the following exercises, determine whether the equation represents continuous growth, continuous decay, or neither. Explain.

39) $y = 3742(e)^{0.75t}$

Answer:

continuous growth; the growth rate is greater than 0.

40) $y = 150(e)^{\frac{3.25}{t}}$

41) $y = 2.25(e)^{-2t}$

Answer:

continuous decay; the growth rate is less than 0.

42) Suppose an investment account is opened with an initial deposit of $\$12,000$ earning 7.2% interest compounded continuously. How much will the account be worth after 30 years?

43) How much less would the account from Exercise 42 be worth after 30 years if it were compounded monthly instead?

Answer:

$\$669.42$

Numeric

For the following exercises, evaluate each function. Round answers to four decimal places, if necessary.

44) $f(x) = 2(5)^x$ for $f(-3)$

45) $f(x) = -4^{2x+3}$ for $f(-1)$

Answer:

$f(-1) = -4$

46) $f(x) = e^x$, for $f(3)$

47) $f(x) = -2e^{x-1}$, for $f(-1)$

Answer:

$f(-1) \approx -0.2707$

48) $f(x) = 2.7(4)^{-x+1} + 1.5$, for $f(-2)$

49) $f(x) = 1.2e^{2x} - 0.3$, for $f(3)$

Answer:

$f(3) \approx 483.8146$

50) $f(x) = -\frac{3}{2}(3)^{-x} + \frac{3}{2}$, for $f(2)$

Technology

For the following exercises, use a graphing calculator to find the equation of an exponential function given the points on the curve.

51) $(0, 3)$ and $(3, 375)$

Answer:

$y = 3 \cdot 5^x$

52) $(3, 222.62)$and $(10, 77.456)$

53) $(20, 29.495)$and $(150, 730.89)$

Answer:

$y \approx 18 \cdot 1.025^x$

54) $(5, 2.909)$and $(13, 0.005)$

55) ((11,310.035)\) and $(25, 356.3652)$

Answer:

$y \approx 0.2 \cdot 1.95^x$

Extensions

56) The annual percentage yield (APY) of an investment account is a representation of the actual interest rate earned on a compounding account. It is based on a compounding period of one year. Show that the APY of an account that compounds monthly can be found with the formula $APY = \left(1 + \frac{r}{12}\right)^{12} - 1$

57) Repeat the previous exercise to find the formula for the APY of an account that compounds daily. Use the results from this and the previous exercise to develop a function $I(n)$for the APY of any account that compounds $n$times per year.

Answer:

$$\begin{aligned} APY &= \frac{A(t) - a}{a} \\ &= \frac{a\left(1 + \frac{r}{365}\right)^{365(1)} - a}{a} \\ &= \frac{a\left[\left(1 + \frac{r}{365}\right)^{365} - 1\right]}{a} \\ &= \left(1 + \frac{r}{365}\right)^{365} - 1 \end{aligned} \quad ; I(n) = \left(1 + \frac{r}{n}\right)^{n} - 1$$

58) Recall that an exponential function is any equation written in the form $f(x) = a \cdot b^x$ such that a and b are positive numbers and $b \neq 1$. Any positive number $b$can be written as $b = e^n$ for some value of n. Use this fact to rewrite the formula for an exponential function that uses the number $e$as a base.

59) In an exponential decay function, the base of the exponent is a value between 0 and 1. Thus, for some number $b > 1$,the exponential decay function can be written as $f(x) = a \cdot \left(\frac{1}{b}\right)^x$ Use this formula, along with the fact that $b = e^n$ to show that an exponential decay function takes the form $f(x) = a \cdot (e)^{-nx}$ for some positive number n.

Answer:

Let f be the exponential decay function $f(x) = a \cdot \left(\frac{1}{b}\right)^x$ such that $b > 1$. Then for some number $n > 0$,

$$\begin{aligned} f(x) &= a \cdot \left(\frac{1}{b}\right)^x \\ &= a\left(b^{-1}\right)^x \\ &= a\left(\left(e^n\right)^{-1}\right)^x \\ &= a\left(e^{-n}\right)^x \\ &= a(e)^{-nx} \end{aligned}$$

60) The formula for the amount A in an investment account with a nominal interest rate r at any time t is given by $A(t) = a(e)^{rt}$, where $a$is the amount of principal initially deposited into an account that compounds continuously. Prove that the percentage of interest earned to principal at any time t can be calculated with the formula $I(t) = e^{rt} - 1$

Real-World Applications

61) The fox population in a certain region has an annual growth rate of 9% per year. In the year 2012, there were $23,900$ fox counted in the area. What is the fox population predicted to be in the year 2020?

Answer:

$47,622$fox

62) A scientist begins with 100 milligrams of a radioactive substance that decays exponentially. After 35 hours, 50mg of the substance remains. How many milligrams will remain after 54 hours?

63) In the year 1985, a house was valued at $\$110,000$ By the year 2005, the value had appreciated to $\$145,000$ What was the annual growth rate between 1985 and 2005? Assume that the value continued to grow by the same percentage. What was the value of the house in the year 2010?

Answer:

1.39%; $\$155,368.09$

64) A car was valued at $\$38,000$ in the year 2007. By 2013, the value had depreciated to $\$11,000$ If the car's value continues to drop by the same percentage, what will it be worth by 2017?

65) Jamal wants to save $\$54,000$ for a down payment on a home. How much will he need to invest in an account with 8.2% APR, compounding daily, in order to reach his goal in 5 years?

Answer:

$\$35,838.76$

66) Kyoko has $\$10,000$ that she wants to invest. Her bank has several investment accounts to choose from, all compounding daily. Her goal is to have $\$15,000$ by the time she finishes graduate school in 6 years. To the nearest hundredth of a percent, what should her minimum annual interest rate be in order to reach her goal? (Hint: solve the compound interest formula for the interest rate.)

67) Alyssa opened a retirement account with 7.25% APR in the year 2000. Her initial deposit was $\$13,500$ How much will the account be worth in 2025 if interest compounds monthly? How much more would she make if interest compounded continuously?

Answer:

$\$82,247.78$; $\$449.75$

68) An investment account with an annual interest rate of 7% was opened with an initial deposit of $\$4,000$ Compare the values of the account after 9 years when the interest is compounded annually, quarterly, monthly, and continuously.

Graphs of Exponential Functions

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool.

Verbal

1) What role does the horizontal asymptote of an exponential function play in telling us about the end behavior of the graph?

Answer:

An asymptote is a line that the graph of a function approaches, as x either increases or decreases without bound. The horizontal asymptote of an exponential function tells us the limit of the function's values as the independent variable gets either extremely large or extremely small.

2) What is the advantage of knowing how to recognize transformations of the graph of a parent function algebraically?

Algebraic

3) The graph of $f(x)=3^x$ is reflected about the y-axis and stretched vertically by a factor of 4 What is the equation of the new function, $g(x)$. State its y-intercept, domain, and range.

Answer:

$g(x)=4(3)^{-x}$ $(0,4)$ Domain: all real numbers; Range: all real numbers greater than 0.

4) The graph of $f(x)=\left(\frac{1}{2}\right)^{-x}$ is reflected about the y-axis and compressed vertically by a factor of $\left(\frac{1}{5}\right)$ What is the equation of the new function, $g(x)$ State its y-intercept, domain, and range.

5) The graph of $f(x)=10^x$ is reflected about the x-axis and shifted upward 7 units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

Answer:

$g(x)=-10^x+7$; y-intercept: $(0,6)$; Domain: all real numbers; Range: all real numbers less than 7.

6) The graph of $f(x)=(1.68)^x$ is shifted right 3 units, stretched vertically by a factor of 2 reflected about the x-axis, and then shifted downward 3 units. What is the equation of the new function, $g(x)$? State its y-intercept (to the nearest thousandth), domain, and range.

7) The graph of $f(x)=2\left(\frac{1}{4}\right)^{x-20}$ is shifted left 2 units, stretched vertically by a factor of 4 reflected about the x-axis, and then shifted downward 4 units. What is the equation of the new function, $g(x)$? State its y-intercept, domain, and range.

Answer:

$g(x)=2\left(\frac{1}{4}\right)^x$; y-intercept: $(0,2)$; Domain: all real numbers; Range: all real numbers greater than 0.

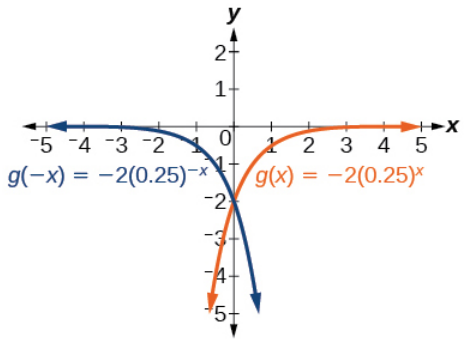
Graphical

For the following exercises, graph the function and its reflection about the y-axis on the same axes, and give the y-intercept.

8) $f(x)=3\left(\frac{1}{2}\right)^x$

9) $g(x)=-2(0.25)^x$

Answer:



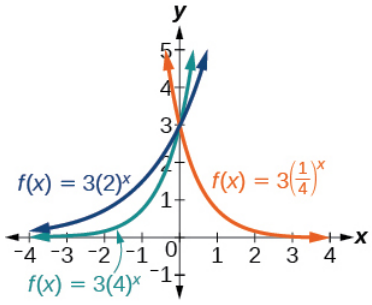
y-intercept: $(0, -2)$

10) $h(x) = 6(1.75)^{-x}$

For the following exercises, graph each set of functions on the same axes.

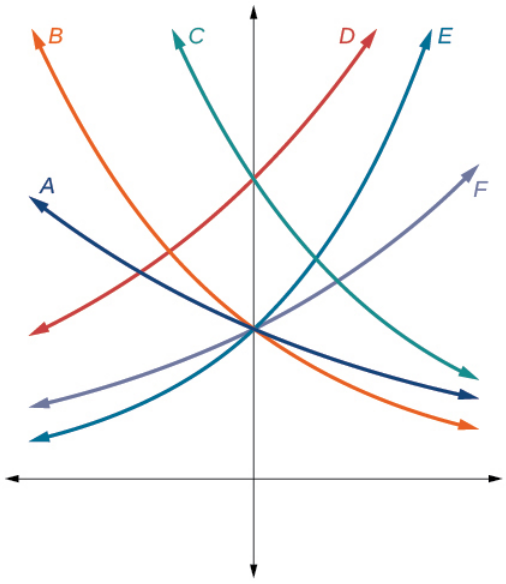
11) $f(x) = 3\left(\frac{1}{4}\right)^x, g(x) = 3(2)^x, h(x) = 3(4)^x$

Answer:



12) $f(x) = \frac{1}{4}(3)^x, g(x) = 2(3)^x, h(x) = 4(3)^x$

For the following exercises, match each function with one of the graphs in Figure below.



13) $f(x) = 2(0.69)^x$

Answer:

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B

14) $f(x) = 2(1.28)^x$

15) $f(x) = 2(0.81)^x$

Answer:

A

16) $f(x) = 4(1.28)^x$

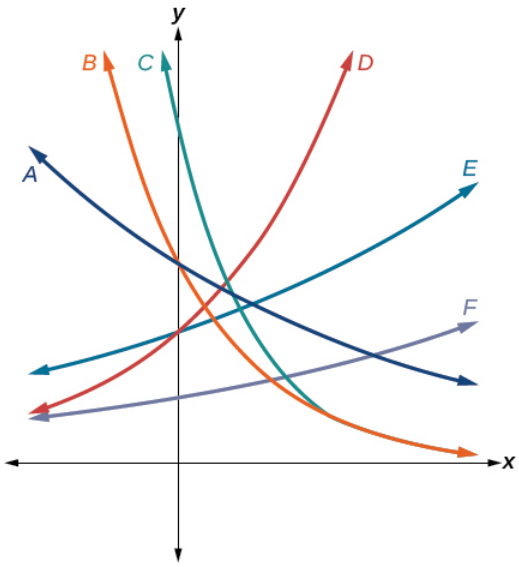
17) $f(x) = 2(1.59)^x$

Answer:

E

18) $f(x) = 4(0.69)^x$

For the following exercises, use the graphs shown in Figure below. All have the form $f(x) = ab^x$.



19) Which graph has the largest value for b?

Answer:

D

20) Which graph has the smallest value for b?

21) Which graph has the largest value for a?

Answer:

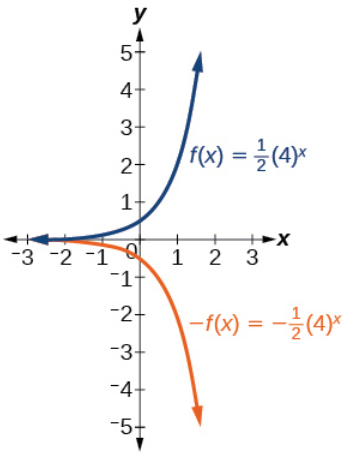
C

22) Which graph has the smallest value for a?

For the following exercises, graph the function and its reflection about the x-axis on the same axes.

23) $f(x) = \frac{1}{2}(4)^x$

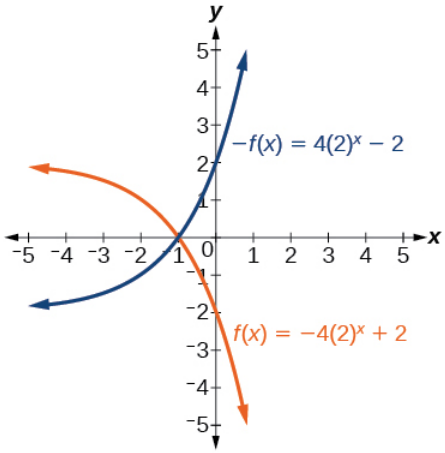
Answer:



24) $f(x) = 3(0.75)^x - 1$

25) $f(x) = -4(2)^x + 2$

Answer:

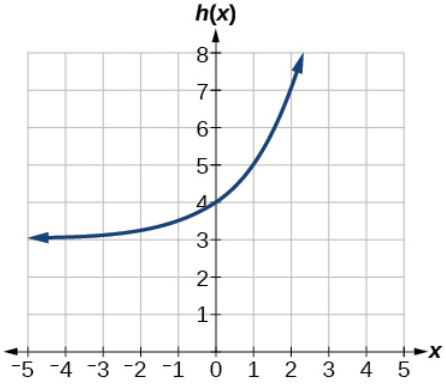


For the following exercises, graph the transformation of $f(x) = 2^x$ Give the horizontal asymptote, the domain, and the range.

26) $f(x) = 2^{-x}$

27) $h(x) = 2^x + 3$

Answer:



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Horizontal asymptote: $h(x) = 3$ Domain: all real numbers; Range: all real numbers strictly greater than 3.

28) $f(x) = 2^{x-2}$

For the following exercises, describe the end behavior of the graphs of the functions.

29) $f(x) = -5(4)^x - 1$

Answer:

As $x \to \infty$, $f(x) \to -\infty$

As $x \to -\infty$, $f(x) \to -1$

30) $f(x) = 3\left(\frac{1}{2}\right)^x - 2$

31) $f(x) = 3(4)^{-x} + 2$

Answer:

As $x \to \infty$, $f(x) \to 2$

As $x \to -\infty$, $f(x) \to \infty$

For the following exercises, start with the graph of $f(x) = 4^x$, then write a function that results from the given transformation.

32) Shift $f(x)$ 4 units upward

33) Shift $f(x)$ 3 units downward

Answer:

$f(x) = 4^x - 3$

34) Shift $f(x)$ 2 units left

35) Shift $f(x)$ 5 units right

Answer:

$f(x) = 4^{x-5}$

36) Reflect $f(x)$ about the x-axis

37) Reflect $f(x)$ about the y-axis

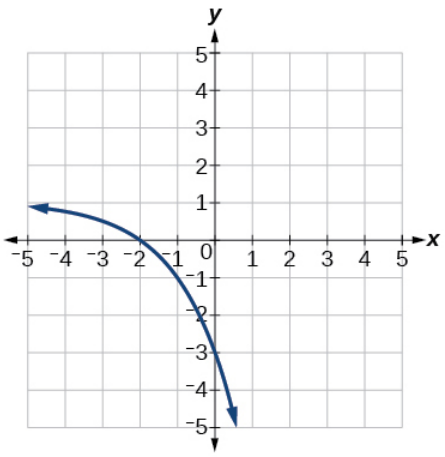
Answer:

$f(x) = 4^{-x}$

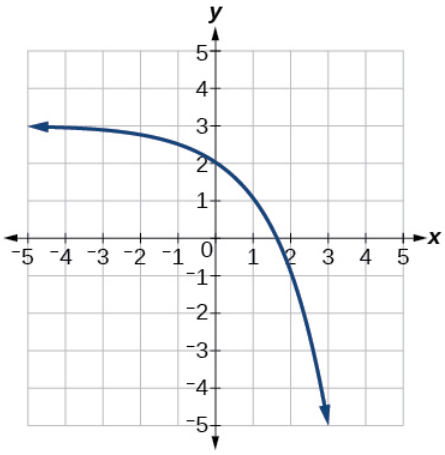
For the following exercises, each graph is a transformation of $f(x) = 2^x$ Write an equation describing the transformation.

38)

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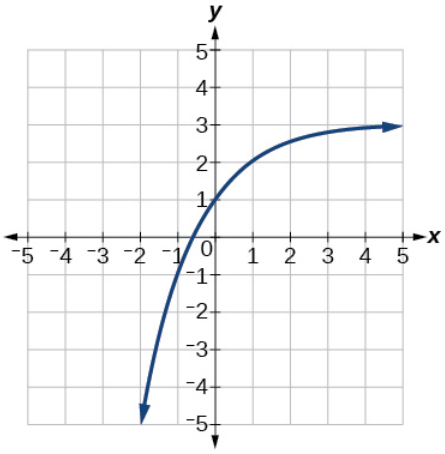
39)



Answer:

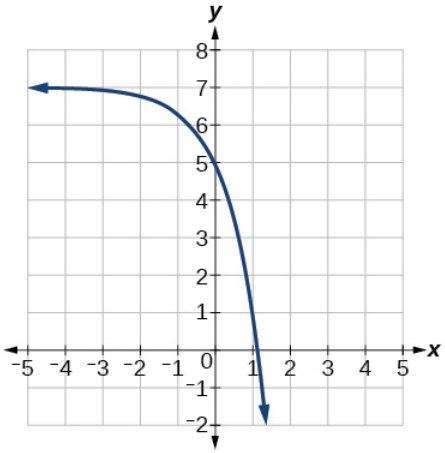
$y = -2^x + 3$

40)



For the following exercises, find an exponential equation for the graph.

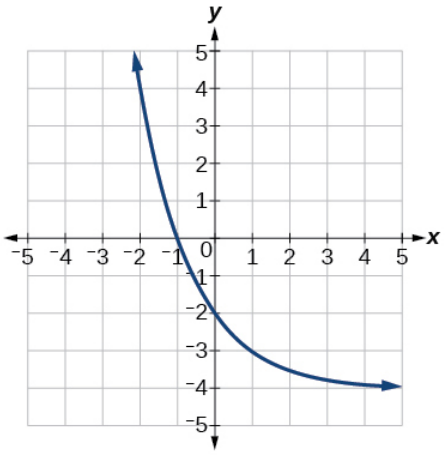
41)



Answer:

$y = -2(3)^x + 7$

42)



Numeric

For the following exercises, evaluate the exponential functions for the indicated value of x.

43) $g(x) = \frac{1}{3}(7)^{x-2}$ for $g(6)$.

Answer:

$g(6) = 800 + \frac{1}{3} \approx 800.3333$

44) $f(x) = 4(2)^{x-1} - 2$ for $f(5)$.

45) $h(x) = -\frac{1}{2}\left(\frac{1}{2}\right)^x + 6$ for $h(-7)$.

Answer:

$h(-7) = -58$

Technology

For the following exercises, use a graphing calculator to approximate the solutions of the equation. Round to the nearest thousandth.

46) $-50 = \left(\frac{1}{2}\right)^{-x}$

47) $116 = \left(\frac{1}{4}\right)\left(\frac{1}{8}\right)^{x}$

Answer:

$x \approx -2.953$

48) $12 = 2(3)^{x} + 1$

49) $5 = 3\left(\frac{1}{2}\right)^{x-1} - 2$

Answer:

$x \approx -0.222$

50) $-30 = -4(2)^{x+2} + 2$

Extensions

51) Explore and discuss the graphs of $F(x) = (b)^{x}$ and $G(x) = \left(\frac{1}{b}\right)^{x}$. Then make a conjecture about the relationship between the graphs of the functions b^{x} and $\left(\frac{1}{b}\right)^{x}$ for any real number $b > 0$.

Answer:

The graph of $G(x) = \left(\frac{1}{b}\right)^{x}$ is the refelction about the y-axis of the graph of $F(x) = (b)^{x}$; For any real number $b > 0$ and function $f(x) = (b)^{x}$ the graph of $\left(\frac{1}{b}\right)^{x}$ is the the reflection about the y-axis, $F(-x)$.

52) Prove the conjecture made in the previous exercise.

53) Explore and discuss the graphs of $f(x) = 4^{x}$, $g(x) = 4^{x-2}$, and $h(x) = \left(\frac{1}{16}\right)4^{x}$ Then make a conjecture about the relationship between the graphs of the functions b^{x} and $\left(\frac{1}{b^{n}}\right)b^{x}$ for any real number n and real number $b > 0$.

Answer:

The graphs of $g(x)$ and $h(x)$ are the same and are a horizontal shift to the right of the graph of $f(x)$; For any real number n, real number $b > 0$, and function $f(x) = b^{x}$ the graph of $\left(\frac{1}{b^{n}}\right)b^{x}$ is the horizontal shift $f(x - n)$.

54) Prove the conjecture made in the previous exercise.

Logarithmic Functions

The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.

Verbal

1) What is a base b logarithm? Discuss the meaning by interpreting each part of the equivalent equations $b^{y} = x$ and $\log_{b} x = y$ for $b > 0, b \neq 1$.

Answer

A logarithm is an exponent. Specifically, it is the exponent to which a base b is raised to produce a given value. In the expressions given, the base b has the same value. The exponent, $y$in the expression b^{y} can also be written as the logarithm, $\log_{b} x = y$ and the value of x is the result of raising b to the power of y.

2) How is the logarithmic function $f(x) = \log_{b} x$ related to the exponential function $g(x) = b^{x}$? What is the result of composing these two functions?

3) How can the logarithmic equation $\log_{b} x = y$ be solved for x using the properties of exponents?

Answer

Since the equation of a logarithm is equivalent to an exponential equation, the logarithm can be converted to the exponential equation $b^y = x$ and then properties of exponents can be applied to solve for x

4) Discuss the meaning of the common logarithm. What is its relationship to a logarithm with base b and how does the notation differ?

5) Discuss the meaning of the natural logarithm. What is its relationship to a logarithm with base b and how does the notation differ?

Answer

The natural logarithm is a special case of the logarithm with base b in that the natural log always has base e.Rather than notating the natural logarithm as $\log_e(x)$the notation used is $\ln(x)$

Algebraic

For the following exercises, rewrite each equation in exponential form.

6) $\log_4(q) = m$

7) $\log_a(b) = c$

Answer

$a^c = b$

8) $\log_{16}(y) = x$

9) $\log_x(64) = y$

Answer

$x^y = 64$

10) $\log_y(x) = -11$

11) $\log_{15}(a) = b$

Answer

$15^b = a$

12) $\log_y(137) = x$

13) $\log_{13}(142) = a$

Answer

$13^a = 142$

14) $\log(v) = t$

15) $\ln(w) = n$

Answer

$e^n = w$

For the following exercises, rewrite each equation in logarithmic form.

16) $4^x = y$

17) $c^d = k$

Answer

$\log_c(k) = d$

18) $m^{-7} = n$

19) $19^{x} = y$

Answer

$\log_{19}(y) = x$

20) $x^{-\frac{10}{13}} = y$

21) $n^{4} = 103$

Answer

$\log_{n}(103) = 4$

22) $\left(\frac{7}{5}\right)^{m} = n$

23) $y^{x} = \frac{39}{100}$

Answer

$\log_{y}\left(\frac{39}{100}\right) = x$

24) $10^{a} = b$

25) $e^{k} = h$

Answer

$\ln(w) = n$

For the following exercises, solve for x by converting the logarithmic equation to exponential form.

26) $\log_{3}(x) = 2$

27) $\log_{2}(x) = -3$

Answer

$x = 2^{-3} = \frac{1}{8}$

28) $\log_{5}(x) = 2$

29) $\log_{3}(x) = 3$

Answer

$x = 3^{3} = 27$

30) $\log_{2}(x) = 6$

31) $\log_{9}(x) = \frac{1}{2}$

Answer

$x = 9^{\frac{1}{2}} = 3$

32) $\log_{18}(x) = 2$

33) $\log_{6}(x) = -3$

Answer

$$x = 6^{-3} = \frac{1}{216}$$

34) $\log(x) = 3$

35) $\ln(x) = 2$

Answer

$$x = e^2$$

For the following exercises, use the definition of common and natural logarithms to simplify.

36) $\log(100^8)$

37) $10^{\log(32)}$

Answer

32

38) $2\log(.0001)$

39) $e^{\ln(1.06)}$

Answer

1.06

40) $\ln(e^{-5.03})$

41) $e^{\ln(10.125)} + 4$

Answer

14.125

Numeric

For the following exercises, evaluate the base b logarithmic expression without using a calculator.

42) $\log_3\left(\frac{1}{27}\right)$

43) $\log_6(\sqrt{6})$

Answer

$$\frac{1}{2}$$

44) $\log_2\left(\frac{1}{8}\right) + 4$

45) $6\log_8(4)$

Answer

4

For the following exercises, evaluate the common logarithmic expression without using a calculator.

46) $\log(10,000)$

47) $\log(0.001)$

Answer

-3

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48) $\log(1)+7$

49) $2\log(100^{-3})$

Answer

-12

For the following exercises, evaluate the natural logarithmic expression without using a calculator.

50) $\ln\left(e^{\frac{1}{3}}\right)$

51) $\ln(1)$

Answer

0

52) $\ln\left(e^{-0.225}\right)-3$

53) $25\ln\left(e^{\frac{2}{5}}\right)$

Answer

10

Technology

For the following exercises, evaluate each expression using a calculator. Round to the nearest thousandth.

54) $\log(0.04)$

55) $\ln(15)$

Answer

2.708

56) $\ln\left(\frac{4}{5}\right)$

57) $\log(\sqrt{2})$

Answer

0.151

58) $\ln(\sqrt{2})$

Extensions

59) Is $x=0$ in the domain of the function $f(x)=\log x$? If so, what is the value of the function when $x=0$? Verify the result.

Answer

No, the function has no defined value for $x=0$ To verify, suppose $x=0$ is in the domain of the function $f(x)=\log(x)$ Then there is some number n such that $n=\log(0)$ Rewriting as an exponential equation gives: $10^n=0$ which is impossible since no such real number n exists. Therefore, $x=0$ is not the domain of the function $f(x)=\log(x)$.

60) Is $f(x)=0$ in the range of the function $f(x)=\log(x)$?If so, for what value of x?Verify the result.

61) Is there a number x such that $\ln x=2$? If so, what is that number? Verify the result.

Answer

Yes. Suppose there exists a real number x such that $\ln x=2$ Rewriting as an exponential equation gives $x=e^2$ which is a real number. To verify, let $x=e^2$ Then, by definition, $\ln(x)=\ln\left(e^2\right)=2$

62) Is the following true: $\frac{\log_3(27)}{\log_4\left(\frac{1}{64}\right)} = -1$ log3(27)log4(164)=−1? log3(27)log4(164)=−1? Verify the result.

63) Is the following true: $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$ Verify the result.

Answer

No; $\ln(1) = 0$, so $\frac{\ln(e^{1.725})}{\ln(1)} = 1.725$ is undefined.

Real-World Applications

64) The exposure index EI for a 35 millimeter camera is a measurement of the amount of light that hits the film. It is determined by the equation $EI = \log_2\left(\frac{f^2}{t}\right)$ where f is the "f-stop" setting on the camera, and t is the exposure time in seconds. Suppose the f-stop setting is 8 and the desired exposure time is 2seconds. What will the resulting exposure index be?

65) Refer to the previous exercise. Suppose the light meter on a camera indicates an EI of -2 and the desired exposure time is 16 seconds. What should the f-stop setting be?

Answer

2

66) The intensity levels I of two earthquakes measured on a seismograph can be compared by the formula $\log\left(\frac{I_1}{I_2}\right) = M_1 - M_2$ where M is the magnitude given by the Richter Scale. In August 2009, an earthquake of magnitude 6.1 hit Honshu, Japan. In March 2011, that same region experienced yet another, more devastating earthquake, this time with a magnitude of 9.0. How many times greater was the intensity of the 2011 earthquake? Round to the nearest whole number.

Graphs of Logarithmic Functions

In this section we will discuss the values for which a logarithmic function is defined, and then turn our attention to graphing the family of logarithmic functions.

Verbal

1) The inverse of every logarithmic function is an exponential function and vice-versa. What does this tell us about the relationship between the coordinates of the points on the graphs of each?

Answer

Since the functions are inverses, their graphs are mirror images about the line $y - x$ So for every point (a, b) on the graph of a logarithmic function, there is a corresponding point (b, a) on the graph of its inverse exponential function.

2) What type(s) of translation(s), if any, affect the range of a logarithmic function?

3) What type(s) of translation(s), if any, affect the domain of a logarithmic function?

Answer

Shifting the function right or left and reflecting the function about the y-axis will affect its domain.

4) Consider the general logarithmic function $f(x) = \log_b(x)$Why can't x be zero?

5) Does the graph of a general logarithmic function have a horizontal asymptote? Explain.

Answer

No. A horizontal asymptote would suggest a limit on the range, and the range of any logarithmic function in general form is all real numbers.

Algebraic

For the following exercises, state the domain and range of the function.

6) $f(x) = \log_3(x+4)$

7) $h(x) = \ln\left(\frac{1}{2} - x\right)$

Answer

Domain: $\left(-\infty, \frac{1}{2}\right)$; Range: $(-\infty, \infty)$

8) $g(x) = \log_5(2x+9) - 2$

9) $h(x) = \ln(4x+17) - 5$

Answer

Domain: $\left(-\frac{17}{4}, \infty\right)$; Range: $(-\infty, \infty)$

10) $f(x) = \log_2(12 - 3x) - 3$

For the following exercises, state the domain and the vertical asymptote of the function.

11) $f(x) = \log_b(x-5)$

Answer

Domain: $(5, \infty)$; Vertical asymptote: $x = 5$

12) $g(x) = \ln(3-x)$

13) $f(x) = \log(3x+1)$

Answer

Domain: $\left(-\frac{1}{3}, \infty\right)$; Vertical asymptote: $x = -\frac{1}{3}$

14) $f(x) = 3\log(-x) + 2$

15) $g(x) = -\ln(3x+9) - 7$

Answer

Domain: $(-3, \infty)$; Vertical asymptote: $x = -3$

For the following exercises, state the domain, vertical asymptote, and end behavior of the function.

16) $f(x) = \ln(2-x)$

17) $f(x) = \log\left(x - \frac{3}{7}\right)$

Answer

Domain: $\left(\frac{3}{7}, \infty\right)$

Vertical asymptote: $x = \frac{3}{7}$

End behavior: as $x \to \left(\frac{3}{7}\right)^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$

18) $h(x) = -\log(3x-4) + 3$

19) $g(x) = \ln(2x+6) - 5$

Answer

Domain: $(-3, \infty)$

Vertical asymptote: $x = -3$

End behavior: as $x \to -3^+$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$

20) $f(x) = \log_3(15 - 5x) + 6$

For the following exercises, state the domain, range, and x- and y-intercepts, if they exist. If they do not exist, write DNE.

21) $h(x) = \log_4(x - 1) + 1$

Answer

Domain: $(1, \infty)$

Range: $-\infty, \infty$

Vertical asymptote: $x = 1$

x-intercept: $\left(\frac{5}{4}, 0\right)$

y-intercept: DNE

22) $f(x) = \log(5x + 10) + 3$

23) $g(x) = \ln(-x) - 2$

Answer

Domain: $(-\infty, 0)$

Range: $-\infty, \infty$

Vertical asymptote: $x = 0$

x-intercept: $\left(-e^2, 0\right)$

y-intercept: DNE

24) $f(x) = \log_2(x + 2) - 5$

25) $h(x) = 3\ln(x) - 9$

Answer

Domain: $(0, \infty)$

Range: $-\infty, \infty$

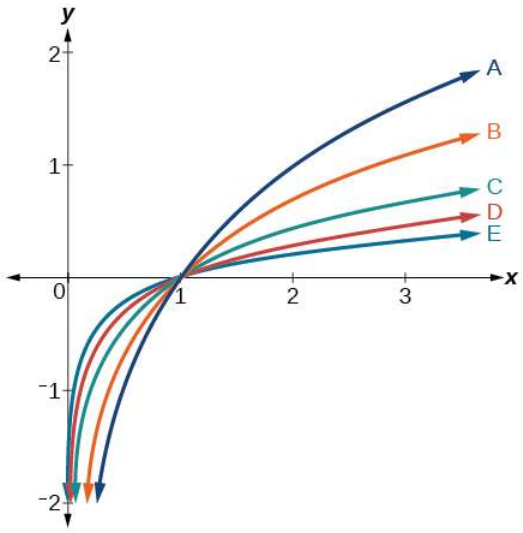
Vertical asymptote: $x = 0$

x-intercept: $\left(e^3, 0\right)$

y-intercept: DNE

Graphical

For the following exercises, match each function in Figure below with the letter corresponding to its graph.



26) $d(x) = \log(x)$

27) $f(x) = \ln(x)$

Answer

B

28) $g(x) = \log_2(x)$

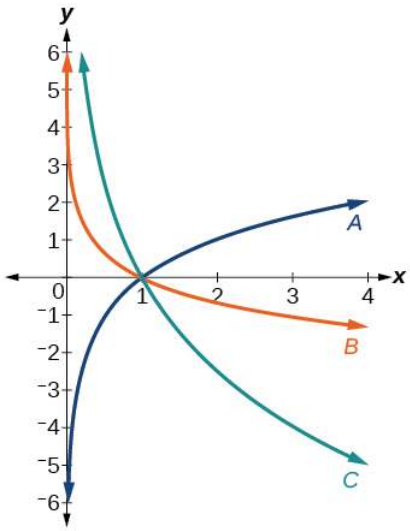
29) $h(x) = \log_5(x)$

Answer

C

30) $j(x) = \log_{25}(x)$

For the following exercises, match each function in Figure with the letter corresponding to its graph.



31) $f(x) = \log_{\frac{1}{3}}(x)$

Answer

B

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32) $g(x) = \log_2(x)$

33) $h(x) = \log_{\frac{3}{4}}(x)$

Answer

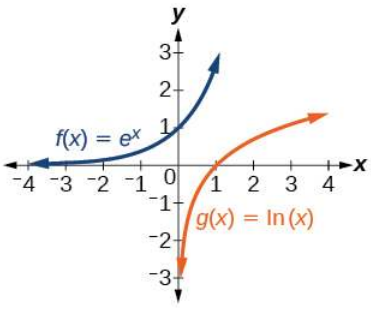
C

For the following exercises, sketch the graphs of each pair of functions on the same axis.

34) $f(x) = \log(x)$ and $g(x) = 10^x$

35) $f(x) = e^x$ and $g(x) = \ln(x)$

Answer



For the following exercises, match each function in Figure with the letter corresponding to its graph.

36) $f(x) = \log_4(-x+2)$

37) $g(x) = -\log_4(x+2)$

Answer

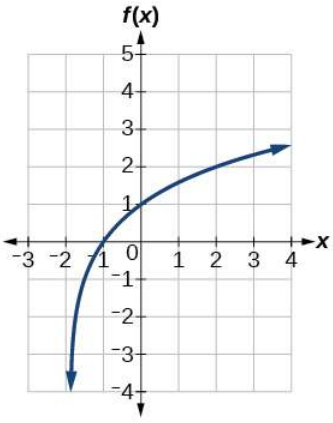
C

38) $h(x) = \log_4(x+2)$

For the following exercises, sketch the graph of the indicated function.

39) $f(x) = \log_2(x+2)$

Answer

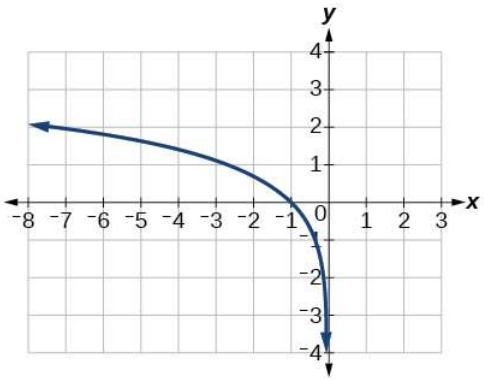


40) $f(x) = 2\log(x)$

41) $f(x) = \ln(-x)$

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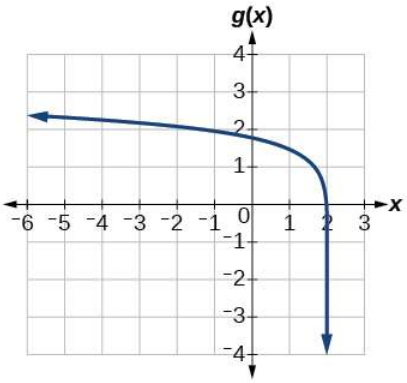
Answer



42) $g(x) = \log(4x + 16) + 4$

43) $g(x) = \log(6 - 3x) + 1$

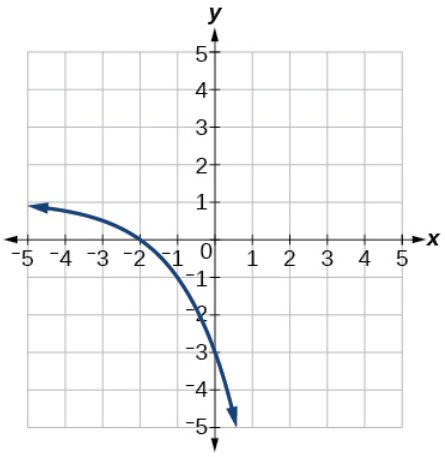
Answer



44) $h(x) = -\frac{1}{2}\log(x + 1) - 3$

For the following exercises, write a logarithmic equation corresponding to the graph shown.

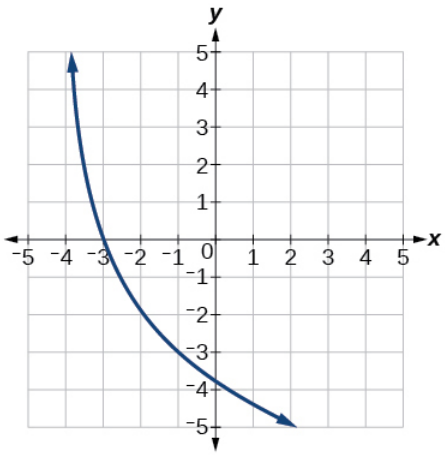
45) Use $y = \log_2(x)$ as the parent function.



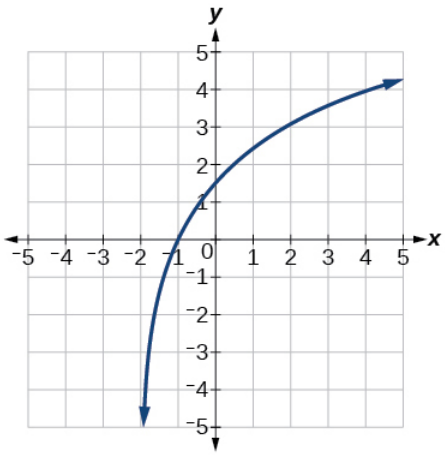
Answer

$f(x) = \log_2(-(x - 1))$

46) Use $f(x) = \log_3(x)$ as the parent function.



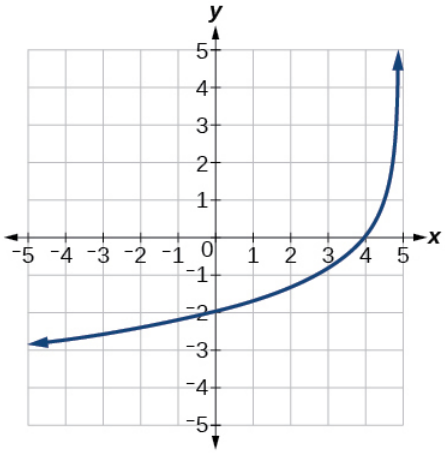
47) Use $f(x) = \log_4(x)$ as the parent function.



Answer

$f(x) = 3\log_4(x+2)$

48) Use $f(x) = \log_5(x)$ as the parent function.



Technology

For the following exercises, use a graphing calculator to find approximate solutions to each equation.

49) $\log(x-1)+2=\ln(x-1)+2$

Answer

$x=2$

50) $\log(2x-3)+2=-\log(2x-3)+5$

51) $\ln(x-2)+2=-\ln(x+1)$

Answer

$x\approx 2.303$

52) $2\ln(5x+1)=\dfrac{1}{2}\ln(-5x)+1$

53) $\dfrac{1}{3}\log(1-x)=\log(x+1)+\dfrac{1}{3}$

Answer

$x\approx -0.472$

Extensions

54) Let b be any positive real number such that $b\neq 1$. What must $\log_b 1$be equal to? Verify the result.

55) Explore and discuss the graphs of $f(x)=\log_{\frac{1}{2}}(x)$ and $g(x)=-\log_2(x)$. Make a conjecture based on the result.

Answer

The graphs of $f(x)=\log_{\frac{1}{2}}(x)$ and $g(x)=-\log_2(x)$ appear to be the same;

Conjecture: for any positive base $b\neq 1$, $\log_b(x)=\log_{\frac{1}{b}}(x)$

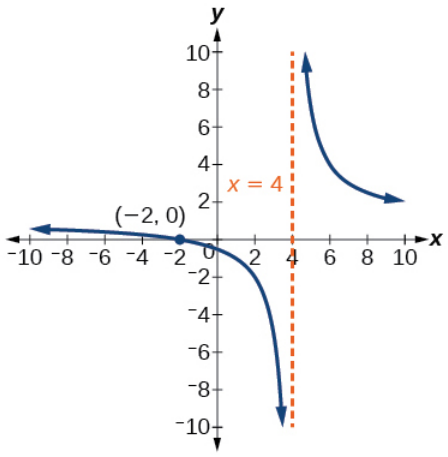
56) Prove the conjecture made in the previous exercise.

57) What is the domain of the function $f(x)=\ln\left(\frac{x+2}{x-4}\right)$ Discuss the result.

Answer

Recall that the argument of a logarithmic function must be positive, so we determine where $\frac{x+2}{x-4}>0$. From the graph of the function $f(x)=\frac{x+2}{x-4}$ note that the graph lies above the x-axis on the interval $(-\infty,-2)$ and again to the right of the vertical asymptote, that is $(4,\infty)$ Therefore, the domain is $(-\infty,-2)\cup(4,\infty)$

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58) Use properties of exponents to find the x-intercepts of the function $f(x) = \log\left(x^2 + 4x + 4\right)$ algebraically. Show the steps for solving, and then verify the result by graphing the function.

Logarithmic Properties

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here.

Verbal

1) How does the power rule for logarithms help when solving logarithms with the form $\log_b(\sqrt[n]{x})$

Answer

Any root expression can be rewritten as an expression with a rational exponent so that the power rule can be applied, making the logarithm easier to calculate. Thus, $\log_b\left(x^{\frac{1}{n}}\right) = \frac{1}{n}\log_b(x)$.

2) What does the change-of-base formula do? Why is it useful when using a calculator?

Algebraic

For the following exercises, expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

3) $\log_b(7x \cdot 2y)$

Answer

$\log_b(2) + \log_b(7) + \log_b(x) + \log_b(y)$

4) $\ln(3ab \cdot 5c)$

5) $\log_b\left(\frac{13}{17}\right)$

Answer

$\log_b(13) - \log_b(17)$

6) $\log_4\left(\frac{\frac{x}{z}}{w}\right)$

7) $\ln\left(\frac{1}{4^k}\right)$

Answer

$-k\ln(4)$

8) $\log_2(y^x)$

For the following exercises, condense to a single logarithm if possible.

9) $\ln(7)+\ln(x)+\ln(y)$

Answer

$\ln(7xy)$

10) $\log_3(2)+\log_3(a)+\log_3(11)+\log_3(b)$

11) $\log_b(28)-\log_b(7)$

Answer

$\log_b(4)$

12) $\ln(a)-\ln(d)-\ln(c)$

13) $-\log_b\left(\frac{1}{7}\right)$

Answer

$\log_b(7)$

14) $\frac{1}{3}\ln(8)$

For the following exercises, use the properties of logarithms to expand each logarithm as much as possible. Rewrite each expression as a sum, difference, or product of logs.

15) $\log\left(\frac{x^{15}y^{13}}{z^{19}}\right)$

Answer

$15\log(x)+13\log(y)-19\log(z)$

16) $\ln\left(\frac{a^{-2}}{b^{-4}c^5}\right)$

17) $\log\left(\sqrt{x^3y^{-4}}\right)$

Answer

$\frac{3}{2}\log(x)-2\log(y)$

18) $\ln\left(y\sqrt{\frac{y}{1-y}}\right)$

19) $\log\left(x^2y^3\sqrt[3]{x^2y^5}\right)$

Answer

$\frac{8}{3}\log(x)+\frac{14}{3}\log(y)$

For the following exercises, condense each expression to a single logarithm using the properties of logarithms.

20) $\log\left(2x^4\right)+\log\left(3x^5\right)$

21) $\ln\left(6x^9\right)-\ln\left(3x^2\right)$

Answer

$\ln(2x^7)$

22) $2\log(x)+3\log(x+1)$

23) $\log(x)-\frac{1}{2}\log(y)+3\log(z)$

Answer

$\log\left(\frac{xz^3}{\sqrt{y}}\right)$

24) $4\log_7(c)+\frac{\log_7(a)}{3}+\frac{\log_7(b)}{3}$

For the following exercises, rewrite each expression as an equivalent ratio of logs using the indicated base.

25) $\log_7(15)$ to base e

Answer

$\log_7(15)=\frac{\ln(15)}{\ln(7)}$

For the following exercises, suppose $\log_5(6)=a$ and $\log_5(11)=b$ Use the change-of-base formula along with properties of logarithms to rewrite each expression in terms of a and b Show the steps for solving.

27) $\log_{11}(5)$

Answer

$\log_{11}(5)=\frac{\log_5(5)}{\log_5(11)}=\frac{1}{b}$

28) $\log_6(55)$

29) $\log_{11}\left(\frac{6}{11}\right)$

Answer

$\log_{11}\left(\frac{6}{11}\right)=\frac{\log_{11}\left(\frac{6}{11}\right)}{\log_5(11)}=\frac{\log_5(6)-\log_5(11)}{\log_5(11)}=\frac{a-b}{b}=\frac{a}{b}-1$

Numeric

For the following exercises, use properties of logarithms to evaluate without using a calculator.

30) $\log_3\left(\frac{1}{9}\right)-3\log_3(3)$

31) $6\log_8(2)+\frac{\log_8(64)}{3\log_8(4)}$

Answer

3

32) $2\log_9(3)-4\log_9(3)+\log_9\left(\frac{1}{729}\right)$

For the following exercises, use the change-of-base formula to evaluate each expression as a quotient of natural logs. Use a calculator to approximate each to five decimal places.

33) $\log_3(22)$

Answer

2.81359

34) $\log_8(65)$

35) $\log_6(5.38)$

Answer

0.93913

36) $\log_4\left(\frac{15}{2}\right)$

37) $\log_{\frac{1}{2}}(4.7)$

Answer

-2.23266

Extensions

38) Use the product rule for logarithms to find all x values such that $\log_{12}(2x+6)+\log_{12}(x+2)=2$ Show the steps for solving.

39) Use the quotient rule for logarithms to find all x values such that $\log_6(x+2)-\log_6(x-3)=1$ Show the steps for solving.

Answer

Rewriting as an exponential equation and solving for x:

$$6^1=\frac{x+2}{x-3}$$
$$0=\frac{x+2}{x-3}-6$$
$$0=\frac{x+2}{x-3}-\frac{6(x-3)}{(x-3)}$$
$$0=\frac{x+2-6x+18}{x-3}$$
$$0=\frac{x-4}{x-3}$$
$$x=4$$

Checking, we find that $\log_6(4+2)-\log_6(4-3)=\log_6(6)-\log_6(1)$ is defined, so $x=4$

40) Can the power property of logarithms be derived from the power property of exponents using the equation $b^x=m$ If not, explain why. If so, show the derivation.

41) Prove that $\log_b(n)=\frac{1}{\log_b(n)}$ for any positive integers $b>1$ and $n>1$

Answer

Let b and n be positive integers greater than 1Then, by the change-of-base formula, $\log_b(n)=\frac{\log_n(n)}{\log_n(b)}=\frac{1}{\log_n(b)}$

42) Does $\log_{81}(2401)=\log_3(7)$Verify the claim algebraically.

Exponential and Logarithmic Equations

Uncontrolled population growth can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

Verbal

1) How can an exponential equation be solved?

Answer

Determine first if the equation can be rewritten so that each side uses the same base. If so, the exponents can be set equal to each other. If the equation cannot be rewritten so that each side uses the same base, then apply the logarithm to each side and use properties of logarithms to solve.

2) When does an extraneous solution occur? How can an extraneous solution be recognized?

3) When can the one-to-one property of logarithms be used to solve an equation? When can it not be used?

Answer

The one-to-one property can be used if both sides of the equation can be rewritten as a single logarithm with the same base. If so, the arguments can be set equal to each other, and the resulting equation can be solved algebraically. The one-to-one property cannot be used when each side of the equation cannot be rewritten as a single logarithm with the same base.

Algebraic

For the following exercises, use like bases to solve the exponential equation.

4) $4^{-3v-2} = 4^{-v}$

5) $64 \cdot 4^{3x} = 16$

Answer

$x = -\frac{1}{3}$

6) $3^{2x+1} \cdot 3^{x} = 243$

7) $2^{-3n} \cdot \frac{1}{4} = 2^{n+2}$

Answer

$n = -1$

8) $625 \cdot 5^{3x+3} = 125$

9) $\frac{36^{3b}}{36^{2b}} = 216^{2-b}$

Answer

$b = \frac{6}{5}$

10) $\left(\frac{1}{64}\right)^{3n} \cdot 8 = 2^{6}$

For the following exercises, use logarithms to solve.

11) $9^{x-10} = 1$

Answer

$x = 10$

12) $2e^{6x} = 13$

13) $e^{r+10} - 10 = -42$

Answer

No solution

14) $2 \cdot 10^{9a} = 29$

15) $-8 \cdot 10^{p+7} - 7 = -24$

Answer

$p = \log\left(\frac{17}{8}\right) - 7$

16) $7e^{3n-5} + 5 = -89$

17) $e^{-3k} + 6 = 44$

Answer

$k = -\frac{\ln(38)}{3}$

18) $-5e^{9x-8} - 8 = -62$

19) $-6e^{9x+8} + 2 = -74$

Answer

$x = \frac{\frac{\ln(38)}{3} - 8}{9}$

20) $2^{x+1} = 5^{2x-1}$

21) $e^{2x} - e^{x} - 132 = 0$

Answer

$x = \ln 12$

22) $7e^{8x+8} - 5 = -95$

23) $10e^{8x+3} + 2 = 8$

Answer

$x = \frac{\frac{\ln(3)}{5} - 3}{8}$

24) $4e^{3x+3} - 7 = 53$

25) $8e^{-5x-2} - 4 = -90$

Answer

No solution

26) $3^{2x+1} = 7^{x-2}$

27) $e^{2x} - e^{x} - 6 = 0$

Answer

$x = \ln 3$

28) $3e^{3-3x} + 6 = -31$

For the following exercises, use the definition of a logarithm to rewrite the equation as an exponential equation.

29) $\log\left(\frac{1}{100}\right) = -2$

Answer

$$10^{-2} = \frac{1}{100}$$

30) $\log_{324}(18) = \frac{1}{2}$

For the following exercises, use the definition of a logarithm to solve the equation.

31) $5 \log_7 n = 10$

Answer

$$n = 49$$

32) $-8 \log_9 x = 16$

33) $4 + \log_2(9k) = 2$

Answer

$$k = \frac{1}{36}$$

34) $2 \log(8n + 4) + 6 = 10$

35) $10 - 4 \ln(9 - 8x) = 6$

Answer

$$x = \frac{9 - e}{8}$$

For the following exercises, use the one-to-one property of logarithms to solve.

36) $\ln(10 - 3x) = \ln(-4x)$

37) $\log_{13}(5n - 2) = \log_{13}(8 - 5n)$

Answer

$$n = 1$$

38) $\log(x + 3) - \log(x) = \log(74)$

39) $\ln(-3x) = \ln(x^2 - 6x)$

Answer

No solution

40) $\log_4(6 - m) = \log_4(3m)$

41) $\ln(x - 2) - \ln(x) = \ln(54)$

Answer

No solution

42) $\log_9(2n^2 - 14n) = \log_9(-45 + n^2)$

43) $\ln(x^2 - 10) + \ln(9) = \ln(10)$

Answer

$$x = \pm\frac{10}{3}$$

For the following exercises, solve each equation for x.

44) $\log(x + 12) = \log(x) + \log(12)$

45) $\ln(x)+\ln(x-3)=\ln(7x)$

Answer

$x=10$

46) $\log_2(7x+6)=3$

47) $\ln(7)+\ln(2-4x^2)=\ln(14)$

Answer

$x=0$

48) $\log_8(x+6)-\log_8(x)=\log_8(58)$

49) $\ln(3)-\ln(3-3x)=\ln(4)$

Answer

$x=\frac{3}{4}$

50) $\log_3(3x)-\log_3(6)=\log_3(77)$

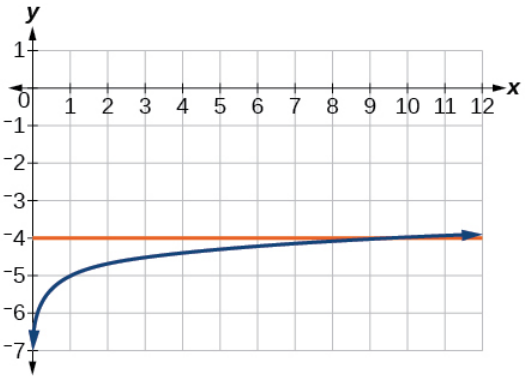
Graphical

For the following exercises, solve the equation for x if there is a solution. Then graph both sides of the equation, and observe the point of intersection (if it exists) to verify the solution.

51) $\log_9(x)-5=-4$

Answer

$x=9$



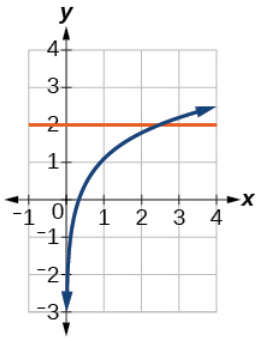
52) $\log_3(x)+3=2$

53) $\ln(3x)=2$

Answer

$x=\frac{e^2}{3}\approx 2.5$

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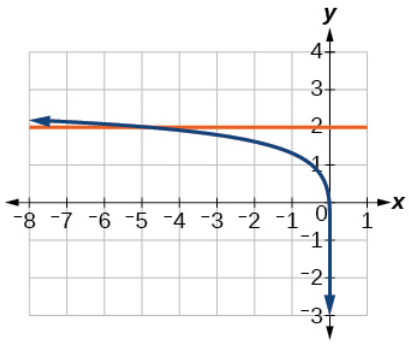


54) $\ln(x-5)=1$

55) $\log(4)+\log(-5x)=2$

Answer

$x=-5$

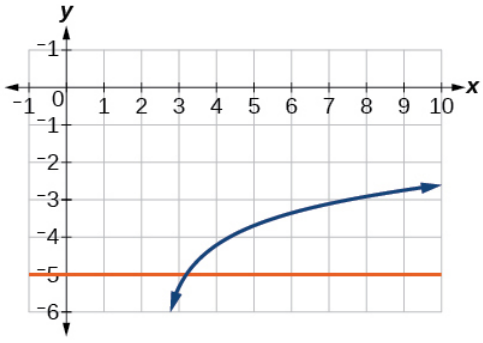


56) $-7+\log_3(4-x)=-6$

57) $\ln(4x-10)-6=-5$

Answer

$x=\dfrac{e+10}{4}\approx 3.2$



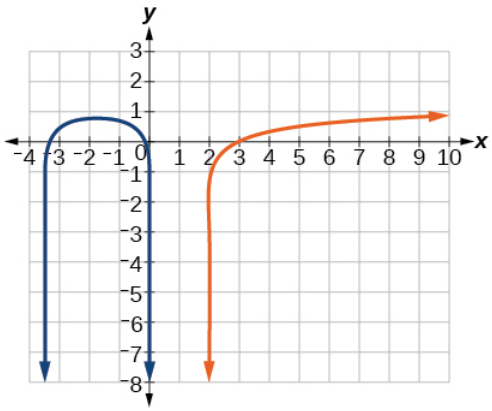
58) $\log(4-2x)=\log(-4x)$

59) $\log_{11}(-2x^2-7x)=\log_{11}(x-2)$

Answer

No solution

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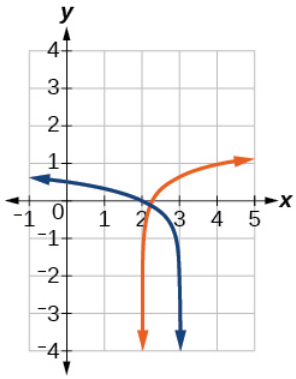


60) $\ln(2x+9) = \ln(-5x)$

61) $\log_9(3-x) = \log_9(4x-8)$

Answer

$x = \frac{11}{5} \approx 2.2$

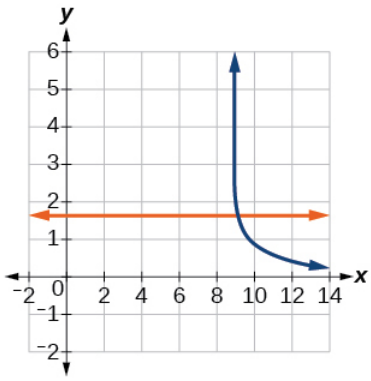


62) $\log(x^2+13) = \log(7x+3)$

63) $\frac{3}{\log_2(10)} - \log(x-9) = \log(44)$

Answer

$x = \frac{101}{11} \approx 9.2$



64) $\ln(x) - \ln(x+3) = \ln(6)$

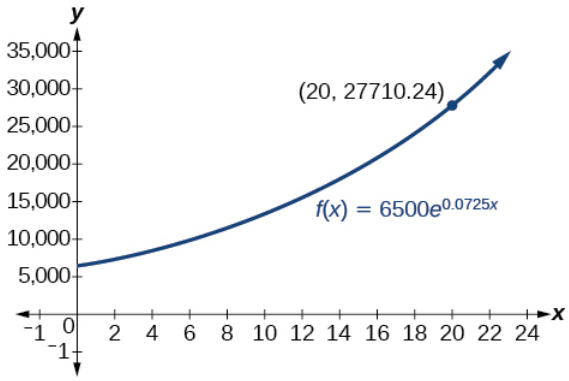
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For the following exercises, solve for the indicated value, and graph the situation showing the solution point.

65) An account with an initial deposit of $\$6,500$ earns 7.25% annual interest, compounded continuously. How much will the account be worth after 20 years?

Answer

about $\$27,710.24$

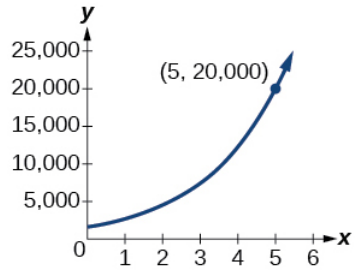


66) The formula for measuring sound intensity in decibels D is defined by the equation $D = 10 \log\left(\frac{I}{I_0}\right)$ where I is the intensity of the sound in watts per square meter and $I_0 = 10^{-12}$ is the lowest level of sound that the average person can hear. How many decibels are emitted from a jet plane with a sound intensity of $8 \cdot 3 \cdot 10^2$ watts per square meter?

67) The population of a small town is modeled by the equation $P = 1650e^{0.5t}$ where t is measured in years. In approximately how many years will the town's population reach $20,000$?

Answer

about 5 years



Technology

For the following exercises, solve each equation by rewriting the exponential expression using the indicated logarithm. Then use a calculator to approximate the variable to 3 decimal places.

68) $1000(1.03)^t = 5000$ using the common log.

69) $e^{5x} = 17$ using the natural log.

Answer

$$\frac{\ln(17)}{5} \approx 0.567$$

70) $3(1.04)^{3t} = 8$ using the common log

71) $3^{4x-5} = 38$ using the common log

Answer

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$$x = \frac{\log(38+5\log(3))}{4\log(3)} \approx 2.078$$

72) $50e^{-0.12t} = 10$ using the natural log

For the following exercises, use a calculator to solve the equation. Unless indicated otherwise, round all answers to the nearest ten-thousandth.

73) $7e^{3x-5} + 7.9 = 47$

Answer

$x \approx 2.2401$

74) $\ln(3) + \ln(4.4x + 6.8) = 2$

75) $\log(-0.7x - 9) = 1 + 5\log(5)$

Answer

$x \approx -44655.7143$

76) Atmospheric pressure P in pounds per square inch is represented by the formula $P = 14.7e^{-0.21x}$ where x is the number of miles above sea level. To the nearest foot, how high is the peak of a mountain with an atmospheric pressure of 8.369 pounds per square inch? (Hint: there are 5280 feet in a mile)

77) The magnitude M of an earthquake is represented by the equation $M = \frac{2}{3}\log\left(\frac{E}{E_0}\right)$ where E is the amount of energy released by the earthquake in joules $E_0 = 10^{4.4}$ and is the assigned minimal measure released by an earthquake. To the nearest hundredth, what would the magnitude be of an earthquake releasing $1.4 \cdot 10^{13}$ joules of energy?

Answer

about 5.83

Extensions

78) Use the definition of a logarithm along with the one-to-one property of logarithms to prove that $b^{\log_b x} = x$.

79) Recall the formula for continually compounding interest, $y = Ae^{kt}$ Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

Answer

$$t = \ln\left(\left(\frac{y}{A}\right)^{\frac{1}{k}}\right)$$

80) Recall the compound interest formula $A = a\left(1 + \frac{r}{k}\right)^{kt}$ Use the definition of a logarithm along with properties of logarithms to solve the formula for time t.

81) Newton's Law of Cooling states that the temperature T of an object at any time t can be described by the equation $T = T_s + (T_0 - T_s)e^{-kt}$ where T_s is the temperature of the surrounding environment, T_0 is the initial temperature of the object, and k is the cooling rate. Use the definition of a logarithm along with properties of logarithms to solve the formula for time t such that t is equal to a single logarithm.

Answer

$$t = \ln\left(\left(\frac{T-T_s}{T_0-T_s}\right)^{-\frac{1}{k}}\right)$$

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Exponential and Logarithmic Models

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth, including radioactive isotopes and Newton's Law of Cooling.

Verbal

1) With what kind of exponential model would *half-life* be associated? What role does half-life play in these models?

Answer

Half-life is a measure of decay and is thus associated with exponential decay models. The half-life of a substance or quantity is the amount of time it takes for half of the initial amount of that substance or quantity to decay.

2) What is carbon dating? Why does it work? Give an example in which carbon dating would be useful.

3) With what kind of exponential model would *doubling time* be associated? What role does doubling time play in these models?

Answer

Doubling time is a measure of growth and is thus associated with exponential growth models. The doubling time of a substance or quantity is the amount of time it takes for the initial amount of that substance or quantity to double in size.

4) Define Newton's Law of Cooling. Then name at least three real-world situations where Newton's Law of Cooling would be applied.

5) What is an order of magnitude? Why are orders of magnitude useful? Give an example to explain.

Answer

An order of magnitude is the nearest power of ten by which a quantity exponentially grows. It is also an approximate position on a logarithmic scale; Sample response: Orders of magnitude are useful when making comparisons between numbers that differ by a great amount. For example, the mass of Saturn is 95 times greater than the mass of Earth. This is the same as saying that the mass of Saturn is about 10^2 times, or 2 orders of magnitude greater, than the mass of Earth.

Numeric

6) The temperature of an object in degrees Fahrenheit after t minutes is represented by the equation $T(t) = 68e^{-0.0174t} + 72$ To the nearest degree, what is the temperature of the object after one and a half hours?

For the following exercises, use the logistic growth model $f(x) = \dfrac{150}{1+8e^{-2x}}$

7) Find and interpret $f(0)$Round to the nearest tenth.

Answer

$f(0) \approx 16.7$; The amount initially present is about 16.7units.

8) Find and interpret $f(4)$Round to the nearest tenth.

9) Find the carrying capacity.

Answer

150

10) Graph the model.

11) Determine whether the data from the table could best be represented as a function that is linear, exponential, or logarithmic. Then write a formula for a model that represents the data.

x	$f(x)$
–2	0.694

x	$f(x)$
–1	0.833
0	1
1	1.2
2	1.44
3	1.728
4	2.074
5	2.488

Answer

exponential; $f(x) = 1.2^x$

12) Rewrite $f(x) = 1.68(0.65)^x$ as an exponential equation with base e to five significant digits.

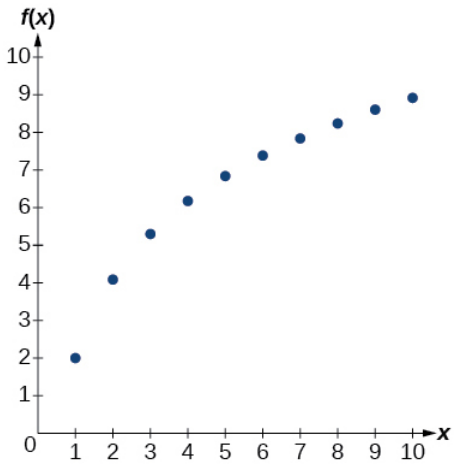
Technology

For the following exercises, enter the data from each table into a graphing calculator and graph the resulting scatter plots. Determine whether the data from the table could represent a function that is linear, exponential, or logarithmic.

13)

x	$f(x)$
1	2
2	4.079
3	5.296
4	6.159
5	6.828
6	7.375
7	7.838
8	8.238
9	8.592
10	8.908

Answer



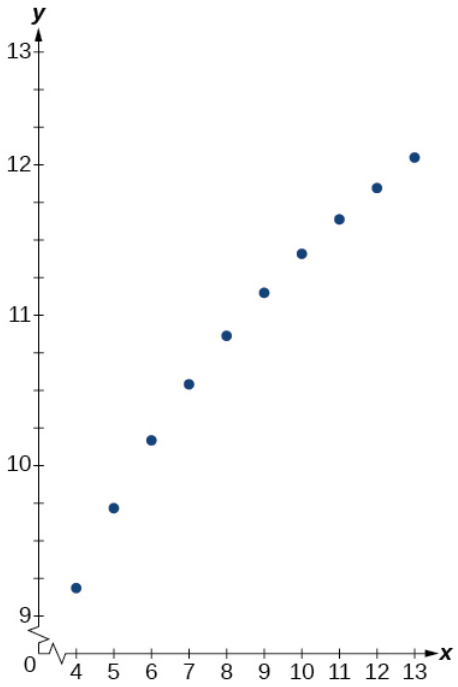
14)

x	$f(x)$
1	2.4
2	2.88
3	3.456
4	4.147
5	4.977
6	5.972
7	7.166
8	8.6
9	10.32
10	12.383

15)

x	$f(x)$
4	9.429
5	9.972
6	10.415
7	10.79
8	11.115
9	11.401
10	11.657
11	11.889
12	12.101
13	12.295

Answer



16)

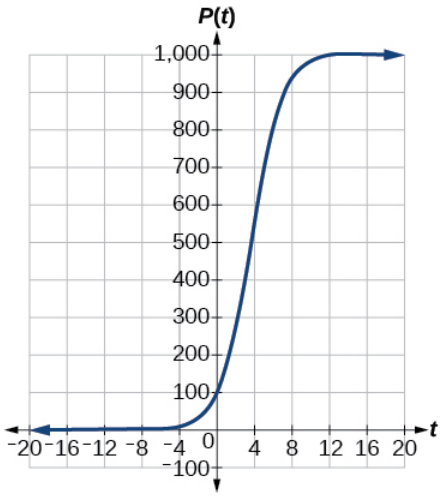
x	$f(x)$
1.25	5.75
2.25	8.75
3.56	12.68
4.2	14.6
5.65	18.95
6.75	22.25
7.25	23.75
8.6	27.8
9.25	29.75
10.5	33.5

For the following exercises, use a graphing calculator and this scenario: the population of a fish farm in t years is modeled by the equation $P(t) = \dfrac{1000}{1+9e^{-0.6t}}$

17) Graph the function.

Answer

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18) What is the initial population of fish?

19) To the nearest tenth, what is the doubling time for the fish population?

Answer

about 1.4 years

20) To the nearest whole number, what will the fish population be after 2 years?

21) To the nearest tenth, how long will it take for the population to reach 900?

Answer

about 7.3 years

22) What is the carrying capacity for the fish population? Justify your answer using the graph of P.

Extensions

23) A substance has a half-life of 2.045 minutes. If the initial amount of the substance was 132.8 grams, how many half-lives will have passed before the substance decays to 8.3 grams? What is the total time of decay?

Answer

4 half-lives; 8.18 minutes

24) The formula for an increasing population is given by $P(t) = P_0 e^{rt}$ where P_0 is the initial population and $r > 0$ Derive a general formula for the time t it takes for the population to increase by a factor of M.

25) Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$ Show each step for solving this equation algebraically for the seismic moment S.

Answer

$$M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$$
$$\log\left(\frac{S}{S_0}\right) = \frac{3}{2}M$$
$$\frac{S}{S_0} = 10^{\frac{3M}{2}}$$
$$S = S_0 10^{\frac{3M}{2}}$$

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26) What is the y-intercept of the logistic growth model $y = \dfrac{c}{1+ae^{-rx}}$ Show the steps for calculation. What does this point tell us about the population?

27) Prove that $b^x = e^{x\ln(b)}$ for positive $b \neq 1$.

Answer

Let $y = b^x$ for some non-negative real number b such that $b \neq 1$ Then,

$$\ln(y) = \ln(b^x)$$
$$\ln(y) = x\ln(b)$$
$$e^{\ln(y)} = e^{x\ln(b)}$$
$$y = e^{x\ln(b)}$$

Real-World Applications

For the following exercises, use this scenario: A doctor prescribes 125 milligrams of a therapeutic drug that decays by about 30% each hour.

28) To the nearest hour, what is the half-life of the drug?

29) Write an exponential model representing the amount of the drug remaining in the patient's system after t hours. Then use the formula to find the amount of the drug that would remain in the patient's system after 3 hours. Round to the nearest milligram.

Answer

$A = 125e^{(-0.3567t)}$; $A \approx 43$ mg

30) Using the model found in the previous exercise, find $f(10$ and interpret the result. Round to the nearest hundredth.

For the following exercises, use this scenario: A tumor is injected with 0.5 grams of Iodine-125, which has a decay rate of 1.15% per day.

31) To the nearest day, how long will it take for half of the Iodine-125 to decay?

Answer

about 60 days

32) Write an exponential model representing the amount of Iodine-125 remaining in the tumor after t days. Then use the formula to find the amount of Iodine-125 that would remain in the tumor after 60 days. Round to the nearest tenth of a gram.

33) A scientist begins with 250 grams of a radioactive substance. After 250 minutes, the sample has decayed to 32 grams. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest minute, what is the half-life of this substance?

Answer

$f(t) = 250e^{(-0.00914t)}$half-life: about 76 minutes

34) The half-life of Radium-226 is 1590 years. What is the annual decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

35) The half-life of Erbium-165 is 10.4 hours. What is the hourly decay rate? Express the decimal result to four significant digits and the percentage to two significant digits.

Answer

$r \approx -0.0667$ So the hourly decay rate is about 6.67%

36) A wooden artifact from an archeological dig contains 60 percent of the carbon-14 that is present in living trees. To the nearest year, about how many years old is the artifact? (The half-life of carbon-14 is 5730 years.)

37) A research student is working with a culture of bacteria that doubles in size every twenty minutes. The initial population count was 1350 bacteria. Rounding to five significant digits, write an exponential equation representing this situation. To the nearest whole number, what is the population size after 3 hours?

Answer

$f(t) = 1350e^{(0.03466t)}$; after 3 hours: $P(180) \approx 691,200$

For the following exercises, use this scenario: A biologist recorded a count of 360 bacteria present in a culture after 5 minutes and 1000 bacteria present after 20 minutes.

38) To the nearest whole number, what was the initial population in the culture?

39) Rounding to six significant digits, write an exponential equation representing this situation. To the nearest minute, how long did it take the population to double?

Answer

$f(t) = 256e^{(0.068110t)}$; doubling time: about 10 minutes

For the following exercises, use this scenario: A pot of boiling soup with an internal temperature of 100° Fahrenheit was taken off the stove to cool in a 69° F room. After fifteen minutes, the internal temperature of the soup was 95° F.

40) Use Newton's Law of Cooling to write a formula that models this situation.

41) To the nearest minute, how long will it take the soup to cool to 80° F?

Answer

about 88 minutes

42) To the nearest degree, what will the temperature be after 2 and a half hours?

For the following exercises, use this scenario: A turkey is taken out of the oven with an internal temperature of 165° F and is allowed to cool in a 75° F room. After half an hour, the internal temperature of the turkey is 145° F.

43) Write a formula that models this situation.

Answer

$T(t) = 90e^{(-0.008377t)} + 75$, where t is in minutes.

44) To the nearest degree, what will the temperature be after 50 minutes?

45) To the nearest minute, how long will it take the turkey to cool to 110° F?

Answer

about 113 minutes

For the following exercises, find the value of the number shown on each logarithmic scale. Round all answers to the nearest thousandth.



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Answer

$\log(x) = 1.5; x \approx 31.623$

48) Plot each set of approximate values of intensity of sounds on a logarithmic scale: Whisper: $10^{-10}\frac{W}{m^2}$ Vacuum: $10^{-4}\frac{W}{m^2}$ Jet: $10^{2}\frac{W}{m^2}$

49) Recall the formula for calculating the magnitude of an earthquake, $M = \frac{2}{3}\log\left(\frac{S}{S_0}\right)$ One earthquake has magnitude 3.9 on the MMS scale. If a second earthquake has 750 times as much energy as the first, find the magnitude of the second quake. Round to the nearest hundredth.

Answer

MMS magnitude: 5.82

For the following exercises, use this scenario: The equation $N(t) = \frac{500}{1+49e^{-0.7t}}$ models the number of people in a town who have heard a rumor after t days.

50) How many people started the rumor?

51) To the nearest whole number, how many people will have heard the rumor after 3 days?

Answer

$N(3) \approx 71$

52) As t increases without bound, what value does $N(t)$ approach? Interpret your answer.

For the following exercise, choose the correct answer choice.

53) A doctor and injects a patient with 13 milligrams of radioactive dye that decays exponentially. After 12 minutes, there are 4.75 milligrams of dye remaining in the patient's system. Which is an appropriate model for this situation?

a. $f(t) = 13(0.0805)^t$
b. $f(t) = 13e^{0.9195t}$
c. $f(t) = 13e^{(-0.0839t)}$
d. $f(t) = \frac{4.75}{1+13e^{-0.83925t}}$

Answer

c

Fitting Exponential Models to Data

We will concentrate on three types of regression models in this section: exponential, logarithmic, and logistic. Having already worked with each of these functions gives us an advantage. Knowing their formal definitions, the behavior of their graphs, and some of their real-world applications gives us the opportunity to deepen our understanding. As each regression model is presented, key features and definitions of its associated function are included for review.

Verbal

1) What situations are best modeled by a logistic equation? Give an example, and state a case for why the example is a good fit.

Answer

Logistic models are best used for situations that have limited values. For example, populations cannot grow indefinitely since resources such as food, water, and space are limited, so a logistic model best describes populations.

2) What is a carrying capacity? What kind of model has a carrying capacity built into its formula? Why does this make sense?

3) What is regression analysis? Describe the process of performing regression analysis on a graphing utility.

Answer

Regression analysis is the process of finding an equation that best fits a given set of data points. To perform a regression analysis on a graphing utility, first list the given points using the STAT then EDIT menu. Next graph the scatter plot using the STAT PLOT feature. The shape of the data points on the scatter graph can help determine which regression feature to use. Once this is determined, select the appropriate regression analysis command from the STAT then CALC menu.

4) What might a scatterplot of data points look like if it were best described by a logarithmic model?

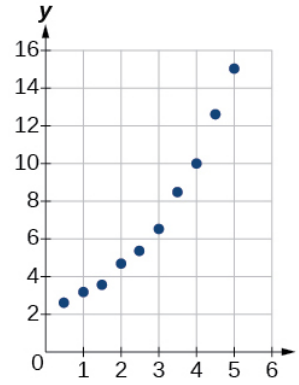
5) What does the y-intercept on the graph of a logistic equation correspond to for a population modeled by that equation?

Answer

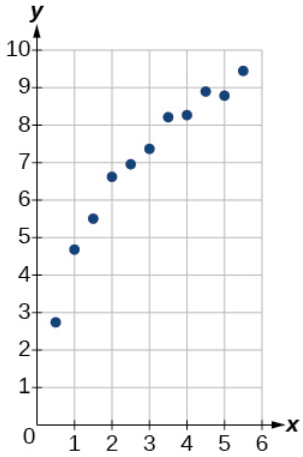
The y-intercept on the graph of a logistic equation corresponds to the initial population for the population model.

Graphical

For the following exercises, match the given function of best fit with the appropriate scatterplot in Figure (a) through Figure (e). Answer using the letter beneath the matching graph.



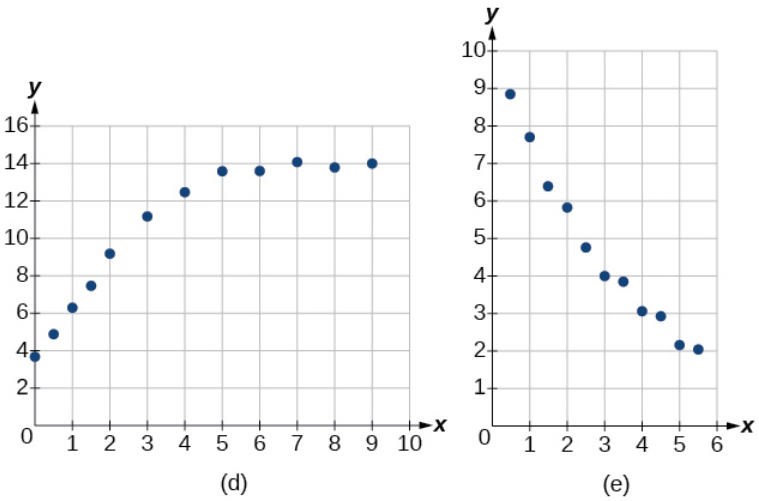
(a)



(b)

Figure_04_08_203.jpg" src="/@api/deki/files/13803/CNX_PreCalc_Figure_04_08_203.jpg" />

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6) $y = 10.209e^{-0.294x}$

7) $y = 5.598 - 1.912\ln(x)$

Answer

c

8) $y = 2.104(1.479)^x$

9) $y = 4.607 + 2.733\ln(x)$

Answer

b

10) $y = \dfrac{14.005}{1 + 2.79e^{-0.812x}}$

Numeric

11) To the nearest whole number, what is the initial value of a population modeled by the logistic equation $P(t) = \dfrac{175}{1 + 6.995e^{-0.68t}}$ What is the carrying capacity?

Answer

$P(0) = 22;\ 175$

12) Rewrite the exponential model $A(t) = 1550(1.085)^x$ as an equivalent model with base eExpress the exponent to four significant digits.

13) A logarithmic model is given by the equation $h(p) = 67.682 - 5.792\ln(p)$To the nearest hundredth, for what value of p does $h(p) = 62$?

Answer

$p \approx 2.67$

14) A logistic model is given by the equation $P(t) = \dfrac{90}{1 + 5e^{-0.42t}}$ To the nearest hundredth, for what value of t does $P(t) = 45$

15) What is the y-intercept on the graph of the logistic model given in the previous exercise?

Answer

y-intercept: $(0, 15)$

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Technology

For the following exercises, use this scenario: The population P of a koi pond over x months is modeled by the function $P(x) = \frac{68}{1+16e^{-0.28x}}$.

16) Graph the population model to show the population over a span of 3 years.

17) What was the initial population of koi?

Answer

\(4)\ koi

18) How many koi will the pond have after one and a half years?

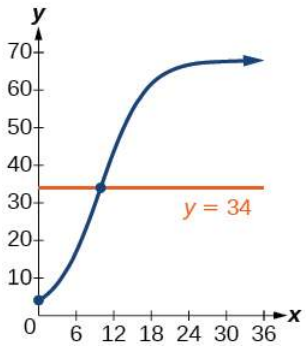
19) How many months will it take before there are 20 koi in the pond?

Answer

about 6.8 months

20) Use the intersect feature to approximate the number of months it will take before the population of the pond reaches half its carrying capacity.

Answer



For the following exercises, use this scenario: The population P of an endangered species habitat for wolves is modeled by the function $P(x) = \frac{558}{1+54.8e^{-0.462x}}$ where x is given in years.

21) Graph the population model to show the population over a span of 10 years.

22) What was the initial population of wolves transported to the habitat?

Answer

\(10)\ wolves

23) How many wolves will the habitat have after 3 years?

24) How many years will it take before there are 100 wolves in the habitat?

Answer

about 5.4 years

25) Use the intersect feature to approximate the number of years it will take before the population of the habitat reaches half its carrying capacity.

For the following exercises, refer to Table below.

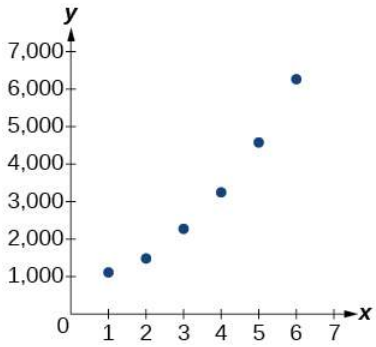
x	$f(x)$

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x	$f(x)$
1	1125
2	1495
3	2310
4	3294
5	4650
6	6361

26) Use a graphing calculator to create a scatter diagram of the data.

Answer



27) Use the regression feature to find an exponential function that best fits the data in the table.

28) Write the exponential function as an exponential equation with base e

Answer

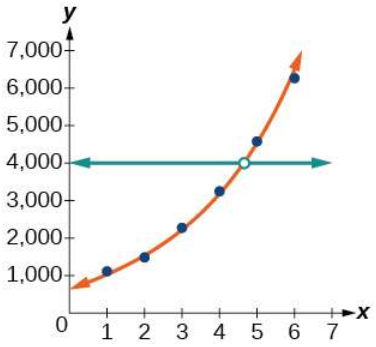
$f(x) = 776.682e^{0.3549x}$

29) Graph the exponential equation on the scatter diagram.

30) Use the intersect feature to find the value of x for which $f(x) = 4000$.

Answer

When $f(x) = 4000$, $x \approx 4.6$



For the following exercises, refer to Table below.

x	$f(x)$
1	555

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x	$f(x)$
2	383
3	307
4	210
5	158
6	122

31) Use a graphing calculator to create a scatter diagram of the data.

32) Use the regression feature to find an exponential function that best fits the data in the table.

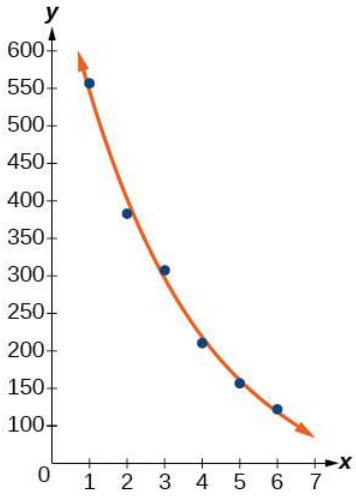
Answer

$f(x) = 731.92(0.738)^x$

33) Write the exponential function as an exponential equation with base e

34) Graph the exponential equation on the scatter diagram.

Answer



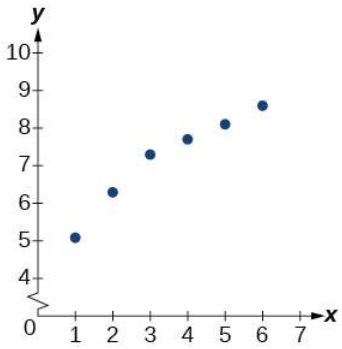
35) Use the intersect feature to find the value of x for which $f(x) = 250$

For the following exercises, refer to Table below.

x	$f(x)$
1	5.1
2	6.3
3	7.3
4	7.7
5	8.1
6	8.6

36) Use a graphing calculator to create a scatter diagram of the data.

Answer



37) Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

38) Use the logarithmic function to find the value of the function when $x = 10$

Answer

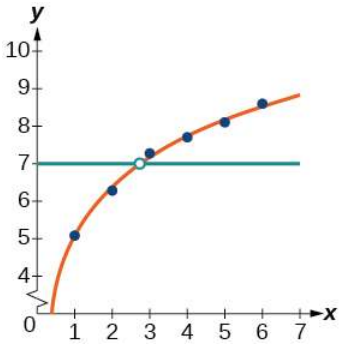
$f(10) \approx 9.5$

39) Graph the logarithmic equation on the scatter diagram.

40) Use the intersect feature to find the value of x for which $f(x) = 7$

Answer

When $f(x) = 7$, $x \approx 2.7$



For the following exercises, refer to Table below.

x	$f(x)$
1	7.5
2	6
3	5.2
4	4.3
5	3.9
6	3.4
7	3.1
8	2.9

41) Use a graphing calculator to create a scatter diagram of the data.

42) Use the LOGarithm option of the REGression feature to find a logarithmic function of the form $y = a + b\ln(x)$ that best fits the data in the table.

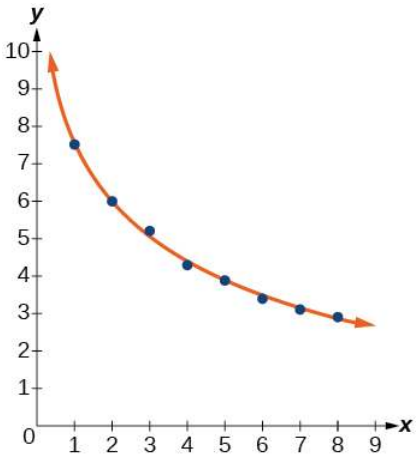
Answer

$f(x) = 7.544 - 2.268\ln(x)$

43) Use the logarithmic function to find the value of the function when $x = 10$

44) Graph the logarithmic equation on the scatter diagram.

Answer



45) Use the intersect feature to find the value of x for which $f(x) = 8$

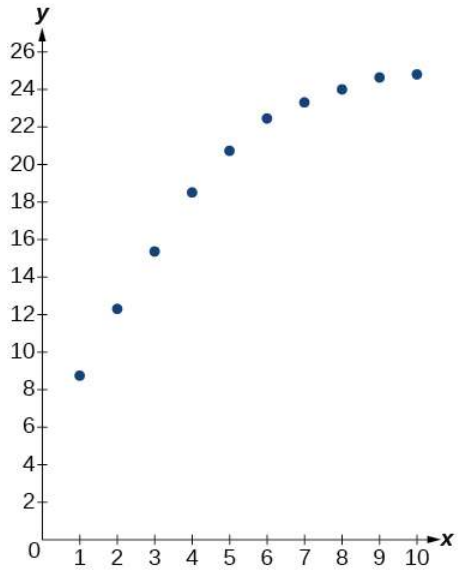
For the following exercises, refer to Table below.

x	$f(x)$
1	8.7
2	12.3
3	15.4
4	18.5
5	20.7
6	22.5
7	23.3
8	24
9	24.6
10	24.8

46) Use a graphing calculator to create a scatter diagram of the data.

Answer

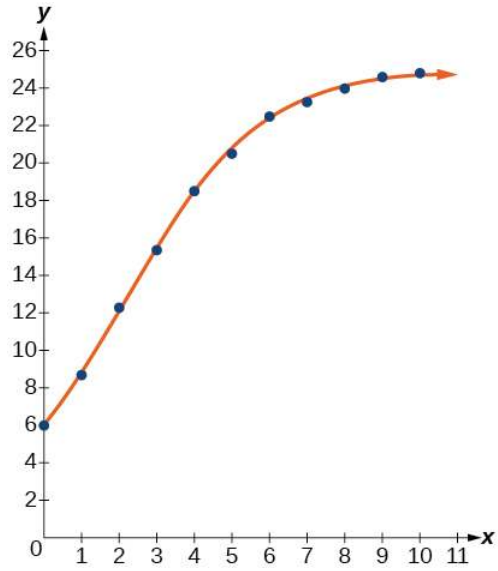
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47) Use the LOGISTIC regression option to find a logistic growth model of the form $y = \dfrac{c}{1 + ae^{-bx}}$ that best fits the data in the table.

48) Graph the logistic equation on the scatter diagram.

Answer



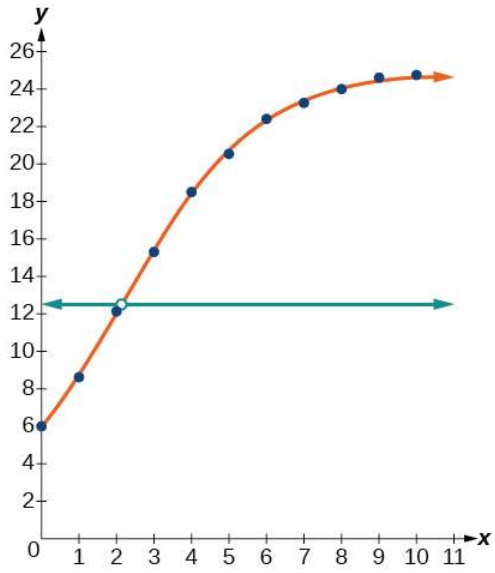
49) To the nearest whole number, what is the predicted carrying capacity of the model?

50) Use the intersect feature to find the value of x for which the model reaches half its carrying capacity.

Answer

When $f(x) = 12.5$, $x \approx 2.1$

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For the following exercises, refer to Table below.

x	$f(x)$
0	12
2	28.6
4	52.8
5	70.3
7	99.9
8	112.5
10	125.8
11	127.9
15	135.1
17	135.9

51) Use a graphing calculator to create a scatter diagram of the data.

52) Use the LOGISTIC regression option to find a logistic growth model of the form $y = \dfrac{c}{1 + ae^{-bx}}$ that best fits the data in the table.

Answer

$$y = \frac{136.068}{1 + 10.324e^{-0.480x}}$$

53) Graph the logistic equation on the scatter diagram.

54) To the nearest whole number, what is the predicted carrying capacity of the model?

Answer

about 136

55) Use the intersect feature to find the value of x for which the model reaches half its carrying capacity.

Extensions

56) Recall that the general form of a logistic equation for a population is given by $P(t)=\dfrac{c}{1+ae^{-bt}}$ such that the initial population at time $t=0$ is $P(0)=P_0$ Show algebraically that $\dfrac{c-P(t)}{P(t)}=\dfrac{c-P_0}{P_0}e^{-bt}$

Answer

Working with the left side of the equation, we see that it can be rewritten as ae^{-bt}

$$\begin{aligned}\frac{c-P(t)}{P(t)} &= \frac{c-\frac{c}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}}\\ &= \frac{\frac{c(1+ae^{-bt})-c}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}}\\ &= \frac{\frac{c(1+ae^{-bt}-1)}{1+ae^{-bt}}}{\frac{c}{1+ae^{-bt}}}\\ &= 1+ae^{-bt}-1\\ &= ae^{-bt}\end{aligned}$$

$$\begin{aligned}P_0 &= \frac{c}{1+ae^{-b(0)}}\\ &= \frac{c}{1+a}\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{c-P_0}{P_0}e^{-bt} &= \frac{c-\frac{c}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= \frac{\frac{c(1+a)-c}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= \frac{\frac{c(1+a-1)}{1+a}}{\frac{c}{1+a}}e^{-bt}\\ &= (1+a-1)e^{-bt}\\ &= ae^{-bt}\end{aligned}$$

Thus,

$$\frac{c-P(t)}{P(t)}=\frac{c-P_0}{P_0}e^{-bt}$$

57) Use a graphing utility to find an exponential regression formula $f(x)$ and a logarithmic regression formula $g(x)$ for the points $(1.5,1.5)$ and $(8.5,8.5$Round all numbers to 6 decimal places. Graph the points and both formulas along with the line $y=x$ on the same axis. Make a conjecture about the relationship of the regression formulas.

58) Verify the conjecture made in the previous exercise. Round all numbers to six decimal places when necessary.

Answer

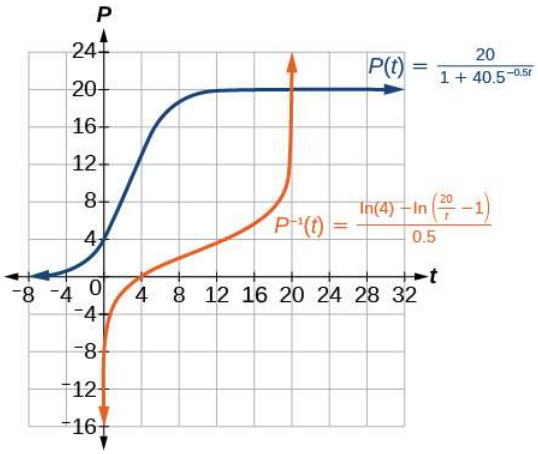
First rewrite the exponential with base e: $f(x)=1.034341e^{0.247800x}$Then test to verify that $f(g(x))=x$ taking rounding error into consideration:

$$
\begin{aligned}
g(f(x)) &= 4.035510\ln\left(1.034341e^{0.247800x}\right) - 0.136259 \\
&= 4.03551\left(\ln(1.034341) + \ln\left(e^{0.2478x}\right)\right) - 0.136259 \\
&= 4.03551(\ln(1.034341) + 0.2478x) - 0.136259 \\
&= 0.136257 + 0.999999x - 0.136259 \\
&= -0.000002 + 0.999999x \\
&\approx 0 + x \\
&= x
\end{aligned}
$$

59) Find the inverse function $f^{-1}(x)$ for the logistic function $f(x) = \dfrac{c}{1+ae^{-bx}}$ Show all steps.

60) Use the result from the previous exercise to graph the logistic model $P(t) = \dfrac{20}{1+4e^{-0.5t}}$ along with its inverse on the same axis. What are the intercepts and asymptotes of each function?

Answer



The graph of $P(t)$ has a y-intercept at $(0, 4)$ and horizontal asymptotes at $y = 0$ and $y = 20$. The graph of $P^{-1}(t)$ has an x-intercept at $(4, 0)$ and vertical asymptotes at $x = 0$ and $x = 20$.

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4.R: Exponential and Logarithmic Functions (Review)

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