DIFFERENTIAL EQUATIONS AND LINEAR ALGEBRA



Steven Zook
Reedley College



Reedley College

Differential Equations and Linear Algebra

Steven Zook

This text is disseminated via the Open Education Resource (OER) LibreTexts Project (https://LibreTexts.org) and like the thousands of other texts available within this powerful platform, it is freely available for reading, printing, and "consuming."

The LibreTexts mission is to bring together students, faculty, and scholars in a collaborative effort to provide an accessible, and comprehensive platform that empowers our community to develop, curate, adapt, and adopt openly licensed resources and technologies; through these efforts we can reduce the financial burden born from traditional educational resource costs, ensuring education is more accessible for students and communities worldwide.

Most, but not all, pages in the library have licenses that may allow individuals to make changes, save, and print this book. Carefully consult the applicable license(s) before pursuing such effects. Instructors can adopt existing LibreTexts texts or Remix them to quickly build course-specific resources to meet the needs of their students. Unlike traditional textbooks, LibreTexts' web based origins allow powerful integration of advanced features and new technologies to support learning.



LibreTexts is the adaptable, user-friendly non-profit open education resource platform that educators trust for creating, customizing, and sharing accessible, interactive textbooks, adaptive homework, and ancillary materials. We collaborate with individuals and organizations to champion open education initiatives, support institutional publishing programs, drive curriculum development projects, and more.

The LibreTexts libraries are Powered by NICE CXone Expert and was supported by the Department of Education Open Textbook Pilot Project, the California Education Learning Lab, the UC Davis Office of the Provost, the UC Davis Library, the California State University Affordable Learning Solutions Program, and Merlot. This material is based upon work supported by the National Science Foundation under Grant No. 1246120, 1525057, and 1413739.

Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation nor the US Department of Education.

Have questions or comments? For information about adoptions or adaptions contact info@LibreTexts.org or visit our main website at https://LibreTexts.org.

This text was compiled on 10/13/2025

TABLE OF CONTENTS

Licensing

1: Systems of Equations

- 1.1: Systems of Equations, Geometry
- 1.2: Systems of Equations, Algebraic Procedures
- 1.3: Gaussian Elimination
- 1.4: Uniqueness of the Reduced Row-Echelon Form
- 1.5: Rank and Homogeneous Systems
- 1.6: Exercises

2: Matrices

- 2.1: Matrix Arithmetic
- 2.2: Multiplication of Matrices
- 2.3: The ijth Entry of a Product
- 2.4: Properties of Matrix Multiplication
- 2.5: The Transpose
- 2.6: The Identity and Inverses
- 2.7: Finding the Inverse of a Matrix
- 2.8: Elementary Matrices
- 2.9: More on Matrix Inverses
- 2.10: LU Factorization
- 2.11: Exercises

3: Determinants

- 3.1: Basic Techniques
- 3.2: Properties of Determinants
- 3.3: Finding Determinants using Row Operations
- 3.4: Applications of the Determinant
- 3.5: Exercises

4: R^n

- 4.1: Vectors in R^n
- 4.2: Vector Algebra
- 4.3: Geometric Meaning of Vector Addition
- 4.4: Length of a Vector
- 4.5: Geometric Meaning of Scalar Multiplication
- 4.6: Parametric Lines
- 4.7: Spanning, Linear Independence and Basis in R^n
- 4.8: Orthogonality
- 4.9: Applications
- 4.10: Exercises

5: Linear Transformations

- 5.1: Linear Transformations
- 5.2: The Matrix of a Linear Transformation I
- 5.3: Properties of Linear Transformations
- 5.4: Special Linear Transformations in R^2
- 5.5: One-to-One and Onto Transformations
- 5.6: The Kernel and Image of A Linear Map
- 5.7: The Matrix of a Linear Transformation II
- 5.8: The General Solution of a Linear System
- 5.9: Exercises

6: Spectral Theory

- 6.1: Eigenvalues and Eigenvectors of a Matrix
- 6.2: Diagonalization
- 6.3: Applications of Spectral Theory
- 6.4: Orthogonality
- 6.5: Exercises

7: What are Differential Equations?

- 7.1: Applications Leading to Differential Equations
- 7.2: Basic Concepts
  - 7.2.1: Basic Concepts (Exercises)
- 7.3: Direction Fields for First Order Equations
  - 7.3.1: Direction Fields for First Order Equations (Exercises)

8: First Order Equations

- 8.1: Linear First Order Equations
  - 8.1.1: Linear First Order Equations (Exercises)
- 8.2: Separable Equations
  - 8.2.1: Separable Equations (Exercises)
- 8.3: Existence and Uniqueness of Solutions of Nonlinear Equations
  - 8.3.1: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises)
- 8.4: Transformation of Nonlinear Equations into Separable Equations
  - 8.4.1: Transformation of Nonlinear Equations into Separable Equations (Exercises)
- 8.5: Exact Equations
  - 8.5.1: Exact Equations (Exercises)
- 8.6: Integrating Factors
  - 8.6.1: Integrating Factors (Exercises)

9: Applications of First Order Equations

- 9.1: Growth and Decay
  - 9.1.1: Growth and Decay (Exercises)
- 9.2: Cooling and Mixing
  - 9.2.1: Cooling and Mixing (Exercises)

- 9.3: Elementary Mechanics
  - 9.3.1: Elementary Mechanics (Exercises)

10: Linear Second Order Equations

- 10.1: Homogeneous Linear Equations
  - 10.1.1: Homogeneous Linear Equations (Exercises)
- 10.2: Constant Coefficient Homogeneous Equations
  - 10.2.1: Constant Coefficient Homogeneous Equations (Exercises)
- 10.3: Nonhomogeneous Linear Equations
  - 10.3.1: Nonhomgeneous Linear Equations (Exercises)
- 10.4: The Method of Undetermined Coefficients I
  - 10.4.1: The Method of Undetermined Coefficients I (Exercises)
- 10.5: The Method of Undetermined Coefficients II
  - 10.5.1: The Method of Undetermined Coefficients II (Exercises)
- 10.6: Reduction of Order
  - 10.6.1: Reduction of Order (Exercises)

11: Applications of Linear Second Order Equations

- 11.1: Spring Problems I
  - 11.1.1: Spring Problems I (Exercises)
- 11.2: Spring Problems II
  - 11.2.1: Spring Problems II (Exercises)

12: Series Solutions of Linear Second Order Equations

- 12.1: Prelude to Series Solutions of Linear Second Order Equations
- 12.2: Review of Power Series
  - 12.2.1: Review of Power Series (Exercises)
- 12.3: Series Solutions Near an Ordinary Point I
  - 12.3.1: Series Solutions Near an Ordinary Point I (Exercises)

13: Laplace Transforms

- 13.1: Introduction to the Laplace Transform
  - 13.1.1: Introduction to the Laplace Transform (Exercises)
- 13.2: The Inverse Laplace Transform
  - 13.2.1: The Inverse Laplace Transform (Exercises)
- 13.3: Solution of Initial Value Problems
  - 13.3.1: Solution of Initial Value Problems (Exercises)
- 13.4: The Unit Step Function
  - 13.4.1: The Unit Step Function (Exercises)
- 13.5: Constant Coefficient Equations with Piecewise Continuous Forcing Functions
  - 13.5.1: Constant Coefficient Equations with Piecewise Continuous Forcing Functions (Exercises)
- 13.6: A Brief Table of Laplace Transforms

14: Linear Higher Order Differential Equations

- 14.1: Introduction to Linear Higher Order Equations
  - 14.1.1: Introduction to Linear Higher Order Equations (Exercises)
- 14.2: Higher Order Constant Coefficient Homogeneous Equations
  - 14.2.1: Higher Order Constant Coefficient Homogeneous Equations (Exercises)

15: Linear Systems of Differential Equations

- 15.1: Introduction to Systems of Differential Equations
  - 15.1.1: Introduction to Systems of Differential Equations (Exercises)
- 15.2: Linear Systems of Differential Equations
  - 15.2.1: Linear Systems of Differential Equations (Exercises)
- 15.3: Basic Theory of Homogeneous Linear Systems
  - 15.3.1: Basic Theory of Homogeneous Linear Systems (Exercises)
- 15.4: Constant Coefficient Homogeneous Systems I
  - 15.4.1: Constant Coefficient Homogeneous Systems I (Exercises)

16: Appendices

- 16.1: Trigonometric Identities
- 16.2: Review of Derivative Rules
- 16.3: Table of Derivatives
- 16.4: Table of Integrals

Detailed Licensing

Index

Glossary

Detailed Licensing

Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

1: Systems of Equations

1.1: Systems of Equations, Geometry

1.2: Systems of Equations, Algebraic Procedures

1.3: Gaussian Elimination

1.4: Uniqueness of the Reduced Row-Echelon Form

1.5: Rank and Homogeneous Systems

1.6: Exercises

This page titled 1: Systems of Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

1.1: Systems of Equations, Geometry

Systems of Equations, Geometry

Outcomes

A. Relate the types of solution sets of a system of two (three) variables to the intersections of lines in a plane (the intersection of planes in three space)

As you may remember, linear equations like $2x + 3y = 6$ can be graphed as straight lines in the coordinate plane. We say that this equation is in two variables, in this case x and y. Suppose you have two such equations, each of which can be graphed as a straight line, and consider the resulting graph of two lines. What would it mean if there exists a point of intersection between the two lines? This point, which lies on *both* graphs, gives x and y values for which both equations are true. In other words, this point gives the ordered pair (x, y) that satisfy both equations. If the point (x, y) is a point of intersection, we say that (x, y) is a **solution** to the two equations. In linear algebra, we often are concerned with finding the solution(s) to a system of equations, if such solutions exist. First, we consider graphical representations of solutions and later we will consider the algebraic methods for finding solutions.

When looking for the intersection of two lines in a graph, several situations may arise. The following picture demonstrates the possible situations when considering two equations (two lines in the graph) involving two variables.

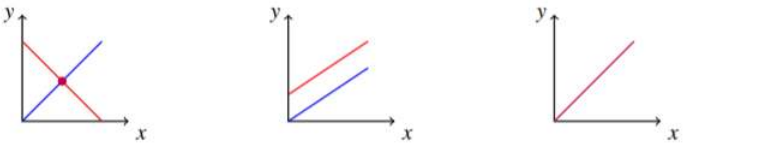


Figure 1.1.1

In the first diagram, there is a unique point of intersection, which means that there is only one (unique) solution to the two equations. In the second, there are no points of intersection and no solution. When no solution exists, this means that the two lines are parallel and they never intersect. The third situation which can occur, as demonstrated in diagram three, is that the two lines are really the same line. For example, $x + y = 1$ and $2x + 2y = 2$ are equations which when graphed yield the same line. In this case there are infinitely many points which are solutions of these two equations, as every ordered pair which is on the graph of the line satisfies both equations. When considering linear systems of equations, there are always three types of solutions possible; exactly one (unique) solution, infinitely many solutions, or no solution.

Example 1.1.1: A Graphical Solution

Use a graph to find the solution to the following system of equations

$$x + y = 3$$
$$y - x = 5$$

Solution

Through graphing the above equations and identifying the point of intersection, we can find the solution(s). Remember that we must have either one solution, infinitely many, or no solutions at all. The following graph shows the two equations, as well as the intersection. Remember, the point of intersection represents the solution of the two equations, or the (x, y) which satisfy both equations. In this case, there is one point of intersection at $(-1, 4)$ which means we have one unique solution, $x = -1, y = 4$.

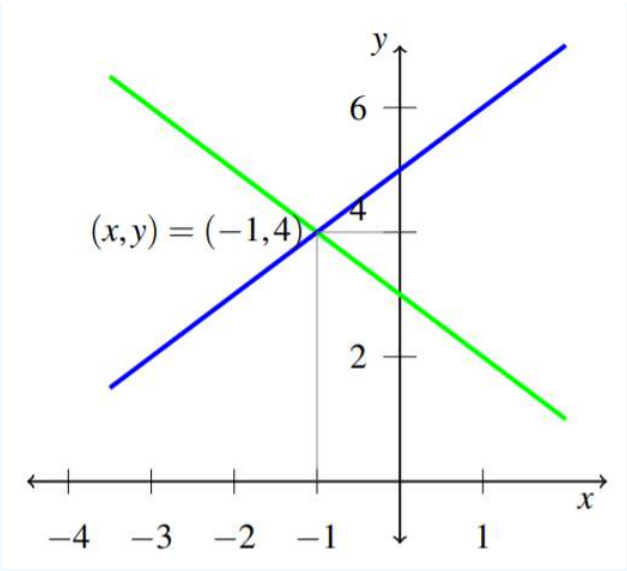


Figure 1.1.2

In the above example, we investigated the intersection point of two equations in two variables, x and y. Now we will consider the graphical solutions of three equations in two variables.

Consider a system of three equations in two variables. Again, these equations can be graphed as straight lines in the plane, so that the resulting graph contains three straight lines. Recall the three possible types of solutions; no solution, one solution, and infinitely many solutions. There are now more complex ways of achieving these situations, due to the presence of the third line. For example, you can imagine the case of three intersecting lines having no common point of intersection. Perhaps you can also imagine three intersecting lines which do intersect at a single point. These two situations are illustrated below.

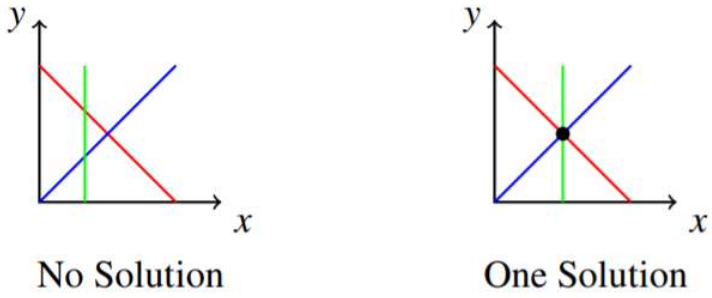


Figure 1.1.3

Consider the first picture above. While all three lines intersect with one another, there is no common point of intersection where all three lines meet at one point. Hence, there is no solution to the three equations. Remember, a solution is a point (x, y) which satisfies **all** three equations. In the case of the second picture, the lines intersect at a common point. This means that there is one solution to the three equations whose graphs are the given lines. You should take a moment now to draw the graph of a system which results in three parallel lines. Next, try the graph of three identical lines. Which type of solution is represented in each of these graphs?

We have now considered the graphical solutions of systems of two equations in two variables, as well as three equations in two variables. However, there is no reason to limit our investigation to equations in two variables. We will now consider equations in three variables.

You may recall that equations in three variables, such as $2x + 4y - 5z = 8$, form a plane. Above, we were looking for intersections of lines in order to identify any possible solutions. When graphically solving systems of equations in three variables, we look for intersections of planes. These points of intersection give the (x, y, z) that satisfy all the equations in the system. What types of solutions are possible when working with three variables? Consider the following picture involving two planes, which are given by two equations in three variables.

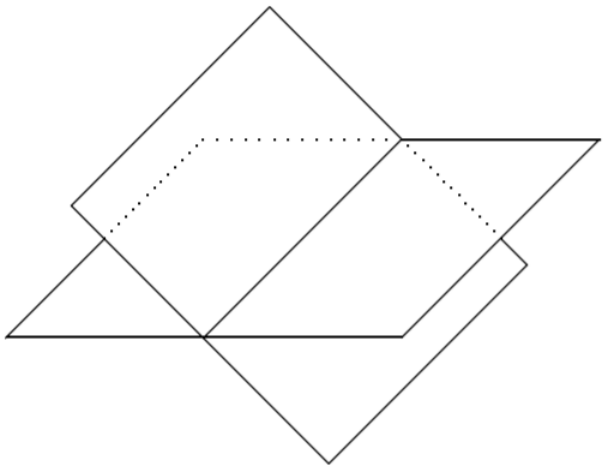


Figure **1.1.4**

Notice how these two planes intersect in a line. This means that the points (x, y, z) on this line satisfy both equations in the system. Since the line contains infinitely many points, this system has infinitely many solutions.

It could also happen that the two planes fail to intersect. However, is it possible to have two planes intersect at a single point? Take a moment to attempt drawing this situation, and convince yourself that it is not possible! This means that when we have only two equations in three variables, there is no way to have a unique solution! Hence, the types of solutions possible for two equations in three variables are no solution or infinitely many solutions.

Now imagine adding a third plane. In other words, consider three equations in three variables. What types of solutions are now possible? Consider the following diagram.

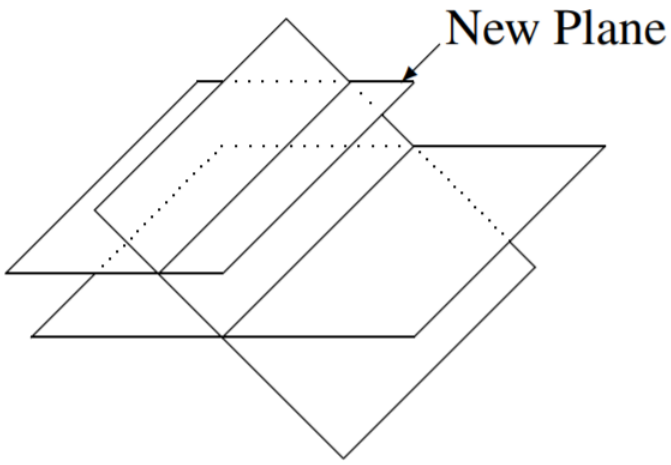


Figure **1.1.5**

In this diagram, there is no point which lies in all three planes. There is no intersection between **all** planes so there is no solution. The picture illustrates the situation in which the line of intersection of the new plane with one of the original planes forms a line parallel to the line of intersection of the first two planes. However, in three dimensions, it is possible for two lines to fail to intersect even though they are not parallel. Such lines are called **skew lines**.

Recall that when working with two equations in three variables, it was not possible to have a unique solution. Is it possible when considering three equations in three variables? In fact, it is possible, and we demonstrate this situation in the following picture.

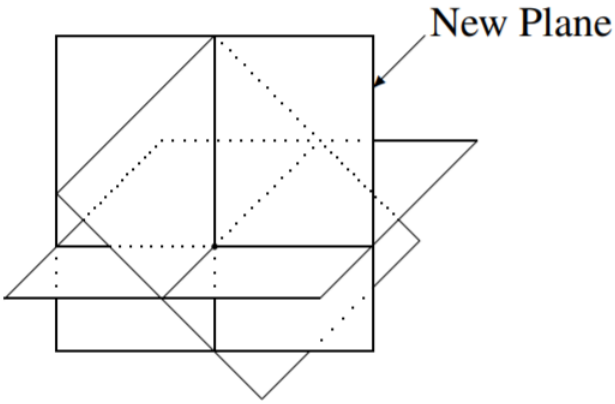


Figure 1.1.6

In this case, the three planes have a single point of intersection. Can you think of other types of solutions possible? Another is that the three planes could intersect in a line, resulting in infinitely many solutions, as in the following diagram.

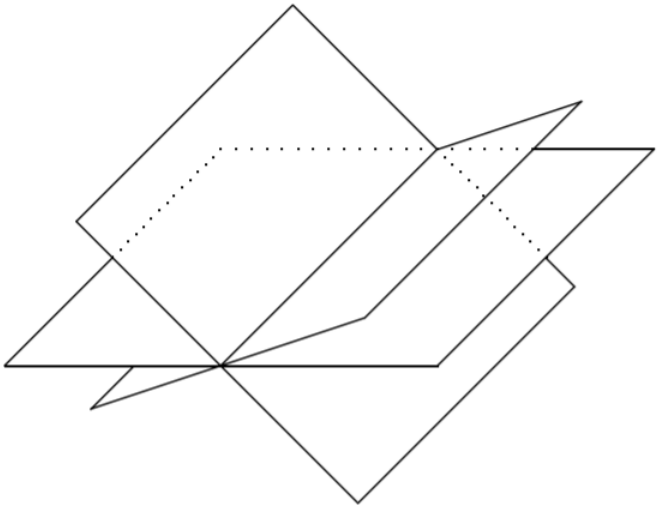


Figure 1.1.7

We have now seen how three equations in three variables can have no solution, a unique solution, or intersect in a line resulting in infinitely many solutions. It is also possible that the three equations graph the same plane, which also leads to infinitely many solutions.

You can see that when working with equations in three variables, there are many more ways to achieve the different types of solutions than when working with two variables. It may prove enlightening to spend time imagining (and drawing) many possible scenarios, and you should take some time to try a few.

You should also take some time to imagine (and draw) graphs of systems in more than three variables. Equations like $x + y - 2z + 4w = 8$ with more than three variables are often called **hyper-planes**. You may soon realize that it is tricky to draw the graphs of hyper-planes! Through the tools of linear algebra, we can algebraically examine these types of systems which are difficult to graph. In the following section, we will consider these algebraic tools.

? Exercise 1.1.1

Graphically, find the point (x_1, y_1) which lies on both lines, $x + 3y = 1$ and $4x - y = 3$. That is, graph each line and see where they intersect.

Exercise 1.1.2

Graphically, find the point of intersection of the two lines, $3x + y = 3$ and $x + 2y = 1$. That is, graph each line and see where they intersect

Exercise 1.1.3

You have a system of k equations in two variables, $k \geq 2$. Explain the geometric significance of

a. No solution.
b. A unique solution.
c. An infinite number of solutions.

This page titled 1.1: Systems of Equations, Geometry is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.1: Systems of Equations, Geometry** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

1.2: Systems of Equations, Algebraic Procedures

Algebraic Procedures

Outcomes

A. Use elementary operations to find the solution to a linear system of equations.
B. Find the row-echelon form and reduced row-echelon form of a matrix.
C. Determine whether a system of linear equations has no solution, a unique solution or an infinite number of solutions from its .
D. Solve a system of equations using Gaussian Elimination and Gauss-Jordan Elimination.
E. Model a physical system with linear equations and then solve.

We have taken an in depth look at graphical representations of systems of equations, as well as how to find possible solutions graphically. Our attention now turns to working with systems algebraically.

Definition 1.2.1: System of Linear Equations

A **system of linear equations** is a list of equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where a_{ij} and b_j are real numbers. The above is a system of m equations in the n variables, $x_1, x_2 \cdots, x_n$. Written more simply in terms of summation notation, the above can be written in the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \ i = 1, 2, 3, \cdots, m$$

The relative size of m and n is not important here. Notice that we have allowed a_{ij} and b_j to be any real number. We can also call these numbers **scalars** . We will use this term throughout the text, so keep in mind that the term **scalar** just means that we are working with real numbers.

Now, suppose we have a system where $b_i = 0$ for all i. In other words every equation equals 0. This is a special type of system.

Definition 1.2.2: Homogeneous System of Equations

A system of equations is called **homogeneous** if each equation in the system is equal to 0. A homogeneous system has the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

where a_{ij} are scalars and x_i are variables.

Recall from the previous section that our goal when working with systems of linear equations was to find the point of intersection of the equations when graphed. In other words, we looked for the solutions to the system. We now wish to find these solutions algebraically. We want to find values for $x_1, \cdots, x_n$ which solve all of the equations. If such a set of values exists, we call $(x_1, \cdots, x_n)$ the **solution set**.

Recall the above discussions about the types of solutions possible. We will see that systems of linear equations will have one unique solution, infinitely many solutions, or no solution. Consider the following definition.

Definition 1.2.3: Consistent and Inconsistent Systems

A system of linear equations is called **consistent** if there exists at least one solution. It is called **inconsistent** if there is no solution.

If you think of each equation as a condition which must be satisfied by the variables, consistent would mean there is some choice of variables which can satisfy **all** the conditions. Inconsistent would mean there is no choice of the variables which can satisfy all of the conditions.

The following sections provide methods for determining if a system is consistent or inconsistent, and finding solutions if they exist.

Elementary Operations

We begin this section with an example. Recall from Example 1.1.1 that the solution to the given system was $(x, y) = (-1, 4)$.

Example 1.2.1: Verifying an Ordered Pair is a Solution

Algebraically verify that $(x, y) = (-1, 4)$ is a solution to the following system of equations.

$$\begin{aligned} x + y &= 3 \\ y - x &= 5 \end{aligned}$$

Solution

By graphing these two equations and identifying the point of intersection, we previously found that $(x, y) = (-1, 4)$ is the unique solution.

We can verify algebraically by substituting these values into the original equations, and ensuring that the equations hold. First, we substitute the values into the first equation and check that it equals 3.

$$x + y = (-1) + (4) = 3$$

This equals 3 as needed, so we see that $(-1, 4)$ is a solution to the first equation. Substituting the values into the second equation yields

$$y - x = (4) - (-1) = 4 + 1 = 5$$

which is true. For $(x, y) = (-1, 4)$ each equation is true and therefore, this is a solution to the system.

Now, the interesting question is this: If you were not given these numbers to verify, how could you algebraically determine the solution? Linear algebra gives us the tools needed to answer this question. The following basic operations are important tools that we will utilize.

Definition 1.2.4: Elementary Operations

Elementary operations are those operations consisting of the following.

1. Interchange the order in which the equations are listed.
2. Multiply any equation by a nonzero number.
3. Replace any equation with itself added to a multiple of another equation.

It is important to note that none of these operations will change the set of solutions of the system of equations. In fact, elementary operations are the *key tool* we use in linear algebra to find solutions to systems of equations.

Consider the following example.

Example 1.2.2: Effects of an Elementary Operation

Show that the system

$$x+y=7$$
$$2x-y=8$$

has the same solution as the system

$$x+y=7$$
$$-3y=-6$$

Solution

Notice that the second system has been obtained by taking the second equation of the first system and adding -2 times the first equation, as follows:

$$2x-y+(-2)(x+y)=8+(-2)(7)$$

By simplifying, we obtain

$$-3y=-6$$

which is the second equation in the second system. Now, from here we can solve for y and see that $y=2$. Next, we substitute this value into the first equation as follows

$$x+y=x+2=7$$

Hence $x=5$ and so $(x,y)=(5,2)$ is a solution to the second system. We want to check if $(5,2)$ is also a solution to the first system. We check this by substituting $(x,y)=(5,2)$ into the system and ensuring the equations are true.

$$x+y=(5)+(2)=7$$
$$2x-y=2(5)-(2)=8$$

Hence, $(5,2)$ is also a solution to the first system.

This example illustrates how an elementary operation applied to a system of two equations in two variables does not affect the solution set. However, a linear system may involve many equations and many variables and there is no reason to limit our study to small systems. For any size of system in any number of variables, the solution set is still the collection of solutions to the equations. In every case, the above operations of Definition 1.2.4 do not change the set of solutions to the system of linear equations.

In the following theorem, we use the notation E_i to represent an equation, while b_i denotes a constant.

Theorem 1.2.1: Elementary Operations and Solutions

Suppose you have a system of two linear equations

$$E_1=b_1$$
$$E_2=b_2 \tag{1.2.1}$$

Then the following systems have the same solution set as (1.2.1):

1. $$E_2=b_2$$
$$E_1=b_1 \tag{1.2.2}$$

2. $$E_1=b_1$$
$$kE_2=kb_2 \tag{1.2.3}$$

   for any scalar k, provided $k\neq 0$.

3. $$E_1=b_1$$
$$E_2+kE_1=b_2+kb_1 \tag{1.2.4}$$

   for any scalar k (including $k=0$).

Before we proceed with the proof of Theorem 1.2.1, let us consider this theorem in context of Example 1.2.2. Then,

$$E_1 = x + y, \quad b_1 = 7$$
$$E_2 = 2x - y, \quad b_2 = 8$$

Recall the elementary operations that we used to modify the system in the solution to the example. First, we added (-2) times the first equation to the second equation. In terms of Theorem 1.2.1, this action is given by

$$E_2 + (-2)\, E_1 = b_2 + (-2)\, b_1$$

or

$$2x - y + (-2)\,(x + y) = 8 + (-2)\, 7$$

This gave us the second system in Example 1.2.2, given by

$$E_1 = b_1$$
$$E_2 + (-2)\, E_1 = b_2 + (-2)\, b_1$$

From this point, we were able to find the solution to the system. Theorem 1.2.1 tells us that the solution we found is in fact a solution to the original system.

We will now prove Theorem 1.2.1.

Proof

1. The proof that the systems (1.2.1) and (1.2.2) have the same solution set is as follows. Suppose that $(x_1, \cdots, x_n)$ is a solution to $E_1 = b_1, E_2 = b_2$. We want to show that this is a solution to the system in (1.2.2) above. This is clear, because the system in (1.2.2) is the original system, but listed in a different order. Changing the order does not effect the solution set, so $(x_1, \cdots, x_n)$ is a solution to (1.2.2).
2. Next we want to prove that the systems (1.2.1) and (1.2.3) have the same solution set. That is $E_1 = b_1, E_2 = b_2$ has the same solution set as the system $E_1 = b_1, kE_2 = kb_2$ provided $k \neq 0$. Let $(x_1, \cdots, x_n)$ be a solution of $E_1 = b_1, E_2 = b_2,$. We want to show that it is a solution to $E_1 = b_1, kE_2 = kb_2$. Notice that the only difference between these two systems is that the second involves multiplying the equation, $E_2 = b_2$ by the scalar k. Recall that when you multiply both sides of an equation by the same number, the sides are still equal to each other. Hence if $(x_1, \cdots, x_n)$ is a solution to $E_2 = b_2$, then it will also be a solution to $kE_2 = kb_2$. Hence, $(x_1, \cdots, x_n)$ is also a solution to (1.2.3).
   Similarly, let $(x_1, \cdots, x_n)$ be a solution of $E_1 = b_1, kE_2 = kb_2$. Then we can multiply the equation $kE_2 = kb_2$ by the scalar $1/k$, which is possible only because we have required that $k \neq 0$. Just as above, this action preserves equality and we obtain the equation $E_2 = b_2$. Hence $(x_1, \cdots, x_n)$ is also a solution to $E_1 = b_1, E_2 = b_2$.
3. Finally, we will prove that the systems (1.2.1) and (1.2.4) have the same solution set. We will show that any solution of $E_1 = b_1, E_2 = b_2$ is also a solution of (1.2.4). Then, we will show that any solution of (1.2.4) is also a solution of $E_1 = b_1, E_2 = b_2$. Let $(x_1, \cdots, x_n)$ be a solution to $E_1 = b_1, E_2 = b_2$. Then in particular it solves $E_1 = b_1$. Hence, it solves the first equation in (1.2.4). Similarly, it also solves $E_2 = b_2$. By our proof of (1.2.3), it also solves $kE_1 = kb_1$. Notice that if we add E_2 and kE_1, this is equal to $b_2 + kb_1$. Therefore, if $(x_1, \cdots, x_n)$ solves $E_1 = b_1, E_2 = b_2$ it must also solve $E_2 + kE_1 = b_2 + kb_1$.
   Now suppose $(x_1, \cdots, x_n)$ solves the system $E_1 = b_1, E_2 + kE_1 = b_2 + kb_1$. Then in particular it is a solution of $E_1 = b_1$. Again by our proof of (1.2.3), it is also a solution to $kE_1 = kb_1$. Now if we subtract these equal quantities from both sides of $E_2 + kE_1 = b_2 + kb_1$ we obtain $E_2 = b_2$, which shows that the solution also satisfies $E_1 = b_1, E_2 = b_2$.

Stated simply, the above theorem shows that the elementary operations do not change the solution set of a system of equations.

We will now look at an example of a system of three equations and three variables. Similarly to the previous examples, the goal is to find values for x, y, z such that each of the given equations are satisfied when these values are substituted in.

Example 1.2.3: Solving a System of Equations with Elementary Operations

Find the solutions to the system,

$$\begin{aligned} x+3y+6z &= 25 \\ 2x+7y+14z &= 58 \\ 2y+5z &= 19 \end{aligned} \tag{1.2.5}$$

Solution

We can relate this system to Theorem 1.2.1 above. In this case, we have

$$\begin{aligned} E_1 &= x+3y+6z, & b_1 &= 25 \\ E_2 &= 2x+7y+14z, & b_2 &= 58 \\ E_3 &= 2y+5z, & b_3 &= 19 \end{aligned}$$

Theorem 1.2.1 claims that if we do elementary operations on this system, we will not change the solution set. Therefore, we can solve this system using the elementary operations given in Definition 1.2.4. First, replace the second equation by (-2) times the first equation added to the second. This yields the system

$$\begin{aligned} x+3y+6z &= 25 \\ y+2z &= 8 \\ 2y+5z &= 19 \end{aligned} \tag{1.2.6}$$

Now, replace the third equation with (-2) times the second added to the third. This yields the system

$$\begin{aligned} x+3y+6z &= 25 \\ y+2z &= 8 \\ z &= 3 \end{aligned} \tag{1.2.7}$$

At this point, we can easily find the solution. Simply take $z=3$ and substitute this back into the previous equation to solve for y, and similarly to solve for x.

$$\begin{aligned} x+3y+6\left(3\right) = x+3y+18 &= 25 \\ y+2\left(3\right) = y+6 &= 8 \\ z &= 3 \end{aligned}$$

The second equation is now

$$y+6=8$$

You can see from this equation that $y=2$. Therefore, we can substitute this value into the first equation as follows:

$$x+3\left(2\right)+18=25$$

By simplifying this equation, we find that $x=1$. Hence, the solution to this system is $(x,y,z)=(1,2,3)$. This process is called **back substitution**.

Alternatively, in (1.2.7) you could have continued as follows. Add (-2) times the third equation to the second and then add (-6) times the second to the first. This yields

$$\begin{aligned} x+3y &= 7 \\ y &= 2 \\ z &= 3 \end{aligned}$$

Now add (-3) times the second to the first. This yields

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

a system which has the same solution set as the original system. This avoided back substitution and led to the same solution set. It is your decision which you prefer to use, as both methods lead to the correct solution, $(x,y,z)=(1,2,3)$.

This page titled 1.2: Systems of Equations, Algebraic Procedures is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.2: Systems of Equations, Algebraic Procedures** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

1.3: Gaussian Elimination

The work we did in the previous section will always find the solution to the system. In this section, we will explore a less cumbersome way to find the solutions. First, we will represent a linear system with an **augmented matrix.** A **matrix** is simply a rectangular array of numbers. The size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. In order to construct an augmented matrix from a linear system, we create a **coefficient matrix** from the coefficients of the variables in the system, as well as a **constant matrix** from the constants. The coefficients from one equation of the system create one row of the augmented matrix.

For example, consider the linear system in Example 1.2.3

$$\begin{aligned} x + 3y + 6z &= 25 \\ 2x + 7y + 14z &= 58 \\ 2y + 5z &= 19 \end{aligned}$$

This system can be written as an augmented matrix, as follows

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Notice that it has exactly the same information as the original system. Here it is understood that the first column contains the coefficients from x in each equation, in order, $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Similarly, we create a column from the coefficients on y in each equation, $\begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$ and a column from the coefficients on z in each equation, $\begin{bmatrix} 6 \\ 14 \\ 5 \end{bmatrix}$. For a system of more than three variables, we would continue in this way constructing a column for each variable. Similarly, for a system of less than three variables, we simply construct a column for each variable.

Finally, we construct a column from the constants of the equations, $\begin{bmatrix} 25 \\ 58 \\ 19 \end{bmatrix}$.

The rows of the augmented matrix correspond to the equations in the system. For example, the top row in the augmented matrix, $\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \end{array}\right]$ corresponds to the equation

$$x + 3y + 6z = 25.$$

Consider the following definition.

Definition 1.3.1: Augmented Matrix of a Linear System

For a linear system of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the x_i are variables and the a_{ij} and b_i are constants, the augmented matrix of this system is given by

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

Now, consider elementary operations in the context of the augmented matrix. The elementary operations in Definition 1.2.4 can be used on the rows just as we used them on equations previously. Changes to a system of equations in as a result of an elementary operation are equivalent to changes in the augmented matrix resulting from the corresponding row operation. Note that Theorem

1.2.1 implies that any elementary row operations used on an augmented matrix will not change the solution to the corresponding system of equations. We now formally define elementary row operations. These are the *key tool* we will use to find solutions to systems of equations.

Definition 1.3.2: Elementary Row Operations

The **elementary row operations** (also known as **row operations**) consist of the following

1. Switch two rows.
2. Multiply a row by a nonzero number.
3. Replace a row by any multiple of another row added to it.

Recall how we solved Example 1.2.3. We can do the exact same steps as above, except now in the context of an augmented matrix and using row operations. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Thus the first step in solving the system given by (1.2.5) would be to take (-2) times the first row of the augmented matrix and add it to the second row,

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{array}\right]$$

Note how this corresponds to (1.2.6). Next take (-2) times the second row and add to the third,

$$\left[\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

This augmented matrix corresponds to the system

$$\begin{aligned} x + 3y + 6z &= 25 \\ y + 2z &= 8 \\ z &= 3 \end{aligned}$$

which is the same as (1.2.7). By back substitution you obtain the solution $x = 1, y = 2,$ and $z = 3.$

Through a systematic procedure of row operations, we can simplify an augmented matrix and carry it to **row-echelon form** or **reduced row-echelon form**, which we define next. These forms are used to find the solutions of the system of equations corresponding to the augmented matrix.

In the following definitions, the term **leading entry** refers to the first nonzero entry of a row when scanning the row from left to right.

Definition 1.3.3: Row-Echelon Form

An augmented matrix is in **row-echelon form** if

1. All nonzero rows are above any rows of zeros.
2. Each leading entry of a row is in a column to the right of the leading entries of any row above it.
3. Each leading entry of a row is equal to 1.

We also consider another reduced form of the augmented matrix which has one further condition.

Definition 1.3.4: Reduced Row-Echelon Form

An augmented matrix is in **reduced row-echelon form** if

1. All nonzero rows are above any rows of zeros.
2. Each leading entry of a row is in a column to the right of the leading entries of any rows above it.
3. Each leading entry of a row is equal to **1**.
4. All entries in a column above and below a leading entry are zero.

Notice that the first three conditions on a reduced row-echelon form matrix are the same as those for row-echelon form.

Hence, every reduced row-echelon form matrix is also in row-echelon form. The converse is not necessarily true; we cannot assume that every matrix in row-echelon form is also in reduced row-echelon form. However, it often happens that the row-echelon form is sufficient to provide information about the solution of a system.

The following examples describe matrices in these various forms. As an exercise, take the time to carefully verify that they are in the specified form.

Example 1.3.1: Not in Row-Echelon Form

The following augmented matrices are not in row-echelon form (and therefore also not in reduced row-echelon form).

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & -6 \\ 4 & 0 & 7 \end{array}\right], \left[\begin{array}{ccc|c} 0 & 2 & 3 & 3 \\ 1 & 5 & 0 & 2 \\ 7 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Example 1.3.2: Matrices in Row-Echelon Form

The following augmented matrices are in row-echelon form, but not in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 6 & 5 & 8 & 2 \\ 0 & 0 & 1 & 2 & 7 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 3 & 5 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Notice that we could apply further row operations to these matrices to carry them to reduced row-echelon form. Take the time to try that on your own. Consider the following matrices, which are in reduced row-echelon form.

Example 1.3.3: Matrices in Reduced Row-Echelon Form

The following augmented matrices are in reduced row-echelon form.

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

One way in which the row-echelon form of a matrix is useful is in identifying the pivot positions and pivot columns of the matrix.

Definition 1.3.5: Pivot Position and Pivot Column

A **pivot position** in a matrix is the location of a leading entry in the row-echelon form of a matrix.

A **pivot column** is a column that contains a pivot position.

For example consider the following.

Example 1.3.4: Pivot Position

Let

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array}\right]$$

Where are the pivot positions and pivot columns of the augmented matrix A?

Solution

The row-echelon form of this matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is all we need in this example, but note that this matrix is not in reduced row-echelon form.

In order to identify the pivot positions in the original matrix, we look for the leading entries in the row-echelon form of the matrix. Here, the entry in the first row and first column, as well as the entry in the second row and second column are the leading entries. Hence, these locations are the pivot positions. We identify the pivot positions in the original matrix, as in the following:

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 4 \\ 3 & \boxed{2} & 1 & 6 \\ 4 & 4 & 4 & 10 \end{array}\right]$$

Thus the pivot columns in the matrix are the first two columns.

The following is an algorithm for carrying a matrix to row-echelon form and reduced row-echelon form. You may wish to use this algorithm to carry the above matrix to row-echelon form or reduced row-echelon form yourself for practice.

Algorithm 1.3.1: Reduced Row-Echelon Form Algorithm

This algorithm provides a method for using row operations to take a matrix to its reduced row-echelon form. We begin with the matrix in its original form.

1. Starting from the left, find the first nonzero column. This is the first pivot column, and the position at the top of this column is the first pivot position. Switch rows if necessary to place a nonzero number in the first pivot position.
2. Use row operations to make the entries below the first pivot position (in the first pivot column) equal to zero.
3. Ignoring the row containing the first pivot position, repeat steps 1 and 2 with the remaining rows. Repeat the process until there are no more rows to modify.
4. Divide each nonzero row by the value of the leading entry, so that the leading entry becomes 1. The matrix will then be in row-echelon form.

   The following step will carry the matrix from row-echelon form to reduced row-echelon form.

5. Moving from right to left, use row operations to create zeros in the entries of the pivot columns which are above the pivot positions. The result will be a matrix in reduced row-echelon form.

Most often we will apply this algorithm to an augmented matrix in order to find the solution to a system of linear equations. However, we can use this algorithm to compute the reduced row-echelon form of any matrix which could be useful in other applications.

Consider the following example of Algorithm 1.3.1.

Example 1.3.5: Finding Row-Echelon Form and Reduced Row-Echelon Form of a Matrix

Let

$$A = \begin{bmatrix} 0 & -5 & -4 \\ 1 & 4 & 3 \\ 5 & 10 & 7 \end{bmatrix}$$

Find the row-echelon form of A. Then complete the process until A is in reduced row-echelon form.

Solution

In working through this example, we will use the steps outlined in Algorithm 1.3.1.

1. The first pivot column is the first column of the matrix, as this is the first nonzero column from the left. Hence the first pivot position is the one in the first row and first column. Switch the first two rows to obtain a nonzero entry in the first pivot position, outlined in a box below.

$$\begin{bmatrix} \boxed{1} & 4 & 3 \\ 0 & -5 & -4 \\ 5 & 10 & 7 \end{bmatrix}$$

2. Step two involves creating zeros in the entries below the first pivot position. The first entry of the second row is already a zero. All we need to do is subtract 5 times the first row from the third row. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 10 & 8 \end{bmatrix}$$

3. Now ignore the top row. Apply steps 1 and 2 to the smaller matrix

$$\begin{bmatrix} -5 & -4 \\ 10 & 8 \end{bmatrix}$$

   In this matrix, the first column is a pivot column, and -5 is in the first pivot position. Therefore, we need to create a zero below it. To do this, add 2 times the first row (of this matrix) to the second. The resulting matrix is

$$\begin{bmatrix} -5 & -4 \\ 0 & 0 \end{bmatrix}$$

   Our original matrix now looks like

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & -5 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

   We can see that there are no more rows to modify.

4. Now, we need to create leading 1s in each row. The first row already has a leading 1 so no work is needed here. Divide the second row by -5 to create a leading 1. The resulting matrix is

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

   This matrix is now in row-echelon form.

5. Now create zeros in the entries above pivot positions in each column, in order to carry this matrix all the way to reduced row-echelon form. Notice that there is no pivot position in the third column so we do not need to create any zeros in this column! The column in which we need to create zeros is the second. To do so, subtract 4 times the second row from the first row. The resulting matrix is

$$\begin{bmatrix} 1 & 0 & -\frac{1}{5} \\ 0 & 1 & \frac{4}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix is now in reduced row-echelon form.

The above algorithm gives you a simple way to obtain the row-echelon form and reduced row-echelon form of a matrix. The main idea is to do row operations in such a way as to end up with a matrix in row-echelon form or reduced row-echelon form. This process is important because the resulting matrix will allow you to describe the solutions to the corresponding linear system of equations in a meaningful way.

In the next example, we look at how to solve a system of equations using the corresponding augmented matrix.

✔ Example 1.3.6: Finding the Solution to a System

Give the complete solution to the following system of equations

$$\begin{aligned} 2x + 4y - 3z &= -1 \\ 5x + 10y - 7z &= -2 \\ 3x + 6y + 5z &= 9 \end{aligned}$$

Solution

The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 5 & 10 & -7 & -2 \\ 3 & 6 & 5 & 9 \end{array}\right]$$

In order to find the solution to this system, we wish to carry the augmented matrix to reduced row-echelon form. We will do so using Algorithm 1.3.1. Notice that the first column is nonzero, so this is our first pivot column. The first entry in the first row, 2, is the first leading entry and it is in the first pivot position. We will use row operations to create zeros in the entries below the 2. First, replace the second row with -5 times the first row plus 2 times the second row. This yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 6 & 5 & 9 \end{array}\right]$$

Now, replace the third row with -3 times the first row plus to 2 times the third row. This yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 21 \end{array}\right]$$

Now the entries in the first column below the pivot position are zeros. We now look for the second pivot column, which in this case is column three. Here, the 1 in the second row and third column is in the pivot position. We need to do just one row operation to create a zero below the 1.

Taking -1 times the second row and adding it to the third row yields

$$\left[\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

We could proceed with the algorithm to carry this matrix to row-echelon form or reduced row-echelon form. However, remember that we are looking for the solutions to the system of equations. Take another look at the third row of the matrix. Notice that it corresponds to the equation

$$0x + 0y + 0z = 20$$

There is no solution to this equation because for all x, y, z, the left side will equal 0 and $0 \neq 20$. This shows there is no solution to the given system of equations. In other words, this system is inconsistent.

The following is another example of how to find the solution to a system of equations by carrying the corresponding augmented matrix to reduced row-echelon form.

Example 1.3.7: An Infinite Set of Solutions

Give the complete solution to the system of equations

$$\begin{aligned} 3x - y - 5z &= 9 \\ y - 10z &= 0 \\ -2x + y &= -6 \end{aligned} \tag{1.3.1}$$

Solution

The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ -2 & 1 & 0 & -6 \end{array}\right]$$

In order to find the solution to this system, we will carry the augmented matrix to reduced row-echelon form, using Algorithm 1.3.1. The first column is the first pivot column. We want to use row operations to create zeros beneath the first entry in this column, which is in the first pivot position. Replace the third row with 2 times the first row added to 3 times the third row. This gives

$$\left[\begin{array}{ccc|c} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 1 & -10 & 0 \end{array}\right]$$

Now, we have created zeros beneath the 3 in the first column, so we move on to the second pivot column (which is the second column) and repeat the procedure. Take -1 times the second row and add to the third row.

$$\left[\begin{array}{ccc|c} 3 & -1 & -5 & 9 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The entry below the pivot position in the second column is now a zero. Notice that we have no more pivot columns because we have only two leading entries.

At this stage, we also want the leading entries to be equal to one. To do so, divide the first row by 3.

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{3} & -\frac{5}{3} & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This matrix is now in row-echelon form.

Let's continue with row operations until the matrix is in reduced row-echelon form. This involves creating zeros above the pivot positions in each pivot column. This requires only one step, which is to add $\frac{1}{3}$ times the second row to the first row.

$$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 3 \\ 0 & 1 & -10 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is in reduced row-echelon form, which you should verify using Definition 1.3.4. The equations corresponding to this reduced row-echelon form are

$$\begin{aligned} x - 5z &= 3 \\ y - 10z &= 0 \end{aligned}$$

or

$$\begin{aligned} x &= 3 + 5z \\ y &= 10z \end{aligned}$$

Observe that z is not restrained by any equation. In fact, z can equal any number. For example, we can let $z = t$, where we can choose t to be any number. In this context t is called a **parameter** . Therefore, the solution set of this system is

$$\begin{aligned} x &= 3 + 5t \\ y &= 10t \\ z &= t \end{aligned}$$

where t is arbitrary. The system has an infinite set of solutions which are given by these equations. For any value of t we select, $x, y,$ and z will be given by the above equations. For example, if we choose $t = 4$ then the corresponding solution would be

$$\begin{aligned} x &= 3 + 5(4) = 23 \\ y &= 10(4) = 40 \\ z &= 4 \end{aligned}$$

In Example 1.3.7 the solution involved one parameter. It may happen that the solution to a system involves more than one parameter, as shown in the following example.

Example 1.3.8: A Two Parameter Set of Solutions

Find the solution to the system

$$\begin{aligned} x + 2y - z + w &= 3 \\ x + y - z + w &= 1 \\ x + 3y - z + w &= 5 \end{aligned}$$

Solution

The augmented matrix is

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array}\right]$$

We wish to carry this matrix to row-echelon form. Here, we will outline the row operations used. However, make sure that you understand the steps in terms of Algorithm 1.3.1.

Take -1 times the first row and add to the second. Then take -1 times the first row and add to the third. This yields

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 2 \end{array}\right]$$

Now add the second row to the third row and divide the second row by -1.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{1.3.2}$$

This matrix is in row-echelon form and we can see that x and y correspond to pivot columns, while z and w do not. Therefore, we will assign parameters to the variables z and w. Assign the parameter s to z and the parameter t to w. Then the first row

yields the equation $x+2y-s+t=3$, while the second row yields the equation $y=2$. Since $y=2$, the first equation becomes $x+4-s+t=3$ showing that the solution is given by

$$\begin{aligned} x &= -1+s-t \\ y &= 2 \\ z &= s \\ w &= t \end{aligned}$$

It is customary to write this solution in the form

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1+s-t \\ 2 \\ s \\ t \end{bmatrix} \tag{1.3.3}$$

This example shows a system of equations with an infinite solution set which depends on two parameters. It can be less confusing in the case of an infinite solution set to first place the augmented matrix in reduced row-echelon form rather than just row-echelon form before seeking to write down the description of the solution.

In the above steps, this means we don't stop with the row-echelon form in equation (1.3.2). Instead we first place it in reduced row-echelon form as follows.

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Then the solution is $y=2$ from the second row and $x=-1+z-w$ from the first. Thus letting $z=s$ and $w=t$, the solution is given by (1.3.3).

You can see here that there are two paths to the correct answer, which both yield the same answer. Hence, either approach may be used. The process which we first used in the above solution is called **Gaussian Elimination** This process involves carrying the matrix to row-echelon form, converting back to equations, and using back substitution to find the solution. When you do row operations until you obtain reduced row-echelon form, the process is called **Gauss-Jordan Elimination**.

We have now found solutions for systems of equations with no solution and infinitely many solutions, with one parameter as well as two parameters. Recall the three types of solution sets which we discussed in the previous section; no solution, one solution, and infinitely many solutions. Each of these types of solutions could be identified from the graph of the system. It turns out that we can also identify the type of solution from the reduced row-echelon form of the augmented matrix.

- *No Solution:* In the case where the system of equations has no solution, the row-echelon form of the augmented matrix will have a row of the form

  $$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right]$$

  This row indicates that the system is inconsistent and has no solution.
- *One Solution:* In the case where the system of equations has one solution, every column of the coefficient matrix is a pivot column. The following is an example of an augmented matrix in reduced row-echelon form for a system of equations with one solution.

  $$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

- *Infinitely Many Solutions:* In the case where the system of equations has infinitely many solutions, the solution contains parameters. There will be columns of the coefficient matrix which are not pivot columns. The following are examples of augmented matrices in reduced row-echelon form for systems of equations with infinitely many solutions.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

or

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -3 \end{array}\right]$$

This page titled 1.3: Gaussian Elimination is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.3: Gaussian Elimination** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

1.4: Uniqueness of the Reduced Row-Echelon Form

As we have seen in earlier sections, we know that every matrix can be brought into reduced row-echelon form by a sequence of elementary row operations. Here we will prove that the resulting matrix is unique; in other words, the resulting matrix in reduced row-echelon does not depend upon the particular sequence of elementary row operations or the order in which they were performed.

Let A be the augmented matrix of a homogeneous system of linear equations in the variables $x_1, x_2, \cdots, x_n$ which is also in reduced row-echelon form. The matrix A divides the set of variables in two different types. We say that x_i is a *basic variable* whenever A has a leading 1 in column number i, in other words, when column i is a pivot column. Otherwise we say that x_i is a *free variable*.

Recall Example 1.3.8.

Example 1.4.1: Basic and Free Variables

Find the basic and free variables in the system

$$\begin{aligned} x + 2y - z + w &= 3 \\ x + y - z + w &= 1 \\ x + 3y - z + w &= 5 \end{aligned}$$

Solution

Recall from the solution of Example 1.3.8 that the row-echelon form of the augmented matrix of this system is given by

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

You can see that columns 1 and 2 are pivot columns. These columns correspond to variables x and y, making these the basic variables. Columns 3 and 4 are not pivot columns, which means that z and w are free variables.

We can write the solution to this system as

$$\begin{aligned} x &= -1 + s - t \\ y &= 2 \\ z &= s \\ w &= t \end{aligned}$$

Here the free variables are written as parameters, and the basic variables are given by linear functions of these parameters.

In general, all solutions can be written in terms of the free variables. In such a description, the free variables can take any values (they become parameters), while the basic variables become simple linear functions of these parameters. Indeed, a basic variable x_i is a linear function of *only* those free variables x_j with $j > i$. This leads to the following observation.

Proposition 1.4.1: Basic and Free Variables

If x_i is a basic variable of a homogeneous system of linear equations, then any solution of the system with $x_j = 0$ for all those free variables x_j with $j > i$ must also have $x_i = 0$.

Using this proposition, we prove a lemma which will be used in the proof of the main result of this section below.

Lemma 1.4.1: Solutions and the Reduced Row-Echelon Form of a Matrix

Let A and B be two distinct augmented matrices for two homogeneous systems of m equations in n variables, such that A and B are each in reduced row-echelon. Then, the two systems do not have exactly the same solutions.

Proof

With respect to the linear systems associated with the matrices A and B, the corresponding variables

With respect to the linear systems associated with the matrices A and B, there are two cases to consider:

- Case 1: the two systems have the same basic variables
- Case 2: the two systems do not have the same basic variables

In case 1, the two matrices will have exactly the same pivot positions. However, since A and B are not identical, there is some row of A which is different from the corresponding row of B and yet the rows each have a pivot in the same column position. Let i be the index of this column position. Since the matrices are in reduced row-echelon form, the two rows must differ at some entry in a column $j > i$. Let these entries be a in A and b in B, where $a \neq b$. Since A is in reduced row-echelon form, if x_j were a basic variable for its linear system, we would have $a = 0$. Similarly, if x_j were a basic variable for the linear system of the matrix B, we would have $b = 0$. Since a and b are unequal, they cannot both be equal to 0, and hence x_j cannot be a basic variable for both linear systems. However, since the systems have the same basic variables, x_j must then be a free variable for each system. We now look at the solutions of the systems in which x_j is set equal to 1 and all other free variables are set equal to 0. For this choice of parameters, the solution of the system for matrix A has $x_j = -a$, while the solution of the system for matrix B has $x_j = -b$, so that the two systems have different solutions.

In case 2, there is a variable x_i which is a basic variable for one matrix, let's say A, and a free variable for the other matrix B. The system for matrix B has a solution in which $x_i = 1$ and $x_j = 0$ for all other free variables x_j. However, by Proposition 1.4.1this cannot be a solution of the system for the matrix A. This completes the proof of case 2.

Now, we say that the matrix B is **equivalent** to the matrix A provided that B can be obtained from A by performing a sequence of elementary row operations beginning with A. The importance of this concept lies in the following result.

Theorem 1.4.1: Equivalent Matrices

The two linear systems of equations corresponding to two equivalent augmented matrices have exactly the same solutions.

Proof

The proof of this theorem is left as an exercise.

Now, we can use Lemma 1.4.1 and Theorem 1.4.1 to prove the main result of this section.

Theorem 1.4.2: Uniqueness of the Reduced Row-Echelon Form

Every matrix A is equivalent to a unique matrix in reduced row-echelon form.

Proof

Let A be an $m \times n$ matrix and let B and C be matrices in reduced row-echelon form, each equivalent to A. It suffices to show that $B = C$.

Let A^+ be the matrix A augmented with a new rightmost column consisting entirely of zeros. Similarly, augment matrices B and C each with a rightmost column of zeros to obtain B^+ and C^+. Note that B^+ and C^+ are matrices in reduced row-echelon form which are obtained from A^+ by respectively applying the same sequence of elementary row operations which were used to obtain B and C from A.

Now, A^+, B^+, and C^+ can all be considered as augmented matrices of homogeneous linear systems in the variables $x_1, x_2, \cdots, x_n$. Because B^+ and C^+ are each equivalent to A^+, Theorem 1.4.1 ensures that all three homogeneous linear systems have exactly the same solutions. By Lemma 1.4.1 we conclude that $B^+ = C^+$. By construction, we must also have $B = C$.

According to this theorem we can say that each matrix A has a unique reduced row-echelon form.

This page titled 1.4: Uniqueness of the Reduced Row-Echelon Form is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.4: Uniqueness of the Reduced Row-Echelon Form** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

1.5: Rank and Homogeneous Systems

Rank and Homogeneous Systems

There is a special type of system which requires additional study. This type of system is called a homogeneous system of equations, which we defined above in Definition 1.2.3. Our focus in this section is to consider what types of solutions are possible for a homogeneous system of equations.

Consider the following definition.

Definition 1.5.1: Trivial Solution

Consider the homogeneous system of equations given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Then, $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ is always a solution to this system. We call this the **trivial solution** .

If the system has a solution in which not all of the $x_1, \cdots, x_n$ are equal to zero, then we call this solution **nontrivial** . The trivial solution does not tell us much about the system, as it says that $0 = 0$! Therefore, when working with homogeneous systems of equations, we want to know when the system has a nontrivial solution.

Suppose we have a homogeneous system of m equations, using n variables, and suppose that $n > m$. In other words, there are more variables than equations. Then, it turns out that this system always has a nontrivial solution. Not only will the system have a nontrivial solution, but it also will have infinitely many solutions. It is also possible, but not required, to have a nontrivial solution if $n = m$ and $n < m$.

Consider the following example.

Example 1.5.1: Solutions to a Homogeneous System of Equations

Find the nontrivial solutions to the following homogeneous system of equations

$$\begin{aligned} 2x + y - z &= 0 \\ x + 2y - 2z &= 0 \end{aligned}$$

Solution

Notice that this system has $m = 2$ equations and $n = 3$ variables, so $n > m$. Therefore by our previous discussion, we expect this system to have infinitely many solutions.

The process we use to find the solutions for a homogeneous system of equations is the same process we used in the previous section. First, we construct the augmented matrix, given by

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right]$$

Then, we carry this matrix to its reduced row-echelon form, given below.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

The corresponding system of equations is

$$\begin{aligned} x &= 0 \\ y - z &= 0 \end{aligned}$$

Since z is not restrained by any equation, we know that this variable will become our parameter. Let $z = t$ where t is any number. Therefore, our solution has the form

$$\begin{aligned} x &= 0 \\ y &= z = t \\ z &= t \end{aligned}$$

Hence this system has infinitely many solutions, with one parameter t.

Suppose we were to write the solution to the previous example in another form. Specifically,

$$\begin{aligned} x &= 0 \\ y &= 0 + t \\ z &= 0 + t \end{aligned}$$

can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Notice that we have constructed a column from the constants in the solution (all equal to 0), as well as a column corresponding to the coefficients on t in each equation. While we will discuss this form of solution more in further chapters, for now consider the column of coefficients of the parameter t. In this case, this is the column $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

There is a special name for this column, which is **basic solution**. The basic solutions of a system are columns constructed from the coefficients on parameters in the solution. We often denote basic solutions by X_1, X_2 etc., depending on how many solutions occur. Therefore, Example 1.5.1 has the basic solution $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

We explore this further in the following example.

Example 1.5.2: Basic Solutions of a Homogeneous System

Consider the following homogeneous system of equations.

$$\begin{aligned} x + 4y + 3z &= 0 \\ 3x + 12y + 9z &= 0 \end{aligned}$$

Find the basic solutions to this system.

Solution

The augmented matrix of this system and the resulting reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 3 & 12 & 9 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

When written in equations, this system is given by

$$x + 4y + 3z = 0$$

Notice that only x corresponds to a pivot column. In this case, we will have two parameters, one for y and one for z. Let $y = s$ and $z = t$ for any numbers s and t. Then, our solution becomes

$$\begin{aligned} x &= -4s - 3t \\ y &= s \\ z &= t \end{aligned}$$

which can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

You can see here that we have two columns of coefficients corresponding to parameters, specifically one for s and one for t. Therefore, this system has two basic solutions! These are

$$X_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

We now present a new definition.

Definition 1.5.2: Linear Combination

Let $X_1, \cdots, X_n, V$ be column matrices. Then V is said to be a **linear combination** of the columns $X_1, \cdots, X_n$ if there exist scalars, $a_1, \cdots, a_n$ such that

$$V = a_1 X_1 + \cdots + a_n X_n$$

A remarkable result of this section is that a linear combination of the basic solutions is again a solution to the system. Even more remarkable is that every solution can be written as a linear combination of these solutions. Therefore, if we take a linear combination of the two solutions to Example 1.5.2, this would also be a solution. For example, we could take the following linear combination

$$3 \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

You should take a moment to verify that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

is in fact a solution to the system in Example 1.5.2.

Another way in which we can find out more information about the solutions of a homogeneous system is to consider the **rank** of the associated coefficient matrix. We now define what is meant by the rank of a matrix.

Definition 1.5.3: Rank of a Matrix

Let A be a matrix and consider any row-echelon form of A. Then, the number r of leading entries of A does not depend on the row-echelon form you choose, and is called the **rank** of A. We denote it by Rank(A).

Similarly, we could count the number of pivot positions (or pivot columns) to determine the rank of A.

Example 1.5.3: Finding the Rank of a Matrix

Consider the matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 9 \\ 2 & 4 & 6 \end{bmatrix}$$

What is its rank?

Solution

First, we need to find the reduced row-echelon form of A. Through the usual algorithm, we find that this is

$$\begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Here we have two leading entries, or two pivot positions, shown above in boxes.The rank of A is $r = 2$.

Notice that we would have achieved the same answer if we had found the row-echelon form of A instead of the reduced row-echelon form.

Suppose we have a homogeneous system of m equations in n variables, and suppose that $n > m$. From our above discussion, we know that this system will have infinitely many solutions. If we consider the rank of the coefficient matrix of this system, we can find out even more about the solution. Note that we are looking at just the coefficient matrix, not the entire augmented matrix.

Theorem 1.5.1: Rank and Solutions to a Homogeneous System

Let A be the $m \times n$ coefficient matrix corresponding to a homogeneous system of equations, and suppose A has rank r. Then, the solution to the corresponding system has $n - r$ parameters.

Consider our above Example 1.5.2 in the context of this theorem. The system in this example has $m = 2$ equations in $n = 3$ variables. First, because $n > m$, we know that the system has a nontrivial solution, and therefore infinitely many solutions. This tells us that the solution will contain at least one parameter. The rank of the coefficient matrix can tell us even more about the solution! The rank of the coefficient matrix of the system is 1, as it has one leading entry in row-echelon form. Theorem 1.5.1 tells us that the solution will have $n - r = 3 - 1 = 2$ parameters. You can check that this is true in the solution to Example 1.5.2.

Notice that if $n = m$ or $n < m$, it is possible to have either a unique solution (which will be the trivial solution) or infinitely many solutions.

We are not limited to homogeneous systems of equations here. The rank of a matrix can be used to learn about the solutions of any system of linear equations. In the previous section, we discussed that a system of equations can have no solution, a unique solution, or infinitely many solutions. Suppose the system is consistent, whether it is homogeneous or not. The following theorem tells us how we can use the rank to learn about the type of solution we have.

Theorem 1.5.2: Rank and Solutions to a Consistent System of Equations

Let A be the $m \times (n+1)$ augmented matrix corresponding to a consistent system of equations in n variables, and suppose A has rank r. Then

1. the system has a unique solution if $r = n$
2. the system has infinitely many solutions if $r < n$

We will not present a formal proof of this, but consider the following discussions.

1. *No Solution* The above theorem assumes that the system is consistent, that is, that it has a solution. It turns out that it is possible for the augmented matrix of a system with no solution to have any rank r as long as $r > 1$. Therefore, we must know that the system is consistent in order to use this theorem!
2. *Unique Solution* Suppose $r = n$. Then, there is a pivot position in every column of the coefficient matrix of A. Hence, there is a unique solution.
3. *Infinitely Many Solutions* Suppose $r < n$. Then there are infinitely many solutions. There are less pivot positions (and hence less leading entries) than columns, meaning that not every column is a pivot column. The columns which are ***not*** pivot columns correspond to parameters. In fact, in this case we have $n - r$ parameters.

This page titled 1.5: Rank and Homogeneous Systems is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.5: Rank and Homogeneous Systems** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

1.6: Exercises

Exercises

Exercise 1.6.1

Graphically, find the point (x_1, y_1) which lies on both lines, $x + 3y = 1$ and $4x - y = 3$. That is, graph each line and see where they intersect.

Answer

$\begin{matrix} x + 3y = 1 \\ 4x - y = 3 \end{matrix}$, Solution is: $\left[x = \frac{10}{13}, y = \frac{1}{13}\right]$.

Exercise 1.6.2

Graphically, find the point of intersection of the two lines $3x + y = 3$ and $x + 2y = 1$. That is, graph each line and see where they intersect.

Answer

$\begin{matrix} 3x + y = 3 \\ x + 2y = 1 \end{matrix}$, Solution is: $[x = 1, y = 0]$

Exercise 1.6.3

Do the three lines, $x + 2y = 1$, $2x - y = 1$, and $4x + 3y = 3$ have a common point of intersection? If so, find the point and if not, tell why they don't have such a common point of intersection.

Answer

$\begin{matrix} x + 2y = 1 \\ 2x - y = 1 \\ 4x + 3y = 3 \end{matrix}$, Solution is: $\left[x = \frac{3}{5}, y = \frac{1}{5}\right]$

Exercise 1.6.4

Do the three planes, $x + y - 3z = 2$, $2x + y + z = 1$, and $3x + 2y - 2z = 0$ have a common point of intersection? If so, find one and if not, tell why there is no such point.

Answer

No solution exists. You can see this by writing the augmented matrix and doing row operations. $\begin{bmatrix} 1 & 1 & -3 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & -2 & 0 \end{bmatrix}$, row echelon form: $\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Thus one of the equation says $0 = 1$ in an equivalent system of equations.

Exercise 1.6.5

Four times the weight of Gaston is 150 pounds more than the weight of Ichabod. Four times the weight of Ichabod is 660 pounds less than seventeen times the weight of Gaston. Four times the weight of Gaston plus the weight of Siegfried equals 290 pounds. Brunhilde would balance all three of the others. Find the weights of the four people.

Answer

$$\begin{matrix} 4g - I = 150 \\ 4I - 17g = -660 \\ 4g + s = 290 \\ g + I + s - b = 0 \end{matrix}$$, Solution is: $\{g = 60, I = 90, b = 200, s = 50\}$

? Exercise 1.6.6

Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right]$$

Answer

The solution exists but is not unique.

? Exercise 1.6.7

Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right]$$

Answer

A solution exists and is unique.

? Exercise 1.6.8

Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & 0 & * & 0 & * \\ 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right]$$

? Exercise 1.6.9

Consider the following augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a nonzero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left[\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & * & \blacksquare \end{array}\right]$$

Answer

There might be a solution. If so, there are infinitely many.

? Exercise 1.6.10

Suppose a system of equations has fewer equations than variables. Will such a system necessarily be consistent? If so, explain why and if not, give an example which is not consistent.

Answer

No. Consider $x+y+z=2$ and $x+y+z=1$.

? Exercise 1.6.11

If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, tell why not.

Answer

These can have a solution. For example, $x+y=1, 2x+2y=2, 3x+3y=3$ even has an infinite set of solutions.

? Exercise 1.6.12

Find h such that

$$\left[\begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array}\right]$$

is the augmented matrix of an *inconsistent* system.

Answer

$h=4$

? Exercise 1.6.13

Find h such that

$$\left[\begin{array}{cc|c} 1 & h & 3 \\ 2 & 4 & 6 \end{array}\right]$$

is the augmented matrix of a *consistent* system.

Answer

Any h will work.

? Exercise 1.6.14

Find h such that

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & h & 12 \end{array}\right]$$

is the augmented matrix of a *consistent* system.

Answer

Any h will work.

Exercise 1.6.15

Choose h and k such that the augmented matrix shown has each of the following:

a. one solution
b. no solution
c. infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 2 & 4 & k \end{array}\right]$$

Answer

If $h \neq 2$ there will be a unique solution for any k. If $h = 2$ and $k \neq 4$, there are no solutions. If $h = 2$ and $k = 4$, then there are infinitely many solutions.

Exercise 1.6.16

Choose h and k such that the augmented matrix shown has each of the following:

a. one solution
b. no solution
c. infinitely many solutions

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array}\right]$$

Answer

If $h \neq 4$, then there is exactly one solution. If $h = 4$ and $k \neq 4$, then there are no solutions. If $h = 4$ and $k = 4$, then there are infinitely many solutions.

Exercise 1.6.17

Determine if the system is consistent. If so, is the solution unique?

$$\begin{aligned} x + 2y + z - w &= 2 \\ x - y + z + w &= 1 \\ 2x + y - z &= 1 \\ 4x + 2y + z &= 5 \end{aligned}$$

Answer

There is no solution. The system is inconsistent. You can see this from the augmented matrix. $\left[\begin{array}{ccccc} 1 & 2 & 1 & -1 & 2 \\ 1 & -1 & 1 & 1 & 1 \\ 2 & 1 & -1 & 0 & 1 \\ 4 & 2 & 1 & 0 & 5 \end{array}\right]$, reduced row-echelon form: $\left[\begin{array}{ccccc} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$.

Exercise 1.6.18

Determine if the system is consistent. If so, is the solution unique?

$$x+2y+z-w=2$$
$$x-y+z+w=0$$
$$2x+y-z=1$$
$$4x+2y+z=3$$

Answer

Solution is: $[w=\frac{3}{2}y-1, x=\frac{2}{3}-\frac{1}{2}y, z=\frac{1}{3}]$

? Exercise 1.6.19

Determine which matrices are in reduced row-echelon form.

a. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 7 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}$

Answer

a. This one is not.

b. This one is.

c. This one is.

? Exercise 1.6.20

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 2 & -1 & 3 & -1 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & 1 & -2 \end{bmatrix}$$

? Exercise 1.6.21

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

? Exercise 1.6.22

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 3 & -6 & -7 & -8 \\ 1 & -2 & -2 & -2 \\ 1 & -2 & -3 & -4 \end{bmatrix}$$

? Exercise 1.6.23

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 2 & 4 & 5 & 15 \\ 1 & 2 & 3 & 9 \\ 1 & 2 & 2 & 6 \end{bmatrix}$$

Exercise 1.6.24

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 4 & -1 & 7 & 10 \\ 1 & 0 & 3 & 3 \\ 1 & -1 & -2 & 1 \end{bmatrix}$$

Exercise 1.6.25

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} 3 & 5 & -4 & 2 \\ 1 & 2 & -1 & 1 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

Exercise 1.6.26

Row reduce the following matrix to obtain the row-echelon form. Then continue to obtain the reduced row-echelon form.

$$\begin{bmatrix} -2 & 3 & -8 & 7 \\ 1 & -2 & 5 & -5 \\ 1 & -3 & 7 & -8 \end{bmatrix}$$

Exercise 1.6.27

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 \end{array}\right]$$

Exercise 1.6.28

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{array}\right]$$

Answer

The reduced row-echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{array}\right]$. Therefore, the solution is of the form $z = t, y = \frac{3}{4} + t\left(\frac{1}{4}\right), x = \frac{1}{2} - \frac{1}{2}t$ where $t \in \mathbb{R}$.

Exercise 1.6.29

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 4 & 2 \end{array}\right]$$

Answer

The reduced row-echelon form is $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & -4 & -1 \end{array}\right]$ and so the solution is $z = t, y = 4t, x = 2 - 4t$.

Exercise 1.6.30

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 & 2 \end{array}\right]$$

Answer

The reduced row-echelon form is $\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 9 & 3 \\ 0 & 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & -7 & -1 \\ 0 & 0 & 0 & 1 & 6 & 1 \end{array}\right]$ and so $x_5 = t, x_4 = 1 - 6t, x_3 = -1 + 7t, x_2 = 4t, x_1 = 3 - 9t$.

Exercise 1.6.31

Find the solution of the system whose augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 1 & -1 & 2 & 2 & 2 & 0 \end{array}\right]$$

Answer

The reduced row-echelon form is $\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. Therefore, let $x_5 = t, x_3 = s$. Then the other variables are given by $x_4 = -\frac{1}{2} - \frac{3}{2}t, x_2 = \frac{3}{2} - t\frac{1}{2}, , x_1 = \frac{5}{2} + \frac{1}{2}t - 2s$.

Exercise 1.6.32

Find the solution to the system of equations, $7x + 14y + 15z = 22$, $2x + 4y + 3z = 5$, and $3x + 6y + 10z = 13$.

Answer

Solution is: $[x = 1 - 2t, z = 1, y = t]$

Exercise 1.6.33

Find the solution to the system of equations, $3x - y + 4z = 6$, $y + 8z = 0$, and $-2x + y = -4$.

Answer

Solution is: $[x = 2 - 4t, y = -8t, z = t]$

? Exercise 1.6.34

Find the solution to the system of equations, $9x - 2y + 4z = -17$, $13x - 3y + 6z = -25$, and $-2x - z = 3$.

Answer

Solution is: $[x = -1, y = 2, z = -1]$

? Exercise 1.6.35

Find the solution to the system of equations, $65x + 84y + 16z = 546$, $81x + 105y + 20z = 682$, and $84x + 110y + 21z = 713$.

Answer

Solution is: $[x = 2, y = 4, z = 5]$

? Exercise 1.6.36

Find the solution to the system of equations, $8x + 2y + 3z = -3$, $8x + 3y + 3z = -1$, and $4x + y + 3z = -9$.

Answer

Solution is: $[x = 1, y = 2, z = -5]$

? Exercise 1.6.37

Find the solution to the system of equations, $-8x + 2y + 5z = 18$, $-8x + 3y + 5z = 13$, and $-4x + y + 5z = 19$.

Answer

Solution is: $[x = -1, y = -5, z = 4]$

? Exercise 1.6.38

Find the solution to the system of equations, $3x - y - 2z = 3$, $y - 4z = 0$, and $-2x + y = -2$.

Answer

Solution is: $[x = 2t + 1, y = 4t, z = t]$

? Exercise 1.6.39

Find the solution to the system of equations, $-9x + 15y = 66$, $-11x + 18y = 79$, $-x + y = 4$, and $z = 3$.

Answer

Solution is: $[x = 1, y = 5, z = 3]$

? Exercise 1.6.40

Find the solution to the system of equations, $-19x + 8y = -108$, $-71x + 30y = -404$, $-2x + y = -12$, $4x + z = 14$.

Answer

Solution is: $[x = 4, y = -4, z = -2]$

Exercise 1.6.41

Suppose a system of equations has fewer equations than variables and you have found a solution to this system of equations. Is it possible that your solution is the only one? Explain.

Answer

No. Consider $x+y+z=2$ and $x+y+z=1$.

Exercise 1.6.42

Suppose a system of linear equations has a 2×4 augmented matrix and the last column is a pivot column. Could the system of linear equations be consistent? Explain.

Answer

No. This would lead to $0=1$.

Exercise 1.6.43

Suppose the coefficient matrix of a system of n equations with n variables has the property that every column is a pivot column. Does it follow that the system of equations must have a solution? If so, must the solution be unique? Explain.

Answer

Yes. It has a unique solution.

Exercise 1.6.44

Suppose there is a unique solution to a system of linear equations. What must be true of the pivot columns in the augmented matrix?

Answer

The last column must not be a pivot column. The remaining columns must each be pivot columns.

Exercise 1.6.45

The steady state temperature, u, of a plate solves Laplace's equation, $\Delta u=0$. One way to approximate the solution is to divide the plate into a square mesh and require the temperature at each node to equal the average of the temperature at the four adjacent nodes. In the following picture, the numbers represent the observed temperature at the indicated nodes. Find the temperature at the interior nodes, indicated by x, y, z, and w. One of the equations is $z=0.05in\frac{1}{4}(10+0+w+x)$.

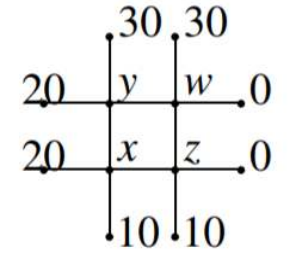


Figure 1.6.1

Answer

You need $\begin{array}{l} \frac{1}{4}(20+30+w+x)-y=0 \\ \frac{1}{4}(y+30+0+z)-w=0 \\ \frac{1}{4}(20+y+z+10)-x=0 \\ \frac{1}{4}(x+w+0+10)-z=0 \end{array}$, Solution is: $[w=15, x=15, y=20, z=10]$.

? Exercise 1.6.46

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & -16 & -1 & -5 \\ 1 & -4 & 0 & -1 \\ 1 & -4 & -1 & -2 \end{bmatrix}$$

? Exercise 1.6.47

Find the rank of the following matrix.

$$\begin{bmatrix} 3 & 6 & 5 & 12 \\ 1 & 2 & 2 & 5 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

? Exercise 1.6.48

Find the rank of the following matrix.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 3 \\ 1 & 4 & 1 & 0 & -8 \\ 1 & 4 & 0 & 1 & 2 \\ -1 & -4 & 0 & -1 & -2 \end{bmatrix}$$

? Exercise 1.6.49

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & -4 & 3 & -9 \\ 1 & -1 & 1 & -2 \\ 1 & -1 & 0 & -3 \end{bmatrix}$$

? Exercise 1.6.50

Find the rank of the following matrix.

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 7 \\ 1 & 0 & 0 & 1 & 7 \end{bmatrix}$$

? Exercise 1.6.51

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 15 & 29 \\ 1 & 4 & 8 \\ 1 & 3 & 5 \\ 3 & 9 & 15 \end{bmatrix}$$

Exercise 1.6.52

Find the rank of the following matrix.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 & -18 \\ 1 & 2 & 2 & -1 & -11 \\ -1 & -2 & -2 & 1 & 11 \end{bmatrix}$$

Exercise 1.6.53

Find the rank of the following matrix.

$$\begin{bmatrix} 1 & -2 & 0 & 3 & 11 \\ 1 & -2 & 0 & 4 & 15 \\ 1 & -2 & 0 & 3 & 11 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exercise 1.6.54

Find the rank of the following matrix.

$$\begin{bmatrix} -2 & -3 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ -3 & 0 & -3 \end{bmatrix}$$

Exercise 1.6.55

Find the rank of the following matrix.

$$\begin{bmatrix} 4 & 4 & 20 & -1 & 17 \\ 1 & 1 & 5 & 0 & 5 \\ 1 & 1 & 5 & -1 & 2 \\ 3 & 3 & 15 & -3 & 6 \end{bmatrix}$$

Exercise 1.6.56

Find the rank of the following matrix.

$$\begin{bmatrix} -1 & 3 & 4 & -3 & 8 \\ 1 & -3 & -4 & 2 & -5 \\ 1 & -3 & -4 & 1 & -2 \\ -2 & 6 & 8 & -2 & 4 \end{bmatrix}$$

Exercise 1.6.57

Suppose A is an $m \times n$ matrix. Explain why the rank of A is always no larger than $\min(m, n)$.

Answer

It is because you cannot have more than $\min(m, n)$ nonzero rows in the reduced row-echelon form. Recall that the number of pivot columns is the same as the number of nonzero rows from the description of this reduced row-echelon form.

? Exercise 1.6.58

State whether each of the following sets of data are possible for the matrix equation $AX = B$. If possible, describe the solution set. That is, tell whether there exists a unique solution, no solution or infinitely many solutions. Here, $[A|B]$ denotes the augmented matrix.

a. A is a 5×6 matrix, $rank(A) = 4$ and $rank[A|B] = 4$.
b. A is a 3×4 matrix, $rank(A) = 3$ and $rank[A|B] = 2$.
c. A is a 4×2 matrix, $rank(A) = 4$ and $rank[A|B] = 4$.
d. A is a 5×5 matrix, $rank(A) = 4$ and $rank[A|B] = 5$.
e. A is a 4×2 matrix, $rank(A) = 2$ and $rank[A|B] = 2$.

Answer

a. This says B is in the span of four of the columns. Thus the columns are not independent. Infinite solution set.
b. This surely can't happen. If you add in another column, the rank does not get smaller.
c. This says B is in the span of the columns and the columns must be independent. You can't have the rank equal 4 if you only have two columns.
d. This says B is not in the span of the columns. In this case, there is no solution to the system of equations represented by the augmented matrix.
e. In this case, there is a unique solution since the columns of A are independent.

? Exercise 1.6.59

Consider the system $-5x + 2y - z = 0$ and $-5x - 2y - z = 0$. Both equations equal zero and so $-5x + 2y - z = -5x - 2y - z$ which is equivalent to $y = 0$. Does it follow that x and z can equal anything? Notice that when $x = 1$, $z = -4$, and $y = 0$ are plugged in to the equations, the equations do not equal 0. Why?

Answer

These are not legitimate row operations. They do not preserve the solution set of the system.

? Exercise 1.6.60

Balance the following chemical reactions.

a. $KNO_3 + H_2CO_3 \rightarrow K_2CO_3 + HNO_3$
b. $AgI + Na_2S \rightarrow Ag_2S + NaI$
c. $Ba_3N_2 + H_2O \rightarrow Ba(OH)_2 + NH_3$
d. $CaCl_2 + Na_3PO_4 \rightarrow Ca_3(PO_4)_2 + NaCl$

? Exercise 1.6.61

In the section on dimensionless variables it was observed that $\rho V^2 AB$ has the units of force. Describe a systematic way to obtain such combinations of the variables which will yield something which has the units of force.

? Exercise 1.6.62

Consider the following diagram of four circuits.

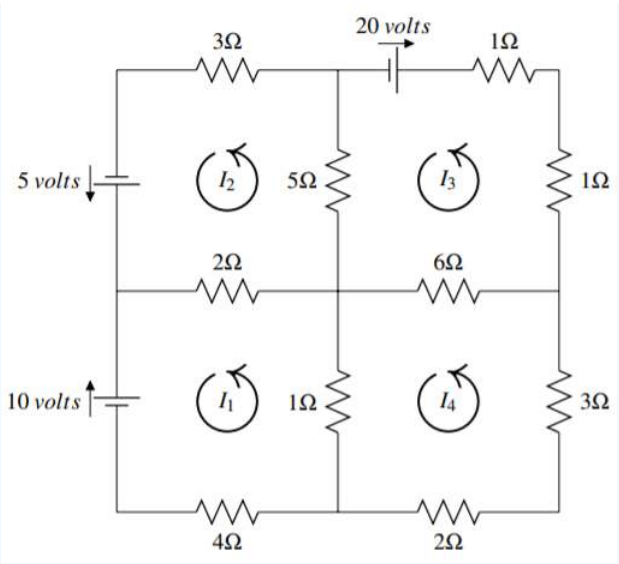


Figure 1.6.2

The current in amps in the four circuits is denoted by I_1, I_2, I_3, I_4 and it is understood that the motion is in the counter clockwise direction. If I_k ends up being negative, then it just means the current flows in the clockwise direction.

In the above diagram, the top left circuit should give the equation

$$2I_2 - 2I_1 + 5I_2 - 5I_3 + 3I_2 = 5$$

For the circuit on the lower left, you should have

$$4I_1 + I_1 - I_4 + 2I_1 - 2I_2 = -10$$

Write equations for each of the other two circuits and then give a solution to the resulting system of equations.

Answer

The other two equations are

$$\begin{aligned} 6I_3 - 6I_4 + I_3 + I_3 + 5I_3 - 5I_2 &= -20 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 &= 0 \end{aligned} \tag{1.6.1}$$

Then the system is

$$\begin{aligned} 2I_2 - 2I_1 + 5I_2 - 5I_3 + 3I_2 &= 5 \\ 4I_1 + I_1 - I_4 + 2I_1 - 2I_2 &= -10 \\ 6I_3 - 6I_4 + I_3 + I_3 + 5I_3 - 5I_2 &= -20 \\ 2I_4 + 3I_4 + 6I_4 - 6I_3 + I_4 - I_1 &= 0 \end{aligned}$$

The solution is:

$$\begin{aligned} I_1 &= -\frac{750}{373} \\ I_2 &= -\frac{1421}{1119} \\ I_3 &= -\frac{3061}{1119} \\ I_4 &= -\frac{1718}{1119} \end{aligned} \tag{1.6.2}$$

? Exercise 1.6.63

Consider the following diagram of three circuits.

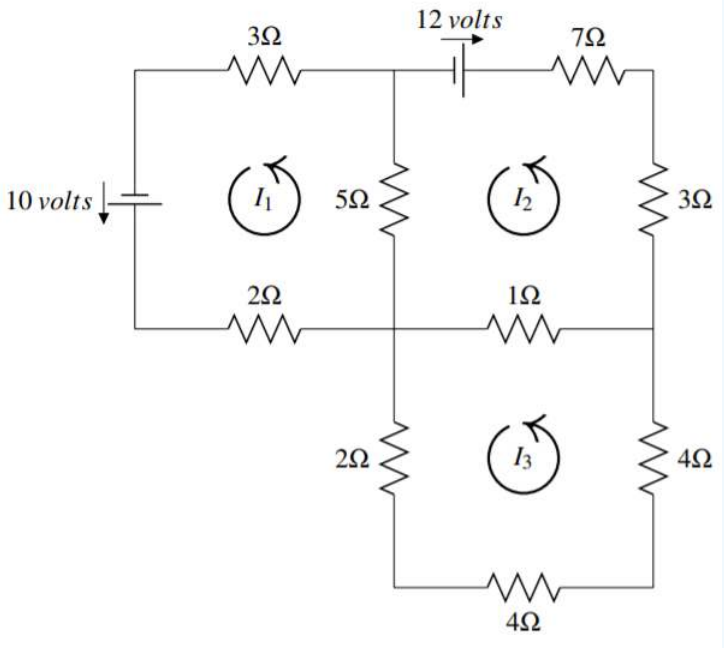


Figure **1.6.3**

The current in amps in the four circuits is denoted by I_1, I_2, I_3 and it is understood that the motion is in the counter clockwise direction. If I_k ends up being negative, then it just means the current flows in the clockwise direction.

Find I_1, I_2, I_3.

Answer

You have

$$\begin{aligned} 2I_1 + 5I_1 + 3I_1 - 5I_2 &= 10 \\ I_2 - I_3 + 3I_2 + 7I_2 + 5I_2 - 5I_1 &= -12 \\ 2I_3 + 4I_3 + 4I_3 + I_3 - I_2 &= 0 \end{aligned} \tag{1.6.3}$$

Simplifying this yields

$$\begin{aligned} 10I_1 - 5I_2 &= 10 \\ -5I_1 + 16I_2 - I_3 &= -12 \\ -I_2 + 11I_3 &= 0 \end{aligned} \tag{1.6.4}$$

The solution is given by

$$I_1 = \frac{218}{295}, I_2 = -\frac{154}{295}, I_3 = -\frac{14}{295}$$

This page titled 1.6: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **1.9: Exercises** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

CHAPTER OVERVIEW

2: Matrices

2.1: Matrix Arithmetic
2.2: Multiplication of Matrices
2.3: The ijth Entry of a Product
2.4: Properties of Matrix Multiplication
2.5: The Transpose
2.6: The Identity and Inverses
2.7: Finding the Inverse of a Matrix
2.8: Elementary Matrices
2.9: More on Matrix Inverses
2.10: LU Factorization
2.11: Exercises

This page titled 2: Matrices is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

2.1: Matrix Arithmetic

Outcomes

A. Perform the matrix operations of matrix addition, scalar multiplication, transposition and matrix multiplication. Identify when these operations are not defined. Represent these operations in terms of the entries of a matrix.
B. Prove algebraic properties for matrix addition, scalar multiplication, transposition, and matrix multiplication. Apply these properties to manipulate an algebraic expression involving matrices.
C. Compute the inverse of a matrix using row operations, and prove identities involving matrix inverses.
D. Solve a linear system using matrix algebra.
E. Use multiplication by an elementary matrix to apply row operations.
F. Write a matrix as a product of elementary matrices.

You have now solved systems of equations by writing them in terms of an augmented matrix and then doing row operations on this augmented matrix. It turns out that matrices are important not only for systems of equations but also in many applications.

Recall that a **matrix** is a rectangular array of numbers. Several of them are referred to as **matrices**. For example, here is a matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} \tag{2.1.1}$$

Recall that the size or dimension of a matrix is defined as $m \times n$ where m is the number of rows and n is the number of columns. The above matrix is a 3×4 matrix because there are three rows and four columns. You can remember the columns are like columns in a Greek temple. They stand upright while the rows lay flat like rows made by a tractor in a plowed field.

When specifying the size of a matrix, you always list the number of rows before the number of columns.You might remember that you always list the rows before the columns by using the phrase **Row**man **C**atholic.

Consider the following definition.

Definition 2.1.1: Square Matrix

A matrix A which has size $n \times n$ is called a **square matrix** . In other words, A is a square matrix if it has the same number of rows and columns.

There is some notation specific to matrices which we now introduce. We denote the columns of a matrix A by A_j as follows

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$$

Therefore, A_j is the jth column of A, when counted from left to right.

The individual elements of the matrix are called **entries** or **components** of A. Elements of the matrix are identified according to their position. The (i,j)**-entry** of a matrix is the entry in the ith row and j^{th} column. For example, in the matrix (2.1.1) above, 8 is in position $(2,3)$ (and is called the $(2,3)$-entry) because it is in the second row and the third column.

In order to remember which matrix we are speaking of, we will denote the entry in the ith row and the jth column of matrix A by a_{ij}. Then, we can write A in terms of its entries, as $A = [a_{ij}]$. Using this notation on the matrix in (2.1.1), $a_{23} = 8, a_{32} = -9, a_{12} = 2$, etc.

There are various operations which are done on matrices of appropriate sizes. Matrices can be added to and subtracted from other matrices, multiplied by a scalar, and multiplied by other matrices. We will never divide a matrix by another matrix, but we will see later how matrix inverses play a similar role.

In doing arithmetic with matrices, we often define the action by what happens in terms of the entries (or components) of the matrices. Before looking at these operations in depth, consider a few general definitions.

Definition 2.1.2: The Zero Matrix

The $m \times n$ **zero matrix** is the $m \times n$ matrix having every entry equal to zero. It is denoted by $\mathbf{0}$.

One possible zero matrix is shown in the following example.

Example 2.1.1: The Zero Matrix

The 2×3 zero matrix is $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Note there is a 2×3 zero matrix, a 3×4 zero matrix, etc. In fact there is a zero matrix for every size!

Definition 2.1.3: Equality of Matrices

Let A and B be two $m \times n$ matrices. Then $A = B$ means that for $A = [a_{ij}]$ and $B = [b_{ij}]$, $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

In other words, two matrices are equal exactly when they are the same size and the corresponding entries are identical. Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

because they are different sizes. Also,

$$\begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

because, although they are the same size, their corresponding entries are not identical.

In the following section, we explore addition of matrices.

Addition of Matrices

When adding matrices, all matrices in the sum need have the same size. For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & 4 & 8 \\ 2 & 8 & 5 \end{bmatrix}$$

cannot be added, as one has size 3×2 while the other has size 2×3.

However, the addition

$$\begin{bmatrix} 4 & 6 & 3 \\ 5 & 0 & 4 \\ 11 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 0 \\ 4 & -4 & 14 \\ 1 & 2 & 6 \end{bmatrix}$$

is possible.

The formal definition is as follows.

Definition 2.1.4: Addition of Matrices

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then $A + B = C$ where C is the $m \times n$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{ij} + b_{ij}$$

This definition tells us that when adding matrices, we simply add corresponding entries of the matrices. This is demonstrated in the next example.

Example 2.1.2: Addition of Matrices of Same Size

Add the following matrices, if possible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix}$$

Solution

Notice that both A and B are of size 2×3. Since A and B are of the same size, the addition is possible. Using Definition 2.1.4, the addition is done as follows.

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 2 & 3 \\ -6 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+2 & 3+3 \\ 1+-6 & 0+2 & 4+1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 6 \\ -5 & 2 & 5 \end{bmatrix}$$

Addition of matrices obeys very much the same properties as normal addition with numbers. Note that when we write for example $A + B$ then we assume that both matrices are of equal size so that the operation is indeed possible.

Proposition 2.1.1: Properties of Matrix Addition

Let A, B and C be matrices. Then, the following properties hold.

- Commutative Law of Addition

$$A + B = B + A \tag{2.1.2}$$

- Associative Law of Addition

$$(A + B) + C = A + (B + C) \tag{2.1.3}$$

- Existence of an Additive Identity

$$\begin{gathered} \text{There exists a zero matrix } 0 \text{ such that} \\ A + 0 = A \end{gathered} \tag{2.1.4}$$

- Existence of an Additive Inverse

$$\begin{gathered} \text{There exists a matrix } -A \text{ such that} \\ A + (-A) = 0 \end{gathered} \tag{2.1.5}$$

Proof

Consider the Commutative Law of Addition given in (2.1.2). Let A, B, C, and D be matrices such that $A + B = C$ and $B + A = D$. We want to show that $D = C$. To do so, we will use the definition of matrix addition given in Definition 2.1.4. Now,

$$c_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij} = d_{ij}$$

Therefore, $C = D$ because the ij^{th} entries are the same for all i and j. Note that the conclusion follows from the commutative law of addition of numbers, which says that if a and b are two numbers, then $a + b = b + a$. The proof of the other results are similar, and are left as an exercise.

We call the zero matrix in (2.1.4) the **additive identity**. Similarly, we call the matrix $-A$ in (2.1.5) the **additive inverse**. $-A$ is defined to equal $(-1)A = [-a_{ij}]$. In other words, every entry of A is multiplied by -1.

In the next section we will study scalar multiplication in more depth to understand what is meant by $(-1)A$.

Scalar Multiplication of Matrices

Recall that we use the word *scalar* when referring to numbers. Therefore, *scalar multiplication of a matrix* is the multiplication of a matrix by a number. To illustrate this concept, consider the following example in which a matrix is multiplied by the scalar **3**.

$$3\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 2 & 8 & 7 \\ 6 & -9 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 & 12 \\ 15 & 6 & 24 & 21 \\ 18 & -27 & 3 & 6 \end{bmatrix}$$

The new matrix is obtained by multiplying every entry of the original matrix by the given scalar.

The formal definition of scalar multiplication is as follows.

Definition 2.1.5: Scalar Multiplication of Matrices

If $A = [a_{ij}]$ and k is a scalar, then $kA = [ka_{ij}]$.

Consider the following example.

Example 2.1.3: Effect of Multiplication by a Scalar

Find the result of multiplying the following matrix A by 7.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$$

Solution

By Definition 2.1.5, we multiply each element of A by 7. Therefore,

$$7A = 7\begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(0) \\ 7(1) & 7(-4) \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 7 & -28 \end{bmatrix}$$

Similarly to addition of matrices, there are several properties of scalar multiplication which hold.

Proposition 2.1.2: Properties of Scalar Multiplication

Let A, B be matrices, and k, p be scalars. Then, the following properties hold.

- Distributive Law over Matrix Addition

$$k(A+B) = kA + kB$$

- Distributive Law over Scalar Addition

$$(k+p)A = kA + pA$$

- Associative Law for Scalar Multiplication

$$k(pA) = (kp)A$$

- Rule for Multiplication by 1

$$1A = A$$

Proof

The proof of this proposition is similar to the proof of Proposition 2.1.1 and is left an exercise to the reader.

This page titled 2.1: Matrix Arithmetic is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.1: Matrix Arithmetic** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

https://math.libretexts.org/@go/page/117908

2.2: Multiplication of Matrices

The next important matrix operation we will explore is multiplication of matrices. The operation of matrix multiplication is one of the most important and useful of the matrix operations. Throughout this section, we will also demonstrate how matrix multiplication relates to linear systems of equations.

First, we provide a formal definition of row and column vectors.

Definition 2.2.1: Row and Column Vectors

Matrices of size $n \times 1$ or $1 \times n$ are called **vectors**. If X is such a matrix, then we write x_i to denote the entry of X in the ith row of a column matrix, or the ith column of a row matrix.

The $n \times 1$ matrix

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called a **column vector.** The $1 \times n$ matrix

$$X = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}$$

is called a **row vector**.

We may simply use the term **vector** throughout this text to refer to either a column or row vector. If we do so, the context will make it clear which we are referring to.

In this chapter, we will again use the notion of linear combination of vectors as in Definition 9.2.2. In this context, a linear combination is a sum consisting of vectors multiplied by scalars. For example,

$$\begin{bmatrix} 50 \\ 122 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9\begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

is a linear combination of three vectors.

It turns out that we can express any system of linear equations as a linear combination of vectors. In fact, the vectors that we will use are just the columns of the corresponding augmented matrix!

Definition 2.2.2: The Vector Form of a System of Linear Equations

Suppose we have a system of equations given by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

We can express this system in **vector form** which is as follows:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Notice that each vector used here is one column from the corresponding augmented matrix. There is one vector for each variable in the system, along with the constant vector.

The first important form of matrix multiplication is multiplying a matrix by a vector. Consider the product given by

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

We will soon see that this equals

$$7\begin{bmatrix} 1 \\ 4 \end{bmatrix} + 8\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 9\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix}$$

In general terms,

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ &= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} \end{aligned} \tag{2.2.1}$$

Thus you take x_1 times the first column, add to x_2 times the second column, and finally x_3 times the third column. The above sum is a linear combination of the columns of the matrix. When you multiply a matrix on the left by a vector on the right, the numbers making up the vector are just the scalars to be used in the linear combination of the columns as illustrated above.

Here is the formal definition of how to multiply an $m \times n$ matrix by an $n \times 1$ column vector.

Definition 2.2.3: Multiplication of Vector by Matrix

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let X be an $n \times 1$ matrix given by

$$A = [A_1 \cdots A_n], X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then the product AX is the $m \times 1$ column vector which equals the following linear combination of the columns of A:

$$x_1A_1 + x_2A_2 + \cdots + x_nA_n = \sum_{j=1}^{n} x_jA_j$$

If we write the columns of A in terms of their entries, they are of the form

$$A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

Then, we can write the product AX as

$$AX = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Note that multiplication of an $m \times n$ matrix and an $n \times 1$ vector produces an $m \times 1$ vector.

Here is an example.

Example 2.2.1: A Vector Multiplied by a Matrix

Compute the product AX for

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & 2 & 1 & -2 \\ 2 & 1 & 4 & 1 \end{bmatrix}, X = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Solution

We will use Definition 2.2.3 to compute the product. Therefore, we compute the product AX as follows.

$$\begin{aligned} & 1\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + 1\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ & = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ & = \begin{bmatrix} 8 \\ 2 \\ 5 \end{bmatrix} \end{aligned} \tag{2.2.2}$$

Using the above operation, we can also write a system of linear equations in **matrix form**. In this form, we express the system as a matrix multiplied by a vector. Consider the following definition.

Definition 2.2.4: The Matrix Form of a System of Linear Equations

Suppose we have a system of equations given by

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Then we can express this system in **matrix form** as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The expression $AX = B$ is also known as the **Matrix Form** of the corresponding system of linear equations. The matrix A is simply the coefficient matrix of the system, the vector X is the column vector constructed from the variables of the system, and finally the vector B is the column vector constructed from the constants of the system. It is important to note that any system of linear equations can be written in this form.

Notice that if we write a homogeneous system of equations in matrix form, it would have the form $AX = 0$, for the zero vector 0.

You can see from this definition that a vector

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

will satisfy the equation $AX = B$ only when the entries $x_1, x_2, \cdots, x_n$ of the vector X are solutions to the original system.

Now that we have examined how to multiply a matrix by a vector, we wish to consider the case where we multiply two matrices of more general sizes, although these sizes still need to be appropriate as we will see. For example, in Example 2.2.1, we multiplied a 3×4 matrix by a 4×1 vector. We want to investigate how to multiply other sizes of matrices.

We have not yet given any conditions on when matrix multiplication is possible! For matrices A and B, in order to form the product AB, the number of columns of A must equal the number of rows of B. Consider a product AB where A has size $m \times n$ and B has size $n \times p$. Then, the product in terms of size of matrices is given by

$$(m \times \overbrace{n)\,(n}^{\text{these must match!}} \times p\ ) = m \times p$$

Note the two outside numbers give the size of the product. One of the most important rules regarding matrix multiplication is the following. If the two middle numbers don't match, you can't multiply the matrices!

When the number of columns of A equals the number of rows of B the two matrices are said to be **conformable** and the product AB is obtained as follows.

Definition 2.2.5: Multiplication of Two Matrices

Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix of the form

$$B = [B_1 \cdots B_p]$$

where $B_1, \ldots, B_p$ are the $n \times 1$ columns of B. Then the $m \times p$ matrix AB is defined as follows:

$$AB = A[B_1 \cdots B_p] = [(AB)_1 \cdots (AB)_p]$$

where $(AB)_k$ is an $m \times 1$ matrix or column vector which gives the kth column of AB.

Consider the following example.

Example 2.2.2: Multiplying Two Matrices

Find AB if possible.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}$$

Solution

The first thing you need to verify when calculating a product is whether the multiplication is possible. The first matrix has size 2×3 and the second matrix has size 3×3. The inside numbers are equal, so A and B are conformable matrices. According to the above discussion AB will be a 2×3 matrix. Definition 2.2.5 gives us a way to calculate each column of AB, as follows.

$$\begin{bmatrix} \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}}^{\text{First column}}, & \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}}^{\text{Second column}}, & \overbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}^{\text{Third column}} \end{bmatrix}$$

You know how to multiply a matrix times a vector, using Definition 2.2.3 for each of the three columns. Thus

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 3 \\ -2 & 7 & 3 \end{bmatrix}$$

Since vectors are simply $n \times 1$ or $1 \times m$ matrices, we can also multiply a vector by another vector.

Example 2.2.3: Vector Times Vector Multiplication

Multiply if possible $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix}$.

Solution

In this case we are multiplying a matrix of size 3×1 by a matrix of size 1×4. The inside numbers match so the product is defined. Note that the product will be a matrix of size 3×4. Using Definition 2.2.5, we can compute this product as follows

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}}^{\text{First column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix}}^{\text{Second column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}}^{\text{Third column}}, \overbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}}^{\text{Fourth column}} \end{bmatrix}$$

You can use Definition 2.2.3 to verify that this product is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 4 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

Example 2.2.4: A Multiplication Which is Not Defined

Find BA if possible.

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ -2 & 1 & 1 \end{bmatrix}, A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Solution

First check if it is possible. This product is of the form $(3 \times 3)(2 \times 3)$. The inside numbers do not match and so you can't do this multiplication.

In this case, we say that the multiplication is not defined. Notice that these are the same matrices which we used in Example 2.2.2. In this example, we tried to calculate BA instead of AB. This demonstrates another property of matrix multiplication. While the product AB maybe be defined, we cannot assume that the product BA will be possible. Therefore, it is important to always check that the product is defined before carrying out any calculations.

Earlier, we defined the zero matrix 0 to be the matrix (of appropriate size) containing zeros in all entries. Consider the following example for multiplication by the zero matrix.

Example 2.2.5: Multiplication by the Zero Matrix

Compute the product $A0$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and the 2×2 zero matrix given by

$$0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solution

In this product, we compute

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Hence, $A0 = 0$.

Notice that we could also multiply A by the 2×1 zero vector given by $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The result would be the 2×1 zero vector. Therefore, it is always the case that $A0 = 0$, for an appropriately sized zero matrix or vector.

This page titled 2.2: Multiplication of Matrices is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.2: Multiplication of Matrices** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

2.3: The ijth Entry of a Product

In previous sections, we used the entries of a matrix to describe the action of matrix addition and scalar multiplication. We can also study matrix multiplication using the entries of matrices.

What is the ij^{th} entry of AB? It is the entry in the ith row and the jth column of the product AB.

Now if A is $m \times n$ and B is $n \times p$, then we know that the product AB has the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

The jth column of AB is of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

which is an $m \times 1$ column vector. It is calculated by

$$b_{1j} \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + b_{2j} \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + b_{nj} \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Therefore, the ij^{th} entry is the entry in row i of this vector. This is computed by

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

The following is the formal definition for the ij^{th} entry of a product of matrices.

Definition 2.3.1: The ij^{th} Entry of a Product

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let $B = [b_{ij}]$ be an $n \times p$ matrix. Then AB is an $m \times p$ matrix and the (i, j)-entry of AB is defined as

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Another way to write this is

$$(AB)_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

In other words, to find the (i, j)-entry of the product AB, or $(AB)_{ij}$, you multiply the ith row of A, on the left by the jth column of B. To express AB in terms of its entries, we write $AB = [(AB)_{ij}]$.

Consider the following example.

Example 2.3.1: The Entries of a Product

Compute AB if possible. If it is, find the $(3,2)$-entry of AB using Definition 2.3.1.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \end{bmatrix}$$

Solution

First check if the product is possible. It is of the form $(3 \times 2)(2 \times 3)$ and since the inside numbers match, it is possible to do the multiplication. The result should be a 3×3 matrix. We can first compute AB:

$$\left[\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right]$$

where the commas separate the columns in the resulting product. Thus the above product equals

$$\begin{bmatrix} 16 & 15 & 5 \\ 13 & 15 & 5 \\ 46 & 42 & 14 \end{bmatrix}$$

which is a 3×3 matrix as desired. Thus, the $(3,2)$-entry equals 42.

Now using Definition 2.3.1, we can find that the $(3,2)$-entry equals

$$\begin{aligned} \sum_{k=1}^{2} a_{3k} b_{k2} &= a_{31} b_{12} + a_{32} b_{22} \\ &= 2 \times 3 + 6 \times 6 = 42 \end{aligned} \tag{2.3.1}$$

Consulting our result for AB above, this is correct!

You may wish to use this method to verify that the rest of the entries in AB are correct.

Here is another example.

Example 2.3.2: Finding the Entries of a Product

Determine if the product AB is defined. If it is, find the $(2,1)$-entry of the product.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 7 & 6 & 2 \\ 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}$$

Solution

This product is of the form $(3 \times 3)(3 \times 2)$. The middle numbers match so the matrices are conformable and it is possible to compute the product.

We want to find the $(2,1)$-entry of AB, that is, the entry in the second row and first column of the product. We will use Definition 2.3.1, which states

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

In this case, $n = 3$, $i = 2$ and $j = 1$. Hence the $(2,1)$-entry is found by computing

$$(AB)_{21} = \sum_{k=1}^{3} a_{2k} b_{k1} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}$$

Substituting in the appropriate values, this product becomes

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = \begin{bmatrix} 7 & 6 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 1 \times 7 + 3 \times 6 + 2 \times 2 = 29$$

Hence, $(AB)_{21} = 29$.

You should take a moment to find a few other entries of AB. You can multiply the matrices to check that your answers are correct. The product AB is given by

$$AB = \begin{bmatrix} 13 & 13 \\ 29 & 32 \\ 0 & 0 \end{bmatrix}$$

This page titled 2.3: The ijth Entry of a Product is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.3: The ijth Entry of a Product** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

2.4: Properties of Matrix Multiplication

As pointed out above, it is sometimes possible to multiply matrices in one order but not in the other order. However, even if both AB and BA are defined, they may not be equal.

Example 2.4.1: Matrix Multiplication is Not Commutative

Compare the products AB and BA, for matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Solution

First, notice that A and B are both of size 2×2. Therefore, both products AB and BA are defined. The first product, AB is

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

The second product, BA is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Therefore, $AB \neq BA$.

This example illustrates that you cannot assume $AB = BA$ even when multiplication is defined in both orders. If for some matrices A and B it is true that $AB = BA$, then we say that A and B **commute**. This is one important property of matrix multiplication.

The following are other important properties of matrix multiplication. Notice that these properties hold only when the size of matrices are such that the products are defined.

Proposition 2.4.1: Properties of Matrix Multiplication

The following hold for matrices A, B, and C and for scalars r and s,

$$A(rB + sC) = r(AB) + s(AC) \tag{2.4.1}$$

$$(B + C)A = BA + CA \tag{2.4.2}$$

$$A(BC) = (AB)C \tag{2.4.3}$$

Proof

First we will prove (2.4.1). We will use Definition 2.3.1 and prove this statement using the ij^{th} entries of a matrix. Therefore,

$$\begin{aligned} (A(rB + sC))_{ij} &= \sum_k a_{ik}(rB + sC)_{kj} \\ &= \sum_k a_{ik}(rb_{kj} + sc_{kj}) \\ &= r\sum_k a_{ik}b_{kj} + s\sum_k a_{ik}c_{kj} \\ &= r(AB)_{ij} + s(AC)_{ij} \\ &= (r(AB) + s(AC))_{ij} \end{aligned} \tag{2.4.4}$$

Thus $A(rB + sC) = r(AB) + s(AC)$ as claimed.

The proof of Equation (2.4.2) follows the same pattern and is left as an exercise.

Statement Equation (2.4.3) is the **associative law of multiplication**. Using Definition 2.3.1,

$$\begin{aligned}(A(BC))_{ij} &= \sum_k a_{ik}(BC)_{kj} \\ &= \sum_k a_{ik} \sum_l b_{kl} c_{lj} \\ &= \sum_l (AB)_{il} c_{lj} = ((AB)C)_{ij}.\end{aligned}$$

This proves (2.4.3).

This page titled 2.4: Properties of Matrix Multiplication is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.4: Properties of Matrix Multiplication** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

2.5: The Transpose

Another important operation on matrices is that of taking the **transpose**. For a matrix A, we denote the **transpose** of A by A^T. Before formally defining the transpose, we explore this operation on the following matrix.

$$\begin{bmatrix} 1 & 4 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 6 \end{bmatrix}$$

What happened? The first column became the first row and the second column became the second row. Thus the 3×2 matrix became a 2×3 matrix. The number 4 was in the first row and the second column and it ended up in the second row and first column.

The definition of the transpose is as follows.

Definition 2.5.1: The Transpose of a Matrix

Let A be an $m \times n$ matrix. Then A^T, the **transpose** of A, denotes the $n \times m$ matrix given by

$$A^T = [a_{ij}]^T = [a_{ji}]$$

The (i,j)-entry of A becomes the (j,i)-entry of A^T.

Consider the following example.

Example 2.5.1: The Transpose of a Matrix

Calculate A^T for the following matrix

$$A = \begin{bmatrix} 1 & 2 & -6 \\ 3 & 5 & 4 \end{bmatrix}$$

Solution

By Definition 2.5.1, we know that for $A = [a_{ij}]$, $A^T = [a_{ji}]$. In other words, we switch the row and column location of each entry. The $(1,2)$-entry becomes the $(2,1)$-entry.

Thus,

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ -6 & 4 \end{bmatrix}$$

Notice that A is a 2×3 matrix, while A^T is a 3×2 matrix.

The transpose of a matrix has the following important properties.

Lemma 2.5.1: Properties of the Transpose of a Matrix

Let A be an $m \times n$ matrix, B an $n \times p$ matrix, and r and s scalars. Then

1. $$(A^T)^T = A$$
2. $$(AB)^T = B^T A^T$$
3. $$(rA + sB)^T = rA^T + sB^T$$

Proof

First we prove 2. From Definition 2.5.1,

$$(AB)^T = [(AB)_{ij}]^T = [(AB)_{ji}] = \sum_k a_{jk}b_{ki} = \sum_k b_{ki}a_{jk}$$
$$= \sum_k [b_{ik}]^T [a_{kj}]^T = [b_{ij}]^T [a_{ij}]^T = B^T A^T \tag{2.5.1}$$

The proof of Formula 3 is left as an exercise.

The transpose of a matrix is related to other important topics. Consider the following definition.

Definition 2.5.2: Symmetric and Skew Symmetric Matrices

An $n \times n$ matrix A is said to be **symmetric** if $A = A^T$. It is said to be **skew symmetric** if $A = -A^T$.

We will explore these definitions in the following examples.

Example 2.5.2: Symmetric Matrices

Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Use Definition 2.5.2 to show that A is symmetric.

Solution

By Definition 2.5.2, we need to show that $A = A^T$. Now, using Definition 2.5.1,

$$A^T = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & -3 \\ 3 & -3 & 7 \end{bmatrix}$$

Hence, $A = A^T$, so A is symmetric.

Example 2.5.3: A Skew Symmetric Matrix

Let

$$A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$$

Show that A is skew symmetric.

Solution

By Definition 2.5.2,

$$A^T = \begin{bmatrix} 0 & -1 & -3 \\ 1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

You can see that each entry of A^T is equal to -1 times the same entry of A. Hence, $A^T = -A$ and so by Definition 2.5.2, A is skew symmetric.

This page titled 2.5: The Transpose is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.5: The Transpose** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

2.6: The Identity and Inverses

There is a special matrix, denoted I, which is called to as the **identity matrix**. The identity matrix is always a square matrix, and it has the property that there are ones down the main diagonal and zeroes elsewhere. Here are some identity matrices of various sizes.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first is the 1×1 identity matrix, the second is the 2×2 identity matrix, and so on. By extension, you can likely see what the $n \times n$ identity matrix would be. When it is necessary to distinguish which size of identity matrix is being discussed, we will use the notation I_n for the $n \times n$ identity matrix.

The identity matrix is so important that there is a special symbol to denote the ij^{th} entry of the identity matrix. This symbol is given by $I_{ij} = \delta_{ij}$ where δ_{ij} is the **Kronecker symbol** defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

I_n is called the **identity matrix** because it is a **multiplicative identity** in the following sense.

Lemma 2.6.1: Multiplication by the Identity Matrix

Suppose A is an $m \times n$ matrix and I_n is the $n \times n$ identity matrix**.** Then $AI_n = A$**.** If I_m is the $m \times m$ identity matrix, it also follows that $I_m A = A$.

Proof

The (i, j)-entry of AI_n is given by:

$$\sum_k a_{ik}\delta_{kj} = a_{ij}$$

and so $AI_n = A$**.** The other case is left as an exercise for you.

We now define the matrix operation which in some ways plays the role of division.

Definition 2.6.1: The Inverse of a Matrix

A square $n \times n$ matrix A is said to have an **inverse** A^{-1} if and only if

$$AA^{-1} = A^{-1}A = I_n$$

In this case, the matrix A is called **invertible**.

Such a matrix A^{-1} will have the same size as the matrix A. It is very important to observe that the inverse of a matrix, if it exists, is unique. Another way to think of this is that if it acts like the inverse, then it **is** the inverse.

Theorem 2.6.1: Uniqueness of Inverse

Suppose A is an $n \times n$ matrix such that an inverse A^{-1} exists. Then there is only one such inverse matrix. That is, given any matrix B such that $AB = BA = I$, $B = A^{-1}$.

Proof

In this proof, it is assumed that I is the $n \times n$ identity matrix. Let A, B be $n \times n$ matrices such that A^{-1} exists and $AB = BA = I$. We want to show that $A^{-1} = B$. Now using properties we have seen, we get:

$$A^{-1} = A^{-1}I = A^{-1}(AB) = \left(A^{-1}A\right)B = IB = B$$

Hence, $A^{-1} = B$ which tells us that the inverse is unique.

The next example demonstrates how to check the inverse of a matrix.

Example 2.6.1: Verifying the Inverse of a Matrix

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Show $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ is the inverse of A.

Solution

To check this, multiply

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

and

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

showing that this matrix is indeed the inverse of A.

Unlike ordinary multiplication of numbers, it can happen that $A \neq 0$ but A may fail to have an inverse. This is illustrated in the following example.

Example 2.6.2: A Nonzero Matrix With No Inverse

Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Show that A does not have an inverse.

Solution

One might think A would have an inverse because it does not equal zero. However, note that

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If A^{-1} existed, we would have the following

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= A^{-1}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \\ &= A^{-1}\left(A\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) \\ &= \left(A^{-1}A\right)\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= I\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{aligned} \tag{2.6.1}$$

This says that

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which is impossible! Therefore, A does not have an inverse.

In the next section, we will explore how to find the inverse of a matrix, if it exists.

This page titled 2.6: The Identity and Inverses is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.6: The Identity and Inverses** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

2.7: Finding the Inverse of a Matrix

In Example 2.6.1, we were given A^{-1} and asked to verify that this matrix was in fact the inverse of A. In this section, we explore how to find A^{-1}.

Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

as in Example 2.6.1. In order to find A^{-1}, we need to find a matrix $\begin{bmatrix} x & z \\ y & w \end{bmatrix}$ such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We can multiply these two matrices, and see that in order for this equation to be true, we must find the solution to the systems of equations,

$$\begin{aligned} x + y &= 1 \\ x + 2y &= 0 \end{aligned}$$

and

$$\begin{aligned} z + w &= 0 \\ z + 2w &= 1 \end{aligned}$$

Writing the augmented matrix for these two systems gives

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 0 \end{array}\right]$$

for the first system and

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 1 \end{array}\right] \tag{2.7.1}$$

for the second.

Let's solve the first system. Take -1 times the first row and add to the second to get

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \end{array}\right]$$

Now take -1 times the second row and add to the first to get

$$\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right]$$

Writing in terms of variables, this says $x = 2$ and $y = -1$.

Now solve the second system, (2.7.1) to find z and w. You will find that $z = -1$ and $w = 1$.

If we take the values found for x, y, z, and w and put them into our inverse matrix, we see that the inverse is

$$A^{-1} = \begin{bmatrix} x & z \\ y & w \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

After taking the time to solve the second system, you may have noticed that exactly the same row operations were used to solve both systems. In each case, the end result was something of the form $[I|X]$ where I is the identity and X gave a column of the inverse. In the above,

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

the first column of the inverse was obtained by solving the first system and then the second column

$$\begin{bmatrix} z \\ w \end{bmatrix}$$

To simplify this procedure, we could have solved both systems at once! To do so, we could have written

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right]$$

and row reduced until we obtained

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right]$$

and read off the inverse as the 2×2 matrix on the right side.

This exploration motivates the following important algorithm.

Algorithm 2.7.1: Matrix Inverse Algorithm

Suppose A is an $n \times n$ matrix. To find A^{-1} if it exists, form the augmented $n \times 2n$ matrix

$$[A|I]$$

If possible do row operations until you obtain an $n \times 2n$ matrix of the form

$$[I|B]$$

When this has been done, $B = A^{-1}$. In this case, we say that A is **invertible**. If it is impossible to row reduce to a matrix of the form $[I|B]$, then A has no inverse.

This algorithm shows how to find the inverse if it exists. It will also tell you if A does not have an inverse.

Consider the following example.

Example 2.7.1: Finding the Inverse

Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$. Find A^{-1} if it exists.

Solution

Set up the augmented matrix

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Now we row reduce, with the goal of obtaining the 3×3 identity matrix on the left hand side. First, take -1 times the first row and add to the second followed by -3 times the first row added to the third row. This yields

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right]$$

Then take 5 times the second row and add to -2 times the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right]$$

Next take the third row and add to -7 times the first row. This yields

$$\left[\begin{array}{ccc|ccc} -7 & -14 & 0 & -6 & 5 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right]$$

Now take $-\frac{7}{5}$ times the second row and add to the first row.

$$\left[\begin{array}{ccc|ccc} -7 & 0 & 0 & 1 & -2 & -2 \\ 0 & -10 & 0 & -5 & 5 & 0 \\ 0 & 0 & 14 & 1 & 5 & -2 \end{array}\right]$$

Finally divide the first row by -7, the second row by -10 and the third row by 14 which yields

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right]$$

Notice that the left hand side of this matrix is now the 3×3 identity matrix I_3. Therefore, the inverse is the 3×3 matrix on the right hand side, given by

$$\left[\begin{array}{ccc} -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right]$$

It may happen that through this algorithm, you discover that the left hand side cannot be row reduced to the identity matrix. Consider the following example of this situation.

Example 2.7.2: A Matrix Which Has No Inverse

Let $A = \left[\begin{array}{ccc} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{array}\right]$. Find A^{-1} if it exists.

Solution

Write the augmented matrix $[A|I]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

and proceed to do row operations attempting to obtain $[I|A^{-1}]$. Take -1 times the first row and add to the second. Then take -2 times the first row and add to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 1 \end{array}\right]$$

Next add -1 times the second row to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

At this point, you can see there will be no way to obtain I on the left side of this augmented matrix. Hence, there is no way to complete this algorithm, and therefore the inverse of A does not exist. In this case, we say that A is not invertible.

If the algorithm provides an inverse for the original matrix, it is always possible to check your answer. To do so, use the method demonstrated in Example 2.6.1. Check that the products AA^{-1} and $A^{-1}A$ both equal the identity matrix. Through this method, you can always be sure that you have calculated A^{-1} properly!

One way in which the inverse of a matrix is useful is to find the solution of a system of linear equations. Recall from Definition 2.2.4 that we can write a system of equations in matrix form, which is of the form $AX = B$. Suppose you find the inverse of the matrix A^{-1}. Then you could multiply both sides of this equation on the left by A^{-1} and simplify to obtain

$$\begin{aligned} (A^{-1})\,AX &= A^{-1}B \\ (A^{-1}A)\,X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned}$$

Therefore we can find X, the solution to the system, by computing $X = A^{-1}B$. Note that once you have found A^{-1}, you can easily get the solution for different right hand sides (different B). It is always just $A^{-1}B$.

We will explore this method of finding the solution to a system in the following example.

Example 2.7.3: Using the Inverse to Solve a System of Equations

Consider the following system of equations. Use the inverse of a suitable matrix to give the solutions to this system.

$$\begin{aligned} x + z &= 1 \\ x - y + z &= 3 \\ x + y - z &= 2 \end{aligned}$$

Solution

First, we can write the system of equations in matrix form

$$AX = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = B \tag{2.7.2}$$

The inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

is

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Verifying this inverse is left as an exercise.

From here, the solution to the given system (2.7.2) is found by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -2 \\ -\frac{3}{2} \end{bmatrix}$$

What if the right side, B, of (2.7.2) had been $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$? In other words, what would be the solution to

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}?$$

By the above discussion, the solution is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -1 & 0 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

This illustrates that for a system $AX = B$ where A^{-1} exists, it is easy to find the solution when the vector B is changed.

We conclude this section with some important properties of the inverse.

Theorem 2.7.1: Inverses of Transposes and Products

Let A, B, and A_i for $i = 1, \ldots, k$ be $n \times n$ matrices.

1. If A is an invertible matrix, then $(A^T)^{-1} = (A^{-1})^T$
2. If A and B are invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
3. If $A_1, A_2, \ldots, A_k$ are invertible, then the product $A_1 A_2 \cdots A_k$ is invertible, and $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$

Consider the following theorem.

Theorem 2.7.2: Properties of the Inverse

Let A be an $n \times n$ matrix and I the usual identity matrix.

1. I is invertible and $I^{-1} = I$
2. If A is invertible then so is A^{-1}, and $(A^{-1})^{-1} = A$
3. If A is invertible then so is A^k, and $(A^k)^{-1} = (A^{-1})^k$
4. If A is invertible and p is a nonzero real number, then pA is invertible and $(pA)^{-1} = \frac{1}{p}A^{-1}$

This page titled 2.7: Finding the Inverse of a Matrix is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.7: Finding the Inverse of a Matrix** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

2.8: Elementary Matrices

We now turn our attention to a special type of matrix called an **elementary matrix**. An elementary matrix is always a square matrix. Recall the row operations given in Definition 1.3.2. Any elementary matrix, which we often denote by E, is obtained from applying *one* row operation to the identity matrix of the same size.

For example, the matrix

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is the elementary matrix obtained from switching the two rows. The matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the elementary matrix obtained from multiplying the second row of the 3×3 identity matrix by 3. The matrix

$$E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

is the elementary matrix obtained from adding -3 times the first row to the third row.

You may construct an elementary matrix from any row operation, but remember that you can only apply one operation.

Consider the following definition.

Definition 2.8.1: Elementary Matrices and Row Operations

Let E be an $n \times n$ matrix. Then E is an **elementary matrix** if it is the result of applying one row operation to the $n \times n$ identity matrix I_n.

Those which involve switching rows of the identity matrix are called permutation matrices.

Therefore, E constructed above by switching the two rows of I_2 is called a permutation matrix.

Elementary matrices can be used in place of row operations and therefore are very useful. It turns out that multiplying (on the left hand side) by an elementary matrix E will have the same effect as doing the row operation used to obtain E.

The following theorem is an important result which we will use throughout this text.

Theorem 2.8.1: Multiplication by an Elementary Matrix and Row Operations

To perform any of the three row operations on a matrix A it suffices to take the product EA, where E is the elementary matrix obtained by using the desired row operation on the identity matrix.

Therefore, instead of performing row operations on a matrix A, we can row reduce through matrix multiplication with the appropriate elementary matrix. We will examine this theorem in detail for each of the three row operations given in Definition 1.3.2.

First, consider the following lemma.

Lemma 2.8.1: Action of Permutation Matrix

Let P^{ij} denote the elementary matrix which involves switching the ith and the jth rows. Then P^{ij} is a permutation matrix and

$$P^{ij}A = B$$

where B is obtained from A by switching the ith and the jth rows.

We will explore this idea more in the following example.

Example 2.8.1: Switching Rows with an Elementary Matrix

Let

$$P^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ g & d \\ e & f \end{bmatrix}$$

Find B where $B = P^{12}A$.

Solution

You can see that the matrix P^{12} is obtained by switching the first and second rows of the 3×3 identity matrix I.

Using our usual procedure, compute the product $P^{12}A = B$. The result is given by

$$B = \begin{bmatrix} g & d \\ a & b \\ e & f \end{bmatrix}$$

Notice that B is the matrix obtained by switching rows 1 and 2 of A. Therefore by multiplying A by P^{12}, the row operation which was applied to I to obtain P^{12} is applied to A to obtain B.

Theorem 2.8.1 applies to all three row operations, and we now look at the row operation of multiplying a row by a scalar. Consider the following lemma.

Lemma 2.8.2: Multiplication by a Scalar and Elementary Matrices

Let $E(k,i)$ denote the elementary matrix corresponding to the row operation in which the ith row is multiplied by the nonzero scalar, k. Then

$$E(k,i)A = B$$

where B is obtained from A by multiplying the ith row of A by k.

We will explore this lemma further in the following example.

Example 2.8.2: Multiplication of a Row by 5 Using Elementary Matrix

Let

$$E(5,2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Find the matrix B where $B = E(5,2)A$

Solution

You can see that $E(5,2)$ is obtained by multiplying the second row of the identity matrix by 5.

Using our usual procedure for multiplication of matrices, we can compute the product $E(5,2)A$. The resulting matrix is given by

$$B = \begin{bmatrix} a & b \\ 5c & 5d \\ e & f \end{bmatrix}$$

Notice that B is obtained by multiplying the second row of A by the scalar 5.

There is one last row operation to consider. The following lemma discusses the final operation of adding a multiple of a row to another row.

Lemma 2.8.3: Adding Multiples of Rows and Elementary Matrices

Let $E(k \times i + j)$ denote the elementary matrix obtained from I by adding k times the ith row to the jth. Then

$$E(k \times i + j) A = B$$

where B is obtained from A by adding k times the ith row to the jth row of A.

Consider the following example.

Example 2.8.3: Adding Two Times the First Row to the Last

Let

$$E(2 \times 1 + 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

Find B where $B = E(2 \times 1 + 3) A$.

Solution

You can see that the matrix $E(2 \times 1 + 3)$ was obtained by adding 2 times the first row of I to the third row of I.

Using our usual procedure, we can compute the product $E(2 \times 1 + 3) A$. The resulting matrix B is given by

$$B = \begin{bmatrix} a & b \\ c & d \\ 2a + e & 2b + f \end{bmatrix}$$

You can see that B is the matrix obtained by adding 2 times the first row of A to the third row.

Suppose we have applied a row operation to a matrix A. Consider the row operation required to return A to its original form, to undo the row operation. It turns out that this action is how we find the inverse of an elementary matrix E.

Consider the following theorem.

Theorem 2.8.2: Elementary Matrices and Inverses

Every elementary matrix is invertible and its inverse is also an elementary matrix.

In fact, the inverse of an elementary matrix is constructed by doing the *reverse* row operation on I. E^{-1} will be obtained by performing the row operation which would carry E back to I.

- If E is obtained by switching rows i and j, then E^{-1} is also obtained by switching rows i and j.
- If E is obtained by multiplying row i by the scalar k, then E^{-1} is obtained by multiplying row i by the scalar $\frac{1}{k}$.
- If E is obtained by adding k times row i to row j, then E^{-1} is obtained by subtracting k times row i from row j.

Consider the following example.

Example 2.8.4: Inverse of an Elementary Matrix

Let

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Find E^{-1}.

Solution

Consider the elementary matrix E given by

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Here, E is obtained from the 2×2 identity matrix by multiplying the second row by 2. In order to carry E back to the identity, we need to multiply the second row of E by $\frac{1}{2}$. Hence,

E^{-1} is given by

$$E^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

We can verify that $EE^{-1} = I$. Take the product EE^{-1}, given by

$$EE^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This equals I so we know that we have compute E^{-1} properly.

Suppose an $m \times n$ matrix A is row reduced to its reduced row-echelon form. By tracking each row operation completed, this row reduction can be completed through multiplication by elementary matrices.

Consider the following definition.

Definition 2.8.2: The Form $B = UA$

Let A be an $m \times n$ matrix and let B be the reduced row-echelon form of A. Then we can write $B = UA$ where U is the product of all elementary matrices representing the row operations done to A to obtain B.

Consider the following example.

Example 2.8.5: The Form $B = UA$

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}$. Find B, the reduced row-echelon form of A and write it in the form $B = UA$.

Solution

To find B, row reduce A. For each step, we will record the appropriate elementary matrix. First, switch rows 1 and 2.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

The resulting matrix is equivalent to finding the product of $P^{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and A.

Next, add (-2) times row 1 to row 3.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

This is equivalent to multiplying by the matrix $E(-2 \times 1 + 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. Notice that the resulting matrix is B, the required reduced row-echelon form of A.

We can then write

$$\begin{aligned} B &= E(-2\times 1+2)\left(P^{12}A\right) \\ &= \left(E(-2\times 1+2)P^{12}\right)A \\ &= UA \end{aligned} \tag{2.8.1}$$

It remains to find the matrix U.

$$\begin{aligned} U &= E(-2\times 1+2)P^{12} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix} \end{aligned} \tag{2.8.2}$$

We can verify that $B=UA$ holds for this matrix U:

$$\begin{aligned} UA &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= B \end{aligned} \tag{2.8.3}$$

While the process used in the above example is reliable and simple when only a few row operations are used, it becomes cumbersome in a case where many row operations are needed to carry A to B. The following theorem provides an alternate way to find the matrix U.

Theorem 2.8.3: Finding the Matrix U

Let A be an $m\times n$ matrix and let B be its reduced row-echelon form. Then $B=UA$ where U is an invertible $m\times m$ matrix found by forming the matrix $[A|I_m]$ and row reducing to $[B|U]$.

Let's revisit the above example using the process outlined in Theorem 2.8.3.

Example 2.8.6: The Form $B=UA$, Revisited

Let $A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \end{bmatrix}$. Using the process outlined in Theorem 2.8.3, find U such that $B=UA$.

Solution

First, set up the matrix $[A|I_m]$.

$$\left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right]$$

Now, row reduce this matrix until the left side equals the reduced row-echelon form of A.

$$\begin{aligned} \left[\begin{array}{cc|ccc} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right] &\rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 \end{array}\right] \\ &\rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right] \end{aligned} \tag{2.8.4}$$

The left side of this matrix is B, and the right side is U. Comparing this to the matrix U found above in Example 2.8.5, you can see that the same matrix is obtained regardless of which process is used.

Recall from Algorithm 2.7.1 that an $n \times n$ matrix A is invertible if and only if A can be carried to the $n \times n$ identity matrix using the usual row operations. This leads to an important consequence related to the above discussion.

Suppose A is an $n \times n$ invertible matrix. Then, set up the matrix $[A|I_n]$ as done above, and row reduce until it is of the form $[B|U]$. In this case, $B = I_n$ because A is invertible.

$$\begin{aligned} B &= UA \\ I_n &= UA \\ U^{-1} &= A \end{aligned} \tag{2.8.5}$$

Now suppose that $U = E_1 E_2 \cdots E_k$ where each E_i is an elementary matrix representing a row operation used to carry A to I. Then,

$$U^{-1} = (E_1 E_2 \cdots E_k)^{-1} = E_k^{-1} \cdots E_2^{-1} E_1 - 1$$

Remember that if E_i is an elementary matrix, so too is E_i^{-1}. It follows that

$$\begin{aligned} A &= U^{-1} \\ &= E_k^{-1} \cdots E_2^{-1} E_1 - 1 \end{aligned} \tag{2.8.6}$$

and A can be written as a product of elementary matrices.

Theorem 2.8.4: Product of Elementary Matrices

Let A be an $n \times n$ matrix. Then A is invertible if and only if it can be written as a product of elementary matrices.

Consider the following example.

Example 2.8.7: Product of Elementary Matrices

Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. Write A as a product of elementary matrices.

Solution

We will use the process outlined in Theorem 2.8.3 to write A as a product of elementary matrices. We will set up the matrix $[A|I]$ and row reduce, recording each row operation as an elementary matrix.

First:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

represented by the elementary matrix $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Secondly:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

represented by the elementary matrix $E_2 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Finally:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right]$$

represented by the elementary matrix $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$.

Notice that the reduced row-echelon form of A is I. Hence $I = UA$ where U is the product of the above elementary matrices. It follows that $A = U^{-1}$. Since we want to write A as a product of elementary matrices, we wish to express U^{-1} as a product of elementary matrices.

$$\begin{aligned} U^{-1} &= (E_3E_2E_1)^{-1} \\ &= E_1^{-1}E_2^{-1}E_3^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \\ &= A \end{aligned} \tag{2.8.7}$$

This gives A written as a product of elementary matrices. By Theorem 2.8.4 it follows that A is invertible.

This page titled 2.8: Elementary Matrices is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.8: Elementary Matrices** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

2.9: More on Matrix Inverses

In this section, we will prove three theorems which will clarify the concept of matrix inverses. In order to do this, first recall some important properties of elementary matrices.

Recall that an elementary matrix is a square matrix obtained by performing an elementary operation on an identity matrix. Each elementary matrix is invertible, and its inverse is also an elementary matrix. If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the product EA is the result of applying to A the same elementary row operation that was applied to the $m \times m$ identity matrix in order to obtain E.

Let R be the reduced row-echelon form of an $m \times n$ matrix A. R is obtained by iteratively applying a sequence of elementary row operations to A. Denote by $E_1, E_2, \cdots, E_k$ the elementary matrices associated with the elementary row operations which were applied, in order, to the matrix A to obtain the resulting R. We then have that $R = (E_k \cdots (E_2 (E_1 A))) = E_k \cdots E_2 E_1 A$. Let E denote the product matrix $E_k \cdots E_2 E_1$ so that we can write $R = EA$ where E is an invertible matrix whose inverse is the product $(E_1)^{-1}(E_2)^{-1} \cdots (E_k)^{-1}$.

Now, we will consider some preliminary lemmas.

Lemma 2.9.1: Invertible Matrix and Zeros

Suppose that A and B are matrices such that the product AB is an identity matrix. Then the reduced row-echelon form of A does not have a row of zeros.

Proof

Let R be the reduced row-echelon form of A. Then $R = EA$ for some invertible square matrix E as described above. By hypothesis $AB = I$ where I is an identity matrix, so we have a chain of equalities

$$R(BE^{-1}) = (EA)(BE^{-1}) = E(AB)E^{-1} = EIE^{-1} = EE^{-1} = I$$

If R would have a row of zeros, then so would the product $R(BE^{-1})$. But since the identity matrix I does not have a row of zeros, neither can R have one.

We now consider a second important lemma.

Lemma 2.9.2: Size of Invertible Matrix

Suppose that A and B are matrices such that the product AB is an identity matrix. Then A has at least as many columns as it has rows.

Proof

Let R be the reduced row-echelon form of A. By Lemma 2.9.1, we know that R does not have a row of zeros, and therefore each row of R has a leading 1. Since each column of R contains at most one of these leading 1s, R must have at least as many columns as it has rows.

An important theorem follows from this lemma.

Theorem 2.9.1: Invertible Matrices are Square

Only square matrices can be invertible.

Proof

Suppose that A and B are matrices such that both products AB and BA are identity matrices. We will show that A and B must be square matrices of the same size. Let the matrix A have m rows and n columns, so that A is an $m \times n$ matrix. Since

the product AB exists, B must have n rows, and since the product BA exists, B must have m columns so that B is an $n \times m$ matrix. To finish the proof, we need only verify that $m = n$.

We first apply Lemma 2.9.2 with A and B, to obtain the inequality $m \leq n$. We then apply Lemma 2.9.2 again (switching the order of the matrices), to obtain the inequality $n \leq m$. It follows that $m = n$, as we wanted.

Of course, not all square matrices are invertible. In particular, zero matrices are not invertible, along with many other square matrices.

The following proposition will be useful in proving the next theorem.

Proposition 2.9.1: Reduced Row-Echelon Form of a Square Matrix

If R is the reduced row-echelon form of a square matrix, then either R has a row of zeros or R is an identity matrix.

The proof of this proposition is left as an exercise to the reader. We now consider the second important theorem of this section.

Theorem 2.9.2: Unique Inverse of a Matrix

Suppose A and B are square matrices such that $AB = I$ where I is an identity matrix. Then it follows that $BA = I$. Further, both A and B are invertible and $B = A^{-1}$ and $A = B^{-1}$.

Proof

Let R be the reduced row-echelon form of a square matrix A. Then, $R = EA$ where E is an invertible matrix. Since $AB = I$, Lemma 2.9.1 gives us that R does not have a row of zeros. By noting that R is a square matrix and applying Proposition 2.9.1, we see that $R = I$. Hence, $EA = I$.

Using both that $EA = I$ and $AB = I$, we can finish the proof with a chain of equalities as given by

$$\begin{aligned} BA = IBIA &= (EA)B(E^{-1}E)A \\ &= E(AB)E^{-1}(EA) \\ &= EIE^{-1}I \\ &= EE^{-1} = I \end{aligned} \tag{2.9.1}$$

It follows from the definition of the inverse of a matrix that $B = A^{-1}$ and $A = B^{-1}$.

This theorem is very useful, since with it we need only test one of the products AB or BA in order to check that B is the inverse of A. The hypothesis that A and B are square matrices is very important, and without this the theorem does not hold.

We will now consider an example.

Example 2.9.1: Non Square Matrices

Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

Show that $A^T A = I$ but $AA^T \neq I$.

Solution

Consider the product $A^T A$ given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.9.2}$$

Therefore, $A^TA = I_2$, where I_2 is the 2×2 identity matrix. However, the product AA^T is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.9.3}$$

Hence AA^T is not the 3×3 identity matrix. This shows that for Theorem 2.9.2, it is essential that both matrices be square and of the same size.

Is it possible to have matrices A and B such that $AB = I$, while $BA = 0$? This question is left to the reader to answer, and you should take a moment to consider the answer.

We conclude this section with an important theorem.

Theorem 2.9.3: The reduced row-echelon form of an Invertible Matrix

For any matrix A the following conditions are equivalent:

- A is invertible
- The reduced row-echelon form of A is an identity matrix

Proof

In order to prove this, we show that for any given matrix A, each condition implies the other. We first show that if A is invertible, then its reduced row-echelon form is an identity matrix, then we show that if the reduced row-echelon form of A is an identity matrix, then A is invertible.

If A is invertible, there is some matrix B such that $AB = I$. By Lemma 2.9.1, we get that the of A does not have a row of zeros. Then by Theorem 2.9.1, it follows that A and the reduced row-echelon form of A are square matrices. Finally, by Proposition 2.9.1, this reduced row-echelon form of A must be an identity matrix. This proves the first implication.

Now suppose the reduced row-echelon form of A is an identity matrix I. Then $I = EA$ for some product E of elementary matrices. By Theorem 2.9.2, we can conclude that A is invertible.

Theorem 2.9.3 corresponds to Algorithm 2.7.1, which claims that A^{-1} is found by row reducing the augmented matrix $[A|I]$ to the form $[I|A^{-1}]$. This will be a matrix product $E[A|I]$ where E is a product of elementary matrices. By the rules of matrix multiplication, we have that $E[A|I] = [EA|EI] = [EA|E]$.

It follows that the reduced row-echelon form of $[A|I]$ is $[EA|E]$, where EA gives the reduced row-echelon form of A. By Theorem 2.9.3, if $EA \neq I$, then A is not invertible, and if $EA = I$, A is invertible. If $EA = I$, then by Theorem 2.9.2, $E = A^{-1}$. This proves that Algorithm 2.7.1 does in fact find A^{-1}.

This page titled 2.9: More on Matrix Inverses is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.9: More on Matrix Inverses** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

2.10: LU Factorization

An LU factorization of a matrix involves writing the given matrix as the product of a lower triangular matrix L which has the main diagonal consisting entirely of ones, and an upper triangular matrix U in the indicated order. This is the version discussed here but it is sometimes the case that the L has numbers other than 1 down the main diagonal. It is still a useful concept. The L goes with "lower" and the U with "upper".

It turns out many matrices can be written in this way and when this is possible, people get excited about slick ways of solving the system of equations, $AX = B$. It is for this reason that you want to study the LU factorization. It allows you to work only with triangular matrices. It turns out that it takes about half as many operations to obtain an LU factorization as it does to find the row reduced echelon form.

First it should be noted not all matrices have an LU factorization and so we will emphasize the techniques for achieving it rather than formal proofs.

Example 2.10.1: A Matrix with NO LU factorization

Can you write $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in the form LU as just described?

Solution

To do so you would need

$$\begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & b \\ xa & xb+c \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, $b = 1$ and $a = 0$. Also, from the bottom rows, $xa = 1$ which can't happen and have $a = 0$. Therefore, you can't write this matrix in the form It has no LU factorization. This is what we mean above by saying the method lacks generality.

Nevertheless the method is often extremely useful, and we will describe below one the many methods used to produce an LU factorization when possible.

Finding An LU Factorization By Inspection

Which matrices have an LU factorization? It turns out it is those whose row-echelon form can be achieved without switching rows. In other words matrices which only involve using row operations of type 2 or 3 to obtain the row-echelon form.

Example 2.10.2: An LU factorization

Find an LU factorization of $A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 3 & 4 & 0 \end{bmatrix}$.

Solution

One way to find the LU factorization is to simply look for it directly. You need

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 3 & 2 & 1 \\ 2 & 3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix}\begin{bmatrix} a & d & h & j \\ 0 & b & e & i \\ 0 & 0 & c & f \end{bmatrix}.$$

Then multiplying these you get

$$\begin{bmatrix} a & d & h & j \\ xa & xd+b & xh+e & xj+i \\ ya & yd+zb & yh+ze+c & yj+iz+f \end{bmatrix}$$

and so you can now tell what the various quantities equal. From the first column, you need $a = 1, x = 1, y = 2$. Now go to the second column. You need $d = 2, xd + b = 3$ so $b = 1, yd + zb = 3$ so $z = -1$. From the third column, $h = 0, e = 2, c = 6$. Now from the fourth column, $j = 2, i = -1, f = -5$. Therefore, an LU factorization is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 6 & -5 \end{bmatrix}.$$

You can check whether you got it right by simply multiplying these two.

LU Factorization, Multiplier Method

Remember that for a matrix A to be written in the form $A = LU$, you must be able to reduce it to its row-echelon form without interchanging rows. The following method gives a process for calculating the LU factorization of such a matrix A.

Example 2.10.3: LU factorization

Find an LU factorization for

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & 3 & -2 \end{bmatrix}$$

Solution

Write the matrix as the following product.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -2 & 3 & -2 \end{bmatrix}$$

In the matrix on the right, begin with the left row and zero out the entries below the top using the row operation which involves adding a multiple of a row to another row. You do this and also update the matrix on the left so that the product will be unchanged. Here is the first step. Take -2 times the top row and add to the second. Then take 2 times the top row and add to the second in the matrix on the left.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ -2 & 3 & -2 \end{bmatrix}$$

The next step is to take 2 times the top row and add to the bottom in the matrix on the right. To ensure that the product is unchanged, you place a in the bottom left in the matrix on the left. Thus the next step yields

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 7 & 4 \end{bmatrix}$$

Next take 7 times the middle row on right and add to bottom row. Updating the matrix on the left in a similar manner to what was done earlier,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & -31 \end{bmatrix}$$

At this point, stop. You are done.

The method just described is called the multiplier method.

Solving Systems using LU Factorization

One reason people care about the LU factorization is it allows the quick solution of systems of equations. Here is an example.

Example 2.10.4: LU factorization to Solve Equations

Suppose you want to find the solutions to

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 3 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

Solution

Of course one way is to write the augmented matrix and grind away. However, this involves more row operations than the computation of the LU factorization and it turns out that the LU factorization can give the solution quickly. Here is how. The following is an LU factorization for the matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 4 & 3 & 1 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Let $UX = Y$ and consider $LY = B$ where in this case, $B = [1, 2, 3]^T$. Thus

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

which yields very quickly that $Y = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$.

Now you can find X by solving $UX = Y$. Thus in this case,

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -11 & -7 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

which yields

$$X = \begin{bmatrix} -\frac{3}{5} + \frac{7}{5}t \\ \frac{9}{5} - \frac{11}{5}t \\ t \\ -1 \end{bmatrix}, \ t \in \mathbb{R}.$$

Justification for the Multiplier Method

Why does the multiplier method work for finding the LU factorization? Suppose A is a matrix which has the property that the row-echelon form for A may be achieved without switching rows. Thus every row which is replaced using this row operation in obtaining the row-echelon form may be modified by using a row which is above it.

Lemma 2.10.1: Multiplier Method and Triangular Matrices

Let L be a lower (upper) triangular matrix $m \times m$ which has ones down the main diagonal. Then L^{-1} also is a lower (upper) triangular matrix which has ones down the main diagonal. In the case that L is of the form

$$L = \begin{bmatrix} 1 & & & \\ a_1 & 1 & & \\ \vdots & & \ddots & \\ a_n & & & 1 \end{bmatrix} \tag{2.10.1}$$

where all entries are zero except for the left column and main diagonal, it is also the case that L^{-1} is obtained from L by simply multiplying each entry below the main diagonal in L with -1. The same is true if the single nonzero column is in another

position.

Proof

Consider the usual setup for finding the inverse $\left[\begin{array}{cc} L & I \end{array}\right]$. Then each row operation done to L to reduce to row reduced echelon form results in changing only the entries in I below the main diagonal. In the special case of L given in (2.10.1) or the single nonzero column is in another position, multiplication by -1 as described in the lemma clearly results in L^{-1}.

For a simple illustration of the last claim,

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & a & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -a & 1 \end{array}\right]$$

Now let A be an $m \times n$ matrix, say

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right]$$

and assume A can be row reduced to an upper triangular form using only row operation 3. Thus, in particular, $a_{11} \neq 0$. Multiply on the left by $E_1 =$

$$\left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a_{m1}}{a_{11}} & 0 & \cdots & 1 \end{array}\right]$$

This is the product of elementary matrices which make modifications in the first column only. It is equivalent to taking $-a_{21}/a_{11}$ times the first row and adding to the second. Then taking $-a_{31}/a_{11}$ times the first row and adding to the third and so forth. The quotients in the first column of the above matrix are the multipliers. Thus the result is of the form

$$E_1 A = \left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{m2} & \cdots & a'_{mn} \end{array}\right]$$

By assumption, $a'_{22} \neq 0$ and so it is possible to use this entry to zero out all the entries below it in the matrix on the right by multiplication by a matrix of the form $E_2 = \left[\begin{array}{cc} 1 & 0 \\ 0 & E \end{array}\right]$ where E is an $[m-1] \times [m-1]$ matrix of the form

$$E = \left[\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ -\frac{a'_{32}}{a'_{22}} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{a'_{m2}}{a'_{22}} & 0 & \cdots & 1 \end{array}\right]$$

Again, the entries in the first column below the 1 are the multipliers. Continuing this way, zeroing out the entries below the diagonal entries, finally leads to

$$E_{m-1} E_{n-2} \cdots E_1 A = U$$

where U is upper triangular. Each E_j has all ones down the main diagonal and is lower triangular. Now multiply both sides by the inverses of the E_j in the reverse order. This yields

$$A = E_1^{-1} E_2^{-1} \cdots E_{m-1}^{-1} U$$

By Lemma 2.10.1, this implies that the product of those E_j^{-1} is a lower triangular matrix having all ones down the main diagonal.

The above discussion and lemma gives the justification for the multiplier method. The expressions

$$\frac{-a_{21}}{a_{11}}, \frac{-a_{31}}{a_{11}}, \cdots, \frac{-a_{m1}}{a_{11}}$$

denoted respectively by $M_{21}, \cdots, M_{m1}$ to save notation which were obtained in building E_1 are the multipliers. Then according to the lemma, to find E_1^{-1} you simply write

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -M_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -M_{m1} & 0 & \cdots & 1 \end{bmatrix}$$

Similar considerations apply to the other E_j^{-1}. Thus L is a product of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ -M_{21} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -M_{m1} & 0 & \cdots & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & -M_{m[m-1]} & 1 \end{bmatrix}$$

each factor having at most one nonzero column, the position of which moves from left to right in scanning the above product of matrices from left to right. It follows from what we know about the effect of multiplying on the left by an elementary matrix that the above product is of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -M_{21} & 1 & \cdots & 0 & 0 \\ \vdots & -M_{32} & \ddots & \vdots & \vdots \\ -M_{[m-1]1} & \vdots & \cdots & 1 & 0 \\ -M_{m1} & -M_{m2} & \cdots & -M_{m[m-1]} & 1 \end{bmatrix}$$

In words, beginning at the left column and moving toward the right, you simply insert, into the corresponding position in the identity matrix, -1 times the multiplier which was used to zero out an entry in that position below the main diagonal in A, while retaining the main diagonal which consists entirely of ones. This is L.

This page titled 2.10: LU Factorization is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.10: LU Factorization** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

2.11: Exercises

Exercise 2.11.1

For the following pairs of matrices, determine if the sum $A+B$ is defined. If so, find the sum.

a. $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}, B=\begin{bmatrix}0&1\\1&0\end{bmatrix}$

b. $A=\begin{bmatrix}2&1&2\\1&1&0\end{bmatrix}, B=\begin{bmatrix}-1&0&3\\0&1&4\end{bmatrix}$

c. $A=\begin{bmatrix}1&0\\-2&3\\4&2\end{bmatrix}, B=\begin{bmatrix}2&7&-1\\0&3&4\end{bmatrix}$

Exercise 2.11.2

For each matrix A, find the matrix $-A$ such that $A+(-A)=0$.

a. $A=\begin{bmatrix}1&2\\2&1\end{bmatrix}$

b. $A=\begin{bmatrix}-2&3\\0&2\end{bmatrix}$

c. $A=\begin{bmatrix}0&1&2\\1&-1&3\\4&2&0\end{bmatrix}$

Exercise 2.11.3

In the context of Proposition 2.1.1, describe $-A$ and 0.

Answer

To get $-A$, just replace every entry of A with its additive inverse. The 0 matrix is the one which has all zeros in it.

2.1.2: Scalar Multiplication of Matrices

Exercise 2.11.4

For each matrix A, find the product $(-2)A, 0A$, and $3A$.

a. $A=\begin{bmatrix}1&2\\2&1\end{bmatrix}$

b. $A=\begin{bmatrix}-2&3\\0&2\end{bmatrix}$

c. $A=\begin{bmatrix}0&1&2\\1&-1&3\\4&2&0\end{bmatrix}$

Exercise 2.11.5

Using only the properties given in Proposition 2.1.1 and Proposition 2.1.2, show $-A$ is unique.

Answer

Suppose B also works. Then

$$-A=-A+(A+B)=(-A+A)+B=0+B=B$$

Exercise 2.11.6

Using only the properties given in Proposition 2.1.1 and Proposition 2.1.2, show $\mathbf{0}$ is unique.

Answer

Suppose $0'$ also works. Then $0' = 0' + 0 = 0$.

Exercise 2.11.7

Using only the properties given in Proposition 2.1.1 and Proposition 2.1.2 show $0A = 0$. Here the 0 on the left is the scalar 0 and the $\mathbf{0}$ on the right is the zero matrix of appropriate size.

Answer

$0A = (0+0)A = 0A + 0A$. Now add $-(0A)$ to both sides. Then $\mathbf{0} = 0A$.

Exercise 2.11.8

Using only the properties given in Proposition 2.1.1 and Proposition 2.1.2, as well as previous problems show $(-1)A = -A$.

Answer

$A + (-1)A = (1 + (-1))A = 0A = \mathbf{0}$. Therefore, from the uniqueness of the additive inverse proved in the above Problem 2.11.7, it follows that $-A = (-1)A$.

2.2

Exercise 2.11.9

Consider the matrices $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 7 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}, E = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Find the following if possible. If it is not possible explain why.

a. $-3A$
b. $3B - A$
c. AC
d. CB
e. AE
f. EA

Answer

a. $\begin{bmatrix} -3 & -6 & -9 \\ -6 & -3 & -21 \end{bmatrix}$
b. $\begin{bmatrix} 8 & -5 & 3 \\ -11 & 5 & -4 \end{bmatrix}$
c. Not possible
d. $\begin{bmatrix} -3 & 3 & 4 \\ 6 & -1 & 7 \end{bmatrix}$
e. Not possible
f. Not possible

Exercise 2.11.10

Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Find the following if possible. If it is not possible explain why.

a. $-3A$
b. $3B - A$
c. AC
d. CA
e. AE
f. EA
g. BE
h. DE

Answer

a. $\begin{bmatrix} -3 & -6 \\ -9 & -6 \\ -3 & 3 \end{bmatrix}$
b. Not possible.
c. $\begin{bmatrix} 11 & 2 \\ 13 & 6 \\ -4 & 2 \end{bmatrix}$
d. Not possible.
e. $\begin{bmatrix} 7 \\ 9 \\ -2 \end{bmatrix}$
f. Not possible.
g. Not possible.
h. $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$

Exercise 2.11.11

Let $A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & -2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 & -3 \\ -1 & 2 & 0 \\ -3 & -1 & 0 \end{bmatrix}$. Find the following if possible.

a. AB
b. BA
c. AC
d. CA
e. CB
f. BC

Answer

a. $\begin{bmatrix} 3 & 0 & -4 \\ -4 & 1 & 6 \\ 5 & 1 & -6 \end{bmatrix}$
b. $\begin{bmatrix} 1 & -2 \\ -2 & -3 \end{bmatrix}$
c. Not possible

d. $\begin{bmatrix} -4 & -6 \\ -5 & -3 \\ -1 & -2 \end{bmatrix}$

e. $\begin{bmatrix} 8 & 1 & -3 \\ 7 & 6 & -6 \end{bmatrix}$

? Exercise 2.11.12

Let $A = \begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$. Find all 2×2 matrices, B such that $AB = 0$.

Answer

$$\begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} -x - z & -w - y \\ 3x + 3z & 3w + 3y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.11.1}$$

Solution is: $w = -y, x = -z$ so the matrices are of the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$.

? Exercise 2.11.13

Let $X = \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$. Find $X^T Y$ and XY^T if possible.

Answer

$$X^T Y = \begin{bmatrix} 0 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix}, XY^T = 1$$

? Exercise 2.11.14

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix}$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?

Answer

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 7 & 2k + 2 \\ 15 & 4k + 6 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 3 & k \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3k + 3 & 4k + 6 \end{bmatrix} \tag{2.11.2}$$

Thus you must have $\begin{matrix} 3k + 3 = 15 \\ 2k + 2 = 10 \end{matrix}$, Solution is: $[k = 4]$

? Exercise 2.11.15

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix}$. Is it possible to choose k such that $AB = BA$? If so, what should k equal?

Answer

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix} = \begin{bmatrix} 3 & 2k + 2 \\ 7 & 4k + 6 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 1 & k \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 3k + 1 & 4k + 2 \end{bmatrix} \tag{2.11.3}$$

However, $7 \neq 3$ and so there is no possible choice of k which will make these matrices commute.

Exercise 2.11.16

Find 2×2 matrices, A, B, and C such that $A \neq 0, C \neq B$, but $AC = AB$.

Answer

Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.11.4}$$

Exercise 2.11.17

Give an example of matrices (of any size), A, B, C such that $B \neq C$, $A \neq 0$, and yet $AB = AC$.

Exercise 2.11.18

Find 2×2 matrices A and B such that $A \neq 0$ and $B \neq 0$ but $AB = 0$.

Answer

Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Exercise 2.11.19

Give an example of matrices (of any size), A, B such that $A \neq 0$ and $B \neq 0$ but $AB = 0$.

Exercise 2.11.20

Find 2×2 matrices A and B such that $A \neq 0$ and $B \neq 0$ with $AB \neq BA$.

Answer

Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \tag{2.11.5}$$

Exercise 2.11.21

Write the system

$$\begin{gathered} x_1 - x_2 + 2x_3 \\ 2x_3 + x_1 \\ 3x_3 \\ 3x_4 + 3x_2 + x_1 \end{gathered}$$

in the form $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where A is an appropriate matrix.

Answer

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 3 & 0 & 3 \end{bmatrix}$$

? Exercise 2.11.22

Write the system

$$\begin{aligned} x_1 + 3x_2 + 2x_3 \\ 2x_3 + x_1 \\ 6x_3 \\ x_4 + 3x_2 + x_1 \end{aligned}$$

in the form $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where A is an appropriate matrix.

Answer

$$A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 6 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix}$$

? Exercise 2.11.23

Write the system

$$\begin{aligned} x_1 + x_2 + x_3 \\ 2x_3 + x_1 + x_2 \\ x_3 - x_1 \\ 3x_4 + x_1 \end{aligned}$$

in the form $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ where A is an appropriate matrix.

Answer

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$$

? Exercise 2.11.24

A matrix A is called *idempotent* if $A^2 = A$. Let

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

and show that A is idempotent.

2.3

? Exercise 2.11.25

For each pair of matrices, find the $(1,2)$-entry and $(2,3)$-entry of the product AB.

1. $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ 2 & 5 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & 6 & -2 \\ 7 & 2 & 1 \\ -1 & 0 & 0 \end{bmatrix}$
2. $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 4 \\ 1 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 0 \\ -4 & 16 & 1 \\ 0 & 2 & 2 \end{bmatrix}$

2.4

? Exercise 2.11.26

Suppose A and B are square matrices of the same size. Which of the following are necessarily true?

a. $(A-B)^2 = A^2 - 2AB + B^2$
b. $(AB)^2 = A^2B^2$
c. $(A+B)^2 = A^2 + 2AB + B^2$
d. $(A+B)^2 = A^2 + AB + BA + B^2$
e. $A^2B^2 = A(AB)B$
f. $(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
g. $(A+B)(A-B) = A^2 - B^2$

Answer

a. Not necessarily true.
b. Not necessarily true.
c. Not necessarily true.
d. Necessarily true.
e. Necessarily true.
f. Not necessarily true.
g. Not necessarily true.

2.5

? Exercise 2.11.27

Consider the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -5 & 2 \\ -3 & 2 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ 5 & 0 \end{bmatrix}$,
$D = \begin{bmatrix} -1 & 1 \\ 4 & -3 \end{bmatrix}, E = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Find the following if possible. If it is not possible explain why.

a. $-3A^T$
b. $3B - A^T$

c. E^TB
d. EE^T
e. B^TB
f. CA^T
g. D^TBE

Answer

a. $\begin{bmatrix} -3 & -9 & -3 \\ -6 & -6 & 3 \end{bmatrix}$
b. $\begin{bmatrix} 5 & -18 & 5 \\ -11 & 4 & 4 \end{bmatrix}$
c. $\begin{bmatrix} -7 & 1 & 5 \end{bmatrix}$
d. $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$
e. $\begin{bmatrix} 13 & -16 & 1 \\ -16 & 29 & -8 \\ 1 & -8 & 5 \end{bmatrix}$
f. $\begin{bmatrix} 5 & 7 & -1 \\ 5 & 15 & 5 \end{bmatrix}$
g. Not possible.

? Exercise 2.11.28

Let A be an $n \times n$ matrix. Show A equals the sum of a symmetric and a skew symmetric matrix.

Hint

Show that $\frac{1}{2}\left(A^T + A\right)$ is symmetric and then consider using this as one of the matrices.

? Exercise 2.11.29

Show that the main diagonal of every skew symmetric matrix consists of only zeros. Recall that the main diagonal consists of every entry of the matrix which is of the form a_{ii}.

Answer

If A is symmetric then $A = -A^T$. It follows that $a_{ii} = -a_{ii}$ and so each $a_{ii} = 0$.

? Exercise 2.11.30

Prove 3 from Lemma 2.5.1. That is, show that for an $m \times n$ matrix A, an $m \times n$ matrix B, and scalars r, s, the following holds:

$$(rA + sB)^T = rA^T + sB^T$$

2.6

? Exercise 2.11.31

Prove that $I_m A = A$ where A is an $m \times n$ matrix.

Answer

$(I_m A)_{ij} \equiv \sum_j \delta_{ik} A_{kj} = A_{ij}$

Exercise 2.11.32

Suppose $AB = AC$ and A is an invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

Answer

Yes $B = C$. Multiply $AB = AC$ on the left by A^{-1}.

Exercise 2.11.33

Suppose $AB = AC$ and A is a non invertible $n \times n$ matrix. Does it follow that $B = C$? Explain why or why not.

Exercise 2.11.34

Give an example of a matrix A such that $A^2 = I$ and yet $A \neq I$ and $A \neq -I$.

Answer

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.7

Exercise 2.11.35

Let

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{bmatrix}$$

Exercise 2.11.36

Let

$$A = \begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer

$$\begin{bmatrix} 0 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{3}{5} & \frac{1}{5} \\ 1 & 0 \end{bmatrix}$$

Exercise 2.11.37

Add exercises text here.Let

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer

$$\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{2}{3} \end{bmatrix}$$

? Exercise 2.11.38

Let

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer

$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}^{-1}$ does not exist. The of this matrix is $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$

? Exercise 2.11.39

Let A be a 2×2 invertible matrix, with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find a formula for A^{-1} in terms of a, b, c, d.

Answer

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

? Exercise 2.11.40

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

? Exercise 2.11.41

Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 3 \\ 0 & \frac{1}{3} & -\frac{2}{3} \\ 1 & 0 & -1 \end{bmatrix}$$

Exercise 2.11.42

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 4 & 5 & 10 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer

The reduced row-echelon form is $\begin{bmatrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$. There is no inverse.

Exercise 2.11.43

Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Find A^{-1} if possible. If A^{-1} does not exist, explain why.

Answer

$$\begin{bmatrix} 1 & 2 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 2 & 1 & -3 & 2 \\ 1 & 2 & 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 3 & \frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} \\ -1 & 0 & 0 & 1 \\ -2 & -\frac{3}{4} & \frac{1}{4} & \frac{9}{4} \end{bmatrix}$$

Exercise 2.11.44

Using the inverse of the matrix, find the solution to the systems:

a. $$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

b. $$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Now give the solution in terms of a and b to

$$\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

Exercise 2.11.45

Using the inverse of the matrix, find the solution to the systems:

a. $$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

b. $$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$$

Now give the solution in terms of a, b, and c to the following:

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Answer

a. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{2}{3} \\ 0 \end{bmatrix}$

b. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -12 \\ 1 \\ 5 \end{bmatrix}$

c. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3c-2a \\ \frac{1}{3}b-\frac{2}{3}c \\ a-c \end{bmatrix}$

Exercise 2.11.46

Show that if A is an $n \times n$ invertible matrix and X is a $n \times 1$ matrix such that $AX = B$ for B an $n \times 1$ matrix, then $X = A^{-1}B$.

Answer

Multiply both sides of $AX = B$ on the left by A^{-1}.

Exercise 2.11.47

Prove that if A^{-1} exists and $AX = 0$ then $X = 0$.

Answer

Multiply on both sides on the left by A^{-1}. Thus

$$0 = A^{-1}0 = A^{-1}(AX) = \left(A^{-1}A\right)X = IX = X$$

Exercise 2.11.48: Inverse Product

Show that if A^{-1} exists for an $n \times n$ matrix, then it is unique. That is, if $BA = I$ and $AB = I$, then $B = A^{-1}$.

Answer

$A^{-1} = A^{-1}I = A^{-1}(AB) = \left(A^{-1}A\right)B = IB = B$.

Exercise 2.11.49

Show that if A is an invertible $n \times n$ matrix, then so is A^T and $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.

Answer

You need to show that $(A^{-1})^T$ acts like the inverse of A^T because from uniqueness in the above problem, this will imply it is the inverse. From properties of the transpose,

$$\begin{aligned} A^T(A^{-1})^T &= (A^{-1}A)^T = I^T = I \\ (A^{-1})^T A^T &= (AA^{-1})^T = I^T = I \end{aligned} \tag{2.11.6}$$

Hence $(A^{-1})^T = (A^T)^{-1}$ and this last matrix exists.

? Exercise 2.11.50

Show $(AB)^{-1} = B^{-1}A^{-1}$ by verifying that

$$AB\left(B^{-1}A^{-1}\right) = I$$

and

$$B^{-1}A^{-1}(AB) = I$$

Hint: Use Problem 2.11.48.

Answer

$(AB)\,B^{-1}A^{-1} = A\,(BB^{-1})\,A^{-1} = AA^{-1} = I\;B^{-1}A^{-1}\,(AB) = B^{-1}\,(A^{-1}A)\,B = B^{-1}IB = B^{-1}B = I$

? Exercise 2.11.51

Show that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ by verifying that

$$(ABC)\left(C^{-1}B^{-1}A^{-1}\right) = I$$

and

$$\left(C^{-1}B^{-1}A^{-1}\right)(ABC) = I$$

Hint: Use Problem 2.11.48.

Answer

The proof of this exercise follows from the previous one.

? Exercise 2.11.52

If A is invertible, show $(A^2)^{-1} = (A^{-1})^2$. **Hint:** Use Problem 2.11.48.

Answer

$A^2(A^{-1})^2 = AAA^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I\;(A^{-1})^2A^2 = A^{-1}A^{-1}AA = A^{-1}IA = A^{-1}A = I$

? Exercise 2.11.53

If A is invertible, show $(A^{-1})^{-1} = A$. **Hint:** Use Problem 2.11.48.

Answer

$A^{-1}A = AA^{-1} = I$ and so by uniqueness, $(A^{-1})^{-1} = A$.

Exercise 2.11.54

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.

Exercise 2.11.55

Let $A = \begin{bmatrix} 4 & 0 \\ 2 & 1 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 8 & 0 \\ 2 & 1 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.

Exercise 2.11.56

Let $A = \begin{bmatrix} 1 & -3 \\ 0 & 5 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$. Find the elementary matrix E that represents this row operation.

Exercise 2.11.57

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 4 \\ 0 & 5 & 1 \end{bmatrix}$.

a. Find the elementary matrix E such that $EA = B$.

b. Find the inverse of E, E^{-1}, such that $E^{-1}B = A$.

Exercise 2.11.58

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 10 & 2 \\ 2 & -1 & 4 \end{bmatrix}$.

a. Find the elementary matrix E such that $EA = B$.

b. Find the inverse of E, E^{-1}, such that $E^{-1}B = A$.

Exercise 2.11.59

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 1 & -\frac{1}{2} & 2 \end{bmatrix}$.

a. Find the elementary matrix E such that $EA = B$.

b. Find the inverse of E, E^{-1}, such that $E^{-1}B = A$.

Exercise 2.11.60

Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 2 & -1 & 4 \end{bmatrix}$. Suppose a row operation is applied to A and the result is $B = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 5 \\ 2 & -1 & 4 \end{bmatrix}$.

a. Find the elementary matrix E such that $EA = B$.

b. Find the inverse of E, E^{-1}, such that $E^{-1}B = A$.

Exercise 2.11.61

Find an LU factorization of $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

Answer

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

Exercise 2.11.62

Find an LU factorization of $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 \\ 5 & 0 & 1 & 3 \end{bmatrix}$.

Answer

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 1 \\ 5 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 5 & -10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -24 & -17 \end{bmatrix}$$

Exercise 2.11.63

Find an LU factorization of the matrix $\begin{bmatrix} 1 & -2 & -5 & 0 \\ -2 & 5 & 11 & 3 \\ 3 & -6 & -15 & 1 \end{bmatrix}$.

Answer

$$\begin{bmatrix} 1 & -2 & -5 & 0 \\ -2 & 5 & 11 & 3 \\ 3 & -6 & -15 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & -5 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 2.11.64

Find an LU factorization of the matrix $\begin{bmatrix} 1 & -1 & -3 & -1 \\ -1 & 2 & 4 & 3 \\ 2 & -3 & -7 & -3 \end{bmatrix}$.

Answer

$$\begin{bmatrix} 1 & -1 & -3 & -1 \\ -1 & 2 & 4 & 3 \\ 2 & -3 & -7 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -3 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Exercise 2.11.65

Find an LU factorization of the matrix $\begin{bmatrix} 1 & -3 & -4 & -3 \\ -3 & 10 & 10 & 10 \\ 1 & -6 & 2 & -5 \end{bmatrix}$.

Answer

$$\begin{bmatrix} 1 & -3 & -4 & -3 \\ -3 & 10 & 10 & 10 \\ 1 & -6 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & -4 & -3 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

? Exercise 2.11.66

Find an LU factorization of the matrix $\begin{bmatrix} 1 & 3 & 1 & -1 \\ 3 & 10 & 8 & -1 \\ 2 & 5 & -3 & -3 \end{bmatrix}$.

Answer

$$\begin{bmatrix} 1 & 3 & 1 & -1 \\ 3 & 10 & 8 & -1 \\ 2 & 5 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

? Exercise 2.11.67

Find an LU factorization of the matrix $\begin{bmatrix} 3 & -2 & 1 \\ 9 & -8 & 6 \\ -6 & 2 & 2 \\ 3 & 2 & -7 \end{bmatrix}$.

Answer

$$\begin{bmatrix} 3 & -2 & 1 \\ 9 & -8 & 6 \\ -6 & 2 & 2 \\ 3 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 & 1 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

? Exercise 2.11.68

Find an LU factorization of the matrix $\begin{bmatrix} -3 & -1 & 3 \\ 9 & 9 & -12 \\ 3 & 19 & -16 \\ 12 & 40 & -26 \end{bmatrix}$.

? Exercise 2.11.69

Find an LU factorization of the matrix $\begin{bmatrix} -1 & -3 & -1 \\ 1 & 3 & 0 \\ 3 & 9 & 0 \\ 4 & 12 & 16 \end{bmatrix}$.

Answer

$$\begin{bmatrix} -1 & -3 & -1 \\ 1 & 3 & 0 \\ 3 & 9 & 0 \\ 4 & 12 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

? Exercise 2.11.70

Find the LU factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.

$$x + 2y = 5$$
$$2x + 3y = 6$$

Answer

An LU factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$$

First solve

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

which gives $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$. Then solve

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

which says that $y = 4$ and $x = -3$.

? Exercise 2.11.71

Find the LU factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.

$$\begin{aligned} x + 2y + z &= 1 \\ y + 3z &= 2 \\ 2x + 3y &= 6 \end{aligned}$$

Answer

An LU factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

First solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

which yields $u = 1, v = 2, w = 6$. Next solve

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

This yields $z = 6, y = -16, x = 27$.

? Exercise 2.11.72

Find the LU factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.

$$\begin{aligned} x + 2y + 3z &= 5 \\ 2x + 3y + z &= 6 \\ x - y + z &= 2 \end{aligned}$$

Exercise 2.11.73

Find the LU factorization of the coefficient matrix using Dolittle's method and use it to solve the system of equations.

$$\begin{aligned} x+2y+3z &= 5 \\ 2x+3y+z &= 6 \\ 3x+5y+4z &= 11 \end{aligned}$$

Answer

An LU factorization of the coefficient matrix is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

First solve

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 11 \end{bmatrix}$$

Solution is: $\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$. Next solve

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$$

Solution is: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7t-3 \\ 4-5t \\ t \end{bmatrix}, t \in \mathbb{R}$.

Exercise 2.11.74

Is there only one LU factorization for a given matrix? **Hint:** Consider the equation

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Look for all possible LU factorizations.

Answer

Sometimes there is more than one LU factorization as is the case in this example. The given equation clearly gives an LU factorization. However, it appears that the following equation gives another LU factorization.

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

This page titled 2.11: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **2.E: Exercises** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

CHAPTER OVERVIEW

3: Determinants

3.1: Basic Techniques

3.2: Properties of Determinants

3.3: Finding Determinants using Row Operations

3.4: Applications of the Determinant

3.5: Exercises

This page titled 3: Determinants is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

3.1: Basic Techniques

Cofactors and 2 x 2 Determinants

Outcomes

A. Evaluate the determinant of a square matrix using either Laplace Expansion or row operations.
B. Demonstrate the effects that row operations have on determinants.
C. Verify the following:
   a. The determinant of a product of matrices is the product of the determinants.
   b. The determinant of a matrix is equal to the determinant of its transpose.

Let A be an $n \times n$ matrix. That is, let A be a square matrix. The **determinant** of A, denoted by $\det(A)$ is a very important number which we will explore throughout this section.

If A is a 2×2 matrix, the determinant is given by the following formula.

Definition 3.1.1: Determinant of a Two By Two Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$\det(A) = ad - cb$$

The determinant is also often denoted by enclosing the matrix with two vertical lines. Thus

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The following is an example of finding the determinant of a 2×2 matrix.

Example 3.1.1: A Two by Two Determinant

Find $\det(A)$ for the matrix $A = \begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix}$.

Solution

From Definition 3.1.1,

$$\det(A) = (2)(6) - (-1)(4) = 12 + 4 = 16$$

The 2×2 determinant can be used to find the determinant of larger matrices. We will now explore how to find the determinant of a 3×3 matrix, using several tools including the 2×2 determinant.

We begin with the following definition.

Definition 3.1.2: The ij^{th} Minor of a Matrix

Let A be a 3×3 matrix. The ij^{th} **minor** of A, denoted as $minor(A)_{ij}$, is the determinant of the 2×2 matrix which results from deleting the ith row and the jth column of A.

In general, if A is an $n \times n$ matrix, then the ij^{th} minor of A is the determinant of the $n-1 \times n-1$ matrix which results from deleting the ith row and the jth column of A.

Hence, there is a minor associated with each entry of A. Consider the following example which demonstrates this definition.

Example 3.1.2: Finding Minors of a Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $minor(A)_{12}$ and $minor(A)_{23}$.

Solution

First we will find $minor(A)_{12}$. By Definition 3.1.2, this is the determinant of the 2×2 matrix which results when you delete the first row and the second column. This minor is given by

$$minor(A)_{12} = \det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

Using Definition 3.1.1, we see that

$$\det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = (4)(1) - (3)(2) = 4 - 6 = -2$$

Therefore $minor(A)_{12} = -2$.

Similarly, $minor(A)_{23}$ is the determinant of the 2×2 matrix which results when you delete the second row and the third column. This minor is therefore

$$minor(A)_{23} = \det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

Finding the other minors of A is left as an exercise.

The ij^{th} minor of a matrix A is used in another important definition, given next.

Definition 3.1.3: The ij^{th} Cofactor of a Matrix

Suppose A is an $n \times n$ matrix. The ij^{th} **cofactor**, denoted by $\mathrm{cof}(A)_{ij}$ is defined to be

$$\mathrm{cof}(A)_{ij} = (-1)^{i+j} minor(A)_{ij}$$

It is also convenient to refer to the cofactor of an entry of a matrix as follows. If a_{ij} is the ij^{th} entry of the matrix, then its cofactor is just $\mathrm{cof}(A)_{ij}$.

Example 3.1.3: Finding Cofactors of a Matrix

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\mathrm{cof}(A)_{12}$ and $\mathrm{cof}(A)_{23}$.

Solution

We will use Definition 3.1.3 to compute these cofactors.

First, we will compute $\mathrm{cof}(A)_{12}$. Therefore, we need to find $minor(A)_{12}$. This is the determinant of the 2×2 matrix which results when you delete the first row and the second column. Thus $minor(A)_{12}$ is given by

$$\det \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} = -2$$

Then,

$$\text{cof}(A)_{12} = (-1)^{1+2} minor(A)_{12} = (-1)^{1+2}(-2) = 2$$

Hence, $\text{cof}(A)_{12} = 2$.

Similarly, we can find $\text{cof}(A)_{23}$. First, find $minor(A)_{23}$, which is the determinant of the 2×2 matrix which results when you delete the second row and the third column. This minor is therefore

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} = -4$$

Hence,

$$\text{cof}(A)_{23} = (-1)^{2+3} minor(A)_{23} = (-1)^{2+3}(-4) = 4$$

You may wish to find the remaining cofactors for the above matrix. Remember that there is a cofactor for every entry in the matrix.

We have now established the tools we need to find the determinant of a 3×3 matrix.

Definition 3.1.4: The Determinant of a Three By Three Matrix

Let A be a 3×3 matrix. Then, $\det(A)$ is calculated by picking a row (or column) and taking the product of each entry in that row (column) with its cofactor and adding these products together.

This process when applied to the ith row (column) is known as **expanding along the ith row (column)** as is given by

$$\det(A) = a_{i1}\text{cof}(A)_{i1} + a_{i2}\text{cof}(A)_{i2} + a_{i3}\text{cof}(A)_{i3}$$

When calculating the determinant, you can choose to expand any row or any column. Regardless of your choice, you will always get the same number which is the determinant of the matrix A. This method of evaluating a determinant by expanding along a row or a column is called **Laplace Expansion** or **Cofactor Expansion**.

Consider the following example.

Example 3.1.4: Finding the Determinant of a Three by Three Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Find $\det(A)$ using the method of Laplace Expansion.

Solution

First, we will calculate $\det(A)$ by expanding along the first column. Using Definition 3.1.4, we take the 1 in the first column and multiply it by its cofactor,

$$1(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = (1)(1)(-1) = -1$$

Similarly, we take the 4 in the first column and multiply it by its cofactor, as well as with the 3 in the first column. Finally, we add these numbers together, as given in the following equation.

$$\det(A) = \overbrace{1(-1)^{1+1}\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{11}} + \overbrace{4(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{21}} + \overbrace{3(-1)^{3+1}\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix}}^{\text{cof}(A)_{31}}$$

Calculating each of these, we obtain

$$\det(A) = 1(1)(-1) + 4(-1)(-4) + 3(1)(-5) = -1 + 16 + -15 = 0$$

Hence, $\det(A) = 0$.

As mentioned in Definition 3.1.4, we can choose to expand along any row or column. Let's try now by expanding along the second row. Here, we take the 4 in the second row and multiply it to its cofactor, then add this to the 3 in the second row multiplied by its cofactor, and the 2 in the second row multiplied by its cofactor. The calculation is as follows.

$$\det(A) = 4\overbrace{(-1)^{2+1}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix}}^{\text{cof}(A)_{21}} + 3\overbrace{(-1)^{2+2}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix}}^{\text{cof}(A)_{22}} + 2\overbrace{(-1)^{2+3}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}}^{\text{cof}(A)_{23}}$$

Calculating each of these products, we obtain

$$\det(A) = 4(-1)(-2) + 3(1)(-8) + 2(-1)(-4) = 0$$

You can see that for both methods, we obtained $\det(A) = 0$.

As mentioned above, we will always come up with the same value for $\det(A)$ regardless of the row or column we choose to expand along. You should try to compute the above determinant by expanding along other rows and columns. This is a good way to check your work, because you should come up with the same number each time!

We present this idea formally in the following theorem.

Theorem 3.1.1: The Determinant is Well Defined

Expanding the $n \times n$ matrix along any row or column always gives the same answer, which is the determinant.

We have now looked at the determinant of 2×2 and 3×3 matrices. It turns out that the method used to calculate the determinant of a 3×3 matrix can be used to calculate the determinant of any sized matrix. Notice that Definition 3.1.2, Definition 3.1.3 and Definition 3.1.4 can all be applied to a matrix of any size.

For example, the ij^{th} minor of a 4×4 matrix is the determinant of the 3×3 matrix you obtain when you delete the ith row and the jth column. Just as with the 3×3 determinant, we can compute the determinant of a 4×4 matrix by Laplace Expansion, along any row or column

Consider the following example.

Example 3.1.5: Determinant of a Four by Four Matrix

Find $\det(A)$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 3 & 4 & 5 \\ 3 & 4 & 3 & 2 \end{bmatrix}$$

Solution

As in the case of a 3×3 matrix, you can expand this along any row or column. Lets pick the third column. Then, using Laplace Expansion,

$$\det(A) = 3(-1)^{1+3}\begin{vmatrix} 5 & 4 & 3 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} + 2(-1)^{2+3}\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} +$$

$$4(-1)^{3+3}\begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 3 & 4 & 2 \end{vmatrix} + 3(-1)^{4+3}\begin{vmatrix} 1 & 2 & 4 \\ 5 & 4 & 3 \\ 1 & 3 & 5 \end{vmatrix}$$

Now, you can calculate each 3×3 determinant using Laplace Expansion, as we did above. You should complete these as an exercise and verify that $\det(A) = -12$.

The following provides a formal definition for the determinant of an $n \times n$ matrix. You may wish to take a moment and consider the above definitions for 2×2 and 3×3 determinants in context of this definition.

Definition 3.1.5: The Determinant of an $n \times n$ Matrix

Let A be an $n \times n$ matrix where $n \geq 2$ and suppose the determinant of an $(n-1) \times (n-1)$ has been defined. Then

$$\det(A) = \sum_{j=1}^{n} a_{ij}\text{cof}(A)_{ij} = \sum_{i=1}^{n} a_{ij}\text{cof}(A)_{ij}$$

The first formula consists of expanding the determinant along the ith row and the second expands the determinant along the jth column.

In the following sections, we will explore some important properties and characteristics of the determinant.

The Determinant of a Triangular Matrix

There is a certain type of matrix for which finding the determinant is a very simple procedure. Consider the following definition.

Definition 3.1.6: Triangular Matrices

A matrix A is upper triangular if $a_{ij} = 0$ whenever $i > j$. Thus the entries of such a matrix below the main diagonal equal 0, as shown. Here, $*$ refers to any nonzero number.

$$\begin{bmatrix} * & * & \cdots & * \\ 0 & * & \cdots & \vdots \\ \vdots & \vdots & \ddots & * \\ 0 & \cdots & 0 & * \end{bmatrix}$$

A lower triangular matrix is defined similarly as a matrix for which all entries above the main diagonal are equal to zero.

The following theorem provides a useful way to calculate the determinant of a triangular matrix.

Theorem 3.1.2: Determinant of a Triangular Matrix

Let A be an upper or lower triangular matrix. Then $\det(A)$ is obtained by taking the product of the entries on the main diagonal.

The verification of this Theorem can be done by computing the determinant using Laplace Expansion along the first row or column.

Consider the following example.

Example 3.1.6: Determinant of a Triangular Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 77 \\ 0 & 2 & 6 & 7 \\ 0 & 0 & 3 & 33.7 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Find $\det(A)$.

Solution

From Theorem 3.1.2, it suffices to take the product of the elements on the main diagonal. Thus $\det(A) = 1 \times 2 \times 3 \times (-1) = -6$.

Without using Theorem 3.1.2, you could use Laplace Expansion. We will expand along the first column. This gives

$$\begin{aligned} \det(A) = & 1\begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{2+1}\begin{vmatrix} 2 & 3 & 77 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix} + \\ & 0(-1)^{3+1}\begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 0 & -1 \end{vmatrix} + 0(-1)^{4+1}\begin{vmatrix} 2 & 3 & 77 \\ 2 & 6 & 7 \\ 0 & 3 & 33.7 \end{vmatrix} \end{aligned} \tag{3.1.1}$$

and the only nonzero term in the expansion is

$$1\begin{vmatrix} 2 & 6 & 7 \\ 0 & 3 & 33.7 \\ 0 & 0 & -1 \end{vmatrix}$$

Now find the determinant of this 3×3 matrix, by expanding along the first column to obtain

$$\begin{aligned} \det(A) &= 1 \times \left(2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{2+1}\begin{vmatrix} 6 & 7 \\ 0 & -1 \end{vmatrix} + 0(-1)^{3+1}\begin{vmatrix} 6 & 7 \\ 3 & 33.7 \end{vmatrix}\right) \\ &= 1 \times 2 \times \begin{vmatrix} 3 & 33.7 \\ 0 & -1 \end{vmatrix} \end{aligned}$$

Next use Definition 3.1.1 to find the determinant of this 2×2 matrix, which is just $3 \times -1 - 0 \times 33.7 = -3$. Putting all these steps together, we have

$$\det(A) = 1 \times 2 \times 3 \times (-1) = -6$$

which is just the product of the entries down the main diagonal of the original matrix!

You can see that while both methods result in the same answer, Theorem 3.1.2 provides a much quicker method.

In the next section, we explore some important properties of determinants.

This page titled 3.1: Basic Techniques is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **3.1: Basic Techniques** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

3.2: Properties of Determinants

Properties of Determinants I: Examples

There are many important properties of determinants. Since many of these properties involve the row operations discussed in Chapter 1, we recall that definition now.

Definition 3.2.1: Row Operations

The row operations consist of the following

1. Switch two rows.
2. Multiply a row by a nonzero number.
3. Replace a row by a multiple of another row added to itself.

We will now consider the effect of row operations on the determinant of a matrix. In future sections, we will see that using the following properties can greatly assist in finding determinants. This section will use the theorems as motivation to provide various examples of the usefulness of the properties.

The first theorem explains the affect on the determinant of a matrix when two rows are switched.

Theorem 3.2.1: Switching Rows

Let A be an $n \times n$ matrix and let B be a matrix which results from switching two rows of A. Then $\det(B) = -\det(A)$.

When we switch two rows of a matrix, the determinant is multiplied by -1. Consider the following example.

Example 3.2.1: Switching Two Rows

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$. Knowing that $\det(A) = -2$, find $\det(B)$.

Solution

By Definition 3.1.1, $\det(A) = 1 \times 4 - 3 \times 2 = -2$. Notice that the rows of B are the rows of A but switched. By Theorem 3.2.1 since two rows of A have been switched, $\det(B) = -\det(A) = -(-2) = 2$. You can verify this using Definition 3.1.1.

The next theorem demonstrates the effect on the determinant of a matrix when we multiply a row by a scalar.

Theorem 3.2.2: Multiplying a Row by a Scalar

Let A be an $n \times n$ matrix and let B be a matrix which results from multiplying some row of A by a scalar k. Then $\det(B) = k\det(A)$.

Notice that this theorem is true when we multiply *one* row of the matrix by k. If we were to multiply *two* rows of A by k to obtain B, we would have $\det(B) = k^2\det(A)$. Suppose we were to multiply all n rows of A by k to obtain the matrix B, so that $B = kA$. Then, $\det(B) = k^n\det(A)$. This gives the next theorem.

Theorem 3.2.3: Scalar Multiplication

Let A and B be $n \times n$ matrices and k a scalar, such that $B = kA$. Then $\det(B) = k^n\det(A)$.

Consider the following example.

Example 3.2.2: Multiplying a Row by 5

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 10 \\ 3 & 4 \end{bmatrix}$. Knowing that $\det(A) = -2$, find $\det(B)$.

Solution

By Definition 3.1.1, $\det(A) = -2$. We can also compute $\det(B)$ using Definition 3.1.1, and we see that $\det(B) = -10$.

Now, let's compute $\det(B)$ using Theorem 3.2.2 and see if we obtain the same answer. Notice that the first row of B is 5 times the first row of A, while the second row of B is equal to the second row of A. By Theorem 3.2.2, $\det(B) = 5 \times \det(A) = 5 \times -2 = -10$.

You can see that this matches our answer above.

Finally, consider the next theorem for the last row operation, that of adding a multiple of a row to another row.

Theorem 3.2.4: Adding a Multiple of a Row to Another Row

Let A be an $n \times n$ matrix and let B be a matrix which results from adding a multiple of a row to another row. Then $\det(A) = \det(B)$.

Therefore, when we add a multiple of a row to another row, the determinant of the matrix is unchanged. Note that if a matrix A contains a row which is a multiple of another row, $\det(A)$ will equal 0. To see this, suppose the first row of A is equal to -1 times the second row. By Theorem 3.2.4, we can add the first row to the second row, and the determinant will be unchanged. However, this row operation will result in a row of zeros. Using Laplace Expansion along the row of zeros, we find that the determinant is 0.

Consider the following example.

Example 3.2.3: Adding a Row to Another Row

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix}$. Find $\det(B)$.

Solution

By Definition 3.1.1, $\det(A) = -2$. Notice that the second row of B is two times the first row of A added to the second row. By Theorem 3.2.1, $\det(B) = \det(A) = -2$. As usual, you can verify this answer using Definition 3.1.1.

Example 3.2.4: Multiple of a Row

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Show that $\det(A) = 0$.

Solution

Using Definition 3.1.1, the determinant is given by

$$\det(A) = 1 \times 4 - 2 \times 2 = 0$$

However notice that the second row is equal to 2 times the first row. Then by the discussion above following Theorem 3.2.4 the determinant will equal 0.

Until now, our focus has primarily been on row operations. However, we can carry out the same operations with columns, rather than rows. The three operations outlined in Definition 3.2.1 can be done with columns instead of rows. In this case, in Theorems 3.2.1, 3.2.2, and 3.2.4 you can replace the word, "row" with the word "column".

There are several other major properties of determinants which do not involve row (or column) operations. The first is the determinant of a product of matrices.

Theorem 3.2.5: Determinant of a Product

Let A and B be two $n \times n$ matrices. Then

$$\det(AB) = \det(A)\det(B)$$

In order to find the determinant of a product of matrices, we can simply take the product of the determinants.

Consider the following example.

Example 3.2.5: The Determinant of a Product

Compare $\det(AB)$ and $\det(A)\det(B)$ for

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

Solution

First compute AB, which is given by

$$AB = \begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix}$$

and so by Definition 3.1.1

$$\det(AB) = \det\begin{bmatrix} 11 & 4 \\ -1 & -4 \end{bmatrix} = -40$$

Now

$$\det(A) = \det\begin{bmatrix} 1 & 2 \\ -3 & 2 \end{bmatrix} = 8$$

and

$$\det(B) = \det\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} = -5$$

Computing $\det(A) \times \det(B)$ we have $8 \times -5 = -40$. This is the same answer as above and you can see that $\det(A)\det(B) = 8 \times (-5) = -40 = \det(AB)$.

Consider the next important property.

Theorem 3.2.6: Determinant of the Transpose

Let A be a matrix where A^T is the transpose of A. Then,

$$\det(A^T) = \det(A)$$

This theorem is illustrated in the following example.

Example 3.2.6: Determinant of the Transpose

Let

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 3 \end{bmatrix}$$

Find $\det(A^T)$.

Solution

First, note that

$$A^T = \begin{bmatrix} 2 & 4 \\ 5 & 3 \end{bmatrix}$$

Using Definition 3.1.1, we can compute $\det(A)$ and $\det(A^T)$. It follows that $\det(A) = 2 \times 3 - 4 \times 5 = -14$ and $\det(A^T) = 2 \times 3 - 5 \times 4 = -14$. Hence, $\det(A) = \det(A^T)$.

The following provides an essential property of the determinant, as well as a useful way to determine if a matrix is invertible.

Theorem 3.2.7: Determinant of the Inverse

Let A be an $n \times n$ matrix. Then A is invertible if and only if $\det(A) \neq 0$. If this is true, it follows that

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Consider the following example.

Example 3.2.7: Determinant of an Invertible Matrix

Let $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$. For each matrix, determine if it is invertible. If so, find the determinant of the inverse.

Solution

Consider the matrix A first. Using Definition 3.1.1 we can find the determinant as follows:

$$\det(A) = 3 \times 4 - 2 \times 6 = 12 - 12 = 0$$

By Theorem 3.2.7 A is not invertible.

Now consider the matrix B. Again by Definition 3.1.1 we have

$$\det(B) = 2 \times 1 - 5 \times 3 = 2 - 15 = -13$$

By Theorem 3.2.7 B is invertible and the determinant of the inverse is given by

$$\begin{aligned} \det(A^{-1}) &= \frac{1}{\det(A)} \\ &= \frac{1}{-13} \\ &= -\frac{1}{13} \end{aligned} \tag{3.2.1}$$

Properties of Determinants II: Some Important Proofs

This section includes some important proofs on determinants and cofactors.

First we recall the definition of a determinant. If $A = [a_{ij}]$ is an $n \times n$ matrix, then $\det A$ is defined by computing the expansion along the first row:

$$\det A = \sum_{i=1}^{n} a_{1,i} \operatorname{cof}(A)_{1,i}. \tag{3.2.2}$$

If $n = 1$ then $\det A = a_{1,1}$.

The following example is straightforward and strongly recommended as a means for getting used to definitions.

Example 3.2.8:

(1) Let E_{ij} be the elementary matrix obtained by interchanging ith and jth rows of I. Then $\det E_{ij} = -1$.

(2) Let E_{ik} be the elementary matrix obtained by multiplying the ith row of I by k. Then $\det E_{ik} = k$.

(3) Let E_{ijk} be the elementary matrix obtained by multiplying ith row of I by k and adding it to its jth row. Then $\det E_{ijk} = 1$.

(4) If C and B are such that CB is defined and the ith row of C consists of zeros, then the ith row of CB consists of zeros.

(5) If E is an elementary matrix, then $\det E = \det E^T$.

Many of the proofs in section use the Principle of Mathematical Induction. This concept is discussed in Appendix A.2 and is reviewed here for convenience. First we check that the assertion is true for $n = 2$ (the case $n = 1$ is either completely trivial or meaningless).

Next, we assume that the assertion is true for $n - 1$ (where $n \geq 3$) and prove it for n. Once this is accomplished, by the Principle of Mathematical Induction we can conclude that the statement is true for all $n \times n$ matrices for every $n \geq 2$.

If A is an $n \times n$ matrix and $1 \leq j \leq n$, then the matrix obtained by removing 1st column and jth row from A is an $n-1 \times n-1$ matrix (we shall denote this matrix by $A(j)$ below). Since these matrices are used in computation of cofactors $\text{cof}(A)_{1,i}$, for $1 \leq i \neq n$, the inductive assumption applies to these matrices.

Consider the following lemma.

Lemma 3.2.1:

If A is an $n \times n$ matrix such that one of its rows consists of zeros, then $\det A = 0$.

Proof

We will prove this lemma using Mathematical Induction.

If $n = 2$ this is easy (check!).

Let $n \geq 3$ be such that every matrix of size $n-1 \times n-1$ with a row consisting of zeros has determinant equal to zero. Let i be such that the ith row of A consists of zeros. Then we have $a_{ij} = 0$ for $1 \leq j \leq n$.

Fix $j \in \{1, 2, \ldots, n\}$ such that $j \neq i$. Then matrix $A(j)$ used in computation of $\text{cof}(A)_{1,j}$ has a row consisting of zeros, and by our inductive assumption $\text{cof}(A)_{1,j} = 0$.

On the other hand, if $j = i$ then $a_{1,j} = 0$. Therefore $a_{1,j}\text{cof}(A)_{1,j} = 0$ for all j and by (3.2.2) we have

$$\det A = \sum_{j=1}^{n} a_{1,j}\text{cof}(A)_{1,j} = 0$$

as each of the summands is equal to 0.

Lemma 3.2.2:

Assume A, B and C are $n \times n$ matrices that for some $1 \leq i \leq n$ satisfy the following.

1. jth rows of all three matrices are identical, for $j \neq i$.
2. Each entry in the jth row of A is the sum of the corresponding entries in jth rows of B and C.

Then $\det A = \det B + \det C$.

Proof

This is not difficult to check for $n = 2$ (do check it!).

Now assume that the statement of Lemma is true for $n-1 \times n-1$ matrices and fix A, B and C as in the statement. The assumptions state that we have $a_{l,j} = b_{l,j} = c_{l,j}$ for $j \neq i$ and for $1 \leq l \leq n$ and $a_{l,i} = b_{l,i} + c_{l,i}$ for all $1 \leq l \leq n$. Therefore $A(i)$

$= B(i) = C(i)$, and $A(j)$ has the property that its ith row is the sum of ith rows of $B(j)$ and $C(j)$ for $j \neq i$ while the other rows of all three matrices are identical. Therefore by our inductive assumption we have $\operatorname{cof}(A)_{1j} = \operatorname{cof}(B)_{1j} + \operatorname{cof}(C)_{1j}$ for $j \neq i$.

By (3.2.2) we have (using all equalities established above)

$$\begin{aligned} \det A &= \sum_{l=1}^{n} a_{1,l}\operatorname{cof}(A)_{1,l} \\ &= \sum_{l\neq i} a_{1,l}(\operatorname{cof}(B)_{1,l} + \operatorname{cof}(C)_{1,l}) + (b_{1,i} + c_{1,i})\operatorname{cof}(A)_{1,i} \\ &= \det B + \det C \end{aligned} \tag{3.2.3}$$

This proves that the assertion is true for all n and completes the proof.

Theorem 3.2.8:

Let A and B be $n \times n$ matrices.

1. If A is obtained by interchanging ith and jth rows of B (with $i \neq j$), then $\det A = -\det B$.
2. If A is obtained by multiplying ith row of B by k then $\det A = k \det B$.
3. If two rows of A are identical then $\det A = 0$.
4. If A is obtained by multiplying ith row of B by k and adding it to jth row of B ($i \neq j$) then $\det A = \det B$.

Proof

We prove all statements by induction. The case $n = 2$ is easily checked directly (and it is strongly suggested that you do check it).

We assume $n \geq 3$ and (1)–(4) are true for all matrices of size $n-1 \times n-1$.

(1) We prove the case when $j = i + 1$, i.e., we are interchanging two consecutive rows.

Let $l \in \{1, \ldots, n\} \setminus \{i, j\}$. Then $A(l)$ is obtained from $B(l)$ by interchanging two of its rows (draw a picture) and by our assumption

$$\operatorname{cof}(A)_{1,l} = -\operatorname{cof}(B)_{1,l}. \tag{3.2.4}$$

Now consider $a_{1,i}\operatorname{cof}(A)_{1,l}$. We have that $a_{1,i} = b_{1,j}$ and also that $A(i) = B(j)$. Since $j = i + 1$, we have

$$(-1)^{1+j} = (-1)^{1+i+1} = -(-1)^{1+i}$$

and therefore $a_{1i}\operatorname{cof}(A)_{1i} = -b_{1j}\operatorname{cof}(B)_{1j}$ and $a_{1j}\operatorname{cof}(A)_{1j} = -b_{1i}\operatorname{cof}(B)_{1i}$. Putting this together with (3.2.4) into (3.2.2) we see that if in the formula for $\det A$ we change the sign of each of the summands we obtain the formula for $\det B$.

$$\det A = \sum_{l=1}^{n} a_{1l}\operatorname{cof}(A)_{1l} = -\sum_{l=1}^{n} b_{1l}B_{1l} = \det B.$$

We have therefore proved the case of (1) when $j = i + 1$. In order to prove the general case, one needs the following fact. If $i < j$, then in order to interchange ith and jth row one can proceed by interchanging two adjacent rows $2(j-i)+1$ times: First swap ith and $i+1$st, then $i+1$st and $i+2$nd, and so on. After one interchanges $j-1$st and jth row, we have ith row in position of jth and lth row in position of $l-1$st for $i+1 \leq l \leq j$. Then proceed backwards swapping adjacent rows until everything is in place.

Since $2(j-i)+1$ is an odd number $(-1)^{2(j-i)+1} = -1$ and we have that $\det A = -\det B$.

(2) This is like (1)… but much easier. Assume that (2) is true for all $n-1 \times n-1$ matrices. We have that $a_{ji} = kb_{ji}$ for $1 \leq j \leq n$. In particular $a_{1i} = kb_{1i}$, and for $l \neq i$ matrix $A(l)$ is obtained from $B(l)$ by multiplying one of its rows by k. Therefore $\operatorname{cof}(A)_{1l} = k\operatorname{cof}(B)_{1l}$ for $l \neq i$, and for all l we have $a_{1l}\operatorname{cof}(A)_{1l} = kb_{1l}\operatorname{cof}(B)_{1l}$. By (3.2.2), we have $\det A = k \det B$.

(3) This is a consequence of (1). If two rows of A are identical, then A is equal to the matrix obtained by interchanging those two rows and therefore by (1) $\det A = -\det A$. This implies $\det A = 0$.

(4) Assume (4) is true for all $n-1 \times n-1$ matrices and fix A and B such that A is obtained by multiplying ith row of B by k and adding it to jth row of B ($i \neq j$) then $\det A = \det B$. If $k = 0$ then $A = B$ and there is nothing to prove, so we may assume $k \neq 0$.

Let C be the matrix obtained by replacing the jth row of B by the ith row of B multiplied by k. By Lemma 3.2.2, we have that

$$\det A = \det B + \det C$$

and we 'only' need to show that $\det C = 0$. But ith and jth rows of C are proportional. If D is obtained by multiplying the j th row of C by $\frac{1}{k}$ then by (2) we have $\det C = \frac{1}{k} \det D$ (recall that $k \neq 0$!). But ith and jth rows of D are identical, hence by (3) we have $\det D = 0$ and therefore $\det C = 0$.

Theorem 3.2.9:

Let A and B be two $n \times n$ matrices. Then

$$\det(AB) = \det(A) \det(B)$$

Proof

If A is an elementary matrix of either type, then multiplying by A on the left has the same effect as performing the corresponding elementary row operation. Therefore the equality $\det(AB) = \det A \det B$ in this case follows by Example 3.2.8 and Theorem 3.2.8.

If C is the reduced row-echelon form of A then we can write $A = E_1 \cdot E_2 \cdot \cdots \cdot E_m \cdot C$ for some elementary matrices $E_1, \ldots, E_m$.

Now we consider two cases.

Assume first that $C = I$. Then $A = E_1 \cdot E_2 \cdot \cdots \cdot E_m$ and $AB = E_1 \cdot E_2 \cdot \cdots \cdot E_m B$. By applying the above equality m times, and then $m-1$ times, we have that

$$\begin{aligned} \det AB &= \det E_1 \det E_2 \cdot \det E_m \cdot \det B \\ &= \det(E_1 \cdot E_2 \cdot \cdots \cdot E_m) \det B \\ &= \det A \det B. \end{aligned} \tag{3.2.5}$$

Now assume $C \neq I$. Since it is in reduced row-echelon form, its last row consists of zeros and by (4) of Example 3.2.8 the last row of CB consists of zeros. By Lemma 3.2.1 we have $\det C = \det(CB) = 0$ and therefore

$$\det A = \det(E_1 \cdot E_2 \cdot E_m) \cdot \det(C) = \det(E_1 \cdot E_2 \cdot E_m) \cdot 0 = 0$$

and also

$$\det AB = \det(E_1 \cdot E_2 \cdot E_m) \cdot \det(CB) = \det(E_1 \cdot E_2 \cdot \cdots \cdot E_m) 0 = 0$$

hence $\det AB = 0 = \det A \det B$.

The same 'machine' used in the previous proof will be used again.

Theorem 3.2.10:

Let A be a matrix where A^T is the transpose of A. Then,

$$\det\left(A^T\right) = \det(A)$$

Proof

Note first that the conclusion is true if A is elementary by (5) of Example 3.2.8.

Let C be the reduced row-echelon form of A. Then we can write $A = E_1 \cdot E_2 \cdot \cdots \cdot E_m C$. Then $A^T = C^T \cdot E_m^T \cdot \cdots \cdot E_2^T \cdot E_1$. By Theorem 3.2.9 we have

$$\det(A^T) = \det(C^T) \cdot \det(E_m^T) \cdot \cdots \cdot \det(E_2^T) \cdot \det(E_1).$$

By (5) of Example 3.2.8 we have that $\det E_j = \det E_j^T$ for all j. Also, $\det C$ is either 0 or 1 (depending on whether $C = I$ or not) and in either case $\det C = \det C^T$. Therefore $\det A = \det A^T$.

The above discussions allow us to now prove Theorem 3.1.1. It is restated below.

Theorem 3.2.11:

Expanding an $n \times n$ matrix along any row or column always gives the same result, which is the determinant.

Proof

We first show that the determinant can be computed along any row. The case $n = 1$ does not apply and thus let $n \geq 2$.

Let Abe an $n \times n$ matrix and fix $j > 1$. We need to prove that

$$\det A = \sum_{i=1}^{n} a_{j,i}\text{cof}(A)_{j,i}.$$

Let us prove the case when $j = 2$.

Let B be the matrix obtained from A by interchanging its 1st and 2nd rows. Then by Theorem 3.2.8 we have

$$\det A = -\det B.$$

Now we have

$$\det B = \sum_{i=1}^{n} b_{1,i}\text{cof}(B)_{1,i}.$$

Since B is obtained by interchanging the 1st and 2nd rows of A we have that $b_{1,i} = a_{2,i}$ for all i and one can see that $minor(B)_{1,i} = minor(A)_{2,i}$.

Further,

$$\text{cof}(B)_{1,i} = (-1)^{1+i} minor B_{1,i} = -(-1)^{2+i} minor(A)_{2,i} = -\text{cof}(A)_{2,i}$$

hence $\det B = -\sum_{i=1}^{n} a_{2,i}\text{cof}(A)_{2,i}$, and therefore $\det A = -\det B = \sum_{i=1}^{n} a_{2,i}\text{cof}(A)_{2,i}$ as desired.

The case when $j > 2$ is very similar; we still have $minor(B)_{1,i} = minor(A)_{j,i}$ but checking that $\det B = -\sum_{i=1}^{n} a_{j,i}\text{cof}(A)_{j,i}$ is slightly more involved.

Now the cofactor expansion along column j of A is equal to the cofactor expansion along row j of A^T, which is by the above result just proved equal to the cofactor expansion along row 1 of A^T, which is equal to the cofactor expansion along column 1 of A. Thus the cofactor cofactor along any column yields the same result.

Finally, since $\det A = \det A^T$ by Theorem 3.2.10, we conclude that the cofactor expansion along row 1 of A is equal to the cofactor expansion along row 1 of A^T, which is equal to the cofactor expansion along column 1 of A. Thus the proof is complete.

This page titled 3.2: Properties of Determinants is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **3.2: Properties of Determinants** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

3.3: Finding Determinants using Row Operations

Theorems 3.2.1, 3.2.2 and 3.2.4 illustrate how row operations affect the determinant of a matrix. In this section, we look at two examples where row operations are used to find the determinant of a large matrix. Recall that when working with large matrices, Laplace Expansion is effective but timely, as there are many steps involved. This section provides useful tools for an alternative method. By first applying row operations, we can obtain a simpler matrix to which we apply Laplace Expansion.

While working through questions such as these, it is useful to record your row operations as you go along. Keep this in mind as you read through the next example.

Example 3.3.1: Finding a Determinant

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 4 & 5 & 4 & 3 \\ 2 & 2 & -4 & 5 \end{bmatrix}$$

Solution

We will use the properties of determinants outlined above to find $\det(A)$. First, add -5 times the first row to the second row. Then add -4 times the first row to the third row, and -2 times the first row to the fourth row. This yields the matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -9 & -13 & -17 \\ 0 & -3 & -8 & -13 \\ 0 & -2 & -10 & -3 \end{bmatrix}$$

Notice that the only row operation we have done so far is adding a multiple of a row to another row. Therefore, by Theorem 3.2.4, $\det(B) = \det(A)$.

At this stage, you could use Laplace Expansion to find $\det(B)$. However, we will continue with row operations to find an even simpler matrix to work with.

Add -3 times the third row to the second row. By Theorem 3.2.4 this does not change the value of the determinant. Then, multiply the fourth row by -3. This results in the matrix

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 11 & 22 \\ 0 & -3 & -8 & -13 \\ 0 & 6 & 30 & 9 \end{bmatrix}$$

Here, $\det(C) = -3\det(B)$, which means that $\det(B) = \left(-\frac{1}{3}\right)\det(C)$

Since $\det(A) = \det(B)$, we now have that $\det(A) = \left(-\frac{1}{3}\right)\det(C)$. Again, you could use Laplace Expansion here to find $\det(C)$. However, we will continue with row operations.

Now replace the add 2 times the third row to the fourth row. This does not change the value of the determinant by Theorem 3.2.4. Finally switch the third and second rows. This causes the determinant to be multiplied by -1. Thus $\det(C) = -\det(D)$ where

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -8 & -13 \\ 0 & 0 & 11 & 22 \\ 0 & 0 & 14 & -17 \end{bmatrix}$$

Hence, $\det(A) = \left(-\frac{1}{3}\right)\det(C) = \left(\frac{1}{3}\right)\det(D)$

You could do more row operations or you could note that this can be easily expanded along the first column. Then, expand the resulting 3×3 matrix also along the first column. This results in

$$\det(D) = 1(-3)\begin{vmatrix} 11 & 22 \\ 14 & -17 \end{vmatrix} = 1485$$

and so $\det(A) = \left(\frac{1}{3}\right)(1485) = 495$.

You can see that by using row operations, we can simplify a matrix to the point where Laplace Expansion involves only a few steps. In Example 3.3.1, we also could have continued until the matrix was in upper triangular form, and taken the product of the entries on the main diagonal. Whenever computing the determinant, it is useful to consider all the possible methods and tools.

Consider the next example.

Example 3.3.2: Find the Determinant

Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & -3 & 2 & 1 \\ 2 & 1 & 2 & 5 \\ 3 & -4 & 1 & 2 \end{bmatrix}$$

Solution

Once again, we will simplify the matrix through row operations. Add -1 times the first row to the second row. Next add -2 times the first row to the third and finally take -3 times the first row and add to the fourth row. This yields

$$B = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -5 & -1 & -1 \\ 0 & -3 & -4 & 1 \\ 0 & -10 & -8 & -4 \end{bmatrix}$$

By Theorem 3.2.4, $\det(A) = \det(B)$.

Remember you can work with the columns also. Take -5 times the fourth column and add to the second column. This yields

$$C = \begin{bmatrix} 1 & -8 & 3 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{bmatrix}$$

By Theorem 3.2.4 $\det(A) = \det(C)$.

Now take -1 times the third row and add to the top row. This gives.

$$D = \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & -8 & -4 & 1 \\ 0 & 10 & -8 & -4 \end{bmatrix}$$

which by Theorem 3.2.4 has the same determinant as A.

Now, we can find $\det(D)$ by expanding along the first column as follows. You can see that there will be only one non zero term.

$$\det(D) = 1 \det \begin{bmatrix} 0 & -1 & -1 \\ -8 & -4 & 1 \\ 10 & -8 & -4 \end{bmatrix} + 0 + 0 + 0$$

Expanding again along the first column, we have

$$\det(D) = 1\left(0 + 8\det\begin{bmatrix} -1 & -1 \\ -8 & -4 \end{bmatrix} + 10\det\begin{bmatrix} -1 & -1 \\ -4 & 1 \end{bmatrix}\right) = -82$$

Now since $\det(A) = \det(D)$, it follows that $\det(A) = -82$.

Remember that you can verify these answers by using Laplace Expansion on A. Similarly, if you first compute the determinant using Laplace Expansion, you can use the row operation method to verify.

This page titled 3.3: Finding Determinants using Row Operations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **3.3: Finding Determinants using Row Operations** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

3.4: Applications of the Determinant

Outcomes

A. Use determinants to determine whether a matrix has an inverse, and evaluate the inverse using cofactors.
B. Apply Cramer's Rule to solve a 2×2 or a 3×3 linear system.
C. Given data points, find an appropriate interpolating polynomial and use it to estimate points.

A Formula for the Inverse

The determinant of a matrix also provides a way to find the inverse of a matrix. Recall the definition of the inverse of a matrix in Definition 2.6.1. We say that A^{-1}, an $n \times n$ matrix, is the inverse of A, also $n \times n$, if $AA^{-1} = I$ and $A^{-1}A = I$.

We now define a new matrix called the **cofactor matrix** of A. The cofactor matrix of A is the matrix whose ij^{th} entry is the ij^{th} cofactor of A. The formal definition is as follows.

Definition 3.4.1: The Cofactor Matrix

Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then the **cofactor matrix of A**, denoted $\mathrm{cof}(A)$, is defined by $\mathrm{cof}(A) = \left[\mathrm{cof}(A)_{ij}\right]$ where $\mathrm{cof}(A)_{ij}$ is the ij^{th} cofactor of A.

Note that $\mathrm{cof}(A)_{ij}$ denotes the ij^{th} entry of the cofactor matrix.

We will use the cofactor matrix to create a formula for the inverse of A. First, we define the **adjugate** of A to be the transpose of the cofactor matrix. We can also call this matrix the **classical adjoint** of A, and we denote it by $adj(A)$.

In the specific case where A is a 2×2 matrix given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $adj(A)$ is given by

$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In general, $adj(A)$ can always be found by taking the transpose of the cofactor matrix of A. The following theorem provides a formula for A^{-1} using the determinant and adjugate of A.

Theorem 3.4.1: The Inverse and the Determinant

Let A be an $n \times n$ matrix. Then

$$A\, adj(A) = adj(A)\, A = \det(A) I$$

Moreover A is invertible if and only if $\det(A) \neq 0$. In this case we have:

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

Notice that the first formula holds for any $n \times n$ matrix A, and in the case A is invertible we actually have a formula for A^{-1}.

Consider the following example.

Example 3.4.1: Find Inverse Using the Determinant

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

using the formula in Theorem 3.4.1.

Solution

According to Theorem 3.4.1,

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

First we will find the determinant of this matrix. Using Theorems 3.2.1, 3.2.2, and 3.2.4, we can first simplify the matrix through row operations. First, add -3 times the first row to the second row. Then add -1 times the first row to the third row to obtain

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & -8 \\ 0 & 0 & -2 \end{bmatrix}$$

By Theorem 3.2.4, $\det(A) = \det(B)$. By Theorem 3.1.2, $\det(B) = 1 \times -6 \times -2 = 12$. Hence, $\det(A) = 12$.

Now, we need to find $adj(A)$. To do so, first we will find the cofactor matrix of A. This is given by

$$\text{cof}(A) = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix}$$

Here, the ij^{th} entry is the ij^{th} cofactor of the original matrix A which you can verify. Therefore, from Theorem 3.4.1, the inverse of A is given by

$$A^{-1} = \frac{1}{12} \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix}^T = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

Remember that we can always verify our answer for A^{-1}. Compute the product AA^{-1} and $A^{-1}A$ and make sure each product is equal to I.

Compute $A^{-1}A$ as follows

$$A^{-1}A = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

You can verify that $AA^{-1} = I$ and hence our answer is correct.

We will look at another example of how to use this formula to find A^{-1}.

Example 3.4.2: Find the Inverse From a Formula

Find the inverse of the matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix}$$

using the formula given in Theorem 3.4.1.

Solution

First we need to find $\det(A)$. This step is left as an exercise and you should verify that $\det(A) = \frac{1}{6}$. The inverse is therefore equal to

$$A^{-1} = \frac{1}{(1/6)}\, adj(A) = 6\, adj(A)$$

We continue to calculate as follows. Here we show the 2×2 determinants needed to find the cofactors.

$$A^{-1} = 6 \begin{bmatrix} \begin{vmatrix} \frac{1}{3} & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} -\frac{1}{6} & -\frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} -\frac{1}{6} & \frac{1}{3} \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ -\begin{vmatrix} 0 & \frac{1}{2} \\ \frac{2}{3} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{5}{6} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{5}{6} & \frac{2}{3} \end{vmatrix} \\ \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} \end{vmatrix} & -\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{2} \end{vmatrix} & \begin{vmatrix} \frac{1}{2} & 0 \\ -\frac{1}{6} & \frac{1}{3} \end{vmatrix} \end{bmatrix}^T$$

Expanding all the 2×2 determinants, this yields

$$A^{-1} = 6 \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}^T = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$$

Again, you can always check your work by multiplying $A^{-1}A$ and AA^{-1} and ensuring these products equal I.

$$A^{-1}A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This tells us that our calculation for A^{-1} is correct. It is left to the reader to verify that $AA^{-1} = I$.

The verification step is very important, as it is a simple way to check your work! If you multiply $A^{-1}A$ and AA^{-1} and these products are not both equal to I, be sure to go back and double check each step. One common error is to forget to take the transpose of the cofactor matrix, so be sure to complete this step.

We will now prove Theorem 3.4.1.

Theorem 3.4.1: The Inverse and the Determinant

Proof

(of Theorem 3.4.1) Recall that the (i,j)-entry of $adj(A)$ is equal to $\text{cof}(A)_{ji}$. Thus the (i,j)-entry of $B = A \cdot adj(A)$ is :

$$B_{ij} = \sum_{k=1}^{n} a_{ik} adj(A)_{kj} = \sum_{k=1}^{n} a_{ik} \text{cof}(A)_{jk}$$

By the cofactor expansion theorem, we see that this expression for B_{ij} is equal to the determinant of the matrix obtained from A by replacing its jth row by $a_{i1}, a_{i2}, \ldots a_{in}$ — i.e., its ith row.

If $i = j$ then this matrix is A itself and therefore $B_{ii} = \det A$. If on the other hand $i \neq j$, then this matrix has its ith row equal to its jth row, and therefore $B_{ij} = 0$ in his case. Thus we obtain:

$$A\, adj(A) = \det(A) I$$

Similarly we can verify that:

$$adj(A)\, A = \det(A) I$$

And this proves the first part of the theorem.

Further if A is invertible, then by Theorem 3.2.5 we have:

$$1 = \det(I) = \det\left(AA^{-1}\right) = \det(A)\det\left(A^{-1}\right)$$

and thus $\det(A) \neq 0$. Equivalently, if $\det(A) = 0$, then A is not invertible.

Finally if $\det(A) \neq 0$, then the above formula shows that A is invertible and that:

$$A^{-1} = \frac{1}{\det(A)} adj(A)$$

This completes the proof.

This method for finding the inverse of A is useful in many contexts. In particular, it is useful with complicated matrices where the entries are functions, rather than numbers.

Consider the following example.

Example 3.4.3: Inverse for Non-Constant Matrix

Suppose

$$A(t) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}$$

Show that $A(t)^{-1}$ exists and then find it.

Solution

First note $\det(A(t)) = e^t(\cos^2 t + \sin^2 t) = e^t \neq 0$ so $A(t)^{-1}$ exists.

The cofactor matrix is

$$C(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix}$$

and so the inverse is

$$\frac{1}{e^t}\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix}^T = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$$

Cramer's Rule

Another context in which the formula given in Theorem 3.4.1 is important is **Cramer's Rule**. Recall that we can represent a system of linear equations in the form $AX = B$, where the solutions to this system are given by X. Cramer's Rule gives a formula for the solutions X in the special case that A is a square invertible matrix. Note this rule does not apply if you have a system of equations in which there is a different number of equations than variables (in other words, when A is not square), or when A is not invertible.

Suppose we have a system of equations given by $AX = B$, and we want to find solutions X which satisfy this system. Then recall that if A^{-1} exists,

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B \\ \left(A^{-1}A\right)X &= A^{-1}B \\ IX &= A^{-1}B \\ X &= A^{-1}B \end{aligned} \tag{3.4.1}$$

Hence, the solutions X to the system are given by $X = A^{-1}B$. Since we assume that A^{-1} exists, we can use the formula for A^{-1} given above. Substituting this formula into the equation for X, we have

$$X = A^{-1}B = \frac{1}{\det(A)} adj(A)\, B$$

Let x_i be the ith entry of X and b_j be the jth entry of B. Then this equation becomes

$$x_i = \sum_{j=1}^{n} [a_{ij}]^{-1} b_j = \sum_{j=1}^{n} \frac{1}{\det(A)} adj(A)_{ij} b_j$$

where $adj(A)_{ij}$ is the ij^{th} entry of $adj(A)$.

By the formula for the expansion of a determinant along a column,

$$x_i = \frac{1}{\det(A)} \det \begin{bmatrix} * & \cdots & b_1 & \cdots & * \\ \vdots & & \vdots & & \vdots \\ * & \cdots & b_n & \cdots & * \end{bmatrix}$$

where here the ith column of A is replaced with the column vector $[b_1 \cdots, b_n]^T$. The determinant of this modified matrix is taken and divided by $\det(A)$. This formula is known as Cramer's rule.

We formally define this method now.

Procedure 3.4.1: Using Cramer's Rule

Suppose A is an $n \times n$ invertible matrix and we wish to solve the system $AX = B$ for $X = [x_1, \cdots, x_n]^T$. Then Cramer's rule says

$$x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is the matrix obtained by replacing the ith column of A with the column matrix

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

We illustrate this procedure in the following example.

Example 3.4.4: Using Cramer's Rule

Find x, y, z if

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Solution

We will use method outlined in Procedure 3.4.1 to find the values for x, y, z which give the solution to this system. Let

$$B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

In order to find x, we calculate

$$x = \frac{\det(A_1)}{\det(A)}$$

where A_1 is the matrix obtained from replacing the first column of A with B.

Hence, A_1 is given by

$$A_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & -3 & 2 \end{bmatrix}$$

Therefore,

$$x = \frac{\det(A_1)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 3 & -3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = \frac{1}{2}$$

Similarly, to find y we construct A_2 by replacing the second column of A with B. Hence, A_2 is given by

$$A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$$

Therefore,

$$y = \frac{\det(A_2)}{\det(A)} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = -\frac{1}{7}$$

Similarly, A_3 is constructed by replacing the third column of A with B. Then, A_3 is given by

$$A_3 = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & -3 & 3 \end{bmatrix}$$

Therefore, z is calculated as follows.

$$z = \frac{\det(A_3)}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 2 \\ 2 & -3 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & -3 & 2 \end{vmatrix}} = \frac{11}{14}$$

Cramer's Rule gives you another tool to consider when solving a system of linear equations.

We can also use Cramer's Rule for systems of non linear equations. Consider the following system where the matrix A has functions rather than numbers for entries.

Using Cramer's Rule

Example 3.4.5: Use Cramer's Rule for Non-Constant Matrix

Solve for z if

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ t \\ 0.05int^2 \end{bmatrix}$$

Solution

We are asked to find the value of z in the solution. We will solve using Cramer's rule. Thus

$$z = .05in \frac{\begin{vmatrix} 1 & 0 & 1 \\ 0 & e^t \cos t & t \\ 0 & -e^t \sin t & t^2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & e^t \cos t & e^t \sin t \\ 0 & -e^t \sin t & e^t \cos t \end{vmatrix}} = t\left((\cos t)\, t + \sin t\right) e^{-t}$$

Polynomial Interpolation

In studying a set of data that relates variables x and y, it may be the case that we can use a polynomial to "fit" to the data. If such a polynomial can be established, it can be used to estimate values of x and y which have not been provided.

Consider the following example.

Example 3.4.6: Polynomial Interpolation

Given data points $(1,4),(2,9),(3,12)$, find an interpolating polynomial $p(x)$ of degree at most 2 and then estimate the value corresponding to $x = \frac{1}{2}$.

Solution

We want to find a polynomial given by

$$p(x) = r_0 + r_1 x_1 + r_2 x_2^2$$

such that $p(1) = 4, p(2) = 9$ and $p(3) = 12$. To find this polynomial, substitute the known values in for x and solve for r_0, r_1, and r_2.

$$\begin{aligned} p(1) &= r_0 + r_1 + r_2 = 4 \\ p(2) &= r_0 + 2r_1 + 4r_2 = 9 \\ p(3) &= r_0 + 3r_1 + 9r_2 = 12 \end{aligned} \tag{3.4.2}$$

Writing the augmented matrix, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 9 \\ 1 & 3 & 9 & 12 \end{array}\right]$$

After row operations, the resulting matrix is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Therefore the solution to the system is $r_0 = -3, r_1 = 8, r_2 = -1$ and the required interpolating polynomial is

$$p(x) = -3 + 8x - x^2$$

To estimate the value for $x = \frac{1}{2}$, we calculate $p(\frac{1}{2})$:

$$\begin{aligned} p(\frac{1}{2}) &= -3 + 8(\frac{1}{2}) - (\frac{1}{2})^2 \\ &= -3 + 4 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned} \tag{3.4.3}$$

This procedure can be used for any number of data points, and any degree of polynomial. The steps are outlined below.

Procedure 3.4.2: Finding an Interpolation Polynomial

Suppose that values of x and corresponding values of y are given, such that the actual relationship between x and y is unknown. Then, values of y can be estimated using an **interpolating polynomial** $p(x)$. If given $x_1, \ldots, x_n$ and the corresponding $y_1, \ldots, y_n$, the procedure to find $p(x)$ is as follows:

1. The desired polynomial $p(x)$ is given by

$$p(x) = r_0 + r_1 x + r_2 x^2 + \ldots + r_{n-1} x^{n-1}$$

2. $p(x_i) = y_i$ for all $i = 1, 2, \ldots, n$ so that

$$\begin{aligned} r_0 + r_1 x_1 + r_2 x_1^2 + \ldots + r_{n-1} x_1^{n-1} &= y_1 \\ r_0 + r_1 x_2 + r_2 x_2^2 + \ldots + r_{n-1} x_2^{n-1} &= y_2 \\ &\vdots \\ r_0 + r_1 x_n + r_2 x_n^2 + \ldots + r_{n-1} x_n^{n-1} &= y_n \end{aligned}$$

3. Set up the augmented matrix of this system of equations

$$\left[\begin{array}{ccccc|c} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & y_1 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & y_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & y_n \end{array}\right]$$

4. Solving this system will result in a unique solution $r_0, r_1, \cdots, r_{n-1}$. Use these values to construct $p(x)$, and estimate the value of $p(a)$ for any $x = a$.

This procedure motivates the following theorem.

Theorem 3.4.2: Polynomial Interpolation

Given n data points $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial $p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$ such that $p(x_i) = y_i$ for $i = 1, 2, \cdots, n$. The resulting polynomial $p(x)$ is called the **interpolating polynomial** for the data points.

We conclude this section with another example.

Example 3.4.7: Polynomial Interpolation

Consider the data points $(0, 1), (1, 2), (3, 22), (5, 66)$. Find an interpolating polynomial $p(x)$ of degree at most three, and estimate the value of $p(2)$.

Solution

The desired polynomial $p(x)$ is given by:

$$p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$$

Using the given points, the system of equations is

$$\begin{aligned} p(0) &= r_0 = 1 \\ p(1) &= r_0 + r_1 + r_2 + r_3 = 2 \\ p(3) &= r_0 + 3r_1 + 9r_2 + 27r_3 = 22 \\ p(5) &= r_0 + 5r_1 + 25r_2 + 125r_3 = 66 \end{aligned} \tag{3.4.4}$$

The augmented matrix is given by:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 3 & 9 & 27 & 22 \\ 1 & 5 & 25 & 125 & 66 \end{array}\right]$$

The resulting matrix is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, $r_0 = 1, r_1 = -2, r_2 = 3, r_3 = 0$ and $p(x) = 1 - 2x + 3x^2$. To estimate the value of $p(2)$, we compute $p(2) = 1 - 2(2) + 3(2^2) = 1 - 4 + 12 = 9$.

This page titled 3.4: Applications of the Determinant is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **3.4: Applications of the Determinant** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

3.5: Exercises

Exercise 3.5.1

Find the determinants of the following matrices.

a. $\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 3 \\ 0 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix}$

Exercise 3.5.2

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \\ -2 & 5 & 1 \end{bmatrix}$. Find the following.

a. $minor(A)_{11}$

b. $minor(A)_{21}$

c. $minor(A)_{32}$

d. $cof(A)_{11}$

e. $cof(A)_{21}$

f. $cof(A)_{32}$

Exercise 3.5.3

Find the determinants of the following matrices.

a. $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{bmatrix}$

b. $\begin{bmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & -9 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 1 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$

Answer

a. The answer is **31**.

b. The answer is **375**.

c. The answer is **−2**.

Exercise 3.5.4

Find the following determinant by expanding along the first row and second column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

Answer

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 6$$

Exercise 3.5.5

Find the following determinant by expanding along the first column and third row.

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

Answer

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 2$$

Exercise 3.5.6

Find the following determinant by expanding along the second row and first column.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix}$$

Answer

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 6$$

Exercise 3.5.7

Compute the determinant by cofactor expansion. Pick the easiest row or column to use.

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix}$$

Answer

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 1 & 3 & 1 \end{vmatrix} = -4$$

Exercise 3.5.8

Find the determinant of the following matrices.

a. $A = \begin{bmatrix} 1 & -34 \\ 0 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 4 & 3 & 14 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 3 & 15 & 0 \\ 0 & 4 & 1 & 7 \\ 0 & 0 & -3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

? Exercise 3.5.9

An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \cdots \to \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Answer

It does not change the determinant. This was just taking the transpose.

? Exercise 3.5.10

An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \cdots \to \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

Answer

In this case two rows were switched and so the resulting determinant is -1 times the first

? Exercise 3.5.11

An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \cdots \to \begin{bmatrix} a & b \\ a+c & b+d \end{bmatrix}$$

Answer

The determinant is unchanged. It was just the first row added to the second.

? Exercise 3.5.12

An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \cdots \to \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$$

Answer

The second row was multiplied by 2 so the determinant of the result is 2 times the original determinant.

? Exercise 3.5.13

An operation is done to get from the first matrix to the second. Identify what was done and tell how it will affect the value of the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \cdots \to \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

Answer

In this case the two columns were switched so the determinant of the second is -1 times the determinant of the first.

? Exercise 3.5.14

Let A be an $r \times r$ matrix and suppose there are $r-1$ rows (columns) such that all rows (columns) are linear combinations of these $r-1$ rows (columns). Show $\det(A) = 0$.

Answer

If the determinant is nonzero, then it will remain nonzero with row operations applied to the matrix. However, by assumption, you can obtain a row of zeros by doing row operations. Thus the determinant must have been zero after all.

? Exercise 3.5.15

Show $\det(aA) = a^n \det(A)$ for an $n \times n$ matrix A and scalar a.

Answer

$\det(aA) = \det(aIA) = \det(aI)\det(A) = a^n \det(A)$. The matrix which has a down the main diagonal has determinant equal to a^n.

? Exercise 3.5.16

Construct 2×2 matrices A and B to show that the $\det A \det B = \det(AB)$.

Answer

$$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix}\right) = -8$$

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\det\begin{bmatrix} -1 & 2 \\ -5 & 6 \end{bmatrix} = -2 \times 4 = -8$$

? Exercise 3.5.17

Is it true that $\det(A+B) = \det(A) + \det(B)$? If this is so, explain why. If it is not so, give a counter example.

Answer

This is not true at all. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

? Exercise 3.5.18

An $n \times n$ matrix is called **nilpotent** if for some positive integer, k it follows $A^k = 0$. If A is a nilpotent matrix and k is the smallest possible integer such that $A^k = 0$, what are the possible values of $\det(A)$?

Answer

It must be 0 because $0 = \det(0) = \det(A^k) = (\det(A))^k$.

? Exercise 3.5.19

A matrix is said to be **orthogonal** if $A^TA = I$. Thus the inverse of an orthogonal matrix is just its transpose. What are the possible values of $\det(A)$ if A is an orthogonal matrix?

Answer

You would need $\det(AA^T) = \det(A)\det(A^T) = \det(A)^2 = 1$ and so $\det(A) = 1$, or -1.

? Exercise 3.5.20

Let A and B be two $n \times n$ matrices. $A \sim B$ (A is **similar** to B) means there exists an invertible matrix P such that $A = P^{-1}BP$. Show that if $A \sim B$, then $\det(A) = \det(B)$.

Answer

$\det(A) = \det(S^{-1}BS) = \det(S^{-1})\det(B)\det(S) = \det(B)\det(S^{-1}S) = \det(B)$.

? Exercise 3.5.21

Tell whether each statement is true or false. If true, provide a proof. If false, provide a counter example.

a. If A is a 3×3 matrix with a zero determinant, then one column must be a multiple of some other column.
b. If any two columns of a square matrix are equal, then the determinant of the matrix equals zero.
c. For two $n \times n$ matrices A and B, $\det(A+B) = \det(A) + \det(B)$.
d. For an $n \times n$ matrix A, $\det(3A) = 3\det(A)$
e. If A^{-1} exists then $\det(A^{-1}) = \det(A)^{-1}$.
f. If B is obtained by multiplying a single row of A by 4 then $\det(B) = 4\det(A)$.
g. For A an $n \times n$ matrix, $\det(-A) = (-1)^n \det(A)$.
h. If A is a real $n \times n$ matrix, then $\det(A^TA) \geq 0$.
i. If $A^k = 0$ for some positive integer k, then $\det(A) = 0$.
j. If $AX = 0$ for some $X \neq 0$, then $\det(A) = 0$.

Answer

a. False. Consider $\begin{bmatrix} 1 & 1 & 2 \\ -1 & 5 & 4 \\ 0 & 3 & 3 \end{bmatrix}$
b. True.
c. False.
d. False.
e. True.
f. False.
g. True.
h. True.
i. True.
j. True.

Exercise 3.5.22

Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{vmatrix}$$

Answer

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{vmatrix} = -6$$

Exercise 3.5.23

Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix}$$

Answer

$$\begin{vmatrix} 2 & 1 & 3 \\ 2 & 4 & 2 \\ 1 & 4 & -5 \end{vmatrix} = -32$$

Exercise 3.5.24

Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 3 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -2 \end{vmatrix}$$

Answer

One can row reduce this using only row operation 3 to

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -5 & -5 & -3 \\ 0 & 0 & 2 & \frac{9}{5} \\ 0 & 0 & 0 & -\frac{63}{10} \end{bmatrix}$$

and therefore, the determinant is -63.

$$\begin{vmatrix} 1 & 2 & 1 & 2 \\ 3 & 1 & -2 & 3 \\ -1 & 0 & 3 & 1 \\ 2 & 3 & 2 & -2 \end{vmatrix} = 63$$

Exercise 3.5.25

Find the determinant using row operations to first simplify.

$$\begin{vmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & -2 & 3 \\ -1 & 0 & 3 & 3 \\ 2 & 1 & 2 & -2 \end{vmatrix}$$

Answer

One can row reduce this using only row operation 3 to

$$\begin{bmatrix} 1 & 4 & 1 & 2 \\ 0 & -10 & -5 & -3 \\ 0 & 0 & 2 & \frac{19}{5} \\ 0 & 0 & 0 & -\frac{211}{20} \end{bmatrix}$$

Thus the determinant is given by

$$\begin{vmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & -2 & 3 \\ -1 & 0 & 3 & 3 \\ 2 & 1 & 2 & -2 \end{vmatrix} = 211$$

? Exercise 3.5.26

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse which involves the cofactor matrix.

Answer

$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix} = -13$ and so it has an inverse. This inverse is

$$\frac{1}{-13}\begin{bmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T = \frac{1}{-13}\begin{bmatrix} -1 & 3 & -6 \\ 3 & -9 & 5 \\ -4 & -1 & 2 \end{bmatrix}^T \tag{3.5.1}$$

$$= \begin{bmatrix} \frac{1}{13} & -\frac{3}{13} & \frac{4}{13} \\ -\frac{3}{13} & \frac{9}{13} & \frac{1}{13} \\ \frac{6}{13} & -\frac{5}{13} & -\frac{2}{13} \end{bmatrix}$$

? Exercise 3.5.27

Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

Answer

$\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix} = 7$ so it has an inverse. This inverse is $\frac{1}{7}\begin{bmatrix} 1 & 3 & -6 \\ -2 & 1 & 5 \\ 2 & -1 & 2 \end{bmatrix}^T = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} & \frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} & -\frac{1}{7} \\ -\frac{6}{7} & \frac{5}{7} & \frac{2}{7} \end{bmatrix}$

Exercise 3.5.28

Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

Answer

$$\det \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 3$$

so it has an inverse which is

$$\begin{bmatrix} 1 & 0 & -3 \\ -\frac{2}{3} & \frac{1}{3} & \frac{5}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Exercise 3.5.29

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

Exercise 3.5.30

Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$$

Determine whether the matrix A has an inverse by finding whether the determinant is non zero. If the determinant is nonzero, find the inverse using the formula for the inverse.

Answer

$$\det \begin{bmatrix} 1 & 0 & 3 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix} = 2$$

and so it has an inverse. The inverse turns out to equal

$$\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} & 0 \\ \frac{3}{2} & -\frac{9}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

Exercise 3.5.31

For the following matrices, determine if they are invertible. If so, use the formula for the inverse in terms of the cofactor matrix to find each inverse. If the inverse does not exist, explain why.

a. $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

Answer

a. $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$

b. $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 4 & 1 & 1 \end{vmatrix} = -15$

c. $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 0$

Exercise 3.5.32

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

Answer

No. It has a nonzero determinant for all t

Exercise 3.5.33

Consider the matrix

$$A = \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{bmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

Answer

$$\det \begin{bmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{bmatrix} = t^3 + 2$$

and so it has no inverse when $t = -\sqrt[3]{2}$

Exercise 3.5.34

Consider the matrix

$$A = \begin{bmatrix} e^t & \cosh t & \sinh t \\ e^t & \sinh t & \cosh t \\ e^t & \cosh t & \sinh t \end{bmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

Answer

$$\det \begin{bmatrix} e^t & \cosh t & \sinh t \\ e^t & \sinh t & \cosh t \\ e^t & \cosh t & \sinh t \end{bmatrix} = 0$$

and so this matrix fails to have a nonzero determinant at any value of t.

Exercise 3.5.35

Consider the matrix

$$A = \begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

Answer

$$\det \begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix} = 5e^{-t} \neq 0$$

and so this matrix is always invertible.

Exercise 3.5.36

Show that if $\det(A) \neq 0$ for A an $n \times n$ matrix, it follows that if $AX = 0$, then $X = 0$.

Answer

If $\det(A) \neq 0$, then A^{-1} exists and so you could multiply on both sides on the left by A^{-1} and obtain that $X = 0$.

Exercise 3.5.37

Suppose A, B are $n \times n$ matrices and that $AB = I$. Show that then $BA = I$. **Hint**: First explain why $\det(A)$, $\det(B)$ are both nonzero. Then $(AB)A = A$ and then show $BA(BA - I) = 0$. From this use what is given to conclude $A(BA - I) = 0$. Then use Exercise 3.5.36.

Answer

You have $1 = \det(A)\det(B)$. Hence both A and B have inverses. Letting X be given,

$$A(BA - I)X = (AB)AX - AX = AX - AX = 0$$

and so it follows from the above problem that $(BA-I)X=0$. Since X is arbitrary, it follows that $BA=I$.

Exercise 3.5.38

Use the formula for the inverse in terms of the cofactor matrix to find the inverse of the matrix

$$A=\begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t\cos t & e^t\sin t \\ 0 & e^t\cos t-e^t\sin t & e^t\cos t+e^t\sin t \end{bmatrix}$$

Answer

$$\det\begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t\cos t & e^t\sin t \\ 0 & e^t\cos t-e^t\sin t & e^t\cos t+e^t\sin t \end{bmatrix}=e^{3t}.$$

Hence the inverse is

$$e^{-3t}\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t}\cos t+e^{2t}\sin t & -(e^{2t}\cos t-e^{2t}\sin)t \\ 0 & -e^{2t}\sin t & e^{2t}\cos(t) \end{bmatrix}^T$$
$$=\begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t}(\cos t+\sin t) & -(\sin t)e^{-t} \\ 0 & -e^{-t}(\cos t-\sin t) & (\cos t)e^{-t} \end{bmatrix}$$

Exercise 3.5.39

Find the inverse, if it exists, of the matrix

$$A=\begin{bmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{bmatrix}$$

Answer

$$\begin{bmatrix} e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \\ e^t & -\cos t & -\sin t \end{bmatrix}^{-1}$$
$$=\begin{bmatrix} \frac{1}{2}e^{-t} & 0 & \frac{1}{2}e^{-t} \\ \frac{1}{2}\cos t+\frac{1}{2}\sin t & -\sin t & \frac{1}{2}\sin t-\frac{1}{2}\cos t \\ \frac{1}{2}\sin t-\frac{1}{2}\cos t & \cos t & -\frac{1}{2}\cos t-\frac{1}{2}\sin t \end{bmatrix}$$

Exercise 3.5.40

Suppose A is an upper triangular matrix. Show that A^{-1} exists if and only if all elements of the main diagonal are non zero. Is it true that A^{-1} will also be upper triangular? Explain. Could the same be concluded for lower triangular matrices?

Answer

The given condition is what it takes for the determinant to be non zero. Recall that the determinant of an upper triangular matrix is just the product of the entries on the main diagonal

Exercise 3.5.41

If A, B, and C are each $n \times n$ matrices and ABC is invertible, show why each of A, B, and C are invertible.

Answer

This follows because $\det(ABC) = \det(A)\det(B)\det(C)$ and if this product is nonzero, then each determinant in the product is nonzero and so each of these matrices is invertible.

Exercise 3.5.42

Decide if this statement is true or false: Cramer's rule is useful for finding solutions to systems of linear equations in which there is an infinite set of solutions.

Answer

False.

Exercise 3.5.43

Use Cramer's rule to find the solution to

$$\begin{aligned} x + 2y &= 1 \\ 2x - y &= 2 \end{aligned} \tag{3.5.2}$$

Answer

Solution is: $[x = 1, y = 0]$

Exercise 3.5.44

Use Cramer's rule to find the solution to

$$\begin{aligned} x + 2y + z &= 1 \\ 2x - y - z &= 2 \\ x + z &= 1 \end{aligned}$$

Answer

Solution is: $[x = 1, y = 0, z = 0]$. For example,

$$y = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 2 & -1 & -1 \\ 1 & 0 & 1 \end{vmatrix}} = 0$$

This page titled 3.5: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

CHAPTER OVERVIEW

4: R^n

4.1: Vectors in R^n
4.2: Vector Algebra
4.3: Geometric Meaning of Vector Addition
4.4: Length of a Vector
4.5: Geometric Meaning of Scalar Multiplication
4.6: Parametric Lines
4.7: Spanning, Linear Independence and Basis in R^n
4.8: Orthogonality
4.9: Applications
4.10: Exercises

This page titled 4: R^n is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

4.1: Vectors in R^n

Outcomes

A. Find the position vector of a point in $\mathbb{R}^n$.

The notation $\mathbb{R}^n$ refers to the collection of ordered lists of n real numbers, that is

$$\mathbb{R}^n = \{(x_1 \cdots x_n) : x_j \in \mathbb{R} \text{ for } j = 1, \cdots, n\}$$

In this chapter, we take a closer look at vectors in $\mathbb{R}^n$. First, we will consider what $\mathbb{R}^n$ looks like in more detail. Recall that the point given by $0 = (0, \cdots, 0)$ is called the **origin**.

Now, consider the case of $\mathbb{R}^n$ for $n = 1$. Then from the definition we can identify $\mathbb{R}$ with points in $\mathbb{R}^1$ as follows:

$$\mathbb{R} = \mathbb{R}^1 = \{(x_1) : x_1 \in \mathbb{R}\}$$

Hence, $\mathbb{R}$ is defined as the set of all real numbers and geometrically, we can describe this as all the points on a line.

Now suppose $n = 2$. Then, from the definition,

$$\mathbb{R}^2 = \{(x_1, x_2) : x_j \in \mathbb{R} \text{ for } j = 1, 2\}$$

Consider the familiar coordinate plane, with an x axis and a y axis. Any point within this coordinate plane is identified by where it is located along the x axis, and also where it is located along the y axis. Consider as an example the following diagram.

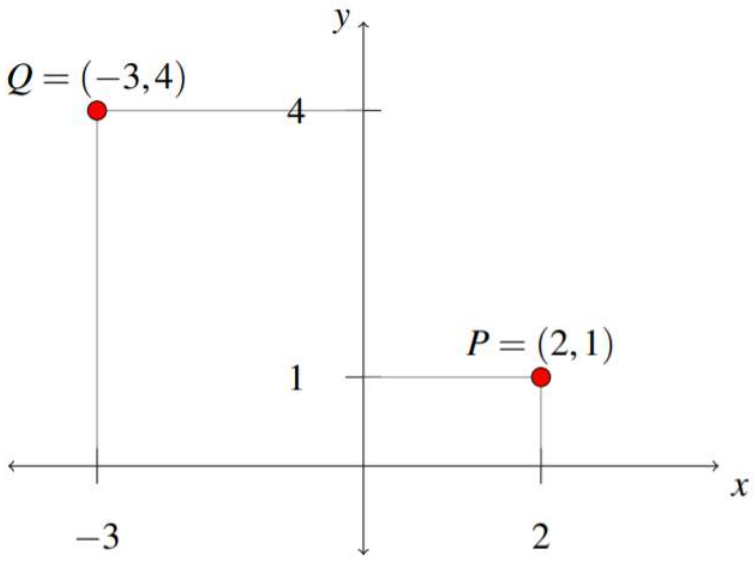


Figure 4.1.1

Hence, every element in $\mathbb{R}^2$ is identified by two components, x and y, in the usual manner. The coordinates x, y (or x_1, x_2) uniquely determine a point in the plan. Note that while the definition uses x_1 and x_2 to label the coordinates and you may be used to x and y, these notations are equivalent.

Now suppose $n = 3$. You may have previously encountered the 3-dimensional coordinate system, given by

$$\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_j \in \mathbb{R} \text{ for } j = 1, 2, 3\}$$

Points in $\mathbb{R}^3$ will be determined by three coordinates, often written (x, y, z) which correspond to the x, y, and z axes. We can think as above that the first two coordinates determine a point in a plane. The third component determines the height above or below the plane, depending on whether this number is positive or negative, and all together this determines a point in space. You see that the ordered triples correspond to points in space just as the ordered pairs correspond to points in a plane and single real numbers correspond to points on a line.

The idea behind the more general $\mathbb{R}^n$ is that we can extend these ideas beyond $n = 3$. This discussion regarding points in $\mathbb{R}^n$ leads into a study of vectors in $\mathbb{R}^n$. While we consider $\mathbb{R}^n$ for all n, we will largely focus on $n = 2, 3$ in this section.

Consider the following definition.

Definition 4.1.1 THe Position Vector

Let $P = (p_1, \cdots, p_n)$ be the coordinates of a point in $\mathbb{R}^n$. Then the vector $\overrightarrow{0P}$ with its tail at $0 = (0, \cdots, 0)$ and its tip at P is called the **position vector** of the point P. We write

$$\overrightarrow{0P} = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$$

For this reason we may write both $P = (p_1, \cdots, p_n) \in \mathbb{R}^n$ and $\overrightarrow{0P} = [p_1 \cdots p_n]^T \in \mathbb{R}^n$.

This definition is illustrated in the following picture for the special case of $\mathbb{R}^3$.

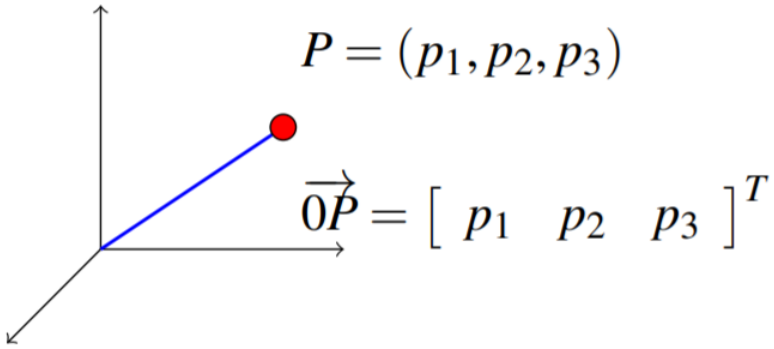


Figure 4.1.2

Thus every point P in $\mathbb{R}^n$ determines its position vector $\overrightarrow{0P}$. Conversely, every such position vector $\overrightarrow{0P}$ which has its tail at 0 and point at P determines the point P of $\mathbb{R}^n$.

Now suppose we are given two points, P, Q whose coordinates are $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$ respectively. We can also determine the **position vector from P to Q** (also called the **vector from P to Q**) defined as follows.

$$\overrightarrow{PQ} = \begin{bmatrix} q_1 - p_1 \\ \vdots \\ q_n - p_n \end{bmatrix} = \overrightarrow{0Q} - \overrightarrow{0P}$$

Now, imagine taking a vector in $\mathbb{R}^n$ and moving it around, always keeping it pointing in the same direction as shown in the following picture.

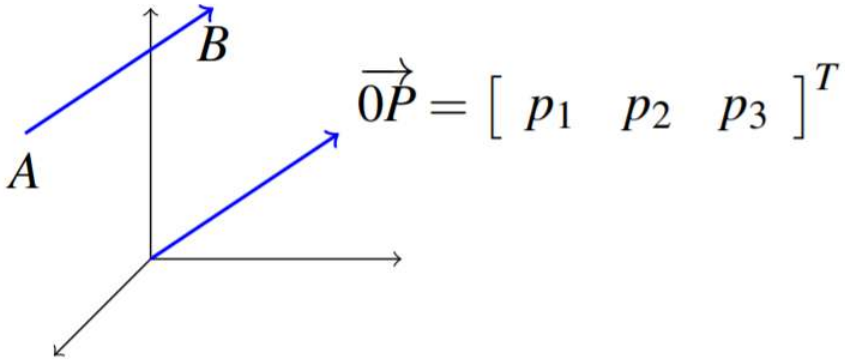


Figure 4.1.3

After moving it around, it is regarded as the same vector. Each vector, $\overrightarrow{0P}$ and $\overrightarrow{AB}$ has the same length (or magnitude) and direction. Therefore, they are equal.

Consider now the general definition for a vector in $\mathbb{R}^n$.

Definition 4.1.2 Vectors in $\mathbb{R}^n$

Let $\mathbb{R}^n = \{(x_1, \cdots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, \cdots, n\}$. Then,

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is called a **vector**. Vectors have both size (magnitude) and direction. The numbers x_j are called the **components** of $\vec{x}$.

Using this notation, we may use $\vec{p}$ to denote the position vector of point P. Notice that in this context, $\vec{p} = \overrightarrow{0P}$. These notations may be used interchangeably.

You can think of the components of a vector as directions for obtaining the vector. Consider $n = 3$. Draw a vector with its tail at the point $(0,0,0)$ and its tip at the point (a,b,c). This vector it is obtained by starting at $(0,0,0)$, moving parallel to the x axis to $(a,0,0)$ and then from here, moving parallel to the y axis to $(a,b,0)$ and finally parallel to the z axis to (a,b,c). Observe that the same vector would result if you began at the point (d,e,f), moved parallel to the x axis to $(d+a,e,f)$, then parallel to the y axis to $(d+a,e+b,f)$, and finally parallel to the z axis to $(d+a,e+b,f+c)$. Here, the vector would have its tail sitting at the point determined by $A = (d,e,f)$ and its point at $B = (d+a,e+b,f+c)$. It is the **same vector** because it will point in the same direction and have the same length. It is like you took an actual arrow, and moved it from one location to another keeping it pointing the same direction.

We conclude this section with a brief discussion regarding notation. In previous sections, we have written vectors as columns, or $n \times 1$ matrices. For convenience in this chapter we may write vectors as the transpose of row vectors, or $1 \times n$ matrices. These are of course equivalent and we may move between both notations. Therefore, recognize that

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \end{bmatrix}^T$$

Notice that two vectors $\vec{u} = [u_1 \cdots u_n]^T$ and $\vec{v} = [v_1 \cdots v_n]^T$ are equal if and only if all corresponding components are equal. Precisely,

$$\begin{gathered} \vec{u} = \vec{v} \text{ if and only if} \\ u_j = v_j \text{ for all } j = 1, \cdots, n \end{gathered}$$

Thus $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T \in \mathbb{R}^3$ and $\begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T \in \mathbb{R}^3$ but $\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T \neq \begin{bmatrix} 2 & 1 & 4 \end{bmatrix}^T$ because, even though the same numbers are involved, the order of the numbers is different.

For the specific case of $\mathbb{R}^3$, there are three special vectors which we often use. They are given by

$$\vec{i} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$

$$\vec{j} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$$

$$\vec{k} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

We can write any vector $\vec{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ as a linear combination of these vectors, written as $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$. This notation will be used throughout this chapter.

This page titled 4.1: Vectors in Rⁿ is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **4.1: Vectors in Rⁿ** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

4.2: Vector Algebra

Outcomes

A. Understand vector addition and scalar multiplication, algebraically.
B. Introduce the notion of linear combination of vectors.

Addition and scalar multiplication are two important algebraic operations done with vectors. Notice that these operations apply to vectors in $\mathbb{R}^n$, for any value of n. We will explore these operations in more detail in the following sections.

Addition of Vectors in $\mathbb{R}^n$

Addition of vectors in $\mathbb{R}^n$ is defined as follows.

Definition 4.2.1: Addition of Vectors in $\mathbb{R}^n$

If $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ then $\vec{u} + \vec{v} \in \mathbb{R}^n$ and is defined by

$$\begin{aligned} \vec{u} + \vec{v} &= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\ &= \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \end{aligned} \tag{4.2.1}$$

To add vectors, we simply add corresponding components. Therefore, in order to add vectors, they must be the same size.

Addition of vectors satisfies some important properties which are outlined in the following theorem.

Theorem 4.2.1: Properties of Vector Addition

The following properties hold for vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$.

- The Commutative Law of Addition

$$\vec{u} + \vec{v} = \vec{v} + \vec{u}$$

- The Associative Law of Addition

$$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

- The Existence of an Additive Identity

$$\vec{u} + \vec{0} = \vec{u} \tag{4.2.2}$$

- The Existence of an Additive Inverse

$$\vec{u} + (-\vec{u}) = \vec{0}$$

The additive identity shown in Equation (4.2.2) is also called the **zero vector**, the $n \times 1$ vector in which all components are equal to 0. Further, $-\vec{u}$ is simply the vector with all components having same value as those of $\vec{u}$ but opposite sign; this is just $(-1)\vec{u}$. This will be made more explicit in the next section when we explore scalar multiplication of vectors. Note that subtraction is defined as $\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$.

Scalar Multiplication of Vectors in $\mathbb{R}^n$

Scalar multiplication of vectors in $\mathbb{R}^n$ is defined as follows.

Definition 4.2.2: Scalar Multiplication of Vectors in $\mathbb{R}^n$

If $\vec{u} \in \mathbb{R}^n$ and $k \in \mathbb{R}$ is a scalar, then $k\vec{u} \in \mathbb{R}^n$ is defined by

$$k\vec{u} = k \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix}$$

Just as with addition, scalar multiplication of vectors satisfies several important properties. These are outlined in the following theorem.

Theorem 4.2.2: Properties of Scalar Multiplication

The following properties hold for vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ and k, p scalars.

- The Distributive Law over Vector Addition

$$k\left(\vec{u} + \vec{v}\right) = k\vec{u} + k\vec{v}$$

- The Distributive Law over Scalar Addition

$$\left(k + p\right)\vec{u} = k\vec{u} + p\vec{u}$$

- The Associative Law for Scalar Multiplication

$$k\left(p\vec{u}\right) = \left(kp\right)\vec{u}$$

- Rule for Multiplication by $\mathbf{1}$

$$1\vec{u} = \vec{u}$$

Proof

We will show the proof of:

$$k\left(\vec{u} + \vec{v}\right) = k\vec{u} + k\vec{v}$$

Note that:

$$\begin{aligned} k\left(\vec{u} + \vec{v}\right) &= k[u_1 + v_1 \cdots u_n + v_n]^T \\ &= [k\left(u_1 + v_1\right) \cdots k\left(u_n + v_n\right)]^T \\ &= [ku_1 + kv_1 \cdots ku_n + kv_n]^T \\ &= [ku_1 \cdots ku_n]^T + [kv_1 \cdots kv_n]^T \\ &= k\vec{u} + k\vec{v} \end{aligned}$$

We now present a useful notion you may have seen earlier combining vector addition and scalar multiplication

Definition 4.2.3: Linear Combination

A vector $\vec{v}$ is said to be a **linear combination** of the vectors $\vec{u}_1, \cdots, \vec{u}_n$ if there exist scalars, $a_1, \cdots, a_n$ such that

$$\vec{v} = a_1\vec{u}_1 + \cdots + a_n\vec{u}_n$$

For example,

$$3\begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}.$$

Thus we can say that

$$\vec{v} = \begin{bmatrix} -18 \\ 3 \\ 2 \end{bmatrix}$$

is a linear combination of the vectors

$$\vec{u}_1 = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} \text{ and } \vec{u}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

This page titled 4.2: Vector Algebra is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **4.2: Vector Algebra** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

4.3: Geometric Meaning of Vector Addition

Learning Objectives

A. Understand vector addition, geometrically.

Recall that an element of $\mathbb{R}^n$ is an ordered list of numbers. For the specific case of $n = 2, 3$ this can be used to determine a point in two or three dimensional space. This point is specified relative to some coordinate axes.

Consider the case $n = 3$. Recall that taking a vector and moving it around without changing its length or direction does not change the vector. This is important in the geometric representation of vector addition.

Suppose we have two vectors, $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$. Each of these can be drawn geometrically by placing the tail of each vector at $\mathbf{0}$ and its point at (u_1, u_2, u_3) and (v_1, v_2, v_3) respectively. Suppose we slide the vector $\vec{v}$ so that its tail sits at the point of $\vec{u}$. We know that this does not change the vector $\vec{v}$. Now, draw a new vector from the tail of $\vec{u}$ to the point of $\vec{v}$. This vector is $\vec{u} + \vec{v}$.

The geometric significance of vector addition in $\mathbb{R}^n$ for any n is given in the following definition.

Definition 4.3.1: Geometry of Vector Addition

Let $\vec{u}$ and $\vec{v}$ be two vectors. Slide $\vec{v}$ so that the tail of $\vec{v}$ is on the point of $\vec{u}$. Then draw the arrow which goes from the tail of $\vec{u}$ to the point of $\vec{v}$. This arrow represents the vector $\vec{u} + \vec{v}$.

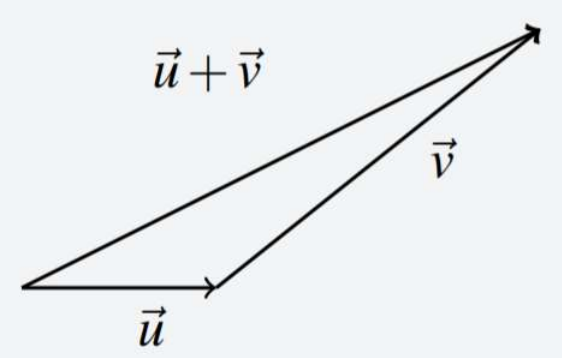


Figure 4.3.1

This definition is illustrated in the following picture in which $\vec{u} + \vec{v}$ is shown for the special case $n = 3$.

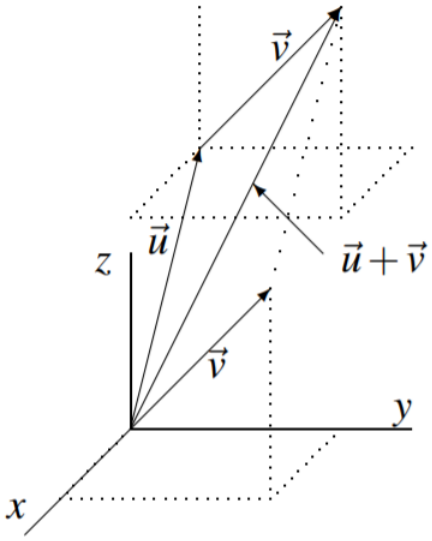


Figure 4.3.2

Notice the parallelogram created by $\vec{u}$ and $\vec{v}$ in the above diagram. Then $\vec{u}+\vec{v}$ is the directed diagonal of the parallelogram determined by the two vectors $\vec{u}$ and $\vec{v}$.

When you have a vector $\vec{v}$, its additive inverse $-\vec{v}$ will be the vector which has the same magnitude as $\vec{v}$ but the opposite direction. When one writes $\vec{u}-\vec{v}$, the meaning is $\vec{u}+(-\vec{v})$ as with real numbers. The following example illustrates these definitions and conventions.

Example 4.3.1: Graphing Vector Addition

Consider the following picture of vectors $\vec{u}$ and $\vec{v}$.



Figure 4.3.3

Sketch a picture of $\vec{u}+\vec{v}, \vec{u}-\vec{v}$.

Solution

We will first sketch $\vec{u}+\vec{v}$. Begin by drawing $\vec{u}$ and then at the point of $\vec{u}$, place the tail of $\vec{v}$ as shown. Then $\vec{u}+\vec{v}$ is the vector which results from drawing a vector from the tail of $\vec{u}$ to the tip of $\vec{v}$.

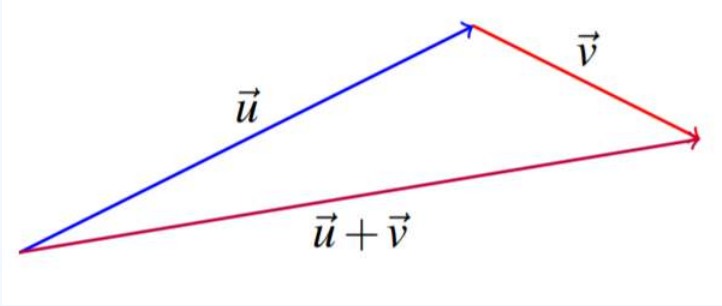


Figure 4.3.4

Next consider $\vec{u}-\vec{v}$. This means $\vec{u}+(-\vec{v})$. From the above geometric description of vector addition, $-\vec{v}$ is the vector which has the same length but which points in the opposite direction to $\vec{v}$. Here is a picture.

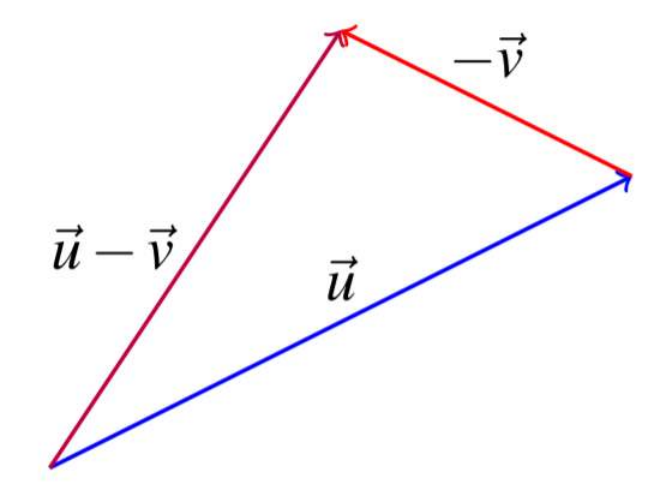


Figure 4.3.5

This page titled 4.3: Geometric Meaning of Vector Addition is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **4.3: Geometric Meaning of Vector Addition** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

4.4: Length of a Vector

Learning Objectives

- Find the length of a vector and the distance between two points in $\mathbb{R}^n$.
- Find the corresponding unit vector to a vector in $\mathbb{R}^n$.

In this section, we explore what is meant by the length of a vector in $\mathbb{R}^n$. We develop this concept by first looking at the distance between two points in $\mathbb{R}^n$.

First, we will consider the concept of distance for $\mathbb{R}$, that is, for points in $\mathbb{R}^1$. Here, the distance between two points P and Q is given by the absolute value of their difference. We denote the distance between P and Q by $d(P,Q)$ which is defined as

$$d(P,Q) = \sqrt{(P-Q)^2} \tag{4.4.1}$$

Consider now the case for $n = 2$, demonstrated by the following picture.

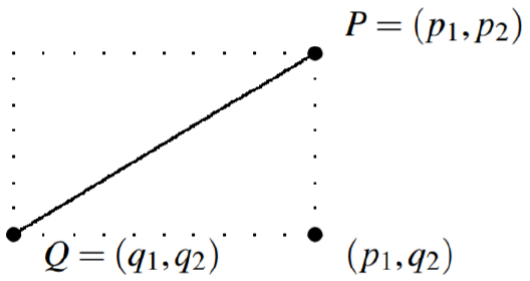


Figure 4.4.1

There are two points $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ in the plane. The distance between these points is shown in the picture as a solid line. Notice that this line is the hypotenuse of a right triangle which is half of the rectangle shown in dotted lines. We want to find the length of this hypotenuse which will give the distance between the two points. Note the lengths of the sides of this triangle are $|p_1 - q_1|$ and $|p_2 - q_2|$, the absolute value of the difference in these values. Therefore, the Pythagorean Theorem implies the length of the hypotenuse (and thus the distance between P and Q) equals

$$\left(|p_1 - q_1|^2 + |p_2 - q_2|^2\right)^{1/2} = \left((p_1 - q_1)^2 + (p_2 - q_2)^2\right)^{1/2} \tag{4.4.2}$$

Now suppose $n = 3$ and let $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ be two points in $\mathbb{R}^3$. Consider the following picture in which the solid line joins the two points and a dotted line joins the points (q_1, q_2, q_3) and (p_1, p_2, q_3).

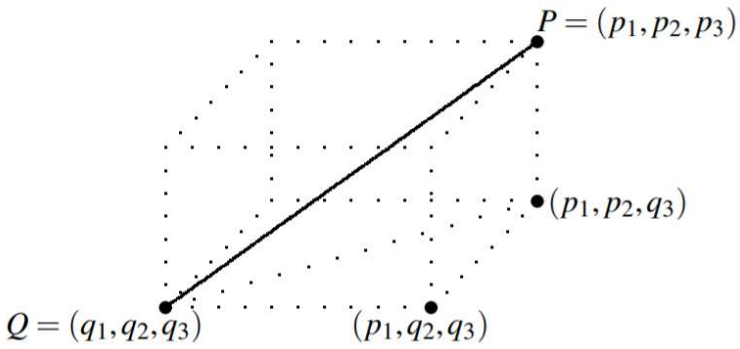


Figure 4.4.2

Here, we need to use Pythagorean Theorem twice in order to find the length of the solid line. First, by the Pythagorean Theorem, the length of the dotted line joining (q_1, q_2, q_3) and (p_1, p_2, q_3) equals

$$\left((p_1-q_1)^2+(p_2-q_2)^2\right)^{1/2}$$

while the length of the line joining (p_1,p_2,q_3) to (p_1,p_2,p_3) is just $|p_3-q_3|$. Therefore, by the Pythagorean Theorem again, the length of the line joining the points $P=(p_1,p_2,p_3)$ and $Q=(q_1,q_2,q_3)$ equals

$$\begin{aligned}&\left(\left(\left((p_1-q_1)^2+(p_2-q_2)^2\right)^{1/2}\right)^2+(p_3-q_3)^2\right)^{1/2}\\&=\left((p_1-q_1)^2+(p_2-q_2)^2+(p_3-q_3)^2\right)^{1/2}\end{aligned} \tag{4.4.3}$$

This discussion motivates the following definition for the distance between points in $\mathbb{R}^n$.

Definition 4.4.1: Distance Between Points

Let $P=(p_1,\cdots,p_n)$ and $Q=(q_1,\cdots,q_n)$ be two points in $\mathbb{R}^n$. Then the distance between these points is defined as

$$\text{distance between } P \text{ and } Q \;=d(P,Q)=\left(\sum_{k=1}^{n}|p_k-q_k|^2\right)^{1/2}$$

This is called the **distance formula**. We may also write $|P-Q|$ as the distance between P and Q.

From the above discussion, you can see that Definition 4.4.1 holds for the special cases $n=1,2,3$, as in Equations (4.4.1), (4.4.2), (4.4.3). In the following example, we use Definition 4.4.1 to find the distance between two points in $\mathbb{R}^4$.

Example 4.4.1: Distance Between Points

Find the distance between the points P and Q in $\mathbb{R}^4$, where P and Q are given by

$$P=(1,2,-4,6)$$

and

$$Q=(2,3,-1,0)$$

Solution

We will use the formula given in Definition 4.4.1 to find the distance between P and Q. Use the distance formula and write

$$d(P,Q)=\left((1-2)^2+(2-3)^2+(-4-(-1))^2+(6-0)^2\right)^{\frac{1}{2}}=47$$

Therefore, $d(P,Q)=\sqrt{47}$.

There are certain properties of the distance between points which are important in our study. These are outlined in the following theorem.

Theorem 4.4.1: Properties of Distance

Let P and Q be points in $\mathbb{R}^n$, and let the distance between them, $d(P,Q)$, be given as in Definition 4.4.1. Then, the following properties hold.

- $d(P,Q)=d(Q,P)$
- $d(P,Q)\geq 0$, and equals 0 exactly when $P=Q$.

There are many applications of the concept of distance. For instance, given two points, we can ask what collection of points are all the same distance between the given points. This is explored in the following example.

Example 4.4.2: The Plane Between Two Points

Describe the points in $\mathbb{R}^3$ which are at the same distance between $(1,2,3)$ and $(0,1,2)$.

Solution

Let $P=(p_1,p_2,p_3)$ be such a point. Therefore, P is the same distance from $(1,2,3)$ and $(0,1,2)$. Then by Definition 4.4.1,

$$\sqrt{(p_1-1)^2+(p_2-2)^2+(p_3-3)^2}=\sqrt{(p_1-0)^2+(p_2-1)^2+(p_3-2)^2}$$

Squaring both sides we obtain

$$(p_1-1)^2+(p_2-2)^2+(p_3-3)^2=p_1^2+(p_2-1)^2+(p_3-2)^2$$

and so

$$p_1^2-2p_1+14+p_2^2-4p_2+p_3^2-6p_3=p_1^2+p_2^2-2p_2+5+p_3^2-4p_3$$

Simplifying, this becomes

$$-2p_1+14-4p_2-6p_3=-2p_2+5-4p_3$$

which can be written as

$$2p_1+2p_2+2p_3=-9 \tag{4.4.4}$$

Therefore, the points $P=(p_1,p_2,p_3)$ which are the same distance from each of the given points form a plane whose equation is given by (4.4.4).

We can now use our understanding of the distance between two points to define what is meant by the length of a vector. Consider the following definition.

Definition 4.4.2: Length of a Vector

Let $\vec{u}=[u_1\cdots u_n]^T$ be a vector in $\mathbb{R}^n$. Then, the length of $\vec{u}$, written $\|\vec{u}\|$ is given by

$$\|\vec{u}\|=\sqrt{u_1^2+\cdots+u_n^2}$$

This definition corresponds to Definition 4.4.1, if you consider the vector $\vec{u}$ to have its tail at the point $0=(0,\cdots,0)$ and its tip at the point $U=(u_1,\cdots,u_n)$. Then the length of $\vec{u}$ is equal to the distance between 0 and U, $d(0,U)$. In general, $d(P,Q)=||\vec{PQ}||$.

Consider Example 4.4.1. By Definition 4.4.2, we could also find the distance between P and Q as the length of the vector connecting them. Hence, if we were to draw a vector $\overrightarrow{PQ}$ with its tail at P and its point at Q, this vector would have length equal to $\sqrt{47}$.

We conclude this section with a new definition for the special case of vectors of length 1.

Definition 4.4.3: Unit Vector

Let $\vec{u}$ be a vector in $\mathbb{R}^n$. Then, we call $\vec{u}$ a **unit vector** if it has length 1, that is if

$$\|\vec{u}\|=1$$

Let $\vec{v}$ be a vector in $\mathbb{R}^n$. Then, the vector $\vec{u}$ which has the same direction as $\vec{v}$ but length equal to 1 is the corresponding unit vector of $\vec{v}$. This vector is given by

$$\vec{u}=\frac{1}{\|\vec{v}\|}\vec{v}$$

We often use the term **normalize** to refer to this process. When we **normalize** a vector, we find the corresponding unit vector of length 1. Consider the following example.

Example 4.4.3: Finding a Unit Vector

Let $\vec{v}$ be given by

$$\vec{v} = \begin{bmatrix} 1 & -3 & 4 \end{bmatrix}^T$$

Find the unit vector $\vec{u}$ which has the same direction as $\vec{v}$.

Solution

We will use Definition 4.4.3 to solve this. Therefore, we need to find the length of $\vec{v}$ which, by Definition 4.4.2 is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Using the corresponding values we find that

$$\begin{aligned} \|\vec{v}\| &= \sqrt{1^2 + (-3)^2 + 4^2} \\ &= \sqrt{1 + 9 + 16} \\ &= \sqrt{26} \end{aligned} \tag{4.4.5}$$

In order to find $\vec{u}$, we divide $\vec{v}$ by $\sqrt{26}$. The result is

$$\begin{aligned} \vec{u} &= \frac{1}{\|\vec{v}\|}\vec{v} \\ &= \frac{1}{\sqrt{26}}\begin{bmatrix} 1 & -3 & 4 \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{26}} & -\frac{3}{\sqrt{26}} & \frac{4}{\sqrt{26}} \end{bmatrix}^T \end{aligned} \tag{4.4.6}$$

You can verify using the Definition 4.4.1 that $\|\vec{u}\| = 1$.

This page titled 4.4: Length of a Vector is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **4.4: Length of a Vector** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

4.5: Geometric Meaning of Scalar Multiplication

Outcomes

A. Understand scalar multiplication, geometrically.

Recall that the point $P = (p_1, p_2, p_3)$ determines a vector $\vec{p}$ from 0 to P. The length of $\vec{p}$, denoted $\|\vec{p}\|$, is equal to $\sqrt{p_1^2 + p_2^2 + p_3^2}$ by Definition 4.4.1.

Now suppose we have a vector $\vec{u} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^T$ and we multiply $\vec{u}$ by a scalar k. By Definition 4.2.2, $k\vec{u} = \begin{bmatrix} ku_1 & ku_2 & ku_3 \end{bmatrix}^T$. Then, by using Definition 4.4.1, the length of this vector is given by

$$\sqrt{\left((ku_1)^2 + (ku_2)^2 + (ku_3)^2\right)} = |k| \sqrt{u_1^2 + u_2^2 + u_3^2}$$

Thus the following holds.

$$\|k\vec{u}\| = |k| \, \|\vec{u}\|$$

In other words, multiplication by a scalar magnifies or shrinks the length of the vector by a factor of $|k|$. If $|k| > 1$, the length of the resulting vector will be magnified. If $|k| < 1$, the length of the resulting vector will shrink. Remember that by the definition of the absolute value, $|k| > 0$.

What about the direction? Draw a picture of $\vec{u}$ and $k\vec{u}$ where k is negative. Notice that this causes the resulting vector to point in the opposite direction while if $k > 0$ it preserves the direction the vector points. Therefore the direction can either reverse, if $k < 0$, or remain preserved, if $k > 0$.

Consider the following example.

Example 4.5.1: Graphing Scalar Multiplication

Consider the vectors $\vec{u}$ and $\vec{v}$ drawn below.



Figure 4.5.1

Draw $-\vec{u}$, $2\vec{v}$, and $-\frac{1}{2}\vec{v}$.

Solution

In order to find $-\vec{u}$, we preserve the length of $\vec{u}$ and simply reverse the direction. For $2\vec{v}$, we double the length of $\vec{v}$, while preserving the direction. Finally $-\frac{1}{2}\vec{v}$ is found by taking half the length of $\vec{v}$ and reversing the direction. These vectors are shown in the following diagram.



Figure 4.5.2

https://math.libretexts.org/@go/page/117930

Now that we have studied both vector addition and scalar multiplication, we can combine the two actions. Recall Definition 9.2.2 of linear combinations of column matrices. We can apply this definition to vectors in $\mathbb{R}^n$. A linear combination of vectors in $\mathbb{R}^n$ is a sum of vectors multiplied by scalars.

In the following example, we examine the geometric meaning of this concept.

Example 4.5.2: Graphing a Linear Combination of Vectors

Consider the following picture of the vectors $\vec{u}$ and $\vec{v}$



Figure 4.5.3

Sketch a picture of $\vec{u}+2\vec{v}, \vec{u}-\frac{1}{2}\vec{v}$.

Solution

The two vectors are shown below.

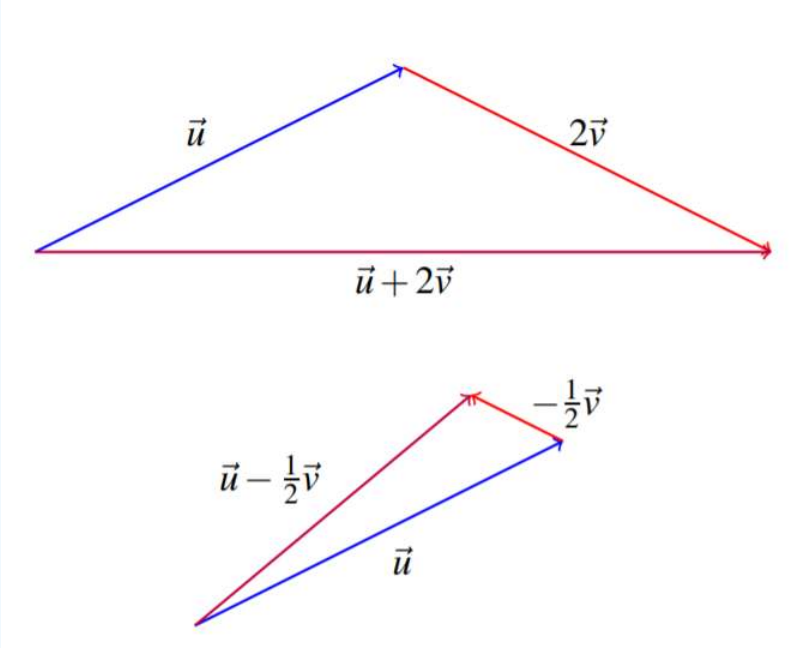


Figure 4.5.4

This page titled 4.5: Geometric Meaning of Scalar Multiplication is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **4.5: Geometric Meaning of Scalar Multiplication** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

4.6: Parametric Lines

Outcomes

A. Find the vector and parametric equations of a line.

We can use the concept of vectors and points to find equations for arbitrary lines in $\mathbb{R}^n$, although in this section the focus will be on lines in $\mathbb{R}^3$.

To begin, consider the case $n = 1$ so we have $\mathbb{R}^1 = \mathbb{R}$. There is only one line here which is the familiar number line, that is $\mathbb{R}$ itself. Therefore it is not necessary to explore the case of $n = 1$ further.

Now consider the case where $n = 2$, in other words $\mathbb{R}^2$. Let P and P_0 be two different points in $\mathbb{R}^2$ which are contained in a line L. Let $\vec{p}$ and $\vec{p_0}$ be the position vectors for the points P and P_0 respectively. Suppose that Q is an arbitrary point on L. Consider the following diagram.

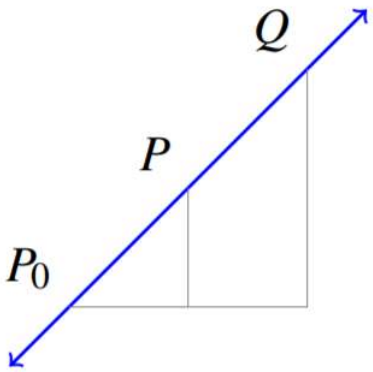


Figure 4.6.1

Our goal is to be able to define Q in terms of P and P_0. Consider the vector $\overrightarrow{P_0P} = \vec{p} - \vec{p_0}$ which has its tail at P_0 and point at P. If we add $\vec{p} - \vec{p_0}$ to the position vector $\vec{p_0}$ for P_0, the sum would be a vector with its point at P. In other words,

$$\vec{p} = \vec{p_0} + (\vec{p} - \vec{p_0})$$

Now suppose we were to add $t(\vec{p} - \vec{p_0})$ to $\vec{p}$ where t is some scalar. You can see that by doing so, we could find a vector with its point at Q. In other words, we can find t such that

$$\vec{q} = \vec{p_0} + t(\vec{p} - \vec{p_0})$$

This equation determines the line L in $\mathbb{R}^2$. In fact, it determines a line L in $\mathbb{R}^n$. Consider the following definition.

Definition 4.6.1: Vector Equation of a Line

Suppose a line L in $\mathbb{R}^n$ contains the two different points P and P_0. Let $\vec{p}$ and $\vec{p_0}$ be the position vectors of these two points, respectively. Then, L is the collection of points Q which have the position vector $\vec{q}$ given by

$$\vec{q} = \vec{p_0} + t(\vec{p} - \vec{p_0})$$

where $t \in \mathbb{R}$.

Let $\vec{d} = \vec{p} - \vec{p_0}$. Then $\vec{d}$ is the **direction vector for** L and the **vector equation** for L is given by

$$\vec{p} = \vec{p_0} + t\vec{d}, t \in \mathbb{R}$$

Note that this definition agrees with the usual notion of a line in two dimensions and so this is consistent with earlier concepts. Consider now points in $\mathbb{R}^3$. If a point $P \in \mathbb{R}^3$ is given by $P = (x, y, z)$, $P_0 \in \mathbb{R}^3$ by $P_0 = (x_0, y_0, z_0)$, then we can write

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

where $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. This is the vector equation of L written in **component form** .

The following theorem claims that such an equation is in fact a line.

Proposition 4.6.1: Algebraic Description of a Straight Line

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$ with $\vec{b} \neq \vec{0}$. Then $\vec{x} = \vec{a} + t\vec{b},\ t \in \mathbb{R}$, is a line.

Proof

Let $\vec{x_1}, \vec{x_2} \in \mathbb{R}^n$. Define $\vec{x_1} = \vec{a}$ and let $\vec{x_2} - \vec{x_1} = \vec{b}$. Since $\vec{b} \neq \vec{0}$, it follows that $\vec{x_2} \neq \vec{x_1}$. Then $\vec{a} + t\vec{b} = \vec{x_1} + t(\vec{x_2} - \vec{x_1})$. It follows that $\vec{x} = \vec{a} + t\vec{b}$ is a line containing the two different points X_1 and X_2 whose position vectors are given by $\vec{x}_1$ and $\vec{x}_2$ respectively.

We can use the above discussion to find the equation of a line when given two distinct points. Consider the following example.

Example 4.6.1: A Line From Two Points

Find a vector equation for the line through the points $P_0 = (1, 2, 0)$ and $P = (2, -4, 6)$.

Solution

We will use the definition of a line given above in Definition 4.6.1 to write this line in the form

$$\vec{q} = \vec{p_0} + t(\vec{p} - \vec{p_0})$$

Let $\vec{q} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B$. Then, we can find $\vec{p}$ and $\vec{p_0}$ by taking the position vectors of points P and P_0 respectively. Then,

$$\vec{q} = \vec{p_0} + t(\vec{p} - \vec{p_0})$$

can be written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} B + t \begin{bmatrix} 1 \\ -6 \\ 6 \end{bmatrix} B,\ t \in \mathbb{R}$$

Here, the direction vector $\begin{bmatrix} 1 \\ -6 \\ 6 \end{bmatrix} B$ is obtained by $\vec{p} - \vec{p_0} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} B - \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} B$ as indicated above in Definition 4.6.1.

Notice that in the above example we said that we found "a" vector equation for the line, not "the" equation. The reason for this terminology is that there are infinitely many different vector equations for the same line. To see this, replace t with another parameter, say $3s$. Then you obtain a different vector equation for the same line because the same set of points is obtained.

In Example 4.6.1, the vector given by $\begin{bmatrix} 1 \\ -6 \\ 6 \end{bmatrix} B$ is the direction vector defined in Definition 4.6.1. If we know the direction vector of a line, as well as a point on the line, we can find the vector equation.

Consider the following example.

Example 4.6.2: A Line From a Point and a Direction Vector

Find a vector equation for the line which contains the point $P_0 = (1,2,0)$ and has direction vector $\vec{d} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} B$

Solution

We will use Definition 4.6.1 to write this line in the form $\vec{p} = \vec{p_0} + t\vec{d},\ t \in \mathbb{R}$. We are given the direction vector $\vec{d}$. In order to find $\vec{p_0}$, we can use the position vector of the point P_0. This is given by $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} B$. Letting $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} B$, the equation for the line is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} B = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} B + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} B,\ t \in \mathbb{R} \tag{4.6.1}$$

We sometimes elect to write a line such as the one given in (4.6.1) in the form

$$\left.\begin{array}{l} x = 1 + t \\ y = 2 + 2t \\ z = t \end{array}\right\} \text{ where } t \in \mathbb{R} \tag{4.6.2}$$

This set of equations give the same information as (4.6.1), and is called the **parametric equation of the line**.

Consider the following definition.

Definition 4.6.2: Parametric Equation of a Line

Let L be a line in $\mathbb{R}^3$ which has direction vector $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} B$ and goes through the point $P_0 = (x_0, y_0, z_0)$. Then, letting t be a parameter, we can write L as

$$\left.\begin{array}{l} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{array}\right\} \text{ where } t \in \mathbb{R}$$

This is called a **parametric equation** of the line L.

You can verify that the form discussed following Example 4.6.2 in equation (4.6.2) is of the form given in Definition 4.6.2.

There is one other form for a line which is useful, which is the **symmetric form**. Consider the line given by (4.6.2). You can solve for the parameter t to write

$$\begin{array}{l} t = x - 1 \\ t = \frac{y-2}{2} \\ t = z \end{array}$$

Therefore,

$$x - 1 = \frac{y-2}{2} = z$$

This is the **symmetric form** of the line.

In the following example, we look at how to take the equation of a line from symmetric form to parametric form.

Example 4.6.3: Change Symmetric Form to Parametric Form

Suppose the **symmetric form of a line** is

$$\frac{x-2}{3}=\frac{y-1}{2}=z+3$$

Write the line in parametric form as well as vector form.

Solution

We want to write this line in the form given by Definition 4.6.2. This is of the form

$$\left.\begin{array}{l} x=x_0+ta \\ y=y_0+tb \\ z=z_0+tc \end{array}\right\} \text{ where } t\in\mathbb{R}$$

Let $t=\frac{x-2}{3}, t=\frac{y-1}{2}$ and $t=z+3$, as given in the symmetric form of the line. Then solving for x, y, z, yields

$$\left.\begin{array}{l} x=2+3t \\ y=1+2t \\ z=-3+t \end{array}\right\} \text{ with } t\in\mathbb{R}$$

This is the parametric equation for this line.

Now, we want to write this line in the form given by Definition 4.6.1. This is the form

$$\vec{p}=\vec{p_0}+t\vec{d}$$

where $t\in\mathbb{R}$. This equation becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} B=\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} B+t\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} B,\ t\in\mathbb{R}$$

This page titled 4.6: Parametric Lines is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **4.6: Parametric Lines** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

4.7: Spanning, Linear Independence and Basis in R^n

Outcomes

A. Determine the span of a set of vectors, and determine if a vector is contained in a specified span.
B. Determine if a set of vectors is linearly independent.
C. Understand the concepts of subspace, basis, and dimension.
D. Find the row space, column space, and null space of a matrix.

By generating all linear combinations of a set of vectors one can obtain various subsets of $\mathbb{R}^n$ which we call subspaces. For example what set of vectors in $\mathbb{R}^3$ generate the XY-plane? What is the smallest such set of vectors can you find? The tools of spanning, linear independence and basis are exactly what is needed to answer these and similar questions and are the focus of this section. The following definition is essential.

Definition 4.7.1: Subset

Let U and W be sets of vectors in $\mathbb{R}^n$. If all vectors in U are also in W, we say that U is a **subset** of W, denoted

$$U \subseteq W$$

Spanning Set of Vectors

We begin this section with a definition.

Definition 4.7.2: Span of a Set of Vectors

The collection of all linear combinations of a set of vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is known as the span of these vectors and is written as $\mathbf{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

Consider the following example.

Example 4.7.1: Span of Vectors

Describe the span of the vectors $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T \in \mathbb{R}^3$.

Solution

You can see that any linear combination of the vectors $\vec{u}$ and $\vec{v}$ yields a vector of the form $\begin{bmatrix} x & y & 0 \end{bmatrix}^T$ in the XY-plane.

Moreover every vector in the XY-plane is in fact such a linear combination of the vectors $\vec{u}$ and $\vec{v}$. That's because

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = (-2x + 3y) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (x - y) \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Thus $\mathbf{span}\{\vec{u}, \vec{v}\}$ is precisely the XY-plane.

You can convince yourself that no single vector can span the XY-plane. In fact, take a moment to consider what is meant by the span of a single vector.

However you can make the set larger if you wish. For example consider the larger set of vectors $\{\vec{u}, \vec{v}, \vec{w}\}$ where $\vec{w} = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix}^T$. Since the first two vectors already span the entire XY-plane, the span is once again precisely the XY-plane and nothing has been gained. Of course if you add a new vector such as $\vec{w} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$ then it does span a different space. What is the span of $\vec{u}, \vec{v}, \vec{w}$ in this case?

The distinction between the sets $\{\vec{u}, \vec{v}\}$ and $\{\vec{u}, \vec{v}, \vec{w}\}$ will be made using the concept of linear independence.

Consider the vectors $\vec{u}, \vec{v}$, and $\vec{w}$ discussed above. In the next example, we will show how to formally demonstrate that $\vec{w}$ is in the span of $\vec{u}$ and $\vec{v}$.

Example 4.7.2: Vector in a Span

Let $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T \in \mathbb{R}^3$. Show that $\vec{w} = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix}^T$ is in $\text{span}\{\vec{u}, \vec{v}\}$.

Solution

For a vector to be in $\text{span}\{\vec{u}, \vec{v}\}$, it must be a linear combination of these vectors. If $\vec{w} \in \text{span}\{\vec{u}, \vec{v}\}$, we must be able to find scalars a, b such that

$$\vec{w} = a\vec{u} + b\vec{v}$$

We proceed as follows.

$$\begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

This is equivalent to the following system of equations

$$\begin{aligned} a + 3b &= 4 \\ a + 2b &= 5 \end{aligned} \tag{4.7.1}$$

We solving this system the usual way, constructing the augmented matrix and row reducing to find the reduced row-echelon form.

$$\left[\begin{array}{cc|c} 1 & 3 & 4 \\ 1 & 2 & 5 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -1 \end{array}\right]$$

The solution is $a = 7, b = -1$. This means that

$$\vec{w} = 7\vec{u} - \vec{v}$$

Therefore we can say that $\vec{w}$ is in $\text{span}\{\vec{u}, \vec{v}\}$.

Linearly Independent Set of Vectors

We now turn our attention to the following question: what linear combinations of a given set of vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ yields the zero vector? Clearly $0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_k = \vec{0}$, but is it possible to have $\sum_{i=1}^k a_i\vec{u}_i = \vec{0}$ without all coefficients being zero?

You can create examples where this easily happens. For example if $\vec{u}_1 = \vec{u}_2$, then $1\vec{u}_1 - \vec{u}_2 + 0\vec{u}_3 + \cdots + 0\vec{u}_k = \vec{0}$, no matter the vectors $\{\vec{u}_3, \cdots, \vec{u}_k\}$. 0But sometimes it can be more subtle.

Example 4.7.3: Linearly Dependent Set of Vectors

Consider the vectors

$$\vec{u}_1 = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix}^T, \vec{u}_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, \vec{u}_3 = \begin{bmatrix} -2 & 3 & 2 \end{bmatrix}^T, \text{ and } \vec{u}_4 = \begin{bmatrix} 1 & -2 & 0 \end{bmatrix}^T$$

in $\mathbb{R}^3$.

Then verify that

$$1\vec{u}_1 + 0\vec{u}_2 + -\vec{u}_3 - 2\vec{u}_4 = \vec{0}$$

You can see that the linear combination does yield the zero vector but has some non-zero coefficients. Thus we define a set of vectors to be *linearly dependent* if this happens.

Definition 4.7.3: Linearly Dependent Set of Vectors

A set of non-zero vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is said to be **linearly dependent** if a linear combination of these vectors without all coefficients being zero does yield the zero vector.

Note that if $\sum_{i=1}^{k} a_i \vec{u}_i = \vec{0}$ and some coefficient is non-zero, say $a_1 \neq 0$, then

$$\vec{u}_1 = \frac{-1}{a_1} \sum_{i=2}^{k} a_i \vec{u}_i$$

and thus $\vec{u}_1$ is in the span of the other vectors. And the converse clearly works as well, so we get that a set of vectors is linearly dependent precisely when one of its vector is in the span of the other vectors of that set.

In particular, you can show that the vector $\vec{u}_1$ in the above example is in the span of the vectors $\{\vec{u}_2, \vec{u}_3, \vec{u}_4\}$.

If a set of vectors is NOT linearly dependent, then it must be that any linear combination of these vectors which yields the zero vector must use all zero coefficients. This is a very important notion, and we give it its own name of *linear independence*.

Definition 4.7.4: Linearly Independent Set of Vectors

A set of non-zero vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if whenever

$$\sum_{i=1}^{k} a_i \vec{u}_i = \vec{0}$$

it follows that each $a_i = 0$.

Note also that we require all vectors to be non-zero to form a linearly independent set.

To view this in a more familiar setting, form the $n \times k$ matrix A having these vectors as columns. Then all we are saying is that the set $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent precisely when $AX = 0$ has only the trivial solution.

Here is an example.

Example 4.7.4: Linearly Independent Vectors

Consider the vectors $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$, $\vec{v} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T$, and $\vec{w} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$ in $\mathbb{R}^3$. Verify whether the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

Solution

So suppose that we have a linear combinations $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}$. Then you can see that this can only happen with $a = b = c = 0$.

As mentioned above, you can equivalently form the 3×3 matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, and show that $AX = 0$ has only the trivial solution.

Thus this means the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

In terms of spanning, a set of vectors is linearly independent if it does not contain unnecessary vectors, that is not vector is in the span of the others.

Thus we put all this together in the following important theorem.

Theorem 4.7.1: Linear Independence as a Linear Combination

Let $\{\vec{u}_1, \cdots, \vec{u}_k\}$ be a collection of vectors in $\mathbb{R}^n$. Then the following are equivalent:

1. It is linearly independent, that is whenever

$$\sum_{i=1}^{k} a_i \vec{u}_i = \vec{0}$$

it follows that each coefficient $a_i = 0$.

2. No vector is in the span of the others.
3. The system of linear equations $AX = 0$ has only the trivial solution, where A is the $n \times k$ matrix having these vectors as columns.

The last sentence of this theorem is useful as it allows us to use the reduced row-echelon form of a matrix to determine if a set of vectors is linearly independent. Let the vectors be columns of a matrix A. Find the reduced row-echelon form of A. If each column has a leading one, then it follows that the vectors are linearly independent.

Sometimes we refer to the condition regarding sums as follows: The set of vectors, $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent if and only if there is no nontrivial linear combination which equals the zero vector. A nontrivial linear combination is one in which not all the scalars equal zero. Similarly, a trivial linear combination is one in which all scalars equal zero.

Here is a detailed example in $\mathbb{R}^4$.

Example 4.7.5: Linear Independence

Determine whether the set of vectors given by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$$

is linearly independent. If it is linearly dependent, express one of the vectors as a linear combination of the others.

Solution

In this case the matrix of the corresponding homogeneous system of linear equations is

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 3 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{array} \right]$$

The reduced row-echelon form is

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

and so every column is a pivot column and the corresponding system $AX = 0$ only has the trivial solution. Therefore, these vectors are linearly independent and there is no way to obtain one of the vectors as a linear combination of the others.

Consider another example.

Example 4.7.6: Linear Independence

Determine whether the set of vectors given by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix} \right\}$$

is linearly independent. If it is linearly dependent, express one of the vectors as a linear combination of the others.

Solution

Form the 4×4 matrix A having these vectors as columns:

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 1 & 2 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

Then by Theorem 4.7.1, the given set of vectors is linearly independent exactly if the system $AX = 0$ has only the trivial solution.

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 3 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Not all the columns of the coefficient matrix are pivot columns and so the vectors are not linearly independent. In this case, we say the vectors are linearly dependent.

It follows that there are infinitely many solutions to $AX = 0$, one of which is

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

Therefore we can write

$$1\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This can be rearranged as follows

$$1\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

This gives the last vector as a linear combination of the first three vectors.

Notice that we could rearrange this equation to write any of the four vectors as a linear combination of the other three.

When given a linearly independent set of vectors, we can determine if related sets are linearly independent.

Example 4.7.7: Related Sets of Vectors

Let $\{\vec{u}, \vec{v}, \vec{w}\}$ be an independent set of $\mathbb{R}^n$. Is $\{\vec{u} + \vec{v}, 2\vec{u} + \vec{w}, \vec{v} - 5\vec{w}\}$ linearly independent?

Solution

Suppose $a(\vec{u} + \vec{v}) + b(2\vec{u} + \vec{w}) + c(\vec{v} - 5\vec{w}) = \vec{0}_n$ for some $a, b, c \in \mathbb{R}$. Then

$$(a + 2b)\vec{u} + (a + c)\vec{v} + (b - 5c)\vec{w} = \vec{0}_n.$$

Since $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent,

$$\begin{aligned} a + 2b &= 0 \\ a + c &= 0 \\ b - 5c &= 0 \end{aligned} \tag{4.7.2}$$

This system of three equations in three variables has the unique solution $a = b = c = 0$. Therefore, $\{\vec{u} + \vec{v}, 2\vec{u} + \vec{w}, \vec{v} - 5\vec{w}\}$ is independent.

The following corollary follows from the fact that if the augmented matrix of a homogeneous system of linear equations has more columns than rows, the system has infinitely many solutions.

Corollary 4.7.1: Linear Dependence in $\mathbb{R}''$

Let $\{\vec{u}_1, \cdots, \vec{u}_k\}$ be a set of vectors in $\mathbb{R}^n$. If $k > n$, then the set is linearly dependent (i.e. NOT linearly independent).

Proof

Form the $n \times k$ matrix A having the vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ as its columns and suppose $k > n$. Then A has rank $r \leq n < k$, so the system $AX = 0$ has a nontrivial solution and thus not linearly independent by Theorem 4.7.1.

Example 4.7.8: Linear Dependence

Consider the vectors

$$\left\{\begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$$

Are these vectors linearly independent?

Solution

This set contains three vectors in $\mathbb{R}^2$. By Corollary 4.7.1 these vectors are linearly dependent. In fact, we can write

$$(-1)\begin{bmatrix} 1 \\ 4 \end{bmatrix} + (2)\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

showing that this set is linearly dependent.

The third vector in the previous example is in the span of the first two vectors. We could find a way to write this vector as a linear combination of the other two vectors. It turns out that the linear combination which we found is the **only** one, provided that the set is linearly independent.

Theorem 4.7.2: Unique Linear Combination

Let $U \subseteq \mathbb{R}^n$ be an independent set. Then any vector $\vec{x} \in \text{span}(U)$ can be written uniquely as a linear combination of vectors of U.

Proof

To prove this theorem, we will show that two linear combinations of vectors in U that equal $\vec{x}$ must be the same. Let $U = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$. Suppose that there is a vector $\vec{x} \in \text{span}(U)$ such that

$$\begin{aligned} \vec{x} &= s_1\vec{u}_1 + s_2\vec{u}_2 + \cdots + s_k\vec{u}_k, \text{ for some } s_1, s_2, \ldots, s_k \in \mathbb{R}, \text{ and} \\ \vec{x} &= t_1\vec{u}_1 + t_2\vec{u}_2 + \cdots + t_k\vec{u}_k, \text{ for some } t_1, t_2, \ldots, t_k \in \mathbb{R}. \end{aligned} \tag{4.7.3}$$

Then $\vec{0}_n = \vec{x} - \vec{x} = (s_1 - t_1)\vec{u}_1 + (s_2 - t_2)\vec{u}_2 + \cdots + (s_k - t_k)\vec{u}_k$.

Since U is independent, the only linear combination that vanishes is the trivial one, so $s_i - t_i = 0$ for all i, $1 \leq i \leq k$.

Therefore, $s_i = t_i$ for all i, $1 \leq i \leq k$, and the representation is unique.Let $U \subseteq \mathbb{R}^n$ be an independent set. Then any vector $\vec{x} \in \text{span}(U)$ can be written uniquely as a linear combination of vectors of U.

Suppose that $\vec{u}, \vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^3$, and that $\{\vec{v}, \vec{w}\}$ is independent. Consider the set $\{\vec{u}, \vec{v}, \vec{w}\}$. When can we know that this set is independent? It turns out that this follows exactly when $\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}$.

Example 4.7.9

Suppose that $\vec{u}, \vec{v}$ and $\vec{w}$ are nonzero vectors in $\mathbb{R}^3$, and that $\{\vec{v}, \vec{w}\}$ is independent. Prove that $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent if and only if $\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}$.

Solution

If $\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}$, then there exist $a, b \in \mathbb{R}$ so that $\vec{u} = a\vec{v} + b\vec{w}$. This implies that $\vec{u} - a\vec{v} - b\vec{w} = \vec{0}_3$, so $\vec{u} - a\vec{v} - b\vec{w}$ is a nontrivial linear combination of $\{\vec{u}, \vec{v}, \vec{w}\}$ that vanishes, and thus $\{\vec{u}, \vec{v}, \vec{w}\}$ is dependent.

Now suppose that $\vec{u} \notin \text{span}\{\vec{v}, \vec{w}\}$, and suppose that there exist $a, b, c \in \mathbb{R}$ such that $a\vec{u} + b\vec{v} + c\vec{w} = \vec{0}_3$. If $a \neq 0$, then $\vec{u} = -\frac{b}{a}\vec{v} - \frac{c}{a}\vec{w}$, and $\vec{u} \in \text{span}\{\vec{v}, \vec{w}\}$, a contradiction. Therefore, $a = 0$, implying that $b\vec{v} + c\vec{w} = \vec{0}_3$. Since $\{\vec{v}, \vec{w}\}$ is independent, $b = c = 0$, and thus $a = b = c = 0$, i.e., the only linear combination of $\vec{u}, \vec{v}$ and $\vec{w}$ that vanishes is the trivial one.

Therefore, $\{\vec{u}, \vec{v}, \vec{w}\}$ is independent.

Consider the following useful theorem.

Theorem 4.7.3: Invertible Matrices

Let A be an invertible $n \times n$ matrix. Then the columns of A are independent and span $\mathbb{R}^n$. Similarly, the rows of A are independent and span the set of all $1 \times n$ vectors.

This theorem also allows us to determine if a matrix is invertible. If an $n \times n$ matrix A has columns which are independent, or span $\mathbb{R}^n$, then it follows that A is invertible. If it has rows that are independent, or span the set of all $1 \times n$ vectors, then A is invertible.

A Short Application to Chemistry

The following section applies the concepts of spanning and linear independence to the subject of chemistry.

When working with chemical reactions, there are sometimes a large number of reactions and some are in a sense redundant. Suppose you have the following chemical reactions.

$$\begin{aligned} CO + \tfrac{1}{2}O_2 &\to CO_2 \\ H_2 + \tfrac{1}{2}O_2 &\to H_2O \\ CH_4 + \tfrac{3}{2}O_2 &\to CO + 2H_2O \\ CH_4 + 2O_2 &\to CO_2 + 2H_2O \end{aligned}$$

There are four chemical reactions here but they are not independent reactions. There is some redundancy. What are the independent reactions? Is there a way to consider a shorter list of reactions? To analyze this situation, we can write the reactions in a matrix as follows

$$\begin{bmatrix} CO & O_2 & CO_2 & H_2 & H_2O & CH_4 \\ 1 & 1/2 & -1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1 & -1 & 0 \\ -1 & 3/2 & 0 & 0 & -2 & 1 \\ 0 & 2 & -1 & 0 & -2 & 1 \end{bmatrix}$$

Each row contains the coefficients of the respective elements in each reaction. For example, the top row of numbers comes from $CO + \frac{1}{2}O_2 - CO_2 = 0$ which represents the first of the chemical reactions.

We can write these coefficients in the following matrix

$$\begin{bmatrix} 1 & 1/2 & -1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1 & -1 & 0 \\ -1 & 3/2 & 0 & 0 & -2 & 1 \\ 0 & 2 & -1 & 0 & -2 & 1 \end{bmatrix}$$

Rather than listing all of the reactions as above, it would be more efficient to only list those which are independent by throwing out that which is redundant. We can use the concepts of the previous section to accomplish this.

First, take the reduced row-echelon form of the above matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 3 & -1 & -1 \\ 0 & 1 & 0 & 2 & -2 & 0 \\ 0 & 0 & 1 & 4 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The top three rows represent “independent" reactions which come from the original four reactions. One can obtain each of the original four rows of the matrix given above by taking a suitable linear combination of rows of this reduced row-echelon matrix.

With the redundant reaction removed, we can consider the simplified reactions as the following equations

$$\begin{aligned} CO + 3H_2 - 1H_2O - 1CH_4 = 0 \\ O_2 + 2H_2 - 2H_2O = 0 \\ CO_2 + 4H_2 - 2H_2O - 1CH_4 = 0 \end{aligned}$$

In terms of the original notation, these are the reactions

$$\begin{aligned} CO + 3H_2 \rightarrow H_2O + CH_4 \\ O_2 + 2H_2 \rightarrow 2H_2O \\ CO_2 + 4H_2 \rightarrow 2H_2O + CH_4 \end{aligned}$$

These three reactions provide an equivalent system to the original four equations. The idea is that, in terms of what happens chemically, you obtain the same information with the shorter list of reactions. Such a simplification is especially useful when dealing with very large lists of reactions which may result from experimental evidence.

Subspaces and Basis

The goal of this section is to develop an understanding of a subspace of $\mathbb{R}^n$. Before a precise definition is considered, we first examine the subspace test given below.

Theorem 4.7.4: Subspace Test

A subset V of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

1. the zero vector of $\mathbb{R}^n$, $\vec{0}_n$, is in V;
2. V is closed under addition, i.e., for all $\vec{u}, \vec{w} \in V$, $\vec{u} + \vec{w} \in V$;
3. V is closed under scalar multiplication, i.e., for all $\vec{u} \in V$ and $k \in \mathbb{R}$, $k\vec{u} \in V$.

This test allows us to determine if a given set is a subspace of $\mathbb{R}^n$. Notice that the subset $V = \left\{\vec{0}\right\}$ is a subspace of $\mathbb{R}^n$ (called the zero subspace), as is $\mathbb{R}^n$ itself. A subspace which is not the zero subspace of $\mathbb{R}^n$ is referred to as a proper subspace.

A subspace is simply a set of vectors with the property that linear combinations of these vectors remain in the set. Geometrically in $\mathbb{R}^3$, it turns out that a subspace can be represented by either the origin as a single point, lines and planes which contain the origin, or the entire space $\mathbb{R}^3$.

Consider the following example of a line in $\mathbb{R}^3$.

Example 4.7.10: Subspace of $\mathbb{R}^3$

In $\mathbb{R}^3$, the line L through the origin that is parallel to the vector $\vec{d} = \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}$ has (vector) equation $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -5 \\ 1 \\ -4 \end{bmatrix}, t \in \mathbb{R}$, so

$$L = \left\{ t\vec{d} \mid t \in \mathbb{R} \right\}.$$

Then L is a subspace of $\mathbb{R}^3$.

Solution

Using the subspace test given above we can verify that L is a subspace of $\mathbb{R}^3$.

- First: $\vec{0}_3 \in L$ since $0\vec{d} = \vec{0}_3$.
- Suppose $\vec{u}, \vec{v} \in L$. Then by definition, $\vec{u} = s\vec{d}$ and $\vec{v} = t\vec{d}$, for some $s, t \in \mathbb{R}$. Thus

$$\vec{u} + \vec{v} = s\vec{d} + t\vec{d} = (s+t)\vec{d}.$$

  Since $s + t \in \mathbb{R}$, $\vec{u} + \vec{v} \in L$; i.e., L is closed under addition.
- Suppose $\vec{u} \in L$ and $k \in \mathbb{R}$ (k is a scalar). Then $\vec{u} = t\vec{d}$, for some $t \in \mathbb{R}$, so

$$k\vec{u} = k(t\vec{d}) = (kt)\vec{d}.$$

  Since $kt \in \mathbb{R}$, $k\vec{u} \in L$; i.e., L is closed under scalar multiplication.

Since L satisfies all conditions of the subspace test, it follows that L is a subspace.

Note that there is nothing special about the vector $\vec{d}$ used in this example; the same proof works for any nonzero vector $\vec{d} \in \mathbb{R}^3$, so any line through the origin is a subspace of $\mathbb{R}^3$.

We are now prepared to examine the precise definition of a subspace as follows.

Definition 4.7.5: Subspace

Let V be a nonempty collection of vectors in $\mathbb{R}^n$. Then V is called a subspace if whenever a and b are scalars and $\vec{u}$ and $\vec{v}$ are vectors in V, the linear combination $a\vec{u} + b\vec{v}$ is also in V.

More generally this means that a subspace contains the span of any finite collection vectors in that subspace. It turns out that in $\mathbb{R}^n$, a subspace is exactly the span of finitely many of its vectors.

Theorem 4.7.5: Subspaces are Spans

Let V be a nonempty collection of vectors in $\mathbb{R}^n$. Then V is a subspace of $\mathbb{R}^n$ if and only if there exist vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in V such that

$$V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$$

Furthermore, let W be another subspace of $\mathbb{R}^n$ and suppose $\{\vec{u}_1, \cdots, \vec{u}_k\} \in W$. Then it follows that V is a subset of W.

Note that since W is arbitrary, the statement that $V \subseteq W$ means that any other subspace of $\mathbb{R}^n$ that contains these vectors will also contain V.

Proof

We first show that if V is a subspace, then it can be written as $V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$. Pick a vector $\vec{u}_1$ in V. If $V = \text{span}\{\vec{u}_1\}$, then you have found your list of vectors and are done. If $V \neq \text{span}\{\vec{u}_1\}$, then there exists $\vec{u}_2$ a vector of V which is not in $\text{span}\{\vec{u}_1\}$. Consider $\text{span}\{\vec{u}_1, \vec{u}_2\}$. If $V = \text{span}\{\vec{u}_1, \vec{u}_2\}$, we are done. Otherwise, pick $\vec{u}_3$ not in $\text{span}\{\vec{u}_1, \vec{u}_2\}$. Continue this way. Note that since V is a subspace, these spans are each contained in V. The process must stop with $\vec{u}_k$ for some $k \leq n$ by Corollary 4.7.1, and thus $V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

Now suppose $V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$, we must show this is a subspace. So let $\sum_{i=1}^k c_i\vec{u}_i$ and $\sum_{i=1}^k d_i\vec{u}_i$ be two vectors in V, and let a and b be two scalars. Then

$$a\sum_{i=1}^k c_i\vec{u}_i + b\sum_{i=1}^k d_i\vec{u}_i = \sum_{i=1}^k (ac_i + bd_i)\vec{u}_i$$

which is one of the vectors in $\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$ and is therefore contained in V. This shows that $\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$ has the properties of a subspace.

To prove that $V \subseteq W$, we prove that if $\vec{u}_i \in V$, then $\vec{u}_i \in W$.

Suppose $\vec{u} \in V$. Then $\vec{u} = a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_k\vec{u}_k$ for some $a_i \in \mathbb{R}$, $1 \leq i \leq k$. Since W contain each $\vec{u}_i$ and W is a vector space, it follows that $a_1\vec{u}_1 + a_2\vec{u}_2 + \cdots + a_k\vec{u}_k \in W$.

Since the vectors $\vec{u}_i$ we constructed in the proof above are not in the span of the previous vectors (by definition), they must be linearly independent and thus we obtain the following corollary.

Corollary 4.7.2: Subspaces are Spans of Independent Vectors

If V is a subspace of $\mathbb{R}^n$, then there exist linearly independent vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in V such that $V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.

In summary, subspaces of $\mathbb{R}^n$ consist of spans of finite, linearly independent collections of vectors of $\mathbb{R}^n$. Such a collection of vectors is called a basis.

Definition 4.7.6: Basis of a Subspace

Let V be a subspace of $\mathbb{R}^n$. Then $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is a **basis** for V if the following two conditions hold.

1. $\text{span}\{\vec{u}_1, \cdots, \vec{u}_k\} = V$
2. $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is linearly independent

Note the plural of basis is **bases**.

The following is a simple but very useful example of a basis, called the standard basis.

Definition 4.7.7: Standard Basis of $\mathbb{R}^n$

Let $\vec{e}_i$ be the vector in $\mathbb{R}^n$ which has a 1 in the ith entry and zeros elsewhere, that is the ith column of the identity matrix. Then the collection $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$ and is called the standard basis of $\mathbb{R}^n$.

The main theorem about bases is not only they exist, but that they must be of the same size. To show this, we will need the the following fundamental result, called the Exchange Theorem.

Theorem 4.7.6: Exchange Theorem

Suppose $\{\vec{u}_1, \cdots, \vec{u}_r\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, and each $\vec{u}_k$ is contained in $\text{span}\{\vec{v}_1, \cdots, \vec{v}_s\}$ Then $s \geq r$.
In words, spanning sets have at least as many vectors as linearly independent sets.

Proof

Since each $\vec{u}_j$ is in $\text{span}\{\vec{v}_1, \cdots, \vec{v}_s\}$, there exist scalars a_{ij} such that

$$\vec{u}_j = \sum_{i=1}^{s} a_{ij}\vec{v}_i$$

Suppose for a contradiction that $s < r$. Then the matrix $A = [a_{ij}]$ has fewer rows, s than columns, r. Then the system $AX = 0$ has a non trivial solution $\vec{d}$, that is there is a $\vec{d} \neq \vec{0}$ such that $A\vec{d} = \vec{0}$. In other words,

$$\sum_{j=1}^{r} a_{ij}d_j = 0, \ i = 1, 2, \cdots, s$$

Therefore,

$$\begin{aligned} \sum_{j=1}^{r} d_j\vec{u}_j &= \sum_{j=1}^{r} d_j \sum_{i=1}^{s} a_{ij}\vec{v}_i \\ &= \sum_{i=1}^{s} \left(\sum_{j=1}^{r} a_{ij}d_j \right) \vec{v}_i = \sum_{i=1}^{s} 0\vec{v}_i = 0 \end{aligned} \tag{4.7.4}$$

which contradicts the assumption that $\{\vec{u}_1, \cdots, \vec{u}_r\}$ is linearly independent, because not all the d_j are zero. Thus this contradiction indicates that $s \geq r$.

We are now ready to show that any two bases are of the same size.

Theorem 4.7.7: Bases of $\mathbb{R}^n$ are of the Same Size

Let V be a subspace of $\mathbb{R}^n$ with two bases B_1 and B_2. Suppose B_1 contains s vectors and B_2 contains r vectors. Then $s = r$.

Proof

This follows right away from Theorem 9.4.4. Indeed observe that $B_1 = \{\vec{u}_1, \cdots, \vec{u}_s\}$ is a spanning set for V while $B_2 = \{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent, so $s \geq r$. Similarly $B_2 = \{\vec{v}_1, \cdots, \vec{v}_r\}$ is a spanning set for V while $B_1 = \{\vec{u}_1, \cdots, \vec{u}_s\}$ is linearly independent, so $r \geq s$.

The following definition can now be stated.

Definition 4.7.8: Dimension of a Subspace

Let V be a subspace of $\mathbb{R}^n$. Then the **dimension** of V, written $\dim(V)$ is defined to be the number of vectors in a basis.

The next result follows.

Corollary 4.7.3: Dimension of $\mathbb{R}^n$

The dimension of $\mathbb{R}^n$ is n.

Proof

You only need to exhibit a basis for $\mathbb{R}^n$ which has n vectors. Such a basis is the standard basis $\{\vec{e}_1, \cdots, \vec{e}_n\}$.

Consider the following example.

Example 4.7.11: Basis of Subspace

Let

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \ : \ a - b = d - c \right\}.$$

Show that V is a subspace of $\mathbb{R}^4$, find a basis of V, and find $\dim(V)$.

Solution

The condition $a - b = d - c$ is equivalent to the condition $a = b - c + d$, so we may write

$$V = \left\{ \begin{bmatrix} b - c + d \\ b \\ c \\ d \end{bmatrix} \ : \ b, c, d \in \mathbb{R} \right\} = \left\{ b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \ : \ b, c, d \in \mathbb{R} \right\}$$

This shows that V is a subspace of $\mathbb{R}^4$, since $V = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ where

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Furthermore,

$$\left\{\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}$$

is linearly independent, as can be seen by taking the reduced row-echelon form of the matrix whose columns are $\vec{u}_1, \vec{u}_2$ and $\vec{u}_3$.

$$\begin{bmatrix}1&-1&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

Since every column of the reduced row-echelon form matrix has a leading one, the columns are linearly independent.

Therefore $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is linearly independent and spans V, so is a basis of V. Hence V has dimension three.

We continue by stating further properties of a set of vectors in $\mathbb{R}^n$.

Corollary 4.7.4: Linearly Independent and Spanning Sets in $\mathbb{R}^n$

The following properties hold in $\mathbb{R}^n$:

- Suppose $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent. Then $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is a basis for $\mathbb{R}^n$.
- Suppose $\{\vec{u}_1, \cdots, \vec{u}_m\}$ spans $\mathbb{R}^n$. Then $m \geq n$.
- If $\{\vec{u}_1, \cdots, \vec{u}_n\}$ spans $\mathbb{R}^n$, then $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent.

Proof

Assume first that $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is linearly independent, and we need to show that this set spans $\mathbb{R}^n$. To do so, let $\vec{v}$ be a vector of $\mathbb{R}^n$, and we need to write $\vec{v}$ as a linear combination of $\vec{u}_i$'s. Consider the matrix A having the vectors $\vec{u}_i$ as columns:

$$A = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}$$

By linear independence of the $\vec{u}_i$'s, the reduced row-echelon form of A is the identity matrix. Therefore the system $A\vec{x} = \vec{v}$ has a (unique) solution, so $\vec{v}$ is a linear combination of the $\vec{u}_i$'s.

To establish the second claim, suppose that $m < n$. Then letting $\vec{u}_{i_1}, \cdots, \vec{u}_{i_k}$ be the pivot columns of the matrix

$$\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}$$

it follows $k \leq m < n$ and these k pivot columns would be a basis for $\mathbb{R}^n$ having fewer than n vectors, contrary to Corollary 4.7.3.

Finally consider the third claim. If $\{\vec{u}_1, \cdots, \vec{u}_n\}$ is not linearly independent, then replace this list with $\{\vec{u}_{i_1}, \cdots, \vec{u}_{i_k}\}$ where these are the pivot columns of the matrix

$$\begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_n \end{bmatrix}$$

Then $\{\vec{u}_{i_1}, \cdots, \vec{u}_{i_k}\}$ spans $\mathbb{R}^n$ and is linearly independent, so it is a basis having less than n vectors again contrary to Corollary 4.7.3.

The next theorem follows from the above claim.

Theorem 4.7.8: Existence of Basis

Let V be a subspace of $\mathbb{R}^n$. Then there exists a basis of V with $\dim(V) \leq n$.

Consider Corollary 4.7.4 together with Theorem 4.7.8. Let $\dim(V) = r$. Suppose there exists an independent set of vectors in V. If this set contains r vectors, then it is a basis for V. If it contains less than r vectors, then vectors can be added to the set to create a

basis of V. Similarly, any spanning set of V which contains more than r vectors can have vectors removed to create a basis of V.

We illustrate this concept in the next example.

Example 4.7.12: Extending an Independent Set

Consider the set U given by

$$U = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \;\middle|\; a - b = d - c \right\}$$

Then U is a subspace of $\mathbb{R}^4$ and $\dim(U) = 3$.

Then

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix} \right\},$$

is an independent subset of U. Therefore S can be extended to a basis of U.

Solution

To extend S to a basis of U, find a vector in U that is **not** in $\operatorname{span}(S)$.

$$\begin{bmatrix} 1 & 2 & ? \\ 1 & 3 & ? \\ 1 & 3 & ? \\ 1 & 2 & ? \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 0 \\ 1 & 3 & -1 \\ 1 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore, S can be extended to the following basis of U:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\},$$

Next we consider the case of removing vectors from a spanning set to result in a basis.

Theorem 4.7.9: Finding a Basis from a Span

Let W be a subspace. Also suppose that $W = span\{\vec{w}_1, \cdots, \vec{w}_m\}$. Then there exists a subset of $\{\vec{w}_1, \cdots, \vec{w}_m\}$ which is a basis for W.

Proof

Let S denote the set of positive integers such that for $k \in S$, there exists a subset of $\{\vec{w}_1, \cdots, \vec{w}_m\}$ consisting of exactly k vectors which is a spanning set for W. Thus $m \in S$. Pick the smallest positive integer in S. Call it k. Then there exists $\{\vec{u}_1, \cdots, \vec{u}_k\} \subseteq \{\vec{w}_1, \cdots, \vec{w}_m\}$ such that $\operatorname{span}\{\vec{u}_1, \cdots, \vec{u}_k\} = W$. If

$$\sum_{i=1}^{k} c_i \vec{w}_i = \vec{0}$$

and not all of the $c_i = 0$, then you could pick $c_j \neq 0$, divide by it and solve for $\vec{u}_j$ in terms of the others,

$$\vec{w}_j = \sum_{i \neq j} \left(-\frac{c_i}{c_j} \right) \vec{w}_i$$

Then you could delete $\vec{w}_j$ from the list and have the same span. Any linear combination involving $\vec{w}_j$ would equal one in which $\vec{w}_j$ is replaced with the above sum, showing that it could have been obtained as a linear combination of $\vec{w}_i$ for $i \neq j$. Thus $k-1 \in S$ contrary to the choice of k. Hence each $c_i = 0$ and so $\{\vec{u}_1, \cdots, \vec{u}_k\}$ is a basis for W consisting of vectors of $\{\vec{w}_1, \cdots, \vec{w}_m\}$.

The following example illustrates how to carry out this shrinking process which will obtain a subset of a span of vectors which is linearly independent.

Example 4.7.13: Subset of a Span

Let W be the subspace

$$span \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 8 \\ 19 \\ -8 \\ 8 \end{bmatrix}, \begin{bmatrix} -6 \\ -15 \\ 6 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Find a basis for W which consists of a subset of the given vectors.

Solution

You can use the reduced row-echelon form to accomplish this reduction. Form the matrix which has the given vectors as columns.

$$\begin{bmatrix} 1 & 1 & 8 & -6 & 1 & 1 \\ 2 & 3 & 19 & -15 & 3 & 5 \\ -1 & -1 & -8 & 6 & 0 & 0 \\ 1 & 1 & 8 & -6 & 1 & 1 \end{bmatrix}$$

Then take the reduced row-echelon form

$$\begin{bmatrix} 1 & 0 & 5 & -3 & 0 & -2 \\ 0 & 1 & 3 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It follows that a basis for W is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Since the first, second, and fifth columns are obviously a basis for the column space of the , the same is true for the matrix having the given vectors as columns.

Consider the following theorems regarding a subspace contained in another subspace.

Theorem 4.7.10: Subset of a Subspace

Let V and W be subspaces of $\mathbb{R}^n$, and suppose that $W \subseteq V$. Then $\dim(W) \leq \dim(V)$ with equality when $W = V$.

Theorem 4.7.11: Extending a Basis

Let W be any non-zero subspace $\mathbb{R}^n$ and let $W \subseteq V$ where V is also a subspace of $\mathbb{R}^n$. Then every basis of W can be extended to a basis for V.

The proof is left as an exercise but proceeds as follows. Begin with a basis for $W, \{\vec{w}_1, \cdots, \vec{w}_s\}$ and add in vectors from V until you obtain a basis for V. Not that the process will stop because the dimension of V is no more than n.

Consider the following example.

Example 4.7.14: Extending a Basis

Let $V = \mathbb{R}^4$ and let

$$W = \text{span}\left\{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}\right\}$$

Extend this basis of W to a basis of $\mathbb{R}^n$.

Solution

An easy way to do this is to take the reduced row-echelon form of the matrix

$$\begin{bmatrix}1&0&1&0&0&0\\0&1&0&1&0&0\\1&0&0&0&1&0\\1&1&0&0&0&1\end{bmatrix} \tag{4.7.5}$$

Note how the given vectors were placed as the first two columns and then the matrix was extended in such a way that it is clear that the span of the columns of this matrix yield all of $\mathbb{R}^4$. Now determine the pivot columns. The reduced row-echelon form is

$$\begin{bmatrix}1&0&0&0&1&0\\0&1&0&0&-1&1\\0&0&1&0&-1&0\\0&0&0&1&1&-1\end{bmatrix} \tag{4.7.6}$$

Therefore the pivot columns are

$$\begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

and now this is an extension of the given basis for W to a basis for $\mathbb{R}^4$.

Why does this work? The columns of (4.7.5) obviously span $\mathbb{R}^4$. In fact the span of the first four is the same as the span of all six.

Consider another example.

Example 4.7.15: Extending a Basis

Let W be the span of $\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$ in $\mathbb{R}^4$. Let V consist of the span of the vectors

$$\left[\begin{array}{r}1\\0\\1\\0\end{array}\right], \left[\begin{array}{r}0\\1\\1\\1\end{array}\right], \left[\begin{array}{r}7\\-6\\1\\-6\end{array}\right], \left[\begin{array}{r}-5\\7\\2\\7\end{array}\right], \left[\begin{array}{r}0\\0\\0\\1\end{array}\right]$$

Find a basis for V which extends the basis for W.

Solution

Note that the above vectors are not linearly independent, but their span, denoted as V is a subspace which does include the subspace W.

Using the process outlined in the previous example, form the following matrix

$$\left[\begin{array}{rrrrr}1&0&7&-5&0\\0&1&-6&7&0\\1&1&1&2&0\\0&1&-6&7&1\end{array}\right]$$

Next find its reduced row-echelon form

$$\left[\begin{array}{rrrrr}1&0&7&-5&0\\0&1&-6&7&0\\0&0&0&0&1\\0&0&0&0&0\end{array}\right]$$

It follows that a basis for V consists of the first two vectors and the last.

$$\left\{\left[\begin{array}{r}1\\0\\1\\0\end{array}\right], \left[\begin{array}{r}0\\1\\1\\1\end{array}\right], \left[\begin{array}{r}0\\0\\0\\1\end{array}\right]\right\}$$

Thus V is of dimension 3 and it has a basis which extends the basis for W.

Row Space, Column Space, and Null Space of a Matrix

We begin this section with a new definition.

Definition 4.7.9: Row and Column Space

Let A be an $m \times n$ matrix. The **column space** of A, written $\text{col}(A)$, is the span of the columns. The **row space** of A, written $\text{row}(A)$, is the span of the rows.

Using the reduced row-echelon form, we can obtain an efficient description of the row and column space of a matrix. Consider the following lemma.

Lemma 4.7.1: Effect of Row Operations on Row Space

Let A and B be $m \times n$ matrices such that A can be carried to B by elementary row [column] operations. Then $\text{row}(A) = \text{row}(B)$ [$\text{col}(A) = \text{col}(B)$].

Proof

We will prove that the above is true for row operations, which can be easily applied to column operations.

Let $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m$ denote the rows of A.

- If B is obtained from A by a interchanging two rows of A, then A and B have exactly the same rows, so $\text{row}(B) = \text{row}(A)$.
- Suppose $p \neq 0$, and suppose that for some j, $1 \leq j \leq m$, B is obtained from A by multiplying row j by p. Then

$$\text{row}(B) = \text{span}\{\vec{r}_1, \ldots, p\vec{r}_j, \ldots, \vec{r}_m\}.$$

Since

$$\{\vec{r}_1, \ldots, p\vec{r}_j, \ldots, \vec{r}_m\} \subseteq \text{row}(A),$$

it follows that $\text{row}(B) \subseteq \text{row}(A)$. Conversely, since

$$\{\vec{r}_1, \ldots, \vec{r}_m\} \subseteq \text{row}(B),$$

it follows that $\text{row}(A) \subseteq \text{row}(B)$. Therefore, $\text{row}(B) = \text{row}(A)$.

Suppose $p \neq 0$, and suppose that for some i and j, $1 \leq i, j \leq m$, B is obtained from A by adding p time row j to row i. Without loss of generality, we may assume $i < j$.

Then

$$\text{row}(B) = \text{span}\{\vec{r}_1, \ldots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \ldots, \vec{r}_j, \ldots, \vec{r}_m\}.$$

Since

$$\{\vec{r}_1, \ldots, \vec{r}_{i-1}, \vec{r}_i + p\vec{r}_j, \ldots, \vec{r}_m\} \subseteq \text{row}(A),$$

it follows that $\text{row}(B) \subseteq \text{row}(A)$.

Conversely, since

$$\{\vec{r}_1, \ldots, \vec{r}_m\} \subseteq \text{row}(B),$$

it follows that $\text{row}(A) \subseteq \text{row}(B)$. Therefore, $\text{row}(B) = \text{row}(A)$.

Consider the following lemma.

Lemma 4.7.2: Row Space of a reduced row-echelon form Matrix

Let A be an $m \times n$ matrix and let R be its reduced row-echelon form. Then the nonzero rows of R form a basis of $\text{row}(R)$, and consequently of $\text{row}(A)$.

This lemma suggests that we can examine the reduced row-echelon form of a matrix in order to obtain the row space. Consider now the column space. The column space can be obtained by simply saying that it equals the span of all the columns. However, you can often get the column space as the span of fewer columns than this. A variation of the previous lemma provides a solution. Suppose A is row reduced to its reduced row-echelon form R. Identify the pivot columns of R (columns which have leading ones), and take the corresponding columns of A. It turns out that this forms a basis of $\text{col}(A)$.

Before proceeding to an example of this concept, we revisit the definition of rank.

Definition 4.7.10: Rank of a Matrix

Previously, we defined $\text{rank}(A)$ to be the number of leading entries in the row-echelon form of A. Using an understanding of dimension and row space, we can now define rank as follows:

$$\text{rank}(A) = \dim(\text{row}(A))$$

Consider the following example.

Example 4.7.16: Rank, Column and Row Space

Find the rank of the following matrix and describe the column and row spaces.

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 3 & 7 & 8 & 6 & 6 \end{bmatrix}$$

Solution

The reduced row-echelon form of A is

$$\begin{bmatrix} 1 & 0 & -9 & 9 & 2 \\ 0 & 1 & 5 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the rank is 2.

Notice that the first two columns of R are pivot columns. By the discussion following Lemma 4.7.2, we find the corresponding columns of A, in this case the first two columns. Therefore a basis for $\text{col}(A)$ is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix} \right\}$$

For example, consider the third column of the original matrix. It can be written as a linear combination of the first two columns of the original matrix as follows.

$$\begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix} = -9 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$$

What about an efficient description of the row space? By Lemma 4.7.2 we know that the nonzero rows of R create a basis of $\text{row}(A)$. For the above matrix, the row space equals

$$\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1 & 0 & -9 & 9 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 5 & -3 & 0 \end{bmatrix} \right\}$$

Notice that the column space of A is given as the span of columns of the original matrix, while the row space of A is the span of rows of the reduced row-echelon form of A.

Consider another example.

Example 4.7.17: Rank, Column and Row Space

Find the rank of the following matrix and describe the column and row spaces.

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 6 & 0 & 2 \\ 1 & 2 & 1 & 3 & 2 \\ 1 & 3 & 2 & 4 & 0 \end{bmatrix}$$

Solution

The reduced row-echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{13}{2} \\ 0 & 1 & 0 & 2 & -\frac{5}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and so the rank is 3. The row space is given by

$$\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{13}{2} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 2 & -\frac{5}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 & \frac{1}{2} \end{bmatrix} \right\}$$

Notice that the first three columns of the reduced row-echelon form are pivot columns. The column space is the span of the first three columns in the **original matrix**,

$$\text{col}(A) = \text{span}\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\3\\2\\3\end{bmatrix}, \begin{bmatrix}1\\6\\1\\2\end{bmatrix}\right\}$$

Consider the solution given above for Example 4.7.17, where the rank of A equals 3. Notice that the row space and the column space each had dimension equal to 3. It turns out that this is not a coincidence, and this essential result is referred to as the Rank Theorem and is given now. Recall that we defined $\text{rank}(A) = \dim(\text{row}(A))$.

Theorem 4.7.12: Rank Theorem

Let A be an $m \times n$ matrix. Then $\dim(\text{col}(A))$, the dimension of the column space, is equal to the dimension of the row space, $\dim(\text{row}(A))$.

The following statements all follow from the Rank Theorem.

Corollary 4.7.5: Results of the Rank Theorem

Let A be a matrix. Then the following are true:

1. $\text{rank}(A) = \text{rank}(A^T)$.
2. For A of size $m \times n$, $\text{rank}(A) \leq m$ and $\text{rank}(A) \leq n$.
3. For A of size $n \times n$, A is invertible if and only if $\text{rank}(A) = n$.
4. For invertible matrices B and C of appropriate size, $\text{rank}(A) = \text{rank}(BA) = \text{rank}(AC)$.

Consider the following example.

Example 4.7.18: Rank of the Transpose

Let

$$A = \begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix}$$

Find $\text{rank}(A)$ and $\text{rank}(A^T)$.

Solution

To find $\text{rank}(A)$ we first row reduce to find the reduced row-echelon form.

$$A = \begin{bmatrix}1 & 2\\-1 & 1\end{bmatrix} \to \cdots \to \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Therefore the rank of A is 2. Now consider A^T given by

$$A^T = \begin{bmatrix}1 & -1\\2 & 1\end{bmatrix}$$

Again we row reduce to find the reduced row-echelon form.

$$\begin{bmatrix}1 & -1\\2 & 1\end{bmatrix} \to \cdots \to \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

You can see that $\text{rank}(A^T) = 2$, the same as $\text{rank}(A)$.

We now define what is meant by the null space of a general $m \times n$ matrix.

Definition 4.7.11: Null Space, or Kernel, of A

The null space of a matrix A, also referred to as the kernel of A, is defined as follows.

$$\mathbf{null}\,(A) = \left\{\vec{x} : A\vec{x} = \vec{0}\right\}$$

It can also be referred to using the notation $\ker(A)$. Similarly, we can discuss the image of A, denoted by $\mathbf{im}\,(A)$. The image of A consists of the vectors of $\mathbb{R}^m$ which "get hit" by A. The formal definition is as follows.

Definition 4.7.12: Image of A

The image of A, written $\mathbf{im}\,(A)$ is given by

$$\mathbf{im}\,(A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$$

Consider A as a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ whose action is given by multiplication. The following diagram displays this scenario.

$$\underset{\mathbb{R}^n}{\overset{\mathbf{null}(A)}{}} \quad \underset{\rightarrow}{\overset{A}{}} \quad \underset{\mathbb{R}^m}{\overset{\mathbf{im}(A)}{}}$$

As indicated, $\mathbf{im}\,(A)$ is a subset of $\mathbb{R}^m$ while $\mathbf{null}\,(A)$ is a subset of $\mathbb{R}^n$.

It turns out that the null space and image of A are both subspaces. Consider the following example.

Example 4.7.19: Null Space

Let A be an $m \times n$ matrix. Then the null space of A, $\mathbf{null}(A)$ is a subspace of $\mathbb{R}^n$.

Solution

- Since $A\vec{0}_n = \vec{0}_m$, $\vec{0}_n \in \mathbf{null}(A)$.
- Let $\vec{x}, \vec{y} \in \mathbf{null}(A)$. Then $A\vec{x} = \vec{0}_m$ and $A\vec{y} = \vec{0}_m$, so

  $$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}_m + \vec{0}_m = \vec{0}_m,$$

  and thus $\vec{x} + \vec{y} \in \mathbf{null}(A)$.
- Let $\vec{x} \in \mathbf{null}(A)$ and $k \in \mathbb{R}$. Then $A\vec{x} = \vec{0}_m$, so

  $$A(k\vec{x}) = k(A\vec{x}) = k\vec{0}_m = \vec{0}_m,$$

  and thus $k\vec{x} \in \mathbf{null}(A)$.

Therefore by the subspace test, $\mathbf{null}(A)$ is a subspace of $\mathbb{R}^n$.

The proof that $\mathbf{im}(A)$ is a subspace of $\mathbb{R}^m$ is similar and is left as an exercise to the reader.

We now wish to find a way to describe $\mathbf{null}(A)$ for a matrix A. However, finding $\mathbf{null}\,(A)$ is not new! There is just some new terminology being used, as $\mathbf{null}\,(A)$ is simply the solution to the system $A\vec{x} = \vec{0}$.

Theorem 4.7.13: Basis of null(A)

Let A be an $m \times n$ matrix such that $\mathbf{rank}(A) = r$. Then the system $A\vec{x} = \vec{0}_m$ has $n - r$ basic solutions, providing a basis of $\mathbf{null}(A)$ with $\mathbf{dim}(\mathbf{null}(A)) = n - r$.

Consider the following example.

Example 4.7.20: Null Space of A

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$

Find $\mathbf{null}\,(A)$ and $\mathbf{im}\,(A)$.

Solution

In order to find $\mathbf{null}\,(A)$, we simply need to solve the equation $A\vec{x} = \vec{0}$. This is the usual procedure of writing the augmented matrix, finding the reduced row-echelon form and then the solution. The augmented matrix and corresponding reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 3 & 3 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The third column is not a pivot column, and therefore the solution will contain a parameter. The solution to the system $A\vec{x} = \vec{0}$ is given by

$$\begin{bmatrix} -3t \\ t \\ t \end{bmatrix} : t \in \mathbb{R}$$

which can be written as

$$t \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R}$$

Therefore, the null space of A is all multiples of this vector, which we can write as

$$\mathbf{null}(A) = \mathbf{span}\left\{\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}\right\}$$

Finally $\mathbf{im}\,(A)$ is just $\{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ and hence consists of the span of all columns of A, that is $\mathbf{im}\,(A) = \mathbf{col}(A)$.

Notice from the above calculation that that the first two columns of the reduced row-echelon form are pivot columns. Thus the column space is the span of the first two columns in the **original matrix**, and we get

$$\mathbf{im}\,(A) = \mathbf{col}(A) = \mathbf{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\right\}$$

Here is a larger example, but the method is entirely similar.

✔ Example 4.7.21: Null Space of A

Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 1 \\ 2 & -1 & 1 & 3 & 0 \\ 3 & 1 & 2 & 3 & 1 \\ 4 & -2 & 2 & 6 & 0 \end{bmatrix}$$

Find the null space of A.

Solution

To find the null space, we need to solve the equation $AX = 0$. The augmented matrix and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 3 & 0 & 0 \\ 3 & 1 & 2 & 3 & 1 & 0 \\ 4 & -2 & 2 & 6 & 0 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrrr|r} 1 & 0 & \frac{3}{5} & \frac{6}{5} & \frac{1}{5} & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{3}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It follows that the first two columns are pivot columns, and the next three correspond to parameters. Therefore, $\mathbf{null}\,(A)$ is given by

$$\left[\begin{array}{c} \left(-\frac{3}{5}\right)s+\left(-\frac{6}{5}\right)t+\left(\frac{1}{5}\right)r \\ \left(-\frac{1}{5}\right)s+\left(\frac{3}{5}\right)t+\left(-\frac{2}{5}\right)r \\ s \\ t \\ r \end{array}\right] : s,t,r \in \mathbb{R}.$$

We write this in the form

$$s\left[\begin{array}{r} -\frac{3}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \\ 0 \end{array}\right]+t\left[\begin{array}{r} -\frac{6}{5} \\ \frac{3}{5} \\ 0 \\ 1 \\ 0 \end{array}\right]+r\left[\begin{array}{r} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \\ 1 \end{array}\right] : s,t,r \in \mathbb{R}.$$

In other words, the null space of this matrix equals the span of the three vectors above. Thus

$$\mathbf{null}\,(A) = \mathbf{span}\left\{\left[\begin{array}{r} -\frac{3}{5} \\ -\frac{1}{5} \\ 1 \\ 0 \\ 0 \end{array}\right], \left[\begin{array}{r} -\frac{6}{5} \\ \frac{3}{5} \\ 0 \\ 1 \\ 0 \end{array}\right], \left[\begin{array}{r} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \\ 1 \end{array}\right]\right\}$$

Notice also that the three vectors above are linearly independent and so the dimension of $\mathbf{null}\,(A)$ is 3. The following is true in general, the number of parameters in the solution of $AX = 0$ equals the dimension of the null space. Recall also that the number of leading ones in the reduced row-echelon form equals the number of pivot columns, which is the rank of the matrix, which is the same as the dimension of either the column or row space.

Before we proceed to an important theorem, we first define what is meant by the nullity of a matrix.

Definition 4.7.13: Nullity

The dimension of the null space of a matrix is called the **nullity**, denoted $\mathbf{dim}(\mathbf{null}\,(A))$.

From our observation above we can now state an important theorem.

Theorem 4.7.14: Rank and Nullity

Let A be an $m \times n$ matrix. Then $\mathbf{rank}\,(A) + \mathbf{dim}(\mathbf{null}\,(A)) = n$.

Example 4.7.22: Rank and Nullity

Let

$$A = \left[\begin{array}{rrr} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & 3 \end{array}\right]$$

Find $\mathbf{rank}\,(A)$ and $\mathbf{dim}(\mathbf{null}\,(A))$.

Solution

In the above Example 4.7.20 we determined that the reduced row-echelon form of A is given by

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the rank of A is 2. We also determined that the null space of A is given by

$$\text{null}(A) = \text{span}\left\{\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}\right\}$$

Therefore the nullity of A is 1. It follows from Theorem 4.7.14 that $\text{rank}\,(A) + \dim(\text{null}\,(A)) = 2 + 1 = 3$, which is the number of columns of A.

We conclude this section with two similar, and important, theorems.

Theorem 4.7.15

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}(A) = n$.
2. $\text{row}(A) = \mathbb{R}^n$, i.e., the rows of A span $\mathbb{R}^n$.
3. The columns of A are independent in $\mathbb{R}^m$.
4. The $n \times n$ matrix A^TA is invertible.
5. There exists an $n \times m$ matrix C so that $CA = I_n$.
6. If $A\vec{x} = \vec{0}_m$ for some $\vec{x} \in \mathbb{R}^n$, then $\vec{x} = \vec{0}_n$.

Theorem 4.7.16

Let A be an $m \times n$ matrix. The following are equivalent.

1. $\text{rank}(A) = m$.
2. $\text{col}(A) = \mathbb{R}^m$, i.e., the columns of A span $\mathbb{R}^m$.
3. The rows of A are independent in $\mathbb{R}^n$.
4. The $m \times m$ matrix AA^T is invertible.
5. There exists an $n \times m$ matrix C so that $AC = I_m$.
6. The system $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$.

This page titled 4.7: Spanning, Linear Independence and Basis in R^n is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **4.10: Spanning, Linear Independence and Basis in R^n** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

4.8: Orthogonality

Outcomes

A. Determine if a given set is orthogonal or orthonormal.
B. Determine if a given matrix is orthogonal.
C. Given a linearly independent set, use the Gram-Schmidt Process to find corresponding orthogonal and orthonormal sets.
D. Find the orthogonal projection of a vector onto a subspace.
E. Find the least squares approximation for a collection of points.

In this section, we examine what it means for vectors (and sets of vectors) to be **orthogonal** and **orthonormal**. First, it is necessary to review some important concepts. You may recall the definitions for the span of a set of vectors and a linear independent set of vectors. We include the definitions and examples here for convenience.

Definition 4.8.1: Span of a Set of Vectors and Subspace

The collection of all linear combinations of a set of vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is known as the span of these vectors and is written as $\mathbf{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$.
We call a collection of the form $\mathbf{span}\{\vec{u}_1, \cdots, \vec{u}_k\}$ a subspace of $\mathbb{R}^n$.

Consider the following example.

Example 4.8.1: Spanning Vectors

Describe the span of the vectors $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T \in \mathbb{R}^3$.

Solution

You can see that any linear combination of the vectors $\vec{u}$ and $\vec{v}$ yields a vector $\begin{bmatrix} x & y & 0 \end{bmatrix}^T$ in the XY-plane.

Moreover every vector in the XY-plane is in fact such a linear combination of the vectors $\vec{u}$ and $\vec{v}$. That's because

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = (-2x + 3y) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (x - y) \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Thus span$\{\vec{u}, \vec{v}\}$ is precisely the XY-plane.

The span of a set of a vectors in $\mathbb{R}^n$ is what we call a **subspace of $\mathbb{R}^n$**. A subspace W is characterized by the feature that any linear combination of vectors of W is again a vector contained in W.

Another important property of sets of vectors is called **linear independence**.

Definition 4.8.2: Linear Independence

A set of non-zero vectors $\{\vec{u}_1, \cdots, \vec{u}_k\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if no vector in that set is in the span of the other vectors of that set.

Here is an example.

Example 4.8.2: Linearly Independent Vectors

Consider vectors $\vec{u} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$, $\vec{v} = \begin{bmatrix} 3 & 2 & 0 \end{bmatrix}^T$, and $\vec{w} = \begin{bmatrix} 4 & 5 & 0 \end{bmatrix}^T \in \mathbb{R}^3$. Verify whether the set $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

Solution

We already verified in Example 4.8.1 that $\text{span}\{\vec{u},\vec{v}\}$ is the XY-plane. Since $\vec{w}$ is clearly also in the XY-plane, then the set $\{\vec{u},\vec{v},\vec{w}\}$ is **not** linearly independent.

In terms of spanning, a set of vectors is linearly independent if it does not contain unnecessary vectors. In the previous example you can see that the vector $\vec{w}$ does not help to span any new vector not already in the span of the other two vectors. However you can verify that the set $\{\vec{u},\vec{v}\}$ is linearly independent, since you will not get the XY-plane as the span of a single vector.

We can also determine if a set of vectors is linearly independent by examining linear combinations. A set of vectors is linearly independent if and only if whenever a linear combination of these vectors equals zero, it follows that all the coefficients equal zero. It is a good exercise to verify this equivalence, and this latter condition is often used as the (equivalent) definition of linear independence.

If a subspace is spanned by a linearly independent set of vectors, then we say that it is a basis for the subspace.

Definition 4.8.3: Basis

Let V be a subspace of $\mathbb{R}^n$. Then $\{\vec{u}_1,\cdots,\vec{u}_k\}$ is a **basis** for V if the following two conditions hold.

1. $\text{span}\{\vec{u}_1,\cdots,\vec{u}_k\}=V$
2. $\{\vec{u}_1,\cdots,\vec{u}_k\}$ is linearly independent

Thus the set of vectors $\{\vec{u},\vec{v}\}$ from Example 4.8.2 is a basis for XY-plane in $\mathbb{R}^3$ since it is both linearly independent and spans the XY-plane.

Recall from the properties of the dot product of vectors that two vectors $\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u}\cdot\vec{v}=0$. Suppose a vector is orthogonal to a spanning set of $\mathbb{R}^n$. What can be said about such a vector? This is the discussion in the following example.

Example 4.8.3: Orthogonal Vector to a Spanning Set

Let $\{\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_k\}\in\mathbb{R}^n$ and suppose $\mathbb{R}^n=\text{span}\{\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_k\}$. Furthermore, suppose that there exists a vector $\vec{u}\in\mathbb{R}^n$ for which $\vec{u}\cdot\vec{x}_j=0$ for all j, $1\le j\le k$. What type of vector is $\vec{u}$?

Solution

Write $\vec{u}=t_1\vec{x}_1+t_2\vec{x}_2+\cdots+t_k\vec{x}_k$ for some $t_1,t_2,\ldots,t_k\in\mathbb{R}$ (this is possible because $\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_k$ span $\mathbb{R}^n$).

Then

$$\begin{aligned}\|\vec{u}\|^2&=\vec{u}\cdot\vec{u}\\&=\vec{u}\cdot(t_1\vec{x}_1+t_2\vec{x}_2+\cdots+t_k\vec{x}_k)\\&=\vec{u}\cdot(t_1\vec{x}_1)+\vec{u}\cdot(t_2\vec{x}_2)+\cdots+\vec{u}\cdot(t_k\vec{x}_k)\\&=t_1(\vec{u}\cdot\vec{x}_1)+t_2(\vec{u}\cdot\vec{x}_2)+\cdots+t_k(\vec{u}\cdot\vec{x}_k)\\&=t_1(0)+t_2(0)+\cdots+t_k(0)=0.\end{aligned}\tag{4.8.1}$$

Since $\|\vec{u}\|^2=0$, $\|\vec{u}\|=0$. We know that $\|\vec{u}\|=0$ if and only if $\vec{u}=\vec{0}_n$. Therefore, $\vec{u}=\vec{0}_n$. In conclusion, the only vector orthogonal to every vector of a spanning set of $\mathbb{R}^n$ is the zero vector.

We can now discuss what is meant by an orthogonal set of vectors.

Definition 4.8.4: Orthogonal Set of Vectors

Let $\{\vec{u}_1,\vec{u}_2,\cdots,\vec{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. Then this set is called an **orthogonal set** if the following conditions hold:

1. $\vec{u}_i\cdot\vec{u}_j=0$ for all $i\ne j$
2. $\vec{u}_i\ne\vec{0}$ for all i

If we have an orthogonal set of vectors and normalize each vector so they have length 1, the resulting set is called an **orthonormal set** of vectors. They can be described as follows.

Definition 4.8. :5 Orthonormal Set of Vectors

A set of vectors, $\{\vec{w}_1, \cdots, \vec{w}_m\}$ is said to be an **orthonormal** set if

$$\vec{w}_i \cdot \vec{w}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Note that all orthonormal sets are orthogonal, but the reverse is not necessarily true since the vectors may not be normalized. In order to normalize the vectors, we simply need divide each one by its length.

Definition 4.8.6: Normalizing an Orthogonal Set

Normalizing an orthogonal set is the process of turning an orthogonal (but not orthonormal) set into an orthonormal set. If $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is an orthogonal subset of $\mathbb{R}^n$, then

$$\left\{ \frac{1}{\|\vec{u}_1\|}\vec{u}_1, \frac{1}{\|\vec{u}_2\|}\vec{u}_2, \ldots, \frac{1}{\|\vec{u}_k\|}\vec{u}_k \right\}$$

is an orthonormal set.

We illustrate this concept in the following example.

Example 4.8.4: Orthonormal Set

Consider the set of vectors given by

$$\{\vec{u}_1, \vec{u}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

Show that it is an orthogonal set of vectors but not an orthonormal one. Find the corresponding orthonormal set.

Solution

One easily verifies that $\vec{u}_1 \cdot \vec{u}_2 = 0$ and $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal set of vectors. On the other hand one can compute that $\|\vec{u}_1\| = \|\vec{u}_2\| = \sqrt{2} \neq 1$ and thus it is not an orthonormal set.

Thus to find a corresponding orthonormal set, we simply need to normalize each vector. We will write $\{\vec{w}_1, \vec{w}_2\}$ for the corresponding orthonormal set. Then,

$$\begin{aligned} \vec{w}_1 &= \frac{1}{\|\vec{u}_1\|}\vec{u}_1 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned} \tag{4.8.2}$$

Similarly,

$$\begin{aligned} \vec{w}_2 &= \frac{1}{\|\vec{u}_2\|}\vec{u}_2 \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned} \tag{4.8.3}$$

Therefore the corresponding orthonormal set is

$$\{\vec{w}_1, \vec{w}_2\} = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$$

You can verify that this set is orthogonal.

Consider an orthogonal set of vectors in $\mathbb{R}^n$, written $\{\vec{w}_1, \cdots, \vec{w}_k\}$ with $k \leq n$. The span of these vectors is a subspace W of $\mathbb{R}^n$. If we could show that this orthogonal set is also linearly independent, we would have a basis of W. We will show this in the next theorem.

Theorem 4.8.1: Orthogonal Basis of a Subspace

Let $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$ be an orthonormal set of vectors in $\mathbb{R}^n$. Then this set is linearly independent and forms a basis for the subspace $W = \text{span}\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$.

Proof

To show it is a linearly independent set, suppose a linear combination of these vectors equals $\vec{0}$, such as:

$$a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_k\vec{w}_k = \vec{0}, a_i \in \mathbb{R}$$

We need to show that all $a_i = 0$. To do so, take the dot product of each side of the above equation with the vector $\vec{w}_i$ and obtain the following.

$$\begin{aligned} \vec{w}_i \cdot (a_1\vec{w}_1 + a_2\vec{w}_2 + \cdots + a_k\vec{w}_k) &= \vec{w}_i \cdot \vec{0} \\ a_1(\vec{w}_i \cdot \vec{w}_1) + a_2(\vec{w}_i \cdot \vec{w}_2) + \cdots + a_k(\vec{w}_i \cdot \vec{w}_k) &= 0 \end{aligned} \tag{4.8.4}$$

Now since the set is orthogonal, $\vec{w}_i \cdot \vec{w}_m = 0$ for all $m \neq i$, so we have:

$$a_1(0) + \cdots + a_i(\vec{w}_i \cdot \vec{w}_i) + \cdots + a_k(0) = 0$$

$$a_i\|\vec{w}_i\|^2 = 0$$

Since the set is orthogonal, we know that $\|\vec{w}_i\|^2 \neq 0$. It follows that $a_i = 0$. Since the a_i was chosen arbitrarily, the set $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$ is linearly independent.

Finally since $W = \text{span}\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_k\}$, the set of vectors also spans W and therefore forms a basis of W.

If an orthogonal set is a basis for a subspace, we call this an orthogonal basis. Similarly, if an orthonormal set is a basis, we call this an orthonormal basis.

We conclude this section with a discussion of Fourier expansions. Given any orthogonal basis B of $\mathbb{R}^n$ and an arbitrary vector $\vec{x} \in \mathbb{R}^n$, how do we express $\vec{x}$ as a linear combination of vectors in B? The solution is Fourier expansion.

Theorem 4.8.2: Fourier Expansion

Let V be a subspace of $\mathbb{R}^n$ and suppose $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_m\}$ is an orthogonal basis of V. Then for any $\vec{x} \in V$,

$$\vec{x} = \left(\frac{\vec{x} \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\right)\vec{u}_1 + \left(\frac{\vec{x} \cdot \vec{u}_2}{\|\vec{u}_2\|^2}\right)\vec{u}_2 + \cdots + \left(\frac{\vec{x} \cdot \vec{u}_m}{\|\vec{u}_m\|^2}\right)\vec{u}_m$$

This expression is called the **Fourier expansion** of $\vec{x}$, and

$$\frac{\vec{x} \cdot \vec{u}_j}{\|\vec{u}_j\|^2},$$

$j = 1, 2, \ldots, m$ are the Fourier coefficients.

Consider the following example.

Example 4.8.5: Fourier Expansion

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and $\vec{u}_3 = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$, and let $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then $B = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis of $\mathbb{R}^3$.

Compute the Fourier expansion of $\vec{x}$, thus writing $\vec{x}$ as a linear combination of the vectors of B.

Solution

Since B is a basis (verify!) there is a unique way to express $\vec{x}$ as a linear combination of the vectors of B. Moreover since B is an orthogonal basis (verify!), then this can be done by computing the Fourier expansion of $\vec{x}$.

That is:

$$\vec{x} = \left(\frac{\vec{x} \cdot \vec{u}_1}{\|\vec{u}_1\|^2}\right)\vec{u}_1 + \left(\frac{\vec{x} \cdot \vec{u}_2}{\|\vec{u}_2\|^2}\right)\vec{u}_2 + \left(\frac{\vec{x} \cdot \vec{u}_3}{\|\vec{u}_3\|^2}\right)\vec{u}_3.$$

We readily compute:

$$\frac{\vec{x} \cdot \vec{u}_1}{\|\vec{u}_1\|^2} = \frac{2}{6}, \frac{\vec{x} \cdot \vec{u}_2}{\|\vec{u}_2\|^2} = \frac{3}{5}, \text{ and } \frac{\vec{x} \cdot \vec{u}_3}{\|\vec{u}_3\|^2} = \frac{4}{30}.$$

Therefore,

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + \frac{2}{15}\begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}.$$

Orthogonal Matrices

Recall that the process to find the inverse of a matrix was often cumbersome. In contrast, it was very easy to take the transpose of a matrix. Luckily for some special matrices, the transpose equals the inverse. When an $n \times n$ matrix has all real entries and its transpose equals its inverse, the matrix is called an **orthogonal matrix**.

The precise definition is as follows.

Definition 4.8.7: Orthogonal Matrices

A real $n \times n$ matrix U is called an **orthogonal** matrix if

$$UU^T = U^TU = I.$$

Note since U is assumed to be a square matrix, it suffices to verify only one of these equalities $UU^T = I$ or $U^TU = I$ holds to guarantee that U^T is the inverse of U.

Consider the following example.

Example 4.8.6: Orthogonal Matrix

Orthogonal Matrix Show the matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

is orthogonal.

Solution

All we need to do is verify (one of the equations from) the requirements of Definition 4.8.7.

$$UU^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $UU^T = I$, this matrix is orthogonal.

Here is another example.

Example 4.8.7: Orthogonal Matrix

Orthogonal Matrix Let $U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$. Is U orthogonal?

Solution

Again the answer is yes and this can be verified simply by showing that $U^TU = I$:

$$\begin{aligned} U^TU &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{4.8.5}$$

When we say that U is orthogonal, we are saying that $UU^T = I$, meaning that

$$\sum_j u_{ij}u^T_{jk} = \sum_j u_{ij}u_{kj} = \delta_{ik}$$

where δ_{ij} is the **Kronecker symbol** defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

In words, the product of the ith row of U with the kth row gives 1 if $i = k$ and 0 if $i \neq k$. The same is true of the columns because $U^TU = I$ also. Therefore,

$$\sum_j u^T_{ij}u_{jk} = \sum_j u_{ji}u_{jk} = \delta_{ik}$$

which says that the product of one column with another column gives 1 if the two columns are the same and 0 if the two columns are different.

More succinctly, this states that if $\vec{u}_1, \cdots, \vec{u}_n$ are the columns of U, an orthogonal matrix, then

$$\vec{u}_i \cdot \vec{u}_j = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

We will say that the columns form an orthonormal set of vectors, and similarly for the rows. Thus a matrix is **orthogonal** if its rows (or columns) form an **orthonormal** set of vectors. Notice that the convention is to call such a matrix orthogonal rather than orthonormal (although this may make more sense!).

Proposition 4.8.1: Orthonormal Basis

The rows of an $n \times n$ orthogonal matrix form an orthonormal basis of $\mathbb{R}^n$. Further, any orthonormal basis of $\mathbb{R}^n$ can be used to construct an $n \times n$ orthogonal matrix.

Proof

Recall from Theorem 4.8.1 that an orthonormal set is linearly independent and forms a basis for its span. Since the rows of an $n \times n$ orthogonal matrix form an orthonormal set, they must be linearly independent. Now we have n linearly independent vectors, and it follows that their span equals $\mathbb{R}^n$. Therefore these vectors form an orthonormal basis for $\mathbb{R}^n$.

Suppose now that we have an orthonormal basis for $\mathbb{R}^n$. Since the basis will contain n vectors, these can be used to construct an $n \times n$ matrix, with each vector becoming a row. Therefore the matrix is composed of orthonormal rows, which by our above discussion, means that the matrix is orthogonal. Note we could also have construct a matrix with each vector becoming a column instead, and this would again be an orthogonal matrix. In fact this is simply the transpose of the previous matrix.

Consider the following proposition.

Proposition 4.8.2: Determinant of Orthogonal Matrices

Det Suppose U is an orthogonal matrix. Then $\det(U) = \pm 1$.

Proof

This result follows from the properties of determinants. Recall that for any matrix A, $\det(A)^T = \det(A)$. Now if U is orthogonal, then:

$$(\det(U))^2 = \det(U^T)\det(U) = \det(U^T U) = \det(I) = 1$$

Therefore $(\det(U))^2 = 1$ and it follows that $\det(U) = \pm 1$.

Orthogonal matrices are divided into two classes, proper and improper. The proper orthogonal matrices are those whose determinant equals 1 and the improper ones are those whose determinant equals -1. The reason for the distinction is that the improper orthogonal matrices are sometimes considered to have no physical significance. These matrices cause a change in orientation which would correspond to material passing through itself in a non physical manner. Thus in considering which coordinate systems must be considered in certain applications, you only need to consider those which are related by a proper orthogonal transformation. Geometrically, the linear transformations determined by the proper orthogonal matrices correspond to the composition of rotations.

We conclude this section with two useful properties of orthogonal matrices.

Example 4.8.8: Product and Inverse of Orthogonal Matrices

Suppose A and B are orthogonal matrices. Then AB and A^{-1} both exist and are orthogonal.

Solution

First we examine the product AB.

$$(AB)(B^T A^T) = A(BB^T)A^T = AA^T = I$$

Since AB is square, $B^T A^T = (AB)^T$ is the inverse of AB, so AB is invertible, and $(AB)^{-1} = (AB)^T$ Therefore, AB is orthogonal.

Next we show that $A^{-1} = A^T$ is also orthogonal.

$$(A^{-1})^{-1} = A = (A^T)^T = (A^{-1})^T$$

Therefore A^{-1} is also orthogonal.

Gram-Schmidt Process

The Gram-Schmidt process is an algorithm to transform a set of vectors into an orthonormal set spanning the same subspace, that is generating the same collection of linear combinations (see Definition 9.2.2).

The goal of the Gram-Schmidt process is to take a linearly independent set of vectors and transform it into an orthonormal set with the same span. The first objective is to construct an orthogonal set of vectors with the same span, since from there an orthonormal set can be obtained by simply dividing each vector by its length.

Algorithm 4.8.1: Gram-Schmidt Process

Let $\{\vec{u}_1, \cdots, \vec{u}_n\}$ be a set of linearly independent vectors in $\mathbb{R}^n$.

I: Construct a new set of vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$ as follows:

$$\begin{aligned}
\vec{v}_1 &= \vec{u}_1 \\
\vec{v}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 \\
\vec{v}_3 &= \vec{u}_3 - \left(\frac{\vec{u}_3 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 - \left(\frac{\vec{u}_3 \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\right)\vec{v}_2 \\
&\vdots \\
\vec{v}_n &= \vec{u}_n - \left(\frac{\vec{u}_n \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\right)\vec{v}_1 - \left(\frac{\vec{u}_n \cdot \vec{v}_2}{\|\vec{v}_2\|^2}\right)\vec{v}_2 - \cdots - \left(\frac{\vec{u}_n \cdot \vec{v}_{n-1}}{\|\vec{v}_{n-1}\|^2}\right)\vec{v}_{n-1}
\end{aligned}$$

II: Now let $\vec{w}_i = \dfrac{\vec{v}_i}{\|\vec{v}_i\|}$ for $i = 1, \cdots, n$.

Then

1. $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is an orthogonal set.
2. $\{\vec{w}_1, \cdots, \vec{w}_n\}$ is an orthonormal set.
3. $\text{span}\{\vec{u}_1, \cdots, \vec{u}_n\} = \text{span}\{\vec{v}_1, \cdots, \vec{v}_n\} = \text{span}\{\vec{w}_1, \cdots, \vec{w}_n\}$.

Solution

The full proof of this algorithm is beyond this material, however here is an indication of the arguments.

To show that $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is an orthogonal set, let

$$a_2 = \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}$$

then:

$$\begin{aligned}
\vec{v}_1 \cdot \vec{v}_2 &= \vec{v}_1 \cdot (\vec{u}_2 - a_2\vec{v}_1) \\
&= \vec{v}_1 \cdot \vec{u}_2 - a_2(\vec{v}_1 \cdot \vec{v}_1 \\
&= \vec{v}_1 \cdot \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}\|\vec{v}_1\|^2 \\
&= (\vec{v}_1 \cdot \vec{u}_2) - (\vec{u}_2 \cdot \vec{v}_1) = 0
\end{aligned}$$

Now that you have shown that $\{\vec{v}_1, \vec{v}_2\}$ is orthogonal, use the same method as above to show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is also orthogonal, and so on.

Then in a similar fashion you show that $\text{span}\{\vec{u}_1, \cdots, \vec{u}_n\} = \text{span}\{\vec{v}_1, \cdots, \vec{v}_n\}$.

Finally defining $\vec{w}_i = \dfrac{\vec{v}_i}{\|\vec{v}_i\|}$ for $i = 1, \cdots, n$ does not affect orthogonality and yields vectors of length 1, hence an orthonormal set. You can also observe that it does not affect the span either and the proof would be complete.

Consider the following example.

Example 4.8.9: Find Orthonormal Set with Same Span

Consider the set of vectors $\{\vec{u}_1, \vec{u}_2\}$ given as in Example 4.8.1. That is

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \in \mathbb{R}^3$$

Use the Gram-Schmidt algorithm to find an orthonormal set of vectors $\{\vec{w}_1, \vec{w}_2\}$ having the same span.

Solution

We already remarked that the set of vectors in $\{\vec{u}_1, \vec{u}_2\}$ is linearly independent, so we can proceed with the Gram-Schmidt algorithm:

$$\begin{aligned} \vec{v}_1 &= \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ \vec{v}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \right) \vec{v}_1 \\ &= \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} \end{aligned} \tag{4.8.6}$$

Now to normalize simply let

$$\begin{aligned} \vec{w}_1 &= \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \\ \vec{w}_2 &= \frac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \end{aligned} \tag{4.8.7}$$

You can verify that $\{\vec{w}_1, \vec{w}_2\}$ is an orthonormal set of vectors having the same span as $\{\vec{u}_1, \vec{u}_2\}$, namely the XY-plane.

In this example, we began with a linearly independent set and found an orthonormal set of vectors which had the same span. It turns out that if we start with a basis of a subspace and apply the Gram-Schmidt algorithm, the result will be an orthogonal basis of the same subspace. We examine this in the following example.

Example 4.8.10: Find a Corresponding Orthogonal Basis

Let

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

and let $U = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$. Use the Gram-Schmidt Process to construct an orthogonal basis B of U.

Solution

First $\vec{f}_1 = \vec{x}_1$.

Next,

$$\vec{f}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Finally,

$$\vec{f}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{0}{1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \\ 0 \end{bmatrix}.$$

Therefore,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \\ 0 \end{bmatrix} \right\}$$

is an orthogonal basis of U. However, it is sometimes more convenient to deal with vectors having integer entries, in which case we take

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

Orthogonal Projections

An important use of the Gram-Schmidt Process is in orthogonal projections, the focus of this section.

You may recall that a subspace of $\mathbb{R}^n$ is a set of vectors which contains the zero vector, and is closed under addition and scalar multiplication. Let's call such a subspace W. In particular, a plane in $\mathbb{R}^n$ which contains the origin, $(0, 0, \cdots, 0)$, is a subspace of $\mathbb{R}^n$.

Suppose a point Y in $\mathbb{R}^n$ is not contained in W, then what point Z in W is closest to Y? Using the Gram-Schmidt Process, we can find such a point. Let $\vec{y}, \vec{z}$ represent the position vectors of the points Y and Z respectively, with $\vec{y} - \vec{z}$ representing the vector connecting the two points Y and Z. It will follow that if Z is the point on W closest to Y, then $\vec{y} - \vec{z}$ will be perpendicular to W (can you see why?); in other words, $\vec{y} - \vec{z}$ is orthogonal to W (and to every vector contained in W) as in the following diagram.

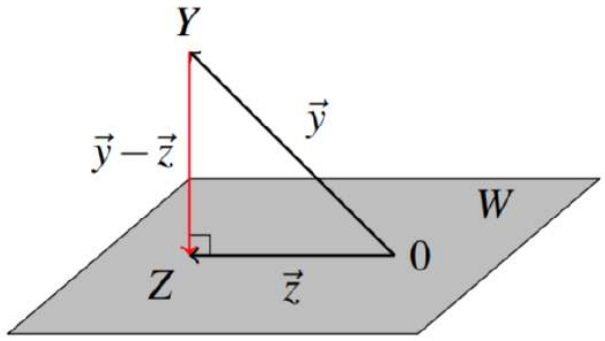


Figure 4.8.1

The vector $\vec{z}$ is called the **orthogonal projection** of $\vec{y}$ on W. The definition is given as follows.

Definition 4.8.8: Orthogonal Projection

Let W be a subspace of $\mathbb{R}^n$, and Y be any point in $\mathbb{R}^n$. Then the orthogonal projection of Y onto W is given by

$$\vec{z} = \text{proj}_W(\vec{y}) = \left(\frac{\vec{y} \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 + \left(\frac{\vec{y} \cdot \vec{w}_2}{\|\vec{w}_2\|^2}\right)\vec{w}_2 + \cdots + \left(\frac{\vec{y} \cdot \vec{w}_m}{\|\vec{w}_m\|^2}\right)\vec{w}_m$$

where $\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_m\}$ is any orthogonal basis of W.

Therefore, in order to find the orthogonal projection, we must first find an orthogonal basis for the subspace. Note that one could use an orthonormal basis, but it is not necessary in this case since as you can see above the normalization of each vector is included in the formula for the projection.

Before we explore this further through an example, we show that the orthogonal projection does indeed yield a point Z (the point whose position vector is the vector $\vec{z}$ above) which is the point of W closest to Y.

Theorem 4.8.3: Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$ and Y any point in $\mathbb{R}^n$. Let Z be the point whose position vector is the orthogonal projection of Y onto W.

Then, Z is the point in W closest to Y.

Proof

First Z is certainly a point in W since it is in the span of a basis of W.

To show that Z is the point in W closest to Y, we wish to show that $|\vec{y}-\vec{z}_1| > |\vec{y}-\vec{z}|$ for all $\vec{z}_1 \neq \vec{z} \in W$. We begin by writing $\vec{y}-\vec{z}_1 = (\vec{y}-\vec{z})+(\vec{z}-\vec{z}_1)$. Now, the vector $\vec{y}-\vec{z}$ is orthogonal to W, and $\vec{z}-\vec{z}_1$ is contained in W. Therefore these vectors are orthogonal to each other. By the Pythagorean Theorem, we have that

$$\|\vec{y}-\vec{z}_1\|^2 = \|\vec{y}-\vec{z}\|^2 + \|\vec{z}-\vec{z}_1\|^2 > \|\vec{y}-\vec{z}\|^2$$

This follows because $\vec{z} \neq \vec{z}_1$ so $\|\vec{z}-\vec{z}_1\|^2 > 0$.

Hence, $\|\vec{y}-\vec{z}_1\|^2 > \|\vec{y}-\vec{z}\|^2$. Taking the square root of each side, we obtain the desired result.

Consider the following example.

Example 4.8.11: Orthogonal Projection

Let W be the plane through the origin given by the equation $x-2y+z=0$. Find the point in W closest to the point $Y=(1,0,3)$.

Solution

We must first find an orthogonal basis for W. Notice that W is characterized by all points (a,b,c) where $c=2b-a$. In other words,

$$W = \begin{bmatrix} a \\ b \\ 2b-a \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},\ a,b \in \mathbb{R}$$

We can thus write W as

$$\begin{aligned} W &= \text{span}\{\vec{u}_1, \vec{u}_2\} \\ &= \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\} \end{aligned} \tag{4.8.8}$$

Notice that this span is a basis of W as it is linearly independent. We will use the Gram-Schmidt Process to convert this to an orthogonal basis, $\{\vec{w}_1, \vec{w}_2\}$. In this case, as we remarked it is only necessary to find an orthogonal basis, and it is not required that it be orthonormal.

$$\vec{w}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{aligned}\vec{w}_2 &= \vec{u}_2 - \left(\frac{\vec{u}_2 \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 \\ &= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \left(\frac{-2}{2}\right)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\end{aligned} \tag{4.8.9}$$

Therefore an orthogonal basis of W is

$$\{\vec{w}_1, \vec{w}_2\} = \left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$$

We can now use this basis to find the orthogonal projection of the point $Y = (1, 0, 3)$ on the subspace W. We will write the position vector $\vec{y}$ of Y as $\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$. Using Definition 4.8.8, we compute the projection as follows:

$$\begin{aligned}\vec{z} &= \text{proj}_W(\vec{y}) \\ &= \left(\frac{\vec{y} \cdot \vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 + \left(\frac{\vec{y} \cdot \vec{w}_2}{\|\vec{w}_2\|^2}\right)\vec{w}_2 \\ &= \left(\frac{-2}{2}\right)\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \left(\frac{4}{3}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \\ \frac{7}{3} \end{bmatrix}\end{aligned} \tag{4.8.10}$$

Therefore the point Z on W closest to the point $(1, 0, 3)$ is $\left(\frac{1}{3}, \frac{4}{3}, \frac{7}{3}\right)$.

Recall that the vector $\vec{y} - \vec{z}$ is perpendicular (orthogonal) to all the vectors contained in the plane W. Using a basis for W, we can in fact find all such vectors which are perpendicular to W. We call this set of vectors the **orthogonal complement** of W and denote it $W^\perp$.

Definition 4.8.9: Orthogonal Complement

Let W be a subspace of $\mathbb{R}^n$. Then the orthogonal complement of W, written $W^\perp$, is the set of all vectors $\vec{x}$ such that $\vec{x} \cdot \vec{z} = 0$ for all vectors $\vec{z}$ in W.

$$W^\perp = \{\vec{x} \in \mathbb{R}^n \text{ such that } \vec{x} \cdot \vec{z} = 0 \text{ for all } \vec{z} \in W\}$$

The orthogonal complement is defined as the set of all vectors which are orthogonal to all vectors in the original subspace. It turns out that it is sufficient that the vectors in the orthogonal complement be orthogonal to a spanning set of the original space.

Proposition 4.8.3: Orthogonal to Spanning Set

Let W be a subspace of $\mathbb{R}^n$ such that $W = \text{span}\{\vec{w}_1, \vec{w}_2, \cdots, \vec{w}_m\}$. Then $W^\perp$ is the set of all vectors which are orthogonal to each $\vec{w}_i$ in the spanning set.

The following proposition demonstrates that the orthogonal complement of a subspace is itself a subspace.

Proposition 4.8.4: The Orthogonal Complement

Let W be a subspace of $\mathbb{R}^n$. Then the orthogonal complement $W^\perp$ is also a subspace of $\mathbb{R}^n$.

Consider the following proposition.

Proposition 4.8.5: Orthogonal Complement of $\mathbb{R}^n$

The complement of $\mathbb{R}^n$ is the set containing the zero vector:

$$(\mathbb{R}^n)^\perp = \left\{\vec{0}\right\}$$

Similarly,

$$\left\{\vec{0}\right\}^\perp = (\mathbb{R}^n).$$

Proof

Here, $\vec{0}$ is the zero vector of $\mathbb{R}^n$. Since $\vec{x} \cdot \vec{0} = 0$ for all $\vec{x} \in \mathbb{R}^n$, $\mathbb{R}^n \subseteq \{\vec{0}\}^\perp$. Since $\{\vec{0}\}^\perp \subseteq \mathbb{R}^n$, the equality follows, i.e., $\{\vec{0}\}^\perp = \mathbb{R}^n$.

Again, since $\vec{x} \cdot \vec{0} = 0$ for all $\vec{x} \in \mathbb{R}^n$, $\vec{0} \in (\mathbb{R}^n)^\perp$, so $\{\vec{0}\} \subseteq (\mathbb{R}^n)^\perp$. Suppose $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$. Since $\vec{x} \cdot \vec{x} = ||\vec{x}||^2$ and $\vec{x} \neq \vec{0}$, $\vec{x} \cdot \vec{x} \neq 0$, so $\vec{x} \notin (\mathbb{R}^n)^\perp$. Therefore $(\mathbb{R}^n)^\perp \subseteq \{\vec{0}\}$, and thus $(\mathbb{R}^n)^\perp = \{\vec{0}\}$.

In the next example, we will look at how to find $W^\perp$.

Example 4.8.12: Orthogonal Complement

Let W be the plane through the origin given by the equation $x - 2y + z = 0$. Find a basis for the orthogonal complement of W.

Solution

From Example 4.8.11 we know that we can write W as

$$W = \text{span}\left\{\vec{u}_1, \vec{u}_2\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}$$

In order to find $W^\perp$, we need to find all $\vec{x}$ which are orthogonal to every vector in this span.

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. In order to satisfy $\vec{x} \cdot \vec{u}_1 = 0$, the following equation must hold.

$$x_1 - x_3 = 0$$

In order to satisfy $\vec{x} \cdot \vec{u}_2 = 0$, the following equation must hold.

$$x_2 + 2x_3 = 0$$

Both of these equations must be satisfied, so we have the following system of equations.

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

To solve, set up the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

Using Gaussian Elimination, we find that $W^{\perp} = \text{span}\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$, and hence $\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right\}$ is a basis for $W^{\perp}$.

The following results summarize the important properties of the orthogonal projection.

Theorem 4.8.4: Orthogonal Projection

Let W be a subspace of $\mathbb{R}^n$, Y be any point in $\mathbb{R}^n$, and let Z be the point in W closest to Y. Then,

1. The position vector $\vec{z}$ of the point Z is given by $\vec{z} = \text{proj}_W(\vec{y})$
2. $\vec{z} \in W$ and $\vec{y} - \vec{z} \in W^{\perp}$
3. $|Y - Z| < |Y - Z_1|$ for all $Z_1 \neq Z \in W$

Consider the following example of this concept.

Example 4.8.13: Find a Vector Closest to a Given Vector

Let

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \vec{v} = \begin{bmatrix} 4 \\ 3 \\ -2 \\ 5 \end{bmatrix}.$$

We want to find the vector in $W = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ closest to $\vec{y}$.

Solution

We will first use the Gram-Schmidt Process to construct the orthogonal basis, B, of W:

$$B = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}\right\}.$$

By Theorem 4.8.4,

$$\text{proj}_U(\vec{v}) = \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{1}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \frac{12}{6}\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

is the vector in U closest to $\vec{y}$.

Consider the next example.

Example 4.8.14: Vector Written as a Sum of Two Vectors

Let W be a subspace given by $W = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}\right\}$, and $Y = (1, 2, 3, 4)$.

Find the point Z in W closest to Y, and moreover write $\vec{y}$ as the sum of a vector in W and a vector in $W^{\perp}$.

Solution

From Theorem 4.8.3 the point Z in W closest to Y is given by $\vec{z} = \text{proj}_W(\vec{y})$.

Notice that since the above vectors already give an orthogonal basis for W, we have:

$$\begin{aligned}\vec{z} &= \text{proj}_W(\vec{y}) \\ &= \left(\frac{\vec{y}\cdot\vec{w}_1}{\|\vec{w}_1\|^2}\right)\vec{w}_1 + \left(\frac{\vec{y}\cdot\vec{w}_2}{\|\vec{w}_2\|^2}\right)\vec{w}_2 \\ &= \left(\frac{4}{2}\right)\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + \left(\frac{10}{5}\right)\begin{bmatrix}0\\1\\0\\2\end{bmatrix} \\ &= \begin{bmatrix}2\\2\\2\\4\end{bmatrix}\end{aligned} \tag{4.8.11}$$

Therefore the point in W closest to Y is $Z = (2,2,2,4)$.
Now, we need to write $\vec{y}$ as the sum of a vector in W and a vector in $W^\perp$. This can easily be done as follows:

$$\vec{y} = \vec{z} + (\vec{y} - \vec{z})$$

since $\vec{z}$ is in W and as we have seen $\vec{y} - \vec{z}$ is in $W^\perp$.
The vector $\vec{y} - \vec{z}$ is given by

$$\vec{y} - \vec{z} = \begin{bmatrix}1\\2\\3\\4\end{bmatrix} - \begin{bmatrix}2\\2\\2\\4\end{bmatrix} = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

Therefore, we can write $\vec{y}$ as

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \begin{bmatrix}2\\2\\2\\4\end{bmatrix} + \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

Example 4.8.15: Point in a Plane Closest to a Given Point

Find the point Z in the plane $3x + y - 2z = 0$ that is closest to the point $Y = (1,1,1)$.

Solution

The solution will proceed as follows.

1. Find a basis X of the subspace W of $\mathbb{R}^3$ defined by the equation $3x + y - 2z = 0$.
2. Orthogonalize the basis X to get an orthogonal basis B of W.
3. Find the projection on W of the position vector of the point Y.

We now begin the solution.

1. $3x + y - 2z = 0$ is a system of one equation in three variables. Putting the augmented matrix in reduced row-echelon form:

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{1}{3} & -\frac{2}{3} & 0 \end{array}\right]$$

gives general solution $x = \frac{1}{3}s + \frac{2}{3}t$, $y = s$, $z = t$ for any $s, t \in \mathbb{R}$. Then

$$W = \text{span}\left\{\begin{bmatrix}-\frac{1}{3}\\1\\0\end{bmatrix}, \begin{bmatrix}\frac{2}{3}\\0\\1\end{bmatrix}\right\}$$

Let $X = \left\{\begin{bmatrix}-1\\3\\0\end{bmatrix}, \begin{bmatrix}2\\0\\3\end{bmatrix}\right\}$. Then X is linearly independent and $\text{span}(X) = W$, so X is a basis of W.

2. Use the Gram-Schmidt Process to get an orthogonal basis of W:

$$\vec{f}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \text{ and } \vec{f}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} - \frac{-2}{10} \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 9 \\ 3 \\ 15 \end{bmatrix}.$$

Therefore $B = \left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \right\}$ is an orthogonal basis of W.

3. To find the point Z on W closest to $Y = (1,1,1)$, compute

$$\begin{aligned} \text{proj}_W \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= \frac{2}{10} \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + \frac{9}{35} \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 4 \\ 6 \\ 9 \end{bmatrix}. \end{aligned} \tag{4.8.12}$$

Therefore, $Z = \left(\frac{4}{7}, \frac{6}{7}, \frac{9}{7}\right)$.

Least Squares Approximation

It should not be surprising to hear that many problems do not have a perfect solution, and in these cases the objective is always to try to do the best possible. For example what does one do if there are no solutions to a system of linear equations $A\vec{x} = \vec{b}$? It turns out that what we do is find $\vec{x}$ such that $A\vec{x}$ is as close to $\vec{b}$ as possible. A very important technique that follows from orthogonal projections is that of the least square approximation, and allows us to do exactly that.

We begin with a lemma.

Recall that we can form the image of an $m \times n$ matrix A by $\text{im}(A) == \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$. Rephrasing Theorem 4.8.4 using the subspace $W = \text{im}(A)$ gives the equivalence of an orthogonality condition with a minimization condition. The following picture illustrates this orthogonality condition and geometric meaning of this theorem.

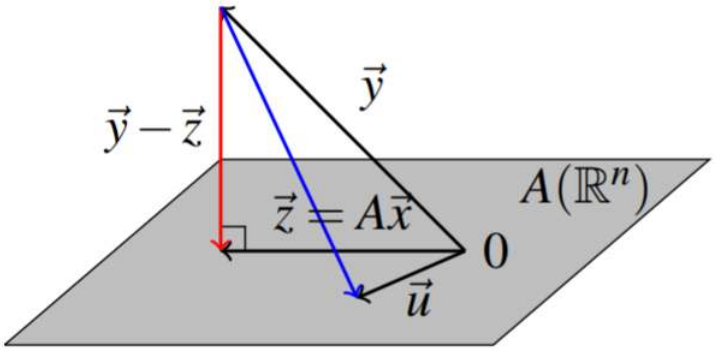


Figure 4.8.2

Theorem 4.8.5: Existence of Minimizers

Let $\vec{y} \in \mathbb{R}^m$ and let A be an $m \times n$ matrix.

Choose $\vec{z} \in W = \text{im}(A)$ given by $\vec{z} = \text{proj}_W(\vec{y})$, and let $\vec{x} \in \mathbb{R}^n$ such that $\vec{z} = A\vec{x}$.

Then

1. $\vec{y} - A\vec{x} \in W^\perp$
2. $\|\vec{y} - A\vec{x}\| < \|\vec{y} - \vec{u}\|$ for all $\vec{u} \neq \vec{z} \in W$

We note a simple but useful observation.

Lemma 4.8.1: Transpose and Dot Product

Let A be an $m \times n$ matrix. Then

$$A\vec{x} \cdot \vec{y} = \vec{x} \cdot A^T\vec{y}$$

Proof

This follows from the definitions:

$$A\vec{x} \cdot \vec{y} = \sum_{i,j} a_{ij}x_j y_i = \sum_{i,j} x_j a_{ji} y_i = \vec{x} \cdot A^T\vec{y}$$

The next corollary gives the technique of least squares.

Corollary 4.8.1: Least Squares and Normal Equation

A specific value of $\vec{x}$ which solves the problem of Theorem 4.8.5 is obtained by solving the equation

$$A^T A\vec{x} = A^T\vec{y}$$

Furthermore, there always exists a solution to this system of equations.

Proof

For $\vec{x}$ the minimizer of Theorem 4.8.5, $(\vec{y} - A\vec{x}) \cdot A\vec{u} = 0$ for all $\vec{u} \in \mathbb{R}^n$ and from Lemma 4.8.1, this is the same as saying

$$A^T(\vec{y} - A\vec{x}) \cdot \vec{u} = 0$$

for all $u \in \mathbb{R}^n$. This implies

$$A^T\vec{y} - A^T A\vec{x} = \vec{0}.$$

Therefore, there is a solution to the equation of this corollary, and it solves the minimization problem of Theorem 4.8.5.

Note that $\vec{x}$ might not be unique but $A\vec{x}$, the closest point of $A(\mathbb{R}^n)$ to $\vec{y}$ is unique as was shown in the above argument.

Consider the following example.

Example 4.8.16: Least Squares Solution to a System

Find a least squares solution to the system

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Solution

First, consider whether there exists a real solution. To do so, set up the augmnented matrix given by

$$\left[\begin{array}{cc|c} 2 & 1 & 2 \\ -1 & 3 & 1 \\ 4 & 5 & 1 \end{array}\right]$$

The reduced row-echelon form of this augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

It follows that there is no real solution to this system. Therefore we wish to find the least squares solution. The normal equations are

$$A^TA\vec{x} = A^T\vec{y}$$

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \tag{4.8.13}$$

and so we need to solve the system

$$\begin{bmatrix} 21 & 19 \\ 19 & 35 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

This is a familiar exercise and the solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{34} \\ \frac{7}{34} \end{bmatrix}$$

Consider another example.

Example 4.8.17: Least Squares Solution to a System

Find a least squares solution to the system

$$\begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$$

Solution

First, consider whether there exists a real solution. To do so, set up the augmnented matrix given by

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ -1 & 3 & 2 \\ 4 & 5 & 9 \end{array}\right]$$

The reduced row-echelon form of this augmented matrix is

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

It follows that the system has a solution given by $x = y = 1$. However we can also use the normal equations and find the least squares solution.

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -1 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$$

Then

$$\begin{bmatrix} 21 & 19 \\ 19 & 35 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40 \\ 54 \end{bmatrix}$$

The least squares solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which is the same as the solution found above.

An important application of Corollary 4.8.1 is the problem of finding the least squares regression line in statistics. Suppose you are given points in the xy plane

$$\{(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\}$$

and you would like to find constants m and b such that the line $\vec{y} = m\vec{x} + b$ goes through all these points. Of course this will be impossible in general. Therefore, we try to find m, b such that the line will be as close as possible. The desired system is

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix}$$

which is of the form $\vec{y} = A\vec{x}$. It is desired to choose m and b to make

$$\left\| A \begin{bmatrix} m \\ b \end{bmatrix} - \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \right\|^2$$

as small as possible. According to Theorem 4.8.5 and Corollary 4.8.1, the best values for m and b occur as the solution to

$$A^T A \begin{bmatrix} m \\ b \end{bmatrix} = A^T \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix}$$

Thus, computing $A^T A$,

$$\begin{bmatrix} \sum_{i=1}^n x_i^2 & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & n \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n x_i y_i \\ \sum_{i=1}^n y_i \end{bmatrix}$$

Solving this system of equations for m and b (using Cramer's rule for example) yields:

$$m = \frac{-\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n y_i\right) + \left(\sum_{i=1}^n x_i y_i\right) n}{\left(\sum_{i=1}^n x_i^2\right) n - \left(\sum_{i=1}^n x_i\right)^2}$$

and

$$b = \frac{-\left(\sum_{i=1}^n x_i\right)\sum_{i=1}^n x_i y_i + \left(\sum_{i=1}^n y_i\right)\sum_{i=1}^n x_i^2}{\left(\sum_{i=1}^n x_i^2\right) n - \left(\sum_{i=1}^n x_i\right)^2}.$$

Consider the following example.

Example 4.8.18: Least Squares Regression

Find the least squares regression line $\vec{y} = m\vec{x} + b$ for the following set of data points:

$$\{(0,1),(1,2),(2,2),(3,4),(4,5)\}$$

Solution

In this case we have $n = 5$ data points and we obtain:

$$\sum_{i=1}^5 x_i = 10 \qquad \sum_{i=1}^5 y_i = 14$$

$$\sum_{i=1}^5 x_i y_i = 38 \qquad \sum_{i=1}^5 x_i^2 = 30$$

and hence

$$\begin{aligned} m &= \frac{-10 * 14 + 5 * 38}{5 * 30 - 10^2} = 1.00 \\ b &= \frac{-10 * 38 + 14 * 30}{5 * 30 - 10^2} = 0.80 \end{aligned} \tag{4.8.14}$$

The least squares regression line for the set of data points is:

$$\vec{y} = \vec{x} + .8$$

One could use this line to approximate other values for the data. For example for $x = 6$ one could use $y(6) = 6 + .8 = 6.8$ as an approximate value for the data.

The following diagram shows the data points and the corresponding regression line.

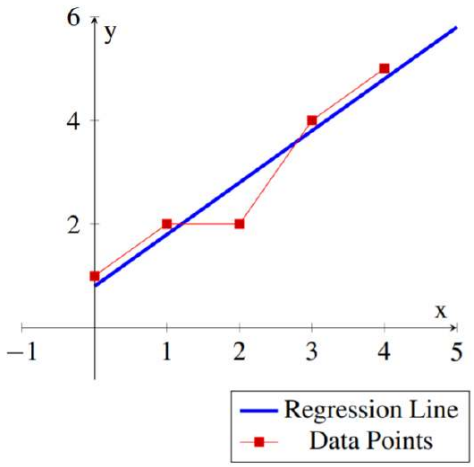


Figure 4.8.3

One could clearly do a least squares fit for curves of the form $y = ax^2 + bx + c$ in the same way. In this case you want to solve as well as possible for $a, b,$ and c the system

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

and one would use the same technique as above. Many other similar problems are important, including many in higher dimensions and they are all solved the same way.

This page titled 4.8: Orthogonality is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **4.11: Orthogonality** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

4.9: Applications

Applications

Outcomes

A. Apply the concepts of vectors in $\mathbb{R}^n$ to the applications of physics and work.

Vectors and Physics

Suppose you push on something. Then, your push is made up of two components, how hard you push and the direction you push. This illustrates the concept of force.

Definition 4.9.1: Force

Force is a vector. The magnitude of this vector is a measure of how hard it is pushing. It is measured in units such as Newtons or pounds or tons. The direction of this vector is the direction in which the push is taking place.

Vectors are used to model force and other physical vectors like velocity. As with all vectors, a vector modeling force has two essential ingredients, its magnitude and its direction.

Recall the special vectors which point along the coordinate axes. These are given by

$$\vec{e}_i = [0 \cdots 0\ 1\ 0 \cdots 0]^T$$

where the 1 is in the ith slot and there are zeros in all the other spaces. The direction of $\vec{e}_i$ is referred to as the ith direction.

Consider the following picture which illustrates the case of $\mathbb{R}^3$. Recall that in $\mathbb{R}^3$, we may refer to these vectors as $\vec{i}, \vec{j}$, and $\vec{k}$.

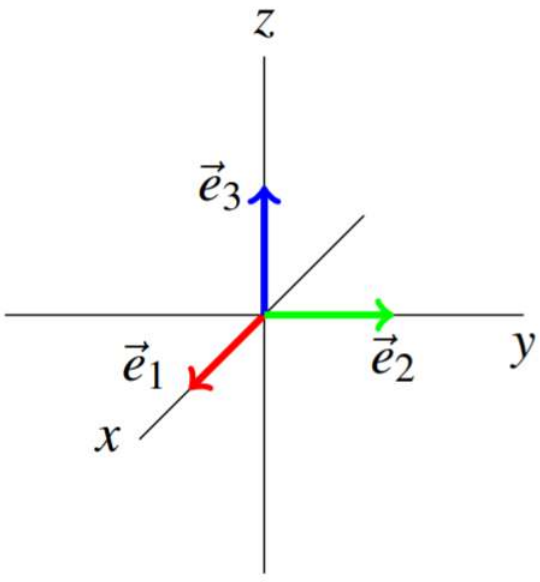


Figure 4.9.1

Given a vector $\vec{u} = [u_1 \cdots u_n]^T$, it follows that

$$\vec{u} = u_1\vec{e}_1 + \cdots + u_n\vec{e}_n = \sum_{k=1}^{n} u_i\vec{e}_i$$

What does addition of vectors mean physically? Suppose two forces are applied to some object. Each of these would be represented by a force vector and the two forces acting together would yield an overall force acting on the object which would also be a force vector known as the resultant. Suppose the two vectors are $\vec{u} = \sum_{k=1}^{n} u_i\vec{e}_i$ and $\vec{v} = \sum_{k=1}^{n} v_i\vec{e}_i$. Then the vector $\vec{u}$ involves a

component in the ith direction given by $u_i\vec{e}_i$, while the component in the ith direction of $\vec{v}$ is $v_i\vec{e}_i$. Then the vector $\vec{u}+\vec{v}$ should have a component in the ith direction equal to $(u_i+v_i)\vec{e}_i$. This is exactly what is obtained when the vectors, $\vec{u}$ and $\vec{v}$ are added.

$$\begin{aligned}\vec{u}+\vec{v} &= [u_1+v_1\cdots u_n+v_n]^T \\ &= \sum_{i=1}^{n}(u_i+v_i)\vec{e}_i\end{aligned} \tag{4.9.1}$$

Thus the addition of vectors according to the rules of addition in $\mathbb{R}^n$ which were presented earlier, yields the appropriate vector which duplicates the cumulative effect of all the vectors in the sum.

Consider now some examples of vector addition.

Example 4.9.1: The Resultant of Three Forces

There are three ropes attached to a car and three people pull on these ropes. The first exerts a force of $\vec{F}_1 = 2\vec{i}+3\vec{j}-2\vec{k}$ Newtons, the second exerts a force of $\vec{F}_2 = 3\vec{i}+5\vec{j}+\vec{k}$ Newtons and the third exerts a force of $5\vec{i}-\vec{j}+2\vec{k}$ Newtons. Find the total force in the direction of $\vec{i}$.

Solution

To find the total force, we add the vectors as described above. This is given by

$$\begin{aligned}&(2\vec{i}+3\vec{j}-2\vec{k})+(3\vec{i}+5\vec{j}+\vec{k})+(5\vec{i}-\vec{j}+2\vec{k}) \\ &=(2+3+5)\vec{i}+(3+5+-1)\vec{j}+(-2+1+2)\vec{k} \\ &=10\vec{i}+7\vec{j}+\vec{k}\end{aligned} \tag{4.9.2}$$

Hence, the total force is $10\vec{i}+7\vec{j}+\vec{k}$ Newtons. Therefore, the force in the $\vec{i}$ direction is 10 Newtons.

Consider another example.

Example 4.9.2: Finding a Vector from Geometric Description

An airplane flies North East at 100 miles per hour. Write this as a vector.

Solution

A picture of this situation follows.

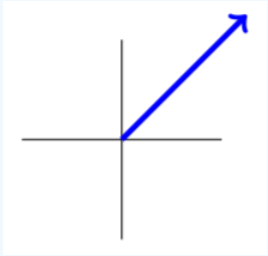


Figure 4.9.2

Therefore, we need to find the vector $\vec{u}$ which has length 100 and direction as shown in this diagram. We can consider the vector $\vec{u}$ as the hypotenuse of a right triangle having equal sides, since the direction of $\vec{u}$ corresponds with the $45°$ line. The sides, corresponding to the $\vec{i}$ and $\vec{j}$ directions, should be each of length $100/\sqrt{2}$. Therefore, the vector is given by

$$\vec{u} = \frac{100}{\sqrt{2}}\vec{i}+\frac{100}{\sqrt{2}}\vec{j} = \begin{bmatrix} 0.05in\frac{100}{\sqrt{2}} & 0.05in\frac{100}{\sqrt{2}} \end{bmatrix}^T$$

This example also motivates the concept of **velocity**, defined below.

Definition 4.9.2: Speed and Velocity

The **speed** of an object is a measure of how fast it is going. It is measured in units of length per unit time. For example, miles per hour, kilometers per minute, feet per second. The **velocity** is a vector having the speed as the magnitude but also specifying the direction.

Thus the velocity vector in the above example is $0.05in\frac{100}{\sqrt{2}}\vec{i} + 0.05in\frac{100}{\sqrt{2}}\vec{j}$, while the speed is 100 miles per hour.

Consider the following example.

The velocity of an airplane is $100\vec{i} + \vec{j} + \vec{k}$ measured in kilometers per hour and at a certain instant of time its position is $(1,2,1)$.

Find the position of this airplane one minute later.

Solution

Here imagine a Cartesian coordinate system in which the third component is altitude and the first and second components are measured on a line from West to East and a line from South to North.

Consider the vector $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$, which is the initial position vector of the airplane. As the plane moves, the position vector changes according to the velocity vector. After one minute (considered as $\frac{1}{60}$ of an hour) the airplane has moved in the $\vec{i}$ direction a distance of $100 \times \frac{1}{60} = \frac{5}{3}$ kilometer. In the $\vec{j}$ direction it has moved $\frac{1}{60}$ kilometer during this same time, while it moves $\frac{1}{60}$ kilometer in the $\vec{k}$ direction. Therefore, the new displacement vector for the airplane is

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T + \begin{bmatrix} \frac{5}{3} & \frac{1}{60} & \frac{1}{60} \end{bmatrix}^T = \begin{bmatrix} \frac{8}{3} & \frac{121}{60} & \frac{121}{60} \end{bmatrix}^T$$

Now consider an example which involves combining two velocities.

Example 4.9.4: Sum of Two Velocities

A certain river is one half kilometer wide with a current flowing at 4 kilometers per hour from East to West. A man swims directly toward the opposite shore from the South bank of the river at a speed of 3 kilometers per hour. How far down the river does he find himself when he has swam across? How far does he end up swimming?

Solution

Consider the following picture which demonstrates the above scenario.

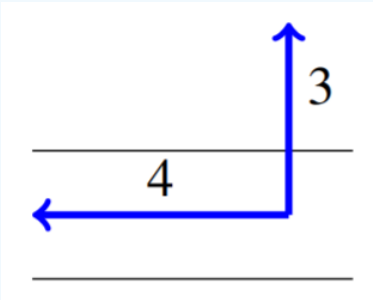


Figure 4.9.3

First we want to know the total time of the swim across the river. The velocity in the direction across the river is 3 kilometers per hour, and the river is $\frac{1}{2}$ kilometer wide. It follows the trip takes $1/6$ hour or 10 minutes.

Now, we can compute how far downstream he will end up. Since the river runs at a rate of 4 kilometers per hour, and the trip takes $1/6$ hour, the distance traveled downstream is given by $4\left(\frac{1}{6}\right) = \frac{2}{3}$ kilometers.

The distance traveled by the swimmer is given by the hypotenuse of a right triangle. The two arms of the triangle are given by the distance across the river, $\frac{1}{2}$km, and the distance traveled downstream, $\frac{2}{3}$ km. Then, using the Pythagorean Theorem, we can calculate the total distance d traveled.

$$d = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{5}{6}\text{km}$$

Therefore, the swimmer travels a total distance of $\frac{5}{6}$ kilometers.

Work

The mathematical concept of work is an application of vectors in $\mathbb{R}^n$. The physical concept of work differs from the notion of work employed in ordinary conversation. For example, suppose you were to slide a 150 pound weight off a table which is three feet high and shuffle along the floor for 50 yards, keeping the height always three feet and then deposit this weight on another three foot high table. The physical concept of work would indicate that the force exerted by your arms did no work during this project. The reason for this definition is that even though your arms exerted considerable force on the weight, the direction of motion was at right angles to the force they exerted. The only part of a force which does work in the sense of physics is the component of the force in the direction of motion.

Work is defined to be the magnitude of the component of this force times the distance over which it acts, when the component of force points in the direction of motion. In the case where the force points in exactly the opposite direction of motion work is given by (-1) times the magnitude of this component times the distance. Thus the work done by a force on an object as the object moves from one point to another is a measure of the extent to which the force contributes to the motion. This is illustrated in the following picture in the case where the given force contributes to the motion.

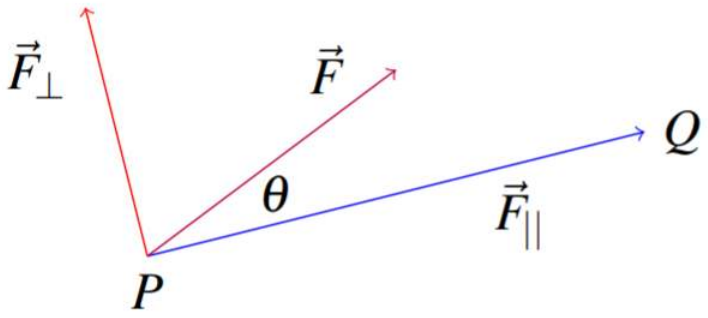


Figure 4.9.4

Recall that for any vector $\vec{u}$ in $\mathbb{R}^n$, we can write $\vec{u}$ as a sum of two vectors, as in

$$\vec{u} = \vec{u}_{||} + \vec{u}_\perp$$

For any force $\vec{F}$, we can write this force as the sum of a vector in the direction of the motion and a vector perpendicular to the motion. In other words,

$$\vec{F} = \vec{F}_{||} + \vec{F}_\perp$$

In the above picture the force, $\vec{F}$ is applied to an object which moves on the straight line from P to Q. There are two vectors shown, $\vec{F}_{||}$ and $\vec{F}_\perp$ and the picture is intended to indicate that when you add these two vectors you get $\vec{F}$. In other words, $\vec{F} = \vec{F}_{||} + \vec{F}_\perp$. Notice that $\vec{F}_{||}$ acts in the direction of motion and $\vec{F}_\perp$ acts perpendicular to the direction of motion. Only $\vec{F}_{||}$ contributes to the work done by $\vec{F}$ on the object as it moves from P to Q. $\vec{F}_{||}$ is called the component of the force in the direction of motion. From trigonometry, you see the magnitude of $\vec{F}_{||}$ should equal $\|\vec{F}\|\,|\cos\theta|$. Thus, since $\vec{F}_{||}$ points in the direction of the vector from P to Q, the total work done should equal

$$\|\vec{F}\|\|\vec{PQ}\|\cos\theta = \|\vec{F}\|\|\vec{q} - \vec{p}\|\cos\theta$$

Now, suppose the included angle had been obtuse. Then the work done by the force $\vec{F}$ on the object would have been negative because $\vec{F}_{||}$ would point in -1 times the direction of the motion. In this case, $\cos\theta$ would also be negative and so it is still the case that the work done would be given by the above formula. Thus from the geometric description of the dot product given above, the work equals

$$\|\vec{F}\|\|\vec{q} - \vec{p}\|\cos\theta = \vec{F} \bullet (\vec{q} - \vec{p})$$

This explains the following definition.

Definition 4.9.3: Work Done on an Object by a Force

Let $\vec{F}$ be a force acting on an object which moves from the point P to the point Q, which have position vectors given by $\vec{p}$ and $\vec{q}$ respectively. Then the **work** done on the object by the given force equals $\vec{F} \bullet (\vec{q} - \vec{p})$.

Consider the following example.

Example 4.9.4: Finding Work

Let $\vec{F} = \begin{bmatrix} 2 & 7 & -3 \end{bmatrix}^T$ Newtons. Find the work done by this force in moving from the point $(1,2,3)$ to the point $(-9,-3,4)$ along the straight line segment joining these points where distances are measured in meters.

Solution

First, compute the vector $\vec{q} - \vec{p}$, given by

$$\begin{bmatrix} -9 & -3 & 4 \end{bmatrix}^T - \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} -10 & -5 & 1 \end{bmatrix}^T$$

According to Definition 4.9.3 the work done is

$$\begin{aligned} \begin{bmatrix} 2 & 7 & 3 \end{bmatrix}^T \bullet \begin{bmatrix} -10 & -5 & 1 \end{bmatrix}^T &= -20 + (-35) + (-3) \\ &= -58 \text{ Newton meters} \end{aligned} \tag{4.9.3}$$

Note that if the force had been given in pounds and the distance had been given in feet, the units on the work would have been foot pounds. In general, work has units equal to units of a force times units of a length. Recall that 1 Newton meter is equal to 1 Joule. Also notice that the work done by the force can be negative as in the above example.

This page titled 4.9: Applications is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **4.12: Applications** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

4.10: Exercises

Exercise 4.10.1

Find $-3\begin{bmatrix} 5 \\ -1 \\ 2 \\ -3 \end{bmatrix} + 5\begin{bmatrix} -8 \\ 2 \\ -3 \\ 6 \end{bmatrix}$.

Answer

$$\begin{bmatrix} -55 \\ 13 \\ -21 \\ 39 \end{bmatrix}$$

Exercise 4.10.2

Find $-7\begin{bmatrix} 6 \\ 0 \\ 4 \\ -1 \end{bmatrix} + 6\begin{bmatrix} -13 \\ -1 \\ 1 \\ 6 \end{bmatrix}$.

Exercise 4.10.3

Decide whether

$$\vec{v} = \begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix}$$

is a linear combination of the vectors

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Answer

$$\begin{bmatrix} 4 \\ 4 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Exercise 4.10.4

Decide whether

$$\vec{v} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

is a linear combination of the vectors

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \vec{u}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Answer

The system

$$\begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = a_1 \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

has no solution.

Exercise 4.10.5

Find the vector equation for the line through $(-7, 6, 0)$ and $(-1, 1, 4)$. Then, find the parametric equations for this line.

Exercise 4.10.6

Find parametric equations for the line through the point $(7, 7, 1)$ with a direction vector $\vec{d} = \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix}$.

Exercise 4.10.7

Parametric equations of the line are

$$\begin{aligned} x &= t + 2 \\ y &= 6 - 3t \\ x &= -t = 6 \end{aligned} \tag{4.10.1}$$

Find a direction vector for the line and a point on the line.

Exercise 4.10.8

Find the vector equation for the line through the two points $(-5, 5, 1)$, $(2, 2, 4)$. Then, find the parametric equations.

Exercise 4.10.9

The equation of a line in two dimensions is written as $y = x - 5$. Find parametric equations for this line.

Exercise 4.10.10

Find parametric equations for the line through $(6, 5, -2)$ and $(5, 1, 2)$.

Exercise 4.10.11

Find the vector equation and parametric equations for the line through the point $(-7, 10, -6)$ with a direction vector $\vec{d} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

Exercise 4.10.12

Parametric equations of the line are

$$\begin{aligned} x &= 2t + 2 \\ y &= 5 - 4t \\ z &= -t - 3 \end{aligned} \tag{4.10.2}$$

Find a direction vector for the line and a point on the line, and write the vector equation of the line.

Exercise 4.10.13

Find the vector equation and parametric equations for the line through the two points $(4,10,0),\ (1,-5,-6)$.

Exercise 4.10.14

Find the point on the line segment from $P=(-4,7,5)$ to $Q=(2,-2,-3)$ which is $\frac{1}{7}$ of the way from P to Q.

Exercise 4.10.15

Suppose a triangle in $\mathbb{R}^n$ has vertices at P_1, P_2, and P_3. Consider the lines which are drawn from a vertex to the mid point of the opposite side. Show these three lines intersect in a point and find the coordinates of this point.

Exercise 4.10.16

Here are some vectors.

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ -10 \end{bmatrix}, \begin{bmatrix} 12 \\ 17 \\ -24 \end{bmatrix}$$

Describe the span of these vectors as the span of as few vectors as possible.

Exercise 4.10.17

Here are some vectors.

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 12 \\ 29 \\ -24 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 9 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 12 \\ -10 \end{bmatrix}.$$

Describe the span of these vectors as the span of as few vectors as possible.

Exercise 4.10.18

Here are some vectors.

$$\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Describe the span of these vectors as the span of as few vectors as possible.

Exercise 4.10.19

Here are some vectors.

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Exercise 4.10.20

Here are some vectors.

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Exercise 4.10.21

Here are some vectors.

$$\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} 1 \\ 9 \\ 1 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Exercise 4.10.22

Here are some vectors,

$$\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Exercise 4.10.23

Here are some vectors.

$$\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Exercise 4.10.24

Here are some vectors.

$$\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

Now here is another vector:

$$\begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}$$

Is this vector in the span of the first four vectors? If it is, exhibit a linear combination of the first four vectors which equals this vector, using as few vectors as possible in the linear combination.

Exercise 4.10.25

Suppose $\{\vec{x}_1, \cdots, \vec{x}_k\}$ is a set of vectors from $\mathbb{R}^n$. Show that $\vec{0}$ is in $span\{\vec{x}_1, \cdots, \vec{x}_k\}$.

Answer

$$\sum_{i=1}^{k} 0\vec{x}_k = \vec{0}$$

Exercise 4.10.26

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ 2 \\ 1 \end{bmatrix}$$

Exercise 4.10.27

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} -1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 6 \\ 4 \end{bmatrix}$$

? Exercise 4.10.28

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ -2 \\ 1 \end{bmatrix}$$

? Exercise 4.10.29

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 34 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 8 \\ 1 \end{bmatrix}$$

? Exercise 4.10.30

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others.

$$\begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -10 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

? Exercise 4.10.31

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 10 \\ -14 \\ 1 \end{bmatrix}$$

? Exercise 4.10.32

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 34 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 7 \\ 1 \end{bmatrix}$$

? Exercise 4.10.33

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 1 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}$$

? Exercise 4.10.34

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others.

$$\begin{bmatrix} 1 \\ 2 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \\ 5 \end{bmatrix}$$

? Exercise 4.10.35

Are the following vectors linearly independent? If they are, explain why and if they are not, exhibit one of them as a linear combination of the others. Also give a linearly independent set of vectors which has the same span as the given vectors.

$$\begin{bmatrix} 2 \\ 3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ -6 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 4 \end{bmatrix}$$

? Exercise 4.10.36

Here are some vectors in $\mathbb{R}^4$.

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

These vectors can't possibly be linearly independent. Tell why. Next obtain a linearly independent subset of these vectors which has the same span as these vectors. In other words, find a basis for the span of these vectors.

? Exercise 4.10.37

Here are some vectors in $\mathbb{R}^4$.

$$\begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

These vectors can't possibly be linearly independent. Tell why. Next obtain a linearly independent subset of these vectors which has the same span as these vectors. In other words, find a basis for the span of these vectors.

Exercise 4.10.38

Here are some vectors in $\mathbb{R}^4$.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -7 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

These vectors can't possibly be linearly independent. Tell why. Next obtain a linearly independent subset of these vectors which has the same span as these vectors. In other words, find a basis for the span of these vectors.

Exercise 4.10.39

Here are some vectors in $\mathbb{R}^4$.

$$\begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

These vectors can't possibly be linearly independent. Tell why. Next obtain a linearly independent subset of these vectors which has the same span as these vectors. In other words, find a basis for the span of these vectors.

Exercise 4.10.40

Here are some vectors in $\mathbb{R}^4$.

$$\begin{bmatrix} 1 \\ 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 11 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ -3 \\ 1 \end{bmatrix}$$

These vectors can't possibly be linearly independent. Tell why. Next obtain a linearly independent subset of these vectors which has the same span as these vectors. In other words, find a basis for the span of these vectors.

Exercise 4.10.41

Here are some vectors in $\mathbb{R}^4$.

$$\begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{3}{2} \\ -\frac{9}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

These vectors can't possibly be linearly independent. Tell why. Next obtain a linearly independent subset of these vectors which has the same span as these vectors. In other words, find a basis for the span of these vectors.

Exercise 4.10.42

Here are some vectors in $\mathbb{R}^4$.

$$\begin{bmatrix} 1 \\ 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

These vectors can't possibly be linearly independent. Tell why. Next obtain a linearly independent subset of these vectors which has the same span as these vectors. In other words, find a basis for the span of these vectors

Exercise 4.10.44

Here are some vectors in $\mathbb{R}^4$.

$$\left[\begin{array}{r}1\\4\\-2\\1\end{array}\right], \left[\begin{array}{r}1\\5\\-3\\1\end{array}\right], \left[\begin{array}{r}1\\1\\1\\1\end{array}\right], \left[\begin{array}{r}2\\1\\3\\2\end{array}\right], \left[\begin{array}{r}1\\5\\-2\\1\end{array}\right]$$

These vectors can't possibly be linearly independent. Tell why. Next obtain a linearly independent subset of these vectors which has the same span as these vectors. In other words, find a basis for the span of these vectors.

Exercise 4.10.45

Here are some vectors in $\mathbb{R}^4$.

$$\left[\begin{array}{r}1\\-1\\3\\1\end{array}\right], \left[\begin{array}{r}1\\0\\7\\1\end{array}\right], \left[\begin{array}{r}1\\0\\8\\1\end{array}\right], \left[\begin{array}{r}4\\-9\\-6\\4\end{array}\right], \left[\begin{array}{r}1\\0\\8\\1\end{array}\right]$$

These vectors can't possibly be linearly independent. Tell why. Next obtain a linearly independent subset of these vectors which has the same span as these vectors. In other words, find a basis for the span of these vectors.

Exercise 4.10.46

Here are some vectors in $\mathbb{R}^4$.

$$\left[\begin{array}{r}1\\-1\\-1\\1\end{array}\right], \left[\begin{array}{r}-3\\3\\3\\-3\end{array}\right], \left[\begin{array}{r}1\\0\\-1\\1\end{array}\right], \left[\begin{array}{r}2\\-9\\-2\\2\end{array}\right], \left[\begin{array}{r}1\\0\\0\\1\end{array}\right]$$

These vectors can't possibly be linearly independent. Tell why. Next obtain a linearly independent subset of these vectors which has the same span as these vectors. In other words, find a basis for the span of these vectors.

Exercise 4.10.47

Here are some vectors in $\mathbb{R}^4$.

$$\left[\begin{array}{r}1\\b+1\\a\\1\end{array}\right], \left[\begin{array}{r}3\\3b+3\\3a\\3\end{array}\right], \left[\begin{array}{r}1\\b+2\\2a+1\\1\end{array}\right], \left[\begin{array}{r}2\\2b-5\\-5a-7\\2\end{array}\right], \left[\begin{array}{r}1\\b+2\\2a+2\\1\end{array}\right]$$

These vectors can't possibly be linearly independent. Tell why. Next obtain a linearly independent subset of these vectors which has the same span as these vectors. In other words, find a basis for the span of these vectors.

Exercise 4.10.48

Let $H = span\left\{\left[\begin{array}{r}2\\1\\1\\1\end{array}\right], \left[\begin{array}{r}-1\\0\\-1\\-1\end{array}\right], \left[\begin{array}{r}5\\2\\3\\3\end{array}\right], \left[\begin{array}{r}-1\\1\\-2\\-2\end{array}\right]\right\}$. Find the dimension of H and determine a basis.

Exercise 4.10.49

Let H denote $span\left\{\begin{bmatrix}0\\1\\1\\-1\end{bmatrix},\begin{bmatrix}-1\\-1\\-2\\2\end{bmatrix},\begin{bmatrix}2\\3\\5\\-5\end{bmatrix},\begin{bmatrix}0\\1\\2\\-2\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.10.50

Let H denote $span\left\{\begin{bmatrix}-2\\1\\1\\-3\end{bmatrix},\begin{bmatrix}-9\\4\\3\\-9\end{bmatrix},\begin{bmatrix}-33\\15\\12\\-36\end{bmatrix},\begin{bmatrix}-22\\10\\8\\-24\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.10.51

Let H denote $span\left\{\begin{bmatrix}-1\\1\\-1\\-2\end{bmatrix},\begin{bmatrix}-4\\3\\-2\\-4\end{bmatrix},\begin{bmatrix}-3\\2\\-1\\-2\end{bmatrix},\begin{bmatrix}-1\\1\\-2\\-4\end{bmatrix},\begin{bmatrix}-7\\5\\-3\\-6\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.10.52

Let H denote $span\left\{\begin{bmatrix}2\\3\\2\\1\end{bmatrix},\begin{bmatrix}8\\15\\6\\3\end{bmatrix},\begin{bmatrix}3\\6\\2\\1\end{bmatrix},\begin{bmatrix}4\\6\\6\\3\end{bmatrix},\begin{bmatrix}8\\15\\6\\3\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.10.53

Let H denote $span\left\{\begin{bmatrix}0\\2\\0\\-1\end{bmatrix},\begin{bmatrix}-1\\6\\0\\-2\end{bmatrix},\begin{bmatrix}-2\\16\\0\\-6\end{bmatrix},\begin{bmatrix}-3\\22\\0\\-8\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.10.54

Let H denote $span\left\{\begin{bmatrix}5\\1\\1\\4\end{bmatrix},\begin{bmatrix}14\\3\\2\\8\end{bmatrix},\begin{bmatrix}38\\8\\6\\24\end{bmatrix},\begin{bmatrix}47\\10\\7\\28\end{bmatrix},\begin{bmatrix}10\\2\\3\\12\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.10.55

Let H denote $span\left\{\begin{bmatrix}6\\1\\1\\5\end{bmatrix},\begin{bmatrix}17\\3\\2\\10\end{bmatrix},\begin{bmatrix}52\\9\\7\\35\end{bmatrix},\begin{bmatrix}18\\3\\4\\20\end{bmatrix}\right\}$. Find the dimension of H and determine a basis.

Exercise 4.10.56

Let $M=\left\{\vec{u}=\begin{bmatrix}u_1\\u_2\\u_3\\u_4\end{bmatrix}\in\mathbb{R}^4:\sin(u_1)=1\right\}$. Is M a subspace? Explain.

Answer

No. Let $\vec{u} = \begin{bmatrix} \frac{\pi}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$. Then $2\vec{u} \notin M$ although $\vec{u} \in M$.

? Exercise 4.10.57

Let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : ||u_1|| \leq 4 \right\}$. Is M a subspace? Explain.

Answer

No. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in M$ but $10 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \notin M$.

? Exercise 4.10.58

Let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_1 \geq 0 \text{ for each } i = 1, 2, 3, 4 \right\}$. Is M a subspace? Explain.

Answer

This is not a subspace. $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is in it. However, $(-1) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is not.

? Exercise 4.10.59

Let $\vec{w}, \vec{w}_1$ be given vectors in $\mathbb{R}^4$ and define

$$M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : \vec{w} \bullet \vec{u} = 0 \text{ and } \vec{w}_1 \bullet \vec{u} = 0 \right\}.$$

Is M a subspace? Explain.

Answer

This is a subspace because it is closed with respect to vector addition and scalar multiplication.

? Exercise 4.10.60

Let $\vec{w} \in \mathbb{R}^4$ and let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : \vec{w} \bullet \vec{u} = 0 \right\}$. Is M a subspace? Explain.

Answer

Yes, this is a subspace because it is closed with respect to vector addition and scalar multiplication.

Exercise 4.10.61

Let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_3 \geq u_1 \right\}$. Is M a subspace? Explain.

Answer

This is not a subspace. $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ is in it. However $(-1)\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$ is not.

Exercise 4.10.62

Let $M = \left\{ \vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \in \mathbb{R}^4 : u_3 = u_1 = 0 \right\}$. Is M a subspace? Explain.

Answer

This is a subspace. It is closed with respect to vector addition and scalar multiplication.

Exercise 4.10.63

Consider the set of vectors S given by

$$S = \left\{ \begin{bmatrix} 4u + v - 5w \\ 12u + 6v - 6w \\ 4u + 4v + 4w \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is S a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.10.64

Consider the set of vectors S given by

$$S = \left\{ \begin{bmatrix} 2u + 6v + 7w \\ -3u - 9v - 12w \\ 2u + 6v + 6w \\ u + 3v + 3w \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is S a subspace of $\mathbb{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.10.65

Consider the set of vectors S given by

$$S = \left\{ \begin{bmatrix} 2u + v \\ 6v - 3u + 3w \\ 3v - 6u + 3w \end{bmatrix} : u, v, w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.10.66

Consider the vectors of the form

$$\left\{ \begin{bmatrix} 2u+v+7w \\ u-2v+w \\ -6v-6w \end{bmatrix} : u,v,w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.10.67

Consider the vectors of the form

$$\left\{ \begin{bmatrix} 3u+v+11w \\ 18u+6v+66w \\ 28u+8v+100w \end{bmatrix} : u,v,w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.10.68

Consider the vectors of the form

$$\left\{ \begin{bmatrix} 3u+v \\ 2w-4u \\ 2w-2v-8u \end{bmatrix} : u,v,w \in \mathbb{R} \right\}.$$

Is this set of vectors a subspace of $\mathbb{R}^3$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.10.69

Consider the set of vectors $\mathcal{S}$ given by

$$\left\{ \begin{bmatrix} u+v+w \\ 2u+2v+4w \\ u+v+w \\ 0 \end{bmatrix} : u,v,w \in \mathbb{R} \right\}.$$

Is $\mathcal{S}$ is a subspace of $\mathbb{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.10.70

Consider the set of vectors $\mathcal{S}$ given by

$$\left\{ \begin{bmatrix} v \\ -3u-3w \\ 8u-4v+4w \end{bmatrix} : u,v,w \in \mathbb{R} \right\}.$$

Is $\mathcal{S}$ is a subspace of $\mathbb{R}^4$? If so, explain why, give a basis for the subspace and find its dimension.

Exercise 4.10.71

If you have 5 vectors in $\mathbb{R}^5$ and the vectors are linearly independent, can it always be concluded they span $\mathbb{R}^5$? Explain.

Answer

Yes. If not, there would exist a vector not in the span. But then you could add in this vector and obtain a linearly independent set of vectors with more vectors than a basis.

Exercise 4.10.72

If you have 6 vectors in $\mathbb{R}^5$, is it possible they are linearly independent? Explain.

Answer

They can't be.

Exercise 4.10.73

Suppose A is an $m \times n$ matrix and $\{\vec{w}_1, \cdots, \vec{w}_k\}$ is a linearly independent set of vectors in $A(\mathbb{R}^n) \subseteq \mathbb{R}^m$. Now suppose $A\vec{z}_i = \vec{w}_i$. Show $\{\vec{z}_1, \cdots, \vec{z}_k\}$ is also independent.

Answer

Say $\sum_{i=1}^{k} c_i \vec{z}_i = \vec{0}$. Then apply A to it as follows.

$$\sum_{i=1}^{k} c_a A\vec{z}_i = \sum_{i=1}^{k} c_i \vec{w}_i = \vec{0}$$

and so, by linear independence of the $\vec{w}_i$, it follows that each $c_i = 0$.

Exercise 4.10.74

Suppose V, W are subspaces of $\mathbb{R}^n$. Let $V \cap W$ be all vectors which are in both V and W. Show that $V \cap W$ is a subspace also.

Answer

If $\vec{x}, \vec{y} \in V \cap W$, then for scalars α, β, the linear combination $\alpha\vec{x} + \beta\vec{y}$ must be in both V and W since they are both subspaces.

Exercise 4.10.75

Suppose V and W both have dimension equal to 7 and they are subspaces of $\mathbb{R}^{10}$. What are the possibilities for the dimension of $V \cap W$? Hint: Remember that a linear independent set can be extended to form a basis.

Exercise 4.10.76

Suppose V has dimension p and W has dimension q and they are each contained in a subspace, U which has dimension equal to n where $n > \max(p, q)$. What are the possibilities for the dimension of $V \cap W$? Hint: Remember that a linearly independent set can be extended to form a basis.

Answer

Let $\{x_1, \cdots, x_k\}$ be a basis for $V \cap W$. Then there is a basis for V and W which are respectively

$$\{x_1, \cdots, x_k, y_{k+1}, \cdots, y_p\},\ \{x_1, \cdots, x_k, z_{k+1}, \cdots z_q\}$$

It follows that you must have $k + p - k + q - k \leq n$ and so you must have

$$p + q - n \leq k$$

Exercise 4.10.77

Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Show that

$$\dim(\ker(AB)) \leq \dim(\ker(A)) + \dim(\ker(B)).$$

Consider the subspace, $B(\mathbb{R}^p) \cap \ker(A)$ and suppose a basis for this subspace is $\{\vec{w}_1, \cdots, \vec{w}_k\}$. Now suppose $\{\vec{u}_1, \cdots, \vec{u}_r\}$ is a basis for $\ker(B)$. Let $\{\vec{z}_1, \cdots, \vec{z}_k\}$ be such that $B\vec{z}_1 = \vec{w}_i$ and argue that

$$\ker(AB) \subseteq span\{\vec{u}_1, \cdots, \vec{u}_r, \vec{z}_1, \cdots, \vec{z}_k\}.$$

Answer

Here is how you do this. Suppose $AB\vec{x} = \vec{0}$. Then $B\vec{x} \in \ker(A) \cap B(\mathbb{R}^p)$ and so $B\vec{x} = \sum_{i=1}^{k} B\vec{z}_i$ showing that

$$\vec{x} - \sum_{i=1}^{k} \vec{z}_i \in \ker(B)$$

Consider $B(\mathbb{R}^p) \cap \ker(A)$ and let a basis be $\{\vec{w}_1, \cdots, \vec{w}_k\}$. Then each $\vec{w}_i$ is of the form $B\vec{z}_i = \vec{w}_i$. Therefore, $\{\vec{z}_1, \cdots, \vec{z}_k\}$ is linearly independent and $AB\vec{z}_i = 0$. Now let $\{\vec{u}_1, \cdots, \vec{u}_r\}$ be a basis for $\ker(B)$. If $AB\vec{x} = \vec{0}$, then $B\vec{x} \in \ker(A) \cap B(\mathbb{R}^p)$ and so $B\vec{x} = \sum_{i=1}^{k} c_i B\vec{z}_1$ which implies

$$\vec{x} - \sum_{i=1}^{k} c_i \vec{z}_i \in \ker(B)$$

and so it is of the form

$$\vec{x} - \sum_{i=1}^{k} c_i \vec{z}_i = \sum_{j=1}^{r} d_j \vec{u}_j$$

It follows that if $AB\vec{x} = \vec{0}$ so that $\vec{x} \in \ker(AB)$, then

$$\vec{x} \in span(\vec{z}_1, \cdots, \vec{z}_k, \vec{u}_1, \cdots, \vec{u}_r).$$

Therefore,

$$\begin{aligned} \dim(\ker(AB)) &\leq k + r = \dim(B(\mathbb{R}^p) \cap \ker(A)) + \dim(\ker(B)) \\ &\leq \dim(\ker(A)) + \dim(\ker(B)) \end{aligned} \tag{4.10.3}$$

Exercise 4.10.78

Show that if A is an $m \times n$ matrix, then $\ker(A)$ is a subspace of $\mathbb{R}^n$.

Answer

If $\vec{x}, \vec{y} \in \ker(A)$ then

$$A(a\vec{x} + b\vec{y}) = aA\vec{x} + bA\vec{y} = a\vec{0} + b\vec{0} = \vec{0}$$

and so $\ker(A)$ is closed under linear combinations. Hence it is a subspace.

Exercise 4.10.79

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 3 & 0 & -2 & 0 & 3 \\ 3 & 9 & 1 & -7 & 0 & 8 \\ 1 & 3 & 1 & -3 & 1 & -1 \\ 1 & 3 & -1 & -1 & -2 & 10 \end{bmatrix}$$

Exercise 4.10.80

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 3 & 0 & -2 & 7 & 3 \\ 3 & 9 & 1 & -7 & 23 & 8 \\ 1 & 3 & 1 & -3 & 9 & 2 \\ 1 & 3 & -1 & -1 & 5 & 4 \end{bmatrix}$$

Exercise 4.10.81

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 7 & 0 \\ 3 & 1 & 10 & 0 & 23 & 0 \\ 1 & 1 & 4 & 1 & 7 & 0 \\ 1 & -1 & 2 & -2 & 9 & 1 \end{bmatrix}$$

Exercise 4.10.82

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{bmatrix}$$

Exercise 4.10.83

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 1 & 2 & 3 & -2 & -18 \\ 1 & 2 & 2 & -1 & -11 \\ -1 & -2 & -2 & 1 & 11 \end{bmatrix}$$

Exercise 4.10.84

Find the rank of the following matrix. Also find a basis for the row and column spaces.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 3 & 1 & 10 & 0 \\ -1 & 1 & -2 & 1 \\ 1 & -1 & 2 & -2 \end{bmatrix}$$

Exercise 4.10.85

Find $\ker(A)$ for the following matrices.

a. $A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 4 & 0 \\ 3 & 6 & -2 \\ 1 & 2 & -2 \end{bmatrix}$

d. $A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ 2 & 0 & 1 & 2 \\ 6 & 4 & -5 & -6 \\ 0 & 2 & -4 & -6 \end{bmatrix}$

Exercise 4.10.86

Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} \frac{1}{6}\sqrt{2}\sqrt{3} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \\ -\frac{1}{6}\sqrt{2}\sqrt{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ 0 \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

Exercise 4.10.87

Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

Exercise 4.10.88

Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

Exercise 4.10.89

Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

Exercise 4.10.90

Determine whether the following set of vectors is orthogonal. If it is orthogonal, determine whether it is also orthonormal.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

If the set of vectors is orthogonal but not orthonormal, give an orthonormal set of vectors which has the same span.

Exercise 4.10.91

Here are some matrices. Label according to whether they are symmetric, skew symmetric, or orthogonal.

a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

b. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & 4 \\ -3 & 4 & 7 \end{bmatrix}$

c. $\begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$

Answer

a. Orthogonal
b. Symmetric
c. Skew Symmetric

Exercise 4.10.92

For U an orthogonal matrix, explain why $||U\vec{x}|| = ||\vec{x}||$ for any vector $\vec{x}$. Next explain why if U is an $n \times n$ matrix with the property that $||U\vec{x}|| = ||\vec{x}||$ for all vectors, $\vec{x}$, then U must be orthogonal. Thus the orthogonal matrices are exactly those which preserve length.

Answer

$||U\vec{x}||^2 = U\vec{x} \bullet U\vec{x} = U^T U\vec{x} \bullet \vec{x} = I\vec{x} \bullet \vec{x} = ||\vec{x}||^2$. Next suppose distance is preserved by U. Then

$$\begin{aligned} (U(\vec{x}+\vec{y})) \bullet (U(\vec{x}+\vec{y})) &= ||Ux||^2 + ||Uy||^2 + 2(Ux \bullet Uy) \\ &= ||\vec{x}||^2 + ||\vec{y}||^2 + 2(U^T U\vec{x} \bullet \vec{y}) \end{aligned} \tag{4.10.4}$$

But since U preserves distances, it is also the case that

$$(U(\vec{x}+\vec{y}) \bullet U(\vec{x}+\vec{y})) = ||\vec{x}||^2 + ||\vec{y}||^2 + 2(\vec{x} \bullet \vec{y})$$

Hence

$$\vec{x} \bullet \vec{y} = U^T U\vec{x} \bullet \vec{y}$$

and so

$$((U^T U - I)\vec{x}) \bullet \vec{y} = 0$$

Since y is arbitrary, it follows that $U^T U - I = 0$. Thus U is orthogonal.

Exercise 4.10.93

Suppose U is an orthogonal $n \times n$ matrix. Explain why $rank(U) = n$.

Answer

You could observe that $\det(UU^T) = (\det(U))^2 - 1$ so $\det(U) \neq 0$.

Exercise 4.10.94

Fill in the missing entries to make the matrix orthogonal.

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & - & - \\ - & \frac{\sqrt{6}}{3} & - \end{bmatrix}.$$

Answer

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & a \\ 0 & \frac{\sqrt{6}}{3} & b \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & a \\ 0 & \frac{\sqrt{6}}{3} & b \end{bmatrix}^T = \begin{bmatrix} 1 & \frac{1}{3}\sqrt{3}a - \frac{1}{3} & \frac{1}{3}\sqrt{3}b - \frac{1}{3} \\ \frac{1}{3}\sqrt{3}a - \frac{1}{3} & a^2 + \frac{2}{3} & ab - \frac{1}{3} \\ \frac{1}{3}\sqrt{3}b - \frac{1}{3} & ab - \frac{1}{3} & b^2 + \frac{2}{3} \end{bmatrix} \tag{4.10.5}$$

This requires, $a = 1/\sqrt{3}, b = 1/\sqrt{3}$.

$$\begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 1/\sqrt{3} \\ 0 & \frac{\sqrt{6}}{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & 1/\sqrt{3} \\ 0 & \frac{\sqrt{6}}{3} & 1/\sqrt{3} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 4.10.95

Fill in the missing entries to make the matrix orthogonal.

$$\begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & - & - \\ - & 0 & - \end{bmatrix}$$

Answer

$$\begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & a \\ -\frac{1}{3} & 0 & b \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & a \\ -\frac{1}{3} & 0 & b \end{bmatrix}^T = \begin{bmatrix} 1 & \frac{1}{6}\sqrt{2}a - \frac{1}{18} & \frac{1}{6}\sqrt{2}b - \frac{2}{9} \\ \frac{1}{6}\sqrt{2}a - \frac{1}{18} & a^2 + \frac{17}{18} & ab - \frac{2}{9} \\ \frac{1}{6}\sqrt{2}b - \frac{2}{9} & ab - \frac{2}{9} & b^2 + \frac{1}{9} \end{bmatrix}$$

This requires $a = \frac{1}{3\sqrt{2}}$, $b = \frac{4}{3\sqrt{2}}$.

$$\begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{\sqrt{2}}{2} & \frac{1}{6}\sqrt{2} \\ \frac{2}{3} & \frac{-\sqrt{2}}{2} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 4.10.96

Fill in the missing entries to make the matrix orthogonal.

$$\begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & - \\ \frac{2}{3} & 0 & - \\ - & - & \frac{4}{15}\sqrt{5} \end{bmatrix}$$

https://math.libretexts.org/@go/page/138173

Answer

Try

$$\begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & c \\ \frac{2}{3} & 0 & d \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & c \\ \frac{2}{3} & 0 & d \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{bmatrix}^T = \begin{bmatrix} c^2 + \frac{41}{45} & cd + \frac{2}{9} & \frac{4}{15}\sqrt{5}c - \frac{8}{45} \\ cd + \frac{2}{9} & d^2 + \frac{4}{9} & \frac{4}{15}\sqrt{5}d + \frac{4}{9} \\ \frac{4}{15}\sqrt{5}c - \frac{8}{45} & \frac{4}{15}\sqrt{5}d + \frac{4}{9} & 1 \end{bmatrix} \tag{4.10.6}$$

This requires that $c = \frac{2}{3\sqrt{5}}, d = \frac{-5}{3\sqrt{5}}$.

$$\begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{-5}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{-5}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{15}\sqrt{5} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercise 4.10.97

Find an orthonormal basis for the span of each of the following sets of vectors.

a. $\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 11 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}$

c. $\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 10 \end{bmatrix}, \begin{bmatrix} -7 \\ 1 \\ 1 \end{bmatrix}$

Answer

a. $\begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} \frac{3}{5} \\ 0 \\ -\frac{4}{5} \end{bmatrix}, \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

c. $\begin{bmatrix} \frac{3}{5} \\ 0 \\ -\frac{4}{5} \end{bmatrix}, \begin{bmatrix} \frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

Exercise 4.10.98

Using the Gram Schmidt process find an orthonormal basis for the following span:

$$span\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

Answer

A solution is

$$\begin{bmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}, \begin{bmatrix} \frac{3}{10}\sqrt{2} \\ -\frac{2}{5}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} \frac{7}{15}\sqrt{3} \\ -\frac{1}{15}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \end{bmatrix}$$

Exercise 4.10.99

Using the Gram Schmidt process find an orthonormal basis for the following span:

$$span\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

Answer

Then a solution is

$$\begin{bmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{6}\sqrt{2}\sqrt{3} \\ -\frac{2}{9}\sqrt{2}\sqrt{3} \\ \frac{5}{18}\sqrt{2}\sqrt{3} \\ \frac{1}{9}\sqrt{2}\sqrt{3} \end{bmatrix}, \begin{bmatrix} \frac{5}{111}\sqrt{3}\sqrt{37} \\ \frac{1}{133}\sqrt{3}\sqrt{37} \\ -\frac{17}{333}\sqrt{3}\sqrt{37} \\ \frac{22}{333}\sqrt{3}\sqrt{37} \end{bmatrix}$$

Exercise 4.10.100

The set $V = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : 2x + 3y - z = 0\right\}$ is a subspace of $\mathbb{R}^3$. Find an orthonormal basis for this subspace.

Answer

The subspace is of the form

$$\begin{bmatrix} x \\ y \\ 2x + 3y \end{bmatrix}$$

and a basis is $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$. Therefore, an orthonormal basis is

$$\begin{bmatrix} \frac{1}{5}\sqrt{5} \\ 0 \\ \frac{2}{5}\sqrt{5} \end{bmatrix}, \begin{bmatrix} -\frac{3}{35}\sqrt{5}\sqrt{14} \\ \frac{1}{14}\sqrt{5}\sqrt{14} \\ \frac{3}{70}\sqrt{5}\sqrt{14} \end{bmatrix}$$

Exercise 4.10.101

Consider the following scalar equation of a plane.

$$2x - 3y + z = 0$$

Find the orthogonal complement of the vector $\vec{v} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$. Also find the point on the plane which is closest to $(3, 4, 1)$.

Exercise 4.10.102

Consider the following scalar equation of a plane.

$$x+3y+z=0$$

Find the orthogonal complement of the vector $\vec{v}=\begin{bmatrix}1\\2\\1\end{bmatrix}$. Also find the point on the plane which is closest to $(3,4,1)$.

Exercise 4.10.103

Let $\vec{v}$ be a vector and let $\vec{n}$ be a normal vector for a plane through the origin. Find the equation of the line through the point determined by $\vec{v}$ which has direction vector $\vec{n}$. Show that it intersects the plane at the point determined by $\vec{v}-proj_{\vec{n}}\vec{v}$. Hint: The line:$\vec{v}+t\vec{n}$. It is in the plane if $\vec{n}\bullet(\vec{v}+t\vec{n})=0$. Determine t. Then substitute in to the equation of the line.

Exercise 4.10.104

As shown in the above problem, one can find the closest point to~v in a plane through the origin by finding the intersection of the line through $\vec{v}$ having direction vector equal to the normal vector to the plane with the plane. If the plane does not pass through the origin, this will still work to find the point on the plane closest to the point determined by $\vec{v}$. Here is a relation which defines a plane

$$2x+y+z=11$$

and here is a point: $(1,1,2)$. Find the point on the plane which is closest to this point. Then determine the distance from the point to the plane by taking the distance between these two points. Hint: Line: $(x,y,z)=(1,1,2)+t(2,1,1)$. Now require that it intersect the plane.

Exercise 4.10.105

In general, you have a point (x_0,y_0,z_0) and a scalar equation for a plane $ax+by+cz=d$ where $a^2+b^2+c^2>0$. Determine a formula for the closest point on the plane to the given point. Then use this point to get a formula for the distance from the given point to the plane. Hint: Find the line perpendicular to the plane which goes through the given point: $(x,y,z)=(x_0,y_0,z_0)+t(a,b,c)$. Now require that this point satisfy the equation for the plane to determine t.

Exercise 4.10.106

Find the least squares solution to the following system.

$$\begin{aligned}x+2y&=1\\2x+3y&=2\\3x+5y&=4\end{aligned}\tag{4.10.7}$$

Answer

$$\begin{aligned}\begin{bmatrix}1&2\\2&3\\3&5\end{bmatrix}^T\begin{bmatrix}1&2\\2&3\\3&5\end{bmatrix}&=\begin{bmatrix}14&23\\23&38\end{bmatrix}\begin{bmatrix}14&23\\23&38\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}\\&=\begin{bmatrix}1&2\\2&3\\3&5\end{bmatrix}^T\begin{bmatrix}1\\2\\4\end{bmatrix}=\begin{bmatrix}17\\28\end{bmatrix}\end{aligned}\tag{4.10.8}$$

$$\begin{bmatrix} 14 & 23 \\ 23 & 38 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 28 \end{bmatrix} \tag{4.10.9}$$

$$\begin{bmatrix} 14 & 23 \\ 23 & 38 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 28 \end{bmatrix},$$

Solution is: $\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$

? Exercise 4.10.107

You are doing experiments and have obtained the ordered pairs,

$$(0,1),(1,2),(2,3.5),(3,4)$$

Find m and b such that $\vec{y} = m\vec{x} + b$ approximates these four points as well as possible.

? Exercise 4.10.108

Suppose you have several ordered triples, (x_i, y_i, z_i). Describe how to find a polynomial such as

$$z = a + bx + cy + dxy + ex^2 + fy^2$$

giving the best fit to the given ordered triples.

? Exercise 4.10.109

The wind blows from the South at 20 kilometers per hour and an airplane which flies at 600 kilometers per hour in still air is heading East. Find the velocity of the airplane and its location after two hours.

? Exercise 4.10.110

The wind blows from the West at 30 kilometers per hour and an airplane which flies at 400 kilometers per hour in still air is heading North East. Find the velocity of the airplane and its position after two hours.

? Exercise 4.10.111

The wind blows from the North at 10 kilometers per hour. An airplane which flies at 300 kilometers per hour in still air is supposed to go to the point whose coordinates are at $(100,100)$. In what direction should the airplane fly?

? Exercise 4.10.112

Three forces act on an object. Two are $\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$ Newtons. Find the third force if the object is not to move.

? Exercise 4.10.113

Three forces act on an object. Two are $\begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ Newtons. Find the third force if the total force on the object is to be $\begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}$.

Exercise 4.10.114

A river flows West at the rate of b miles per hour. A boat can move at the rate of 8 miles per hour. Find the smallest value of b such that it is not possible for the boat to proceed directly across the river.

Exercise 4.10.115

The wind blows from West to East at a speed of 50 miles per hour and an airplane which travels at 400 miles per hour in still air is heading North West. What is the velocity of the airplane relative to the ground? What is the component of this velocity in the direction North?

Answer

The velocity is the sum of two vectors. $50\vec{i} + \frac{300}{\sqrt{2}}\left(\vec{i}+\vec{j}\right) = \left(50+\frac{300}{\sqrt{2}}\right)\vec{i} + \frac{300}{\sqrt{2}}\vec{j}$. The component in the direction of North is then $\frac{300}{\sqrt{2}} = 150\sqrt{2}$ and the velocity relative to the ground is

$$\left(50+\frac{300}{\sqrt{2}}\right)\vec{i} + \frac{300}{\sqrt{2}}\vec{j}$$

Exercise 4.10.116

The wind blows from West to East at a speed of 60 miles per hour and an airplane can travel travels at 100 miles per hour in still air. How many degrees West of North should the airplane head in order to travel exactly North?

Exercise 4.10.117

The wind blows from West to East at a speed of 50 miles per hour and an airplane which travels at 400 miles per hour in still air heading somewhat West of North so that, with the wind, it is flying due North. It uses 30.0 gallons of gas every hour. If it has to travel 600.0 miles due North, how much gas will it use in flying to its destination?

Exercise 4.10.118

An airplane is flying due north at 150.0 miles per hour but it is not actually going due North because there is a wind which is pushing the airplane due east at 40.0 miles per hour. After one hour, the plane starts flying $30°$ East of North. Assuming the plane starts at $(0,0)$, where is it after 2 hours? Let North be the direction of the positive y axis and let East be the direction of the positive x axis.

Answer

Velocity of plane for the first hour: $\begin{bmatrix} 0 & 150 \end{bmatrix} + \begin{bmatrix} 40 & 0 \end{bmatrix} = \begin{bmatrix} 40 & 150 \end{bmatrix}$. After one hour it is at $(40,150)$. Next the velocity of the plane is $150\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} + \begin{bmatrix} 40 & 0 \end{bmatrix}$ in miles per hour. After two hours it is then at $(40,150) + 150\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} + \begin{bmatrix} 40 & 0 \end{bmatrix} = \begin{bmatrix} 155 & 75\sqrt{3}+150 \end{bmatrix} = \begin{bmatrix} 155.0 & 279.9 \end{bmatrix}$

Exercise 4.10.119

City A is located at the origin $(0,0)$ while city B is located at $(300,500)$ where distances are in miles. An airplane flies at 250 miles per hour in still air. This airplane wants to fly from city A to city B but the wind is blowing in the direction of the positive y axis at a speed of 50 miles per hour. Find a unit vector such that if the plane heads in this direction, it will end up at city B having flown the shortest possible distance. How long will it take to get there?

Answer

Wind: $\begin{bmatrix} 0 & 50 \end{bmatrix}$. Direction it needs to travel: $(3,5)\frac{1}{\sqrt{34}}$. Then you need $250\begin{bmatrix} a & b \end{bmatrix} + \begin{bmatrix} 0 & 50 \end{bmatrix}$ to have this direction where $\begin{bmatrix} a & b \end{bmatrix}$ is an appropriate unit vector. Thus you need

$$\begin{aligned} a^2+b^2&=1 \\ \frac{250b+50}{250a}&=\frac{5}{3} \end{aligned} \tag{4.10.10}$$

Thus $a=\frac{3}{5}, b=\frac{4}{5}$. The velocity of the plane relative to the ground is $\begin{bmatrix} 150 & 250 \end{bmatrix}$. The speed of the plane relative to the ground is given by

$$\sqrt{(150)^2+(250)^2}=291.55 \text{ miles per hour}$$

It has to go a distance of $\sqrt{(300)^2+(500)^2}=583.10$ miles. Therefore, it takes

$$\frac{583.1}{291.55}=2 \text{ hours}$$

? Exercise 4.10.120

A certain river is one half mile wide with a current flowing at 2 miles per hour from East to West. A man swims directly toward the opposite shore from the South bank of the river at a speed of 3 miles per hour. How far down the river does he find himself when he has swam across? How far does he end up traveling?

Answer

Water: $\begin{bmatrix} -2 & 0 \end{bmatrix}$ Swimmer: $\begin{bmatrix} 0 & 3 \end{bmatrix}$ Speed relative to earth: $\begin{bmatrix} -2 & 3 \end{bmatrix}$. It takes him $1/6$ of an hour to get across. Therefore, he ends up traveling $\frac{1}{6}\sqrt{4+9}=\frac{1}{6}\sqrt{13}$ miles. He ends up $1/3$ mile down stream.

? Exercise 4.10.121

A certain river is one half mile wide with a current flowing at 2 miles per hour from East to West. A man can swim at 3 miles per hour in still water. In what direction should he swim in order to travel directly across the river? What would the answer to this problem be if the river flowed at 3 miles per hour and the man could swim only at the rate of 2 miles per hour?

Answer

Man: $3\begin{bmatrix} a & b \end{bmatrix}$ Water: $\begin{bmatrix} -2 & 0 \end{bmatrix}$ Then you need $3a=2$ and so $a=2/3$ and hence $b=\sqrt{5}/3$. The vector is then $\begin{bmatrix} \frac{2}{3} & \frac{\sqrt{5}}{3} \end{bmatrix}$.

In the second case, he could not do it. You would need to have a unit vector $\begin{bmatrix} a & b \end{bmatrix}$ such that $2a=3$ which is not possible.

? Exercise 4.10.122

Three forces are applied to a point which does not move. Two of the forces are $2\vec{i}+2\vec{j}-6\vec{k}$ Newtons and $8\vec{i}+8\vec{j}+3\vec{k}$ Newtons. Find the third force.

? Exercise 4.10.123

The total force acting on an object is to be $4\vec{i}+2\vec{j}-3\vec{k}$ Newtons. A force of $-3\vec{i}-1\vec{j}+8\vec{k}$ Newtons is being applied. What other force should be applied to achieve the desired total force?

? Exercise 4.10.124

A bird flies from its nest 8 km in the direction $\frac{5}{6}\pi$ north of east where it stops to rest on a tree. It then flies 1 km in the direction due southeast and lands atop a telephone pole. Place an xy coordinate system so that the origin is the bird's nest, and the positive x axis points east and the positive y axis points north. Find the displacement vector from the nest to the telephone pole.

Exercise 4.10.125

If $\vec{F}$ is a force and $\vec{D}$ is a vector, show $\text{proj}_{\vec{D}}\left(\vec{F}\right) = \left(\|\vec{F}\|\cos\theta\right)\vec{u}$ where $\vec{u}$ is the unit vector in the direction of $\vec{D}$, where $\vec{u} = \vec{D}/\|\vec{D}\|$ and θ is the included angle between the two vectors, $\vec{F}$ and $\vec{D}$. $\|\vec{F}\|\cos\theta$ is sometimes called the component of the force, $\vec{F}$ in the direction, $\vec{D}$.

Answer

$\text{proj}_{\vec{D}}\left(\vec{F}\right) = \frac{\vec{F}\bullet\vec{D}}{\|\vec{D}\|}\frac{\vec{D}}{\|\vec{D}\|} = \left(\|\vec{F}\|\cos\theta\right)\frac{\vec{D}}{\|\vec{D}\|} = \left(\|\vec{F}\|\cos\theta\right)\vec{u}$

Exercise 4.10.126

A boy drags a sled for 100 feet along the ground by pulling on a rope which is 20 degrees from the horizontal with a force of 40 pounds. How much work does this force do?

Answer

$40\cos\left(\frac{20}{180}\pi\right)100 = 3758.8$

Exercise 4.10.127

A girl drags a sled for 200 feet along the ground by pulling on a rope which is 30 degrees from the horizontal with a force of 20 pounds. How much work does this force do?

Answer

$20\cos\left(\frac{\pi}{6}\right)200 = 3464.1$

Exercise 4.10.128

A large dog drags a sled for 300 feet along the ground by pulling on a rope which is 45 degrees from the horizontal with a force of 20 pounds. How much work does this force do?

Answer

$20\left(\cos\frac{\pi}{4}\right)300 = 4242.6$

Exercise 4.10.129

How much work does it take to slide a crate 20 meters along a loading dock by pulling on it with a 200 Newton force at an angle of $30°$ from the horizontal? Express your answer in Newton meters.

Answer

$200\left(\cos\left(\frac{\pi}{6}\right)\right)20 = 3464.1$

Exercise 4.10.130

An object moves 10 meters in the direction of $\vec{j}$. There are two forces acting on this object, $\vec{F}_1 = \vec{i} + \vec{j} + 2\vec{k}$, and $\vec{F}_2 = -5\vec{i} + 2\vec{j} - 6\vec{k}$. Find the total work done on the object by the two forces. **Hint:** You can take the work done by the resultant of the two forces or you can add the work done by each force. Why?

Answer

$\begin{bmatrix} -4 \\ 3 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times 10 = 30$ You can consider the resultant of the two forces because of the properties of the dot product.

Exercise 4.10.131

An object moves 10 meters in the direction of $\vec{j} + \vec{i}$. There are two forces acting on this object, $\vec{F}_1 = \vec{i} + 2\vec{j} + 2\vec{k}$, and $\vec{F}_2 = 5\vec{i} + 2\vec{j} - 6\vec{k}$. Find the total work done on the object by the two forces. **Hint:** You can take the work done by the resultant of the two forces or you can add the work done by each force. Why?

Answer

$$\begin{aligned} \vec{F}_1 \bullet \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} 10 + \vec{F}_2 \bullet \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} 10 &= \left(\vec{F}_1 + \vec{F}_2\right) \bullet \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} 10 \\ &= \begin{bmatrix} 6 \\ 4 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} 10 \\ &= 50\sqrt{2} \end{aligned} \tag{4.10.11}$$

Exercise 4.10.132

An object moves 20 meters in the direction of $\vec{k} + \vec{j}$. There are two forces acting on this object, $\vec{F}_1 = \vec{i} + \vec{j} + 2\vec{k}$, and $\vec{F}_2 = \vec{i} + 2\vec{j} - 6\vec{k}$. Find the total work done on the object by the two forces. **Hint:** You can take the work done by the resultant of the two forces or you can add the work done by each force.

Answer

$$\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} 20 = -10\sqrt{2}$$

This page titled 4.10: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **4.12: Applications** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

CHAPTER OVERVIEW

5: Linear Transformations

5.1: Linear Transformations
5.2: The Matrix of a Linear Transformation I
5.3: Properties of Linear Transformations
5.4: Special Linear Transformations in R^2
5.5: One-to-One and Onto Transformations
5.6: The Kernel and Image of A Linear Map
5.7: The Matrix of a Linear Transformation II
5.8: The General Solution of a Linear System
5.9: Exercises

This page titled 5: Linear Transformations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

5.1: Linear Transformations

Outcomes

A. Understand the definition of a linear transformation, and that all linear transformations are determined by matrix multiplication.

Recall that when we multiply an $m \times n$ matrix by an $n \times 1$ column vector, the result is an $m \times 1$ column vector. In this section we will discuss how, through matrix multiplication, an $m \times n$ matrix **transforms** an $n \times 1$ column vector into an $m \times 1$ column vector.

Recall that the $n \times 1$ vector given by

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

is said to belong to $\mathbb{R}^n$, which is the set of all $n \times 1$ vectors. In this section, we will discuss transformations of vectors in $\mathbb{R}^n$.

Consider the following example.

Example 5.1.1: A Function Which Transforms Vectors

Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$. Show that by matrix multiplication A transforms vectors in $\mathbb{R}^3$ into vectors in $\mathbb{R}^2$.

Solution

First, recall that vectors in $\mathbb{R}^3$ are vectors of size 3×1, while vectors in $\mathbb{R}^2$ are of size 2×1. If we multiply A, which is a 2×3 matrix, by a 3×1 vector, the result will be a 2×1 vector. This what we mean when we say that A *transforms* vectors.

Now, for $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$, multiply on the left by the given matrix to obtain the new vector. This product looks like

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y \\ 2x+y \end{bmatrix}$$

The resulting product is a 2×1 vector which is determined by the choice of x and y. Here are some numerical examples.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

Here, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in $\mathbb{R}^3$ was transformed by the matrix into the vector $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ in $\mathbb{R}^2$.

Here is another example:

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \end{bmatrix}$$

The idea is to define a function which takes vectors in $\mathbb{R}^3$ and delivers new vectors in $\mathbb{R}^2$. In this case, that function is multiplication by the matrix A.

Let T denote such a function. The notation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ means that the function T transforms vectors in $\mathbb{R}^n$ into vectors in $\mathbb{R}^m$. The notation $T(\vec{x})$ means the transformation T applied to the vector $\vec{x}$. The above example demonstrated a transformation achieved by

matrix multiplication. In this case, we often write

$$T_A(\vec{x}) = A\vec{x}$$

Therefore, T_A is the transformation determined by the matrix A. In this case we say that T is a matrix transformation.

Recall the property of matrix multiplication that states that for k and p scalars,

$$A(kB + pC) = kAB + pAC$$

In particular, for A an $m \times n$ matrix and B and C, $n \times 1$ vectors in $\mathbb{R}^n$, this formula holds.

In other words, this means that matrix multiplication gives an example of a linear transformation, which we will now define.

Definition 5.1.1: Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a function, where for each $\vec{x} \in \mathbb{R}^n, T(\vec{x}) \in \mathbb{R}^m$. Then T is a **linear transformation** if whenever k, p are scalars and $\vec{x}_1$ and $\vec{x}_2$ are vectors in $\mathbb{R}^n$ ($n \times 1$ vectors),

$$T(k\vec{x}_1 + p\vec{x}_2) = kT(\vec{x}_1) + pT(\vec{x}_2)$$

Consider the following example.

Example 5.1.2: Linear Transformation

Let T be a transformation defined by $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + y \\ x - z \end{bmatrix} \text{ for all } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$$

Show that T is a linear transformation.

Solution

By Definition 5.1.1 we need to show that $T(k\vec{x}_1 + p\vec{x}_2) = kT(\vec{x}_1) + pT(\vec{x}_2)$ for all scalars k, p and vectors $\vec{x}_1, \vec{x}_2$. Let

$$\vec{x}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$$

Then

$$
\begin{aligned}
T\left(k\vec{x}_1+p\vec{x}_2\right) &= T\left(k\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + p\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} kx_1 \\ ky_1 \\ kz_1 \end{bmatrix} + \begin{bmatrix} px_2 \\ py_2 \\ pz_2 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} kx_1+px_2 \\ ky_1+py_2 \\ kz_1+pz_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} (kx_1+px_2)+(ky_1+py_2) \\ (kx_1+px_2)-(kz_1+pz_2) \end{bmatrix} \\
&= \begin{bmatrix} (kx_1+ky_1)+(px_2+py_2) \\ (kx_1-kz_1)+(px_2-pz_2) \end{bmatrix} \\
&= \begin{bmatrix} kx_1+ky_1 \\ kx_1-kz_1 \end{bmatrix} + \begin{bmatrix} px_2+py_2 \\ px_2-pz_2 \end{bmatrix} \\
&= k\begin{bmatrix} x_1+y_1 \\ x_1-z_1 \end{bmatrix} + p\begin{bmatrix} x_2+y_2 \\ x_2-z_2 \end{bmatrix} \\
&= kT(\vec{x}_1)+pT(\vec{x}_2)
\end{aligned}
$$

Therefore T is a linear transformation.

Two important examples of linear transformations are the zero transformation and identity transformation. The zero transformation defined by $T\left(\vec{x}\right)=\vec{0}$ for all $\vec{x}$ is an example of a linear transformation. Similarly the identity transformation defined by $T\left(\vec{x}\right)=\vec{x}$ is also linear. Take the time to prove these using the method demonstrated in Example 5.1.2.

We began this section by discussing matrix transformations, where multiplication by a matrix transforms vectors. These matrix transformations are in fact linear transformations.

Theorem 5.1.1: Matrix Transformations are Linear Transformations

Let $T:\mathbb{R}^n\mapsto\mathbb{R}^m$ be a transformation defined by $T(\vec{x})=A\vec{x}$. Then T is a linear transformation.

It turns out that every linear transformation can be expressed as a matrix transformation, and thus linear transformations are exactly the same as matrix transformations.

This page titled 5.1: Linear Transformations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **5.1: Linear Transformations** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

5.2: The Matrix of a Linear Transformation I

Outcomes

A. Find the matrix of a linear transformation with respect to the standard basis.
B. Determine the action of a linear transformation on a vector in $\mathbb{R}^n$.

In the above examples, the action of the linear transformations was to multiply by a matrix. It turns out that this is always the case for linear transformations. If T is **any** linear transformation which maps $\mathbb{R}^n$ to $\mathbb{R}^m$, there is **always** an $m \times n$ matrix A with the property that

$$T(\vec{x}) = A\vec{x} \tag{5.2.1}$$

for all $\vec{x} \in \mathbb{R}^n$.

Theorem 5.2.1: Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then we can find a matrix A such that $T(\vec{x}) = A\vec{x}$. In this case, we say that T is *determined* or *induced* by the matrix A.

Here is why. Suppose $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation and you want to find the matrix defined by this linear transformation as described in (5.2.1). Note that

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \sum_{i=1}^{n} x_i \vec{e}_i$$

where $\vec{e}_i$ is the ith column of I_n, that is the $n \times 1$ vector which has zeros in every slot but the ith and a 1 in this slot.

Then since T is linear,

$$\begin{aligned} T(\vec{x}) &= \sum_{i=1}^{n} x_i T(\vec{e}_i) \\ &= \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \end{aligned} \tag{5.2.2}$$

The desired matrix is obtained from constructing the ith column as $T(\vec{e}_i)$. Recall that the set $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$ is called the standard basis of $\mathbb{R}^n$. Therefore the matrix of T is found by applying T to the standard basis. We state this formally as the following theorem.

Theorem 5.2.2: Matrix of a Linear Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then the matrix A satisfying $T(\vec{x}) = A\vec{x}$ is given by

$$A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$$

where $\vec{e}_i$ is the ith column of I_n, and then $T(\vec{e}_i)$ is the ith column of A.

The following Corollary is an essential result.

Corollary 5.2.1: Matrix and Linear Transformation

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if it is a matrix transformation.

Consider the following example.

Example 5.2.1: The Matrix of a Linear Transformation

Suppose T is a linear transformation, $T : \mathbb{R}^3 \to \mathbb{R}^2$ where

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}, T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}9\\-3\end{bmatrix}, T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

Find the matrix A of T such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$.

Solution

By Theorem 5.2.2 we construct A as follows:

$$A = \begin{bmatrix} | & & | \\ T(\vec{e}_1) & \cdots & T(\vec{e}_n) \\ | & & | \end{bmatrix}$$

In this case, A will be a 2×3 matrix, so we need to find $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$. Luckily, we have been given these values so we can fill in A as needed, using these vectors as the columns of A. Hence,

$$A = \begin{bmatrix}1 & 9 & 1\\2 & -3 & 1\end{bmatrix}$$

In this example, we were given the resulting vectors of $T(\vec{e}_1), T(\vec{e}_2)$, and $T(\vec{e}_3)$. Constructing the matrix A was simple, as we could simply use these vectors as the columns of A. The next example shows how to find A when we are not given the $T(\vec{e}_i)$ so clearly.

Example 5.2.2: The Matrix of Linear Transformation: Inconveniently Defined

Suppose T is a linear transformation, $T : \mathbb{R}^2 \to \mathbb{R}^2$ and

$$T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}, T\begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}3\\2\end{bmatrix}$$

Find the matrix A of T such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$.

Solution

By Theorem 5.2.2 to find this matrix, we need to determine the action of T on $\vec{e}_1$ and $\vec{e}_2$. In Example 9.9.2, we were given these resulting vectors. However, in this example, we have been given T of two different vectors. How can we find out the action of T on $\vec{e}_1$ and $\vec{e}_2$? In particular for $\vec{e}_1$, suppose there exist x and y such that

$$\begin{bmatrix}1\\0\end{bmatrix} = x\begin{bmatrix}1\\1\end{bmatrix} + y\begin{bmatrix}0\\-1\end{bmatrix} \tag{5.2.3}$$

Then, since T is linear,

$$T\begin{bmatrix}1\\0\end{bmatrix} = xT\begin{bmatrix}1\\1\end{bmatrix} + yT\begin{bmatrix}0\\-1\end{bmatrix}$$

Substituting in values, this sum becomes

$$T\begin{bmatrix}1\\0\end{bmatrix} = x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}3\\2\end{bmatrix} \tag{5.2.4}$$

Therefore, if we know the values of x and y which satisfy (5.2.3), we can substitute these into equation (5.2.4). By doing so, we find $T(\vec{e}_1)$ which is the first column of the matrix A.

We proceed to find x and y. We do so by solving (5.2.3), which can be done by solving the system

$$\begin{aligned} x &= 1 \\ x - y &= 0 \end{aligned}$$

We see that $x = 1$ and $y = 1$ is the solution to this system. Substituting these values into equation (5.2.4), we have

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

Therefore $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ is the first column of A.

Computing the second column is done in the same way, and is left as an exercise.

The resulting matrix A is given by

$$A = \begin{bmatrix} 4 & -3 \\ 4 & -2 \end{bmatrix}$$

This example illustrates a very long procedure for finding the matrix of A. While this method is reliable and will always result in the correct matrix A, the following procedure provides an alternative method.

Procedure 5.2.1: Finding the Matrix of Inconveniently Defined Linear Transformation

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Suppose there exist vectors $\{\vec{a}_1, \cdots, \vec{a}_n\}$ in $\mathbb{R}^n$ such that $\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1}$ exists, and

$$T(\vec{a}_i) = \vec{b}_i$$

Then the matrix of T must be of the form

$$\begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1}$$

We will illustrate this procedure in the following example. You may also find it useful to work through Example 5.2.2 using this procedure.

Example 5.2.3: Matrix of a Linear Transformation Given Inconveniently

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, T\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Find the matrix of this linear transformation.

Solution

By Procedure 5.2.1, $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1}$ and $B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$

Then, Procedure 5.2.1 claims that the matrix of T is

$$C = BA^{-1} = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}$$

Indeed you can first verify that $T(\vec{x}) = C\vec{x}$ for the 3 vectors above:

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 0 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

But more generally $T(\vec{x}) = C\vec{x}$ for any $\vec{x}$. To see this, let $\vec{y} = A^{-1}\vec{x}$ and then using linearity of T:

$$T(\vec{x}) = T(A\vec{y}) = T\left(\sum_i \vec{y}_i \vec{a}_i\right) = \sum \vec{y}_i T(\vec{a}_i) \sum \vec{y}_i \vec{b}_i = B\vec{y} = BA^{-1}\vec{x} = C\vec{x}$$

Recall the dot product discussed earlier. Consider the map $\vec{v} \mapsto \text{proj}_{\vec{u}}(\vec{v})$ which takes a vector a transforms it to its projection onto a given vector $\vec{u}$. It turns out that this map is linear, a result which follows from the properties of the dot product. This is shown as follows.

$$\begin{aligned} \text{proj}_{\vec{u}}(k\vec{v} + p\vec{w}) &= \left(\frac{(k\vec{v} + p\vec{w}) \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\right)\vec{u} \\ &= k\left(\frac{\vec{v} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\right)\vec{u} + p\left(0.05in\frac{\vec{w} \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\right)\vec{u} \\ &= k\,\text{proj}_{\vec{u}}(\vec{v}) + p\,\text{proj}_{\vec{u}}(\vec{w}) \end{aligned} \tag{5.2.5}$$

Consider the following example.

Example 5.2.4: Matrix of a Projection Map

Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and let T be the projection map $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by

$$T(\vec{v}) = \text{proj}_{\vec{u}}(\vec{v})$$

for any $\vec{v} \in \mathbb{R}^3$.

1. Does this transformation come from multiplication by a matrix?
2. If so, what is the matrix?

Solution

1. First, we have just seen that $T(\vec{v}) = \text{proj}_{\vec{u}}(\vec{v})$ is linear. Therefore by Theorem 5.2.1, we can find a matrix A such that $T(\vec{x}) = A\vec{x}$.
2. The columns of the matrix for T are defined above as $T(\vec{e}_i)$. It follows that $T(\vec{e}_i) = \text{proj}_{\vec{u}}(\vec{e}_i)$ gives the ith column of the desired matrix. Therefore, we need to find

$$\text{proj}_{\vec{u}}(\vec{e}_i) = \left(\frac{\vec{e}_i \bullet \vec{u}}{\vec{u} \bullet \vec{u}}\right)\vec{u}$$

For the given vector $\vec{u}$, this implies the columns of the desired matrix are

$$\frac{1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \frac{2}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \frac{3}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

which you can verify. Hence the matrix of T is

$$\frac{1}{14}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

This page titled 5.2: The Matrix of a Linear Transformation I is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **5.2: The Matrix of a Linear Transformation I** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

5.3: Properties of Linear Transformations

Outcomes

A. Use properties of linear transformations to solve problems.

B. Find the composite of transformations and the inverse of a transformation.

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then there are some important properties of T which will be examined in this section. Consider the following theorem.

Theorem 5.3.1: Properties of Linear Transformations

Properties of Linear Transformationsproperties Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation and let $\vec{x} \in \mathbb{R}^n$.

- T preserves the zero vector.

$$T(0\vec{x}) = 0T(\vec{x}). \text{ Hence } T(\vec{0}) = \vec{0}$$

- T preserves the negative of a vector:

$$T((-1)\vec{x}) = (-1)T(\vec{x}). \text{ Hence } T(-\vec{x}) = -T(\vec{x}).$$

- T preserves linear combinations:

$$\text{Let } \vec{x}_1, \ldots, \vec{x}_k \in \mathbb{R}^n \text{ and } a_1, \ldots, a_k \in \mathbb{R}.$$

$$\text{Then if } \vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + \ldots + a_k\vec{x}_k, \text{ it follows that}$$

$$T(\vec{y}) = T(a_1\vec{x}_1 + a_2\vec{x}_2 + \ldots + a_k\vec{x}_k) = a_1T(\vec{x}_1) + a_2T(\vec{x}_2) + \ldots + a_kT(\vec{x}_k).$$

These properties are useful in determining the action of a transformation on a given vector. Consider the following example.

Example 5.3.1: Linear Combination

Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^4$ be a linear transformation such that

$$T\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix}, T\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix}$$

Find $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$.

Solution

Using the third property in Theorem 9.6.1, we can find $T\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ by writing $\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$.

Therefore we want to find $a, b \in \mathbb{R}$ such that

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = a\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + b\begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

The necessary augmented matrix and resulting reduced row-echelon form are given by:

$$\left[\begin{array}{cc|c} 1 & 4 & -7 \\ 3 & 0 & 3 \\ 1 & 5 & -9 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

Hence $a = 1, b = -2$ and

$$\begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix}$$

Now, using the third property above, we have

$$\begin{aligned} T \begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} &= T \left(1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + (-2) \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \right) \\ &= 1T \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} - 2T \begin{bmatrix} 4 \\ 0 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 4 \\ 0 \\ -2 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ 5 \\ -1 \\ 5 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix} \end{aligned} \tag{5.3.1}$$

Therefore, $T \begin{bmatrix} -7 \\ 3 \\ -9 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \\ -12 \end{bmatrix}$.

Suppose two linear transformations act in the same way on $\vec{x}$ for all vectors. Then we say that these transformations are equal.

Definition 5.3.1: Equal Transformations

Let S and T be linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then $S = T$ if and only if for every $\vec{x} \in \mathbb{R}^n$,

$$S(\vec{x}) = T(\vec{x})$$

Suppose two linear transformations act on the same vector $\vec{x}$, first the transformation T and then a second transformation given by S. We can find the **composite** transformation that results from applying both transformations.

Definition 5.3.2: Composition of Linear Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations. Then the **composite** of S and T is

$$S \circ T : \mathbb{R}^k \mapsto \mathbb{R}^m$$

The action of $S \circ T$ is given by

$$(S \circ T)(\vec{x}) = S(T(\vec{x})) \text{ for all } \vec{x} \in \mathbb{R}^k$$

Notice that the resulting vector will be in $\mathbb{R}^m$. Be careful to observe the order of transformations. We write $S \circ T$ but apply the transformation T first, followed by S.

Theorem 5.3.2: Composition of Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations such that T is induced by the matrix A and S is induced by the matrix B. Then $S \circ T$ is a linear transformation which is induced by the matrix BA.

Consider the following example.

Example 5.3.2: Composition of Transformations

Let T be a linear transformation induced by the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

and S a linear transformation induced by the matrix

$$B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Find the matrix of the composite transformation $S \circ T$. Then, find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Solution

By Theorem 5.3.2, the matrix of $S \circ T$ is given by BA.

$$BA = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 2 & 0 \end{bmatrix}$$

To find $(S \circ T)(\vec{x})$, multiply $\vec{x}$ by BA as follows

$$\begin{bmatrix} 8 & 4 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \end{bmatrix}$$

To check, first determine $T(\vec{x})$:

$$\begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$$

Then, compute $S(T(\vec{x}))$ as follows:

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 9 \\ 2 \end{bmatrix} = \begin{bmatrix} 24 \\ 2 \end{bmatrix}$$

Consider a composite transformation $S \circ T$, and suppose that this transformation acted such that $(S \circ T)(\vec{x}) = \vec{x}$. That is, the transformation S took the vector $T(\vec{x})$ and returned it to $\vec{x}$. In this case, S and T are inverses of each other. Consider the following definition.

Definition 5.3.3: Inverse of a Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^n$ be linear transformations. Suppose that for each $\vec{x} \in \mathbb{R}^n$,

$$(S \circ T)(\vec{x}) = \vec{x}$$

and

$$(T \circ S)(\vec{x}) = \vec{x}$$

Then, S is called an inverse of T and T is called an inverse of S. Geometrically, they reverse the action of each other.

The following theorem is crucial, as it claims that the above inverse transformations are unique.

Theorem 5.3.3: Inverse of a Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear transformation induced by the matrix A. Then T has an inverse transformation if and only if the matrix A is invertible. In this case, the inverse transformation is unique and denoted $T^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^n$. T^{-1} is induced by the matrix A^{-1}.

Consider the following example.

Example 5.3.3: Inverse of a Transformation

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation induced by the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Show that T^{-1} exists and find the matrix B which it is induced by.

Solution

Since the matrix A is invertible, it follows that the transformation T is invertible. Therefore, T^{-1} exists.

You can verify that A^{-1} is given by:

$$A^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$$

Therefore the linear transformation T^{-1} is induced by the matrix A^{-1}.

This page titled 5.3: Properties of Linear Transformations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **5.3: Properties of Linear Transformations** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

5.4: Special Linear Transformations in R²

Outcomes

A. Find the matrix of rotations and reflections in $\mathbb{R}^2$ and determine the action of each on a vector in $\mathbb{R}^2$.

In this section, we will examine some special examples of linear transformations in $\mathbb{R}^2$ including rotations and reflections. We will use the geometric descriptions of vector addition and scalar multiplication discussed earlier to show that a rotation of vectors through an angle and reflection of a vector across a line are examples of linear transformations.

More generally, denote a transformation given by a rotation by T. Why is such a transformation linear? Consider the following picture which illustrates a rotation. Let $\vec{u}, \vec{v}$ denote vectors.

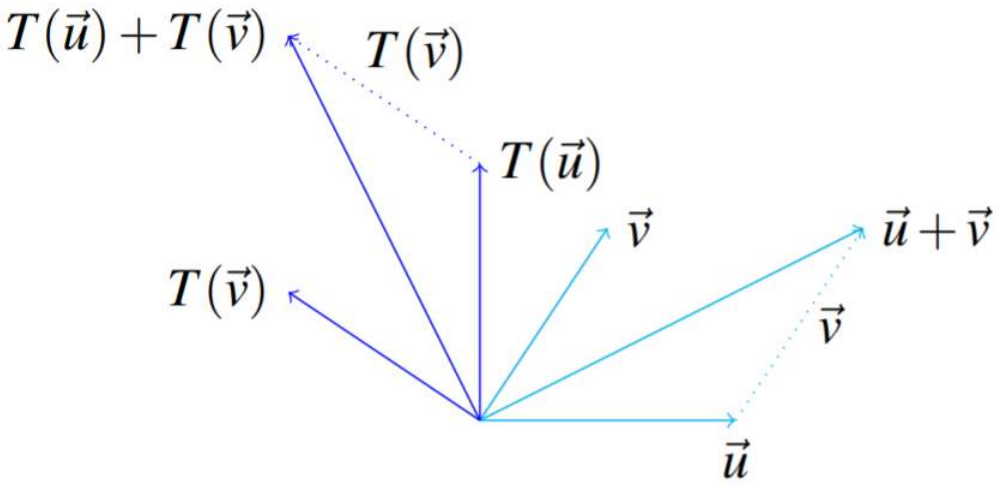


Figure 5.4.1

Let's consider how to obtain $T[\vec{u}+\vec{v}]$. Simply, you add $T(\vec{u})$ and $T(\vec{v})$. Here is why. If you add $T(\vec{u})$ to $T(\vec{v})$ you get the diagonal of the parallelogram determined by $T(\vec{u})$ and $T(\vec{v})$, as this action is our usual vector addition. Now, suppose we first add $\vec{u}$ and $\vec{v}$, and then apply the transformation T to $\vec{u}+\vec{v}$. Hence, we find $T(\vec{u}+\vec{v})$. As shown in the diagram, this will result in the same vector. In other words, $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$.

This is because the rotation preserves all angles between the vectors as well as their lengths. In particular, it preserves the shape of this parallelogram. Thus both $T[\vec{u}] + T[\vec{v}]$ and $T[\vec{u}+\vec{v}]$ give the same vector. It follows that T distributes across addition of the vectors of $\mathbb{R}^2$.

Similarly, if k is a scalar, it follows that $T[k\vec{u}] = kT[\vec{u}]$. Thus rotations are an example of a linear transformation by Definition 9.6.1.

The following theorem gives the matrix of a linear transformation which rotates all vectors through an angle of θ.

Theorem 5.4.1: Rotation

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by rotating vectors through an angle of θ. Then the matrix A of R_θ is given by

$$\begin{bmatrix} \cos[\theta] & -\sin[\theta] \\ \sin[\theta] & \cos[\theta] \end{bmatrix}$$

Proof

Let $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These identify the geometric vectors which point along the positive x axis and positive y axis as shown.

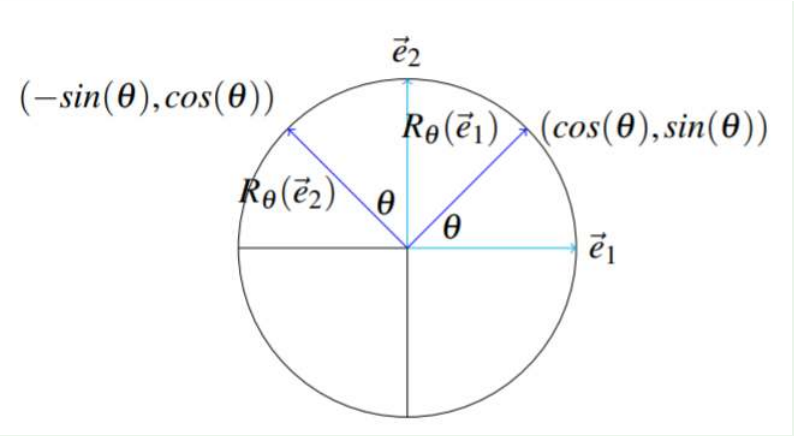


Figure 5.4.2

From Theorem 5.2.2, we need to find $R_\theta(\vec{e}_1)$ and $R_\theta(\vec{e}_2)$, and use these as the columns of the matrix A of T. We can use $\cos, \sin$ of the angle θ to find the coordinates of $R_\theta(\vec{e}_1)$ as shown in the above picture. The coordinates of $R_\theta(\vec{e}_2)$ also follow from trigonometry. Thus

$$R_\theta(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, R_\theta(\vec{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Therefore, from Theorem 5.2.2,

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

We can also prove this algebraically without the use of the above picture. The definition of $[\cos[\theta], \sin[\theta]]$ is as the coordinates of the point of $R_\theta(\vec{e}_1)$. Now the point of the vector $\vec{e}_2$ is exactly $\pi/2$ further along the unit circle from the point of $\vec{e}_1$, and therefore after rotation through an angle of θ the coordinates x and y of the point of $R_\theta(\vec{e}_2)$ are given by

$$[x, y] = [\cos[\theta + \pi/2], \sin[\theta + \pi/2]] = [-\sin\theta, \cos\theta]$$

Consider the following example.

Example 5.4.1: Rotation in $\mathbb{R}^2$

Let $R_{\frac{\pi}{2}} : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation through $\pi/2$. Find the matrix of $R_{\frac{\pi}{2}}$. Then, find $R_{\frac{\pi}{2}}(\vec{x})$ where $\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Solution

By Theorem 5.4.1, the matrix of $R_{\frac{\pi}{2}}$ is given by

$$\begin{bmatrix} \cos[\theta] & -\sin[\theta] \\ \sin[\theta] & \cos[\theta] \end{bmatrix} = \begin{bmatrix} \cos[\pi/2] & -\sin[\pi/2] \\ \sin[\pi/2] & \cos[\pi/2] \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

To find $R_{\frac{\pi}{2}}(\vec{x})$, we multiply the matrix of $R_{\frac{\pi}{2}}$ by $\vec{x}$ as follows

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We now look at an example of a linear transformation involving two angles.

Example 5.4.2: The Rotation Matrix of the Sum of Two Angles

Find the matrix of the linear transformation which is obtained by first rotating all vectors through an angle of ϕ and then through an angle θ. Hence the linear transformation rotates all vectors through an angle of $\theta + \phi$.

Solution

Let $R_{\theta+\phi}$ denote the linear transformation which rotates every vector through an angle of $\theta+\phi$. Then to obtain $R_{\theta+\phi}$, we first apply R_ϕ and then R_θ where R_ϕ is the linear transformation which rotates through an angle of ϕ and R_θ is the linear transformation which rotates through an angle of θ. Denoting the corresponding matrices by $A_{\theta+\phi}$, A_ϕ, and A_θ, it follows that for every $\vec{u}$

$$R_{\theta+\phi}\left[\vec{u}\right] = A_{\theta+\phi}\vec{u} = A_\theta A_\phi \vec{u} = R_\theta R_\phi\left[\vec{u}\right]$$

Notice the order of the matrices here!

Consequently, you must have

$$\begin{aligned} A_{\theta+\phi} &= \begin{bmatrix} \cos\left[\theta+\phi\right] & -\sin\left[\theta+\phi\right] \\ \sin\left[\theta+\phi\right] & \cos\left[\theta+\phi\right] \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = A_\theta A_\phi \end{aligned} \tag{5.4.1}$$

The usual matrix multiplication yields

$$\begin{aligned} A_{\theta+\phi} &= \begin{bmatrix} \cos\left[\theta+\phi\right] & -\sin\left[\theta+\phi\right] \\ \sin\left[\theta+\phi\right] & \cos\left[\theta+\phi\right] \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix} \\ &= A_\theta A_\phi \end{aligned} \tag{5.4.2}$$

Don't these look familiar? They are the usual trigonometric identities for the sum of two angles derived here using linear algebra concepts.

Here we have focused on rotations in two dimensions. However, you can consider rotations and other geometric concepts in any number of dimensions. This is one of the major advantages of linear algebra. You can break down a difficult geometrical procedure into small steps, each corresponding to multiplication by an appropriate matrix. Then by multiplying the matrices, you can obtain a single matrix which can give you numerical information on the results of applying the given sequence of simple procedures.

Linear transformations which reflect vectors across a line are a second important type of transformations in $\mathbb{R}^2$. Consider the following theorem.

Theorem 5.4.2: Reflection

Let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by reflecting vectors over the line $\vec{y} = m\vec{x}$. Then the matrix of Q_m is given by

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

Consider the following example.

Example 5.4.3: Reflection in $\mathbb{R}^2$

Let $Q_2 : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection over the line $\vec{y} = 2\vec{x}$. Then Q_2 is a linear transformation. Find the matrix of Q_2. Then, find $Q_2(\vec{x})$ where $\vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Solution

By Theorem 5.4.2, the matrix of Q_2 is given by

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} = \frac{1}{1+(2)^2}\begin{bmatrix} 1-(2)^2 & 2(2) \\ 2(2) & (2)^2-1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3 & 8 \\ 8 & 3 \end{bmatrix}$$

To find $Q_2(\vec{x})$ we multiply $\vec{x}$ by the matrix of Q_2 as follows:

$$\frac{1}{5}\begin{bmatrix} -3 & 8 \\ 8 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{19}{5} \\ \frac{2}{5} \end{bmatrix}$$

Consider the following example which incorporates a reflection as well as a rotation of vectors.

Example 5.4.4: Rotation Followed by a Reflection

Find the matrix of the linear transformation which is obtained by first rotating all vectors through an angle of $\pi/6$ and then reflecting through the x axis.

Solution

By Theorem 5.4.1, the matrix of the transformation which involves rotating through an angle of $\pi/6$ is

$$\begin{bmatrix} \cos[\pi/6] & -\sin[\pi/6] \\ \sin[\pi/6] & \cos[\pi/6] \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

Reflecting across the x axis is the same action as reflecting vectors over the line $\vec{y} = m\vec{x}$ with $m = 0$. By Theorem 5.4.2, the matrix for the transformation which reflects all vectors through the x axis is

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix} = \frac{1}{1+(0)^2}\begin{bmatrix} 1-(0)^2 & 2(0) \\ 2(0) & (0)^2-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Therefore, the matrix of the linear transformation which first rotates through $\pi/6$ and then reflects through the x axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{bmatrix}$$

This page titled 5.4: Special Linear Transformations in R^2 is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **5.4: Special Linear Transformations in R^2** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

5.5: One-to-One and Onto Transformations

Outcomes

A. Determine if a linear transformation is onto or one to one.

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. We define the **range** or **image** of T as the set of vectors of $\mathbb{R}^m$ which are of the form $T(\vec{x})$ (equivalently, $A\vec{x}$) for some $\vec{x} \in \mathbb{R}^n$. It is common to write $T\mathbb{R}^n$, $T(\mathbb{R}^n)$, or $\text{Im}(T)$ to denote these vectors.

Lemma 5.5.1: Range of a Matrix Transformation

Let A be an $m \times n$ matrix where $A_1, \cdots, A_n$ denote the columns of A. Then, for a vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n$,

$$A\vec{x} = \sum_{k=1}^{n} x_k A_k$$

Therefore, $A(\mathbb{R}^n)$ is the collection of all linear combinations of these products.

Proof

This follows from the definition of matrix multiplication.

This section is devoted to studying two important characterizations of linear transformations, called one to one and onto. We define them now.

Definition 5.5.1: One to One

Suppose $\vec{x}_1$ and $\vec{x}_2$ are vectors in $\mathbb{R}^n$. A linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is called **one to one** (often written as $1-1$) if whenever $\vec{x}_1 \neq \vec{x}_2$ it follows that :

$$T(\vec{x}_1) \neq T(\vec{x}_2)$$

Equivalently, if $T(\vec{x}_1) = T(\vec{x}_2)$, then $\vec{x}_1 = \vec{x}_2$. Thus, T is one to one if it never takes two different vectors to the same vector.

The second important characterization is called onto.

Definition 5.5.2: Onto

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then T is called **onto** if whenever $\vec{x}_2 \in \mathbb{R}^m$ there exists $\vec{x}_1 \in \mathbb{R}^n$ such that $T(\vec{x}_1) = \vec{x}_2$.

We often call a linear transformation which is one-to-one an **injection**. Similarly, a linear transformation which is onto is often called a **surjection**.

The following proposition is an important result.

Proposition 5.5.1: One to One

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Then T is one to one if and only if $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$.

Proof

We need to prove two things here. First, we will prove that if T is one to one, then $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = \vec{0}$. Second, we will show that if $T(\vec{x}) = \vec{0}$ implies that $\vec{x} = \vec{0}$, then it follows that T is one to one. Recall that a linear transformation has the property that $T(\vec{0}) = \vec{0}$.

Suppose first that T is one to one and consider $T(\vec{0})$.

$$T(\vec{0}) = T\left(\vec{0}+\vec{0}\right) = T(\vec{0}) + T(\vec{0})$$

and so, adding the additive inverse of $T(\vec{0})$ to both sides, one sees that $T(\vec{0}) = \vec{0}$. If $T(\vec{x}) = \vec{0}$ it must be the case that $\vec{x} = \vec{0}$ because it was just shown that $T(\vec{0}) = \vec{0}$ and T is assumed to be one to one.

Now assume that if $T(\vec{x}) = \vec{0}$, then it follows that $\vec{x} = \vec{0}$. If $T(\vec{v}) = T(\vec{u})$, then

$$T(\vec{v}) - T(\vec{u}) = T(\vec{v} - \vec{u}) = \vec{0}$$

which shows that $\vec{v} - \vec{u} = 0$. In other words, $\vec{v} = \vec{u}$, and T is one to one.

Note that this proposition says that if $A = \begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix}$ then A is one to one if and only if whenever

$$0 = \sum_{k=1}^{n} c_k A_k$$

it follows that each scalar $c_k = 0$.

We will now take a look at an example of a one to one and onto linear transformation.

Example 5.5.1: A One to One and Onto Linear Transformation

Suppose

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Then, $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. Is T onto? Is it one to one?

Solution

Recall that because T can be expressed as matrix multiplication, we know that T is a linear transformation. We will start by looking at onto. So suppose $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. Does there exist $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$? If so, then since $\begin{bmatrix} a \\ b \end{bmatrix}$ is an arbitrary vector in $\mathbb{R}^2$, it will follow that T is onto.

This question is familiar to you. It is asking whether there is a solution to the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

This is the same thing as asking for a solution to the following system of equations.

$$\begin{aligned} x + y &= a \\ x + 2y &= b \end{aligned}$$

Set up the augmented matrix and row reduce.

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 2 & b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2a-b \\ 0 & 1 & b-a \end{array}\right] \tag{5.5.1}$$

You can see from this point that the system has a solution. Therefore, we have shown that for any a, b, there is a $\begin{bmatrix} x \\ y \end{bmatrix}$ such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. Thus T is onto.

Now we want to know if T is one to one. By Proposition 5.5.1 it is enough to show that $A\vec{x} = 0$ implies $\vec{x} = 0$. Consider the system $A\vec{x} = 0$ given by:

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is the same as the system given by

$$x+y=0$$
$$x+2y=0$$

We need to show that the solution to this system is $x=0$ and $y=0$. By setting up the augmented matrix and row reducing, we end up with

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

This tells us that $x=0$ and $y=0$. Returning to the original system, this says that if

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In other words, $A\vec{x}=0$ implies that $\vec{x}=0$. By Proposition 5.5.1, A is one to one, and so T is also one to one.

We also could have seen that T is one to one from our above solution for onto. By looking at the matrix given by (5.5.1), you can see that there is a **unique** solution given by $x=2a-b$ and $y=b-a$. Therefore, there is only one vector, specifically $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2a-b \\ b-a \end{bmatrix}$ such that $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. Hence by Definition 5.5.1, T is one to one.

Example 5.5.2: An Onto Transformation

Let $T:\mathbb{R}^4 \mapsto \mathbb{R}^2$ be a linear transformation defined by

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+d \\ b+c \end{bmatrix} \text{ for all } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$$

Prove that T is onto but not one to one.

Solution

You can prove that T is in fact linear.

To show that T is onto, let $\begin{bmatrix} x \\ y \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^2$. Taking the vector $\begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^4$ we have

$$T\begin{bmatrix} x \\ y \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x+0 \\ y+0 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

This shows that T is onto.

By Proposition 5.5.1 T is one to one if and only if $T(\vec{x})=\vec{0}$ implies that $\vec{x}=\vec{0}$. Observe that

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1+-1 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

There exists a nonzero vector $\vec{x}$ in $\mathbb{R}^4$ such that $T(\vec{x}) = \vec{0}$. It follows that T is not one to one.

The above examples demonstrate a method to determine if a linear transformation T is one to one or onto. It turns out that the matrix A of T can provide this information.

Theorem 5.5.1: Matrix of a One to One or Onto Transformation

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation induced by the $m \times n$ matrix A. Then T is one to one if and only if the rank of A is n. T is onto if and only if the rank of A is m.

Consider Example 5.5.2. Above we showed that T was onto but not one to one. We can now use this theorem to determine this fact about T.

Example 5.5.3: An Onto Transformation

Let $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ be a linear transformation defined by

$$T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a+d \\ b+c \end{bmatrix} \text{ for all } \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$$

Prove that T is onto but not one to one.

Solution

Using Theorem 5.5.1 we can show that T is onto but not one to one from the matrix of T. Recall that to find the matrix A of T, we apply T to each of the standard basis vectors $\vec{e}_i$ of $\mathbb{R}^4$. The result is the 2×4 matrix A given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Fortunately, this matrix is already in reduced row-echelon form. The rank of A is 2. Therefore by the above theorem T is onto but not one to one.

Recall that if S and T are linear transformations, we can discuss their composite denoted $S \circ T$. The following examines what happens if both S and T are onto.

Example 5.5.4: Composite of Onto Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations. If T and S are onto, then $S \circ T$ is onto.

Solution

Let $\vec{z} \in \mathbb{R}^m$. Since S is onto, there exists a vector $\vec{y} \in \mathbb{R}^n$ such that $S(\vec{y}) = \vec{z}$. Furthermore, since T is onto, there exists a vector $\vec{x} \in \mathbb{R}^k$ such that $T(\vec{x}) = \vec{y}$. Thus

$$\vec{z} = S(\vec{y}) = S(T(\vec{x})) = (ST)(\vec{x}),$$

showing that for each $\vec{z} \in \mathbb{R}^m$ there exists and $\vec{x} \in \mathbb{R}^k$ such that $(ST)(\vec{x}) = \vec{z}$. Therefore, $S \circ T$ is onto.

The next example shows the same concept with regards to one-to-one transformations.

Example 5.5.5: Composite of One to One Transformations

Let $T : \mathbb{R}^k \mapsto \mathbb{R}^n$ and $S : \mathbb{R}^n \mapsto \mathbb{R}^m$ be linear transformations. Prove that if T and S are one to one, then $S \circ T$ is one-to-one.

Solution

To prove that $S \circ T$ is one to one, we need to show that if $S(T(\vec{v})) = \vec{0}$ it follows that $\vec{v} = \vec{0}$. Suppose that $S(T(\vec{v})) = \vec{0}$. Since S is one to one, it follows that $T(\vec{v}) = \vec{0}$. Similarly, since T is one to one, it follows that $\vec{v} = \vec{0}$. Hence $S \circ T$ is one to one.

This page titled 5.5: One-to-One and Onto Transformations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **5.5: One-to-One and Onto Transformations** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

5.6: The Kernel and Image of A Linear Map

Outcomes

A. Describe the kernel and image of a linear transformation, and find a basis for each.

In this section we will consider the case where the linear transformation is not necessarily an isomorphism. First consider the following important definition.

Definition 5.6.1: Kernel and Image

Let V and W be subspaces of $\mathbb{R}^n$ and let $T : V \mapsto W$ be a linear transformation. Then the image of T denoted as $\mathbf{im}(T)$ is defined to be the set

$$\mathbf{im}(T) = \{T(\vec{v}) : \vec{v} \in V\}$$

In words, it consists of all vectors in W which equal $T(\vec{v})$ for some $\vec{v} \in V$.

The kernel of T, written $\mathbf{ker}(T)$, consists of all $\vec{v} \in V$ such that $T(\vec{v}) = \vec{0}$. That is,

$$\mathbf{ker}(T) = \left\{\vec{v} \in V : T(\vec{v}) = \vec{0}\right\}$$

It follows that $\mathbf{im}(T)$ and $\mathbf{ker}(T)$ are subspaces of W and V respectively.

Proposition 5.6.1: Kernel and Image as Subspaces

Let V, W be subspaces of $\mathbb{R}^n$ and let $T : V \to W$ be a linear transformation. Then $\mathbf{ker}(T)$ is a subspace of V and $\mathbf{im}(T)$ is a subspace of W.

Proof

First consider $\mathbf{ker}(T)$. It is necessary to show that if $\vec{v}_1, \vec{v}_2$ are vectors in $\mathbf{ker}(T)$ and if a, b are scalars, then $a\vec{v}_1 + b\vec{v}_2$ is also in $\mathbf{ker}(T)$. But

$$T(a\vec{v}_1 + b\vec{v}_2) = aT(\vec{v}_1) + bT(\vec{v}_2) = a\vec{0} + b\vec{0} = \vec{0}$$

Thus $\mathbf{ker}(T)$ is a subspace of V.

Next suppose $T(\vec{v}_1), T(\vec{v}_2)$ are two vectors in $\mathbf{im}(T)$. Then if a, b are scalars,

$$aT(\vec{v}_2) + bT(\vec{v}_2) = T(a\vec{v}_1 + b\vec{v}_2)$$

and this last vector is in $\mathbf{im}(T)$ by definition.

We will now examine how to find the kernel and image of a linear transformation and describe the basis of each.

Example 5.6.1: Kernel and Image of a Linear Transformation

Let $T : \mathbb{R}^4 \mapsto \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix}$$

Then T is a linear transformation. Find a basis for $\mathbf{ker}(T)$ and $\mathbf{im}(T)$.

Solution

You can verify that T is a linear transformation.

First we will find a basis for $\ker(T)$. To do so, we want to find a way to describe all vectors $\vec{x} \in \mathbb{R}^4$ such that $T(\vec{x}) = \vec{0}$. Let $\vec{x} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ be such a vector. Then

$$T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-b \\ c+d \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The values of a, b, c, d that make this true are given by solutions to the system

$$\begin{aligned} a - b &= 0 \\ c + d &= 0 \end{aligned} \tag{5.6.1}$$

The solution to this system is $a = s, b = s, c = t, d = -t$ where s, t are scalars. We can describe $\ker(T)$ as follows.

$$\ker(T) = \left\{ \begin{bmatrix} s \\ s \\ t \\ -t \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

Notice that this set is linearly independent and therefore forms a basis for $\ker(T)$.

We move on to finding a basis for $\text{im}(T)$. We can write the image of T as

$$\text{im}(T) = \left\{ \begin{bmatrix} a-b \\ c+d \end{bmatrix} \right\}$$

We can write this in the form

$$\text{span} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

This set is clearly not linearly independent. By removing unnecessary vectors from the set we can create a linearly independent set with the same span. This gives a basis for $\text{im}(T)$ as

$$\text{im}(T) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Recall that a linear transformation T is called one to one if and only if $T(\vec{x}) = \vec{0}$ implies $\vec{x} = \vec{0}$. Using the concept of kernel, we can state this theorem in another way.

Theorem 5.6.1: One to One and Kernel

Let T be a linear transformation where $\ker(T)$ is the kernel of T. Then T is one to one if and only if $\ker(T)$ consists of **only** the zero vector.

A major result is the relation between the dimension of the kernel and dimension of the image of a linear transformation. In the previous example $\ker(T)$ had dimension 2, and $\text{im}(T)$ also had dimension of 2. Is it a coincidence that the dimension of $\mathbb{M}_{22}$ is $4 = 2 + 2$? Consider the following theorem.

Theorem 5.6.2: Dimension of Kernel and Image

Let $T: V \to W$ be a linear transformation where V, W are subspaces of $\mathbb{R}^n$. Suppose the dimension of V is m. Then

$$m = \dim(\ker(T)) + \dim(\text{im}(T))$$

Proof

From Proposition 5.6.1, $\text{im}(T)$ is a subspace of W. We know that there exists a basis for $\text{im}(T)$, $\{T(\vec{v}_1), \cdots, T(\vec{v}_r)\}$. Similarly, there is a basis for $\ker(T)$, $\{\vec{u}_1, \cdots, \vec{u}_s\}$. Then if $\vec{v} \in V$, there exist scalars c_i such that

$$T(\vec{v}) = \sum_{i=1}^{r} c_i T(\vec{v}_i)$$

Hence $T\left(\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i\right) = 0$. It follows that $\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i$ is in $\ker(T)$. Hence there are scalars a_i such that

$$\vec{v} - \sum_{i=1}^{r} c_i \vec{v}_i = \sum_{j=1}^{s} a_j \vec{u}_j$$

Hence $\vec{v} = \sum_{i=1}^{r} c_i \vec{v}_i + \sum_{j=1}^{s} a_j \vec{u}_j$. Since $\vec{v}$ is arbitrary, it follows that

$$V = \text{span}\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$$

If the vectors $\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$ are linearly independent, then it will follow that this set is a basis. Suppose then that

$$\sum_{i=1}^{r} c_i \vec{v}_i + \sum_{j=1}^{s} a_j \vec{u}_j = 0$$

Apply T to both sides to obtain

$$\sum_{i=1}^{r} c_i T(\vec{v}_i) + \sum_{j=1}^{s} a_j T(\vec{u})_j = \sum_{i=1}^{r} c_i T(\vec{v}_i) = 0$$

Since $\{T(\vec{v}_1), \cdots, T(\vec{v}_r)\}$ is linearly independent, it follows that each $c_i = 0$. Hence $\sum_{j=1}^{s} a_j \vec{u}_j = 0$ and so, since the $\{\vec{u}_1, \cdots, \vec{u}_s\}$ are linearly independent, it follows that each $a_j = 0$ also. Therefore $\{\vec{u}_1, \cdots, \vec{u}_s, \vec{v}_1, \cdots, \vec{v}_r\}$ is a basis for V and so

$$n = s + r = \dim(\ker(T)) + \dim(\text{im}(T))$$

The above theorem leads to the next corollary.

Corollary 5.6.1

Let $T: V \to W$ be a linear transformation where V, W are subspaces of $\mathbb{R}^n$. Suppose the dimension of V is m. Then

$$\dim(\ker(T)) \leq m$$

$$\dim(\text{im}(T)) \leq m$$

This follows directly from the fact that $n = \dim(\ker(T)) + \dim(\text{im}(T))$.

Consider the following example.

Example 5.6.2

Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$$

Then $\text{im}(T) = V$ is a subspace of $\mathbb{R}^3$ and T is an isomorphism of $\mathbb{R}^2$ and V. Find a 2×3 matrix A such that the restriction of multiplication by A to $V = \text{im}(T)$ equals T^{-1}.

Solution

Since the two columns of the above matrix are linearly independent, we conclude that $\dim(\text{im}(T)) = 2$ and therefore $\dim(\ker(T)) = 2 - \dim(\text{im}(T)) = 2 - 2 = 0$ by Theorem 5.6.2. Then by Theorem 5.6.1 it follows that T is one to one.

Thus T is an isomorphism of $\mathbb{R}^2$ and the two dimensional subspace of $\mathbb{R}^3$ which is the span of the columns of the given matrix. Now in particular,

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; T(\vec{e}_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus

$$T^{-1}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \vec{e}_1, \; T^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{e}_2$$

Extend T^{-1} to all of $\mathbb{R}^3$ by defining

$$T^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{e}_1$$

Notice that the vectors

$$\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

are linearly independent so T^{-1} can be extended linearly to yield a linear transformation defined on $\mathbb{R}^3$. The matrix of T^{-1} denoted as A needs to satisfy

$$A\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and so

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so the restriction to V of matrix multiplication by this matrix yields T^{-1}.

This page titled 5.6: The Kernel and Image of A Linear Map is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **5.7: The Kernel and Image of A Linear Map** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

5.7: The Matrix of a Linear Transformation II

Outcomes

A. Find the matrix of a linear transformation with respect to general bases.

We begin this section with an important lemma.

Lemma 5.7.1: Mapping of a Basis

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be an isomorphism. Then T maps any basis of $\mathbb{R}^n$ to another basis for $\mathbb{R}^n$.

Conversely, if $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear transformation which maps a basis of $\mathbb{R}^n$ to another basis of $\mathbb{R}^n$, then it is an isomorphism.

Proof

First, suppose $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear transformation which is one to one and onto. Let $\{\vec{v}_1, \cdots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$. We wish to show that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is also a basis for $\mathbb{R}^n$.

First consider why it is linearly independent. Suppose $\sum_{k=1}^n a_k T(\vec{v}_k) = \vec{0}$. Then by linearity we have $T\left(\sum_{k=1}^n a_k \vec{v}_k\right) = \vec{0}$ and since T is one to one, it follows that $\sum_{k=1}^n a_k \vec{v}_k = \vec{0}$. This requires that each $a_k = 0$ because $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is independent, and it follows that $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is linearly independent.

Next take $\vec{w} \in \mathbb{R}^n$. Since T is onto, there exists $\vec{v} \in \mathbb{R}^n$ such that $T(\vec{v}) = \vec{w}$. Since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is a basis, in particular it is a spanning set and there are scalars b_k such that $T\left(\sum_{k=1}^n b_k \vec{v}_k\right) = T(\vec{v}) = \vec{w}$. Therefore $\vec{w} = \sum_{k=1}^n b_k T(\vec{v}_k)$ which is in the $\operatorname{span}\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$. Therefore, $\{T(\vec{v}_1), \cdots, T(\vec{v}_n)\}$ is a basis as claimed.

Suppose now that $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a linear transformation such that $T(\vec{v}_i) = \vec{w}_i$ where $\{\vec{v}_1, \cdots, \vec{v}_n\}$ and $\{\vec{w}_1, \cdots, \vec{w}_n\}$ are two bases for $\mathbb{R}^n$.

To show that T is one to one, let $T\left(\sum_{k=1}^n c_k \vec{v}_k\right) = \vec{0}$. Then $\sum_{k=1}^n c_k T(\vec{v}_k) = \sum_{k=1}^n c_k \vec{w}_k = \vec{0}$. It follows that each $c_k = 0$ because it is given that $\{\vec{w}_1, \cdots, \vec{w}_n\}$ is linearly independent. Hence $T\left(\sum_{k=1}^n c_k \vec{v}_k\right) = \vec{0}$ implies that $\sum_{k=1}^n c_k \vec{v}_k = \vec{0}$ and so T is one to one.

To show that T is onto, let $\vec{w}$ be an arbitrary vector in $\mathbb{R}^n$. This vector can be written as $\vec{w} = \sum_{k=1}^n d_k \vec{w}_k = \sum_{k=1}^n d_k T(\vec{v}_k) = T\left(\sum_{k=1}^n d_k \vec{v}_k\right)$. Therefore, T is also onto.

Consider now an important definition.

Definition 5.7.1: Coordinate Vector

Let $B = \{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n\}$ be a basis for $\mathbb{R}^n$ and let $\vec{x}$ be an arbitrary vector in $\mathbb{R}^n$. Then $\vec{x}$ is uniquely represented as $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n$ for scalars $a_1, \cdots, a_n$.

The **coordinate vector** of $\vec{x}$ with respect to the basis B, written $C_B(\vec{x})$ or $[\vec{x}]_B$, is given by

$$C_B(\vec{x}) = C_B\left(a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n\right) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

Consider the following example.

Example 5.7.1: Coordinate Vector

Let $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ be a basis of $\mathbb{R}^2$ and let $\vec{x} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ be a vector in $\mathbb{R}^2$. Find $C_B(\vec{x})$.

Solution

First, note the order of the basis is important so label the vectors in the basis B as

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\} = \{\vec{v}_1, \vec{v}_2\}$$

Now we need to find a_1, a_2 such that $\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2$, that is:

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Solving this system gives $a_1 = 2, a_2 = -1$. Therefore the coordinate vector of $\vec{x}$ with respect to the basis B is

$$C_B(\vec{x}) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Given any basis B, one can easily verify that the coordinate function is actually an isomorphism.

Theorem 5.7.1: C_BTransformation is a Linear

For any basis B of $\mathbb{R}^n$, the coordinate function

$$C_B : \mathbb{R}^n \to \mathbb{R}^n$$

is a linear transformation, and moreover an isomorphism.

We now discuss the main result of this section, that is how to represent a linear transformation with respect to different bases.

Theorem 5.7.2: The Matrix of a Linear

Let $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation, and let B_1 and B_2 be bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Then the following holds

$$C_{B_2} T = M_{B_2 B_1} C_{B_1} \tag{5.7.1}$$

where $M_{B_2 B_1}$ is a unique $m \times n$ matrix.

If the basis B_1 is given by $B_1 = \{\vec{v}_1, \cdots, \vec{v}_n\}$ in this order, then

$$M_{B_2 B_1} = \left[C_{B_2}(T(\vec{v}_1)) \; C_{B_2}(T(\vec{v}_2)) \; \cdots C_{B_2}(T(\vec{v}_n)) \right]$$

Proof

The above equation (5.7.1) can be represented by the following diagram.

$$\begin{array}{ccc} & T & \\ \mathbb{R}^n & \to & \mathbb{R}^m \\ C_{B_1} \downarrow & \circ & \downarrow C_{B_2} \\ \mathbb{R}^n & \to & \mathbb{R}^m \\ & M_{B_2 B_1} & \end{array}$$

Since C_{B_1} is an isomorphism, then the matrix we are looking for is the matrix of the linear transformation

$$C_{B_2} T C_{B_1}^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^m.$$

By Theorem 5.2.2, the columns are given by the image of the standard basis $\{\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n\}$. But since $C_{B_1}^{-1}(\vec{e}_i) = \vec{v}_i$, we readily obtain that

$$\begin{aligned} M_{B_2B_1} &= \left[C_{B_2}TC_{B_1}^{-1}(\vec{e}_1)\ C_{B_2}TC_{B_1}^{-1}(\vec{2}_2)\ \cdots\ C_{B_2}TC_{B_1}^{-1}(\vec{e}_n)\right] \\ &= [C_{B_2}(T(\vec{v}_1))\ C_{B_2}(T(\vec{v}_2))\ \cdots\ C_{B_2}(T(\vec{v}_n))] \end{aligned}$$

and this completes the proof.

Consider the following example.

Example 5.7.2: Matrix of a Linear

Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be a linear transformation defined by $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} b \\ a \end{bmatrix}$.

Consider the two bases

$$B_1 = \{\vec{v}_1, \vec{v}_2\} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$$

and

$$B_2 = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

Find the matrix M_{B_2,B_1} of T with respect to the bases B_1 and B_2.

Solution

By Theorem 5.7.2, the columns of $M_{B_2B_1}$ are the coordinate vectors of $T(\vec{v}_1), T(\vec{v}_2)$ with respect to B_2.

Since

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

a standard calculation yields

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(-\frac{1}{2}\right)\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

the first column of $M_{B_2B_1}$ is $\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$.

The second column is found in a similar way. We have

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and with respect to B_2 calculate:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence the second column of $M_{B_2B_1}$ is given by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We thus obtain

$$M_{B_2B_1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

We can verify that this is the correct matrix $M_{B_2B_1}$ on the specific example

$$\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

First applying T gives

$$T(\vec{v}) = T\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

and one can compute that

$$C_{B_2}\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

On the other hand, one compute $C_{B_1}(\vec{v})$ as

$$C_{B_1}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix},$$

and finally applying $M_{B_1B_2}$ gives

$$\begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

as above.

We see that the same vector results from either method, as suggested by Theorem 5.7.2.

If the bases B_1 and B_2 are equal, say B, then we write M_B instead of M_{BB}. The following example illustrates how to compute such a matrix. Note that this is what we did earlier when we considered only $B_1 = B_2$ to be the standard basis.

Example 5.7.3: Matrix of a Linear Transformation with respect to an Arbitrary

Consider the basis B of $\mathbb{R}^3$ given by

$$B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

And let $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ be the linear transformation defined on B as:

$$T\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, T\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, T\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

1. Find the matrix M_B of T relative to the basis B.
2. Then find the usual matrix of T with respect to the standard basis of $\mathbb{R}^3$.

Solution

Equation (5.7.1) gives $C_B T = M_B C_B$, and thus $M_B = C_B T C_B^{-1}$.

Now $C_B(\vec{v}_i) = \vec{e}_i$, so the matrix of C_B^{-1} (with respect to the standard basis) is given by

$$\left[C_B^{-1}(\vec{e}_1) \;\; C_B^{-1}(\vec{e}_2) \;\; C_B^{-1}(\vec{e}_2)\right] = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Moreover the matrix of TC_B^{-1} is given by

$$\left[TC_B^{-1}(\vec{e}_1) \;\; TC_B^{-1}(\vec{e}_2) \;\; TC_B^{-1}(\vec{e}_2)\right] = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Thus

$$\begin{aligned} M_B &= C_B T C_B^{-1} = [C_B^{-1}]^{-1}[TC_B^{-1}] \\ &= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -5 & 1 \\ -1 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \end{aligned}$$

Consider how this works. Let $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ be an arbitrary vector in $\mathbb{R}^3$.

Apply C_B^{-1} to $\vec{b}$ to get

$$b_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Apply T to this linear combination to obtain

$$b_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 + b_2 \\ -b_1 + 2b_2 + b_3 \\ b_1 - b_2 + b_3 \end{bmatrix}$$

Now take the matrix M_B of the transformation (as found above) and multiply it by $\vec{b}$.

$$\begin{bmatrix} 2 & -5 & 1 \\ -1 & 4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2b_1 - 5b_2 + b_3 \\ -b_1 + 4b_2 \\ -2b_2 + b_3 \end{bmatrix}$$

Is this the coordinate vector of the above relative to the given basis? We check as follows.

$$\begin{aligned} &(2b_1 - 5b_2 + b_3) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (-b_1 + 4b_2) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-2b_2 + b_3) \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} b_1 + b_2 \\ -b_1 + 2b_2 + b_3 \\ b_1 - b_2 + b_3 \end{bmatrix} \end{aligned}$$

You see it is the same thing.

Now lets find the matrix of T with respect to the standard basis. Let A be this matrix. That is, multiplication by A is the same as doing T. Thus

$$A \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

Hence

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & -3 \\ -3 & -2 & 4 \end{bmatrix}$$

Of course this is a very different matrix than the matrix of the linear transformation with respect to the non standard basis.

This page titled 5.7: The Matrix of a Linear Transformation II is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **5.8: The Matrix of a Linear Transformation II** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

5.8: The General Solution of a Linear System

Outcomes

A. Use linear transformations to determine the particular solution and general solution to a system of equations.
B. Find the kernel of a linear transformation.

Recall the definition of a linear transformation discussed above. T is a **linear transformation** if whenever $\vec{x}, \vec{y}$ are vectors and k, p are scalars,

$$T\left(k\vec{x}+p\vec{y}\right)=kT\left(\vec{x}\right)+pT\left(\vec{y}\right)$$

Thus linear transformations distribute across addition and pass scalars to the outside.

It turns out that we can use linear transformations to solve linear systems of equations. Indeed given a system of linear equations of the form $A\vec{x}=\vec{b}$, one may rephrase this as $T(\vec{x})=\vec{b}$ where T is the linear transformation T_A induced by the coefficient matrix A. With this in mind consider the following definition.

Definition 5.8.1: Particular Solution of a System of Equations

Suppose a linear system of equations can be written in the form

$$T\left(\vec{x}\right)=\vec{b}$$

If $T\left(\vec{x}_p\right)=\vec{b}$, then $\vec{x}_p$ is called a **particular solution** of the linear system.

Recall that a system is called homogeneous if every equation in the system is equal to 0. Suppose we represent a homogeneous system of equations by $T\left(\vec{x}\right)=0$. It turns out that the $\vec{x}$ for which $T\left(\vec{x}\right)=0$ are part of a special set called the **null space** of T. We may also refer to the null space as the **kernel** of T, and we write $ker\left(T\right)$.

Consider the following definition.

Definition 5.8.2: Null Space or Kernel of a Linear Transformation

Let T be a linear transformation. Define

$$\ker\left(T\right)=\left\{\vec{x}:T\left(\vec{x}\right)=\vec{0}\right\}$$

The kernel, $\ker\left(T\right)$ consists of the set of all vectors $\vec{x}$ for which $T(\vec{x})=\vec{0}$. This is also called the **null space** of T.

We may also refer to the kernel of T as the **solution space** of the equation $T\left(\vec{x}\right)=\vec{0}$.

Consider the following example.

Example 5.8.1: The Kernel of the Derivative

Let $\frac{d}{dx}$ denote the linear transformation defined on f, the functions which are defined on $\mathbb{R}$ and have a continuous derivative. Find $\ker\left(\frac{d}{dx}\right)$.

Solution

The example asks for functions f which the property that $\frac{df}{dx}=0$. As you may know from calculus, these functions are the constant functions. Thus $\ker\left(\frac{d}{dx}\right)$ is the set of constant functions.

Definition 5.8.2 states that $\ker\left(T\right)$ is the set of solutions to the equation,

$$T\left(\vec{x}\right)=\vec{0}$$

Since we can write $T(\vec{x})$ as $A\vec{x}$, you have been solving such equations for quite some time.

We have spent a lot of time finding solutions to systems of equations in general, as well as homogeneous systems. Suppose we look at a system given by $A\vec{x} = \vec{b}$, and consider the related homogeneous system. By this, we mean that we replace $\vec{b}$ by $\vec{0}$ and look at $A\vec{x} = \vec{0}$. It turns out that there is a very important relationship between the solutions of the original system and the solutions of the associated homogeneous system. In the following theorem, we use linear transformations to denote a system of equations. Remember that $T(\vec{x}) = A\vec{x}$.

Theorem 5.8.1: Particular Solution and General Solution

Suppose $\vec{x}_p$ is a solution to the linear system given by ,

$$T(\vec{x}) = \vec{b}$$

Then if $\vec{y}$ is any other solution to $T(\vec{x}) = \vec{b}$, there exists $\vec{x}_0 \in \ker(T)$ such that

$$\vec{y} = \vec{x}_p + \vec{x}_0$$

Hence, every solution to the linear system can be written as a sum of a particular solution, $\vec{x}_p$, and a solution $\vec{x}_0$ to the associated homogeneous system given by $T(\vec{x}) = \vec{0}$.

Proof

Consider $\vec{y} - \vec{x}_p = \vec{y} + (-1)\vec{x}_p$. Then $T(\vec{y} - \vec{x}_p) = T(\vec{y}) - T(\vec{x}_p)$. Since $\vec{y}$ and $\vec{x}_p$ are both solutions to the system, it follows that $T(\vec{y}) = \vec{b}$ and $T(\vec{x}_p) = \vec{b}$.

Hence, $T(\vec{y}) - T(\vec{x}_p) = \vec{b} - \vec{b} = \vec{0}$. Let $\vec{x}_0 = \vec{y} - \vec{x}_p$. Then, $T(\vec{x}_0) = \vec{0}$ so $\vec{x}_0$ is a solution to the associated homogeneous system and so is in $\ker(T)$.

Sometimes people remember the above theorem in the following form. The solutions to the system $T(\vec{x}) = \vec{b}$ are given by $\vec{x}_p + \ker(T)$ where $\vec{x}_p$ is a particular solution to $T(\vec{x}) = \vec{b}$.

For now, we have been speaking about the kernel or null space of a linear transformation T. However, we know that every linear transformation T is determined by some matrix A. Therefore, we can also speak about the null space of a matrix. Consider the following example.

Example 5.8.2: The Null Space of a Matrix

Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix}$$

Find $\text{null}(A)$. Equivalently, find the solutions to the system of equations $A\vec{x} = \vec{0}$.

Solution

We are asked to find $\left\{\vec{x} : A\vec{x} = \vec{0}\right\}$. In other words we want to solve the system, $A\vec{x} = \vec{0}$. Let $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$. Then this amounts to solving

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is the linear system

$$x+2y+3z=0$$
$$2x+y+z+2w=0$$
$$4x+5y+7z+2w=0$$

To solve, set up the augmented matrix and row reduce to find the reduced row-echelon form.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 0 \\ 2 & 1 & 1 & 2 & 0 \\ 4 & 5 & 7 & 2 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cccc|c} 1 & 0 & -\frac{1}{3} & \frac{4}{3} & 0 \\ 0 & 1 & \frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This yields $x=\frac{1}{3}z-\frac{4}{3}w$ and $y=\frac{2}{3}w-\frac{5}{3}z$. Since $\mathbf{null}\,(A)$ consists of the solutions to this system, it consists vectors of the form,

$$\left[\begin{array}{c} \frac{1}{3}z-\frac{4}{3}w \\ \frac{2}{3}w-\frac{5}{3}z \\ z \\ w \end{array}\right] = z\left[\begin{array}{c} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{array}\right] + w\left[\begin{array}{c} -\frac{4}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{array}\right]$$

Consider the following example.

Example 5.8.3: A General Solution

The **general solution** of a linear system of equations is the set of all possible solutions. Find the general solution to the linear system,

$$\left[\begin{array}{cccc} 1 & 2 & 3 & 0 \\ 2 & 1 & 1 & 2 \\ 4 & 5 & 7 & 2 \end{array}\right]\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] = \left[\begin{array}{c} 9 \\ 7 \\ 25 \end{array}\right]$$

given that $\left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \end{array}\right]$ is one solution.

Solution

Note the matrix of this system is the same as the matrix in Example 5.8.2. Therefore, from Theorem 5.8.1, you will obtain all solutions to the above linear system by adding a particular solution $\vec{x}_p$ to the solutions of the associated homogeneous system, $\vec{x}$. One particular solution is given above by

$$\vec{x}_p = \left[\begin{array}{c} x \\ y \\ z \\ w \end{array}\right] = \left[\begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \end{array}\right]$$

Using this particular solution along with the solutions found in Example 5.8.2, we obtain the following solutions,

$$z\left[\begin{array}{c} \frac{1}{3} \\ -\frac{5}{3} \\ 1 \\ 0 \end{array}\right] + w\left[\begin{array}{c} -\frac{4}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{array}\right] + \left[\begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \end{array}\right]$$

Hence, any solution to the above linear system is of this form.

This page titled 5.8: The General Solution of a Linear System is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **5.9: The General Solution of a Linear System** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

5.9: Exercises

Exercise 5.9.1

Show the map T: $\mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$ where A is an $m \times n$ matrix and $\vec{x}$ is an $m \times 1$ column vector is a linear transformation.

Answer

This result follows from the properties of matrix multiplication.

Exercise 5.9.2

Show that the function $T_{\vec{u}}$ defined by $T_{\vec{u}}(\vec{v}) = \vec{v} - proj_{\vec{u}}(\vec{v})$ is also a linear transformation.

Answer

$$\begin{aligned} T_{\vec{u}}(a\vec{v} + b\vec{w}) &= a\vec{v} + b\vec{w} - \frac{(a\vec{v} + b\vec{w} \bullet \vec{u})}{||\vec{u}||^2}\vec{u} \\ &= a\vec{v} - a\frac{(\vec{v} \bullet \vec{u})}{||\vec{u}||^2}\vec{u} + b\vec{w} - b\frac{(\vec{w} \bullet \vec{u})}{||\vec{u}||^2}\vec{u} \\ &= aT_{\vec{u}}(\vec{v}) + bT_{\vec{u}}(\vec{w}) \end{aligned} \tag{5.9.1}$$

Exercise 5.9.3

Let $\vec{u}$ be a fixed vector. The function $T_{\vec{u}}$ defined by $T_{\vec{u}}\vec{v} = \vec{u} + \vec{v}$ has the effect of translating all vectors by adding $\vec{u} \neq \vec{0}$. Show this is not a linear transformation. Explain why it is not possible to represent $T_{\vec{u}}$ in $\mathbb{R}^3$ by multiplying by a 3×3 matrix.

Answer

Linear transformations take $\vec{0}$ to $\vec{0}$ which T does not. Also $T_{\vec{a}}(\vec{u} + \vec{v}) \neq T_{\vec{a}}\vec{u} + T_{\vec{a}}\vec{v}$.

Exercise 5.9.4

Consider the following functions which map $\mathbb{R}^n$ to $\mathbb{R}^n$.

a. T multiplies the jth component of $\vec{x}$ by a nonzero number b.
b. T replaces the ith component of $\vec{x}$ with b times the jth component added to the ith component.
c. T switches the ith and jth components.

Show these functions are linear transformations and describe their matrices A such that $T(\vec{x}) = A\vec{x}$.

Answer

a. The matrix of T is the elementary matrix which multiplies the jth diagonal entry of the identity matrix by b.
b. The matrix of T is the elementary matrix which takes b times the jth row and adds to the ith row.
c. The matrix of T is the elementary matrix which switches the ith and the jth rows where the two components are in the i th and jth positions.

Exercise 5.9.5

You are given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ and you know that

$$T(A_i) = B_i$$

where $[A_1 \cdots a_n]^{-1}$ exists. Show that the matrix of T is of the form

$$[B_1 \cdots B_n][A_1 \cdots A_n]^{-1}$$

Answer

Suppose

$$\begin{bmatrix} \vec{c}_1^T \\ \vdots \\ \vec{c}_n^T \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1}$$

Thus $\vec{c}_i^T \vec{a}_j = \delta_{ij}$. Therefore

$$\begin{aligned} \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1} \vec{a}_i &= \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} \vec{c}_1^T \\ \vdots \\ \vec{c}_n^T \end{bmatrix} \vec{a}_i \\ &= \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \vec{e}_i \\ &= \vec{b}_i \end{aligned} \tag{5.9.2}$$

Thus $T\vec{a}_i = \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \end{bmatrix} \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}^{-1} \vec{a}_i = A\vec{a}_i$. If $\vec{x}$ is arbitrary, then since the matrix $\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$ is invertible, there exists a unique $\vec{y}$ such that $\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix} \vec{y} = \vec{x}$ Hence

$$T\vec{x} = T\left(\sum_{i=1}^n y_i \vec{a}_i\right) = \sum_{i=1}^n y_i T\vec{a}_i = \sum_{i=1}^n y_i A\vec{a}_i = A\left(\sum_{i=1}^n y_i \vec{a}_i\right) = A\vec{x}$$

Exercise 5.9.6

Suppose T is a linear transformation such that

$$\begin{aligned} T\begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix} &= \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} \\ T\begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \\ T\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \end{aligned} \tag{5.9.3}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Answer

$$\begin{bmatrix} 5 & 1 & 5 \\ 1 & 1 & 3 \\ 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 4 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 37 & 17 & 11 \\ 17 & 7 & 5 \\ 11 & 14 & 6 \end{bmatrix}$$

Exercise 5.9.7

Suppose T is a linear transformation such that

$$T\begin{bmatrix}1\\1\\-8\end{bmatrix}=\begin{bmatrix}1\\3\\1\end{bmatrix}$$
$$T\begin{bmatrix}-1\\0\\6\end{bmatrix}=\begin{bmatrix}2\\4\\1\end{bmatrix} \tag{5.9.4}$$
$$T\begin{bmatrix}0\\-1\\3\end{bmatrix}=\begin{bmatrix}6\\1\\-1\end{bmatrix}$$

Find the matrix of T. That is find A such that $T(\vec{x})=A\vec{x}$.

Answer

$$\begin{bmatrix}1&2&6\\3&4&1\\1&1&-1\end{bmatrix}\begin{bmatrix}6&3&1\\5&3&1\\6&2&1\end{bmatrix}=\begin{bmatrix}52&21&9\\44&23&8\\5&4&1\end{bmatrix}$$

Exercise 5.9.8

Suppose T is a linear transformation such that

$$T\begin{bmatrix}1\\3\\-7\end{bmatrix}=\begin{bmatrix}-3\\1\\3\end{bmatrix}$$
$$T\begin{bmatrix}-1\\-2\\6\end{bmatrix}=\begin{bmatrix}1\\3\\-3\end{bmatrix} \tag{5.9.5}$$
$$T\begin{bmatrix}0\\-1\\2\end{bmatrix}=\begin{bmatrix}5\\3\\-3\end{bmatrix}$$

Find the matrix of T. That is find A such that $T(\vec{x})=A\vec{x}$.

Answer

$$\begin{bmatrix}-3&1&5\\1&3&3\\3&-3&-3\end{bmatrix}\begin{bmatrix}2&2&1\\1&2&1\\4&1&1\end{bmatrix}=\begin{bmatrix}15&1&3\\17&11&7\\-9&-3&-3\end{bmatrix}$$

Exercise 5.9.9

Suppose T is a linear transformation such that

$$T\begin{bmatrix}1\\1\\-7\end{bmatrix}=\begin{bmatrix}3\\3\\3\end{bmatrix}$$
$$T\begin{bmatrix}-1\\0\\6\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix} \tag{5.9.6}$$
$$T\begin{bmatrix}0\\-1\\2\end{bmatrix}=\begin{bmatrix}1\\3\\-1\end{bmatrix}$$

Find the matrix of T. That is find A such that $T(\vec{x})=A\vec{x}$.

Answer

$$\begin{bmatrix} 3 & 1 & 1 \\ 3 & 2 & 3 \\ 3 & 3 & -1 \end{bmatrix} \begin{bmatrix} 6 & 2 & 1 \\ 5 & 2 & 1 \\ 6 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 29 & 9 & 5 \\ 46 & 13 & 8 \\ 27 & 11 & 5 \end{bmatrix}$$

Exercise 5.9.10

Suppose T is a linear transformation such that

$$\begin{aligned} T\begin{bmatrix} 1 \\ 2 \\ -18 \end{bmatrix} &= \begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix} \\ T\begin{bmatrix} -1 \\ -1 \\ 15 \end{bmatrix} &= \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \\ T\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} &= \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix} \end{aligned} \tag{5.9.7}$$

Find the matrix of T. That is find A such that $T(\vec{x}) = A\vec{x}$.

Answer

$$\begin{bmatrix} 5 & 3 & 2 \\ 2 & 3 & 5 \\ 5 & 5 & -2 \end{bmatrix} \begin{bmatrix} 11 & 4 & 1 \\ 10 & 4 & 1 \\ 12 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 109 & 38 & 10 \\ 112 & 35 & 10 \\ 81 & 34 & 8 \end{bmatrix}$$

Exercise 5.9.11

Consider the following functions $T : \mathbb{R}^3 \to \mathbb{R}^2$. Show that each is a linear transformation and determine for each the matrix A such that $T(\vec{x}) = A\vec{x}$.

a. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z \\ 2y - 3x + z \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7x + 2y + z \\ 3x - 11y + 2z \end{bmatrix}$

c. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x + 2y + z \\ x + 2y + 6z \end{bmatrix}$

d. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y - 5x + z \\ x + y + z \end{bmatrix}$

Exercise 5.9.12

Consider the following functions $T : \mathbb{R}^3 \to \mathbb{R}^2$. Explain why each of these functions T is not linear.

a. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + 3z + 1 \\ 2y - 3x + z \end{bmatrix}$

b. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y^2 + 3z \\ 2y + 3x + z \end{bmatrix}$

c. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sin x + 2y + 3z \\ 2y + 3x + z \end{bmatrix}$

d. $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+2y+3z \\ 2y+3x-\ln z \end{bmatrix}$

Exercise 5.9.13

Suppose

$$[A_1 \cdots A_n]^{-1}$$

exists where each $A_j \in \mathbb{R}^n$ and let vectors $\{B_1, \cdots, B_n\}$ in $\mathbb{R}^m$ be given. Show that there **always** exists a linear transformation T such that $T(A_i) = B_i$.

Exercise 5.9.14

Find the matrix for $T(\vec{w}) = proj_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T$.

Answer

Recall that $proj_{\vec{u}}(\vec{v}) = \frac{\vec{v} \bullet \vec{u}}{\|\vec{u}\|^2}\vec{u}$ and so the desired matrix has ith column equal to $proj_{\vec{u}}(\vec{e}_i)$. Therefore, the matrix desired is

$$\frac{1}{14}\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}$$

Exercise 5.9.15

Find the matrix for $T(\vec{w}) = proj_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & 5 & 3 \end{bmatrix}^T$.

Answer

$$\frac{1}{35}\begin{bmatrix} 1 & 5 & 3 \\ 5 & 25 & 15 \\ 3 & 15 & 9 \end{bmatrix}$$

Exercise 5.9.16

Find the matrix for $T(\vec{w}) = proj_{\vec{v}}(\vec{w})$ where $\vec{v} = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}^T$.

Answer

$$\frac{1}{10}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 9 \end{bmatrix}$$

Exercise 5.9.17

Show that if a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then it is always the case that $T(\vec{0}) = \vec{0}$.

Exercise 5.9.18

Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ and S a linear transformation induced by $B = \begin{bmatrix} 0 & -2 \\ 4 & 2 \end{bmatrix}$. Find matrix of $S \circ T$ and find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Answer

The matrix of $S \circ T$ is given by BA.

$$\begin{bmatrix} 0 & -2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 10 & 8 \end{bmatrix}$$

Now, $(S \circ T)(\vec{x}) = (BA)\vec{x}$.

$$\begin{bmatrix} 2 & -4 \\ 10 & 8 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}$$

? Exercise 5.9.19

Let T be a linear transformation and suppose $T\left(\begin{bmatrix} 1 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Suppose S is a linear transformation induced by the matrix $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. Find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$.

Answer

To find $(S \circ T)(\vec{x})$ we compute $S(T(\vec{x}))$.

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ -11 \end{bmatrix}$$

? Exercise 5.9.20

Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ and S a lienar transformation induced by $B = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$. Find matrix of $S \circ T$ and find $(S \circ T)(\vec{x})$ for $\vec{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

? Exercise 5.9.21

Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$. Find the matrix of T^{-1}.

Answer

The matrix of T^{-1} is A^{-1}.

$$\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}$$

? Exercise 5.9.22

Let T be a linear transformation induced by the matrix $A = \begin{bmatrix} 4 & -3 \\ 2 & -2 \end{bmatrix}$. Find the matrix of T^{-1}.

? Exercise 5.9.23

Let T be a linear transformation and suppose $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 9 \\ 8 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$. Find the matrix of T^{-1}.

Exercise 5.9.24

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$.

Answer

$$\begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

Exercise 5.9.25

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/4$.

Answer

$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

Exercise 5.9.26

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $-\pi/3$.

Answer

$$\begin{bmatrix} \cos\left(-\frac{\pi}{3}\right) & -\sin\left(-\frac{\pi}{3}\right) \\ \sin\left(-\frac{\pi}{3}\right) & \cos\left(-\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

Exercise 5.9.27

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $2\pi/3$.

Answer

$$\begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}$$

Exercise 5.9.28

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/12$. Hint: Note that $\pi/12 = \pi/3 - \pi/4$.

Answer

$$\begin{aligned} &\begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} \begin{bmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4}\sqrt{2}\sqrt{3} + \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2} - \frac{1}{4}\sqrt{2}\sqrt{3} \\ \frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2}\sqrt{3} + \frac{1}{4}\sqrt{2} \end{bmatrix} \end{aligned} \tag{5.9.8}$$

Exercise 5.9.29

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $2\pi/3$ and then reflects across the x axis.

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{bmatrix}$$

? Exercise 5.9.30

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/3$ and then reflects across the x axis.

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}$$

? Exercise 5.9.31

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/4$ and then reflects across the x axis.

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

? Exercise 5.9.32

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $\pi/6$ and then reflects across the x axis followed by a reflection across the y axis.

Answer

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

? Exercise 5.9.33

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ across the x axis and then rotates every vector through an angle of $\pi/4$.

Answer

$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix}$$

? Exercise 5.9.34

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ across the y axis and then rotates every vector through an angle of $\pi/4$.

Answer

$$\begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

Exercise 5.9.35

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ across the x axis and then rotates every vector through an angle of $\pi/6$.

Answer

$$\begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\sqrt{3} \end{bmatrix}$$

Exercise 5.9.36

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ across the y axis and then rotates every vector through an angle of $\pi/6$.

Answer

$$\begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\sqrt{3} \end{bmatrix}$$

Exercise 5.9.37

Find the matrix for the linear transformation which rotates every vector in $\mathbb{R}^2$ through an angle of $5\pi/12$. Hint: Note that $5\pi/12 = 2\pi/3 - \pi/4$.

Answer

$$\begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} \begin{bmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{4}\sqrt{2} & -\frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{4}\sqrt{2} \\ \frac{1}{4}\sqrt{2}\sqrt{3} + \frac{1}{4}\sqrt{2} & \frac{1}{4}\sqrt{2}\sqrt{3} - \frac{1}{4}\sqrt{2} \end{bmatrix}$$

Note that it doesn't matter about the order in this case.

Exercise 5.9.38

Find the matrix of the linear transformation which rotates every vector in $\mathbb{R}^3$ counter clockwise about the z axis when viewed from the positive z axis through an angle of $30°$ and then reflects through the xy plane.

Answer

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) & 0 \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2}\sqrt{3} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Exercise 5.9.39

Let $\vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ be a unit vector in $\mathbb{R}^2$. Find the matrix which reflects all vectors across this vector, as shown in the following picture.

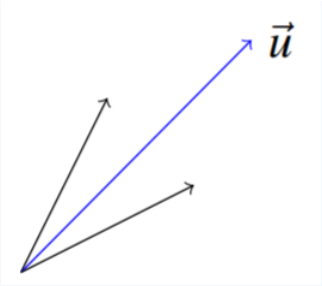


Figure 5.9.1

Hint: Notice that $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ for some θ. First rotate through $-\theta$. Next reflect through the x axis. Finally rotate through θ.

Answer

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\cos\theta\sin\theta \\ 2\cos\theta\sin\theta & \sin^2\theta - \cos^2\theta \end{bmatrix} \tag{5.9.9}$$

Now to write in terms of (a,b), note that $a/\sqrt{a^2+b^2} = \cos\theta$, $b/\sqrt{a^2+b^2} = \sin\theta$. Now plug this in to the above. The result is

$$\begin{bmatrix} \frac{a^2-b^2}{a^2+b^2} & 2\frac{ab}{a^2+b^2} \\ 2\frac{ab}{a^2+b^2} & \frac{b^2-a^2}{a^2+b^2} \end{bmatrix} = \frac{1}{a^2+b^2}\begin{bmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{bmatrix}$$

Since this is a unit vector, $a^2+b^2=1$ and so you get

$$\begin{bmatrix} a^2-b^2 & 2ab \\ 2ab & b^2-a^2 \end{bmatrix}$$

? Exercise 5.9.40

Let T be a linear transformation given by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Is T one to one? Is T onto?

? Exercise 5.9.41

Let T be a linear transformation given by

$$T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Is T one to one? Is T onto?

? Exercise 5.9.42

Let T be a linear transformation given by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Is T one to one? Is T onto?

Exercise 5.9.43

Let T be a linear transformation given by

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & -5 \\ 2 & 0 & 2 \\ 2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Is T one to one? Is T onto?

Exercise 5.9.44

Give an example of a 3×2 matrix with the property that the linear transformation determined by this matrix is one to one but not onto.

Answer

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Exercise 5.9.45

Suppose A is an $m \times n$ matrix in which $m \leq n$. Suppose also that the rank of A equals m. Show that the transformation T determined by A maps $\mathbb{R}^n$ onto $\mathbb{R}^m$. Hint: The vectors $\vec{e}_1, \cdots, \vec{e}_m$ occur as columns in the reduced row-echelon form for A.

Answer

This says that the columns of A have a subset of m vectors which are linearly independent. Therefore, this set of vectors is a basis for $\mathbb{R}^m$. It follows that the span of the columns is all of $\mathbb{R}^m$. Thus A is onto.

Exercise 5.9.46

Suppose A is an $m \times n$ matrix in which $m \geq n$. Suppose also that the rank of A equals n. Show that A is one to one. Hint: If not, there exists a vector, $\vec{x}$ such that $A\vec{x} = 0$, and this implies at least one column of A is a linear combination of the others. Show this would require the rank to be less than n.

Answer

The columns are independent. Therefore, A is one to one.

Exercise 5.9.47

Explain why an $n \times n$ matrix A is both one to one and onto if and only if its rank is n.

Answer

The rank is n is the same as saying the columns are independent which is the same as saying A is one to one which is the same as saying the columns are a basis. Thus the span of the columns of A is all of $\mathbb{R}^n$ and so A is onto. If A is onto, then the columns must be linearly independent since otherwise the span of these columns would have dimension less than n and so the dimension of $\mathbb{R}^n$ would be less than n.

Exercise 5.9.48

Let V and W be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and let $T : V \to W$ be a linear transformation. Suppose that $\{T\vec{v}_1, \cdots, T\vec{v}_r\}$ is linearly independent. Show that it must be the case that $\{T\vec{v}_1, \cdots, T\vec{v}_r\}$ is also linearly independent.

Answer

If $\sum_i^r a_i\vec{v}_r = 0$, then using linearity properties of T we get

$$0 = T(0) = T\left(\sum_i^r a_i\vec{v}_r\right) = \sum_i^r a_i T(\vec{v}_r).$$

Since we assume that $\{T\vec{v}_a, \cdots, T\vec{v}_r\}$ is linearly independent, we must have all $a_i = 0$, and therefore we conclude that $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is also linearly independent.

Exercise 5.9.49

Let

$$V = span\left\{\begin{bmatrix}1\\1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}\right\}$$

Let $T\vec{x} = A\vec{x}$ where A is the matrix

$$\begin{bmatrix}1&1&1&1\\0&1&1&0\\0&1&2&1\\1&1&1&2\end{bmatrix}$$

Give a basis for $im(T)$.

Exercise 5.9.50

Let

$$V = span\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\4\\4\\1\end{bmatrix}\right\}$$

Let $T\vec{x} = A\vec{x}$ where A is the matrix

$$\begin{bmatrix}1&1&1&1\\0&1&1&0\\0&1&2&1\\1&1&1&2\end{bmatrix}$$

Find a basis for $im(T)$. In this case, the original vectors do not form an independent set.

Answer

Since the third vector is a linear combinations of the first two, then the image of the third vector will also be a linear combinations of the image of the first two. However the image of the first two vectors are linearly independent (check!), and hence form a basis of the image.

Thus a basis for $im(T)$ is:

$$V = span\left\{\begin{bmatrix}2\\0\\1\\3\end{bmatrix}, \begin{bmatrix}4\\2\\4\\5\end{bmatrix}\right\}$$

Exercise 5.9.51

If $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent and T is a one to one linear transformation, show that $\{T\vec{v}_1, \cdots, T\vec{v}_r\}$ is also linearly independent. Give an example which shows that if T is only linear, it can happen that, although $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is linearly independent, $\{T\vec{v}_1, \cdots, T\vec{v}_r\}$ is not. In fact, show that it can happen that each of the $T\vec{v}_j$ equals 0.

Exercise 5.9.52

Let V and W be subspaces of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and let $T : V \to W$ be a linear transformation. Show that if T is onto W and if $\{\vec{v}_1, \cdots, \vec{v}_r\}$ is a basis for V, then $span\{T\vec{v}_1, \cdots, T\vec{v}_r\} = W$.

Exercise 5.9.53

Define T: $\mathbb{R}^4 \to \mathbb{R}^3$ as follows.

$$T\vec{x} = \begin{bmatrix}3 & 2 & 1 & 8\\2 & 2 & -2 & 6\\1 & 1 & -1 & 3\end{bmatrix}\vec{x}$$

Find a basis for $im(T)$. Also find a basis for $\ker(T)$.

Exercise 5.9.54

Define T: $\mathbb{R}^3 \to \mathbb{R}^3$ as follows.

$$T\vec{x} = \begin{bmatrix}1 & 2 & 0\\1 & 1 & 1\\0 & 1 & 1\end{bmatrix}\vec{x}$$

where on the right, it is just matrix multiplication of the vector $\vec{x}$ which is meant. Explain why T is an isomorphism of $\mathbb{R}^3$ to $\mathbb{R}^3$.

Exercise 5.9.55

Suppose T: $\mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation given by

$$T\vec{x} = A\vec{x}$$

where A is a 3×3 matrix. Show that T is an isomorphism if and only if A is invertible.

Exercise 5.9.56

Suppose T: $\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation given by

$$T\vec{x} = A\vec{x}$$

where A is an $m \times n$ matrix. Show that T is never an ismorphism if $m \neq n$. In particular, show that if $m > n$, T cannot be onto and if $m < n$, then T cannot be one to one.

Exercise 5.9.57

Define T: $\mathbb{R}^2 \to \mathbb{R}^3$ as follows.

$$T\vec{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \vec{x}$$

where on the right, it is just matrix multiplication of the vector $\vec{x}$ which is meant. Show that T is one to one. Next let $W = im(T)$. Show that T is an isomorphism of $\mathbb{R}^2$ and $im(T)$.

Exercise 5.9.58

In the above problem, find a 2×3 matrix A such that the restriction of A to $im(T)$ gives the same result as T^{-1} on $im(T)$. Hint: You might let A be such that

$$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, A \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

now find another vector $\vec{v} \in \mathbb{R}^3$ such that

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v} \right\}$$

is a basis. You could pick

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

for example. Explain why this one works or one of your choice works. Then you could define $A\vec{v}$ to equal some vector in $\mathbb{R}^2$. Explain why there will be more than one such matrix A which will deliver the inverse isomorphism T^{-1} on $im(T)$.

Exercise 5.9.59

Now let V equan $span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ and let T: $V \to W$ be a linear transformation where

$$W = span \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

and

$$T \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Explain why T is an isomorphism. Determine a matrix A which, when multiplied on the left gives the same result as T on V and a matrix B which delivers T^{-1} on W. Hint: You need to have

$$A\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\\1&1\\0&1\end{bmatrix}$$

Now enlarge $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$ to obtain a basis for $\mathbb{R}^3$. You could add in $\begin{bmatrix}0\\0\\1\end{bmatrix}$ for example, and then pick another vector in $\mathbb{R}^4$ and let $A\begin{bmatrix}0\\0\\1\end{bmatrix}$ equal this other vector. Then you would have

$$A\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\1&1&0\\0&1&1\end{bmatrix}$$

This would involve picking for the new vector in $\mathbb{R}^4$ the vector $\begin{bmatrix}0&0&0&1\end{bmatrix}^T$. Then you could find A. You can do something similar to find a matrix for T^{-1} denoted as B.

? Exercise 5.9.60

Let $V=\mathbb{R}^3$ and let

$$W=span(S),\ \text{where } S=\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix},\begin{bmatrix}-2\\2\\-2\end{bmatrix},\begin{bmatrix}-1\\1\\1\end{bmatrix},\begin{bmatrix}1\\-1\\3\end{bmatrix}\right\}$$

Find a basis of W consisting of vectors in S.

Answer

In this case $\dim(W)=1$ and a basis for W consisting of vectors in S can be obtained by taking any (nonzero) vector from S.

? Exercise 5.9.61

Let T be a linear transformation given by

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Find a basis for $\ker(T)$ and $im(T)$.

Answer

A basis for $\ker(T)$ is $\left\{\begin{bmatrix}1\\-1\end{bmatrix}\right\}$ and a basis for $im(T)$ is $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$. There are many other possibilities for the specific bases, but in this case $\dim(\ker(T))=1$ and $\dim(im(T))=1$.

? Exercise 5.9.62

Let T be a linear transformation given by

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Find a basis for $\ker(T)$ and $im(T)$.

Answer

In this case $\ker(T) = \{0\}$ and $im(T) = \mathbb{R}^2$ (pick any basis of $\mathbb{R}^2$).

Exercise 5.9.63

Let $V = \mathbb{R}^3$ and let

$$W = span\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\-1\end{bmatrix}\right\}$$

Extend this basis of W to a basis of V.

Answer

There are many possible such extensions, one is (how do we know?):

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

Exercise 5.9.64

Let T be a linear transformation given by

$$T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

What is $\dim(\ker(T))$?

Answer

We can easily see that $\dim(im(T)) = 1$, and thus $\dim(\ker(T)) = 3 - \dim(im(T)) = 3 - 1 = 2$.

Exercise 5.9.65

Let $B = \left\{\begin{bmatrix}2\\-1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}\right\}$ be a basis of $\mathbb{R}^2$ and let $\vec{x} = \begin{bmatrix}5\\-7\end{bmatrix}$ be a vector in $\mathbb{R}^2$. Find $C_B(\vec{x})$.

Exercise 5.9.66

Let $B = \left\{\begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}2\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\0\\2\end{bmatrix}\right\}$ be a basis of $\mathbb{R}^3$ and let $\vec{x} = \begin{bmatrix}5\\-1\\4\end{bmatrix}$ be a vector in $\mathbb{R}^2$. Find $C_B(\vec{x})$.

Answer

$$C_B(\vec{x}) = \begin{bmatrix}2\\1\\-1\end{bmatrix}$$

Exercise 5.9.67

Let T: $\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by $T\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = \begin{bmatrix}a+b\\a-b\end{bmatrix}$.

Consider the two bases

$$B_1 = \{\vec{v}_1, \vec{v}_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

and

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Find the matrix M_{B_2,B_1} of T with respect to the bases B_1 and B_2.

Answer

$M_{B_2B_1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

? Exercise 5.9.68

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} -3\hat{t} \\ -\hat{t} \\ \hat{t} \end{bmatrix}$, $\hat{t}_3 \in \mathbb{R}$. A basis for the solution space is $\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$

? Exercise 5.9.69

Using Exercise 5.9.68 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 3 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Answer

Note that this has the same matrix as the above problem. Solution is: $\begin{bmatrix} -3\hat{t}_3 \\ -\hat{t}_3 \\ \hat{t}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $\hat{t}_3 \in \mathbb{R}$

? Exercise 5.9.70

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} 3\hat{t} \\ 2\hat{t} \\ \hat{t} \end{bmatrix}$, A basis is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Exercise 5.9.71

Using Exercise **5.9.70** find the general solution to the following linear system.

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} 3\hat{t} \\ 2\hat{t} \\ \hat{t} \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \\ 0 \end{bmatrix}, \hat{t} \in \mathbb{R}$

Exercise 5.9.72

Write the solution set of the following system as a linear combination of vectors.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} -4\hat{t} \\ -2\hat{t} \\ \hat{t} \end{bmatrix}$. A basis is $\begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}$

Exercise 5.9.73

Using Exercise 5.9.72 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} -4\hat{t} \\ -2\hat{t} \\ \hat{t} \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \hat{t} \in \mathbb{R}$

Exercise 5.9.74

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} -\hat{t} \\ 2\hat{t} \\ \hat{t} \end{bmatrix}, \hat{t} \in \mathbb{R}$

Exercise 5.9.75

Using Exercise 5.9.74 find the general solution to the following linear system.

$$\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ 1 & -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} -\hat{t} \\ 2\hat{t} \\ \hat{t} \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

Exercise 5.9.76

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & -1 & 3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} 0 \\ -\hat{t} \\ -\hat{t} \\ \hat{t} \end{bmatrix}, \hat{t} \in \mathbb{R}$

Exercise 5.9.77

Using Exercise 5.9.76 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & -1 & 3 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} 0 \\ -\hat{t} \\ -\hat{t} \\ \hat{t} \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$

Exercise 5.9.78

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} -s-t \\ s \\ s \\ t \end{bmatrix}$, $s,t \in \mathbb{R}$. A basis is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exercise 5.9.79

Using Exercise 5.9.78 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 0 \end{bmatrix}$$

Answer

Solution is:

$$\begin{bmatrix} -\hat{t} \\ \hat{t} \\ \hat{t} \\ 0 \end{bmatrix} + \begin{bmatrix} -8 \\ 5 \\ 0 \\ 5 \end{bmatrix}$$

Exercise 5.9.80

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer

Solution is:

$$\begin{bmatrix} -\frac{1}{2}s - \frac{1}{2}t \\ \frac{1}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix}$$

for $s,t \in \mathbb{R}$. A basis is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exercise 5.9.81

Using Exercise 5.9.80 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 1 & 2 \\ 3 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 3 \end{bmatrix}$$

Answer

Solution is:

$$\begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2}s - \frac{1}{2}t \\ \frac{1}{2}s - \frac{1}{2}t \\ s \\ t \end{bmatrix}$$

Exercise 5.9.82

Write the solution set of the following system as a linear combination of vectors

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} -\hat{t} \\ \hat{t} \\ \hat{t} \\ 0 \end{bmatrix}$, a basis is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

Exercise 5.9.83

Using Exercise 5.9.82 find the general solution to the following linear system.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}$$

Answer

Solution is: $\begin{bmatrix} -\hat{t} \\ \hat{t} \\ \hat{t} \\ 0 \end{bmatrix} + \begin{bmatrix} -9 \\ 5 \\ 0 \\ 6 \end{bmatrix}, t \in \mathbb{R}$.

Exercise 5.9.84

Suppose $A\vec{x} = \vec{b}$ has a solution. Explain why the solution is unique precisely when $A\vec{x} = \vec{0}$ has only the trivial solution.

Answer

If not, then there would be a infintely many solutions to $A\vec{x} = \vec{0}$ and each of these added to a solution to $A\vec{x} = \vec{b}$ would be a solution to $A\vec{x} = \vec{b}$.

This page titled 5.9: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) .

CHAPTER OVERVIEW

6: Spectral Theory

6.1: Eigenvalues and Eigenvectors of a Matrix
6.2: Diagonalization
6.3: Applications of Spectral Theory
6.4: Orthogonality
6.5: Exercises

This page titled 6: Spectral Theory is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

6.1: Eigenvalues and Eigenvectors of a Matrix

Outcomes

A. Describe eigenvalues geometrically and algebraically.
B. Find eigenvalues and eigenvectors for a square matrix.

Spectral Theory refers to the study of eigenvalues and eigenvectors of a matrix. It is of fundamental importance in many areas and is the subject of our study for this chapter.

Definition of Eigenvectors and Eigenvalues

In this section, we will work with the entire set of complex numbers, denoted by $\mathbb{C}$. Recall that the real numbers, $\mathbb{R}$ are contained in the complex numbers, so the discussions in this section apply to both real and complex numbers.

To illustrate the idea behind what will be discussed, consider the following example.

Example 6.1.1: Eigenvectors and Eigenvalues

Let

$$A = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix}$$

Compute the product AX for

$$X = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}, X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

What do you notice about AX in each of these products?

Solution

First, compute AX for

$$X = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

This product is given by

$$AX = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -50 \\ -40 \\ 30 \end{bmatrix} = 10 \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

In this case, the product AX resulted in a vector which is equal to 10 times the vector X. In other words, $AX = 10X$.

Let's see what happens in the next product. Compute AX for the vector

$$X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This product is given by

$$AX = \begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

In this case, the product AX resulted in a vector equal to 0 times the vector X, $AX = 0X$.

Perhaps this matrix is such that AX results in kX, for every vector X. However, consider

$$\begin{bmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 38 \\ -11 \end{bmatrix}$$

In this case, AX did not result in a vector of the form kX for some scalar k.

There is something special about the first two products calculated in Example 6.1.1. Notice that for each, $AX = kX$ where k is some scalar. When this equation holds for some X and k, we call the scalar k an **eigenvalue** of A. We often use the special symbol λ instead of k when referring to eigenvalues. In Example 6.1.1, the values 10 and 0 are eigenvalues for the matrix A and we can label these as $\lambda_1 = 10$ and $\lambda_2 = 0$.

When $AX = \lambda X$ for some $X \neq 0$, we call such an X an **eigenvector** of the matrix A. The eigenvectors of A are associated to an eigenvalue. Hence, if λ_1 is an eigenvalue of A and $AX = \lambda_1 X$, we can label this eigenvector as X_1. Note again that in order to be an eigenvector, X must be nonzero.

There is also a geometric significance to eigenvectors. When you have a **nonzero** vector which, when multiplied by a matrix results in another vector which is parallel to the first or equal to $\mathbf{0}$, this vector is called an eigenvector of the matrix. This is the meaning when the vectors are in $\mathbb{R}^n$.

The formal definition of eigenvalues and eigenvectors is as follows.

Definition 6.1.1: Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix and let $X \in \mathbb{C}^n$ be a **nonzero vector** for which

$$AX = \lambda X \tag{6.1.1}$$

for some scalar λ. Then λ is called an **eigenvalue** of the matrix A and X is called an **eigenvector** of A associated with λ, or a λ-eigenvector of A.

The set of all eigenvalues of an $n \times n$ matrix A is denoted by $\sigma(A)$ and is referred to as the **spectrum** of A.

The eigenvectors of a matrix A are those vectors X for which multiplication by A results in a vector in the same direction or opposite direction to X. Since the zero vector 0 has no direction this would make no sense for the zero vector. As noted above, $\mathbf{0}$ is never allowed to be an eigenvector.

Let's look at eigenvectors in more detail. Suppose X satisfies (6.1.1). Then

$$\begin{gathered} AX - \lambda X = 0 \\ \text{or} \\ (A - \lambda I) X = 0 \end{gathered}$$

for some $X \neq 0$. Equivalently you could write $(\lambda I - A) X = 0$, which is more commonly used. Hence, when we are looking for eigenvectors, we are looking for nontrivial solutions to this homogeneous system of equations!

Recall that the solutions to a homogeneous system of equations consist of basic solutions, and the linear combinations of those basic solutions. In this context, we call the basic solutions of the equation $(\lambda I - A) X = 0$ **basic eigenvectors**. It follows that any (nonzero) linear combination of basic eigenvectors is again an eigenvector.

Suppose the matrix $(\lambda I - A)$ is invertible, so that $(\lambda I - A)^{-1}$ exists. Then the following equation would be true.

$$\begin{aligned} X &= IX \\ &= \left((\lambda I - A)^{-1} (\lambda I - A) \right) X \\ &= (\lambda I - A)^{-1} ((\lambda I - A) X) \\ &= (\lambda I - A)^{-1} 0 \\ &= 0 \end{aligned} \tag{6.1.2}$$

This claims that $X = 0$. However, we have required that $X \neq 0$. Therefore $(\lambda I - A)$ cannot have an inverse!

Recall that if a matrix is not invertible, then its determinant is equal to 0. Therefore we can conclude that

$$\det(\lambda I - A) = 0 \tag{6.1.3}$$

Note that this is equivalent to $\det(A - \lambda I) = 0$.

The expression $\det(\lambda I - A)$ is a polynomial (in the variable x) called the **characteristic polynomial of** A, and $\det(\lambda I - A) = 0$ is called the **characteristic equation**. For this reason we may also refer to the eigenvalues of A as **characteristic values**, but the former is often used for historical reasons.

The following theorem claims that the roots of the characteristic polynomial are the eigenvalues of A. Thus when [eigen2] holds, A has a nonzero eigenvector.

Theorem 6.1.1: The Existence of an Eigenvector

Let A be an $n \times n$ matrix and suppose $\det(\lambda I - A) = 0$ for some $\lambda \in \mathbb{C}$.
Then λ is an eigenvalue of A and thus there exists a nonzero vector $X \in \mathbb{C}^n$ such that $AX = \lambda X$.

Proof

For A an $n \times n$ matrix, the method of Laplace Expansion demonstrates that $\det(\lambda I - A)$ is a polynomial of degree n. As such, the equation (6.1.3) has a solution $\lambda \in \mathbb{C}$ by the Fundamental Theorem of Algebra. The fact that λ is an eigenvalue is left as an exercise.

Finding Eigenvectors and Eigenvalues

Now that eigenvalues and eigenvectors have been defined, we will study how to find them for a matrix A.

First, consider the following definition.

Definition 6.1.2: Multiplicity of an Eigenvalue

Let A be an $n \times n$ matrix with characteristic polynomial given by $\det(\lambda I - A)$. Then, the multiplicity of an eigenvalue λ of A is the number of times λ occurs as a root of that characteristic polynomial.

For example, suppose the characteristic polynomial of A is given by $(\lambda - 2)^2$. Solving for the roots of this polynomial, we set $(\lambda - 2)^2 = 0$ and solve for λ. We find that $\lambda = 2$ is a root that occurs twice. Hence, in this case, $\lambda = 2$ is an eigenvalue of A of multiplicity equal to 2.

We will now look at how to find the eigenvalues and eigenvectors for a matrix A in detail. The steps used are summarized in the following procedure.

Procedure 6.1.1: Finding Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix.

1. First, find the eigenvalues λ of A by solving the equation $\det(\lambda I - A) = 0$.
2. For each λ, find the basic eigenvectors $X \neq 0$ by finding the basic solutions to $(\lambda I - A)X = 0$.

To verify your work, make sure that $AX = \lambda X$ for each λ and associated eigenvector X.

We will explore these steps further in the following example.

Example 6.1.2: Find the Eigenvalues and Eigenvectors

Let $A = \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}$. Find its eigenvalues and eigenvectors.

Solution

We will use Procedure 6.1.1. First we find the eigenvalues of A by solving the equation

$$\det(\lambda I - A) = 0$$

This gives

$$\det\left(\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\right) = 0$$

$$\det \begin{bmatrix} \lambda + 5 & -2 \\ 7 & \lambda - 4 \end{bmatrix} = 0 \tag{6.1.4}$$

Computing the determinant as usual, the result is

$$\lambda^2 + \lambda - 6 = 0$$

Solving this equation, we find that $\lambda_1 = 2$ and $\lambda_2 = -3$.

Now we need to find the basic eigenvectors for each λ. First we will find the eigenvectors for $\lambda_1 = 2$. We wish to find all vectors $X \neq 0$ such that $AX = 2X$. These are the solutions to $(2I - A)X = 0$.

$$\left(2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -2 \\ 7 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.1.5}$$

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{cc|c} 7 & -2 & 0 \\ 7 & -2 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{cc|c} 1 & -\frac{2}{7} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The solution is any vector of the form

$$\begin{bmatrix} \frac{2}{7}s \\ s \end{bmatrix} = s\begin{bmatrix} \frac{2}{7} \\ 1 \end{bmatrix}$$

Multiplying this vector by 7 we obtain a simpler description for the solution to this system, given by

$$t\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

This gives the basic eigenvector for $\lambda_1 = 2$ as

$$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

To check, we verify that $AX = 2X$ for this basic eigenvector.

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

This is what we wanted, so we know this basic eigenvector is correct.

Next we will repeat this process to find the basic eigenvector for $\lambda_2 = -3$. We wish to find all vectors $X \neq 0$ such that $AX = -3X$. These are the solutions to $((-3)I - A)X = 0$.

$$\left((-3)\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}-5&2\\-7&4\end{bmatrix}\right)\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$
$$\begin{bmatrix}2&-2\\7&-7\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \tag{6.1.6}$$

The augmented matrix for this system and corresponding reduced row-echelon form are given by

$$\left[\begin{array}{cc|c}2&-2&0\\7&-7&0\end{array}\right]\to\cdots\to\left[\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right]$$

The solution is any vector of the form

$$\begin{bmatrix}s\\s\end{bmatrix}=s\begin{bmatrix}1\\1\end{bmatrix}$$

This gives the basic eigenvector for $\lambda_2=-3$ as

$$\begin{bmatrix}1\\1\end{bmatrix}$$

To check, we verify that $AX=-3X$ for this basic eigenvector.

$$\begin{bmatrix}-5&2\\-7&4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-3\\-3\end{bmatrix}=-3\begin{bmatrix}1\\1\end{bmatrix}$$

This is what we wanted, so we know this basic eigenvector is correct.

The following is an example using Procedure 6.1.1 for a 3×3 matrix.

Example 6.1.3: Find the Eigenvalues and Eigencectors

Find the eigenvalues and eigenvectors for the matrix

$$A=\begin{bmatrix}5&-10&-5\\2&14&2\\-4&-8&6\end{bmatrix}$$

Solution

We will use Procedure 6.1.1. First we need to find the eigenvalues of A. Recall that they are the solutions of the equation

$$\det(\lambda I-A)=0$$

In this case the equation is

$$\det\left(\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}5&-10&-5\\2&14&2\\-4&-8&6\end{bmatrix}\right)=0$$

which becomes

$$\det\begin{bmatrix}\lambda-5&10&5\\-2&\lambda-14&-2\\4&8&\lambda-6\end{bmatrix}=0$$

Using Laplace Expansion, compute this determinant and simplify. The result is the following equation.

$$(\lambda-5)\left(\lambda^2-20\lambda+100\right)=0$$

Solving this equation, we find that the eigenvalues are $\lambda_1=5,\lambda_2=10$ and $\lambda_3=10$. Notice that 10 is a root of multiplicity two due to

$$\lambda^2-20\lambda+100=(\lambda-10)^2$$

Therefore, $\lambda_2 = 10$ is an eigenvalue of multiplicity two.

Now that we have found the eigenvalues for A, we can compute the eigenvectors.

First we will find the basic eigenvectors for $\lambda_1 = 5$. In other words, we want to find all non-zero vectors X so that $AX = 5X$. This requires that we solve the equation $(5I - A)X = 0$ for X as follows.

$$\left(5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}5&-10&-5\\2&14&2\\-4&-8&6\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

That is you need to find the solution to

$$\begin{bmatrix}0&10&5\\-2&-9&-2\\4&8&-1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

By now this is a familiar problem. You set up the augmented matrix and row reduce to get the solution. Thus the matrix you must row reduce is

$$\left[\begin{array}{ccc|c}0&10&5&0\\-2&-9&-2&0\\4&8&-1&0\end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c}1&0&-\frac{5}{4}&0\\0&1&\frac{1}{2}&0\\0&0&0&0\end{array}\right]$$

and so the solution is any vector of the form

$$\begin{bmatrix}\frac{5}{4}s\\-\frac{1}{2}s\\s\end{bmatrix} = s\begin{bmatrix}\frac{5}{4}\\-\frac{1}{2}\\1\end{bmatrix}$$

where $s \in \mathbb{R}$. If we multiply this vector by 4, we obtain a simpler description for the solution to this system, as given by

$$t\begin{bmatrix}5\\-2\\4\end{bmatrix} \tag{6.1.7}$$

where $t \in \mathbb{R}$. Here, the basic eigenvector is given by

$$X_1 = \begin{bmatrix}5\\-2\\4\end{bmatrix}$$

Notice that we cannot let $t = 0$ here, because this would result in the zero vector and eigenvectors are never equal to 0! Other than this value, every other choice of t in (6.1.7) results in an eigenvector.

It is a good idea to check your work! To do so, we will take the original matrix and multiply by the basic eigenvector X_1. We check to see if we get $5X_1$.

$$\begin{bmatrix}5&-10&-5\\2&14&2\\-4&-8&6\end{bmatrix}\begin{bmatrix}5\\-2\\4\end{bmatrix} = \begin{bmatrix}25\\-10\\20\end{bmatrix} = 5\begin{bmatrix}5\\-2\\4\end{bmatrix}$$

This is what we wanted, so we know that our calculations were correct.

Next we will find the basic eigenvectors for $\lambda_2, \lambda_3 = 10$. These vectors are the basic solutions to the equation,

$$\left(10\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}5&-10&-5\\2&14&2\\-4&-8&6\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

That is you must find the solutions to

$$\begin{bmatrix}5&10&5\\-2&-4&-2\\4&8&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Consider the augmented matrix

$$\left[\begin{array}{ccc|c}5&10&5&0\\-2&-4&-2&0\\4&8&4&0\end{array}\right]$$

The reduced row-echelon form for this matrix is

$$\left[\begin{array}{ccc|c}1&2&1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

and so the eigenvectors are of the form

$$\begin{bmatrix}-2s-t\\s\\t\end{bmatrix}=s\begin{bmatrix}-2\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Note that you can't pick t and s both equal to zero because this would result in the zero vector and eigenvectors are never equal to zero.

Here, there are two basic eigenvectors, given by

$$X_2=\begin{bmatrix}-2\\1\\0\end{bmatrix},X_3=\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

Taking any (nonzero) linear combination of X_2 and X_3 will also result in an eigenvector for the eigenvalue $\lambda=10$. As in the case for $\lambda=5$, always check your work! For the first basic eigenvector, we can check $AX_2=10X_2$ as follows.

$$\begin{bmatrix}5&-10&-5\\2&14&2\\-4&-8&6\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix}=\begin{bmatrix}-10\\0\\10\end{bmatrix}=10\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

This is what we wanted. Checking the second basic eigenvector, X_3, is left as an exercise.

It is important to remember that for any eigenvector X, $X\neq 0$. However, it is possible to have eigenvalues equal to zero. This is illustrated in the following example.

Example 6.1.4: A Zero Eigenvalue

Let

$$A=\begin{bmatrix}2&2&-2\\1&3&-1\\-1&1&1\end{bmatrix}$$

Find the eigenvalues and eigenvectors of A.

Solution

First we find the eigenvalues of A. We will do so using Definition 6.1.1.

In order to find the eigenvalues of A, we solve the following equation.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 2 & -2 & 2 \\ -1 & \lambda - 3 & 1 \\ 1 & -1 & \lambda - 1 \end{bmatrix} = 0$$

This reduces to $\lambda^3 - 6\lambda^2 + 8\lambda = 0$. You can verify that the solutions are $\lambda_1 = 0, \lambda_2 = 2, \lambda_3 = 4$. Notice that while eigenvectors can never equal 0, it is possible to have an eigenvalue equal to 0.

Now we will find the basic eigenvectors. For $\lambda_1 = 0$, we need to solve the equation $(0I - A)X = 0$. This equation becomes $-AX = 0$, and so the augmented matrix for finding the solutions is given by

$$\left[\begin{array}{ccc|c} -2 & -2 & 2 & 0 \\ -1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the eigenvectors are of the form $t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ where $t \neq 0$ and the basic eigenvector is given by

$$X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

We can verify that this eigenvector is correct by checking that the equation $AX_1 = 0X_1$ holds. The product AX_1 is given by

$$AX_1 = \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This clearly equals $0X_1$, so the equation holds. Hence, $AX_1 = 0X_1$ and so 0 is an eigenvalue of A.

Computing the other basic eigenvectors is left as an exercise.

In the following sections, we examine ways to simplify this process of finding eigenvalues and eigenvectors by using properties of special types of matrices.

Eigenvalues and Eigenvectors for Special Types of Matrices

There are three special kinds of matrices which we can use to simplify the process of finding eigenvalues and eigenvectors. Throughout this section, we will discuss similar matrices, elementary matrices, as well as triangular matrices.

We begin with a definition.

Definition 6.1.3: Similar Matrices

Let A and B be $n \times n$ matrices. Suppose there exists an invertible matrix P such that

$$A = P^{-1}BP$$

Then A and B are called **similar matrices**.

It turns out that we can use the concept of similar matrices to help us find the eigenvalues of matrices. Consider the following lemma.

Lemma 6.1.1: Similar Matrices and Eigenvalues

Let A and B be similar matrices, so that $A = P^{-1}BP$ where A, B are $n \times n$ matrices and P is invertible. Then A, B have the same eigenvalues.

Proof

We need to show two things. First, we need to show that if $A = P^{-1}BP$, then A and B have the same eigenvalues. Secondly, we show that if A and B have the same eigenvalues, then $A = P^{-1}BP$.

Here is the proof of the first statement. Suppose $A = P^{-1}BP$ and λ is an eigenvalue of A, that is $AX = \lambda X$ for some $X \neq 0$. Then

$$P^{-1}BPX = \lambda X$$

and so

$$BPX = \lambda PX$$

Since P is one to one and $X \neq 0$, it follows that $PX \neq 0$. Here, PX plays the role of the eigenvector in this equation. Thus λ is also an eigenvalue of B. One can similarly verify that any eigenvalue of B is also an eigenvalue of A, and thus both matrices have the same eigenvalues as desired.

Proving the second statement is similar and is left as an exercise.

Note that this proof also demonstrates that the eigenvectors of A and B will (generally) be *different*. We see in the proof that $AX = \lambda X$, while $B(PX) = \lambda(PX)$. Therefore, for an eigenvalue λ, A will have the eigenvector X while B will have the eigenvector PX.

The second special type of matrices we discuss in this section is elementary matrices. Recall from Definition 2.8.1 that an elementary matrix E is obtained by applying one row operation to the identity matrix.

It is possible to use elementary matrices to simplify a matrix before searching for its eigenvalues and eigenvectors. This is illustrated in the following example.

Example 6.1.5: Simplify Using Elementary Matrices

Find the eigenvalues for the matrix

$$A = \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix}$$

Solution

This matrix has big numbers and therefore we would like to simplify as much as possible before computing the eigenvalues.

We will do so using row operations. First, add 2 times the second row to the third row. To do so, left multiply A by $E(2,2)$. Then right multiply A by the inverse of $E(2,2)$ as illustrated.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 33 & 105 & 105 \\ 10 & 28 & 30 \\ -20 & -60 & -62 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

By Lemma 6.1.1, the resulting matrix has the same eigenvalues as A where here, the matrix $E(2,2)$ plays the role of P.

We do this step again, as follows. In this step, we use the elementary matrix obtained by adding -3 times the second row to the first row.

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix} \tag{6.1.8}$$

Again by Lemma 6.1.1, this resulting matrix has the same eigenvalues as A. At this point, we can easily find the eigenvalues. Let

$$B = \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

Then, we find the eigenvalues of B (and therefore of A) by solving the equation $\det(\lambda I - B) = 0$. You should verify that this equation becomes

$$(\lambda + 2)(\lambda + 2)(\lambda - 3) = 0$$

Solving this equation results in eigenvalues of $\lambda_1 = -2, \lambda_2 = -2$, and $\lambda_3 = 3$. Therefore, these are also the eigenvalues of A.

Through using elementary matrices, we were able to create a matrix for which finding the eigenvalues was easier than for A. At this point, you could go back to the original matrix A and solve $(\lambda I - A)X = 0$ to obtain the eigenvectors of A.

Notice that when you multiply on the right by an elementary matrix, you are doing the column operation defined by the elementary matrix. In (6.1.8) multiplication by the elementary matrix on the right merely involves taking three times the first column and adding to the second. Thus, without referring to the elementary matrices, the transition to the new matrix in (6.1.8) can be illustrated by

$$\begin{bmatrix} 33 & -105 & 105 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -9 & 15 \\ 10 & -32 & 30 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & 15 \\ 10 & -2 & 30 \\ 0 & 0 & -2 \end{bmatrix}$$

The third special type of matrix we will consider in this section is the triangular matrix. Recall Definition 3.1.6 which states that an upper (lower) triangular matrix contains all zeros below (above) the main diagonal. Remember that finding the determinant of a triangular matrix is a simple procedure of taking the product of the entries on the main diagonal.. It turns out that there is also a simple way to find the eigenvalues of a triangular matrix.

In the next example we will demonstrate that the eigenvalues of a triangular matrix are the entries on the main diagonal.

Example 6.1.6: Eigenvalues for a Triangular Matrix

Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix}$. Find the eigenvalues of A.

Solution

We need to solve the equation $\det(\lambda I - A) = 0$ as follows

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -2 & -4 \\ 0 & \lambda - 4 & -7 \\ 0 & 0 & \lambda - 6 \end{bmatrix} = (\lambda - 1)(\lambda - 4)(\lambda - 6) = 0 \tag{6.1.9}$$

Solving the equation $(\lambda - 1)(\lambda - 4)(\lambda - 6) = 0$ for λ results in the eigenvalues $\lambda_1 = 1, \lambda_2 = 4$ and $\lambda_3 = 6$. Thus the eigenvalues are the entries on the main diagonal of the original matrix.

The same result is true for lower triangular matrices. For any triangular matrix, the eigenvalues are equal to the entries on the main diagonal. To find the eigenvectors of a triangular matrix, we use the usual procedure.

In the next section, we explore an important process involving the eigenvalues and eigenvectors of a matrix.

This page titled 6.1: Eigenvalues and Eigenvectors of a Matrix is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **7.1: Eigenvalues and Eigenvectors of a Matrix** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

6.2: Diagonalization

Outcomes

A. Determine when it is possible to diagonalize a matrix.
B. When possible, diagonalize a matrix.

Similarity and Diagonalization

We begin this section by recalling the definition of similar matrices. Recall that if A, B are two $n \times n$ matrices, then they are **similar** if and only if there exists an invertible matrix P such that

$$A = P^{-1}BP$$

In this case we write $A \sim B$. The concept of similarity is an example of an **equivalence relation**.

Lemma 6.2.1: Similarity is an Equivalence Relation

Similarity is an equivalence relation, i.e. for $n \times n$ matrices A, B, and C,

1. $A \sim A$ (reflexive)
2. If $A \sim B$, then $B \sim A$ (symmetric)
3. If $A \sim B$ and $B \sim C$, then $A \sim C$ (transitive)

Proof

It is clear that $A \sim A$, taking $P = I$.

Now, if $A \sim B$, then for some P invertible,

$$A = P^{-1}BP$$

and so

$$PAP^{-1} = B$$

But then

$$\left(P^{-1}\right)^{-1} AP^{-1} = B$$

which shows that $B \sim A$.

Now suppose $A \sim B$ and $B \sim C$. Then there exist invertible matrices P, Q such that

$$A = P^{-1}BP, \ B = Q^{-1}CQ$$

Then,

$$A = P^{-1}\left(Q^{-1}CQ\right)P = (QP)^{-1}C(QP)$$

showing that A is similar to C.

Another important concept necessary to this section is the trace of a matrix. Consider the definition.

Definition 6.2.1: Trace of a Matrix

If $A = [a_{ij}]$ is an $n \times n$ matrix, then the trace of A is

$$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}.$$

In words, the trace of a matrix is the sum of the entries on the main diagonal.

Lemma 6.2.2: Properties of Trace

For $n \times n$ matrices A and B, and any $k \in \mathbb{R}$,

1. $\text{trace}(A+B) = \text{trace}(A) + \text{trace}(B)$
2. $\text{trace}(kA) = k \cdot \text{trace}(A)$
3. $\text{trace}(AB) = \text{trace}(BA)$

The following theorem includes a reference to the characteristic polynomial of a matrix. Recall that for any $n \times n$ matrix A, the characteristic polynomial of A is $c_A(x) = \det(xI - A)$.

Theorem 6.2.1: Properties of Similar Matrices

If A and B are $n \times n$ matrices and $A \sim B$, then

1. $\det(A) = \det(B)$
2. $\text{rank}(A) = \text{rank}(B)$
3. $\text{trace}(A) = \text{trace}(B)$
4. $c_A(x) = c_B(x)$
5. A and B have the same eigenvalues

We now proceed to the main concept of this section. When a matrix is similar to a diagonal matrix, the matrix is said to be diagonalizable. We define a diagonal matrix D as a matrix containing a zero in every entry except those on the main diagonal. More precisely, if d_{ij} is the ij^{th} entry of a diagonal matrix D, then $d_{ij} = 0$ unless $i = j$. Such matrices look like the following.

$$D = \begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$$

where $*$ is a number which might not be zero.

The following is the formal definition of a diagonalizable matrix.

Definition 6.2.2: Diagonalizable

Let A be an $n \times n$ matrix. Then A is said to be **diagonalizable** if there exists an invertible matrix P such that

$$P^{-1}AP = D$$

where D is a diagonal matrix.

Notice that the above equation can be rearranged as $A = PDP^{-1}$. Suppose we wanted to compute A^{100}. By diagonalizing A first it suffices to then compute $(PDP^{-1})^{100}$, which reduces to $PD^{100}P^{-1}$. This last computation is much simpler than A^{100}. While this process is described in detail later, it provides motivation for diagonalization.

Diagonalizing a Matrix

The most important theorem about diagonalizability is the following major result.

Theorem 6.2.2: Eigenvectors and Diagonalizable Matrices

An $n \times n$ matrix A is diagonalizable if and only if there is an invertible matrix P given by

$$P = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$$

where the X_k are eigenvectors of A.

Moreover if A is diagonalizable, the corresponding eigenvalues of A are the diagonal entries of the diagonal matrix D.

Proof

Suppose P is given as above as an invertible matrix whose columns are eigenvectors of A. Then P^{-1} is of the form

$$P^{-1} = \begin{bmatrix} W_1^T \\ W_2^T \\ \vdots \\ W_n^T \end{bmatrix}$$

where $W_k^T X_j = \delta_{kj}$, which is the Kronecker's symbol defined by

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

Then

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} W_1^T \\ W_2^T \\ \vdots \\ W_n^T \end{bmatrix} \begin{bmatrix} AX_1 & AX_2 & \cdots & AX_n \end{bmatrix} \\ &= \begin{bmatrix} W_1^T \\ W_2^T \\ \vdots \\ W_n^T \end{bmatrix} \begin{bmatrix} \lambda_1 X_1 & \lambda_2 X_2 & \cdots & \lambda_n X_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \end{aligned} \tag{6.2.1}$$

Conversely, suppose A is diagonalizable so that $P^{-1}AP = D$. Let

$$P = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix}$$

where the columns are the X_k and

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Then

$$AP = PD = \begin{bmatrix} X_1 & X_2 & \cdots & X_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

and so

$$\begin{bmatrix} AX_1 & AX_2 & \cdots & AX_n \end{bmatrix} = \begin{bmatrix} \lambda_1 X_1 & \lambda_2 X_2 & \cdots & \lambda_n X_n \end{bmatrix}$$

showing the X_k are eigenvectors of A and the λ_k are eigenvectors.

Notice that because the matrix P defined above is invertible it follows that the set of eigenvectors of A, $\{X_1, X_2, \cdots, X_n\}$, form a basis of $\mathbb{R}^n$.

We demonstrate the concept given in the above theorem in the next example. Note that not only are the columns of the matrix P formed by eigenvectors, but P must be invertible so must consist of a wide variety of eigenvectors. We achieve this by using basic eigenvectors for the columns of P.

Example 6.2.1: Diagonalize a Matrix

Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

Solution

By Theorem 6.2.2 we use the eigenvectors of A as the columns of P, and the corresponding eigenvalues of A as the diagonal entries of D.

First, we will find the eigenvalues of A. To do so, we solve $\det(\lambda I - A) = 0$ as follows.

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}\right) = 0$$

This computation is left as an exercise, and you should verify that the eigenvalues are $\lambda_1 = 2, \lambda_2 = 2$, and $\lambda_3 = 6$.

Next, we need to find the eigenvectors. We first find the eigenvectors for $\lambda_1, \lambda_2 = 2$. Solving $(2I - A)X = 0$ to find the eigenvectors, we find that the eigenvectors are

$$t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

where t, s are scalars. Hence there are two basic eigenvectors which are given by

$$X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

You can verify that the basic eigenvector for $\lambda_3 = 6$ is $X_3 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

Then, we construct the matrix P as follows.

$$P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

That is, the columns of P are the basic eigenvectors of A. Then, you can verify that

$$P^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

Thus,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \end{aligned} \tag{6.2.2}$$

You can see that the result here is a diagonal matrix where the entries on the main diagonal are the eigenvalues of A. We expected this based on Theorem 6.2.2. Notice that eigenvalues on the main diagonal *must* be in the same order as the corresponding eigenvectors in P.

Consider the next important theorem.

Theorem 6.2.3: Linearly Independent Eigenvectors

Let A be an $n \times n$ matrix, and suppose that A has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. For each i, let X_i be a λ_i-eigenvector of A. Then $\{X_1, X_2, \ldots, X_m\}$ is linearly independent.

The corollary that follows from this theorem gives a useful tool in determining if A is diagonalizable.

Corollary 6.2.1: Distinct Eigenvalues

Let A be an $n \times n$ matrix and suppose it has n distinct eigenvalues. Then it follows that A is diagonalizable.

It is possible that a matrix A cannot be diagonalized. In other words, we cannot find an invertible matrix P so that $P^{-1}AP = D$.

Consider the following example.

Example 6.2.2: A Matrix which cannot be Diagonalized

Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

If possible, find an invertible matrix P and diagonal matrix D so that $P^{-1}AP = D$.

Solution

Through the usual procedure, we find that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$. To find the eigenvectors, we solve the equation $(\lambda I - A)X = 0$. The matrix $(\lambda I - A)$ is given by

$$\begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 1 \end{bmatrix}$$

Substituting in $\lambda = 1$, we have the matrix

$$\begin{bmatrix} 1-1 & -1 \\ 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$$

Then, solving the equation $(\lambda I - A)X = 0$ involves carrying the following augmented matrix to its reduced row-echelon form.

$$\left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Then the eigenvectors are of the form

$$t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the basic eigenvector is

$$X_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

In this case, the matrix A has one eigenvalue of multiplicity two, but only one basic eigenvector. In order to diagonalize A, we need to construct an invertible 2×2 matrix P. However, because A only has one basic eigenvector, we cannot construct this P.

Notice that if we were to use X_1 as both columns of P, P would not be invertible. For this reason, we cannot repeat eigenvectors in P.

Hence this matrix cannot be diagonalized.

The idea that a matrix may not be diagonalizable suggests that conditions exist to determine when it is possible to diagonalize a matrix. We saw earlier in Corollary 6.2.1 that an $n \times n$ matrix with n distinct eigenvalues is diagonalizable. It turns out that there are other useful diagonalizability tests.

First we need the following definition.

Definition 6.2.3: Eigenspace

Let A be an $n \times n$ matrix and $\lambda \in \mathbb{R}$. The eigenspace of A corresponding to λ, written $E_\lambda(A)$ is the set of all eigenvectors corresponding to λ.

In other words, the eigenspace $E_\lambda(A)$ is all X such that $AX = \lambda X$. Notice that this set can be written $E_\lambda(A) = \text{null}(\lambda I - A)$, showing that $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$.

Recall that the multiplicity of an eigenvalue λ is the number of times that it occurs as a root of the characteristic polynomial.

Consider now the following lemma.

Lemma 6.2.3: Dimension of the Eigenspace

If A is an $n \times n$ matrix, then

$$\dim(E_\lambda(A)) \leq m$$

where λ is an eigenvalue of A of multiplicity m.

This result tells us that if λ is an eigenvalue of A, then the number of linearly independent λ-eigenvectors is never more than the multiplicity of λ. We now use this fact to provide a useful diagonalizability condition.

Theorem 6.2.4: Diagonalizability Condition

Let A be an $n \times n$ matrix A. Then A is diagonalizable if and only if for each eigenvalue λ of A, $\dim(E_\lambda(A))$ is equal to the multiplicity of λ.

Complex Eigenvalues

In some applications, a matrix may have eigenvalues which are complex numbers. For example, this often occurs in differential equations. These questions are approached in the same way as above.

Consider the following example.

Example 6.2.3: A Real Matrix with Complex Eigenvalues

Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of A.

Solution

We will first find the eigenvalues as usual by solving the following equation.

$$\det\left(\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&2&-1\\0&1&2\end{bmatrix}\right)=0$$

This reduces to $(\lambda-1)(\lambda^2-4\lambda+5)=0$. The solutions are $\lambda_1=1, \lambda_2=2+i$ and $\lambda_3=2-i$.

There is nothing new about finding the eigenvectors for $\lambda_1=1$ so this is left as an exercise.

Consider now the eigenvalue $\lambda_2=2+i$. As usual, we solve the equation $(\lambda I-A)X=0$ as given by

$$\left((2+i)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&0&0\\0&2&-1\\0&1&2\end{bmatrix}\right)X=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

In other words, we need to solve the system represented by the augmented matrix

$$\left[\begin{array}{ccc|c}1+i&0&0&0\\0&i&1&0\\0&-1&i&0\end{array}\right]$$

We now use our row operations to solve the system. Divide the first row by $(1+i)$ and then take $-i$ times the second row and add to the third row. This yields

$$\left[\begin{array}{ccc|c}1&0&0&0\\0&i&1&0\\0&0&0&0\end{array}\right]$$

Now multiply the second row by $-i$ to obtain the reduced row-echelon form, given by

$$\left[\begin{array}{ccc|c}1&0&0&0\\0&1&-i&0\\0&0&0&0\end{array}\right]$$

Therefore, the eigenvectors are of the form

$$t\begin{bmatrix}0\\i\\1\end{bmatrix}$$

and the basic eigenvector is given by

$$X_2=\begin{bmatrix}0\\i\\1\end{bmatrix}$$

As an exercise, verify that the eigenvectors for $\lambda_3=2-i$ are of the form

$$t\begin{bmatrix}0\\-i\\1\end{bmatrix}$$

Hence, the basic eigenvector is given by

$$X_3=\begin{bmatrix}0\\-i\\1\end{bmatrix}$$

As usual, be sure to check your answers! To verify, we check that $AX_3=(2-i)X_3$ as follows.

$$\begin{bmatrix}1&0&0\\0&2&-1\\0&1&2\end{bmatrix}\begin{bmatrix}0\\-i\\1\end{bmatrix}=\begin{bmatrix}0\\-1-2i\\2-i\end{bmatrix}=(2-i)\begin{bmatrix}0\\-i\\1\end{bmatrix}$$

> Therefore, we know that this eigenvector and eigenvalue are correct.

Notice that in Example 6.2.3, two of the eigenvalues were given by $\lambda_2 = 2 + i$ and $\lambda_3 = 2 - i$. You may recall that these two complex numbers are **conjugates**. It turns out that whenever a matrix containing real entries has a complex eigenvalue λ, it also has an eigenvalue equal to $\overline{\lambda}$, the conjugate of λ.

This page titled 6.2: Diagonalization is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **7.2: Diagonalization** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

6.3: Applications of Spectral Theory

Outcomes

A. Use diagonalization to find a high power of a matrix.
B. Use diagonalization to solve dynamical systems.

Raising a Matrix to a High Power

Suppose we have a matrix A and we want to find A^{50}. One could try to multiply A with itself 50 times, but this is computationally extremely intensive (try it!). However diagonalization allows us to compute high powers of a matrix relatively easily. Suppose A is diagonalizable, so that $P^{-1}AP = D$. We can rearrange this equation to write $A = PDP^{-1}$.

Now, consider A^2. Since $A = PDP^{-1}$, it follows that

$$A^2 = \left(PDP^{-1}\right)^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

Similarly,

$$A^3 = \left(PDP^{-1}\right)^3 = PDP^{-1}PDP^{-1}PDP^{-1} = PD^3P^{-1}$$

In general,

$$A^n = \left(PDP^{-1}\right)^n = PD^nP^{-1}$$

Therefore, we have reduced the problem to finding D^n. In order to compute D^n, then because D is diagonal we only need to raise every entry on the main diagonal of D to the power of n.

Through this method, we can compute large powers of matrices. Consider the following example.

Example 6.3.1: Raising a Matrix to a High Power

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$. Find A^{50}.

Solution

We will first diagonalize A. The steps are left as an exercise and you may wish to verify that the eigenvalues of A are $\lambda_1 = 1, \lambda_2 = 1$, and $\lambda_3 = 2$.

The basic eigenvectors corresponding to $\lambda_1, \lambda_2 = 1$ are

$$X_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

The basic eigenvector corresponding to $\lambda_3 = 2$ is

$$X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now we construct P by using the basic eigenvectors of A as the columns of P. Thus

$$P = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Then also

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

which you may wish to verify.

Then,

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= D \end{aligned} \tag{6.3.1}$$

Now it follows by rearranging the equation that

$$A = PDP^{-1} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}$$

Therefore,

$$\begin{aligned} A^{50} &= PD^{50}P^{-1} \\ &= \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{50} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \end{aligned} \tag{6.3.2}$$

By our discussion above, D^{50} is found as follows.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{50} = \begin{bmatrix} 1^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & 2^{50} \end{bmatrix}$$

It follows that

$$\begin{aligned} A^{50} &= \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1^{50} & 0 & 0 \\ 0 & 1^{50} & 0 \\ 0 & 0 & 2^{50} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2^{50} & -1+2^{50} & 0 \\ 0 & 1 & 0 \\ 1-2^{50} & 1-2^{50} & 1 \end{bmatrix} \end{aligned} \tag{6.3.3}$$

Through diagonalization, we can efficiently compute a high power of A. Without this, we would be forced to multiply this by hand!

The next section explores another interesting application of diagonalization.

Raising a Symmetric Matrix to a High Power

We already have seen how to use matrix diagonalization to compute powers of matrices. This requires computing eigenvalues of the matrix A, and finding an invertible matrix of eigenvectors P such that $P^{-1}AP$ is diagonal. In this section we will see that if the matrix A is symmetric (see Definition 2.5.2), then we can actually find such a matrix P that is an orthogonal matrix of eigenvectors. Thus P^{-1} is simply its transpose P^T, and P^TAP is diagonal. When this happens we say that A is **orthogonally diagonalizable**

In fact this happens if and only if A is a symmetric matrix as shown in the following important theorem.

Theorem 6.3.1: Principal Axis Theorem

The following conditions are equivalent for an $n \times n$ matrix A:

1. A is symmetric.
2. A has an orthonormal set of eigenvectors.
3. A is orthogonally diagonalizable.

Proof

The complete proof is beyond this course, but to give an idea assume that A has an orthonormal set of eigenvectors, and let P consist of these eigenvectors as columns. Then $P^{-1} = P^T$, and $P^T AP = D$ a diagonal matrix. But then $A = PDP^T$, and

$$A^T = (PDP^T)^T = (P^T)^T D^T P^T = PDP^T = A$$

so A is symmetric.

Now given a symmetric matrix A, one shows that eigenvectors corresponding to different eigenvalues are always orthogonal. So it suffices to apply the Gram-Schmidt process on the set of basic eigenvectors of each eigenvalue to obtain an orthonormal set of eigenvectors.

We demonstrate this in the following example.

Example 6.3.2: Orthogonal Diagonalization of a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$. Find an orthogonal matrix P such that $P^T AP$ is a diagonal matrix.

Solution

In this case, verify that the eigenvalues are 2 and 1. First we will find an eigenvector for the eigenvalue 2. This involves row reducing the following augmented matrix.

$$\left[\begin{array}{ccc|c} 2-1 & 0 & 0 & 0 \\ 0 & 2-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 2-\frac{3}{2} & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so an eigenvector is

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Finally to obtain an eigenvector of length one (unit eigenvector) we simply divide this vector by its length to yield:

$$\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Next consider the case of the eigenvalue 1. To obtain basic eigenvectors, the matrix which needs to be row reduced in this case is

$$\left[\begin{array}{ccc|c} 1-1 & 0 & 0 & 0 \\ 0 & 1-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1-\frac{3}{2} & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the eigenvectors are of the form

$$\begin{bmatrix} s \\ -t \\ t \end{bmatrix}$$

Note that all these vectors are automatically orthogonal to eigenvectors corresponding to the first eigenvalue. This follows from the fact that A is symmetric, as mentioned earlier.

We obtain basic eigenvectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Since they are themselves orthogonal (by luck here) we do not need to use the Gram-Schmidt process and instead simply normalize these vectors to obtain

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

An orthogonal matrix P to orthogonally diagonalize A is then obtained by letting these basic vectors be the columns.

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

We verify this works. $P^T AP$ is of the form

$$\begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

which is the desired diagonal matrix.

We can now apply this technique to efficiently compute high powers of a symmetric matrix.

Example 6.3.3: Powers of a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$. Compute A^7.

Solution

We found in Example 6.3.2 that $P^TAP = D$ is diagonal, where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Thus $A = PDP^T$ and $A^7 = PDP^T\ PDP^T \cdots PDP^T = PD^7P^T$ which gives:

$$\begin{aligned} A^7 &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^7 \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^7 \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{2^7}{\sqrt{2}} & \frac{2^7}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{2^7+1}{2} & \frac{2^7-1}{2} \\ 0 & \frac{2^7-1}{2} & \frac{2^7+1}{2} \end{bmatrix} \end{aligned}$$

Markov Matrices

There are applications of great importance which feature a special type of matrix. Matrices whose columns consist of non-negative numbers that sum to one are called **Markov matrices**. An important application of Markov matrices is in population migration, as illustrated in the following definition.

Definition 6.3.1: Migration Matrices

Let m locations be denoted by the numbers $1, 2, \cdots, m$. Suppose it is the case that each year the proportion of residents in location j which move to location i is a_{ij}. Also suppose no one escapes or emigrates from without these m locations. This last assumption requires $\sum_i a_{ij} = 1$, and means that the matrix A, such that $A = [a_{ij}]$, is a Markov matrix. In this context, A is also called a **migration matrix**.

Consider the following example which demonstrates this situation.

Example 6.3.4: Migration Matrix

Let A be a Markov matrix given by

$$A = \begin{bmatrix} .4 & .2 \\ .6 & .8 \end{bmatrix}$$

Verify that A is a Markov matrix and describe the entries of A in terms of population migration.

Solution

The columns of A are comprised of non-negative numbers which sum to 1. Hence, A is a Markov matrix.

Now, consider the entries a_{ij} of A in terms of population. The entry $a_{11} = .4$ is the proportion of residents in location one which stay in location one in a given time period. Entry $a_{21} = .6$ is the proportion of residents in location 1 which move to location 2 in the same time period. Entry $a_{12} = .2$ is the proportion of residents in location 2 which move to location 1. Finally, entry $a_{22} = .8$ is the proportion of residents in location 2 which stay in location 2 in this time period.

Considered as a Markov matrix, these numbers are usually identified with probabilities. Hence, we can say that the probability that a resident of location one will stay in location one in the time period is $.4$.

Observe that in Example 6.3.4 if there was initially say 15 thousand people in location 1 and 10 thousands in location 2, then after one year there would be $.4 \times 15 + .2 \times 10 = 8$ thousands people in location 1 the following year, and similarly there would be $.6 \times 15 + .8 \times 10 = 17$ thousands people in location 2 the following year.

More generally let $X_n = [x_{1n} \cdots x_{mn}]^T$ where x_{in} is the population of location i at time period n. We call X_n the **state vector at period** n. In particular, we call X_0 the initial state vector. Letting A be the migration matrix, we compute the population in each location i one time period later by AX_n. In order to find the population of location i after k years, we compute the ith component of $A^k X$. This discussion is summarized in the following theorem.

Theorem 6.3.2: State Vector

Let A be the migration matrix of a population and let X_n be the vector whose entries give the population of each location at time period n. Then X_n is the state vector at period n and it follows that

$$X_{n+1} = AX_n$$

The sum of the entries of X_n will equal the sum of the entries of the initial vector X_0. Since the columns of A sum to 1, this sum is preserved for every multiplication by A as demonstrated below.

$$\sum_i \sum_j a_{ij} x_j = \sum_j x_j \left(\sum_i a_{ij} \right) = \sum_j x_j$$

Consider the following example.

Example 6.3.5: Using a Migration Matrix

Consider the migration matrix

$$A = \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix}$$

for locations $1, 2,$ and 3. Suppose initially there are 100 residents in location 1, 200 in location 2 and 400 in location 3. Find the population in the three locations after $1, 2,$ and 10 units of time.

Solution

Using Theorem 6.3.2 we can find the population in each location using the equation $X_{n+1} = AX_n$. For the population after 1 unit, we calculate $X_1 = AX_0$ as follows.

$$\begin{aligned} X_1 &= AX_0 \\ \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} &= \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix} \begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix} \\ &= \begin{bmatrix} 100 \\ 180 \\ 420 \end{bmatrix} \end{aligned} \tag{6.3.4}$$

Therefore after one time period, location 1 has 100 residents, location 2 has 180, and location 3 has 420. Notice that the **total** population is unchanged, it simply migrates within the given locations. We find the locations after two time periods in the same way.

$$
\begin{aligned}
X_2 &= AX_1 \\
\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} &= \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix} \begin{bmatrix} 100 \\ 180 \\ 420 \end{bmatrix} \\
&= \begin{bmatrix} 102 \\ 164 \\ 434 \end{bmatrix}
\end{aligned} \tag{6.3.5}
$$

We could progress in this manner to find the populations after 10 time periods. However from our above discussion, we can simply calculate $(A^n X_0)_i$, where n denotes the number of time periods which have passed. Therefore, we compute the populations in each location after 10 units of time as follows.

$$
\begin{aligned}
X_{10} &= A^{10} X_0 \\
\begin{bmatrix} x_{110} \\ x_{210} \\ x_{310} \end{bmatrix} &= \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix}^{10} \begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix} \\
&= \begin{bmatrix} 115.085\,829\,22 \\ 120.130\,672\,44 \\ 464.783\,498\,34 \end{bmatrix}
\end{aligned} \tag{6.3.6}
$$

Since we are speaking about populations, we would need to round these numbers to provide a logical answer. Therefore, we can say that after 10 units of time, there will be 115 residents in location one, 120 in location two, and 465 in location three.

A second important application of Markov matrices is the concept of random walks. Suppose a walker has m locations to choose from, denoted $1, 2, \cdots, m$. Let a_{ij} refer to the probability that the person will travel **to** location i **from** location j. Again, this requires that

$$\sum_{i=1}^{k} a_{ij} = 1$$

In this context, the vector $X_n = [x_{1n} \cdots x_{mn}]^T$ contains the probabilities x_{in} the walker ends up in location $i, 1 \leq i \leq m$ at time n.

Example 6.3.6: Random Walks

Suppose three locations exist, referred to as locations 1, 2 and 3. The Markov matrix of probabilities $A = [a_{ij}]$ is given by

$$\begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix}$$

If the walker starts in location 1, calculate the probability that he ends up in location 3 at time $n = 2$.

Solution

Since the walker begins in location 1, we have

$$X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The goal is to calculate x_{32}. To do this we calculate X_2, using $X_{n+1} = AX_n$.

$$
\begin{aligned}
X_1 &= AX_0 \\
&= \begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix}
\end{aligned} \tag{6.3.7}
$$

$$\begin{aligned} X_2 &= AX_1 \\ &= \begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.3 & 0.4 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.4 \\ 0.2 \end{bmatrix} \\ &= \begin{bmatrix} 0.3 \\ 0.42 \\ 0.28 \end{bmatrix} \end{aligned} \tag{6.3.8}$$

This gives the probabilities that our walker ends up in locations 1, 2, and 3. For this example we are interested in location 3, with a probability on 0.28.

Returning to the context of migration, suppose we wish to know how many residents will be in a certain location after a very long time. It turns out that if some power of the migration matrix has all positive entries, then there is a vector X_s such that A^nX_0 approaches X_s as n becomes very large. Hence as more time passes and n increases, A^nX_0 will become closer to the vector X_s.

Consider Theorem 6.3.2. Let n increase so that X_n approaches X_s. As X_n becomes closer to X_s, so too does X_{n+1}. For sufficiently large n, the statement $X_{n+1} = AX_n$ can be written as $X_s = AX_s$.

This discussion motivates the following theorem.

Theorem 6.3.3: Steady State Vector

Let A be a migration matrix. Then there exists a **steady state vector** written X_s such that

$$X_s = AX_s$$

where X_s has positive entries which have the same sum as the entries of X_0.

As n increases, the state vectors X_n will approach X_s.

Note that the condition in Theorem 6.3.3 can be written as $(I - A)X_s = 0$, representing a homogeneous system of equations.

Consider the following example. Notice that it is the same example as the Example 6.3.5 but here it will involve a longer time frame.

Example 6.3.7: Populations over the Long Run

Consider the migration matrix

$$A = \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix}$$

for locations $1, 2,$ and 3. Suppose initially there are 100 residents in location 1, 200 in location 2 and 400 in location 4. Find the population in the three locations after a long time.

Solution

By Theorem 6.3.3 the steady state vector X_s can be found by solving the system $(I - A)X_s = 0$.

Thus we need to find a solution to

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} .6 & 0 & .1 \\ .2 & .8 & 0 \\ .2 & .2 & .9 \end{bmatrix} \right) \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix and the resulting reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} 0.4 & 0 & -0.1 & 0 \\ -0.2 & 0.2 & 0 & 0 \\ -0.2 & -0.2 & 0.1 & 0 \end{array} \right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -0.25 & 0 \\ 0 & 1 & -0.25 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

Therefore, the eigenvectors are

$$t\begin{bmatrix} 0.25 \\ 0.25 \\ 1 \end{bmatrix}$$

The initial vector X_0 is given by

$$\begin{bmatrix} 100 \\ 200 \\ 400 \end{bmatrix}$$

Now all that remains is to choose the value of t such that

$$0.25t + 0.25t + t = 100 + 200 + 400$$

Solving this equation for t yields $t = \frac{1400}{3}$. Therefore the population in the long run is given by

$$\frac{1400}{3}\begin{bmatrix} 0.25 \\ 0.25 \\ 1 \end{bmatrix} = \begin{bmatrix} 116.666\,666\,666\,666\,7 \\ 116.666\,666\,666\,666\,7 \\ 466.666\,666\,666\,666\,7 \end{bmatrix}$$

Again, because we are working with populations, these values need to be rounded. The steady state vector X_s is given by

$$\begin{bmatrix} 117 \\ 117 \\ 466 \end{bmatrix}$$

We can see that the numbers we calculated in Example 6.3.5 for the populations after the 10th unit of time are not far from the long term values.

Consider another example.

Example 6.3.8: Populations After a Long Time

Suppose a migration matrix is given by

$$A = \begin{bmatrix} \frac{1}{5} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{11}{20} & \frac{1}{4} & \frac{3}{10} \end{bmatrix}$$

Find the comparison between the populations in the three locations after a long time.

Solution

In order to compare the populations in the long term, we want to find the steady state vector X_s. Solve

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{5} & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{11}{20} & \frac{1}{4} & \frac{3}{10} \end{bmatrix}\right)\begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix and the resulting reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} \frac{4}{5} & -\frac{1}{2} & -\frac{1}{5} & 0 \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{2} & 0 \\ -\frac{11}{20} & -\frac{1}{4} & \frac{7}{10} & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{16}{19} & 0 \\ 0 & 1 & -\frac{18}{19} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so an eigenvector is

$$\begin{bmatrix} 16 \\ 18 \\ 19 \end{bmatrix}$$

Therefore, the proportion of population in location 2 to location 1 is given by $\frac{18}{16}$. The proportion of population 3 to location 2 is given by $\frac{19}{18}$.

Eigenvalues of Markov Matrices

The following is an important proposition.

Proposition 6.3.1: Eigenvalues of a Migration Matrix

Let $A = [a_{ij}]$ be a migration matrix. Then 1 is always an eigenvalue for A.

Proof

Remember that the determinant of a matrix always equals that of its transpose. Therefore,

$$\det(\lambda I - A) = \det\left((\lambda I - A)^T\right) = \det\left(\lambda I - A^T\right)$$

because $I^T = I$. Thus the characteristic equation for A is the same as the characteristic equation for A^T. Consequently, A and A^T have the same eigenvalues. We will show that 1 is an eigenvalue for A^T and then it will follow that 1 is an eigenvalue for A.

Remember that for a migration matrix, $\sum_i a_{ij} = 1$. Therefore, if $A^T = [b_{ij}]$ with $b_{ij} = a_{ji}$, it follows that

$$\sum_j b_{ij} = \sum_j a_{ji} = 1$$

Therefore, from matrix multiplication,

$$A^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \sum_j b_{ij} \\ \vdots \\ \sum_j b_{ij} \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Notice that this shows that $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is an eigenvector for A^T corresponding to the eigenvalue, $\lambda = 1$. As explained above, this shows that $\lambda = 1$ is an eigenvalue for A because A and A^T have the same eigenvalues.

Dynamical Systems

The migration matrices discussed above give an example of a discrete dynamical system. We call them discrete because they involve discrete values taken at a sequence of points rather than on a continuous interval of time.

An example of a situation which can be studied in this way is a predator prey model. Consider the following model where x is the number of prey and y the number of predators in a certain area at a certain time. These are functions of $n \in \mathbb{N}$ where $n = 1, 2, \cdots$ are the ends of intervals of time which may be of interest in the problem. In other words, $x(n)$ is the number of prey at the end of the nth interval of time. An example of this situation may be modeled by the following equation

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

This says that from time period n to $n+1$, x increases if there are more x and decreases as there are more y. In the context of this example, this means that as the number of predators increases, the number of prey decreases. As for y, it increases if there are more y and also if there are more x.

This is an example of a matrix recurrence which we define now.

Definition 6.3.2: Matrix Recurrence

Suppose a dynamical system is given by

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned} \tag{6.3.9}$$

This system can be expressed as $V_{n+1} = AV_n$ where $V_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

In this section, we will examine how to find solutions to a dynamical system given certain initial conditions. This process involves several concepts previously studied, including matrix diagonalization and Markov matrices. The procedure is given as follows. Recall that when diagonalized, we can write $A^n = PD^nP^{-1}$.

Procedure 6.3.1: Solving a Dynamical System

Suppose a dynamical system is given by

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned} \tag{6.3.10}$$

Given initial conditions x_0 and y_0, the solutions to the system are found as follows:

1. Express the dynamical system in the form $V_{n+1} = AV_n$.
2. Diagonalize A to be written as $A = PDP^{-1}$.
3. Then $V_n = PD^nP^{-1}V_0$ where V_0 is the vector containing the initial conditions.
4. If given specific values for n, substitute into this equation. Otherwise, find a general solution for n.

We will now consider an example in detail.

Example 6.3.9: Solutions of a Discrete Dynamical System

Suppose a dynamical system is given by

$$\begin{aligned} x_{n+1} &= 1.5x_n - 0.5y_n \\ y_{n+1} &= 1.0x_n \end{aligned} \tag{6.3.11}$$

Express this system as a matrix recurrence and find solutions to the dynamical system for initial conditions $x_0 = 20, y_0 = 10$.

Solution

First, we express the system as a matrix recurrence.

$$\begin{aligned} V_{n+1} &= AV_n \\ \begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} &= \begin{bmatrix} 1.5 & -0.5 \\ 1.0 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} \end{aligned} \tag{6.3.12}$$

Then

$$A = \begin{bmatrix} 1.5 & -0.5 \\ 1.0 & 0 \end{bmatrix}$$

You can verify that the eigenvalues of A are 1 and $.5$. By diagonalizing, we can write A in the form

$$P^{-1}DP = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Now given an initial condition

$$V_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

the solution to the dynamical system is given by

$$
\begin{aligned}
V_n &= PD^nP^{-1}V_0 \\
\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .5 \end{bmatrix}^n \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (.5)^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\
&= \begin{bmatrix} y_0((.5)^n - 1) - x_0((.5)^n - 2) \\ y_0(2(.5)^n - 1) - x_0(2(.5)^n - 2) \end{bmatrix}
\end{aligned} \tag{6.3.13}
$$

If we let n become arbitrarily large, this vector approaches

$$\begin{bmatrix} 2x_0 - y_0 \\ 2x_0 - y_0 \end{bmatrix}$$

Thus for large n,

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} \approx \begin{bmatrix} 2x_0 - y_0 \\ 2x_0 - y_0 \end{bmatrix}$$

Now suppose the initial condition is given by

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$$

Then, we can find solutions for various values of n. Here are the solutions for values of n between 1 and 5

$$n = 1 : \begin{bmatrix} 25.0 \\ 20.0 \end{bmatrix}, n = 2 : \begin{bmatrix} 27.5 \\ 25.0 \end{bmatrix}, n = 3 : \begin{bmatrix} 28.75 \\ 27.5 \end{bmatrix}$$

$$n = 4 : \begin{bmatrix} 29.375 \\ 28.75 \end{bmatrix}, n = 5 : \begin{bmatrix} 29.688 \\ 29.375 \end{bmatrix}$$

Notice that as n increases, we approach the vector given by

$$\begin{bmatrix} 2x_0 - y_0 \\ 2x_0 - y_0 \end{bmatrix} = \begin{bmatrix} 2(20) - 10 \\ 2(20) - 10 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \end{bmatrix}$$

These solutions are graphed in the following figure.

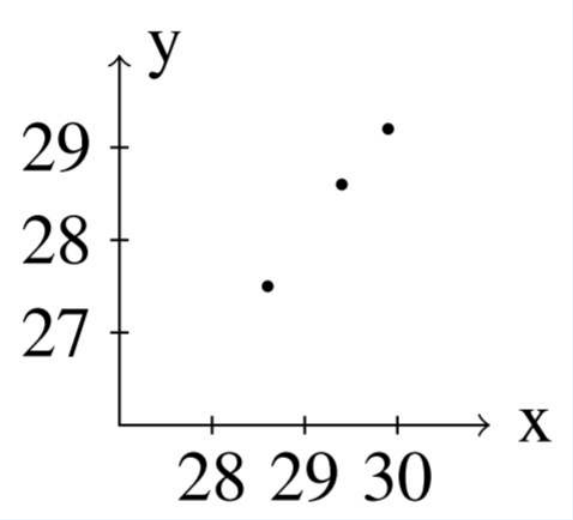


Figure 6.3.1

The following example demonstrates another system which exhibits some interesting behavior. When we graph the solutions, it is possible for the ordered pairs to spiral around the origin.

Example 6.3.10: Finding Solutions to a Dynamical System

Suppose a dynamical system is of the form

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

Find solutions to the dynamical system for given initial conditions.

Solution

Let

$$A = \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix}$$

To find solutions, we must diagonalize A. You can verify that the eigenvalues of A are complex and are given by $\lambda_1 = .7 + .7i$ and $\lambda_2 = .7 - .7i$. The eigenvector for $\lambda_1 = .7 + .7i$ is

$$\begin{bmatrix} 1 \\ i \end{bmatrix}$$

and that the eigenvector for $\lambda_2 = .7 - .7i$ is

$$\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Thus the matrix A can be written in the form

$$\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} .7+.7i & 0 \\ 0 & .7-.7i \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{bmatrix}$$

and so,

$$\begin{aligned} V_n &= PD^nP^{-1}V_0 \\ \begin{bmatrix} x(n) \\ y(n) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \begin{bmatrix} (.7+.7i)^n & 0 \\ 0 & (.7-.7i)^n \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \end{aligned} \tag{6.3.14}$$

The explicit solution is given by

$$\begin{bmatrix} x_0\left(\frac{1}{2}(0.7-0.7i)^n + \frac{1}{2}(0.7+0.7i)^n\right) + y_0\left(\frac{1}{2}i(0.7-0.7i)^n - \frac{1}{2}i(0.7+0.7i)^n\right) \\ y_0\left(\frac{1}{2}(0.7-0.7i)^n + \frac{1}{2}(0.7+0.7i)^n\right) - x_0\left(\frac{1}{2}i(0.7-0.7i)^n - \frac{1}{2}i(0.7+0.7i)^n\right) \end{bmatrix}$$

Suppose the initial condition is

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Then one obtains the following sequence of values which are graphed below by letting $n = 1, 2, \cdots, 20$

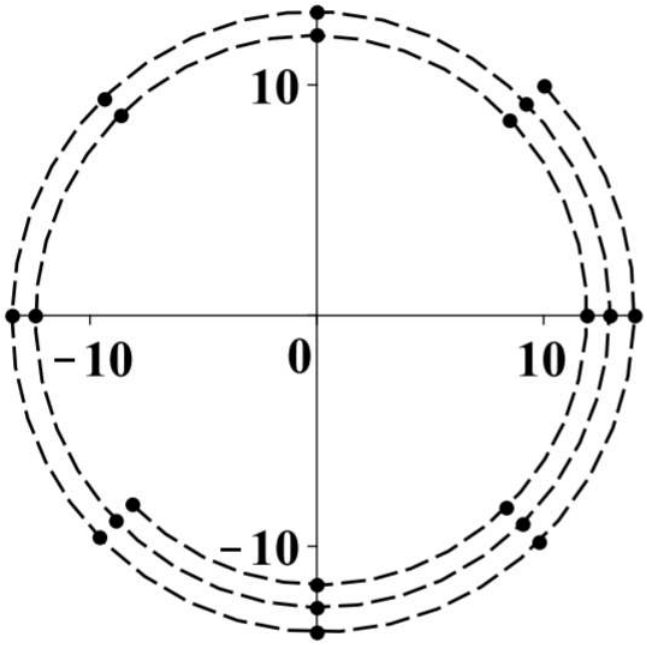


Figure 6.3.2

In this picture, the dots are the values and the dashed line is to help to picture what is happening.

These points are getting gradually closer to the origin, but they are circling the origin in the clockwise direction as they do so. As n increases, the vector $\begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$ approaches $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

This type of behavior along with complex eigenvalues is typical of the deviations from an equilibrium point in the Lotka Volterra system of differential equations which is a famous model for predator-prey interactions. These differential equations are given by

$$\begin{aligned} x' &= x(a - by) \\ y' &= -y(c - dx) \end{aligned} \tag{6.3.15}$$

where a, b, c, d are positive constants. For example, you might have X be the population of moose and Y the population of wolves on an island.

Note that these equations make logical sense. The top says that the rate at which the moose population increases would be aX if there were no predators Y. However, this is modified by multiplying instead by $(a - bY)$ because if there are predators, these will militate against the population of moose. The more predators there are, the more pronounced is this effect. As to the predator equation, you can see that the equations predict that if there are many prey around, then the rate of growth of the predators would seem to be high. However, this is modified by the term $-cY$ because if there are many predators, there would be competition for the available food supply and this would tend to decrease Y'.

The behavior near an equilibrium point, which is a point where the right side of the differential equations equals zero, is of great interest. In this case, the equilibrium point is

$$x = \frac{c}{d}, y = \frac{a}{b}$$

Then one defines new variables according to the formula

$$x + \frac{c}{d} = x, \; y = y + \frac{a}{b}$$

In terms of these new variables, the differential equations become

$$\begin{aligned} x' &= \left(x + \frac{c}{d}\right)\left(a - b\left(y + \frac{a}{b}\right)\right) \\ y' &= -\left(y + \frac{a}{b}\right)\left(c - d\left(x + \frac{c}{d}\right)\right) \end{aligned} \tag{6.3.16}$$

Multiplying out the right sides yields

$$\begin{aligned} x' &= -bxy - b\frac{c}{d}y \\ y' &= dxy + \frac{a}{b}dx \end{aligned} \tag{6.3.17}$$

The interest is for x, y small and so these equations are essentially equal to

$$x' = -b\frac{c}{d}y,\ y' = \frac{a}{b}dx$$

Replace x' with the difference quotient $\frac{x(t+h)-x(t)}{h}$ where h is a small positive number and y' with a similar difference quotient. For example one could have h correspond to one day or even one hour. Thus, for h small enough, the following would seem to be a good approximation to the differential equations.

$$\begin{aligned} x(t+h) &= x(t) - hb\frac{c}{d}y \\ y(t+h) &= y(t) + h\frac{a}{b}dx \end{aligned} \tag{6.3.18}$$

Let $1, 2, 3, \cdots$ denote the ends of discrete intervals of time having length h chosen above. Then the above equations take the form

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{hbc}{d} \\ \frac{had}{b} & 1 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

Note that the eigenvalues of this matrix are always complex.

We are not interested in time intervals of length h for h very small. Instead, we are interested in much longer lengths of time. Thus, replacing the time interval with mh,

$$\begin{bmatrix} x(n+m) \\ y(n+m) \end{bmatrix} = \begin{bmatrix} 1 & -\frac{hbc}{d} \\ \frac{had}{b} & 1 \end{bmatrix}^m \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

For example, if $m = 2$, you would have

$$\begin{bmatrix} x(n+2) \\ y(n+2) \end{bmatrix} = \begin{bmatrix} 1 - ach^2 & -2b\frac{c}{d}h \\ 2\frac{a}{b}dh & 1 - ach^2 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

Note that most of the time, the eigenvalues of the new matrix will be complex.

You can also notice that the upper right corner will be negative by considering higher powers of the matrix. Thus letting $1, 2, 3, \cdots$ denote the ends of discrete intervals of time, the desired discrete dynamical system is of the form

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} a & -b \\ c & d \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}$$

where a, b, c, d are positive constants and the matrix will likely have complex eigenvalues because it is a power of a matrix which has complex eigenvalues.

You can see from the above discussion that if the eigenvalues of the matrix used to define the dynamical system are less than 1 in absolute value, then the origin is stable in the sense that as $n \to \infty$, the solution converges to the origin. If either eigenvalue is larger than 1 in absolute value, then the solutions to the dynamical system will usually be unbounded, unless the initial condition is chosen very carefully. The next example exhibits the case where one eigenvalue is larger than 1 and the other is smaller than 1.

The following example demonstrates a familiar concept as a dynamical system.

Example 6.3.11: The Fibonacci Sequence

The Fibonacci sequence is the sequence given by

$$1, 1, 2, 3, 5, \cdots$$

which is defined recursively in the form

$$x(0)=1=x(1),\ x(n+2)=x(n+1)+x(n)$$

Show how the Fibonacci Sequence can be considered a dynamical system.

Solution

This sequence is extremely important in the study of reproducing rabbits. It can be considered as a dynamical system as follows. Let $y(n)=x(n+1)$. Then the above recurrence relation can be written as

$$\begin{bmatrix} x(n+1) \\ y(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x(n) \\ y(n) \end{bmatrix}, \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues of the matrix A are $\lambda_1 = \frac{1}{2} - \frac{1}{2}\sqrt{5}$ and $\lambda_2 = \frac{1}{2}\sqrt{5} + \frac{1}{2}$. The corresponding eigenvectors are, respectively,

$$X_1 = \begin{bmatrix} -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{bmatrix}, X_2 = \begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 \end{bmatrix}$$

You can see from a short computation that one of the eigenvalues is smaller than 1 in absolute value while the other is larger than 1 in absolute value. Now, diagonalizing A gives us

$$\begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}\sqrt{5} + \frac{1}{2} & 0 \\ 0 & \frac{1}{2} - \frac{1}{2}\sqrt{5} \end{bmatrix}$$

Then it follows that for a given initial condition, the solution to this dynamical system is of the form

$$\begin{bmatrix} x(n) \\ y(n) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\sqrt{5} - \frac{1}{2} & -\frac{1}{2}\sqrt{5} - \frac{1}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\right)^n & 0 \\ 0 & \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^n \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{5}\sqrt{5} & \frac{1}{10}\sqrt{5} + \frac{1}{2} \\ -\frac{1}{5}\sqrt{5} & \frac{1}{5}\sqrt{5}\left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6.3.19}$$

It follows that

$$x(n) = \left(\frac{1}{2}\sqrt{5} + \frac{1}{2}\right)^n \left(\frac{1}{10}\sqrt{5} + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{2}\sqrt{5}\right)^n \left(\frac{1}{2} - \frac{1}{10}\sqrt{5}\right)$$

Here is a picture of the ordered pairs $(x(n), y(n))$ for $n = 0, 1, \cdots, n$.

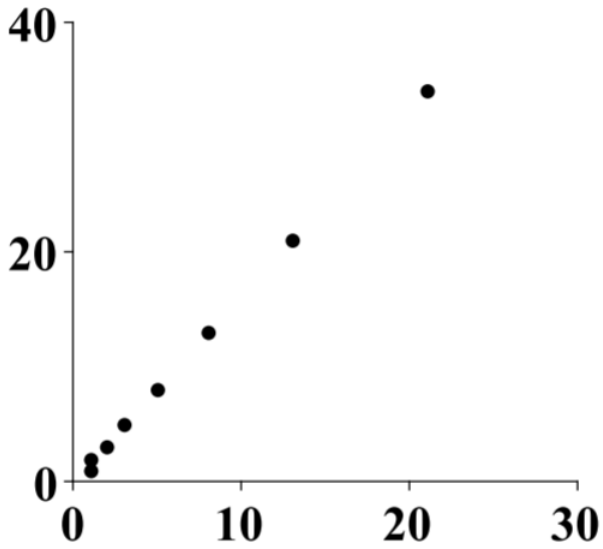


Figure **6.3.3**

There is so much more that can be said about dynamical systems. It is a major topic of study in differential equations and what is given above is just an introduction.

The Matrix Exponential

The goal of this section is to use the concept of the matrix exponential to solve first order linear differential equations. We begin by proving the matrix exponential.

Suppose A is a diagonalizable matrix. Then the **matrix exponential**, written e^A, can be easily defined. Recall that if D is a diagonal matrix, then

$$P^{-1}AP = D$$

D is of the form

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \tag{6.3.20}$$

and it follows that

$$D^m = \begin{bmatrix} \lambda_1^m & & 0 \\ & \ddots & \\ 0 & & \lambda_n^m \end{bmatrix}$$

Since A is diagonalizable,

$$A = PDP^{-1}$$

and

$$A^m = PD^mP^{-1}$$

Recall why this is true.

$$A = PDP^{-1}$$

and so

$$\begin{aligned} A^m &= \overbrace{PDP^{-1}PDP^{-1}PDP^{-1}\cdots PDP^{-1}}^{\text{m times}} \\ &= PD^mP^{-1} \end{aligned} \tag{6.3.21}$$

We now will examine what is meant by the matrix exponential e^A. Begin by formally writing the following power series for e^A:

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!} = P\left(\sum_{k=0}^{\infty} \frac{D^k}{k!}\right)P^{-1}$$

If D is given above in (6.3.20), the above sum is of the form

$$P\left(\sum_{k=0}^{\infty}\begin{bmatrix} \frac{1}{k!}\lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \frac{1}{k!}\lambda_n^k \end{bmatrix}\right)P^{-1}$$

This can be rearranged as follows:

$$\begin{aligned} e^A &= P\begin{bmatrix} \sum_{k=0}^{\infty}\frac{1}{k!}\lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \sum_{k=0}^{\infty}\frac{1}{k!}\lambda_n^k \end{bmatrix}P^{-1} \\ &= P\begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}P^{-1} \end{aligned}$$

This justifies the following theorem.

Theorem 6.3.4: The Matrix Exponential

Let A be a diagonalizable matrix, with eigenvalues $\lambda_1, \ldots, \lambda_n$and corresponding matrix of eigenvectors P. Then the matrix exponential, e^A, is given by

$$e^A = P\begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}P^{-1}$$

Example 6.3.12: Compute e^A for a Matrix A

Let

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

Find e^A.

Solution

The eigenvalues work out to be $1, 2, 3$ and eigenvectors associated with these eigenvalues are

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \leftrightarrow 1, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \leftrightarrow 2, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \leftrightarrow 3$$

Then let

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, P = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

and so

$$P^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then the matrix exponential is

$$e^{At} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^1 & 0 & 0 \\ 0 & e^2 & 0 \\ 0 & 0 & e^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} e^2 & e^2 - e^3 & e^2 - e^3 \\ e^2 - e & e^2 & e^2 - e \\ -e^2 + e & -e^2 + e^3 & -e^2 + e + e^3 \end{bmatrix}$$

The matrix exponential is a useful tool to solve autonomous systems of first order linear differential equations. These are equations which are of the form

$$X' = AX, X(0) = C$$

where A is a diagonalizable $n \times n$ matrix and C is a constant vector. X is a vector of functions in one variable, t:

$$X = X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Then X' refers to the first derivative of X and is given by

$$X' = X'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix}, \; x_i'(t) = \text{the derivative of } x_i(t)$$

Then it turns out that the solution to the above system of equations is $X(t) = e^{At}C$. To see this, suppose A is diagonalizable so that

$$A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1}$$

Then

$$e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

$$e^{At}C = P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}C$$

Differentiating $e^{At}C$ yields

$$X' = \left(e^{At}C\right)' = P \begin{bmatrix} \lambda_1 e^{\lambda_1 t} & & \\ & \ddots & \\ & & \lambda_n e^{\lambda_n t} \end{bmatrix} P^{-1}C$$

$$= P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1}C$$

$$\begin{aligned} &= P\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^{-1} P \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} P^{-1} C \\ &= A\left(e^{At} C\right) = AX \end{aligned} \tag{6.3.22}$$

Therefore $X = X(t) = e^{At}C$ is a solution to $X' = AX$.

To prove that $X(0) = C$ if $X(t) = e^{At}C$:

$$X(0) = e^{A0}C = P\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} P^{-1}C = C$$

Example 6.3.13: Solving an Initial Value Problem

Solve the initial value problem

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solution

The matrix is diagonalizable and can be written as

$$\begin{aligned} A &= PDP^{-1} \\ \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix} \end{aligned} \tag{6.3.23}$$

Therefore, the matrix exponential is of the form

$$e^{At} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix}$$

The solution to the initial value problem is

$$\begin{aligned} X(t) &= e^{At}C \\ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4e^t - 3e^{2t} \\ 3e^{2t} - 2e^t \end{bmatrix} \end{aligned} \tag{6.3.24}$$

We can check that this works:

$$\begin{aligned} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} &= \begin{bmatrix} 4e^0 - 3e^{2(0)} \\ 3e^{2(0)} - 2e^0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned} \tag{6.3.25}$$

Lastly,

$$X' = \begin{bmatrix} 4e^t - 3e^{2t} \\ 3e^{2t} - 2e^t \end{bmatrix}' = \begin{bmatrix} 4e^t - 6e^{2t} \\ 6e^{2t} - 2e^t \end{bmatrix}$$

and

$$AX = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 4e^t - 3e^{2t} \\ 3e^{2t} - 2e^t \end{bmatrix} = \begin{bmatrix} 4e^t - 6e^{2t} \\ 6e^{2t} - 2e^t \end{bmatrix}$$

which is the same thing. Thus this is the solution to the initial value problem.

This page titled 6.3: Applications of Spectral Theory is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **7.3: Applications of Spectral Theory** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

6.4: Orthogonality

Orthogonal Diagonalization

We begin this section by recalling some important definitions. Recall from Definition 4.11.4 that non-zero vectors are called orthogonal if their dot product equals $\mathbf{0}$. A set is orthonormal if it is orthogonal and each vector is a unit vector.

An orthogonal matrix U, from Definition 4.11.7, is one in which $UU^T = I$. In other words, the transpose of an orthogonal matrix is equal to its inverse. A key characteristic of orthogonal matrices, which will be essential in this section, is that the columns of an orthogonal matrix form an orthonormal set.

We now recall another important definition.

Definition 6.4.1: Symmetric and Skew Symmetric Matrices

A real $n \times n$ matrix A, is **symmetric** if $A^T = A$. If $A = -A^T$, then A is called **skew symmetric.**

Before proving an essential theorem, we first examine the following lemma which will be used below.

Lemma 6.4.1: The Dot Product

Let $A = (a_{ij})$ be a real symmetric $n \times n$ matrix, and let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then

$$A\vec{x} \cdot \vec{y} = \vec{x} \cdot A\vec{y}$$

Proof

This result follows from the definition of the dot product together with properties of matrix multiplication, as follows:

$$\begin{aligned} A\vec{x} \cdot \vec{y} &= \sum_{k,l} a_{kl} x_l y_k \\ &= \sum_{k,l} (a_{lk})^T x_l y_k \\ &= \vec{x} \cdot A^T \vec{y} \\ &= \vec{x} \cdot A\vec{y} \end{aligned} \tag{6.4.1}$$

The last step follows from $A^T = A$, since A is symmetric.

We can now prove that the eigenvalues of a real symmetric matrix are real numbers. Consider the following important theorem.

Theorem 6.4.1: Orthogonal Eigenvectors

Let A be a real symmetric matrix. Then the eigenvalues of A are real numbers and eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof

Recall that for a complex number $a + ib$, the complex conjugate, denoted by $\overline{a+ib}$ is given by $\overline{a+ib} = a - ib$. The notation, $\overline{\vec{x}}$ will denote the vector which has every entry replaced by its complex conjugate.

Suppose A is a real symmetric matrix and $A\vec{x} = \lambda\vec{x}$. Then

$$\overline{\lambda\vec{x}}^T \vec{x} = \left(\overline{A\vec{x}}\right)^T \vec{x} = \overline{\vec{x}}^T A^T \vec{x} = \overline{\vec{x}}^T A\vec{x} = \lambda \overline{\vec{x}}^T \vec{x}$$

Dividing by $\overline{\vec{x}}^T \vec{x}$ on both sides yields $\overline{\lambda} = \lambda$ which says λ is real. To do this, we need to ensure that $\overline{\vec{x}}^T \vec{x} \neq 0$. Notice that $\overline{\vec{x}}^T \vec{x} = 0$ if and only if $\vec{x} = \vec{0}$. Since we chose $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$, $\vec{x}$ is an eigenvector and therefore must be nonzero.

Now suppose A is real symmetric and $A\vec{x} = \lambda\vec{x}$, $A\vec{y} = \mu\vec{y}$ where $\mu \neq \lambda$. Then since A is symmetric, it follows from Lemma 6.4.1 about the dot product that

$$\lambda\vec{x}\cdot\vec{y} = A\vec{x}\cdot\vec{y} = \vec{x}\cdot A\vec{y} = \vec{x}\cdot\mu\vec{y} = \mu\vec{x}\cdot\vec{y}$$

Hence $(\lambda-\mu)\vec{x}\cdot\vec{y} = 0$. It follows that, since $\lambda-\mu \neq 0$, it must be that $\vec{x}\cdot\vec{y} = 0$. Therefore the eigenvectors form an orthogonal set.

The following theorem is proved in a similar manner.

Theorem 6.4.2: Eigenvalues of Skew Symmetric Matrix

The eigenvalues of a real skew symmetric matrix are either equal to 0 or are pure imaginary numbers.

Proof

First, note that if $A = 0$ is the zero matrix, then A is skew symmetric and has eigenvalues equal to 0.

Suppose $A = -A^T$ so A is skew symmetric and $A\vec{x} = \lambda\vec{x}$. Then

$$\overline{\lambda\vec{x}}^T\vec{x} = \left(\overline{A\vec{x}}\right)^T\vec{x} = \overline{\vec{x}}^T A^T\vec{x} = -\overline{\vec{x}}^T A\vec{x} = -\lambda\overline{\vec{x}}^T\vec{x}$$

and so, dividing by $\overline{\vec{x}}^T\vec{x}$ as before, $\overline{\lambda} = -\lambda$. Letting $\lambda = a+ib$, this means $a-ib = -a-ib$ and so $a = 0$. Thus λ is pure imaginary.

Consider the following example.

Example 6.4.1: Eigenvalues of a Skew Symmetric Matrix

Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Find its eigenvalues.

Solution

First notice that A is skew symmetric. By Theorem 6.4.2, the eigenvalues will either equal 0 or be pure imaginary. The eigenvalues of A are obtained by solving the usual equation

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda & 1 \\ -1 & \lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

Hence the eigenvalues are $\pm i$, pure imaginary.

Consider the following example.

Example 6.4.2: Eigenvalues of a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$. Find its eigenvalues.

Solution

First, notice that A is symmetric. By Theorem 6.4.1, the eigenvalues will all be real. The eigenvalues of A are obtained by solving the usual equation

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda-1 & -2 \\ -2 & \lambda-3 \end{bmatrix} = \lambda^2 - 4\lambda - 1 = 0$$

The eigenvalues are given by $\lambda_1 = 2+\sqrt{5}$ and $\lambda_2 = 2-\sqrt{5}$ which are both real.

Recall that a diagonal matrix $D = (d_{ij})$ is one in which $d_{ij} = 0$ whenever $i \neq j$. In other words, all numbers not on the main diagonal are equal to zero.

Consider the following important theorem.

Theorem 6.4.3: Orthogonal Diagonalization

Let A be a real symmetric matrix. Then there exists an orthogonal matrix U such that

$$U^T AU = D$$

where D is a diagonal matrix. Moreover, the diagonal entries of D are the eigenvalues of A.

We can use this theorem to diagonalize a symmetric matrix, using orthogonal matrices. Consider the following corollary.

Corollary 6.4.1: Orthonormal Set of Eigenvectors

If A is a real $n \times n$ symmetric matrix, then there exists an orthonormal set of eigenvectors, $\{\vec{u}_1, \cdots, \vec{u}_n\}$.

Proof

Since A is symmetric, then by Theorem 6.4.3, there exists an orthogonal matrix U such that $U^T AU = D$, a diagonal matrix whose diagonal entries are the eigenvalues of A. Therefore, since A is symmetric and all the matrices are real,

$$\overline{D} = \overline{D^T} = \overline{U^T A^T U} = U^T A^T U = U^T AU = D$$

showing D is real because each entry of D equals its complex conjugate.

Now let

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_n \end{bmatrix}$$

where the $\vec{u}_i$ denote the columns of U and

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

The equation, $U^T AU = D$ implies $AU = UD$ and

$$\begin{aligned} AU &= \begin{bmatrix} A\vec{u}_1 & A\vec{u}_2 & \cdots & A\vec{u}_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 \vec{u}_1 & \lambda_2 \vec{u}_2 & \cdots & \lambda_n \vec{u}_n \end{bmatrix} \\ &= UD \end{aligned} \tag{6.4.2}$$

where the entries denote the columns of AU and UD respectively. Therefore, $A\vec{u}_i = \lambda_i \vec{u}_i$. Since the matrix U is orthogonal, the ij^{th} entry of $U^T U$ equals δ_{ij} and so

$$\delta_{ij} = \vec{u}_i^T \vec{u}_j = \vec{u}_i \cdot \vec{u}_j$$

This proves the corollary because it shows the vectors $\{\vec{u}_i\}$ form an orthonormal set.

Definition 6.4.2: Principal Axes

Let A be an $n \times n$ matrix. Then the principal axes of A is a set of orthonormal eigenvectors of A.

In the next example, we examine how to find such a set of orthonormal eigenvectors.

Example 6.4.3: Find an Orthonormal Set of Eigenvectors

Find an orthonormal set of eigenvectors for the symmetric matrix

$$A = \begin{bmatrix} 17 & -2 & -2 \\ -2 & 6 & 4 \\ -2 & 4 & 6 \end{bmatrix}$$

Solution

Recall Procedure 7.1.1 for finding the eigenvalues and eigenvectors of a matrix. You can verify that the eigenvalues are $18, 9, 2$. First find the eigenvector for 18 by solving the equation $(18I - A)X = 0$. The appropriate augmented matrix is given by

$$\left[\begin{array}{ccc|c} 18-17 & 2 & 2 & 0 \\ 2 & 18-6 & -4 & 0 \\ 2 & -4 & 18-6 & 0 \end{array}\right]$$

The reduced row-echelon form is

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore an eigenvector is

$$\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$$

Next find the eigenvector for $\lambda = 9$. The augmented matrix and resulting reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 9-17 & 2 & 2 & 0 \\ 2 & 9-6 & -4 & 0 \\ 2 & -4 & 9-6 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus an eigenvector for $\lambda = 9$ is

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Finally find an eigenvector for $\lambda = 2$. The appropriate augmented matrix and reduced row-echelon form are

$$\left[\begin{array}{ccc|c} 2-17 & 2 & 2 & 0 \\ 2 & 2-6 & -4 & 0 \\ 2 & -4 & 2-6 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus an eigenvector for $\lambda = 2$ is

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The set of eigenvectors for A is given by

$$\left\{\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$$

You can verify that these eigenvectors form an orthogonal set. By dividing each eigenvector by its magnitude, we obtain an orthonormal set:

$$\left\{\frac{1}{\sqrt{18}}\begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$$

Consider the following example.

Example 6.4.4: Repeated Eigenvalues

Find an orthonormal set of three eigenvectors for the matrix

$$A = \begin{bmatrix} 10 & 2 & 2 \\ 2 & 13 & 4 \\ 2 & 4 & 13 \end{bmatrix}$$

Solution

You can verify that the eigenvalues of A are 9 (with multiplicity two) and 18 (with multiplicity one). Consider the eigenvectors corresponding to $\lambda = 9$. The appropriate augmented matrix and reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} 9-10 & -2 & -2 & 0 \\ -2 & 9-13 & -4 & 0 \\ -2 & -4 & 9-13 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so eigenvectors are of the form

$$\begin{bmatrix} -2y-2z \\ y \\ z \end{bmatrix}$$

We need to find two of these which are orthogonal. Let one be given by setting $z = 0$ and $y = 1$, giving $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

In order to find an eigenvector orthogonal to this one, we need to satisfy

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2y-2z \\ y \\ z \end{bmatrix} = 5y + 4z = 0$$

The values $y = -4$ and $z = 5$ satisfy this equation, giving another eigenvector corresponding to $\lambda = 9$ as

$$\begin{bmatrix} -2(-4) - 2(5) \\ (-4) \\ 5 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}$$

Next find the eigenvector for $\lambda = 18$. The augmented matrix and the resulting reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} 18-10 & -2 & -2 & 0 \\ -2 & 18-13 & -4 & 0 \\ -2 & -4 & 18-13 & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so an eigenvector is

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Dividing each eigenvector by its length, the orthonormal set is

$$\left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \frac{\sqrt{5}}{15} \begin{bmatrix} -2 \\ -4 \\ 5 \end{bmatrix}, \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$$

In the above solution, the repeated eigenvalue implies that there would have been many other orthonormal bases which could have been obtained. While we chose to take $z = 0, y = 1$, we could just as easily have taken $y = 0$ or even $y = z = 1$. Any such change would have resulted in a different orthonormal set.

Recall the following definition.

Definition 6.4.3: Diagonalizable

An $n \times n$ matrix A is said to be **non defective** or **diagonalizable** if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix.

As indicated in Theorem 6.4.3 if A is a real symmetric matrix, there exists an orthogonal matrix U such that $U^T AU = D$ where D is a diagonal matrix. Therefore, every symmetric matrix is diagonalizable because if U is an orthogonal matrix, it is invertible and its inverse is U^T. In this case, we say that A is **orthogonally diagonalizable**. Therefore every symmetric matrix is in fact orthogonally diagonalizable. The next theorem provides another way to determine if a matrix is orthogonally diagonalizable.

Theorem 6.4.4: Orthogonally Diagonalizable

Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable if and only if A has an orthonormal set of eigenvectors.

Recall from Corollary 6.4.1 that every symmetric matrix has an orthonormal set of eigenvectors. In fact these three conditions are equivalent.

In the following example, the orthogonal matrix U will be found to orthogonally diagonalize a matrix.

Example 6.4.5: Diagonalize a Symmetric Matrix

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$. Find an orthogonal matrix U such that $U^T AU$ is a diagonal matrix.

Solution

In this case, the eigenvalues are 2 (with multiplicity one) and 1 (with multiplicity two). First we will find an eigenvector for the eigenvalue 2. The appropriate augmented matrix and resulting reduced row-echelon form are given by

$$\left[\begin{array}{ccc|c} 2-1 & 0 & 0 & 0 \\ 0 & 2-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 2-\frac{3}{2} & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and so an eigenvector is

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

However, it is desired that the eigenvectors be unit vectors and so dividing this vector by its length gives

$$\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Next find the eigenvectors corresponding to the eigenvalue equal to 1. The appropriate augmented matrix and resulting reduced row-echelon form are given by:

$$\left[\begin{array}{ccc|c} 1-1 & 0 & 0 & 0 \\ 0 & 1-\frac{3}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1-\frac{3}{2} & 0 \end{array}\right] \to \cdots \to \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the eigenvectors are of the form

$$\begin{bmatrix} s \\ -t \\ t \end{bmatrix}$$

Two of these which are orthonormal are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, choosing $s = 1$ and $t = 0$, and $\begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$, letting $s = 0$, $t = 1$ and normalizing the resulting vector.

To obtain the desired orthogonal matrix, we let the orthonormal eigenvectors computed above be the columns.

$$\begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

To verify, compute $U^T AU$ as follows:

$$U^T AU = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = D$$

the desired diagonal matrix. Notice that the eigenvectors, which construct the columns of U, are in the same order as the eigenvalues in D.

We conclude this section with a Theorem that generalizes earlier results.

Theorem 6.4.5: Triangulation of a Matrix

Let A be an $n \times n$ matrix. If A has n real eigenvalues, then an orthogonal matrix U can be found to result in the upper triangular matrix $U^T AU$.

triangulation

This Theorem provides a useful Corollary.

Corollary 6.4.2: Determinant and Trace

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$. Then it follows that $\det(A)$ is equal to the product of the λ_i, while $trace(A)$ is equal to the sum of the λ_i.

Proof

By Theorem 6.4.5, there exists an orthogonal matrix U such that $U^T AU = P$, where P is an upper triangular matrix. Since P is similar to A, the eigenvalues of P are $\lambda_1, \lambda_2, \ldots, \lambda_n$. Furthermore, since P is (upper) triangular, the entries on the main diagonal of P are its eigenvalues, so $\det(P) = \lambda_1 \lambda_2 \cdots \lambda_n$ and $trace(P) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Since P and A are similar, $\det(A) = \det(P)$ and $trace(A) = trace(P)$, and therefore the results follow.

The Singular Value Decomposition

We begin this section with an important definition.

Definition 6.4.4: Singular Values

Let A be an $m \times n$ matrix. The singular values of A are the square roots of the positive eigenvalues of $A^T A$.

Singular Value Decomposition (SVD) can be thought of as a generalization of orthogonal diagonalization of a symmetric matrix to an arbitrary $m \times n$ matrix. This decomposition is the focus of this section.

The following is a useful result that will help when computing the SVD of matrices.

Proposition 6.4.1: Same Nonzero Eigenvalues

Let A be an $m \times n$ matrix. Then A^TA and AA^T have the same **nonzero eigenvalues.**

Proof

Suppose A is an $m \times n$ matrix, and suppose that λ is a nonzero eigenvalue of A^TA. Then there exists a nonzero vector $X \in \mathbb{R}^n$ such that

$$(A^TA)X = \lambda X. \tag{6.4.3}$$

Multiplying both sides of this equation by A yields:

$$\begin{aligned} A(A^TA)X &= A\lambda X \\ (AA^T)(AX) &= \lambda(AX). \end{aligned} \tag{6.4.4}$$

Since $\lambda \neq 0$ and $X \neq 0_n$, $\lambda X \neq 0_n$, and thus by equation (6.4.3), $(A^TA)X \neq 0_m$; thus $A^T(AX) \neq 0_m$, implying that $AX \neq 0_m$.

Therefore AX is an eigenvector of AA^T corresponding to eigenvalue λ. An analogous argument can be used to show that every nonzero eigenvalue of AA^T is an eigenvalue of A^TA, thus completing the proof.

Given an $m \times n$ matrix A, we will see how to express A as a product

$$A = U\Sigma V^T$$

where

- U is an $m \times m$ orthogonal matrix whose columns are eigenvectors of AA^T.
- V is an $n \times n$ orthogonal matrix whose columns are eigenvectors of A^TA.
- Σ is an $m \times n$ matrix whose only nonzero values lie on its main diagonal, and are the singular values of A.

How can we find such a decomposition? We are aiming to decompose A in the following form:

$$A = U \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} V^T$$

where σ is of the form

$$\sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{bmatrix}$$

Thus $A^T = V \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} U^T$ and it follows that

$$A^TA = V \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} U^TU \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix} V^T = V \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix} V^T$$

and so $A^TAV = V \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$. Similarly, $AA^TU = U \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}$. Therefore, you would find an orthonormal basis of eigenvectors for AA^T make them the columns of a matrix such that the corresponding eigenvalues are decreasing. This gives U. You could then do the same for A^TA to get V.

We formalize this discussion in the following theorem.

Theorem 6.4.6: Singular Value Decomposition

Let A be an $m \times n$ matrix. Then there exist orthogonal matrices U and V of the appropriate size such that $A = U\Sigma V^T$ where Σ is of the form

$$\Sigma = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}$$

and σ is of the form

$$\sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_k \end{bmatrix}$$

for the σ_i the singular values of A.

Proof

There exists an orthonormal basis, $\{\vec{v}_i\}_{i=1}^n$ such that $A^TA\vec{v}_i = \sigma_i^2\vec{v}_i$ where $\sigma_i^2 > 0$ for $i = 1, \cdots, k, (\sigma_i > 0)$ and equals zero if $i > k$. Thus for $i > k$, $A\vec{v}_i = \vec{0}$ because

$$A\vec{v}_i \cdot A\vec{v}_i = A^TA\vec{v}_i \cdot \vec{v}_i = \vec{0} \cdot \vec{v}_i = 0.$$

For $i = 1, \cdots, k$, define $\vec{u}_i \in \mathbb{R}^m$ by

$$\vec{u}_i = \sigma_i^{-1}A\vec{v}_i.$$

Thus $A\vec{v}_i = \sigma_i\vec{u}_i$. Now

$$\begin{aligned} \vec{u}_i \cdot \vec{u}_j &= \sigma_i^{-1}A\vec{v}_i \cdot \sigma_j^{-1}A\vec{v}_j = \sigma_i^{-1}\vec{v}_i \cdot \sigma_j^{-1}A^TA\vec{v}_j \\ &= \sigma_i^{-1}\vec{v}_i \cdot \sigma_j^{-1}\sigma_j^2\vec{v}_j = \frac{\sigma_j}{\sigma_i}(\vec{v}_i \cdot \vec{v}_j) = \delta_{ij}. \end{aligned} \tag{6.4.5}$$

Thus $\{\vec{u}_i\}_{i=1}^k$ is an orthonormal set of vectors in $\mathbb{R}^m$. Also,

$$AA^T\vec{u}_i = AA^T\sigma_i^{-1}A\vec{v}_i = \sigma_i^{-1}AA^TA\vec{v}_i = \sigma_i^{-1}A\sigma_i^2\vec{v}_i = \sigma_i^2\vec{u}_i.$$

Now extend $\{\vec{u}_i\}_{i=1}^k$ to an orthonormal basis for all of $\mathbb{R}^m$, $\{\vec{u}_i\}_{i=1}^m$ and let

$$U = \begin{bmatrix} \vec{u}_1 & \cdots & \vec{u}_m \end{bmatrix}$$

while $V = (\vec{v}_1 \cdots \vec{v}_n)$. Thus U is the matrix which has the $\vec{u}_i$ as columns and V is defined as the matrix which has the $\vec{v}_i$ as columns. Then

$$U^TAV = \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_k^T \\ \vdots \\ \vec{u}_m^T \end{bmatrix} A \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_n \end{bmatrix}$$

$$= \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_k^T \\ \vdots \\ \vec{u}_m^T \end{bmatrix} \begin{bmatrix} \sigma_1\vec{u}_1 & \cdots & \sigma_k\vec{u}_k & \vec{0} & \cdots & \vec{0} \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}$$

where σ is given in the statement of the theorem.

The singular value decomposition has as an immediate corollary which is given in the following interesting result.

Corollary 6.4.3: Rank and Singular Values

Let A be an $m \times n$ matrix. Then the rank of A and A^Tequals the number of singular values.

Let's compute the Singular Value Decomposition of a simple matrix.

Example 6.4.6: Singular Value Decomposition

Let $A = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. Find the Singular Value Decomposition (SVD) of A.

Solution

To begin, we compute AA^T and A^TA.

$$AA^T = \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 5 \\ 5 & 11 \end{bmatrix}.$$

$$A^TA = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 6 \\ 2 & 2 & -2 \\ 6 & -2 & 10 \end{bmatrix}.$$

Since AA^T is 2×2 while A^TA is 3×3, and AA^T and A^TA have the same *nonzero* eigenvalues (by Proposition 6.4.1), we compute the characteristic polynomial $c_{AA^T}(x)$ (because it's easier to compute than $c_{A^TA}(x)$).

$$\begin{aligned} c_{AA^T}(x) = \det(xI - AA^T) &= \begin{vmatrix} x-11 & -5 \\ -5 & x-11 \end{vmatrix} \\ &= (x-11)^2 - 25 \\ &= x^2 - 22x + 121 - 25 \\ &= x^2 - 22x + 96 \\ &= (x-16)(x-6) \end{aligned} \tag{6.4.6}$$

Therefore, the eigenvalues of AA^T are $\lambda_1 = 16$ and $\lambda_2 = 6$.

The eigenvalues of A^TA are $\lambda_1 = 16$, $\lambda_2 = 6$, and $\lambda_3 = 0$, and the singular values of A are $\sigma_1 = \sqrt{16} = 4$ and $\sigma_2 = \sqrt{6}$. By convention, we list the eigenvalues (and corresponding singular values) in non increasing order (i.e., from largest to smallest).

To find the matrix V:

To construct the matrix V we need to find eigenvectors for A^TA. Since the eigenvalues of AA^T are distinct, the corresponding eigenvectors are orthogonal, and we need only normalize them.

$\lambda_1 = 16$: solve $(16I - A^TA)Y = 0$.

$$\left[\begin{array}{ccc|c} 6 & -2 & -6 & 0 \\ -2 & 14 & 2 & 0 \\ -6 & 2 & 6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } Y = \begin{bmatrix} t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, t \in \mathbb{R}.$$

$\lambda_2 = 6$: solve $(6I - A^TA)Y = 0$.

$$\left[\begin{array}{ccc|c} -4 & -2 & -6 & 0 \\ -2 & 4 & 2 & 0 \\ -6 & 2 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } Y = \begin{bmatrix} -s \\ -s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, s \in \mathbb{R}.$$

$\lambda_3 = 0$: solve $(-A^TA)Y = 0$.

$$\left[\begin{array}{ccc|c} -10 & -2 & -6 & 0 \\ -2 & -2 & 2 & 0 \\ -6 & 2 & -10 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ so } Y = \begin{bmatrix} -r \\ 2r \\ r \end{bmatrix} = r \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, r \in \mathbb{R}.$$

Let

$$V_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, V_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, V_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Then

$$V = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & -\sqrt{2} & -1 \\ 0 & -\sqrt{2} & 2 \\ \sqrt{3} & \sqrt{2} & 1 \end{bmatrix}.$$

Also,

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix},$$

and we use A, V^T, and Σ to find U.

Since V is orthogonal and $A = U\Sigma V^T$, it follows that $AV = U\Sigma$. Let $V = \begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix}$, and let $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, where U_1 and U_2 are the two columns of U.

Then we have

$$\begin{aligned} A\begin{bmatrix} V_1 & V_2 & V_3 \end{bmatrix} &= \begin{bmatrix} U_1 & U_2 \end{bmatrix}\Sigma \\ \begin{bmatrix} AV_1 & AV_2 & AV_3 \end{bmatrix} &= \begin{bmatrix} \sigma_1 U_1 + 0U_2 & 0U_1 + \sigma_2 U_2 & 0U_1 + 0U_2 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1 U_1 & \sigma_2 U_2 & 0 \end{bmatrix} \end{aligned} \tag{6.4.7}$$

which implies that $AV_1 = \sigma_1 U_1 = 4U_1$ and $AV_2 = \sigma_2 U_2 = \sqrt{6}U_2$.

Thus,

$$U_1 = \frac{1}{4}AV_1 = \frac{1}{4}\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{4\sqrt{2}}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and

$$U_2 = \frac{1}{\sqrt{6}}AV_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix}\frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 3 \\ -3 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore,

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 & 3 \\ 3 & 1 & 1 \end{bmatrix} \\ &= \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} 4 & 0 & 0 \\ 0 & \sqrt{6} & 0 \end{bmatrix}\left(\frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{3} & 0 & \sqrt{3} \\ -\sqrt{2} & -\sqrt{2} & \sqrt{2} \\ -1 & 2 & 1 \end{bmatrix}\right). \end{aligned} \tag{6.4.8}$$

Here is another example.

Example 6.4.7: Finding the SVD

Find an SVD for $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

Solution

Since A is 3×1, A^TA is a 1×1 matrix whose eigenvalues are easier to find than the eigenvalues of the 3×3 matrix AA^T.

$$A^TA = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \end{bmatrix}.$$

Thus A^TA has eigenvalue $\lambda_1 = 9$, and the eigenvalues of AA^T are $\lambda_1 = 9$, $\lambda_2 = 0$, and $\lambda_3 = 0$. Furthermore, A has only one singular value, $\sigma_1 = 3$.

To find the matrix V: To do so we find an eigenvector for A^TA and normalize it. In this case, finding a unit eigenvector is trivial: $V_1 = \begin{bmatrix} 1 \end{bmatrix}$, and

$$V = \begin{bmatrix} 1 \end{bmatrix}.$$

Also, $\Sigma = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, and we use A, V^T, and Σ to find U.

Now $AV = U\Sigma$, with $V = \begin{bmatrix} V_1 \end{bmatrix}$, and $U = \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix}$, where U_1, U_2, and U_3 are the columns of U. Thus

$$\begin{aligned} A \begin{bmatrix} V_1 \end{bmatrix} &= \begin{bmatrix} U_1 & U_2 & U_3 \end{bmatrix} \Sigma \\ \begin{bmatrix} AV_1 \end{bmatrix} &= \begin{bmatrix} \sigma_1 U_1 + 0U_2 + 0U_3 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1 U_1 \end{bmatrix} \end{aligned} \tag{6.4.9}$$

This gives us $AV_1 = \sigma_1 U_1 = 3U_1$, so

$$U_1 = \frac{1}{3} AV_1 = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

The vectors U_2 and U_3 are eigenvectors of AA^T corresponding to the eigenvalue $\lambda_2 = \lambda_3 = 0$. Instead of solving the system $(0I - AA^T)X = 0$ and then using the Gram-Schmidt process on the resulting set of two basic eigenvectors, the following approach may be used.

Find vectors U_2 and U_3 by first extending $\{U_1\}$ to a basis of $\mathbb{R}^3$, then using the Gram-Schmidt algorithm to orthogonalize the basis, and finally normalizing the vectors.

Starting with $\{3U_1\}$ instead of $\{U_1\}$ makes the arithmetic a bit easier. It is easy to verify that

$$\left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^3$. Set

$$E_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and apply the Gram-Schmidt algorithm to $\{E_1, X_2, X_3\}$.

This gives us

$$E_2 = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \text{ and } E_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Therefore,

$$U_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}, U_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

and

$$U = \begin{bmatrix} -\frac{1}{3} & \frac{4}{\sqrt{18}} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Finally,

$$A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{\sqrt{18}} & 0 \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & \frac{1}{\sqrt{2}} \\ \frac{2}{3} & \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}.$$

Consider another example.

Example 6.4.8: Find the SVD

Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \end{bmatrix}$$

Solution

First consider $A^T A$

$$\begin{bmatrix} \frac{16}{5} & \frac{32}{5} & 0 \\ \frac{32}{5} & \frac{64}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

What are some eigenvalues and eigenvectors? Some computing shows these are

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \\ 0 \end{bmatrix} \right\} \leftrightarrow 0, \left\{ \begin{bmatrix} \frac{1}{5}\sqrt{5} \\ \frac{2}{5}\sqrt{5} \\ 0 \end{bmatrix} \right\} \leftrightarrow 16$$

Thus the matrix V is given by

$$V = \begin{bmatrix} \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next consider AA^T

$$\begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

Eigenvectors and eigenvalues are

$$\left\{ \begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \right\} \leftrightarrow 0, \left\{ \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \right\} \leftrightarrow 16$$

Thus you can let U be given by

$$U = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$$

Lets check this. $U^T AV =$

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{2}\sqrt{5} & \frac{4}{5}\sqrt{2}\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} & 0 \\ \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This illustrates that if you have a good way to find the eigenvectors and eigenvalues for a Hermitian matrix which has nonnegative eigenvalues, then you also have a good way to find the singular value decomposition of an arbitrary matrix.

Positive Definite Matrices

Positive definite matrices are often encountered in applications such mechanics and statistics.

We begin with a definition.

Definition 6.4.5: Positive Definite Matrix

Let A be an $n \times n$ symmetric matrix. Then A is positive definite if all of its eigenvalues are positive.

The relationship between a negative definite matrix and positive definite matrix is as follows.

Lemma 6.4.2: Negative Definite Matrix

An $n \times n$ matrix A is negative definite if and only if $-A$ is positive definite

Consider the following lemma.

Lemma 6.4.3: Positive Definite Matrix and Invertibility

If A is positive definite, then it is invertible.

Proof

If $A\vec{v} = \vec{0}$, then 0 is an eigenvalue if $\vec{v}$ is nonzero, which does not happen for a positive definite matrix. Hence $\vec{v} = \vec{0}$ and so A is one to one. This is sufficient to conclude that it is invertible.

Notice that this lemma implies that if a matrix A is positive definite, then $\det(A) > 0$.

The following theorem provides another characterization of positive definite matrices. It gives a useful test for verifying if a matrix is positive definite.

Theorem 6.4.7: Positive Definite Matrix

Let A be a symmetric matrix. Then A is positive definite if and only if $\vec{x}^T A\vec{x}$ is positive for all nonzero $\vec{x} \in \mathbb{R}^n$.

Proof

Since A is symmetric, there exists an orthogonal matrix U so that

$$U^T AU = diag(\lambda_1, \lambda_2, \ldots, \lambda_n) = D,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A. Let $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$, and define $\vec{y} = U^T\vec{x}$. Then

$$\vec{x}^T A\vec{x} = \vec{x}^T(UDU^T)\vec{x} = (\vec{x}^T U)D(U^T\vec{x}) = \vec{y}^T D\vec{y}.$$

Writing $\vec{y}^T = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}$,

$$\begin{aligned} \vec{x}^T A\vec{x} &= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} diag(\lambda_1, \lambda_2, \ldots, \lambda_n) \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_n y_n^2. \end{aligned} \tag{6.4.10}$$

($\Rightarrow$) First we will assume that A is positive definite and prove that $\vec{x}^T A\vec{x}$ is positive.

Suppose A is positive definite, and $\vec{x} \in \mathbb{R}^n$, $\vec{x} \neq \vec{0}$. Since U^T is invertible, $\vec{y} = U^T\vec{x} \neq \vec{0}$, and thus $y_j \neq 0$ for some j, implying $y_j^2 > 0$ for some j. Furthermore, since all eigenvalues of A are positive, $\lambda_i y_i^2 \geq 0$ for all i and $\lambda_j y_j^2 > 0$. Therefore, $\vec{x}^T A\vec{x} > 0$.

($\Leftarrow$) Now we will assume $\vec{x}^T A\vec{x}$ is positive and show that A is positive definite.

If $\vec{x}^T A\vec{x} > 0$ whenever $\vec{x} \neq \vec{0}$, choose $\vec{x} = U\vec{e}_j$, where $\vec{e}_j$ is the jth column of I_n. Since U is invertible, $\vec{x} \neq \vec{0}$, and thus

$$\vec{y} = U^T\vec{x} = U^T(U\vec{e}_j) = \vec{e}_j.$$

Thus $y_j = 1$ and $y_i = 0$ when $i \neq j$, so

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots \lambda_n y_n^2 = \lambda_j,$$

i.e., $\lambda_j = \vec{x}^T A\vec{x} > 0$. Therefore, A is positive definite.

There are some other very interesting consequences which result from a matrix being positive definite. First one can note that the property of being positive definite is transferred to each of the principal submatrices which we will now define.

Definition 6.4.6: The Submatrix A_k

Let A be an $n \times n$ matrix. Denote by A_k the $k \times k$ matrix obtained by deleting the $k+1, \cdots, n$ columns and the $k+1, \cdots, n$ rows from A. Thus $A_n = A$ and A_k is the $k \times k$ submatrix of A which occupies the upper left corner of A.

Lemma 6.4.4: Positive Definite and Submatrices

Let A be an $n \times n$ positive definite matrix. Then each submatrix A_k is also positive definite.

Proof

This follows right away from the above definition. Let $\vec{x} \in \mathbb{R}^k$ be nonzero. Then

$$\vec{x}^T A_k\vec{x} = \begin{bmatrix} \vec{x}^T & 0 \end{bmatrix} A \begin{bmatrix} \vec{x} \\ 0 \end{bmatrix} > 0$$

by the assumption that A is positive definite.

There is yet another way to recognize whether a matrix is positive definite which is described in terms of these submatrices. We state the result, the proof of which can be found in more advanced texts.

Theorem 6.4.8: Positive Matrix and Determinant of A_k

Let A be a symmetric matrix. Then A is positive definite if and only if $\det(A_k)$ is greater than 0 for every submatrix A_k, $k = 1, \cdots, n$.

Proof

We prove this theorem by induction on n. It is clearly true if $n = 1$. Suppose then that it is true for $n-1$ where $n \geq 2$. Since $\det(A) = \det(A_n) > 0$, it follows that all the eigenvalues are nonzero. We need to show that they are all positive. Suppose

not. Then there is some even number of them which are negative, even because the product of all the eigenvalues is known to be positive, equaling $\det(A)$. Pick two, λ_1 and λ_2 and let $A\vec{u}_i = \lambda_i \vec{u}_i$ where $\vec{u}_i \neq \vec{0}$ for $i = 1, 2$ and $\vec{u}_1 \cdot \vec{u}_2 = 0$. Now if $\vec{y} \equiv \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2$ is an element of $span\{\vec{u}_1, \vec{u}_2\}$, then since these are eigenvalues and $\vec{u}_1 \cdot \vec{u}_2 = 0$, a short computation shows

$$\begin{aligned} &(\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2)^T A (\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2) \\ &= |\alpha_1|^2 \lambda_1 \vec{u}_1^2 + |\alpha_2|^2 \lambda_2 \vec{u}_2^2 < 0. \end{aligned}$$

Now letting $\vec{x} \in \mathbb{R}^{n-1}$, we can use the induction hypothesis to write

$$\begin{bmatrix} x^T & 0 \end{bmatrix} A \begin{bmatrix} \vec{x} \\ 0 \end{bmatrix} = \vec{x}^T A_{n-1} \vec{x} > 0.$$

Now the dimension of $\{\vec{z} \in \mathbb{R}^n : z_n = 0\}$ is $n-1$ and the dimension of $span\{\vec{u}_1, \vec{u}_2\} = 2$ and so there must be some nonzero $\vec{x} \in \mathbb{R}^n$ which is in both of these subspaces of $\mathbb{R}^n$. However, the first computation would require that $\vec{x}^T A \vec{x} < 0$ while the second would require that $\vec{x}^T A \vec{x} > 0$. This contradiction shows that all the eigenvalues must be positive. This proves the if part of the theorem. The converse can also be shown to be correct, but it is the direction which was just shown which is of most interest.

Corollary 6.4.4: Symmetric and Negative Definite Matrix

Let A be symmetric. Then A is negative definite if and only if

$$(-1)^k \det(A_k) > 0$$

for every $k = 1, \cdots, n$.

Proof

This is immediate from the above theorem when we notice, that A is negative definite if and only if $-A$ is positive definite. Therefore, if $\det(-A_k) > 0$ for all $k = 1, \cdots, n$, it follows that A is negative definite. However, $\det(-A_k) = (-1)^k \det(A_k)$.

The Cholesky Factorization

Another important theorem is the existence of a specific factorization of positive definite matrices. It is called the Cholesky Factorization and factors the matrix into the product of an upper triangular matrix and its transpose.

Theorem 6.4.9: Cholesky Factorization

Let A be a positive definite matrix. Then there exists an upper triangular matrix U whose main diagonal entries are positive, such that A can be written

$$A = U^T U$$

This factorization is unique.

The process for finding such a matrix U relies on simple row operations.

Procedure 6.4.1: Finding the Cholesky Factorization

Let A be a positive definite matrix. The matrix U that creates the Cholesky Factorization can be found through two steps.

1. Using only type **3** elementary row operations (multiples of rows added to other rows) put A in upper triangular form. Call this matrix $\hat{U}$. Then $\hat{U}$ has positive entries on the main diagonal.
2. Divide each row of $\hat{U}$ by the square root of the diagonal entry in that row. The result is the matrix U.

Of course you can always verify that your factorization is correct by multiplying U and U^T to ensure the result is the original matrix A.

Consider the following example.

Example 6.4.9: Cholesky Factorization

Show that $A = \begin{bmatrix} 9 & -6 & 3 \\ -6 & 5 & -3 \\ 3 & -3 & 6 \end{bmatrix}$ is positive definite, and find the Cholesky factorization of A.

Solution

First we show that A is positive definite. By Theorem 6.4.8 it suffices to show that the determinant of each submatrix is positive.

$$A_1 = \begin{bmatrix} 9 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 9 & -6 \\ -6 & 5 \end{bmatrix},$$

so $\det(A_1) = 9$ and $\det(A_2) = 9$. Since $\det(A) = 36$, it follows that A is positive definite.

Now we use Procedure 6.4.1 to find the Cholesky Factorization. Row reduce (using only type 3 row operations) until an upper triangular matrix is obtained.

$$\begin{bmatrix} 9 & -6 & 3 \\ -6 & 5 & -3 \\ 3 & -3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & -6 & 3 \\ 0 & 1 & -1 \\ 0 & -1 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & -6 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{bmatrix}$$

Now divide the entries in each row by the square root of the diagonal entry in that row, to give

$$U = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

You can verify that $U^T U = A$.

Example 6.4.10: Cholesky Factorization

Let A be a positive definite matrix given by

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix}$$

Determine its Cholesky factorization.

Solution

You can verify that A is in fact positive definite.

To find the Cholesky factorization we first row reduce to an upper triangular matrix.

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{11}{3} & \frac{5}{3} \\ 0 & \frac{5}{3} & \frac{14}{5} \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 1 & 1 \\ 0 & \frac{11}{3} & \frac{5}{3} \\ 0 & 0 & \frac{43}{11} \end{bmatrix}$$

Now divide the entries in each row by the square root of the diagonal entry in that row and simplify.

$$U = \begin{bmatrix} \sqrt{3} & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ 0 & \frac{1}{3}\sqrt{3}\sqrt{11} & \frac{5}{33}\sqrt{3}\sqrt{11} \\ 0 & 0 & \frac{1}{11}\sqrt{11}\sqrt{43} \end{bmatrix}$$

QR Factorization

In this section, a reliable factorization of matrices is studied. Called the QR factorization of a matrix, it *always* exists. While much can be said about the QR factorization, this section will be limited to real matrices. Therefore we assume the dot product used below is the usual dot product. We begin with a definition.

Definition 6.4.7: QR Factorization

Let A be a real $m \times n$ matrix. Then a QR factorization of A consists of two matrices, Q orthogonal and R upper triangular, such that $A = QR$.

qrfactorization

The following theorem claims that such a factorization exists.

Theorem 6.4.10: Existence of QR Factorization

Let A be any real $m \times n$ matrix with linearly independent columns. Then there exists an orthogonal matrix Q and an upper triangular matrix R having non-negative entries on the main diagonal such that

$$A = QR$$

The procedure for obtaining the QR factorization for any matrix A is as follows.

Procedure 6.4.2: QR Factorization

Let A be an $m \times n$ matrix given by $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$ where the A_i are the linearly independent columns of A.

1. Apply the Gram-Schmidt Process 4.11.1 to the columns of A, writing B_i for the resulting columns.
2. Normalize the B_i, to find $C_i = \frac{1}{B_i} B_i$.
3. Construct the orthogonal matrix Q as $Q = \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix}$.
4. Construct the upper triangular matrix R as

$$R = \begin{bmatrix} B_1 & A_2 \cdot C_1 & A_3 \cdot C_1 & \cdots & A_n \cdot C_1 \\ 0 & B_2 & A_3 \cdot C_2 & \cdots & A_n \cdot C_2 \\ 0 & 0 & B_3 & \cdots & A_n \cdot C_3 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & B_n \end{bmatrix}$$

5. Finally, write $A = QR$ where Q is the orthogonal matrix and R is the upper triangular matrix obtained above.

Notice that Q is an orthogonal matrix as the C_i form an orthonormal set. Since $B_i > 0$ for all i (since the length of a vector is always positive), it follows that R is an upper triangular matrix with positive entries on the main diagonal.

Consider the following example.

Example 6.4.11: Finding a QR Factorization

Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find an orthogonal matrix Q and upper triangular matrix R such that $A = QR$.

Solution

First, observe that A_1, A_2, the columns of A, are linearly independent. Therefore we can use the Gram-Schmidt Process to create a corresponding orthogonal set $\{B_1, B_2\}$ as follows:

$$\begin{aligned} B_1 = A_1 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ B_2 = A_2 - \frac{A_2 \cdot B_1}{B_1^2} B_1 &= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \end{aligned} \tag{6.4.11}$$

Normalize each vector to create the set $\{C_1, C_2\}$ as follows:

$$\begin{aligned} C_1 = \frac{1}{B_1} B_1 &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ C_2 = \frac{1}{B_2} B_2 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \end{aligned} \tag{6.4.12}$$

Now construct the orthogonal matrix Q as

$$\begin{aligned} Q &= \begin{bmatrix} C_1 & C_2 & \cdots & C_n \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \end{aligned} \tag{6.4.13}$$

Finally, construct the upper triangular matrix R as

$$\begin{aligned} R &= \begin{bmatrix} B_1 & A_2 \cdot C_1 \\ 0 & B_2 \end{bmatrix} \\ &= \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix} \end{aligned} \tag{6.4.14}$$

It is left to the reader to verify that $A = QR$.

The QR Factorization and Eigenvalues

The QR factorization of a matrix has a very useful application. It turns out that it can be used repeatedly to estimate the eigenvalues of a matrix. Consider the following procedure.

Procedure 6.4.3: Using the QR Factorization to Estimate Eigenvalues

Let A be an invertible matrix. Define the matrices $A_1, A_2, \cdots$ as follows:

1. $A_1 = A$ factored as $A_1 = Q_1 R_1$
2. $A_2 = R_1 Q_1$ factored as $A_2 = Q_2 R_2$
3. $A_3 = R_2 Q_2$ factored as $A_3 = Q_3 R_3$

Continue in this manner, where in general $A_k = Q_k R_k$ and $A_{k+1} = R_k Q_k$.

Then it follows that this sequence of A_i converges to an upper triangular matrix which is similar to A. Therefore the eigenvalues of A can be approximated by the entries on the main diagonal of this upper triangular matrix.

Power Methods

While the QR algorithm can be used to compute eigenvalues, there is a useful and fairly elementary technique for finding the eigenvector and associated eigenvalue nearest to a given complex number which is called the *shifted inverse power method*. It tends to work extremely well provided you start with something which is fairly close to an eigenvalue.

Power methods are based the consideration of powers of a given matrix. Let $\{\vec{x}_1, \cdots, \vec{x}_n\}$ be a basis of eigenvectors for $\mathbb{C}^n$ such that $A\vec{x}_n = \lambda_n \vec{x}_n$. Now let $\vec{u}_1$ be some nonzero vector. Since $\{\vec{x}_1, \cdots, \vec{x}_n\}$ is a basis, there exists unique scalars, c_i such that

$$\vec{u}_1 = \sum_{k=1}^{n} c_k \vec{x}_k$$

Assume you have not been so unlucky as to pick $\vec{u}_1$ in such a way that $c_n = 0$. Then let $A\vec{u}_k = \vec{u}_{k+1}$ so that

$$\vec{u}_m = A^m \vec{u}_1 = \sum_{k=1}^{n-1} c_k \lambda_k^m \vec{x}_k + \lambda_n^m c_n \vec{x}_n. \tag{6.4.15}$$

For large m the last term, $\lambda_n^m c_n \vec{x}_n$, determines quite well the direction of the vector on the right. This is because $|\lambda_n|$ is larger than $|\lambda_k|$ for $k < n$ and so for a large m, the sum, $\sum_{k=1}^{n-1} c_k \lambda_k^m \vec{x}_k$, on the right is fairly insignificant. Therefore, for large m, $\vec{u}_m$ is essentially a multiple of the eigenvector $\vec{x}_n$, the one which goes with λ_n. The only problem is that there is no control of the size of the vectors $\vec{u}_m$. You can fix this by scaling. Let S_2 denote the entry of $A\vec{u}_1$ which is largest in absolute value. We call this a **scaling factor**. Then $\vec{u}_2$ will not be just $A\vec{u}_1$ but $A\vec{u}_1/S_2$. Next let S_3 denote the entry of $A\vec{u}_2$ which has largest absolute value and define $\vec{u}_3 \equiv A\vec{u}_2/S_3$. Continue this way. The scaling just described does not destroy the relative insignificance of the term involving a sum in (6.4.15). Indeed it amounts to nothing more than changing the units of length. Also note that from this scaling procedure, the absolute value of the largest element of $\vec{u}_k$ is always equal to 1. Therefore, for large m,

$$\vec{u}_m = \frac{\lambda_n^m c_n \vec{x}_n}{S_2 S_3 \cdots S_m} + (\text{relatively insignificant term}).$$

Therefore, the entry of $A\vec{u}_m$ which has the largest absolute value is essentially equal to the entry having largest absolute value of

$$A\left(\frac{\lambda_n^m c_n \vec{x}_n}{S_2 S_3 \cdots S_m}\right) = \frac{\lambda_n^{m+1} c_n \vec{x}_n}{S_2 S_3 \cdots S_m} \approx \lambda_n \vec{u}_m$$

and so for large m, it must be the case that $\lambda_n \approx S_{m+1}$. This suggests the following procedure.

Procedure 6.4.4: Finding the Largest Eigenvalue with its Eigenvector

1. Start with a vector $\vec{u}_1$ which you hope has a component in the direction of $\vec{x}_n$. The vector $(1, \cdots, 1)^T$ is usually a pretty good choice.
2. If $\vec{u}_k$ is known,
$$\vec{u}_{k+1} = \frac{A\vec{u}_k}{S_{k+1}}$$
where S_{k+1} is the entry of $A\vec{u}_k$ which has largest absolute value.
3. When the scaling factors, S_k are not changing much, S_{k+1} will be close to the eigenvalue and $\vec{u}_{k+1}$ will be close to an eigenvector.
4. Check your answer to see if it worked well.

The shifted inverse power method involves finding the eigenvalue closest to a given complex number along with the associated eigenvalue. If μ is a complex number and you want to find λ which is closest to μ, you could consider the eigenvalues and eigenvectors of $(A - \mu I)^{-1}$. Then $A\vec{x} = \lambda\vec{x}$ if and only if

$$(A - \mu I)\vec{x} = (\lambda - \mu)\vec{x}$$

If and only if

$$\frac{1}{\lambda - \mu}\vec{x} = (A - \mu I)^{-1}\vec{x}$$

Thus, if λ is the closest eigenvalue of A to μ then out of all eigenvalues of $(A-\mu I)^{-1}$, you would have $\frac{1}{\lambda-\mu}$ would be the largest. Thus all you have to do is apply the power method to $(A-\mu I)^{-1}$ and the eigenvector you get will be the eigenvector which corresponds to λ where λ is the closest to μ of all eigenvalues of A. You could use the eigenvector to determine this directly.

Example 6.4.12: Finding Eigenvalue and Eigenvector

Find the eigenvalue and eigenvector for

$$\begin{bmatrix} 3 & 2 & 1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix}$$

which is closest to $.9+.9i$.

Solution

$$\left(\begin{bmatrix} 3 & 2 & 1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} - (.9+.9i)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^{-1}$$

$$= \begin{bmatrix} -0.619\,19 - 10.545i & -5.524\,9 - 4.972\,4i & -0.370\,57 - 5.821\,3i \\ 5.524\,9 + 4.972\,4i & 5.276\,2 + 0.248\,62i & 2.762\,4 + 2.486\,2i \\ 0.741\,14 + 11.643i & 5.524\,9 + 4.972\,4i & 0.492\,52 + 6.918\,9i \end{bmatrix}$$

Then pick an initial guess an multiply by this matrix raised to a large power.

$$= \begin{bmatrix} -0.619\,19 - 10.545i & -5.524\,9 - 4.972\,4i & -0.370\,57 - 5.821\,3i \\ 5.524\,9 + 4.972\,4i & 5.276\,2 + 0.248\,62i & 2.762\,4 + 2.486\,2i \\ 0.741\,14 + 11.643i & 5.524\,9 + 4.972\,4i & 0.492\,52 + 6.918\,9i \end{bmatrix}^{15} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

This equals

$$\begin{bmatrix} 1.562\,9 \times 10^{13} - 3.899\,3 \times 10^{12}i \\ -5.864\,5 \times 10^{12} + 9.764\,2 \times 10^{12}i \\ -1.562\,9 \times 10^{13} + 3.899\,9 \times 10^{12}i \end{bmatrix}$$

Now divide by an entry to make the vector have reasonable size. This yields

$$\begin{bmatrix} -0.999\,99 - 3.614\,0 \times 10^{-5}i \\ 0.499\,99 - 0.499\,99i \\ 1.0 \end{bmatrix}$$

which is close to

$$\begin{bmatrix} -1 \\ 0.5 - 0.5i \\ 1.0 \end{bmatrix}$$

Then

$$\begin{bmatrix} 3 & 2 & 1 \\ -2 & 0 & -1 \\ -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0.5 - 0.5i \\ 1.0 \end{bmatrix} = \begin{bmatrix} -1.0 - 1.0i \\ 1.0 \\ 1.0 + 1.0i \end{bmatrix}$$

Now to determine the eigenvalue, you could just take the ratio of corresponding entries. Pick the two corresponding entries which have the largest absolute values. In this case, you would get the eigenvalue is $1+i$ which happens to be the exact eigenvalue. Thus an eigenvector and eigenvalue are

$$\begin{bmatrix} -1 \\ 0.5 - 0.5i \\ 1.0 \end{bmatrix}, 1+i$$

Usually it won't work out so well but you can still find what is desired. Thus, once you have obtained approximate eigenvalues using the QR algorithm, you can find the eigenvalue more exactly along with an eigenvector associated with it by using the shifted inverse power method.

Quadratic Forms

One of the applications of orthogonal diagonalization is that of quadratic forms and graphs of level curves of a quadratic form. This section has to do with rotation of axes so that with respect to the new axes, the graph of the level curve of a quadratic form is oriented parallel to the coordinate axes. This makes it much easier to understand. For example, we all know that $x_1^2 + x_2^2 = 1$ represents the equation in two variables whose graph in $\mathbb{R}^2$ is a circle of radius 1. But how do we know what the graph of the equation $5x_1^2 + 4x_1x_2 + 3x_2^2 = 1$ represents?

We first formally define what is meant by a quadratic form. In this section we will work with only *real* quadratic forms, which means that the coefficients will all be real numbers.

Definition 6.4.8: Quadratic Form

A **quadratic form** is a polynomial of degree two in n variables $x_1, x_2, \cdots, x_n$, written as a linear combination of x_i^2 terms and x_ix_j terms.

Consider the quadratic form $q = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots$. We can write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ as the vector whose entries are the variables contained in the quadratic form.

Similarly, let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ be the matrix whose entries are the coefficients of x_i^2 and x_ix_j from q. Note that the matrix A is not unique, and we will consider this further in the example below. Using this matrix A, the quadratic form can be written as $q = \vec{x}^T A\vec{x}$.

$$\begin{aligned} q &= \vec{x}^T A\vec{x} \\ &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11}x_1 + a_{21}x_2 + \cdots + a_{n1}x_n \\ a_{12}x_1 + a_{22}x_2 + \cdots + a_{n2}x_n \\ \vdots \\ a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{nn}x_n \end{bmatrix} \\ &= a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots \end{aligned} \tag{6.4.16}$$

Let's explore how to find this matrix A. Consider the following example.

Example 6.4.13: Matrix of a Quadratic Form

Let a quadratic form q be given by

$$q = 6x_1^2 + 4x_1x_2 + 3x_2^2$$

Write q in the form $\vec{x}^T A\vec{x}$.

Solution

First, let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

Then, writing $q = \vec{x}^T A\vec{x}$ gives

$$\begin{aligned} q &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= a_{11}x_1^2 + a_{21}x_1x_2 + a_{12}x_1x_2 + a_{22}x_2^2 \end{aligned} \tag{6.4.17}$$

Notice that we have an x_1x_2 term as well as an x_2x_1 term. Since multiplication is commutative, these terms can be combined. This means that q can be written

$$q = a_{11}x_1^2 + (a_{21} + a_{12})\, x_1x_2 + a_{22}x_2^2$$

Equating this to q as given in the example, we have

$$a_{11}x_1^2 + (a_{21} + a_{12})\, x_1x_2 + a_{22}x_2^2 = 6x_1^2 + 4x_1x_2 + 3x_2^2$$

Therefore,

$$\begin{aligned} a_{11} &= 6 \\ a_{22} &= 3 \\ a_{21} + a_{12} &= 4 \end{aligned} \tag{6.4.18}$$

This demonstrates that the matrix A is not unique, as there are several correct solutions to $a_{21} + a_{12} = 4$. However, we will *always* choose the coefficients such that $a_{21} = a_{12} = \frac{1}{2}(a_{21} + a_{12})$. This results in $a_{21} = a_{12} = 2$. This choice is key, as it will ensure that A turns out to be a symmetric matrix.

Hence,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

You can verify that $q = \vec{x}^T A\vec{x}$ holds for this choice of A.

The above procedure for choosing A to be symmetric applies for any quadratic form q. We will *always* choose coefficients such that $a_{ij} = a_{ji}$.

We now turn our attention to the focus of this section. Our goal is to start with a quadratic form q as given above and find a way to rewrite it to eliminate the x_ix_j terms. This is done through a change of variables. In other words, we wish to find y_i such that

$$q = d_{11}y_1^2 + d_{22}y_2^2 + \cdots + d_{nn}y_n^2$$

Letting $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ and $D = [d_{ij}]$, we can write $q = \vec{y}^T D\vec{y}$ where D is the matrix of coefficients from q. There is something special about this matrix D that is crucial. Since no y_iy_j terms exist in q, it follows that $d_{ij} = 0$ for all $i \neq j$. Therefore, D is a diagonal matrix. Through this change of variables, we find the **principal axes** $y_1, y_2, \cdots, y_n$ of the quadratic form.

This discussion sets the stage for the following essential theorem.

Theorem 6.4.11: Diagonalizing a Quadratic Form

Let q be a quadratic form in the variables $x_1, \cdots, x_n$. It follows that q can be written in the form $q = \vec{x}^T A\vec{x}$ where

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and $A = [a_{ij}]$ is the symmetric matrix of coefficients of q.

New variables $y_1, y_2, \cdots, y_n$ can be found such that $q = \vec{y}^T D\vec{y}$ where

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

and $D = [d_{ij}]$ is a diagonal matrix. The matrix D contains the eigenvalues of A and is found by orthogonally diagonalizing A.

While not a formal proof, the following discussion should convince you that the above theorem holds. Let q be a quadratic form in the variables $x_1, \cdots, x_n$. Then, q can be written in the form $q = \vec{x}^T A \vec{x}$ for a symmetric matrix A. By Theorem 6.4.3 we can orthogonally diagonalize the matrix A such that $U^T AU = D$ for an orthogonal matrix U and diagonal matrix D.

Then, the vector $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ is found by $\vec{y} = U^T \vec{x}$. To see that this works, rewrite $\vec{y} = U^T \vec{x}$ as $\vec{x} = U\vec{y}$. Letting $q = \vec{x}^T A \vec{x}$, proceed as follows:

$$\begin{aligned} q &= \vec{x}^T A \vec{x} \\ &= (U\vec{y})^T A (U\vec{y}) \\ &= \vec{y}^T (U^T AU) \vec{y} \\ &= \vec{y}^T D \vec{y} \end{aligned} \tag{6.4.19}$$

The following procedure details the steps for the change of variables given in the above theorem.

Procedure 6.4.5: Diagonalizing a Quadratic Form

Let q be a quadratic form in the variables $x_1, \cdots, x_n$ given by

$$q = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots$$

Then, q can be written as $q = d_{11}y_1^2 + \cdots + d_{nn}y_n^2$ as follows:

1. Write $q = \vec{x}^T A \vec{x}$ for a symmetric matrix A.
2. Orthogonally diagonalize A to be written as $U^T AU = D$ for an orthogonal matrix U and diagonal matrix D.
3. Write $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$. Then, $\vec{x} = U\vec{y}$.
4. The quadratic form q will now be given by

$$q = d_{11}y_1^2 + \cdots + d_{nn}y_n^2 = \vec{y}^T D \vec{y}$$

where $D = [d_{ij}]$ is the diagonal matrix found by orthogonally diagonalizing A.

Consider the following example.

Example 6.4.14: Choosing New Axes to Simplify a Quadratic Form

Consider the following level curve

$$6x_1^2 + 4x_1x_2 + 3x_2^2 = 7$$

shown in the following graph.

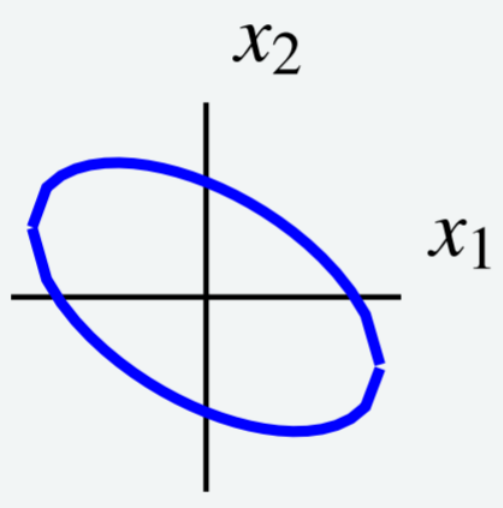


Figure 6.4.1

Use a change of variables to choose new axes such that the ellipse is oriented parallel to the new coordinate axes. In other words, use a change of variables to rewrite q to eliminate the x_1x_2 term.

Solution

Notice that the level curve is given by $q = 7$ for $q = 6x_1^2 + 4x_1x_2 + 3x_2^2$. This is the same quadratic form that we examined earlier in Example 6.4.13. Therefore we know that we can write $q = \vec{x}^T A\vec{x}$ for the matrix

$$A = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix}$$

Now we want to orthogonally diagonalize A to write $U^T AU = D$ for an orthogonal matrix U and diagonal matrix D. The details are left to the reader, and you can verify that the resulting matrices are

$$\begin{aligned} U &= \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \\ D &= \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned} \tag{6.4.20}$$

Next we write $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. It follows that $\vec{x} = U\vec{y}$.

We can now express the quadratic form q in terms of y, using the entries from D as coefficients as follows:

$$\begin{aligned} q &= d_{11}y_1^2 + d_{22}y_2^2 \\ &= 7y_1^2 + 2y_2^2 \end{aligned} \tag{6.4.21}$$

Hence the level curve can be written $7y_1^2 + 2y_2^2 = 7$. The graph of this equation is given by:

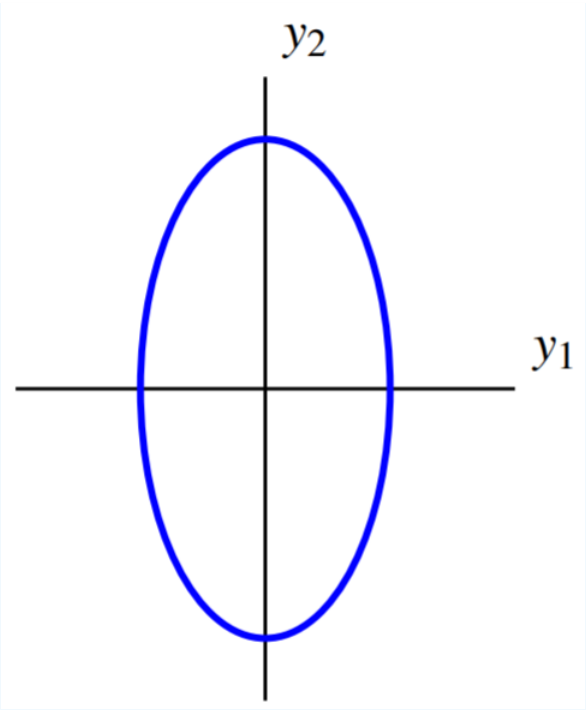


Figure 6.4.2

The change of variables results in new axes such that with respect to the new axes, the ellipse is oriented parallel to the coordinate axes. These are called the **principal axes** of the quadratic form.

The following is another example of diagonalizing a quadratic form.

Example 6.4.15: Choosing New Axes to Simplify a Quadratic Form

Consider the level curve

$$5x_1^2 - 6x_1x_2 + 5x_2^2 = 8$$

shown in the following graph.

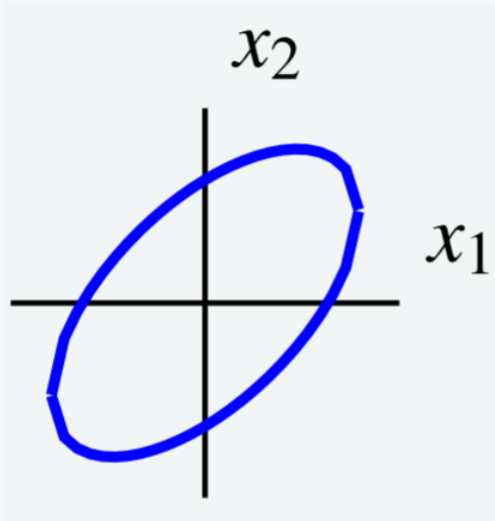


Figure 6.4.3

Use a change of variables to choose new axes such that the ellipse is oriented parallel to the new coordinate axes. In other words, use a change of variables to rewrite q to eliminate the x_1x_2 term.

Solution

First, express the level curve as $\vec{x}^T A\vec{x}$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and A is symmetric. Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then $q = \vec{x}^T A\vec{x}$ is given by

$$\begin{aligned} q &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2 \end{aligned} \tag{6.4.22}$$

Equating this to the given description for q, we have

$$5x_1^2 - 6x_1x_2 + 5x_2^2 = a_{11}x_1^2 + (a_{12} + a_{21})x_1x_2 + a_{22}x_2^2$$

This implies that $a_{11} = 5, a_{22} = 5$ and in order for A to be symmetric, $a_{12} = a_{22} = \frac{1}{2}(a_{12} + a_{21}) = -3$. The result is $A = \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$. We can write $q = \vec{x}^T A\vec{x}$ as

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8$$

Next, orthogonally diagonalize the matrix A to write $U^T AU = D$. The details are left to the reader and the necessary matrices are given by

$$\begin{aligned} U &= \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \end{bmatrix} \\ D &= \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix} \end{aligned} \tag{6.4.23}$$

Write $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, such that $\vec{x} = U\vec{y}$. Then it follows that q is given by

$$\begin{aligned} q &= d_{11}y_1^2 + d_{22}y_2^2 \\ &= 2y_1^2 + 8y_2^2 \end{aligned} \tag{6.4.24}$$

Therefore the level curve can be written as $2y_1^2 + 8y_2^2 = 8$.

This is an ellipse which is parallel to the coordinate axes. Its graph is of the form

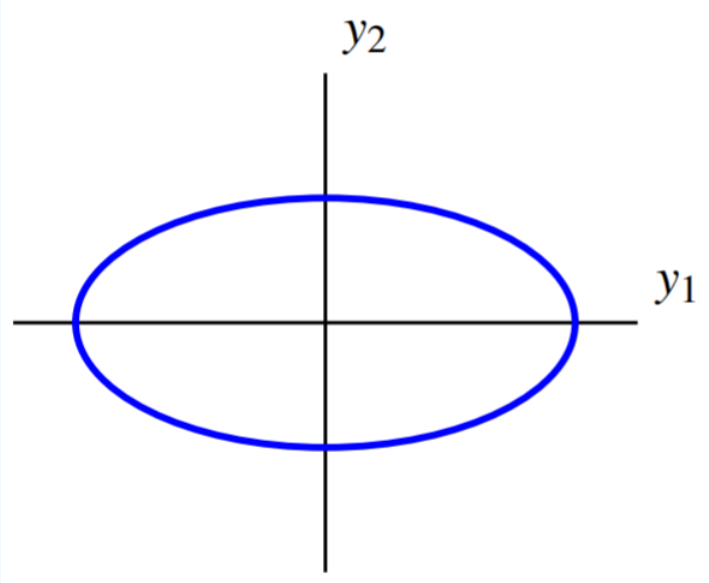


Figure 6.4.4

Thus this change of variables chooses new axes such that with respect to these new axes, the ellipse is oriented parallel to the coordinate axes.

This page titled 6.4: Orthogonality is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) via source content that was edited to the style and standards of the LibreTexts platform.

- **7.4: Orthogonality** by Ken Kuttler is licensed CC BY 4.0. Original source: https://lyryx.com/first-course-linear-algebra.

6.5: Exercises

Exercise 6.5.1

If A is an invertible $n \times n$ matrix, compare the eigenvalues of A and A^{-1}. More generally, for m an arbitrary integer, compare the eigenvalues of A and A^m.

Answer

$A^m X = \lambda^m X$ for any integer. In the case of -1, $A^{-1}\lambda X = AA^{-1}X = X$ so $A^{-1}X = \lambda^{-1}X$. Thus the eigenvalues of A^{-1} are just λ^{-1} where λ is an eigenvalue of A.

Exercise 6.5.2

If A is an $n \times n$ matrix and c is a nonzero constant, compare the eigenvalues of A and cA.

Answer

Say $AX = \lambda X$. Then $cAX = c\lambda X$ and so the eigenvalues of cA are just $c\lambda$ where λ is an eigenvalue of A.

Exercise 6.5.3

Let A, B be invertible $n \times n$ matrices which commute. That is, $AB = BA$. Suppose X is an eigenvector of B. Show that then AX must also be an eigenvector for B.

Answer

$BAX = ABX = A\lambda X = \lambda AX$. Here it is assumed that $BX = \lambda X$.

Exercise 6.5.4

Suppose A is an $n \times n$ matrix and it satisfies $A^m = A$ for some m a positive integer larger than 1. Show that if λ is an eigenvalue of A then $|\lambda|$ equals either 0 or 1.

Answer

Let X be the eigenvector. Then $A^m X = \lambda^m X$, $A^m X = AX = \lambda X$ and so

$$\lambda^m = \lambda$$

Hence if $\lambda \neq 0$, then

$$\lambda^{m-1} = 1$$

and so $|\lambda| = 1$.

Exercise 6.5.5

Show that if $AX = \lambda X$ and $AY = \lambda Y$, then whenever k, p are scalars,

$$A(kX + pY) = \lambda(kX + pY)$$

Does this imply that $kX + pY$ is an eigenvector? Explain.

Answer

The formula follows from properties of matrix multiplications. However, this vector might not be an eigenvector because it might equal $\mathbf{0}$ and eigenvectors cannot equal $\mathbf{0}$.

? Exercise 6.5.6

Suppose A is a 3×3 matrix and the following information is available.

$$\begin{aligned} A\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} &= 0\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \\ A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= -2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ A\begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix} &= -2\begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix} \end{aligned} \tag{6.5.1}$$

Find $A\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$.

? Exercise 6.5.7

Suppose A is a 3×3 matrix and the following information is available.

$$\begin{aligned} A\begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} &= 1\begin{bmatrix} -1 \\ -2 \\ -2 \end{bmatrix} \\ A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= 0\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ A\begin{bmatrix} -1 \\ -4 \\ -3 \end{bmatrix} &= 2\begin{bmatrix} -1 \\ -4 \\ -3 \end{bmatrix} \end{aligned} \tag{6.5.2}$$

Find $A\begin{bmatrix} 3 \\ -4 \\ 3 \end{bmatrix}$.

? Exercise 6.5.8

Suppose A is a 3×3 matrix and the following information is available.

$$\begin{aligned} A\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} &= 2\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \\ A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} &= 1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ A\begin{bmatrix} -3 \\ -5 \\ -4 \end{bmatrix} &= -3\begin{bmatrix} -3 \\ -5 \\ -4 \end{bmatrix} \end{aligned} \tag{6.5.3}$$

Find $A\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$.

? Exercise 6.5.9

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -6 & -92 & 12 \\ 0 & 0 & 0 \\ -2 & -31 & 4 \end{bmatrix}$$

One eigenvalue is -2.

? Exercise 6.5.10

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -2 & -17 & -6 \\ 0 & 0 & 0 \\ 1 & 9 & 3 \end{bmatrix}$$

One eigenvalue is 1.

? Exercise 6.5.11

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 9 & 2 & 8 \\ 2 & -6 & -2 \\ -8 & 2 & -5 \end{bmatrix}$$

One eigenvalue is -3.

? Exercise 6.5.12

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 6 & 76 & 16 \\ -2 & -21 & -4 \\ 2 & 64 & 17 \end{bmatrix}$$

One eigenvalue is -2.

? Exercise 6.5.13

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 5 & 2 \\ -8 & -11 & -4 \\ 10 & 11 & 3 \end{bmatrix}$$

One eigenvalue is -3.

? Exercise 6.5.14

Is it possible for a nonzero matrix to have only 0 as an eigenvalue?

Answer

Yes. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ works.

Exercise 6.5.15

If A is the matrix of a linear transformation which rotates all vectors in $\mathbb{R}^2$ through $60°$, explain why A cannot have any real eigenvalues. Is there an angle such that rotation through this angle would have a real eigenvalue? What eigenvalues would be obtainable in this way?

Exercise 6.5.16

Let A be the 2×2 matrix of the linear transformation which rotates all vectors in $\mathbb{R}^2$ through an angle of θ. For which values of θ does A have a real eigenvalue?

Answer

When you think of this geometrically, it is clear that the only two values of θ are 0 and π or these added to integer multiples of 2π.

Exercise 6.5.17

Let T be the linear transformation which reflects vectors about the x axis. Find a matrix for T and then find its eigenvalues and eigenvectors.

Answer

The matrix of T is $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \leftrightarrow 1$$

Exercise 6.5.18

Let T be the linear transformation which rotates all vectors in $\mathbb{R}^2$ counterclockwise through an angle of $\pi/2$. Find a matrix of T and then find eigenvalues and eigenvectors.

Answer

The matrix of T is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -i \\ 1 \end{bmatrix}\right\} \leftrightarrow -i, \left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\} \leftrightarrow i$$

Exercise 6.5.19

Let T be the linear transformation which reflects all vectors in $\mathbb{R}^3$ through the xy plane. Find a matrix for T and then obtain its eigenvalues and eigenvectors.

Answer

The matrix of T is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\} \leftrightarrow 1$$

Exercise 6.5.20

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 5 & -18 & -32 \\ 0 & 5 & 4 \\ 2 & -5 & -11 \end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible.

Answer

The eigenvalues are $-1, -1, 1$. The eigenvectors corresponding to the eigenvalues are:

$$\{\begin{bmatrix} 10 & -2 & 3 \end{bmatrix}\} \leftrightarrow -1, \left\{\begin{bmatrix} 7 \\ -2 \\ 2 \end{bmatrix}\right\} \leftrightarrow 1$$

Therefore this matrix is not diagonalizable.

Exercise 6.5.21

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -12 & -28 & 28 \\ 4 & 9 & -8 \\ -4 & -8 & 9 \end{bmatrix}$$

One eigenvalue is 3. Diagonalize if possible.

Answer

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} 7 \\ -2 \\ 2 \end{bmatrix}\right\} \leftrightarrow 3$$

The matrix P needed to diagonalize the above matrix is

$$\begin{bmatrix} 2 & -2 & 7 \\ 0 & 1 & -2 \\ 1 & 0 & 2 \end{bmatrix}$$

and the diagonal matrix D is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Exercise 6.5.22

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 89 & 38 & 268 \\ 14 & 2 & 40 \\ -30 & -12 & -90 \end{bmatrix}$$

One eigenvalue is -3. Diagonalize if possible.

Answer

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix}-6\\-1\\-2\end{bmatrix}\right\} \leftrightarrow 6, \left\{\begin{bmatrix}-5\\-2\\2\end{bmatrix}\right\} \leftrightarrow -3, \left\{\begin{bmatrix}-8\\-2\\3\end{bmatrix}\right\} \leftrightarrow 2$$

The matrix P needed to diagonalize the above matrix is

$$\begin{bmatrix}-6 & -5 & -8\\-1 & -2 & -2\\2 & 2 & 3\end{bmatrix}$$

and the diagonal matrix D is

$$\begin{bmatrix}6 & 0 & 0\\0 & -3 & 0\\0 & 0 & -2\end{bmatrix}$$

? Exercise 6.5.23

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix}1 & 90 & 0\\0 & -2 & 0\\3 & 89 & -2\end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible.

? Exercise 6.5.24

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix}11 & 45 & 30\\10 & 26 & 20\\-20 & -60 & -44\end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible.

? Exercise 6.5.25

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix}95 & 25 & 24\\-196 & -53 & -48\\-164 & -42 & -43\end{bmatrix}$$

One eigenvalue is 5. Diagonalize if possible.

? Exercise 6.5.26

Suppose A is an $n \times n$ matrix and let V be an eigenvector such that $AV = \lambda V$. Also suppose the characteristic polynomial of A is

$$\det(xI - A) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Explain why

$$(A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I)V = 0$$

If A is diagonalizable, give a proof of the Cayley Hamilton theorem based on this. This theorem says A satisfies its characteristic equation

$$A^n + a_{n-1}A^{n-1} + \cdots + a_1 A + a_0 I = 0$$

? Exercise 6.5.27

Suppose the characteristic polynomial of an $n \times n$ matrix A is $1 - X^n$. Find A^{mn} where m is an integer.

Answer

The eigenvalues are distinct because they are the nth roots of 1. Hence if X is a given vector with

$$X = \sum_{j=1}^{n} a_j V_j$$

then

$$A^{nm}X = A^{nm}\sum_{j=1}^{n} a_j V_j = \sum_{j=1}^{n} a_j A^{nm} V_j = \sum_{j=1}^{n} a_j V_j = X$$

so $A^{nm} = I$.

? Exercise 6.5.28

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 15 & -24 & 7 \\ -6 & 5 & -1 \\ -58 & 76 & -20 \end{bmatrix}$$

One eigenvalue is -2. Diagonalize if possible. Hint: This one has some complex eigenvalues.

? Exercise 6.5.29

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 15 & -25 & 6 \\ -13 & 23 & -4 \\ -91 & 155 & -30 \end{bmatrix}$$

One eigenvalue is 2. Diagonalize if possible. Hint: This one has some complex eigenvalues.

? Exercise 6.5.30

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} -11 & -12 & 4 \\ 8 & 17 & -4 \\ -4 & 28 & -3 \end{bmatrix}$$

One eigenvalue is 1. Diagonalize if possible. Hint: This one has some complex eigenvalues.

? Exercise 6.5.31

Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 14 & -12 & 5 \\ -6 & 2 & -1 \\ -69 & 51 & -21 \end{bmatrix}$$

One eigenvalue is -3. Diagonalize if possible. Hint: This one has some complex eigenvalues.

? Exercise 6.5.32

Suppose A is an $n \times n$ matrix consisting entirely of real entries but $a+ib$ is a complex eigenvalue having the eigenvector, $X+iY$ Here X and Y are real vectors. Show that then $a-ib$ is also an eigenvalue with the eigenvector, $X-iY$. Hint: You should remember that the conjugate of a product of complex numbers equals the product of the conjugates. Here $a+ib$ is a complex number whose conjugate equals $a-ib$.

Answer

$AX=(a+ib)X$. Now take conjugates of both sides. Since A is real,

$$A\overline{X}=(a-ib)\overline{X}$$

? Exercise 6.5.33

Let $A=\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Diagonalize A to find A^{10}.

Answer

First we write $A=PDP^{-1}$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}=\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore $A^{10}=PD^{10}P^{-1}$.

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{10} &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}^{10}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^{10} & 0 \\ 0 & 3^{10} \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 29525 & 29524 \\ 29524 & 29525 \end{bmatrix} \end{aligned} \tag{6.5.4}$$

? Exercise 6.5.34

Let $A=\begin{bmatrix} 1 & 4 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{bmatrix}$. Diagonalize A to find A^{50}.

? Exercise 6.5.35

Let $A=\begin{bmatrix} 1 & -2 & -1 \\ 2 & -1 & 1 \\ -2 & 3 & 1 \end{bmatrix}$. Diagonalize A to find A^{100}.

Exercise 6.5.36

The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{7}{10} & \frac{1}{9} & \frac{1}{5} \\ \frac{1}{10} & \frac{7}{9} & \frac{2}{5} \\ \frac{1}{5} & \frac{1}{9} & \frac{2}{5} \end{bmatrix}$$

a. Initially, there are 90 people in location 1, 81 in location 2, and 85 in location 3. How many are in each location after one time period?

b. The total number of individuals in the migration process is 256. After a long time, how many are in each location?

Answer

a. Multiply the given matrix by the initial state vector given by $\begin{bmatrix} 90 \\ 81 \\ 85 \end{bmatrix}$. After one time period there are 89 people in location 1, 106 in location 2, and 61 in location 3.

b. Solve the system given by $(I-A)X_s = 0$ where A is the migration matrix and $X_s = \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix}$ is the steady state vector.

The solution to this system is given by

$$\begin{aligned} x_{1s} &= \frac{8}{5}x_{3s} \\ x_{2s} &= \frac{63}{25}x_{3s} \end{aligned} \tag{6.5.5}$$

Letting $x_{3s} = t$ and using the fact that there are a total of 256 individuals, we must solve

$$\frac{8}{5}t + \frac{63}{25}t + t = 256$$

We find that $t = 50$. Therefore after a long time, there are 80 people in location 1, 126 in location 2, and 50 in location 3.

Exercise 6.5.37

The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{2}{5} & \frac{2}{5} \end{bmatrix}$$

a. Initially, there are 130 individuals in location 1, 300 in location 2, and 70 in location 3. How many are in each location after two time periods?

b. The total number of individuals in the migration process is 500. After a long time, how many are in each location?

Exercise 6.5.38

The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{3}{10} & \frac{3}{8} & \frac{1}{3} \\ \frac{1}{10} & \frac{3}{8} & \frac{1}{3} \\ \frac{3}{5} & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$$

The total number of individuals in the migration process is 480. After a long time, how many are in each location?

Answer

We solve $(I-A)X_s = 0$ to find the steady state vector $X_s = \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \end{bmatrix}$. The solution to the system is given by

$$\begin{aligned} x_{1s} &= \frac{5}{6}x_{3s} \\ x_{2s} &= \frac{2}{3}x_{3s} \end{aligned} \tag{6.5.6}$$

Letting $x_{3s} = t$ and using the fact that there are a total of 480 individuals, we must solve

$$\frac{5}{6}t + \frac{2}{3}t + t = 480$$

We find that $t = 192$. Therefore after a long time, there are 160 people in location 1, 128 in location 2, and 192 in location 3.

Exercise 6.5.39

The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{3}{10} & \frac{1}{3} & \frac{1}{5} \\ \frac{3}{10} & \frac{1}{3} & \frac{7}{10} \\ \frac{2}{5} & \frac{1}{3} & \frac{1}{10} \end{bmatrix}$$

The total number of individuals in the migration process is 1155. After a long time, how many are in each location?

Exercise 6.5.40

The following is a Markov (migration) matrix for three locations

$$\begin{bmatrix} \frac{2}{5} & \frac{1}{10} & \frac{1}{8} \\ \frac{3}{10} & \frac{2}{5} & \frac{5}{8} \\ \frac{3}{10} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

The total number of individuals in the migration process is 704. After a long time, how many are in each location?

Exercise 6.5.41

A person sets off on a random walk with three possible locations. The Markov matrix of probabilities $A = [a_{ij}]$ is given by

$$\begin{bmatrix} 0.1 & 0.3 & 0.7 \\ 0.1 & 0.3 & 0.2 \\ 0.8 & 0.4 & 0.1 \end{bmatrix}$$

If the walker starts in location 2, what is the probability of ending back in location 2 at time $n = 3$?

Answer

$$X_3 = \begin{bmatrix} 0.38 \\ 0.18 \\ 0.44 \end{bmatrix}$$

Therefore the probability of ending up back in location 2 is 0.18.

Exercise 6.5.42

A person sets off on a random walk with three possible locations. The Markov matrix of probabilities $A = [a_{ij}]$ is given by

$$\begin{bmatrix} 0.5 & 0.1 & 0.6 \\ 0.2 & 0.9 & 0.2 \\ 0.3 & 0 & 0.2 \end{bmatrix}$$

It is unknown where the walker starts, but the probability of starting in each location is given by

$$X_0 = \begin{bmatrix} 0.2 \\ 0.25 \\ 0.55 \end{bmatrix}$$

What is the probability of the walker being in location 1 at time $n = 2$?

Answer

$$X_2 = \begin{bmatrix} 0.367 \\ 0.4625 \\ 0.1705 \end{bmatrix}$$

Therefore the probability of ending up in location 1 is 0.367.

Exercise 6.5.43

You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE, NE, NW, and SW respectively. Suppose that the following table is observed to take place.

Table 6.5.1

	SE	NE	NW	SW
SE	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{1}{3}$	$\frac{7}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{2}{9}$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{1}{9}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailer starting at NE ends in NW is $1/10$, the probability that a trailer starting at SW ends in NW is $1/5$, and so forth. Approximately how many will you have in each location after a long time if the total number of trailers is 413?

Answer

The migration matrix is

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{10} & \frac{1}{10} & \frac{1}{5} \\ \frac{1}{3} & \frac{7}{10} & \frac{1}{5} & \frac{1}{10} \\ \frac{2}{9} & \frac{1}{10} & \frac{3}{5} & \frac{1}{5} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{10} & \frac{1}{2} \end{bmatrix}$$

To find the number of trailers in each location after a long time we solve system $(I - A)X_s = 0$ for the steady state vector $X_s = \begin{bmatrix} x_{1s} \\ x_{2s} \\ x_{3s} \\ x_{4s} \end{bmatrix}$. The solution to the system is

$$\begin{aligned} x_{1s} &= \frac{9}{10}x_{4s} \\ x_{2s} &= \frac{12}{5}x_{4s} \\ x_{3s} &= \frac{8}{5}x_{4s} \end{aligned} \tag{6.5.7}$$

Letting $x_{4s} = t$ and using the fact that there are a total of 413 trailers we must solve

$$\frac{9}{10}t + \frac{12}{5}t + \frac{8}{5}t + t = 413$$

We find that $t = 70$. Therefore after a long time, there are 63 trailers in the SE, 168 in the NE, 112 in the NW and 70 in the SW.

Exercise 6.5.44

You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE, NE, NW, and SW respectively. Suppose that the following table is observed to take place.

Table 6.5.2

	SE	NE	NW	SW
SE	$\frac{1}{7}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{2}{7}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{1}{7}$	$\frac{1}{4}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{3}{7}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailer starting at NE ends in NW is 1/10, the probability that a trailer starting at SW ends in NW is 1/5, and so forth. Approximately how many will you have in each location after a long time if the total number of trailers is 1469.

Exercise 6.5.45

The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{5}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{2}{5}$ and so forth.

Table 6.5.3

	rains	sunny	p.c.
rains	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{3}$
sunny	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{3}$
p.c.	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

Exercise 6.5.46

The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{10}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{2}{5}$ and so forth.

Table 6.5.4

	rains	sunny	p.c.
rains	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{3}$
sunny	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{4}{9}$
p.c.	$\frac{7}{10}$	$\frac{2}{5}$	$\frac{2}{9}$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

Exercise 6.5.47

You own a trailer rental company in a large city and you have four locations, one in the South East, one in the North East, one in the North West, and one in the South West. Denote these locations by SE, NE, NW, and SW respectively. Suppose that the following table is observed to take place.

Table 6.5.5

	SE	NE	NW	SW
SE	$\frac{5}{11}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$
NE	$\frac{1}{11}$	$\frac{7}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
NW	$\frac{2}{11}$	$\frac{1}{10}$	$\frac{3}{5}$	$\frac{1}{5}$
SW	$\frac{3}{11}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$

In this table, the probability that a trailer starting at NE ends in NW is 1/10, the probability that a trailer starting at SW ends in NW is 1/5, and so forth. Approximately how many will you have in each location after a long time if the total number of trailers is 407?

Exercise 6.5.48

The University of Poohbah offers three degree programs, scouting education (SE), dance appreciation (DA), and engineering (E). It has been determined that the probabilities of transferring from one program to another are as in the following table.

Table 6.5.6

	SE	DA	E
SE	.8	.1	.3
DA	.1	.7	.5
E	.1	.2	.2

where the number indicates the probability of transferring from the top program to the program on the left. Thus the probability of going from DA to E is .2. Find the probability that a student is enrolled in the various programs.

Exercise 6.5.49

In the city of Nabal, there are three political persuasions, republicans (R), democrats (D), and neither one (N). The following table shows the transition probabilities between the political parties, the top row being the initial political party and the side row being the political affiliation the following year.

Table 6.5.7

	R	D	N
R	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{2}{7}$
D	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{4}{7}$
N	$\frac{3}{5}$	$\frac{1}{2}$	$\frac{1}{7}$

Find the probabilities that a person will be identified with the various political persuasions. Which party will end up being most important?

Exercise 6.5.50

The following table describes the transition probabilities between the states rainy, partly cloudy and sunny. The symbol p.c. indicates partly cloudy. Thus if it starts off p.c. it ends up sunny the next day with probability $\frac{1}{5}$. If it starts off sunny, it ends up sunny the next day with probability $\frac{2}{7}$ and so forth.

Table 6.5.8

	rains	sunny	p.c.
rains	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{5}{9}$
sunny	$\frac{1}{5}$	$\frac{2}{7}$	$\frac{1}{3}$
p.c.	$\frac{3}{5}$	$\frac{3}{7}$	$\frac{1}{9}$

Given this information, what are the probabilities that a given day is rainy, sunny, or partly cloudy?

Exercise 6.5.51

Find the solution to the initial value problem

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 0 & -1 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{6.5.8}$$

Hint: form the matrix exponential e^{At} and then the solution is $e^{At}C$ where C is the initial vector,

Answer

The solution is

$$e^{At}C = \begin{bmatrix} 8e^{2t} - 6e^{3t} \\ 18e^{3t} - 16e^{2t} \end{bmatrix}$$

Exercise 6.5.52

Find the solution to the initial value problem

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \tag{6.5.9}$$

Hint: form the matrix exponential e^{At} and then the solution is $e^{At}C$ where C is the initial vector.

Exercise 6.5.53

Find the solution to the initial value problem

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -1 & 2 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \tag{6.5.10}$$

Hint: form the matrix exponential e^{At} and then the solution is $e^{At}C$ where C is the initial vector.

Exercise 6.5.54

Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{bmatrix} 11 & -1 & -4 \\ -1 & 11 & -4 \\ -4 & -4 & 14 \end{bmatrix}$$

Hint: Two eigenvalues are 12 and 18.

Answer

The eigenvectors and eigenvalues are:

$$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow 6, \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} \leftrightarrow 12, \left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\} \leftrightarrow 18$$

Exercise 6.5.55

Find the eigenvalues and an orthonormal basis of eigenvectors for A.

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 4 & -2 \\ -2 & -2 & 7 \end{bmatrix}$$

Hint: One eigenvalue is 3.

Answer

The eigenvectors and eigenvalues are:

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \leftrightarrow 3, \left\{ \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\} \leftrightarrow 9$$

Exercise 6.5.56

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^TAU = D$.

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Hint: One eigenvalue is -2.

Answer

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow -2$$

$$\begin{bmatrix} \sqrt{3}/3 & -\sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & 0 & \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}^T \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3}/3 & -\sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & \sqrt{2}/2 & -\sqrt{6}/6 \\ \sqrt{3}/3 & 0 & \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Exercise 6.5.57

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^TAU = D$.

$$A = \begin{bmatrix} 17 & -7 & -4 \\ -7 & 17 & -4 \\ -4 & -4 & 14 \end{bmatrix}$$

Hint: Two eigenvalues are 18 and 24.

Answer

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow 6, \left\{\begin{bmatrix} -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow 18, \left\{\begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{bmatrix}\right\} \leftrightarrow 24$$

The matrix U has these as its columns.

Exercise 6.5.58

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^TAU = D$.

$$A = \begin{bmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{bmatrix}$$

Hint: Two eigenvalues are 12 and 18.

Answer

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -\frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow 6, \left\{\begin{bmatrix} -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{bmatrix}\right\} \leftrightarrow 12, \left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow 18.$$

The matrix U has these as its columns.

Exercise 6.5.59

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{bmatrix} -\frac{5}{3} & \frac{1}{15}\sqrt{6}\sqrt{5} & \frac{8}{15}\sqrt{5} \\ \frac{1}{15}\sqrt{6}\sqrt{5} & -\frac{14}{5} & -\frac{1}{15}\sqrt{6} \\ \frac{8}{15}\sqrt{5} & -\frac{1}{15}\sqrt{6} & \frac{7}{15} \end{bmatrix}$$

Hint: The eigenvalues are $-3, -2, 1$.

Answer

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} \frac{1}{6}\sqrt{6} \\ 0 \\ \frac{1}{6}\sqrt{5}\sqrt{6} \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} -\frac{1}{3}\sqrt{2}\sqrt{3} \\ -\frac{1}{5}\sqrt{5} \\ \frac{1}{15}\sqrt{2}\sqrt{15} \end{bmatrix}\right\} - 2, \left\{\begin{bmatrix} -\frac{1}{6}\sqrt{6} \\ \frac{2}{5}\sqrt{5} \\ \frac{1}{30}\sqrt{30} \end{bmatrix}\right\} \leftrightarrow -3$$

These vectors are the columns of U.

Exercise 6.5.60

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Answer

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} 0 \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} 0 \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}\right\} \leftrightarrow 2, \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \leftrightarrow 3.$$

These vectors are the columns of the matrix U.

Exercise 6.5.61

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

Answer

The eigenvectors and eigenvalues are:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \leftrightarrow 2, \left\{ \begin{bmatrix} 0 \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \right\} \leftrightarrow 4, \left\{ \begin{bmatrix} 0 \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix} \right\} \leftrightarrow 6.$$

These vectors are the columns of U.

Exercise 6.5.62

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{bmatrix} \frac{4}{3} & \frac{1}{3}\sqrt{3}\sqrt{2} & \frac{1}{3}\sqrt{2} \\ \frac{1}{3}\sqrt{3}\sqrt{2} & 1 & -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{2} & -\frac{1}{3}\sqrt{3} & \frac{5}{3} \end{bmatrix}$$

Hint: The eigenvalues are $0, 2, 2$ where 2 is listed twice because it is a root of multiplicity 2.

Answer

The eigenvectors and eigenvalues are:

$$\left\{ \begin{bmatrix} -\frac{1}{5}\sqrt{2}\sqrt{5} \\ \frac{1}{5}\sqrt{3}\sqrt{5} \\ \frac{1}{5}\sqrt{5} \end{bmatrix} \right\} \leftrightarrow 0, \left\{ \begin{bmatrix} \frac{1}{3}\sqrt{3} \\ 0 \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{5}\sqrt{2}\sqrt{5} \\ \frac{1}{5}\sqrt{3}\sqrt{5} \\ -\frac{1}{5}\sqrt{5} \end{bmatrix} \right\} \leftrightarrow 2.$$

The columns are these vectors.

Exercise 6.5.63

Find the eigenvalues and an orthonormal basis of eigenvectors for A. Diagonalize A by finding an orthogonal matrix U and a diagonal matrix D such that $U^T AU = D$.

$$A = \begin{bmatrix} 1 & \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{1}{6}\sqrt{3}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{3}{2} & \frac{1}{12}\sqrt{2}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{6} & \frac{1}{12}\sqrt{2}\sqrt{6} & \frac{1}{2} \end{bmatrix}$$

Hint: The eigenvalues are $2, 1, 0$.

Answer

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -\frac{1}{3}\sqrt{3} \\ 0 \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow 0, \left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}\right\} \leftrightarrow 2.$$

The columns are these vectors.

Exercise 6.5.64

Find the eigenvalues and an orthonormal basis of eigenvectors for the matrix

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{6}\sqrt{3}\sqrt{2} & -\frac{7}{18}\sqrt{3}\sqrt{6} \\ \frac{1}{6}\sqrt{3}\sqrt{2} & \frac{3}{2} & -\frac{1}{12}\sqrt{2}\sqrt{6} \\ -\frac{7}{18}\sqrt{3}\sqrt{6} & -\frac{1}{12}\sqrt{2}\sqrt{6} & -\frac{5}{6} \end{bmatrix}$$

Hint: The eigenvalues are $1, 2, -2$.

Answer

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -\frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}\right\} \leftrightarrow 1, \left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ 0 \\ \frac{1}{3}\sqrt{2}\sqrt{3} \end{bmatrix}\right\} \leftrightarrow -2, \left\{\begin{bmatrix} \frac{1}{3}\sqrt{3} \\ \frac{1}{2}\sqrt{2} \\ -\frac{1}{6}\sqrt{6} \end{bmatrix}\right\} \leftrightarrow 2.$$

Then the columns of U are these vectors.

Exercise 6.5.65

Find the eigenvalues and an orthonormal basis of eigenvectors for the matrix

$$A = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{1}{10}\sqrt{5} & \\ -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{7}{5} & -\frac{1}{5}\sqrt{6} & \\ \frac{1}{10} & \sqrt{5} & -\frac{1}{5}\sqrt{6} & -\frac{9}{10} \end{bmatrix}$$

Hint: The eigenvalues are $-1, 2, -1$ where -1 is listed twice because it has multiplicity 2 as a zero of the characteristic equation.

Answer

The eigenvectors and eigenvalues are:

$$\left\{\begin{bmatrix} -\frac{1}{6}\sqrt{6} \\ 0 \\ \frac{1}{6}\sqrt{5}\sqrt{6} \end{bmatrix}, \begin{bmatrix} \frac{1}{3}\sqrt{2}\sqrt{3} \\ \frac{1}{5}\sqrt{5} \\ \frac{1}{15}\sqrt{2}\sqrt{15} \end{bmatrix}\right\} \leftrightarrow -1, \left\{\begin{bmatrix} \frac{1}{6}\sqrt{6} \\ -\frac{2}{5}\sqrt{5} \\ \frac{1}{30}\sqrt{30} \end{bmatrix}\right\} \leftrightarrow 2.$$

The columns of U are these vectors.

$$\begin{bmatrix} -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{2}\sqrt{3} & \frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} \\ \frac{1}{6}\sqrt{5}\sqrt{6} & \frac{1}{15}\sqrt{2}\sqrt{15} & \frac{1}{30}\sqrt{30} \end{bmatrix}^T \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{1}{10}\sqrt{5} \\ -\frac{1}{5}\sqrt{6}\sqrt{5} & \frac{7}{5} & -\frac{1}{5}\sqrt{6} \\ \frac{1}{10}\sqrt{5} & -\frac{1}{5}\sqrt{6} & -\frac{9}{10} \end{bmatrix}.$$

$$\begin{bmatrix} -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{2}\sqrt{3} & \frac{1}{6}\sqrt{6} \\ 0 & \frac{1}{5}\sqrt{5} & -\frac{2}{5}\sqrt{5} \\ \frac{1}{6}\sqrt{5}\sqrt{6} & \frac{1}{15}\sqrt{2}\sqrt{15} & \frac{1}{30}\sqrt{30} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

? Exercise 6.5.66

Explain why a matrix A is symmetric if and only if there exists an orthogonal matrix U such that $A = U^T DU$ for D a diagonal matrix.

Answer

If A is given by the formula, then

$$A^T = U^T D^T U = U^T DU = A$$

Next suppose $A = A^T$. Then by the theorems on symmetric matrices, there exists an orthogonal matrix U such that

$$UAU^T = D$$

for D diagonal. Hence

$$A = U^T DU$$

? Exercise 6.5.67

Show that if A is a real symmetric matrix and λ and μ are two different eigenvalues, then if X is an eigenvector for λ and Y is an eigenvector for μ, then $X \bullet Y = 0$. Also all eigenvalues are real. Supply reasons for each step in the following argument. First

$$\lambda X^T \overline{X} = (AX)^T \overline{X} = X^T A\overline{X} = X^T \overline{AX} = X^T \overline{\lambda X} = \overline{\lambda} X^T \overline{X}$$

and so $\lambda = \overline{\lambda}$. This shows that all eigenvalues are real. It follows all the eigenvectors are real. Why? Now let X, Y, μ and λ be given as above.

$$\lambda(X \bullet Y) = \lambda X \bullet Y = AX \bullet Y = X \bullet AY = X \bullet \mu Y = \mu(X \bullet Y) = \mu(X \bullet Y)$$

and so

$$(\lambda - \mu) X \bullet Y = 0$$

Why does it follow that $X \bullet Y = 0$?

Answer

Since $\lambda \neq \mu$, it follows $X \bullet Y = 0$.

? Exercise 6.5.68

Find the Cholesky factorization for the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 10 \end{bmatrix}$$

? Exercise 6.5.69

Find the Cholesky factorization for the matrix

$$\begin{bmatrix} 4 & 8 & 0 \\ 8 & 17 & 2 \\ 0 & 2 & 13 \end{bmatrix}$$

Exercise 6.5.70

Find the Cholesky factorization for the matrix

$$\begin{bmatrix} 4 & 8 & 0 \\ 8 & 20 & 8 \\ 0 & 8 & 20 \end{bmatrix}$$

Exercise 6.5.71

Find the Cholesky factorization for the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 8 & 10 \\ 1 & 10 & 18 \end{bmatrix}$$

Exercise 6.5.72

Find the Cholesky factorization for the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 8 & 10 \\ 1 & 10 & 26 \end{bmatrix}$$

Exercise 6.5.73

Suppose you have a lower triangular matrix L and it is invertible. Show that LL^T must be positive definite.

Exercise 6.5.74

Using the Gram Schmidt process or the QR factorization, find an orthonormal basis for the following span:

$$span\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$$

Answer

Using the QR factorization, we have:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{6}\sqrt{6} & \frac{3}{10}\sqrt{2} & \frac{7}{15}\sqrt{3} \\ \frac{1}{3}\sqrt{6} & -\frac{2}{5}\sqrt{2} & -\frac{1}{15}\sqrt{3} \\ \frac{1}{6}\sqrt{6} & \frac{1}{2}\sqrt{2} & -\frac{1}{3}\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{6} & \frac{1}{2}\sqrt{6} & \frac{1}{6}\sqrt{6} \\ 0 & \frac{5}{2}\sqrt{2} & \frac{3}{10}\sqrt{2} \\ 0 & 0 & \frac{7}{15}\sqrt{3} \end{bmatrix}$$

A solution is then

$$\begin{bmatrix} \frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \\ \frac{1}{6}\sqrt{6} \end{bmatrix}, \begin{bmatrix} \frac{3}{10}\sqrt{2} \\ -\frac{2}{5}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \end{bmatrix}, \begin{bmatrix} \frac{7}{15}\sqrt{3} \\ -\frac{1}{15}\sqrt{3} \\ -\frac{1}{3}\sqrt{3} \end{bmatrix}$$

Exercise 6.5.75

Using the Gram Schmidt process or the QR factorization, find an orthonormal basis for the following span:

$$span\left\{\begin{bmatrix}1\\2\\1\\0\end{bmatrix},\begin{bmatrix}2\\-1\\3\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\right\}$$

Answer

$$\begin{bmatrix}1&2&1\\2&-1&0\\1&3&0\\0&1&1\end{bmatrix}=\begin{bmatrix}\frac{1}{6}\sqrt{6}&\frac{1}{6}\sqrt{2}\sqrt{3}&\frac{5}{111}\sqrt{3}\sqrt{37}&\frac{7}{111}\sqrt{111}\\\frac{1}{3}\sqrt{6}&-\frac{2}{9}\sqrt{2}\sqrt{3}&\frac{1}{333}\sqrt{3}\sqrt{37}&-\frac{2}{111}\sqrt{111}\\\frac{1}{6}\sqrt{6}&\frac{5}{18}\sqrt{2}\sqrt{3}&-\frac{17}{333}\sqrt{3}\sqrt{37}&-\frac{1}{37}\sqrt{111}\\0&\frac{1}{9}\sqrt{2}\sqrt{3}&\frac{22}{333}\sqrt{3}\sqrt{37}&-\frac{7}{111}\sqrt{111}\end{bmatrix}$$

$$\begin{bmatrix}\sqrt{6}&\frac{1}{2}\sqrt{6}&\frac{1}{6}\sqrt{6}\\0&\frac{3}{2}\sqrt{2}\sqrt{3}&\frac{5}{18}\sqrt{2}\sqrt{3}\\0&0&\frac{1}{9}\sqrt{3}\sqrt{37}\\0&0&0\end{bmatrix}$$

Then a solution is

$$\begin{bmatrix}\frac{1}{6}\sqrt{6}&\\\frac{1}{3}\sqrt{6}&\\\frac{1}{6}\sqrt{6}&0\end{bmatrix},\begin{bmatrix}\frac{1}{6}\sqrt{2}\sqrt{3}\\-\frac{2}{9}\sqrt{2}\sqrt{3}\\\frac{5}{18}\sqrt{2}\sqrt{3}\\\frac{1}{9}\sqrt{2}\sqrt{3}\end{bmatrix},\begin{bmatrix}\frac{5}{111}\sqrt{3}\sqrt{37}\\\frac{1}{333}\sqrt{3}\sqrt{37}\\-\frac{17}{333}\sqrt{3}\sqrt{37}\\\frac{22}{333}\sqrt{3}\sqrt{37}\end{bmatrix}$$

Exercise 6.5.76

a. $\begin{bmatrix}1&2&3\\0&3&4\\0&0&1\end{bmatrix}$

b. $\begin{bmatrix}2&1\\2&1\end{bmatrix}$

c. $\begin{bmatrix}1&2\\-1&2\end{bmatrix}$

d. $\begin{bmatrix}1&1\\2&3\end{bmatrix}$

e. $\begin{bmatrix}\sqrt{11}&1&3\sqrt{6}\\\sqrt{11}&7&-\sqrt{6}\\2\sqrt{11}&-4&-\sqrt{6}\end{bmatrix}$ Hint: Notice that the columns are orthogonal.

Exercise 6.5.77

Using a computer algebra system, find a QR factorization for the following matrices.

a. $\begin{bmatrix}1&1&2\\3&-2&3\\2&1&1\end{bmatrix}$

b. $\begin{bmatrix}1&2&1&3\\4&5&-4&3\\2&1&2&1\end{bmatrix}$

c. $\begin{bmatrix}1&2\\3&2\\1&-4\end{bmatrix}$ Find the thin QR factorization of this one.

Exercise 6.5.78

A quadratic form in three variables is an expression of the form $a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz$. Show that every such quadratic form may be written as

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where A is a symmetric matrix.

Answer

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a_1 & a_4/2 & a_5/2 \\ a_4/2 & a_2 & a_6/2 \\ a_5/2 & a_6/2 & a_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Exercise 6.5.79

Given a quadratic form in three variables, x, y, and z, show there exists an orthogonal matrix U and variables x', y', z' such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = U \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

with the property that in terms of the new varaibles, the quadratic form is

$$\lambda_1(x')^2 + \lambda_2(y')^2 + \lambda_3(z')^2$$

where the numbers, λ_1, λ_2, and λ_3 are the eigenvalues of the matrix A in Exercise 6.5.78.

Answer

The quadratic form may be written as

$$\vec{x}^T A \vec{x}$$

where $A = A^T$. By the theorem about diagonalizing a symmetric matrix, there exists an orthogonal matrix U such that

$$U^T AU = D,\ A = UDU^T$$

Then the quadratic form is

$$\vec{x}^T UDU^T \vec{x} = (U^T \vec{x})^T D(U^T \vec{x})$$

where D is a diagonal matrix having the real eigenvalues of A down the main diagonal. Now simply let

$$\vec{x}' = U^T \vec{x}$$

Exercise 6.5.80

Consider the quadratic form q given by $q = 3x_1^2 - 12x_1x_2 - 2x_2^2$.

a. Write q in the form $\vec{x}^T A \vec{x}$ for an appropriate symmetric matrix A.
b. Use a change of variables to rewrite q to eliminate the x_1x_2 term.

Exercise 6.5.81

Consider the quadratic form q given by $q = -2x_1^2 + 2x_1x_2 - 2x_2^2$.

a. Write q in the form $\vec{x}^T A\vec{x}$ for an appropriate symmetric matrix A.
b. Use a change of variables to rewrite q to eliminate the x_1x_2 term.

Exercise 6.5.82

Consider the quadratic form q given by $q = 7x_1^2 + 6x_1x_2 - x_2^2$.

a. Write q in the form $\vec{x}^T A\vec{x}$ for an appropriate symmetric matrix A.
b. Use a change of variables to rewrite q to eliminate the x_1x_2 term.

This page titled 6.5: Exercises is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by Ken Kuttler (Lyryx) .

CHAPTER OVERVIEW

7: What are Differential Equations?

In this chapter, we begin our study of differential equations.

7.1: Applications Leading to Differential Equations

7.2: Basic Concepts

7.2.1: Basic Concepts (Exercises)

7.3: Direction Fields for First Order Equations

7.3.1: Direction Fields for First Order Equations (Exercises)

This page titled 7: What are Differential Equations? is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

7.1: Applications Leading to Differential Equations

In order to apply mathematical methods to a physical or “real life” problem, we must formulate the problem in mathematical terms; that is, we must construct a mathematical model for the problem. Many physical problems concern relationships between changing quantities. Since rates of change are represented mathematically by derivatives, mathematical models often involve equations relating an unknown function and one or more of its derivatives. Such equations are differential equations. They are the subject of this book.

Much of calculus is devoted to learning mathematical techniques that are applied in later courses in mathematics and the sciences; you wouldn’t have time to learn much calculus if you insisted on seeing a specific application of every topic covered in the course. Similarly, much of this book is devoted to methods that can be applied in later courses. Only a relatively small part of the book is devoted to the derivation of specific differential equations from mathematical models, or relating the differential equations that we study to specific applications. In this section we mention a few such applications. The mathematical model for an applied problem is almost always simpler than the actual situation being studied, since simplifying assumptions are usually required to obtain a mathematical problem that can be solved. For example, in modeling the motion of a falling object, we might neglect air resistance and the gravitational pull of celestial bodies other than Earth, or in modeling population growth we might assume that the population grows continuously rather than in discrete steps.

A good mathematical model has two important properties:

- It’s sufficiently simple so that the mathematical problem can be solved.
- It represents the actual situation sufficiently well so that the solution to the mathematical problem predicts the outcome of the real problem to within a useful degree of accuracy. If results predicted by the model don’t agree with physical observations,the underlying assumptions of the model must be revised until satisfactory agreement is obtained.

We will now give examples of mathematical models involving differential equations. We willreturn to these problems at the appropriate times, as we learn how to solve the various types of differential equations that occur in the models. All the examples in this section deal with functions of time, which we denote by t. If y is a function of t, y' denotes the derivative of y with respect to t; thus,

$$y' = \frac{dy}{dt}.$$

Population Growth and Decay

Although the number of members of a population (people in a given country, bacteria in a laboratory culture, wildflowers in a forest, etc.) at any given time t is necessarily an integer, models that use differential equations to describe the growth and decay of populations usually rest on the simplifying assumption that the number of members of the population can be regarded as a differentiable function $P = P(t)$. In most models it is assumed that the differential equation takes the form

$$P' = a(P)P \tag{7.1.1}$$

where a is a continuous function of P that represents the rate of change of population per unit time per individual. In the **Malthusian model**, it is assumed that $a(P)$ is a constant, so Equation 7.1.1 becomes

$$P' = aP. \tag{7.1.2}$$

(When you see a name in blue italics, just click on it for information about the person.) This model assumes that the numbers of births and deaths per unit time are both proportional to the population. The constants of proportionality are the birth rate (births per unit time per individual) and the death rate (deaths per unit time per individual); a is the birth rate minus the death rate. You learned in calculus that if c is any constant then

$$P = ce^{at} \tag{7.1.3}$$

satisfies Equation 7.1.2, so Equation 7.1.2 has infinitely many solutions. To select the solution of the specific problem that we are considering, we must know the population P_0 at an initial time, say $t = 0$. Setting $t = 0$ in Equation 7.1.3 yields $c = P(0) = P_0$, so the applicable solution is

$$P(t) = P_0 e^{at}.$$

This implies that

$$\lim_{t\to\infty} P(t) = \begin{cases} \infty & \text{if } a > 0, \\ 0 & \text{if } a < 0; \end{cases}$$

that is, the population approaches infinity if the birth rate exceeds the death rate, or zero if the death rate exceeds the birth rate.

To see the limitations of the Malthusian model, suppose we are modeling the population of a country, starting from a time $t = 0$ when the birth rate exceeds the death rate (so $a > 0$), and the country's resources in terms of space, food supply, and other necessities of life can support the existing population. Then the prediction $P = P_0 e^{at}$ may be reasonably accurate as long as it remains within limits that the country's resources can support. However, the model must inevitably lose validity when the prediction exceeds these limits. (If nothing else, eventually there will not be enough space for the predicted population!) This flaw in the Malthusian model suggests the need for a model that accounts for limitations of space and resources that tend to oppose the rate of population growth as the population increases.

Perhaps the most famous model of this kind is the **Verhulst model**, where Equation 7.1.2 is replaced by

$$P' = aP(1 - \alpha P), \tag{7.1.4}$$

where α is a positive constant. As long as P is small compared to $1/\alpha$, the ratio P'/P is approximately equal to a. Therefore the growth is approximately exponential; however, as P increases, the ratio P'/P decreases as opposing factors become significant.

Equation 7.1.4 is the **logistic equation**. You will learn how to solve it in Section 1.2. (See **Exercise 2.2.28**.) The solution is

$$P = \frac{P_0}{\alpha P_0 + (1 - \alpha P_0)e^{-at}},$$

where $P_0 = P(0) > 0$. Therefore $\lim_{t\to\infty} P(t) = 1/\alpha$, independent of P_0.

Figure 1.1.1 shows typical graphs of P versus t for various values of P_0.

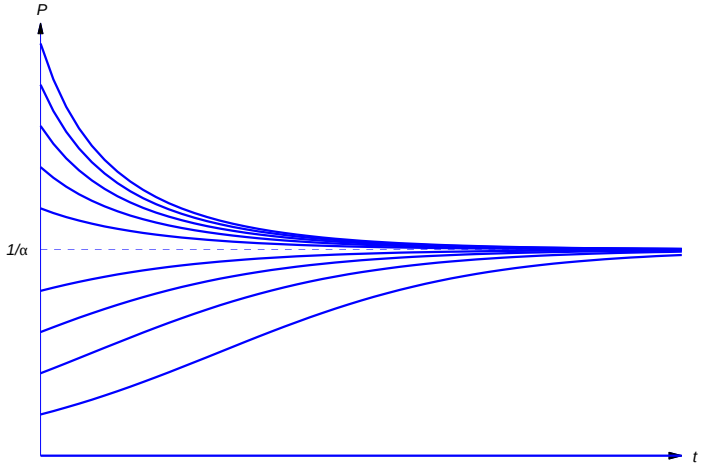


Figure 1.1.1 : Solutions of the logistic equation

Newton's Law of Cooling

According to **Newton's law of cooling**, the temperature of a body changes at a rate proportional to the difference between the temperature of the body and the temperature of the surrounding medium. Thus, if T_m is the temperature of the medium and $T = T(t)$ is the temperature of the body at time t, then

$$T' = -k(T - T_m) \tag{7.1.5}$$

where k is a positive constant and the minus sign indicates; that the temperature of the body increases with time if it is less than the temperature of the medium, or decreases if it is greater. We will see in Section 4.2 that if T_m is constant then the solution of Equation 7.1.5 is

$$T = T_m + (T_0 - T_m)e^{-kt} \tag{7.1.6}$$

where T_0 is the temperature of the body when $t = 0$. Therefore

$$\lim_{t\to\infty} T(t) = T_m$$

independent of T_0 (Common sense suggests this. Why?). Figure 1.1.2 shows typical graphs of T versus t for various values of T_0.

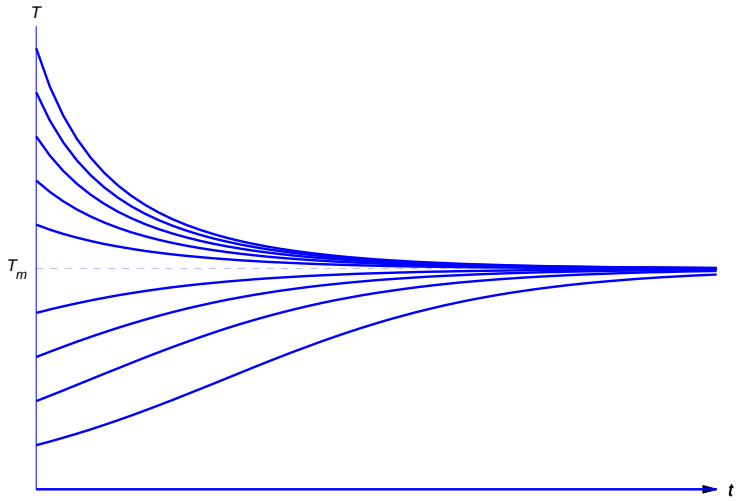


Figure 1.1.2 : Temperature according to Newton's Law of Cooling

Assuming that the medium remains at constant temperature seems reasonable if we are considering a cup of coffee cooling in a room, but not if we are cooling a huge cauldron of molten metal in the same room. The difference between the two situations is that the heat lost by the coffee isn't likely to raise the temperature of the room appreciably, but the heat lost by the cooling metal is. In this second situation we must use a model that accounts for the heat exchanged between the object and the medium. Let $T = T(t)$ and $T_m = T_m(t)$ be the temperatures of the object and the medium respectively, and let T_0 and T_m0 be their initial values. Again, we assume that T and Tm are related by Equation 7.1.5. We also assume that the change in heat of the object as its temperature changes from T_0 to T is $a(T - T_0)$ and the change in heat of the medium as its temperature changes from T_{m0} to T_m is $a_m(T_m - T_{m0})$, where a and am are positive constants depending upon the masses and thermal properties of the object and medium respectively. If we assume that the total heat of the in the object and the medium remains constant (that is, energy is conserved), then

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0.$$

Solving this for T_m and substituting the result into Equation 7.1.6 yields the differential equation

$$T' = -k\left(1 + \frac{a}{a_m}\right)T + k\left(T_{m0} + \frac{a}{a_m}T_0\right)$$

for the temperature of the object. After learning to solve linear first order equations, you'll be able to show (**Exercise 4.2.17**) that

$$T = \frac{aT_0 + a_mT_{m0}}{a + a_m} + \frac{a_m\,(T_0 - T_{m0})}{a + a_m}e^{-k(1+a/a_m)t}$$

Glucose Absorption by the Body

Glucose is absorbed by the body at a rate proportional to the amount of glucose present in the blood stream. Let λ denote the (positive) constant of proportionality. Suppose there are G_0 units of glucose in the bloodstream when $t = 0$, and let $G = G(t)$ be the number of units in the bloodstream at time $t > 0$. Then, since the glucose being absorbed by the body is leaving the bloodstream, G satisfies the equation

$$G' = -\lambda G. \tag{7.1.7}$$

From calculus you know that if c is any constant then

$$G = ce^{-\lambda t} \tag{7.1.8}$$

satisfies Equation (1.1.7), so Equation 7.1.7 has infinitely many solutions. Setting $t = 0$ in Equation 7.1.8 and requiring that $G(0) = G_0$ yields $c = G_0$, so

$$G(t) = G_0 e^{-\lambda t}.$$

Now let's complicate matters by injecting glucose intravenously at a constant rate of r units of glucose per unit of time. Then the rate of change of the amount of glucose in the bloodstream per unit time is

$$G' = -\lambda G + r \tag{7.1.9}$$

where the first term on the right is due to the absorption of the glucose by the body and the second term is due to the injection. After you've studied Section 2.1, you'll be able to show that the solution of Equation 7.1.9 that satisfies $G(0) = G_0$ is

$$G = \frac{r}{\lambda} + \left(G_0 - \frac{r}{\lambda}\right) e^{-\lambda t}$$

Graphs of this function are similar to those in Figure 1.1.2 . (Why?)

Spread of Epidemics

One model for the spread of epidemics assumes that the number of people infected changes at a rate proportional to the product of the number of people already infected and the number of people who are susceptible, but not yet infected. Therefore, if S denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time t, then $S - I$ is the number of people who are susceptible, but not yet infected. Thus,

$$I' = rI(S - I)$$

, where r is a positive constant. Assuming that $I(0) = I_0$, the solution of this equation is

$$I = \frac{SI_0}{I_0 + (S - I_0)e^{-rSt}}$$

(**Exercise 2.2.29**). Graphs of this function are similar to those in Figure 1.1.1. (Why?) Since $\lim_{t\to\infty} I(t) = S$, this model predicts that all the susceptible people eventually become infected.

Newton's Second Law of Motion

According to Newton's second law of motion, the instantaneous acceleration a of an object with constant mass m is related to the force F acting on the object by the equation $F = ma$. For simplicity, let's assume that $m = 1$ and the motion of the object is along a vertical line. Let y be the displacement of the object from some reference point on Earth's surface, measured positive upward. In many applications, there are three kinds of forces that may act on the object:

a. A force such as gravity that depends only on the position y, which we write as $-p(y)$, where $p(y) > 0$ if $y \geq 0$.
b. A force such as atmospheric resistance that depends on the position and velocity of the object, which we write as $-q(y, y')y'$, where q is a nonnegative function and we've put y' "outside" to indicate that the resistive force is always in the direction opposite to the velocity.
c. A force $f = f(t)$, exerted from an external source (such as a towline from a helicopter) that depends only on t.

In this case, Newton's second law implies that

$$y'' = -q(y, y')y' - p(y) + f(t),$$

which is usually rewritten as

$$y'' + q(y, y')y' + p(y) = f(t).$$

Since the second (and no higher) order derivative of y occurs in this equation, we say that it is a second order differential equation.

Interacting Species: Competition

Let $P = P(t)$ and $Q = Q(t)$ be the populations of two species at time t, and assume that each population would grow exponentially if the other did not exist; that is, in the absence of competition we would have

$$P' = aP \quad \text{and} \quad Q' = bQ, \tag{7.1.10}$$

where a and b are positive constants. One way to model the effect of competition is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so Equation 7.1.10 is replaced by

$$P' = aP - \alpha Q$$
$$Q' = -\beta P + bQ,$$

where α and β are positive constants. (Since negative population doesn't make sense, this system works only while P and Q are both positive.) Now suppose $P(0) = P_0 > 0$ and $Q(0) = Q_0 > 0$. It can be shown (**Exercise 10.4.42**) that there's a positive constant ρ such that if (P_0, Q_0) is above the line L through the origin with slope ρ, then the species with population P becomes extinct in finite time, but if (P_0, Q_0) is below L, the species with population Q becomes extinct in finite time. Figure 1.1.3 illustrates this. The curves shown there are given parametrically by $P = P(t), Q = Q(t)$, $t > 0$. The arrows indicate direction along the curves with increasing t.

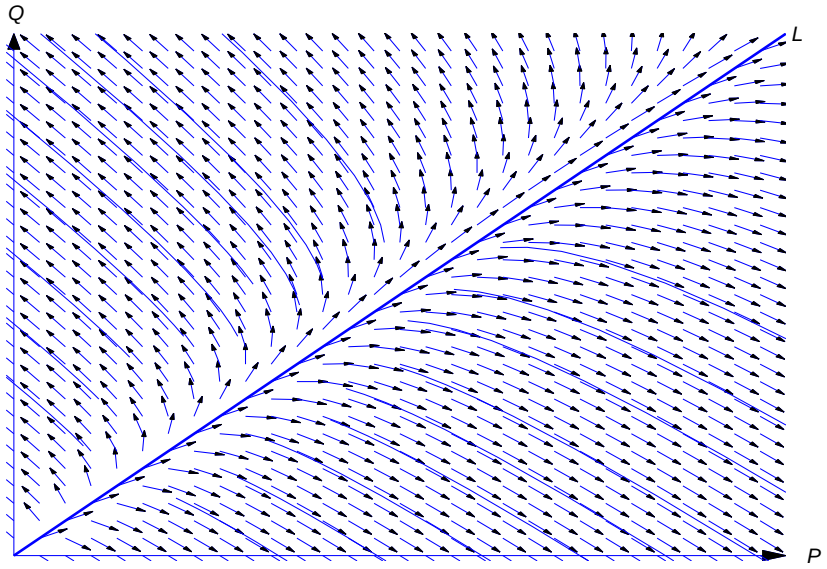


Figure 1.1.3 : Populations of competing species

This page titled 7.1: Applications Leading to Differential Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **1.1: Applications Leading to Differential Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

7.2: Basic Concepts

A **differential equation** is an equation that contains one or more derivatives of an unknown function. The **order** of a differential equation is the order of the highest derivative that it contains. A differential equation is an **ordinary differential equation** if it involves an unknown function of only one variable, or a **partial differential equation** if it involves partial derivatives of a function of more than one variable. For now we'll consider only ordinary differential equations, and we'll just call them **differential equations.**

> Throughout this text, all variables and constants are real unless it is stated otherwise. We'll usually use x for the independent variable unless the independent variable is time; then we'll use t.

The simplest differential equations are first order equations of the form

$$\frac{dy}{dx} = f(x)$$

or equivalently

$$y' = f(x),$$

where f is a known function of x. We already know from calculus how to find functions that satisfy this kind of equation. For example, if

$$y' = x^3,$$

then

$$y = \int x^3\,dx = \frac{x^4}{4} + c,$$

where c is an arbitrary constant. If $n > 1$ we can find functions y that satisfy equations of the form

$$y^{(n)} = f(x) \tag{7.2.1}$$

by repeated integration. Again, this is a calculus problem.

Except for illustrative purposes in this section, there's no need to consider differential equations like Equation 7.2.1. We'll usually consider differential equations that can be written as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \tag{7.2.2}$$

where at least one of the functions y, y', …, $y^{(n-1)}$ actually appears on the right. Here are some examples:

$$\begin{aligned}
\tfrac{dy}{dx} - x^2 &= 0 && \text{(first order)},\\
\tfrac{dy}{dx} + 2xy^2 &= -2 && \text{(first order)},\\
\tfrac{d^2y}{dx^2} + 2\tfrac{dy}{dx} + y &= 2x && \text{(second order)},\\
xy''' + y^2 &= \sin x && \text{(third order)},\\
y^{(n)} + xy' + 3y &= x && \text{(n-th order)}.
\end{aligned}$$

Although none of these equations is written as in Equation 7.2.2, all of them **can** be written in this form:

$$
\begin{aligned}
y' &= x^2, \\
y' &= -2 - 2xy^2, \\
y'' &= 2x - 2y' - y, \\
y''' &= \frac{\sin x - y^2}{x}, \\
y^{(n)} &= x - xy' - 3y.
\end{aligned}
$$

Solutions of Differential Equations

A **solution** of a differential equation is a function that satisfies the differential equation on some open interval; thus, y is a solution of Equation 7.2.2 if y is n times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

for all x in some open interval (a, b). In this case, we also say that **y is a solution of Equation 7.2.2 on** (a, b). Functions that satisfy a differential equation at isolated points are not interesting. For example, $y = x^2$ satisfies

$$xy' + x^2 = 3x$$

if and only if $x = 0$ or $x = 1$, but it is not a solution of this differential equation because it does not satisfy the equation on an open interval.

The graph of a solution of a differential equation is a **solution curve**. More generally, a curve C is said to be an **integral curve** of a differential equation if every function $y = y(x)$ whose graph is a segment of C is a solution of the differential equation. Thus, any solution curve of a differential equation is an integral curve, but an integral curve need not be a solution curve.

Example 1.2.1

If a is any positive constant, the circle

$$x^2 + y^2 = a^2 \tag{7.2.3}$$

is an integral curve of

$$y' = -\frac{x}{y}. \tag{7.2.4}$$

Solution

To see this, note that the only functions whose graphs are segments of Equation 7.2.3 are

$$y_1 = \sqrt{a^2 - x^2} \quad \text{and} \quad y_2 = -\sqrt{a^2 - x^2}.$$

We leave it to you to verify that these functions both satisfy Equation 7.2.4 on the open interval $(-a, a)$. However, Equation 7.2.3 is not a solution curve of Equation 7.2.4, since it is not the graph of a function.

Example 1.2.2

Verify that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{7.2.5}$$

is a solution of

$$xy' + y = x^2 \tag{7.2.6}$$

on $(0, \infty)$ and on $(-\infty, 0)$.

Solution

Substituting Equation 7.2.5 and

$$y' = \frac{2x}{3} - \frac{1}{x^2}$$

into Equation 7.2.6 yields

$$xy'(x) + y(x) = x\left(\frac{2x}{3} - \frac{1}{x^2}\right) + \left(\frac{x^2}{3} + \frac{1}{x}\right) = x^2$$

for all $x \neq 0$. Therefore y is a solution of Equation 7.2.6 on $(-\infty, 0)$ and $(0, \infty)$. However, y isn't a solution of the differential equation on any open interval that contains $x = 0$, since y is not defined at $x = 0$.

Figure 1.2.2 shows the graph of Equation 7.2.5. The part of the graph of Equation 7.2.5 on $(0, \infty)$ is a solution curve of Equation 7.2.6, as is the part of the graph on $(-\infty, 0)$.

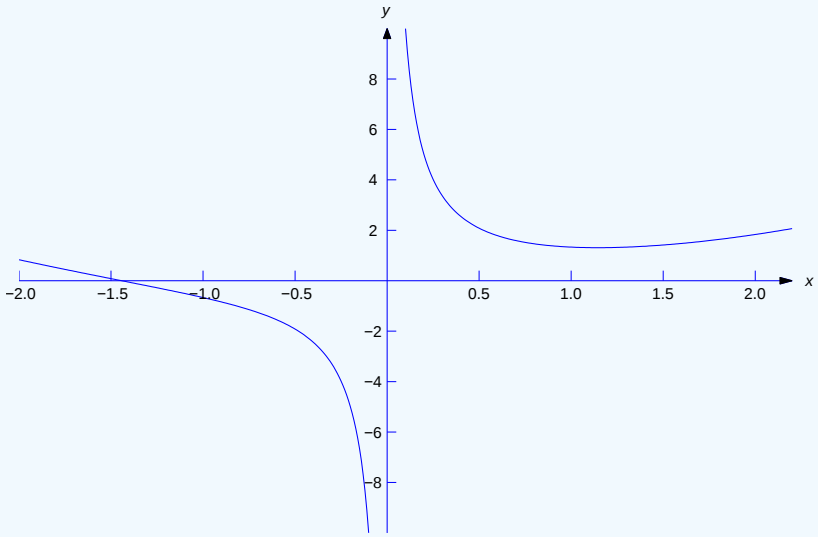


Figure 1.2.2 : $y = \frac{x^2}{3} + \frac{1}{x}$

Example 1.2.3

Show that if c_1 and c_2 are constants then

$$y = (c_1 + c_2 x)e^{-x} + 2x - 4 \tag{7.2.7}$$

is a solution of

$$y'' + 2y' + y = 2x \tag{7.2.8}$$

on $(-\infty, \infty)$.

Solution

Differentiating Equation 7.2.7 twice yields

$$y' = -(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2$$

and

$$y'' = (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x},$$

so

$$\begin{aligned} y'' + 2y' + y &= (c_1 + c_2 x)e^{-x} - 2c_2 e^{-x} + 2\left[-(c_1 + c_2 x)e^{-x} + c_2 e^{-x} + 2\right] \\ &\quad + (c_1 + c_2 x)e^{-x} + 2x - 4 \\ &= (1 - 2 + 1)(c_1 + c_2 x)e^{-x} + (-2 + 2)c_2 e^{-x} + 4 + 2x - 4 \\ &= 2x \end{aligned}$$

for all values of x. Therefore y is a solution of Equation 7.2.8 on $(-\infty, \infty)$.

Example 1.2.4

Find all solutions of

$$y^{(n)} = e^{2x}. \tag{7.2.9}$$

Solution

Integrating Equation 7.2.9 yields

$$y^{(n-1)} = \frac{e^{2x}}{2} + k_1,$$

where k_1 is a constant. If $n \geq 2$, integrating again yields

$$y^{(n-2)} = \frac{e^{2x}}{4} + k_1 x + k_2.$$

If $n \geq 3$, repeatedly integrating yields

$$y = \frac{e^{2x}}{2^n} + k_1 \frac{x^{n-1}}{(n-1)!} + k_2 \frac{x^{n-2}}{(n-2)!} + \cdots + k_n, \tag{7.2.10}$$

where k_1, k_2, …, k_n are constants. This shows that every solution of Equation 7.2.9 has the form Equation 7.2.10 for some choice of the constants k_1, k_2, …, k_n. On the other hand, differentiating Equation 7.2.10 n times shows that if k_1, k_2, …, k_n are arbitrary constants, then the function y in Equation 7.2.10 satisfies Equation 7.2.9.

Since the constants k_1, k_2, …, k_n in Equation 7.2.10 are arbitrary, so are the constants

$$\frac{k_1}{(n-1)!}, \frac{k_2}{(n-2)!}, \cdots, k_n.$$

Therefore Example 1.2.4 actually shows that all solutions of Equation 7.2.9 can be written as

$$y = \frac{e^{2x}}{2^n} + c_1 + c_2 x + \cdots + c_n x^{n-1},$$

where we renamed the arbitrary constants in Equation 7.2.10 to obtain a simpler formula. As a general rule, arbitrary constants appearing in solutions of differential equations should be simplified if possible. You'll see examples of this throughout the text.

Initial Value Problems

In Example 1.2.4 we saw that the differential equation $y^{(n)} = e^{2x}$ has an infinite family of solutions that depend upon the n arbitrary constants c_1, c_2, …, c_n. In the absence of additional conditions, there's no reason to prefer one solution of a differential equation over another. However, we'll often be interested in finding a solution of a differential equation that satisfies one or more specific conditions. The next example illustrates this.

Example 1.2.5

Find a solution of

$$y' = x^3$$

such that $y(1) = 2$.

Solution

At the beginning of this section we saw that the solutions of $y' = x^3$ are

$$y = \frac{x^4}{4} + c.$$

To determine a value of c such that $y(1) = 2$, we set $x = 1$ and $y = 2$ here to obtain

$$2 = y(1) = \frac{1}{4} + c$$

so

$$c = \frac{7}{4}.$$

Therefore the required solution is

$$y = \frac{x^4 + 7}{4}.$$

Figure 1.2.2 shows the graph of this solution. Note that imposing the condition $y(1) = 2$ is equivalent to requiring the graph of y to pass through the point $(1, 2)$.

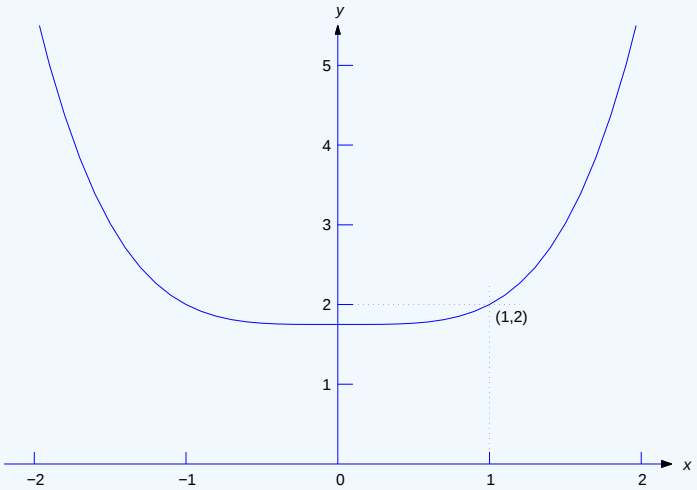


Figure 1.2.2 : $y = \frac{x^4+7}{4}$

We can rewrite the problem considered in Example 1.2.5 more briefly as

$$y' = x^3, \quad y(1) = 2.$$

We call this an **initial value problem**. The requirement $y(1) = 2$ is an **initial condition**. Initial value problems can also be posed for higher order differential equations. For example,

$$y'' - 2y' + 3y = e^x, \quad y(0) = 1, \quad y'(0) = 2 \tag{7.2.11}$$

is an initial value problem for a second order differential equation where y and y' are required to have specified values at $x = 0$. In general, an initial value problem for an n-th order differential equation requires y and its first $n - 1$ derivatives to have specified values at some point x_0. These requirements are the **initial conditions**.We'll denote an initial value problem for a differential equation by writing the initial conditions after the equation, as in Equation 7.2.11. For example, we would write an initial value problem for Equation 7.2.2 as

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)}), \; y(x_0) = k_0, \; y'(x_0) = k_1, \; \dots, \; y^{(n-1)} = k_{n-1}. \tag{7.2.12}$$

Consistent with our earlier definition of a solution of the differential equation in Equation 7.2.12, we say that y is a solution of the initial value problem Equation 7.2.12 if y is n times differentiable and

$$y^{(n)}(x) = f(x, y(x), y'(x), \dots, y^{(n-1)}(x))$$

for all x in some open interval (a, b) that contains x_0, and y satisfies the initial conditions in Equation 7.2.12. The largest open interval that contains x_0 on which y is defined and satisfies the differential equation is the **interval of validity** of y.

✔ Example 1.2.6

In Example 1.2.5 we saw that

$$y = \frac{x^4 + 7}{4} \tag{7.2.13}$$

is a solution of the initial value problem

$$y' = x^3, \quad y(1) = 2.$$

Since the function in Equation 7.2.13 is defined for all x, the interval of validity of this solution is $(-\infty, \infty)$.

Example 1.2.7

In Example 1.2.2 we verified that

$$y = \frac{x^2}{3} + \frac{1}{x} \tag{7.2.14}$$

is a solution of

$$xy' + y = x^2$$

on $(0, \infty)$ and on $(-\infty, 0)$. By evaluating Equation 7.2.14 at $x = \pm 1$, you can see that Equation 7.2.14 is a solution of the initial value problems

$$xy' + y = x^2, \quad y(1) = \frac{4}{3} \tag{7.2.15}$$

and

$$xy' + y = x^2, \quad y(-1) = -\frac{2}{3}. \tag{7.2.16}$$

The interval of validity of Equation 7.2.14 as a solution of Equation 7.2.15 is $(0, \infty)$, since this is the largest interval that contains $x_0 = 1$ on which Equation 7.2.14 is defined. Similarly, the interval of validity of Equation 7.2.14 as a solution of Equation 7.2.16 is $(-\infty, 0)$, since this is the largest interval that contains $x_0 = -1$ on which Equation 7.2.14 is defined.

Free Fall Under Constant Gravity

The term **initial value problem** originated in problems of motion where the independent variable is t (representing elapsed time), and the initial conditions are the position and velocity of an object at the initial (starting) time of an experiment.

Example 1.2.8

An object falls under the influence of gravity near Earth's surface, where it can be assumed that the magnitude of the acceleration due to gravity is a constant g.

a. Construct a mathematical model for the motion of the object in the form of an initial value problem for a second order differential equation, assuming that the altitude and velocity of the object at time $t = 0$ are known. Assume that gravity is the only force acting on the object.
b. Solve the initial value problem derived above to obtain the altitude as a function of time.

Solution a

Let $y(t)$ be the altitude of the object at time t. Since the acceleration of the object has constant magnitude g and is in the downward (negative) direction, y satisfies the second order equation

$$y'' = -g,$$

where the prime now indicates differentiation with respect to t. If y_0 and v_0 denote the altitude and velocity when $t = 0$, then y is a solution of the initial value problem

$$y'' = -g, \quad y(0) = y_0, \quad y'(0) = v_0. \tag{7.2.17}$$

Solution b

Integrating Equation 7.2.17 twice yields

$$y' = -gt + c_1,$$
$$y = -\frac{gt^2}{2} + c_1 t + c_2.$$

Imposing the initial conditions $y(0) = y_0$ and $y'(0) = v_0$ in these two equations shows that $c_1 = v_0$ and $c_2 = y_0$. Therefore the solution of the initial value problem Equation 7.2.17 is

$$y = -\frac{gt^2}{2} + v_0 t + y_0.$$

This page titled 7.2: Basic Concepts is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **1.2: Basic Concepts** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

7.2.1: Basic Concepts (Exercises)

1. Find the order of the equation.

a. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\ \frac{d^3y}{dx^3} + x = 0$

b. $y'' - 3y' + 2y = x^7$

c. $y' - y^7 = 0$

d. $y''y - (y')^2 = 2$

2. Verify that the function is a solution of the differential equation on some interval, for any choice of the arbitrary constants appearing in the function.

a. $y = ce^{2x}; \quad y' = 2y$

b. $y = \frac{x^2}{3} + \frac{c}{x}; \quad xy' + y = x^2$

c. $y = \frac{1}{2} + ce^{-x^2}; \quad y' + 2xy = x$

d. $y = (1 + ce^{-x^2/2})(1 - ce^{-x^2/2})^{-1}; \quad 2y' + x(y^2 - 1) = 0$

e. $y = \tan\left(\frac{x^3}{3} + c\right); \quad y' = x^2(1 + y^2)$

f. $y = (c_1 + c_2x)e^x + \sin x + x^2; \quad y'' - 2y' + y = -2\cos x + x^2 - 4x + 2$

g. $y = c_1e^x + c_2x + \frac{2}{x}; \quad (1 - x)y'' + xy' - y = 4(1 - x - x^2)x^{-3}$

h. $y = x^{-1/2}(c_1 \sin x + c_2 \cos x) + 4x + 8; x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 4x^3 + 8x^2 + 3x - 2$

3. Find all solutions of the equation.

a. $y' = -x$

b. $y' = -x \sin x$

c. $y' = x \ln x$

d. $y'' = x \cos x$

e. $y'' = 2xe^x$

f. $y'' = 2x + \sin x + e^x$

g. $y''' = -\cos x$

h. $y''' = -x^2 + e^x$

i. $y''' = 7e^{4x}$

4. Solve the initial value problem.

a. $y' = -xe^x, \quad y(0) = 1$

b. $y' = x \sin x^2, \quad y\left(\sqrt{\frac{\pi}{2}}\right) = 1$

c. $y' = \tan x, \quad y(\pi/4) = 3$

d. $y'' = x^4, \quad y(2) = -1, \quad y'(2) = -1$

e. $y'' = xe^{2x}, \quad y(0) = 7, \quad y'(0) = 1$

f. $y'' = -x \sin x, \quad y(0) = 1, \quad y'(0) = -3$

g. $y''' = x^2e^x, \quad y(0) = 1, \quad y'(0) = -2, \quad y''(0) = 3$

h. $y''' = 2 + \sin 2x, \quad y(0) = 1, \quad y'(0) = -6, \quad y''(0) = 3$

i. $y''' = 2x + 1, \quad y(2) = 1, \quad y'(2) = -4, \quad y''(2) = 7$

5. Verify that the function is a solution of the initial value problem.

a. $y = x\cos x; \quad y' = \cos x - y\tan x, \quad y(\pi/4) = \frac{\pi}{4\sqrt{2}}$

b. $y = \frac{1+2\ln x}{x^2} + \frac{1}{2}; \quad y' = \frac{x^2-2x^2y+2}{x^3}, \quad y(1) = \frac{3}{2}$

c. $y = \tan\left(\frac{x^2}{2}\right); \quad y' = x(1+y^2), \quad y(0) = 0$

d. $y = \frac{2}{x-2}; \quad y' = \frac{-y(y+1)}{x}, \quad y(1) = -2$

6. Verify that the function is a solution of the initial value problem.

a. $y = x^2(1+\ln x); \quad y'' = \frac{3xy'-4y}{x^2}, \quad y(e) = 2e^2, \quad y'(e) = 5e$

b. $y = \frac{x^2}{3} + x - 1; \quad y'' = \frac{x^2-xy'+y+1}{x^2}, \quad y(1) = \frac{1}{3}, \quad y'(1) = \frac{5}{3}$

c. $y = (1+x^2)^{-1/2}; \quad y'' = \frac{(x^2-1)y-x(x^2+1)y'}{(x^2+1)^2}, \quad y(0) = 1,\ y'(0) = 0$

d. $y = \frac{x^2}{1-x}; \quad y'' = \frac{2(x+y)(xy'-y)}{x^3}, \quad y(1/2) = 1/2, \quad y'(1/2) = 3$

7. Suppose an object is launched from a point 320 feet above the earth with an initial velocity of 128 ft/sec upward, and the only force acting on it thereafter is gravity. Take $g = 32$ ft/sec^2.

a. Find the highest altitude attained by the object.
b. Determine how long it takes for the object to fall to the ground.

8. Let a be a nonzero real number.

a. Verify that if c is an arbitrary constant then

$$y = (x-c)^a \tag{A}$$

is a solution of

$$y' = ay^{(a-1)/a} \tag{B}$$

on (c, ∞).
b. Suppose $a < 0$ or $a > 1$. Can you think of a solution of (B) that isn't of the form (A)?

9. Verify that

$$y = \begin{cases} e^x - 1, & x \ge 0, \\ 1 - e^{-x}, & x < 0, \end{cases}$$

is a solution of

$$y' = |y| + 1$$

on $(-\infty, \infty)$.

10.

(a) Verify that if c is any real number then

$$y = c^2 + cx + 2c + 1 \tag{A}$$

satisfies

$$y' = \frac{-(x+2) + \sqrt{x^2+4x+4y}}{2} \tag{B}$$

on some open interval. Identify the open interval.

(b) Verify that

$$y_1 = \frac{-x(x+4)}{4}$$

also satisfies (B) on some open interval, and identify the open interval. (Note that y_1 can't be obtained by selecting a value of c in (A).)

This page titled 7.2.1: Basic Concepts (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **1.2E: Basic Concepts (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

7.3: Direction Fields for First Order Equations

It's impossible to find explicit formulas for solutions of some differential equations. Even if there are such formulas, they may be so complicated that they're useless. In this case we may resort to graphical or numerical methods to get some idea of how the solutions of the given equation behave.

In Section 2.3 we'll take up the question of existence of solutions of a first order equation

$$y' = f(x, y). \tag{7.3.1}$$

In this section we'll simply assume that Equation 7.3.1 has solutions and discuss a graphical method for approximating them. In Chapter 3 we discuss numerical methods for obtaining approximate solutions of Equation 7.3.1. Recall that a solution of Equation 7.3.1 is a function $y = y(x)$ such that

$$y'(x) = f(x, y(x))$$

for all values of x in some interval, and an integral curve is either the graph of a solution or is made up of segments that are graphs of solutions. Therefore, not being able to solve Equation 7.3.1 is equivalent to not knowing the equations of integral curves of Equation 7.3.1. However, it is easy to calculate the slopes of these curves. To be specific, the slope of an integral curve of Equation 7.3.1 through a given point (x_0, y_0) is given by the number $f(x_0, y_0)$. This is the basis of **the method of direction fields**.

If f is defined on a set R, we can construct a **direction field** for Equation 7.3.1 in R by drawing a short line segment through each point (x, y) in R with slope $f(x, y)$. Of course, as a practical matter, we can't actually draw line segments through **every** point in R; rather, we must select a finite set of points in R. For example, suppose f is defined on the closed rectangular region

$$R : \{a \le x \le b, c \le y \le d\}.$$

Let

$$a = x_0 < x_1 < \cdots < x_m = b$$

be equally spaced points in $[a, b]$ and

$$c = y_0 < y_1 < \cdots < y_n = d$$

form a **rectangular grid** (Figure 1.3.1). Through each point in the grid we draw a short line segment with slope $f(x_i, y_j)$. The result is an approximation to a direction field for Equation 7.3.1 in R. If the grid points are sufficiently numerous and close together, we can draw approximate integral curves of Equation 7.3.1 by drawing curves through points in the grid tangent to the line segments associated with the points in the grid.

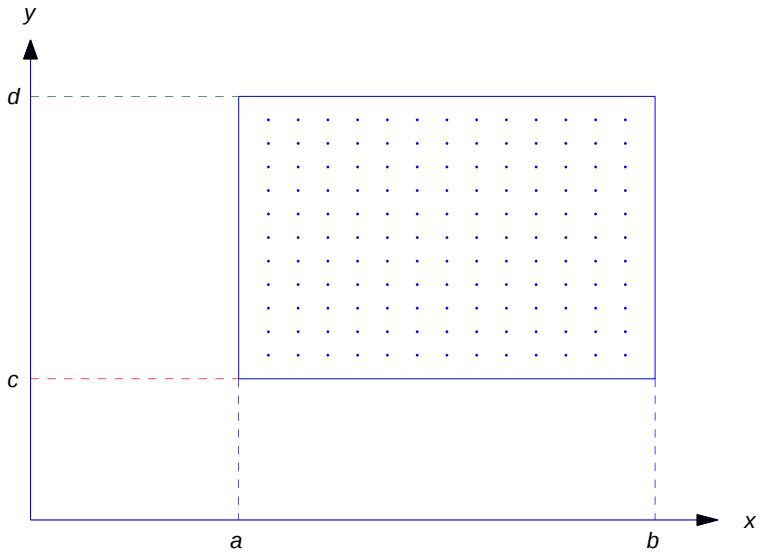


Figure 1.3.1 : A rectangular grid

Unfortunately, approximating a direction field and graphing integral curves in this way is too tedious to be done effectively by hand. However, there is software for doing this. As you'll see, the combination of direction fields and integral curves gives useful insights into the behavior of the solutions of the differential equation even if we can't obtain exact solutions.

We'll study numerical methods for solving a single first order equation Equation 7.3.1 in Chapter 3. These methods can be used to plot solution curves of Equation 7.3.1 in a rectangular region R **if** f **is continuous on** R. Figures 1.3.2 , 1.3.3 , and 1.3.4 show direction fields and solution curves for the differential equations:

- $y' = \dfrac{x^2 - y^2}{1 + x^2 + y^2}$,
- $y' = 1 + xy^2$, and
- $y' = \dfrac{x - y}{1 + x^2}$.

which are all of the form Equation 7.3.1 with f continuous for all (x, y).

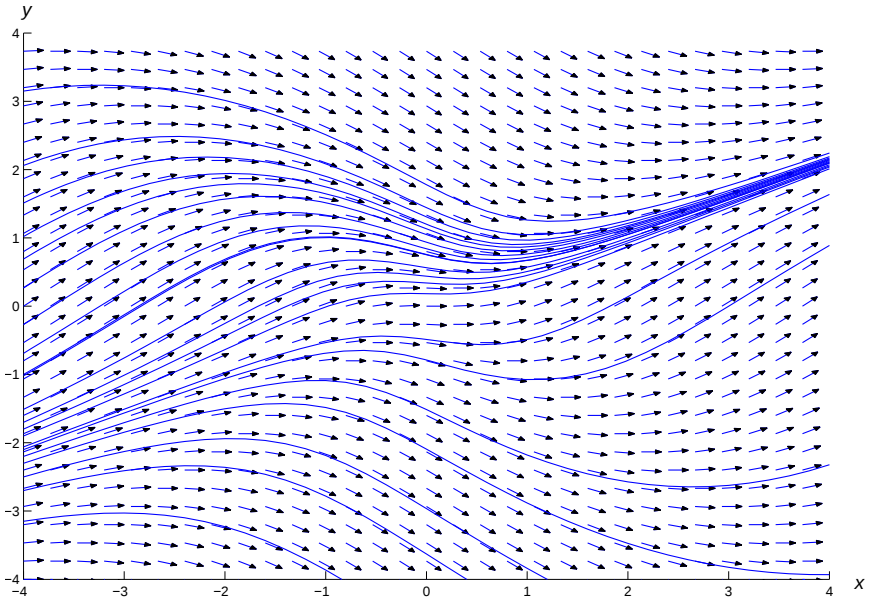


Figure 1.3.2 : A direction and integral curves for $y' = \frac{x^2-y^2}{1+x^2+y^2}$.

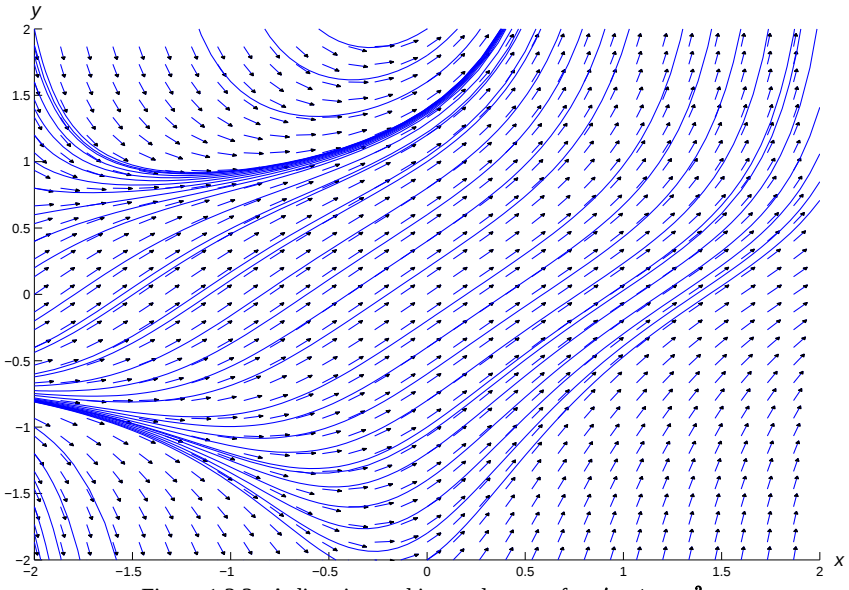


Figure 1.3.3 : A direction and integral curves for $y' = 1 + xy^2$.

https://math.libretexts.org/@go/page/121633

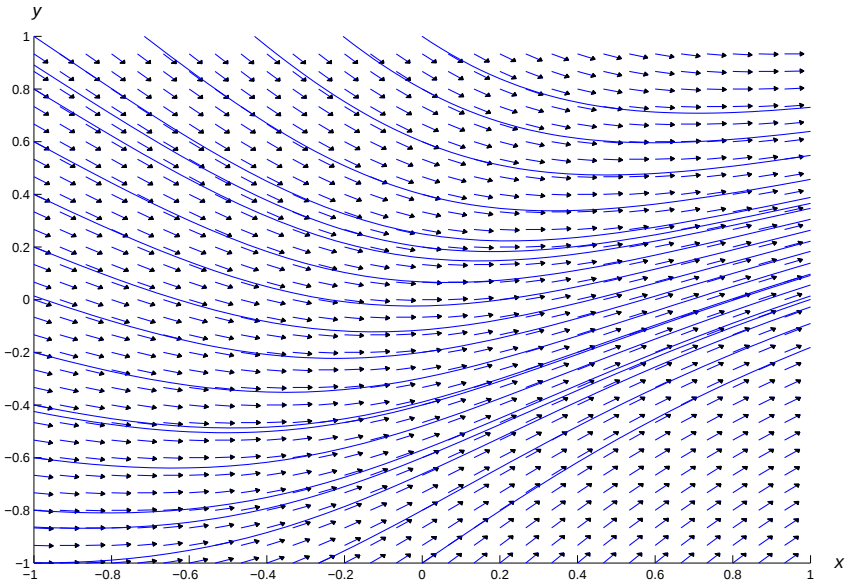


Figure 1.3.4 : A direction and integral curves for $y' = \frac{x-y}{1+x^2}$.

The methods of Chapter 3 will not work for the equation

$$y' = -x/y \tag{7.3.2}$$

if R contains part of the x-axis, since $f(x,y) = -x/y$ is undefined when $y = 0$. Similarly, they will not work for the equation

$$y' = \frac{x^2}{1-x^2-y^2} \tag{7.3.3}$$

if R contains any part of the unit circle $x^2 + y^2 = 1$, because the right side of Equation 7.3.3 is undefined if $x^2 + y^2 = 1$. However, Equation 7.3.2 and Equation 7.3.3 can written as

$$y' = \frac{A(x,y)}{B(x,y)} \tag{7.3.4}$$

where A and B are continuous on any rectangle R. Because of this, some differential equation software is based on numerically solving pairs of equations of the form

$$\frac{dx}{dt} = B(x,y), \quad \frac{dy}{dt} = A(x,y) \tag{7.3.5}$$

where x and y are regarded as functions of a parameter t. If $x = x(t)$ and $y = y(t)$ satisfy these equations, then

$$y' = \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{A(x,y)}{B(x,y)},$$

so $y = y(x)$ satisfies Equation 7.3.4.

Equations 7.3.2 and 7.3.3 can be reformulated as in Equation 7.3.4 with

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x$$

and

$$\frac{dx}{dt} = 1 - x^2 - y^2, \quad \frac{dy}{dt} = x^2,$$

respectively. Even if f is continuous and otherwise "nice" throughout R, your software may require you to reformulate the equation $y' = f(x,y)$ as

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = f(x,y),$$

which is of the form Equation 7.3.5 with $A(x,y)=f(x,y)$ and $B(x,y)=1$.

Figure 1.3.5 shows a direction field and some integral curves for Equation 7.3.2. As we saw in Example [example:1.2.1} and will verify again in Section 2.2, the integral curves of Equation 7.3.2 are circles centered at the origin.

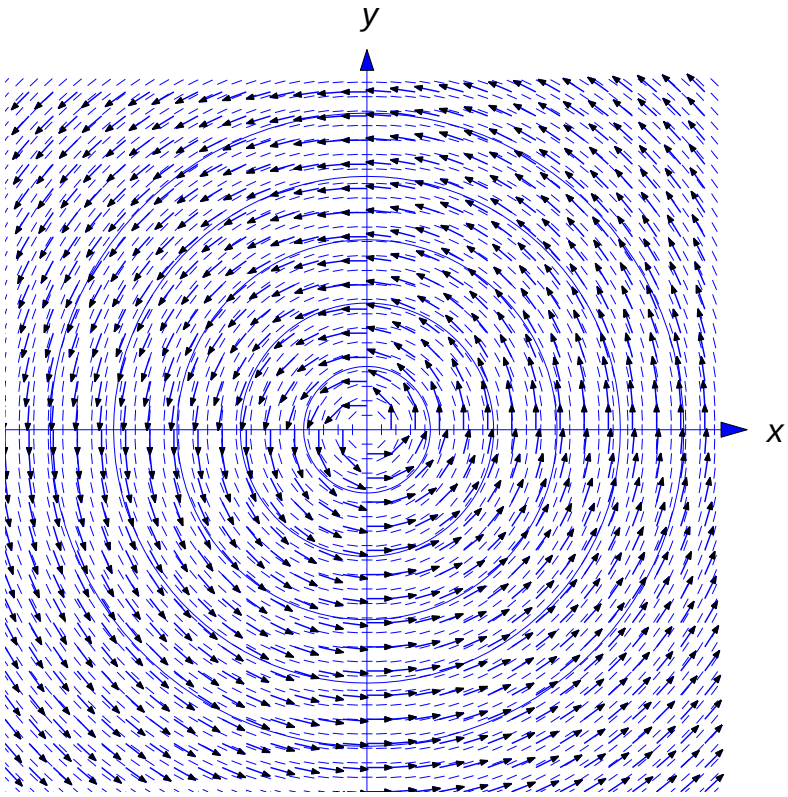


Figure 1.3.5 : A direction field and integral curves for $y'=-\frac{x}{y}$

Figure 1.3.6 shows a direction field and some integral curves for Equation 7.3.3. The integral curves near the top and bottom are solution curves. However, the integral curves near the middle are more complicated.

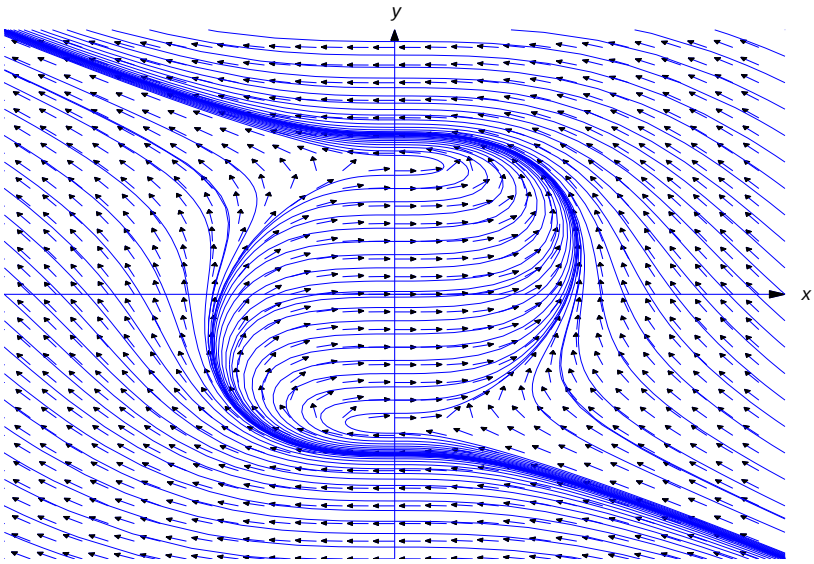


Figure 1.3.6 : A direction field and integral curves for $y'=\dfrac{x^2}{1-x^2-y^2}$.

For example, Figure 1.3.7 shows the integral curve through the origin. The vertices of the dashed rectangle are on the circle $x^2+y^2=1$ ($a\approx .846$, $b\approx .533$), where all integral curves of Equation 7.3.3 have infinite slope. There are three solution curves of Equation 7.3.3 on the integral curve in the figure: the segment above the level $y=b$ is the graph of a solution on $(-\infty,a)$, the segment below

the level $y = -b$ is the graph of a solution on $(-a, \infty)$, and the segment between these two levels is the graph of a solution on $(-a, a)$.

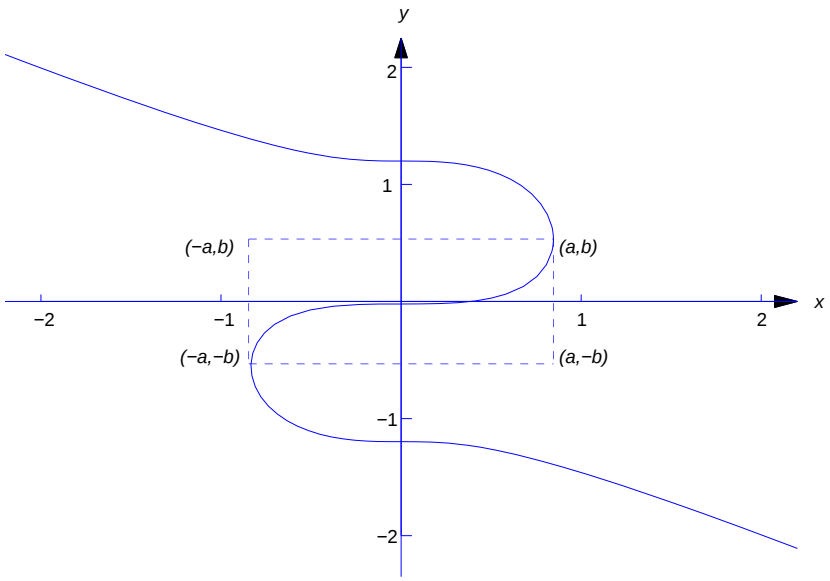


Figure 1.3.7 : An integral curve for $y' = \dfrac{x^2}{1 - x^2 - y^2}$.

Using Technology

As you study from this book, you'll often be asked to use computer software and graphics. Exercises with this intent are marked as (computer or calculator required), (computer and/or graphics required), or (laboratory work requiring software and/or graphics). Often you may not completely understand how the software does what it does. This is similar to the situation most people are in when they drive automobiles or watch television, and it doesn't decrease the value of using modern technology as an aid to learning. Just be careful that you use the technology as a supplement to thought rather than a substitute for it.

This page titled 7.3: Direction Fields for First Order Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **1.3: Direction Fields for First Order Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

7.3.1: Direction Fields for First Order Equations (Exercises)

Exercises for Section 1.3

In Exercises 1–11 a direction field is drawn for the given equation. Sketch some integral curves.

1.

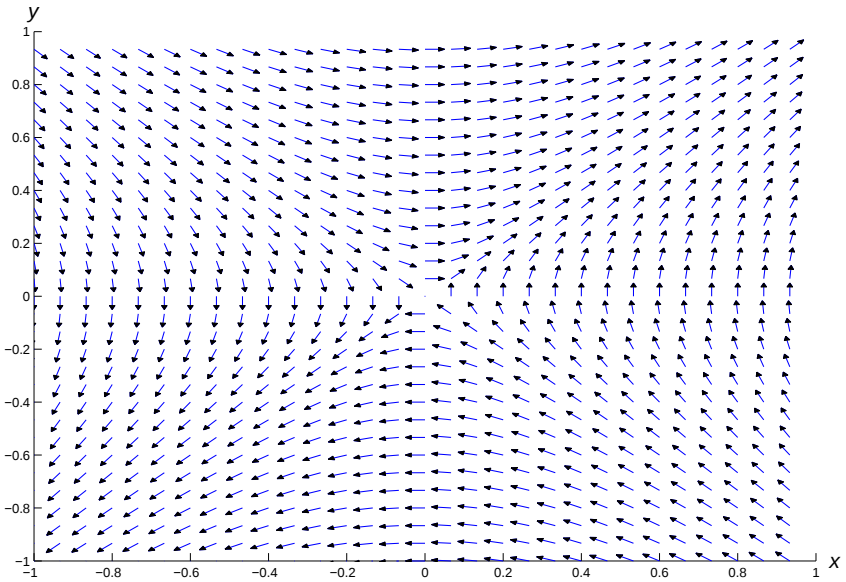


Figure 1.3E.1 : A direction field for $y' = \frac{x}{y}$

2.

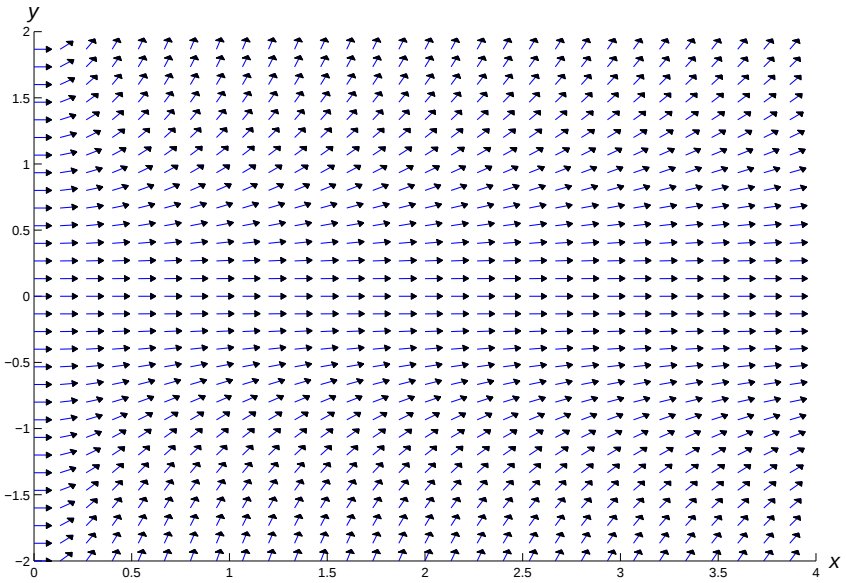


Figure 1.3E.2 : A direction field for $y' = \frac{2xy^2}{1+x^2}$

3.

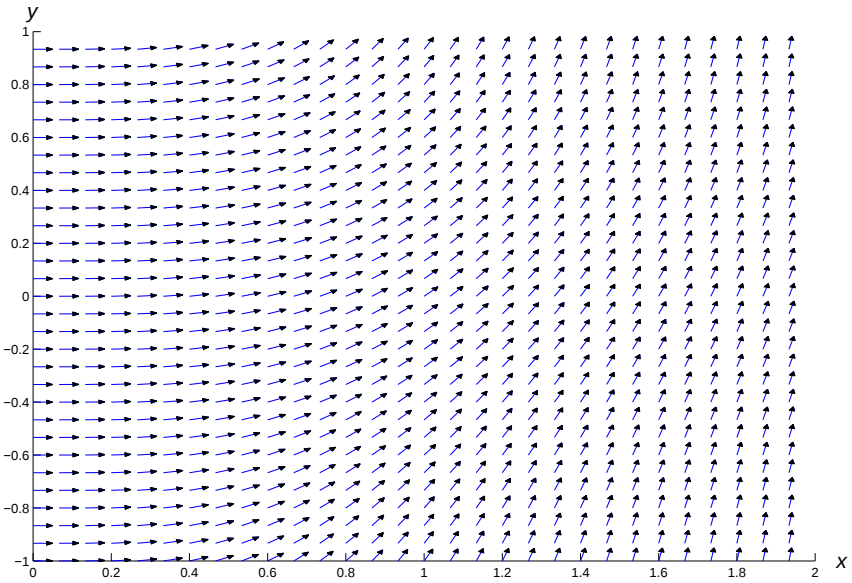


Figure 1.3E.3 : A direction field for $y' = x^2(1 + y^2)$

4.

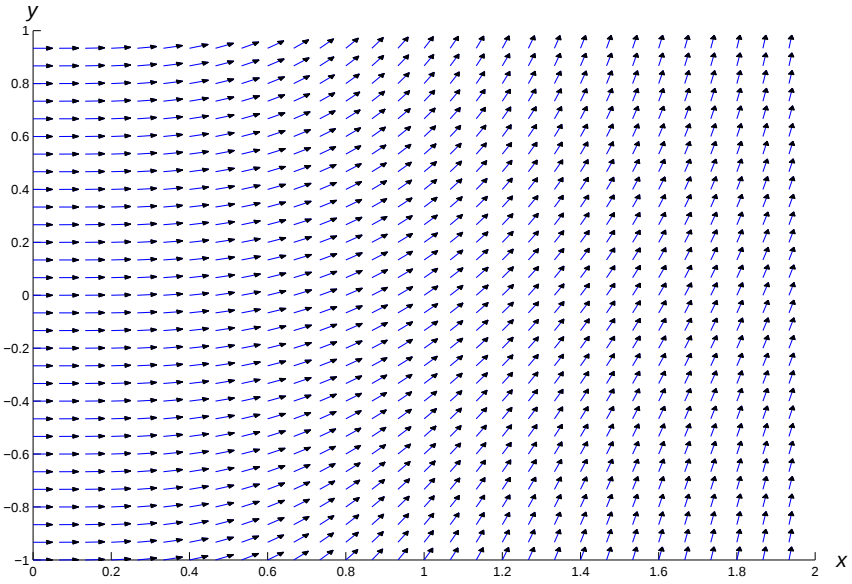


Figure 1.3E.4 : A direction field for $y' = \frac{1}{1+x^2+y^2}$

5.

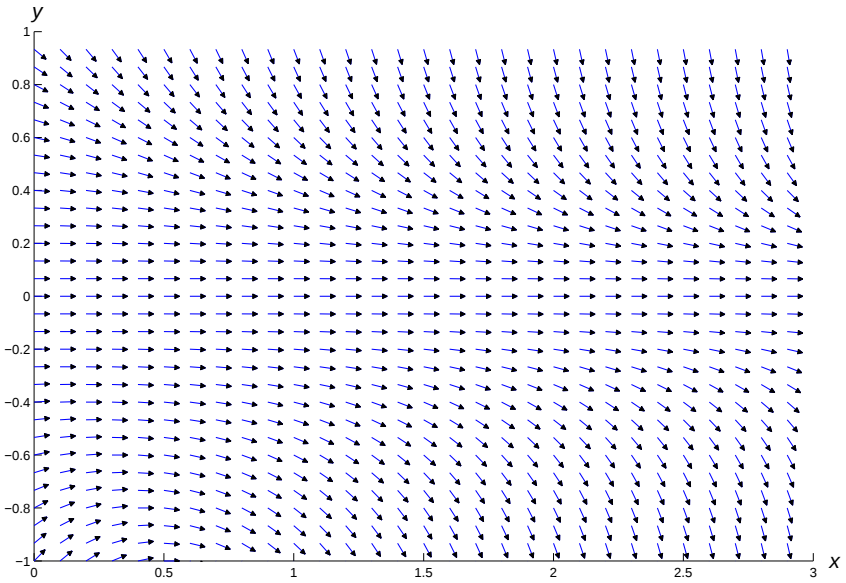


Figure 1.3E.5 : A direction field for $y' = -(2xy^2 + y^3)$

6.

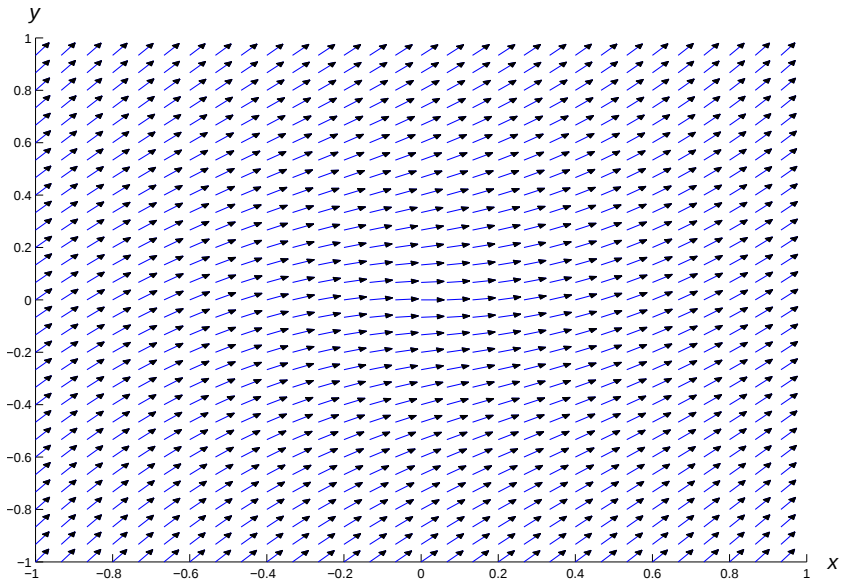


Figure 1.3E.6 : A direction field for $y' = (x^2 + y^2)^{1/2}$

7.

https://math.libretexts.org/@go/page/138142

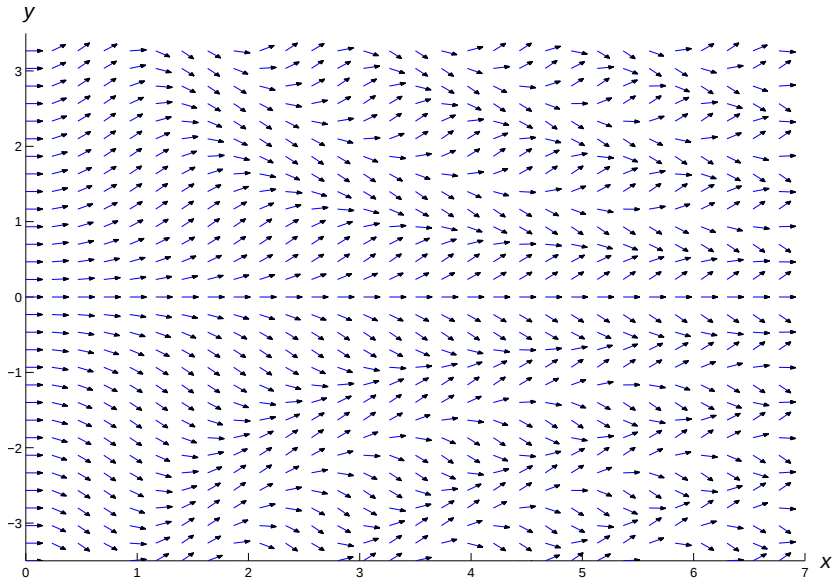


Figure 1.3E.7 : A direction field for $y' = \sin xy$

8.

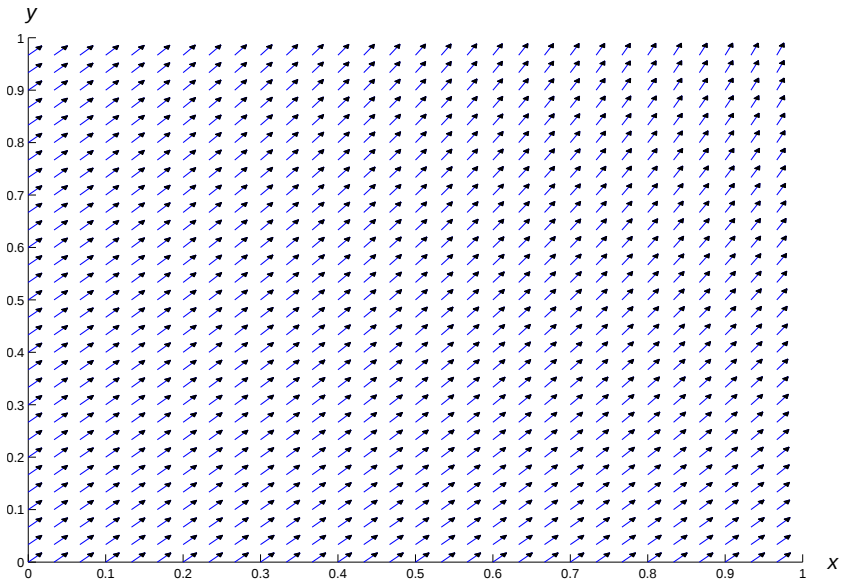


Figure 1.3E.8 : A direction field for $y' = e^{xy}$

9.

https://math.libretexts.org/@go/page/138142

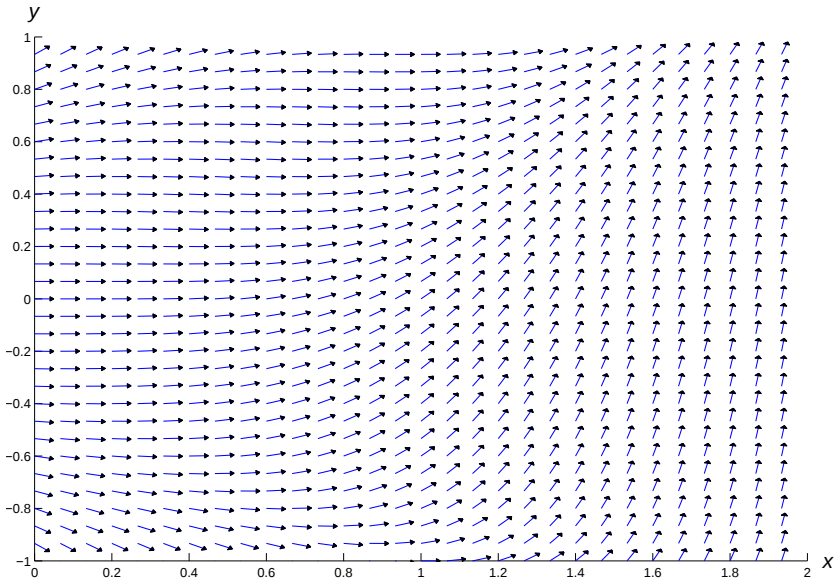


Figure 1.3E.9 : A direction field for $y' = (x - y^2)(x^2 - y)$

10.

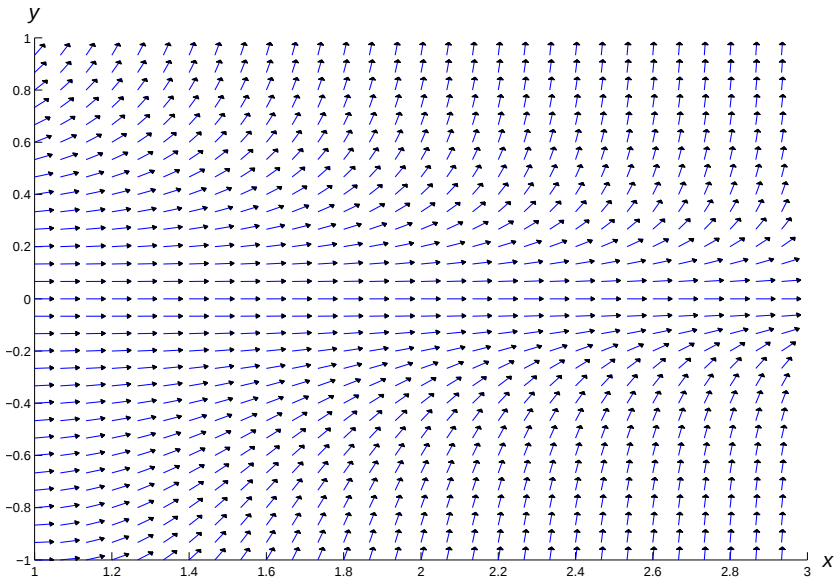


Figure 1.3E.10 : A direction field for $y' = x^3y^2 + xy^3$

11.

https://math.libretexts.org/@go/page/138142

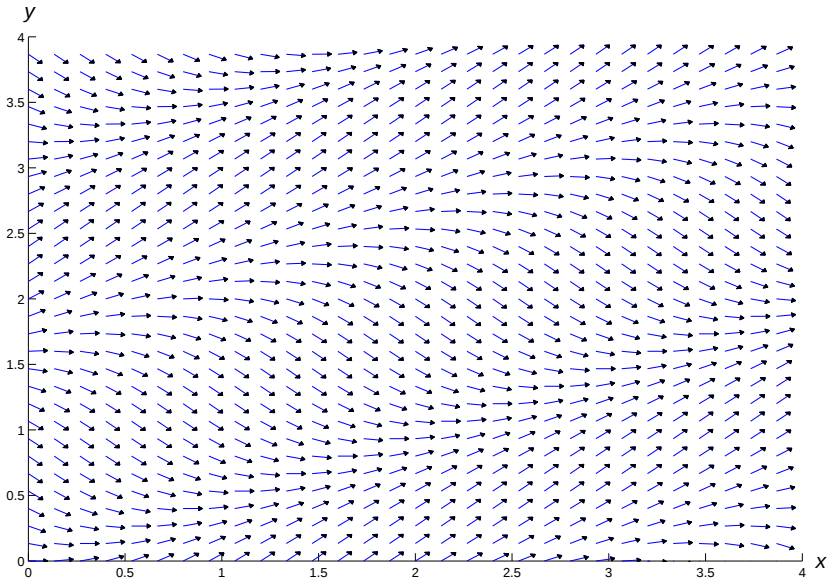


Figure 1.3E.11 : A direction field for $y' = \sin(x - 2y)$

In Exercises 12 - 22 construct a direction field and plot some integral curves in the indicated rectangular region.

12. $y' = y(y-1); \quad \{-1 \le x \le 2, \ -2 \le y \le 2\}$

13. $y' = 2 - 3xy; \quad \{-1 \le x \le 4, \ -4 \le y \le 4\}$

14. $y' = xy(y-1); \quad \{-2 \le x \le 2, \ -4 \le y \le 4\}$

15. $y' = 3x + y; \quad \{-2 \le x \le 2, \ 0 \le y \le 4\}$

16. $y' = y - x^3; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

17. $y' = 1 - x^2 - y^2; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

18. $y' = x(y^2 - 1); \quad \{-3 \le x \le 3, \ -3 \le y \le 2\}$

19. $y' = \frac{x}{y(y^2-1)}; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

20. $y' = \frac{xy^2}{y-1}; \quad \{-2 \le x \le 2, \ -1 \le y \le 4\}$

21. $y' = \frac{x(y^2-1)}{y}; \quad \{-1 \le x \le 1, \ -2 \le y \le 2\}$

22. $y' = -\frac{x^2+y^2}{1-x^2-y^2}; \quad \{-2 \le x \le 2, \ -2 \le y \le 2\}$

23. By suitably renaming the constants and dependent variables in the equations

$$T' = -k(T - T_m) \tag{A}$$

and

$$G' = -\lambda G + r \tag{B}$$

discussed in Section 1.2 in connection with Newton's law of cooling and absorption of glucose in the body, we can write both as

$$y' = -ay + b, \tag{C}$$

where a is a positive constant and b is an arbitrary constant. Thus, (A) is of the form (C) with $y = T$, $a = k$, and $b = kT_m$, and (B) is of the form (C) with $y = G$, $a = \lambda$, and $b = r$. We'll encounter equations of the form (C) in many other applications in Chapter 2.

Choose a positive a and an arbitrary b. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T, \ c \le y \le d\}$$

of the ty-plane. Vary T, c, and d until you discover a common property of all the solutions of (C). Repeat this experiment with various choices of a and b until you can state this property precisely in terms of a and b.

24. By suitably renaming the constants and dependent variables in the equations

$$P' = aP(1 - \alpha P) \tag{A}$$

and

$$I' = rI(S - I) \tag{B}$$

discussed in Section 1.1 in connection with Verhulst's population model and the spread of an epidemic, we can write both in the form

$$y' = ay - by^2, \tag{C}$$

where a and b are positive constants. Thus, (A) is of the form (C) with $y = P$, $a = a$, and $b = a\alpha$, and (B) is of the form (C) with $y = I$, $a = rS$, and $b = r$. In Chapter 2 we'll encounter equations of the form (C) in other applications..

Choose positive numbers a and b. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T,\ 0 \le y \le d\}$$

of the ty-plane. Vary T and d until you discover a common property of all solutions of (C) with $y(0) > 0$. Repeat this experiment with various choices of a and b until you can state this property precisely in terms of a and b.

Choose positive numbers a and b. Construct a direction field and plot some integral curves for (C) in a rectangular region of the form

$$\{0 \le t \le T,\ c \le y \le 0\}$$

of the ty-plane. Vary a, b, T and c until you discover a common property of all solutions of (C) with $y(0) < 0$.

You can verify your results later by doing Exercise 2.2.27.

This page titled 7.3.1: Direction Fields for First Order Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **1.3E: Direction Fields for First Order Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

CHAPTER OVERVIEW

8: First Order Equations

In this chapter, we study first-order differential equations for which there are general methods of solution.

8.1: Linear First Order Equations

8.1.1: Linear First Order Equations (Exercises)

8.2: Separable Equations

8.2.1: Separable Equations (Exercises)

8.3: Existence and Uniqueness of Solutions of Nonlinear Equations

8.3.1: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises)

8.4: Transformation of Nonlinear Equations into Separable Equations

8.4.1: Transformation of Nonlinear Equations into Separable Equations (Exercises)

8.5: Exact Equations

8.5.1: Exact Equations (Exercises)

8.6: Integrating Factors

8.6.1: Integrating Factors (Exercises)

This page titled 8: First Order Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

8.1: Linear First Order Equations

A first order differential equation is said to be **linear** if it can be written as

$$y' + p(x)y = f(x). \tag{8.1.1}$$

A first order differential equation that cannot be written like this is **nonlinear**. We say that Equation 8.1.1 is homogeneous if $f \equiv 0$; otherwise it is **nonhomogeneous**. Since $y \equiv 0$ is obviously a solution of the homgeneous equation

$$y' + p(x)y = 0,$$

we call it the **trivial solution**. Any other solution is **nontrivial**.

Example 2.1.1

The first order equations

$$x^2y' + 3y = x^2$$
$$xy' - 8x^2y = \sin x$$
$$xy' + (\ln x)y = 0$$
$$y' = x^2y - 2$$

are not in the form in Equation 8.1.1, but they are linear, since they can be rewritten as

$$y' + \frac{3}{x^2}y = 1$$
$$y' - 8xy = \frac{\sin x}{x}$$
$$y' + \frac{\ln x}{x}y = 0$$
$$y' - x^2y = -2$$

Example 2.1.2

Here are some nonlinear first order equations:

$$xy' + 3y^2 = 2x \qquad \text{(because } y \text{ is squared)}$$
$$yy' = 3 \qquad \text{(because of the product } yy')$$
$$y' + xe^y = 12 \qquad \text{(because of } e^y)$$

General Solution of a Linear First Order Equation

To motivate a definition that we'll need, consider the simple linear first order equation

$$y' = \frac{1}{x^2}. \tag{8.1.2}$$

From calculus we know that y satisfies this equation if and only if

$$y = -\frac{1}{x} + c, \tag{8.1.3}$$

where c is an arbitrary constant. We call c a **parameter** and say that Equation 8.1.3 defines a **one–parameter family** of functions. For each real number c, the function defined by Equation 8.1.3 is a solution of Equation 8.1.2 on $(-\infty, 0)$ and $(0, \infty)$; moreover, every solution of Equation 8.1.2 on either of these intervals is of the form Equation 8.1.3 for some choice of c. We say that Equation 8.1.3 is **the general solution** of Equation 8.1.2.

We'll see that a similar situation occurs in connection with any first order linear equation

$$y' + p(x)y = f(x); \tag{8.1.4}$$

that is, if p and f are continuous on some open interval (a, b) then there's a unique formula $y = y(x, c)$ analogous to Equation 8.1.3 that involves x and a parameter c and has the these properties:

- For each fixed value of c, the resulting function of x is a solution of Equation 8.1.4 on (a, b).
- If y is a solution of Equation 8.1.4 on (a, b), then y can be obtained from the formula by choosing c appropriately.

We'll call $y = y(x, c)$ the **general solution** of Equation 8.1.4.

When this has been established, it will follow that an equation of the form

$$P_0(x)y' + P_1(x)y = F(x) \tag{8.1.5}$$

has a general solution on any open interval (a, b) on which P_0, P_1, and F are all continuous and P_0 has no zeros, since in this case we can rewrite Equation 8.1.5 in the form Equation 8.1.4 with $p = P_1/P_0$ and $f = F/P_0$, which are both continuous on (a, b).

To avoid awkward wording in examples and exercises, we will not specify the interval (a, b) when we ask for the general solution of a specific linear first order equation. Let's agree that this always means that we want the general solution on every open interval on which p and f are continuous if the equation is of the form Equation 8.1.4, or on which P_0, P_1, and F are continuous and P_0 has no zeros, if the equation is of the form Equation 8.1.5. We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0, P_1, and F are all continuous on an open interval (a, b), but P_0 **does** have a zero in (a, b), then Equation 8.1.5 may fail to have a general solution on (a, b) in the sense just defined. Since this isn't a major point that needs to be developed in depth, we will not discuss it further; however, see **Exercise 2.1.44** for an example.

Homogeneous Linear First Order Equations

We begin with the problem of finding the general solution of a homogeneous linear first order equation. The next example recalls a familiar result from calculus.

Example 2.1.3

Let a be a constant.

a. Find the general solution of

$$y' - ay = 0. \tag{8.1.6}$$

b. Solve the initial value problem

$$y' - ay = 0, \quad y(x_0) = y_0.$$

Solution a

(a) You already know from calculus that if c is any constant, then $y = ce^{ax}$ satisfies Equation 8.1.6. However, let's pretend you've forgotten this, and use this problem to illustrate a general method for solving a homogeneous linear first order equation.

We know that Equation 8.1.6 has the trivial solution $y \equiv 0$. Now suppose y is a nontrivial solution of Equation 8.1.6. Then, since a differentiable function must be continuous, there must be some open interval I on which y has no zeros. We rewrite Equation 8.1.6 as

$$\frac{y'}{y} = a$$

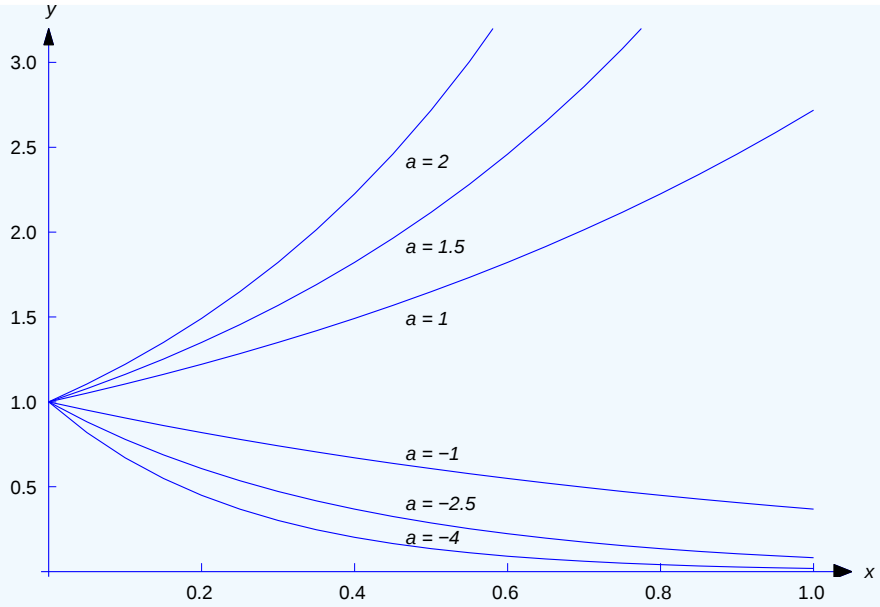


Figure 2.1.1 : Solutions of $y' - ay = 0$, $y(0) = 1$

for x in I. Integrating this shows that

$$\ln|y| = ax + k, \quad \text{so} \quad |y| = e^k e^{ax},$$

where k is an arbitrary constant. Since e^{ax} can never equal zero, y has no zeros, so y is either always positive or always negative. Therefore we can rewrite y as

$$y = ce^{ax} \tag{8.1.7}$$

where

$$c = \begin{cases} e^k & \text{if } y > 0, \\ -e^k & \text{if } y < 0. \end{cases}$$

This shows that every nontrivial solution of Equation 8.1.6 is of the form $y = ce^{ax}$ for some nonzero constant c. Since setting $c = 0$ yields the trivial solution, **all** solutions of Equation 8.1.6 are of the form Equation 8.1.7. Conversely, Equation 8.1.7 is a solution of Equation 8.1.6 for every choice of c, since differentiating Equation 8.1.7 yields $y' = ace^{ax} = ay$.

Solution b

Imposing the initial condition $y(x_0) = y_0$ yields $y_0 = ce^{ax_0}$, so $c = y_0 e^{-ax_0}$ and

$$y = y_0 e^{-ax_0} e^{ax} = y_0 e^{a(x-x_0)}.$$

Figure 2.1.1 show the graphs of this function with $x_0 = 0$, $y_0 = 1$, and various values of a.

Example 2.1.4

a. Find the general solution of

$$xy' + y = 0. \tag{8.1.8}$$

b. Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3. \tag{8.1.9}$$

Solution a

We rewrite Equation 8.1.8 as

$$y' + \frac{1}{x}y = 0, \tag{8.1.10}$$

where x is restricted to either $(-\infty, 0)$ or $(0, \infty)$. If y is a nontrivial solution of Equation 8.1.10, there must be some open interval I on which y has no zeros. We can rewrite Equation 8.1.10 as

$$\frac{y'}{y} = -\frac{1}{x}$$

for x in I. Integrating shows that

$$\ln|y| = -\ln|x| + k, \quad so \quad |y| = \frac{e^k}{|x|}.$$

Since a function that satisfies the last equation can't change sign on either $(-\infty, 0)$ or $(0, \infty)$, we can rewrite this result more simply as

$$y = \frac{c}{x} \tag{8.1.11}$$

where

$$c = \begin{cases} e^k & \text{if} y > 0, \\ -e^k & \text{if} y < 0. \end{cases}$$

We have now shown that every solution of Equation 8.1.10 is given by Equation 8.1.11 for some choice of c. (Even though we assumed that y was nontrivial to derive Equation 8.1.11, we can get the trivial solution by setting $c = 0$ in Equation 8.1.11.) Conversely, any function of the form Equation 8.1.11 is a solution of Equation 8.1.10, since differentiating Equation 8.1.11 yields

$$y' = -\frac{c}{x^2},$$

and substituting this and Equation 8.1.11 into Equation 8.1.10 yields

$$\begin{aligned} y' + \frac{1}{x}y = & \quad -\frac{c}{x^2} + \frac{1}{x}\frac{c}{x} \\ = & \quad -\frac{c}{x^2} + \frac{c}{x^2} = 0. \end{aligned}$$

Figure 2.1.2 shows the graphs of some solutions corresponding to various values of c

Solution b

Imposing the initial condition $y(1) = 3$ in Equation 8.1.11 yields $c = 3$. Therefore the solution of Equation 8.1.9 is

$$y = \frac{3}{x}.$$

The interval of validity of this solution is $(0, \infty)$.

The results in Examples **8.1.3***a* and **8.1.4***b* are special cases of the next theorem.

Theorem 2.1.1

If p is continuous on (a, b), then the general solution of the homogeneous equation

$$y' + p(x)y = 0 \tag{8.1.12}$$

on (a, b) is

$$y = ce^{-P(x)},$$

where

$$P(x) = \int p(x)\,dx \tag{8.1.13}$$

is any antiderivative of p on (a, b); that is ,

$$P'(x) = p(x), \quad a < x < b \tag{8.1.14}$$

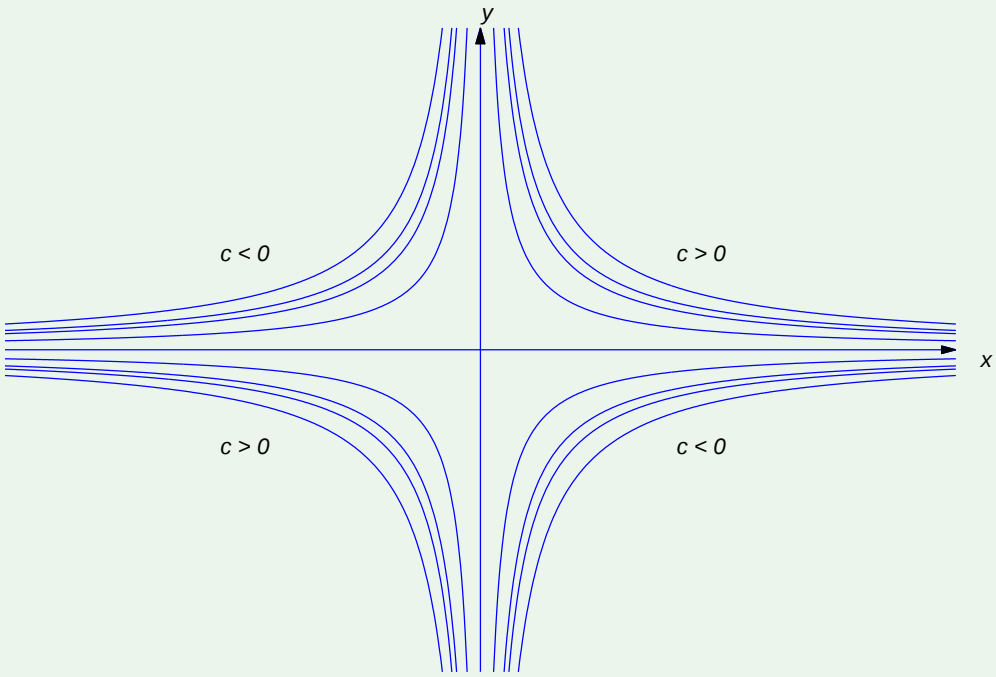


Figure 2.1.2 : Solutions of $xy' + y = 0$ on $(0, \infty)$ and $(-\infty, 0)$

Proof

If $y = ce^{-P(x)}$, differentiating y and using Equation 8.1.14 shows that

$$y' = -P'(x)ce^{-P(x)} = -p(x)ce^{-P(x)} = -p(x)y,$$

so $y' + p(x)y = 0$; that is, y is a solution of Equation 8.1.12, for any choice of c.

Now we'll show that any solution of Equation 8.1.12 can be written as $y = ce^{-P(x)}$ for some constant c. The trivial solution can be written this way, with $c = 0$. Now suppose y is a nontrivial solution. Then there's an open subinterval I of (a, b) on which y has no zeros. We can rewrite Equation 8.1.12 as

$$\frac{y'}{y} = -p(x) \tag{8.1.15}$$

for x in I. Integrating Equation 8.1.15 and recalling Equation 8.1.13 yields

$$\ln|y| = -P(x) + k,$$

where k is a constant. This implies that

$$|y| = e^k e^{-P(x)}.$$

Since P is defined for all x in (a, b) and an exponential can never equal zero, we can take $I = (a, b)$, so y has zeros on (a, b) (a, b), so we can rewrite the last equation as $y = ce^{-P(x)}$, where

$$c = \begin{cases} e^k & \text{if } y > 0 \text{ on } (a, b), \\ -e^k & \text{if } y < 0 \text{ on } (a, b). \end{cases}$$

REMARK: Rewriting a first order differential equation so that one side depends only on y and y' and the other depends only on x is called separation of variables. We did this in Examples 2.1.3 and 2.1.4 , and in rewriting Equation 8.1.12 and Equation 8.1.15. We will apply this method to nonlinear equations in Section 2.2.

Linear Nonhomogeneous First Order Equations

We'll now solve the nonhomogeneous equation

$$y' + p(x)y = f(x). \tag{8.1.16}$$

When considering this equation we call

$$y' + p(x)y = 0$$

the **complementary equation**.

We'll find solutions of Equation 8.1.16 in the form $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation and u is to be determined. This method of using a solution of the complementary equation to obtain solutions of a nonhomogeneous equation is a special case of a method called **variation of parameters**, which you'll encounter several times in this book. (Obviously, u can't be constant, since if it were, the left side of Equation 8.1.16 would be zero. Recognizing this, the early users of this method viewed u as a "parameter" that varies; hence, the name "variation of parameters.")

If

$$y = uy_1, \quad \text{then} \quad y' = u'y_1 + uy_1'.$$

Substituting these expressions for y and y' into Equation 8.1.16 yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x),$$

which reduces to

$$u'y_1 = f(x), \tag{8.1.17}$$

since y_1 is a solution of the complementary equation; that is,

$$y_1' + p(x)y_1 = 0.$$

In the proof of Theorem 2.2.1 we saw that y_1 has no zeros on an interval where p is continuous. Therefore we can divide Equation 8.1.17 through by y_1 to obtain

$$u' = f(x)/y_1(x).$$

We can integrate this (introducing a constant of integration), and multiply the result by y_1 to get the general solution of Equation 8.1.16. Before turning to the formal proof of this claim, let's consider some examples.

Example 2.1.5

Find the general solution of

$$y' + 2y = x^3e^{-2x}. \tag{8.1.18}$$

By applying a of Example 2.1.3 with $a = -2$, we see that $y_1 = e^{-2x}$ is a solution of the complementary equation $y' + 2y = 0$. Therefore we seek solutions of Equation 8.1.18 in the form $y = ue^{-2x}$, so that

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y' + 2y = u'e^{-2x} - 2ue^{-2x} + 2ue^{-2x} = u'e^{-2x}. \tag{8.1.19}$$

Therefore y is a solution of Equation 8.1.18 if and only if

$$u'e^{-2x} = x^3e^{-2x} \quad \text{or, equivalently,} \quad u' = x^3.$$

Therefore

$$u = \frac{x^4}{4} + c,$$

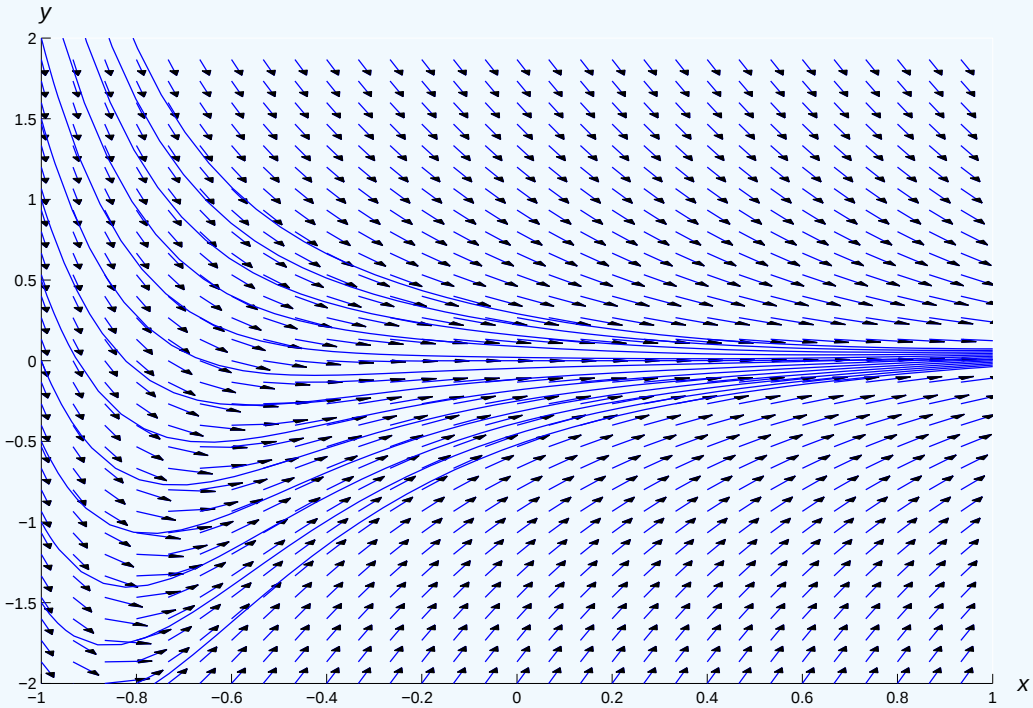


Figure 2.1.3 : A direction field and integral curves for $y' + 2y = x^2 e^{-2x}$

and

$$y = ue^{-2x} = e^{-2x}\left(\frac{x^4}{4} + c\right)$$

is the general solution of Equation 8.1.18.

Figure 2.1.3 shows a direction field and some integral curves for Equation 8.1.18.

Example 2.1.6

Find the general solution

$$y' + (\cot x)y = x \csc x. \tag{8.1.20}$$

Solve the initial value problem

$$y' + (\cot x)y = x \csc x, \quad y(\pi/2) = 1. \tag{8.1.21}$$

Solution

Here $p(x) = \cot x$ and $f(x) = x \csc x$ are both continuous except at the points $x = r\pi$, where r is an integer. Therefore we seek solutions of Equation 8.1.20 on the intervals $(r\pi, (r+1)\pi)$. We need a nontrival solution y_1 of the complementary equation; thus, y_1 must satisfy $y_1' + (\cot x)y_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = -\cot x = -\frac{\cos x}{\sin x}. \tag{8.1.22}$$

Integrating this yields

$$\ln|y_1| = -\ln|\sin x|,$$

where we take the constant of integration to be zero since we need only **one** function that satisfies Equation 8.1.22. Clearly $y_1 = 1/\sin x$ is a suitable choice. Therefore we seek solutions of Equation 8.1.20 in the form

$$y = \frac{u}{\sin x},$$

so that

$$y' = \frac{u'}{\sin x} - \frac{u\cos x}{\sin^2 x} \tag{8.1.23}$$

and

$$\begin{aligned} y' + (\cot x)y &= \tfrac{u'}{\sin x} - \tfrac{u\cos x}{\sin^2 x} + \tfrac{u\cot x}{\sin x} \\ &= \tfrac{u'}{\sin x} - \tfrac{u\cos x}{\sin^2 x} + \tfrac{u\cos x}{\sin^2 x} \\ &= \tfrac{u'}{\sin x}. \end{aligned} \tag{8.1.24}$$

Therefore y is a solution of Equation 8.1.20 if and only if

$$u'/\sin x = x\csc x = x/\sin x \quad \text{or, equivalently,} \quad u' = x.$$

Integrating this yields

$$u = \frac{x^2}{2} + c, \quad \text{and} \quad y = \frac{u}{\sin x} = \frac{x^2}{2\sin x} + \frac{c}{\sin x}. \tag{8.1.25}$$

is the general solution of Equation 8.1.20 on every interval $(r\pi, (r+1)\pi)$ (r =integer).

b. Imposing the initial condition $y(\pi/2) = 1$ in Equation 8.1.25 yields

$$1 = \frac{\pi^2}{8} + c \text{ or } c = 1 - \frac{\pi^2}{8}.$$

Thus,

$$y = \frac{x^2}{2\sin x} + \frac{(1-\pi^2/8)}{\sin x}$$

is a solution of Equation 8.1.20. The interval of validity of this solution is $(0, \pi)$; Figure 2.1.4 shows its graph.

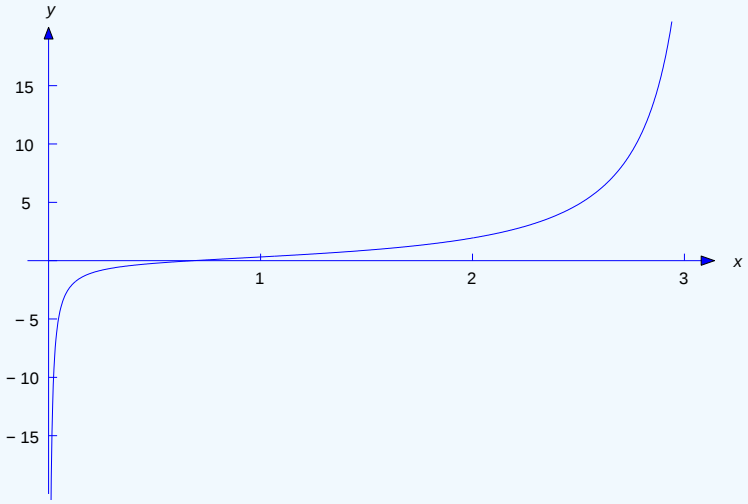


Figure 2.1.4 : Solution of $y' + (\cot x)y = x\csc x, y(\pi/2) = 1$

REMARK: It wasn't necessary to do the computations 8.1.23 and 8.1.24 in Example 2.1.6 , since we showed in the discussion preceding Example 2.1.5 that if $y = uy_1$ where $y_1' + p(x)y_1 = 0$, then $y' + p(x)y = u'y_1$. We did these computations so you would see this happen in this specific example. We recommend that you include these "unnecesary" computations in doing exercises, until you're confident that you really understand the method. After that, omit them.

We summarize the method of variation of parameters for solving

$$y' + p(x)y = f(x) \tag{8.1.26}$$

as follows:

a. Find a function y_1 such that

$$\frac{y_1'}{y_1} = -p(x).$$

For convenience, take the constant of integration to be zero.

b. Write

$$y = uy_1 \tag{8.1.27}$$

to remind yourself of what you're doing.

c. Write $u'y_1 = f$ and solve for u'; thus, $u' = f/y_1$.

d. Integrate u' to obtain u, with an arbitrary constant of integration.

e. Substitute u into Equation 8.1.27 to obtain y.

To solve an equation written as

$$P_0(x)y' + P_1(x)y = F(x),$$

we recommend that you divide through by $P_0(x)$ to obtain an equation of the form Equation 8.1.26 and then follow this procedure.

Solutions in Integral Form

Sometimes the integrals that arise in solving a linear first order equation can't be evaluated in terms of elementary functions. In this case the solution must be left in terms of an integral.

Example 2.1.7

Find the general solution of

$$y' - 2xy = 1.$$

Solve the initial value problem

$$y' - 2xy = 1, \quad y(0) = y_0. \tag{8.1.28}$$

Solution

a. To apply variation of parameters, we need a nontrivial solution y_1 of the complementary equation; thus, $y_1' - 2xy_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = 2x.$$

Integrating this and taking the constant of integration to be zero yields

$$\ln|y_1| = x^2, \quad \text{so} \quad |y_1| = e^{x^2}.$$

We choose $y_1 = e^{x^2}$ and seek solutions of Equation 8.1.28 in the form $y = ue^{x^2}$, where

$$u'e^{x^2} = 1, \quad \text{so} \quad u' = e^{-x^2}.$$

Therefore

$$u = c + \int e^{-x^2}\,dx,$$

but we can't simplify the integral on the right because there's no elementary function with derivative equal to e^{-x^2}. Therefore the best available form for the general solution of Equation 8.1.28 is

$$y = ue^{x^2} = e^{x^2}\left(c + \int e^{-x^2}\,dx\right). \tag{8.1.29}$$

b. Since the initial condition in Equation 8.1.28 is imposed at $x_0 = 0$, it is convenient to rewrite Equation 8.1.29 as

$$y = e^{x^2}\left(c + \int_0^x e^{-t^2}\,dt\right), \quad \text{since} \quad \int_0^0 e^{-t^2}\,dt = 0$$

Setting $x = 0$ and $y = y_0$ here shows that $c = y_0$. Therefore the solution of the initial value problem is

$$y = e^{x^2}\left(y_0 + \int_0^x e^{-t^2}\,dt\right). \tag{8.1.30}$$

For a given value of y_0 and each fixed x, the integral on the right can be evaluated by numerical methods. An alternate procedure is to apply the numerical integration procedures discussed in Chapter 3 directly to the initial value problem Equation 8.1.28. Figure 2.1.5 shows graphs of of Equation 8.1.30 for several values of y_0.

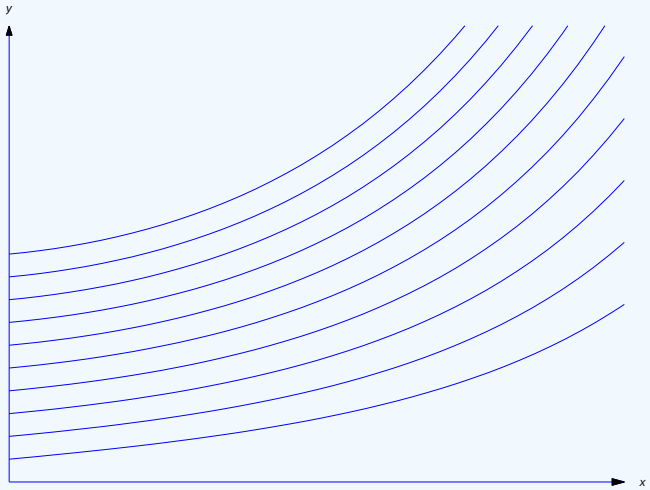


Figure 2.1.5 : Solution of $y' - 2xy = 1, y(0) = y_0$

An Existence and Uniqueness Theorem

The method of variation of parameters leads to this theorem.

Theorem 2.1.2 Existence and Uniqueness Theorem

Suppose p and f are continuous on an open interval (a, b), and let y_1 be any nontrivial solution of the complementary equation

$$y' + p(x)y = 0$$

on (a, b). Then:

a. The general solution of the nonhomogeneous equation

$$y' + p(x)y = f(x) \tag{8.1.31}$$

on (a, b) is

$$y = y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right). \tag{8.1.32}$$

b. If x_0 is an arbitrary point in (a, b) and y_0 is an arbitrary real number , then the initial value problem

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has the unique solution

$$y = y_1(x)\left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^x \frac{f(t)}{y_1(t)}\,dt\right)$$

on (a, b).

Proof

(a) To show that Equation 8.1.32 is the general solution of Equation 8.1.31 on (a,b), we must prove that:

i. If c is any constant, the function y in Equation 8.1.32 is a solution of Equation 8.1.31 on (a,b).
ii. If y is a solution of Equation 8.1.31 on (a,b) then y is of the form Equation 8.1.32 for some constant c.

To prove (i), we first observe that any function of the form Equation 8.1.32 is defined on (a,b), since p and f are continuous on (a,b). Differentiating Equation 8.1.32 yields

$$y' = y_1'(x)\left(c + \int f(x)/y_1(x)\,dx\right) + f(x).$$

Since $y_1' = -p(x)y_1$, this and Equation 8.1.32 imply that

$$\begin{aligned} y' &= -p(x)y_1(x)\left(c + \int f(x)/y_1(x)\,dx\right) + f(x) \\ &= -p(x)y(x) + f(x), \end{aligned}$$

which implies that y is a solution of Equation 8.1.31.

To prove (ii), suppose y is a solution of Equation 8.1.31 on (a,b). From the proof of Theorem 2.1.1, we know that y_1 has no zeros on (a,b), so the function $u = y/y_1$ is defined on (a,b). Moreover, since

$$y' = -py + f \quad \text{and} \quad y_1' = -py_1,$$

$$\begin{aligned} u' &= \frac{y_1y' - y_1'y}{y_1^2} \\ &= \frac{y_1(-py+f) - (-py_1)y}{y_1^2} = \frac{f}{y_1}. \end{aligned}$$

Integrating $u' = f/y_1$ yields

$$u = \left(c + \int f(x)/y_1(x)\,dx\right),$$

which implies Equation 8.1.32, since $y = uy_1$.

(b) We've proved (a), where $\int f(x)/y_1(x)\,dx$ in Equation 8.1.32 is an arbitrary antiderivative of f/y_1. Now it is convenient to choose the antiderivative that equals zero when $x = x_0$, and write the general solution of Equation 8.1.31 as

$$y = y_1(x)\left(c + \int_{x_0}^{x} \frac{f(t)}{y_1(t)}\,dt\right).$$

Since

$$y(x_0) = y_1(x_0)\left(c + \int_{x_0}^{x_0} \frac{f(t)}{y_1(t)}\,dt\right) = cy_1(x_0),$$

we see that $y(x_0) = y_0$ if and only if $c = y_0/y_1(x_0)$.

This page titled 8.1: Linear First Order Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.1: Linear First Order Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

8.1.1: Linear First Order Equations (Exercises)

Q2.1.1

In **Exercises 2.1.1-2.1.5** find the general solution.

1. $y' + ay = 0$ (a=constant)
2. $y' + 3x^2y = 0$
3. $xy' + (\ln x)y = 0$
4. $xy' + 3y = 0$
5. $x^2y' + y = 0$

Q2.1.2

In **Exercises 2.1.6-2.1.11** solve the initial value problem.

6. $y' + \left(\frac{1+x}{x}\right)y = 0, \quad y(1) = 1$
7. $xy' + \left(1 + \frac{1}{\ln x}\right)y = 0, \quad y(e) = 1$
8. $xy' + (1 + x\cot x)y = 0, \quad y\left(\frac{\pi}{2}\right) = 2$
9. $y' - \left(\frac{2x}{1+x^2}\right)y = 0, \quad y(0) = 2$
10. $y' + \frac{k}{x}y = 0, \quad y(1) = 3 \quad (k = \text{constant})$
11. $y' + (\tan kx)y = 0, \quad y(0) = 2 \quad (k = \text{constant})$

Q2.1.3

In **Exercises 2.1.12-2.1.15** find the general solution. Also, plot a direction field and some integral curves on the rectangular region $\{-2 \le x \le 2, -2 \le y \le 2\}$.

12. $y' + 3y = 1$
13. $y' + \left(\frac{1}{x} - 1\right)y = -\frac{2}{x}$
14. $y' + 2xy = xe^{-x^2}$
15. $y' + \frac{2x}{1+x^2}y = \frac{e^{-x}}{1+x^2}$

Q2.1.4

In **Exercises 2.1.16-2.1.24** find the general solution.

16. $y' + \frac{1}{x}y = \frac{7}{x^2} + 3$
17. $y' + \frac{4}{x-1}y = \frac{1}{(x-1)^5} + \frac{\sin x}{(x-1)^4}$
18. $xy' + (1 + 2x^2)y = x^3e^{-x^2}$
19. $xy' + 2y = \frac{2}{x^2} + 1$
20. $y' + (\tan x)y = \cos x$
21. $(1 + x)y' + 2y = \frac{\sin x}{1+x}$
22. $(x - 2)(x - 1)y' - (4x - 3)y = (x - 2)^3$
23. $y' + (2\sin x\cos x)y = e^{-\sin^2 x}$
24. $x^2y' + 3xy = e^x$

Q2.1.5

In **Exercises 2.1.25-2.1.29** solve the initial value problem and sketch the graph of the solution.

25. $y' + 7y = e^{3x}, \quad y(0) = 0$

26. $(1 + x^2)y' + 4xy = \frac{2}{1+x^2}, \quad y(0) = 1$

27. $xy' + 3y = \frac{2}{x(1+x^2)}, \quad y(-1) = 0$

28. $y' + (\cot x)y = \cos x, \quad y\left(\frac{\pi}{2}\right) = 1$

29. $y' + \frac{1}{x}y = \frac{2}{x^2} + 1, \quad y(-1) = 0$

Q2.1.6

In **Exercises 2.1.30-2.1.37**, solve the initial value problem.

30. $(x-1)y' + 3y = \frac{1}{(x-1)^3} + \frac{\sin x}{(x-1)^2}, \quad y(0) = 1$

31. $xy' + 2y = 8x^2, \quad y(1) = 3$

32. $xy' - 2y = -x^2, \quad y(1) = 1$

33. $y' + 2xy = x, \quad y(0) = 3$

34. $(x-1)y' + 3y = \frac{1+(x-1)\sec^2 x}{(x-1)^3}, \quad y(0) = -1$

35. $(x+2)y' + 4y = \frac{1+2x^2}{x(x+2)^3}, \quad y(-1) = 2$

36. $(x^2-1)y' - 2xy = x(x^2-1), \quad y(0) = 4$

37. $(x^2-5)y' - 2xy = -2x(x^2-5), \quad y(2) = 7$

Q2.1.7

In **Excercises 2.1.28-2.1.42** solve the initial value problem and leave the answer in a form involving a definite integral. (You can solve these problems numerically by methods discussed in Chapter 3.)

38. $y' + 2xy = x^2, \quad y(0) = 3$

39. $y' + \frac{1}{x}y = \frac{\sin x}{x^2}, \quad y(1) = 2$

40. $y' + y = \frac{e^{-x}\tan x}{x}, \quad y(1) = 0$

41. $y' + \frac{2x}{1+x^2}y = \frac{e^x}{(1+x^2)^2}, \quad y(0) = 1$

42. $xy' + (x+1)y = e^{x^2}, \quad y(1) = 2$

43. Experiments indicate that glucose is absorbed by the body at a rate proportional to the amount of glucose present in the bloodstream. Let λ denote the (positive) constant of proportionality. Now suppose glucose is injected into a patient's bloodstream at a constant rate of r units per unit of time. Let $G = G(t)$ be the number of units in the patient's bloodstream at time $t > 0$. Then

$$G' = -\lambda G + r,$$

where the first term on the right is due to the absorption of the glucose by the patient's body and the second term is due to the injection. Determine G for $t > 0$, given that $G(0) = G_0$. Also, find $\lim_{t\to\infty} G(t)$.

44.

(a) Plot a direction field and some integral curves for

$$xy' - 2y = -1 \qquad \text{(A)}$$

on the rectangular region $\{-1 \le x \le 1, -.5 \le y \le 1.5\}$. What do all the integral curves have in common?

(b) Show that the general solution of (A) on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{1}{2} + cx^2.$$

(c) Show that y is a solution of (A) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} \frac{1}{2} + c_1 x^2, & x \geq 0, \\ \frac{1}{2} + c_2 x^2, & x < 0, \end{cases}$$

where c_1 and c_2 are arbitrary constants.

(d) Conclude from c that all solutions of (A) on $(-\infty, \infty)$ are solutions of the initial value problem

$$xy' - 2y = -1, \quad y(0) = \frac{1}{2}.$$

(e) Use (b) to show that if $x_0 \neq 0$ and y_0 is arbitrary, then the initial value problem

$$xy' - 2y = -1, \quad y(x_0) = y_0$$

has infinitely many solutions on ($-\infty, \infty$). Explain why this doesn't contradict Theorem 2.1.1.

45. Suppose f is continuous on an open interval (a, b) and α is a constant.

(a) Derive a formula for the solution of the initial value problem

$$y' + \alpha y = f(x), \quad y(x_0) = y_0, \tag{A}$$

where x_0 is in (a, b) and y_0 is an arbitrary real number.

(b) Suppose $(a, b) = (a, \infty)$, $\alpha > 0$ and $\lim_{x \to \infty} f(x) = L$. Show that if y is the solution of (A), then $\lim_{x \to \infty} y(x) = L/\alpha$.

46. Assume that all functions in this exercise are defined on a common interval (a, b).

(a) Prove: If y_1 and y_2 are solutions of

$$y' + p(x)y = f_1(x)$$

and

$$y' + p(x)y = f_2(x)$$

respectively, and c_1 and c_2 are constants, then $y = c_1 y_1 + c_2 y_2$ is a solution of

$$y' + p(x)y = c_1 f_1(x) + c_2 f_2(x).$$

(This is the **principle of superposition.)**

(b) Use (a) to show that if y_1 and y_2 are solutions of the nonhomogeneous equation

$$y' + p(x)y = f(x), \quad \text{(A)}$$

then $y_1 - y_2$ is a solution of the homogeneous equation

$$y' + p(x)y = 0. \quad \text{(B)}$$

(c) Use (a) to show that if y_1 is a solution of (A) and y_2 is a solution of (B), then $y_1 + y_2$ is a solution of (A).

47. Some nonlinear equations can be transformed into linear equations by changing the dependent variable. Show that if

$$g'(y)y' + p(x)g(y) = f(x)$$

where y is a function of x and g is a function of y, then the new dependent variable $z = g(y)$ satisfies the linear equation

$$z' + p(x)z = f(x).$$

48. Solve by the method discussed in Exercise 47.

(a) $(\sec^2 y)y' - 3\tan y = -1$

(b) $e^{y^2}\left(2yy'+\frac{2}{x}\right)=\frac{1}{x^2}$

(c) $\frac{xy'}{y}+2\ln y=4x^2$

(d) $\frac{y'}{(1+y)^2}-\frac{1}{x(1+y)}=-\frac{3}{x^2}$

49. We've shown that if p and f are continuous on (a,b) then every solution of

$$y'+p(x)y=f(x) \tag{A}$$

on (a,b) can be written as $y=uy_1$, where y_1 is a nontrivial solution of the complementary equation for (A) and $u'=f/y_1$. Now suppose f, f', …, $f^{(m)}$ and p, p', …, $p^{(m-1)}$ are continuous on (a,b), where m is a positive integer, and define

$$\begin{aligned} f_0&=f,\\ f_j&=f'_{j-1}+pf_{j-1},\quad 1\le j\le m. \end{aligned}$$

Show that

$$u^{(j+1)}=\frac{f_j}{y_1},\quad 0\le j\le m.$$

This page titled 8.1.1: Linear First Order Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.1E: Linear First Order Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

8.2: Separable Equations

A first order differential equation is **separable** if it can be written as

$$h(y)y' = g(x), \tag{8.2.1}$$

where the left side is a product of y' and a function of y and the right side is a function of x. Rewriting a separable differential equation in this form is called **separation of variables.** In Section 2.1, we used separation of variables to solve homogeneous linear equations. In this section we'll apply this method to nonlinear equations.

To see how to solve Equation 8.2.1, let's first assume that y is a solution. Let $G(x)$ and $H(y)$ be antiderivatives of $g(x)$ and $h(y)$; that is,

$$H'(y) = h(y) \quad \text{and} \quad G'(x) = g(x). \tag{8.2.2}$$

Then, from the chain rule,

$$\frac{d}{dx}H(y(x)) = H'(y(x))y'(x) = h(y)y'(x).$$

Therefore Equation 8.2.1 is equivalent to

$$\frac{d}{dx}H(y(x)) = \frac{d}{dx}G(x).$$

Integrating both sides of this equation and combining the constants of integration yields

$$H(y(x)) = G(x) + c. \tag{8.2.3}$$

Although we derived this equation on the assumption that y is a solution of Equation 8.2.1, we can now view it differently: Any differentiable function y that satisfies Equation 8.2.3 for some constant c is a solution of Equation 8.2.1. To see this, we differentiate both sides of Equation 8.2.3, using the chain rule on the left, to obtain

$$H'(y(x))y'(x) = G'(x),$$

which is equivalent to

$$h(y(x))y'(x) = g(x)$$

because of Equation 8.2.2.

In conclusion, to solve Equation 8.2.1 it suffices to find functions $G = G(x)$ and $H = H(y)$ that satisfy Equation 8.2.2. Then any differentiable function $y = y(x)$ that satisfies Equation 8.2.3 is a solution of Equation 8.2.1.

✔ Example 2.2.1

Solve the equation

$$y' = x(1 + y^2).$$

Solution

Separating variables yields

$$\frac{y'}{1 + y^2} = x.$$

Integrating yields

$$\tan^{-1} y = \frac{x^2}{2} + c$$

Therefore

$$y = \tan\left(\frac{x^2}{2} + c\right).$$

Example 2.2.2

a. Solve the equation

$$y' = -\frac{x}{y}. \tag{8.2.4}$$

b. Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = 1. \tag{8.2.5}$$

c. Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = -2. \tag{8.2.6}$$

Solution a

Separating variables in Equation 8.2.4 yields

$$yy' = -x.$$

Integrating yields

$$\frac{y^2}{2} = -\frac{x^2}{2} + c, \quad \text{or equivalently} \quad x^2 + y^2 = 2c.$$

The last equation shows that c must be positive if y is to be a solution of Equation 8.2.4 on an open interval. Therefore we let $2c = a^2$ (with $a > 0$) and rewrite the last equation as

$$x^2 + y^2 = a^2. \tag{8.2.7}$$

This equation has two differentiable solutions for y in terms of x:

$$y = \sqrt{a^2 - x^2}, \quad -a < x < a, \tag{8.2.8}$$

and

$$y = -\sqrt{a^2 - x^2}, \quad -a < x < a. \tag{8.2.9}$$

The solution curves defined by Equation 8.2.8 are semicircles above the x-axis and those defined by Equation 8.2.9 are semicircles below the x-axis (Figure 2.2.1).

Solution b

The solution of Equation 8.2.5 is positive when $x = 1$; hence, it is of the form Equation 8.2.8. Substituting $x = 1$ and $y = 1$ into Equation 8.2.7 to satisfy the initial condition yields $a^2 = 2$; hence, the solution of Equation 8.2.5 is

$$y = \sqrt{2 - x^2}, \quad -\sqrt{2} < x < \sqrt{2}.$$

Solution c

The solution of Equation 8.2.6 is negative when $x = 1$ and is therefore of the form Equation 8.2.9. Substituting $x = 1$ and $y = -2$ into Equation 8.2.7 to satisfy the initial condition yields $a^2 = 5$. Hence, the solution of Equation 8.2.6 is

$$y = -\sqrt{5 - x^2}, \quad -\sqrt{5} < x < \sqrt{5}.$$

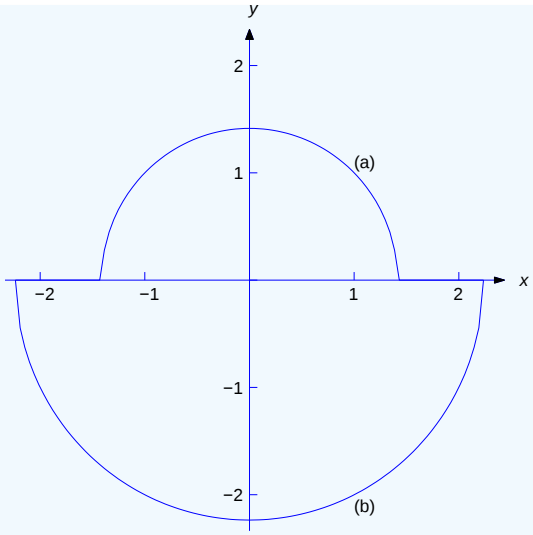


Figure 2.2.1 : (a) $y = \sqrt{2 - x^2},\ -\sqrt{2} < x < \sqrt{2}$; (b) $y = -\sqrt{5 - x^2},\ -\sqrt{5} < x < \sqrt{5}$

Implicit Solutions of Separable Equations

In Examples 2.2.1 and 2.2.2 we were able to solve the equation $H(y) = G(x) + c$ to obtain explicit formulas for solutions of the given separable differential equations. As we'll see in the next example, this isn't always possible. In this situation we must broaden our definition of a solution of a separable equation. The next theorem provides the basis for this modification. We omit the proof, which requires a result from advanced calculus called as the **implicit function theorem**.

Theorem 2.2.1 : Implicit Function Theorem

Suppose $g = g(x)$ is continuous on (a, b) and $h = h(y)$ are continuous on (c, d). Let G be an antiderivative of g on (a, b) and let H be an antiderivative of h on (c, d). Let x_0 be an arbitrary point in (a, b), let y_0 be a point in (c, d) such that $h(y_0) \neq 0$, and define

$$c = H(y_0) - G(x_0). \tag{8.2.10}$$

Then there's a function $y = y(x)$ defined on some open interval (a_1, b_1), where $a \le a_1 < x_0 < b_1 \le b$, such that $y(x_0) = y_0$ and

$$H(y) = G(x) + c \tag{8.2.11}$$

for $a_1 < x < b_1$. Therefore y is a solution of the initial value problem

$$h(y)y' = g(x), \quad y(x_0) = y_0. \tag{8.2.12}$$

It's convenient to say that Equation 8.2.11 with c arbitrary is an **implicit solution** of $h(y)y' = g(x)$. Curves defined by Equation 8.2.11 are integral curves of $h(y)y' = g(x)$. If c satisfies Equation 8.2.10, we'll say that Equation 8.2.11 is an **implicit solution of the initial value problem** Equation 8.2.12. However, keep these points in mind:

- For some choices of c there may not be any differentiable functions y that satisfy Equation 8.2.11.
- The function y in Equation 8.2.11 (not Equation 8.2.11 itself) is a solution of $h(y)y' = g(x)$.

Example 2.2.3

a. Find implicit solutions of

$$y' = \frac{2x + 1}{5y^4 + 1}. \tag{8.2.13}$$

b. Find an implicit solution of

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \tag{8.2.14}$$

Solution a

Separating variables yields

$$(5y^4+1)y' = 2x+1.$$

Integrating yields the implicit solution

$$y^5 + y = x^2 + x + c. \tag{8.2.15}$$

of Equation 8.2.13.

Solution b

Imposing the initial condition $y(2) = 1$ in Equation 8.2.15 yields $1+1 = 4+2+c$, so $c = -4$. Therefore

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem Equation 8.2.14. Although more than one differentiable function $y = y(x)$ satisfies Equation 8.2.13 near $x = 1$, it can be shown that there is only one such function that satisfies the initial condition $y(1) = 2$. Figure 2.2.2 shows a direction field and some integral curves for Equation 8.2.13.

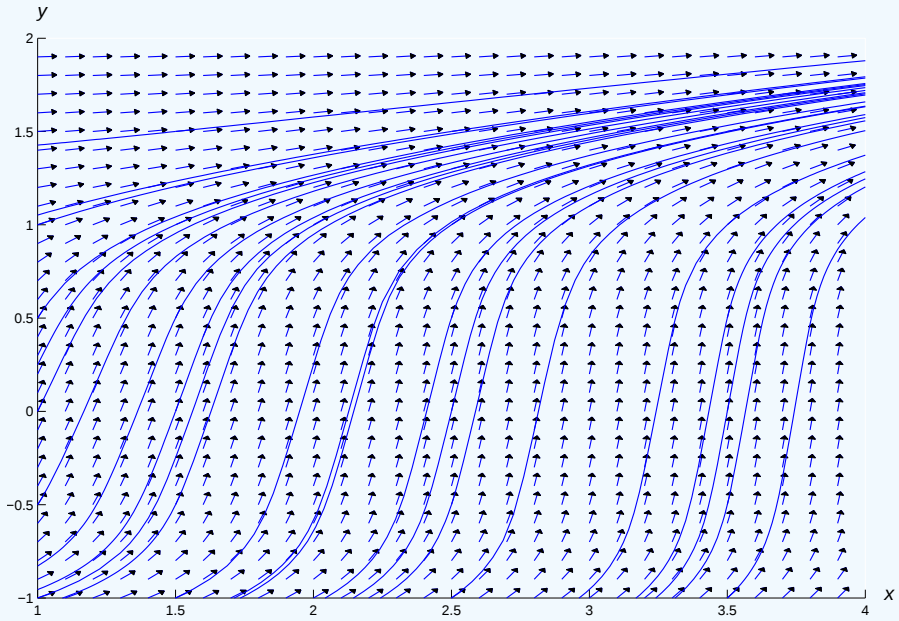


Figure 2.2.2 : A direction field and integral curves for $y' = \frac{2x+1}{5y^4+1}$

Constant Solutions of Separable Equations

An equation of the form

$$y' = g(x)p(y)$$

is separable, since it can be rewritten as

$$\frac{1}{p(y)}y' = g(x).$$

However, the division by $p(y)$ is not legitimate if $p(y) = 0$ for some values of y. The next two examples show how to deal with this problem.

Example 2.2.4

Find all solutions of

$$y' = 2xy^2. \tag{8.2.16}$$

Solution

Here we must divide by $p(y) = y^2$ to separate variables. This isn't legitimate if y is a solution of Equation 8.2.16 that equals zero for some value of x. One such solution can be found by inspection: $y \equiv 0$. Now suppose y is a solution of Equation 8.2.16 that isn't identically zero. Since y is continuous there must be an interval on which y is never zero. Since division by y^2 is legitimate for x in this interval, we can separate variables in Equation 8.2.16 to obtain

$$\frac{y'}{y^2} = 2x.$$

Integrating this yields

$$-\frac{1}{y} = x^2 + c,$$

which is equivalent to

$$y = -\frac{1}{x^2 + c}. \tag{8.2.17}$$

We've now shown that if y is a solution of Equation 8.2.16 that is not identically zero, then y must be of the form Equation 8.2.17. By substituting Equation 8.2.17 into Equation 8.2.16, you can verify that Equation 8.2.17 is a solution of Equation 8.2.16. Thus, solutions of Equation 8.2.16 are $y \equiv 0$ and the functions of the form Equation 8.2.17. Note that the solution $y \equiv 0$ isn't of the form Equation 8.2.17 for any value of c.

Figure 2.2.3 shows a direction field and some integral curves for Equation 8.2.16

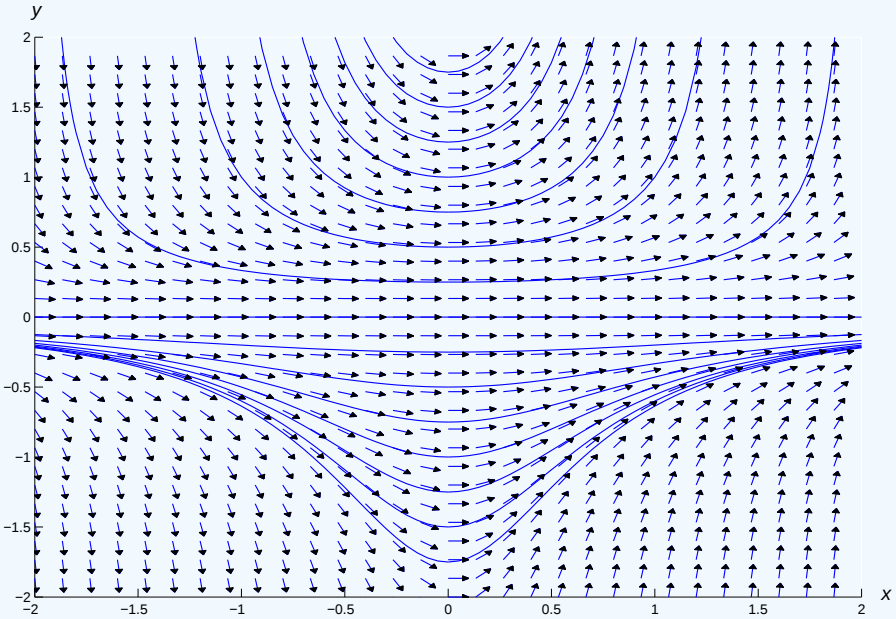


Figure 2.2.3 : A direction field and integral curves for $y' = 2xy^2$

Example 2.2.5

Find all solutions of

$$y' = \frac{1}{2}x(1 - y^2). \tag{8.2.18}$$

Here we must divide by $p(y) = 1 - y^2$ to separate variables. This isn't legitimate if y is a solution of Equation 8.2.18 that equals ± 1 for some value of x. Two such solutions can be found by inspection: $y \equiv 1$ and $y \equiv -1$. Now suppose y is a solution of Equation 8.2.18 such that $1 - y^2$ isn't identically zero. Since $1 - y^2$ is continuous there must be an interval on which $1 - y^2$ is never zero. Since division by $1 - y^2$ is legitimate for x in this interval, we can separate variables in Equation 8.2.18 to obtain

$$\frac{2y'}{y^2-1} = -x.$$

A partial fraction expansion on the left yields

$$\left[\frac{1}{y-1} - \frac{1}{y+1}\right] y' = -x,$$

and integrating yields

$$\ln\left|\frac{y-1}{y+1}\right| = -\frac{x^2}{2} + k;$$

hence,

$$\left|\frac{y-1}{y+1}\right| = e^k e^{-x^2/2}.$$

Since $y(x) \neq \pm 1$ for x on the interval under discussion, the quantity $(y-1)/(y+1)$ cannot change sign in this interval. Therefore we can rewrite the last equation as

$$\frac{y-1}{y+1} = ce^{-x^2/2},$$

where $c = \pm e^k$, depending upon the sign of $(y-1)/(y+1)$ on the interval. Solving for y yields

$$y = \frac{1+ce^{-x^2/2}}{1-ce^{-x^2/2}}. \tag{8.2.19}$$

We've now shown that if y is a solution of Equation 8.2.18 that is not identically equal to ± 1, then y must be as in Equation 8.2.19. By substituting Equation 8.2.19 into Equation 8.2.18 you can verify that Equation 8.2.19 is a solution of Equation 8.2.18. Thus, the solutions of Equation 8.2.18 are $y \equiv 1$, $y \equiv -1$ and the functions of the form Equation 8.2.19. Note that the constant solution $y \equiv 1$ can be obtained from this formula by taking $c = 0$; however, the other constant solution, $y \equiv -1$, cannot be obtained in this way.

Figure 2.2.4 shows a direction field and some integrals for Equation 8.2.18.

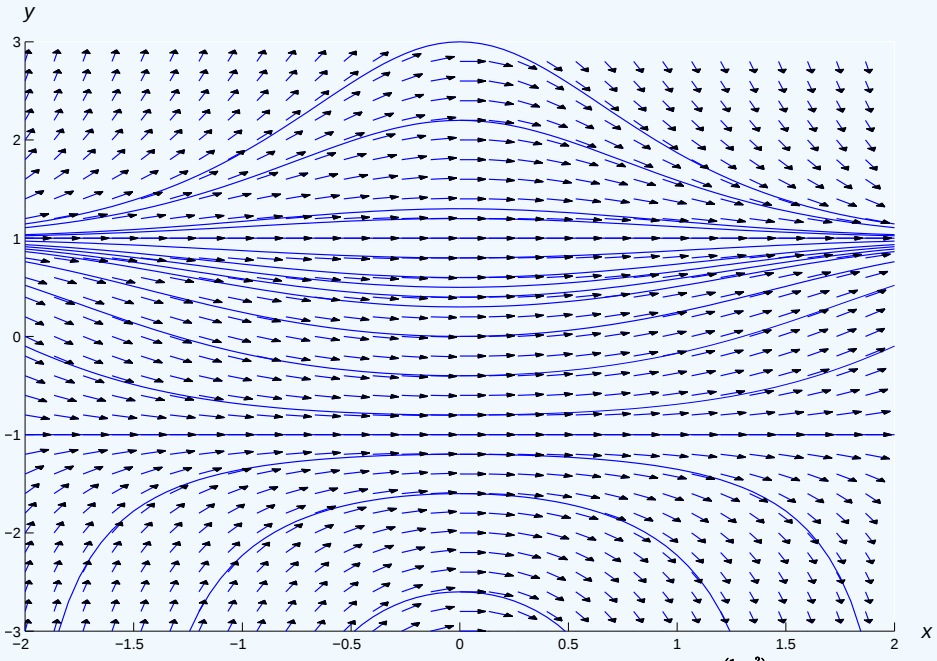


Figure 2.2.4 : A direction field and integral curves for $y' = \frac{x(1-y^2)}{2}$

https://math.libretexts.org/@go/page/121638

Differences Between Linear and Nonlinear Equations

Theorem 2.1.2 states that if p and f are continuous on (a,b) then every solution of

$$y' + p(x)y = f(x)$$

on (a,b) can be obtained by choosing a value for the constant c in the general solution, and if x_0 is any point in (a,b) and y_0 is arbitrary, then the initial value problem

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has a solution on (a,b).

The not true for nonlinear equations. First, we saw in Examples 2.2.4 and 2.2.5 that a nonlinear equation may have solutions that cannot be obtained by choosing a specific value of a constant appearing in a one-parameter family of solutions. Second, it is in general impossible to determine the interval of validity of a solution to an initial value problem for a nonlinear equation by simply examining the equation, since the interval of validity may depend on the initial condition. For instance, in Example 2.2.2 we saw that the solution of

$$\frac{dy}{dx} = -\frac{x}{y}, \quad y(x_0) = y_0$$

is valid on $(-a,a)$, where $a = \sqrt{x_0^2 + y_0^2}$.

Example 2.2.6

Solve the initial value problem

$$y' = 2xy^2, \quad y(0) = y_0$$

and determine the interval of validity of the solution.

Solution

First suppose $y_0 \neq 0$. From Example 2.2.4 , we know that y must be of the form

$$y = -\frac{1}{x^2 + c}. \tag{8.2.20}$$

Imposing the initial condition shows that $c = -1/y_0$. Substituting this into Equation 8.2.20 and rearranging terms yields the solution

$$y = \frac{y_0}{1 - y_0 x^2}.$$

This is also the solution if $y_0 = 0$. If $y_0 < 0$, the denominator isn't zero for any value of x, so the the solution is valid on $(-\infty, \infty)$. If $y_0 > 0$, the solution is valid only on $(-1/\sqrt{y_0}, 1/\sqrt{y_0})$.

This page titled 8.2: Separable Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.2: Separable Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

8.2.1: Separable Equations (Exercises)

Q2.2.1

In **Exercises 2.2.1-2.2.6** find all solutions.

1. $y' = \frac{3x^2+2x+1}{y-2}$
2. $(\sin x)(\sin y) + (\cos y)y' = 0$
3. $xy' + y^2 + y = 0$
4. $y' \ln|y| + x^2 y = 0$
5. $(3y^3 + 3y\cos y + 1)y' + \frac{(2x+1)y}{1+x^2} = 0$
6. $x^2 y y' = (y^2 - 1)^{3/2}$

Q2.2.2

In **Exercises 2.2.7-2.2.10** find all solutions. Also, plot a direction field and some integral curves on the indicated rectangular region.

7. $y' = x^2(1 + y^2);\ \{-1 \le x \le 1,\ -1 \le y \le 1\}$
8. $y'(1 + x^2) + xy = 0;\ \{-2 \le x \le 2,\ -1 \le y \le 1\}$
9. $y' = (x - 1)(y - 1)(y - 2);\ \{-2 \le x \le 2,\ -3 \le y \le 3\}$
10. $(y - 1)^2 y' = 2x + 3;\ \{-2 \le x \le 2,\ -2 \le y \le 5\}$

Q2.2.3

In **Exercises 2.2.11** and *2.2.12* solve the initial value problem.

11. $y' = \frac{x^2+3x+2}{y-2},\quad y(1) = 4$
12. $y' + x(y^2 + y) = 0,\quad y(2) = 1$

Q2.2.4

In **Exercises 2.2.13-2.2.16** solve the initial value problem and graph the solution.

13. $(3y^2 + 4y)y' + 2x + \cos x = 0,\quad y(0) = 1$
14. $y' + \frac{(y+1)(y-1)(y-2)}{x+1} = 0,\quad y(1) = 0$
15. $y' + 2x(y + 1) = 0,\quad y(0) = 2$
16. $y' = 2xy(1 + y^2),\quad y(0) = 1$

Q2.2.5

In **Exercises 2.2.17-2.2.23** solve the initial value problem and find the interval of validity of the solution.

17. $y'(x^2 + 2) + 4x(y^2 + 2y + 1) = 0,\quad y(1) = -1$
18. $y' = -2x(y^2 - 3y + 2),\quad y(0) = 3$
19. $y' = \frac{2x}{1+2y},\quad y(2) = 0$ &
20. $y' = 2y - y^2,\quad y(0) = 1$
21. $x + yy' = 0,\quad y(3) = -4$
22. $y' + x^2(y + 1)(y - 2)^2 = 0,\quad y(4) = 2$
23. $(x + 1)(x - 2)y' + y = 0,\quad y(1) = -3$

Q2.2.6

24. Solve $y' = \frac{(1+y^2)}{(1+x^2)}$ explicitly.

25. Solve $y'\sqrt{1-x^2} + \sqrt{1-y^2} = 0$ explicitly.

26. Solve $y' = \frac{\cos x}{\sin y}, \quad y(\pi) = \frac{\pi}{2}$ explicitly.

27. Solve the initial value problem

$$y' = ay - by^2, \quad y(0) = y_0.$$

Discuss the behavior of the solution if a $y_0 \geq 0$; b $y_0 < 0$.

28. The population $P = P(t)$ of a species satisfies the logistic equation

$$P' = aP(1 - \alpha P)$$

and $P(0) = P_0 > 0$. Find P for $t > 0$, and find $\lim_{t\to\infty} P(t)$.

29. An epidemic spreads through a population at a rate proportional to the product of the number of people already infected and the number of people susceptible, but not yet infected. Therefore, if S denotes the total population of susceptible people and $I = I(t)$ denotes the number of infected people at time t, then

$$I' = rI(S - I),$$

where r is a positive constant. Assuming that $I(0) = I_0$, find $I(t)$ for $t > 0$, and show that $\lim_{t\to\infty} I(t) = S$.

30. The result of *Exercise 2.2.29* is discouraging: if any susceptible member of the group is initially infected, then in the long run all susceptible members are infected! On a more hopeful note, suppose the disease spreads according to the model of *Exercise 2.2.29*, but there's a medication that cures the infected population at a rate proportional to the number of infected individuals. Now the equation for the number of infected individuals becomes

$$I' = rI(S - I) - qI \tag{A}$$

where q is a positive constant.

a. Choose r and S positive. By plotting direction fields and solutions of (A) on suitable rectangular grids

$$R = \{0 \leq t \leq T,\ 0 \leq I \leq d\}$$

in the (t, I)-plane, verify that if I is any solution of (A) such that $I(0) > 0$, then $\lim_{t\to\infty} I(t) = S - q/r$ if $q < rS$ and $\lim_{t\to\infty} I(t) = 0$ if $q \geq rS$.

b. To verify the experimental results of (a), use separation of variables to solve (A) with initial condition $I(0) = I_0 > 0$, and find $\lim_{t\to\infty} I(t)$.

31. Consider the differential equation

$$y' = ay - by^2 - q, \tag{A}$$

where a, b are positive constants, and q is an arbitrary constant. Suppose y denotes a solution of this equation that satisfies the initial condition $y(0) = y_0$.

a. Choose a and b positive and $q < a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids

$$R = \{0 \leq t \leq T,\ c \leq y \leq d\} \tag{B}$$

in the (t, y)-plane, discover that there are numbers y_1 and y_2 with $y_1 < y_2$ such that if $y_0 > y_1$ then $\lim_{t\to\infty} y(t) = y_2$, and if $y_0 < y_1$ then $y(t) = -\infty$ for some finite value of t. (What happens if $y_0 = y_1$?)

b. Choose a and b positive and $q = a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids of the form (B), discover that there's a number y_1 such that if $y_0 \geq y_1$ then $\lim_{t\to\infty} y(t) = y_1$, while if $y_0 < y_1$ then $y(t) = -\infty$ for some finite value of t.

c. Choose positive a, b and $q > a^2/4b$. By plotting direction fields and solutions of (A) on suitable rectangular grids of the form (B), discover that no matter what y_0 is, $y(t) = -\infty$ for some finite value of t.

d. Verify your results experiments analytically. Start by separating variables in (A) to obtain

$$\frac{y'}{ay - by^2 - q} = 1.$$

To decide what to do next you'll have to use the quadratic formula. This should lead you to see why there are three cases. Take it from there! Because of its role in the transition between these three cases, $q_0 = a^2/4b$ is called a **bifurcation value** of q. In general, if q is a parameter in any differential equation, q_0 is said to be a bifurcation value of q if the nature of the solutions of the equation with $q < q_0$ is qualitatively different from the nature of the solutions with $q > q_0$.

32. By plotting direction fields and solutions of

$$y' = qy - y^3,$$

convince yourself that $q_0 = 0$ is a bifurcation value of q for this equation. Explain what makes you draw this conclusion.

33. Suppose a disease spreads according to the model of **Exercise 2.2.29**, but there's a medication that cures the infected population at a constant rate of q individuals per unit time, where $q > 0$. Then the equation for the number of infected individuals becomes

$$I' = rI(S - I) - q.$$

Assuming that $I(0) = I_0 > 0$, use the results of **Exercise 2.2.31** to describe what happens as $t \to \infty$.

34. Assuming that $p \not\equiv 0$, state conditions under which the linear equation

$$y' + p(x)y = f(x)$$

is separable. If the equation satisfies these conditions, solve it by separation of variables and by the method developed in Section 2.1.

Q2.2.7

Solve the equations in **Exercises 2.2.35-2.2.38** using variation of parameters followed by separation of variables.

35. $y' + y = \frac{2xe^{-x}}{1+ye^x}$&
36. $xy' - 2y = \frac{x^6}{y+x^2}$
37. $y' - y = \frac{(x+1)e^{4x}}{(y+e^x)^2}$&
38. $y' - 2y = \frac{xe^{2x}}{1-ye^{-2x}}$

39. Use variation of parameters to show that the solutions of the following equations are of the form $y = uy_1$, where u satisfies a separable equation $u' = g(x)p(u)$. Find y_1 and g for each equation.

a. $xy' + y = h(x)p(xy)$
b. $xy' - y = h(x)p\left(\frac{y}{x}\right)$
c. $y' + y = h(x)p(e^x y)$
d. $xy' + ry = h(x)p(x^r y)$
e. $y' + \frac{v'(x)}{v(x)}y = h(x)p\left(v(x)y\right)$

This page titled 8.2.1: Separable Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.2E: Separable Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

8.3: Existence and Uniqueness of Solutions of Nonlinear Equations

Although there are methods for solving some nonlinear equations, it is impossible to find useful formulas for the solutions of most. Whether we are looking for exact solutions or numerical approximations, it is useful to know conditions that imply the existence and uniqueness of solutions of initial value problems for nonlinear equations. In this section we state such a condition and illustrate it with examples.

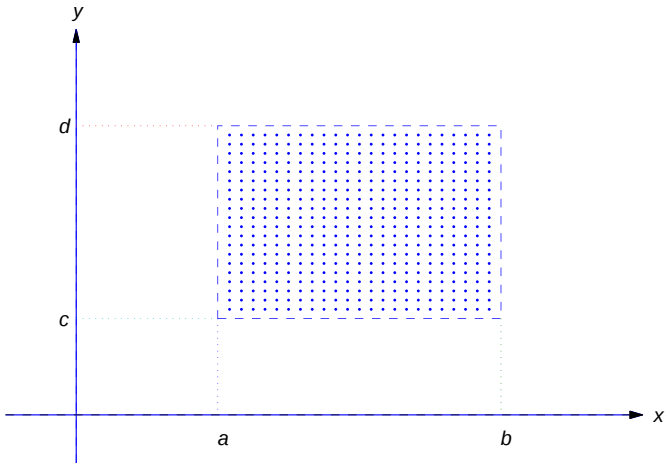


Figure 2.3.1 : An open rectangle

Some terminology: an **open rectangle** R is a set of points (x, y) such that

$$a < x < b \quad \text{and} \quad c < y < d$$

(Figure 2.3.1). We'll denote this set by $R : \{a < x < b, c < y < d\}$. "Open" means that the boundary rectangle (indicated by the dashed lines in Figure 2.3.1) is not included in R.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for first order nonlinear differential equations. We omit the proof, which is beyond the scope of this book.

Theorem 2.3.1 : existence and uniqueness

a. If f is continuous on an open rectangle

$$R : \{a < x < b, c < y < d\}$$

that contains (x_0, y_0) then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{8.3.1}$$

has at least one solution on some open subinterval of (a, b) that contains x_0.

b. If both f and f_y are continuous on R then Equation 8.3.1 has a unique solution on some open subinterval of (a, b) that contains x_0

It's important to understand exactly what Theorem 2.3.1 says.

- (a) is an **existence theorem**. It guarantees that a solution exists on some open interval that contains x_0, but provides no information on how to find the solution, or to determine the open interval on which it exists. Moreover, (a) provides no information on the number of solutions that Equation 8.3.1 may have. It leaves open the possibility that Equation 8.3.1 may have two or more solutions that differ for values of x arbitrarily close to x_0. We will see in Example 2.3.6 that this can happen.
- (b) is a **uniqueness theorem**. It guarantees that Equation 8.3.1 has a unique solution on some open interval (a,b) that contains x_0. However, if $(a, b) \neq (-\infty, \infty)$, Equation 8.3.1 may have more than one solution on a larger interval that contains (a, b). For example, it may happen that $b < \infty$ and all solutions have the same values on (a, b), but two solutions y_1 and y_2 are defined on some interval (a, b_1) with $b_1 > b$, and have different values for $b < x < b_1$; thus, the graphs of the y_1 and y_2 "branch off" in

different directions at $x = b$. (See Example 2.3.7 and Figure 2.3.3). In this case, continuity implies that $y_1(b) = y_2(b)$ (call their common value y), and y_1 and y_2 are both solutions of the initial value problem

$$y = f(x, y), \quad y(b) = \overline{y} \tag{8.3.2}$$

that differ on every open interval that contains b. Therefore f or f_y must have a discontinuity at some point in each open rectangle that contains (b, y), since if this were not so, 8.3.2 would have a unique solution on some open interval that contains b. We leave it to you to give a similar analysis of the case where $a > -\infty$.

Example 2.3.1

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0. \tag{8.3.3}$$

Since

$$f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{2y(1 + 2x^2)}{(1 + x^2 + y^2)^2}$$

are continuous for all (x, y), Theorem 2.3.1 implies that if (x_0, y_0) is arbitrary, then Equation 8.3.3 has a unique solution on some open interval that contains x_0.

Example 2.3.2

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{x^2 + y^2}, \quad y(x_0) = y_0. \tag{8.3.4}$$

Here

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{4x^2 y}{(x^2 + y^2)^2}$$

are continuous everywhere except at $(0, 0)$. If $(x_0, y_0) \neq (0, 0)$, there's an open rectangle R that contains (x_0, y_0) that does not contain $(0, 0)$. Since f and f_y are continuous on R, Theorem 2.3.1 implies that if $(x_0, y_0) \neq (0, 0)$ then Equation 8.3.4 has a unique solution on some open interval that contains x_0.

Example 2.3.3

Consider the initial value problem

$$y' = \frac{x + y}{x - y}, \quad y(x_0) = y_0. \tag{8.3.5}$$

Here

$$f(x, y) = \frac{x + y}{x - y} \quad \text{and} \quad f_y(x, y) = \frac{2x}{(x - y)^2}$$

are continuous everywhere except on the line $y = x$. If $y_0 \neq x_0$, there's an open rectangle R that contains (x_0, y_0) that does not intersect the line $y = x$. Since f and f_y are continuous on R, Theorem 2.3.1 implies that if $y_0 \neq x_0$, Equation 8.3.5 has a unique solution on some open interval that contains x_0.

Example 2.3.4

In Example 2.2.4, we saw that the solutions of

$$y' = 2xy^2 \tag{8.3.6}$$

are

$$y \equiv 0 \quad \text{and} \quad y = -\frac{1}{x^2 + c},$$

where c is an arbitrary constant. In particular, this implies that no solution of Equation 8.3.6 other than $y \equiv 0$ can equal zero for any value of x. Show that Theorem 8.3.1*b* implies this.

We'll obtain a contradiction by assuming that Equation 8.3.6 has a solution y_1 that equals zero for some value of x, but is not identically zero. If y_1 has this property, there's a point x_0 such that $y_1(x_0) = 0$, but $y_1(x) \neq 0$ for some value of x in every open interval that contains x_0. This means that the initial value problem

$$y' = 2xy^2, \quad y(x_0) = 0 \tag{8.3.7}$$

has two solutions $y \equiv 0$ and $y = y_1$ that differ for some value of x on every open interval that contains x_0. This contradicts Theorem 2.3.1 (b), since in Equation 8.3.6 the functions

$$f(x, y) = 2xy^2 \quad \text{and} \quad f_y(x, y) = 4xy.$$

are both continuous for all (x, y), which implies that Equation 8.3.7 has a unique solution on some open interval that contains x_0.

Example 2.3.5

Consider the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(x_0) = y_0. \tag{8.3.8}$$

a. For what points (x_0, y_0) does Theorem 8.3.1*a* imply that Equation 8.3.8 has a solution?
b. For what points (x_0, y_0) does Theorem 8.3.1*b* imply that Equation 8.3.8 has a unique solution on some open interval that contains x_0?

Solution a

Since

$$f(x, y) = \frac{10}{3}xy^{2/5}$$

is continuous for all (x, y), Theorem 2.3.1 implies that Equation 8.3.8 has a solution for every (x_0, y_0).

Solution b

Here

$$f_y(x, y) = \frac{4}{3}xy^{-3/5}$$

is continuous for all (x, y) with $y \neq 0$. Therefore, if $y_0 \neq 0$ there's an open rectangle on which both f and f_y are continuous, and Theorem 2.3.1 implies that Equation 8.3.8 has a unique solution on some open interval that contains x_0.

If $y = 0$ then $f_y(x, y)$ is undefined, and therefore discontinuous; hence, Theorem 2.3.1 does not apply to Equation 8.3.8 if $y_0 = 0$.

Example 2.3.6

Example 2.3.5 leaves open the possibility that the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 0 \tag{8.3.9}$$

has more than one solution on every open interval that contains $x_0 = 0$. Show that this is true.

Solution

By inspection, $y \equiv 0$ is a solution of the differential equation

$$y' = \frac{10}{3}xy^{2/5}. \tag{8.3.10}$$

Since $y \equiv 0$ satisfies the initial condition $y(0) = 0$, it is a solution of Equation 8.3.9.

Now suppose y is a solution of Equation 8.3.10 that is not identically zero. Separating variables in Equation 8.3.10 yields

$$y^{-2/5}y' = \frac{10}{3}x$$

on any open interval where y has no zeros. Integrating this and rewriting the arbitrary constant as $5c/3$ yields

$$\frac{5}{3}y^{3/5} = \frac{5}{3}(x^2 + c).$$

Therefore

$$y = (x^2 + c)^{5/3}. \tag{8.3.11}$$

Since we divided by y to separate variables in Equation 8.3.10, our derivation of Equation 8.3.11 is legitimate only on open intervals where y has no zeros. However, Equation 8.3.11 actually defines y for all x, and differentiating Equation 8.3.11 shows that

$$y' = \frac{10}{3}x(x^2 + c)^{2/3} = \frac{10}{3}xy^{2/5}, \quad -\infty < x < \infty$$

Therefore Equation 8.3.11 satisfies Equation 8.3.10 on $(-\infty, \infty)$ even if $c \le 0$, so that $y(\sqrt{|c|}) = y(-\sqrt{|c|}) = 0$. In particular, taking $c = 0$ in Equation 8.3.11 yields

$$y = x^{10/3}$$

as a second solution of Equation 8.3.9. Both solutions are defined on $(-\infty, \infty)$, and they differ on every open interval that contains $x_0 = 0$ (Figure 2.3.2). In fact, there are **four** distinct solutions of Equation 8.3.9 defined on $(-\infty, \infty)$ that differ from each other on every open interval that contains $x_0 = 0$. Can you identify the other two?

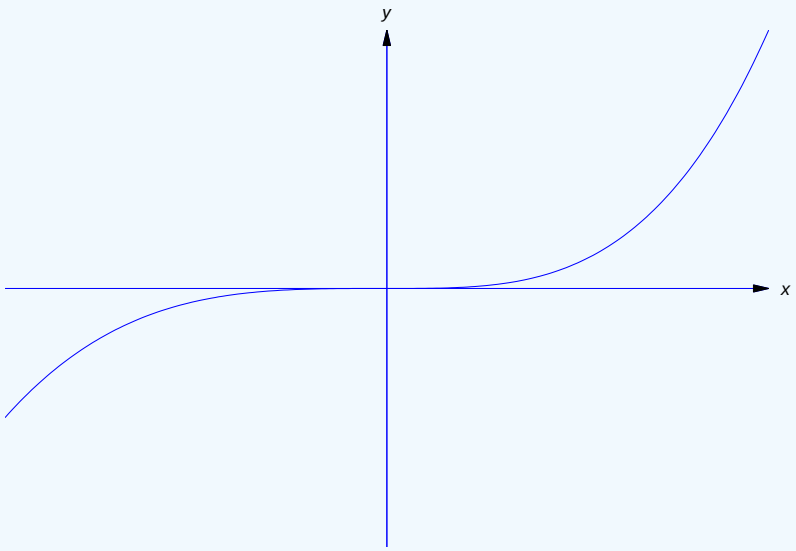


Figure 2.3.2 : Two solutions ($y = 0$ and $y = x^{1/2}$) of Equation 8.3.9 that differ on every interval containing $x_0 = 0$

Example 2.3.7

From Example 2.3.5 , the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1 \tag{8.3.12}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find a solution and determine the largest open interval (a, b) on which it is unique.

Solution

Let y be any solution of Equation 8.3.12. Because of the initial condition $y(0) = -1$ and the continuity of y, there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form Equation 8.3.11. Setting $x = 0$ and $y = -1$ in Equation 8.3.11 yields $c = -1$, so

$$y = (x^2 - 1)^{5/3} \tag{8.3.13}$$

for x in I. Therefore every solution of Equation 8.3.12 differs from zero and is given by Equation 8.3.13 on $(-1, 1)$; that is, Equation 8.3.13 is the unique solution of Equation 8.3.12 on $(-1, 1)$. This is the largest open interval on which Equation 8.3.12 has a unique solution. To see this, note that Equation 8.3.13 is a solution of Equation 8.3.12 on $(-\infty, \infty)$. From **Exercise 2.2.15**, there are infinitely many other solutions of Equation 8.3.12 that differ from Equation 8.3.13 on every open interval larger than $(-1, 1)$. One such solution is

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 \le x \le 1, \\ 0, & |x| > 1. \end{cases}$$

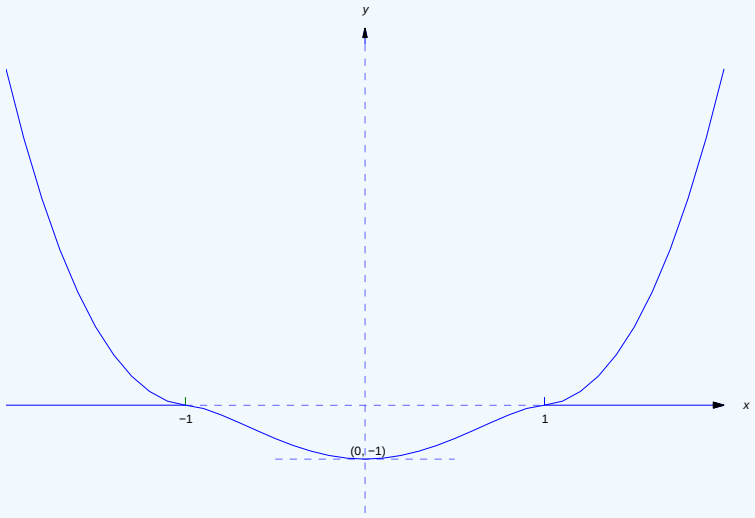


Figure 2.3.3 : Two solutions of Equation 8.3.12 on (−1,1) that coincide on (−1, 1), but on no larger open interval. (right)

Example 2.3.8

From Example 2.3.5), the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 1 \tag{8.3.14}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it is unique.

Solution

Let y be any solution of Equation 8.3.14. Because of the initial condition $y(0) = 1$ and the continuity of y, there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form Equation 8.3.11. Setting $x = 0$ and $y = 1$ in Equation 8.3.11 yields $c = 1$, so

$$y = (x^2 + 1)^{5/3} \tag{8.3.15}$$

for x in I. Therefore every solution of Equation 8.3.14 differs from zero and is given by Equation 8.3.15 on $(-\infty,\infty)$; that is, Equation 8.3.15 is the unique solution of Equation 8.3.14 on $(-\infty,\infty)$. Figure 2.3.4) shows the graph of this solution.

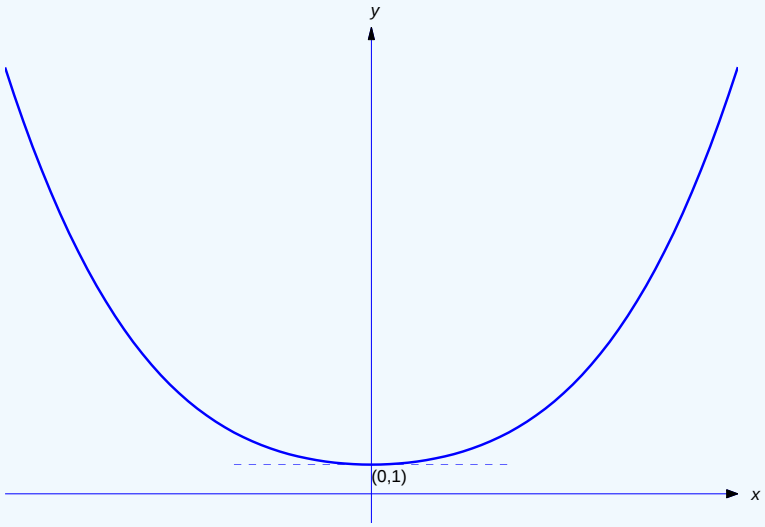


Figure 2.3.4 : The unique solution of Equation 8.3.14.

This page titled 8.3: Existence and Uniqueness of Solutions of Nonlinear Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.3: Existence and Uniqueness of Solutions of Nonlinear Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

8.3.1: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises)

Q2.3.1

In **Exercises 2.3.1-2.3.13**, find all (x_0, y_0) for which Theorem 2.3.1 implies that the initial value problem $y' = f(x, y),\ y(x_0) = y_0$ has (a) a solution and (b) a unique solution on some open interval that contains x_0.

1. $y' = \frac{x^2+y^2}{\sin x}$
2. $y' = \frac{e^x+y}{x^2+y^2}$
3. $y' = \tan xy$
4. $y' = \frac{x^2+y^2}{\ln xy}$
5. $y' = (x^2 + y^2)y^{1/3}$
6. $y' = 2xy$
7. $y' = \ln(1 + x^2 + y^2)$
8. $y' = \frac{2x+3y}{x-4y}$
9. $y' = (x^2 + y^2)^{1/2}$
10. $y' = x(y^2 - 1)^{2/3}$
11. $y' = (x^2 + y^2)^2$
12. $y' = (x + y)^{1/2}$
13. $y' = \frac{\tan y}{x-1}$

Q2.3.2

14. Apply Theorem 2.3.1 to the initial value problem

$$y' + p(x)y = q(x), \quad y(x_0) = y_0$$

for a linear equation, and compare the conclusions that can be drawn from it to those that follow from Theorem 2.1.2.

15.

a. Verify that the function

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 < x < 1, \\ 0, & |x| \geq 1, \end{cases}$$

is a solution of the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1$$

on $(-\infty, \infty)$. HINT: You'll need the definition

$$y'(\overline{x}) = \lim_{x \to \overline{x}} \frac{y(x) - y(\overline{x})}{x - \overline{x}}$$

to verify that y satisfies the differential equation at $\overline{x} = \pm 1$.

b. Verify that if $\epsilon_i = 0$ or 1 for $i = 1, 2$ and $a, b > 1$, then the function

$$y = \begin{cases} \epsilon_1(x^2-a^2)^{5/3}, & -\infty < x < -a, \\ 0, & -a \le x \le -1, \\ (x^2-1)^{5/3}, & -1 < x < 1, \\ 0, & 1 \le x \le b, \\ \epsilon_2(x^2-b^2)^{5/3}, & b < x < \infty, \end{cases}$$

is a solution of the initial value problem of a on $(-\infty,\infty)$.

16. Use the ideas developed in **Exercise 2.3.15** to find infinitely many solutions of the initial value problem

$$y' = y^{2/5}, \quad y(0) = 1$$

on $(-\infty,\infty)$.

17. Consider the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(x_0) = y_0. \tag{A}$$

a. For what points (x_0,y_0) does Theorem 2.3.1 imply that (A) has a solution?
b. For what points (x_0,y_0) does Theorem 2.3.1 imply that (A) has a unique solution on some open interval that contains x_0?

18. Find nine solutions of the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(0) = 1$$

that are all defined on $(-\infty,\infty)$ and differ from each other for values of x in every open interval that contains $x_0 = 0$.

19. From Theorem 2.3.1, the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(0) = 9$$

has a unique solution on an open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it is unique.

20.

a. From Theorem 2.3.1, the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(3) = -7 \tag{A}$$

has a unique solution on some open interval that contains $x_0 = 3$. Determine the largest such open interval, and find the solution on this interval.
b. Find infinitely many solutions of (A), all defined on $(-\infty,\infty)$.

21. Prove:

a. If

$$f(x,y_0) = 0, \quad a < x < b, \tag{A}$$

andx_0 is in (a,b), then $y \equiv y_0$ is a solution of

$$y' = f(x,y), \quad y(x_0) = y_0$$

on (a,b).
b. If f and f_y are continuous on an open rectangle that contains (x_0,y_0) and (A) holds, no solution of $y' = f(x,y)$ other than $y \equiv y_0$ can equal y_0 at any point in (a,b).

This page titled 8.3.1: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.3E: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

8.4: Transformation of Nonlinear Equations into Separable Equations

In Section 2.1, we found that the solutions of a linear nonhomogeneous equation

$$y' + p(x)y = f(x)$$

are of the form $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation

$$y' + p(x)y = 0 \tag{8.4.1}$$

and u is a solution of

$$u'y_1(x) = f(x).$$

Note that this last equation is separable, since it can be rewritten as

$$u' = \frac{f(x)}{y_1(x)}.$$

In this section we'll consider nonlinear differential equations that are not separable to begin with, but can be solved in a similar fashion by writing their solutions in the form $y = uy_1$, where y_1 is a suitably chosen known function and u satisfies a separable equation. We'llsay in this case that we **transformed** the given equation into a separable equation.

Bernoulli Equations

A **Bernoulli equation** is an equation of the form

$$y' + p(x)y = f(x)y^r, \tag{8.4.2}$$

where r can be any real number other than 0 or 1. (Note that Equation 8.4.2 is linear if and only if $r = 0$ or $r = 1$.) We can transform Equation 8.4.2 into a separable equation by variation of parameters: if y_1 is a nontrivial solution of Equation 8.4.1, substituting $y = uy_1$ into Equation 8.4.2 yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x)(uy_1)^r,$$

which is equivalent to the separable equation

$$u'y_1(x) = f(x)(y_1(x))^r u^r \quad \text{or} \quad \frac{u'}{u^r} = f(x)(y_1(x))^{r-1},$$

since $y_1' + p(x)y_1 = 0$.

Example 8.4.1

Solve the Bernoulli equation

$$y' - y = xy^2. \tag{8.4.3}$$

Solution

Since $y_1 = e^x$ is a solution of $y' - y = 0$, we look for solutions of Equation 8.4.3 in the form $y = ue^x$, where

$$u'e^x = xu^2e^{2x} \quad \text{or equivalently} \quad u' = xu^2e^x.$$

Separating variables yields

$$\frac{u'}{u^2} = xe^x,$$

and integrating yields

$$-\frac{1}{u} = (x-1)e^x + c.$$

Hence,

$$u = -\frac{1}{(x-1)e^x + c}$$

and

$$y = -\frac{1}{x - 1 + ce^{-x}}.$$

Figure 8.4.1 shows direction field and some integral curves of Equation 8.4.3.

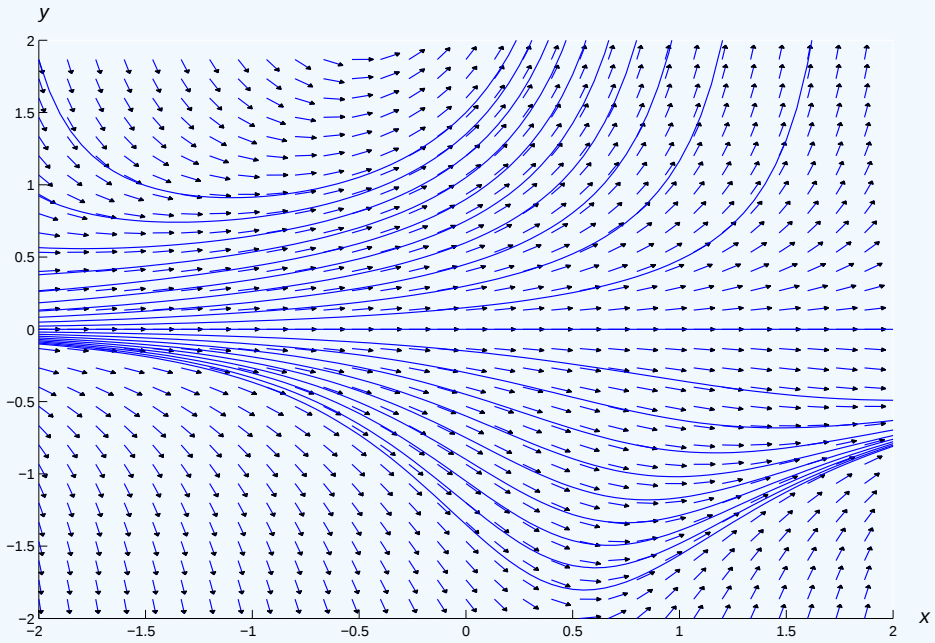


Figure 8.4.1: A direction field and integral curves for $y' - y = xy^2$

Other Nonlinear Equations That Can be Transformed Into Separable Equations

We've seen that the nonlinear Bernoulli equation can be transformed into a separable equation by the substitution $y = uy_1$ if y_1 is suitably chosen. Now let's discover a sufficient condition for a nonlinear first order differential equation

$$y' = f(x, y) \tag{8.4.4}$$

to be transformable into a separable equation in the same way. Substituting $y = uy_1$ into Equation 8.4.4 yields

$$u'y_1(x) + uy_1'(x) = f(x, uy_1(x)),$$

which is equivalent to

$$u'y_1(x) = f(x, uy_1(x)) - uy_1'(x). \tag{8.4.5}$$

If

$$f(x, uy_1(x)) = q(u)y_1'(x)$$

for some function q, then Equation 8.4.5 becomes

$$u'y_1(x) = (q(u) - u)y_1'(x), \tag{8.4.6}$$

which is separable. After checking for constant solutions $u \equiv u_0$ such that $q(u_0) = u_0$, we can separate variables to obtain

$$\frac{u'}{q(u) - u} = \frac{y_1'(x)}{y_1(x)}.$$

Homogeneous Nonlinear Equations

In the text we will consider only the most widely studied class of equations for which the method of the preceding paragraph works. Other types of equations appear in Exercises 8.4.44 - 8.4.51.

The differential equation Equation 8.4.4 is said to be **homogeneous** if x and y occur in f in such a way that $f(x,y)$ depends only on the ratio y/x; that is, Equation 8.4.4 can be written as

$$y' = q(y/x), \tag{8.4.7}$$

where $q = q(u)$ is a function of a single variable. For example,

$$y' = \frac{y + xe^{-y/x}}{x} = \frac{y}{x} + e^{-y/x}$$

and

$$y' = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$$

are of the form Equation 8.4.7, with

$$q(u) = u + e^{-u} \quad \text{and} \quad q(u) = u^2 + u - 1,$$

respectively. The general method discussed above can be applied to Equation 8.4.7 with $y_1 = x$ (and therefore $y_1' = 1$). Thus, substituting $y = ux$ in Equation 8.4.7 yields

$$u'x + u = q(u),$$

and separation of variables (after checking for constant solutions $u \equiv u_0$ such that $q(u_0) = u_0$) yields

$$\frac{u'}{q(u) - u} = \frac{1}{x}.$$

Before turning to examples, we point out something that you may've have already noticed: the definition of **homogeneous equation** given here is not the same as the definition given in Section 2.1, where we said that a linear equation of the form

$$y' + p(x)y = 0$$

is homogeneous. We make no apology for this inconsistency, since we didn't create it historically, **homogeneous** has been used in these two inconsistent ways. The one having to do with linear equations is the most important. This is the only section of the book where the meaning defined here will apply.

Since y/x is in general undefined if $x = 0$, we'll consider solutions of nonhomogeneous equations only on open intervals that do not contain the point $x = 0$.

Example 8.4.2

Solve

$$y' = \frac{y + xe^{-y/x}}{x}. \tag{8.4.8}$$

Solution

Substituting $y = ux$ into Equation 8.4.8 yields

$$u'x + u = \frac{ux + xe^{-ux/x}}{x} = u + e^{-u}.$$

Simplifying and separating variables yields

$$e^u u' = \frac{1}{x}.$$

Integrating yields $e^u = \ln|x| + c$. Therefore $u = \ln(\ln|x| + c)$ and $y = ux = x\ln(\ln|x| + c)$.

Figure 8.4.2 shows a direction field and integral curves for Equation 8.4.8.

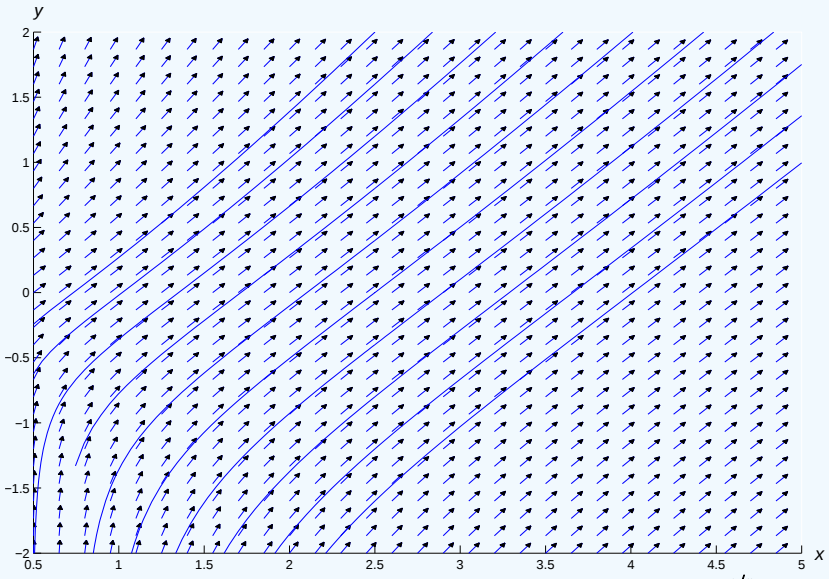


Figure 8.4.2: A direction field and some integral curves for $y' = \dfrac{y + xe^{-y/x}}{x}$

Example 8.4.3

a. Solve

$$x^2y' = y^2 + xy - x^2. \tag{8.4.9}$$

b. Solve the initial value problem

$$x^2y' = y^2 + xy - x^2, \quad y(1) = 2. \tag{8.4.10}$$

Solution a

We first find solutions of Equation 8.4.9 on open intervals that don't contain $x = 0$. We can rewrite Equation 8.4.9 as

$$y' = \frac{y^2 + xy - x^2}{x^2}$$

for x in any such interval. Substituting $y = ux$ yields

$$u'x + u = \frac{(ux)^2 + x(ux) - x^2}{x^2} = u^2 + u - 1,$$

so

$$u'x = u^2 - 1. \tag{8.4.11}$$

By inspection this equation has the constant solutions $u \equiv 1$ and $u \equiv -1$. Therefore $y = x$ and $y = -x$ are solutions of Equation 8.4.9. If u is a solution of Equation 8.4.11 that does not assume the values ± 1 on some interval, separating variables yields

$$\frac{u'}{u^2 - 1} = \frac{1}{x},$$

or, after a partial fraction expansion,

$$\frac{1}{2}\left[\frac{1}{u-1} - \frac{1}{u+1}\right]u' = \frac{1}{x}.$$

Multiplying by 2 and integrating yields

$$\ln\left|\frac{u-1}{u+1}\right| = 2\ln|x| + k,$$

or

$$\left|\frac{u-1}{u+1}\right| = e^k x^2,$$

which holds if

$$\frac{u-1}{u+1} = cx^2 \tag{8.4.12}$$

where c is an arbitrary constant. Solving for u yields

$$u = \frac{1+cx^2}{1-cx^2}.$$

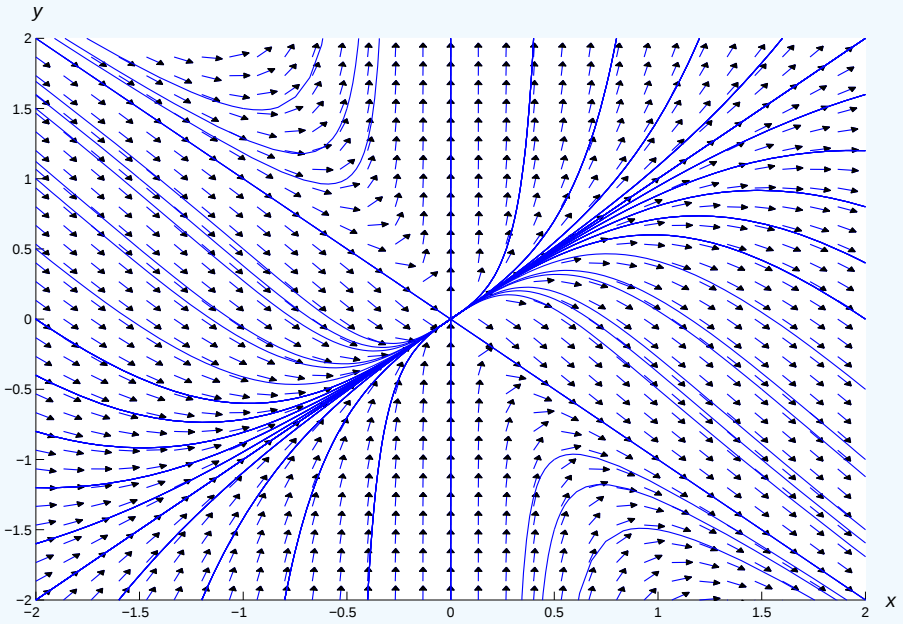


Figure 8.4.3: A direction field and integral curves for $x^2y' = y^2 + xy - x^2$

Therefore

$$y = ux = \frac{x(1+cx^2)}{1-cx^2} \tag{8.4.13}$$

is a solution of Equation 8.4.10 for any choice of the constant c. Setting $c = 0$ in Equation 8.4.13 yields the solution $y = x$. However, the solution $y = -x$ can't be obtained from Equation 8.4.13. Thus, the solutions of Equation 8.4.9 on intervals that don't contain $x = 0$ are $y = -x$ and functions of the form Equation 8.4.13.

The situation is more complicated if $x = 0$ is the open interval. First, note that $y = -x$ satisfies Equation 8.4.9 on $(-\infty, \infty)$. If c_1 and c_2 are arbitrary constants, the function

$$y = \begin{cases} \frac{x(1+c_1x^2)}{1-c_1x^2}, & a < x < 0 \\ \frac{x(1+c_2x^2}{1-c_2x^2}, & 0 \le x < b \end{cases} \tag{8.4.14}$$

is a solution of Equation 8.4.9 on (a, b), where

$$a = \begin{cases} -\frac{1}{\sqrt{c_1}} & \text{if } c_1 > 0, \\ -\infty & \text{if } c_1 \le 0, \end{cases} \quad \text{and} \quad b = \begin{cases} \frac{1}{\sqrt{c_2}} & \text{if } c_2 > 0, \\ \infty & \text{if } c_2 \le 0. \end{cases}$$

We leave it to you to verify this. To do so, note that if y is any function of the form Equation 8.4.13 then $y(0) = 0$ and $y'(0) = 1$.

Figure 8.4.3 shows a direction field and some integral curves for Equation 8.4.9.

Solution b

We could obtain c by imposing the initial condition $y(1) = 2$ in Equation 8.4.13, and then solving for c. However, it is easier to use Equation 8.4.12. Since $u = y/x$, the initial condition $y(1) = 2$ implies that $u(1) = 2$. Substituting this into Equation 8.4.12 yields $c = 1/3$. Hence, the solution of Equation 8.4.10 is

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}.$$

The interval of validity of this solution is $(-\sqrt{3}, \sqrt{3})$. However, the largest interval on which Equation 8.4.10 has a unique solution is $(0, \sqrt{3})$. To see this, note from Equation 8.4.14 that any function of the form

$$y = \begin{cases} \dfrac{x(1 + cx^2)}{1 - cx^2}, & a < x \le 0 \\ \dfrac{x(1 + x^2/3)}{1 - x^2/3} & 0 \le x < \sqrt{3} \end{cases} \tag{8.4.15}$$

is a solution of Equation 8.4.10 on $(a, \sqrt{3})$, where $a = -1/\sqrt{c}$ if $c > 0$ or $a = -\infty$ if $c \le 0$. Why does this not contradict Theorem 2.3.1?

Figure 8.4.4 shows several solutions of the initial value problem Equation 8.4.10. Note that these solutions coincide on $(0, \sqrt{3})$.

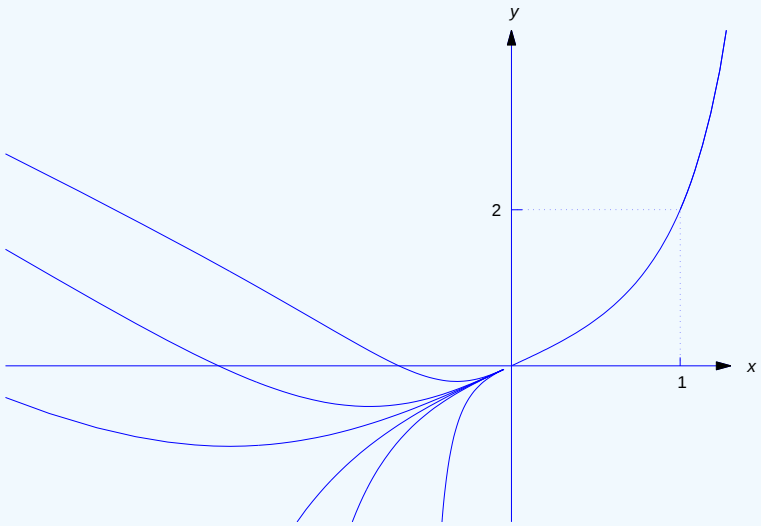


Figure 8.4.4: Solutions of $x^2y' = y^2 + xy - x^2$, $y(1) = 2$

In the last two examples we were able to solve the given equations explicitly. However, this is not always possible, as you'll see in the exercises.

This page titled 8.4: Transformation of Nonlinear Equations into Separable Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.4: Transformation of Nonlinear Equations into Separable Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

8.4.1: Transformation of Nonlinear Equations into Separable Equations (Exercises)

Q2.4.1

In **Exercises 2.4.1-2.4.4** solve the given Bernoulli equation.

1. $y' + y = y^2$
2. $7xy' - 2y = -\frac{x^2}{y^6}$
3. $x^2y' + 2y = 2e^{1/x}y^{1/2}$
4. $(1 + x^2)y' + 2xy = \frac{1}{(1+x^2)y}$

Q2.4.2

In **Exercises 2.4.5** and **2.4.6** find all solutions. Also, plot a direction field and some integral curves on the indicated rectangular region.

5. $y' - xy = x^3y^3; \quad \{-3 \le x \le 3, -2 \le y \le 2\}$
6. $y' - \frac{1+x}{3x}y = y^4; \quad \{-2 \le x \le 2, -2 \le y \le 2\}$

Q2.4.3

In **Exercises 2.4.7-2.4.11** solve the initial value problem.

7. $y' - 2y = xy^3, \quad y(0) = 2\sqrt{2}$
8. $y' - xy = xy^{3/2}, \quad y(1) = 4$
9. $xy' + y = x^4y^4, \quad y(1) = 1/2$
10. $y' - 2y = 2y^{1/2}, \quad y(0) = 1$
11. $y' - 4y = \frac{48x}{y^2}, \quad y(0) = 1$

Q2.4.4

In **Exercises 2.4.12** and **2.4.13** solve the initial value problem and graph the solution.

12. $x^2y' + 2xy = y^3, \quad y(1) = 1/\sqrt{2}$
13. $y' - y = xy^{1/2}, \quad y(0) = 4$

Q2.4.5

14. You may have noticed that the logistic equation

$$P' = aP(1 - \alpha P)$$

from Verhulst's model for population growth can be written in Bernoulli form as

$$P' - aP = -a\alpha P^2.$$

This isn't particularly interesting, since the logistic equation is separable, and therefore solvable by the method studied in Section 2.2. So let's consider a more complicated model, where a is a positive constant and α is a positive continuous function of t on $[0, \infty)$. The equation for this model is

$$P' - aP = -a\alpha(t)P^2,$$

a non-separable Bernoulli equation.

a. Assuming that $P(0) = P_0 > 0$, find P for $t > 0$.
b. Verify that your result reduces to the known results for the Malthusian model where $\alpha = 0$, and the Verhulst model where α is a nonzero constant.
c. Assuming that

c. Assuming that

$$\lim_{t\to\infty} e^{-at}\int_0^t \alpha(\tau)e^{a\tau}\,d\tau = L$$

exists (finite or infinite), find $\lim_{t\to\infty} P(t)$.

Q2.4.6

In **Exercises 2.4.15-2.4.18** solve the equation explicitly.

15. $y' = \frac{y+x}{x}$
16. $y' = \frac{y^2+2xy}{x^2}$
17. $xy^3y' = y^4 + x^4$
18. $y' = \frac{y}{x} + \sec\frac{y}{x}$

Q2.4.7

In **Exercises 2.4.19-2.4.21** solve the equation explicitly. Also, plot a direction field and some integral curves on the indicated rectangular region.

19. $x^2y' = xy + x^2 + y^2; \quad \{-8 \le x \le 8, -8 \le y \le 8\}$
20. $xyy' = x^2 + 2y^2; \quad \{-4 \le x \le 4, -4 \le y \le 4\}$
21. $y' = \frac{2y^2+x^2e^{-(y/x)^2}}{2xy}; \quad \{-8 \le x \le 8, -8 \le y \le 8\}$

Q2.4.8

In **Exercises 2.4.22-2.4.27** solve the initial value problem.

22. $y' = \frac{xy+y^2}{x^2}, \quad y(-1) = 2$
23. $y' = \frac{x^3+y^3}{xy^2}, \quad y(1) = 3$
24. $xyy' + x^2 + y^2 = 0, \quad y(1) = 2$
25. $y' = \frac{y^2-3xy-5x^2}{x^2}, \quad y(1) = -1$
26. $x^2y' = 2x^2 + y^2 + 4xy, \quad y(1) = 1$
27. $xyy' = 3x^2 + 4y^2, \quad y(1) = \sqrt{3}$

Q2.4.9

In **Exercises 2.4.28-2.4.34** solve the given homogeneous equation implicitly.

28. $y' = \frac{x+y}{x-y}$
29. $(y'x - y)(\ln|y| - \ln|x|) = x$
30. $y' = \frac{y^3+2xy^2+x^2y+x^3}{x(y+x)^2}$
31. $y' = \frac{x+2y}{2x+y}$
32. $y' = \frac{y}{y-2x}$
33. $y' = \frac{xy^2+2y^3}{x^3+x^2y+xy^2}$
34. $y' = \frac{x^3+x^2y+3y^3}{x^3+3xy^2}$

Q2.4.10

35.

a. Find a solution of the initial value problem

$$x^2y' = y^2 + xy - 4x^2, \quad y(-1) = 0 \tag{A}$$

on the interval $(-\infty, 0)$. Verify that this solution is actually valid on $(-\infty, \infty)$.

b. Use Theorem 2.3.1 to show that (A) has a unique solution on $(-\infty, 0)$.

c. Plot a direction field for the differential equation in (A) on a square

$$\{-r \le x \le r, -r \le y \le r\},$$

where r is any positive number. Graph the solution you obtained in (a) on this field.

d. Graph other solutions of (A) that are defined on $(-\infty, \infty)$.

e. Graph other solutions of (A) that are defined only on intervals of the form $(-\infty, a)$, where is a finite positive number.

36.

a. Solve the equation

$$xyy' = x^2 - xy + y^2 \tag{A}$$

implicitly.

b. Plot a direction field for (A) on a square

$$\{0 \le x \le r, 0 \le y \le r\}$$

where r is any positive number.

c. Let K be a positive integer. (You may have to try several choices for K.) Graph solutions of the initial value problems

$$xyy' = x^2 - xy + y^2, \quad y(r/2) = \frac{kr}{K},$$

for $k = 1, 2, \ldots, K$. Based on your observations, find conditions on the positive numbers x_0 and y_0 such that the initial value problem

$$xyy' = x^2 - xy + y^2, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution (i) on $(0, \infty)$ or (ii) only on an interval (a, ∞), where $a > 0$?

d. What can you say about the graph of the solution of (B) as $x \to \infty$? (Again, assume that $x_0 > 0$ and $y_0 > 0$.)

37.

a. Solve the equation

$$y' = \frac{2y^2 - xy + 2x^2}{xy + 2x^2} \tag{A}$$

implicitly.

b. Plot a direction field for (A) on a square

$$\{-r \le x \le r, -r \le y \le r\}$$

where r is any positive number. By graphing solutions of (A), determine necessary and sufficient conditions on (x_0, y_0) such that (A) has a solution on (i) $(-\infty, 0)$ or (ii) $(0, \infty)$ such that $y(x_0) = y_0$.

38. Follow the instructions of **Exercise 2.4.37** for the equation

$$y' = \frac{xy + x^2 + y^2}{xy}.$$

39. Pick any nonlinear homogeneous equation $y' = q(y/x)$ you like, and plot direction fields on the square $\{-r \le x \le r, \ -r \le y \le r\}$, where $r > 0$. What happens to the direction field as you vary r? Why?

40. Prove: If $ad - bc \ne 0$, the equation

$$y' = \frac{ax + by + \alpha}{cx + dy + \beta}$$

can be transformed into the homogeneous nonlinear equation

$$\frac{dY}{dX} = \frac{aX + bY}{cX + dY}$$

by the substitution $x = X - X_0,\ y = Y - Y_0$, where X_0 and Y_0 are suitably chosen constants.

Q2.4.11

In **Exercises 2.4.21-2.4.43** use a method suggested by **Exercise 2.4.40** to solve the given equation implicitly.

41. $y' = \frac{-6x+y-3}{2x-y-1}$

42. $y' = \frac{2x+y+1}{x+2y-4}$

43. $y' = \frac{-x+3y-14}{x+y-2}$

Q2.4.12

In **Exercises 2.4.44-2.4.51** find a function y_1 such that the substitution $y = uy_1$ transforms the given equation into a separable equation of the form (2.4.6). Then solve the given equation explicitly.

44. $3xy^2y' = y^3 + x$

45. $xyy' = 3x^6 + 6y^2$

46. $x^3y' = 2(y^2 + x^2y - x^4)$

47. $y' = y^2e^{-x} + 4y + 2e^x$

48. $y' = \frac{y^2 + y\tan x + \tan^2 x}{\sin^2 x}$

49. $x(\ln x)^2y' = -4(\ln x)^2 + y\ln x + y^2$

50. $2x(y + 2\sqrt{x})y' = (y + \sqrt{x})^2$

51. $(y + e^{x^2})y' = 2x(y^2 + ye^{x^2} + e^{2x^2})$

Q2.4.13

52. Solve the initial value problem

$$y' + \frac{2}{x}y = \frac{3x^2y^2 + 6xy + 2}{x^2(2xy + 3)}, \quad y(2) = 2.$$

53. Solve the initial value problem

$$y' + \frac{3}{x}y = \frac{3x^4y^2 + 10x^2y + 6}{x^3(2x^2y + 5)}, \quad y(1) = 1.$$

54. Prove: If y is a solution of a homogeneous nonlinear equation $y' = q(y/x)$, so is $y_1 = y(ax)/a$, where a is any nonzero constant.

55. A **generalized Riccati equation** is of the form

$$y' = P(x) + Q(x)y + R(x)y^2. \tag{A}$$

(If $R \equiv -1$, (A) is a **Riccati equation**.) Let y_1 be a known solution and y an arbitrary solution of (A). Let $z = y - y_1$. Show that z is a solution of a Bernoulli equation with $n = 2$.

Q2.4.14

In **Exercises 2.4.56-2.4.59**, given that y_1 is a solution of the given equation, use the method suggested by **Exercise 2.4.55** to find other solutions.

56. $y' = 1 + x - (1 + 2x)y + xy^2;\ y_1 = 1$

57. $y' = e^{2x} + (1 - 2e^x)y + y^2;\ y_1 = e^x$

58. $xy' = 2 - x + (2x - 2)y - xy^2;\ y_1 = 1$

59. $xy' = x^3 + (1 - 2x^2)y + xy^2$; $y_1 = x$

This page titled 8.4.1: Transformation of Nonlinear Equations into Separable Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.4E: Transformation of Nonlinear Equations into Separable Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

8.5: Exact Equations

In this section it is convenient to write first order differential equations in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0. \tag{8.5.1}$$

This equation can be interpreted as

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0, \tag{8.5.2}$$

where x is the independent variable and y is the dependent variable, or as

$$M(x,y)\,\frac{dx}{dy} + N(x,y) = 0, \tag{8.5.3}$$

where y is the independent variable and x is the dependent variable. Since the solutions of Equation 8.5.2 and Equation 8.5.3 will often have to be left in implicit form we will say that $F(x,y) = c$ is an implicit solution of Equation 8.5.1 if every differentiable function $y = y(x)$ that satisfies $F(x,y) = c$ is a solution of Equation 8.5.2 and every differentiable function $x = x(y)$ that satisfies $F(x,y) = c$ is a solution of Equation 8.5.3

Here are some examples:

Table 8.5.1: Examples of Exact Differential Equations in three forms

Equation 8.5.1	Equation 8.5.2	Equation 8.5.3
$3x^2y^2\,dx + 2x^3y\,dy = 0$	$3x^2y^2 + 2x^3y\,\frac{dy}{dx} = 0$	$3x^2y^2\,\frac{dx}{dy} + 2x^3y = 0$
$(x^2+y^2)\,dx + 2xy\,dy = 0$	$(x^2+y^2) + 2xy\,\frac{dy}{dx} = 0$	$(x^2+y^2)\,\frac{dx}{dy} + 2xy = 0$
$3y\sin x\,dx - 2xy\cos x\,dy = 0$	$3y\sin x - 2xy\cos x\,\frac{dy}{dx} = 0$	$3y\sin x\,\frac{dx}{dy} - 2xy\cos x = 0$

Note that a separable equation can be written as Equation 8.5.1 as

$$M(x)\,dx + N(y)\,dy = 0.$$

we will develop a method for solving Equation 8.5.1 under appropriate assumptions on M and N. This method is an extension of the method of separation of variables. Before stating it we consider an example.

Example 8.5.1

Show that

$$x^4y^3 + x^2y^5 + 2xy = c \tag{8.5.4}$$

is an implicit solution of

$$(4x^3y^3 + 2xy^5 + 2y)\,dx + (3x^4y^2 + 5x^2y^4 + 2x)\,dy = 0. \tag{8.5.5}$$

Solution

Regarding y as a function of x and differentiating Equation 8.5.4 implicitly with respect to x yields

$$(4x^3y^3 + 2xy^5 + 2y) + (3x^4y^2 + 5x^2y^4 + 2x)\,\frac{dy}{dx} = 0.$$

Similarly, regarding x as a function of y and differentiating Equation 8.5.4 implicitly with respect to y yields

$$(4x^3y^3 + 2xy^5 + 2y)\frac{dx}{dy} + (3x^4y^2 + 5x^2y^4 + 2x) = 0.$$

Therefore Equation 8.5.4 is an implicit solution of Equation 8.5.5 in either of its two possible interpretations.

You may think Example 8.5.1 is pointless, since concocting a differential equation that has a given implicit solution is not particularly interesting. However, it illustrates the next important theorem, which we will prove by using implicit differentiation, as in Example 8.5.1.

Theorem 8.5.1

If $F = F(x, y)$ has continuous partial derivatives F_x and F_y, then

$$F(x, y) = c \tag{8.5.6}$$

(with c as a constant) is an implicit solution of the differential equation

$$F_x(x, y)\,dx + F_y(x, y)\,dy = 0. \tag{8.5.7}$$

Proof

Regarding y as a function of x and differentiating Equation 8.5.6 implicitly with respect to x yields

$$F_x(x, y) + F_y(x, y)\,\frac{dy}{dx} = 0.$$

On the other hand, regarding x as a function of y and differentiating Equation 8.5.6 implicitly with respect to y yields

$$F_x(x, y)\,\frac{dx}{dy} + F_y(x, y) = 0.$$

Thus, Equation 8.5.6 is an implicit solution of Equation 8.5.7 in either of its two possible interpretations.

We will say that the equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{8.5.8}$$

is **exact** on an an open rectangle R if there's a function $F = F(x, y)$ such F_x and F_y are continuous, and

$$F_x(x, y) = M(x, y) \quad \text{and} \quad F_y(x, y) = N(x, y) \tag{8.5.9}$$

for all (x, y) in R. This usage of "exact" is related to its usage in calculus, where the expression

$$F_x(x, y)\,dx + F_y(x, y)\,dy$$

(obtained by substituting Equation 8.5.9 into the left side of Equation 8.5.8) is the **exact differential of** F.

Example 8.5.1 shows that it is easy to solve Equation 8.5.8 if it is exact **and** we know a function F that satisfies Equation 8.5.9. The important questions are:

- Question 1. Given an equation Equation 8.5.8, how can we determine whether it is exact?
- Question 2. If Equation 8.5.8 is exact, how do we find a function F satisfying Equation 8.5.9?

To discover the answer to Question 1, assume that there's a function F that satisfies Equation 8.5.9 on some open rectangle R, and in addition that F has continuous mixed partial derivatives F_{xy} and F_{yx}. Then a theorem from calculus implies that

$$F_{xy} = F_{yx}. \tag{8.5.10}$$

If $F_x = M$ and $F_y = N$, differentiating the first of these equations with respect to y and the second with respect to x yields

$$F_{xy} = M_y \quad \text{and} \quad F_{yx} = N_x. \tag{8.5.11}$$

From Equation 8.5.10 and Equation 8.5.11, we conclude that a necessary condition for exactness is that $M_y = N_x$. This motivates the next theorem, which we state without proof.

Theorem 8.5.2: The Exactness Condition

Suppose M and N are continuous and have continuous partial derivatives M_y and N_x on an open rectangle R. Then

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is exact on R if and only if

$$M_y(x,y) = N_x(x,y) \tag{8.5.12}$$

for all (x,y) in R..

To help you remember the exactness condition, observe that the coefficients of dx and dy are differentiated in Equation 8.5.12 with respect to the "opposite" variables; that is, the coefficient of dx is differentiated with respect to y, while the coefficient of dy is differentiated with respect to x.

Example 8.5.2

Show that the equation

$$3x^2y\,dx + 4x^3\,dy = 0$$

is not exact on any open rectangle.

Solution

Here

$$M(x,y) = 3x^2y \quad \text{and} \quad N(x,y) = 4x^3$$

so

$$M_y(x,y) = 3x^2 \quad \text{and} N_x(x,y) = 12x^2.$$

Therefore $M_y = N_x$ on the line $x = 0$, but not on any open rectangle, so there's no function F such that $F_x(x,y) = M(x,y)$ and $F_y(x,y) = N(x,y)$ for all (x,y) on any open rectangle.

The next example illustrates two possible methods for finding a function F that satisfies the condition $F_x = M$ and $F_y = N$ if $M\,dx + N\,dy = 0$ is exact.

Example 8.5.3

Solve

$$(4x^3y^3 + 3x^2)\,dx + (3x^4y^2 + 6y^2)\,dy = 0. \tag{8.5.13}$$

Solution (Method 1)

Here

$$M(x,y) = 4x^3y^3 + 3x^2, \quad N(x,y) = 3x^4y^2 + 6y^2,$$

and

$$M_y(x,y) = N_x(x,y) = 12x^3y^2$$

for all (x,y). Therefore Theorem 8.5.2 implies that there's a function F such that

$$F_x(x,y) = M(x,y) = 4x^3y^3 + 3x^2 \tag{8.5.14}$$

and

$$F_y(x,y) = N(x,y) = 3x^4y^2 + 6y^2 \tag{8.5.15}$$

for all (x,y). To find F, we integrate Equation 8.5.14 with respect to x to obtain

$$F(x,y) = x^4y^3 + x^3 + \phi(y), \tag{8.5.16}$$

where $\phi(y)$ is the "constant" of integration. (Here ϕ is "constant" in that it is independent of x, the variable of integration.) If ϕ is any differentiable function of y then F satisfies Equation 8.5.14. To determine ϕ so that F also satisfies Equation 8.5.15, assume that ϕ is differentiable and differentiate F with respect to y. This yields

$$F_y(x,y) = 3x^4y^2 + \phi'(y).$$

Comparing this with Equation 8.5.15 shows that

$$\phi'(y) = 6y^2.$$

We integrate this with respect to y and take the constant of integration to be zero because we are interested only in finding **some** F that satisfies Equation 8.5.14 and Equation 8.5.15. This yields

$$\phi(y) = 2y^3.$$

Substituting this into Equation 8.5.16 yields

$$F(x,y) = x^4y^3 + x^3 + 2y^3. \tag{8.5.17}$$

Now Theorem 8.5.1 implies that

$$x^4y^3 + x^3 + 2y^3 = c$$

is an implicit solution of Equation 8.5.13. Solving this for y yields the explicit solution

$$y = \left(\frac{c - x^3}{2 + x^4}\right)^{1/3}.$$

Solution (Method 2)

Instead of first integrating Equation 8.5.14 with respect to x, we could begin by integrating Equation 8.5.15 with respect to y to obtain

$$F(x,y) = x^4y^3 + 2y^3 + \psi(x), \tag{8.5.18}$$

where ψ is an arbitrary function of x. To determine ψ, we assume that ψ is differentiable and differentiate F with respect to x, which yields

$$F_x(x,y) = 4x^3y^3 + \psi'(x).$$

Comparing this with Equation 8.5.14 shows that

$$\psi'(x) = 3x^2.$$

Integrating this and again taking the constant of integration to be zero yields

$$\psi(x) = x^3.$$

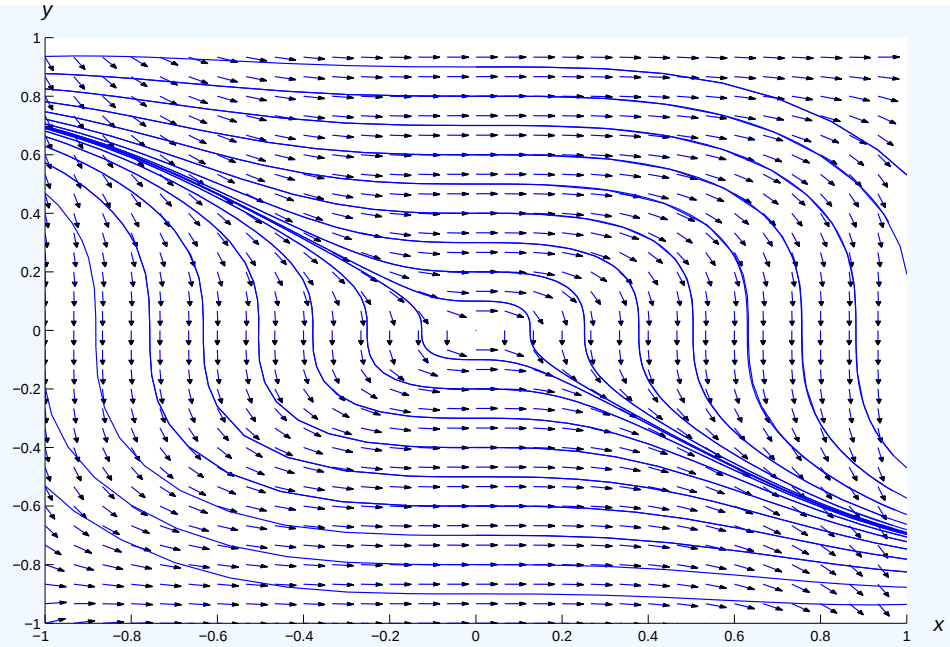


Figure 8.5.1: A direction field and integral curves for $(4x^3y^3+3x^2)\,dx+(3x^4y^2+6y^2)\,dy=0$

Substituting this into Equation 8.5.18 yields Equation 8.5.17.

Figure 8.5.1 shows a direction field and some integral curves of Equation 8.5.13.

Here's a summary of the procedure used in Method 1 of this example. You should summarize procedure used in Method 2.

HOWTO: Procedure For Solving An Exact Equation (Method 1)

- **Step 1.** Check that the equation

$$M(x,y)\,dx+N(x,y)\,dy=0 \tag{8.5.19}$$

  satisfies the exactness condition $M_y=N_x$. If not, don't go further with this procedure.
- **Step 2.** Integrate

$$\frac{\partial F(x,y)}{\partial x}=M(x,y)$$

  with respect to x to obtain

$$F(x,y)=G(x,y)+\phi(y), \tag{8.5.20}$$

  where G is an antiderivative of M with respect to x, and ϕ is an unknown function of y.
- **Step 3.** Differentiate Equation 8.5.20 with respect to y to obtain

$$\frac{\partial F(x,y)}{\partial y}=\frac{\partial G(x,y)}{\partial y}+\phi'(y).$$

- **Step 4.** Equate the right side of this equation to N and solve for ϕ'; thus,

$$\frac{\partial G(x,y)}{\partial y}+\phi'(y)=N(x,y),\quad \text{so}\quad \phi'(y)=N(x,y)-\frac{\partial G(x,y)}{\partial y}.$$

- **Step 5.** Integrate ϕ' with respect to y, taking the constant of integration to be zero, and substitute the result in Equation 8.5.20 to obtain $F(x,y)$.
- **Step 6.** Set $F(x,y)=c$ to obtain an implicit solution of Equation 8.5.19. If possible, solve for y explicitly as a function of x.

It's a common mistake to omit Step 6 in the procedure above. However, it is important to include this step, since F isn't itself a solution of Equation 8.5.19. Many equations can be conveniently solved by either of the two methods used in Example 8.5.3.

However, sometimes the integration required in one approach is more difficult than in the other. In such cases we choose the approach that requires the easier integration.

Example 8.5.4

Solve the equation

$$\left(ye^{xy}\tan x + e^{xy}\sec^2 x\right)dx + xe^{xy}\tan x\,dy = 0 \tag{8.5.21}$$

Solution

We leave it to you to check that $M_y = N_x$ on any open rectangle where $\tan x$ and $\sec x$ are defined. Here we must find a function F such that

$$F_x(x,y) = ye^{xy}\tan x + e^{xy}\sec^2 x \tag{8.5.22}$$

and

$$F_y(x,y) = xe^{xy}\tan x. \tag{8.5.23}$$

It's difficult to integrate Equation 8.5.22 with respect to x, but easy to integrate Equation 8.5.23 with respect to y. This yields

$$F(x,y) = e^{xy}\tan x + \psi(x). \tag{8.5.24}$$

Differentiating this with respect to x yields

$$F_x(x,y) = ye^{xy}\tan x + e^{xy}\sec^2 x + \psi'(x).$$

Comparing this with Equation 8.5.22 shows that $\psi'(x) = 0$. Hence, ψ is a constant, which we can take to be zero in Equation 8.5.24, and

$$e^{xy}\tan x = c,$$

is an implicit solution of Equation 8.5.21.

Attempting to apply our procedure to an differential equation that is not exact will lead to failure in Step 4, since the function

$$N - \frac{\partial G}{\partial y}$$

will not be independent of x if $M_y \neq N_x$, and therefore cannot be the derivative of a function of y alone. Example 8.5.5 illustrates this.

Example 8.5.5

Verify that the equation

$$3x^2y^2\,dx + 6x^3y\,dy = 0 \tag{8.5.25}$$

is not exact, and show that the procedure for solving exact equations fails when applied to Equation 8.5.25.

Solution

Here

$$M_y(x,y) = 6x^2y \quad \text{and} \quad N_x(x,y) = 18x^2y,$$

so Equation 8.5.25 is not exact. Nevertheless, let us try to find a function F such that

$$F_x(x,y) = 3x^2y^2 \tag{8.5.26}$$

and

$$F_y(x,y) = 6x^3y. \tag{8.5.27}$$

Integrating Equation 8.5.26 with respect to x yields

$$F(x,y) = x^3y^2 + \phi(y),$$

and differentiating this with respect to y yields

$$F_y(x,y) = 2x^3y + \phi'(y).$$

For this equation to be consistent with Equation 8.5.27,

$$6x^3y = 2x^3y + \phi'(y),$$

or

$$\phi'(y) = 4x^3y.$$

This is a contradiction, since ϕ' must be independent of x. Therefore the procedure fails.

This page titled 8.5: Exact Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.5: Exact Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

8.5.1: Exact Equations (Exercises)

Q2.5.1

In **Exercises 2.5.1-2.5.17** determine which equations are exact and solve them.

1. $6x^2y^2\,dx + 4x^3y\,dy = 0$

2. $(3y\cos x + 4xe^x + 2x^2e^x)\,dx + (3\sin x + 3)\,dy = 0$

3. $14x^2y^3\,dx + 21x^2y^2\,dy = 0$

4. $(2x - 2y^2)\,dx + (12y^2 - 4xy)\,dy = 0$

5. $(x + y)^2\,dx + (x + y)^2\,dy = 0$

6. $(4x + 7y)\,dx + (3x + 4y)\,dy = 0$

7. $(-2y^2\sin x + 3y^3 - 2x)\,dx + (4y\cos x + 9xy^2)\,dy = 0$

8. $(2x + y)\,dx + (2y + 2x)\,dy = 0$

9. $(3x^2 + 2xy + 4y^2)\,dx + (x^2 + 8xy + 18y)\,dy = 0$

10. $(2x^2 + 8xy + y^2)\,dx + (2x^2 + xy^3/3)\,dy = 0$

11. $\left(\frac{1}{x} + 2x\right)\,dx + \left(\frac{1}{y} + 2y\right)\,dy = 0$

12. $(y\sin xy + xy^2\cos xy)\,dx + (x\sin xy + xy^2\cos xy)\,dy = 0$

13. $\frac{x\,dx}{(x^2+y^2)^{3/2}} + \frac{y\,dy}{(x^2+y^2)^{3/2}} = 0$

14. $(e^x(x^2y^2 + 2xy^2) + 6x)\,dx + (2x^2ye^x + 2)\,dy = 0$

15. $\left(x^2e^{x^2+y}(2x^2 + 3) + 4x\right)\,dx + (x^3e^{x^2+y} - 12y^2)\,dy = 0$

16. $(e^{xy}(x^4y + 4x^3) + 3y)\,dx + (x^5e^{xy} + 3x)\,dy = 0$

17. $(3x^2\cos xy - x^3y\sin xy + 4x)\,dx + (8y - x^4\sin xy)\,dy = 0$

Q2.5.2

In **Exercises 2.5.18-2.5.22** solve the initial value problem.

18. $(4x^3y^2 - 6x^2y - 2x - 3)\,dx + (2x^4y - 2x^3)\,dy = 0, \quad y(1) = 3$

19. $(-4y\cos x + 4\sin x\cos x + \sec^2 x)\,dx + (4y - 4\sin x)\,dy = 0, \quad y(\pi/4) = 0$

20. $(y^3 - 1)e^x\,dx + 3y^2(e^x + 1)\,dy = 0, \quad y(0) = 0$

21. $(\sin x - y\sin x - 2\cos x)\,dx + \cos x\,dy = 0, \quad y(0) = 1$

22. $(2x - 1)(y - 1)\,dx + (x + 2)(x - 3)\,dy = 0, \quad y(1) = -1$

Q2.5.3

23. Solve the exact equation

$$(7x + 4y)\,dx + (4x + 3y)\,dy = 0.$$

Plot a direction field and some integral curves for this equation on the rectangle

$$\{-1 \le x \le 1, -1 \le y \le 1\}.$$

24. Solve the exact equation

$$e^x(x^4y^2 + 4x^3y^2 + 1)\,dx + (2x^4ye^x + 2y)\,dy = 0.$$

Plot a direction field and some integral curves for this equation on the rectangle

$$\{-2 \le x \le 2, -1 \le y \le 1\}.$$

25. Plot a direction field and some integral curves for the exact equation

$$(x^3y^4 + x)\,dx + (x^4y^3 + y)\,dy = 0$$

on the rectangle $\{-1 \le x \le 1, -1 \le y \le 1\}$. (See **Exercise 2.5.37(a)**).

26. Plot a direction field and some integral curves for the exact equation

$$(3x^2 + 2y)\,dx + (2y + 2x)\,dy = 0$$

on the rectangle $\{-2 \le x \le 2, -2 \le y \le 2\}$. (See E**xercise 2.5.37(b)**).

27.

a. Solve the exact equation

$$(x^3y^4 + 2x)\,dx + (x^4y^3 + 3y)\,dy = 0 \tag{A}$$

implicitly.

b. For what choices of (x_0, y_0) does Theorem 2.3.1 imply that the initial value problem

$$(x^3y^4 + 2x)\,dx + (x^4y^3 + 3y)\,dy = 0, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution on an open interval (a, b) that contains x_0?

c. Plot a direction field and some integral curves for (A) on a rectangular region centered at the origin. What is the interval of validity of the solution of (B)?

28.

a. Solve the exact equation

$$(x^2 + y^2)\,dx + 2xy\,dy = 0 \tag{A}$$

implicitly.

b. For what choices of (x_0, y_0) does Theorem 2.3.1 imply that the initial value problem

$$(x^2 + y^2)\,dx + 2xy\,dy = 0, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution $y = y(x)$ on some open interval (a, b) that contains x_0?

c. Plot a direction field and some integral curves for (A). From the plot determine, the interval (a, b) of b, the monotonicity properties (if any) of the solution of (B), and $\lim_{x \to a+} y(x)$ and $\lim_{x \to b-} y(x)$.

29. Find all functions M such that the equation is exact.

a. $M(x, y)\,dx + (x^2 - y^2)\,dy = 0$
b. $M(x, y)\,dx + 2xy \sin x \cos y\,dy = 0$
c. $M(x, y)\,dx + (e^x - e^y \sin x)\,dy = 0$

30. Find all functions N such that the equation is exact.

a. $(x^3y^2 + 2xy + 3y^2)\,dx + N(x, y)\,dy = 0$
b. $(\ln xy + 2y \sin x)\,dx + N(x, y)\,dy = 0$
c. $(x \sin x + y \sin y)\,dx + N(x, y)\,dy = 0$

31. Suppose M, N, and their partial derivatives are continuous on an open rectangle R, and G is an antiderivative of M with respect to x; that is,

$$\frac{\partial G}{\partial x} = M.$$

Show that if $M_y \ne N_x$ in R then the function

$$N - \frac{\partial G}{\partial y}$$

is not independent of x.

32. Prove: If the equations $M_1\,dx + N_1\,dy = 0$ and $M_2\,dx + N_2\,dy = 0$ are exact on an open rectangle R, so is the equation

$$(M_1 + M_2)\,dx + (N_1 + N_2)\,dy = 0.$$

33. Find conditions on the constants A, B, C, and D such that the equation

$$(Ax + By)\,dx + (Cx + Dy)\,dy = 0$$

is exact.

34. Find conditions on the constants A, B, C, D, E, and F such that the equation

$$(Ax^2 + Bxy + Cy^2)\,dx + (Dx^2 + Exy + Fy^2)\,dy = 0$$

is exact.

35. Suppose M and N are continuous and have continuous partial derivatives M_y and N_x that satisfy the exactness condition $M_y = N_x$ on an open rectangle R. Show that if (x, y) is in R and

$$F(x, y) = \int_{x_0}^{x} M(s, y_0)\,ds + \int_{y_0}^{y} N(x, t)\,dt,$$

then $F_x = M$ and $F_y = N$.

36. Under the assumptions of **Exercise 2.5.35**, show that

$$F(x, y) = \int_{y_0}^{y} N(x_0, s)\,ds + \int_{x_0}^{x} M(t, y)\,dt.$$

37. Use the method suggested by **Exercise 2.5.35**, with $(x_0, y_0) = (0, 0)$, to solve the these exact equations:

a. $(x^3y^4 + x)\,dx + (x^4y^3 + y)\,dy = 0$
b. $(x^2 + y^2)\,dx + 2xy\,dy = 0$
c. $(3x^2 + 2y)\,dx + (2y + 2x)\,dy = 0$

38. Solve the initial value problem

$$y' + \frac{2}{x}y = -\frac{2xy}{x^2 + 2x^2y + 1}, \quad y(1) = -2.$$

39. Solve the initial value problem

$$y' - \frac{3}{x}y = \frac{2x^4(4x^3 - 3y)}{3x^5 + 3x^3 + 2y}, \quad y(1) = 1.$$

40. Solve the initial value problem

$$y' + 2xy = -e^{-x^2}\left(\frac{3x + 2ye^{x^2}}{2x + 3ye^{x^2}}\right), \quad y(0) = -1.$$

41. Rewrite the separable equation

$$h(y)y' = g(x) \tag{A}$$

as an exact equation

$$M(x, y)\,dx + N(x, y)\,dy = 0. \tag{B}$$

Show that applying the method of this section to (B) yields the same solutions that would be obtained by applying the method of separation of variables to (A)

42. Suppose all second partial derivatives of $M = M(x, y)$ and $N = N(x, y)$ are continuous and $M\,dx + N\,dy = 0$ and $-N\,dx + M\,dy = 0$ are exact on an open rectangle R. Show that $M_{xx} + M_{yy} = N_{xx} + N_{yy} = 0$ on R.

43. Suppose all second partial derivatives of $F = F(x, y)$ are continuous and $F_{xx} + F_{yy} = 0$ on an open rectangle R. (A function with these properties is said to be **harmonic**; see also **Exercise 2.5.42**.) Show that $-F_y\,dx + F_x\,dy = 0$ is exact on R, and therefore there's a function G such that $G_x = -F_y$ and $G_y = F_x$ in R. (A function G with this property is said to be a **harmonic conjugate** of F.)

44. Verify that the following functions are harmonic, and find all their harmonic conjugates. (See **Exercise 2.5.43**.)

a. $x^2 - y^2$
b. $e^x \cos y$
c. $x^3 - 3xy^2$
d. $\cos x \cosh y$
e. $\sin x \cosh y$

This page titled 8.5.1: Exact Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.5E: Exact Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

8.6: Integrating Factors

In the previous section on Exact Equations we saw that if M, N, M_y and N_x are continuous and $M_y = N_x$ on an open rectangle R then

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{8.6.1}$$

is exact on R. Sometimes an equation that isn't exact can be made exact by multiplying it by an appropriate function. For example,

$$(3x + 2y^2)\,dx + 2xy\,dy = 0 \tag{8.6.2}$$

is not exact, since $M_y(x,y) = 4y \neq N_x(x,y) = 2y$ in Equation 8.6.2. However, multiplying Equation 8.6.2 by x yields

$$(3x^2 + 2xy^2)\,dx + 2x^2y\,dy = 0, \tag{8.6.3}$$

which is exact, since $M_y(x,y) = N_x(x,y) = 4xy$ in Equation 8.6.3. Solving Equation 8.6.3 by the procedure given in Section 2.5 yields the implicit solution

$$x^3 + x^2y^2 = c.$$

A function $\mu = \mu(x,y)$ is an **integrating factor** for Equation 8.6.1 if

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0 \tag{8.6.4}$$

is exact. If we know an integrating factor μ for Equation 8.6.1, we can solve the exact equation Equation 8.6.4 by the method of Section 2.5. It would be nice if we could say that Equation 8.6.1 and Equation 8.6.4 always have the same solutions, but this isn't so. For example, a solution $y = y(x)$ of Equation 8.6.4 such that $\mu(x, y(x)) = 0$ on some interval $a < x < b$ could fail to be a solution of 8.6.1 (**Exercise 2.6.1**), while Equation 8.6.1 may have a solution $y = y(x)$ such that $\mu(x, y(x))$ isn't even defined (**Exercise 2.6.2**). Similar comments apply if y is the independent variable and x is the dependent variable in Equation 8.6.1 and Equation 8.6.4. However, if $\mu(x,y)$ is defined and nonzero for all (x,y), Equation 8.6.1 and Equation 8.6.4 are equivalent; that is, they have the same solutions.

Finding Integrating Factors

By applying Theorem 2.5.2 (with M and N replaced by μM and μN), we see that Equation 8.6.4 is exact on an open rectangle R if μM, μN, $(\mu M)_y$, and $(\mu N)_x$ are continuous and

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)$$

or, equivalently

$$\mu_y M + \mu M_y = \mu_x N + \mu N_x$$

on R. It's better to rewrite the last equation as

$$\mu(M_y - N_x) = \mu_x N - \mu_y M, \tag{8.6.5}$$

which reduces to the known result for exact equations; that is, if $M_y = N_x$ then Equation 8.6.5 holds with $\mu = 1$, so Equation 8.6.1 is exact.

You may think Equation 8.6.5 is of little value, since it involves **partial** derivatives of the unknown integrating factor μ, and we haven't studied methods for solving such equations. However, we'll now show that Equation 8.6.5 is useful if we restrict our search to integrating factors that are products of a function of x and a function of y; that is, $\mu(x,y) = P(x)Q(y)$. We're not saying that **every** equation $M\,dx + N\,dy = 0$ has an integrating factor of this form; rather, we are saying that **some** equations have such integrating factors.We'llnow develop a way to determine whether a given equation has such an integrating factor, and a method for finding the integrating factor in this case.

If $\mu(x,y) = P(x)Q(y)$, then $\mu_x(x,y) = P'(x)Q(y)$ and $\mu_y(x,y) = P(x)Q'(y)$, so Equation 8.6.5 becomes

$$P(x)Q(y)(M_y - N_x) = P'(x)Q(y)N - P(x)Q'(y)M, \tag{8.6.6}$$

or, after dividing through by $P(x)Q(y)$,

$$M_y - N_x = \frac{P'(x)}{P(x)}N - \frac{Q'(y)}{Q(y)}M. \tag{8.6.7}$$

Now let

$$p(x) = \frac{P'(x)}{P(x)} \quad \text{and} \quad q(y) = \frac{Q'(y)}{Q(y)},$$

so Equation 8.6.7 becomes

$$M_y - N_x = p(x)N - q(y)M. \tag{8.6.8}$$

We obtained Equation 8.6.8 by **assuming** that $M\,dx + N\,dy = 0$ has an integrating factor $\mu(x,y) = P(x)Q(y)$. However, we can now view Equation 8.6.7 differently: If there are functions $p = p(x)$ and $q = q(y)$ that satisfy Equation 8.6.8 and we define

$$P(x) = \pm e^{\int p(x)\,dx} \quad \text{and} \quad Q(y) = \pm e^{\int q(y)\,dy}, \tag{8.6.9}$$

then reversing the steps that led from Equation 8.6.6 to Equation 8.6.8 shows that $\mu(x,y) = P(x)Q(y)$ is an integrating factor for $M\,dx + N\,dy = 0$. In using this result, we take the constants of integration in Equation 8.6.9 to be zero and choose the signs conveniently so the integrating factor has the simplest form.

There's no simple general method for ascertaining whether functions $p = p(x)$ and $q = q(y)$ satisfying Equation 8.6.8 exist. However, the next theorem gives simple sufficient conditions for the given equation to have an integrating factor that depends on only one of the independent variables x and y, and for finding an integrating factor in this case.

Theorem 2.6.1

Let M, N, M_y, and N_x be continuous on an open rectangle R. Then:

a. If $(M_y - N_x)/N$ is independent of y on R and we define

$$p(x) = \frac{M_y - N_x}{N}$$

then

$$\mu(x) = \pm e^{\int p(x)\,dx} \tag{8.6.10}$$

is an integrating factor for

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{8.6.11}$$

on R.

b. If $(N_x - M_y)/M$ is independent of x on R and we define

$$q(y) = \frac{N_x - M_y}{M},$$

then

$$\mu(y) = \pm e^{\int q(y)\,dy} \tag{8.6.12}$$

is an integrating factor for Equation 8.6.11 on R.

Proof

(a) If $(M_y - N_x)/N$ is independent of y, then Equation 8.6.8 holds with $p = (M_y - N_x)/N$ and $q \equiv 0$. Therefore

$$P(x) = \pm e^{\int p(x)\,dx} \quad \text{and} \quad Q(y) = \pm e^{\int q(y)\,dy} = \pm e^0 = \pm 1,$$

so Equation 8.6.10 is an integrating factor for Equation 8.6.11 on R.

(b) If $(N_x - M_y)/M$ is independent of x then eqrefeq:2.6.8 holds with $p \equiv 0$ and $q = (N_x - M_y)/M$, and a similar argument shows that Equation 8.6.12 is an integrating factor for Equation 8.6.11 on R.

The next two examples show how to apply Theorem 2.6.1 .

Example 2.6.1

Find an integrating factor for the equation

$$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + (3x^2y^2 + 4y)\,dy = 0 \tag{8.6.13}$$

and solve the equation.

Solution

In Equation 8.6.13

$$M = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x,\ N = 3x^2y^2 + 4y,$$

and

$$M_y - N_x = (6xy^2 - 6x^3y^2 - 8xy) - 6xy^2 = -6x^3y^2 - 8xy,$$

so Equation 8.6.13 isn't exact. However,

$$\frac{M_y - N_x}{N} = -\frac{6x^3y^2 + 8xy}{3x^2y^2 + 4y} = -2x$$

is independent of y, so Theorem 2.6.1 (a) applies with $p(x) = -2x$. Since

$$\int p(x)\,dx = -\int 2x\,dx = -x^2,$$

$\mu(x) = e^{-x^2}$ is an integrating factor. Multiplying Equation 8.6.13 by μ yields the exact equation

$$e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + e^{-x^2}(3x^2y^2 + 4y)\,dy = 0. \tag{8.6.14}$$

To solve this equation, we must find a function F such that

$$F_x(x,y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) \tag{8.6.15}$$

and

$$F_y(x,y) = e^{-x^2}(3x^2y^2 + 4y). \tag{8.6.16}$$

Integrating Equation 8.6.16 with respect to y yields

$$F(x,y) = e^{-x^2}(x^2y^3 + 2y^2) + \psi(x). \tag{8.6.17}$$

Differentiating this with respect to x yields

$$F_x(x,y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2) + \psi'(x).$$

Comparing this with Equation 8.6.15 shows that $\psi'(x) = 2xe^{-x^2}$; therefore, we can let $\psi(x) = -e^{-x^2}$ in Equation 8.6.17 and conclude that

$$e^{-x^2}\left(y^2(x^2y + 2) - 1\right) = c$$

is an implicit solution of Equation 8.6.14. It is also an implicit solution of Equation 8.6.13.

Figure 2.6.1 shows a direction field and some integral curves for Equation 8.6.13

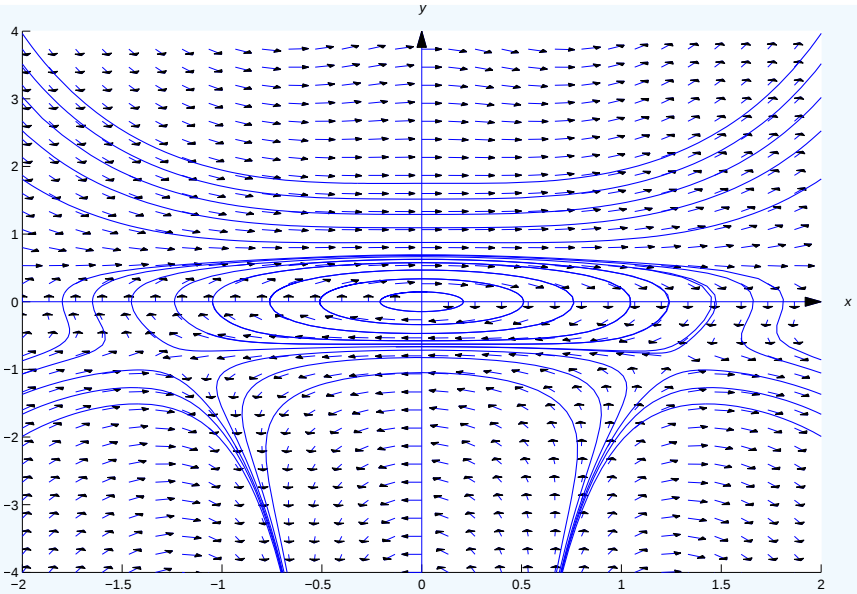


Figure 2.6.1 : A direction field and integral curves for $(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)\,dx + (3x^2y^2 + 4y)\,dy = 0$

Example 2.6.2

Find an integrating factor for

$$2xy^3dx + (3x^2y^2 + x^2y^3 + 1)dy = 0 \tag{8.6.18}$$

and solve the equation.

Solution

In Equation 8.6.18,

$$M = 2xy^3, \quad N = 3x^2y^2 + x^2y^3 + 1,$$

and

$$M_y - N_x = 6x^2 - (6xy^2 + 2xy^3) = -2xy^3,$$

so Equation 8.6.18 isn't exact. Moreover,

$$\frac{M_y - N_x}{N} = -\frac{2xy^3}{3x^2y^2 + x^2y^2 + 1}$$

is not independent of y, so Theorem 2.6.1(a) does not apply. However, Theorem 2.6.1(b) does apply, since

$$\frac{N_x - M_y}{M} = \frac{2xy^3}{2xy^3} = 1$$

is not independent of x, so we can take $q(y) = 1$. Since

$$\int q(y)dy = \int dy = y,$$

$\mu(y) = e^y$ is an integrating factor. Multiplying Equation 8.6.18 by μ yields the exact equation.

$$2xy^3e^ydx + (3x^2y^2 + x^2y^3 + 1)e^ydy = 0. \tag{8.6.19}$$

To solve this equation, we must find a function F such that

$$F_x(x, y) = 2xy^3e^y \tag{8.6.20}$$

and

$$F_y(x,y) = (3x^2y^2 + x^2y^3 + 1)e^y. \tag{8.6.21}$$

Integrating Equation 8.6.20 with respect to x yields

$$F(x,y) = x^2y^3e^y + \phi(y) \tag{8.6.22}$$

Differentiating this with respect to y yields

$$F_y = (3x^2y^2 + x^2y^3)e^y + \phi'(y)$$

and comparing this with Equation 8.6.21 shows that φ 0 (y) = e y . Therefore we set φ(y) = e y in Equation 8.6.22 and conclude that

$$(x^2y^3 + 1)e^y = c$$

is an implicit solution of 8.6.19. It is also an implicit solution of 8.6.18. Figure 2.6.2 shows a direction field and some integral curves for 8.6.18.

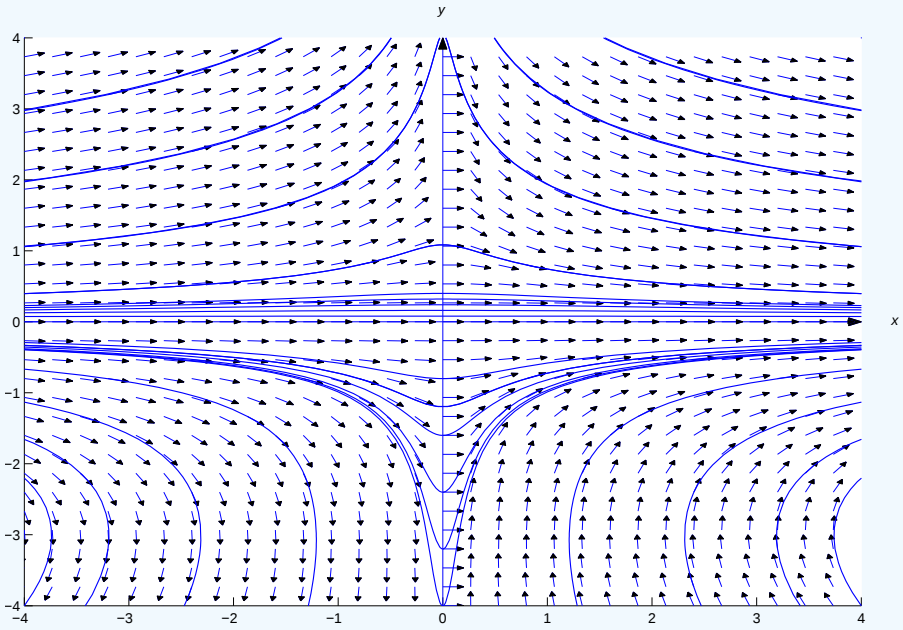


Figure 2.6.2 : A direction field and integral curves for $(2xy^3e^y dx + (3x^2y^2 + x^2y^3 + 1)e^y dy = 0$

Theorem 2.6.1 does not apply in the next example, but the more general argument that led to Theorem 2.6.1 provides an integrating factor.

Example 2.6.3

Find an integrating factor for

$$(3xy + 6y^2)dx + (2x^2 + 9xy)dy = 0 \tag{8.6.23}$$

and solve the equation.

Solution

In Equation 8.6.23

$$M = 3xy + 6y^2, \quad N = 2x^2 + 9xy,$$

and

$$M_y - N_x = (3x + 12y) - (4x + 9y) = -x + 3y.$$

Therefore

$$\frac{M_y - N_x}{M} = \frac{-x+3y}{3xy+6y^2} \quad \text{and} \quad \frac{N_x - M_y}{N} = \frac{x-3y}{2x^2+9xy}$$

so Theorem 2.6.1 does not apply. Following the more general argument that led to Theorem 2.6.1 , we look for functions $p = p(x)$ and $q = q(y)$ such that

$$M_y - N_x = p(x)N - q(y)M;$$

that is,

$$-x+3y = p(x)(2x^2+9xy) - q(y)(3xy+6y^2).$$

Since the left side contains only first degree terms in x and y, we rewrite this equation as

$$xp(x)(2x+9y) - yq(y)(3x+6y) = -x+3y.$$

This will be an identity if

$$xp(x) = A \quad \text{and} yq(y) = B, \tag{8.6.24}$$

where A and B are constants such that

$$-x+3y = A(2x+9y) - B(3x+6y),$$

or, equivalently,

$$-x+3y = (2A-3B)x + (9A-6B)y.$$

Equating the coefficients of x and y on both sides shows that the last equation holds for all (x, y) if

$$2A-3B = -1$$
$$9A-6B = 3$$

which has the solution A = 1, B = 1. Therefore Equation 8.6.24 implies that

$$p(x) = \frac{1}{x} \quad \text{and} \quad q(y) = \frac{1}{y}.$$

Since

$$\int p(x)dx = \ln|x| \quad \text{and} \quad \int q(y)dy = \ln|y|,$$

we can let $P(x) = x$ and $Q(y) = y$; hence, $\mu(x, y) = xy$ is an integrating factor. Multiplying Equation 8.6.23 by μ yields the exact equation

$$(3x^2y^2 + 6xy^3)dx + (2x^3y + 9x^2y^2)dy = 0.$$

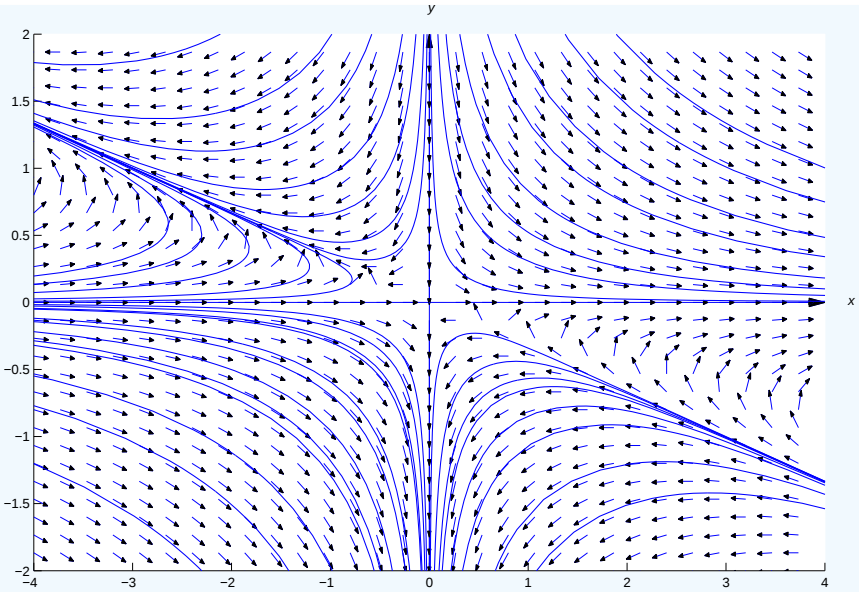


Figure 2.6.3 : A direction field and integral curves for $(3xy+6y^2)dx+(2x^2+9xy)dy=0$

We leave it to you to use the method of Section 2.5 to show that this equation has the implicit solution

$$x^3y^2+3x^2y^3=c. \tag{8.6.25}$$

This is also an implicit solution of Equation 8.6.23. Since x ≡ 0 and y ≡ 0 satisfy Equation 8.6.25, you should check to see that x ≡ 0 and y ≡ 0 are also solutions of Equation 8.6.23. (Why is it necesary to check this?) Figure 2.6.3 shows a direction field and integral curves for Equation 8.6.23. See **Exercise 2.6.28** for a general discussion of equations like Equation 8.6.23.

Example 2.6.4

The separable equation

$$-ydx+(x+x^6)dy=0 \tag{8.6.26}$$

can be converted to the exact equation

$$-\frac{dx}{x+x^6}+\frac{dy}{y}=0 \tag{8.6.27}$$

by multiplying through by the integrating factor

$$\mu(x,y)=\frac{1}{y(x+x^6)}.$$

However, to solve Equation 8.6.27 by the method of Section 2.5 we would have to evaluate the nasty integral

$$\int\frac{dx}{x+x^6}.$$

Instead, we solve Equation 8.6.26 explicitly for y by finding an integrating factor of the form $\mu(x,y)=x^ay^b$.

Solution

In Equation 8.6.26

$$M=-y,\ N=x+x^6,$$

and

$$M_y-N_x=-1-(1+6x^5)=-2-6x^5.$$

We look for functions $p=p(x)$ and $q=q(y)$ such that

$$M_y - N_x = p(x)N - q(y)M;$$

that is,

$$-2 - 6x^5 = p(x)(x + x^6) + q(y)y. \tag{8.6.28}$$

The right side will contain the term $-6x^5$ if $p(x) = -6/x$. Then Equation 8.6.28 becomes

$$-2 - 6x^5 = -6 - 6x^5 + q(y)y,$$

so $q(y) = 4/y$. Since

$$\int p(x)\,dx = -\int \frac{6}{x}\,dx = -6\ln|x| = \ln\frac{1}{x^6},$$

and

$$\int q(y)\,dy = \int \frac{4}{y}\,dy = 4\ln|y| = \ln y^4,$$

we can take $P(x) = x^{-6}$ and $Q(y) = y^4$, which yields the integrating factor $\mu(x, y) = x^{-6}y^4$. Multiplying Equation 8.6.26 by μ yields the exact equation

$$-\frac{y^5}{x^6}\,dx + \left(\frac{y^4}{x^5} + y^4\right)dy = 0.$$

We leave it to you to use the method of the Section 2.5 to show that this equation has the implicit solution

$$\left(\frac{y}{x}\right)^5 + y^5 = k.$$

Solving for y yields

$$y = k^{1/5}x(1 + x^5)^{-1/5},$$

which we rewrite as

$$y = cx(1 + x^5)^{-1/5}$$

by renaming the arbitrary constant. This is also a solution of Equation 8.6.26.

Figure 2.6.4 shows a direction field and some integral curves for Equation 8.6.26.

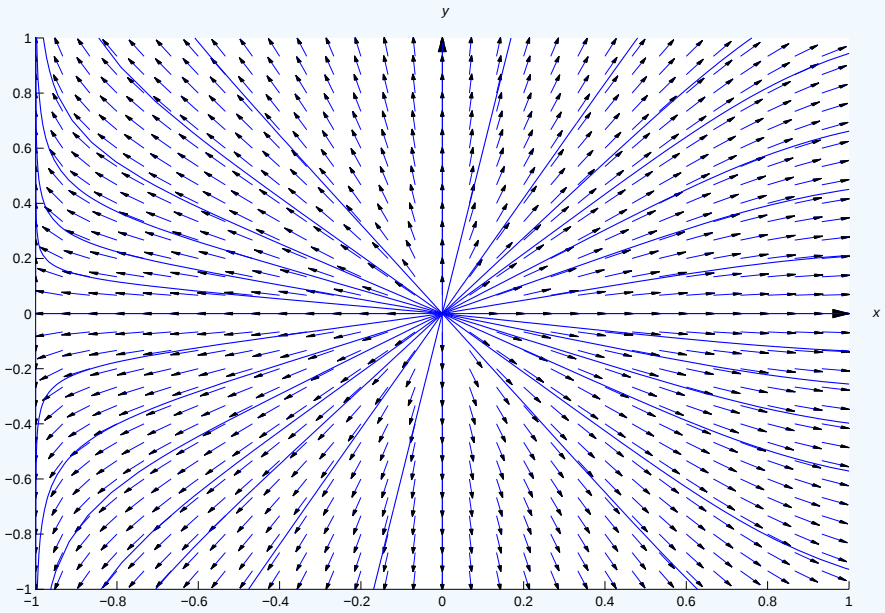


Figure 2.6.4 : A direction field and integral curves for $-ydx + (x + x^6)dy = 0$

https://math.libretexts.org/@go/page/121749

This page titled 8.6: Integrating Factors is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.6: Integrating Factors** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

8.6.1: Integrating Factors (Exercises)

Q2.6.1

1.

a. Verify that $\mu(x,y) = y$ is an integrating factor for

$$ydx + \left(2x + \frac{1}{y}\right) dy = 0 \tag{A}$$

on any open rectangle that does not intersect the x axis or, equivalently, that

$$y^2dx + (2xy + 1)dy = 0 \tag{B}$$

is exact on any such rectangle.

b. Verify that $y \equiv 0$ is a solution of (B), but not of (A).

c. Show that

$$y(xy + 1) = c \tag{C}$$

is an implicit solution of (B), and explain why every differentiable function $y = y(x)$ other than $y \equiv 0$ that satisfies (C) is also a solution of (A).

2.

a. Verify that $\mu(x,y) = 1/(x-y)^2$ is an integrating factor for

$$-y^2dx + x^2dy = 0 \tag{A}$$

on any open rectangle that does not intersect the line $y = x$ or, equivalently, that

$$-\frac{y^2}{(x-y)^2}dx + \frac{x^2}{(x-y)^2}dy = 0 \tag{B}$$

is exact on any such rectangle.

b. Use Theorem 2.2.1 to show that

$$\frac{xy}{(x-y)} = c \tag{C}$$

is an implicit solution of (B), and explain why it is also an implicit solution of (A)

c. Verify that $y = x$ is a solution of (A), even though it can't be obtained from (C).

Q2.6.2

In **Exercises 2.6.3-2.6.16** find an integrating factor; that is a function of only one variable, and solve the given equation.

3. $ydx - xdy = 0$

4. $3x^2ydx + 2x^3dy = 0$

5. $2y^3dx + 3y^2dy = 0$

6. $(5xy + 2y + 5)dx + 2xdy = 0$

7. $(xy + x + 2y + 1)\,dx + (x + 1)\,dy = 0$

8. $(27xy^2 + 8y^3)\,dx + (18x^2y + 12xy^2)\,dy = 0$

9. $(6xy^2 + 2y)\,dx + (12x^2y + 6x + 3)\,dy = 0$

10. $y^2\,dx + \left(xy^2 + 3xy + \frac{1}{y}\right)\,dy = 0$

11. $(12x^3y + 24x^2y^2)\,dx + (9x^4 + 32x^3y + 4y)\,dy = 0$

12. $(x^2y + 4xy + 2y)\,dx + (x^2 + x)\,dy = 0$

13. $-y\,dx+(x^4-x)\,dy=0$

14. $\cos x\cos y\,dx+(\sin x\cos y-\sin x\sin y+y)\,dy=0$

15. $(2xy+y^2)\,dx+(2xy+x^2-2x^2y^2-2xy^3)\,dy=0$

16. $y\sin y\,dx+x(\sin y-y\cos y)\,dy=0$

Q2.6.3

In **Exercises 2.6.17-2.6.23** find an integrating factor of the form $\mu(x,y)=P(x)Q(y)$ and solve the given equation.

17. $y(1+5\ln|x|)\,dx+4x\ln|x|\,dy=0$

18. $(\alpha y+\gamma xy)\,dx+(\beta x+\delta xy)\,dy=0$

19. $(3x^2y^3-y^2+y)\,dx+(-xy+2x)\,dy=0$

20. $2y\,dx+3(x^2+x^2y^3)\,dy=0$

21. $(a\cos xy-y\sin xy)\,dx+(b\cos xy-x\sin xy)\,dy=0$

22. $x^4y^4\,dx+x^5y^3\,dy=0$

23. $y(x\cos x+2\sin x)\,dx+x(y+1)\sin x\,dy=0$

Q2.6.4

In **Exercises 2.6.24-2.6.27** find an integrating factor and solve the equation. Plot a direction field and some integral curves for the equation in the indicated rectangular region.

24. $(x^4y^3+y)\,dx+(x^5y^2-x)\,dy=0;\quad\{-1\le x\le 1,-1\le y\le 1\}$

25. $(3xy+2y^2+y)\,dx+(x^2+2xy+x+2y)\,dy=0;\quad\{-2\le x\le 2,-2\le y\le 2\}$

26. $(12xy+6y^3)\,dx+(9x^2+10xy^2)\,dy=0;\quad\{-2\le x\le 2,-2\le y\le 2\}$

27. $(3x^2y^2+2y)\,dx+2x\,dy=0;\quad\{-4\le x\le 4,-4\le y\le 4\}$

Q2.6.5

28. Suppose a, b, c, and d are constants such that $ad-bc\ne 0$, and let m and n be arbitrary real numbers. Show that

$$(ax^my+by^{n+1})\,dx+(cx^{m+1}+dxy^n)\,dy=0$$

has an integrating factor $\mu(x,y)=x^\alpha y^\beta$.

29. Suppose M, N, M_x, and N_y are continuous for all (x,y), and $\mu=\mu(x,y)$ is an integrating factor for

$$M(x,y)dx+N(x,y)dy=0. \tag{A}$$

Assume that μ_x and μ_y are continuous for all (x,y), and suppose $y=y(x)$ is a differentiable function such that $\mu(x,y(x))=0$ and $\mu_x(x,y(x))\ne 0$ for all x in some interval I. Show that y is a solution of (A) on I.

30. According to Theorem 2.1.2, the general solution of the linear nonhomogeneous equation

$$y'+p(x)y=f(x) \tag{A}$$

is

$$y=y_1x\left(c+\int f(x)/y_1(x)dx\right), \tag{B}$$

where y_1 is any nontrivial solution of the complementary equation $y'+p(x)y=0$. In this exercise we obtain this conclusion in a different way. You may find it instructive to apply the method suggested here to solve some of the exercises in Section 2.1.

a. Rewrite (A) as

$$[p(x)y-f(x)]dx+dy=0, \tag{C}$$

and show that $\mu = \pm e^{\int p(x)\,dx}$ is an integrating factor for (C).

b. Multiply (A) through by $\mu = \pm e^{\int p(x)\,dx}$ and verify that the resulting equation can be rewritten as

$$(\mu(x)y)' = \mu(x)f(x).$$

Then integrate both sides of this equation and solve for y to show that the general solution of (A) is

$$y = \frac{1}{\mu(x)}\left(c + \int f(x)\mu(x)\,dx\right).$$

Why is this form of the general solution equivalent to (B)?

This page titled 8.6.1: Integrating Factors (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **2.6E: Integrating Factors (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

CHAPTER OVERVIEW

9: Applications of First Order Equations

In this chapter, we consider applications of first order differential equations.

9.1: Growth and Decay

9.1.1: Growth and Decay (Exercises)

9.2: Cooling and Mixing

9.2.1: Cooling and Mixing (Exercises)

9.3: Elementary Mechanics

9.3.1: Elementary Mechanics (Exercises)

Thumbnail: False color time-lapse video of E. coli colony growing on microscope slide. This growth can be model with first order logistic equation. Added approximate scale bar based on the approximate length of 2.0 μm of E. coli bacteria. (CC BY-SA 4.0 International; Stewart EJ, Madden R, Paul G, Taddei F).

This page titled 9: Applications of First Order Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

9.1: Growth and Decay

This section begins with a discussion of exponential growth and decay, which you have probably already seen in calculus. We consider applications to radioactive decay, carbon dating, and compound interest. We also consider more complicated problems where the rate of change of a quantity is in part proportional to the magnitude of the quantity, but is also influenced by other other factors for example, a radioactive substance is manufactured at a certain rate, but decays at a rate proportional to its mass, or a saver makes regular deposits in a savings account that draws compound interest.

Since the applications in this section deal with functions of time, we'll denote the independent variable by t. If Q is a function of t, Q' will denote the derivative of Q with respect to t; thus,

$$Q' = \frac{dQ}{dt}.$$

Exponential Growth and Decay

One of the most common mathematical models for a physical process is the **exponential model**, where it is assumed that the rate of change of a quantity Q is proportional to Q; thus

$$Q' = aQ, \tag{9.1.1}$$

where a is the constant of proportionality.

From Example 2.1.3, the general solution of Equation 9.1.1 is

$$Q = ce^{at}$$

and the solution of the initial value problem

$$Q' = aQ, \quad Q(t_0) = Q_0$$

is

$$Q = Q_0 e^{a(t-t_0)}. \tag{9.1.2}$$

Since the solutions of $Q' = aQ$ are exponential functions, we say that a quantity Q that satisfies this equation **grows exponentially** if $a > 0$, or **decays exponentially** if $a < 0$ (Figure 4.1.1).

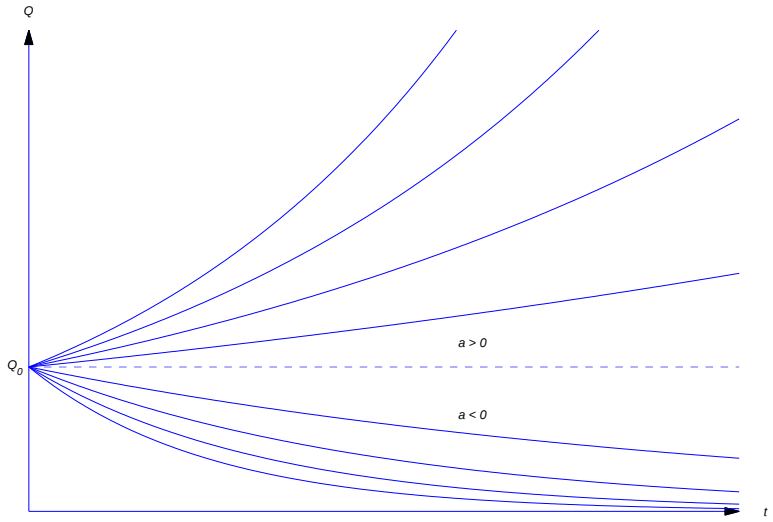


Figure 4.1.1 : Exponential growth and decay

Radioactive Decay

Experimental evidence shows that radioactive material decays at a rate proportional to the mass of the material present. According to this model the mass $Q(t)$ of a radioactive material present at time t satisfies Equation 9.1.1, where a is a negative constant whose

value for any given material must be determined by experimental observation. For simplicity, we'll replace the negative constant a by $-k$, where k is a positive number that we'll call the **decay constant** of the material. Thus, Equation 9.1.1 becomes

$$Q' = -kQ.$$

If the mass of the material present at $t = t_0$ is Q_0, the mass present at time t is the solution of

$$Q' = -kQ, \quad Q(t_0) = Q_0.$$

From Equation 9.1.2 with $a = -k$, the solution of this initial value problem is

$$Q = Q_0 e^{-k(t-t_0)}. \tag{9.1.3}$$

The **half–life** τ of a radioactive material is defined to be the time required for half of its mass to decay; that is, if $Q(t_0) = Q_0$, then

$$Q(\tau + t_0) = \frac{Q_0}{2}. \tag{9.1.4}$$

From Equation 9.1.3 with $t = \tau + t_0$, Equation 9.1.4 is equivalent to

$$Q_0 e^{-k\tau} = \frac{Q_0}{2},$$

so

$$e^{-k\tau} = \frac{1}{2}.$$

Taking logarithms yields

$$-k\tau = \ln \frac{1}{2} = -\ln 2,$$

so the half-life is

$$\tau = \frac{1}{k} \ln 2. \tag{9.1.5}$$

(Figure 4.1.2). The half-life is independent of t_0 and Q_0, since it is determined by the properties of material, not by the amount of the material present at any particular time.

Example 4.1.1

A radioactive substance has a half-life of 1620 years.

a. If its mass is now 4 g (grams), how much will be left 810 years from now?
b. Find the time t_1 when 1.5 g of the substance remain.

Solution a

From Equation 9.1.3 with $t_0 = 0$ and $Q_0 = 4$,

$$Q = 4e^{-kt}, \tag{9.1.6}$$

where we determine k from Equation 9.1.5, with τ= 1620 years:

$$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{1620}.$$

Substituting this in Equation 9.1.6 yields

$$Q = 4e^{-(t \ln 2)/1620}. \tag{9.1.7}$$

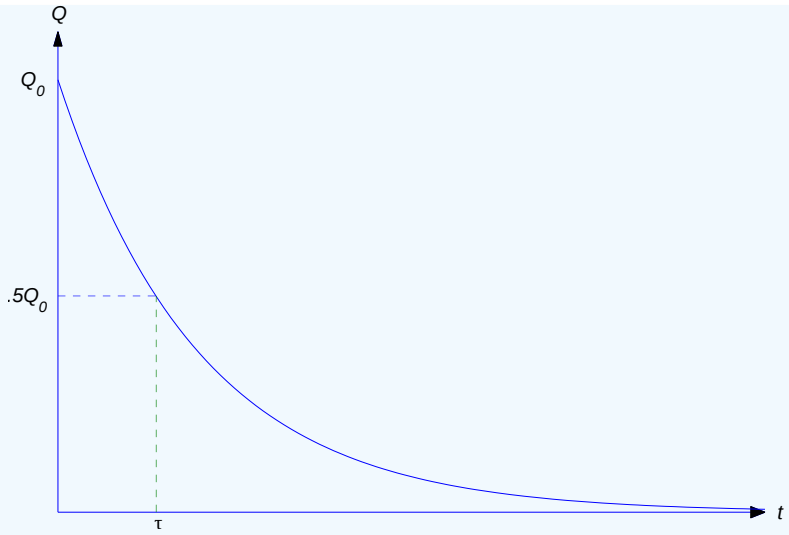


Figure 4.1.2 : Half-life of a radioactive substance

Therefore the mass left after 810 years will be

$$\begin{aligned} Q(810) &= 4e^{-(810\ln 2)/1620} = 4e^{-(\ln 2)/2} \\ &= 2\sqrt{2}\ \text{g}. \end{aligned}$$

Solution b

Setting $t = t_1$ in Equation 9.1.7 and requiring that $Q(t_1) = 1.5$ yields

$$\frac{3}{2} = 4e^{(-t_1 \ln 2)/1620}.$$

Dividing by 4 and taking logarithms yields

$$\ln\frac{3}{8} = -\frac{t_1 \ln 2}{1620}.$$

Since $\ln 3/8 = -\ln 8/3$,

$$t_1 = 1620\frac{\ln 8/3}{\ln 2} \approx 2292.4 \ \text{ years}.$$

Interest Compounded Continuously

Suppose we deposit an amount of money Q_0 in an interest-bearing account and make no further deposits or withdrawals for t years, during which the account bears interest at a constant annual rate r. To calculate the value of the account at the end of t years, we need one more piece of information: how the interest is added to the account, or—as the bankers say—how it is **compounded**. If the interest is compounded annually, the value of the account is multiplied by $1 + r$ at the end of each year. This means that after t years the value of the account is

$$Q(t) = Q_0(1 + r)^t.$$

If interest is compounded semiannually, the value of the account is multiplied by $(1 + r/2)$ every 6 months. Since this occurs twice annually, the value of the account after t years is

$$Q(t) = Q_0\left(1 + \frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded n times per year, the value of the account is multiplied n times per year by $(1 + r/n)$; therefore, the value of the account after t years is

$$Q(t) = Q_0\left(1 + \frac{r}{n}\right)^{nt}. \tag{9.1.8}$$

Thus, increasing the frequency of compounding increases the value of the account after a fixed period of time. Table 4.1.1 shows the effect of increasing the number of compoundings over $t = 5$ years on an initial deposit of $Q_0 = 100$ (dollars), at an annual interest rate of 6%.

Table 4.1.1 : The effect of compound interest

n (number of compoundings per year)	$\$100\left(1+\frac{.06}{n}\right)^{5n}$ (value in dollars after 5 years)
1	$133.82
2	$134.39
4	$134.68
8	$134.83
364	$134.98

You can see from Table 4.1.1 that the value of the account after 5 years is an increasing function of n. Now suppose the maximum allowable rate of interest on savings accounts is restricted by law, but the time intervals between successive compoundings isn't; then competing banks can attract savers by compounding often. The ultimate step in this direction is to **compound continuously**, by which we mean that $n \to \infty$ in Equation 9.1.8. Since we know from calculus that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r,$$

this yields

$$\begin{aligned} Q(t) &= \lim_{n\to\infty} Q_0\left(1+\tfrac{r}{n}\right)^{nt} = Q_0\left[\lim_{n\to\infty}\left(1+\tfrac{r}{n}\right)^n\right]^t \\ &= Q_0 e^{rt}. \end{aligned}$$

Observe that $Q = Q_0 e^{rt}$ is the solution of the initial value problem

$$Q' = rQ, \quad Q(0) = Q_0;$$

that is, with continuous compounding the value of the account grows exponentially.

Example 4.1.2

If \$150 is deposited in a bank that pays $5\frac{1}{2}\%$ annual interest compounded continuously, the value of the account after t years is

$$Q(t) = 150e^{.055t}$$

dollars. (Note that it is necessary to write the interest rate as a decimal; thus, $r = .055$.) Therefore, after $t = 10$ years the value of the account is

$$Q(10) = 150e^{.55} \approx \$259.99.$$

Example 4.1.3

We wish to accumulate \$10,000 in 10 years by making a single deposit in a savings account bearing $5\frac{1}{2}\%$ annual interest compounded continuously. How much must we deposit in the account?

Solution

The value of the account at time t is

$$Q(t) = Q_0 e^{.055t}. \tag{9.1.9}$$

Since we want $Q(10)$ to be \$10,000, the initial deposit Q_0 must satisfy the equation

$$10000 = Q_0 e^{.55}, \tag{9.1.10}$$

obtained by setting $t = 10$ and $Q(10) = 10000$ in Equation 9.1.9. Solving Equation 9.1.10 for Q_0 yields

$$Q_0 = 10000e^{-.55} \approx \$5769.50.$$

Mixed Growth and Decay

Example 4.1.4

A radioactive substance with decay constant k is produced at a constant rate of a units of mass per unit time.

a. Assuming that $Q(0) = Q_0$, find the mass $Q(t)$ of the substance present at time t.
b. Find $\lim_{t\to\infty} Q(t)$.

Solution a:

Here

$$Q' = \text{ rate of increase of } Q - \text{ rate of decrease of } Q.$$

The rate of increase is the constant a. Since Q is radioactive with decay constant k, the rate of decrease is kQ. Therefore

$$Q' = a - kQ.$$

This is a linear first order differential equation. Rewriting it and imposing the initial condition shows that Q is the solution of the initial value problem

$$Q' + kQ = a, \quad Q(0) = Q_0. \tag{9.1.11}$$

Since e^{-kt} is a solution of the complementary equation, the solutions of Equation 9.1.11 are of the form $Q = ue^{-kt}$, where $u'e^{-kt} = a$, so $u' = ae^{kt}$. Hence,

$$u = \frac{a}{k}e^{kt} + c$$

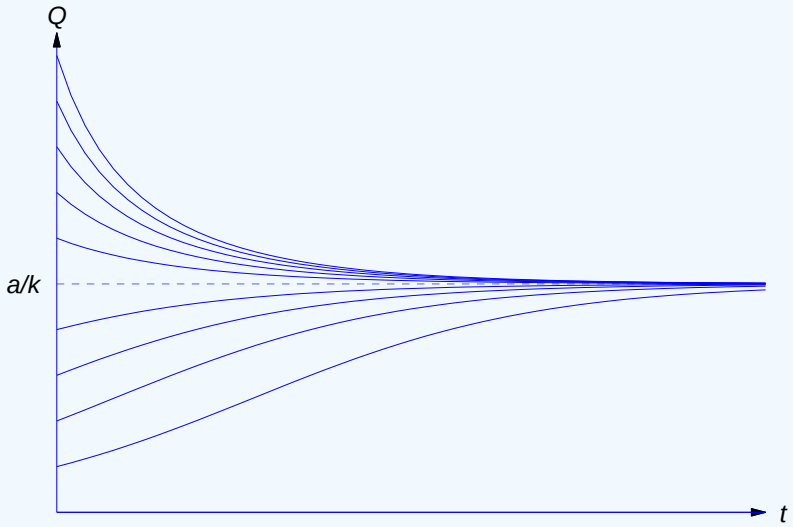


Figure 4.1.3 : $Q(t)$ approaches the steady state value $\frac{a}{k}$ as $t \to \infty$

and

$$Q = ue^{-kt} = \frac{a}{k} + ce^{-kt}.$$

Since $Q(0) = Q_0$, setting $t = 0$ here yields

$$Q_0 = \frac{a}{k} + c \quad \text{or} \quad c = Q_0 - \frac{a}{k}.$$

Therefore

$$Q = \frac{a}{k} + \left(Q_0 - \frac{a}{k}\right) e^{-kt}. \tag{9.1.12}$$

b. Since $k > 0$, $\lim_{t\to\infty} e^{-kt} = 0$, so from Equation 9.1.12

$$\lim_{t\to\infty} Q(t) = \frac{a}{k}.$$

This limit depends only on a and k, and not on Q_0. We say that a/k is the **steady state** value of Q. From Equation 9.1.12 we also see that Q approaches its steady state value from above if $Q_0 > a/k$, or from below if $Q_0 < a/k$. If $Q_0 = a/k$, then Q remains constant (Figure 4.1.3).

Carbon Dating

The fact that Q approaches a steady state value in the situation discussed in Example 4 underlies the method of **carbon dating**, devised by the American chemist and Nobel Prize Winner **W.S. Libby**.

Carbon 12 is stable, but carbon-14, which is produced by cosmic bombardment of nitrogen in the upper atmosphere, is radioactive with a half-life of about 5570 years. Libby assumed that the quantity of carbon-12 in the atmosphere has been constant throughout time, and that the quantity of radioactive carbon-14 achieved its steady state value long ago as a result of its creation and decomposition over millions of years. These assumptions led Libby to conclude that the ratio of carbon-14 to carbon-12 has been nearly constant for a long time. This constant, which we denote by R, has been determined experimentally.

Living cells absorb both carbon-12 and carbon-14 in the proportion in which they are present in the environment. Therefore the ratio of carbon-14 to carbon-12 in a living cell is always R. However, when the cell dies it ceases to absorb carbon, and the ratio of carbon-14 to carbon-12 decreases exponentially as the radioactive carbon-14 decays. This is the basis for the method of carbon dating, as illustrated in the next example.

Example 4.1.5

An archaeologist investigating the site of an ancient village finds a burial ground where the amount of carbon-14 present in individual remains is between 42 and 44% of the amount present in live individuals. Estimate the age of the village and the length of time for which it survived.

Solution

Let $Q = Q(t)$ be the quantity of carbon-14 in an individual set of remains t years after death, and let Q_0 be the quantity that would be present in live individuals. Since carbon-14 decays exponentially with half-life 5570 years, its decay constant is

$$k = \frac{\ln 2}{5570}.$$

Therefore

$$Q = Q_0 e^{-t(\ln 2)/5570}$$

if we choose our time scale so that $t_0 = 0$ is the time of death. If we know the present value of Q we can solve this equation for t, the number of years since death occurred. This yields

$$t = -5570\frac{\ln Q/Q_0}{\ln 2}.$$

It is given that $Q = 0.42Q_0$ in the remains of individuals who died first. Therefore these deaths occurred about

$$t_1 = -5570\frac{\ln .42}{\ln 2} \approx 6971$$

years ago. For the most recent deaths, $Q = .44Q_0$; hence, these deaths occurred about

$$t_2 = -5570\frac{\ln .44}{\ln 2} \approx 6597$$

years ago. Therefore it is reasonable to conclude that the village was founded about 7000 years ago, and lasted for about 400 years.

A Savings Program

Example 4.1.6

A person opens a savings account with an initial deposit of \$1000 and subsequently deposits \$50 per week. Find the value $Q(t)$ of the account at time $t > 0$, assuming that the bank pays 6% interest compounded continuously.

Solution

Observe that Q isn't continuous, since there are 52 discrete deposits per year of \$50 each. To construct a mathematical model for this problem in the form of a differential equation, we make the simplifying assumption that the deposits are made continuously at a rate of \$2600 per year. This is essential, since solutions of differential equations are continuous functions. With this assumption, Q increases continuously at the rate

$$Q' = 2600 + 0.06Q$$

and therefore Q satisfies the differential equation

$$Q' - .06Q = 2600. \tag{9.1.13}$$

(Of course, we must recognize that the solution of this equation is an approximation to the true value of Q at any given time. We'll discuss this further below.) Since $e^{.06t}$ is a solution of the complementary equation, the solutions of Equation 9.1.13 are of the form $Q = ue^{.06t}$, where $u'e^{.06t} = 2600$. Hence, $u' = 2600e^{-.06t}$,

$$u = -\frac{2600}{.06}e^{-0.06t} + c$$

and

$$Q = ue^{.06t} = -\frac{2600}{.06} + ce^{.06t}. \tag{9.1.14}$$

Setting $t = 0$ and $Q = 1000$ here yields

$$c = 1000 + \frac{2600}{0.06},$$

and substituting this into Equation 9.1.14 yields

$$Q = 1000e^{.06t} + \frac{2600}{.06}(e^{.06t} - 1) \tag{9.1.15}$$

where the first term is the value due to the initial deposit and the second is due to the subsequent weekly deposits.

Mathematical models must be tested for validity by comparing predictions based on them with the actual outcome of experiments. Example 6 is unusual in that we can compute the exact value of the account at any specified time and compare it with the approximate value predicted by Equation 9.1.15 (See **Exercise 4.1.21**). The following table gives a comparison for a ten year period. Each exact answer corresponds to the time of the year-end deposit, and each year is assumed to have exactly 52 weeks.

Table 4.1.3

Year	Approximate Value of Q (Example 4.1.6)	Exact Value of P (Exercise 4.1.21)	Error $Q-P$	Percentage Error $(Q-P)/P$
1	$3741.42	$3739.87	$1.55	.0413
2	6652.36	6649.17	3.19	.0479
3	9743.30	9738.37	4.93	.0506
4	13,025.38	13,018.60	6.78	.0521
5	16,510.41	16,501.66	8.75	.0530
6	20,210.94	20,200.11	10.83	.0536
7	24,140.30	24,127.25	13.05	.0541
8	28,312.63	28,297.23	15.40	.0544
9	32,742.97	32,725.07	17.90	.0547
10	37,447.27	37,426.72	20.55	.0549

This page titled 9.1: Growth and Decay is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **4.1: Growth and Decay** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

9.1.1: Growth and Decay (Exercises)

Q4.1.1

1. The half-life of a radioactive substance is 3200 years. Find the quantity $Q(t)$ of the substance left at time $t > 0$ if $Q(0) = 20$ g.

2. The half-life of a radioactive substance is 2 days. Find the time required for a given amount of the material to decay to 1/10 of its original mass.

3. A radioactive material loses 25% of its mass in 10 minutes. What is its half-life?

4. A tree contains a known percentage p_0 of a radioactive substance with half-life τ. When the tree dies the substance decays and isn't replaced. If the percentage of the substance in the fossilized remains of such a tree is found to be p_1, how long has the tree been dead?

5. If t_p and t_q are the times required for a radioactive material to decay to $1/p$ and $1/q$ times its original mass (respectively), how are t_p and t_q related?

6. Find the decay constant k for a radioactive substance, given that the mass of the substance is Q_1 at time t_1 and Q_2 at time t_2.

7. A process creates a radioactive substance at the rate of 2 g/hr and the substance decays at a rate proportional to its mass, with constant of proportionality $k = .1(\text{hr})^{-1}$. If $Q(t)$ is the mass of the substance at time t, find $\lim_{t\to\infty} Q(t)$.

8. A bank pays interest continuously at the rate of 6%. How long does it take for a deposit of Q_0 to grow in value to $2Q_0$?

9. At what rate of interest, compounded continuously, will a bank deposit double in value in 8 years?

10. A savings account pays 5% per annum interest compounded continuously. The initial deposit is Q_0 dollars. Assume that there are no subsequent withdrawals or deposits.

a. How long will it take for the value of the account to triple?
b. What is Q_0 if the value of the account after 10 years is $100,000 dollars?

11. A candymaker makes 500 pounds of candy per week, while his large family eats the candy at a rate equal to $Q(t)/10$ pounds per week, where $Q(t)$ is the amount of candy present at time t.

a. Find $Q(t)$ for $t > 0$ if the candymaker has 250 pounds of candy at $t = 0$.
b. Find $\lim_{t\to\infty} Q(t)$.

12. Suppose a substance decays at a yearly rate equal to half the square of the mass of the substance present. If we start with 50 g of the substance, how long will it be until only 25 g remain?

13. A super bread dough increases in volume at a rate proportional to the volume V present. If V increases by a factor of 10 in 2 hours and $V(0) = V_0$, find V at any time t. How long will it take for V to increase to $100V_0$?

14. A radioactive substance decays at a rate proportional to the amount present, and half the original quantity Q_0 is left after 1500 years. In how many years would the original amount be reduced to $3Q_0/4$? How much will be left after 2000 years?

15. A wizard creates gold continuously at the rate of 1 ounce per hour, but an assistant steals it continuously at the rate of 5% of however much is there per hour. Let $W(t)$ be the number of ounces that the wizard has at time t. Find $W(t)$ and $\lim_{t\to\infty} W(t)$ if $W(0) = 1$.

16. A process creates a radioactive substance at the rate of 1 g/hr, and the substance decays at an hourly rate equal to 1/10 of the mass present (expressed in grams). Assuming that there are initially 20 g, find the mass $S(t)$ of the substance present at time t, and find $\lim_{t\to\infty} S(t)$.

17. A tank is empty at $t = 0$. Water is added to the tank at the rate of 10 gal/min, but it leaks out at a rate (in gallons per minute) equal to the number of gallons in the tank. What is the smallest capacity the tank can have if this process is to continue forever?

18. A person deposits $25,000 in a bank that pays 5% per year interest, compounded continuously. The person continuously withdraws from the account at the rate of $750 per year. Find $V(t)$, the value of the account at time t after the initial deposit.

19. A person has a fortune that grows at rate proportional to the square root of its worth. Find the worth W of the fortune as a function of t if it was \$1 million 6 months ago and is \$4 million today.

20. Let $p = p(t)$ be the quantity of a product present at time t. The product is manufactured continuously at a rate proportional to p, with proportionality constant 1/2, and it is consumed continuously at a rate proportional to p^2, with proportionality constant 1/8. Find $p(t)$ if $p(0) = 100$.

21.

a. In the situation of Example 4.1.6 find the exact value P(t) of the person's account after t years, where t is an integer. Assume that each year has exactly 52 weeks, and include the year-end deposit in the computation.

HINT: At time t the initial \$1000 has been on deposit for t years. There have been $52t$ deposits of \$50 each. The first \$50 has been on deposit for $t - 1/52$ years, the second for $t - 2/52$ years ... in general, the j th \$50 has been on deposit for $t - j/52$ years ($1 \leq j \leq 52t$). Find the present value of each \$50 deposit assuming 6% interest compounded continuously, and use the formula

$$1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x} (x \neq 1)$$

to find their total value.

b. Let

$$p(t) = \frac{Q(t) - P(t)}{P(t)}$$

be the relative error after t years. Find

$$p(\infty) = \lim_{t \to \infty} p(t).$$

22. A homebuyer borrows P_0 dollars at an annual interest rate r, agreeing to repay the loan with equal monthly payments of M dollars per month over N years.

a. Derive a differential equation for the loan principal (amount that the homebuyer owes) $P(t)$ at time $t > 0$, making the simplifying assumption that the homebuyer repays the loan continuously rather than in discrete steps. (See Example 4.1.6.)

b. Solve the equation derived in (a).

c. Use the result of (b) to determine an approximate value for M assuming that each year has exactly 12 months of equal length.

d. It can be shown that the exact value of M is given by

$$M = \frac{rP_0}{12} \left(1 - (1 + r/12)^{-12N}\right)^{-1}.$$

Compare the value of M obtained from the answer in (c) to the exact value if (i) $P_0 = \$50,000$, \(r=7{1\over2}\)%, $N = 20$ (ii) $P_0 = \$150,000$, $r = 9.0\%$, $N = 30$.

23. Assume that the homebuyer of **Exercise 4.1.22** elects to repay the loan continuously at the rate of αM dollars per month, where α is a constant greater than 1. (This is called **accelerated payment**.)

a. Determine the time $T(\alpha)$ when the loan will be paid off and the amount $S(\alpha)$ that the homebuyer will save.
b. Suppose $P_0 = \$50,000$, $r = 8\%$, and $N = 15$. Compute the savings realized by accelerated payments with $\alpha = 1.05, 1.10$, and 1.15.

24. A benefactor wishes to establish a trust fund to pay a researcher's salary for T years. The salary is to start at S_0 dollars per year and increase at a fractional rate of a per year. Find the amount of money P_0 that the benefactor must deposit in a trust fund paying interest at a rate r per year. Assume that the researcher's salary is paid continuously, the interest is compounded continuously, and the salary increases are granted continuously.

25. A radioactive substance with decay constant k is produced at the rate of

$$\frac{at}{1 + btQ(t)}$$

units of mass per unit time, where a and b are positive constants and $Q(t)$ is the mass of the substance present at time t; thus, the rate of production is small at the start and tends to slow when Q is large.

a. Set up a differential equation for Q.
b. Choose your own positive values for a, b, k, and $Q_0 = Q(0)$. Use a numerical method to discover what happens to $Q(t)$ as $t \to \infty$. (Be precise, expressing your conclusions in terms of a, b, k. However, no proof is required.)

26. Follow the instructions of **Exercise 4.1.25**, assuming that the substance is produced at the rate of $at/(1 + bt(Q(t))^2)$ units of mass per unit of time.

27. Follow the instructions of **Exercise 4.1.25**, assuming that the substance is produced at the rate of $at/(1 + bt)$ units of mass per unit of time.

This page titled 9.1.1: Growth and Decay (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **4.1E: Growth and Decay (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

9.2: Cooling and Mixing

Newton's Law of Cooling

Newton's law of cooling states that if an object with temperature $T(t)$ at time t is in a medium with temperature $T_m(t)$, the rate of change of T at time t is proportional to $T(t)-T_m(t)$; thus, T satisfies a differential equation of the form

$$T' = -k(T - T_m). \tag{9.2.1}$$

Here $k > 0$, since the temperature of the object must decrease if $T > T_m$, or increase if $T < T_m$. We'll call k the **temperature decay constant of the medium**.

For simplicity, in this section we'll assume that the medium is maintained at a constant temperature T_m. This is another example of building a simple mathematical model for a physical phenomenon. Like most mathematical models it has its limitations. For example, it is reasonable to assume that the temperature of a room remains approximately constant if the cooling object is a cup of coffee, but perhaps not if it is a huge cauldron of molten metal. (For more on this see **Exercise 4.2.17**.)

To solve Equation 9.2.1, we rewrite it as

$$T' + kT = kT_m.$$

Since e^{-kt} is a solution of the complementary equation, the solutions of this equation are of the form $T = ue^{-kt}$, where $u'e^{-kt} = kT_m$, so $u' = kT_m e^{kt}$. Hence,

$$u = T_m e^{kt} + c,$$

so

$$T = ue^{-kt} = T_m + ce^{-kt}.$$

If $T(0) = T_0$, setting $t = 0$ here yields $c = T_0 - T_m$, so

$$T = T_m + (T_0 - T_m)e^{-kt}. \tag{9.2.2}$$

Note that $T - T_m$ decays exponentially, with decay constant k.

Example 4.2.1

A ceramic insulator is baked at $400°$C and cooled in a room in which the temperature is $25°$C. After 4 minutes the temperature of the insulator is $200°$C. What is its temperature after 8 minutes?

Solution

Here $T_0 = 400$ and $T_m = 25$, so Equation 9.2.2 becomes

$$T = 25 + 375e^{-kt}. \tag{9.2.3}$$

We determine k from the stated condition that $T(4) = 200$; that is,

$$200 = 25 + 375e^{-4k};$$

hence,

$$e^{-4k} = \frac{175}{375} = \frac{7}{15}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{4}\ln\frac{7}{15} = \frac{1}{4}\ln\frac{15}{7}.$$

Substituting this into Equation 9.2.3 yields

$$T = 25 + 375e^{-\frac{t}{4}\ln\frac{15}{7}}$$

(Figure 4.2.1). Therefore the temperature of the insulator after 8 minutes is

$$\begin{aligned} T(8) &= 25 + 375e^{-2\ln\frac{15}{7}} \\ &= 25 + 375\left(\tfrac{7}{15}\right)^2 \approx 107^\circ\text{C}. \end{aligned}$$

Example 4.2.2

An object with temperature 72°F is placed outside, where the temperature is -20°F. At 11:05 the temperature of the object is 60° F and at 11:07 its temperature is 50°F. At what time was the object placed outside?

Solution

Let $T(t)$ be the temperature of the object at time t. For convenience, we choose the origin $t_0 = 0$ of the time scale to be 11:05 so that $T_0 = 60$. We must determine the time τ when $T(\tau) = 72$. Substituting $T_0 = 60$ and $T_m = -20$ into Equation 9.2.2 yields

$$T = -20 + (60 - (-20))\, e^{-kt}$$

or

$$T = -20 + 80e^{-kt}. \tag{9.2.4}$$

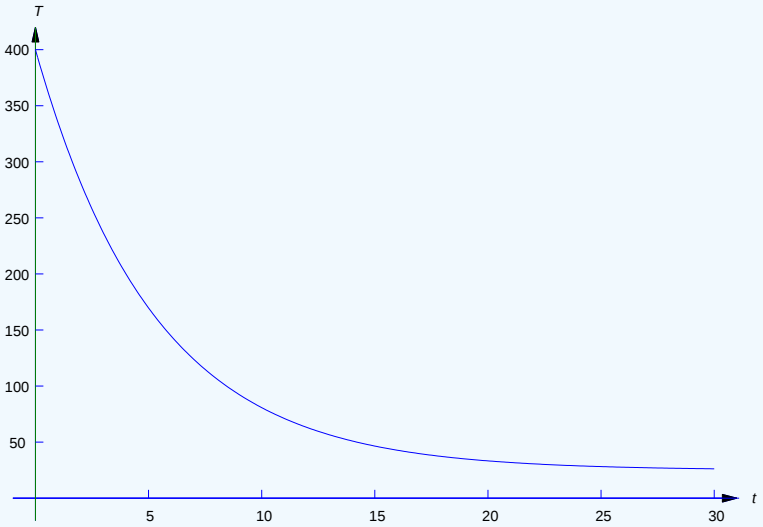


Figure 4.2.1 : $T = 25 + 375e^{-(t/4)\ln 15/7}$

We obtain k from the stated condition that the temperature of the object is 50°F at 11:07. Since 11:07 is $t = 2$ on our time scale, we can determine k by substituting $T = 50$ and $t = 2$ into Equation 9.2.4 to obtain

$$50 = -20 + 80e^{-2k}$$

This is shown in Figure 4.2.2 .

Hence,

$$e^{-2k} = \frac{70}{80} = \frac{7}{8}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{2}\ln\frac{7}{8} = \frac{1}{2}\ln\frac{8}{7}.$$

Substituting this into Equation 9.2.4 yields

$$T = -20 + 80e^{-\frac{t}{2}\ln\frac{8}{7}},$$

and the condition $T(\tau) = 72$ implies that

$$72 = -20 + 80e^{-\frac{\tau}{2}\ln\frac{8}{7}};$$

hence,

$$e^{-\frac{\tau}{2}\ln\frac{8}{7}} = \frac{92}{80} = \frac{23}{20}.$$

Taking logarithms and solving for τ yields

$$\tau = -\frac{2\ln\frac{23}{20}}{\ln\frac{8}{7}} \approx -2.09 \text{ min}.$$

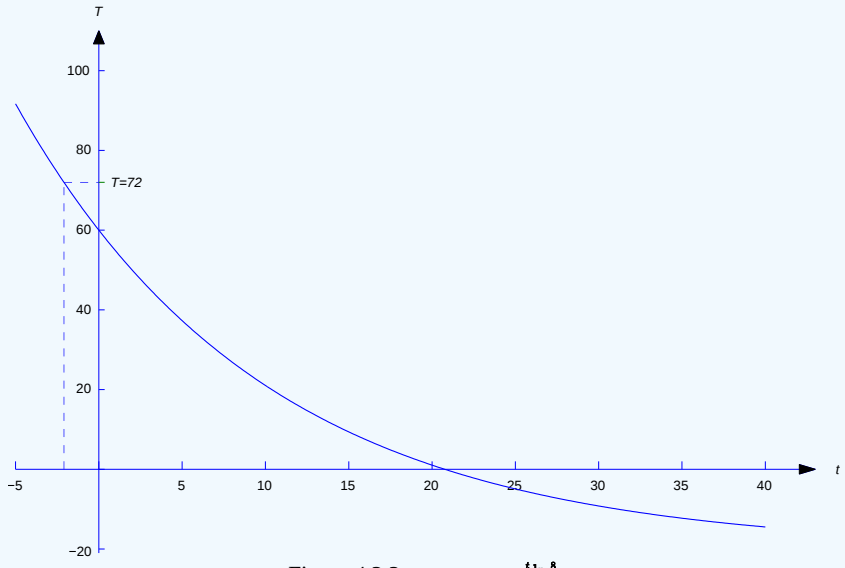


Figure 4.2.2 : $-20 + 80e^{-\frac{t}{2}\ln\frac{8}{7}}$

Therefore the object was placed outside about 2 minutes and 5 seconds before 11:05; that is, at 11:02:55.

Mixing Problems

In the next two examples a saltwater solution with a given concentration (weight of salt per unit volume of solution) is added at a specified rate to a tank that initially contains saltwater with a different concentration. The problem is to determine the quantity of salt in the tank as a function of time. This is an example of a **mixing problem**. To construct a tractable mathematical model for mixing problems we assume in our examples (and most exercises) that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Exercises 4.2.22 and 4.2.23 deal with situations where this isn't so, but the distribution of salt becomes approximately uniform as $t \to \infty$.

Example 4.2.3

A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains 1/2 pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate (Figure 4.2.3).

a. Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $Q(t)$.
b. Find $\lim_{t\to\infty} Q(t)$.

Solution a

To find a differential equation for Q, we must use the given information to derive an expression for Q'. But Q' is the rate of change of the quantity of salt in the tank changes with respect to time; thus, if **rate in** denotes the rate at which salt enters the tank and **rate out** denotes the rate by which it leaves, then

$$Q' = \text{rate in} - \text{rate out}. \tag{9.2.5}$$

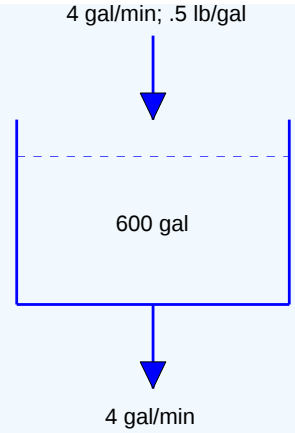


Figure 4.2.3 : A mixing problem

The rate in is

$$\left(\frac{1}{2}\ \text{lb/gal}\right) \times (4\ \text{gal/min}) = 2\ \text{lb/min}.$$

Determining the rate out requires a little more thought. We're removing 4 gallons of the mixture per minute, and there are always 600 gallons in the tank; that is, we are removing $1/150$ of the mixture per minute. Since the salt is evenly distributed in the mixture, we are also removing $1/150$ of the salt per minute. Therefore, if there are $Q(t)$ pounds of salt in the tank at time t, the rate out at any time t is $Q(t)/150$. Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned}\text{rate out} &= (\text{concentration}) \times (\text{rate of flow out})\\ &= (\text{lb/gal}) \times (\text{gal/min})\\ &= \frac{Q(t)}{600} \times 4\\ &= \frac{Q(t)}{150}.\end{aligned}$$

We can now write Equation 9.2.5 as

$$Q' = 2 - \frac{Q}{150}.$$

This first order equation can be rewritten as

$$Q' + \frac{Q}{150} = 2.$$

Since $e^{-t/150}$ is a solution of the complementary equation, the solutions of this equation are of the form $Q = ue^{-t/150}$, where $u'e^{-t/150} = 2$, so $u' = 2e^{t/150}$. Hence,

$$u = 300e^{t/150} + c,$$

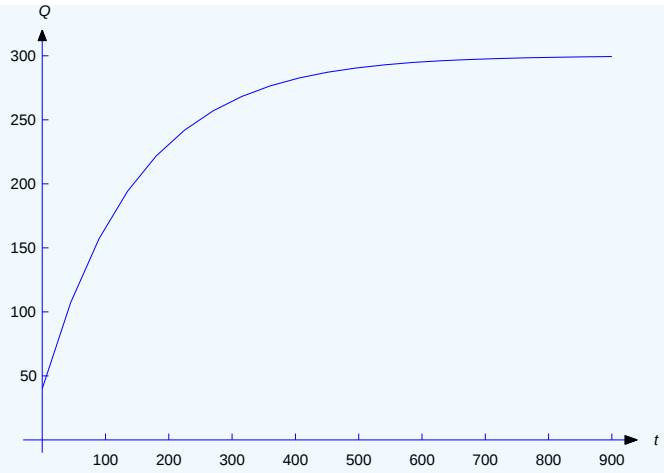


Figure 4.2.4 : $Q = 300 - 260e^{-t/150}$

so

$$Q = ue^{-t/150} = 300 + ce^{-t/150} \tag{9.2.6}$$

(Figure 4.2.4). Since $Q(0) = 40$, $c = -260$; therefore,

$$Q = 300 - 260e^{-t/150}.$$

Solution b

From Equation 9.2.6, we see that that $\lim_{t\to\infty} Q(t) = 300$ for any value of $Q(0)$. This is intuitively reasonable, since the incoming solution contains 1/2 pound of salt per gallon and there are always 600 gallons of water in the tank.

Example 4.2.4

A 500-liter tank initially contains 10 g of salt dissolved in 200 liters of water. Starting at $t_0 = 0$, water that contains 1/4 g of salt per liter is poured into the tank at the rate of 4 liters/min and the mixture is drained from the tank at the rate of 2 liters/min (Figure 4.2.5). Find a differential equation for the quantity $Q(t)$ of salt in the tank at time t prior to the time when the tank overflows and find the concentration $K(t)$ (g/liter) of salt in the tank at any such time.

Solution

We first determine the amount $W(t)$ of solution in the tank at any time t prior to overflow. Since $W(0) = 200$ and we are adding 4 liters/min while removing only 2 liters/min, there's a net gain of 2 liters/min in the tank; therefore,

$$W(t) = 2t + 200.$$

Since $W(150) = 500$ liters (capacity of the tank), this formula is valid for $0 \le t \le 150$.

Now let $Q(t)$ be the number of grams of salt in the tank at time t, where $0 \le t \le 150$. As in Example 4.2.3

$$Q' = \text{rate in} - \text{rate out}. \tag{9.2.7}$$

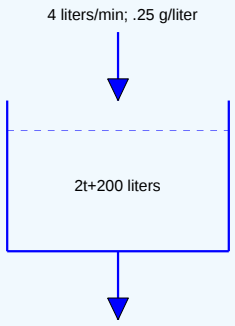


Figure 4.2.5 : Another mixing problem

https://math.libretexts.org/@go/page/121648

The rate in is

$$\left(\frac{1}{4}\ \text{g/liter}\right) \times (4\ \text{liters/min}) = 1\ \text{g/min}. \tag{9.2.8}$$

To determine the rate out, we observe that since the mixture is being removed from the tank at the constant rate of 2 liters/min and there are $2t + 200$ liters in the tank at time t, the fraction of the mixture being removed per minute at time t is

$$\frac{2}{2t+200} = \frac{1}{t+100}.$$

We're removing this same fraction of the salt per minute. Therefore, since there are $Q(t)$ grams of salt in the tank at time t,

$$\text{rate out} = \frac{Q(t)}{t+100}. \tag{9.2.9}$$

Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned}\text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) \\ &= (\text{g/liter}) \times (\text{liters/min}) \\ &= \frac{Q(t)}{2t+200} \times 2 = \frac{Q(t)}{t+100}.\end{aligned}$$

Substituting Equation 9.2.8 and Equation 9.2.9 into Equation 9.2.7 yields

$$Q' = 1 - \frac{Q}{t+100}, \quad \text{so} \quad Q' + \frac{1}{t+100}Q = 1. \tag{9.2.10}$$

By separation of variables, $1/(t+100)$ is a solution of the complementary equation, so the solutions of Equation 9.2.10 are of the form

$$Q = \frac{u}{t+100}, \quad \text{where} \quad \frac{u'}{t+100} = 1, \quad \text{so} \quad u' = t + 100.$$

Hence,

$$u = \frac{(t+100)^2}{2} + c. \tag{9.2.11}$$

Since $Q(0) = 10$ and $u = (t+100)Q$, Equation 9.2.11 implies that

$$(100)(10) = \frac{(100)^2}{2} + c,$$

so

$$c = 100(10) - \frac{(100)^2}{2} = -4000$$

and therefore

$$u = \frac{(t+100)^2}{2} - 4000.$$

Hence,

$$Q = \frac{u}{t+200} = \frac{t+100}{2} - \frac{4000}{t+100}.$$

Now let $K(t)$ be the concentration of salt at time t. Then

$$K(t) = \frac{1}{4} - \frac{2000}{(t+100)^2}$$

This is shown in Figure 4.2.6 .

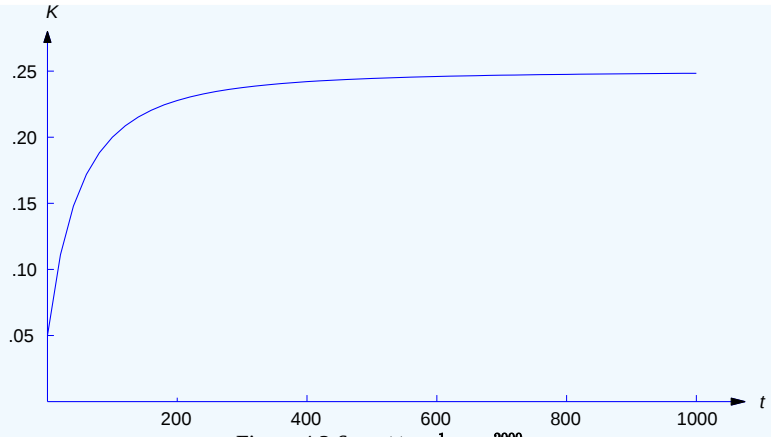


Figure 4.2.6 : $K(t) = \frac{1}{4} - \frac{2000}{(t+100)^2}$

This page titled 9.2: Cooling and Mixing is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **4.2: Cooling and Mixing** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

9.2.1: Cooling and Mixing (Exercises)

Q4.2.1

1. A thermometer is moved from a room where the temperature is 70°F to a freezer where the temperature is $12°F$. After 30 seconds the thermometer reads 40°F. What does it read after 2 minutes?

2. A fluid initially at 100°C is placed outside on a day when the temperature is -10°C, and the temperature of the fluid drops 20°C in one minute. Find the temperature $T(t)$ of the fluid for $t > 0$.

3. At 12:00 pm a thermometer reading 10°F is placed in a room where the temperature is 70°F. It reads 56° when it is placed outside, where the temperature is 5°F, at 12:03. What does it read at 12:05 pm?

4. A thermometer initially reading 212°F is placed in a room where the temperature is 70°F. After 2 minutes the thermometer reads 125°F.

   a. What does the thermometer read after 4 minutes?
   b. When will the thermometer read 72°F?
   c. When will the thermometer read 69°F?

5. An object with initial temperature 150°C is placed outside, where the temperature is 35°C. Its temperatures at 12:15 and 12:20 are 120°C and 90°C, respectively.

   a. At what time was the object placed outside?
   b. When will its temperature be 40°C?

6. An object is placed in a room where the temperature is 20°C. The temperature of the object drops by 5°C in 4 minutes and by 7°C in 8 minutes. What was the temperature of the object when it was initially placed in the room?

7. A cup of boiling water is placed outside at 1:00 pm. One minute later the temperature of the water is 152°F. After another minute its temperature is 112°F. What is the outside temperature?

8. A tank initially contains 40 gallons of pure water. A solution with 1 gram of salt per gallon of water is added to the tank at 3 gal/min, and the resulting solution drains out at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.

9. A tank initially contains a solution of 10 pounds of salt in 60 gallons of water. Water with $1/2$ pound of salt per gallon is added to the tank at 6 gal/min, and the resulting solution leaves at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.

10. A tank initially contains 100 liters of a salt solution with a concentration of $.1$ g/liter. A solution with a salt concentration of $.3$ g/liter is added to the tank at 5 liters/min, and the resulting mixture is drained out at the same rate. Find the concentration $K(t)$ of salt in the tank as a function of t.

11. A 200 gallon tank initially contains 100 gallons of water with 20 pounds of salt. A salt solution with $1/4$ pound of salt per gallon is added to the tank at 4 gal/min, and the resulting mixture is drained out at 2 gal/min. Find the quantity of salt in the tank as it is about to overflow.

12. Suppose water is added to a tank at 10 gal/min, but leaks out at the rate of $1/5$ gal/min for each gallon in the tank. What is the smallest capacity the tank can have if the process is to continue indefinitely?

13. A chemical reaction in a laboratory with volume V (in ft^3) produces q_1 ft^3/min of a noxious gas as a byproduct. The gas is dangerous at concentrations greater than $\bar{c}$, but harmless at concentrations $\leq \bar{c}$. Intake fans at one end of the laboratory pull in fresh air at the rate of q_2 ft^3/min and exhaust fans at the other end exhaust the mixture of gas and air from the laboratory at the same rate. Assuming that the gas is always uniformly distributed in the room and its initial concentration c_0 is at a safe level, find the smallest value of q_2 required to maintain safe conditions in the laboratory for all time.

14. A 1200-gallon tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is added to the tank at the rate of 6 gal/min and the resulting mixture is drained from the tank at 4 gal/min. Find the quantity $Q(t)$ of salt in the tank at any time $t > 0$ prior to overflow.

15. Tank T_1 initially contain 50 gallons of pure water. Starting at $t_0 = 0$, water that contains 1 pound of salt per gallon is poured into T_1 at the rate of 2 gal/min. The mixture is drained from T_1 at the same rate into a second tank T_2, which initially contains 50 gallons of pure water. Also starting at $t_0 = 0$, a mixture from another source that contains 2 pounds of salt per gallon is poured into T_2 at the rate of 2 gal/min. The mixture is drained from T_2 at the rate of 4 gal/min.

a. Find a differential equation for the quantity $Q(t)$ of salt in tank T_2 at time $t > 0$.
b. Solve the equation derived in (a) to determine $Q(t)$.
c. Find $\lim_{t\to\infty} Q(t)$.

16. Suppose an object with initial temperature T_0 is placed in a sealed container, which is in turn placed in a medium with temperature T_m. Let the initial temperature of the container be S_0. Assume that the temperature of the object does not affect the temperature of the container, which in turn does not affect the temperature of the medium. (These assumptions are reasonable, for example, if the object is a cup of coffee, the container is a house, and the medium is the atmosphere.)

a. Assuming that the container and the medium have distinct temperature decay constants k and k_m respectively, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t.
b. Assuming that the container and the medium have the same temperature decay constant k, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t.
c. Find $\lim_{t\to\infty} S(t)$ and $\lim_{t\to\infty} T(t)$.

17. In our previous examples and exercises concerning Newton's law of cooling we assumed that the temperature of the medium remains constant. This model is adequate if the heat lost or gained by the object is insignificant compared to the heat required to cause an appreciable change in the temperature of the medium. If this isn't so, we must use a model that accounts for the heat exchanged between the object and the medium. Let $T = T(t)$ and $T_m = T_m(t)$ be the temperatures of the object and the medium, respectively, and let T_0 and T_{m0} be their initial values. Again, we assume that T and T_m are related by Newton's law of cooling,

$$T' = -k(T - T_m). \tag{A}$$

We also assume that the change in heat of the object as its temperature changes from T_0 to T is $a(T - T_0)$ and that the change in heat of the medium as its temperature changes from T_{m0} to T_m is $a_m(T_m - T_{m0})$, where a and a_m are positive constants depending upon the masses and thermal properties of the object and medium, respectively. If we assume that the total heat of the system consisting of the object and the medium remains constant (that is, energy is conserved), then

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0. \tag{B}$$

a. Equation (A) involves two unknown functions T and T_m. Use (A) and (B) to derive a differential equation involving only T.
b. Find $T(t)$ and $T_m(t)$ for $t > 0$.
c. Find $\lim_{t\to\infty} T(t)$ and $\lim_{t\to\infty} T_m(t)$.

18. Control mechanisms allow fluid to flow into a tank at a rate proportional to the volume V of fluid in the tank, and to flow out at a rate proportional to V^2. Suppose $V(0) = V_0$ and the constants of proportionality are a and b, respectively. Find $V(t)$ for $t > 0$ and find $\lim_{t\to\infty} V(t)$.

19. Identical tanks T_1 and T_2 initially contain W gallons each of pure water. Starting at $t_0 = 0$, a salt solution with constant concentration c is pumped into T_1 at r gal/min and drained from T_1 into T_2 at the same rate. The resulting mixture in T_2 is also drained at the same rate. Find the concentrations $c_1(t)$ and $c_2(t)$ in tanks T_1 and T_2 for $t > 0$.

20. An infinite sequence of identical tanks T_1, T_2, …, T_n, …, initially contain W gallons each of pure water. They are hooked together so that fluid drains from T_n into T_{n+1} $(n = 1, 2, \cdots)$. A salt solution is circulated through the tanks so that it enters and leaves each tank at the constant rate of r gal/min. The solution has a concentration of c pounds of salt per gallon when it enters T_1.

a. Find the concentration $c_n(t)$ in tank T_n for $t > 0$.
b. Find $\lim_{t\to\infty} c_n(t)$ for each n.

21. Tanks T_1 and T_2 have capacities W_1 and W_2 liters, respectively. Initially they are both full of dye solutions with concentrations c_1 and c_2 grams per liter. Starting at $t_0 = 0$, the solution from T_1 is pumped into T_2 at a rate of r liters per minute, and the solution from T_2 is pumped into T_1 at the same rate.

a. Find the concentrations $c_1(t)$ and $c_2(t)$ of the dye in T_1 and T_2 for $t > 0$.
b. Find $\lim_{t\to\infty} c_1(t)$ and $\lim_{t\to\infty} c_2(t)$.

22. Consider the mixing problem of Example 4.2.3, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \to \infty$. In this case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{150}Q = 2$$

where $\lim_{t\to\infty} a(t) = 1$.

a. Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t\to\infty} Q(t)$?.
b. Use numerical methods to confirm your guess in the these cases:

$$\text{(i) } a(t) = t/(1+t) \quad \text{(ii) } a(t) = 1 - e^{-t^2} \quad \text{(iii) } a(t) = 1 - \sin(e^{-t}).$$

23. Consider the mixing problem of Example 4.2.4 in a tank with infinite capacity, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \to \infty$. In this case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{t+100}Q = 1$$

where $\lim_{t\to\infty} a(t) = 1$.

a. Let $K(t)$ be the concentration of salt at time t. Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t\to\infty} K(t)$?
b. Use numerical methods to confirm your guess in the these cases:

$$\text{(i) } a(t) = t/(1+t) \quad \text{(ii) } a(t) = 1 - e^{-t^2} \quad \text{(iii) } a(t) = 1 + \sin(e^{-t}).$$

This page titled 9.2.1: Cooling and Mixing (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **4.2E: Cooling and Mixing (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

9.3: Elementary Mechanics

Newton's Second Law of Motion

In this section we consider an object with constant mass m moving along a line under a force F. Let $y = y(t)$ be the displacement of the object from a reference point on the line at time t, and let $v = v(t)$ and $a = a(t)$ be the velocity and acceleration of the object at time t. Thus, $v = y'$ and $a = v' = y''$, where the prime denotes differentiation with respect to t. Newton's second law of motion asserts that the force F and the acceleration a are related by the equation

$$F = ma. \tag{9.3.1}$$

Note: Units

In applications there are three main sets of units in use for length, mass, force, and time: the cgs, mks, and British systems. All three use the second as the unit of time. Table 4.3.1 shows the other units. Consistent with Equation 9.3.1, the unit of force in each system is defined to be the force required to impart an acceleration of (one unit of length)$/s^2$ to one unit of mass.

Table 4.3.1

Set	Length	Force	Mass
cgs	centimeter (cm)	dyne (d)	gram (g)
mks	meter (m)	newton (N)	kilogram (kg)
British	foot (ft)	pound (lb)	slug (sl)

If we assume that Earth is a perfect sphere with constant mass density, Newton's law of gravitation (discussed later in this section) asserts that the force exerted on an object by Earth's gravitational field is proportional to the mass of the object and inversely proportional to the square of its distance from the center of Earth. However, if the object remains sufficiently close to Earth's surface, we may assume that the gravitational force is constant and equal to its value at the surface. The magnitude of this force is mg, where g is called the **acceleration due to gravity**. (To be completely accurate, g should be called the **magnitude of the acceleration due to gravity at Earth's surface**.) This quantity has been determined experimentally. Approximate values of g are

$$\begin{aligned} g &= 980 \text{ cm/s}^2 && \text{(cgs)} \\ g &= 9.8 \text{ m/s}^2 && \text{(mks)} \\ g &= 32 \text{ ft/s}^2 && \text{(British).} \end{aligned}$$

In general, the force F in Equation 9.3.1 may depend upon t, y, and y'. Since $a = y''$, Equation 9.3.1 can be written in the form

$$my'' = F(t, y, y'), \tag{9.3.2}$$

which is a second order equation. We'll consider this equation with restrictions on F later; however, since Chapter 2 dealt only with first order equations, we consider here only problems in which Equation 9.3.2 can be recast as a first order equation. This is possible if F does not depend on y, so Equation 9.3.2 is of the form

$$my'' = F(t, y').$$

Letting $v = y'$ and $v' = y''$ yields a first order equation for v:

$$mv' = F(t, v). \tag{9.3.3}$$

Solving this equation yields v as a function of t. If we know $y(t_0)$ for some time t_0, we can integrate v to obtain y as a function of t.

Equations of the form Equation 9.3.3 occur in problems involving motion through a resisting medium.

Motion Through a Resisting Medium Under Constant Gravitational Force

Now we consider an object moving vertically in some medium. We assume that the only forces acting on the object are gravity and resistance from the medium. We also assume that the motion takes place close to Earth's surface and take the upward direction to be positive, so the gravitational force can be assumed to have the constant value $-mg$. We'll see that, under reasonable assumptions on the resisting force, the velocity approaches a limit as $t \to \infty$. We call this limit the **terminal velocity**.

Example 4.3.1

An object with mass m moves under constant gravitational force through a medium that exerts a resistance with magnitude proportional to the speed of the object. (Recall that the speed of an object is $|v|$, the absolute value of its velocity v.) Find the velocity of the object as a function of t, and find the terminal velocity. Assume that the initial velocity is v_0.

Solution

The total force acting on the object is

$$F = -mg + F_1, \tag{9.3.4}$$

where $-mg$ is the force due to gravity and F_1 is the resisting force of the medium, which has magnitude $k|v|$, where k is a positive constant. If the object is moving downward ($v \le 0$), the resisting force is upward (Figure 4.3.1a), so

$$F_1 = k|v| = k(-v) = -kv.$$

On the other hand, if the object is moving upward ($v \ge 0$), the resisting force is downward (Figure 4.3.1b , so

$$F_1 = -k|v| = -kv.$$

Thus, Equation 9.3.4 can be written as

$$F = -mg - kv, \tag{9.3.5}$$

regardless of the sign of the velocity.

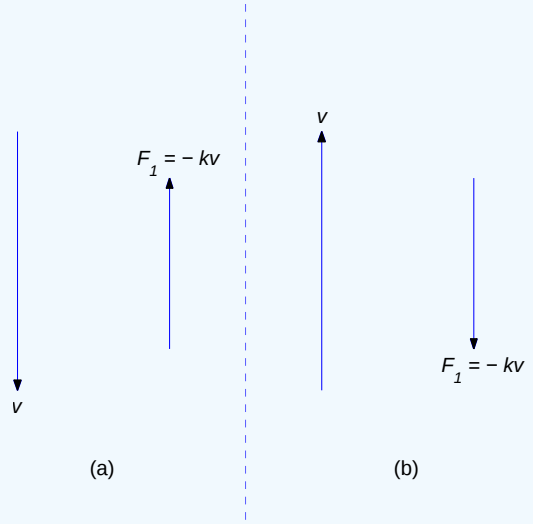


Figure 4.3.1 : Resistive forces

From Newton's second law of motion,

$$F = ma = mv',$$

so Equation 9.3.5 yields

$$mv' = -mg - kv,$$

or

$$v' + \frac{k}{m}v = -g. \tag{9.3.6}$$

Since $e^{-kt/m}$ is a solution of the complementary equation, the solutions of Equation 9.3.6 are of the form $v = ue^{-kt/m}$, where $u'e^{-kt/m} = -g$, so $u' = -ge^{kt/m}$. Hence,

$$u = -\frac{mg}{k}e^{kt/m} + c,$$

so

$$v = ue^{-kt/m} = -\frac{mg}{k} + ce^{-kt/m}. \tag{9.3.7}$$

Since $v(0) = v_0$,

$$v_0 = -\frac{mg}{k} + c,$$

so

$$c = v_0 + \frac{mg}{k}$$

and Equation 9.3.7 becomes

$$v = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}.$$

Letting $t \to \infty$ here shows that the terminal velocity is

$$\lim_{t\to\infty} v(t) = -\frac{mg}{k},$$

which is independent of the initial velocity v_0 (Figure 4.3.2).

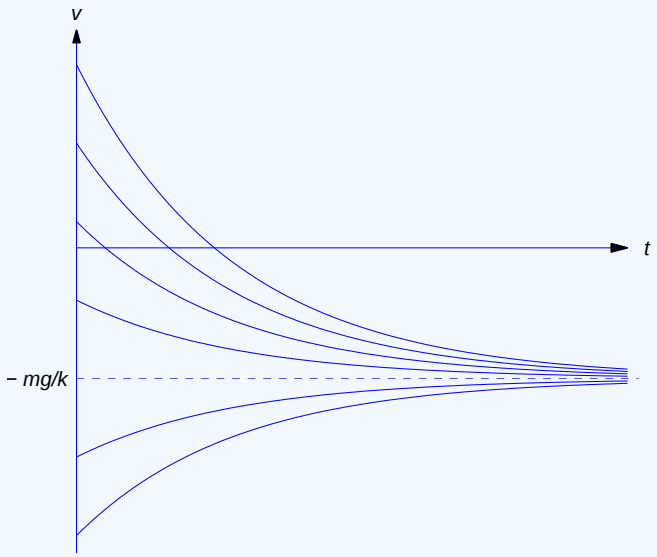


Figure 4.3.2 : Solutions of $mv' = -mg - kv$

Example 4.3.2 : Terminal Velocity

A 960-lb object is given an initial upward velocity of 60 ft/s near the surface of Earth. The atmosphere resists the motion with a force of 3 lb for each ft/s of speed. Assuming that the only other force acting on the object is constant gravity, find its velocity v as a function of t, and find its terminal velocity.

Solution

Since $mg = 960$ and $g = 32$, $m = 960/32 = 30$. The atmospheric resistance is $-3v$ lb if v is expressed in feet per second. Therefore

$$30v' = -960 - 3v,$$

which we rewrite as

$$v' + \frac{1}{10}v = -32.$$

Since $e^{-t/10}$ is a solution of the complementary equation, the solutions of this equation are of the form $v = ue^{-t/10}$, where $u'e^{-t/10} = -32$, so $u' = -32e^{t/10}$. Hence,

$$u = -320e^{t/10} + c,$$

so

$$v = ue^{-t/10} = -320 + ce^{-t/10}. \tag{9.3.8}$$

The initial velocity is 60 ft/s in the upward (positive) direction; hence, $v_0 = 60$. Substituting $t = 0$ and $v = 60$ in Equation 9.3.8 yields

$$60 = -320 + c,$$

so $c = 380$, and Equation 9.3.8 becomes

$$v = -320 + 380e^{-t/10} \text{ ft/s}$$

The terminal velocity is

$$\lim_{t\to\infty} v(t) = -320 \text{ ft/s}.$$

Example 4.3.3

A 10 kg mass is given an initial velocity $v_0 \le 0$ near Earth's surface. The only forces acting on it are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the resistance is 8 N if the speed is 2 m/s, find the velocity of the object as a function of t, and find the terminal velocity.

Solution

Since the object is falling, the resistance is in the upward (positive) direction. Hence,

$$mv' = -mg + kv^2, \tag{9.3.9}$$

where k is a constant. Since the magnitude of the resistance is 8 N when $v = 2$ m/s,

$$k(2^2) = 8,$$

so $k = 2$ N-s^2/m^2. Since $m = 10$ and $g = 9.8$, Equation 9.3.9 becomes

$$10v' = -98 + 2v^2 = 2(v^2 - 49). \tag{9.3.10}$$

If $v_0 = -7$, then $v \equiv -7$ for all $t \ge 0$. If $v_0 \ne -7$, we separate variables to obtain

$$\frac{1}{v^2 - 49}v' = \frac{1}{5}, \tag{9.3.11}$$

which is convenient for the required partial fraction expansion

$$\frac{1}{v^2 - 49} = \frac{1}{(v-7)(v+7)} = \frac{1}{14}\left[\frac{1}{v-7} - \frac{1}{v+7}\right]. \tag{9.3.12}$$

Substituting Equation 9.3.12 into Equation 9.3.11 yields

$$\frac{1}{14}\left[\frac{1}{v-7} - \frac{1}{v+7}\right]v' = \frac{1}{5},$$

so

$$\left[\frac{1}{v-7}-\frac{1}{v+7}\right]v'=\frac{14}{5}.$$

Integrating this yields

$$\ln|v-7|-\ln|v+7|=14t/5+k.$$

Therefore

$$\left|\frac{v-7}{v+7}\right|=e^k e^{14t/5}.$$

Since Theorem 2.3.1 implies that $(v-7)/(v+7)$ cannot change sign (why?), we can rewrite the last equation as

$$\frac{v-7}{v+7}=ce^{14t/5}, \tag{9.3.13}$$

which is an implicit solution of Equation 9.3.10. Solving this for v yields

$$v=-7\frac{c+e^{-14t/5}}{c-e^{-14t/5}}. \tag{9.3.14}$$

Since $v(0)=v_0$, it Equation 9.3.13 implies that

$$c=\frac{v_0-7}{v_0+7}.$$

Substituting this into Equation 9.3.14 and simplifying yields

$$v=-7\frac{v_0(1+e^{-14t/5})-7(1-e^{-14t/5})}{v_0(1-e^{-14t/5})-7(1+e^{-14t/5})}.$$

Since $v_0\le 0$, v is defined and negative for all $t>0$. The terminal velocity is

$$\lim_{t\to\infty}v(t)=-7\text{ m/s},$$

independent of v_0. More generally, it can be shown (**Exercise 4.3.11**) that if v is any solution of Equation 9.3.9 such that $v_0\le 0$ then

$$\lim_{t\to\infty}v(t)=-\sqrt{\frac{mg}{k}}.$$

This is demonstrated in Figure 4.3.3 .

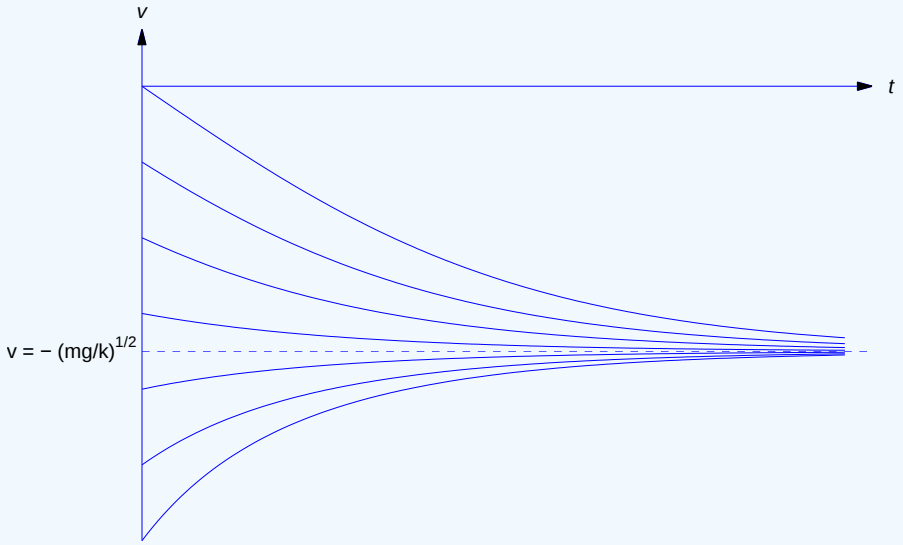


Figure 4.3.3 : Solutions of $mv'=-mg+kv^2,\ v(0)=v_0\le 0$

https://math.libretexts.org/@go/page/121650

Example 4.3.4

A 10-kg mass is launched vertically upward from Earth's surface with an initial velocity of v_0 m/s. The only forces acting on the mass are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the atmospheric resistance is 8 N if the speed is 2 m/s, find the time T required for the mass to reach maximum altitude.

Solution

The mass will climb while $v > 0$ and reach its maximum altitude when $v = 0$. Therefore $v > 0$ for $0 \le t < T$ and $v(T) = 0$; therefore, we replace Equation 9.3.10 by

$$10v' = -98 - 2v^2. \tag{9.3.15}$$

Separating variables yields

$$\frac{5}{v^2 + 49}v' = -1,$$

and integrating this yields

$$\frac{5}{7}\tan^{-1}\frac{v}{7} = -t + c.$$

(Recall that $\tan^{-1} u$ is the number θ such that $-\pi/2 < \theta < \pi/2$ and $\tan\theta = u$.) Since $v(0) = v_0$,

$$c = \frac{5}{7}\tan^{-1}\frac{v_0}{7},$$

so v is defined implicitly by

$$\frac{5}{7}\tan^{-1}\frac{v}{7} = -t + \frac{5}{7}\tan^{-1}\frac{v_0}{7}, \quad 0 \le t \le T. \tag{9.3.16}$$

Solving this for v yields

$$v = 7\tan\left(-\frac{7t}{5} + \tan^{-1}\frac{v_0}{7}\right). \tag{9.3.17}$$

Using the identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

with $A = \tan^{-1}(v_0/7)$ and $B = 7t/5$, and noting that $\tan(\tan^{-1}\theta) = \theta$, we can simplify Equation 9.3.17 to

$$v = 7\frac{v_0 - 7\tan(7t/5)}{7 + v_0\tan(7t/5)}.$$

Since $v(T) = 0$ and $\tan^{-1}(0) = 0$, Equation 9.3.16 implies that

$$-T + \frac{5}{7}\tan^{-1}\frac{v_0}{7} = 0.$$

Therefore

$$T = \frac{5}{7}\tan^{-1}\frac{v_0}{7}.$$

Since $\tan^{-1}(v_0/7) < \pi/2$ for all v_0, the time required for the mass to reach its maximum altitude is less than

$$\frac{5\pi}{14} \approx 1.122 \text{ s}$$

regardless of the initial velocity. Figure 4.3.4 shows graphs of v over $[0, T]$ for various values of v_0.

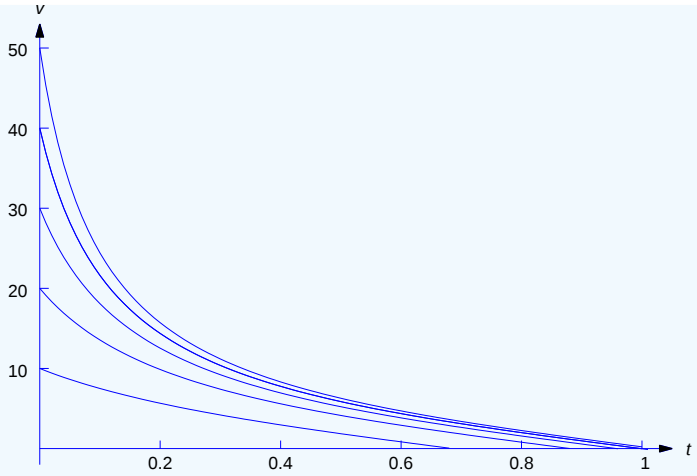


Figure 4.3.4 : Solutions of 4.3.15 for various $v_0 > 0$

Escape Velocity

Suppose a space vehicle is launched vertically and its fuel is exhausted when the vehicle reaches an altitude h above Earth, where h is sufficiently large so that resistance due to Earth's atmosphere can be neglected. Let $t = 0$ be the time when burnout occurs. Assuming that the gravitational forces of all other celestial bodies can be neglected, the motion of the vehicle for $t > 0$ is that of an object with constant mass m under the influence of Earth's gravitational force, which we now assume to vary inversely with the square of the distance from Earth's center; thus, if we take the upward direction to be positive then gravitational force on the vehicle at an altitude y above Earth is

$$F = -\frac{K}{(y+R)^2}, \tag{9.3.18}$$

where R is Earth's radius (Figure 4.3.5).

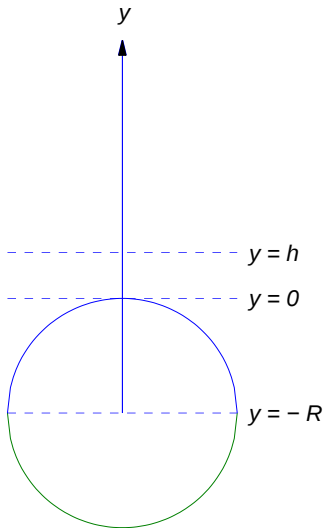


Figure 4.3.5 : Escape velocity

Since $F = -mg$ when $y = 0$, setting $y = 0$ in Equation 9.3.18 yields

$$-mg = -\frac{K}{R^2};$$

therefore $K = mgR^2$ and Equation 9.3.18 can be written more specifically as

$$F = -\frac{mgR^2}{(y+R)^2}. \tag{9.3.19}$$

From Newton's second law of motion,

$$F = m\frac{d^2y}{dt^2},$$

so Equation 9.3.19 implies that

$$\frac{d^2y}{dt^2} = -\frac{gR^2}{(y+R)^2}. \tag{9.3.20}$$

We'll show that there's a number v_e, called the **escape velocity**, with these properties:

1. If $v_0 \geq v_e$ then $v(t) > 0$ for all $t > 0$, and the vehicle continues to climb for all $t > 0$; that is, it "escapes" Earth. (Is it really so obvious that $\lim_{t\to\infty} y(t) = \infty$ in this case? For a proof, see **Exercise 4.3.20**.)
2. If $v_0 < v_e$ then $v(t)$ decreases to zero and becomes negative. Therefore the vehicle attains a maximum altitude y_m and falls back to Earth.

Since Equation 9.3.20 is second order, we cannot solve it by methods discussed so far. However, we are concerned with v rather than y, and v is easier to find. Since $v = y'$ the chain rule implies that

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}.$$

Substituting this into Equation 9.3.20 yields the first order separable equation

$$v\frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}. \tag{9.3.21}$$

When $t = 0$, the velocity is v_0 and the altitude is h. Therefore we can obtain v as a function of y by solving the initial value problem

$$v\frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}, \quad v(h) = v_0.$$

Integrating Equation 9.3.21 with respect to y yields

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + c. \tag{9.3.22}$$

Since $v(h) = v_0$,

$$c = \frac{v_0^2}{2} - \frac{gR^2}{h+R},$$

so Equation 9.3.22 becomes

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right). \tag{9.3.23}$$

If

$$v_0 \geq \left(\frac{2gR^2}{h+R}\right)^{1/2},$$

the parenthetical expression in Equation 9.3.23 is nonnegative, so $v(y) > 0$ for $y > h$. This proves that there's an escape velocity v_e. We'll now prove that

$$v_e = \left(\frac{2gR^2}{h+R}\right)^{1/2}$$

by showing that the vehicle falls back to Earth if

$$v_0 < \left(\frac{2gR^2}{h+R}\right)^{1/2}. \tag{9.3.24}$$

If Equation 9.3.24 holds then the parenthetical expression in Equation 9.3.23 is negative and the vehicle will attain a maximum altitude $y_m > h$ that satisfies the equation

$$0 = \frac{gR^2}{y_m + R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h + R} \right).$$

The velocity will be zero at the maximum altitude, and the object will then fall to Earth under the influence of gravity.

This page titled 9.3: Elementary Mechanics is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **4.3: Elementary Mechanics** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

9.3.1: Elementary Mechanics (Exercises)

Except where directed otherwise, assume that the magnitude of the gravitational force on an object with mass m is constant and equal to mg. In exercises involving vertical motion take the upward direction to be positive.

Q4.3.1

1. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to his speed, with $k = 2.5$ lb-s/ft. Assuming that he starts from rest, find his velocity as a function of time and find his terminal velocity.

2. A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to her speed, with constant of proportionality k. Find k, given that her terminal velocity is -16 ft/s, and then find her velocity v as a function of t. Assume that she starts from rest.

3. A boat weighs $64,000$ lb. Its propellor produces a constant thrust of $50,000$ lb and the water exerts a resistive force with magnitude proportional to the speed, with $k = 2000$ lb-s/ft. Assuming that the boat starts from rest, find its velocity as a function of time, and find its terminal velocity.

4. A constant horizontal force of 10 N pushes a 20 kg-mass through a medium that resists its motion with $.5$ N for every m/s of speed. The initial velocity of the mass is 7 m/s in the direction opposite to the direction of the applied force. Find the velocity of the mass for $t > 0$.

5. A stone weighing $1/2$ lb is thrown upward from an initial height of 5 ft with an initial speed of 32 ft/s. Air resistance is proportional to speed, with $k = 1/128$ lb-s/ft. Find the maximum height attained by the stone.

6. A 3200-lb car is moving at 64 ft/s down a 30-degree grade when it runs out of fuel. Find its velocity after that if friction exerts a resistive force with magnitude proportional to the square of the speed, with $k = 1$ lb-s^2/ft^2. Also find its terminal velocity.

7. A 96 lb weight is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the speed. Find its velocity as a function of time if its terminal velocity is -128 ft/s.

8. An object with mass m moves vertically through a medium that exerts a resistive force with magnitude proportional to the speed. Let $y = y(t)$ be the altitude of the object at time t, with $y(0) = y_0$. Use the results of Example 4.3.1 to show that

$$y(t) = y_0 + \frac{m}{k}(v_0 - v - gt).$$

9. An object with mass m is launched vertically upward with initial velocity v_0 from Earth's surface ($y_0 = 0$) in a medium that exerts a resistive force with magnitude proportional to the speed. Find the time T when the object attains its maximum altitude y_m. Then use the result of **Exercise 4.3.8** to find y_m.

10. An object weighing 256 lb is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the square of the speed. The magnitude of the resisting force is 1 lb when $|v| = 4$ ft/s. Find v for $t > 0$, and find its terminal velocity.

11. An object with mass m is given an initial velocity $v_0 \le 0$ in a medium that exerts a resistive force with magnitude proportional to the square of the speed. Find the velocity of the object for $t > 0$, and find its terminal velocity.

12. An object with mass m is launched vertically upward with initial velocity v_0 in a medium that exerts a resistive force with magnitude proportional to the square of the speed.

a. Find the time T when the object reaches its maximum altitude.
b. Use the result of **Exercise 4.3.11** to find the velocity of the object for $t > T$.

13. An object with mass m is given an initial velocity $v_0 \le 0$ in a medium that exerts a resistive force of the form $a|v|/(1 + |v|)$, where a is positive constant.

a. Set up a differential equation for the speed of the object.
b. Use your favorite numerical method to solve the equation you found in (a), to convince yourself that there's a unique number a_0 such that $\lim_{t\to\infty} s(t) = \infty$ if $a \le a_0$ and $\lim_{t\to\infty} s(t)$ exists (finite) if $a > a_0$. (We say that a_0 is the **bifurcation value** of a.) Try to

find a_0 and $\lim_{t\to\infty} s(t)$ in the case where $a > a_0$.

14. An object of mass m falls in a medium that exerts a resistive force $f = f(s)$, where $s = |v|$ is the speed of the object. Assume that $f(0) = 0$ and f is strictly increasing and differentiable on $(0, \infty)$.

a. Write a differential equation for the speed $s = s(t)$ of the object. Take it as given that all solutions of this equation with $s(0) \geq 0$ are defined for all $t > 0$ (which makes good sense on physical grounds).
b. Show that if $\lim_{s\to\infty} f(s) \leq mg$ then $\lim_{t\to\infty} s(t) = \infty$.
c. Show that if $\lim_{s\to\infty} f(s) > mg$ then $\lim_{t\to\infty} s(t) = s_T$ (terminal speed), where $f(s_T) = mg$..

15. A 100-g mass with initial velocity $v_0 \leq 0$ falls in a medium that exerts a resistive force proportional to the fourth power of the speed. The resistance is .1 N if the speed is 3 m/s.

a. Set up the initial value problem for the velocity v of the mass for $t > 0$.
b. Use **Exercise 4.3.14** (c) to determine the terminal velocity of the object.
c. To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 1]$ (seconds) of the initial value problem of (a) , with initial values $v_0 = 0, -2, -4, \ldots, -12$. Present your results in graphical form similar to Figure 4.3.3.

16. A 64-lb object with initial velocity $v_0 \leq 0$ falls through a dense fluid that exerts a resistive force proportional to the square root of the speed. The resistance is 64 lb if the speed is 16 ft/s.

a. Set up the initial value problem for the velocity v of the mass for $t > 0$.
b. Use **Exercise 4.3.14** (c) to determine the terminal velocity of the object.
c. To confirm your answer to (b), use one of the numerical methods studied in Chapter 3 to compute approximate solutions on $[0, 4]$ (seconds) of the initial value problem of (a), with initial values $v_0 = 0, -5, -10, \ldots, -30$. Present your results in graphical form similar to Figure 4.3.3.

Q4.3.2

In **Exercises 4.3.17-4.3.20**, assume that the force due to gravity is given by Newton's law of gravitation. Take the upward direction to be positive.

17. A space probe is to be launched from a space station 200 miles above Earth. Determine its escape velocity in miles/s. Take Earth's radius to be 3960 miles.

18. A space vehicle is to be launched from the moon, which has a radius of about 1080 miles. The acceleration due to gravity at the surface of the moon is about 5.31 ft/s^2. Find the escape velocity in miles/s.

19.

a. Show that (Equation 4.3.27) can be rewritten as

$$v^2 = \frac{h - y}{y + R} v_e^2 + v_0^2.$$

b. Show that if $v_0 = \rho v_e$ with $0 \leq \rho < 1$, then the maximum altitude y_m attained by the space vehicle is

$$y_m = \frac{h + R\rho^2}{1 - \rho^2}.$$

c. By requiring that $v(y_m) = 0$, use (Equation 4.3.26) to deduce that if $v_0 < v_e$ then

$$|v| = v_e \left[\frac{(1 - \rho^2)(y_m - y)}{y + R} \right]^{1/2},$$

where y_m and ρ are as defined in (b) and $y \geq h$.
d. Deduce from (c) that if $v < v_e$, the vehicle takes equal times to climb from $y = h$ to $y = y_m$ and to fall back from $y = y_m$ to $y = h$.

20. In the situation considered in the discussion of escape velocity, show that $\lim_{t\to\infty} y(t) = \infty$ if $v(t) > 0$ for all $t > 0$.

This page titled 9.3.1: Elementary Mechanics (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **4.3E: Elementary Mechanics (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

CHAPTER OVERVIEW

10: Linear Second Order Equations

In this Chapter, we study a particularly important class of second order equations. Because of their many applications in science and engineering, second order differential equation have historically been the most thoroughly studied class of differential equations. Research on the theory of second order differential equations continues to the present day. This chapter is devoted to second order equations that can be written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x).$$

Such equations are said to be *linear*. As in the case of first order linear equations, this differential equation is said to be *homogeneous* if $F \equiv 0$, or *nonhomogeneous* if $F \not\equiv 0$.

10.1: Homogeneous Linear Equations

10.1.1: Homogeneous Linear Equations (Exercises)

10.2: Constant Coefficient Homogeneous Equations

10.2.1: Constant Coefficient Homogeneous Equations (Exercises)

10.3: Nonhomogeneous Linear Equations

10.3.1: Nonhomgeneous Linear Equations (Exercises)

10.4: The Method of Undetermined Coefficients I

10.4.1: The Method of Undetermined Coefficients I (Exercises)

10.5: The Method of Undetermined Coefficients II

10.5.1: The Method of Undetermined Coefficients II (Exercises)

10.6: Reduction of Order

10.6.1: Reduction of Order (Exercises)

This page titled 10: Linear Second Order Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

10.1: Homogeneous Linear Equations

A second order differential equation is said to be **linear** if it can be written as

$$y'' + p(x)y' + q(x)y = f(x). \tag{10.1.1}$$

We call the function f on the right a **forcing function**, since in physical applications it is often related to a force acting on some system modeled by the differential equation. We say that Equation 10.1.1 is **homogeneous** if $f \equiv 0$ or **nonhomogeneous** if $f \not\equiv 0$. Since these definitions are like the corresponding definitions in Section 2.1 for the linear first order equation

$$y' + p(x)y = f(x), \tag{10.1.2}$$

it is natural to expect similarities between methods of solving Equation 10.1.1 and Equation 10.1.2. However, solving Equation 10.1.1 is more difficult than solving Equation 10.1.2. For example, while Theorem 5.1.1 gives a formula for the general solution of Equation 10.1.2 in the case where $f \equiv 0$ and Theorem 5.1.2 gives a formula for the case where $f \not\equiv 0$, there are no formulas for the general solution of Equation 10.1.1 in either case. Therefore we must be content to solve linear second order equations of special forms.

In Section 2.1 we considered the homogeneous equation $y' + p(x)y = 0$ first, and then used a nontrivial solution of this equation to find the general solution of the nonhomogeneous equation $y' + p(x)y = f(x)$. Although the progression from the homogeneous to the nonhomogeneous case isn't that simple for the linear second order equation, it is still necessary to solve the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{10.1.3}$$

in order to solve the nonhomogeneous equation Equation 10.1.1. This section is devoted to Equation 10.1.3.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for Equation 10.1.3. We omit the proof.

Theorem 5.1.1

Suppose p and q are continuous on an open interval (a, b), let x_0 be any point in (a, b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem

$$y'' + p(x)y' + q(x)y = 0,\ y(x_0) = k_0,\ y'(x_0) = k_1$$

has a unique solution on (a, b).

Since $y \equiv 0$ is obviously a solution of Equation 10.1.3 we call it the **trivial** solution. Any other solution is **nontrivial**. Under the assumptions of Theorem 5.1.1 , the only solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0,\ y(x_0) = 0,\ y'(x_0) = 0$$

on (a, b) is the trivial solution (**Exercise 5.1.24**).

The next three examples illustrate concepts that we'll develop later in this section. You shouldn't be concerned with how to **find** the given solutions of the equations in these examples. This will be explained in later sections.

Example 5.1.1

The coefficients of y' and y in

$$y'' - y = 0 \tag{10.1.4}$$

are the constant functions $p \equiv 0$ and $q \equiv -1$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 5.1.1 implies that every initial value problem for Equation 10.1.4 has a unique solution on $(-\infty, \infty)$.

a. Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of Equation 10.1.4 on $(-\infty, \infty)$.
b. Verify that if c_1 and c_2 are arbitrary constants, $y = c_1e^x + c_2e^{-x}$ is a solution of Equation 10.1.4 on $(-\infty, \infty)$.
c. Solve the initial value problem

c. Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{10.1.5}$$

Solution:

a. If $y_1 = e^x$ then $y_1' = e^x$ and $y_1'' = e^x = y_1$, so $y_1'' - y_1 = 0$. If $y_2 = e^{-x}$, then $y_2' = -e^{-x}$ and $y_2'' = e^{-x} = y_2$, so $y_2'' - y_2 = 0$.

b. If

$$y = c_1 e^x + c_2 e^{-x} \tag{10.1.6}$$

then

$$y' = c_1 e^x - c_2 e^{-x} \tag{10.1.7}$$

and

$$y'' = c_1 e^x + c_2 e^{-x},$$

so

$$\begin{aligned} y'' - y &= (c_1 e^x + c_2 e^{-x}) - (c_1 e^x + c_2 e^{-x}) \\ &= c_1(e^x - e^x) + c_2(e^{-x} - e^{-x}) = 0 \end{aligned}$$

for all x. Therefore $y = c_1 e^x + c_2 e^{-x}$ is a solution of Equation 10.1.4 on $(-\infty, \infty)$.

c.

We can solve Equation 10.1.5 by choosing c_1 and c_2 in Equation 10.1.6 so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in Equation 10.1.6 and Equation 10.1.7 shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3. \end{aligned}$$

Solving these equations yields $c_1 = 2$ and $c_2 = -1$. Therefore $y = 2e^x - e^{-x}$ is the unique solution of Equation 10.1.5 on $(-\infty, \infty)$.

Example 5.1.2

Let ω be a positive constant. The coefficients of y' and y in

$$y'' + \omega^2 y = 0 \tag{10.1.8}$$

are the constant functions $p \equiv 0$ and $q \equiv \omega^2$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 5.1.1 implies that every initial value problem for Equation 10.1.8 has a unique solution on $(-\infty, \infty)$.

a. Verify that $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$ are solutions of Equation 10.1.8 on $(-\infty, \infty)$.
b. Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 \cos \omega x + c_2 \sin \omega x$ is a solution of Equation 10.1.8 on $(-\infty, \infty)$.
c. Solve the initial value problem

$$y'' + \omega^2 y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{10.1.9}$$

Solution:

a. If $y_1 = \cos \omega x$ then $y_1' = -\omega \sin \omega x$ and $y_1'' = -\omega^2 \cos \omega x = -\omega^2 y_1$, so $y_1'' + \omega^2 y_1 = 0$. If $y_2 = \sin \omega x$ then, $y_2' = \omega \cos \omega x$ and $y_2'' = -\omega^2 \sin \omega x = -\omega^2 y_2$, so $y_2'' + \omega^2 y_2 = 0$.

b. If

$$y = c_1 \cos \omega x + c_2 \sin \omega x \tag{10.1.10}$$

then

$$y' = \omega(-c_1 \sin \omega x + c_2 \cos \omega x) \tag{10.1.11}$$

and

$$y'' = -\omega^2(c_1\cos\omega x + c_2\sin\omega x),$$

so

$$\begin{aligned} y'' + \omega^2 y &= -\omega^2(c_1\cos\omega x + c_2\sin\omega x) + \omega^2(c_1\cos\omega x + c_2\sin\omega x) \\ &= c_1\omega^2(-\cos\omega x + \cos\omega x) + c_2\omega^2(-\sin\omega x + \sin\omega x) = 0 \end{aligned}$$

for all x. Therefore $y = c_1\cos\omega x + c_2\sin\omega x$ is a solution of Equation 10.1.8 on $(-\infty,\infty)$.

c. To solve Equation 10.1.9, we must choosing c_1 and c_2 in Equation 10.1.10 so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in Equation 10.1.10 and Equation 10.1.11 shows that $c_1 = 1$ and $c_2 = 3/\omega$. Therefore

$$y = \cos\omega x + \frac{3}{\omega}\sin\omega x$$

is the unique solution of Equation 10.1.9 on $(-\infty,\infty)$.

Theorem 5.1.1 implies that if k_0 and k_1 are arbitrary real numbers then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{10.1.12}$$

has a unique solution on an interval (a,b) that contains x_0, provided that P_0, P_1, and P_2 are continuous and P_0 has no zeros on (a,b). To see this, we rewrite the differential equation in Equation 10.1.12 as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem 5.1.1 with $p = P_1/P_0$ and $q = P_2/P_0$.

Example 5.1.3

The equation

$$x^2y'' + xy' - 4y = 0 \tag{10.1.13}$$

has the form of the differential equation in Equation 10.1.12, with $P_0(x) = x^2$, $P_1(x) = x$, and $P_2(x) = -4$, which are are all continuous on $(-\infty,\infty)$. However, since $P(0) = 0$ we must consider solutions of Equation 10.1.13 on $(-\infty,0)$ and $(0,\infty)$. Since P_0 has no zeros on these intervals, Theorem 5.1.1 implies that the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(0,\infty)$ if $x_0 > 0$, or on $(-\infty,0)$ if $x_0 < 0$.

a. Verify that $y_1 = x^2$ is a solution of Equation 10.1.13 on $(-\infty,\infty)$ and $y_2 = 1/x^2$ is a solution of Equation 10.1.13 on $(-\infty,0)$ and $(0,\infty)$.
b. Verify that if c_1 and c_2 are any constants then $y = c_1x^2 + c_2/x^2$ is a solution of Equation 10.1.13 on $(-\infty,0)$ and $(0,\infty)$.
c. Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 0. \tag{10.1.14}$$

d. Solve the initial value problem

$$x^2y'' + xy' - 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 0. \tag{10.1.15}$$

Solution:

a. If $y_1 = x^2$ then $y_1' = 2x$ and $y_1'' = 2$, so

$$x^2y_1'' + xy_1' - 4y_1 = x^2(2) + x(2x) - 4x^2 = 0$$

for x in $(-\infty,\infty)$. If $y_2 = 1/x^2$, then $y_2' = -2/x^3$ and $y_2'' = 6/x^4$, so

$$x^2y_2'' + xy_2' - 4y_2 = x^2\left(\frac{6}{x^4}\right) - x\left(\frac{2}{x^3}\right) - \frac{4}{x^2} = 0$$

for x in $(-\infty,0)$ or $(0,\infty)$.

b. If

$$y=c_1x^2+\frac{c_2}{x^2} \tag{10.1.16}$$

then

$$y'=2c_1x-\frac{2c_2}{x^3} \tag{10.1.17}$$

and

$$y''=2c_1+\frac{6c_2}{x^4},$$

so

$$\begin{aligned} x^2y''+xy'-4y&=x^2\left(2c_1+\frac{6c_2}{x^4}\right)+x\left(2c_1x-\frac{2c_2}{x^3}\right)-4\left(c_1x^2+\frac{c_2}{x^2}\right)\\ &=c_1(2x^2+2x^2-4x^2)+c_2\left(\frac{6}{x^2}-\frac{2}{x^2}-\frac{4}{x^2}\right)\\ &=c_1\cdot0+c_2\cdot0=0 \end{aligned}$$

for x in $(-\infty,0)$ or $(0,\infty)$.

c. To solve Equation 10.1.14, we choose c_1 and c_2 in Equation 10.1.16 so that $y(1)=2$ and $y'(1)=0$. Setting $x=1$ in Equation 10.1.16 and Equation 10.1.17 shows that this is equivalent to

$$\begin{aligned} c_1+\ \ c_2&=2\\ 2c_1-2c_2&=0. \end{aligned}$$

Solving these equations yields $c_1=1$ and $c_2=1$. Therefore $y=x^2+1/x^2$ is the unique solution of Equation 10.1.14 on $(0,\infty)$.

d. We can solve Equation 10.1.15 by choosing c_1 and c_2 in Equation 10.1.16 so that $y(-1)=2$ and $y'(-1)=0$. Setting $x=-1$ in Equation 10.1.16 and Equation 10.1.17 shows that this is equivalent to

$$\begin{aligned} c_1+\ \ c_2&=2\\ -2c_1+2c_2&=0. \end{aligned}$$

Solving these equations yields $c_1=1$ and $c_2=1$. Therefore $y=x^2+1/x^2$ is the unique solution of Equation 10.1.15 on $(-\infty,0)$.

Although the **formulas** for the solutions of Equation 10.1.14 and Equation 10.1.15 are both $y=x^2+1/x^2$, you should not conclude that these two initial value problems have the same solution. Remember that a solution of an initial value problem is defined **on an interval that contains the initial point**; therefore, the solution of Equation 10.1.14 is $y=x^2+1/x^2$ **on the interval** $(0,\infty)$, which contains the initial point $x_0=1$, while the solution of Equation 10.1.15 is $y=x^2+1/x^2$ **on the interval** $(-\infty,0)$, which contains the initial point $x_0=-1$.

The General Solution of a Homogeneous Linear Second Order Equation

If y_1 and y_2 are defined on an interval (a,b) and c_1 and c_2 are constants, then

$$y=c_1y_1+c_2y_2$$

is a **linear combination of** y_1 and y_2. For example, $y=2\cos x+7\sin x$ is a linear combination of $y_1=\cos x$ and $y_2=\sin x$, with $c_1=2$ and $c_2=7$.

The next theorem states a fact that we've already verified in Examples 5.1.1 , 5.1.2 , 5.1.3 .

Theorem 5.1.2

If y_1 and y_2 are solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{10.1.18}$$

on (a, b), then any linear combination

$$y = c_1 y_1 + c_2 y_2 \tag{10.1.19}$$

of y_1 and y_2 is also a solution of (10.1.18) on (a, b).

Proof

If

$$y = c_1 y_1 + c_2 y_2$$

then

$$y' = c_1 y_1' + c_2 y_2' \quad \text{and} \quad y'' = c_1 y_1'' + c_2 y_2''.$$

Therefore

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (c_1 y_1'' + c_2 y_2'') + p(x)(c_1 y_1' + c_2 y_2') + q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1 \left(y_1'' + p(x)y_1' + q(x)y_1\right) + c_2 \left(y_2'' + p(x)y_2' + q(x)y_2\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0, \end{aligned}$$

since y_1 and y_2 are solutions of Equation 10.1.18.

We say that $\{y_1, y_2\}$ is a **fundamental set of solutions of** (10.1.18) on (a, b) if every solution of Equation 10.1.18 on (a, b) can be written as a linear combination of y_1 and y_2 as in Equation 10.1.19. In this case we say that Equation 10.1.19 is **general solution of** (10.1.18) on (a, b).

Linear Independence

We need a way to determine whether a given set $\{y_1, y_2\}$ of solutions of Equation 10.1.18 is a fundamental set. The next definition will enable us to state necessary and sufficient conditions for this.

We say that two functions y_1 and y_2 defined on an interval (a, b) are **linearly independent on** (a, b) if neither is a constant multiple of the other on (a, b). (In particular, this means that neither can be the trivial solution of Equation 10.1.18, since, for example, if $y_1 \equiv 0$ we could write $y_1 = 0y_2$.) We'll also say that the set $\{y_1, y_2\}$ **is linearly independent on** (a, b).

Theorem 5.1.3

Suppose p and q are continuous on (a, b). Then a set $\{y_1, y_2\}$ of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{10.1.20}$$

on (a, b) is a fundamental set if and only if $\{y_1, y_2\}$ is linearly independent on (a, b).

Proof

We'll present the proof of Theorem 5.1.3 in steps worth regarding as theorems in their own right. However, let's first interpret Theorem 5.1.3 in terms of Examples 5.1.1 , 5.1.2 , 5.1.3 .

Example 5.1.4

Since $e^x/e^{-x} = e^{2x}$ is nonconstant, Theorem 5.1.3 implies that $y = c_1 e^x + c_2 e^{-x}$ is the general solution of $y'' - y = 0$ on $(-\infty, \infty)$.

Since $\cos \omega x / \sin \omega x = \cot \omega x$ is nonconstant, Theorem 5.1.3 implies that $y = c_1 \cos \omega x + c_2 \sin \omega x$ is the general solution of $y'' + \omega^2 y = 0$ on $(-\infty, \infty)$.

Since $x^2/x^{-2} = x^4$ is nonconstant, Theorem 5.1.3 implies that $y = c_1x^2 + c_2/x^2$ is the general solution of $x^2y'' + xy' - 4y = 0$ on $(-\infty, 0)$ and $(0, \infty)$.

The Wronskian and Abel's Formula

To motivate a result that we need in order to prove Theorem 5.1.3 , let's see what is required to prove that $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 10.1.20 on (a, b). Let x_0 be an arbitrary point in (a, b), and suppose y is an arbitrary solution of Equation 10.1.20 on (a, b). Then y is the unique solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1; \tag{10.1.21}$$

that is, k_0 and k_1 are the numbers obtained by evaluating y and y' at x_0. Moreover, k_0 and k_1 can be any real numbers, since Theorem 5.1.1 implies that Equation 10.1.21 has a solution no matter how k_0 and k_1 are chosen. Therefore $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 10.1.20 on (a, b) if and only if it is possible to write the solution of an arbitrary initial value problem Equation 10.1.21 as $y = c_1y_1 + c_2y_2$. This is equivalent to requiring that the system

$$\begin{aligned} c_1y_1(x_0) + c_2y_2(x_0) &= k_0 \\ c_1y_1'(x_0) + c_2y_2'(x_0) &= k_1 \end{aligned} \tag{10.1.22}$$

has a solution (c_1, c_2) for every choice of (k_0, k_1). Let's try to solve Equation 10.1.22.

Multiplying the first equation in Equation 10.1.22 by $y_2'(x_0)$ and the second by $y_2(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_2'(x_0) + c_2y_2(x_0)y_2'(x_0) &= y_2'(x_0)k_0 \\ c_1y_1'(x_0)y_2(x_0) + c_2y_2'(x_0)y_2(x_0) &= y_2(x_0)k_1, \end{aligned}$$

and subtracting the second equation here from the first yields

$$(y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0))\, c_1 = y_2'(x_0)k_0 - y_2(x_0)k_1. \tag{10.1.23}$$

Multiplying the first equation in Equation 10.1.22 by $y_1'(x_0)$ and the second by $y_1(x_0)$ yields

$$\begin{aligned} c_1y_1(x_0)y_1'(x_0) + c_2y_2(x_0)y_1'(x_0) &= y_1'(x_0)k_0 \\ c_1y_1'(x_0)y_1(x_0) + c_2y_2'(x_0)y_1(x_0) &= y_1(x_0)k_1, \end{aligned}$$

and subtracting the first equation here from the second yields

$$(y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0))\, c_2 = y_1(x_0)k_1 - y_1'(x_0)k_0. \tag{10.1.24}$$

If

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) = 0,$$

it is impossible to satisfy Equation 10.1.23 and Equation 10.1.24 (and therefore Equation 10.1.22) unless k_0 and k_1 happen to satisfy

$$\begin{aligned} y_1(x_0)k_1 - y_1'(x_0)k_0 &= 0 \\ y_2'(x_0)k_0 - y_2(x_0)k_1 &= 0. \end{aligned}$$

On the other hand, if

$$y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0) \neq 0 \tag{10.1.25}$$

we can divide Equation 10.1.23 and Equation 10.1.24 through by the quantity on the left to obtain

$$\begin{aligned} c_1 &= \frac{y_2'(x_0)k_0 - y_2(x_0)k_1}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)} \\ c_2 &= \frac{y_1(x_0)k_1 - y_1'(x_0)k_0}{y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)}, \end{aligned} \tag{10.1.26}$$

no matter how k_0 and k_1 are chosen. This motivates us to consider conditions on y_1 and y_2 that imply Equation 10.1.25.

Theorem 5.1.4

Suppose p and q are continuous on (a,b), let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{10.1.27}$$

on (a,b), and define

$$W = y_1y_2' - y_1'y_2. \tag{10.1.28}$$

Let x_0 be any point in (a,b). Then

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p(t)\,dt}, \quad a < x < b \tag{10.1.29}$$

Therefore either W has no zeros in (a,b) or $W \equiv 0$ on (a,b).

Proof

Differentiating Equation 10.1.28 yields

$$W' = y_1'y_2' + y_1y_2'' - y_1'y_2' - y_1''y_2 = y_1y_2'' - y_1''y_2. \tag{10.1.30}$$

Since y_1 and y_2 both satisfy Equation 10.1.27,

$$y_1'' = -py_1' - qy_1 \quad \text{and} \quad y_2'' = -py_2' - qy_2.$$

Substituting these into Equation 10.1.30 yields

$$\begin{aligned} W' &= -y_1\left(py_2' + qy_2\right) + y_2\left(py_1' + qy_1\right) \\ &= -p(y_1y_2' - y_2y_1') - q(y_1y_2 - y_2y_1) \\ &= -p(y_1y_2' - y_2y_1') = -pW. \end{aligned}$$

Therefore $W' + p(x)W = 0$; that is, W is the solution of the initial value problem

$$y' + p(x)y = 0, \quad y(x_0) = W(x_0).$$

We leave it to you to verify by separation of variables that this implies Equation 10.1.29. If $W(x_0) \neq 0$, Equation 10.1.29 implies that W has no zeros in (a,b), since an exponential is never zero. On the other hand, if $W(x_0) = 0$, Equation 10.1.29 implies that $W(x) = 0$ for all x in (a,b).

The function W defined in Equation 10.1.28 is the **Wronskian of** $\{y_1, y_2\}$. Formula Equation 10.1.29 is **Abel's formula**.

The Wronskian of $\{y_1, y_2\}$ is usually written as the determinant

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

The expressions in Equation 10.1.26 for c_1 and c_2 can be written in terms of determinants as

$$c_1 = \frac{1}{W(x_0)} \begin{vmatrix} k_0 & y_2(x_0) \\ k_1 & y_2'(x_0) \end{vmatrix} \quad \text{and} \quad c_2 = \frac{1}{W(x_0)} \begin{vmatrix} y_1(x_0) & k_0 \\ y_1'(x_0) & k_1 \end{vmatrix}.$$

If you've taken linear algebra you may recognize this as **Cramer's rule**.

Example 5.1.5

Verify Abel's formula for the following differential equations and the corresponding solutions, from Examples 5.1.1 , 5.1.2 , 5.1.3 .

a. $y'' - y = 0; \quad y_1 = e^x,\ y_2 = e^{-x}$
b. $y'' + \omega^2 y = 0; \quad y_1 = \cos \omega x,\ y_2 = \sin \omega x$
c. $x^2y'' + xy' - 4y = 0; \quad y_1 = x^2,\ y_2 = 1/x^2$

Solution:

a. Since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^x e^{-x} = -2$$

for all x.

b. Again, since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$\begin{aligned} W(x) &= \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} \\ &= \cos \omega x(\omega \cos \omega x) - (-\omega \sin \omega x) \sin \omega x \\ &= \omega(\cos^2 \omega x + \sin^2 \omega x) = \omega \end{aligned}$$

for all x.

c. Computing the Wronskian of $y_1 = x^2$ and $y_2 = 1/x^2$ directly yields

$$W = \begin{vmatrix} x^2 & 1/x^2 \\ 2x & -2/x^3 \end{vmatrix} = x^2\left(-\frac{2}{x^3}\right) - 2x\left(\frac{1}{x^2}\right) = -\frac{4}{x}. \tag{10.1.31}$$

To verify Abel's formula we rewrite the differential equation as

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$$

to see that $p(x) = 1/x$. If x_0 and x are either both in $(-\infty, 0)$ or both in $(0, \infty)$ then

$$\int_{x_0}^{x} p(t)\,dt = \int_{x_0}^{x} \frac{dt}{t} = \ln\left(\frac{x}{x_0}\right),$$

so Abel's formula becomes

$$\begin{aligned} W(x) &= W(x_0)e^{-\ln(x/x_0)} = W(x_0)\frac{x_0}{x} \\ &= -\left(\frac{4}{x_0}\right)\left(\frac{x_0}{x}\right) \quad \text{from (10.1.31)} \\ &= -\frac{4}{x}, \end{aligned}$$

which is consistent with Equation 10.1.31.

The next theorem will enable us to complete the proof of Theorem 5.1.3 .

Theorem 5.1.5

Suppose p and q are continuous on an open interval (a, b), let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{10.1.32}$$

on (a, b), and let $W = y_1y_2' - y_1'y_2$. Then y_1 and y_2 are linearly independent on (a, b) if and only if W has no zeros on (a, b).

Proof

We first show that if $W(x_0) = 0$ for some x_0 in (a, b), then y_1 and y_2 are linearly dependent on (a, b). Let I be a subinterval of (a, b) on which y_1 has no zeros. (If there's no such subinterval, $y_1 \equiv 0$ on (a, b), so y_1 and y_2 are linearly independent, and we are finished with this part of the proof.) Then y_2/y_1 is defined on I, and

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1y_2' - y_1'y_2}{y_1^2} = \frac{W}{y_1^2}. \tag{10.1.33}$$

However, if $W(x_0) = 0$, Theorem 5.1.4 implies that $W \equiv 0$ on (a, b). Therefore Equation 10.1.33 implies that $(y_2/y_1)' \equiv 0$, so $y_2/y_1 = c$ (constant) on I. This shows that $y_2(x) = cy_1(x)$ for all x in I. However, we want to show that $y_2 = cy_1(x)$ for all x in (a, b). Let $Y = y_2 - cy_1$. Then Y is a solution of Equation 10.1.32 on (a, b) such that $Y \equiv 0$ on I, and therefore $Y' \equiv 0$ on I. Consequently, if x_0 is chosen arbitrarily in I then Y is a solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0,$$

which implies that $Y \equiv 0$ on (a, b), by the paragraph following Theorem 5.1.1 . (See also **Exercise 5.1.24**). Hence, $y_2 - cy_1 \equiv 0$ on (a, b), which implies that y_1 and y_2 are not linearly independent on (a, b).

Now suppose W has no zeros on (a, b). Then y_1 can't be identically zero on (a, b) (why not?), and therefore there is a subinterval I of (a, b) on which y_1 has no zeros. Since Equation 10.1.33 implies that y_2/y_1 is nonconstant on I, y_2 isn't a constant multiple of y_1 on (a, b). A similar argument shows that y_1 isn't a constant multiple of y_2 on (a, b), since

$$\left(\frac{y_1}{y_2}\right)' = \frac{y_1'y_2 - y_1y_2'}{y_2^2} = -\frac{W}{y_2^2}$$

on any subinterval of (a, b) where y_2 has no zeros.

We can now complete the proof of Theorem 5.1.3 . From Theorem 5.1.5 , two solutions y_1 and y_2 of Equation 10.1.32 are linearly independent on (a, b) if and only if W has no zeros on (a, b). From Theorem 5.1.4 and the motivating comments preceding it, $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 10.1.32 if and only if W has no zeros on (a, b). Therefore $\{y_1, y_2\}$ is a fundamental set for Equation 10.1.32 on (a, b) if and only if $\{y_1, y_2\}$ is linearly independent on (a, b).

The next theorem summarizes the relationships among the concepts discussed in this section.

Theorem 5.1.6

Suppose p and q are continuous on an open interval (a, b) and let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{10.1.34}$$

on (a, b). Then the following statements are equivalent; that is, they are either all true or all false.

a. The general solution of (10.1.34) on (a, b) is $y = c_1y_1 + c_2y_2$.
b. $\{y_1, y_2\}$ is a fundamental set of solutions of (10.1.34) on (a, b).
c. $\{y_1, y_2\}$ is linearly independent on (a, b).
d. The Wronskian of $\{y_1, y_2\}$ is nonzero at some point in (a, b).
e. The Wronskian of $\{y_1, y_2\}$ is nonzero at all points in (a, b).

We can apply this theorem to an equation written as

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

on an interval (a, b) where P_0, P_1, and P_2 are continuous and P_0 has no zeros.dd proof here and it will automatically be hidden

Theorem 5.1.7

Suppose c is in (a, b) and α and β are real numbers, not both zero. Under the assumptions of Theorem 5.1.7 , suppose y_1 and y_2 are solutions of Equation 10.1.34 such that

$$\alpha y_1(c) + \beta y_1'(c) = 0 \quad \text{and} \quad \alpha y_2(c) + \beta y_2'(c) = 0. \tag{10.1.35}$$

Then $\{y_1, y_2\}$ isn't linearly independent on (a, b).

Proof

Since α and β are not both zero, Equation 10.1.35 implies that

$$\begin{vmatrix} y_1(c) & y_1'(c) \\ y_2(c) & y_2'(c) \end{vmatrix} = 0, \quad \text{so} \quad \begin{vmatrix} y_1(c) & y_2(c) \\ y_1'(c) & y_2'(c) \end{vmatrix} = 0$$

and Theorem 5.1.6 implies the stated conclusion.

This page titled 10.1: Homogeneous Linear Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.1: Homogeneous Linear Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

10.1.1: Homogeneous Linear Equations (Exercises)

Q5.1.1

1.

a. Verify that $y_1 = e^{2x}$ and $y_2 = e^{5x}$ are solutions of

$$y'' - 7y' + 10y = 0 \tag{A}$$

on $(-\infty, \infty)$.

b. Verify that if c_1 and c_2 are arbitrary constants then $y = c_1e^{2x} + c_2e^{5x}$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = -1, \quad y'(0) = 1.$$

d. Solve the initial value problem

$$y'' - 7y' + 10y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

2.

a. Verify that $y_1 = e^x \cos x$ and $y_2 = e^x \sin x$ are solutions of

$$y'' - 2y' + 2y = 0 \tag{A}$$

on $(-\infty, \infty)$.

b. Verify that if c_1 and c_2 are arbitrary constants then $y = c_1e^x \cos x + c_2e^x \sin x$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = 3, \quad y'(0) = -2.$$

d. Solve the initial value problem

$$y'' - 2y' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

3.

a. Verify that $y_1 = e^x$ and $y_2 = xe^x$ are solutions of

$$y'' - 2y' + y = 0 \tag{A}$$

on $(-\infty, \infty)$.

b. Verify that if c_1 and c_2 are arbitrary constants then $y = e^x(c_1 + c_2x)$ is a solution of (A) on $(-\infty, \infty)$.

c. Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = 7, \quad y'(0) = 4.$$

d. Solve the initial value problem

$$y'' - 2y' + y = 0, \quad y(0) = k_0, \quad y'(0) = k_1.$$

4.

a. Verify that $y_1 = 1/(x-1)$ and $y_2 = 1/(x+1)$ are solutions of

$$(x^2 - 1)y'' + 4xy' + 2y = 0 \tag{A}$$

on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$. What is the general solution of (A) on each of these intervals?

b. Solve the initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = -5, \quad y'(0) = 1.$$

What is the interval of validity of the solution?

c. Graph the solution of the initial value problem.

d. Verify Abel's formula for y_1 and y_2, with $x_0 = 0$.

5. Compute the Wronskians of the given sets of functions.

a. $\{1, e^x\}$
b. $\{e^x, e^x \sin x\}$
c. $\{x + 1, x^2 + 2\}$
d. $\{x^{1/2}, x^{-1/3}\}$
e. $\{\frac{\sin x}{x}, \frac{\cos x}{x}\}$
f. $\{x \ln |x|, x^2 \ln |x|\}$
g. $\{e^x \cos \sqrt{x}, e^x \sin \sqrt{x}\}$

6. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$y'' + 3(x^2 + 1)y' - 2y = 0,$$

given that $W(\pi) = 0$.

7. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

given that $W(0) = 1$. (This is **Legendre's equation**.)

8. Find the Wronskian of a given set $\{y_1, y_2\}$ of solutions of

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

given that $W(1) = 1$. (This is **Bessel's equation**.)

9. (This exercise shows that if you know one nontrivial solution of $y'' + p(x)y' + q(x)y = 0$, you can use Abel's formula to find another.)

Suppose p and q are continuous and y_1 is a solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

that has no zeros on (a, b). Let $P(x) = \int p(x)\,dx$ be any antiderivative of p on (a, b).

a. Show that if K is an arbitrary nonzero constant and y_2 satisfies

$$y_1y_2' - y_1'y_2 = Ke^{-P(x)} \tag{B}$$

on (a, b), then y_2 also satisfies (A) on (a, b), and $\{y_1, y_2\}$ is a fundamental set of solutions on (A) on (a, b).
b. Conclude from (a) that if $y_2 = uy_1$ where $u' = K\frac{e^{-P(x)}}{y_1^2(x)}$, then $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on (a, b).

Q5.1.2

In **Exercises 5.1.10-5.1.23** use the method suggested by **Exercise 5.1.9** to find a second solution y_2 that isn't a constant multiple of the solution y_1. Choose K conveniently to simplify y_2.

10. $y'' - 2y' - 3y = 0$; $y_1 = e^{3x}$

11. $y'' - 6y' + 9y = 0$; $y_1 = e^{3x}$

12. $y'' - 2ay' + a^2y = 0$ (a = constant); $y_1 = e^{ax}$

13. $x^2y'' + xy' - y = 0$; $y_1 = x$

14. $x^2y'' - xy' + y = 0$; $y_1 = x$

15. $x^2y'' - (2a - 1)xy' + a^2y = 0$ (a = nonzero constant); $x > 0$; $y_1 = x^a$

16. $4x^2y'' - 4xy' + (3 - 16x^2)y = 0$; $y_1 = x^{1/2}e^{2x}$

17. $(x - 1)y'' - xy' + y = 0$; $y_1 = e^x$

18. $x^2y'' - 2xy' + (x^2+2)y = 0;\ y_1 = x\cos x$

19. $4x^2(\sin x)y'' - 4x(x\cos x + \sin x)y' + (2x\cos x + 3\sin x)y = 0;\ y_1 = x^{1/2}$

20. $(3x-1)y'' - (3x+2)y' - (6x-8)y = 0;\ y_1 = e^{2x}$

21. $(x^2-4)y'' + 4xy' + 2y = 0;\ y_1 = \frac{1}{x-2}$

22. $(2x+1)xy'' - 2(2x^2-1)y' - 4(x+1)y = 0; y_1 = \frac{1}{x}$

23. $(x^2-2x)y'' + (2-x^2)y' + (2x-2)y = 0; y_1 = e^x$

Q5.1.3

24. Suppose p and q are continuous on an open interval (a,b) and let x_0 be in (a,b). Use Theorem 5.1.1 to show that the only solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0$$

on (a,b) is the trivial solution $y \equiv 0$.

25. Suppose P_0, P_1, and P_2 are continuous on (a,b) and let x_0 be in (a,b). Show that if either of the following statements is true then $P_0(x) = 0$ for some x in (a,b).

a. The initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has more than one solution on (a,b).

b. The initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = 0, \quad y'(x_0) = 0$$

has a nontrivial solution on (a,b).

26. Suppose p and q are continuous on (a,b) and y_1 and y_2 are solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

on (a,b). Let

$$z_1 = \alpha y_1 + \beta y_2 \quad \text{and} \quad z_2 = \gamma y_1 + \delta y_2,$$

where α, β, γ, and δ are constants. Show that if $\{z_1, z_2\}$ is a fundamental set of solutions of (A) on (a,b) then so is $\{y_1, y_2\}$.

27. Suppose p and q are continuous on (a,b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

on (a,b). Let

$$z_1 = \alpha y_1 + \beta y_2 \quad \text{and} \quad z_2 = \gamma y_1 + \delta y_2,$$

where α, β, γ, and δ are constants. Show that $\{z_1, z_2\}$ is a fundamental set of solutions of (A) on (a,b) if and only if $\alpha\gamma - \beta\delta \neq 0$.

28. Suppose y_1 is differentiable on an interval (a,b) and $y_2 = ky_1$, where k is a constant. Show that the Wronskian of $\{y_1, y_2\}$ is identically zero on (a,b).

29. Let

$$y_1 = x^3 \quad \text{and} \quad y_2 = \begin{cases} x^3, & x \geq 0, \\ -x^3, & x < 0. \end{cases}$$

a. Show that the Wronskian of $\{y_1, y_2\}$ is defined and identically zero on $(-\infty, \infty)$.

b. Suppose $a < 0 < b$. Show that $\{y_1, y_2\}$ is linearly independent on (a,b).

c. Use **Exercise 5.1.25b** to show that these results don't contradict Theorem 5.1.5, because neither y_1 nor y_2 can be a solution of an equation

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) if p and q are continuous on (a, b).

30. Suppose p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a set of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b) such that either $y_1(x_0) = y_2(x_0) = 0$ or $y_1'(x_0) = y_2'(x_0) = 0$ for some x_0 in (a, b). Show that $\{y_1, y_2\}$ is linearly dependent on (a, b).

31. Suppose p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on (a, b). Show that if $y_1(x_1) = y_1(x_2) = 0$, where $a < x_1 < x_2 < b$, then $y_2(x) = 0$ for some x in (x_1, x_2).

32. Suppose p and q are continuous on (a, b) and every solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

on (a, b) can be written as a linear combination of the twice differentiable functions $\{y_1, y_2\}$. Use Theorem 5.1.1 to show that y_1 and y_2 are themselves solutions of (A) on (a, b).

33. Suppose p_1, p_2, q_1, and q_2 are continuous on (a, b) and the equations

$$y'' + p_1(x)y' + q_1(x)y = 0 \quad \text{and} \quad y'' + p_2(x)y' + q_2(x)y = 0$$

have the same solutions on (a, b). Show that $p_1 = p_2$ and $q_1 = q_2$ on (a, b).

34. (For this exercise you have to know about 3×3 determinants.) Show that if y_1 and y_2 are twice continuously differentiable on (a, b) and the Wronskian W of $\{y_1, y_2\}$ has no zeros in (a, b) then the equation

$$\frac{1}{W}\begin{vmatrix} y & y_1 & y_2 \\ y' & y_1' & y_2' \\ y'' & y_1'' & y_2'' \end{vmatrix} = 0$$

can be written as

$$y'' + p(x)y' + q(x)y = 0, \tag{A}$$

where p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on (a, b).

35. Use the method suggested by **Exercise 5.1.34** to find a linear homogeneous equation for which the given functions form a fundamental set of solutions on some interval.

a. $e^x \cos 2x, e^x \sin 2x$
b. x, e^{2x}
c. $x, x \ln x$
d. $\cos(\ln x), \sin(\ln x)$
e. $\cosh x, \sinh x$
f. $x^2 - 1, x^2 + 1$

36. Suppose p and q are continuous on (a, b) and $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

on (a, b). Show that if y is a solution of (A) on (a, b), there's exactly one way to choose c_1 and c_2 so that $y = c_1y_1 + c_2y_2$ on (a, b).

37. Suppose p and q are continuous on (a, b) and x_0 is in (a, b). Let y_1 and y_2 be the solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{A}$$

such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \; y_2'(x_0) = 1.$$

(Theorem 5.1.1 implies that each of these initial value problems has a unique solution on (a, b).)

a. Show that $\{y_1, y_2\}$ is linearly independent on (a, b).
b. Show that an arbitrary solution y of (A) on (a, b) can be written as $y = y(x_0)y_1 + y'(x_0)y_2$.
c. Express the solution of the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2.

38. Find solutions y_1 and y_2 of the equation $y'' = 0$ that satisfy the initial conditions

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Then use **Exercise 5.1.37 (c)** to write the solution of the initial value problem

$$y'' = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

as a linear combination of y_1 and y_2.

39. Let x_0 be an arbitrary real number. Given (Example 5.1.1) that e^x and e^{-x} are solutions of $y'' - y = 0$, find solutions y_1 and y_2 of $y'' - y = 0$ such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \; y_2'(x_0) = 1.$$

Then use **Exercise 5.1.37 (c)** to write the solution of the initial value problem

$$y'' - y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2.

40. Let x_0 be an arbitrary real number. Given (Example 5.1.2) that $\cos \omega x$ and $\sin \omega x$ are solutions of $y'' + \omega^2 y = 0$, find solutions of $y'' + \omega^2 y = 0$ such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0 \quad \text{and} \quad y_2(x_0) = 0, \; y_2'(x_0) = 1.$$

Then use **Exercise 5.1.37 (c)** to write the solution of the initial value problem

$$y'' + \omega^2 y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

as a linear combination of y_1 and y_2. Use the identities

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

to simplify your expressions for y_1, y_2, and y.

41. Recall from **Exercise 5.1.4** that $1/(x - 1)$ and $1/(x + 1)$ are solutions of

$$(x^2 - 1)y'' + 4xy' + 2y = 0 \tag{A}$$

on $(-1, 1)$. Find solutions of (A) such that

$$y_1(0) = 1, \quad y_1'(0) = 0 \quad \text{and} \quad y_2(0) = 0, \; y_2'(0) = 1.$$

Then use **Exercise 5.1.37 (c)** to write the solution of initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

as a linear combination of y_1 and y_2.

42.

a. Verify that $y_1 = x^2$ and $y_2 = x^3$ satisfy

$$x^2y'' - 4xy' + 6y = 0 \tag{A}$$

on $(-\infty, \infty)$ and that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(-\infty, 0)$ and $(0, \infty)$.

b. Let a_1, a_2, b_1, and b_2 be constants. Show that

$$y = \begin{cases} a_1x^2 + a_2x^3, & x \geq 0, \\ b_1x^2 + b_2x^3, & x < 0 \end{cases}$$

is a solution of (A) on $(-\infty, \infty)$ if and only if $a_1 = b_1$. From this, justify the statement that y is a solution of (A) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} c_1x^2 + c_2x^3, & x \geq 0, \\ c_1x^2 + c_3x^3, & x < 0, \end{cases}$$

where c_1, c_2, and c_3 are arbitrary constants.

c. For what values of k_0 and k_1 does the initial value problem

$$x^2y'' - 4xy' + 6y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

d. Show that if $x_0 \neq 0$ and k_0, k_1 are arbitrary constants, the initial value problem

$$x^2y'' - 4xy' + 6y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{B}$$

has infinitely many solutions on $(-\infty, \infty)$. On what interval does (B) have a unique solution?

43.

a. Verify that $y_1 = x$ and $y_2 = x^2$ satisfy

$$x^2y'' - 2xy' + 2y = 0 \tag{A}$$

on $(-\infty, \infty)$ and that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(-\infty, 0)$ and $(0, \infty)$.

b. Let a_1, a_2, b_1, and b_2 be constants. Show that

$$y = \begin{cases} a_1x + a_2x^2, & x \geq 0, \\ b_1x + b_2x^2, & x < 0 \end{cases}$$

is a solution of (A) on $(-\infty, \infty)$ if and only if $a_1 = b_1$ and $a_2 = b_2$. From this, justify the statement that the general solution of (A) on $(-\infty, \infty)$ is $y = c_1x + c_2x^2$, where c_1 and c_2 are arbitrary constants.

c. For what values of k_0 and k_1 does the initial value problem

$$x^2y'' - 2xy' + 2y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

d. Show that if $x_0 \neq 0$ and k_0, k_1 are arbitrary constants then the initial value problem

$$x^2y'' - 2xy' + 2y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(-\infty, \infty)$.

44.

a. Verify that $y_1 = x^3$ and $y_2 = x^4$ satisfy

$$x^2y'' - 6xy' + 12y = 0 \tag{A}$$

on $(-\infty, \infty)$, and that $\{y_1, y_2\}$ is a fundamental set of solutions of (A) on $(-\infty, 0)$ and $(0, \infty)$.

b. Show that y is a solution of (A) on $(-\infty, \infty)$ if and only if

$$y = \begin{cases} a_1x^3 + a_2x^4, & x \geq 0, \\ b_1x^3 + b_2x^4, & x < 0, \end{cases}$$

where a_1, a_2, b_1, and b_2 are arbitrary constants.

c. For what values of k_0 and k_1 does the initial value problem

$$x^2y'' - 6xy' + 12y = 0, \quad y(0) = k_0, \quad y'(0) = k_1$$

have a solution? What are the solutions?

d. Show that if $x_0 \neq 0$ and k_0, k_1 are arbitrary constants then the initial value problem

$$x^2y'' - 6xy' + 12y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1 \tag{B}$$

has infinitely many solutions on $(-\infty, \infty)$. On what interval does (B) have a unique solution?

This page titled 10.1.1: Homogeneous Linear Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.1E: Homogeneous Linear Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

10.2: Constant Coefficient Homogeneous Equations

This section deals with homogeneous equations of the special form

$$ay'' + by' + cy = 0, \tag{10.2.1}$$

where a, b, and c are constant ($a \neq 0$). When you've completed this section you'll know everything there is to know about solving such equations.

If a, b, and c are real constants and $a \neq 0$, then

$$ay'' + by' + cy = F(x)$$

is said to be a **constant coefficient equation**. In this section we consider the homogeneous constant coefficient equation

$$ay'' + by' + cy = 0. \tag{10.2.2}$$

As we'll see, all solutions of Equation 10.2.2 are defined on $(-\infty, \infty)$. This being the case, we'll omit references to the interval on which solutions are defined, or on which a given set of solutions is a fundamental set, etc., since the interval will always be $(-\infty, \infty)$.

The key to solving Equation 10.2.2 is that if $y = e^{rx}$ where r is a constant then the left side of Equation 10.2.2 is a multiple of e^{rx}; thus, if $y = e^{rx}$ then $y' = re^{rx}$ and $y'' = r^2e^{rx}$, so

$$ay'' + by' + cy = ar^2e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx}. \tag{10.2.3}$$

The quadratic polynomial

$$p(r) = ar^2 + br + c$$

is the **characteristic polynomial** of Equation 10.2.2, and $p(r) = 0$ is the **characteristic equation**. From Equation 10.2.3 we can see that $y = e^{rx}$ is a solution of Equation 10.2.2 if and only if $p(r) = 0$.

The roots of the characteristic equation are given by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{10.2.4}$$

Consider Three Cases

- Case 1: $b^2 - 4ac > 0$, so the characteristic equation has two distinct real roots.
- Case 2: $b^2 - 4ac = 0$, so the characteristic equation has a repeated real root.
- Case 3: $b^2 - 4ac < 0$, so the characteristic equation has complex roots.

In each case we'll start with an example.

Case 1: Distinct Real Roots

Example 5.2.1

a. Find the general solution of

$$y'' + 6y' + 5y = 0. \tag{10.2.5}$$

b. Solve the initial value problem

$$y'' + 6y' + 5y = 0, \quad y(0) = 3,\ y'(0) = -1. \tag{10.2.6}$$

Solution

a. The characteristic polynomial of Equation 10.2.5 is

$$p(r) = r^2 + 6r + 5 = (r + 1)(r + 5).$$

Since $p(-1) = p(-5) = 0$, $y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are solutions of Equation 10.2.5. Since $y_2/y_1 = e^{-4x}$ is nonconstant, Theorem 5.1.6 implies that the general solution of Equation 10.2.5 is

$$y = c_1 e^{-x} + c_2 e^{-5x}. \tag{10.2.7}$$

b. We must determine c_1 and c_2 in Equation 10.2.7 so that y satisfies the initial conditions in Equation 10.2.6. Differentiating Equation 10.2.7 yields

$$y' = -c_1 e^{-x} - 5c_2 e^{-5x}. \tag{10.2.8}$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ in Equation 10.2.7 and Equation 10.2.8 yields

$$\begin{aligned} c_1 + \ c_2 &= \ \ 3 \\ -c_1 - 5c_2 &= -1. \end{aligned}$$

The solution of this system is $c_1 = 7/2, c_2 = -1/2$. Therefore the solution of Equation 10.2.6 is

$$y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}.$$

Figure 5.2.1 is a graph of this solution.

If the characteristic equation has arbitrary distinct real roots r_1 and r_2, then $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$ are solutions of $ay'' + by' + cy = 0$. Since $y_2/y_1 = e^{(r_2 - r_1)x}$ is nonconstant, Theorem 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of $ay'' + by' + cy = 0$.

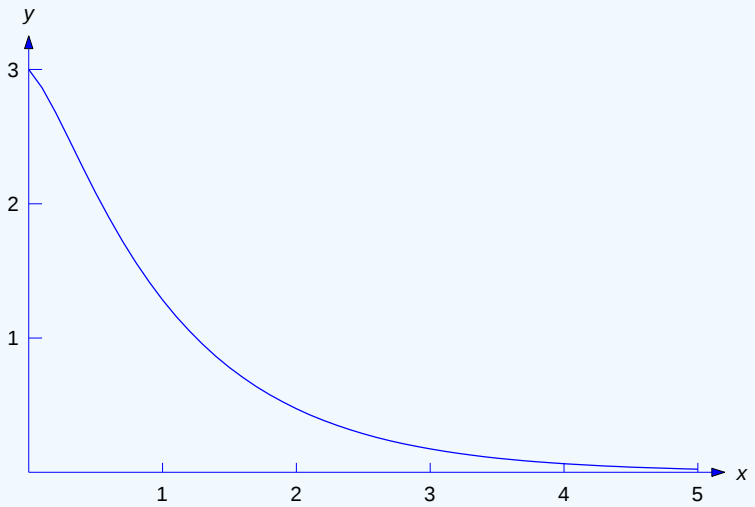


Figure 5.2.1 : $y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}$

Case 2: A Repeated Real Root

Example 5.2.2

a. Find the general solution of

$$y'' + 6y' + 9y = 0. \tag{10.2.9}$$

b. Solve the initial value problem

$$y'' + 6y' + 9y = 0, \quad y(0) = 3,\ y'(0) = -1. \tag{10.2.10}$$

Solution

a. The characteristic polynomial of Equation 10.2.9 is

$$p(r) = r^2 + 6r + 9 = (r+3)^2,$$

so the characteristic equation has the repeated real root $r_1 = -3$. Therefore $y_1 = e^{-3x}$ is a solution of Equation 10.2.9. Since the characteristic equation has no other roots, Equation 10.2.9 has no other solutions of the form e^{rx}. We look for solutions of the form $y = uy_1 = ue^{-3x}$, where u is a function that we'll now determine. (This should remind you of the method of variation of parameters used in Section 2.1 to solve the nonhomogeneous equation $y' + p(x)y = f(x)$, given a solution y_1 of the complementary equation $y' + p(x)y = 0$. It's also a special case of a method called **reduction of order** that we'll study in Section 5.6. For other ways to obtain a second solution of Equation 10.2.9 that's not a multiple of e^{-3x}, see **Exercises 5.1.9, 5.1.12,** and **5.1.33**.

If $y = ue^{-3x}$, then

$$y' = u'e^{-3x} - 3ue^{-3x} \quad \text{and} \quad y'' = u''e^{-3x} - 6u'e^{-3x} + 9ue^{-3x},$$

so

$$\begin{aligned} y'' + 6y' + 9y &= e^{-3x}\left[(u'' - 6u' + 9u) + 6(u' - 3u) + 9u\right] \\ &= e^{-3x}\left[u'' - (6-6)u' + (9 - 18 + 9)u\right] = u''e^{-3x}. \end{aligned}$$

Therefore $y = ue^{-3x}$ is a solution of Equation 10.2.9 if and only if $u'' = 0$, which is equivalent to $u = c_1 + c_2x$, where c_1 and c_2 are constants. Therefore any function of the form

$$y = e^{-3x}(c_1 + c_2x) \tag{10.2.11}$$

is a solution of Equation 10.2.9. Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-3x}$ that we already knew. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = xe^{-3x}$. Since $y_2/y_1 = x$ is nonconstant, Theorem 5.1.6 implies that $\{y_1, y_2\}$ is fundamental set of solutions of Equation 10.2.9, and Equation 10.2.11 is the general solution.

b. Differentiating Equation 10.2.11 yields

$$y' = -3e^{-3x}(c_1 + c_2x) + c_2e^{-3x}. \tag{10.2.12}$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ in Equation 10.2.11 and Equation 10.2.12 yields $c_1 = 3$ and $-3c_1 + c_2 = -1$, so $c_2 = 8$. Therefore the solution of Equation 10.2.10 is

$$y = e^{-3x}(3 + 8x).$$

Figure 5.2.2 is a graph of this solution.



Figure 5.2.2 : $y = e^{-3x}(3 + 8x)$

If the characteristic equation of $ay'' + by' + cy = 0$ has an arbitrary repeated root r_1, the characteristic polynomial must be

$$p(r) = a(r - r_1)^2 = a(r^2 - 2r_1r + r_1^2).$$

Therefore

$$ar^2+br+c=ar^2-(2ar_1)r+ar_1^2,$$

which implies that $b=-2ar_1$ and $c=ar_1^2$. Therefore $ay''+by'+cy=0$ can be written as $a(y''-2r_1y'+r_1^2y)=0$. Since $a\neq 0$ this equation has the same solutions as

$$y''-2r_1y'+r_1^2y=0. \tag{10.2.13}$$

Since $p(r_1)=0$, t $y_1=e^{r_1x}$ is a solution of $ay''+by'+cy=0$, and therefore of Equation 10.2.13. Proceeding as in Example 5.2.1 , we look for other solutions of Equation 10.2.13 of the form $y=ue^{r_1x}$; then

$$y'=u'e^{r_1x}+rue^{r_1x} \quad \text{and} \quad y''=u''e^{r_1x}+2r_1u'e^{r_1x}+r_1^2ue^{r_1x},$$

so

$$\begin{aligned} y''-2r_1y'+r_1^2y&=e^{rx}\left[(u''+2r_1u'+r_1^2u)-2r_1(u'+r_1u)+r_1^2u\right]\\ &=e^{r_1x}\left[u''+(2r_1-2r_1)u'+(r_1^2-2r_1^2+r_1^2)u\right]=u''e^{r_1x}. \end{aligned}$$

Therefore $y=ue^{r_1x}$ is a solution of Equation 10.2.13 if and only if $u''=0$, which is equivalent to $u=c_1+c_2x$, where c_1 and c_2 are constants. Hence, any function of the form

$$y=e^{r_1x}(c_1+c_2x) \tag{10.2.14}$$

is a solution of Equation 10.2.13. Letting $c_1=1$ and $c_2=0$ here yields the solution $y_1=e^{r_1x}$ that we already knew. Letting $c_1=0$ and $c_2=1$ yields the second solution $y_2=xe^{r_1x}$. Since $y_2/y_1=x$ is nonconstant, Theorem 5.1.6 implies that $\{y_1,y_2\}$ is a fundamental set of solutions of Equation 10.2.13, and Equation 10.2.14 is the general solution.

Case 3: Complex Conjugate Roots

Example 5.2.3

a. Find the general solution of

$$y''+4y'+13y=0. \tag{10.2.15}$$

b. Solve the initial value problem

$$y''+4y'+13y=0, \quad y(0)=2,\ y'(0)=-3. \tag{10.2.16}$$

Solution

a. The characteristic polynomial of Equation 10.2.15 is

$$p(r)=r^2+4r+13=r^2+4r+4+9=(r+2)^2+9.$$

The roots of the characteristic equation are $r_1=-2+3i$ and $r_2=-2-3i$. By analogy with Case 1, it is reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions of Equation 10.2.15. This is true (see **Exercise 5.2.34**); however, there are difficulties here, since you are probably not familiar with exponential functions with complex arguments, and even if you are, it is inconvenient to work with them, since they are complex–valued. We'll take a simpler approach, which we motivate as follows: the exponential notation suggests that

$$e^{(-2+3i)x}=e^{-2x}e^{3ix} \quad \text{and} \quad e^{(-2-3i)x}=e^{-2x}e^{-3ix},$$

so even though we haven't defined e^{3ix} and e^{-3ix}, it is reasonable to expect that every linear combination of $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ can be written as $y=ue^{-2x}$, where u depends upon x. To determine u, we note that if $y=ue^{-2x}$ then

$$y'=u'e^{-2x}-2ue^{-2x} \quad \text{and} \quad y''=u''e^{-2x}-4u'e^{-2x}+4ue^{-2x},$$

so

$$\begin{aligned} y''+4y'+13y&=e^{-2x}\left[(u''-4u'+4u)+4(u'-2u)+13u\right]\\ &=e^{-2x}\left[u''-(4-4)u'+(4-8+13)u\right]=e^{-2x}(u''+9u). \end{aligned}$$

Therefore $y=ue^{-2x}$ is a solution of Equation 10.2.15 if and only if

$$u'' + 9u = 0.$$

From Example 5.1.2, the general solution of this equation is

$$u = c_1 \cos 3x + c_2 \sin 3x.$$

Therefore any function of the form

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) \tag{10.2.17}$$

is a solution of Equation 10.2.15. Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-2x} \cos 3x$. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = e^{-2x} \sin 3x$. Since $y_2/y_1 = \tan 3x$ is nonconstant, Theorem 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 10.2.15, and Equation 10.2.17 is the general solution.

b. Imposing the condition $y(0) = 2$ in Equation 10.2.17 shows that $c_1 = 2$. Differentiating Equation 10.2.17 yields

$$y' = -2e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + 3e^{-2x}(-c_1 \sin 3x + c_2 \cos 3x),$$

and imposing the initial condition $y'(0) = -3$ here yields $-3 = -2c_1 + 3c_2 = -4 + 3c_2$, so $c_2 = 1/3$. Therefore the solution of Equation 10.2.16 is

$$y = e^{-2x}(2 \cos 3x + \frac{1}{3} \sin 3x).$$

Figure 5.2.3 is a graph of this function.

Now suppose the characteristic equation of $ay'' + by' + cy = 0$ has arbitrary complex roots; thus, $b^2 - 4ac < 0$ and, from Equation 10.2.4, the roots are

$$r_1 = \frac{-b + i\sqrt{4ac - b^2}}{2a}, \quad r_2 = \frac{-b - i\sqrt{4ac - b^2}}{2a},$$

which we rewrite as

$$r_1 = \lambda + i\omega, \quad r_2 = \lambda - i\omega, \tag{10.2.18}$$

with

$$\lambda = -\frac{b}{2a}, \quad \omega = \frac{\sqrt{4ac - b^2}}{2a}.$$

Don't memorize these formulas. Just remember that r_1 and r_2 are of the form Equation 10.2.18, where λ is an arbitrary real number and ω is positive; λ and ω are the **real** and **imaginary parts**, respectively, of r_1. Similarly, λ and $-\omega$ are the real and imaginary parts of r_2. We say that r_1 and r_2 are **complex conjugates**,

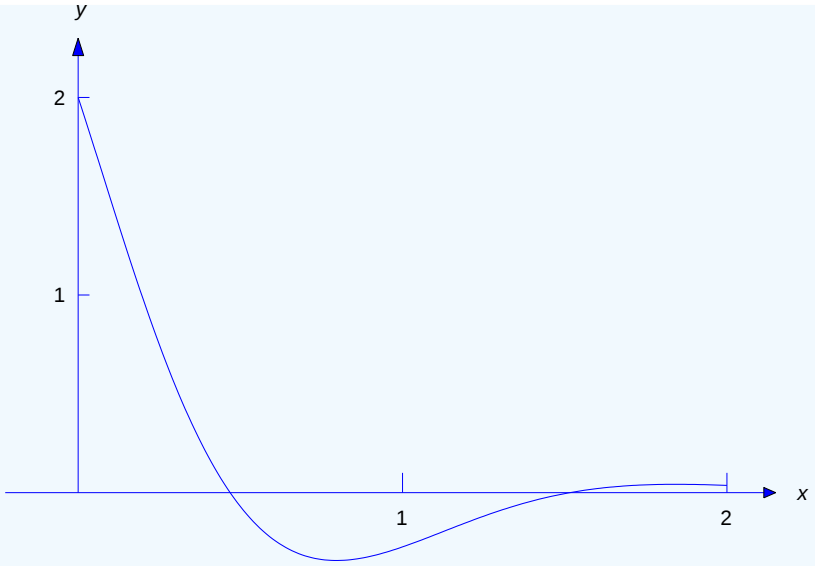


Figure 5.2.3 : $y = e^{-2x}(2\cos 3x + \frac{1}{3}\sin 3x)$

which means that they have the same real part and their imaginary parts have the same absolute values, but opposite signs.

As in Example 5.2.3 , it is reasonable to to expect that the solutions of $ay'' + by' + cy = 0$ are linear combinations of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$. Again, the exponential notation suggests that

$$e^{(\lambda+i\omega)x} = e^{\lambda x}e^{i\omega x} \quad \text{and} \quad e^{(\lambda-i\omega)x} = e^{\lambda x}e^{-i\omega x},$$

so even though we haven't defined $e^{i\omega x}$ and $e^{-i\omega x}$, it is reasonable to expect that every linear combination of $e^{(\lambda+i\omega)x}$ and $e^{(\lambda-i\omega)x}$ can be written as $y = ue^{\lambda x}$, where u depends upon x. To determine u we first observe that since $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ are the roots of the characteristic equation, p must be of the form

$$\begin{aligned} p(r) &= a(r - r_1)(r - r_2) \\ &= a(r - \lambda - i\omega)(r - \lambda + i\omega) \\ &= a\left[(r - \lambda)^2 + \omega^2\right] \\ &= a(r^2 - 2\lambda r + \lambda^2 + \omega^2). \end{aligned}$$

Therefore $ay'' + by' + cy = 0$ can be written as

$$a\left[y'' - 2\lambda y' + (\lambda^2 + \omega^2)y\right] = 0.$$

Since $a \neq 0$ this equation has the same solutions as

$$y'' - 2\lambda y' + (\lambda^2 + \omega^2)y = 0. \tag{10.2.19}$$

To determine u we note that if $y = ue^{\lambda x}$ then

$$y' = u'e^{\lambda x} + \lambda u e^{\lambda x} \quad \text{and} \quad y'' = u''e^{\lambda x} + 2\lambda u'e^{\lambda x} + \lambda^2 u e^{\lambda x}.$$

Substituting these expressions into Equation 10.2.19 and dropping the common factor $e^{\lambda x}$ yields

$$(u'' + 2\lambda u' + \lambda^2 u) - 2\lambda(u' + \lambda u) + (\lambda^2 + \omega^2)u = 0,$$

which simplifies to

$$u'' + \omega^2 u = 0.$$

From Example 5.1.2, the general solution of this equation is

$$u = c_1 \cos \omega x + c_2 \sin \omega x.$$

Therefore any function of the form

$$y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x) \tag{10.2.20}$$

is a solution of Equation 10.2.19. Letting $c_1 = 1$ and $c_2 = 0$ here yields the solution $y_1 = e^{\lambda x} \cos \omega x$. Letting $c_1 = 0$ and $c_2 = 1$ yields a second solution $y_2 = e^{\lambda x} \sin \omega x$. Since $y_2/y_1 = \tan \omega x$ is nonconstant, so Theorem 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of Equation 10.2.19, and Equation 10.2.20 is the general solution.

Summary

The next theorem summarizes the results of this section.

Theorem 5.2.1

Let $p(r) = ar^2 + br + c$ be the characteristic polynomial of

$$ay'' + by' + cy = 0. \tag{10.2.21}$$

Then:

a. If $p(r) = 0$ has distinct real roots r_1 and r_2, then the general solution of Equation 10.2.21 is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

b. If $p(r) = 0$ has a repeated root r_1, then the general solution of Equation 10.2.21 is

$$y = e^{r_1 x}(c_1 + c_2 x).$$

c. If $p(r) = 0$ has complex conjugate roots $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ (where $\omega > 0$), then the general solution of Equation 10.2.21 is

$$y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x).$$

This page titled 10.2: Constant Coefficient Homogeneous Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.2: Constant Coefficient Homogeneous Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

10.2.1: Constant Coefficient Homogeneous Equations (Exercises)

Q5.2.1

In **Exercises 5.2.1-5.2.12** find the general solution.

1. $y'' + 5y' - 6y = 0$
2. $y'' - 4y' + 5y = 0$
3. $y'' + 8y' + 7y = 0$
4. $y'' - 4y' + 4y = 0$
5. $y'' + 2y' + 10y = 0$
6. $y'' + 6y' + 10y = 0$
7. $y'' - 8y' + 16y = 0$
8. $y'' + y' = 0$
9. $y'' - 2y' + 3y = 0$
10. $y'' + 6y' + 13y = 0$
11. $4y'' + 4y' + 10y = 0$
12. $10y'' - 3y' - y = 0$

Q5.2.2

In **Exercises 5.2.13-5.2.17** solve the initial value problem.

13. $y'' + 14y' + 50y = 0, \quad y(0) = 2, \quad y'(0) = -17$
14. $6y'' - y' - y = 0, \quad y(0) = 10, \quad y'(0) = 0$
15. $6y'' + y' - y = 0, \quad y(0) = -1, \quad y'(0) = 3$
16. $4y'' - 4y' - 3y = 0, \quad y(0) = \frac{13}{12}, \quad y'(0) = \frac{23}{24}$
17. $4y'' - 12y' + 9y = 0, \quad y(0) = 3, \quad y'(0) = \frac{5}{2}$

Q5.2.3

In **Exercises 5.2.18-5.2.21** solve the initial value problem and graph the solution.

18. $y'' + 7y' + 12y = 0, \quad y(0) = -1, \quad y'(0) = 0$
19. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$
20. $36y'' - 12y' + y = 0, \quad y(0) = 3, \quad y'(0) = \frac{5}{2}$
21. $y'' + 4y' + 10y = 0, \quad y(0) = 3, \quad y'(0) = -2$

Q5.2.4

22.

a. Suppose y is a solution of the constant coefficient homogeneous equation

$$ay'' + by' + cy = 0. \tag{A}$$

Let $z(x) = y(x - x_0)$, where x_0 is an arbitrary real number. Show that

$$az'' + bz' + cz = 0.$$

b. Let $z_1(x) = y_1(x - x_0)$ and $z_2(x) = y_2(x - x_0)$, where $\{y_1, y_2\}$ is a fundamental set of solutions of (A). Show that $\{z_1, z_2\}$ is also a fundamental set of solutions of (A).

c. The statement of Theorem 5.2.1 is convenient for solving an initial value problem

$$ay'' + by' + cy = 0, \quad y(0) = k_0, \quad y'(0) = k_1,$$

where the initial conditions are imposed at $x_0 = 0$. However, if the initial value problem is

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1, \tag{B}$$

where $x_0 \neq 0$, then determining the constants in

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}, \quad y = e^{r_1 x}(c_1 + c_2 x), \text{ or } y = e^{\lambda x}(c_1 \cos \omega x + c_2 \sin \omega x)$$

(whichever is applicable) is more complicated. Use (b) to restate Theorem 5.2.1 in a form more convenient for solving (B).

Q5.2.5

In **Exercises 5.2.23-5.2.28** use a method suggested by **Exercise 5.2.22** to solve the initial value problem.

23. $y'' + 3y' + 2y = 0, \quad y(1) = -1, \quad y'(1) = 4$

24. $y'' - 6y' - 7y = 0, \quad y(2) = -\frac{1}{3}, \quad y'(2) = -5$

25. $y'' - 14y' + 49y = 0, \quad y(1) = 2, \quad y'(1) = 11$

26. $9y'' + 6y' + y = 0, \quad y(2) = 2, \quad y'(2) = -\frac{14}{3}$

27. $9y'' + 4y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2$

28. $y'' + 3y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -1$

Q5.2.6

29. Prove: If the characteristic equation of

$$ay'' + by' + cy = 0 \tag{A}$$

has a repeated negative root or two roots with negative real parts, then every solution of (A) approaches zero as $x \to \infty$.

30. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has distinct real roots r_1 and r_2. Use a method suggested by **Exercise 5.2.22** to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

31 Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has a repeated real root r_1. Use a method suggested by **Exercise 5.2.22** to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

32. Suppose the characteristic polynomial of $ay'' + by' + cy = 0$ has complex conjugate roots $\lambda \pm i\omega$. Use a method suggested by **Exercise 5.2.22** to find a formula for the solution of

$$ay'' + by' + cy = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1.$$

33. Suppose the characteristic equation of

$$ay'' + by' + cy = 0 \tag{A}$$

has a repeated real root r_1. Temporarily, think of e^{rx} as a function of two real variables x and r.

a. Show that

$$a\frac{\partial^2}{\partial^2 x}(e^{rx}) + b\frac{\partial}{\partial x}(e^{rx}) + ce^{rx} = a(r - r_1)^2 e^{rx}. \tag{B}$$

b. Differentiate (B) with respect to r to obtain

$$a\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial^2 x}(e^{rx})\right)+b\frac{\partial}{\partial r}\left(\frac{\partial}{\partial x}(e^{rx})\right)+c(xe^{rx})=[2+(r-r_1)x]a(r-r_1)e^{rx}. \tag{C}$$

c. Reverse the orders of the partial differentiations in the first two terms on the left side of (C) to obtain

$$a\frac{\partial^2}{\partial x^2}(xe^{rx})+b\frac{\partial}{\partial x}(xe^{rx})+c(xe^{rx})=[2+(r-r_1)x]a(r-r_1)e^{rx}. \tag{D}$$

d. Set $r=r_1$ in (B) and (D) to see that $y_1=e^{r_1x}$ and $y_2=xe^{r_1x}$ are solutions of (A)

34. In calculus you learned that e^u, $\cos u$, and $\sin u$ can be represented by the infinite series

$$e^u=\sum_{n=0}^{\infty}\frac{u^n}{n!}=1+\frac{u}{1!}+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots+\frac{u^n}{n!}+\cdots \tag{A}$$

$$\cos u=\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n}}{(2n)!}=1-\frac{u^2}{2!}+\frac{u^4}{4!}+\cdots+(-1)^n\frac{u^{2n}}{(2n)!}+\cdots, \tag{B}$$

and

$$\sin u=\sum_{n=0}^{\infty}(-1)^n\frac{u^{2n+1}}{(2n+1)!}=u-\frac{u^3}{3!}+\frac{u^5}{5!}+\cdots+(-1)^n\frac{u^{2n+1}}{(2n+1)!}+\cdots \tag{C}$$

for all real values of u. Even though you have previously considered (A) only for real values of u, we can set $u=i\theta$, where θ is real, to obtain

$$e^{i\theta}=\sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}. \tag{D}$$

Given the proper background in the theory of infinite series with complex terms, it can be shown that the series in (D) converges for all real θ.

a. Recalling that $i^2=-1$, write enough terms of the sequence $\{i^n\}$ to convince yourself that the sequence is repetitive:

$$1,i,-1,-i,1,i,-1,-i,1,i,-1,-i,1,i,-1,-i,\cdots.$$

Use this to group the terms in (D) as

$$\begin{aligned}e^{i\theta}&=\left(1-\frac{\theta^2}{2}+\frac{\theta^4}{4}+\cdots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\cdots\right)\\&=\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n}}{(2n)!}+i\sum_{n=0}^{\infty}(-1)^n\frac{\theta^{2n+1}}{(2n+1)!}.\end{aligned}$$

By comparing this result with (B) and (C), conclude that

$$e^{i\theta}=\cos\theta+i\sin\theta. \tag{E}$$

This is **Euler's identity**.

b. Starting from

$$e^{i\theta_1}e^{i\theta_2}=(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2),$$

collect the real part (the terms not multiplied by i) and the imaginary part (the terms multiplied by i) on the right, and use the trigonometric identities

$$\begin{aligned}\cos(\theta_1+\theta_2)&=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\\\sin(\theta_1+\theta_2)&=\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2\end{aligned}$$

to verify that

$$e^{i(\theta_1+\theta_2)}=e^{i\theta_1}e^{i\theta_2},$$

as you would expect from the use of the exponential notation $e^{i\theta}$.

c. If α and β are real numbers, define

$$e^{\alpha+i\beta} = e^{\alpha}e^{i\beta} = e^{\alpha}(\cos\beta + i\sin\beta). \tag{F}$$

Show that if $z_1 = \alpha_1 + i\beta_1$ and $z_2 = \alpha_2 + i\beta_2$ then

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

d. Let a, b, and c be real numbers, with $a \neq 0$. Let $z = u + iv$ where u and v are real-valued functions of x. Then we say that z is a solution of

$$ay'' + by' + cy = 0 \tag{G}$$

if u and v are both solutions of (G). Use Theorem 5.2.1 (c) to verify that if the characteristic equation of (G) has complex conjugate roots $\lambda \pm i\omega$ then $z_1 = e^{(\lambda+i\omega)x}$ and $z_2 = e^{(\lambda-i\omega)x}$ are both solutions of (G).

This page titled 10.2.1: Constant Coefficient Homogeneous Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.2E: Constant Coefficient Homogeneous Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

10.3: Nonhomogeneous Linear Equations

We'll now consider the nonhomogeneous linear second order equation

$$y'' + p(x)y' + q(x)y = f(x), \tag{10.3.1}$$

where the forcing function f isn't identically zero. The next theorem, an extension of Theorem 5.1.1, gives sufficient conditions for existence and uniqueness of solutions of initial value problems for Equation 10.3.1. We omit the proof, which is beyond the scope of this book.

Theorem 5.3.1 : Uniqueness

Suppose p, , q and f are continuous on an open interval (a, b), let x_0 be any point in (a, b), and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on (a, b).

To find the general solution of Equation 10.3.1 on an interval (a, b) where p, q, and f are continuous, it is necessary to find the general solution of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{10.3.2}$$

on (a, b). We call Equation 10.3.2 the **complementary equation** for Equation 10.3.1.

The next theorem shows how to find the general solution of Equation 10.3.1 if we know one solution y_p of Equation 10.3.1 and a fundamental set of solutions of Equation 10.3.2. We call y_p a **particular solution** of Equation 10.3.1 ; it can be any solution that we can find, one way or another.

Theorem 5.3.2

Suppose p, q, and f are continuous on (a, b). Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \tag{10.3.3}$$

on (a, b), and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation

$$y'' + p(x)y' + q(x)y = 0 \tag{10.3.4}$$

on (a, b). Then y is a solution of (10.3.3) on (a, b) if and only if

$$y = y_p + c_1y_1 + c_2y_2, \tag{10.3.5}$$

where c_1 and c_2 are constants.

Proof

We first show that y in Equation 10.3.5 is a solution of Equation 10.3.3 for any choice of the constants c_1 and c_2. Differentiating Equation 10.3.5 twice yields

$$y' = y_p' + c_1y_1' + c_2y_2' \quad \text{and} \quad y'' = y_p'' + c_1y_1'' + c_2y_2'',$$

so

$$\begin{aligned} y'' + p(x)y' + q(x)y &= (y_p'' + c_1y_1'' + c_2y_2'') + p(x)(y_p' + c_1y_1' + c_2y_2') + q(x)(y_p + c_1y_1 + c_2y_2) \\ &= (y_p'' + p(x)y_p' + q(x)y_p) + c_1(y_1'' + p(x)y_1' + q(x)y_1) + c_2(y_2'' + p(x)y_2' + q(x)y_2) \\ &= f + c_1 \cdot 0 + c_2 \cdot 0 = f, \end{aligned}$$

since y_p satisfies Equation 10.3.3 and y_1 and y_2 satisfy Equation 10.3.4.

Now we'll show that every solution of Equation 10.3.3 has the form Equation 10.3.5 for some choice of the constants c_1 and c_2. Suppose y is a solution of Equation 10.3.3. We'll show that $y-y_p$ is a solution of Equation 10.3.4, and therefore of the form $y-y_p=c_1y_1+c_2y_2$, which implies Equation 10.3.5. To see this, we compute

$$\begin{aligned}(y-y_p)''+p(x)(y-y_p)'+q(x)(y-y_p)&=(y''-y_p'')+p(x)(y'-y_p')+q(x)(y-y_p)\\&=(y''+p(x)y'+q(x)y)-(y_p''+p(x)y_p'+q(x)y_p)\\&=f(x)-f(x)=0,\end{aligned}$$

since y and y_p both satisfy Equation 10.3.3.

We say that Equation 10.3.5 is the **general solution of** (10.3.3) on (a,b).

If P_0, P_1, and F are continuous and P_0 has no zeros on (a,b), then Theorem 5.3.2 implies that the general solution of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=F(x) \tag{10.3.6}$$

on (a,b) is $y=y_p+c_1y_1+c_2y_2$, where y_p is a particular solution of Equation 10.3.6 on (a,b) and $\{y_1,y_2\}$ is a fundamental set of solutions of

$$P_0(x)y''+P_1(x)y'+P_2(x)y=0$$

on (a,b). To see this, we rewrite Equation 10.3.6 as

$$y''+\frac{P_1(x)}{P_0(x)}y'+\frac{P_2(x)}{P_0(x)}y=\frac{F(x)}{P_0(x)}$$

and apply Theorem 5.3.2 with $p=P_1/P_0$, $q=P_2/P_0$, and $f=F/P_0$.

To avoid awkward wording in examples and exercises, we will not specify the interval (a,b) when we ask for the general solution of a specific linear second order equation, or for a fundamental set of solutions of a homogeneous linear second order equation. Let's agree that this always means that we want the general solution (or a fundamental set of solutions, as the case may be) on every open interval on which p, q, and f are continuous if the equation is of the form Equation 10.3.3, or on which P_0, P_1, P_2, and F are continuous and P_0 has no zeros, if the equation is of the form Equation 10.3.6. We leave it to you to identify these intervals in specific examples and exercises.

For completeness, we point out that if P_0, P_1, P_2, and F are all continuous on an open interval (a,b), but P_0 **does** have a zero in (a,b), then Equation 10.3.6 may fail to have a general solution on (a,b) in the sense just defined. **Exercises 5.1.42-5.1.44** illustrate this point for a homogeneous equation.

In this section we to limit ourselves to applications of Theorem 5.3.2 where we can guess at the form of the particular solution.

Example 5.3.1

a. Find the general solution of

$$y''+y=1. \tag{10.3.7}$$

b. Solve the initial value problem

$$y''+y=1,\quad y(0)=2,\quad y'(0)=7. \tag{10.3.8}$$

Solution

a. We can apply Theorem 5.3.2 with $(a,b)=(-\infty,\infty)$, since the functions $p\equiv0$, $q\equiv1$, and $f\equiv1$ in Equation 10.3.7 are continuous on $(-\infty,\infty)$. By inspection we see that $y_p\equiv1$ is a particular solution of Equation 10.3.7. Since $y_1=\cos x$ and $y_2=\sin x$ form a fundamental set of solutions of the complementary equation $y''+y=0$, the general solution of Equation 10.3.7 is

$$y=1+c_1\cos x+c_2\sin x. \tag{10.3.9}$$

b. Imposing the initial condition $y(0) = 2$ in Equation 10.3.9 yields $2 = 1 + c_1$, so $c_1 = 1$. Differentiating Equation 10.3.9 yields

$$y' = -c_1 \sin x + c_2 \cos x.$$

Imposing the initial condition $y'(0) = 7$ here yields $c_2 = 7$, so the solution of Equation 10.3.8 is

$$y = 1 + \cos x + 7 \sin x.$$

Figure 5.3.1 is a graph of this function.

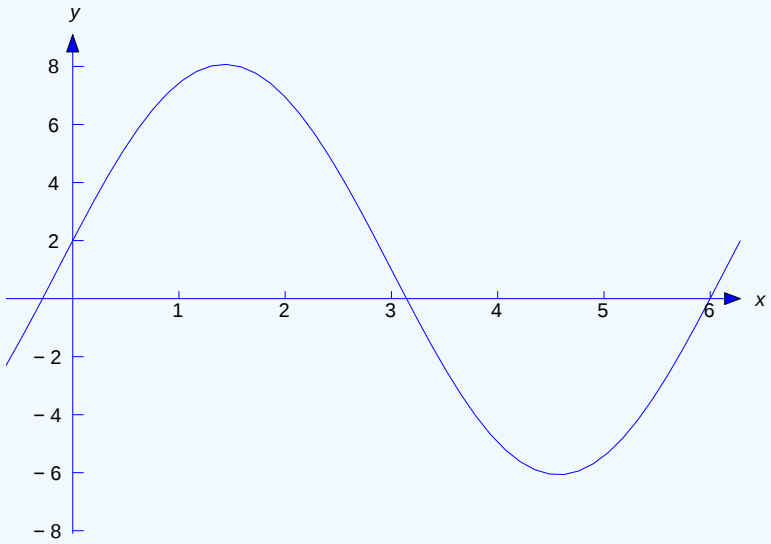


Figure 5.3.1 : $y = 1 + \cos x + 7 \sin x$

Example 5.3.2

a. Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \tag{10.3.10}$$

b. Solve the initial value problem

$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2, \quad y'(0) = 1. \tag{10.3.11}$$

Solution

a. The characteristic polynomial of the complementary equation

$$y'' - 2y' + y = 0$$

is $r^2 - 2r + 1 = (r - 1)^2$, so $y_1 = e^x$ and $y_2 = xe^x$ form a fundamental set of solutions of the complementary equation. To guess a form for a particular solution of Equation 10.3.10, we note that substituting a second degree polynomial $y_p = A + Bx + Cx^2$ into the left side of Equation 10.3.10 will produce another second degree polynomial with coefficients that depend upon A, B, and C. The trick is to choose A, B, and C so the polynomials on the two sides of Equation 10.3.10 have the same coefficients; thus, if

$$y_p = A + Bx + Cx^2 \quad \text{then} \quad y_p' = B + 2Cx \quad \text{and} \quad y_p'' = 2C,$$

so

$$\begin{aligned} y_p'' - 2y_p' + y_p &= 2C - 2(B + 2Cx) + (A + Bx + Cx^2) \\ &= (2C - 2B + A) + (-4C + B)x + Cx^2 = -3 - x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$C = 1$$
$$B - 4C = -1$$
$$A - 2B + 2C = -3,$$

so $C = 1$, $B = -1 + 4C = 3$, and $A = -3 - 2C + 2B = 1$. Therefore $y_p = 1 + 3x + x^2$ is a particular solution of Equation 10.3.10 and Theorem 5.3.2 implies that

$$y = 1 + 3x + x^2 + e^x(c_1 + c_2x) \tag{10.3.12}$$

is the general solution of Equation 10.3.10.

b. Imposing the initial condition $y(0) = -2$ in Equation 10.3.12 yields $-2 = 1 + c_1$, so $c_1 = -3$. Differentiating Equation 10.3.12 yields

$$y' = 3 + 2x + e^x(c_1 + c_2x) + c_2e^x,$$

and imposing the initial condition $y'(0) = 1$ here yields $1 = 3 + c_1 + c_2$, so $c_2 = 1$. Therefore the solution of Equation 10.3.11 is

$$y = 1 + 3x + x^2 - e^x(3 - x).$$

Figure 5.3.2 is a graph of this solution.

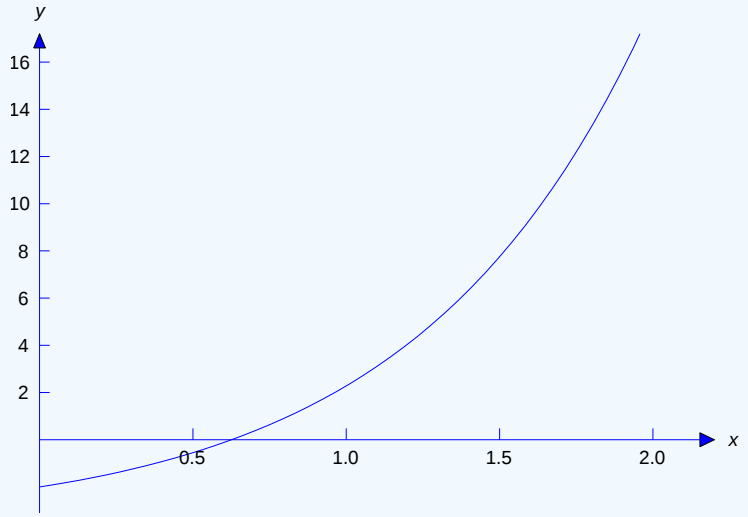


Figure 5.3.2 : $y = 1 + 3x + x^2 - e^x(3 - x)$

Example 5.3.3

Find the general solution of

$$x^2y'' + xy' - 4y = 2x^4 \tag{10.3.13}$$

on $(-\infty, 0)$ and $(0, \infty)$.

Solution

In Example 5.1.3, we verified that $y_1 = x^2$ and $y_2 = 1/x^2$ form a fundamental set of solutions of the complementary equation

$$x^2y'' + xy' - 4y = 0$$

on $(-\infty, 0)$ and $(0, \infty)$. To find a particular solution of Equation 10.3.13, we note that if $y_p = Ax^4$, where A is a constant then both sides of Equation 10.3.13 will be constant multiples of x^4 and we may be able to choose A so the two sides are equal. This is true in this example, since if $y_p = Ax^4$ then

$$x^2y_p'' + xy_p' - 4y_p = x^2(12Ax^2) + x(4Ax^3) - 4Ax^4 = 12Ax^4 = 2x^4$$

if $A = 1/6$; therefore, $y_p = x^4/6$ is a particular solution of Equation 10.3.13 on $(-\infty, \infty)$. Theorem 5.3.2 implies that the general solution of Equation 10.3.13 on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{x^4}{6} + c_1x^2 + \frac{c_2}{x^2}.$$

The Principle of Superposition

The next theorem enables us to break a nonhomogeneous equation into simpler parts, find a particular solution for each part, and then combine their solutions to obtain a particular solution of the original problem.

Theorem 5.3.3 the principle of superposition

Suppose y_{p_1} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on (a, b). Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

on (a, b).

Proof

If $y_p = y_{p_1} + y_{p_2}$ then

$$\begin{aligned} y_p'' + p(x)y_p' + q(x)y_p &= (y_{p_1} + y_{p_2})'' + p(x)(y_{p_1} + y_{p_2})' + q(x)(y_{p_1} + y_{p_2}) \\ &= (y_{p_1}'' + p(x)y_{p_1}' + q(x)y_{p_1}) + (y_{p_2}'' + p(x)y_{p_2}' + q(x)y_{p_2}) \\ &= f_1(x) + f_2(x). \end{aligned}$$

It's easy to generalize Theorem 5.3.3 to the equation

$$y'' + p(x)y' + q(x)y = f(x) \tag{10.3.14}$$

where

$$f = f_1 + f_2 + \cdots + f_k;$$

thus, if y_{p_i} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_i(x)$$

on (a, b) for $i = 1, 2, \ldots, k$, then $y_{p_1} + y_{p_2} + \cdots + y_{p_k}$ is a particular solution of Equation 10.3.14 on (a, b). Moreover, by a proof similar to the proof of Theorem 5.3.3 we can formulate the principle of superposition in terms of a linear equation written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$$

that is, if y_{p_1} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_2(x)$$

on (a, b), then $y_{p_1} + y_{p_2}$ is a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x) + F_2(x)$$

on (a, b).

Example 5.3.4

The function $y_{p_1} = x^4/15$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 \tag{10.3.15}$$

on $(-\infty, \infty)$ and $y_{p_2} = x^2/3$ is a particular solution of

$$x^2y'' + 4xy' + 2y = 4x^2 \tag{10.3.16}$$

on $(-\infty, \infty)$. Use the principle of superposition to find a particular solution of

$$x^2y'' + 4xy' + 2y = 2x^4 + 4x^2 \tag{10.3.17}$$

on $(-\infty, \infty)$.

Solution

The right side $F(x) = 2x^4 + 4x^2$ in Equation 10.3.17 is the sum of the right sides

$$F_1(x) = 2x^4 \quad \text{and} \quad F_2(x) = 4x^2.$$

in Equation 10.3.15 and Equation 10.3.16. Therefore the principle of superposition implies that

$$y_p = y_{p_1} + y_{p_2} = \frac{x^4}{15} + \frac{x^2}{3}$$

is a particular solution of Equation 10.3.17.

This page titled 10.3: Nonhomogeneous Linear Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.3: Nonhomogeneous Linear Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

10.3.1: Nonhomgeneous Linear Equations (Exercises)

Q5.3.1

In **Exercises 5.3.1-5.3.6** find a particular solution by the method used in Example 5.3.2. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

1. $y'' + 5y' - 6y = 22 + 18x - 18x^2$

2. $y'' - 4y' + 5y = 1 + 5x$

3. $y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3$

4. $y'' - 4y' + 4y = 2 + 8x - 4x^2$

5. $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3, \quad y(0) = 2, \quad y'(0) = 9$

6. $y'' + 6y' + 10y = 22 + 20x, \quad y(0) = 2,\ y'(0) = -2$

Q5.3.2

7. Show that the method used in Example 5.3.2 will not yield a particular solution of

$$y'' + y' = 1 + 2x + x^2; \tag{A}$$

that is, (A) does'nt have a particular solution of the form $y_p = A + Bx + Cx^2$, where A, B, and C are constants.

Q5.3.3

In **Exercises 5.3.8-5.3.13** find a particular solution by the method used in Example 5.3.3.

8. $x^2y'' + 7xy' + 8y = \frac{6}{x}$

9. $x^2y'' - 7xy' + 7y = 13x^{1/2}$

10. $x^2y'' - xy' + y = 2x^3$

11. $x^2y'' + 5xy' + 4y = \frac{1}{x^3}$

12. $x^2y'' + xy' + y = 10x^{1/3}$

13. $x^2y'' - 3xy' + 13y = 2x^4$

Q5.3.4

14. Show that the method suggested for finding a particular solution in **Exercises 5.3.8-5.3.13** will not yield a particular solution of

$$x^2y'' + 3xy' - 3y = \frac{1}{x^3}; \tag{A}$$

that is, (A) doesn't have a particular solution of the form $y_p = A/x^3$.

15. Prove: If a, b, c, α, and M are constants and $M \neq 0$ then

$$ax^2y'' + bxy' + cy = Mx^\alpha$$

has a particular solution $y_p = Ax^\alpha$ ($A =$ constant) if and only if $a\alpha(\alpha - 1) + b\alpha + c \neq 0$.

Q5.3.5

If $a, b, c,$ and α are constants, then

$$\alpha(e^{\alpha x})'' + b(e^{\alpha x})' + ce^{\alpha x} = (a\alpha^2 + b\alpha + c)e^{\alpha x}.$$

Use this in **Exercises 5.3.16-5.3.21** to find a particular solution. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

16. $y'' + 5y' - 6y = 6e^{3x}$

17. $y'' - 4y' + 5y = e^{2x}$

18. $y'' + 8y' + 7y = 10e^{-2x}, \quad y(0) = -2,\ y'(0) = 10$

19. $y'' - 4y' + 4y = e^{x}, \quad y(0) = 2, \quad y'(0) = 0$

20. $y'' + 2y' + 10y = e^{x/2}$

21. $y'' + 6y' + 10y = e^{-3x}$

Q5.3.6

22. Show that the method suggested for finding a particular solution in **Exercises 5.3.16-5.3.21** will not yield a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}; \tag{A}$$

that is, (A) doesn't have a particular solution of the form $y_p = Ae^{4x}$.

23. Prove: If α and M are constants and $M \neq 0$ then constant coefficient equation

$$ay'' + by' + cy = Me^{\alpha x}$$

has a particular solution $y_p = Ae^{\alpha x}$ ($A =$ constant) if and only if $e^{\alpha x}$ isn't a solution of the complementary equation.

Q5.3.7

If ω is a constant, differentiating a linear combination of $\cos \omega x$ and $\sin \omega x$ with respect to x yields another linear combination of $\cos \omega x$ and $\sin \omega x$. In **Exercises 5.3.24-5.3.29** use this to find a particular solution of the equation. Then find the general solution and, where indicated, solve the initial value problem and graph the solution.

24. $y'' - 8y' + 16y = 23 \cos x - 7 \sin x$

25. $y'' + y' = -8 \cos 2x + 6 \sin 2x$

26. $y'' - 2y' + 3y = -6 \cos 3x + 6 \sin 3x$

27. $y'' + 6y' + 13y = 18 \cos x + 6 \sin x$

28. $y'' + 7y' + 12y = -2 \cos 2x + 36 \sin 2x, \quad y(0) = -3, \quad y'(0) = 3$

29. $y'' - 6y' + 9y = 18 \cos 3x + 18 \sin 3x, \quad y(0) = 2, \quad y'(0) = 2$

Q5.3.8

30. Find the general solution of

$$y'' + \omega_0^2 y = M \cos \omega x + N \sin \omega x,$$

where M and N are constants and ω and ω_0 are distinct positive numbers.

31. Show that the method suggested for finding a particular solution in **Exercises 5.3.24-5.3.29** will not yield a particular solution of

$$y'' + y = \cos x + \sin x; \tag{A}$$

that is, (A) does not have a particular solution of the form $y_p = A \cos x + B \sin x$.

32. Prove: If M, N are constants (not both zero) and $\omega > 0$, the constant coefficient equation

$$ay'' + by' + cy = M \cos \omega x + N \sin \omega x \tag{A}$$

has a particular solution that's a linear combination of $\cos \omega x$ and $\sin \omega x$ if and only if the left side of (A) is not of the form $a(y'' + \omega^2 y)$, so that $\cos \omega x$ and $\sin \omega x$ are solutions of the complementary equation.

Q5.3.9

In **Exercises 5.3.33-5.3.38** refer to the cited exercises and use the principal of superposition to find a particular solution. Then find the general solution.

33. $y'' + 5y' - 6y = 22 + 18x - 18x^2 + 6e^{3x}$ (See **Exercises 5.3.1** and **5.3.16**.)

34. $y'' - 4y' + 5y = 1 + 5x + e^{2x}$ (See **Exercises 5.3.2** and **5.3.17.**)

35. $y'' + 8y' + 7y = -8 - x + 24x^2 + 7x^3 + 10e^{-2x}$ (See **Exercises 5.3.3** and **5.3.18**.)

36. $y'' - 4y' + 4y = 2 + 8x - 4x^2 + e^x$ (See **Exercises 5.3.4** and **5.3.19**.)

37. $y'' + 2y' + 10y = 4 + 26x + 6x^2 + 10x^3 + e^{x/2}$ (See **Exercises 5.3.5** and **5.3.20**.)

38. $y'' + 6y' + 10y = 22 + 20x + e^{-3x}$ (See **Exercises 5.3.6** and **5.3.21**.)

Q5.3.10

39. Prove: If y_{p_1} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_2(x)$$

on (a, b), then $y_p = y_{p_1} + y_{p_2}$ is a solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F_1(x) + F_2(x)$$

on (a, b).

40. Suppose p, q, and f are continuous on (a, b). Let y_1, y_2, and y_p be twice differentiable on (a, b), such that $y = c_1y_1 + c_2y_2 + y_p$ is a solution of

$$y'' + p(x)y' + q(x)y = f$$

on (a, b) for every choice of the constants c_1, c_2. Show that y_1 and y_2 are solutions of the complementary equation on (a, b).

This page titled 10.3.1: Nonhomgeneous Linear Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.3E: Nonhomgeneous Linear Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

10.4: The Method of Undetermined Coefficients I

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x}G(x), \tag{10.4.1}$$

where α is a constant and G is a polynomial.

From Theorem 5.3.2, the general solution of Equation 10.4.1 is $y = y_p + c_1y_1 + c_2y_2$, where y_p is a particular solution of Equation 10.4.1 and $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation

$$ay'' + by' + cy = 0.$$

In Section 5.2 we showed how to find $\{y_1, y_2\}$. In this section we'll show how to find y_p. The procedure that we'll use is called **the method of undetermined coefficients**. Our first example is similar to **Exercises 5.3.16-5.3.21**.

Example 5.4.1

Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{10.4.2}$$

Then find the general solution.

Solution

Substituting $y_p = Ae^{2x}$ for y in Equation 10.4.2 will produce a constant multiple of Ae^{2x} on the left side of Equation 10.4.2, so it may be possible to choose A so that y_p is a solution of Equation 10.4.2. Let's try it; if $y_p = Ae^{2x}$ then

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 2Ae^{2x} = 4e^{2x}$$

if $A = 2$. Therefore $y_p = 2e^{2x}$ is a particular solution of Equation 10.4.2. To find the general solution, we note that the characteristic polynomial of the complementary equation

$$y'' - 7y' + 12y = 0 \tag{10.4.3}$$

is $p(r) = r^2 - 7r + 12 = (r - 3)(r - 4)$, so $\{e^{3x}, e^{4x}\}$ is a fundamental set of solutions of Equation 10.4.3. Therefore the general solution of Equation 10.4.2 is

$$y = 2e^{2x} + c_1e^{3x} + c_2e^{4x}.$$

Example 5.4.2

Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{10.4.4}$$

Then find the general solution.

Solution

Fresh from our success in finding a particular solution of Equation 10.4.2 — where we chose $y_p = Ae^{2x}$ because the right side of Equation 10.4.2 is a constant multiple of e^{2x} — it may seem reasonable to try $y_p = Ae^{4x}$ as a particular solution of Equation 10.4.4. However, this will not work, since we saw in Example 5.4.1 that e^{4x} is a solution of the complementary equation Equation 10.4.3, so substituting $y_p = Ae^{4x}$ into the left side of Equation 10.4.4) produces zero on the left, no matter how we choose$A$. To discover a suitable form for y_p, we use the same approach that we used in Section 5.2 to find a second solution of

$$ay'' + by' + cy = 0$$

in the case where the characteristic equation has a repeated real root: we look for solutions of Equation 10.4.4 in the form $y = ue^{4x}$, where u is a function to be determined. Substituting

$$y = ue^{4x}, \quad y' = u'e^{4x} + 4ue^{4x}, \quad \text{and} \quad y'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x} \tag{10.4.5}$$

into Equation 10.4.4 and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 7(u' + 4u) + 12u = 5,$$

or

$$u'' + u' = 5.$$

By inspection we see that $u_p = 5x$ is a particular solution of this equation, so \(y_p=5xe^{4x}\) is a particular solution of Equation 10.4.4. Therefore

$$y = 5xe^{4x} + c_1e^{3x} + c_2e^{4x}$$

is the general solution.

Example 5.4.3

Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{10.4.6}$$

Solution

Since the characteristic polynomial of the complementary equation

$$y'' - 8y' + 16y = 0 \tag{10.4.7}$$

is $p(r) = r^2 - 8r + 16 = (r-4)^2$, both $y_1 = e^{4x}$ and $y_2 = xe^{4x}$ are solutions of Equation 10.4.7. Therefore Equation 10.4.6) does not have a solution of the form $y_p = Ae^{4x}$ or $y_p = Axe^{4x}$. As in Example 5.4.2 , we look for solutions of Equation 10.4.6 in the form $y = ue^{4x}$, where u is a function to be determined. Substituting from Equation 10.4.5 into Equation 10.4.6 and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 8(u' + 4u) + 16u = 2,$$

or

$$u'' = 2.$$

Integrating twice and taking the constants of integration to be zero shows that $u_p = x^2$ is a particular solution of this equation, so $y_p = x^2e^{4x}$ is a particular solution of Equation 10.4.4. Therefore

$$y = e^{4x}(x^2 + c_1 + c_2x)$$

is the general solution.

The preceding examples illustrate the following facts concerning the form of a particular solution y_p of a constant coefficent equation

$$ay'' + by' + cy = ke^{\alpha x},$$

where k is a nonzero constant:

a. If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay'' + by' + cy = 0, \tag{10.4.8}$$

then $y_p = Ae^{\alpha x}$, where A is a constant. (See Example 5.4.1).

b. If $e^{\alpha x}$ is a solution of Equation 10.4.8 but $xe^{\alpha x}$ is not, then $y_p = Axe^{\alpha x}$, where A is a constant. (See Example 5.4.2 .)

c. If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of Equation 10.4.8, then $y_p = Ax^2e^{\alpha x}$, where A is a constant. (See Example 5.4.3 .)

See **Exercise 5.4.30** for the proofs of these facts.

In all three cases you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p'' + by_p' + cy_p = ke^{\alpha x},$$

and solve for the constant A, as we did in Example 5.4.1 . (See **Exercises 5.4.31-5.4.33**.) However, if the equation is

$$ay'' + by' + cy = ke^{\alpha x}G(x),$$

where G is a polynomial of degree greater than zero, we recommend that you use the substitution $y = ue^{\alpha x}$ as we did in Examples 5.4.2 and 5.4.3 . The equation for u will turn out to be

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x), \tag{10.4.9}$$

where $p(r) = ar^2 + br + c$ is the characteristic polynomial of the complementary equation and $p'(r) = 2ar + b$ (**Exercise 5.4.30**); however, you shouldn't memorize this since it is easy to derive the equation for u in any particular case. Note, however, that if $e^{\alpha x}$ is a solution of the complementary equation then $p(\alpha) = 0$, so Equation 10.4.9 reduces to

$$au'' + p'(\alpha)u' = G(x),$$

while if both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation then $p(r) = a(r-\alpha)^2$ and $p'(r) = 2a(r-\alpha)$, so $p(\alpha) = p'(\alpha) = 0$ and Equation 10.4.9) reduces to

$$au'' = G(x).$$

Example 5.4.4

Find a particular solution of

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2). \tag{10.4.10}$$

Solution

Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and} y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into Equation 10.4.10) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 3(u' + 3u) + 2u = -1 + 2x + x^2,$$

or

$$u'' + 3u' + 2u = -1 + 2x + x^2. \tag{10.4.11}$$

As in Example 5.3.2, in order to guess a form for a particular solution of Equation 10.4.11), we note that substituting a second degree polynomial $u_p = A + Bx + Cx^2$ for u in the left side of Equation 10.4.11) produces another second degree polynomial with coefficients that depend upon A, B, and C; thus,

$$\text{if} \quad u_p = A + Bx + Cx^2 \quad \text{then} \quad u_p' = B + 2Cx \quad \text{and} \quad u_p'' = 2C.$$

If u_p is to satisfy Equation 10.4.11), we must have

$$\begin{aligned} u_p'' + 3u_p' + 2u_p &= 2C + 3(B + 2Cx) + 2(A + Bx + Cx^2) \\ &= (2C + 3B + 2A) + (6C + 2B)x + 2Cx^2 = -1 + 2x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} 2C &= 1 \\ 2B + 6C &= 2 \\ 2A + 3B + 2C &= -1. \end{aligned}$$

Solving these equations for C, B, and A (in that order) yields $C = 1/2, B = -1/2, A = -1/4$. Therefore

$$u_p = -\frac{1}{4}(1 + 2x - 2x^2)$$

is a particular solution of Equation 10.4.11, and

$$y_p = u_p e^{3x} = -\frac{e^{3x}}{4}(1 + 2x - 2x^2)$$

is a particular solution of Equation 10.4.10.

Example 5.4.5

Find a particular solution of

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2). \tag{10.4.12}$$

Solution

Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and } y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into Equation 10.4.12) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 4(u' + 3u) + 3u = 6 + 8x + 12x^2,$$

or

$$u'' + 2u' = 6 + 8x + 12x^2. \tag{10.4.13}$$

There's no u term in this equation, since e^{3x} is a solution of the complementary equation for Equation 10.4.12). (See **Exercise 5.4.30**.) Therefore Equation 10.4.13) does not have a particular solution of the form $u_p = A + Bx + Cx^2$ that we used successfully in Example 5.4.4 , since with this choice of u_p,

$$u_p'' + 2u_p' = 2C + (B + 2Cx)$$

can't contain the last term ($12x^2$) on the right side of Equation 10.4.13). Instead, let's try $u_p = Ax + Bx^2 + Cx^3$ on the grounds that

$$u_p' = A + 2Bx + 3Cx^2 \quad \text{and} \quad u_p'' = 2B + 6Cx$$

together contain all the powers of x that appear on the right side of Equation 10.4.13).

Substituting these expressions in place of u' and u'' in Equation 10.4.13) yields

$$(2B + 6Cx) + 2(A + 2Bx + 3Cx^2) = (2B + 2A) + (6C + 4B)x + 6Cx^2 = 6 + 8x + 12x^2.$$

Comparing coefficients of like powers of x on the two sides of the last equality shows that u_p satisfies Equation 10.4.13) if

$$\begin{aligned} 6C &= 12 \\ 4B + 6C &= 8 \\ 2A + 2B &= 6. \end{aligned}$$

Solving these equations successively yields $C = 2$, $B = -1$, and $A = 4$. Therefore

$$u_p = x(4 - x + 2x^2)$$

is a particular solution of Equation 10.4.13), and

$$y_p = u_p e^{3x} = xe^{3x}(4 - x + 2x^2)$$

is a particular solution of Equation 10.4.12).

Example 5.4.6

Find a particular solution of

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2). \tag{10.4.14}$$

Solution

Substituting

$$y = ue^{-x/2}, \quad y' = u'e^{-x/2} - \frac{1}{2}ue^{-x/2}, \quad \text{and} \quad y'' = u''e^{-x/2} - u'e^{-x/2} + \frac{1}{4}ue^{-x/2}$$

into Equation 10.4.14) and canceling $e^{-x/2}$ yields

$$4\left(u'' - u' + \frac{u}{4}\right) + 4\left(u' - \frac{u}{2}\right) + u = 4u'' = -8 + 48x + 144x^2,$$

or

$$u'' = -2 + 12x + 36x^2, \tag{10.4.15}$$

which does not contain u or u' because $e^{-x/2}$ and $xe^{-x/2}$ are both solutions of the complementary equation. (See **Exercise 5.4.30**.) To obtain a particular solution of Equation 10.4.15) we integrate twice, taking the constants of integration to be zero; thus,

$$u_p' = -2x + 6x^2 + 12x^3 \quad \text{and} \quad u_p = -x^2 + 2x^3 + 3x^4 = x^2(-1 + 2x + 3x^2).$$

Therefore

$$y_p = u_p e^{-x/2} = x^2 e^{-x/2}(-1 + 2x + 3x^2)$$

is a particular solution of Equation 10.4.14).

Summary

The preceding examples illustrate the following facts concerning particular solutions of a constant coefficent equation of the form

$$ay'' + by' + cy = e^{\alpha x}G(x),$$

where G is a polynomial (see **Exercise 5.4.30**):

a. If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay'' + by' + cy = 0, \tag{10.4.16}$$

then $y_p = e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 5.4.4).

b. If $e^{\alpha x}$ is a solution of Equation 10.4.16 but $xe^{\alpha x}$ is not, then $y_p = xe^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 5.4.5 .)

c. If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of Equation 10.4.16, then $y_p = x^2e^{\alpha x}Q(x)$, where Q is a polynomial of the same degree as G. (See Example 5.4.6 .)

In all three cases, you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p'' + by_p' + cy_p = e^{\alpha x}G(x),$$

and solve for the coefficients of the polynomial Q. However, if you try this you will see that the computations are more tedious than those that you encounter by making the substitution $y = ue^{\alpha x}$ and finding a particular solution of the resulting equation for u. (See **Exercises 5.4.34-5.4.36**.) In Case (a) the equation for u will be of the form

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x),$$

with a particular solution of the form $u_p = Q(x)$, a polynomial of the same degree as G, whose coefficients can be found by the method used in Example 5.4.4 . In Case (b) the equation for u will be of the form

$$au'' + p'(\alpha)u' = G(x)$$

(no u term on the left), with a particular solution of the form $u_p = xQ(x)$, where Q is a polynomial of the same degree as G whose coefficents can be found by the method used in Example 5.4.5 . In Case (c), the equation for u will be of the form

$$au'' = G(x)$$

with a particular solution of the form $u_p = x^2Q(x)$ that can be obtained by integrating $G(x)/a$ twice and taking the constants of integration to be zero, as in Example 5.4.6 .

Using the Principle of Superposition

The next example shows how to combine the method of undetermined coefficients and Theorem 5.3.3, the principle of superposition.

Example 5.4.7

Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}. \tag{10.4.17}$$

Solution

In Example 5.4.1 we found that $y_{p_1} = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x},$$

and in Example 5.4.2 we found that $y_{p_2} = 5xe^{4x}$ is a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

Therefore the principle of superposition implies that $y_p = 2e^{2x} + 5xe^{4x}$ is a particular solution of Equation 10.4.17).

This page titled 10.4: The Method of Undetermined Coefficients I is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.4: The Method of Undetermined Coefficients I** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

10.4.1: The Method of Undetermined Coefficients I (Exercises)

Q5.4.1

In **Exercises 5.4.1-5.4.14** find a particular solution.

1. $y'' - 3y' + 2y = e^{3x}(1 + x)$
2. $y'' - 6y' + 5y = e^{-3x}(35 - 8x)$
3. $y'' - 2y' - 3y = e^{x}(-8 + 3x)$
4. $y'' + 2y' + y = e^{2x}(-7 - 15x + 9x^2)$
5. $y'' + 4y = e^{-x}(7 - 4x + 5x^2)$
6. $y'' - y' - 2y = e^{x}(9 + 2x - 4x^2)$
7. $y'' - 4y' - 5y = -6xe^{-x}$
8. $y'' - 3y' + 2y = e^{x}(3 - 4x)$
9. $y'' + y' - 12y = e^{3x}(-6 + 7x)$
10. $2y'' - 3y' - 2y = e^{2x}(-6 + 10x)$
11. $y'' + 2y' + y = e^{-x}(2 + 3x)$
12. $y'' - 2y' + y = e^{x}(1 - 6x)$
13. $y'' - 4y' + 4y = e^{2x}(1 - 3x + 6x^2)$
14. $9y'' + 6y' + y = e^{-x/3}(2 - 4x + 4x^2)$

Q5.4.2

In **Exercises 5.4.15-5.4.19** find the general solution.

15. $y'' - 3y' + 2y = e^{3x}(1 + x)$
16. $y'' - 6y' + 8y = e^{x}(11 - 6x)$
17. $y'' + 6y' + 9y = e^{2x}(3 - 5x)$
18. $y'' + 2y' - 3y = -16xe^{x}$
19. $y'' - 2y' + y = e^{x}(2 - 12x)$

Q5.4.3

In **Exercises 5.4.20-5.4.23** solve the initial value problem and plot the solution.

20. $y'' - 4y' - 5y = 9e^{2x}(1 + x), \quad y(0) = 0, \quad y'(0) = -10$
21. $y'' + 3y' - 4y = e^{2x}(7 + 6x), \quad y(0) = 2, \quad y'(0) = 8$
22. $y'' + 4y' + 3y = -e^{-x}(2 + 8x), \quad y(0) = 1, \quad y'(0) = 2$
23. $y'' - 3y' - 10y = 7e^{-2x}, \quad y(0) = 1, \quad y'(0) = -17$

Q5.4.4

In **Exercises 5.4.24-5.4.29** use the principle of superposition to find a particular solution.

24. $y'' + y' + y = xe^{x} + e^{-x}(1 + 2x)$
25. $y'' - 7y' + 12y = -e^{x}(17 - 42x) - e^{3x}$
26. $y'' - 8y' + 16y = 6xe^{4x} + 2 + 16x + 16x^2$

27. $y'' - 3y' + 2y = -e^{2x}(3 + 4x) - e^x$

28. $y'' - 2y' + 2y = e^x(1 + x) + e^{-x}(2 - 8x + 5x^2)$

29. $y'' + y = e^{-x}(2 - 4x + 2x^2) + e^{3x}(8 - 12x - 10x^2)$

Q5.4.5

30.

a. Prove that y is a solution of the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x}G(x) \tag{A}$$

if and only if $y = ue^{\alpha x}$, where u satisfies

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x) \tag{B}$$

and $p(r) = ar^2 + br + c$ is the characteristic polynomial of the complementary equation

$$ay'' + by' + cy = 0.$$

For the rest of this exercise, let G be a polynomial. Give the requested proofs for the case where

$$G(x) = g_0 + g_1x + g_2x^2 + g_3x^3.$$

b. Prove that if $e^{\alpha x}$ isn't a solution of the complementary equation then (B) has a particular solution of the form $u_p = A(x)$, where A is a polynomial of the same degree as G, as in Example 5.4.4. Conclude that (A) has a particular solution of the form $y_p = e^{\alpha x}A(x)$.

c. Show that if $e^{\alpha x}$ is a solution of the complementary equation and $xe^{\alpha x}$ isn't , then (B) has a particular solution of the form $u_p = xA(x)$, where A is a polynomial of the same degree as G, as in Example 5.4.5. Conclude that (A) has a particular solution of the form $y_p = xe^{\alpha x}A(x)$.

d. Show that if $e^{\alpha x}$ and $xe^{\alpha x}$ are both solutions of the complementary equation then (B) has a particular solution of the form $u_p = x^2A(x)$, where A is a polynomial of the same degree as G, and $x^2A(x)$ can be obtained by integrating G/a twice, taking the constants of integration to be zero, as in Example 5.4.6. Conclude that (A) has a particular solution of the form $y_p = x^2e^{\alpha x}A(x)$.

Q5.4.6

Exercises 5.4.31–5.4.36 treat the equations considered in Examples 5.4.1–5.4.6. Substitute the suggested form of y_p into the equation and equate the resulting coefficients of like functions on the two sides of the resulting equation to derive a set of simultaneous equations for the coefficients in y_p. Then solve for the coefficients to obtain y_p. Compare the work you've done with the work required to obtain the same results in Examples 5.4.1–5.4.6.

31. Compare with Example 5.4.1:

$$y'' - 7y' + 12y = 4e^{2x}; \quad y_p = Ae^{2x}$$

32. Compare with Example 5.4.2:

$$y'' - 7y' + 12y = 5e^{4x}; \quad y_p = Axe^{4x}$$

33. Compare with Example 5.4.3:

$$y'' - 8y' + 16y = 2e^{4x}; \quad y_p = Ax^2e^{4x}$$

34. Compare with Example 5.4.4:

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2), \quad y_p = e^{3x}(A + Bx + Cx^2)$$

35. Compare with Example 5.4.5:

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2), \quad y_p = e^{3x}(Ax + Bx^2 + Cx^3)$$

36. Compare with Example 5.4.6:

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2), \quad y_p = e^{-x/2}(Ax^2 + Bx^3 + Cx^4)$$

Q5.4.7

37. Write $y = ue^{\alpha x}$ to find the general solution.

a. $y'' + 2y' + y = \frac{e^{-x}}{\sqrt{x}}$
b. $y'' + 6y' + 9y = e^{-3x} \ln x$
c. $y'' - 4y' + 4y = \frac{e^{2x}}{1+x}$
d. $4y'' + 4y' + y = 4e^{-x/2}\left(\frac{1}{x} + x\right)$

38. Suppose $\alpha \neq 0$ and k is a positive integer. In most calculus books integrals like $\int x^k e^{\alpha x}\, dx$ are evaluated by integrating by parts k times. This exercise presents another method. Let

$$y = \int e^{\alpha x} P(x)\, dx$$

with

$$P(x) = p_0 + p_1 x + \cdots + p_k x^k$$

(where $p_k \neq 0$).

a. Show that $y = e^{\alpha x} u$, where

$$u' + \alpha u = P(x). \tag{A}$$

b. Show that (A) has a particular solution of the form

$$u_p = A_0 + A_1 x + \cdots + A_k x^k,$$

where A_k, A_{k-1}, …, A_0 can be computed successively by equating coefficients of $x^k, x^{k-1}, \ldots, 1$ on both sides of the equation

$$u_p' + \alpha u_p = P(x).$$

c. Conclude that

$$\int e^{\alpha x} P(x)\, dx = \left(A_0 + A_1 x + \cdots + A_k x^k\right) e^{\alpha x} + c,$$

where c is a constant of integration.

39. Use the method of **Exercise 5.4.38** to evaluate the integral.

a. $\int e^x (4 + x) dx$
b. $\int e^{-x}(-1 + x^2) dx$
c. $\int x^3 e^{-2x} dx$
d. $\int e^x (1 + x)^2 dx$
e. $\int e^{3x}(-14 + 30x + 27x^2) dx$
f. $\int e^{-x}(1 + 6x^2 - 14x^3 + 3x^4) dx$

40. Use the method suggested in **Exercise 5.4.38** to evaluate $\int x^k e^{\alpha x}\, dx$, where k is an arbitrary positive integer and $\alpha \neq 0$.

This page titled 10.4.1: The Method of Undetermined Coefficients I (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.4E: The Method of Undetermined Coefficients I (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

10.5: The Method of Undetermined Coefficients II

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{10.5.1}$$

where λ and ω are real numbers, $\omega \neq 0$, and P and Q are polynomials. We want to find a particular solution of Equation 10.5.1. As in Section 5.4, the procedure that we will use is called **the method of undetermined coefficients**.

Forcing Functions Without Exponential Factors

We begin with the case where $\lambda = 0$ in Equation 10.5.1 ; thus, we we want to find a particular solution of

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x, \tag{10.5.2}$$

where P and Q are polynomials.

Differentiating $x^r\cos\omega x$ and $x^r\sin\omega x$ yields

$$\frac{d}{dx}x^r\cos\omega x = -\omega x^r\sin\omega x + rx^{r-1}\cos\omega x$$

and

$$\frac{d}{dx}x^r\sin\omega x = \quad \omega x^r\cos\omega x + rx^{r-1}\sin\omega x.$$

This implies that if

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x$$

where A and B are polynomials, then

$$ay_p'' + by_p' + cy_p = F(x)\cos\omega x + G(x)\sin\omega x,$$

where F and G are polynomials with coefficients that can be expressed in terms of the coefficients of A and B. This suggests that we try to choose A and B so that $F = P$ and $G = Q$, respectively. Then y_p will be a particular solution of Equation 10.5.2. The next theorem tells us how to choose the proper form for y_p. For the proof see *Exercise 5.5.37*.

Theorem 5.5.1

Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x, \tag{10.5.3}$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \text{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

provided that $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation. The solutions of

$$a(y'' + \omega^2 y) = P(x)\cos\omega x + Q(x)\sin\omega x$$

for which $\cos\omega x$ and $\sin\omega x$ are solutions of the complementary equation are of the form of Equation 10.5.3, where

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \text{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}.$$

For an analog of this theorem that's applicable to Equation 10.5.1, see **Exercise 5.5.38**.

Example 5.5.1

Find a particular solution of

$$y'' - 2y' + y = 5\cos 2x + 10\sin 2x. \tag{10.5.4}$$

Solution

In Equation 10.5.4 the coefficients of $\cos 2x$ and $\sin 2x$ are both zero degree polynomials (constants). Therefore Theorem 5.5.1 implies that Equation 10.5.4 has a particular solution

$$y_p = A\cos 2x + B\sin 2x.$$

Since

$$y_p' = -2A\sin 2x + 2B\cos 2x \quad \text{and} \quad y_p'' = -4(A\cos 2x + B\sin 2x),$$

replacing y by y_p in Equation 10.5.4 yields

$$\begin{aligned} y_p'' - 2y_p' + y_p &= -4(A\cos 2x + B\sin 2x) - 4(-A\sin 2x + B\cos 2x) + (A\cos 2x + B\sin 2x) \\ &= (-3A - 4B)\cos 2x + (4A - 3B)\sin 2x. \end{aligned}$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ here with the corresponding coefficients on the right side of Equation 10.5.4 shows that y_p is a solution of Equation 10.5.4 if

$$\begin{aligned} -3A - 4B &= \ \ 5 \\ 4A - 3B &= 10. \end{aligned}$$

Solving these equations yields $A = 1$, $B = -2$. Therefore

$$y_p = \cos 2x - 2\sin 2x$$

is a particular solution of Equation 10.5.4.

Example 5.5.2

Find a particular solution of

$$y'' + 4y = 8\cos 2x + 12\sin 2x. \tag{10.5.5}$$

Solution

The procedure used in Example 5.5.1 doesn't work here; substituting $y_p = A\cos 2x + B\sin 2x$ for y in Equation 10.5.5 yields

$$y_p'' + 4y_p = -4(A\cos 2x + B\sin 2x) + 4(A\cos 2x + B\sin 2x) = 0$$

for any choice of A and B, since $\cos 2x$ and $\sin 2x$ are both solutions of the complementary equation for Equation 10.5.5. We're dealing with the second case mentioned in Theorem 5.5.1 , and should therefore try a particular solution of the form

$$y_p = x(A\cos 2x + B\sin 2x). \tag{10.5.6}$$

Then

$$\begin{aligned} y_p' &= A\cos 2x + B\sin 2x + 2x(-A\sin 2x + B\cos 2x) \quad \text{and} \\ y_p'' &= -4A\sin 2x + 4B\cos 2x - 4x(A\cos 2x + B\sin 2x) \\ &= -4A\sin 2x + 4B\cos 2x - 4y_p \ \text{(see (10.5.6))}, \end{aligned}$$

so

$$y_p'' + 4y_p = -4A\sin 2x + 4B\cos 2x.$$

Therefore y_p is a solution of Equation 10.5.5 if

$$-4A\sin 2x + 4B\cos 2x = 8\cos 2x + 12\sin 2x,$$

which holds if $A = -3$ and $B = 2$. Therefore

$$y_p = -x(3\cos 2x - 2\sin 2x)$$

is a particular solution of Equation 10.5.5.

Example 5.5.3

Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x)\cos x + 10\sin x. \tag{10.5.7}$$

Solution

The coefficients of $\cos x$ and $\sin x$ in Equation 10.5.7 are polynomials of degree one and zero, respectively. Therefore Theorem 5.5.1 tells us to look for a particular solution of Equation 10.5.7 of the form

$$y_p = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x. \tag{10.5.8}$$

Then

$$y_p' = (A_1 + B_0 + B_1x)\cos x + (B_1 - A_0 - A_1x)\sin x \tag{10.5.9}$$

and

$$y_p'' = (2B_1 - A_0 - A_1x)\cos x - (2A_1 + B_0 + B_1x)\sin x, \tag{10.5.10}$$

so

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= [A_0 + 3A_1 + 3B_0 + 2B_1 + (A_1 + 3B_1)x]\cos x \\ &\quad + [B_0 + 3B_1 - 3A_0 - 2A_1 + (B_1 - 3A_1)x]\sin x. \end{aligned} \tag{10.5.11}$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in Equation 10.5.7 shows that y_p is a solution of Equation 10.5.7 if

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + B_0 - 2A_1 + 3B_1 &= 10. \end{aligned}$$

Solving the first two equations yields $A_1 = 2$, $B_1 = 6$. Substituting these into the last two equations yields

$$\begin{aligned} A_0 + 3B_0 &= 16 - 3A_1 - 2B_1 = -2 \\ -3A_0 + B_0 &= 10 + 2A_1 - 3B_1 = -4. \end{aligned}$$

Solving these equations yields $A_0 = 1$, $B_0 = -1$. Substituting $A_0 = 1$, $A_1 = 2$, $B_0 = -1$, $B_1 = 6$ into Equation 10.5.8 shows that

$$y_p = (1 + 2x)\cos x - (1 - 6x)\sin x$$

is a particular solution of Equation 10.5.7.

A Useful Observation

In Equations 10.5.9, 10.5.10, and 10.5.11 the polynomials multiplying $\sin x$ can be obtained by replacing A_0, A_1, B_0, and B_1 by B_0, B_1, $-A_0$, and $-A_1$, respectively, in the polynomials multiplying $\cos x$. An analogous result applies in general, as follows (**Exercise 5.5.36**).

Theorem 5.5.2

If

$$y_p = A(x)\cos \omega x + B(x)\sin \omega x,$$

where $A(x)$ and $B(x)$ are polynomials with coefficients A_0 …, A_k and B_0, …, B_k, then the polynomials multiplying $\sin \omega x$ in

$$y_p', \quad y_p'', \quad ay_p'' + by_p' + cy_p \quad \text{and} \quad y_p'' + \omega^2 y_p$$

can be obtained by replacing A_0, …, A_k by B_0, …, B_k and B_0, …, B_k by $-A_0$, …, $-A_k$ in the corresponding polynomials multiplying $\cos \omega x$.

We will not use this theorem in our examples, but we recommend that you use it to check your manipulations when you work the exercises.

Example 5.5.4

Find a particular solution of

$$y'' + y = (8 - 4x)\cos x - (8 + 8x)\sin x. \tag{10.5.12}$$

Solution

According to Theorem 5.5.1 , we should look for a particular solution of the form

$$y_p = (A_0 x + A_1 x^2)\cos x + (B_0 x + B_1 x^2)\sin x, \tag{10.5.13}$$

since $\cos x$ and $\sin x$ are solutions of the complementary equation. However, let's try

$$y_p = (A_0 + A_1 x)\cos x + (B_0 + B_1 x)\sin x \tag{10.5.14}$$

first, so you can see why it doesn't work. From Equation 10.5.10,

$$y_p'' = (2B_1 - A_0 - A_1 x)\cos x - (2A_1 + B_0 + B_1 x)\sin x,$$

which together with Equation 10.5.14 implies that

$$y_p'' + y_p = 2B_1 \cos x - 2A_1 \sin x.$$

Since the right side of this equation does not contain $x\cos x$ or $x \sin x$, Equation 10.5.14 can't satisfy Equation 10.5.12 no matter how we choose A_0, A_1, B_0, and B_1.

Now let y_p be as in Equation 10.5.13. Then

$$\begin{aligned} y_p' = &\left[A_0 + (2A_1 + B_0)x + B_1 x^2\right]\cos x \\ &+ \left[B_0 + (2B_1 - A_0)x - A_1 x^2\right]\sin x \end{aligned}$$

and

$$\begin{aligned} y_p'' = &\left[2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1 x^2\right]\cos x \\ &+ \left[2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1 x^2\right]\sin x, \end{aligned}$$

so

$$y_p'' + y_p = (2A_1 + 2B_0 + 4B_1 x)\cos x + (2B_1 - 2A_0 - 4A_1 x)\sin x.$$

Comparing the coefficients of $\cos x$ and $\sin x$ here with the corresponding coefficients in Equation 10.5.12 shows that y_p is a solution of Equation 10.5.12 if

$$\begin{aligned} 4B_1 &= -4 \\ -4A_1 &= -8 \\ 2B_0 + 2A_1 &= 8 \\ -2A_0 + 2B_1 &= -8. \end{aligned}$$

The solution of this system is $A_1 = 2$, $B_1 = -1$, $A_0 = 3$, $B_0 = 2$. Therefore

$$y_p = x\left[(3 + 2x)\cos x + (2 - x)\sin x\right]$$

is a particular solution of Equation 10.5.12.

Forcing Functions with Exponential Factors

To find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{10.5.15}$$

when $\lambda \neq 0$, we recall from Section 5.4 that substituting $y = ue^{\lambda x}$ into Equation 10.5.15 will produce a constant coefficient equation for u with the forcing function $P(x)\cos\omega x + Q(x)\sin\omega x$. We can find a particular solution u_p of this equation by the procedure that we used in Examples 5.5.1 -5.5.4 . Then $y_p = u_p e^{\lambda x}$ is a particular solution of Equation 10.5.15.

Example 5.5.5

Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right]. \tag{10.5.16}$$

Let $y = ue^{-2x}$. Then

$$\begin{aligned} y'' - 3y' + 2y &= e^{-2x}\left[(u'' - 4u' + 4u) - 3(u' - 2u) + 2u\right] \\ &= e^{-2x}(u'' - 7u' + 12u) \\ &= e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right] \end{aligned}$$

if

$$u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x. \tag{10.5.17}$$

Since $\cos 3x$ and $\sin 3x$ aren't solutions of the complementary equation

$$u'' - 7u' + 12u = 0,$$

Theorem 5.5.1 tells us to look for a particular solution of Equation 10.5.17 of the form

$$u_p = (A_0 + A_1 x)\cos 3x + (B_0 + B_1 x)\sin 3x. \tag{10.5.18}$$

Then

$$\begin{aligned} u_p' &= (A_1 + 3B_0 + 3B_1 x)\cos 3x + (B_1 - 3A_0 - 3A_1 x)\sin 3x \\ \text{and} \quad u_p'' &= (-9A_0 + 6B_1 - 9A_1 x)\cos 3x - (9B_0 + 6A_1 + 9B_1 x)\sin 3x, \end{aligned}$$

so

$$\begin{aligned} u_p'' - 7u_p' + 12u_p = {} & [3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x]\cos 3x \\ & + [21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x]\sin 3x. \end{aligned}$$

Comparing the coefficients of $x\cos 3x$, $x\sin 3x$, $\cos 3x$, and $\sin 3x$ here with the corresponding coefficients on the right side of Equation 10.5.17 shows that u_p is a solution of Equation 10.5.17 if

$$\begin{aligned} 3A_1 - 21B_1 &= 0 \\ 21A_1 + 3B_1 &= 150 \\ 3A_0 - 21B_0 - 7A_1 + 6B_1 &= 2 \\ 21A_0 + 3B_0 - 6A_1 - 7B_1 &= -34. \end{aligned} \tag{10.5.19}$$

Solving the first two equations yields $A_1 = 7$, $B_1 = 1$. Substituting these values into the last two equations of Equation 10.5.19 yields

$$\begin{aligned} 3A_0 - 21B_0 &= 2 + 7A_1 - 6B_1 = 45 \\ 21A_0 + 3B_0 &= -34 + 6A_1 + 7B_1 = 15. \end{aligned}$$

Solving this system yields $A_0 = 1$, $B_0 = -2$. Substituting $A_0 = 1$, $A_1 = 7$, $B_0 = -2$, and $B_1 = 1$ into Equation 10.5.18 shows that

$$u_p = (1 + 7x)\cos 3x - (2 - x)\sin 3x$$

is a particular solution of Equation 10.5.17. Therefore

$$y_p = e^{-2x}\left[(1+7x)\cos 3x - (2-x)\sin 3x\right]$$

is a particular solution of Equation 10.5.16.

Example 5.5.6

Find a particular solution of

$$y'' + 2y' + 5y = e^{-x}\left[(6-16x)\cos 2x - (8+8x)\sin 2x\right]. \tag{10.5.20}$$

Solution

Let $y = ue^{-x}$. Then

$$\begin{aligned} y'' + 2y' + 5y &= e^{-x}\left[(u'' - 2u' + u) + 2(u' - u) + 5u\right] \\ &= e^{-x}(u'' + 4u) \\ &= e^{-x}\left[(6-16x)\cos 2x - (8+8x)\sin 2x\right] \end{aligned}$$

if

$$u'' + 4u = (6-16x)\cos 2x - (8+8x)\sin 2x. \tag{10.5.21}$$

Since $\cos 2x$ and $\sin 2x$ are solutions of the complementary equation

$$u'' + 4u = 0,$$

Theorem 5.5.1 tells us to look for a particular solution of Equation 10.5.21 of the form

$$u_p = (A_0x + A_1x^2)\cos 2x + (B_0x + B_1x^2)\sin 2x.$$

Then

$$\begin{aligned} u_p' = &\left[A_0 + (2A_1 + 2B_0)x + 2B_1x^2\right]\cos 2x \\ &+ \left[B_0 + (2B_1 - 2A_0)x - 2A_1x^2\right]\sin 2x \end{aligned}$$

and

$$\begin{aligned} u_p'' = &\left[2A_1 + 4B_0 - (4A_0 - 8B_1)x - 4A_1x^2\right]\cos 2x \\ &+ \left[2B_1 - 4A_0 - (4B_0 + 8A_1)x - 4B_1x^2\right]\sin 2x, \end{aligned}$$

so

$$u_p'' + 4u_p = (2A_1 + 4B_0 + 8B_1x)\cos 2x + (2B_1 - 4A_0 - 8A_1x)\sin 2x.$$

Equating the coefficients of $x\cos 2x$, $x\sin 2x$, $\cos 2x$, and $\sin 2x$ here with the corresponding coefficients on the right side of Equation 10.5.21 shows that u_p is a solution of Equation 10.5.21 if

$$\begin{aligned} 8B_1 &= -16 \\ -8A_1 &= -\ 8 \\ 4B_0 + 2A_1 &= 6 \\ -4A_0 + 2B_1 &= -8. \end{aligned} \tag{10.5.22}$$

The solution of this system is $A_1 = 1$, $B_1 = -2$, $B_0 = 1$, $A_0 = 1$. Therefore

$$u_p = x[(1+x)\cos 2x + (1-2x)\sin 2x]$$

is a particular solution of Equation 10.5.21, and

$$y_p = xe^{-x}\left[(1+x)\cos 2x + (1-2x)\sin 2x\right]$$

is a particular solution of Equation 10.5.20.

You can also find a particular solution of Equation 10.5.20 by substituting

$$y_p = xe^{-x}\left[(A_0 + A_1x)\cos 2x + (B_0 + B_1x)\sin 2x\right]$$

for y in Equation 10.5.20 and equating the coefficients of $xe^{-x}\cos 2x$, $xe^{-x}\sin 2x$, $e^{-x}\cos 2x$, and $e^{-x}\sin 2x$ in the resulting expression for

$$y_p'' + 2y_p' + 5y_p$$

with the corresponding coefficients on the right side of Equation 10.5.20. (See **Exercise 5.5.38**). This leads to the same system Equation 10.5.22 of equations for A_0, A_1, B_0, and B_1 that we obtained in Example 5.5.6 . However, if you try this approach you'll see that deriving Equation 10.5.22 this way is much more tedious than the way we did it in Example 5.5.6 .

This page titled 10.5: The Method of Undetermined Coefficients II is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.5: The Method of Undetermined Coefficients II** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

10.5.1: The Method of Undetermined Coefficients II (Exercises)

Q5.5.1

In **Exercises 5.5.1-5.5.17** find a particular solution.

1. $y'' + 3y' + 2y = 7\cos x - \sin x$
2. $y'' + 3y' + y = (2 - 6x)\cos x - 9\sin x$
3. $y'' + 2y' + y = e^x(6\cos x + 17\sin x)$
4. $y'' + 3y' - 2y = -e^{2x}(5\cos 2x + 9\sin 2x)$
5. $y'' - y' + y = e^x(2 + x)\sin x$
6. $y'' + 3y' - 2y = e^{-2x}\left[(4 + 20x)\cos 3x + (26 - 32x)\sin 3x\right]$
7. $y'' + 4y = -12\cos 2x - 4\sin 2x$
8. $y'' + y = (-4 + 8x)\cos x + (8 - 4x)\sin x$
9. $4y'' + y = -4\cos x/2 - 8x\sin x/2$
10. $y'' + 2y' + 2y = e^{-x}(8\cos x - 6\sin x)$
11. $y'' - 2y' + 5y = e^x\left[(6 + 8x)\cos 2x + (6 - 8x)\sin 2x\right]$
12. $y'' + 2y' + y = 8x^2\cos x - 4x\sin x$
13. $y'' + 3y' + 2y = (12 + 20x + 10x^2)\cos x + 8x\sin x$
14. $y'' + 3y' + 2y = (1 - x - 4x^2)\cos 2x - (1 + 7x + 2x^2)\sin 2x$
15. $y'' - 5y' + 6y = -e^x\left[(4 + 6x - x^2)\cos x - (2 - 4x + 3x^2)\sin x\right]$
16. $y'' - 2y' + y = -e^x\left[(3 + 4x - x^2)\cos x + (3 - 4x - x^2)\sin x\right]$
17. $y'' - 2y' + 2y = e^x\left[(2 - 2x - 6x^2)\cos x + (2 - 10x + 6x^2)\sin x\right]$

Q5.5.2

In **Exercises 5.5.18-5.5.21** find a particular solution and graph it.

18. $y'' + 2y' + y = e^{-x}\left[(5 - 2x)\cos x - (3 + 3x)\sin x\right]$
19. $y'' + 9y = -6\cos 3x - 12\sin 3x$
20. $y'' + 3y' + 2y = (1 - x - 4x^2)\cos 2x - (1 + 7x + 2x^2)\sin 2x$
21. $y'' + 4y' + 3y = e^{-x}\left[(2 + x + x^2)\cos x + (5 + 4x + 2x^2)\sin x\right]$

Q5.5.3

In **Exercises 5.5.22-5.5.26** solve the initial value problem.

22. $y'' - 7y' + 6y = -e^x(17\cos x - 7\sin x), \quad y(0) = 4,\ y'(0) = 2$
23. $y'' - 2y' + 2y = -e^x(6\cos x + 4\sin x), \quad y(0) = 1,\ y'(0) = 4$
24. $y'' + 6y' + 10y = -40e^x\sin x, \quad y(0) = 2, \quad y'(0) = -3$
25. $y'' - 6y' + 10y = -e^{3x}(6\cos x + 4\sin x), \quad y(0) = 2, \quad y'(0) = 7$
26. $y'' - 3y' + 2y = e^{3x}\left[21\cos x - (11 + 10x)\sin x\right],\ y(0) = 0, \quad y'(0) = 6$

Q5.5.4

In **Exercises 5.5.27-5.5.32** use the principle of superposition to find a particular solution. Where indicated, solve the initial value problem.

27. $y'' - 2y' - 3y = 4e^{3x} + e^x(\cos x - 2\sin x)$

28. $y'' + y = 4\cos x - 2\sin x + xe^x + e^{-x}$

29. $y'' - 3y' + 2y = xe^x + 2e^{2x} + \sin x$

30. $y'' - 2y' + 2y = 4xe^x \cos x + xe^{-x} + 1 + x^2$

31. $y'' - 4y' + 4y = e^{2x}(1 + x) + e^{2x}(\cos x - \sin x) + 3e^{3x} + 1 + x$

32. $y'' - 4y' + 4y = 6e^{2x} + 25\sin x, \quad y(0) = 5,\ y'(0) = 3$

Q5.5.5

In **Exercises 5.5.33-5.5.35** solve the initial value problem and graph the solution.

33. $y'' + 4y = -e^{-2x}\left[(4 - 7x)\cos x + (2 - 4x)\sin x\right],\ y(0) = 3, \quad y'(0) = 1$

34. $y'' + 4y' + 4y = 2\cos 2x + 3\sin 2x + e^{-x}, \quad y(0) = -1,\ y'(0) = 2$

35. $y'' + 4y = e^x(11 + 15x) + 8\cos 2x - 12\sin 2x, \quad y(0) = 3,\ y'(0) = 5$

Q5.5.6

36.

a. Verify that if

$$y_p = A(x)\cos \omega x + B(x)\sin \omega x$$

where A and B are twice differentiable, then

$$y_p' = (A' + \omega B)\cos \omega x + (B' - \omega A)\sin \omega x \quad \text{and}$$
$$y_p'' = (A'' + 2\omega B' - \omega^2 A)\cos \omega x + (B'' - 2\omega A' - \omega^2 B)\sin \omega x.$$

b. Use the results of (a) to verify that

$$ay_p'' + by_p' + cy_p = \left[(c - a\omega^2)A + b\omega B + 2a\omega B' + bA' + aA''\right]\cos \omega x + \left[-b\omega A + (c - a\omega^2)B - 2a\omega A' + bB' + aB''\right]\sin \omega x.$$

c. Use the results of (a) to verify that

$$y_p'' + \omega^2 y_p = (A'' + 2\omega B')\cos \omega x + (B'' - 2\omega A')\sin \omega x.$$

d. Prove Theorem 5.5.2.

37. Let a, b, c, and ω be constants, with $a \neq 0$ and $\omega > 0$, and let

$$P(x) = p_0 + p_1 x + \cdots + p_k x^k \quad \text{and} \quad Q(x) = q_0 + q_1 x + \cdots + q_k x^k,$$

where at least one of the coefficients p_k, q_k is nonzero, so k is the larger of the degrees of P and Q.

a. Show that if $\cos \omega x$ and $\sin \omega x$ are not solutions of the complementary equation

$$ay'' + by' + cy = 0,$$

then there are polynomials

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k \quad \text{and} \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k \tag{A}$$

such that

$$(c-a\omega^2)A+b\omega B+2a\omega B'+bA'+aA'' \quad = P$$
$$-b\omega A+(c-a\omega^2)B-2a\omega A'+bB'+aB'' \quad = Q,$$

where (A_k, B_k), (A_{k-1}, B_{k-1}), …,(A_0, B_0) can be computed successively by solving the systems

$$(c-a\omega^2)A_k+b\omega B_k \quad = p_k$$
$$-b\omega A_k+(c-a\omega^2)B_k \quad = q_k,$$

and, if $1 \le r \le k$,

$$(c-a\omega^2)A_{k-r}+b\omega B_{k-r} \quad = p_{k-r}+\cdots$$
$$-b\omega A_{k-r}+(c-a\omega^2)B_{k-r} \quad = q_{k-r}+\cdots,$$

where the terms indicated by "…" depend upon the previously computed coefficients with subscripts greater than $k-r$. Conclude from this and **Exercise 5.5.36b** that

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x \tag{B}$$

is a particular solution of

$$ay''+by'+cy = P(x)\cos\omega x + Q(x)\sin\omega x.$$

b. Conclude from **Exercise 5.5.36c** that the equation

$$a(y''+\omega^2 y) = P(x)\cos\omega x + Q(x)\sin\omega x \tag{C}$$

does not have a solution of the form (B) with A and B as in (A). Then show that there are polynomials

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \text{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}$$

such that

$$a(A''+2\omega B') \quad = P$$
$$a(B''-2\omega A') \quad = Q,$$

where the pairs (A_k, B_k), (A_{k-1}, B_{k-1}), …, (A_0, B_0) can be computed successively as follows:

$$A_k = -\frac{q_k}{2a\omega(k+1)}$$
$$B_k = \quad \frac{p_k}{2a\omega(k+1)},$$

and, if $k \ge 1$,

$$A_{k-j} = -\frac{1}{2\omega}\left[\frac{q_{k-j}}{a(k-j+1)} - (k-j+2)B_{k-j+1}\right]$$
$$B_{k-j} = \quad \frac{1}{2\omega}\left[\frac{p_{k-j}}{a(k-j+1)} - (k-j+2)A_{k-j+1}\right]$$

for $1 \le j \le k$. Conclude that (B) with this choice of the polynomials A and B is a particular solution of (C).

38. Show that Theorem 5.5.1 implies the next theorem:

Theorem 5.5E.1

Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay''+by'+cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$$

has a particular solution

$$y_p = e^{\lambda x}\left(A(x)\cos\omega x + B(x)\sin\omega x\right), \tag{A}$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \text{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

provided that $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are not solutions of the complementary equation. The equation

$$a\left[y'' - 2\lambda y' + (\lambda^2 + \omega^2)y\right] = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$$

(for which $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are solutions of the complementary equation) has a particular solution of the form (A), where

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \text{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}.$$

39. This exercise presents a method for evaluating the integral

$$y = \int e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)\,dx$$

where $\omega \neq 0$ and

$$P(x) = p_0 + p_1x + \cdots + p_kx^k, \quad Q(x) = q_0 + q_1x + \cdots + q_kx^k.$$

a. Show that $y = e^{\lambda x}u$, where

$$u' + \lambda u = P(x)\cos\omega x + Q(x)\sin\omega x. \tag{A}$$

b. Show that (A) has a particular solution of the form

$$u_p = A(x)\cos\omega x + B(x)\sin\omega x,$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k, \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

and the pairs of coefficients (A_k, B_k), (A_{k-1}, B_{k-1}), …,(A_0, B_0) can be computed successively as the solutions of pairs of equations obtained by equating the coefficients of $x^r\cos\omega x$ and $x^r\sin\omega x$ for $r = k,\ k-1,\ \ldots,\ 0$.

c. Conclude that

$$\int e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)\,dx = e^{\lambda x}\left(A(x)\cos\omega x + B(x)\sin\omega x\right) + c,$$

where c is a constant of integration.

40. Use the method of **Exercise 5.5.39** to evaluate the integral.

a. $\int x^2\cos x\,dx$
b. $\int x^2e^x\cos x\,dx$
c. $\int xe^{-x}\sin 2x\,dx$
d. $\int x^2e^{-x}\sin x\,dx$
e. $\int x^3e^x\sin x\,dx$
f. $\int e^x[x\cos x - (1+3x)\sin x]dx$
g. $\int e^{-x}[(1+x^2)\cos x + (1^2_x)\sin x]dx$

This page titled 10.5.1: The Method of Undetermined Coefficients II (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.5E: The Method of Undetermined Coefficients II (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

10.6: Reduction of Order

In this section we give a method for finding the general solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{10.6.1}$$

if we know a nontrivial solution y_1 of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{10.6.2}$$

The method is called **reduction of order** because it reduces the task of solving Equation 10.6.1 to solving a first order equation. Unlike the method of undetermined coefficients, it does not require P_0, P_1, and P_2 to be constants, or F to be of any special form.

By now you shoudn't be surprised that we look for solutions of Equation 10.6.1 in the form

$$y = uy_1 \tag{10.6.3}$$

where u is to be determined so that y satisfies Equation 10.6.1. Substituting Equation 10.6.3 and

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

into Equation 10.6.1 yields

$$P_0(x)(u''y_1 + 2u'y_1' + uy_1'') + P_1(x)(u'y_1 + uy_1') + P_2(x)uy_1 = F(x).$$

Collecting the coefficients of u, u', and u'' yields

$$(P_0y_1)u'' + (2P_0y_1' + P_1y_1)u' + (P_0y_1'' + P_1y_1' + P_2y_1)u = F. \tag{10.6.4}$$

However, the coefficient of u is zero, since y_1 satisfies Equation 10.6.2. Therefore Equation 10.6.4 reduces to

$$Q_0(x)u'' + Q_1(x)u' = F, \tag{10.6.5}$$

with

$$Q_0 = P_0y_1 \quad \text{and} \quad Q_1 = 2P_0y_1' + P_1y_1.$$

(It isn't worthwhile to memorize the formulas for Q_0 and Q_1!) Since Equation 10.6.5 is a linear first order equation in u', we can solve it for u' by variation of parameters as in Section 1.2, integrate the solution to obtain u, and then obtain y from Equation 10.6.3.

Example 5.6.1

a. Find the general solution of

$$xy'' - (2x+1)y' + (x+1)y = x^2, \tag{10.6.6}$$

given that $y_1 = e^x$ is a solution of the complementary equation

$$xy'' - (2x+1)y' + (x+1)y = 0. \tag{10.6.7}$$

b. As a byproduct of (a), find a fundamental set of solutions of Equation 10.6.7.

Solution

a. If $y = ue^x$, then $y' = u'e^x + ue^x$ and $y'' = u''e^x + 2u'e^x + ue^x$, so

$$\begin{aligned} xy'' - (2x+1)y' + (x+1)y &= x(u''e^x + 2u'e^x + ue^x) - (2x+1)(u'e^x + ue^x) + (x+1)ue^x \\ &= (xu'' - u')e^x. \end{aligned}$$

Therefore $y = ue^x$ is a solution of Equation 10.6.6 if and only if

$$(xu'' - u')e^x = x^2,$$

which is a first order equation in u'. We rewrite it as

$$u'' - \frac{u'}{x} = xe^{-x}. \tag{10.6.8}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that Equation 10.6.8 becomes

$$z' - \frac{z}{x} = xe^{-x}. \tag{10.6.9}$$

We leave it to you to show (by separation of variables) that $z_1 = x$ is a solution of the complementary equation

$$z' - \frac{z}{x} = 0$$

for Equation 10.6.9. By applying variation of parameters as in Section 1.2, we can now see that every solution of Equation 10.6.9 is of the form

$$z = vx \quad \textbf{where} \quad v'x = xe^{-x}, \quad \textbf{so} \quad v' = e^{-x} \quad \textbf{and} \quad v = -e^{-x} + C_1.$$

Since $u' = z = vx$, u is a solution of Equation 10.6.8 if and only if

$$u' = vx = -xe^{-x} + C_1x.$$

Integrating this yields

$$u = (x+1)e^{-x} + \frac{C_1}{2}x^2 + C_2.$$

Therefore the general solution of Equation 10.6.6 is

$$y = ue^x = x + 1 + \frac{C_1}{2}x^2e^x + C_2e^x. \tag{10.6.10}$$

b. By letting $C_1 = C_2 = 0$ in Equation 10.6.10, we see that $y_{p_1} = x + 1$ is a solution of Equation 10.6.6. By letting $C_1 = 2$ and $C_2 = 0$, we see that $y_{p_2} = x + 1 + x^2e^x$ is also a solution of Equation 10.6.6. Since the difference of two solutions of Equation 10.6.6 is a solution of Equation 10.6.7, $y_2 = y_{p_1} - y_{p_2} = x^2e^x$ is a solution of Equation 10.6.7. Since y_2/y_1 is nonconstant and we already know that $y_1 = e^x$ is a solution of Equation 10.6.6, Theorem 5.1.6 implies that $\{e^x, x^2e^x\}$ is a fundamental set of solutions of Equation 10.6.7.

Although Equation 10.6.10 is a correct form for the general solution of Equation 10.6.6, it is silly to leave the arbitrary coefficient of x^2e^x as $C_1/2$ where C_1 is an arbitrary constant. Moreover, it is sensible to make the subscripts of the coefficients of $y_1 = e^x$ and $y_2 = x^2e^x$ consistent with the subscripts of the functions themselves. Therefore we rewrite Equation 10.6.10 as

$$y = x + 1 + c_1e^x + c_2x^2e^x$$

by simply renaming the arbitrary constants. We'll also do this in the next two examples, and in the answers to the exercises.

Example 5.6.2

a. Find the general solution of

$$x^2y'' + xy' - y = x^2 + 1,$$

given that $y_1 = x$ is a solution of the complementary equation

$$x^2y'' + xy' - y = 0. \tag{10.6.11}$$

As a byproduct of this result, find a fundamental set of solutions of Equation 10.6.11.

b. Solve the initial value problem

$$x^2y'' + xy' - y = x^2 + 1, \quad y(1) = 2,\ y'(1) = -3. \tag{10.6.12}$$

Solution

a. If $y = ux$, then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$x^2y'' + xy' - y = x^2(u''x + 2u') + x(u'x + u) - ux$$
$$= x^3u'' + 3x^2u'.$$

Therefore $y = ux$ is a solution of Equation 10.6.12 if and only if

$$x^3u'' + 3x^2u' = x^2 + 1,$$

which is a first order equation in u'. We rewrite it as

$$u'' + \frac{3}{x}u' = \frac{1}{x} + \frac{1}{x^3}. \tag{10.6.13}$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that Equation 10.6.13 becomes

$$z' + \frac{3}{x}z = \frac{1}{x} + \frac{1}{x^3}. \tag{10.6.14}$$

We leave it to you to show by separation of variables that $z_1 = 1/x^3$ is a solution of the complementary equation

$$z' + \frac{3}{x}z = 0$$

for Equation 10.6.14. By variation of parameters, every solution of Equation 10.6.14 is of the form

$$z = \frac{v}{x^3} \quad \text{where} \quad \frac{v'}{x^3} = \frac{1}{x} + \frac{1}{x^3}, \quad \text{so} \quad v' = x^2 + 1 \quad \text{and} \quad v = \frac{x^3}{3} + x + C_1.$$

Since $u' = z = v/x^3$, u is a solution of Equation 10.6.14 if and only if

$$u' = \frac{v}{x^3} = \frac{1}{3} + \frac{1}{x^2} + \frac{C_1}{x^3}.$$

Integrating this yields

$$u = \frac{x}{3} - \frac{1}{x} - \frac{C_1}{2x^2} + C_2.$$

Therefore the general solution of Equation 10.6.12 is

$$y = ux = \frac{x^2}{3} - 1 - \frac{C_1}{2x} + C_2x. \tag{10.6.15}$$

Reasoning as in the solution of Example 10.6.1*a*, we conclude that $y_1 = x$ and $y_2 = 1/x$ form a fundamental set of solutions for Equation 10.6.11.

As we explained above, we rename the constants in Equation 10.6.15 and rewrite it as

$$y = \frac{x^2}{3} - 1 + c_1x + \frac{c_2}{x}. \tag{10.6.16}$$

b. Differentiating Equation 10.6.16 yields

$$y' = \frac{2x}{3} + c_1 - \frac{c_2}{x^2}. \tag{10.6.17}$$

Setting $x = 1$ in Equation 10.6.16 and Equation 10.6.17 and imposing the initial conditions $y(1) = 2$ and $y'(1) = -3$ yields

$$c_1 + c_2 = \frac{8}{3}$$
$$c_1 - c_2 = -\frac{11}{3}.$$

Solving these equations yields $c_1 = -1/2$, $c_2 = 19/6$. Therefore the solution of Equation 10.6.12 is

$$y = \frac{x^2}{3} - 1 - \frac{x}{2} + \frac{19}{6x}.$$

Using reduction of order to find the general solution of a homogeneous linear second order equation leads to a homogeneous linear first order equation in u' that can be solved by separation of variables. The next example illustrates this.

Example 5.6.3

Find the general solution and a fundamental set of solutions of

$$x^2y'' - 3xy' + 3y = 0, \tag{10.6.18}$$

given that $y_1 = x$ is a solution.

Solution

If $y = ux$ then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(u''x + 2u') - 3x(u'x + u) + 3ux \\ &= x^3u'' - x^2u'. \end{aligned}$$

Therefore $y = ux$ is a solution of Equation 10.6.18 if and only if

$$x^3u'' - x^2u' = 0.$$

Separating the variables u' and x yields

$$\frac{u''}{u'} = \frac{1}{x},$$

so

$$\ln|u'| = \ln|x| + k, \quad \text{or equivalently} \quad u' = C_1x.$$

Therefore

$$u = \frac{C_1}{2}x^2 + C_2,$$

so the general solution of Equation 10.6.18 is

$$y = ux = \frac{C_1}{2}x^3 + C_2x,$$

which we rewrite as

$$y = c_1x + c_2x^3.$$

Therefore $\{x, x^3\}$ is a fundamental set of solutions of Equation 10.6.18.

This page titled 10.6: Reduction of Order is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.6: Reduction of Order** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

10.6.1: Reduction of Order (Exercises)

Q5.6.1

In **Exercises 5.6.1-5.6.17** find the general solution, given that y_1 satisfies the complementary equation. As a byproduct, find a fundamental set of solutions of the complementary equation.

1. $(2x+1)y'' - 2y' - (2x+3)y = (2x+1)^2; \quad y_1 = e^{-x}$
2. $x^2y'' + xy' - y = \frac{4}{x^2}; \quad y_1 = x$
3. $x^2y'' - xy' + y = x; \quad y_1 = x$
4. $y'' - 3y' + 2y = \frac{1}{1+e^{-x}}; \quad y_1 = e^{2x}$
5. $y'' - 2y' + y = 7x^{3/2}e^x; \quad y_1 = e^x$
6. $4x^2y'' + (4x - 8x^2)y' + (4x^2 - 4x - 1)y = 4x^{1/2}e^x(1+4x); \quad y_1 = x^{1/2}e^x$
7. $y'' - 2y' + 2y = e^x \sec x; \quad y_1 = e^x \cos x$
8. $y'' + 4xy' + (4x^2+2)y = 8e^{-x(x+2)}; \quad y_1 = e^{-x^2}$
9. $x^2y'' + xy' - 4y = -6x - 4; \quad y_1 = x^2$
10. $x^2y'' + 2x(x-1)y' + (x^2 - 2x + 2)y = x^3e^{2x}; \quad y_1 = xe^{-x}$
11. $x^2y'' - x(2x-1)y' + (x^2 - x - 1)y = x^2e^x; \quad y_1 = xe^x$
12. $(1-2x)y'' + 2y' + (2x-3)y = (1 - 4x + 4x^2)e^x; \quad y_1 = e^x$
13. $x^2y'' - 3xy' + 4y = 4x^4; \quad y_1 = x^2$
14. $2xy'' + (4x+1)y' + (2x+1)y = 3x^{1/2}e^{-x}; \quad y_1 = e^{-x}$
15. $xy'' - (2x+1)y' + (x+1)y = -e^x; \quad y_1 = e^x$
16. $4x^2y'' - 4x(x+1)y' + (2x+3)y = 4x^{5/2}e^{2x}; \quad y_1 = x^{1/2}$
17. $x^2y'' - 5xy' + 8y = 4x^2; \quad y_1 = x^2$

Q5.6.2

In **Exercises 5.6.18-5.6.30** find a fundamental set of solutions, given that y_1 is a solution.

18. $xy'' + (2-2x)y' + (x-2)y = 0; \quad y_1 = e^x$
19. $x^2y'' - 4xy' + 6y = 0; \quad y_1 = x^2$
20. $x^2(\ln|x|)^2y'' - (2x\ln|x|)y' + (2 + \ln|x|)y = 0; \quad y_1 = \ln|x|$
21. $4xy'' + 2y' + y = 0; \quad y_1 = \sin\sqrt{x}$
22. $xy'' - (2x+2)y' + (x+2)y = 0; \quad y_1 = e^x$
23. $x^2y'' - (2a-1)xy' + a^2y = 0; \quad y_1 = x^a$
24. $x^2y'' - 2xy' + (x^2+2)y = 0; \quad y_1 = x\sin x$
25. $xy'' - (4x+1)y' + (4x+2)y = 0; \quad y_1 = e^{2x}$
26. $4x^2(\sin x)y'' - 4x(x\cos x + \sin x)y' + (2x\cos x + 3\sin x)y = 0; \quad y_1 = x^{1/2}$
27. $4x^2y'' - 4xy' + (3 - 16x^2)y = 0; \quad y_1 = x^{1/2}e^{2x}$
28. $(2x+1)xy'' - 2(2x^2-1)y' - 4(x+1)y = 0; \quad y_1 = 1/x$
29. $(x^2 - 2x)y'' + (2 - x^2)y' + (2x-2)y = 0; \quad y_1 = e^x$
30. $xy'' - (4x+1)y' + (4x+2)y = 0; \quad y_1 = e^{2x}$

Q5.6.3

In **Exercises 5.6.31-5.6.33** solve the initial value problem, given that y_1 satisfies the complementary equation.

31. $x^2y'' - 3xy' + 4y = 4x^4, \quad y(-1) = 7, \quad y'(-1) = -8; \quad y_1 = x^2$

32. $(3x-1)y'' - (3x+2)y' - (6x-8)y = 0, \quad y(0) = 2, \; y'(0) = 3; \quad y_1 = e^{2x}$

33. $(x+1)^2y'' - 2(x+1)y' - (x^2+2x-1)y = (x+1)^3e^x, \quad y(0) = 1, \quad y'(0) = \; -1; \quad y_1 = (x+1)e^x$

Q5.6.4

In **Exercises 5.6.34** and **5.6.35** solve the initial value problem and graph the solution, given that y_1 satisfies the complementary equation.

34. $x^2y'' + 2xy' - 2y = x^2, \quad y(1) = \frac{5}{4}, \; y'(1) = \frac{3}{2}; \quad y_1 = x$

35. $(x^2-4)y'' + 4xy' + 2y = x + 2, \quad y(0) = -\frac{1}{3}, \quad y'(0) = -1; \quad y_1 = \frac{1}{x-2}$

Q5.6.5

36. Suppose p_1 and p_2 are continuous on (a,b). Let y_1 be a solution of

$$y'' + p_1(x)y' + p_2(x)y = 0 \tag{A}$$

that has no zeros on (a,b), and let x_0 be in (a,b). Use reduction of order to show that y_1 and

$$y_2(x) = y_1(x)\int_{x_0}^{x} \frac{1}{y_1^2(t)} \exp\left(-\int_{x_0}^{t} p_1(s)\,ds\right) dt$$

form a fundamental set of solutions of (A) on (a,b).

37. The nonlinear first order equation

$$y' + y^2 + p(x)y + q(x) = 0 \tag{A}$$

is a **Riccati equation**. (See **Exercise 2.4.55**.) Assume that p and q are continuous.

a. Show that y is a solution of (A) if and only if $y = z'/z$, where

$$z'' + p(x)z' + q(x)z = 0. \tag{B}$$

b. Show that the general solution of (A) is

$$y = \frac{c_1z_1' + c_2z_2'}{c_1z_1 + c_2z_2}, \tag{C}$$

where $\{z_1, z_2\}$ is a fundamental set of solutions of (B) and c_1 and c_2 are arbitrary constants.

c. Does the formula (C) imply that the first order equation (A) has a two–parameter family of solutions? Explain your answer.

38. Use a method suggested by *Exercise 5.6.37* to find all solutions. of the equation.

a. $y' + y^2 + k^2 = 0$
b. $y' + y^2 - 3y + 2 = 0$
c. $y' + y^2 + 5y - 6 = 0$
d. $y' + y^2 + 8y + 7 = 0$
e. $y' + y^2 + 14y + 50 = 0$
f. $6y' + 6y^2 - y - 1 = 0$
g. $36y' + 36y^2 - 12y + 1 = 0$

39. Use a method suggested by **Exercise 5.6.37** and reduction of order to find all solutions of the equation, given that y_1 is a solution.

a. $x^2(y' + y^2) - x(x+2)y + x + 2 = 0; \quad y_1 = 1/x$
b. $y' + y^2 + 4xy + 4x^2 + 2 = 0; \quad y_1 = -2x$

c. $(2x+1)(y'+y^2)-2y-(2x+3)=0; \quad y_1=-1$
d. $(3x-1)(y'+y^2)-(3x+2)y-6x+8=0; \quad y_1=2$
e. $x^2(y'+y^2)+xy+x^2-\frac{1}{4}=0; \quad y_1=-\tan x-\frac{1}{2x}$
f. $x^2(y'+y^2)-7xy+7=0; \quad y_1=1/x$

40. The nonlinear first order equation

$$y'+r(x)y^2+p(x)y+q(x)=0 \tag{A}$$

is the **generalized Riccati equation**. (See **Exercise 2.4.55**.) Assume that p and q are continuous and r is differentiable.

a. Show that y is a solution of (A) if and only if $y=z'/rz$, where

$$z''+\left[p(x)-\frac{r'(x)}{r(x)}\right]z'+r(x)q(x)z=0. \tag{B}$$

b. Show that the general solution of (A) is

$$y=\frac{c_1z_1'+c_2z_2'}{r(c_1z_1+c_2z_2)},$$

where $\{z_1,z_2\}$ is a fundamental set of solutions of (B) and c_1 and c_2 are arbitrary constants.

This page titled 10.6.1: Reduction of Order (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **5.6E: Reduction of Order (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

CHAPTER OVERVIEW

11: Applications of Linear Second Order Equations

In this Chapter, we study applications of linear second order equations.

11.1: Spring Problems I

11.1.1: Spring Problems I (Exercises)

11.2: Spring Problems II

11.2.1: Spring Problems II (Exercises)

Thumbnail: A series RLC network: a resistor, an inductor, and a capacitor. (CC BY-SA 3.0; Spinningspark at Wikipedia)

This page titled 11: Applications of Linear Second Order Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

11.1: Spring Problems I

We consider the motion of an object of mass m, suspended from a spring of negligible mass. We say that the spring–mass system is in **equilibrium** when the object is at rest and the forces acting on it sum to zero. The position of the object in this case is the **equilibrium position**. We define y to be the displacement of the object from its equilibrium position (Figure 6.1.1), measured positive upward.

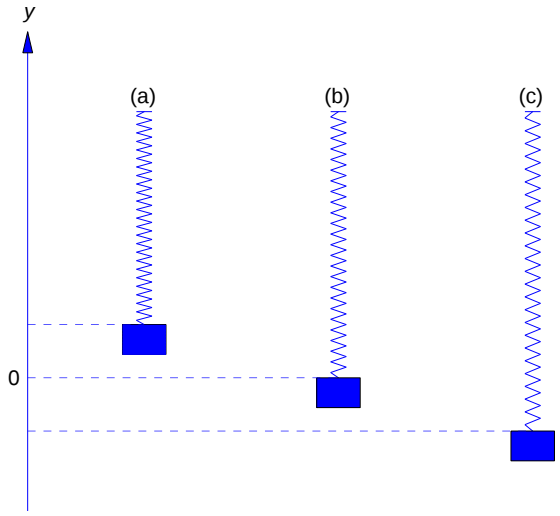


Figure 6.1.1 : (a) $y > 0$ (b) $y = 0$, (c) $y < 0$

Our model accounts for the following kinds of forces acting on the object:

- The force $-mg$, due to gravity.
- A force F_s exerted by the spring resisting change in its length. The **natural length** of the spring is its length with no mass attached. We assume that the spring obeys **Hooke's law**: If the length of the spring is changed by an amount ΔL from its natural length, then the spring exerts a force $F_s = k\Delta L$, where k is a positive number called the **spring constant**. If the spring is stretched then $\Delta L > 0$ and $F_s > 0$, so the spring force is upward, while if the spring is compressed then $\Delta L < 0$ and $F_s < 0$, so the spring force is downward.
- A **damping force** $F_d = -cy'$ that resists the motion with a force proportional to the velocity of the object. It may be due to air resistance or friction in the spring. However, a convenient way to visualize a damping force is to assume that the object is rigidly attached to a piston with negligible mass immersed in a cylinder (called a **dashpot**) filled with a viscous liquid (Figure 6.1.1). As the piston moves, the liquid exerts a damping force. We say that the motion is **undamped** if $c = 0$, or **damped** if $c > 0$.
- An external force F, other than the force due to gravity, that may vary with t, but is independent of displacement and velocity. We say that the motion is **free** if $F \equiv 0$, or **forced** if $F \not\equiv 0$.

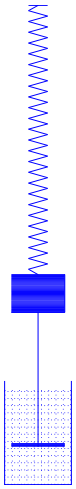


Figure 6.1.2 : A spring – mass system with damping

From Newton's second law of motion,

$$my'' = -mg + F_d + F_s + F = -mg - cy' + F_s + F. \tag{11.1.1}$$

We must now relate F_s to y. In the absence of external forces the object stretches the spring by an amount Δl to assume its equilibrium position (Figure 6.1.3). Since the sum of the forces acting on the object is then zero, Hooke's Law implies that $mg = k\Delta l$. If the object is displaced y units from its equilibrium position, the total change in the length of the spring is $\Delta L = \Delta l - y$, so Hooke's law implies that

$$F_s = k\Delta L = k\Delta l - ky.$$

Substituting this into Equation 11.1.1 yields

$$my'' = -mg - cy' + k\Delta L - ky + F.$$

Since $mg = k\Delta l$ this can be written as

$$my'' + cy' + ky = F. \tag{11.1.2}$$

We call this **the equation of motion**.

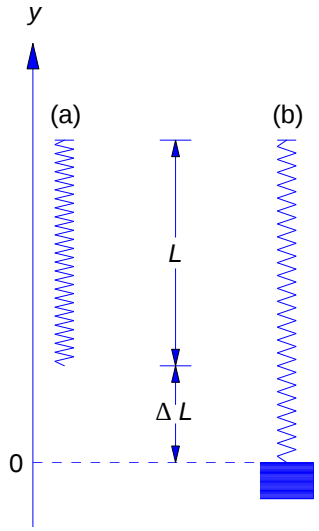


Figure 6.1.3 : (a) Natural length of spring (b) Spring stretched by mass

Simple Harmonic Motion

Throughout the rest of this section we'll consider spring–mass systems without damping; that is, $c = 0$. We'll consider systems with damping in the next section. We first consider the case where the motion is also free; that is, F=0. We begin with an example.

Example 6.1.1

An object stretches a spring 6 inches in equilibrium.

a. Set up the equation of motion and find its general solution.
b. Find the displacement of the object for $t > 0$ if it is initially displaced 18 inches above equilibrium and given a downward velocity of 3 ft/s.

Solution a:

Setting $c = 0$ and $F = 0$ in Equation 11.1.2 yields the equation of motion

$$my'' + ky = 0,$$

which we rewrite as

$$y'' + \frac{k}{m}y = 0. \tag{11.1.3}$$

Although we would need the weight of the object to obtain k from the equation $mg = k\Delta l$ we can obtain k/m from Δl alone; thus, $k/m = g/\Delta l$. Consistent with the units used in the problem statement, we take $g = 32$ ft/s^2. Although Δl is stated in inches, we must convert it to feet to be consistent with this choice of g; that is, $\Delta l = 1/2$ ft. Therefore

$$\frac{k}{m} = \frac{32}{1/2} = 64$$

and Equation 11.1.3 becomes

$$y'' + 64y = 0. \tag{11.1.4}$$

The characteristic equation of Equation 11.1.4 is

$$r^2 + 64 = 0,$$

which has the zeros $r = \pm 8i$. Therefore the general solution of Equation 11.1.4 is

$$y = c_1 \cos 8t + c_2 \sin 8t. \tag{11.1.5}$$

Solution b:

The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus,

$$y(0) = \frac{3}{2} \quad \text{and} \quad y'(0) = -3.$$

Differentiating Equation 11.1.5 yields

$$y' = -8c_1 \sin 8t + 8c_2 \cos 8t. \tag{11.1.6}$$

Setting $t = 0$ in Equation 11.1.5 and Equation 11.1.6 and imposing the initial conditions shows that $c_1 = 3/2$ and $c_2 = -3/8$. Therefore

$$y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t,$$

where y is in feet (Figure 6.1.4).

We'll now consider the equation

$$my'' + ky = 0$$

where m and k are arbitrary positive numbers. Dividing through by m and defining $\omega_0 = \sqrt{k/m}$ yields

$$y'' + \omega_0^2 y = 0.$$

The general solution of this equation is

$$y = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t. \tag{11.1.7}$$

We can rewrite this in a more useful form by defining

$$R = \sqrt{c_1^2 + c_2^2}, \tag{11.1.8}$$

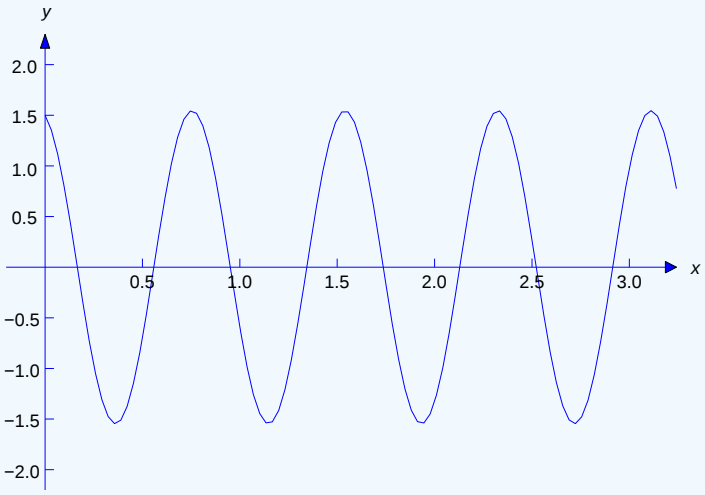


Figure 6.1.4 : $y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t$

and

$$c_1 = R\cos\phi \quad \text{and} \quad c_2 = R\sin\phi. \tag{11.1.9}$$

Substituting from Equation 11.1.9 into Equation 11.1.7 and applying the identity

$$\cos\omega_0 t\cos\phi + \sin\omega_0 t\sin\phi = \cos(\omega_0 t - \phi)$$

yields

$$y = R\cos(\omega_0 t - \phi). \tag{11.1.10}$$

From Equation 11.1.8 and Equation 11.1.9 we see that the R and ϕ can be interpreted as polar coordinates of the point with rectangular coordinates (c_1, c_2) (Figure 6.1.5). Given c_1 and c_2, we can compute R from Equation 11.1.8. From Equation 11.1.8 and Equation 11.1.9, we see that ϕ is related to c_1 and c_2 by

$$\cos\phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \quad \text{and} \quad \sin\phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}.$$

There are infinitely many angles ϕ, differing by integer multiples of 2π, that satisfy these equations. We will always choose ϕ so that $-\pi \le \phi < \pi$.

The motion described by Equation 11.1.7 or Equation 11.1.10 is **simple harmonic motion**. We see from either of these equations that the motion is periodic, with period

$$T = 2\pi/\omega_0.$$

This is the time required for the object to complete one full cycle of oscillation (for example, to move from its highest position to its lowest position and back to its highest position). Since the highest and lowest positions of the object are $y = R$ and $y = -R$, we say that R is the **amplitude** of the oscillation. The angle ϕ in Equation 11.1.10 is the **phase angle**. It's measured in radians. Equation Equation 11.1.10 is the **amplitude–phase form** of the displacement. If t is in seconds then ω_0 is in radians per

second (rad/s); it is the **frequency** of the motion. It is also called the **natural frequency** of the spring–mass system without damping.

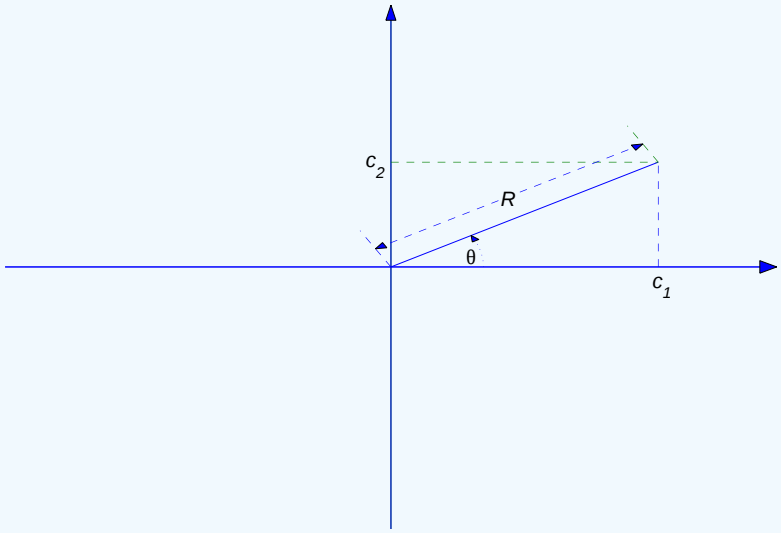


Figure 6.1.5 : $R = \sqrt{c_1^2 + c_2^2}$; $c_1 = R\cos\phi$; $c_2 = R\sin\phi$

Example 6.1.2

We found the displacement of the object in Example 6.1.1 to be

$$y = \frac{3}{2}\cos 8t - \frac{3}{8}\sin 8t.$$

Find the frequency, period, amplitude, and phase angle of the motion.

Solution:

The frequency is $\omega_0 = 8$ rad/s, and the period is $T = 2\pi/\omega_0 = \pi/4$ s. Since $c_1 = 3/2$ and $c_2 = -3/8$, the amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{8}\right)^2} = \frac{3}{8}\sqrt{17}.$$

The phase angle is determined by

$$\cos\phi = \frac{\frac{3}{2}}{\frac{3}{8}\sqrt{17}} = \frac{4}{\sqrt{17}} \tag{11.1.11}$$

and

$$\sin\phi = \frac{-\frac{3}{8}}{\frac{3}{8}\sqrt{17}} = -\frac{1}{\sqrt{17}}. \tag{11.1.12}$$

Using a calculator, we see from Equation 11.1.11 that

$$\phi \approx \pm .245 \text{ rad.}$$

Since $\sin\phi < 0$ (see Equation 11.1.12), the minus sign applies here; that is,

$$\phi \approx -.245 \text{ rad.}$$

Example 6.1.3

The natural length of a spring is 1 m. An object is attached to it and the length of the spring increases to 102 cm when the object is in equilibrium. Then the object is initially displaced downward 1 cm and given an upward velocity of 14 cm/s. Find

the displacement for $t > 0$. Also, find the natural frequency, period, amplitude, and phase angle of the resulting motion. Express the answers in cgs units.

Solution:

In cgs units $g = 980$ cm/s^2. Since $\Delta l = 2$ cm, $\omega_0^2 = g/\Delta l = 490$. Therefore

$$y'' + 490y = 0, \quad y(0) = -1, \quad y'(0) = 14.$$

The general solution of the differential equation is

$$y = c_1 \cos 7\sqrt{10}t + c_2 \sin 7\sqrt{10}t,$$

so

$$y' = 7\sqrt{10}\left(-c_1 \sin 7\sqrt{10}t + c_2 \cos 7\sqrt{10}t\right).$$

Substituting the initial conditions into the last two equations yields $c_1 = -1$ and $c_2 = 2/\sqrt{10}$. Hence,

$$y = -\cos 7\sqrt{10}t + \frac{2}{\sqrt{10}} \sin 7\sqrt{10}t.$$

The frequency is $7\sqrt{10}$ rad/s, and the period is $T = 2\pi/(7\sqrt{10})$ s. The amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + \left(\frac{2}{\sqrt{10}}\right)^2} = \sqrt{\frac{7}{5}}\text{cm}$$

The phase angle is determined by

$$\cos\phi = \frac{c_1}{R} = -\sqrt{\frac{5}{7}} \quad \text{and} \quad \sin\phi = \frac{c_2}{R} = \sqrt{\frac{2}{7}}$$

Therefore ϕ is in the second quadrant and

$$\phi = \cos^{-1}\left(-\sqrt{\frac{5}{7}}\right) \approx 2.58\text{rad}$$

Undamped Forced Oscillation

In many mechanical problems a device is subjected to periodic external forces. For example, soldiers marching in cadence on a bridge cause periodic disturbances in the bridge, and the engines of a propeller driven aircraft cause periodic disturbances in its wings. In the absence of sufficient damping forces, such disturbances – even if small in magnitude – can cause structural breakdown if they are at certain critical frequencies. To illustrate, this we'll consider the motion of an object in a spring–mass system without damping, subject to an external force

$$F(t) = F_0 \cos \omega t$$

where F_0 is a constant. In this case the equation of motion Equation 11.1.2 is

$$my'' + ky = F_0 \cos \omega t,$$

which we rewrite as

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t \tag{11.1.13}$$

with $\omega_0 = \sqrt{k/m}$. We'll see from the next two examples that the solutions of Equation 11.1.13 with $\omega \neq \omega_0$ behave very differently from the solutions with $\omega = \omega_0$.

Example 6.1.4

Solve the initial value problem

$$y'' + \omega_0^2 y = \frac{F_0}{m}\cos\omega t, \quad y(0) = 0, \quad y'(0) = 0, \tag{11.1.14}$$

given that $\omega \neq \omega_0$.

Solution:

We first obtain a particular solution of Equation 11.1.13 by the method of undetermined coefficients. Since $\omega \neq \omega_0$, $\cos\omega t$ isn't a solution of the complementary equation

$$y'' + \omega_0^2 y = 0.$$

Therefore Equation 11.1.13 has a particular solution of the form

$$y_p = A\cos\omega t + B\sin\omega t.$$

Since

$$y_p'' = -\omega^2(A\cos\omega t + B\sin\omega t),$$

$$y_p'' + \omega_0^2 y_p = \frac{F_0}{m}\cos\omega t$$

if and only if

$$(\omega_0^2 - \omega^2)(A\cos\omega t + B\sin\omega t) = \frac{F_0}{m}\cos\omega t.$$

This holds if and only if

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad B = 0,$$

so

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t.$$

The general solution of Equation 11.1.13 is

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t + c_1\cos\omega_0 t + c_2\sin\omega_0 t, \tag{11.1.15}$$

so

$$y' = \frac{-\omega F_0}{m(\omega_0^2 - \omega^2)}\sin\omega t + \omega_0(-c_1\sin\omega_0 t + c_2\cos\omega_0 t).$$

The initial conditions $y(0) = 0$ and $y'(0) = 0$ in Equation 11.1.14 imply that

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad c_2 = 0.$$

Substituting these into Equation 11.1.15 yields

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t). \tag{11.1.16}$$

It is revealing to write this in a different form. We start with the trigonometric identities

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Subtracting the second identity from the first yields

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2\sin\alpha\sin\beta \tag{11.1.17}$$

Now let

$$\alpha - \beta = \omega t \quad \text{and} \quad \alpha + \beta = \omega_0 t, \tag{11.1.18}$$

so that

$$\alpha = \frac{(\omega_0 + \omega)t}{2} \quad \text{and} \quad \beta = \frac{(\omega_0 - \omega)t}{2}. \tag{11.1.19}$$

Substituting Equation 11.1.18 and Equation 11.1.19 into Equation 11.1.17 yields

$$\cos \omega t - \cos \omega_0 t = 2 \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2},$$

and substituting this into Equation 11.1.16 yields

$$y = R(t) \sin \frac{(\omega_0 + \omega)t}{2}, \tag{11.1.20}$$

where

$$R(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}. \tag{11.1.21}$$

From Equation 11.1.20 we can regard y as a sinusoidal variation with frequency $(\omega_0 + \omega)/2$ and variable amplitude $|R(t)|$. In Figure 6.1.6 the dashed curve above the t axis is $y = |R(t)|$, the dashed curve below the t axis is $y = -|R(t)|$, and the displacement y appears as an oscillation bounded by them. The oscillation of y for t on an interval between successive zeros of $R(t)$ is called a **beat**.

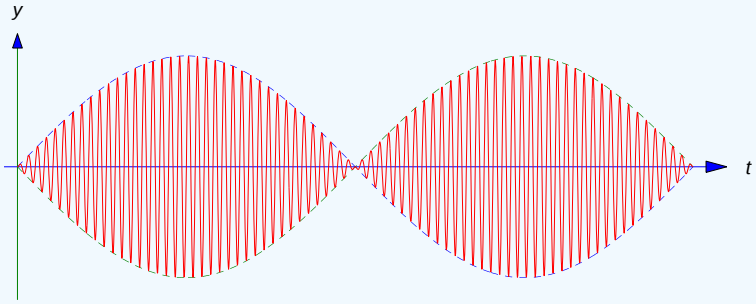


Figure 6.1.6 : Undamped oscillation with beats

You can see from Equation 11.1.20 and Equation 11.1.21 that

$$|y(t)| \le \frac{2|F_0|}{m|\omega_0^2 - \omega^2|};$$

moreover, if $\omega + \omega_0$ is sufficiently large compared with $\omega - \omega_0$, then $|y|$ assumes values close to (perhaps equal to) this upper bound during each beat. However, the oscillation remains bounded for all t. (This assumes that the spring can withstand deflections of this size and continue to obey Hooke's law.) The next example shows that this isn't so if $\omega = \omega_0$.

Example 6.1.5

Find the general solution of

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t. \tag{11.1.22}$$

Solution:

We first obtain a particular solution y_p of Equation 11.1.22. Since $\cos \omega_0 t$ is a solution of the complementary equation, the form for y_p is

$$y_p = t(A \cos \omega_0 t + B \sin \omega_0 t). \tag{11.1.23}$$

Then

$$y_p' = A \cos \omega_0 t + B \sin \omega_0 t + \omega_0 t(-A \sin \omega_0 t + B \cos \omega_0 t)$$

and

$$y_p'' = 2\omega_0(-A\sin\omega_0 t + B\cos\omega_0 t) - \omega_0^2 t(A\cos\omega_0 t + B\sin\omega_0 t). \tag{11.1.24}$$

From Equation 11.1.23 and Equation 11.1.24, we see that y_p satisfies Equation 11.1.22 if

$$-2A\omega_0\sin\omega_0 t + 2B\omega_0\cos\omega_0 t = \frac{F_0}{m}\cos\omega_0 t;$$

that is, if

$$A = 0 \quad \text{and} \quad B = \frac{F_0}{2m\omega_0}.$$

Therefore

$$y_p = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t$$

is a particular solution of Equation 11.1.22. The general solution of Equation 11.1.22 is

$$y = \frac{F_0 t}{2m\omega_0}\sin\omega_0 t + c_1\cos\omega_0 t + c_2\sin\omega_0 t.$$

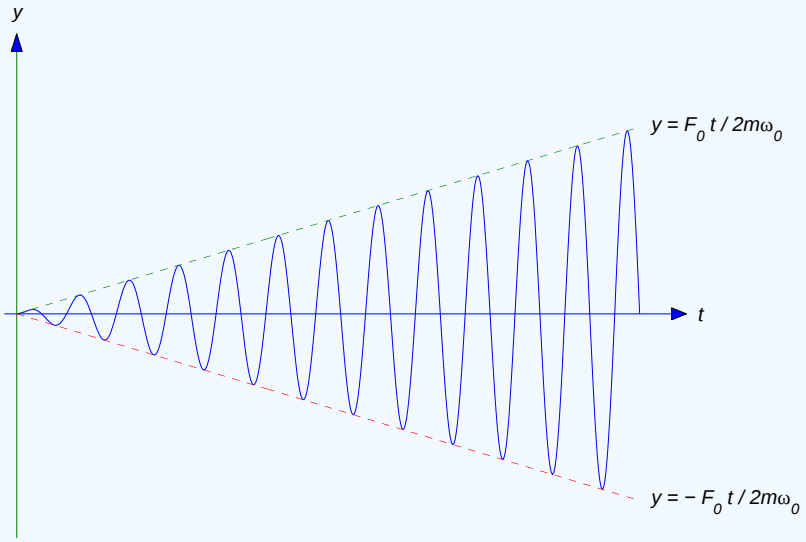


Figure 6.1.7 : Undamped displacement due to resonance.

The graph of y_p is shown in Figure 6.1.7 , where it can be seen that y_p oscillates between the dashed lines

$$y = \frac{F_0 t}{2m\omega_0} \quad \text{and} \quad y = -\frac{F_0 t}{2m\omega_0}$$

with increasing amplitude that approaches ∞ as $t \to \infty$. Of course, this means that the spring must eventually fail to obey Hooke's law or break.

This phenomenon of unbounded displacements of a spring–mass system in response to a periodic forcing function at its natural frequency is called **resonance**. More complicated mechanical structures can also exhibit resonance–like phenomena. For example, rhythmic oscillations of a suspension bridge by wind forces or of an airplane wing by periodic vibrations of reciprocating engines can cause damage or even failure if the frequencies of the disturbances are close to critical frequencies determined by the parameters of the mechanical system in question.

This page titled 11.1: Spring Problems I is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **6.1: Spring Problems I** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

11.1.1: Spring Problems I (Exercises)

In the following exercises assume that there's no damping.

Q6.1.1

1. An object stretches a spring 4 inches in equilibrium. Find and graph its displacement for $t > 0$ if it is initially displaced 36 inches above equilibrium and given a downward velocity of 2 ft/s.

2. An object stretches a string 1.2 inches in equilibrium. Find its displacement for $t > 0$ if it is initially displaced 3 inches below equilibrium and given a downward velocity of 2 ft/s.

3. A spring with natural length $.5$ m has length 50.5 cm with a mass of 2 gm suspended from it. The mass is initially displaced 1.5 cm below equilibrium and released with zero velocity. Find its displacement for $t > 0$.

4. An object stretches a spring 6 inches in equilibrium. Find its displacement for $t > 0$ if it is initially displaced 3 inches above equilibrium and given a downward velocity of 6 inches/s. Find the frequency, period, amplitude and phase angle of the motion.

5. An object stretches a spring 5 cm in equilibrium. It is initially displaced 10 cm above equilibrium and given an upward velocity of $.25$ m/s. Find and graph its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

6. A 10 kg mass stretches a spring 70 cm in equilibrium. Suppose a 2 kg mass is attached to the spring, initially displaced 25 cm below equilibrium, and given an upward velocity of 2 m/s. Find its displacement for $t > 0$. Find the frequency, period, amplitude, and phase angle of the motion.

7. A weight stretches a spring 1.5 inches in equilibrium. The weight is initially displaced 8 inches above equilibrium and given a downward velocity of 4 ft/s. Find its displacement for $t > 0$.

8. A weight stretches a spring 6 inches in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/s. Find its displacement for $t > 0$.

9. A spring–mass system has natural frequency $7\sqrt{10}$ rad/s. The natural length of the spring is $.7$ m. What is the length of the spring when the mass is in equilibrium?

10. A 64 lb weight is attached to a spring with constant $k = 8$ lb/ft and subjected to an external force $F(t) = 2\sin t$. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 2 ft/s. Find its displacement for $t > 0$.

11. A unit mass hangs in equilibrium from a spring with constant $k = 1/16$. Starting at $t = 0$, a force $F(t) = 3\sin t$ is applied to the mass. Find its displacement for $t > 0$.

12. A 4 lb weight stretches a spring 1 ft in equilibrium. An external force $F(t) = .25\sin 8t$ lb is applied to the weight, which is initially displaced 4 inches above equilibrium and given a downward velocity of 1 ft/s. Find and graph its displacement for $t > 0$.

13. A 2 lb weight stretches a spring 6 inches in equilibrium. An external force $F(t) = \sin 8t$ lb is applied to the weight, which is released from rest 2 inches below equilibrium. Find its displacement for $t > 0$.

14. A 10 gm mass suspended on a spring moves in simple harmonic motion with period 4 s. Find the period of the simple harmonic motion of a 20 gm mass suspended from the same spring.

15. A 6 lb weight stretches a spring 6 inches in equilibrium. Suppose an external force $F(t) = \frac{3}{16}\sin \omega t + \frac{3}{8}\cos \omega t$ lb is applied to the weight. For what value of ω will the displacement be unbounded? Find the displacement if ω has this value. Assume that the motion starts from equilibrium with zero initial velocity.

16. A 6 lb weight stretches a spring 4 inches in equilibrium. Suppose an external force $F(t) = 4\sin \omega t - 6\cos \omega t$ lb is applied to the weight. For what value of ω will the displacement be unbounded? Find and graph the displacement if ω has this value. Assume that the motion starts from equilibrium with zero initial velocity.

17. A mass of one kg is attached to a spring with constant $k = 4$ N/m. An external force $F(t) = -\cos \omega t - 2\sin \omega t$ n is applied to the mass. Find the displacement y for $t > 0$ if ω equals the natural frequency of the spring–mass system. Assume that the mass is initially displaced 3 m above equilibrium and given an upward velocity of 450 cm/s.

18. An object is in simple harmonic motion with frequency ω_0, with $y(0) = y_0$ and $y'(0) = v_0$. Find its displacement for $t > 0$. Also, find the amplitude of the oscillation and give formulas for the sine and cosine of the initial phase angle.

19. Two objects suspended from identical springs are set into motion. The period of one object is twice the period of the other. How are the weights of the two objects related?

20. Two objects suspended from identical springs are set into motion. The weight of one object is twice the weight of the other. How are the periods of the resulting motions related?

21. Two identical objects suspended from different springs are set into motion. The period of one motion is 3 times the period of the other. How are the two spring constants related?

This page titled 11.1.1: Spring Problems I (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **6.1E: Spring Problems I (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

11.2: Spring Problems II

Free Vibrations with Damping

In this section we consider the motion of an object in a spring–mass system with damping. We start with unforced motion, so the equation of motion is

$$my'' + cy' + ky = 0. \tag{11.2.1}$$

Now suppose the object is displaced from equilibrium and given an initial velocity. Intuition suggests that if the damping force is sufficiently weak the resulting motion will be oscillatory, as in the undamped case considered in the previous section, while if it is sufficiently strong the object may just move slowly toward the equilibrium position without ever reaching it. We'll now confirm these intuitive ideas mathematically. The characteristic equation of Equation 11.2.1 is

$$mr^2 + cr + k = 0.$$

The roots of this equation are

$$r_1 = \frac{-c - \sqrt{c^2 - 4mk}}{2m} \quad \text{and} \quad r_2 = \frac{-c + \sqrt{c^2 - 4mk}}{2m}. \tag{11.2.2}$$

We saw in Section 5.2 that the form of the solution of Equation 11.2.1 depends upon whether $c^2 - 4mk$ is positive, negative, or zero. We'll now consider these three cases.

Underdamped Motion

We say the motion is **underdamped** if $c < \sqrt{4mk}$. In this case r_1 and r_2 in Equation 11.2.2 are complex conjugates, which we write as

$$r_1 = -\frac{c}{2m} - \omega_1 i \quad \text{and} \quad r_2 = -\frac{c}{2m} + \omega_1 i,$$

where

$$\omega_1 = \frac{\sqrt{4mk - c^2}}{2m}.$$

The general solution of Equation 11.2.1 in this case is

$$y = e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t).$$

By the method used in Section 6.1 to derive the amplitude–phase form of the displacement of an object in simple harmonic motion, we can rewrite this equation as

$$y = Re^{-ct/2m} \cos(\omega_1 t - \phi), \tag{11.2.3}$$

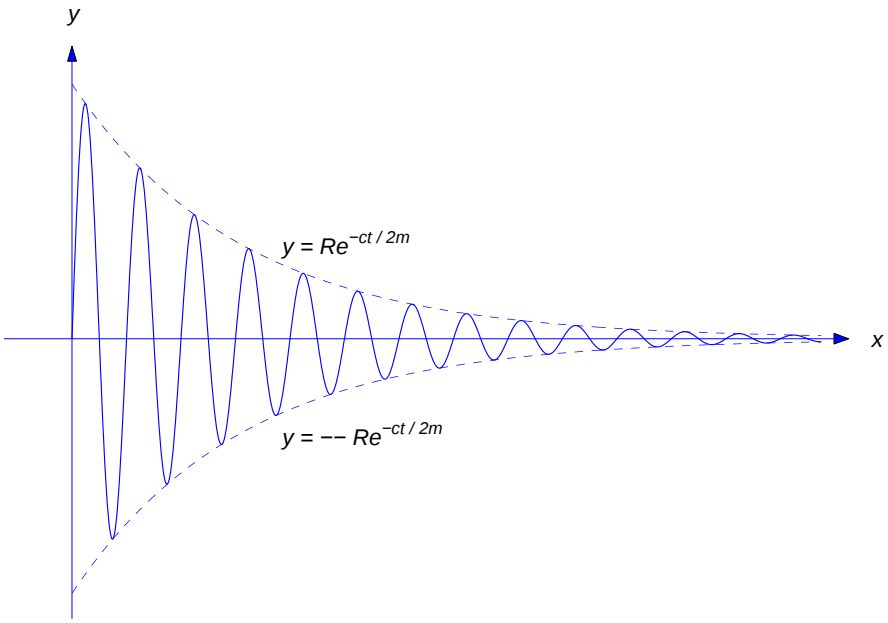


Figure 6.2.1 : Underdamped motion

where

$$R = \sqrt{c_1^2 + c_2^2}, \quad R\cos\phi = c_1, \quad \text{and} \quad R\sin\phi = c_2.$$

The factor $Re^{-ct/2m}$ in Equation 11.2.3 is called the **time–varying amplitude** of the motion, the quantity ω_1 is called the **frequency**, and $T = 2\pi/\omega_1$ (which is the period of the cosine function in Equation 11.2.3 is called the **quasi–period**. A typical graph of Equation 11.2.3 is shown in Figure 6.2.1 . As illustrated in that figure, the graph of y oscillates between the dashed exponential curves $y = \pm Re^{-ct/2m}$.

Overdamped Motion

We say the motion is **overdamped** if $c > \sqrt{4mk}$. In this case the zeros r_1 and r_2 of the characteristic polynomial are real, with $r_1 < r_2 < 0$ (see 11.2.2), and the general solution of 11.2.1 is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Again $\lim_{t\to\infty} y(t) = 0$ as in the underdamped case, but the motion isn't oscillatory, since y can't equal zero for more than one value of t unless $c_1 = c_2 = 0$. (**Exercise 6.2.23**.)

Critically Damped Motion

We say the motion is **critically damped** if $c = \sqrt{4mk}$. In this case $r_1 = r_2 = -c/2m$ and the general solution of Equation 11.2.1 is

$$y = e^{-ct/2m}(c_1 + c_2 t).$$

Again $\lim_{t\to\infty} y(t) = 0$ and the motion is nonoscillatory, since y can't equal zero for more than one value of t unless $c_1 = c_2 = 0$. (**Exercise 6.2.22**).

Example 6.2.1

Suppose a **64** lb weight stretches a spring **6** inches in equilibrium and a dashpot provides a damping force of c lb for each ft/sec of velocity.

a. Write the equation of motion of the object and determine the value of c for which the motion is critically damped.
b. Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = 20$.
c. Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = -20$.

Solution a

Here $m = 2$ slugs and $k = 64/.5 = 128$ lb/ft. Therefore the equation of motion Equation 11.2.1 is

$$2y'' + cy' + 128y = 0. \tag{11.2.4}$$

The characteristic equation is

$$2r^2 + cr + 128 = 0,$$

which has roots

$$r = \frac{-c \pm \sqrt{c^2 - 8 \cdot 128}}{4}.$$

Therefore the damping is critical if

$$c = \sqrt{8 \cdot 128} = 32 \text{ lb-sec/ft}.$$

Solution b

Setting $c = 32$ in Equation 11.2.4 and cancelling the common factor 2 yields

$$y'' + 16y + 64y = 0.$$

The characteristic equation is

$$r^2 + 16r + 64y = (r + 8)^2 = 0.$$

Hence, the general solution is

$$y = e^{-8t}(c_1 + c_2 t). \tag{11.2.5}$$

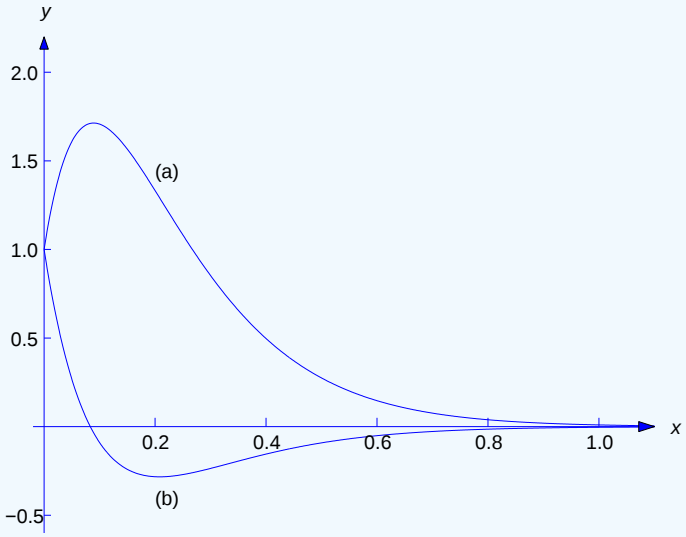


Figure 6.2.2 : (a) $y = e^{-8t}(1 + 28t)$ and (b) $y = e^{-8t}(1 - 12t)$

Differentiating this yields

$$y' = -8y + c_2 e^{-8t}. \tag{11.2.6}$$

6.2.2 Imposing the initial conditions $y(0) = 1$ and $y'(0) = 20$ in the last two equations shows that $1 = c_1$ and $20 = -8 + c_2$. Hence, the solution of the initial value problem is

$$y = e^{-8t}(1 + 28t).$$

Therefore the object approaches equilibrium from above as $t \to \infty$. There's no oscillation.

Solution c

Imposing the initial conditions $y(0)=1$ and $y'(0)=-20$ in Equation 11.2.5 and Equation 11.2.6 yields $1=c_1$ and $-20=-8+c_2$. Hence, the solution of this initial value problem is

$$y=e^{-8t}(1-12t).$$

Therefore the object moves downward through equilibrium just once, and then approaches equilibrium from below as $t\to\infty$. Again, there's no oscillation. The solutions of these two initial value problems are graphed in Figure 6.2.2 .

Example 6.2.2

Find the displacement of the object in Example 6.2.1 if the damping constant is $c=4$ lb–sec/ft and the initial conditions are $y(0)=1.5$ ft and $y'(0)=-3$ ft/sec.

Solution

With $c=4$, the equation of motion Equation 11.2.4 becomes

$$y''+2y'+64y=0 \tag{11.2.7}$$

after cancelling the common factor 2. The characteristic equation

$$r^2+2r+64=0$$

has complex conjugate roots

$$r=\frac{-2\pm\sqrt{4-4\cdot 64}}{2}=-1\pm 3\sqrt{7}i.$$

Therefore the motion is underdamped and the general solution of Equation 11.2.7 is

$$y=e^{-t}(c_1\cos 3\sqrt{7}t+c_2\sin 3\sqrt{7}t).$$

Differentiating this yields

$$y'=-y+3\sqrt{7}e^{-t}(-c_1\sin 3\sqrt{7}t+c_2\cos 3\sqrt{7}t).$$

Imposing the initial conditions $y(0)=1.5$ and $y'(0)=-3$ in the last two equations yields $1.5=c_1$ and $-3=-1.5+3\sqrt{7}c_2$. Hence, the solution of the initial value problem is

$$y=e^{-t}\left(\frac{3}{2}\cos 3\sqrt{7}t-\frac{1}{2\sqrt{7}}\sin 3\sqrt{7}t\right). \tag{11.2.8}$$

The amplitude of the function in parentheses is

$$R=\sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{1}{2\sqrt{7}}\right)^2}=\sqrt{\frac{9}{4}+\frac{1}{4\cdot 7}}=\sqrt{\frac{64}{4\cdot 7}}=\frac{4}{\sqrt{7}}$$

Therefore we can rewrite Equation 11.2.8 as

$$y=\frac{4}{\sqrt{7}}e^{-t}\cos(3\sqrt{7}t-\phi),$$

where

$$\cos\phi=\frac{3}{2R}=\frac{3\sqrt{7}}{8}\quad\text{and}\quad\sin\phi=-\frac{1}{2\sqrt{7}R}=-\frac{1}{8}.$$

Therefore $\phi\cong -0.125$ radians.

Example 6.2.3

Let the damping constant in Example 6.2.1 be $c=40$ lb–sec/ft. Find the displacement y for $t>0$ if $y(0)=1$ and $y'(0)=1$.

Solution

With $c = 40$, the equation of motion Equation 11.2.4 reduces to

$$y'' + 20y' + 64y = 0 \tag{11.2.9}$$

after cancelling the common factor 2. The characteristic equation

$$r^2 + 20r + 64 = (r + 16)(r + 4) = 0$$

has the roots $r_1 = -4$ and $r_2 = -16$. Therefore the general solution of Equation 11.2.9 is

$$y = c_1 e^{-4t} + c_2 e^{-16t}. \tag{11.2.10}$$

Differentiating this yields

$$y' = -4e^{-4t} - 16c_2 e^{-16t}.$$

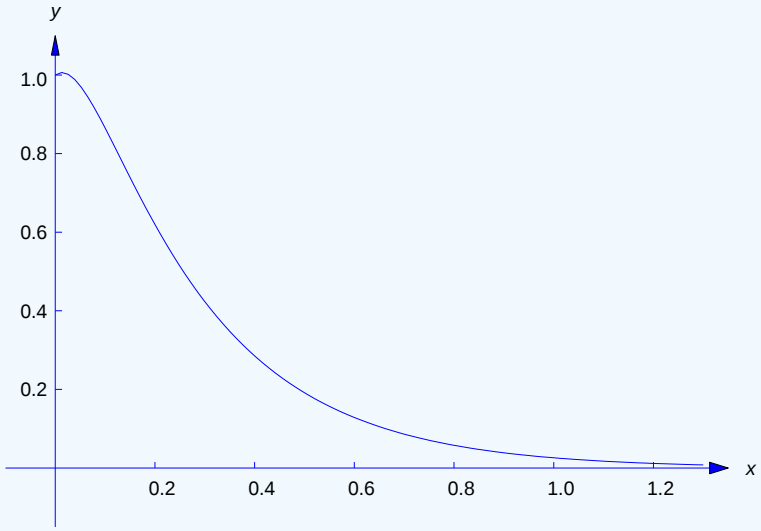


Figure 6.2.3 : $y = \frac{17}{12}e^{-4t} - \frac{5}{12}e^{-16t}$

The last two equations and the initial conditions $y(0) = 1$ and $y'(0) = 1$ imply that

$$\begin{aligned} c_1 \quad + \quad c_2 \quad &= 1 \\ -4c_1 \quad - \quad 16c_2 \quad &= 1. \end{aligned}$$

The solution of this system is $c_1 = 17/12$, $c_2 = -5/12$. Substituting these into Equation 11.2.10 yields

$$y = \frac{17}{12}e^{-4t} - \frac{5}{12}e^{-16t}$$

as the solution of the given initial value problem (Figure 6.2.3).

Forced Vibrations with Damping

Now we consider the motion of an object in a spring-mass system with damping, under the influence of a periodic forcing function $F(t) = F_0 \cos \omega t$, so that the equation of motion is

$$my'' + cy' + ky = F_0 \cos \omega t. \tag{11.2.11}$$

In Section 6.1 we considered this equation with $c = 0$ and found that the resulting displacement y assumed arbitrarily large values in the case of resonance (that is, when $\omega = \omega_0 = \sqrt{k/m}$). Here we'll see that in the presence of damping the displacement remains bounded for all t, and the initial conditions have little effect on the motion as $t \to \infty$. In fact, we'll see that for large t the displacement is closely approximated by a function of the form

$$y = R \cos(\omega t - \phi), \tag{11.2.12}$$

where the amplitude R depends upon m, c, k, F_0, and ω. We're interested in the following question:

QUESTION

Assuming that m, c, k, and F_0 are held constant, what value of ω produces the largest amplitude R in Equation 11.2.12, and what is this largest amplitude?

To answer this question, we must solve Equation 11.2.11 and determine R in terms of F_0, ω_0, ω, and c. We can obtain a particular solution of Equation 11.2.11 by the method of undetermined coefficients. Since $\cos \omega t$ does not satisfy the complementary equation

$$my'' + cy' + ky = 0,$$

we can obtain a particular solution of Equation 11.2.11 in the form

$$y_p = A \cos \omega t + B \sin \omega t. \tag{11.2.13}$$

Differentiating this yields

$$y_p' = \quad \omega(-A \sin \omega t + B \cos \omega t)$$

and

$$y_p'' = -\omega^2 (A \cos \omega t + B \sin \omega t).$$

From the last three equations,

$$my_p'' + cy_p' + ky_p = (-m\omega^2 A + c\omega B + kA) \cos \omega t + (-m\omega^2 B - c\omega A + kB) \sin \omega t,$$

so y_p satisfies Equation 11.2.11 if

$$\begin{aligned} (k - m\omega^2)A + \quad c\omega B \quad &= F_0 \\ -c\omega A \quad + (k - m\omega^2)B \quad &= 0. \end{aligned}$$

Solving for A and B and substituting the results into Equation 11.2.13 yields

$$y_p = \frac{F_0}{(k - m\omega^2)^2 + c^2\omega^2} \left[(k - m\omega^2) \cos \omega t + c\omega \sin \omega t \right],$$

which can be written in amplitude–phase form as

(11.2.14)

where

$$\cos \phi = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \text{and} \quad \sin \phi = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}. \tag{11.2.15}$$

To compare this with the undamped forced vibration that we considered in Section 6.1 it is useful to write

$$k - m\omega^2 = m\left(\frac{k}{m} - \omega^2\right) = m(\omega_0^2 - \omega^2), \tag{11.2.16}$$

where $\omega_0 = \sqrt{k/m}$ is the natural angular frequency of the undamped simple harmonic motion of an object with mass m on a spring with constant k. Substituting Equation 11.2.16 into Equation 11.2.14 yields

$$y_p = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}} \cos(\omega t - \phi). \tag{11.2.17}$$

The solution of an initial value problem

$$my'' + cy' + ky = F_0 \cos \omega t, \quad y(0) = y_0, \quad y'(0) = v_0,$$

is of the form $y = y_c + y_p$, where y_c has one of the three forms

$$y_c = e^{-ct/2m}(c_1 \cos \omega_1 t + c_2 \sin \omega_1 t),$$
$$y_c = e^{-ct/2m}(c_1 + c_2 t),$$
$$y_c = c_1 e^{r_1 t} + c_2 e^{r_2 t} \ (r_1, r_2 < 0).$$

In all three cases $\lim_{t \to \infty} y_c(t) = 0$ for any choice of c_1 and c_2. For this reason we say that y_c is the **transient component** of the solution y. The behavior of y for large t is determined by y_p, which we call the **steady state component** of y. Thus, for large t the motion is like simple harmonic motion at the frequency of the external force.

The amplitude of y_p in Equation 11.2.17 is

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2}}, \tag{11.2.18}$$

which is finite for all ω; that is, the presence of damping precludes the phenomenon of resonance that we encountered in studying undamped vibrations under a periodic forcing function. We'll now find the value $\omega_{\max}$ of ω for which R is maximized. This is the value of ω for which the function

$$\rho(\omega) = m^2(\omega_0^2 - \omega^2)^2 + c^2\omega^2$$

in the denominator of Equation 11.2.18 attains its minimum value. By rewriting this as

$$\rho(\omega) = m^2(\omega_0^4 + \omega^4) + (c^2 - 2m^2\omega_0^2)\omega^2, \tag{11.2.19}$$

you can see that ρ is a strictly increasing function of ω^2 if

$$c \ge \sqrt{2m^2\omega_0^2} = \sqrt{2mk}.$$

(Recall that $\omega_0^2 = k/m$). Therefore $\omega_{\max} = 0$ if this inequality holds. From Equation 11.2.15, you can see that $\phi = 0$ if $\omega = 0$. In this case, Equation 11.2.14 reduces to

$$y_p = \frac{F_0}{\sqrt{m^2\omega_0^4}} = \frac{F_0}{k},$$

which is consistent with Hooke's law: if the mass is subjected to a constant force F_0, its displacement should approach a constant y_p such that $ky_p = F_0$. Now suppose $c < \sqrt{2mk}$. Then, from Equation 11.2.19,

$$\rho'(\omega) = 2\omega(2m^2\omega^2 + c^2 - 2m^2\omega_0^2),$$

and $\omega_{\max}$ is the value of ω for which the expression in parentheses equals zero; that is,

$$\omega_{\max} = \sqrt{\omega_0^2 - \frac{c^2}{2m^2}} = \sqrt{\frac{k}{m}\left(1 - \frac{c^2}{2km}\right)}.$$

(To see that $\rho(\omega_{\max})$ is the minimum value of $\rho(\omega)$, note that $\rho'(\omega) < 0$ if $\omega < \omega_{\max}$ and $\rho'(\omega) > 0$ if $\omega > \omega_{\max}$.) Substituting $\omega = \omega_{\max}$ in Equation 11.2.18 and simplifying shows that the maximum amplitude $R_{\max}$ is

$$R_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}} \quad \text{if} \quad c < \sqrt{2mk}.$$

We summarize our results as follows.

Theorem 6.2.1

Suppose we consider the amplitude R of the steady state component of the solution of

$$my'' + cy' + ky = F_0 \cos \omega t$$

as a function of ω.

a. If $c \ge \sqrt{2mk}$, the maximum amplitude is $R_{\max} = F_0/k$ and it is attained when $\omega = \omega_{\max} = 0$.

b. If $c < \sqrt{2mk}$, the maximum amplitude is

$$R_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}}, \tag{11.2.20}$$

and it is attained when

$$\omega = \omega_{\max} = \sqrt{\frac{k}{m}\left(1 - \frac{c^2}{2km}\right)}. \tag{11.2.21}$$

Note that $R_{\max}$ and $\omega_{\max}$ are continuous functions of c, for $c \geq 0$, since Equation 11.2.20 and Equation 11.2.21 reduce to $R_{\max} = F_0/k$ and $\omega_{\max} = 0$ if $c = \sqrt{2km}$.

This page titled 11.2: Spring Problems II is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **6.2: Spring Problems II** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

11.2.1: Spring Problems II (Exercises)

Q6.2.1

1. A 64 lb object stretches a spring 4 ft in equilibrium. It is attached to a dashpot with damping constant $c = 8$ lb-sec/ft. The object is initially displaced 18 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement and time–varying amplitude for $t > 0$.

2. A 16 lb weight is attached to a spring with natural length 5 ft. With the weight attached, the spring measures 8.2 ft. The weight is initially displaced 3 ft below equilibrium and given an upward velocity of 2 ft/sec. Find and graph its displacement for $t > 0$ if the medium resists the motion with a force of one lb for each ft/sec of velocity. Also, find its time–varying amplitude.

3. An 8 lb weight stretches a spring 1.5 inches. It is attached to a dashpot with damping constant $c = 8$ lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given an upward velocity of 6 ft/sec. Find and graph its displacement for $t > 0$.

4. A 96 lb weight stretches a spring 3.2 ft in equilibrium. It is attached to a dashpot with damping constant c=18 lb-sec/ft. The weight is initially displaced 15 inches below equilibrium and given a downward velocity of 12 ft/sec. Find its displacement for $t > 0$.

5. A 16 lb weight stretches a spring 6 inches in equilibrium. It is attached to a damping mechanism with constant c. Find all values of c such that the free vibration of the weight has infinitely many oscillations.

6. An 8 lb weight stretches a spring .32 ft. The weight is initially displaced 6 inches above equilibrium and given an upward velocity of 4 ft/sec. Find its displacement for $t > 0$ if the medium exerts a damping force of 1.5 lb for each ft/sec of velocity.

7. A 32 lb weight stretches a spring 2 ft in equilibrium. It is attached to a dashpot with constant $c = 8$ lb-sec/ft. The weight is initially displaced 8 inches below equilibrium and released from rest. Find its displacement for $t > 0$.

8. A mass of 20 gm stretches a spring 5 cm. The spring is attached to a dashpot with damping constant 400 dyne sec/cm. Determine the displacement for $t > 0$ if the mass is initially displaced 9 cm above equilibrium and released from rest.

9. A 64 lb weight is suspended from a spring with constant $k = 25$ lb/ft. It is initially displaced 18 inches above equilibrium and released from rest. Find its displacement for $t > 0$ if the medium resists the motion with 6 lb of force for each ft/sec of velocity.

10. A 32 lb weight stretches a spring 1 ft in equilibrium. The weight is initially displaced 6 inches above equilibrium and given a downward velocity of 3 ft/sec. Find its displacement for $t > 0$ if the medium resists the motion with a force equal to 3 times the speed in ft/sec.

11. An 8 lb weight stretches a spring 2 inches. It is attached to a dashpot with damping constant $c = 4$ lb-sec/ft. The weight is initially displaced 3 inches above equilibrium and given a downward velocity of 4 ft/sec. Find its displacement for $t > 0$.

12. A 2 lb weight stretches a spring .32 ft. The weight is initially displaced 4 inches below equilibrium and given an upward velocity of 5 ft/sec. The medium provides damping with constant $c = 1/8$ lb-sec/ft. Find and graph the displacement for $t > 0$.

13. An 8 lb weight stretches a spring 8 inches in equilibrium. It is attached to a dashpot with damping constant $c = .5$ lb-sec/ft and subjected to an external force $F(t) = 4\cos 2t$ lb. Determine the steady state component of the displacement for $t > 0$.

14. A 32 lb weight stretches a spring 1 ft in equilibrium. It is attached to a dashpot with constant $c = 12$ lb-sec/ft. The weight is initially displaced 8 inches above equilibrium and released from rest. Find its displacement for $t > 0$.

15. A mass of one kg stretches a spring 49 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 4 N for each m/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.

16. A mass of 100 grams stretches a spring 98 cm in equilibrium. A dashpot attached to the spring supplies a damping force of 600 dynes for each cm/sec of speed. The mass is initially displaced 10 cm above equilibrium and given a downward velocity of 1 m/sec. Find its displacement for $t > 0$.

17. A 192 lb weight is suspended from a spring with constant $k = 6$ lb/ft and subjected to an external force $F(t) = 8\cos 3t$ lb. Find the steady state component of the displacement for $t > 0$ if the medium resists the motion with a force equal to 8 times the speed in

ft/sec.

18. A 2 gm mass is attached to a spring with constant 20 dyne/cm. Find the steady state component of the displacement if the mass is subjected to an external force $F(t) = 3\cos 4t - 5\sin 4t$ dynes and a dashpot supplies 4 dynes of damping for each cm/sec of velocity.

19. A 96 lb weight is attached to a spring with constant 12 lb/ft. Find and graph the steady state component of the displacement if the mass is subjected to an external force $F(t) = 18\cos t - 9\sin t$ lb and a dashpot supplies 24 lb of damping for each ft/sec of velocity.

20. A mass of one kg stretches a spring 49 cm in equilibrium. It is attached to a dashpot that supplies a damping force of 4 N for each m/sec of speed. Find the steady state component of its displacement if it is subjected to an external force $F(t) = 8\sin 2t - 6\cos 2t$ N.

21. A mass m is suspended from a spring with constant k and subjected to an external force $F(t) = \alpha\cos\omega_0 t + \beta\sin\omega_0 t$, where ω_0 is the natural frequency of the spring–mass system without damping. Find the steady state component of the displacement if a dashpot with constant c supplies damping.

22. Show that if c_1 and c_2 are not both zero then

$$y = e^{r_1 t}(c_1 + c_2 t)$$

can't equal zero for more than one value of t.

23. Show that if c_1 and c_2 are not both zero then

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

can't equal zero for more than one value of t.

24. Find the solution of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0,\ y'(0) = v_0,$$

given that the motion is underdamped, so the general solution of the equation is

$$y = e^{-ct/2m}(c_1\cos\omega_1 t + c_2\sin\omega_1 t).$$

25. Find the solution of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0,\ y'(0) = v_0,$$

given that the motion is overdamped, so the general solution of the equation is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}\ (r_1, r_2 < 0).$$

26. Find the solution of the initial value problem

$$my'' + cy' + ky = 0, \quad y(0) = y_0,\ y'(0) = v_0,$$

given that the motion is critically damped, so that the general solution of the equation is of the form

$$y = e^{r_1 t}(c_1 + c_2 t)\ (r_1 < 0).$$

This page titled 11.2.1: Spring Problems II (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **6.2E: Spring Problems II (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

CHAPTER OVERVIEW

12: Series Solutions of Linear Second Order Equations

In this Chapter, we study a class of second order differential equations that occur in many applications, but cannot be solved in closed form in terms of elementary functions.

12.1: Prelude to Series Solutions of Linear Second Order Equations

12.2: Review of Power Series

12.2.1: Review of Power Series (Exercises)

12.3: Series Solutions Near an Ordinary Point I

12.3.1: Series Solutions Near an Ordinary Point I (Exercises)

This page titled 12: Series Solutions of Linear Second Order Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

12.1: Prelude to Series Solutions of Linear Second Order Equations

In this Chapter, we study a class of second order differential equations that occur in many applications, but cannot be solved in closed form in terms of elementary functions. Here are some examples:

1. **Bessel's equation** which occurs in problems displaying cylindrical symmetry, such as diffraction of light through a circular aperture, propagation of electromagnetic radiation through a coaxial cable, and vibrations of a circular drum head.

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

2. **Airy's equation** which occurs in astronomy and quantum physics.

$$y'' - xy = 0,$$

3. **Legendre's equation** which occurs in problems displaying spherical symmetry, particularly in electromagnetism.

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

These equations and others considered in this chapter can be written in the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \tag{12.1.1}$$

where P_0, P_1, and P_2 are polynomials with no common factor. For most equations that occur in applications, these polynomials are of degree two or less. We'll impose this restriction, although the methods that we'll develop can be extended to the case where the coefficient functions are polynomials of arbitrary degree, or even power series that converge in some circle around the origin in the complex plane.

Since Equation 12.1.1 does not in general have closed form solutions, we seek series representations for solutions. We'll see that if $P_0(0) \neq 0$ then solutions of (A) can be written as power series

$$y = \sum_{n=0}^{\infty} a_n x^n$$

that converge in an open interval centered at $x = 0$.

This page titled 12.1: Prelude to Series Solutions of Linear Second Order Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **7.1: Prelude to Series Solutions of Linear Second Order Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

12.2: Review of Power Series

Many applications give rise to differential equations with solutions that can't be expressed in terms of elementary functions such as polynomials, rational functions, exponential and logarithmic functions, and trigonometric functions. The solutions of some of the most important of these equations can be expressed in terms of power series. We'll study such equations in this chapter. In this section we review relevant properties of power series. We'll omit proofs, which can be found in any standard calculus text.

Definition 7.2.1

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n, \tag{12.2.1}$$

where x_0 and a_0, a_1, …, a_n, …are constants, is called a **power series in** $x-x_0$. We say that the power series Equation 12.2.1 **converges** for a given x if the limit

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(x-x_0)^n$$

exists; otherwise, we say that the power series **diverges** for the given x.

A power series in $x-x_0$ must converge if $x=x_0$, since the positive powers of $x-x_0$ are all zero in this case. This may be the only value of x for which the power series converges. However, the next theorem shows that if the power series converges for some $x \neq x_0$ then the set of all values of x for which it converges forms an interval.

Theorem 7.2.2

For any power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

exactly one of the these statements is true:

I. The power series converges only for $x=x_0$.
II. The power series converges for all values of x.
III. There's a positive number R such that the power series converges if $|x-x_0|<R$ and diverges if $|x-x_0|>R$.

In case (iii) we say that R is the **radius of convergence** of the power series. For convenience, we include the other two cases in this definition by defining $R=0$ in case (i) and $R=\infty$ in case (ii). We define the **open interval of convergence** of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ to be

$$(x_0-R, x_0+R) \quad \text{if} \quad 0<R<\infty, \quad \text{or} \quad (-\infty,\infty) \quad \text{if} \quad R=\infty.$$

If R is finite, no general statement can be made concerning convergence at the endpoints $x=x_0\pm R$ of the open interval of convergence; the series may converge at one or both points, or diverge at both.

Recall from calculus that a series of constants $\sum_{n=0}^{\infty} \alpha_n$ is said to **converge absolutely** if the series of absolute values $\sum_{n=0}^{\infty} |\alpha_n|$ converges. It can be shown that a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ with a positive radius of convergence R converges absolutely in its open interval of convergence; that is, the series

$$\sum_{n=0}^{\infty} |a_n||x-x_0|^n$$

of absolute values converges if $|x-x_0|<R$. However, if $R<\infty$, the series may fail to converge absolutely at an endpoint $x_0\pm R$, even if it converges here.

The next theorem provides a useful method for determining the radius of convergence of a power series. It's derived in calculus by applying the ratio test to the corresponding series of absolute values. For related theorems see **Exercises 7.2.2** and **7.2.4**.

Theorem 7.2.3

Suppose there's an integer N such that $a_n \neq 0$ if $n \geq N$ and

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L,$$

where $0 \leq L \leq \infty$. Then the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is $R = 1/L$, which should be interpreted to mean that $R = 0$ if $L = \infty$, or $R = \infty$ if $L = 0$.

Example 7.2.1

Find the radius of convergence of the series:

a. $\sum_{n=0}^{\infty} n!x^n$
b. $\sum_{n=10}^{\infty} (-1)^n \frac{x^n}{n!}$
c. $\sum_{n=0}^{\infty} 2^n n^2 (x-1)^n$.

Solution a

Here $a_n = n!$, so

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty.$$

Hence, $R = 0$.

Solution b

Here $a_n = (1)^n/n!$ for $n \geq N = 10$, so

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Hence, $R = \infty$.

Solution c

Here $a_n = 2^n n^2$, so

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}(n+1)^2}{2^n n^2} = 2 \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^2 = 2.$$

Hence, $R = 1/2$.

Taylor Series

If a function f has derivatives of all orders at a point $x = x_0$, then the **Taylor series of f about x_0** is defined by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

In the special case where $x_0 = 0$, this series is also called the **Maclaurin series of f**.

Taylor series for most of the common elementary functions converge to the functions on their open intervals of convergence. For example, you are probably familiar with the following Maclaurin series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad -\infty < x < \infty \tag{12.2.2}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad -\infty < x < \infty \tag{12.2.3}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad -\infty < x < \infty \tag{12.2.4}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad -1 < x < 1 \tag{12.2.5}$$

Differentiation of Power Series

A power series with a positive radius of convergence defines a function

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

on its open interval of convergence. We say that the series **represents** f on the open interval of convergence. A function f represented by a power series may be a familiar elementary function as in Equations 12.2.2 - 12.2.5; however, it often happens that f isn't a familiar function, so the series actually **defines** f.

The next theorem shows that a function represented by a power series has derivatives of all orders on the open interval of convergence of the power series, and provides power series representations of the derivatives.

Theorem 7.2.4 : A Power Series

A power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with positive radius of convergence R has derivatives of all orders in its open interval of convergence, and successive derivatives can be obtained by repeatedly differentiating term by term; that is,

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}, \tag{12.2.6}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}, \tag{12.2.7}$$

$$\vdots \tag{12.2.8}$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (x - x_0)^{n-k}. \tag{12.2.9}$$

Moreover, all of these series have the same radius of convergence R.

Example 7.2.2

Let $f(x) = \sin x$. From Equation 12.2.3,

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

From Equation 12.2.6,

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx}\left[\frac{x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

which is the series Equation 12.2.4 for $\cos x$.

Uniqueness of Power Series

The next theorem shows that if f is **defined** by a power series in $x - x_0$ with a positive radius of convergence, then the power series is the Taylor series of f about x_0.

Theorem 7.2.5

If the power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

has a positive radius of convergence, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}; \tag{12.2.10}$$

that is, $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is the Taylor series of f about x_0.

This result can be obtained by setting $x = x_0$ in Equation 12.2.9, which yields

$$f^{(k)}(x_0) = k(k-1)\cdots 1 \cdot a_k = k! a_k.$$

This implies that

$$a_k = \frac{f^{(k)}(x_0)}{k!}.$$

Except for notation, this is the same as Equation 12.2.10.

The next theorem lists two important properties of power series that follow from Theorem 7.2.4 .

Theorem 7.2.6

a. If

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

for all x in an open interval that contains x_0, then $a_n = b_n$ for $n = 0, 1, 2, \ldots$.

b. If

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = 0$$

for all x in an open interval that contains x_0, then $a_n = 0$ for $n = 0, 1, 2, \ldots$.

To obtain (a) we observe that the two series represent the same function f on the open interval; hence, Theorem 7.2.4 implies that

$$a_n = b_n = \frac{f^{(n)}(x_0)}{n!}, \quad n = 0, 1, 2, \ldots.$$

(b) can be obtained from (a) by taking $b_n = 0$ for $n = 0, 1, 2, \ldots$.

Taylor Polynomials

If f has N derivatives at a point x_0, we say that

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

is the **N-th Taylor polynomial of f about x_0**. This definition and Theorem 7.2.4 imply that if

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

where the power series has a positive radius of convergence, then the Taylor polynomials of f about x_0 are given by

$$T_N(x) = \sum_{n=0}^{N} a_n (x - x_0)^n.$$

In numerical applications, we use the Taylor polynomials to approximate f on subintervals of the open interval of convergence of the power series. For example, Equation 12.2.2 implies that the Taylor polynomial T_N of $f(x) = e^x$ is

$$T_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}.$$

The solid curve in Figure 7.2.1 is the graph of $y = e^x$ on the interval $[0, 5]$. The dotted curves in Figure 7.2.1 are the graphs of the Taylor polynomials T_1, …, T_6 of $y = e^x$ about $x_0 = 0$. From this figure, we conclude that the accuracy of the approximation of $y = e^x$ by its Taylor polynomial T_N improves as N increases.

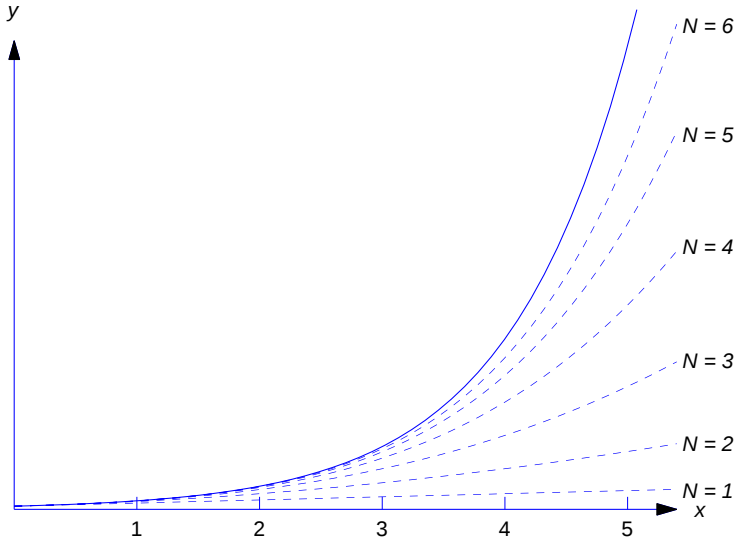


Figure 7.2.1 : Approximation of $y = e^x$ by Taylor polynomials about $x = 0$

Shifting the Summation Index

In Definition 7.2.1 of a power series in $x - x_0$, the n-th term is a constant multiple of $(x - x_0)^n$. This isn't true in Equation 12.2.6, Equation 12.2.7, and Equation 12.2.9, where the general terms are constant multiples of $(x - x_0)^{n-1}$, $(x - x_0)^{n-2}$, and $(x - x_0)^{n-k}$, respectively. However, these series can all be rewritten so that their n-th terms are constant multiples of $(x - x_0)^n$. For example, letting $n = k + 1$ in the series in Equation 12.2.6 yields

$$f'(x) = \sum_{k=0}^{\infty} (k+1) a_{k+1} (x - x_0)^k, \tag{12.2.11}$$

where we start the new summation index k from zero so that the first term in Equation 12.2.11 (obtained by setting $k = 0$) is the same as the first term in Equation 12.2.6 (obtained by setting $n = 1$). However, the sum of a series is independent of the symbol used to denote the summation index, just as the value of a definite integral is independent of the symbol used to denote the variable of integration. Therefore we can replace k by n in Equation 12.2.11 to obtain

$$f'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n, \tag{12.2.12}$$

where the general term is a constant multiple of $(x - x_0)^n$.

It isn't really necessary to introduce the intermediate summation index k. We can obtain Equation 12.2.12 directly from Equation 12.2.6 by replacing n by $n+1$ in the general term of Equation 12.2.6 and subtracting 1 from the lower limit of Equation 12.2.6. More generally, we use the following procedure for shifting indices.

Shifting the Summation Index in a Power Series

For any integer k, the power series

$$\sum_{n=n_0}^{\infty} b_n(x-x_0)^{n-k}$$

can be rewritten as

$$\sum_{n=n_0-k}^{\infty} b_{n+k}(x-x_0)^n$$

that is, replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation leaves the series unchanged.

Example 7.2.3

Rewrite the following power series from Equation 12.2.7 and Equation 12.2.9 so that the general term in each is a constant multiple of $(x-x_0)^n$:

$$(a)\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} \quad (b)\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}.$$

Solution a

Replacing n by $n+2$ in the general term and subtracting 2 from the lower limit of summation yields

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-x_0)^n.$$

Solution b

Replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation yields

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k} = \sum_{n=0}^{\infty}(n+k)(n+k-1)\cdots(n+1)a_{n+k}(x-x_0)^n.$$

Example 7.2.4

Given that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

write the function xf'' as a power series in which the general term is a constant multiple of x^n.

Solution

From Theorem 7.2.4 with $x_0 = 0$,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$xf''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}.$$

Replacing n by $n+1$ in the general term and subtracting 1 from the lower limit of summation yields

$$xf''(x) = \sum_{n=1}^{\infty}(n+1)na_{n+1}x^n.$$

We can also write this as

$$xf''(x) = \sum_{n=0}^{\infty}(n+1)na_{n+1}x^n,$$

since the first term in this last series is zero. (We'll see later that sometimes it is useful to include zero terms at the beginning of a series.)

Linear Combinations of Power Series

If a power series is multiplied by a constant, then the constant can be placed inside the summation; that is,

$$c\sum_{n=0}^{\infty}a_n(x-x_0)^n = \sum_{n=0}^{\infty}ca_n(x-x_0)^n.$$

Two power series

$$f(x) = \sum_{n=0}^{\infty}a_n(x-x_0)^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty}b_n(x-x_0)^n$$

with positive radii of convergence can be added term by term at points common to their open intervals of convergence; thus, if the first series converges for $|x-x_0| < R_1$ and the second converges for $|x-x_0| < R_2$, then

$$f(x)+g(x) = \sum_{n=0}^{\infty}(a_n+b_n)(x-x_0)^n$$

for $|x-x_0| < R$, where R is the smaller of R_1 and R_2. More generally, linear combinations of power series can be formed term by term; for example,

$$c_1f(x)+c_2f(x) = \sum_{n=0}^{\infty}(c_1a_n+c_2b_n)(x-x_0)^n.$$

Example 7.2.5

Find the Maclaurin series for $\cosh x$ as a linear combination of the Maclaurin series for e^x and e^{-x}.

Solution

By definition,

$$\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}.$$

Since

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} \quad \text{and} \quad e^{-x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!},$$

it follows that

$$\cosh x = \sum_{n=0}^{\infty}\frac{1}{2}[1+(-1)^n]\frac{x^n}{n!}. \tag{12.2.13}$$

Since

$$\frac{1}{2}[1+(-1)^n] = \begin{cases} 1 & \text{if } n = 2m, \text{ an even integer,} \\ 0 & \text{if } n = 2m+1, \text{ an odd integer,} \end{cases}$$

we can rewrite Equation 12.2.13 more simply as

$$\cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}.$$

This result is valid on $(-\infty, \infty)$, since this is the open interval of convergence of the Maclaurin series for e^x and e^{-x}.

Example 7.2.6

Suppose

$$y = \sum_{n=0}^{\infty} a_n x^n$$

on an open interval I that contains the origin.

a. Express

$$(2-x)y'' + 2y$$

as a power series in x on I.

b. Use the result of (a) to find necessary and sufficient conditions on the coefficients $\{a_n\}$ for y to be a solution of the homogeneous equation

$$(2-x)y'' + 2y = 0 \tag{12.2.14}$$

on I.

Solution a

From Equation 12.2.7 with $x_0 = 0$,

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$\begin{aligned} (2-x)y'' + 2y &= 2y'' - xy' + 2y \\ &= \textstyle\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n. \end{aligned} \tag{12.2.15}$$

To combine the three series we shift indices in the first two to make their general terms constant multiples of x^n; thus,

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}x^n \tag{12.2.16}$$

and

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1)na_{n+1}x^n = \sum_{n=0}^{\infty} (n+1)na_{n+1}x^n, \tag{12.2.17}$$

where we added a zero term in the last series so that when we substitute from Equation 12.2.16 and Equation 12.2.17 into Equation 12.2.15 all three series will start with $n = 0$; thus,

$$(2-x)y'' + 2y = \sum_{n=0}^{\infty} [2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n]x^n. \tag{12.2.18}$$

Solution b

From Equation 12.2.18 we see that y satisfies Equation 12.2.14 on I if

$$2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n = 0, \quad n = 0, 1, 2, \ldots. \tag{12.2.19}$$

Conversely, Theorem 7.2.5 b implies that if $y = \sum_{n=0}^{\infty} a_n x^n$ satisfies Equation 12.2.14 on I, then Equation 12.2.19 holds.

Example 7.2.7

Suppose

$$y=\sum_{n=0}^{\infty}a_n(x-1)^n$$

on an open interval I that contains $x_0=1$. Express the function

$$(1+x)y''+2(x-1)^2y'+3y \tag{12.2.20}$$

as a power series in $x-1$ on I.

Solution

Since we want a power series in $x-1$, we rewrite the coefficient of y'' in Equation 12.2.20 as $1+x=2+(x-1)$, so Equation 12.2.20 becomes

$$2y''+(x-1)y''+2(x-1)^2y'+3y.$$

From Equation 12.2.6 and Equation 12.2.7 with $x_0=1$,

$$y'=\sum_{n=1}^{\infty}na_n(x-1)^{n-1} \quad \text{and} \quad y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-2}.$$

Therefore

$$2y''=\sum_{n=2}^{\infty}2n(n-1)a_n(x-1)^{n-2},$$

$$(x-1)y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-1)^{n-1},$$

$$2(x-1)^2y'=\sum_{n=1}^{\infty}2na_n(x-1)^{n+1},$$

$$3y=\sum_{n=0}^{\infty}3a_n(x-1)^n.$$

Before adding these four series we shift indices in the first three so that their general terms become constant multiples of $(x-1)^n$. This yields

$$2y''=\sum_{n=0}^{\infty}2(n+2)(n+1)a_{n+2}(x-1)^n, \tag{12.2.21}$$

$$(x-1)y''=\sum_{n=0}^{\infty}(n+1)na_{n+1}(x-1)^n, \tag{12.2.22}$$

$$2(x-1)^2y'=\sum_{n=1}^{\infty}2(n-1)a_{n-1}(x-1)^n, \tag{12.2.23}$$

$$3y=\sum_{n=0}^{\infty}3a_n(x-1)^n, \tag{12.2.24}$$

where we added initial zero terms to the series in Equation 12.2.22 and Equation 12.2.23. Adding Equations 12.2.21–12.2.24 yields

$$\begin{aligned}(1+x)y''+2(x-1)^2y'+3y&=2y''+(x-1)y''+2(x-1)^2y'+3y\\&=\sum_{n=0}^{\infty}b_n(x-1)^n,\end{aligned}$$

where

$$b_0 = 4a_2 + 3a_0, \tag{12.2.25}$$

$$b_n = 2(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + 2(n-1)a_{n-1} + 3a_n,\ n \geq 1. \tag{12.2.26}$$

The formula Equation 12.2.25 for b_0 can't be obtained by setting $n = 0$ in Equation 12.2.26, since the summation in Equation 12.2.23 begins with $n = 1$, while those in Equation 12.2.21, Equation 12.2.22, and Equation 12.2.24 begin with $n = 0$.

This page titled 12.2: Review of Power Series is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **7.2: Review of Power Series** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

12.2.1: Review of Power Series (Exercises)

Q7.1.1

1.For each power series use Theorem 7.1.3 to find the radius of convergence R. If $R > 0$, find the open interval of convergence.

a. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n}(x-1)^n$
b. $\sum_{n=0}^{\infty} 2^n n(x-2)^n$
c. $\sum_{n=0}^{\infty} \frac{n!}{9^n}x^n$
d. $\sum_{n=0}^{\infty} \frac{n(n+1)}{16^n}(x-2)^n$
e. $\sum_{n=0}^{\infty} (-1)^n \frac{7^n}{n!}x^n$
f. $\sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}(n+1)^2}(x+7)^n$

2. Suppose there's an integer M such that $b_m \neq 0$ for $m \geq M$, and

$$\lim_{m\to\infty} \left|\frac{b_{m+1}}{b_m}\right| = L,$$

where $0 \leq L \leq \infty$. Show that the radius of convergence of

$$\sum_{m=0}^{\infty} b_m(x-x_0)^{2m}$$

is $R = 1/\sqrt{L}$, which is interpreted to mean that $R = 0$ if $L = \infty$ or $R = \infty$ if $L = 0$.

3. For each power series, use the result of **Exercise 7.1.2** to find the radius of convergence R. If $R > 0$, find the open interval of convergence.

a. $\sum_{m=0}^{\infty} (-1)^m(3m+1)(x-1)^{2m+1}$
b. $\sum_{m=0}^{\infty} (-1)^m \frac{m(2m+1)}{2^m}(x+2)^{2m}$
c. $\sum_{m=0}^{\infty} \frac{m!}{(2m)!}(x-1)^{2m}$
d. $\sum_{m=0}^{\infty} (-1)^m \frac{m!}{9^m}(x+8)^{2m}$
e. $\sum_{m=0}^{\infty} (-1)^m \frac{(2m-1)}{3^m}x^{2m+1}$
f. $\sum_{m=0}^{\infty} (x-1)^{2m}$

4. Suppose there's an integer M such that $b_m \neq 0$ for $m \geq M$, and

$$\lim_{m\to\infty} \left|\frac{b_{m+1}}{b_m}\right| = L,$$

where $0 \leq L \leq \infty$. Let k be a positive integer. Show that the radius of convergence of

$$\sum_{m=0}^{\infty} b_m(x-x_0)^{km}$$

is $R = 1/\sqrt[k]{L}$, which is interpreted to mean that $R = 0$ if $L = \infty$ or $R = \infty$ if $L = 0$.

5. For each power series use the result of **Exercise 7.1.4** to find the radius of convergence R. If $R > 0$, find the open interval of convergence.

a. $\sum_{m=0}^{\infty} \frac{(-1)^m}{(27)^m}(x-3)^{3m+2}$
b. $\sum_{m=0}^{\infty} \frac{x^{7m+6}}{m}$
c. $\sum_{m=0}^{\infty} \frac{9^m(m+1)}{(m+2)}(x-3)^{4m+2}$
d. $\sum_{m=0}^{\infty} (-1)^m \frac{2^m}{m!}x^{4m+3}$
e. $\sum_{m=0}^{\infty} \frac{m!}{(26)^m}(x+1)^{4m+3}$
f. $\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m m(m+1)}(x-1)^{3m+1}$

6. Graph $y = \sin x$ and the Taylor polynomial

$$T_{2M+1}(x) = \sum_{n=0}^{M} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

on the interval $(-2\pi, 2\pi)$ for $M = 1, 2, 3, \ldots$, until you find a value of M for which there's no perceptible difference between the two graphs.

7. Graph $y = \cos x$ and the Taylor polynomial

$$T_{2M}(x) = \sum_{n=0}^{M} \frac{(-1)^n x^{2n}}{(2n)!}$$

on the interval $(-2\pi, 2\pi)$ for $M = 1, 2, 3, \ldots$, until you find a value of M for which there's no perceptible difference between the two graphs.

8. Graph $y = 1/(1-x)$ and the Taylor polynomial

$$T_N(x) = \sum_{n=0}^{N} x^n$$

on the interval $[0, .95]$ for $N = 1, 2, 3, \ldots$, until you find a value of N for which there's no perceptible difference between the two graphs. Choose the scale on the y-axis so that $0 \le y \le 20$.

9. Graph $y = \cosh x$ and the Taylor polynomial

$$T_{2M}(x) = \sum_{n=0}^{M} \frac{x^{2n}}{(2n)!}$$

on the interval $(-5, 5)$ for $M = 1, 2, 3, \ldots$, until you find a value of M for which there's no perceptible difference between the two graphs. Choose the scale on the y-axis so that $0 \le y \le 75$.

10. Graph $y = \sinh x$ and the Taylor polynomial

$$T_{2M+1}(x) = \sum_{n=0}^{M} \frac{x^{2n+1}}{(2n+1)!}$$

on the interval $(-5, 5)$ for $M = 0, 1, 2, \ldots$, until you find a value of M for which there's no perceptible difference between the two graphs. Choose the scale on the y-axis so that $-75 \le y \le 75$.

Q7.1.2

In **Exercises 7.1.11-7.1.15** find a power series solution \(y(x)=\sum_{n=0}^{\infty} a_{n}x^{n} \nonumber \].

11. $(2+x)y'' + xy' + 3y$

12. $(1+3x^2)y'' + 3x^2y' - 2y$

13. $(1+2x^2)y'' + (2-3x)y' + 4y$

14. $(1+x^2)y'' + (2-x)y' + 3y$

15. $(1+3x^2)y'' - 2xy' + 4y$

Q7.1.3

16. Suppose $y(x) = \sum_{n=0}^{\infty} a_n(x+1)^n$ on an open interval that contains $x_0 = -1$. Find a power series in $x+1$ for

$$xy'' + (4+2x)y' + (2+x)y.$$

17. Suppose $y(x) = \sum_{n=0}^{\infty} a_n(x-2)^n$ on an open interval that contains $x_0 = 2$. Find a power series in $x-2$ for

$$x^2y'' + 2xy' - 3xy.$$

18. Do the following experiment for various choices of real numbers a_0 and a_1.

a. Use differential equations software to solve the initial value problem

$$(2-x)y''+2y=0, \quad y(0)=a_0, \quad y'(0)=a_1,$$

numerically on $(-1.95, 1.95)$. Choose the most accurate method your software package provides. (See Section 10.1 for a brief discussion of one such method.)

b. For $N = 2, 3, 4, \ldots$, compute $a_2, \ldots, a_N$ from Equation 7.1.18 and graph

$$T_N(x)=\sum_{n=0}^{N} a_n x^n$$

and the solution obtained in (a) on the same axes. Continue increasing N until it is obvious that there's no point in continuing. (This sounds vague, but you'll know when to stop.)

19. Follow the directions of **Exercise 7.1.18** for the initial value problem

$$(1+x)y''+2(x-1)^2y'+3y=0, \quad y(1)=a_0, \quad y'(1)=a_1,$$

on the interval $(0, 2)$. Use Equations 7.1.24 and 7.1.25 to compute $\{a_n\}$.

20. Suppose the series $\sum_{n=0}^{\infty} a_n x^n$ converges on an open interval $(-R, R)$, let r be an arbitrary real number, and define

$$y(x)=x^r\sum_{n=0}^{\infty} a_n x^n=\sum_{n=0}^{\infty} a_n x^{n+r}$$

on $(0, R)$. Use Theorem 7.1.4 and the rule for differentiating the product of two functions to show that

$$\begin{aligned} y'(x)&=\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1},\\ y''(x)&=\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2},\\ &\vdots\\ y^{(k)}(x)&=\sum_{n=0}^{\infty}(n+r)(n+r-1)\cdots(n+r-k)a_n x^{n+r-k} \end{aligned}$$

on $(0, R)$

Q7.1.4

21. $x^2(1-x)y''+x(4+x)y'+(2-x)y$

22. $x^2(1+x)y''+x(1+2x)y'-(4+6x)y$

23. $x^2(1+x)y''-x(1-6x-x^2)y'+(1+6x+x^2)y$

24. $x^2(1+3x)y''+x(2+12x+x^2)y'+2x(3+x)y$

25. $x^2(1+2x^2)y''+x(4+2x^2)y'+2(1-x^2)y$

26. $x^2(2+x^2)y''+2x(5+x^2)y'+2(3-x^2)y$

This page titled 12.2.1: Review of Power Series (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **7.2E: Review of Power Series (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

12.3: Series Solutions Near an Ordinary Point I

Many physical applications give rise to second order homogeneous linear differential equations of the form

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \tag{12.3.1}$$

where P_0, P_1, and P_2 are polynomials. Usually the solutions of these equations can't be expressed in terms of familiar elementary functions. Therefore we'll consider the problem of representing solutions of Equation 12.3.1 with series.

We assume throughout that P_0, P_1 and P_2 have no common factors. Then we say that x_0 is an **ordinary point** of Equation 12.3.1 if $P_0(x_0) \neq 0$, or a **singular point** if $P_0(x_0) = 0$. For Legendre's equation,

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0, \tag{12.3.2}$$

$x_0 = 1$ and $x_0 = -1$ are singular points and all other points are ordinary points. For Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0,$$

$x_0 = 0$ is a singular point and all other points are ordinary points. If P_0 is a nonzero constant as in Airy's equation,

$$y'' - xy = 0, \tag{12.3.3}$$

then every point is an ordinary point.

Since polynomials are continuous everywhere, P_1/P_0 and P_2/P_0 are continuous at any point x_0 that isn't a zero of P_0. Therefore, if x_0 is an ordinary point of Equation 12.3.1 and a_0 and a_1 are arbitrary real numbers, then the initial value problem

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0, \quad y(x_0) = a_0, \quad y'(x_0) = a_1 \tag{12.3.4}$$

has a unique solution on the largest open interval that contains x_0 and does not contain any zeros of P_0. To see this, we rewrite the differential equation in Equation 12.3.4 as

$$y'' + \frac{P_1(x)}{P_0(x)}y' + \frac{P_2(x)}{P_0(x)}y = 0$$

and apply Theorem 5.1.1 with $p = P_1/P_0$ and $q = P_2/P_0$. In this section and the next we consider the problem of representing solutions of Equation 12.3.1 by power series that converge for values of x near an ordinary point x_0.

We state the next theorem without proof.

Theorem 7.3.1

Suppose P_0, P_1, and P_2 are polynomials with no common factor and P_0 isn't identically zero. Let x_0 be a point such that $P_0(x_0) \neq 0$, and let ρ be the distance from x_0 to the nearest zero of P_0 in the complex plane. (If P_0 is constant, then $\rho = \infty$.) Then every solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{12.3.5}$$

can be represented by a power series

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n \tag{12.3.6}$$

that converges at least on the open interval $(x_0 - \rho, x_0 + \rho)$. (If P_0 is nonconstant, so that ρ is necessarily finite, then the open interval of convergence of (12.3.6) may be larger than $(x_0 - \rho, x_0 + \rho)$. If P_0 is constant then $\rho = \infty$ and $(x_0 - \rho, x_0 + \rho) = (-\infty, \infty)$.

We call Equation 12.3.6 a **power series solution in $x - x_0$** of Equation 12.3.5. We'll now develop a method for finding power series solutions of Equation 12.3.5. For this purpose we write Equation 12.3.5 as $Ly = 0$, where

$$Ly = P_0y'' + P_1y' + P_2y. \tag{12.3.7}$$

Theorem 7.3.1 implies that every solution of $Ly = 0$ on $(x_0 - \rho, x_0 + \rho)$ can be written as

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Setting $x = x_0$ in this series and in the series

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1}$$

shows that $y(x_0) = a_0$ and $y'(x_0) = a_1$. Since every initial value problem Equation 12.3.4 has a unique solution, this means that a_0 and a_1 can be chosen arbitrarily, and a_2, a_3, …are uniquely determined by them.

To find a_2, a_3, …, we write P_0, P_1, and P_2 in powers of $x - x_0$, substitute

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

$$y' = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1},$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

into Equation 12.3.7, and collect the coefficients of like powers of $x - x_0$. This yields

$$Ly = \sum_{n=0}^{\infty} b_n (x - x_0)^n, \tag{12.3.8}$$

where $\{b_0, b_1, \dots, b_n, \dots\}$ are expressed in terms of $\{a_0, a_1, \dots, a_n, \dots\}$ and the coefficients of P_0, P_1, and P_2, written in powers of $x - x_0$. Since Equation 12.3.8 and (a) of Theorem 7.1.6 imply that $Ly = 0$ if and only if $b_n = 0$ for $n \geq 0$, all power series solutions in $x - x_0$ of $Ly = 0$ can be obtained by choosing a_0 and a_1 arbitrarily and computing a_2, a_3, …, successively so that $b_n = 0$ for $n \geq 0$. For simplicity, we call the power series obtained this way **the power series in $x - x_0$** for the general solution of $Ly = 0$, without explicitly identifying the open interval of convergence of the series.

Example 7.3.1

Let x_0 be an arbitrary real number. Find the power series in $x - x_0$ for the general solution of Equations like Equations 12.3.11, 12.3.12, and 12.3.13, which define a given coefficient in the sequence $\{a_n\}$ in terms of one or more coefficients with lesser indices are called **recurrence relations**. When we use a recurrence relation to compute terms of a sequence we are computing **recursively**.

In the remainder of this section we consider the problem of finding power series solutions in $x - x_0$ for equations of the form

$$\left(1 + \alpha(x - x_0)^2\right) y'' + \beta(x - x_0) y' + \gamma y = 0. \tag{12.3.9}$$

Many important equations that arise in applications are of this form with $x_0 = 0$, including Legendre's equation Equation 12.3.2, Airy's equation Equation 12.3.3, **Chebyshev's equation**,

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0,$$

and **Hermite's equation**,

$$y'' - 2xy' + 2\alpha y = 0.$$

Since

$$P_0(x) = 1 + \alpha(x - x_0)^2$$

in Equation 12.3.9, the point x_0 is an ordinary point of Equation 12.3.9, and Theorem 7.3.1 implies that the solutions of Equation 12.3.9 can be written as power series in $x - x_0$ that converge on the interval $(x_0 - 1/\sqrt{|\alpha|}, x_0 + 1/\sqrt{|\alpha|})$ if $\alpha \neq 0$, or on

$(-\infty,\infty)$ if $\alpha=0$. We'll see that the coefficients in these power series can be obtained by methods similar to the one used in Example 7.3.1 .

To simplify finding the coefficients, we introduce some notation for products:

$$\prod_{j=r}^{s} b_j = b_r b_{r+1}\cdots b_s \quad \text{if} \quad s \geq r.$$

Thus,

$$\prod_{j=2}^{7} b_j = b_2 b_3 b_4 b_5 b_6 b_7,$$

$$\prod_{j=0}^{4}(2j+1) = (1)(3)(5)(7)(9) = 945,$$

and

$$\prod_{j=2}^{2} j^2 = 2^2 = 4.$$

We define

$$\prod_{j=r}^{s} b_j = 1 \quad \text{if} \quad s < r,$$

no matter what the form of b_j.

Example 7.3.2

Find the power series in x for the general solution of

$$y'' + y = 0. \tag{12.3.10}$$

Solution

Here

$$Ly = y'' + y.$$

If

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2},$$

so

$$Ly = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2} + \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

To collect coefficients of like powers of $x - x_0$, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x - x_0)^n + \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

with

$$b_n = (n+2)(n+1)a_{n+2} + a_n.$$

Therefore $Ly = 0$ if and only if

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0, \tag{12.3.11}$$

where a_0 and a_1 are arbitrary. Since the indices on the left and right sides of Equation 12.3.11 differ by two, we write Equation 12.3.11 separately for n even $(n = 2m)$ and n odd $(n = 2m+1)$. This yields

$$a_{2m+2} = \frac{-a_{2m}}{(2m+2)(2m+1)}, \quad m \geq 0, \tag{12.3.12}$$

and

$$a_{2m+3} = \frac{-a_{2m+1}}{(2m+3)(2m+2)}, \quad m \geq 0. \tag{12.3.13}$$

Computing the coefficients of the even powers of $x - x_0$ from Equation 12.3.12 yields

$$\begin{aligned}
a_2 &= -\frac{a_0}{2\cdot 1} \\
a_4 &= -\frac{a_2}{4\cdot 3} = -\frac{1}{4\cdot 3}\left(-\frac{a_0}{2\cdot 1}\right) = \frac{a_0}{4\cdot 3\cdot 2\cdot 1}, \\
a_6 &= -\frac{a_4}{6\cdot 5} = -\frac{1}{6\cdot 5}\left(\frac{a_0}{4\cdot 3\cdot 2\cdot 1}\right) = -\frac{a_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1},
\end{aligned}$$

and, in general,

$$a_{2m} = (-1)^m \frac{a_0}{(2m)!}, \quad m \geq 0. \tag{12.3.14}$$

Computing the coefficients of the odd powers of $x - x_0$ from Equation 12.3.13 yields

$$\begin{aligned}
a_3 &= -\frac{a_1}{3\cdot 2} \\
a_5 &= -\frac{a_3}{5\cdot 4} = -\frac{1}{5\cdot 4}\left(-\frac{a_1}{3\cdot 2}\right) = \frac{a_1}{5\cdot 4\cdot 3\cdot 2}, \\
a_7 &= -\frac{a_5}{7\cdot 6} = -\frac{1}{7\cdot 6}\left(\frac{a_1}{5\cdot 4\cdot 3\cdot 2}\right) = -\frac{a_1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2},
\end{aligned}$$

and, in general,

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!} \quad m \geq 0. \tag{12.3.15}$$

Thus, the general solution of Equation 12.3.10 can be written as

$$y = \sum_{m=0}^{\infty} a_{2m}(x - x_0)^{2m} + \sum_{m=0}^{\infty} a_{2m+1}(x - x_0)^{2m+1},$$

or, from Equation 12.3.14 and Equation 12.3.15, as

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m+1}}{(2m+1)!}. \tag{12.3.16}$$

If we recall from calculus that

$$\sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m}}{(2m)!} = \cos(x - x_0) \quad \text{and} \quad \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m+1}}{(2m+1)!} = \sin(x - x_0),$$

then Equation 12.3.16 becomes

$$y = a_0 \cos(x - x_0) + a_1 \sin(x - x_0),$$

which should look familiar.

$$(1+2x^2)y''+6xy'+2y=0. \tag{12.3.17}$$

Solution

Here

$$Ly=(1+2x^2)y''+6xy'+2y.$$

If

$$y=\sum_{n=0}^{\infty}a_nx^n$$

then

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1} \quad \text{and} \quad y''=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2},$$

so

$$\begin{aligned}Ly&=(1+2x^2)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+6x\sum_{n=1}^{\infty}na_nx^{n-1}+2\sum_{n=0}^{\infty}a_nx^n\\&=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+\sum_{n=0}^{\infty}\left[2n(n-1)+6n+2\right]a_nx^n\\&=\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}+2\sum_{n=0}^{\infty}(n+1)^2a_nx^n.\end{aligned}$$

To collect coefficients of x^n, we shift the summation index in the first sum. This yields

$$Ly=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n+2\sum_{n=0}^{\infty}(n+1)^2a_nx^n=\sum_{n=0}^{\infty}b_nx^n,$$

with

$$b_n=(n+2)(n+1)a_{n+2}+2(n+1)^2a_n, \quad n\ge 0.$$

To obtain solutions of Equation 12.3.17, we set $b_n=0$ for $n\ge 0$. This is equivalent to the recurrence relation

$$a_{n+2}=-2\frac{n+1}{n+2}a_n, \quad n\ge 0. \tag{12.3.18}$$

Since the indices on the left and right differ by two, we write Equation 12.3.18 separately for $n=2m$ and $n=2m+1$, as in Example 7.3.1 . This yields

$$a_{2m+2}=-2\frac{2m+1}{2m+2}a_{2m}=-\frac{2m+1}{m+1}a_{2m}, \quad m\ge 0, \tag{12.3.19}$$

and

$$a_{2m+3}=-2\frac{2m+2}{2m+3}a_{2m+1}=-4\frac{m+1}{2m+3}a_{2m+1}, \quad m\ge 0. \tag{12.3.20}$$

Computing the coefficients of even powers of x from Equation 12.3.19 yields

$$\begin{aligned}a_2&=-\frac{1}{1}a_0,\\a_4&=-\frac{3}{2}a_2=\left(-\frac{3}{2}\right)\left(-\frac{1}{1}\right)a_0=\frac{1\cdot 3}{1\cdot 2}a_0,\\a_6&=-\frac{5}{3}a_4=-\frac{5}{3}\left(\frac{1\cdot 3}{1\cdot 2}\right)a_0=-\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}a_0,\\a_8&=-\frac{7}{4}a_6=-\frac{7}{4}\left(-\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\right)a_0=\frac{1\cdot 3\cdot 5\cdot 7}{1\cdot 2\cdot 3\cdot 4}a_0.\end{aligned}$$

In general,

$$a_{2m} = (-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!} a_0, \quad m \geq 0. \tag{12.3.21}$$

(Note that Equation 12.3.21 is correct for $m = 0$ because we defined $\prod_{j=1}^{0} b_j = 1$ for any b_j.)

Computing the coefficients of odd powers of x from Equation 12.3.20 yields

$$\begin{aligned}
a_3 &= -4\frac{1}{3}a_1,\\
a_5 &= -4\frac{2}{5}a_3 = -4\frac{2}{5}\left(-4\frac{1}{3}\right)a_1 = 4^2\frac{1\cdot 2}{3\cdot 5}a_1,\\
a_7 &= -4\frac{3}{7}a_5 = -4\frac{3}{7}\left(4^2\frac{1\cdot 2}{3\cdot 5}\right)a_1 = -4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}a_1,\\
a_9 &= -4\frac{4}{9}a_7 = -4\frac{4}{9}\left(4^3\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7}\right)a_1 = 4^4\frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9}a_1.
\end{aligned}$$

In general,

$$a_{2m+1} = \frac{(-1)^m 4^m m!}{\prod_{j=1}^{m}(2j+1)} a_1, \quad m \geq 0. \tag{12.3.22}$$

From Equation 12.3.21 and Equation 12.3.22,

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^{m}(2j-1)}{m!} x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{4^m m!}{\prod_{j=1}^{m}(2j+1)} x^{2m+1}.$$

is the power series in x for the general solution of Equation 12.3.17. Since $P_0(x) = 1 + 2x^2$ has no real zeros, Theorem 5.1.1 implies that every solution of Equation 12.3.17 is defined on $(-\infty, \infty)$. However, since $P_0(\pm i/\sqrt{2}) = 0$, Theorem 7.3.1 implies only that the power series converges in $(-1/\sqrt{2}, 1/\sqrt{2})$ for any choice of a_0 and a_1.

The results in Examples 7.3.1 and 7.3.2 are consequences of the following general theorem.

Theorem 7.3.2

The coefficients $\{a_n\}$ in any solution $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ of

$$\left(1 + \alpha(x - x_0)^2\right) y'' + \beta(x - x_0) y' + \gamma y = 0 \tag{12.3.23}$$

satisfy the recurrence relation

$$a_{n+2} = -\frac{p(n)}{(n+2)(n+1)} a_n, \quad n \geq 0, \tag{12.3.24}$$

where

$$p(n) = \alpha n(n-1) + \beta n + \gamma. \tag{12.3.25}$$

Moreover, the coefficients of the even and odd powers of $x - x_0$ can be computed separately as

$$a_{2m+2} = -\frac{p(2m)}{(2m+2)(2m+1)} a_{2m}, \quad m \geq 0 \tag{12.3.26}$$

$$a_{2m+3} = -\frac{p(2m+1)}{(2m+3)(2m+2)} a_{2m+1}, \quad m \geq 0, \tag{12.3.27}$$

where a_0 and a_1 are arbitrary.

Proof

Here

$$Ly = (1 + \alpha(x - x_0)^2) y'' + \beta(x - x_0) y' + \gamma y.$$

If

$$y=\sum_{n=0}^{\infty}a_n(x-x_0)^n,$$

then

$$y'=\sum_{n=1}^{\infty}na_n(x-x_0)^{n-1}\quad\text{and}\quad y''=\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}.$$

Hence,

$$\begin{aligned}Ly &= \textstyle\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}+\sum_{n=0}^{\infty}\left[\alpha n(n-1)+\beta n+\gamma\right]a_n(x-x_0)^n\\ &= \textstyle\sum_{n=2}^{\infty}n(n-1)a_n(x-x_0)^{n-2}+\sum_{n=0}^{\infty}p(n)a_n(x-x_0)^n,\end{aligned}$$

from Equation 12.3.25. To collect coefficients of powers of $x-x_0$, we shift the summation index in the first sum. This yields

$$Ly=\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2}+p(n)a_n\right](x-x_0)^n.$$

Thus, $Ly=0$ if and only if

$$(n+2)(n+1)a_{n+2}+p(n)a_n=0,\quad n\ge 0,$$

which is equivalent to Equation 12.3.24. Writing Equation 12.3.24 separately for the cases where $n=2m$ and $n=2m+1$ yields Equation 12.3.26 and Equation 12.3.27.

Example 7.3.3

Find the power series in $x-1$ for the general solution of

$$(2+4x-2x^2)y''-12(x-1)y'-12y=0. \tag{12.3.28}$$

Solution

We must first write the coefficient $P_0(x)=2+4x-x^2$ in powers of $x-1$. To do this, we write $x=(x-1)+1$ in $P_0(x)$ and then expand the terms, collecting powers of $x-1$; thus,

$$\begin{aligned}2+4x-2x^2 &= 2+4[(x-1)+1]-2[(x-1)+1]^2\\ &= 4-2(x-1)^2.\end{aligned}$$

Therefore we can rewrite Equation 12.3.28 as

$$\left(4-2(x-1)^2\right)y''-12(x-1)y'-12y=0,$$

or, equivalently,

$$\left(1-\frac{1}{2}(x-1)^2\right)y''-3(x-1)y'-3y=0.$$

This is of the form Equation 12.3.23 with $\alpha=-1/2$, $\beta=-3$, and $\gamma=-3$. Therefore, from Equation 12.3.25

$$p(n)=-\frac{n(n-1)}{2}-3n-3=-\frac{(n+2)(n+3)}{2}.$$

Hence, Theorem7.3.2 implies that

$$\begin{aligned}a_{2m+2} &= -\frac{p(2m)}{(2m+2)(2m+1)}a_{2m}\\ &= \frac{(2m+2)(2m+3)}{2(2m+2)(2m+1)}a_{2m}=\frac{2m+3}{2(2m+1)}a_{2m},\quad m\ge 0\end{aligned}$$

and

$$a_{2m+3} = -\frac{p(2m+1)}{(2m+3)(2m+2)}a_{2m+1}$$
$$= \frac{(2m+3)(2m+4)}{2(2m+3)(2m+2)}a_{2m+1} = \frac{m+2}{2(m+1)}a_{2m+1}, \quad m \ge 0.$$

We leave it to you to show that

$$a_{2m} = \frac{2m+1}{2^m}a_0 \quad \text{and} \quad a_{2m+1} = \frac{m+1}{2^m}a_1, \quad m \ge 0,$$

which implies that the power series in $x-1$ for the general solution of Equation 12.3.28 is

$$y = a_0\sum_{m=0}^{\infty}\frac{2m+1}{2^m}(x-1)^{2m} + a_1\sum_{m=0}^{\infty}\frac{m+1}{2^m}(x-1)^{2m+1}.$$

In the examples considered so far we were able to obtain closed formulas for coefficients in the power series solutions. In some cases this is impossible, and we must settle for computing a finite number of terms in the series. The next example illustrates this with an initial value problem.

Example 7.3.4

Compute a_0, a_1, …, a_7 in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1+2x^2)y'' + 10xy' + 8y = 0, \quad y(0) = 2, \quad y'(0) = -3. \tag{12.3.29}$$

Solution

Since $\alpha = 2$, $\beta = 10$, and $\gamma = 8$ in Equation 12.3.29,

$$p(n) = 2n(n-1) + 10n + 8 = 2(n+2)^2.$$

Therefore

$$a_{n+2} = -2\frac{(n+2)^2}{(n+2)(n+1)}a_n = -2\frac{n+2}{n+1}a_n, \quad n \ge 0.$$

Writing this equation separately for $n = 2m$ and $n = 2m+1$ yields

$$a_{2m+2} = -2\frac{(2m+2)}{2m+1}a_{2m} = -4\frac{m+1}{2m+1}a_{2m}, \quad m \ge 0 \tag{12.3.30}$$

$$a_{2m+3} = -2\frac{2m+3}{2m+2}a_{2m+1} = -\frac{2m+3}{m+1}a_{2m+1}, \quad m \ge 0. \tag{12.3.31}$$

Starting with $a_0 = y(0) = 2$, we compute a_2, a_4, and a_6 from Equation 12.3.30 :

$$a_2 = -4\frac{1}{1}2 = -8,$$
$$a_4 = -4\frac{2}{3}(-8) = \frac{64}{3},$$
$$a_6 = -4\frac{3}{5}\left(\frac{64}{3}\right) = -\frac{256}{5}.$$

Starting with $a_1 = y'(0) = -3$, we compute a_3, a_5 and a_7 from Equation 12.3.31 :

$$a_3 = -\frac{3}{1}(-3) = 9,$$
$$a_5 = -\frac{5}{2}9 = -\frac{45}{2},$$
$$a_7 = -\frac{7}{3}\left(-\frac{45}{2}\right) = \frac{105}{2}.$$

Therefore the solution of Equation 12.3.29 is

$$y = 2 - 3x - 8x^2 + 9x^3 + \frac{64}{3}x^4 - \frac{45}{2}x^5 - \frac{256}{5}x^6 + \frac{105}{2}x^7 + \cdots .$$

Using Technology

Computing coefficients recursively as in Example 7.3.4 is tedious. We recommend that you do this kind of computation by writing a short program to implement the appropriate recurrence relation on a calculator or computer. You may wish to do this in verifying examples and doing exercises (identified by the symbol C) in this chapter that call for numerical computation of the coefficients in series solutions. We obtained the answers to these exercises by using software that can produce answers in the form of rational numbers. However, it is perfectly acceptable - and more practical - to get your answers in decimal form. You can always check them by converting our fractions to decimals.

If you're interested in actually using series to compute numerical approximations to solutions of a differential equation, then whether or not there's a simple closed form for the coefficients is essentially irrelevant. For computational purposes it is usually more efficient to start with the given coefficients $a_0 = y(x_0)$ and $a_1 = y'(x_0)$, compute a_2, …, a_N recursively, and then compute approximate values of the solution from the Taylor polynomial

$$T_N(x) = \sum_{n=0}^{N} a_n (x - x_0)^n.$$

The trick is to decide how to choose N so the approximation $y(x) \approx T_N(x)$ is sufficiently accurate on the subinterval of the interval of convergence that you're interested in. In the computational exercises in this and the next two sections, you will often be asked to obtain the solution of a given problem by numerical integration with software of your choice (see Section 10.1 for a brief discussion of one such method), and to compare the solution obtained in this way with the approximations obtained with T_N for various values of N. This is a typical textbook kind of exercise, designed to give you insight into how the accuracy of the approximation $y(x) \approx T_N(x)$ behaves as a function of N and the interval that you're working on. In real life, you would choose one or the other of the two methods (numerical integration or series solution). If you choose the method of series solution, then a practical procedure for determining a suitable value of N is to continue increasing N until the maximum of $|T_N - T_{N-1}|$ on the interval of interest is within the margin of error that you're willing to accept.

In doing computational problems that call for numerical solution of differential equations you should choose the most accurate numerical integration procedure your software supports, and experiment with the step size until you're confident that the numerical results are sufficiently accurate for the problem at hand.

This page titled 12.3: Series Solutions Near an Ordinary Point I is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **7.3: Series Solutions Near an Ordinary Point I** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

12.3.1: Series Solutions Near an Ordinary Point I (Exercises)

Q7.2.1

In **Exercises 7.2.1-7.2.8** find the power series in x for the general solution.

1. $(1+x^2)y'' + 6xy' + 6y = 0$
2. $(1+x^2)y'' + 2xy' - 2y = 0$
3. $(1+x^2)y'' - 8xy' + 20y = 0$
4. $(1-x^2)y'' - 8xy' - 12y = 0$
5. $(1+2x^2)y'' + 7xy' + 2y = 0$
6. $(1+x^2)y'' + 2xy' + \frac{1}{4}y = 0$
7. $(1-x^2)y'' - 5xy' - 4y = 0$
8. $(1+x^2)y'' - 10xy' + 28y = 0$

Q7.2.2

9.

a. Find the power series in x for the general solution of $y'' + xy' + 2y = 0$.
b. For several choices of a_0 and a_1, use differential equations software to solve the initial value problem

$$y'' + xy' + 2y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \tag{A}$$

numerically on $(-5, 5)$.
c. For fixed r in $\{1, 2, 3, 4, 5\}$ graph

$$T_N(x) = \sum_{n=0}^{N} a_n x^n$$

and the solution obtained in (a) on $(-r, r)$. Continue increasing N until there's no perceptible difference between the two graphs.

10. Follow the directions of Exercise [exer:7.2.9} for the differential equation

$$y'' + 2xy' + 3y = 0.$$

Q7.2.3

In **Exercises 7.2.11-7.2.13** find $a_0, \ldots, a_N$ for N at least 7 in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem.

11. $(1+x^2)y'' + xy' + y = 0, \quad y(0) = 2, \quad y'(0) = -1$
12. $(1+2x^2)y'' - 9xy' - 6y = 0, \quad y(0) = 1, \quad y'(0) = -1$
13. $(1+8x^2)y'' + 2y = 0, \quad y(0) = 2, \quad y'(0) = -1$

Q7.2.4

14. Do the next experiment for various choices of real numbers a_0, a_1, and r, with $0 < r < 1/\sqrt{2}$.

a. Use differential equations software to solve the initial value problem

$$(1-2x^2)y'' - xy' + 3y = 0, \quad y(0) = a_0, \quad y'(0) = a_1, \tag{A}$$

numerically on $(-r, r)$.
b. For $N = 2, 3, 4, \ldots$, compute $a_2, \ldots, a_N$ in the power series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of (A), and graph

$$T_N(x) = \sum_{n=0}^{N} a_n x^n$$

and the solution obtained in (a) on $(-r,r)$. Continue increasing N until there's no perceptible difference between the two graphs.

15. Do (a) and (b) for several values of r in $(0,1)$:

a. Use differential equations software to solve the initial value problem

$$(1+x^2)y''+10xy'+14y=0,\quad y(0)=5,\quad y'(0)=1, \tag{A}$$

numerically on $(-r,r)$.

b. For $N=2, 3, 4, \ldots$, compute $a_2, \ldots, a_N$ in the power series solution $y=\sum_{n=0}^{\infty}a_nx^n$ of (A), and graph

$$T_N(x)=\sum_{n=0}^{N}a_nx^n$$

and the solution obtained in (a) on $(-r,r)$. Continue increasing N until there's no perceptible difference between the two graphs. What happens to the required N as $r\to 1$?

c. Try (a) and (b) with $r=1.2$. Explain your results.

Q7.2.5

In **Exercises 7.2.16-7.2.20** find the power series in $x-x_0$ for the general solution.

16. $y''-y=0;\quad x_0=3$

17. $y''-(x-3)y'-y=0;\quad x_0=3$

18. $(1-4x+2x^2)y''+10(x-1)y'+6y=0;\quad x_0=1$

19. $(11-8x+2x^2)y''-16(x-2)y'+36y=0;\quad x_0=2$

20. $(5+6x+3x^2)y''+9(x+1)y'+3y=0;\quad x_0=-1$

Q7.2.6

In **Exercises 7.2.21-7.2.26** find $a_0,\ldots a_N$ for N at least 7 in the power series $y=\sum_{n=0}^{\infty}a_n(x-x_0)^n$ for the solution of the initial value problem. Take x_0 to be the point where the initial conditions are imposed.

21. $(x^2-4)y''-xy'-3y=0,\quad y(0)=-1,\quad y'(0)=2$

22. $y''+(x-3)y'+3y=0,\quad y(3)=-2,\quad y'(3)=3$

23. $(5-6x+3x^2)y''+(x-1)y'+12y=0,\quad y(1)=-1,\quad y'(1)=1$

24. $(4x^2-24x+37)y''+y=0,\quad y(3)=4,\quad y'(3)=-6$

25. $(x^2-8x+14)y''-8(x-4)y'+20y=0,\quad y(4)=3,\quad y'(4)=-4$

26. $(2x^2+4x+5)y''-20(x+1)y'+60y=0,\quad y(-1)=3,\quad y'(-1)=-3$

Q7.2.7

27.

a. Find a power series in x for the general solution of

$$(1+x^2)y''+4xy'+2y=0. \tag{A}$$

b. Use (a) and the formula

$$\frac{1}{1-r}=1+r+r^2+\cdots+r^n+\cdots\quad(-1<r<1)$$

for the sum of a geometric series to find a closed form expression for the general solution of (A) on $(-1,1)$.

c. Show that the expression obtained in (b) is actually the general solution of of (A) on $(-\infty,\infty)$.

28. Use Theorem 7.2.2 to show that the power series in x for the general solution of

$$(1+\alpha x^2)y''+\beta xy'+\gamma y=0$$

is

$$y=a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}p(2j)\right]\frac{x^{2m}}{(2m)!}+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}p(2j+1)\right]\frac{x^{2m+1}}{(2m+1)!}.$$

29. Use **Exercise 7.2.28** to show that all solutions of

$$(1+\alpha x^2)y''+\beta xy'+\gamma y=0$$

are polynomials if and only if

$$\alpha n(n-1)+\beta n+\gamma=\alpha(n-2r)(n-2s-1),$$

where r and s are nonnegative integers.

30.

a. Use Exercise [exer:7.2.28} to show that the power series in x for the general solution of

$$(1-x^2)y''-2bxy'+\alpha(\alpha+2b-1)y=0$$

is $y=a_0y_1+a_1y_2$, where

$$y_1=\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j-\alpha)(2j+\alpha+2b-1)\right]\frac{x^{2m}}{(2m)!}$$

and

$$y_2=\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j+1-\alpha)(2j+\alpha+2b)\right]\frac{x^{2m+1}}{(2m+1)!}$$

b. Suppose $2b$ isn't a negative odd integer and k is a nonnegative integer. Show that y_1 is a polynomial of degree $2k$ such that $y_1(-x)=y_1(x)$ if $\alpha=2k$, while y_2 is a polynomial of degree $2k+1$ such that $y_2(-x)=-y_2(-x)$ if $\alpha=2k+1$. Conclude that if n is a nonnegative integer, then there's a polynomial P_n of degree n such that $P_n(-x)=(-1)^nP_n(x)$ and

$$(1-x^2)P_n''-2bxP_n'+n(n+2b-1)P_n=0. \tag{A}$$

c. Show that (A) implies that

$$[(1-x^2)^bP_n']'=-n(n+2b-1)(1-x^2)^{b-1}P_n,$$

and use this to show that if m and n are nonnegative integers, then

$$[(1-x^2)^bP_n']P_m-[(1-x^2)^bP_m']P_n=[m(m+2b-1)-n(n+2b-1)](1-x^2)^{b-1}P_mP_n \tag{B}$$

d. Now suppose $b>0$. Use (B) and integration by parts to show that if $m\neq n$, then

$$\int_{-1}^{1}(1-x^2)^{b-1}P_m(x)P_n(x)\,dx=0.$$

(We say that P_m and P_n are **orthogonal on** $(-1,1)$ **with respect to the weighting function** $(1-x^2)^{b-1}$.)

31.

a. Use **Exercise 7.2.28** to show that the power series in x for the general solution of Hermite's equation

$$y''-2xy'+2\alpha y=0$$

is $y=a_0y_1+a_1y_1$, where

$$y_1=\sum_{m=0}^{\infty}\left[\prod_{j=0}^{m-1}(2j-\alpha)\right]\frac{2^mx^{2m}}{(2m)!}$$

and

$$y_2 = \sum_{m=0}^{\infty} \left[\prod_{j=0}^{m-1} (2j+1-\alpha) \right] \frac{2^m x^{2m+1}}{(2m+1)!}$$

b. Suppose k is a nonnegative integer. Show that y_1 is a polynomial of degree $2k$ such that $y_1(-x) = y_1(x)$ if $\alpha = 2k$, while y_2 is a polynomial of degree $2k+1$ such that $y_2(-x) = -y_2(-x)$ if $\alpha = 2k+1$. Conclude that if n is a nonnegative integer then there's a polynomial P_n of degree n such that $P_n(-x) = (-1)^n P_n(x)$ and

$$P_n'' - 2xP_n' + 2nP_n = 0. \tag{A}$$

c. Show that (A) implies that

$$[e^{-x^2} P_n']' = -2ne^{-x^2} P_n,$$

and use this to show that if m and n are nonnegative integers, then

$$[e^{-x^2} P_n']' P_m - [e^{-x^2} P_m']' P_n = 2(m-n)e^{-x^2} P_m P_n. \tag{B}$$

d. Use (B) and integration by parts to show that if $m \neq n$, then

$$\int_{-\infty}^{\infty} e^{-x^2} P_m(x) P_n(x)\, dx = 0.$$

(We say that P_m and P_n are **orthogonal on** $(-\infty, \infty)$ **with respect to the weighting function** e^{-x^2}.)

32. Consider the equation

$$(1+\alpha x^3)\, y'' + \beta x^2 y' + \gamma x y = 0, \tag{A}$$

and let $p(n) = \alpha n(n-1) + \beta n + \gamma$. (The special case $y'' - xy = 0$ of (A) is Airy's equation.)

a. Modify the argument used to prove Theorem [thmtype:7.2.2} to show that

$$y = \sum_{n=0}^{\infty} a_n x^n$$

is a solution of (A) if and only if $a_2 = 0$ and

$$a_{n+3} = -\frac{p(n)}{(n+3)(n+2)} a_n, \quad n \geq 0.$$

b. Show from (a) that $a_n = 0$ unless $n = 3m$ or $n = 3m+1$ for some nonnegative integer m, and that

$$a_{3m+3} = -\frac{p(3m)}{(3m+3)(3m+2)} a_{3m}, \quad m \geq 0,$$

and

$$a_{3m+4} = -\frac{p(3m+1)}{(3m+4)(3m+3)} a_{3m+1}, \quad m \geq 0,$$

where a_0 and a_1 may be specified arbitrarily.

c. Conclude from (b) that the power series in x for the general solution of (A) is

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{p(3j)}{3j+2} \right] \frac{x^{3m}}{3^m m!} + a_1 \sum_{m=0}^{\infty} (-1)^m \left[\prod_{j=0}^{m-1} \frac{p(3j+1)}{3j+4} \right] \frac{x^{3m+1}}{3^m m!}.$$

Q7.2.8

In **Exercises 7.2.33-7.2.37** use the method of **Exercise 7.2.32** to find the power series in x for the general solution.

33. $y'' - xy = 0$

34. $(1-2x^3)y'' - 10x^2 y' - 8xy = 0$

35. $(1+x^3)y''+7x^2y'+9xy=0$

36. $(1-2x^3)y''+6x^2y'+24xy=0$

37. $(1-x^3)y''+15x^2y'-63xy=0$

Q7.2.9

38. Consider the equation

$$\left(1+\alpha x^{k+2}\right)y''+\beta x^{k+1}y'+\gamma x^k y=0, \tag{A}$$

where k is a positive integer, and let $p(n)=\alpha n(n-1)+\beta n+\gamma$.

a. Modify the argument used to prove Theorem 7.2.2 to show that

$$y=\sum_{n=0}^{\infty}a_n x^n$$

is a solution of (A) if and only if $a_n=0$ for $2\le n\le k+1$ and

$$a_{n+k+2}=-\frac{p(n)}{(n+k+2)(n+k+1)}a_n,\quad n\ge 0.$$

b. Show from (a) that $a_n=0$ unless $n=(k+2)m$ or $n=(k+2)m+1$ for some nonnegative integer m, and that

$$a_{(k+2)(m+1)}=-\frac{p\left((k+2)m\right)}{(k+2)(m+1)[(k+2)(m+1)-1]}a_{(k+2)m},\quad m\ge 0,$$

and

$$a_{(k+2)(m+1)+1}=-\frac{p\left((k+2)m+1\right)}{[(k+2)(m+1)+1](k+2)(m+1)}a_{(k+2)m+1},\quad m\ge 0,$$

where a_0 and a_1 may be specified arbitrarily.

c. Conclude from (b) that the power series in x for the general solution of (A) is

$$y=a_0\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\frac{p((k+2)j)}{(k+2)(j+1)-1}\right]\frac{x^{(k+2)m}}{(k+2)^m m!}$$
$$+a_1\sum_{m=0}^{\infty}(-1)^m\left[\prod_{j=0}^{m-1}\frac{p((k+2)j+1)}{(k+2)(j+1)+1}\right]\frac{x^{(k+2)m+1}}{(k+2)^m m!}.$$

Q7.2.10

In **Exercises 7.2.39-7.2.44** use the method of **Exercise 7.2.38** to find the power series in x for the general solution.

39. $(1+2x^5)y''+14x^4y'+10x^3y=0$

40. $y''+x^2y=0$

41. $y''+x^6y'+7x^5y=0$

42. $(1+x^8)y''-16x^7y'+72x^6y=0$

43. $(1-x^6)y''-12x^5y'-30x^4y=0$

44. $y''+x^5y'+6x^4y=0$

This page titled 12.3.1: Series Solutions Near an Ordinary Point I (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **7.3E: Series Solutions Near an Ordinary Point I (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

CHAPTER OVERVIEW

13: Laplace Transforms

In this Chapter we study the method of Laplace transforms, which illustrates one of the basic problem solving techniques in mathematics: transform a difficult problem into an easier one, solve the latter, and then use its solution to obtain a solution of the original problem. The method discussed here transforms an initial value problem for a constant coefficient equation into an algebraic equation whose solution can then be used to solve the initial value problem. In some cases this method is merely an alternative procedure for solving problems that can be solved equally well by methods that we considered previously; however, in other cases the method of Laplace transforms is more efficient than the methods previously discussed. This is especially true in physical problems dealing with discontinuous forcing functions.

13.1: Introduction to the Laplace Transform

13.1.1: Introduction to the Laplace Transform (Exercises)

13.2: The Inverse Laplace Transform

13.2.1: The Inverse Laplace Transform (Exercises)

13.3: Solution of Initial Value Problems

13.3.1: Solution of Initial Value Problems (Exercises)

13.4: The Unit Step Function

13.4.1: The Unit Step Function (Exercises)

13.5: Constant Coefficient Equations with Piecewise Continuous Forcing Functions

13.5.1: Constant Coefficient Equations with Piecewise Continuous Forcing Functions (Exercises)

13.6: A Brief Table of Laplace Transforms

This page titled 13: Laplace Transforms is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

13.1: Introduction to the Laplace Transform

Definition of the Laplace Transform

To define the Laplace transform, we first recall the definition of an improper integral. If g is integrable over the interval $[a, T]$ for every $T > a$, then the **improper integral of g over $[a, \infty)$** is defined as

$$\int_a^\infty g(t)\,dt = \lim_{T\to\infty} \int_a^T g(t)\,dt. \tag{13.1.1}$$

We say that the improper integral **converges** if the limit in Equation 13.1.1 exists; otherwise, we say that the improper integral **diverges** or **does not exist**. Here's the definition of the Laplace transform of a function f.

Defintion 8.1.1 : Laplace Transform

Let f be defined for $t \geq 0$ and let s be a real number. Then the **Laplace transform** of f is the function F defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt, \tag{13.1.2}$$

for those values of s for which the improper integral converges.

It is important to keep in mind that the variable of integration in Equation 13.1.2 is t, while s is a parameter independent of t. We use t as the independent variable for f because in applications the Laplace transform is usually applied to functions of time.

The Laplace transform can be viewed as an operator $\mathcal{L}$ that transforms the function $f = f(t)$ into the function $F = F(s)$. Thus, Equation 13.1.2 can be expressed as

$$F = \mathcal{L}(f).$$

The functions f and F form a **transform pair**, which we'll sometimes denote by

$$f(t) \leftrightarrow F(s).$$

It can be shown that if $F(s)$ is defined for $s = s_0$ then it is defined for all $s > s_0$.

Computation of Some Simple Laplace Transforms

Example 8.1.1

Find the Laplace transform of $f(t) = 1$.

Solution

From Equation 13.1.2 with $f(t) = 1$,

$$F(s) = \int_0^\infty e^{-st}\,dt = \lim_{T\to\infty} \int_0^T e^{-st}\,dt.$$

If $s \neq 0$ then

$$\int_0^T e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^T = \frac{1-e^{-sT}}{s}. \tag{13.1.3}$$

Therefore

$$\lim_{T\to\infty} \int_0^T e^{-st}dt = \begin{cases} \frac{1}{s}, & s > 0, \\ \infty, & s < 0. \end{cases} \tag{13.1.4}$$

If $s = 0$ the integrand reduces to the constant 1, and

$$\lim_{T\to\infty}\int_0^T 1\,dt = \lim_{T\to\infty}\int_0^T 1\,dt = \lim_{T\to\infty} T = \infty.$$

Therefore $F(0)$ is undefined, and

$$F(s) = \int_0^\infty e^{-st}dt = \frac{1}{s}, \quad s > 0.$$

This result can be written in operator notation as

$$\mathcal{L}(1) = \frac{1}{s}, \quad s > 0,$$

or as the transform pair

$$1 \leftrightarrow \frac{1}{s}, \quad s > 0.$$

Note

It is convenient to combine the steps of integrating from 0 to T and letting $T \to \infty$. Therefore, instead of writing Equation 13.1.3 and 13.1.4 as separate steps we write

$$\int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \begin{cases} \frac{1}{s}, & s > 0 \\ \infty, & s < 0 \end{cases}$$

We'll follow this practice throughout this chapter.

Example 8.1.2

Find the Laplace transform of $f(t) = t$.

Solution

From Equation 13.1.2 with $f(t) = t$,

$$F(s) = \int_0^\infty e^{-st}t\,dt. \tag{13.1.5}$$

If $s \neq 0$, integrating by parts yields

$$\int_0^\infty e^{-st}t\,dt = -\frac{te^{-st}}{s}\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = -\left[\frac{t}{s} + \frac{1}{s^2}\right]e^{-st}\Big|_0^\infty$$
$$= \begin{cases} \frac{1}{s^2}, & s > 0, \\ \infty, & s < 0. \end{cases}$$

If $s = 0$, the integral in Equation 13.1.5 becomes

$$\int_0^\infty t\,dt = \frac{t^2}{2}\Big|_0^\infty = \infty.$$

Therefore $F(0)$ is undefined and

$$F(s) = \frac{1}{s^2}, \quad s > 0.$$

This result can also be written as

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad s > 0,$$

or as the transform pair

$$t \leftrightarrow \frac{1}{s^2}, \quad s > 0.$$

Example 8.1.3

Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution

From Equation 13.1.2 with $f(t) = e^{at}$,

$$F(s) = \int_0^\infty e^{-st} e^{at}\, dt.$$

Combining the exponentials yields

$$F(s) = \int_0^\infty e^{-(s-a)t}\, dt.$$

However, we know from Example 8.1.1 that

$$\int_0^\infty e^{-st}\, dt = \frac{1}{s}, \quad s > 0.$$

Replacing s by $s - a$ here shows that

$$F(s) = \frac{1}{s-a}, \quad s > a.$$

This can also be written as

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}, \quad s > a, \text{ or } e^{at} \leftrightarrow \frac{1}{s-a}, \quad s > a.$$

Example 8.1.4

[Find the Laplace transforms of $f(t) = \sin \omega t$ and $g(t) = \cos \omega t$, where ω is a constant.

Solution

Define

$$F(s) = \int_0^\infty e^{-st} \sin \omega t\, dt \tag{13.1.6}$$

and

$$G(s) = \int_0^\infty e^{-st} \cos \omega t\, dt. \tag{13.1.7}$$

If $s > 0$, integrating Equation 13.1.6 by parts yields

$$F(s) = -\frac{e^{-st}}{s} \sin \omega t \Big|_0^\infty + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t\, dt,$$

so

$$F(s) = \frac{\omega}{s} G(s). \tag{13.1.8}$$

If $s > 0$, integrating Equation 13.1.7 by parts yields

$$G(s) = -\frac{e^{-st} \cos \omega t}{s} \Big|_0^\infty - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t\, dt,$$

so

$$G(s) = \frac{1}{s} - \frac{\omega}{s}F(s).$$

Now substitute from Equation 13.1.8 into this to obtain

$$G(s) = \frac{1}{s} - \frac{\omega^2}{s^2}G(s).$$

Solving this for $G(s)$ yields

$$G(s) = \frac{s}{s^2+\omega^2}, \quad s > 0.$$

This and Equation 13.1.8 imply that

$$F(s) = \frac{\omega}{s^2+\omega^2}, \quad s > 0.$$

Tables of Laplace Transforms

Extensive tables of Laplace transforms have been compiled and are commonly used in applications. The brief table of Laplace transforms in the Appendix will be adequate for our purposes.

Example 8.1.5

Use the table of Laplace transforms to find $\mathcal{L}(t^3e^{4t})$.

Solution

The table includes the transform pair

$$t^n e^{at} \leftrightarrow \frac{n!}{(s-a)^{n+1}}.$$

Setting $n = 3$ and $a = 4$ here yields

$$\mathcal{L}(t^3e^{4t}) = \frac{3!}{(s-4)^4} = \frac{6}{(s-4)^4}.$$

We'll sometimes write Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms.

Linearity of the Laplace Transform

The next theorem presents an important property of the Laplace transform.

Theorem 8.1.2 Linearity Property

Suppose $\mathcal{L}(f_i)$ is defined for $s > s_i$, $1 \le i \le n$). Let s_0 be the largest of the numbers s_1, s_2, …,s_n, and let c_1, c_2,…, c_n be constants. Then

$$\mathcal{L}(c_1f_1 + c_2f_2 + \cdots + c_nf_n) = c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2) + \cdots + c_n\mathcal{L}(f_n) \text{ for } s > s_0.$$

Proof

We give the proof for the case where $n = 2$. If $s > s_0$ then

$$\begin{aligned}\mathcal{L}(c_1f_1 + c_2f_2) &= \int_0^\infty e^{-st}\left(c_1f_1(t) + c_2f_2(t)\right)\,dt\\ &= c_1\int_0^\infty e^{-st}f_1(t)\,dt + c_2\int_0^\infty e^{-st}f_2(t)\,dt\\ &= c_1\mathcal{L}(f_1) + c_2\mathcal{L}(f_2).\end{aligned}$$

Example 8.1.6

Use Theorem 8.1.2 and the known Laplace transform

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

to find $\mathcal{L}(\cosh bt)$ $(b \neq 0)$.

Solution

By definition,

$$\cosh bt = \frac{e^{bt} + e^{-bt}}{2}.$$

Therefore

$$\begin{aligned} \mathcal{L}(\cosh bt) &= \mathcal{L}\left(\tfrac{1}{2}e^{bt} + \tfrac{1}{2}e^{-bt}\right) \\ &= \tfrac{1}{2}\mathcal{L}(e^{bt}) + \tfrac{1}{2}\mathcal{L}(e^{-bt}) \qquad \text{(linearity property)} \\ &= \tfrac{1}{2}\,\tfrac{1}{s-b} + \tfrac{1}{2}\,\tfrac{1}{s+b}, \end{aligned} \tag{13.1.9}$$

where the first transform on the right is defined for $s > b$ and the second for $s > -b$; hence, both are defined for $s > |b|$. Simplifying the last expression in Equation 13.1.9 yields

$$\mathcal{L}(\cosh bt) = \frac{s}{s^2 - b^2}, \quad s > |b|.$$

The next theorem enables us to start with known transform pairs and derive others. (For other results of this kind, see **Exercises 8.1.6** and **8.1.13**.)

Theorem 8.1.3 First Shifting Theorem

If

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \tag{13.1.10}$$

is the Laplace transform of $f(t)$ for $s > s_0$, then $F(s-a)$ is the Laplace transform of $e^{at}f(t)$ for $s > s_0 + a$.

Proof

Replacing s by $s-a$ in Equation 13.1.10 yields

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt \tag{13.1.11}$$

if $s - a > s_0$; that is, if $s > s_0 + a$. However, Equation 13.1.11 can be rewritten as

$$F(s-a) = \int_0^\infty e^{-st}\left(e^{at}f(t)\right)\,dt,$$

which implies the conclusion.

Example 8.1.7

Use Theorem 8.1.3 and the known Laplace transforms of 1, t, $\cos \omega t$, and $\sin \omega t$ to find

$$\mathcal{L}(e^{at}), \quad \mathcal{L}(te^{at}), \quad \mathcal{L}(e^{\lambda t}\sin \omega t), \text{and } \mathcal{L}(e^{\lambda t}\cos \omega t).$$

Solution

In the following table the known transform pairs are listed on the left and the required transform pairs listed on the right are obtained by applying Theorem 8.1.3 .

Table 8.1.1

$f(t) \leftrightarrow F(s)$	$e^{at}f(t) \leftrightarrow F(s-a)$
$1 \leftrightarrow \frac{1}{s}, \quad s>0$	$e^{at} \leftrightarrow \frac{1}{(s-a)}, \quad s>a$
$t \leftrightarrow \frac{1}{s^2}, \quad s>0$	$te^{at} \leftrightarrow \frac{1}{(s-a)^2}, \quad s>a$
$\sin \omega t \leftrightarrow \frac{\omega}{s^2+\omega^2}, \quad s>0$	$e^{\lambda t}\sin \omega t \leftrightarrow \frac{\omega}{(s-\lambda)^2+\omega^2}, \quad s>\lambda$
$\cos \omega t \leftrightarrow \frac{s}{s^2+\omega^2}, \quad s>0$	$e^{\lambda t}\sin \omega t \leftrightarrow \frac{s-\lambda}{(s-\lambda)^2+\omega^2}, \quad s>\lambda$

Existence of Laplace Transforms

Not every function has a Laplace transform. For example, it can be shown (**Exercise 8.1.3**) that

$$\int_0^\infty e^{-st}e^{t^2}dt = \infty$$

for every real number s. Hence, the function $f(t) = e^{t^2}$ does not have a Laplace transform.

Our next objective is to establish conditions that ensure the existence of the Laplace transform of a function. We first review some relevant definitions from calculus.

Recall that a limit

$$\lim_{t\to t_0} f(t)$$

exists if and only if the one-sided limits

$$\lim_{t\to t_0-} f(t) \quad \text{and} \quad \lim_{t\to t_0+} f(t)$$

both exist and are equal; in this case,

$$\lim_{t\to t_0} f(t) = \lim_{t\to t_0-} f(t) = \lim_{t\to t_0+} f(t).$$

Recall also that f is continuous at a point t_0 in an open interval (a,b) if and only if

$$\lim_{t\to t_0} f(t) = f(t_0),$$

which is equivalent to

$$\lim_{t\to t_0+} f(t) = \lim_{t\to t_0-} f(t) = f(t_0). \tag{13.1.12}$$

For simplicity, we define

$$f(t_0+) = \lim_{t\to t_0+} f(t) \quad \text{and} \quad f(t_0-) = \lim_{t\to t_0-} f(t),$$

so Equation 13.1.12 can be expressed as

$$f(t_0+) = f(t_0-) = f(t_0).$$

If $f(t_0+)$ and $f(t_0-)$ have finite but distinct values, we say that f has a **jump discontinuity** at t_0, and

$$f(t_0+) - f(t_0-)$$

is called the **jump** in f at t_0 (Figure 8.1.1).

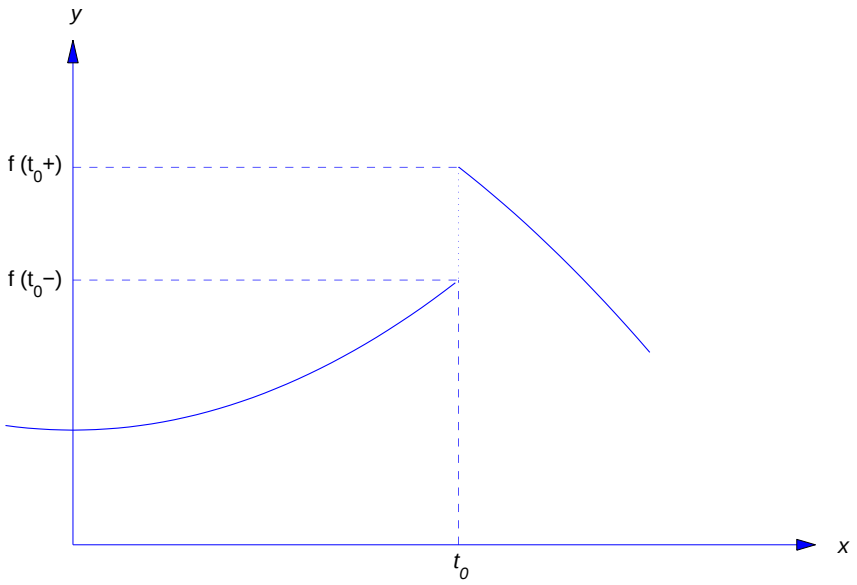


Figure 8.1.1 : A jump discontinuity.

If $f(t_0+)$ and $f(t_0-)$ are finite and equal, but either f isn't defined at t_0 or it is defined but

$$f(t_0) \neq f(t_0+) = f(t_0-),$$

we say that f has a **removable discontinuity** at t_0 (Figure 8.1.2). This terminology is appropriate since a function f with a removable discontinuity at t_0 can be made continuous at t_0 by defining (or redefining)

$$f(t_0) = f(t_0+) = f(t_0-).$$

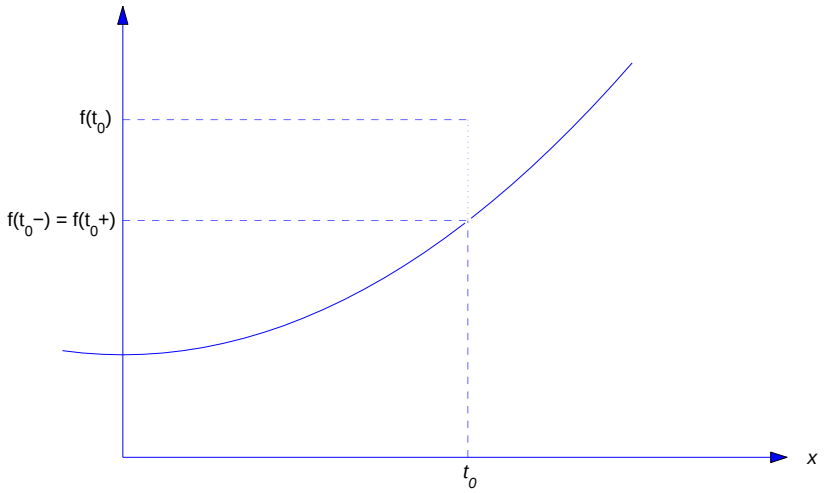


Figure 8.1.2

Note

We know from calculus that a definite integral is not affected by changing the values of its integrand at isolated points. Therefore, redefining a function f to make it continuous at removable discontinuities does not change $\mathcal{L}(f)$.

Definition 8.1.4 : Piecewise Continuous

- A function f is said to be **piecewise continuous** on a finite closed interval $[0,T]$ if $f(0+)$ and $f(T-)$ are finite and f is continuous on the open interval $(0,T)$ except possibly at finitely many points, where f may have jump discontinuities or removable discontinuities.

- A function f is said to be **piecewise continuous** on the infinite interval $[0,\infty)$ if it is piecewise continuous on $[0,T]$ for every $T>0$.

Figure 8.1.3 shows the graph of a typical piecewise continuous function.

It is shown in calculus that if a function is piecewise continuous on a finite closed interval then it is integrable on that interval. But if f is piecewise continuous on $[0,\infty)$, then so is $e^{-st}f(t)$, and therefore

$$\int_0^T e^{-st}f(t)\,dt$$

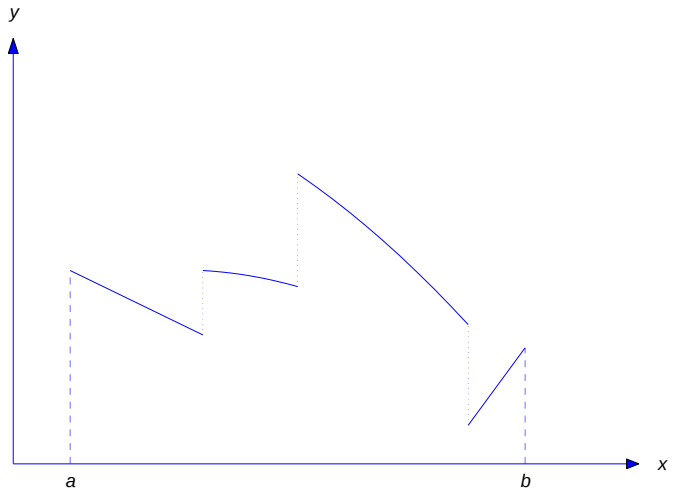


Figure 8.1.3

exists for every $T>0$. However, piecewise continuity alone does not guarantee that the improper integral

$$\int_0^\infty e^{-st}f(t)\,dt=\lim_{T\to\infty}\int_0^T e^{-st}f(t)\,dt \tag{13.1.13}$$

converges for s in some interval (s_0,∞). For example, we noted earlier that Equation 13.1.13 diverges for all s if $f(t)=e^{t^2}$. Stated informally, this occurs because e^{t^2} increases too rapidly as $t\to\infty$. The next definition provides a constraint on the growth of a function that guarantees convergence of its Laplace transform for s in some interval (s_0,∞).

Definition 8.1.5 : of exponential order

A function f is said to be **of exponential order** s_0 if there are constants M and t_0 such that

$$|f(t)|\le Me^{s_0t},\quad t\ge t_0. \tag{13.1.14}$$

In situations where the specific value of s_0 is irrelevant we say simply that f is **of exponential order**.

The next theorem gives useful sufficient conditions for a function f to have a Laplace transform. The proof is sketched in **Exercise 8.1.10**.

Theorem 8.1.6

If f is piecewise continuous on $[0,\infty)$ and of exponential order s_0, then $\mathcal{L}(f)$ is defined for $s>s_0$.

Note

We emphasize that the conditions of Theorem 8.1.6 are sufficient, but **not necessary**, for f to have a Laplace transform. For example, **Exercise 8.1.14(c)** shows that f may have a Laplace transform even though f isn't of exponential order

Example 8.1.8

If f is bounded on some interval $[t_0, \infty)$, say

$$|f(t)| \le M, \quad t \ge t_0,$$

then Equation 13.1.14 holds with $s_0 = 0$, so f is of exponential order zero. Thus, for example, $\sin \omega t$ and $\cos \omega t$ are of exponential order zero, and Theorem 8.1.6 implies that $\mathcal{L}(\sin \omega t)$ and $\mathcal{L}(\cos \omega t)$ exist for $s > 0$. This is consistent with the conclusion of Example 8.1.4 .

Example 8.1.9

It can be shown that if $\lim_{t\to\infty} e^{-s_0 t} f(t)$ exists and is finite then f is of exponential order s_0 (**Exercise 8.1.9**). If α is any real number and $s_0 > 0$ then $f(t) = t^\alpha$ is of exponential order s_0, since

$$\lim_{t\to\infty} e^{-s_0 t} t^\alpha = 0,$$

by L'Hôpital's rule. If $\alpha \ge 0$, f is also continuous on $[0, \infty)$. Therefore **Exercise 8.1.9** and Theorem 8.1.6 imply that $\mathcal{L}(t^\alpha)$ exists for $s \ge s_0$. However, since s_0 is an arbitrary positive number, this really implies that $\mathcal{L}(t^\alpha)$ exists for all $s > 0$. This is consistent with the results of Example 8.1.2 and **Exercises 8.1.6** and **8.1.8**.

Example 8.1.10

Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -3e^{-t}, & t \ge 1. \end{cases}$$

Solution

Since f is defined by different formulas on $[0, 1)$ and $[1, \infty)$, we write

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}(1)\, dt + \int_1^\infty e^{-st}(-3e^{-t})\, dt.$$

Since

$$\int_0^1 e^{-st} dt = \begin{cases} \frac{1-e^{-s}}{s} & s \ne 0 \\ 1 & s = 0 \end{cases}$$

and

$$\int_1^\infty e^{-st}(-3e^{-t})\, dt = -3\int_1^\infty e^{-(s+1)t}\, dt = -\frac{3e^{-(s+1)}}{s+1}, \quad s > -1,$$

it follows that

$$F(s) = \begin{cases} \frac{1-e^{-s}}{s} - 3\frac{e^{-(s+1)}}{s+1} & s > -1, s \ne 0 \\ 1 - \frac{3}{e} & s = 0 \end{cases}$$

This is consistent with Theorem 8.1.6 , since

$$|f(t)| \le 3e^{-t}, \quad t \ge 1,$$

and therefore f is of exponential order $s_0 = -1$.

Note

In Section 8.4 we'll develop a more efficient method for finding Laplace transforms of piecewise continuous functions.

Example 8.1.11

We stated earlier that

$$\int_0^\infty e^{-st}e^{t^2}\,dt = \infty$$

for all s, so Theorem 8.1.6 implies that $f(t) = e^{t^2}$ is not of exponential order, since

$$\lim_{t\to\infty} \frac{e^{t^2}}{Me^{s_0t}} = \lim_{t\to\infty} \frac{1}{M}e^{t^2-s_0t} = \infty,$$

so

$$e^{t^2} > Me^{s_0t}$$

for sufficiently large values of t, for any choice of M and s_0 (**Exercise 8.1.3**).

This page titled 13.1: Introduction to the Laplace Transform is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **8.1: Introduction to the Laplace Transform** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

13.1.1: Introduction to the Laplace Transform (Exercises)

Q8.1.1

1. Find the Laplace transforms of the following functions by evaluating the integral $F(s) = \int_0^\infty e^{-st} f(t)\, dt$.

a. t
b. te^{-t}
c. $\sinh bt$
d. $e^{2t} - 3e^t$
e. t^2

2. Use the table of Laplace transforms to find the Laplace transforms of the following functions.

a. $\cosh t \sin t$
b. $\sin^2 t$
c. $\cos^2 2t$
d. $\cosh^2 t$
e. $t \sinh 2t$
f. $\sin t \cos t$
g. $\sin\left(t + \frac{\pi}{4}\right)$
h. $\cos 2t - \cos 3t$
i. $\sin 2t + \cos 4t$

3. Show that $\int_0^\infty e^{-st} e^{t^2} dt = \infty$ for every real number s.

4. Graph the following piecewise continuous functions and evaluate $f(t+)$, $f(t-)$, and $f(t)$ at each point of discontinuity.

a. $f(t) = \begin{cases} -t, & 0 \le t < 2, \\ t-4, & 2 \le t < 3, \\ 1, & t \ge 3. \end{cases}$

b. $f(t) = \begin{cases} t^2 + 2, & 0 \le t < 1, \\ 4, & t = 1, \\ t, & t > 1. \end{cases}$

c. $f(t) = \begin{cases} \sin t, & 0 \le t < \pi/2, \\ 2\sin t, & \pi/2 \le t < \pi, \\ \cos t, & t \ge \pi. \end{cases}$

d. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 2, & t = 1, \\ 2 - t, & 1 \le t < 2, \\ 3, & t = 2, \\ 6, & t > 2. \end{cases}$

5. Find the Laplace transform:

a. $f(t) = \begin{cases} e^{-t}, & 0 \le t < 1, \\ e^{-2t}, & t \ge 1. \end{cases}$

b. $f(t) = \begin{cases} 1, & 0 \le t < 4, \\ t, & t \ge 4. \end{cases}$

c. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$

d. $f(t) = \begin{cases} te^t, & 0 \le t < 1, \\ e^t, & t \ge 1. \end{cases}$

6. Prove that if $f(t) \leftrightarrow F(s)$ then $t^k f(t) \leftrightarrow (-1)^k F^{(k)}(s)$. HINT: **Assume that it's permissible to differentiate the integral $\int_0^\infty e^{-st} f(t)dt$ with respect to s under the integral sign.**

7. Use the known Laplace transforms

$$\mathcal{L}(e^{\lambda t}\sin\omega t) = \frac{\omega}{(s-\lambda)^2+\omega^2} \quad \text{and} \quad \mathcal{L}(e^{\lambda t}\cos\omega t) = \frac{s-\lambda}{(s-\lambda)^2+\omega^2}$$

and the result of **Exercise 8.1.6** to find $\mathcal{L}(te^{\lambda t}\cos\omega t)$ and $\mathcal{L}(te^{\lambda t}\sin\omega t)$.

8. Use the known Laplace transform $\mathcal{L}(1) = 1/s$ and the result of **Exercise 8.1.6** to show that

$$\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}, \quad n = \text{ integer.}$$

9. Exponential order:

a. Show that if $\lim_{t\to\infty} e^{-s_0 t} f(t)$ exists and is finite then f is of exponential order s_0.
b. Show that if f is of exponential order s_0 then $\lim_{t\to\infty} e^{-st} f(t) = 0$ for all $s > s_0$.
c. Show that if f is of exponential order s_0 and $g(t) = f(t+\tau)$ where $\tau > 0$, then g is also of exponential order s_0.

10. Recall the next theorem from calculus.

Theorem 8.1E.1

Let g be integrable on $[0,T]$ for every $T > 0$. Suppose there's a function w defined on some interval $[\tau,\infty)$ (with $\tau \geq 0$) such that $|g(t)| \leq w(t)$ for $t \geq \tau$ and $\int_\tau^\infty w(t)\,dt$ converges. Then $\int_0^\infty g(t)\,dt$ converges.

Use Theorem 8.1E.1 to show that if f is piecewise continuous on $[0,\infty)$ and of exponential order s_0, then f has a Laplace transform $F(s)$ defined for $s > s_0$.

11. Prove: If f is piecewise continuous and of exponential order then $\lim_{s\to\infty} F(s) = 0$.

12. Prove: If f is continuous on $[0,\infty)$ and of exponential order $s_0 > 0$, then

$$\mathcal{L}\left(\int_0^t f(\tau)\,d\tau\right) = \frac{1}{s}\mathcal{L}(f), \quad s > s_0.$$

HINT: **Use integration by parts to evaluate the transform on the left.**

13. Suppose f is piecewise continuous and of exponential order, and that $\lim_{t\to 0+} f(t)/t$ exists. Show that

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty F(r)\,dr.$$

HINT: **Use the results of Exercises 8.1.6 and 8.1.11.**

14. Suppose f is piecewise continuous on $[0,\infty)$.

a. Prove: If the integral $g(t) = \int_0^t e^{-s_0\tau} f(\tau)\,d\tau$ satisfies the inequality $|g(t)| \leq M$ $(t \geq 0)$, then f has a Laplace transform $F(s)$ defined for $s > s_0$. HINT: **Use integration by parts to show that**

$$\int_0^T e^{-st} f(t)dt = e^{-(s-s_0)T} g(T) + (s-s_0)\int_0^T e^{-(s-s_0)t} g(t)dt$$

b. Show that if $\mathcal{L}(f)$ exists for $s = s_0$ then it exists for $s > s_0$. Show that the function $f(t) = te^{t^2}\cos(e^{t^2})$ has a Laplace transform defined for $s > 0$, even though f isn't of exponential order.
c. Show that the function $f(t) = te^{t^2}\cos(e^{t^2})$ has a Laplace transform defined for $s > 0$, even though f isn't of exponential order.

15. Use the table of Laplace transforms and the result of **Exercise 8.1.13** to find the Laplace transforms of the following functions.

a. $\dfrac{\sin\omega t}{t}$ $(\omega > 0)$
b. $\dfrac{\cos\omega t - 1}{t}$ $(\omega > 0)$

c. $\dfrac{e^{at}-e^{bt}}{t}$

d. $\dfrac{\cosh t-1}{t}$

e. $\dfrac{\sinh^2 t}{t}$

16. The **gamma function** is defined by

$$\Gamma(\alpha)=\int_0^\infty x^{\alpha-1}e^{-x}\,dx,$$

which can be shown to converge if $\alpha>0$.

a. Use integration by parts to show that

$$\Gamma(\alpha+1)=\alpha\Gamma(\alpha),\quad \alpha>0.$$

b. Show that $\Gamma(n+1)=n!$ if $n=1, 2, 3,\ldots$.

c. From (b) and the table of Laplace transforms,

$$\mathcal{L}(t^\alpha)=\frac{\Gamma(\alpha+1)}{s^{\alpha+1}},\quad s>0,$$

if α is a nonnegative integer. Show that this formula is valid for any $\alpha>-1$. HINT: **Change the variable of integration in the integral for $\Gamma(\alpha+1)$.**

17. Suppose f is continuous on $[0,T]$ and $f(t+T)=f(t)$ for all $t\ge 0$. (We say in this case that f is **periodic with period T**.)

a. Conclude from Theorem 8.1.6 that the Laplace transform of f is defined for $s>0$.

b. Show that

$$F(s)=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt,\quad s>0.$$

HINT: **Write**

$$F(s)=\sum_{n=0}^{\infty}\int_{nT}^{(n+1)T} e^{-st}f(t)dt$$

Then show that

$$\int_{nT}^{(n+1)T} e^{-st}f(t)dt=e^{-nsT}\int_0^T e^{-st}f(t)dt$$

and recall the formula for the sum of a geometric series.

18. Use the formula given in **Exercise 8.1.17b** to find the Laplace transforms of the given periodic functions:

a. $f(t)=\begin{cases} t, & 0\le t<1,\\ 2-t, & 1\le t<2,\end{cases}\qquad f(t+2)=f(t),\quad t\ge 0$

b. $f(t)=\begin{cases} 1, & 0\le t<\frac{1}{2},\\ -1, & \frac{1}{2}\le t<1,\end{cases}\qquad f(t+1)=f(t),\quad t\ge 0$

c. $f(t)=|\sin t|$

d. $f(t)=\begin{cases} \sin t, & 0\le t<\pi,\\ 0, & \pi\le t<2\pi,\end{cases}\qquad f(t+2\pi)=f(t)$

This page titled 13.1.1: Introduction to the Laplace Transform (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **8.1E: Introduction to the Laplace Transform (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

13.2: The Inverse Laplace Transform

Definition of the Inverse Laplace Transform

In Section 8.1 we defined the Laplace transform of f by

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\, dt.$$

We'll also say that f is an **inverse Laplace Transform** of F, and write

$$f = \mathcal{L}^{-1}(F).$$

To solve differential equations with the Laplace transform, we must be able to obtain f from its transform F. There's a formula for doing this, but we can't use it because it requires the theory of functions of a complex variable. Fortunately, we can use the table of Laplace transforms to find inverse transforms that we'll need.

Example 8.2.1

Use the table of Laplace transforms to find

a. $\mathcal{L}^{-1}\left(\dfrac{1}{s^2-1}\right)$

b. $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+9}\right)$.

Solution a

Setting $b = 1$ in the transform pair

$$\sinh bt \leftrightarrow \frac{b}{s^2 - b^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2-1}\right) = \sinh t.$$

Solution b

Setting $\omega = 3$ in the transform pair

$$\cos \omega t \leftrightarrow \frac{s}{s^2 + \omega^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t.$$

The next theorem enables us to find inverse transforms of linear combinations of transforms in the table. We omit the proof.

Theorem 8.2.1 : Linearity Property

If $F_1, F_2, \ldots, F_n$ are Laplace transforms and $c_1, c_2, \ldots, c_n$ are constants, then

$$\mathcal{L}^{-1}(c_1F_1 + c_2F_2 + \cdots + c_nF_n) = c_1\mathcal{L}^{-1}(F_1) + c_2\mathcal{L}^{-1}(F_2) + \cdots + c_n\mathcal{L}^{-1}F_n.$$

Example 8.2.2

Find

$$\mathcal{L}^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right).$$

Solution

From the table of Laplace transforms in Section 8.8,,

$$e^{at}\leftrightarrow\frac{1}{s-a}\quad\text{and}\quad\sin\omega t\leftrightarrow\frac{\omega}{s^2+\omega^2}.$$

Theorem 8.2.1 with $a=-5$ and $\omega=\sqrt{3}$ yields

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{8}{s+5}+\frac{7}{s^2+3}\right)&=8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)+7\mathcal{L}^{-1}\left(\frac{1}{s^2+3}\right)\\&=8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right)+\frac{7}{\sqrt{3}}\mathcal{L}^{-1}\left(\frac{\sqrt{3}}{s^2+3}\right)\\&=8e^{-5t}+\frac{7}{\sqrt{3}}\sin\sqrt{3}t.\end{aligned}$$

Example 8.2.3

Find

$$\mathcal{L}^{-1}\left(\frac{3s+8}{s^2+2s+5}\right).$$

Solution

Completing the square in the denominator yields

$$\frac{3s+8}{s^2+2s+5}=\frac{3s+8}{(s+1)^2+4}.$$

Because of the form of the denominator, we consider the transform pairs

$$e^{-t}\cos 2t\leftrightarrow\frac{s+1}{(s+1)^2+4}\quad\text{and}\quad e^{-t}\sin 2t\leftrightarrow\frac{2}{(s+1)^2+4},$$

and write

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{3s+8}{(s+1)^2+4}\right)&=\mathcal{L}^{-1}\left(\frac{3s+3}{(s+1)^2+4}\right)+\mathcal{L}^{-1}\left(\frac{5}{(s+1)^2+4}\right)\\&=3\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right)+\frac{5}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right)\\&=e^{-t}(3\cos 2t+\frac{5}{2}\sin 2t).\end{aligned}$$

Note

We'll often write inverse Laplace transforms of specific functions without explicitly stating how they are obtained. In such cases you should refer to the table of Laplace transforms in Section 8.8.

Inverse Laplace Transforms of Rational Functions

Using the Laplace transform to solve differential equations often requires finding the inverse transform of a rational function

$$F(s)=\frac{P(s)}{Q(s)},$$

where P and Q are polynomials in s with no common factors. Since it can be shown that $\lim_{s\to\infty}F(s)=0$ if F is a Laplace transform, we need only consider the case where $\text{degree}(P)<\text{degree}(Q)$. To obtain $\mathcal{L}^{-1}(F)$, we find the partial fraction expansion of

F, obtain inverse transforms of the individual terms in the expansion from the table of Laplace transforms, and use the linearity property of the inverse transform. The next two examples illustrate this.

Example 8.2.4

Find the inverse Laplace transform of

$$F(s) = \frac{3s+2}{s^2-3s+2}. \tag{13.2.1}$$

Solution

(Method 1)

Factoring the denominator in Equation 13.2.1 yields

$$F(s) = \frac{3s+2}{(s-1)(s-2)}. \tag{13.2.2}$$

The form for the partial fraction expansion is

$$\frac{3s+2}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}. \tag{13.2.3}$$

Multiplying this by $(s-1)(s-2)$ yields

$$3s+2 = (s-2)A + (s-1)B.$$

Setting $s=2$ yields $B=8$ and setting $s=1$ yields $A=-5$. Therefore

$$F(s) = -\frac{5}{s-1} + \frac{8}{s-2}$$

and

$$\mathcal{L}^{-1}(F) = -5\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + 8\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = -5e^t + 8e^{2t}.$$

(Method 2) We don't really have to multiply Equation 13.2.3 by $(s-1)(s-2)$ to compute A and B. We can obtain A by simply ignoring the factor $s-1$ in the denominator of Equation 13.2.2 and setting $s=1$ elsewhere; thus,

$$A = \left.\frac{3s+2}{s-2}\right|_{s=1} = \frac{3\cdot 1+2}{1-2} = -5. \tag{13.2.4}$$

Similarly, we can obtain B by ignoring the factor $s-2$ in the denominator of Equation 13.2.2 and setting $s=2$ elsewhere; thus,

$$B = \left.\frac{3s+2}{s-1}\right|_{s=2} = \frac{3\cdot 2+2}{2-1} = 8. \tag{13.2.5}$$

To justify this, we observe that multiplying Equation 13.2.3 by $s-1$ yields

$$\frac{3s+2}{s-2} = A + (s-1)\frac{B}{s-2},$$

and setting $s=1$ leads to Equation 13.2.4. Similarly, multiplying Equation 13.2.3 by $s-2$ yields

$$\frac{3s+2}{s-1} = (s-2)\frac{A}{s-2} + B$$

and setting $s=2$ leads to Equation 13.2.5. (It isn't necesary to write the last two equations. We wrote them only to justify the shortcut procedure indicated in Equation 13.2.4 and Equation 13.2.5.)

The shortcut employed in the second solution of Example 8.2.4 is **Heaviside's method**. The next theorem states this method formally. For a proof and an extension of this theorem, see **Exercise 8.2.10**.

Theorem 8.2.2

Suppose

$$F(s) = \frac{P(s)}{(s-s_1)(s-s_2)\cdots(s-s_n)}, \tag{13.2.6}$$

where $s_1, s_2, \ldots, s_n$ are distinct and P is a polynomial of degree less than n. Then

$$F(s) = \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \cdots + \frac{A_n}{s-s_n},$$

where A_i can be computed from Equation 13.2.6 by ignoring the factor $s-s_i$ and setting $s = s_i$ elsewhere.

Example 8.2.5

Find the inverse Laplace transform of

$$F(s) = \frac{6 + (s+1)(s^2 - 5s + 11)}{s(s-1)(s-2)(s+1)}. \tag{13.2.7}$$

Solution

The partial fraction expansion of Equation 13.2.7 is of the form

$$F(s) = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{s+1}. \tag{13.2.8}$$

To find A, we ignore the factor s in the denominator of Equation 13.2.7 and set $s = 0$ elsewhere. This yields

$$A = \frac{6 + (1)(11)}{(-1)(-2)(1)} = \frac{17}{2}.$$

Similarly, the other coefficients are given by

$$B = \frac{6 + (2)(7)}{(1)(-1)(2)} = -10,$$

$$C = \frac{6 + 3(5)}{2(1)(3)} = \frac{7}{2},$$

and

$$D = \frac{6}{(-1)(-2)(-3)} = -1.$$

Therefore

$$F(s) = \frac{17}{2}\frac{1}{s} - \frac{10}{s-1} + \frac{7}{2}\frac{1}{s-2} - \frac{1}{s+1}$$

and

$$\begin{aligned}\mathcal{L}^{-1}(F) &= \frac{17}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - 10\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \frac{7}{2}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) \\ &= \frac{17}{2} - 10e^t + \frac{7}{2}e^{2t} - e^{-t}.\end{aligned}$$

Note

We didn't "multiply out" the numerator in Equation 8.2.7 before computing the coefficients in Equation 8.2.8 , since it wouldn't simplify the computations.

Example 8.2.6

Find the inverse Laplace transform of

$$F(s) = \frac{8-(s+2)(4s+10)}{(s+1)(s+2)^2}. \tag{13.2.9}$$

Solution

The form for the partial fraction expansion is

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}. \tag{13.2.10}$$

Because of the repeated factor $(s+2)^2$ in Equation 13.2.9, Heaviside's method doesn't work. Instead, we find a common denominator in Equation 13.2.10. This yields

$$F(s) = \frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{(s+1)(s+2)^2}. \tag{13.2.11}$$

If Equation 13.2.9 and Equation 13.2.11 are to be equivalent, then

$$A(s+2)^2 + B(s+1)(s+2) + C(s+1) = 8-(s+2)(4s+10). \tag{13.2.12}$$

The two sides of this equation are polynomials of degree two. From a theorem of algebra, they will be equal for all s if they are equal for any three distinct values of s. We may determine A, B and C by choosing convenient values of s.

The left side of Equation 13.2.12 suggests that we take $s=-2$ to obtain $C=-8$, and $s=-1$ to obtain $A=2$. We can now choose any third value of s to determine B. Taking $s=0$ yields $4A+2B+C=-12$. Since $A=2$ and $C=-8$ this implies that $B=-6$. Therefore

$$F(s) = \frac{2}{s+1} - \frac{6}{s+2} - \frac{8}{(s+2)^2}$$

and

$$\begin{aligned}\mathcal{L}^{-1}(F) &= 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - 6\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 8\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right)\\ &= 2e^{-t} - 6e^{-2t} - 8te^{-2t}.\end{aligned}$$

Example 8.2.7

Find the inverse Laplace transform of

$$F(s) = \frac{s^2-5s+7}{(s+2)^3}.$$

Solution

The form for the partial fraction expansion is

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}.$$

The easiest way to obtain A, B, and C is to expand the numerator in powers of $s+2$. This yields

$$s^2-5s+7 = [(s+2)-2]^2 - 5[(s+2)-2] + 7 = (s+2)^2 - 9(s+2) + 21.$$

Therefore

$$F(s) = \frac{(s+2)^2 - 9(s+2) + 21}{(s+2)^3}$$
$$= \frac{1}{s+2} - \frac{9}{(s+2)^2} + \frac{21}{(s+2)^3}$$

and

$$\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 9\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) + \frac{21}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+2)^3}\right)$$
$$= e^{-2t}\left(1 - 9t + \frac{21}{2}t^2\right).$$

Example 8.2.8

Find the inverse Laplace transform of

$$F(s) = \frac{1 - s(5+3s)}{s\left[(s+1)^2 + 1\right]}. \tag{13.2.13}$$

Solution

One form for the partial fraction expansion of F is

$$F(s) = \frac{A}{s} + \frac{Bs + C}{(s+1)^2 + 1}. \tag{13.2.14}$$

However, we see from the table of Laplace transforms that the inverse transform of the second fraction on the right of Equation 13.2.14 will be a linear combination of the inverse transforms

$$e^{-t}\cos t \quad \text{and} \quad e^{-t}\sin t$$

of

$$\frac{s+1}{(s+1)^2+1} \quad \text{and} \quad \frac{1}{(s+1)^2+1}$$

respectively. Therefore, instead of Equation 13.2.14 we write

$$F(s) = \frac{A}{s} + \frac{B(s+1) + C}{(s+1)^2 + 1}. \tag{13.2.15}$$

Finding a common denominator yields

$$F(s) = \frac{A\left[(s+1)^2 + 1\right] + B(s+1)s + Cs}{s\left[(s+1)^2 + 1\right]}. \tag{13.2.16}$$

If Equation 13.2.13 and Equation 13.2.16 are to be equivalent, then

$$A\left[(s+1)^2 + 1\right] + B(s+1)s + Cs = 1 - s(5+3s).$$

This is true for all s if it is true for three distinct values of s. Choosing $s = 0$, -1, and 1 yields the system

$$\begin{aligned} 2A &= 1 \\ A - C &= 3 \\ 5A + 2B + C &= -7. \end{aligned}$$

Solving this system yields

$$A = \frac{1}{2}, \quad B = -\frac{7}{2}, \quad C = -\frac{5}{2}.$$

Hence, from Equation 13.2.15,

$$F(s) = \frac{1}{2s} - \frac{7}{2}\frac{s+1}{(s+1)^2+1} - \frac{5}{2}\frac{1}{(s+1)^2+1}.$$

Therefore

$$\begin{aligned}\mathcal{L}^{-1}(F) &= \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{7}{2}\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - \frac{5}{2}\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right)\\ &= \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t.\end{aligned}$$

Example 8.2.9

Find the inverse Laplace transform of

$$F(s) = \frac{8+3s}{(s^2+1)(s^2+4)}. \tag{13.2.17}$$

Solution

The form for the partial fraction expansion is

$$F(s) = \frac{A+Bs}{s^2+1} + \frac{C+Ds}{s^2+4}.$$

The coefficients A, B, C and D can be obtained by finding a common denominator and equating the resulting numerator to the numerator in Equation 13.2.17. However, since there's no first power of s in the denominator of Equation 13.2.17, there's an easier way: the expansion of

$$F_1(s) = \frac{1}{(s^2+1)(s^2+4)}$$

can be obtained quickly by using Heaviside's method to expand

$$\frac{1}{(x+1)(x+4)} = \frac{1}{3}\left(\frac{1}{x+1} - \frac{1}{x+4}\right)$$

and then setting $x = s^2$ to obtain

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right).$$

Multiplying this by $8+3s$ yields

$$F(s) = \frac{8+3s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{8+3s}{s^2+1} - \frac{8+3s}{s^2+4}\right).$$

Therefore

$$\mathcal{L}^{-1}(F) = \frac{8}{3}\sin t + \cos t - \frac{4}{3}\sin 2t - \cos 2t.$$

Using Technology

Some software packages that do symbolic algebra can find partial fraction expansions very easily. We recommend that you use such a package if one is available to you, but only after you've done enough partial fraction expansions on your own to master the technique.

This page titled 13.2: The Inverse Laplace Transform is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **8.2: The Inverse Laplace Transform** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

13.2.1: The Inverse Laplace Transform (Exercises)

Q8.2.1

1. Use the table of Laplace transforms to find the inverse Laplace transform.

a. $\dfrac{3}{(s-7)^4}$

b. $\dfrac{2s-4}{s^2-4s+13}$

c. $\dfrac{1}{s^2+4s+20}$

d. $\dfrac{2}{s^2+9}$

e. $\dfrac{s^2-1}{(s^2+1)^2}$

f. $\dfrac{1}{(s-2)^2-4}$

g. $\dfrac{12s-24}{(s^2-4s+85)^2}$

h. $\dfrac{2}{(s-3)^2-9}$

i. $\dfrac{s^2-4s+3}{(s^2-4s+5)^2}$

2. Use Theorem 8.2.1 and the table of Laplace transforms to find the inverse Laplace transform.

a. $\dfrac{2s+3}{(s-7)^4}$

b. $\dfrac{s^2-1}{(s-2)^6}$

c. $\dfrac{s+5}{s^2+6s+18}$

d. $\dfrac{2s+1}{s^2+9}$

e. $\dfrac{s}{s^2+2s+1}$

f. $\dfrac{s+1}{s^2-9}$

g. $\dfrac{s^3+2s^2-s-3}{(s+1)^4}$

h. $\dfrac{2s+3}{(s-1)^2+4}$

i. $\dfrac{1}{s}-\dfrac{s}{s^2+1}$

j. $\dfrac{3s+4}{s^2-1}$

k. $\dfrac{3}{s-1}+\dfrac{4s+1}{s^2+9}$

l. $\dfrac{3}{(s+2)^2}-\dfrac{2s+6}{s^2+4}$

3. Use Heaviside's method to find the inverse Laplace transform.

a. $\dfrac{3-(s+1)(s-2)}{(s+1)(s+2)(s-2)}$

b. $\dfrac{7+(s+4)(18-3s)}{(s-3)(s-1)(s+4)}$

c. $\dfrac{2+(s-2)(3-2s)}{(s-2)(s+2)(s-3)}$

d. $\dfrac{3-(s-1)(s+1)}{(s+4)(s-2)(s-1)}$

e. $\dfrac{3+(s-2)(10-2s-s^2)}{(s-2)(s+2)(s-1)(s+3)}$

f. $\dfrac{3+(s-3)(2s^2+s-21)}{(s-3)(s-1)(s+4)(s-2)}$

4. Find the inverse Laplace transform.

a. $\dfrac{2+3s}{(s^2+1)(s+2)(s+1)}$

b. $\dfrac{3s^2+2s+1}{(s^2+1)(s^2+2s+2)}$

c. $\dfrac{3s+2}{(s-2)(s^2+2s+5)}$

d. $\dfrac{3s^2+2s+1}{(s-1)^2(s+2)(s+3)}$

e. $\dfrac{2s^2+s+3}{(s-1)^2(s+2)^2}$

f. $\dfrac{3s+2}{(s^2+1)(s-1)^2}$

5. Use the method of Example 8.2.9 to find the inverse Laplace transform.

a. $\dfrac{3s+2}{(s^2+4)(s^2+9)}$

b. $\dfrac{-4s+1}{(s^2+1)(s^2+16)}$

c. $\dfrac{5s+3}{(s^2+1)(s^2+4)}$

d. $\dfrac{-s+1}{(4s^2+1)(s^2+1)}$

e. $\dfrac{17s-34}{(s^2+16)(16s^2+1)}$

f. $\dfrac{2s-1}{(4s^2+1)(9s^2+1)}$

6. Find the inverse Laplace transform.

a. $\dfrac{17s-15}{(s^2-2s+5)(s^2+2s+10)}$

b. $\dfrac{8s+56}{(s^2-6s+13)(s^2+2s+5)}$

c. $\dfrac{s+9}{(s^2+4s+5)(s^2-4s+13)}$

d. $\dfrac{3s-2}{(s^2-4s+5)(s^2-6s+13)}$

e. $\dfrac{3s-1}{(s^2-2s+2)(s^2+2s+5)}$

f. $\dfrac{20s+40}{(4s^2-4s+5)(4s^2+4s+5)}$

7. Find the inverse Laplace transform.

a. $\dfrac{1}{s(s^2+1)}$

b. $\dfrac{1}{(s-1)(s^2-2s+17)}$

c. $\dfrac{3s+2}{(s-2)(s^2+2s+10)}$

d. $\dfrac{34-17s}{(2s-1)(s^2-2s+5)}$

e. $\dfrac{s+2}{(s-3)(s^2+2s+5)}$

f. $\dfrac{2s-2}{(s-2)(s^2+2s+10)}$

8. Find the inverse Laplace transform.

a. $\dfrac{2s+1}{(s^2+1)(s-1)(s-3)}$

b. $\dfrac{s+2}{(s^2+2s+2)(s^2-1)}$

c. $\dfrac{2s-1}{(s^2-2s+2)(s+1)(s-2)}$

d. $\dfrac{s-6}{(s^2-1)(s^2+4)}$

e. $\dfrac{2s-3}{s(s-2)(s^2-2s+5)}$

f. $\dfrac{5s-15}{(s^2-4s+13)(s-2)(s-1)}$

9. Given that $f(t) \leftrightarrow F(s)$, find the inverse Laplace transform of $F(as-b)$, where $a>0$.

10. a. If s_1, s_2, …, s_n are distinct and P is a polynomial of degree less than n, then

$$P(s)(s-s_1)(s-s_2)\cdots(s-s_n) = A_1 s - s_1 + A_2 s - s_2 + \cdots + A_n s - s_n.$$

Multiply through by $s-s_i$ to show that A_i can be obtained by ignoring the factor $s-s_i$ on the left and setting $s=s_i$ elsewhere.

b. Suppose P and Q_1 are polynomials such that $\text{degree}(P) \le \text{degree}(Q_1)$ and $Q_1(s_1) \neq 0$. Show that the coefficient of $1/(s-s_1)$ in the partial fraction expansion of

$$F(s) = P(s)(s-s_1)Q_1(s)$$

is $P(s_1)/Q_1(s_1)$.

c. Explain how the results of (a) and (b) are related.

This page titled 13.2.1: The Inverse Laplace Transform (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench.

- **8.2E: The Inverse Laplace Transform (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0.

13.3: Solution of Initial Value Problems

Laplace Transforms of Derivatives

In the rest of this chapter we'll use the Laplace transform to solve initial value problems for constant coefficient second order equations. To do this, we must know how the Laplace transform of f' is related to the Laplace transform of f. The next theorem answers this question.

Theorem 8.3.1

Suppose f is continuous on $[0,\infty)$ and of exponential order s_0, and f' is piecewise continuous on $[0,\infty)$. Then f and f' have Laplace transforms for $s > s_0$, and

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0). \tag{13.3.1}$$

Proof

We know from Theorem 8.1.6 that $\mathcal{L}(f)$ is defined for $s > s_0$. We first consider the case where f' is continuous on $[0,\infty)$. Integration by parts yields

$$\begin{aligned}\int_0^T e^{-st}f'(t)\,dt &= e^{-st}f(t)\Big|_0^T + s\int_0^T e^{-st}f(t)\,dt\\ &= e^{-sT}f(T) - f(0) + s\int_0^T e^{-st}f(t)\,dt\end{aligned} \tag{13.3.2}$$

for any $T > 0$. Since f is of exponential order s_0, $\lim_{T\to\infty} e^{-sT}f(T) = 0$ and the integral in on the right side of Equation 13.3.2 converges as $T \to \infty$ if $s > s_0$. Therefore

$$\begin{aligned}\int_0^\infty e^{-st}f'(t)\,dt &= -f(0) + s\int_0^\infty e^{-st}f(t)\,dt\\ &= -f(0) + s\mathcal{L}(f),\end{aligned}$$

which proves Equation 13.3.1.

Suppose $T > 0$ and f' is only piecewise continuous on $[0,T]$, with discontinuities at $t_1 < t_2 < \cdots < t_{n-1}$. For convenience, let $t_0 = 0$ and $t_n = T$. Integrating by parts yields

$$\begin{aligned}\int_{t_{i-1}}^{t_i} e^{-st}f'(t)\,dt &= e^{-st}f(t)\Big|_{t_{i-1}}^{t_i} + s\int_{t_{i-1}}^{t_i} e^{-st}f(t)\,dt\\ &= e^{-st_i}f(t_i) - e^{-st_{i-1}}f(t_{i-1}) + s\int_{t_{i-1}}^{t_i} e^{-st}f(t)\,dt.\end{aligned}$$

Summing both sides of this equation from $i = 1$ to n and noting that

$$\begin{aligned}&(e^{-st_1}f(t_1) - e^{-st_0}f(t_0)) + (e^{-st_2}f(t_2) - e^{-st_1}f(t_1)) + \cdots + (e^{-st_N}f(t_N) - e^{-st_{N-1}}f(t_{N-1}))\\ &= e^{-st_N}f(t_N) - e^{-st_0}f(t_0) = e^{-sT}f(T) - f(0)\end{aligned}$$

yields Equation 13.3.2, so Equation 13.3.1 follows as before.

Example 8.3.1

In Example 8.1.4 we saw that

$$\mathcal{L}(\cos\omega t) = \frac{s}{s^2+\omega^2}.$$

Solution

Applying Equation 13.3.1 with $f(t) = \cos\omega t$ shows that

$$\mathcal{L}(-\omega\sin\omega t) = s\frac{s}{s^2+\omega^2} - 1 = -\frac{\omega^2}{s^2+\omega^2}.$$

Therefore

$$\mathcal{L}(\sin\omega t) = \frac{\omega}{s^2+\omega^2},$$

which agrees with the corresponding result obtained in 8.1.4.

In Section 2.1 we showed that the solution of the initial value problem

$$y' = ay, \quad y(0) = y_0, \tag{13.3.3}$$

is $y = y_0e^{at}$. We'll now obtain this result by using the Laplace transform.

Let $Y(s) = \mathcal{L}(y)$ be the Laplace transform of the unknown solution of Equation 13.3.3. Taking Laplace transforms of both sides of Equation 13.3.3 yields

$$\mathcal{L}(y') = \mathcal{L}(ay),$$

which, by Theorem 8.3.1 , can be rewritten as

$$s\mathcal{L}(y) - y(0) = a\mathcal{L}(y),$$

or

$$sY(s) - y_0 = aY(s).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{y_0}{s-a},$$

so

$$y = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{y_0}{s-a}\right) = y_0\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = y_0e^{at},$$

which agrees with the known result.

We need the next theorem to solve second order differential equations using the Laplace transform.

Theorem 8.3.2

Suppose f and f' are continuous on $[0,\infty)$ and of exponential order s_0, and that f'' is piecewise continuous on $[0,\infty)$. Then f, f', and f'' have Laplace transforms for $s > s_0$,

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0), \tag{13.3.4}$$

and

$$\mathcal{L}(f'') = s^2\mathcal{L}(f) - f'(0) - sf(0). \tag{13.3.5}$$

Proof

Theorem 8.3.1 implies that $\mathcal{L}(f')$ exists and satisfies Equation 13.3.4 for $s > s_0$. To prove that $\mathcal{L}(f'')$ exists and satisfies Equation 13.3.5 for $s > s_0$, we first apply Theorem 8.3.1 to $g = f'$. Since g satisfies the hypotheses of Theorem 8.3.1 , we conclude that $\mathcal{L}(g')$ is defined and satisfies

$$\mathcal{L}(g') = s\mathcal{L}(g) - g(0)$$

for $s > s_0$. However, since $g' = f''$, this can be rewritten as

$$\mathcal{L}(f'') = s\mathcal{L}(f') - f'(0).$$

Substituting Equation 13.3.4 into this yields Equation 13.3.5.

Solving Second Order Equations with the Laplace Transform

We'll now use the Laplace transform to solve initial value problems for second order equations.

Example 8.3.2

Use the Laplace transform to solve the initial value problem

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2, \quad y'(0) = 3. \tag{13.3.6}$$

Solution

Taking Laplace transforms of both sides of the differential equation in Equation 13.3.6 yields

$$\mathcal{L}(y'' - 6y' + 5y) = \mathcal{L}\left(3e^{2t}\right) = \frac{3}{s-2},$$

which we rewrite as

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 5\mathcal{L}(y) = \frac{3}{s-2}. \tag{13.3.7}$$

Now denote $\mathcal{L}(y) = Y(s)$. Theorem 8.3.2 and the initial conditions in Equation 13.3.6 imply that

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 2$$

and

$$\mathcal{L}(y'') = s^2Y(s) - y'(0) - sy(0) = s^2Y(s) - 3 - 2s.$$

Substituting from the last two equations into Equation 13.3.7 yields

$$\left(s^2Y(s) - 3 - 2s\right) - 6\left(sY(s) - 2\right) + 5Y(s) = \frac{3}{s-2}.$$

Therefore

$$(s^2 - 6s + 5)Y(s) = \frac{3}{s-2} + (3 + 2s) + 6(-2), \tag{13.3.8}$$

so

$$(s-5)(s-1)Y(s) = \frac{3 + (s-2)(2s-9)}{s-2},$$

and

$$Y(s) = \frac{3 + (s-2)(2s-9)}{(s-2)(s-5)(s-1)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = -\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-5} + \frac{5}{2}\frac{1}{s-1},$$

and taking the inverse transform of this yields

$$y = -e^{2t} + \frac{1}{2}e^{5t} + \frac{5}{2}e^{t}$$

as the solution of Equation 13.3.6.

It isn't necessary to write all the steps that we used to obtain Equation 13.3.8. To see how to avoid this, let's apply the method of Example 8.3.2 to the general initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \tag{13.3.9}$$

Taking Laplace transforms of both sides of the differential equation in Equation 13.3.9 yields

$$a\mathcal{L}(y'') + b\mathcal{L}(y') + c\mathcal{L}(y) = F(s). \tag{13.3.10}$$

Now let $Y(s) = \mathcal{L}(y)$. Theorem 8.3.2 and the initial conditions in Equation 13.3.9 imply that

$$\mathcal{L}(y') = sY(s) - k_0 \quad \text{and} \quad \mathcal{L}(y'') = s^2Y(s) - k_1 - k_0 s.$$

Substituting these into Equation 13.3.10 yields

$$a\left(s^2Y(s) - k_1 - k_0 s\right) + b\left(sY(s) - k_0\right) + cY(s) = F(s). \tag{13.3.11}$$

The coefficient of $Y(s)$ on the left is the characteristic polynomial

$$p(s) = as^2 + bs + c$$

of the complementary equation for Equation 13.3.9. Using this and moving the terms involving k_0 and k_1 to the right side of Equation 13.3.11 yields

$$p(s)Y(s) = F(s) + a(k_1 + k_0 s) + bk_0. \tag{13.3.12}$$

This equation corresponds to Equation 13.3.8 of Example 8.3.2 . Having established the form of this equation in the general case, it is preferable to go directly from the initial value problem to this equation. You may find it easier to remember Equation 13.3.12 rewritten as

$$p(s)Y(s) = F(s) + a\left(y'(0) + sy(0)\right) + by(0). \tag{13.3.13}$$

Example 8.3.3

Use the Laplace transform to solve the initial value problem

$$2y'' + 3y' + y = 8e^{-2t}, \quad y(0) = -4,\ y'(0) = 2. \tag{13.3.14}$$

Solution

The characteristic polynomial is

$$p(s) = 2s^2 + 3s + 1 = (2s+1)(s+1)$$

and

$$F(s) = \mathcal{L}(8e^{-2t}) = \frac{8}{s+2},$$

so Equation 13.3.13 becomes

$$(2s+1)(s+1)Y(s) = \frac{8}{s+2} + 2(2-4s) + 3(-4).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{4\left(1 - (s+2)(s+1)\right)}{(s+1/2)(s+1)(s+2)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = \frac{4}{3}\frac{1}{s+1/2} - \frac{8}{s+1} + \frac{8}{3}\frac{1}{s+2},$$

so the solution of Equation 13.3.14 is

$$y = \mathcal{L}^{-1}(Y(s)) = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$$

(Figure 8.3.1).

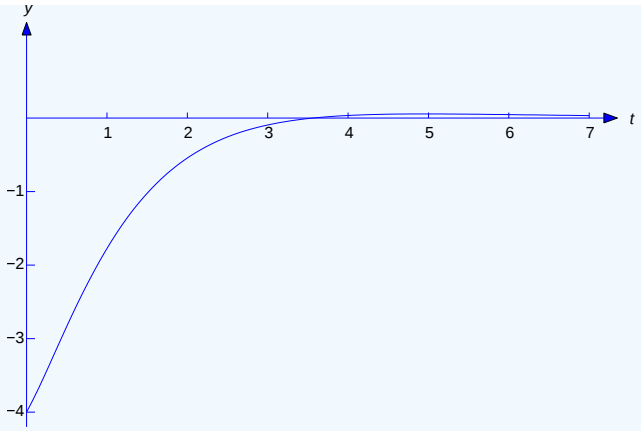


Figure 8.3.1 : $y = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$

Example 8.3.4

Solve the initial value problem

$$y'' + 2y' + 2y = 1, \quad y(0) = -3,\ y'(0) = 1. \tag{13.3.15}$$

Solution

The characteristic polynomial is

$$p(s) = s^2 + 2s + 2 = (s+1)^2 + 1$$

and

$$F(s) = \mathcal{L}(1) = \frac{1}{s},$$

so Equation 13.3.13 becomes

$$\left[(s+1)^2 + 1\right] Y(s) = \frac{1}{s} + 1 \cdot (1 - 3s) + 2(-3).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{1 - s(5 + 3s)}{s\left[(s+1)^2 + 1\right]}.$$

In Example 8.2.8 we found the inverse transform of this function to be

$$y = \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t$$

(Figure 8.3.2), which is therefore the solution of Equation 13.3.15.

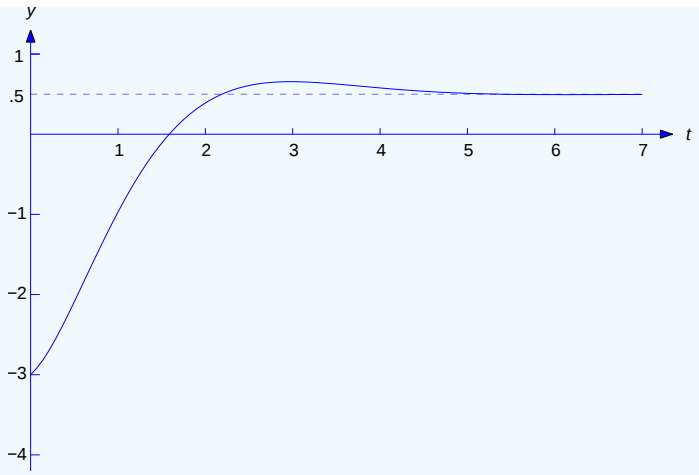


Figure 8.3.2 : $y = \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t$

Note

In our examples we applied Theorems 8.3.1 and 8.3.2 without verifying that the unknown function y satisfies their hypotheses. This is characteristic of the formal manipulative way in which the Laplace transform is used to solve differential equations. Any doubts about the validity of the method for solving a given equation can be resolved by verifying that the resulting function y is the solution of the given problem.

This page titled 13.3: Solution of Initial Value Problems is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **8.3: Solution of Initial Value Problems** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

13.3.1: Solution of Initial Value Problems (Exercises)

Q8.3.1

In **Exercises 8.3.1-8.3.31** use the Laplace transform to solve the initial value problem.

1. $y'' + 3y' + 2y = e^t, \quad y(0) = 1, \quad y'(0) = -6$
2. $y'' - y' - 6y = 2, \quad y(0) = 1, \quad y'(0) = 0$
3. $y'' + y' - 2y = 2e^{3t}, \quad y(0) = -1, \quad y'(0) = 4$
4. $y'' - 4y = 2e^{3t}, \quad y(0) = 1, \quad y'(0) = -1$
5. $y'' + y' - 2y = e^{3t}, \quad y(0) = 1, \quad y'(0) = -1$
6. $y'' + 3y' + 2y = 6e^t, \quad y(0) = 1, \quad y'(0) = -1$
7. $y'' + y = \sin 2t, \quad y(0) = 0, \quad y'(0) = 1$
8. $y'' - 3y' + 2y = 2e^{3t}, \quad y(0) = 1, \quad y'(0) = -1$
9. $y'' - 3y' + 2y = e^{4t}, \quad y(0) = 1, \quad y'(0) = -2$
10. $y'' - 3y' + 2y = e^{3t}, \quad y(0) = -1, \quad y'(0) = -4$
11. $y'' + 3y' + 2y = 2e^t, \quad y(0) = 0, \quad y'(0) = -1$
12. $y'' + y' - 2y = -4, \quad y(0) = 2, \quad y'(0) = 3$
13. $y'' + 4y = 4, \quad y(0) = 0, \quad y'(0) = 1$
14. $y'' - y' - 6y = 2, \quad y(0) = 1, \quad y'(0) = 0$
15. $y'' + 3y' + 2y = e^t, \quad y(0) = 0, \quad y'(0) = 1$
16. $y'' - y = 1, \quad y(0) = 1, \quad y'(0) = 0$
17. $y'' + 4y = 3\sin t, \quad y(0) = 1, \quad y'(0) = -1$
18. $y'' + y' = 2e^{3t}, \quad y(0) = -1, \quad y'(0) = 4$
19. $y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 0$
20. $y'' + y = t, \quad y(0) = 0, \quad y'(0) = 2$
21. $y'' + y = t - 3\sin 2t, \quad y(0) = 1, \quad y'(0) = -3$
22. $y'' + 5y' + 6y = 2e^{-t}, \quad y(0) = 1, \quad y'(0) = 3$
23. $y'' + 2y' + y = 6\sin t - 4\cos t, \quad y(0) = -1, \; y'(0) = 1$
24. $y'' - 2y' - 3y = 10\cos t, \quad y(0) = 2, \quad y'(0) = 7$
25. $y'' + y = 4\sin t + 6\cos t, \quad y(0) = -6, \; y'(0) = 2$
26. $y'' + 4y = 8\sin 2t + 9\cos t, \quad y(0) = 1, \quad y'(0) = 0$
27. $y'' - 5y' + 6y = 10e^t \cos t, \quad y(0) = 2, \quad y'(0) = 1$
28. $y'' + 2y' + 2y = 2t, \quad y(0) = 2, \quad y'(0) = -7$
29. $y'' - 2y' + 2y = 5\sin t + 10\cos t, \quad y(0) = 1, \; y'(0) = 2$
30. $y'' + 4y' + 13y = 10e^{-t} - 36e^t, \quad y(0) = 0, \; y'(0) = -16$
31. $y'' + 4y' + 5y = e^{-t}(\cos t + 3\sin t), \quad y(0) = 0, \quad y'(0) = 4$

Q8.3.2

32. $2y'' - 3y' - 2y = 4e^t, \quad y(0) = 1,\ y'(0) = -2$

33. $6y'' - y' - y = 3e^{2t}, \quad y(0) = 0,\ y'(0) = 0$

34. $2y'' + 2y' + y = 2t, \quad y(0) = 1,\ y'(0) = -1$

35. $4y'' - 4y' + 5y = 4\sin t - 4\cos t, \quad y(0) = 0,\ y'(0) = 11/17$

36. $4y'' + 4y' + y = 3\sin t + \cos t, \quad y(0) = 2,\ y'(0) = -1$

37. $9y'' + 6y' + y = 3e^{3t}, \quad y(0) = 0,\ y'(0) = -3$

38. Suppose a, b, and c are constants and $a \neq 0$. Let

$$y_1 = \mathcal{L}^{-1}\left(\frac{as+b}{as^2+bs+c}\right) \quad \text{and} \quad y_2 = \mathcal{L}^{-1}\left(\frac{a}{as^2+bs+c}\right).$$

Show that

$$y_1(0) = 1, \quad y_1'(0) = 0 \quad \text{and} \quad y_2(0) = 0, \quad y_2'(0) = 1.$$

HINT: **Use the Laplace transform to solve the initial value problems**

$$ay'' + by' + cy = 0, \quad y(0) = 1, \quad y'(0) = 0$$
$$ay'' + by' + cy = 0, \quad y(0) = 0, \quad y'(0) = 1$$

This page titled 13.3.1: Solution of Initial Value Problems (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench.

- **8.3E: Solution of Initial Value Problems (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0.

13.4: The Unit Step Function

In the next section we'll consider initial value problems

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

where a, b, and c are constants and f is piecewise continuous. In this section we'll develop procedures for using the table of Laplace transforms to find Laplace transforms of piecewise continuous functions, and to find the piecewise continuous inverses of Laplace transforms.

Example 8.4.1

Use the table of Laplace transforms to find the Laplace transform of

$$f(t) = \begin{cases} 2t+1, & 0 \le t < 2, \\ 3t, & t \ge 2 \end{cases} \tag{13.4.1}$$

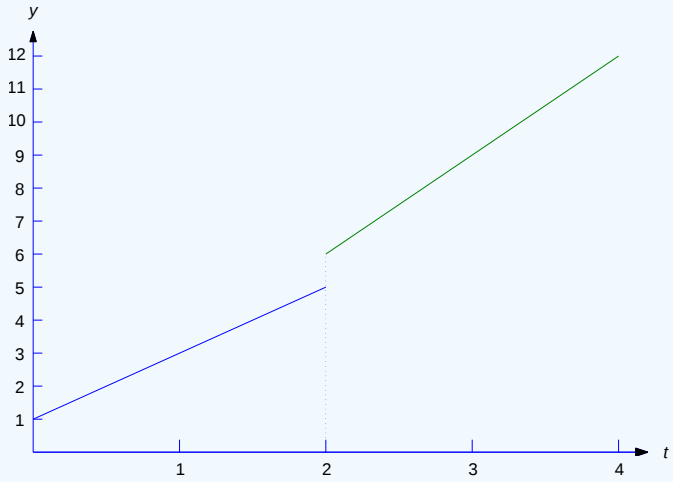


Figure 8.4.1 : The piecewise continuous function Equation 13.4.1.

Solution

Since the formula for f changes at $t = 2$, we write

$$\begin{aligned} \mathcal{L}(f) &= \int_0^\infty e^{-st} f(t)\, dt \\ &= \int_0^2 e^{-st}(2t+1)\, dt + \int_2^\infty e^{-st}(3t)\, dt. \end{aligned} \tag{13.4.2}$$

To relate the first term to a Laplace transform, we add and subtract

$$\int_2^\infty e^{-st}(2t+1)\, dt$$

in Equation 13.4.2 to obtain

$$\begin{aligned} \mathcal{L}(f) &= \int_0^\infty e^{-st}(2t+1)\, dt + \int_2^\infty e^{-st}(3t-2t-1)\, dt \\ &= \int_0^\infty e^{-st}(2t+1)\, dt + \int_2^\infty e^{-st}(t-1)\, dt \\ &= \mathcal{L}(2t+1) + \int_2^\infty e^{-st}(t-1)\, dt. \end{aligned} \tag{13.4.3}$$

To relate the last integral to a Laplace transform, we make the change of variable $x = t-2$ and rewrite the integral as

$$\int_2^\infty e^{-st}(t-1)\,dt = \int_0^\infty e^{-s(x+2)}(x+1)\,dx$$
$$= e^{-2s}\int_0^\infty e^{-sx}(x+1)\,dx.$$

Since the symbol used for the variable of integration has no effect on the value of a definite integral, we can now replace x by the more standard t and write

$$\int_2^\infty e^{-st}(t-1)\,dt = e^{-2s}\int_0^\infty e^{-st}(t+1)\,dt = e^{-2s}\mathcal{L}(t+1).$$

This and Equation 13.4.3 imply that

$$\mathcal{L}(f) = \mathcal{L}(2t+1) + e^{-2s}\mathcal{L}(t+1).$$

Now we can use the table of Laplace transforms to find that

$$\mathcal{L}(f) = \frac{2}{s^2} + \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right).$$

Laplace Transforms of Piecewise Continuous Functions

We'll now develop the method of Example 8.4.1 into a systematic way to find the Laplace transform of a piecewise continuous function. It is convenient to introduce the **unit step function**, defined as

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0. \end{cases} \tag{13.4.4}$$

Thus, $u(t)$ "steps" from the constant value 0 to the constant value 1 at $t = 0$. If we replace t by $t - \tau$ in Equation 13.4.4, then

$$u(t-\tau) = \begin{cases} 0, & t < \tau, \\ 1, & t \geq \tau \end{cases};$$

that is, the step now occurs at $t = \tau$ (Figure 8.4.2).

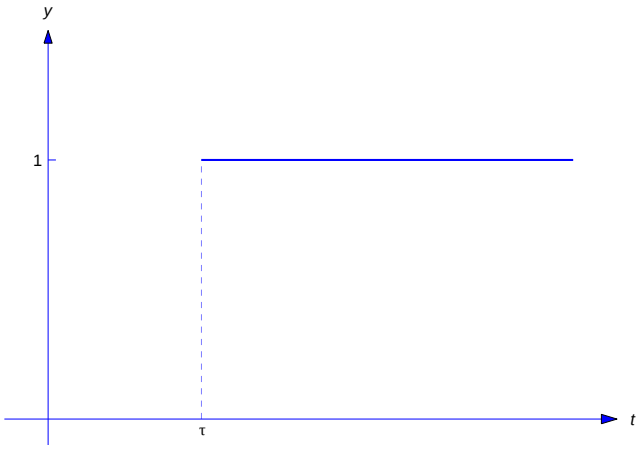


Figure 8.4.2 : $y = u(t-\tau)$

The step function enables us to represent piecewise continuous functions conveniently. For example, consider the function

$$f(t) = \begin{cases} f_0(t), & 0 \leq t < t_1, \\ f_1(t), & t \geq t_1, \end{cases} \tag{13.4.5}$$

where we assume that f_0 and f_1 are defined on $[0, \infty)$, even though they equal f only on the indicated intervals. This assumption enables us to rewrite Equation 13.4.5 as

$$f(t) = f_0(t) + u(t-t_1)\,(f_1(t) - f_0(t))\,. \tag{13.4.6}$$

To verify this, note that if $t < t_1$ then $u(t-t_1) = 0$ and Equation 13.4.6 becomes

$$f(t) = f_0(t) + (0)\,(f_1(t) - f_0(t)) = f_0(t).$$

If $t \geq t_1$ then $u(t-t_1) = 1$ and Equation 13.4.6 becomes

$$f(t) = f_0(t) + (1)\,(f_1(t) - f_0(t)) = f_1(t).$$

We need the next theorem to show how Equation 13.4.6 can be used to find $\mathcal{L}(f)$.

Theorem 8.4.1

Let g be defined on $[0,\infty)$. Suppose $\tau \geq 0$ and $\mathcal{L}\,(g(t+\tau))$ exists for $s > s_0$. Then $\mathcal{L}\,(u(t-\tau)g(t))$ exists for $s > s_0$, and

$$\mathcal{L}(u(t-\tau)g(t)) = e^{-s\tau}\mathcal{L}\,(g(t+\tau))\,.$$

Proof

By definition,

$$\mathcal{L}\,(u(t-\tau)g(t)) = \int_0^\infty e^{-st}u(t-\tau)g(t)\,dt.$$

From this and the definition of $u(t-\tau)$,

$$\mathcal{L}\,(u(t-\tau)g(t)) = \int_0^\tau e^{-st}(0)\,dt + \int_\tau^\infty e^{-st}g(t)\,dt.$$

The first integral on the right equals zero. Introducing the new variable of integration $x = t-\tau$ in the second integral yields

$$\mathcal{L}\,(u(t-\tau)g(t)) = \int_0^\infty e^{-s(x+\tau)}g(x+\tau)\,dx = e^{-s\tau}\int_0^\infty e^{-sx}g(x+\tau)\,dx.$$

Changing the name of the variable of integration in the last integral from x to t yields

$$\mathcal{L}\,(u(t-\tau)g(t)) = e^{-s\tau}\int_0^\infty e^{-st}g(t+\tau)\,dt = e^{-s\tau}\mathcal{L}(g(t+\tau)).$$

Example 8.4.2

Find

$$\mathcal{L}\left(u(t-1)(t^2+1)\right).$$

Solution

Here $\tau = 1$ and $g(t) = t^2+1$, so

$$g(t+1) = (t+1)^2 + 1 = t^2 + 2t + 2.$$

Since

$$\mathcal{L}\,(g(t+1)) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s},$$

Theorem 8.4.1 implies that

$$\mathcal{L}\left(u(t-1)(t^2+1)\right) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s}\right).$$

Example 8.4.3

Use Theorem 8.4.1 to find the Laplace transform of the function

$$f(t) = \begin{cases} 2t+1, & 0 \leq t < 2, \\ 3t, & t \geq 2, \end{cases}$$

from Example 8.4.1 .

Solution

We first write f in the form Equation 13.4.6 as

$$f(t) = 2t + 1 + u(t-2)(t-1).$$

Therefore

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}(2t+1) + \mathcal{L}\left(u(t-2)(t-1)\right) \\ &= \mathcal{L}(2t+1) + e^{-2s}\mathcal{L}(t+1) \quad \text{(from Theorem 13.4.1)} \\ &= \frac{2}{s^2} + \frac{1}{s} + e^{-2s}\left(\frac{1}{s^2} + \frac{1}{s}\right),\end{aligned}$$

which is the result obtained in Example 8.4.1 .

Formula Equation 13.4.6 can be extended to more general piecewise continuous functions. For example, we can write

$$f(t) = \begin{cases} f_0(t), & 0 \le t < t_1, \\ f_1(t), & t_1 \le t < t_2, \\ f_2(t), & t \ge t_2, \end{cases}$$

as

$$f(t) = f_0(t) + u(t-t_1)\left(f_1(t) - f_0(t)\right) + u(t-t_2)\left(f_2(t) - f_1(t)\right)$$

if f_0, f_1, and f_2 are all defined on $[0, \infty)$.

Example 8.4.4

Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2, \\ -2t+1, & 2 \le t < 3, \\ 3t, & 3 \le t < 5, \\ t-1, & t \ge 5 \end{cases} \tag{13.4.7}$$

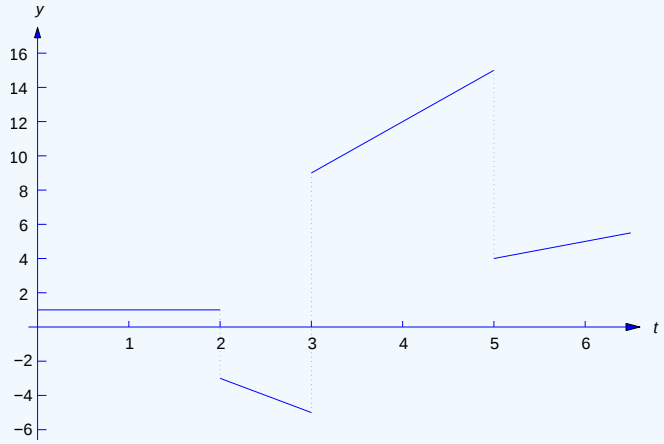


Figure 8.4.3 : The piecewise continuous function Equation 13.4.7

Solution

In terms of step functions,

$$\begin{aligned}f(t) &= 1 + u(t-2)(-2t+1-1) + u(t-3)(3t+2t-1) \\ &= +u(t-5)(t-1-3t),\end{aligned}$$

or

$$f(t) = 1 - 2u(t-2)t + u(t-3)(5t-1) - u(t-5)(2t+1).$$

Now Theorem 8.4.1 implies that

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}(1) - 2e^{-2s}\mathcal{L}(t+2) + e^{-3s}\mathcal{L}\left(5(t+3)-1\right) - e^{-5s}\mathcal{L}\left(2(t+5)+1\right)\\ &= \mathcal{L}(1) - 2e^{-2s}\mathcal{L}(t+2) + e^{-3s}\mathcal{L}(5t+14) - e^{-5s}\mathcal{L}(2t+11)\\ &= \frac{1}{s} - 2e^{-2s}\left(\frac{1}{s^2}+\frac{2}{s}\right) + e^{-3s}\left(\frac{5}{s^2}+\frac{14}{s}\right) - e^{-5s}\left(\frac{2}{s^2}+\frac{11}{s}\right).\end{aligned}$$

The trigonometric identities

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \tag{13.4.8}$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \tag{13.4.9}$$

are useful in problems that involve shifting the arguments of trigonometric functions. We'll use these identities in the next example.

Example 8.4.5

Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \le t < \frac{\pi}{2} \\ \cos t - 3\sin t, & \frac{\pi}{2} \le t < \pi \\ 3\cos t, & t \ge \pi \end{cases} \tag{13.4.10}$$

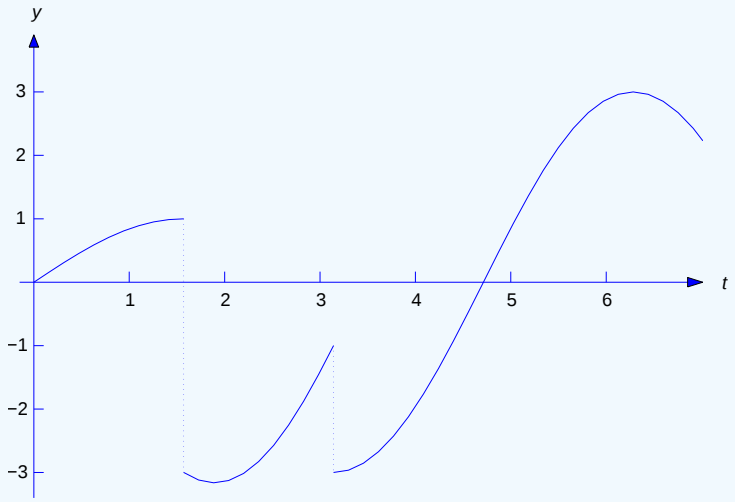


Figure 8.4.4 : The piecewise continuous function Equation 13.4.10.

Solution

In terms of step functions,

$$f(t) = \sin t + u(t-\pi/2)(\cos t - 4\sin t) + u(t-\pi)(2\cos t + 3\sin t).$$

Now Theorem 8.4.1 implies that

$$\begin{aligned}\mathcal{L}(f) \quad = \quad & \mathcal{L}(\sin t) + e^{-\frac{\pi}{2}s}\mathcal{L}\left(\cos\left(t+\tfrac{\pi}{2}\right) - 4\sin\left(t+\tfrac{\pi}{2}\right)\right)\\ & + e^{-\pi s}\mathcal{L}\left(2\cos(t+\pi) + 3\sin(t+\pi)\right).\end{aligned} \tag{13.4.11}$$

Since

$$\cos\left(t+\frac{\pi}{2}\right) - 4\sin\left(t+\frac{\pi}{2}\right) = -\sin t - 4\cos t$$

and

$$2\cos(t+\pi)+3\sin(t+\pi)=-2\cos t-3\sin t,$$

we see from Equation 13.4.11 that

$$\begin{aligned}\mathcal{L}(f)&=\mathcal{L}(\sin t)-e^{-\pi s/2}\mathcal{L}(\sin t+4\cos t)-e^{-\pi s}\mathcal{L}(2\cos t+3\sin t)\\&=\frac{1}{s^2+1}-e^{-\frac{\pi}{2}s}\left(\frac{1+4s}{s^2+1}\right)-e^{-\pi s}\left(\frac{3+2s}{s^2+1}\right).\end{aligned}$$

The Second Shifting Theorem

Replacing $g(t)$ by $g(t-\tau)$ in Theorem 8.4.1 yields the next theorem.

Theorem 8.4.2 : Second Shifting Theorem

If $\tau\ge 0$ and $\mathcal{L}(g)$ exists for $s>s_0$ then $\mathcal{L}\left(u(t-\tau)g(t-\tau)\right)$ exists for $s>s_0$ and

$$\mathcal{L}(u(t-\tau)g(t-\tau))=e^{-s\tau}\mathcal{L}(g(t)),$$

or, equivalently,

$$\text{if } g(t)\leftrightarrow G(s),\ \text{then } u(t-\tau)g(t-\tau)\leftrightarrow e^{-s\tau}G(s). \tag{13.4.12}$$

Note

Recall that the First Shifting Theorem (Theorem 8.1.3) states that multiplying a function by e^{at} corresponds to shifting the argument of its transform by a units. Theorem 8.4.2 states that multiplying a Laplace transform by the exponential $e^{-\tau s}$ corresponds to shifting the argument of the inverse transform by τ units.

Example 8.4.6

Use Equation 13.4.12 to find

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right).$$

Solution

To apply Equation 13.4.12 we let $\tau=2$ and $G(s)=1/s^2$. Then $g(t)=t$ and Equation 13.4.12 implies that

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right)=u(t-2)(t-2).$$

Example 8.4.7

Find the inverse Laplace transform h of

$$H(s)=\frac{1}{s^2}-e^{-s}\left(\frac{1}{s^2}+\frac{2}{s}\right)+e^{-4s}\left(\frac{4}{s^3}+\frac{1}{s}\right),$$

and find distinct formulas for h on appropriate intervals.

Solution

Let

$$G_0(s)=\frac{1}{s^2},\quad G_1(s)=\frac{1}{s^2}+\frac{2}{s},\quad G_2(s)=\frac{4}{s^3}+\frac{1}{s}.$$

Then

$$g_0(t)=t,\ g_1(t)=t+2,\ g_2(t)=2t^2+1.$$

Hence, Equation 13.4.12 and the linearity of $\mathcal{L}^{-1}$ imply that

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}\left(G_0(s)\right) - \mathcal{L}^{-1}\left(e^{-s}G_1(s)\right) + \mathcal{L}^{-1}\left(e^{-4s}G_2(s)\right) \\ &= t - u(t-1)\left[(t-1)+2\right] + u(t-4)\left[2(t-4)^2+1\right] \\ &= t - u(t-1)(t+1) + u(t-4)(2t^2-16t+33), \end{aligned}$$

which can also be written as

$$h(t) = \begin{cases} t, & 0 \le t < 1, \\ -1, & 1 \le t < 4, \\ 2t^2-16t+32, & t \ge 4. \end{cases}$$

Example 8.4.8

Find the inverse transform of

$$H(s) = \frac{2s}{s^2+4} - e^{-\frac{\pi}{2}s}\frac{3s+1}{s^2+9} + e^{-\pi s}\frac{s+1}{s^2+6s+10}.$$

Solution

Let

$$G_0(s) = \frac{2s}{s^2+4}, \quad G_1(s) = -\frac{(3s+1)}{s^2+9},$$

and

$$G_2(s) = \frac{s+1}{s^2+6s+10} = \frac{(s+3)-2}{(s+3)^2+1}.$$

Then

$$g_0(t) = 2\cos 2t, \quad g_1(t) = -3\cos 3t - \frac{1}{3}\sin 3t,$$

and

$$g_2(t) = e^{-3t}(\cos t - 2\sin t).$$

Therefore Equation 13.4.12 and the linearity of $\mathcal{L}^{-1}$ imply that

$$\begin{aligned} h(t) &= 2\cos 2t - u(t-\pi/2)\left[3\cos 3(t-\pi/2) + \frac{1}{3}\sin 3\left(t-\frac{\pi}{2}\right)\right] \\ &= +u(t-\pi)e^{-3(t-\pi)}\left[\cos(t-\pi) - 2\sin(t-\pi)\right]. \end{aligned}$$

Using the trigonometric identities Equation 13.4.8 and Equation 13.4.9, we can rewrite this as

$$h(t) = 2\cos 2t + u(t-\pi/2)\left(3\sin 3t - \frac{1}{3}\cos 3t\right) \tag{13.4.13}$$

$$= -u(t-\pi)e^{-3(t-\pi)}(\cos t - 2\sin t) \tag{13.4.14}$$

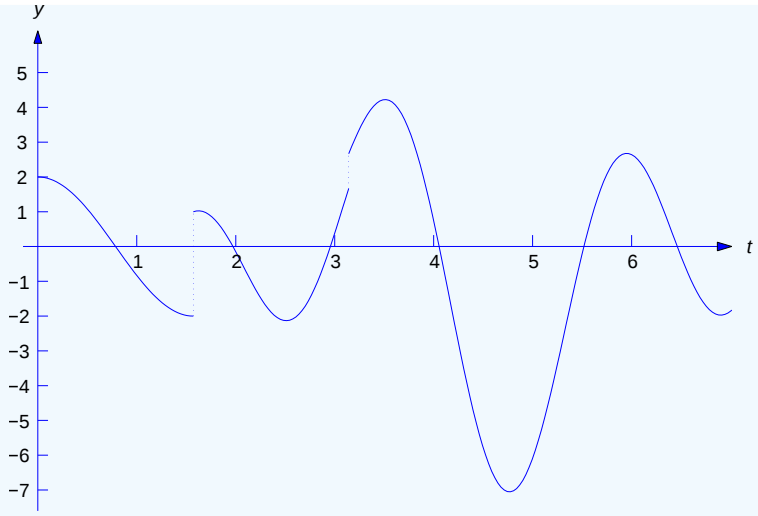


Figure 8.4.5 : The piecewise continuous function Equation **13.4.13**.

This page titled 13.4: The Unit Step Function is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **8.4: The Unit Step Function** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

13.4.1: The Unit Step Function (Exercises)

Q8.4.1

In *Exercises 8.4.1-8.4.6* find the Laplace transform by the method of Example 8.4.1. Then express the given function f in terms of unit step functions as in Equation 8.4.8, and use Theorem 8.4.1 to find $\mathcal{L}(f)$. Graph f for **Exercises 8.4.3** and **8.4.4**.

1. $f(t) = \begin{cases} 1, & 0 \le t < 4, \\ t, & t \ge 4. \end{cases}$
2. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$
3. $f(t) = \begin{cases} 2t-1, & 0 \le t < 2, \\ t, & t \ge 2. \end{cases}$
4. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ t+2, & t \ge 1. \end{cases}$
5. $f(t) = \begin{cases} t-1, & 0 \le t < 2, \\ 4, & t \ge 2. \end{cases}$
6. $f(t) = \begin{cases} t^2, & 0 \le t < 1, \\ 0, & t \ge 1. \end{cases}$

Q8.4.2

In **Exercises 8.4.7-8.4.18** express the given function f in terms of unit step functions and use Theorem 8.4.1 to find $\mathcal{L}(f)$. Graph f for **Exercises 8.4.15-8.4.18**.

7. $f(t) = \begin{cases} 0, & 0 \le t < 2, \\ t^2+3t, & t \ge 2. \end{cases}$
8. $f(t) = \begin{cases} t^2+2, & 0 \le t < 1, \\ t, & t \ge 1. \end{cases}$
9. $f(t) = \begin{cases} te^t, & 0 \le t < 1, \\ e^t, & t \ge 1. \end{cases}$
10. $f(t) = \begin{cases} e^{-t}, & 0 \le t < 1, \\ e^{-2t}, & t \ge 1. \end{cases}$
11. $f(t) = \begin{cases} -t, & 0 \le t < 2, \\ t-4, & 2 \le t < 3, \\ 1, & t \ge 3. \end{cases}$
12. $f(t) = \begin{cases} 0, & 0 \le t < 1, \\ t, & 1 \le t < 2, \\ 0, & t \ge 2. \end{cases}$
13. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ t^2, & 1 \le t < 2, \\ 0, & t \ge 2. \end{cases}$
14. $f(t) = \begin{cases} t, & 0 \le t < 1, \\ 2-t, & 1 \le t < 2, \\ 6, & t > 2. \end{cases}$

15. $f(t)=\begin{cases}\sin t, & 0\le t<\frac{\pi}{2}\\ 2\sin t, & \frac{\pi}{2}\le t<\pi\\ \cos t, & t\ge\pi\end{cases}$

16. $f(t)=\begin{cases}2, & 0\le t<1,\\ -2t+2, & 1\le t<3,\\ 3t, & t\ge 3.\end{cases}$

17. $f(t)=\begin{cases}3, & 0\le t<2,\\ 3t+2, & 2\le t<4,\\ 4t, & t\ge 4.\end{cases}$

18. $f(t)=\begin{cases}(t+1)^2, & 0\le t<1,\\ (t+2)^2, & t\ge 1.\end{cases}$

Q8.4.3

In **Exercises 8.4.19-8.4.28** use Theorem 8.4.2 to express the inverse transforms in terms of step functions, and then find distinct formulas the for inverse transforms on the appropriate intervals, as in Example 8.4.7. Graph the inverse transform for **Exercises 8.4.21, 8.4.22,** and **8.4.25**.

19. $H(s)=\dfrac{e^{-2s}}{s-2}$

20. $H(s)=\dfrac{e^{-s}}{s(s+1)}$

21. $H(s)=\dfrac{e^{-s}}{s^3}+\dfrac{e^{-2s}}{s^2}$

22. $H(s)=\left(\dfrac{2}{s}+\dfrac{1}{s^2}\right)+e^{-s}\left(\dfrac{3}{s}-\dfrac{1}{s^2}\right)+e^{-3s}\left(\dfrac{1}{s}+\dfrac{1}{s^2}\right)$

23. $H(s)=\left(\dfrac{5}{s}-\dfrac{1}{s^2}\right)+e^{-3s}\left(\dfrac{6}{s}+\dfrac{7}{s^2}\right)+\dfrac{3e^{-6s}}{s^3}$

24. $H(s)=\dfrac{e^{-\pi s}(1-2s)}{s^2+4s+5}$

25. $H(s)=\left(\dfrac{1}{s}-\dfrac{s}{s^2+1}\right)+e^{-\frac{\pi}{2}s}\left(\dfrac{3s-1}{s^2+1}\right)$

26. $H(s)=e^{-2s}\left[\dfrac{3(s-3)}{(s+1)(s-2)}-\dfrac{s+1}{(s-1)(s-2)}\right]$

27. $H(s)=\dfrac{1}{s}+\dfrac{1}{s^2}+e^{-s}\left(\dfrac{3}{s}+\dfrac{2}{s^2}\right)+e^{-3s}\left(\dfrac{4}{s}+\dfrac{3}{s^2}\right)$

28. $H(s)=\dfrac{1}{s}-\dfrac{2}{s^3}+e^{-2s}\left(\dfrac{3}{s}-\dfrac{1}{s^3}\right)+\dfrac{e^{-4s}}{s^2}$

Q8.4.4

29. Find $\mathcal{L}\left(u(t-\tau)\right)$.

30. Let $\{t_m\}_{m=0}^{\infty}$ be a sequence of points such that $t_0=0$, $t_{m+1}>t_m$, and $\lim_{m\to\infty}t_m=\infty$. For each nonnegative integer m, let f_m be continuous on $[t_m,\infty)$, and let f be defined on $[0,\infty)$ by

$$f(t)=f_m(t),\ t_m\le t<t_{m+1}\quad (m=0,1,\ldots).$$

Show that f is piecewise continuous on $[0,\infty)$ and that it has the step function representation

$$f(t)=f_0(t)+\sum_{m=1}^{\infty}u(t-t_m)\left(f_m(t)-f_{m-1}(t)\right),\ 0\le t<\infty.$$

How do we know that the series on the right converges for all t in $[0,\infty)$?

31. In addition to the assumptions of **Exercise 8.4.30**, assume that

$$|f_m(t)| \le Me^{s_0 t},\ t \ge t_m,\ m = 0, 1, \dots, \tag{A}$$

and that the series

$$\sum_{m=0}^{\infty} e^{-\rho t_m} \tag{B}$$

converges for some $\rho > 0$. Using the steps listed below, show that $\mathcal{L}(f)$ is defined for $s > s_0$ and

$$\mathcal{L}(f) = \mathcal{L}(f_0) + \sum_{m=1}^{\infty} e^{-st_m}\mathcal{L}(g_m) \tag{C}$$

for $s > s_0 + \rho$, where

$$g_m(t) = f_m(t + t_m) - f_{m-1}(t + t_m).$$

a. Use (A) and Theorem 8.1.6 to show that

$$\mathcal{L}(f) = \sum_{m=0}^{\infty} \int_{t_m}^{t_{m+1}} e^{-st} f_m(t)\, dt \tag{D}$$

is defined for $s > s_0$.

b. Show that (D) can be rewritten as

$$\mathcal{L}(f) = \sum_{m=0}^{\infty} \left(\int_{t_m}^{\infty} e^{-st} f_m(t)\, dt - \int_{t_{m+1}}^{\infty} e^{-st} f_m(t)\, dt \right). \tag{E}$$

c. Use (A), the assumed convergence of (B), and the comparison test to show that the series

$$\sum_{m=0}^{\infty} \int_{t_m}^{\infty} e^{-st} f_m(t)\, dt \quad \text{and} \quad \sum_{m=0}^{\infty} \int_{t_{m+1}}^{\infty} e^{-st} f_m(t)\, dt$$

both converge (absolutely) if $s > s_0 + \rho$.

d. Show that (E) can be rewritten as

$$\mathcal{L}(f) = \mathcal{L}(f_0) + \sum_{m=1}^{\infty} \int_{t_m}^{\infty} e^{-st} \left(f_m(t) - f_{m-1}(t) \right) dt$$

if $s > s_0 + \rho$.

e. Complete the proof of (C).

32. Suppose $\{t_m\}_{m=0}^{\infty}$ and $\{f_m\}_{m=0}^{\infty}$ satisfy the assumptions of **Exercises 8.4.30** and **8.4.31**, and there's a positive constant K such that $t_m \ge Km$ for m sufficiently large. Show that the series (B) of **Exercise 8.4.31** converges for any $\rho > 0$, and conclude from this that (C) of **Exercise 8.4.31** holds for $s > s_0$.

In **Exercises 8.4.33-8.4.36** find the step function representation of f and use the result of **Exercise 8.4.32** to find $\mathcal{L}(f)$. HINT: **You will need formulas related to the formula for the sum of a geometric series.**

33. $f(t) = m + 1,\ m \le t < m + 1\ (m = 0, 1, 2, \dots)$

34. $f(t) = (-1)^m,\ m \le t < m + 1\ (m = 0, 1, 2, \dots)$

35. $f(t) = (m + 1)^2,\ m \le t < m + 1\ (m = 0, 1, 2, \dots)$

36. $f(t) = (-1)^m m,\ m \le t < m + 1\ (m = 0, 1, 2, \dots)$

This page titled 13.4.1: The Unit Step Function (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **8.4E: The Unit Step Function (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

13.5: Constant Coefficient Equations with Piecewise Continuous Forcing Functions

We'll now consider initial value problems of the form

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1, \tag{13.5.1}$$

where a, b, and c are constants ($a \neq 0$) and f is piecewise continuous on $[0, \infty)$. Problems of this kind occur in situations where the input to a physical system undergoes instantaneous changes, as when a switch is turned on or off or the forces acting on the system change abruptly.

It can be shown (*Exercises 8.5.23* and *8.5.24*) that the differential equation in Equation 13.5.1 has no solutions on an open interval that contains a jump discontinuity of f. Therefore we must define what we mean by a solution of Equation 13.5.1 on $[0, \infty)$ in the case where f has jump discontinuities. The next theorem motivates our definition. We omit the proof.

Theorem 13.5.1

Suppose a, b, and c are constants ($a \neq 0$), and f is piecewise continuous on $[0, \infty)$. with jump discontinuities at $t_1, \ldots, t_n$, where

$$0 < t_1 < \cdots < t_n.$$

Let k_0 and k_1 be arbitrary real numbers. Then there is a unique function y defined on $[0, \infty)$ with these properties:

a. $y(0) = k_0$ and $y'(0) = k_1$.
b. y and y' are continuous on $[0, \infty)$.
c. y'' is defined on every open subinterval of $[0, \infty)$ that does not contain any of the points $t_1, \ldots, t_n$, and

$$ay'' + by' + cy = f(t)$$

on every such subinterval.
d. y'' has limits from the right and left at $t_1, \ldots, t_n$.

We define the function y of Theorem 13.5.1 to be the solution of the initial value problem Equation 13.5.1.

We begin by considering initial value problems of the form

$$ay'' + by' + cy = \begin{cases} f_0(t), & 0 \le t < t_1, \\ f_1(t), & t \ge t_1, \end{cases} \qquad y(0) = k_0, \quad y'(0) = k_1, \tag{13.5.2}$$

where the forcing function has a single jump discontinuity at t_1.

Howto: Solve Constant Coefficient Equations with Piecewise Continuous Forcing Functions

We can solve Equation 13.5.2 by the these steps:

- **Step 1.** Find the solution y_0 of the initial value problem

$$ay'' + by' + cy = f_0(t), \quad y(0) = k_0, \quad y'(0) = k_1.$$

- **Step 2.** Compute $c_0 = y_0(t_1)$ and $c_1 = y_0'(t_1)$.
- **Step 3.** Find the solution y_1 of the initial value problem

$$ay'' + by' + cy = f_1(t), \quad y(t_1) = c_0, \quad y'(t_1) = c_1.$$

- **Step 4.** Obtain the solution y of Equation 13.5.2 as

$$y = \begin{cases} y_0(t), & 0 \le t < t_1 \\ y_1(t), & t \ge t_1. \end{cases}$$

It is shown in *Exercise 8.5.23* that y' exists and is continuous at t_1. The next example illustrates this procedure.

Example 13.5.1

Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2,\ y'(0) = -1, \tag{13.5.3}$$

where

$$f(t) = \begin{cases} 1, & 0 \le t < \frac{\pi}{2}, \\ -1, & t \ge \frac{\pi}{2}. \end{cases}$$

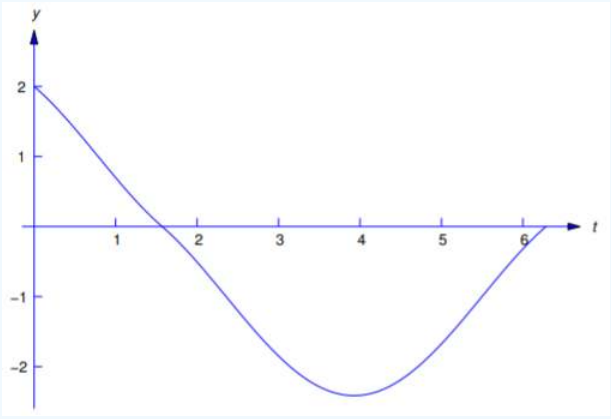


Figure 13.5.1 : Graph of Equation 13.5.4

Solution

The initial value problem in Step 1 is

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = -1.$$

We leave it to you to verify that its solution is

$$y_0 = 1 + \cos t - \sin t.$$

Doing Step 2 yields $y_0(\pi/2) = 0$ and $y_0'(\pi/2) = -1$, so the second initial value problem is

$$y'' + y = -1, \quad y\left(\frac{\pi}{2}\right) = 0,\ y'\left(\frac{\pi}{2}\right) = -1.$$

We leave it to you to verify that the solution of this problem is

$$y_1 = -1 + \cos t + \sin t.$$

Hence, the solution of Equation 13.5.3 is

$$y = \begin{cases} 1 + \cos t - \sin t, & 0 \le t < \frac{\pi}{2}, \\ -1 + \cos t + \sin t, & t \ge \frac{\pi}{2} \end{cases} \tag{13.5.4}$$

If f_0 and f_1 are defined on $[0, \infty)$, we can rewrite Equation 13.5.2 as

$$ay'' + by' + cy = f_0(t) + u(t - t_1)\left(f_1(t) - f_0(t)\right), \quad y(0) = k_0, \quad y'(0) = k_1,$$

and apply the method of Laplace transforms. We'll now solve the problem considered in Example [example:8.5.1} by this method.

Example 13.5.2

Use the Laplace transform to solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2,\ y'(0) = -1, \tag{13.5.5}$$

where

$$f(t) = \begin{cases} 1, & 0 \le t < \frac{\pi}{2}, \\ -1, & t \ge \frac{\pi}{2}. \end{cases}$$

Solution

Here

$$f(t) = 1 - 2u\left(t - \frac{\pi}{2}\right),$$

so Theorem 13.5.1 (with $g(t) = 1$) implies that

$$\mathcal{L}(f) = \frac{1 - 2e^{-\pi s/2}}{s}.$$

Therefore, transforming Equation 13.5.5 yields

$$(s^2 + 1)Y(s) = \frac{1 - 2e^{-\pi s/2}}{s} - 1 + 2s,$$

so

$$Y(s) = (1 - 2e^{-\pi s/2})G(s) + \frac{2s - 1}{s^2 + 1}, \tag{13.5.6}$$

with

$$G(s) = \frac{1}{s(s^2 + 1)}.$$

The form for the partial fraction expansion of G is

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}. \tag{13.5.7}$$

Multiplying through by $s(s^2 + 1)$ yields

$$A(s^2 + 1) + (Bs + C)s = 1,$$

or

$$(A + B)s^2 + Cs + A = 1.$$

Equating coefficients of like powers of s on the two sides of this equation shows that $A = 1$, $B = -A = -1$ and $C = 0$. Hence, from Equation 13.5.7,

$$G(s) = \frac{1}{s} - \frac{s}{s^2 + 1}.$$

Therefore

$$g(t) = 1 - \cos t.$$

From this, Equation 13.5.6, and Theorem 8.4.2,

$$y = 1 - \cos t - 2u\left(t - \frac{\pi}{2}\right)\left(1 - \cos\left(t - \frac{\pi}{2}\right)\right) + 2\cos t - \sin t.$$

Simplifying this (recalling that $\cos(t - \pi/2) = \sin t$) yields

$$y = 1 + \cos t - \sin t - 2u\left(t - \frac{\pi}{2}\right)(1 - \sin t),$$

or

$$y = \begin{cases} 1 + \cos t - \sin t, & 0 \le t < \frac{\pi}{2} \\ -1 + \cos t + \sin t, & t \ge \frac{\pi}{2} \end{cases}$$

which is the result obtained in Example 13.5.1 .

Note

It isn't obvious that using the Laplace transform to solve Equation 13.5.2 as we did in Example 13.5.2 yields a function y with the properties stated in Theorem 13.5.1 ; that is, such that y and y' are continuous on $[0, \infty)$ and y'' has limits from the right and left at t_1. However, this is true if f_0 and f_1 are continuous and of exponential order on $[0, \infty)$. A proof is sketched in *Exercises 8.6.11–8.6.13*.

Example 13.5.3

Solve the initial value problem

$$y'' - y = f(t), \quad y(0) = -1,\ y'(0) = 2, \tag{13.5.8}$$

where

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$$

Solution

Here

$$f(t) = t - u(t-1)(t-1),$$

so

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(t) - \mathcal{L}\left(u(t-1)(t-1)\right) \\ &= \mathcal{L}(t) - e^{-s}\mathcal{L}(t) \qquad \text{(from Theorem 8.4.1)} \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}. \end{aligned}$$

Since transforming Equation 13.5.8 yields

$$(s^2 - 1)Y(s) = \mathcal{L}(f) + 2 - s,$$

we see that

$$Y(s) = (1 - e^{-s})H(s) + \frac{2 - s}{s^2 - 1}, \tag{13.5.9}$$

where

$$H(s) = \frac{1}{s^2(s^2 - 1)} = \frac{1}{s^2 - 1} - \frac{1}{s^2};$$

therefore

$$h(t) = \sinh t - t. \tag{13.5.10}$$

Since

$$\mathcal{L}^{-1}\left(\frac{2 - s}{s^2 - 1}\right) = 2\sinh t - \cosh t,$$

we conclude from Equation 13.5.9, Equation 13.5.10, and Theorem 13.5.1 that

$$y = \sinh t - t - u(t-1)\left(\sinh(t-1) - t + 1\right) + 2\sinh t - \cosh t,$$

or

$$y = 3\sinh t - \cosh t - t - u(t-1)\,(\sinh(t-1) - t + 1) \tag{13.5.11}$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1 = 1$.

Example 13.5.4

Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0,\ y'(0) = 0, \tag{13.5.12}$$

where

$$f(t) = \begin{cases} 0, & 0 \le t < \frac{\pi}{4} \\ \cos 2t & \frac{\pi}{4} \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

Solution

Here

$$f(t) = u(t - \pi/4)\cos 2t - u(t-\pi)\cos 2t,$$

so

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}\left(u(t-\pi/4)\cos 2t\right) - \mathcal{L}\left(u(t-\pi)\cos 2t\right) \\ &= e^{-\pi s/4}\mathcal{L}\left(\cos 2(t+\pi/4)\right) - e^{-\pi s}\mathcal{L}\left(\cos 2(t+\pi)\right) \\ &= -e^{-\pi s/4}\mathcal{L}(\sin 2t) - e^{-\pi s}\mathcal{L}(\cos 2t) \\ &= -\frac{2e^{-\pi s/4}}{s^2+4} - \frac{se^{-\pi s}}{s^2+4}. \end{aligned}$$

Since transforming Equation 13.5.12 yields

$$(s^2+1)Y(s) = \mathcal{L}(f),$$

we see that

$$Y(s) = e^{-\pi s/4}H_1(s) + e^{-\pi s}H_2(s), \tag{13.5.13}$$

where

$$H_1(s) = -\frac{2}{(s^2+1)(s^2+4)} \quad \text{and} \quad H_2(s) = -\frac{s}{(s^2+1)(s^2+4)}. \tag{13.5.14}$$

To simplify the required partial fraction expansions, we first write

$$\frac{1}{(x+1)(x+4)} = \frac{1}{3}\left[\frac{1}{x+1} - \frac{1}{x+4}\right].$$

Setting $x = s^2$ and substituting the result in Equation 13.5.14 yields

$$H_1(s) = -\frac{2}{3}\left[\frac{1}{s^2+1} - \frac{1}{s^2+4}\right] \quad \text{and} \quad H_2(s) = -\frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right].$$

The inverse transforms are

$$h_1(t) = -\frac{2}{3}\sin t + \frac{1}{3}\sin 2t \quad \text{and} \quad h_2(t) = -\frac{1}{3}\cos t + \frac{1}{3}\cos 2t.$$

From Equation 13.5.13 and Theorem 8.4.2,

$$y = u\left(t - \frac{\pi}{4}\right)h_1\left(t - \frac{\pi}{4}\right) + u(t-\pi)h_2(t-\pi). \tag{13.5.15}$$

Since

$$h_1\left(t-\frac{\pi}{4}\right)=-\frac{2}{3}\sin\left(t-\frac{\pi}{4}\right)+\frac{1}{3}\sin 2\left(t-\frac{\pi}{4}\right)$$
$$=-\frac{\sqrt{2}}{3}(\sin t-\cos t)-\frac{1}{3}\cos 2t$$

and

$$h_2(t-\pi)=-\frac{1}{3}\cos(t-\pi)+\frac{1}{3}\cos 2(t-\pi)$$
$$=\frac{1}{3}\cos t+\frac{1}{3}\cos 2t,$$

Equation 13.5.15 can be rewritten as

$$y=-\frac{1}{3}u\left(t-\frac{\pi}{4}\right)\left(\sqrt{2}(\sin t-\cos t)+\cos 2t\right)+\frac{1}{3}u(t-\pi)(\cos t+\cos 2t)$$

or

$$\begin{cases} 0, & 0\le t<\frac{\pi}{4} \\ -\frac{\sqrt{2}}{3}(\sin t-\cos t)-\frac{1}{3}\cos 2t, & \frac{\pi}{4}\le t<\pi, \\ -\frac{\sqrt{2}}{3}\sin t+\frac{1+\sqrt{2}}{3}\cos t, & t\ge\pi \end{cases} \tag{13.5.16}$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1=\pi/4$ and $t_2=\pi$ (Figure 13.5.2).

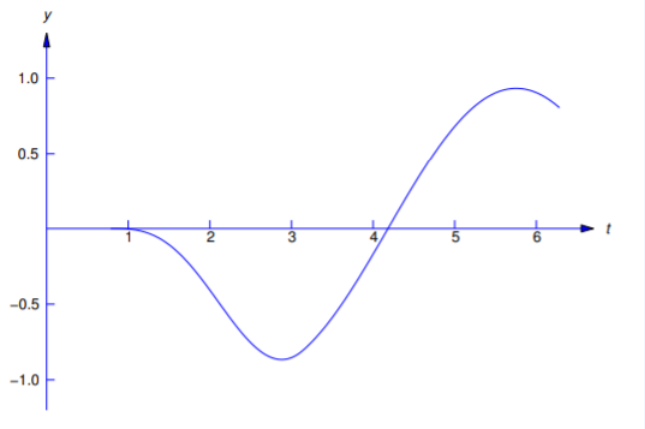


Figure 13.5.2 : Graph of Equation 13.5.16

This page titled 13.5: Constant Coefficient Equations with Piecewise Continuous Forcing Functions is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **8.5: Constant Coefficient Equations with Piecewise Continuous Forcing Functions** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

13.5.1: Constant Coefficient Equations with Piecewise Continuous Forcing Functions (Exercises)

Q8.5.1

In *Exercises 8.5.1-8.5.20* use the Laplace transform to solve the initial value problem. Graph the solution for **Exercise 8.5.6, 8.5.9, 8.5.13**, and **8.5.19**.

1. $y''+y=\begin{cases}3, & 0\le t<\pi,\\ 0, & t\ge\pi,\end{cases}\qquad y(0)=0,\quad y'(0)=0$

2. $y''+y=\begin{cases}3, & 0\le t<4,\\ ;2t-5, & t>4,\end{cases}\qquad y(0)=1,\quad y'(0)=0$

3. $y''-2y'=\begin{cases}4, & 0\le t<1,\\ 6, & t\ge 1,\end{cases}\qquad y(0)=-6,\quad y'(0)=1$

4. $y''-y=\begin{cases}e^{2t}, & 0\le t<2,\\ 1, & t\ge 2,\end{cases}\qquad y(0)=3,\quad y'(0)=-1$

5. $y''-3y'+2y=\begin{cases}0, & 0\le t<1,\\ 1, & 1\le t<2,\\ -1, & t\ge 2,\end{cases}\qquad y(0)=-3,\quad y'(0)=1$

6. $y''+4y=\begin{cases}|\sin t|, & 0\le t<2\pi,\\ 0, & t\ge 2\pi,\end{cases}\qquad y(0)=-3,\quad y'(0)=1$

7. $y''-5y'+4y=\begin{cases}1, & 0\le t<1\\ -1, & 1\le t<2,\\ 0, & t\ge 2,\end{cases}\qquad y(0)=3,\quad y'(0)=-5$

8. $y''+9y=\begin{cases}\cos t, & 0\le t<\frac{3\pi}{2},\\ \sin t, & t\ge\frac{3\pi}{2},\end{cases}\qquad y(0)=0,\ y'(0)=0$

9. $y''+4y=\begin{cases}t, & 0\le t<\frac{\pi}{2},\\ \pi, & t\ge\frac{\pi}{2},\end{cases}\qquad y(0)=0,\ y'(0)=0$

10. $y''+y=\begin{cases}t, & 0\le t<\pi,\\ -t, & t\ge\pi,\end{cases}\qquad y(0)=0,\ y'(0)=0$

11. $y''-3y'+2y=\begin{cases}0, & 0\le t<2,\\ 2t-4, & t\ge 2,\end{cases}\quad,\quad y(0)=0,\quad y'(0)=0$

12. $y''+y=\begin{cases}t, & 0\le t<2\pi,\\ -2t, & t\ge 2\pi,\end{cases}\qquad y(0)=1,\quad y'(0)=2$

13. $y''+3y'+2y=\begin{cases}1, & 0\le t<2,\\ -1, & t\ge 2,\end{cases}\qquad y(0)=0,\ y'(0)=0$

14. $y''-4y'+3y=\begin{cases}-1, & 0\le t<1,\\ 1, & t\ge 1,\end{cases}\qquad y(0)=0,\ y'(0)=0$

15. $y''+2y'+y=\begin{cases}e^t, & 0\le t<1,\\ e^t-1, & t\ge 1,\end{cases}\qquad y(0)=3,\ y'(0)=-1$

16. $y''+2y'+y=\begin{cases}4e^t, & 0\le t<1,\\ 0, & t\ge 1,\end{cases}\qquad y(0)=0,\ y'(0)=0$

17. $y''+3y'+2y=\begin{cases}e^{-t}, & 0\le t<1,\\ 0, & t\ge 1,\end{cases}\qquad y(0)=1,\ y'(0)=-1$

18. $y'' - 4y' + 4y = \begin{cases} e^{2t}, & 0 \le t < 2, \\ -e^{2t}, & t \ge 2, \end{cases} \quad y(0) = 0,\ y'(0) = -1$

19. $y'' = \begin{cases} t^2, & 0 \le t < 1, \\ -t, & 1 \le t < 2, \\ t+1, & t \ge 2, \end{cases} \quad y(0) = 1,\ y'(0) = 0$

20. $y'' + 2y' + 2y = \begin{cases} 1, & 0 \le t < 2\pi, \\ t, & 2\pi \le t < 3\pi, \\ -1, & t \ge 3\pi, \end{cases} \quad y(0) = 2, \quad y'(0) = -1$

Q8.5.2

21. Solve the initial value problem

$$y'' = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where

$$f(t) = m+1, \quad m \le t < m+1, \quad m = 0, 1, 2, \ldots .$$

22. Solve the given initial value problem and find a formula that does not involve step functions and represents y on each interval of continuity of f.

a. $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = m+1, \quad m\pi \le t < (m+1)\pi, \quad m = 0, 1, 2, \ldots.$

b. $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = (m+1)t, \quad 2m\pi \le t < 2(m+1)\pi, \quad m = 0, 1, 2, \ldots$ HINT: **You'll need the formula**

$$1 + 2 + \cdots + m = \frac{m(m+1)}{2}.$$

c. $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = (-1)^m, \quad m\pi \le t < (m+1)\pi, \quad m = 0, 1, 2, \ldots .$

d. $y'' - y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = m+1, \quad m \le t < (m+1), \quad m = 0, 1, 2, \ldots .$
HINT: **You will need the formula**

$$1 + r + \ldots + r^m = \frac{1 - r^{m+1}}{1 - r} (r \ne 1).$$

e. $y'' + 2y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$
$f(t) = (m+1)(\sin t + 2\cos t), \quad 2m\pi \le t < 2(m+1)\pi, \quad m = 0, 1, 2, \ldots .$
(See the hint in d.)

f. $y'' - 3y' + 2y = f(t), \quad y(0) = 0, \quad y'(0) = 0;$

g. $f(t) = m+1, \quad m \le t < m+1, \quad m = 0, 1, 2, \ldots .$
(See the hints in b and d.)

23.

a. Let g be continuous on (α, β) and differentiable on the (α, t_0) and (t_0, β). Suppose $A = \lim_{t \to t_0-} g'(t)$ and $B = \lim_{t \to t_0+} g'(t)$ both exist. Use the mean value theorem to show that

$$\lim_{t \to t_0-} \frac{g(t) - g(t_0)}{t - t_0} = A \quad \text{and} \quad \lim_{t \to t_0+} \frac{g(t) - g(t_0)}{t - t_0} = B.$$

b. Conclude from (a) that $g'(t_0)$ exists and g' is continuous at t_0 if $A = B$.

c. Conclude from (a) that if g is differentiable on (α, β) then g' can't have a jump discontinuity on (α, β).

24.

a. Let a, b, and c be constants, with $a \neq 0$. Let f be piecewise continuous on an interval (α, β), with a single jump discontinuity at a point t_0 in (α, β). Suppose y and y' are continuous on (α, β) and y'' on (α, t_0) and (t_0, β). Suppose also that

$$ay'' + by' + cy = f(t) \tag{A}$$

on (α, t_0) and (t_0, β). Show that

$$y''(t_0+) - y''(t_0-) = \frac{f(t_0+) - f(t_0-)}{a} \neq 0.$$

b. Use (a) and **Exercise 8.5.23c** to show that (A) does not have solutions on any interval (α, β) that contains a jump discontinuity of f.

25. Suppose P_0, P_1, and P_2 are continuous and P_0 has no zeros on an open interval (a, b), and that F has a jump discontinuity at a point t_0 in (a, b). Show that the differential equation

$$P_0(t)y'' + P_1(t)y' + P_2(t)y = F(t)$$

has no solutions on (a, b). HINT: **Generalize the result of Exercise 8.5.24 and use Exercise 8.5.23c.**

26. Let $0 = t_0 < t_1 < \cdots < t_n$. Suppose f_m is continuous on $[t_m, \infty)$ for $m = 1, \dots, n$. Let

$$f(t) = \begin{cases} f_m(t), & t_m \leq t < t_{m+1}, \quad m = 1, \dots, n-1, \\ f_n(t), & t \geq t_n. \end{cases}$$

Show that the solution of

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1,$$

as defined following Theorem 8.5.1, is given by

$$y = \begin{cases} z_0(t), & 0 \leq t < t_1, \\ z_0(t) + z_1(t), & t_1 \leq t < t_2, \\ \vdots & \\ z_0 + \cdots + z_{n-1}(t), & t_{n-1} \leq t < t_n, \\ z_0 + \cdots + z_n(t), & t \geq t_n, \end{cases}$$

where z_0 is the solution of

$$az'' + bz' + cz = f_0(t), \quad z(0) = k_0, \quad z'(0) = k_1$$

and z_m is the solution of

$$az'' + bz' + cz = f_m(t) - f_{m-1}(t), \quad z(t_m) = 0, \quad z'(t_m) = 0$$

for $m = 1, \dots, n$.

This page titled 13.5.1: Constant Coefficient Equations with Piecewise Continuous Forcing Functions (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **8.5E: Constant Coefficient Equations with Piecewise Continuous Forcing Functions (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

13.6: A Brief Table of Laplace Transforms

Table **13.6.1**

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$(s>0)$
t^n	$\frac{n!}{s^{n+1}}$	$(s>0)$
$(n=\text{ integer }>0)$		
$t^p,\ p>-1$	$\frac{\Gamma(p+1)}{s^{(p+1)}}$	$(s>0)$
e^{at}	$\frac{1}{s-a}$	$(s>a)$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$	$(s>0)$
$(n=\text{ integer }>0)$		
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$(s>0)$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$(s>0)$
$e^{\lambda t}\cos \omega t$	$\frac{s-\lambda}{(s-\lambda)^2+\omega^2}$	$(s>\lambda)$
$e^{\lambda t}\sin \omega t$	$\frac{\omega}{(s-\lambda)^2+\omega^2}$	$(s>\lambda)$
$\cosh bt$	$\frac{s}{s^2-b^2}$	$(s>\lvert b\rvert)$
$\sinh bt$	$\frac{b}{s^2-b^2}$	$(s>\lvert b\rvert)$
$t\cos \omega t$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$(s>0)$
$t\sin \omega t$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$(s>0)$
$\sin \omega t-\omega t\cos \omega t$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$	$(s>0)$
$\omega t-\sin \omega t$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$	$(s>0)$
$\frac{1}{t}\sin \omega t$	$\arctan\left(\frac{\omega}{s}\right)$	$(s>0)$
$e^{at}f(t)$	$F(s-a)$	
$t^k f(t)$	$(-1)^k F^{(k)}(s)$	
$f(\omega t)$	$\frac{1}{\omega}F\left(\frac{s}{\omega}\right),\quad \omega>0$	
$u(t-\tau)$	$\frac{e^{-\tau s}}{s}$	$(s>0)$
$u(t-\tau)f(t-\tau)\ (\tau>0)$	$e^{-\tau s}F(s)$	
$\int_o^t f(\tau)g(t-\tau)\,d\tau$	$F(s)\cdot G(s)$	
$\delta(t-a)$	e^{-as}	$(s>0)$

This page titled 13.6: A Brief Table of Laplace Transforms is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **8.8: A Brief Table of Laplace Transforms** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

CHAPTER OVERVIEW

14: Linear Higher Order Differential Equations

This chapter extend the results obtained in Chapter 5 for linear second order equations to linear higher order equations.

14.1: Introduction to Linear Higher Order Equations

14.1.1: Introduction to Linear Higher Order Equations (Exercises)

14.2: Higher Order Constant Coefficient Homogeneous Equations

14.2.1: Higher Order Constant Coefficient Homogeneous Equations (Exercises)

Thumbnail: The Wronskian. In general, for an nth order linear differential equation, if $(n-1)$ solutions are known, the last one can be determined by using the Wronskian.

This page titled 14: Linear Higher Order Differential Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

14.1: Introduction to Linear Higher Order Equations

An nth order differential equation is said to be **linear** if it can be written in the form

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = f(x). \tag{14.1.1}$$

We considered equations of this form with $n = 1$ in Section 2.1 and with $n = 2$ in Chapter 5. In this chapter n is an arbitrary positive integer. In this section we sketch the general theory of linear nth order equations. Since this theory has already been discussed for $n = 2$ in Sections 5.1 and 5.3, we'll omit proofs.

For convenience, we consider linear differential equations written as

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x), \tag{14.1.2}$$

which can be rewritten as Equation 14.1.1 on any interval on which P_0 has no zeros, with $p_1 = P_1/P_0$, …, $p_n = P_n/P_0$ and $f = F/P_0$. For simplicity, throughout this chapter we'll abbreviate the left side of Equation 14.1.2 by Ly; that is,

$$Ly = P_0y^{(n)} + P_1y^{(n-1)} + \cdots + P_ny.$$

We say that the equation $Ly = F$ is **normal** on (a, b) if P_0, P_1, …, P_n and F are continuous on (a, b) and P_0 has no zeros on (a, b). If this is so then $Ly = F$ can be written as Equation 14.1.1 with p_1, …, p_n and f continuous on (a, b).

The next theorem is analogous to Theorem 5.3.1.

Theorem 9.1.1

Suppose $Ly = F$ is normal on (a, b), let x_0 be a point in (a, b), and let k_0, k_1, …, k_{n-1} be arbitrary real numbers. Then the initial value problem

$$Ly = F, \quad y(x_0) = k_0, \quad y'(x_0) = k_1, \dots, \quad y^{(n-1)}(x_0) = k_{n-1}$$

has a unique solution on (a, b).

Homogeneous Equations

Equation 14.1.2 is said to be **homogeneous** if $F \equiv 0$ and **nonhomogeneous** otherwise. Since $y \equiv 0$ is obviously a solution of $Ly = 0$, we call it the **trivial** solution. Any other solution is **nontrivial**.

If y_1, y_2, …, y_n are defined on (a, b) and c_1, c_2, …, c_n are constants, then

$$y = c_1y_1 + c_2y_2 + \cdots + c_ny_n \tag{14.1.3}$$

is a **linear combination** of $\{y_1, y_2 \dots, y_n\}$. It's easy to show that if y_1, y_2, …, y_n are solutions of $Ly = 0$ on (a, b), then so is any linear combination of $\{y_1, y_2, \dots, y_n\}$. (See the proof of Theorem 5.1.2.) We say that $\{y_1, y_2, \dots, y_n\}$ is a **fundamental set of solutions of** $Ly = 0$ on (a, b) if every solution of $Ly = 0$ on (a, b) can be written as a linear combination of $\{y_1, y_2, \dots, y_n\}$, as in Equation 14.1.3. In this case we say that Equation 14.1.3 is the **general solution of** $Ly = 0$ on (a, b).

It can be shown (**Exercises 9.1.14** and **9.1.15**) that if the equation $Ly = 0$ is normal on (a, b) then it has infinitely many fundamental sets of solutions on (a, b). The next definition will help to identify fundamental sets of solutions of $Ly = 0$.

We say that $\{y_1, y_2, \dots, y_n\}$ is **linearly independent** on (a, b) if the only constants c_1, c_2, …, c_n such that

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) = 0, \quad a < x < b, \tag{14.1.4}$$

are $c_1 = c_2 = \cdots = c_n = 0$. If Equation 14.1.4 holds for some set of constants c_1, c_2, …, c_n that are not all zero, then $\{y_1, y_2, \dots, y_n\}$ is **linearly dependent** on (a, b)

The next theorem is analogous to Theorem 5.1.3.

Theorem 9.1.1

If $Ly = 0$ is normal on (a, b), then a set $\{y_1, y_2, \ldots, y_n\}$ of n solutions of $Ly = 0$ on (a, b) is a fundamental set if and only if it is linearly independent on (a, b).

Example 9.1.1

The equation

$$x^3y''' - x^2y'' - 2xy' + 6y = 0 \tag{14.1.5}$$

is normal and has the solutions $y_1 = x^2$, $y_2 = x^3$, and $y_3 = 1/x$ on $(-\infty, 0)$ and $(0, \infty)$. Show that $\{y_1, y_2, y_3\}$ is linearly independent on $(-\infty, 0)$ and $(0, \infty)$. Then find the general solution of Equation 14.1.5 on $(-\infty, 0)$ and $(0, \infty)$.

Solution

Suppose

$$c_1x^2 + c_2x^3 + \frac{c_3}{x} = 0 \tag{14.1.6}$$

on $(0, \infty)$. We must show that $c_1 = c_2 = c_3 = 0$. Differentiating Equation 14.1.6 twice yields the system

$$\begin{aligned} c_1x^2 + c_2x^3 + \frac{c_3}{x} &= 0 \\ 2c_1x + 3c_2x^2 - \frac{c_3}{x^2} &= 0 \\ 2c_1 + 6c_2x + \frac{2c_3}{x^3} &= 0. \end{aligned} \tag{14.1.7}$$

If Equation 14.1.7 holds for all x in $(0, \infty)$, then it certainly holds at $x = 1$; therefore,

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 + 6c_2 + 2c_3 &= 0. \end{aligned} \tag{14.1.8}$$

By solving this system directly, you can verify that it has only the trivial solution $c_1 = c_2 = c_3 = 0$; however, for our purposes it is more useful to recall from linear algebra that a homogeneous linear system of n equations in n unknowns has only the trivial solution if its determinant is nonzero. Since the determinant of Equation 14.1.8 is

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -3 \\ 2 & 4 & 0 \end{vmatrix} = 12,$$

it follows that Equation 14.1.8 has only the trivial solution, so $\{y_1, y_2, y_3\}$ is linearly independent on $(0, \infty)$. Now Theorem 9.1.2 implies that

$$y = c_1x^2 + c_2x^3 + \frac{c_3}{x}$$

is the general solution of Equation 14.1.5 on $(0, \infty)$. To see that this is also true on $(-\infty, 0)$, assume that Equation 14.1.6 holds on $(-\infty, 0)$. Setting $x = -1$ in Equation 14.1.7 yields

$$\begin{aligned} c_1 - c_2 - c_3 &= 0 \\ -2c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 - 6c_2 - 2c_3 &= 0. \end{aligned}$$

Since the determinant of this system is

$$\begin{vmatrix} 1 & -1 & -1 \\ -2 & 3 & -1 \\ 2 & -6 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 2 & -4 & 0 \end{vmatrix} = -12,$$

it follows that $c_1 = c_2 = c_3 = 0$; that is, $\{y_1, y_2, y_3\}$ is linearly independent on $(-\infty, 0)$.

Example 9.1.2

The equation

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0 \tag{14.1.9}$$

is normal and has the solutions $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{2x}$, and $y_4 = e^{-3x}$ on $(-\infty, \infty)$. (Verify.) Show that $\{y_1, y_2, y_3, y_4\}$ is linearly independent on $(-\infty, \infty)$. Then find the general solution of Equation 14.1.9.

Solution

Suppose c_1, c_2, c_3, and c_4 are constants such that

$$c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-3x} = 0 \tag{14.1.10}$$

for all x. We must show that $c_1 = c_2 = c_3 = c_4 = 0$. Differentiating Equation 14.1.10 three times yields the system

$$\begin{aligned} c_1e^x + c_2e^{-x} + \ c_3e^{2x} + \ \ c_4e^{-3x} &= 0 \\ c_1e^x - c_2e^{-x} + 2c_3e^{2x} - \ 3c_4e^{-3x} &= 0 \\ c_1e^x + c_2e^{-x} + 4c_3e^{2x} + \ 9c_4e^{-3x} &= 0 \\ c_1e^x - c_2e^{-x} + 8c_3e^{2x} - 27c_4e^{-3x} &= 0. \end{aligned} \tag{14.1.11}$$

If Equation 14.1.11 holds for all x, then it certainly holds for $x = 0$. Therefore

$$\begin{aligned} c_1 + c_2 + \ c_3 + \ \ c_4 &= 0 \\ c_1 - c_2 + 2c_3 - \ 3c_4 &= 0 \\ c_1 + c_2 + 4c_3 + \ 9c_4 &= 0 \\ c_1 - c_2 + 8c_3 - 27c_4 &= 0. \end{aligned}$$

The determinant of this system is

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -4 \\ 0 & 0 & 3 & 8 \\ 0 & -2 & 7 & -28 \end{vmatrix} = \begin{vmatrix} -2 & 1 & -4 \\ 0 & 3 & 8 \\ -2 & 7 & -28 \end{vmatrix} \\ [5pt] &= \begin{vmatrix} -2 & 1 & -4 \\ 0 & 3 & 8 \\ 0 & 6 & -24 \end{vmatrix} = -2 \begin{vmatrix} 3 & 8 \\ 6 & -24 \end{vmatrix} = 240, \end{aligned} \tag{14.1.12}$$

so the system has only the trivial solution $c_1 = c_2 = c_3 = c_4 = 0$. Now Theorem 9.1.2 implies that

$$y = c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-3x}$$

is the general solution of Equation 14.1.9.

The Wronskian

We can use the method used in Examples 9.1.1 and 9.1.2 to test n solutions $\{y_1, y_2, \dots, y_n\}$ of any nth order equation $Ly = 0$ for linear independence on an interval (a, b) on which the equation is normal. Thus, if c_1, c_2,…, c_n are constants such that

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0, \quad a < b,$$

then differentiating $n - 1$ times leads to the $n \times n$ system of equations

$$\begin{aligned} c_1y_1(x)+c_2y_2(x)+&\quad\cdots\quad+c_ny_n(x)=0\\ c_1y_1'(x)+c_2y_2'(x)+&\quad\cdots\quad+c_ny_n'(x)=0\\ &\quad\vdots\\ c_1y_1^{(n-1)}(x)+c_2y_2^{(n-1)}(x)+&\quad\cdots\quad+c_ny_n^{(n-1)}(x)=0 \end{aligned} \tag{14.1.13}$$

for c_1, c_2, …, c_n. For a fixed x, the determinant of this system is

$$W(x)=\begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x)\\ y_1'(x) & y_2'(x) & \cdots & y_n'(x)\\ \vdots & \vdots & \ddots & \vdots\\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}.$$

We call this determinant the **Wronskian** of $\{y_1,y_2,\dots,y_n\}$. If $W(x)\neq 0$ for some x in (a,b) then the system Equation 14.1.13 has only the trivial solution $c_1=c_2=\cdots=c_n=0$, and Theorem 9.1.2 implies that

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n$$

is the general solution of $Ly=0$ on (a,b).

The next theorem generalizes Theorem 5.1.4. The proof is sketched in **Exercises 9.1.17-9.1.20**.

Theorem 9.1.3 Abel's Formula

Suppose the homogeneous linear nth order equation

$$P_0(x)y^{(n)}+P_1(x)y^{n-1}+\cdots+P_n(x)y=0 \tag{14.1.14}$$

is normal on (a,b), let y_1, y_2, …, y_n be solutions of Equation 14.1.14 on (a,b), and let x_0 be in (a,b). Then the Wronskian of $\{y_1,y_2,\dots,y_n\}$ is given by

$$W(x)=W(x_0)\exp\left\{-\int_{x_0}^{x}\frac{P_1(t)}{P_0(t)}\,dt\right\},\quad a<b. \tag{14.1.15}$$

Therefore, either W has no zeros in (a,b) or $W\equiv 0$ on (a,b).

Formula Equation 14.1.15 is **Abel's formula**.

The next theorem is analogous to Theorem 5.1.6.

Theorem 9.1.4

Suppose $Ly=0$ is normal on (a,b) and let y_1, y_2, …, y_n be n solutions of $Ly=0$ on (a,b). Then the following statements are equivalent; that is, they are either all true or all false:

a. The general solution of $Ly=0$ on (a,b) is $y=c_1y_1+c_2y_2+\cdots+c_ny_n$.
b. $\{y_1,y_2,\dots,y_n\}$ is a fundamental set of solutions of $Ly=0$ on (a,b).
c. $\{y_1,y_2,\dots,y_n\}$ is linearly independent on (a,b).
d. The Wronskian of $\{y_1,y_2,\dots,y_n\}$ is nonzero at some point in (a,b).
e. The Wronskian of $\{y_1,y_2,\dots,y_n\}$ is nonzero at all points in (a,b).

Example 9.1.3

In Example 9.1.1 we saw that the solutions $y_1=x^2$, $y_2=x^3$, and $y_3=1/x$ of

$$x^3y'''-x^2y''-2xy'+6y=0$$

are linearly independent on $(-\infty,0)$ and $(0,\infty)$. Calculate the Wronskian of $\{y_1,y_2,y_3\}$.

Solution

If $x \neq 0$, then

$$W(x) = \begin{vmatrix} x^2 & x^3 & \frac{1}{x} \\ 2x & 3x^2 & -\frac{1}{x^2} \\ 2 & 6x & \frac{2}{x^3} \end{vmatrix} = 2x^3 \begin{vmatrix} 1 & x & \frac{1}{x^3} \\ 2 & 3x & -\frac{1}{x^3} \\ 1 & 3x & \frac{1}{x^3} \end{vmatrix}$$

where we factored x^2, x, and 2 out of the first, second, and third rows of $W(x)$, respectively. Adding the second row of the last determinant to the first and third rows yields

$$W(x) = 2x^3 \begin{vmatrix} 3 & 4x & 0 \\ 2 & 3x & -\frac{1}{x^3} \\ 3 & 6x & 0 \end{vmatrix} = 2x^3 \left(\frac{1}{x^3}\right) \begin{vmatrix} 3 & 4x \\ 3 & 6x \end{vmatrix} = 12x$$

Therefore $W(x) \neq 0$ on $(-\infty, 0)$ and $(0, \infty)$.

Example 9.1.4

In Example 9.1.2 we saw that the solutions $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{2x}$, and $y_4 = e^{-3x}$ of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

are linearly independent on every open interval. Calculate the Wronskian of $\{y_1, y_2, y_3, y_4\}$.

Solution

For all x,

$$W(x) = \begin{vmatrix} e^x & e^{-x} & e^{2x} & e^{-3x} \\ e^x & -e^{-x} & 2e^{2x} & -3e^{-3x} \\ e^x & e^{-x} & 4e^{2x} & 9e^{-3x} \\ e^x & -e^{-x} & 8e^{2x} & -27e^{-3x} \end{vmatrix}.$$

Factoring the exponential common factor from each row yields

$$W(x) = e^{-x} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{vmatrix} = 240e^{-x},$$

from Equation 14.1.12.

Note

Under the assumptions of Theorem 9.1.4 , it isn't necessary to obtain a formula for $W(x)$. Just evaluate $W(x)$ at a convenient point in (a, b), as we did in Examples 9.1.1 and 9.1.2 .

Theorem 9.1.5

Suppose c is in (a, b) and $\alpha_1, \alpha_2, \ldots$, are real numbers, not all zero. Under the assumptions of Theorem 10.3.3, suppose y_1 and y_2 are solutions of Equation 5.1.35 such that

$$\alpha y_i(c) + y_i'(c) + \cdots + y_i^{(n-1)}(c) = 0, \quad 1 \leq i \leq n. \tag{14.1.16}$$

Then $\{y_1, y_2, \ldots y_n\}$ isn't linearly independent on (a, b).

Proof

Since $\alpha_1, \alpha_2, \ldots, \alpha_n$ are not all zero, Equation 14.1.14 implies that

$$\begin{vmatrix} y_1(c) & y_1'(c) & \cdots & y_1^{(n-1)}(c) \\ y_2(c) & y_2'(c) & \cdots & y_2^{(n-1)}(c) \\ \vdots & \vdots & \ddots & \vdots \\ y_n(c) & y_n'(c) & \cdots & y_n^{(n-1)}(c) \end{vmatrix} = 0,$$

so

$$\begin{vmatrix} y_1(c) & y_2(c) & \cdots & y_n(c) \\ y_1'(c) & y_2'(c) & \cdots & y_n'(c) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(c) & y_2^{(n-1)}(c)(c) & \cdots & y_n^{(n-1)}(c)(c) \end{vmatrix} = 0$$

and Theorem 9.1.4 implies the stated conclusion.

General Solution of a Nonhomogeneous Equation

The next theorem is analogous to Theorem 5.3.2. It shows how to find the general solution of $Ly = F$ if we know a particular solution of $Ly = F$ and a fundamental set of solutions of the **complementary equation** $Ly = 0$.

Theorem 9.1.6

The probabilities assigned to events by a distribution function on a sample space are given by.

Proof

Suppose $Ly = F$ is normal on (a, b). Let y_p be a particular solution of $Ly = F$ on (a, b), and let $\{y_1, y_2, \ldots, y_n\}$ be a fundamental set of solutions of the complementary equation $Ly = 0$ on (a, b). Then y is a solution of $Ly = F$ on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

where $c_1, c_2, \ldots, c_n$ are constants.

The next theorem is analogous to Theorem 5.3.2.

Theorem 9.1.7 The Principle of Superposition

Suppose for each $i = 1, 2, \ldots, r$, the function y_{p_i} is a particular solution of $Ly = F_i$ on (a, b). Then

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_r}$$

is a particular solution of

$$Ly = F_1(x) + F_2(x) + \cdots + F_r(x)$$

on (a, b).

We'll apply Theorems 9.1.6 and 9.1.7 throughout the rest of this chapter.

This page titled 14.1: Introduction to Linear Higher Order Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **9.1: Introduction to Linear Higher Order Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

https://math.libretexts.org/@go/page/121712

14.1.1: Introduction to Linear Higher Order Equations (Exercises)

Q9.1.1

1. Verify that the given function is the solution of the initial value problem.

a. $x^3y''' - 3x^2y'' + 6xy' - 6y = \dfrac{-24}{x}, \quad y(-1) = 0, y'(-1) = 0, \quad y''(-1) = 0 \; ; y = -6x - 8x^2 - 3x^3 + \frac{1}{x}$

b. $y''' - \dfrac{1}{x}y'' - y' + \dfrac{1}{x}y = \dfrac{x^2 - 4}{x^4}, \quad y(1) = \dfrac{3}{2}, \quad y'(1) = \dfrac{1}{2}, y''(1) = 1 \; ; y = x + \dfrac{1}{2x}$

c. $xy''' - y'' - xy' + y = x^2, \quad y(1) = 2, \quad y'(1) = 5, \quad y''(1) = -1 \; ; y = -x^2 - 2 + 2e^{(x-1)} - e^{-(x-1)} + 4x$

d. $4x^3y''' + 4x^2y'' - 5xy' + 2y = 30x^2, \quad y(1) = 5, \quad y'(1) = \dfrac{17}{2} \; ; y''(1) = \dfrac{63}{4}; \quad y = 2x^2 \ln x - x^{1/2} + 2x^{-1/2} + 4x^2$

e. $x^4y^{(4)} - 4x^3y''' + 12x^2y'' - 24xy' + 24y = 6x^4, \quad y(1) = -2 \; ; y'(1) = -9, \quad y''(1) = -27, \quad y'''(1) = -52 \; ; y = x^4 \ln x + x - 2x^2 + 3x^3 - 4x^4$

f. $xy^{(4)} - y''' - 4xy'' + 4y' = 96x^2, \quad y(1) = -5, \quad y'(1) = -24 \; ; y''(1) = -36; \quad y'''(1) = -48; \quad y = 9 - 12x + 6x^2 - 8x^3$

2. Solve the initial value problem

$$x^3y''' - x^2y'' - 2xy' + 6y = 0, \quad y(-1) = -4, \quad y'(-1) = -14, \quad y''(-1) = -20.$$

HINT: **See Example 9.1.1.**

3. Solve the initial value problem

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0, \quad y(0) = 5, \quad y'(0) = -6, \quad y''(0) = 10, \quad y'''(0) - 36.$$

HINT: **See Example 9.1.2.**

4. Find solutions y_1, y_2, …, y_n of the equation $y^{(n)} = 0$ that satisfy the initial conditions

$$y_i^{(j)}(x_0) = \begin{cases} 0, & j \neq i - 1, \\ [5pt]1, & j = i - 1, \end{cases} \quad 1 \leq i \leq n.$$

5.

a. Verify that the function

$$y = c_1x^3 + c_2x^2 + \frac{c_3}{x}$$

satisfies

$$x^3y''' - x^2y'' - 2xy' + 6y = 0 \tag{A}$$

if c_1, c_2, and c_3 are constants.

b. Use (a) to find solutions y_1, y_2, and y_3 of (A) such that

$$\begin{aligned} y_1(1) &= 1, \quad y_1'(1) = 0, \quad y_1''(1) = 0 \\ [5pt]y_2(1) &= 0, \quad y_2'(1) = 1, \quad y_2''(1) = 0 \\ [5pt]y_3(1) &= 0, \quad y_3'(1) = 0, \quad y_3''(1) = 1. \end{aligned}$$

c. Use (b) to find the solution of (A) such that

$$y(1) = k_0, \quad y'(1) = k_1, \quad y''(1) = k_2.$$

6. Verify that the given functions are solutions of the given equation, and show that they form a fundamental set of solutions of the equation on any interval on which the equation is normal.

a. $y''' + y'' - y' - y = 0; \quad \{e^x, e^{-x}, xe^{-x}\}$

b. $y''' - 3y'' + 7y' - 5y = 0; \quad \{e^x, e^x \cos 2x, e^x \sin 2x\}.$

c. $xy''' - y'' - xy' + y = 0; \quad \{e^x, e^{-x}, x\}$

d. $x^2y''' + 2xy'' - (x^2 + 2)y = 0; \quad \{e^x/x, e^{-x}/x, 1\}$

e. $(x^2-2x+2)y'''-x^2y''+2xy'-2y=0;\quad \{x,\,x^2,\,e^x\}$
f. $(2x-1)y^{(4)}-4xy'''+(5-2x)y''+4xy'-4y=0;\quad \{x,\,e^x,\,e^{-x},e^{2x}\}$
g. $xy^{(4)}-y'''-4xy'+4y'=0;\quad \{1,x^2,\,e^{2x},\,e^{-2x}\}$

7. Find the Wronskian W of a set of three solutions of

$$y'''+2xy''+e^xy'-y=0,$$

given that $W(0)=2$.

8. Find the Wronskian W of a set of four solutions of

$$y^{(4)}+(\tan x)y'''+x^2y''+2xy=0,$$

given that $W(\pi/4)=K$.

9.

a. Evaluate the Wronskian $W\{e^x,\,xe^x,\,x^2e^x\}$. Evaluate $W(0)$.
b. Verify that y_1, y_2, and y_3 satisfy

$$y'''-3y''+3y'-y=0. \tag{A}$$

c. Use $W(0)$ from (a) and Abel's formula to calculate $W(x)$.
d. What is the general solution of (A)?

10. Compute the Wronskian of the given set of functions.

a. $\{1,\,e^x,\,e^{-x}\}$
b. $\{e^x,\,e^x\sin x,\,e^x\cos x\}$
c. $\{2,\,x+1,\,x^2+2\}$
d. $x,\,x\ln x,\,1/x\}$
e. $\{1,\,x,\,\frac{x^2}{2!},\,\frac{x^3}{3!},\cdots,\,\frac{x^n}{n!}\}$
f. $\{e^x,\,e^{-x},\,x\}$
g. $\{e^x/x,\,e^{-x}/x,\,1\}$
h. $\{x,\,x^2,\,e^x\}$
i. $\{x,\,x^3,\,1/x,\,1/x^2\}$
j. $\{e^x,\,e^{-x},\,x,\,e^{2x}\}$
k. $\{e^{2x},\,e^{-2x},\,1,\,x^2\}$

11. Suppose $Ly=0$ is normal on (a,b) and x_0 is in (a,b). Use Theorem 9.1.1 to show that $y\equiv 0$ is the only solution of the initial value problem

$$Ly=0,\quad y(x_0)=0,\quad y'(x_0)=0,\dots,y^{(n-1)}(x_0)=0,$$

on (a,b).

12. Prove: If y_1, y_2, …, y_n are solutions of $Ly=0$ and the functions

$$z_i=\sum_{j=1}^{n}a_{ij}y_j,\quad 1\le i\le n,$$

form a fundamental set of solutions of $Ly=0$, then so do y_1, y_2, …, y_n.

13. Prove: If

$$y=c_1y_1+c_2y_2+\cdots+c_ky_k+y_p$$

is a solution of a linear equation $Ly=F$ for every choice of the constants c_1, c_2,…, c_k, then $Ly_i=0$ for $1\le i\le k$.

14. Suppose $Ly=0$ is normal on (a,b) and let x_0 be in (a,b). For $1\le i\le n$, let y_i be the solution of the initial value problem

$$Ly_i = 0, \quad y_i^{(j)}(x_0) = \begin{cases} 0, & j \neq i-1, \\ 1, & j = i-1, \end{cases} \quad 1 \leq i \leq n,$$

where x_0 is an arbitrary point in (a, b). Show that any solution of $Ly = 0$ on (a, b), can be written as

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n,$$

with $c_j = y^{(j-1)}(x_0)$.

15. Suppose $\{y_1, y_2, \ldots, y_n\}$ is a fundamental set of solutions of

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0$$

on (a, b), and let

$$\begin{aligned} z_1 &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \\ z_2 &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \\ &\vdots \quad\quad \vdots \quad\quad\quad \vdots \quad\quad \vdots \\ z_n &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n, \end{aligned}$$

where the $\{a_{ij}\}$ are constants. Show that $\{z_1, z_2, \ldots, z_n\}$ is a fundamental set of solutions of (A) if and only if the determinant

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

is nonzero. HINT: **The determinant of a product of $n \times n$ matrices equals the product of the determinants.**

16. Show that $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent on (a, b) if and only if at least one of the functions y_1, y_2, …, y_n can be written as a linear combination of the others on (a, b).

Q9.1.2

Take the following as a hint in **Exercises 9.1.17-9.1.19**:

By the definition of determinant,

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum \pm a_{1i_1} a_{2i_2}, \cdots, a_{ni_n},$$

where the sum is over all permutations $(i_i, i_2, \cdots, i_n)$ of $(1, 2, \cdots, n)$ and the choice of $+$ or $-$ in each term depends only on the permutation associated with that term.

17. Prove: If

$$A(u_1, u_2, \ldots, u_n) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ u_1 & u_2 & \cdots & u_n \end{vmatrix},$$

then

$$A(u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n) = A(u_1, u_2, \ldots, u_n) + A(v_1, v_2, \ldots, v_n).$$

18. Let

$$F = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix},$$

where f_{ij} $(1 \le i,\ j \le n)$ is differentiable. Show that

$$F' = F_1 + F_2 + \cdots + F_n,$$

where F_i is the determinant obtained by differentiating the ith row of F.

19. Use **Exercise 9.1.18** to show that if W is the Wronskian of the n-times differentiable functions y_1, y_2, …, y_n, then

$$W' = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}.$$

Q9.1.3

20. Use **Exercises 9.1.17** and **9.1.19** to show that if W is the Wronskian of solutions $\{y_1, y_2, \ldots, y_n\}$ of the normal equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0, \tag{A}$$

then $W' = -P_1W/P_0$. Derive Abel's formula (Equation 9.1.15) from this.

21. Prove Theorem 9.1.6.

22. Prove Theorem 9.1.7.

23. Show that if the Wronskian of the n-times continuously differentiable functions $\{y_1, y_2, \ldots, y_n\}$ has no zeros in (a, b), then the differential equation obtained by expanding the determinant

$$\begin{vmatrix} y & y_1 & y_2 & \cdots & y_n \\ y' & y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y^{(n)} & y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = 0,$$

in cofactors of its first column is normal and has $\{y_1, y_2, \ldots, y_n\}$ as a fundamental set of solutions on (a, b).

24. Use the method suggested by **Exercise 9.1.23** to find a linear homogeneous equation such that the given set of functions is a fundamental set of solutions on intervals on which the Wronskian of the set has no zeros.

a. $\{x,\ x^2 - 1,\ x^2 + 1\}$
b. $\{e^x,\ e^{-x},\ x\}$
c. $\{e^x,\ xe^{-x},\ 1\}$
d. $\{x,\ x^2,\ e^x\}$
e. $\{x,\ x^2,\ 1/x\}$
f. $\{x + 1,\ e^x,\ e^{3x}\}$
g. $\{x,\ x^3,\ 1/x,\ 1/x^2\}$
h. $\{x,\ x\ln x,\ 1/x,\ x^2\}$
i. $\{e^x,\ e^{-x},\ x,\ e^{2x}\}$
j. $\{e^{2x},\ e^{-2x},\ 1,\ x^2\}$

This page titled 14.1.1: Introduction to Linear Higher Order Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **9.1E: Introduction to Linear Higher Order Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

14.2: Higher Order Constant Coefficient Homogeneous Equations

If a_0, a_1, …, a_n are constants and $a_0 \neq 0$, then

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = F(x)$$

is said to be a **constant coefficient equation**. In this section we consider the homogeneous constant coefficient equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = 0. \tag{14.2.1}$$

Since Equation 14.2.1 is normal on $(-\infty, \infty)$, the theorems in Section 9.1 all apply with $(a, b) = (-\infty, \infty)$.

As in Section 5.2, we call

$$p(r) = a_0r^n + a_1r^{n-1} + \cdots + a_n \tag{14.2.2}$$

the **characteristic polynomial** of Equation 14.2.1. We saw in Section 5.2 that when $n = 2$ the solutions of Equation 14.2.1 are determined by the zeros of the characteristic polynomial. This is also true when $n > 2$, but the situation is more complicated in this case. Consequently, we take a different approach.

If k is a positive integer, let D^k stand for the k-th derivative operator; that is

$$D^ky = y^{(k)}.$$

If

$$q(r) = b_0r^m + b_1r^{m-1} + \cdots + b_m$$

is an arbitrary polynomial, define the operator

$$q(D) = b_0D^m + b_1D^{m-1} + \cdots + b_m$$

such that

$$q(D)y = (b_0D^m + b_1D^{m-1} + \cdots + b_m)y = b_0y^{(m)} + b_1y^{(m-1)} + \cdots + b_my$$

whenever y is a function with m derivatives. We call $q(D)$ a **polynomial operator**.

With p as in Equation 14.2.2,

$$p(D) = a_0D^n + a_1D^{n-1} + \cdots + a_n,$$

so Equation 14.2.1 can be written as $p(D)y = 0$. If r is a constant then

$$\begin{aligned} p(D)e^{rx} &= \left(a_0D^ne^{rx} + a_1D^{n-1}e^{rx} + \cdots + a_ne^{rx}\right) \\ &= (a_0r^n + a_1r^{n-1} + \cdots + a_n)e^{rx}; \end{aligned}$$

that is

$$p(D)(e^{rx}) = p(r)e^{rx}.$$

This shows that $y = e^{rx}$ is a solution of Equation 14.2.1 if $p(r) = 0$. In the simplest case, where p has n distinct real zeros r_1, r_2,…, r_n, this argument yields n solutions

$$y_1 = e^{r_1x}, \quad y_2 = e^{r_2x}, \dots, \quad y_n = e^{r_nx}.$$

It can be shown (**Exercise 9.2.39**) that the Wronskian of $\{e^{r_1x}, e^{r_2x}, \dots, e^{r_nx}\}$ is nonzero if r_1, r_2, …, r_n are distinct; hence, $\{e^{r_1x}, e^{r_2x}, \dots, e^{r_nx}\}$ is a fundamental set of solutions of $p(D)y = 0$ in this case.

Example 9.2.1

a. Find the general solution of

$$y''' - 6y'' + 11y' - 6y = 0. \tag{14.2.3}$$

b. and solve the initial value problem

$$y''' - 6y'' + 11y' - 6y = 0, \quad y(0) = 4, \quad y'(0) = 5, \quad y''(0) = 9. \tag{14.2.4}$$

Solution a

The characteristic polynomial of Equation 14.2.3 is

$$p(r) = r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3).$$

Therefore $\{e^x, e^{2x}, e^{3x}\}$ is a set of solutions of Equation 14.2.3. It is a fundamental set, since its Wronskian is

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x} \neq 0.$$

Therefore the general solution of Equation 14.2.3 is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}. \tag{14.2.5}$$

Solution b

We must determine c_1, c_2 and c_3 in Equation 14.2.5 so that y satisfies the initial conditions in Equation 14.2.4. Differentiating Equation 14.2.5 twice yields

$$\begin{aligned} y' &= c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x} \\ y'' &= c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}. \end{aligned} \tag{14.2.6}$$

Setting $x = 0$ in Equation 14.2.5 and Equation 14.2.6 and imposing the initial conditions yields

$$\begin{aligned} c_1 + c_2 + c_3 &= 4 \\ c_1 + 2c_2 + 3c_3 &= 5 \\ c_1 + 4c_2 + 9c_3 &= 9. \end{aligned}$$

The solution of this system is $c_1 = 4$, $c_2 = -1$, $c_3 = 1$. Therefore the solution of Equation 14.2.4 is

$$y = 4e^x - e^{2x} + e^{3x}$$

(Figure 9.2.1).

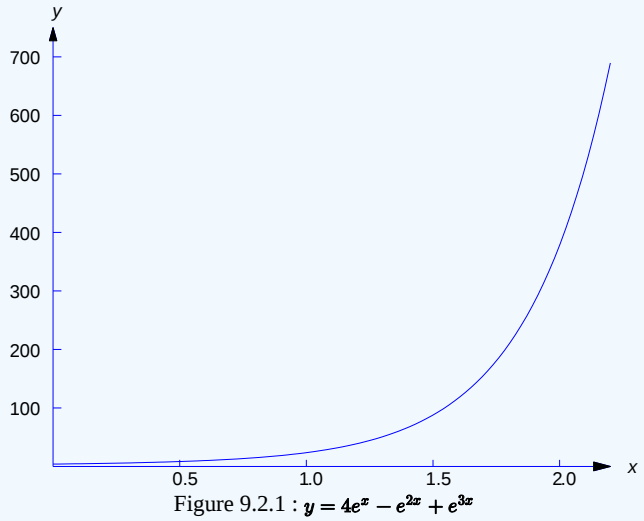


Figure 9.2.1 : $y = 4e^x - e^{2x} + e^{3x}$

Now we consider the case where the characteristic polynomial Equation 14.2.2 does not have n distinct real zeros. For this purpose it is useful to define what we mean by a factorization of a polynomial operator. We begin with an example.

Example 9.2.2

Consider the polynomial

$$p(r) = r^3 - r^2 + r - 1$$

and the associated polynomial operator

$$p(D) = D^3 - D^2 + D - 1.$$

Since $p(r)$ can be factored as

$$p(r) = (r-1)(r^2+1) = (r^2+1)(r-1),$$

it is reasonable to expect that p(D) can be factored as

$$p(D) = (D-1)(D^2+1) = (D^2+1)(D-1). \tag{14.2.7}$$

However, before we can make this assertion we must **define** what we mean by saying that two operators are equal, and what we mean by the products of operators in Equation 14.2.7. We say that two operators are equal if they apply to the same functions and always produce the same result. The definitions of the products in Equation 14.2.7 is this: if y is any three-times differentiable function then

a. $(D-1)(D^2+1)y$ is the function obtained by first applying D^2+1 to y and then applying $D-1$ to the resulting function
b. $(D^2+1)(D-1)y$ is the function obtained by first applying $D-1$ to y and then applying D^2+1 to the resulting function.

From (a):

$$\begin{aligned} (D-1)(D^2+1)y &= (D-1)[(D^2+1)y] \\ &= (D-1)(y''+y) = D(y''+y) - (y''+y) \\ &= (y'''+y') - (y''+y) \\ &= y''' - y'' + y' - y = (D^3 - D^2 + D - 1)y. \end{aligned} \tag{14.2.8}$$

This implies that

$$(D-1)(D^2+1) = (D^3 - D^2 + D - 1).$$

From (b):

$$\begin{aligned} (D^2+1)(D-1)y &= (D^2+1)[(D-1)y] \\ &= (D^2+1)(y'-y) = D^2(y'-y) + (y'-y) \\ &= (y'''-y'') + (y'-y) \\ &= y''' - y'' + y' - y = (D^3 - D^2 + D - 1)y, \end{aligned} \tag{14.2.9}$$

$$(D^2+1)(D-1) = (D^3 - D^2 + D - 1),$$

which completes the justification of Equation 14.2.7.

Example 9.2.3

Use the result of Example 9.2.2 to find the general solution of

$$y''' - y'' + y' - y = 0. \tag{14.2.10}$$

Solution

From Equation 14.2.8, we can rewrite Equation 14.2.10 as

$$(D-1)(D^2+1)y = 0,$$

which implies that any solution of $(D^2+1)y = 0$ is a solution of Equation 14.2.10. Therefore $y_1 = \cos x$ and $y_2 = \sin x$ are solutions of Equation 14.2.10.

From Equation 14.2.9, we can rewrite Equation 14.2.10 as

$$(D^2+1)(D-1)y=0,$$

which implies that any solution of $(D-1)y=0$ is a solution of Equation 14.2.10. Therefore $y_3=e^x$ is solution of Equation 14.2.10.

The Wronskian of $\{e^x,\cos x,\sin x\}$ is

$$W(x)=\begin{vmatrix}\cos x & \sin x & e^x\\ -\sin x & \cos x & e^x\\ -\cos x & -\sin x & e^x\end{vmatrix}.$$

Since

$$W(0)=\begin{vmatrix}1 & 0 & 1\\ 0 & 1 & 1\\ -1 & 0 & 1\end{vmatrix}=2,$$

$\{\cos x,\sin x,e^x\}$ is linearly independent and

$$y=c_1\cos x+c_2\sin x+c_3e^x$$

is the general solution of Equation 14.2.10.

Example 9.2.4

Find the general solution of

$$y^{(4)}-16y=0. \tag{14.2.11}$$

Solution

The characteristic polynomial of Equation 14.2.11 is

$$\begin{aligned}p(r)&=r^4-16\\ &=(r^2-4)(r^2+4)\\ &=(r-2)(r+2)(r^2+4).\end{aligned}$$

By arguments similar to those used in Examples 9.2.2 and 9.2.4 , it can be shown that Equation 14.2.11 can be written as

$$(D^2+4)(D+2)(D-2)y=0$$

or

$$(D^2+4)(D-2)(D+2)y=0$$

or

$$(D-2)(D+2)(D^2+4)y=0.$$

Therefore y is a solution of Equation 14.2.11 if it is a solution of any of the three equations

$$(D-2)y=0,\quad (D+2)y=0,\quad (D^2+4)y=0.$$

Hence, $\{e^{2x},e^{-2x},\cos 2x,\sin 2x\}$ is a set of solutions of Equation 14.2.11. The Wronskian of this set is

$$W(x)=\begin{vmatrix}e^{2x} & e^{-2x} & \cos 2x & \sin 2x\\ 2e^{2x} & -2e^{-2x} & -2\sin 2x & 2\cos 2x\\ 4e^{2x} & 4e^{-2x} & -4\cos 2x & -4\sin 2x\\ 8e^{2x} & -8e^{-2x} & 8\sin 2x & -8\cos 2x\end{vmatrix}.$$

Since

$$W(0) = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 4 & 4 & -4 & 0 \\ 8 & -8 & 0 & -8 \end{vmatrix} = -512,$$

$\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$ is linearly independent, and

$$y_1 = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$$

is the general solution of Equation 14.2.11.

It is known from algebra that every polynomial

$$p(r) = a_0 r^n + a_1 r^{n-1} + \cdots + a_n$$

with real coefficients can be factored as

$$p(r) = a_0 p_1(r) p_2(r) \cdots p_k(r),$$

where no pair of the polynomials p_1, p_2, …, p_k has a commom factor and each is either of the form

$$p_j(r) = (r - r_j)^{m_j}, \tag{14.2.12}$$

where r_j is real and m_j is a positive integer, or

$$p_j(r) = \left[(r - \lambda_j)^2 + \omega_j^2\right]^{m_j}, \tag{14.2.13}$$

where λ_j and ω_j are real, $\omega_j \neq 0$, and m_j is a positive integer. If Equation 14.2.12 holds then r_j is a real zero of p, while if Equation 14.2.13 holds then $\lambda + i\omega$ and $\lambda - i\omega$ are complex conjugate zeros of p. In either case, m_j is the **multiplicity** of the zero(s).

By arguments similar to those used in our examples, it can be shown that

$$p(D) = a_0 p_1(D) p_2(D) \cdots p_k(D) \tag{14.2.14}$$

and that the order of the factors on the right can be chosen arbitrarily. Therefore, if $p_j(D)y = 0$ for some j then $p(D)y = 0$. To see this, we simply rewrite Equation 14.2.14 so that $p_j(D)$ is applied first. Therefore the problem of finding solutions of $p(D)y = 0$ with p as in Equation 14.2.14 reduces to finding solutions of each of these equations

$$p_j(D)y = 0, \quad 1 \leq j \leq k,$$

where p_j is a power of a first degree term or of an irreducible quadratic. To find a fundamental set of solutions $\{y_1, y_2, \ldots, y_n\}$ of $p(D)y = 0$, we find fundamental set of solutions of each of the equations and take $\{y_1, y_2, \ldots, y_n\}$ to be the set of all functions in these separate fundamental sets. In **Exercise 9.2.40** we sketch the proof that $\{y_1, y_2, \ldots, y_n\}$ is linearly independent, and therefore a fundamental set of solutions of $p(D)y = 0$.

To apply this procedure to general homogeneous constant coefficient equations, we must be able to find fundamental sets of solutions of equations of the form

$$(D - a)^m y = 0$$

and

$$\left[(D - \lambda)^2 + \omega^2\right]^m y = 0,$$

where m is an arbitrary positive integer. The next two theorems show how to do this.

Theorem 9.2.1

If m is a positive integer, then

$$\{e^{ax}, xe^{ax}, \ldots, x^{m-1}e^{ax}\} \tag{14.2.15}$$

is a fundamental set of solutions of

$$(D-a)^m y = 0. \tag{14.2.16}$$

Proof

We'll show that if

$$f(x) = c_1 + c_2 x + \cdots + c_m x^{m-1}$$

is an arbitrary polynomial of degree $\le m-1$, then $y = e^{ax} f$ is a solution of Equation 14.2.16. First note that if g is any differentiable function then

$$(D-a)e^{ax}g = De^{ax}g - ae^{ax}g = ae^{ax}g + e^{ax}g' - ae^{ax}g,$$

so

$$(D-a)e^{ax}g = e^{ax}g'. \tag{14.2.17}$$

Therefore

$$\begin{aligned}
(D-a)e^{ax}f &= e^{ax}f' && \text{(from (14.2.17) with } g = f\text{)}\\
(D-a)^2e^{ax}f &= (D-a)e^{ax}f' = e^{ax}f'' && \text{(from (14.2.17) with } g = f'\text{)}\\
(D-a)^3e^{ax}f &= (D-a)e^{ax}f'' = e^{ax}f''' && \text{(from (14.2.17) with } g = f''\text{)}\\
&\vdots\\
(D-a)^me^{ax}f &= (D-a)e^{ax}f^{(m-1)} = e^{ax}f^{(m)} && \text{(from (14.2.17) with } g = f^{(m-1)}\text{).}
\end{aligned}$$

Since $f^{(m)} = 0$, the last equation implies that $y = e^{ax}f$ is a solution of Equation 14.2.16 if f is any polynomial of degree $\le m-1$. In particular, each function in Equation 14.2.15 is a solution of Equation 14.2.16. To see that Equation 14.2.15 is linearly independent (and therefore a fundamental set of solutions of Equation 14.2.16), note that if

$$c_1e^{ax} + c_2xe^{ax} + c\cdots + c_{m-1}x^{m-1}e^{ax} = 0$$

for all x in some interval (a,b), then

$$c_1 + c_2x + c\cdots + c_{m-1}x^{m-1} = 0$$

for all x in (a,b). However, we know from algebra that if this polynomial has more than $m-1$ zeros then $c_1 = c_2 = \cdots = c_n = 0$.

Example 9.2.5

Find the general solution of

$$y''' + 3y'' + 3y' + y = 0. \tag{14.2.18}$$

Solution

The characteristic polynomial of Equation 14.2.18 is

$$p(r) = r^3 + 3r^2 + 3r + 1 = (r+1)^3.$$

Therefore Equation 14.2.18 can be written as

$$(D+1)^3 y = 0,$$

so Theorem 9.2.1 implies that the general solution of Equation 14.2.18 is

$$y = e^{-x}(c_1 + c_2x + c_3x^2).$$

The proof of the next theorem is sketched in **Exercise 9.2.41**.

Theorem 9.2.2

If $\omega \neq 0$ and m is a positive integer, then

$$\{e^{\lambda x}\cos\omega x, xe^{\lambda x}\cos\omega x, \quad \dots, x^{m-1}e^{\lambda x}\cos\omega x,$$
$$e^{\lambda x}\sin\omega x, xe^{\lambda x}\sin\omega x, \quad \dots, x^{m-1}e^{\lambda x}\sin\omega x\}$$

is a fundamental set of solutions of

$$[(D-\lambda)^2+\omega^2]^m y = 0.$$

Example 9.2.6

Find the general solution of

$$(D^2+4D+13)^3 y = 0. \tag{14.2.19}$$

Solution

The characteristic polynomial of Equation 14.2.19 is

$$p(r) = (r^2+4r+13)^3 = \left((r+2)^2+9\right)^3.$$

Therefore Equation 14.2.19 can be be written as

$$[(D+2)^2+9]^3 y = 0,$$

so Theorem 9.2.2 implies that the general solution of Equation 14.2.19 is

$$y = (a_1+a_2x+a_3x^2)e^{-2x}\cos 3x + (b_1+b_2x+b_3x^2)e^{-2x}\sin 3x.$$

Example 9.2.7

Find the general solution of

$$y^{(4)}+4y'''+6y''+4y' = 0. \tag{14.2.20}$$

Solution

The characteristic polynomial of Equation 14.2.20 is

$$\begin{aligned} p(r) &= r^4+4r^3+6r^2+4r \\ &= r(r^3+4r^2+6r+4) \\ &= r(r+2)(r^2+2r+2) \\ &= r(r+2)[(r+1)^2+1]. \end{aligned}$$

Therefore Equation 14.2.20 can be written as

$$[(D+1)^2+1](D+2)Dy = 0.$$

Fundamental sets of solutions of

$$\left[(D+1)^2+1\right]y = 0, \quad (D+2)y = 0, \quad \text{and} \quad Dy = 0.$$

are given by

$$\{e^{-x}\cos x, e^{-x}\sin x\}, \quad \{e^{-2x}\}, \quad \text{and} \quad \{1\},$$

respectively. Therefore the general solution of Equation 14.2.20 is

$$y = e^{-x}(c_1\cos x + c_2\sin x) + c_3e^{-2x} + c_4.$$

Example 9.2.8

Find a fundamental set of solutions of

$$[(D+1)^2+1]^2(D-1)^3(D+1)D^2y=0. \tag{14.2.21}$$

Solution

A fundamental set of solutions of Equation 14.2.21 can be obtained by combining fundamental sets of solutions of

- $[(D+1)^2+1]^2y=0$
- $(D-1)^3y=0$
- $(D+1)y=0$
- $D^2y=0$

Fundamental sets of solutions of these equations are given by

- $\{e^{-x}\cos x, xe^{-x}\cos x, e^{-x}\sin x, xe^{-x}\sin x\}$
- $\{e^x, xe^x, x^2e^x\}$
- $\{e^{-x}\}$,
- $\{1, x\}$

respectively. These ten functions form a fundamental set of solutions of Equation 14.2.21.

This page titled 14.2: Higher Order Constant Coefficient Homogeneous Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **9.2: Higher Order Constant Coefficient Homogeneous Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

14.2.1: Higher Order Constant Coefficient Homogeneous Equations (Exercises)

Q9.2.1

In **Exercises 9.2.1-9.2.14** find the general solution.

1. $y''' - 3y'' + 3y' - y = 0$
2. $y^{(4)} + 8y'' - 9y = 0$
3. $y''' - y'' + 16y' - 16y = 0$
4. $2y''' + 3y'' - 2y' - 3y = 0$
5. $y''' + 5y'' + 9y' + 5y = 0$
6. $4y''' - 8y'' + 5y' - y = 0$
7. $27y''' + 27y'' + 9y' + y = 0$
8. $y^{(4)} + y'' = 0$
9. $y^{(4)} - 16y = 0$
10. $y^{(4)} + 12y'' + 36y = 0$
11. $16y^{(4)} - 72y'' + 81y = 0$
12. $6y^{(4)} + 5y''' + 7y'' + 5y' + y = 0$
13. $4y^{(4)} + 12y''' + 3y'' - 13y' - 6y = 0$
14. $y^{(4)} - 4y''' + 7y'' - 6y' + 2y = 0$

Q9.2.2

In **Exercises 9.2.15-9.2.27** solve the initial value problem. Graph the solution for **Exercises 9.2.17-9.2.19** and **9.2.27**.

15. $y''' - 2y'' + 4y' - 8y = 0, \quad y(0) = 2, \quad y'(0) = -2, \; y''(0) = 0$
16. $y''' + 3y'' - y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = 14, \quad y''(0) = -40$
17. $y''' - y'' - y' + y = 0, \quad y(0) = -2, \quad y'(0) = 9, \quad y''(0) = 4$
18. $y''' - 2y' - 4y = 0, \quad y(0) = 6, \quad y'(0) = 3, \quad y''(0) = 22$
19. $3y''' - y'' - 7y' + 5y = 0, \quad y(0) = \frac{14}{5}, \quad y'(0) = 0, \quad y''(0) = 10$
20. $y''' - 6y'' + 12y' - 8y = 0, \quad y(0) = 1, \quad y'(0) = -1, \quad y''(0) = -4$
21. $2y''' - 11y'' + 12y' + 9y = 0, \quad y(0) = 6, \quad y'(0) = 3, \quad y''(0) = 13$
22. $8y''' - 4y'' - 2y' + y = 0, \quad y(0) = 4, \quad y'(0) = -3, \quad y''(0) = -1$
23. $y^{(4)} - 16y = 0, \quad y(0) = 2, \; y'(0) = 2, \; y''(0) = -2, \; y'''(0) = 0$
24. $y^{(4)} - 6y''' + 7y'' + 6y' - 8y = 0, \quad y(0) = -2, \quad y'(0) = -8, \quad y''(0) = -14, \quad y'''(0) = -62$
25. $4y^{(4)} - 13y'' + 9y = 0, \quad y(0) = 1, \quad y'(0) = 3, \quad y''(0) = 1, \quad y'''(0) = 3$
26. $y^{(4)} + 2y''' - 2y'' - 8y' - 8y = 0, \quad y(0) = 5, \quad y'(0) = -2, \quad y''(0) = 6, \quad y'''(0) = 8$
27. $4y^{(4)} + 8y''' + 19y'' + 32y' + 12y = 0, \quad y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -\frac{7}{2}, \quad y'''(0) = \frac{31}{4}$

Q9.2.3

28. Find a fundamental set of solutions of the given equation, and verify that it is a fundamental set by evaluating its Wronskian at $x = 0$.

a. $(D-1)^2(D-2)y = 0$

b. $(D^2+4)(D-3)y=0$
c. $(D^2+2D+2)(D-1)y=0$
d. $D^3(D-1)y=0$
e. $(D^2-1)(D^2+1)y=0$
f. $(D^2-2D+2)(D^2+1)y=0$

Q9.2.4

In **Exercises 9.2.29-9.2.38** find a fundamental set of solutions.

29. $(D^2+6D+13)(D-2)^2D^3y=0$

30. $(D-1)^2(2D-1)^3(D^2+1)y=0$

31. $(D^2+9)^3D^2y=0$

32. $(D-2)^3(D+1)^2Dy=0$

33. $(D^2+1)(D^2+9)^2(D-2)y=0$

34. $(D^4-16)^2y=0$

35. $(4D^2+4D+9)^3y=0$

36. $D^3(D-2)^2(D^2+4)^2y=0$

37. $(4D^2+1)^2(9D^2+4)^3y=0$

38. $\left[(D-1)^4-16\right]y=0$

Q9.2.5

39 It can be shown that

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n}(a_j-a_i), \tag{A}$$

where the left side is the **Vandermonde determinant** and the right side is the product of all factors of the form (a_j-a_i) with i and j between 1 and n and $i<j$.>

a. Verify (A) for $n=2$ and $n=3$.
b. Find the Wronskian of $\{e^{a_1x}, \quad e^{a_2x},\dots,e^{a_nx}\}$.

40. A theorem from algebra says that if P_1 and P_2 are polynomials with no common factors then there are polynomials Q_1 and Q_2 such that

$$Q_1P_1+Q_2P_2=1.$$

This implies that

$$Q_1(D)P_1(D)y+Q_2(D)P_2(D)y=y$$

for every function y with enough derivatives for the left side to be defined.

a. Use this to show that if P_1 and P_2 have no common factors and

$$P_1(D)y=P_2(D)y=0$$

then $y=0$.

b. Suppose P_1 and P_2 are polynomials with no common factors. Let u_1, …, u_r be linearly independent solutions of $P_1(D)y=0$ and let v_1, …, v_s be linearly independent solutions of $P_2(D)y=0$. Use (a) to show that $\{u_1,\dots,u_r,\ v_1,\dots,v_s\}$ is a linearly independent

set.

c. Suppose the characteristic polynomial of the constant coefficient equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_ny = 0 \tag{A}$$

has the factorization

$$p(r) = a_0p_1(r)p_2(r)\cdots p_k(r),$$

where each p_j is of the form

$$p_j(r) = (r - r_j)^{n_j} \text{ or } p_j(r) = [(r - \lambda_j)^2 + w_j^2]^{m_j} \quad (\omega_j > 0)$$

and no two of the polynomials p_1, p_2, …, p_k have a common factor. Show that we can find a fundamental set of solutions $\{y_1, y_2, \ldots, y_n\}$ of (A) by finding a fundamental set of solutions of each of the equations

$$p_j(D)y = 0, \quad 1 \le j \le k,$$

and taking $\{y_1, y_2, \ldots, y_n\}$ to be the set of all functions in these separate fundamental sets.

41.

a. Show that if

$$z = p(x)\cos\omega x + q(x)\sin\omega x, \tag{A}$$

where p and q are polynomials of degree $\le k$, then

$$(D^2 + \omega^2)z = p_1(x)\cos\omega x + q_1(x)\sin\omega x,$$

where p_1 and q_1 are polynomials of degree $\le k-1$.

b. Apply (a) m times to show that if z is of the form (A) where p and q are polynomial of degree $\le m-1$, then

$$(D^2 + \omega^2)^m z = 0. \tag{B}$$

c. Use Equation 9.2.17 to show that if $y = e^{\lambda x}z$ then

$$[(D - \lambda)^2 + \omega^2]^m y = e^{\lambda x}(D^2 + \omega^2)^m z.$$

d. Conclude from (b) and (c) that if p and q are arbitrary polynomials of degree $\le m-1$ then

$$y = e^{\lambda x}(p(x)\cos\omega x + q(x)\sin\omega x)$$

is a solution of

$$[(D - \lambda)^2 + \omega^2]^m y = 0. \tag{C}$$

e. Conclude from (d) that the functions

$$\begin{aligned} &e^{\lambda x}\cos\omega x, xe^{\lambda x}\cos\omega x, \quad \ldots, x^{m-1}e^{\lambda x}\cos\omega x, \\ &e^{\lambda x}\sin\omega x, xe^{\lambda x}\sin\omega x, \quad \ldots, x^{m-1}e^{\lambda x}\sin\omega x \end{aligned} \tag{D}$$

are all solutions of (C).

f. Complete the proof of Theorem 9.2.2 by showing that the functions in (D) are linearly independent.

42.

a. Use the trigonometric identities

$$\begin{aligned} \cos(A + B) &= \cos A\cos B - \sin A\sin B \\ \sin(A + B) &= \cos A\sin B + \sin A\cos B \end{aligned}$$

to show that

$$(\cos A + i\sin A)(\cos B + i\sin B) = \cos(A + B) + i\sin(A + B).$$

b. Apply (a) repeatedly to show that if n is a positive integer then

$$\prod_{k=1}^{n}(\cos A_k + i\sin A_k) = \cos(A_1 + A_2 + \cdots + A_n) + i\sin(A_1 + A_2 + \cdots + A_n).$$

c. Infer from (b) that if n is a positive integer then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{A}$$

d. Show that (A) also holds if $n = 0$ or a negative integer. HINT: *Verify by direct calculation that*

$$(\cos\theta + i\sin\theta)^{-1} = (\cos\theta - i\sin\theta).$$

Then replace θ by $-\theta$ in (A).

e. Now suppose n is a positive integer. Infer from (A) that if

$$z_k = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1,$$

and

$$\zeta_k = \cos\left(\frac{(2k+1)\pi}{n}\right) + i\sin\left(\frac{(2k+1)\pi}{n}\right), \quad k = 0, 1, \ldots, n-1,$$

then

$$z_k^n = 1 \quad \text{and} \quad \zeta_k^n = -1, \quad k = 0, 1, \ldots, n-1.$$

(Why don't we also consider other integer values for k?)

f. Let ρ be a positive number. Use (e) to show that

$$z^n - \rho = (z - \rho^{1/n}z_0)(z - \rho^{1/n}z_1)\cdots(z - \rho^{1/n}z_{n-1})$$

and

$$z^n + \rho = (z - \rho^{1/n}\zeta_0)(z - \rho^{1/n}\zeta_1)\cdots(z - \rho^{1/n}\zeta_{n-1}).$$

43. Use (e) of **Exercise 9.2.42** to find a fundamental set of solutions of the given equation.

a. $y''' - y = 0$
b. $y''' + y = 0$
c. $y^{(4)} + 64y = 0$
d. $y^{(6)} - y = 0$
e. $y^{(6)} + 64y = 0$
f. $[(D-1)^6 - 1]\,y = 0$
g. $y^{(5)} + y^{(4)} + y''' + y'' + y' + y = 0$

44. An equation of the form

$$a_0x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + \cdots + a_{n-1}xy' + a_ny = 0, \quad x > 0, \tag{A}$$

where a_0, a_1, …, a_n are constants, is an **Euler** or **equidimensional** equation. Show that if

$$x = e^t \quad \text{and} \quad Y(t) = y(x(t)), \tag{B}$$

then

$$x\frac{dy}{dx} = \frac{dY}{dt}$$
$$x^2\frac{d^2y}{dx^2} = \frac{d^2Y}{dt^2} - \frac{dY}{dt}$$
$$x^3\frac{d^3y}{dx^3} = \frac{d^3Y}{dt^3} - 3\frac{d^2Y}{dt^2} + 2\frac{dY}{dt}.$$

In general, it can be shown that if r is any integer ≥ 2 then

$$x^r \frac{d^r y}{dx^r} = \frac{d^r Y}{dt^r} + A_{1r} \frac{d^{r-1} Y}{dt^{r-1}} + \cdots + A_{r-1,r} \frac{dY}{dt}$$

where A_{1r}, …, $A_{r-1,r}$ are integers. Use these results to show that the substitution (B) transforms (A) into a constant coefficient equation for Y as a function of t.

45. Use **Exercise 9.2.44** to show that a function $y = y(x)$ satisfies the equation

$$a_0 x^3 y''' + a_1 x^2 y'' + a_2 x y' + a_3 y = 0, \tag{A}$$

on $(0, \infty)$ if and only if the function $Y(t) = y(e^t)$ satisfies

$$a_0 \frac{d^3 Y}{dt^3} + (a_1 - 3a_0) \frac{d^2 Y}{dt^2} + (a_2 - a_1 + 2a_0) \frac{dY}{dt} + a_3 Y = 0.$$

Assuming that a_0, a_1, a_2, a_3 are real and $a_0 \neq 0$, find the possible forms for the general solution of (A).

This page titled 14.2.1: Higher Order Constant Coefficient Homogeneous Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **9.2E: Higher Order Constant Coefficient Homogeneous Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

CHAPTER OVERVIEW

15: Linear Systems of Differential Equations

In this Chapter we consider systems of differential equations involving more than one unknown function. Such systems arise in many physical applications.SECTION 10.1 presents examples of physical situations that lead to systems of differential equations. SECTION 10.2 discusses linear systems of differential equations. SECTION 10.3 deals with the basic theory of homogeneous linear systems. SECTIONS 10.4, 10.5, AND 10.6 present the theory of constant coefficient homogeneous systems. SECTION 10.7 presents the method of variation of parameters for nonhomogeneous linear systems.

15.1: Introduction to Systems of Differential Equations

15.1.1: Introduction to Systems of Differential Equations (Exercises)

15.2: Linear Systems of Differential Equations

15.2.1: Linear Systems of Differential Equations (Exercises)

15.3: Basic Theory of Homogeneous Linear Systems

15.3.1: Basic Theory of Homogeneous Linear Systems (Exercises)

15.4: Constant Coefficient Homogeneous Systems I

15.4.1: Constant Coefficient Homogeneous Systems I (Exercises)

Thumbnail: Example spiral source vector field. (CC BY-SA 4.0; Jiří Lebl via Source)

This page titled 15: Linear Systems of Differential Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

15.1: Introduction to Systems of Differential Equations

Many physical situations are modelled by systems of n differential equations in n unknown functions, where $n \geq 2$. The next three examples illustrate physical problems that lead to systems of differential equations. In these examples and throughout this chapter we'll denote the independent variable by t.

Example 10.1.1 : Flow

Tanks T_1 and T_2 contain 100 gallons and 300 gallons of salt solutions, respectively. Salt solutions are simultaneously added to both tanks from external sources, pumped from each tank to the other, and drained from both tanks (Figure 10.1.1). A solution with 1 pound of salt per gallon is pumped into T_1 from an external source at 5 gal/min, and a solution with 2 pounds of salt per gallon is pumped into T_2 from an external source at 4 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 3 gal/min. T_1 is drained at 6 gal/min and T_2 is drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2. Assume that both mixtures are well stirred.

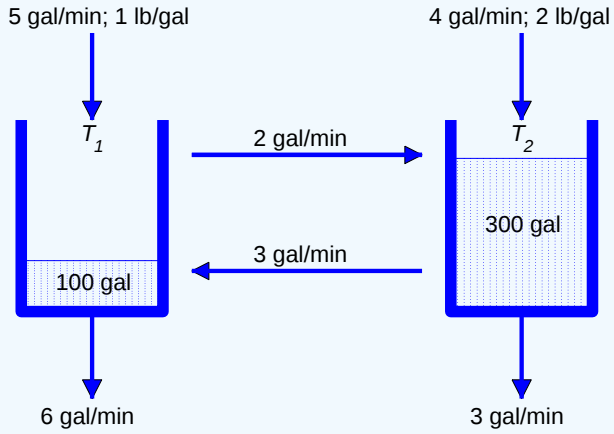


Figure 10.1.1

As in Section 4.2, let **rate in** and **rate out** denote the rates (lb/min) at which salt enters and leaves a tank; thus,

$$Q_1' = (\text{rate in})_1 - (\text{rate out})_1,$$
$$Q_2' = (\text{rate in})_2 - (\text{rate out})_2.$$

Note that the volumes of the solutions in T_1 and T_2 remain constant at 100 gallons and 300 gallons, respectively.

T_1 receives salt from the external source at the rate of

$$(1 \text{ lb/gal}) \times (5 \text{ gal/min}) = 5 \text{ lb/min},$$

and from T_2 at the rate of

$$(\text{lb/gal in } T_2) \times (3 \text{ gal/min}) = \frac{1}{300} Q_2 \times 3$$
$$= \frac{1}{100} Q_2 \text{ lb/min}.$$

Therefore

$$(\text{rate in})_1 = 5 + \frac{1}{100} Q_2. \tag{15.1.1}$$

Solution leaves T_1 at the rate of 8 gal/min, since 6 gal/min are drained and 2 gal/min are pumped to T_2; hence,

$$(\text{rate out})_1 = (\text{ lb/gal in } T_1) \times (8 \text{ gal/min}) = \frac{1}{100} Q_1 \times 8 = \frac{2}{25} Q_1. \tag{15.1.2}$$

Equations 15.1.1 and Equation 15.1.2 imply that

$$Q_1' = 5 + \frac{1}{100}Q_2 - \frac{2}{25}Q_1. \tag{15.1.3}$$

T_2 receives salt from the external source at the rate of

$$(2 \text{ lb/gal}) \times (4 \text{ gal/min}) = 8 \text{ lb/min},$$

and from T_1 at the rate of

$$\begin{aligned}(\text{lb/gal in } T_1) \times (2 \text{ gal/min}) &= \frac{1}{100}Q_1 \times 2 \\ &= \frac{1}{50}Q_1 \text{ lb/min}.\end{aligned}$$

Therefore

$$(\text{rate in})_2 = 8 + \frac{1}{50}Q_1. \tag{15.1.4}$$

Solution leaves T_2 at the rate of 6 gal/min, since 3 gal/min are drained and 3 gal/min are pumped to T_1; hence,

$$(\text{rate out})_2 = (\text{ lb/gal in } T_2) \times (6 \text{ gal/min}) = \frac{1}{300}Q_2 \times 6 = \frac{1}{50}Q_2. \tag{15.1.5}$$

Equations 15.1.4 and 15.1.5 imply that

$$Q_2' = 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2. \tag{15.1.6}$$

We say that Equations 15.1.3 and 15.1.6 form a **system of two first order equations in two unknowns**, and write them together as

$$\begin{aligned}Q_1' &= 5 - \frac{2}{25}Q_1 + \frac{1}{100}Q_2 \\ Q_2' &= 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2.\end{aligned}$$

Example 10.1.2 : Coupled Springs

A mass m_1 is suspended from a rigid support on a spring S_1 and a second mass m_2 is suspended from the first on a spring S_2 (Figure 10.1.2). The springs obey Hooke's law, with spring constants k_1 and k_2. Internal friction causes the springs to exert damping forces proportional to the rates of change of their lengths, with damping constants c_1 and c_2. Let $y_1 = y_1(t)$ and $y_2 = y_2(t)$ be the displacements of the two masses from their equilibrium positions at time t, measured positive upward. Derive a system of differential equations for y_1 and y_2, assuming that the masses of the springs are negligible and that vertical external forces F_1 and F_2 also act on the objects.

Solution

In equilibrium, S_1 supports both m_1 and m_2 and S_2 supports only m_2. Therefore, if $\Delta\ell_1$ and $\Delta\ell_2$ are the elongations of the springs in equilibrium then

$$(m_1 + m_2)g = k_1\Delta\ell_1 \quad \text{and} \quad m_2 g = k_2\Delta\ell_2. \tag{15.1.7}$$

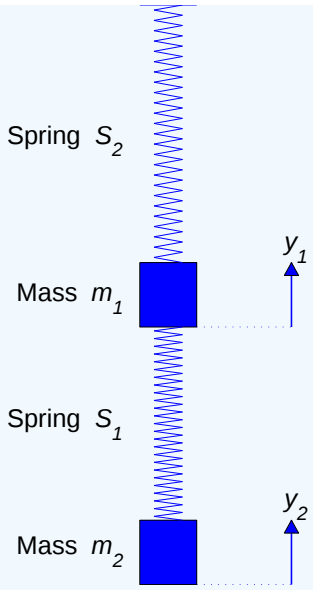


Figure 10.1.2

Let H_1 be the Hooke's law force acting on m_1, and let D_1 be the damping force on m_1. Similarly, let H_2 and D_2 be the Hooke's law and damping forces acting on m_2. According to Newton's second law of motion,

$$\begin{aligned} m_1 y_1'' &= -m_1 g + H_1 + D_1 + F_1, \\ m_2 y_2'' &= -m_2 g + H_2 + D_2 + F_2. \end{aligned} \tag{15.1.8}$$

When the displacements are y_1 and y_2, the change in length of S_1 is $-y_1 + \Delta\ell_1$ and the change in length of S_2 is $-y_2 + y_1 + \Delta\ell_2$. Both springs exert Hooke's law forces on m_1, while only S_2 exerts a Hooke's law force on m_2. These forces are in directions that tend to restore the springs to their natural lengths. Therefore

$$H_1 = k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \quad \text{and} \quad H_2 = k_2(-y_2 + y_1 + \Delta\ell_2). \tag{15.1.9}$$

When the velocities are y_1' and y_2', S_1 and S_2 are changing length at the rates $-y_1'$ and $-y_2' + y_1'$, respectively. Both springs exert damping forces on m_1, while only S_2 exerts a damping force on m_2. Since the force due to damping exerted by a spring is proportional to the rate of change of length of the spring and in a direction that opposes the change, it follows that

$$D_1 = -c_1 y_1' + c_2(y_2' - y_1') \quad \text{and} \quad D_2 = -c_2(y_2' - y_1'). \tag{15.1.10}$$

From Equations 15.1.8, 15.1.9, and 15.1.10,

Missing dimension or its units for m column declaration

and

$$\begin{aligned} m_2 y_2'' &= -m_2 g + k_2(-y_2 + y_1 + \Delta\ell_2) - c_2(y_2' - y_1') + F_2 \\ &= -(m_2 g - k_2\Delta\ell_2) - k_2(y_2 - y_1) - c_2(y_2' - y_1') + F_2. \end{aligned} \tag{15.1.11}$$

From Equation 15.1.7,

$$m_1 g - k_1\Delta\ell_1 + k_2\Delta\ell_2 = -m_2 g + k_2\Delta\ell_2 = 0.$$

Therefore we can rewrite Equation ??? and Equation 15.1.11 as

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2)y_1' + c_2 y_2' - (k_1 + k_2)y_1 + k_2 y_2 + F_1 \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + F_2. \end{aligned}$$

Example 10.1.3 : Gravity

Let $\mathbf{X} = \mathbf{X}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ be the position vector at time t of an object with mass m, relative to a rectangular coordinate system with origin at Earth's center (Figure 10.1.3).

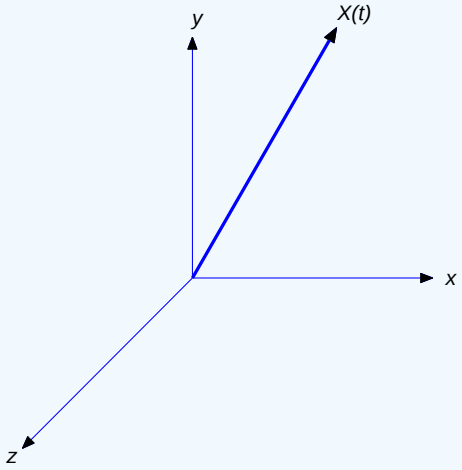


Figure 10.1.3

According to Newton's law of gravitation, Earth's gravitational force $\mathbf{F} = \mathbf{F}(x, y, z)$ on the object is inversely proportional to the square of the distance of the object from Earth's center, and directed toward the center; thus,

$$\mathbf{F} = \frac{\mathbf{K}}{||\mathbf{X}||^2}\left(-\frac{\mathbf{X}}{||\mathbf{X}||}\right) = -\mathbf{K}\frac{\mathbf{x\,i} + \mathbf{y\,j} + \mathbf{z\,k}}{(\mathbf{x^2} + \mathbf{y^2} + \mathbf{z^2})^{3/2}}, \tag{15.1.12}$$

where K is a constant. To determine K, we observe that the magnitude of $\mathbf{F}$ is

$$\|\mathbf{F}\| = K\frac{\|\mathbf{X}\|}{\|\mathbf{X}\|^3} = \frac{K}{\|\mathbf{X}\|^2} = \frac{K}{(x^2 + y^2 + z^2)}.$$

Let R be Earth's radius. Since $\|\mathbf{F}\| = mg$ when the object is at Earth's surface,

$$mg = \frac{K}{R^2}, \quad \text{so} \quad K = mgR^2.$$

Therefore we can rewrite Equation 15.1.12 as

$$\mathbf{F} = -mgR^2\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$

Now suppose $\mathbf{F}$ is the only force acting on the object. According to Newton's second law of motion, $\mathbf{F} = m\mathbf{X}''$; that is,

$$m(x''\,\mathbf{i} + y''\,\mathbf{j} + z''\,\mathbf{k}) = -mgR^2\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$

Cancelling the common factor m and equating components on the two sides of this equation yields the system

$$\begin{aligned} x'' &= -\frac{gR^2x}{(x^2 + y^2 + z^2)^{3/2}} \\ y'' &= -\frac{gR^2y}{(x^2 + y^2 + z^2)^{3/2}} \\ z'' &= -\frac{gR^2z}{(x^2 + y^2 + z^2)^{3/2}}. \end{aligned} \tag{15.1.13}$$

Rewriting Higher Order Systems as First Order Systems

A system of the form

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2, \ldots, y_n) \\ y_2' &= g_2(t, y_1, y_2, \ldots, y_n) \\ &\vdots \\ y_n' &= g_n(t, y_1, y_2, \ldots, y_n) \end{aligned} \tag{15.1.14}$$

is called a **first order system**, since the only derivatives occurring in it are first derivatives. The derivative of each of the unknowns may depend upon the independent variable and all the unknowns, but not on the derivatives of other unknowns. When we wish to emphasize the number of unknown functions in Equation 15.1.14 we will say that Equation 15.1.14 is an $n \times n$ system.

Systems involving higher order derivatives can often be reformulated as first order systems by introducing additional unknowns. The next two examples illustrate this.

Example 10.1.4

Rewrite the system

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2) y_1' + c_2 y_2' - (k_1 + k_2) y_1 + k_2 y_2 + F_1 \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + F_2. \end{aligned} \tag{15.1.15}$$

derived in Example 10.1.2 as a system of first order equations.

Solution

If we define $v_1 = y_1'$ and $v_2 = y_2'$, then $v_1' = y_1''$ and $v_2' = y_2''$, so Equation 15.1.15 becomes

$$\begin{aligned} m_1 v_1' &= -(c_1 + c_2) v_1 + c_2 v_2 - (k_1 + k_2) y_1 + k_2 y_2 + F_1 \\ m_2 v_2' &= c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + F_2. \end{aligned}$$

Therefore $\{y_1, y_2, v_1, v_2\}$ satisfies the 4×4 first order system

$$\begin{aligned} y_1' &= v_1 \\ y_2' &= v_2 \\ v_1' &= \frac{1}{m_1}\left[-(c_1 + c_2) v_1 + c_2 v_2 - (k_1 + k_2) y_1 + k_2 y_2 + F_1\right] \\ v_2' &= \frac{1}{m_2}\left[c_2 v_1 - c_2 v_2 + k_2 y_1 - k_2 y_2 + F_2\right]. \end{aligned} \tag{15.1.16}$$

Note

The difference in form between Equation 15.1.14 and Equation 15.1.16, due to the way in which the unknowns are denoted in the two systems, isn't important; Equation 15.1.16 is a first order system, in that each equation in Equation 15.1.16 expresses the first derivative of one of the unknown functions in a way that does not involve derivatives of any of the other unknowns.

Example 10.1.5

Rewrite the system

$$\begin{aligned} x'' &= f(t, x, x', y, y', y'') \\ y''' &= g(t, x, x', y, y' y'') \end{aligned}$$

as a first order system.

Solution

We regard x, x', y, y', and y'' as unknown functions, and rename them

$$x = x_1,\ x' = x_2, \quad y = y_1, \quad y' = y_2, \quad y'' = y_3.$$

These unknowns satisfy the system

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= f(t, x_1, x_2, y_1, y_2, y_3) \\ y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= g(t, x_1, x_2, y_1, y_2, y_3). \end{aligned}$$

Rewriting Scalar Differential Equations as Systems

In this chapter we'll refer to differential equations involving only one unknown function as **scalar** differential equations. Scalar differential equations can be rewritten as systems of first order equations by the method illustrated in the next two examples.

Example 10.1.6

Rewrite the equation

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0 \tag{15.1.17}$$

as a 4×4 first order system.

Solution

We regard y, y', y'', and y''' as unknowns and rename them

$$y = y_1, \quad y' = y_2, \quad y'' = y_3, \quad \text{and} \quad y''' = y_4.$$

Then $y^{(4)} = y_4'$, so Equation 15.1.17 can be written as

$$y_4' + 4y_4 + 6y_3 + 4y_2 + y_1 = 0.$$

Therefore $\{y_1, y_2, y_3, y_4\}$ satisfies the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= -4y_4 - 6y_3 - 4y_2 - y_1. \end{aligned}$$

Example 10.1.7

Rewrite

$$x''' = f(t, x, x', x'')$$

as a system of first order equations.

Solution

We regard x, x', and x'' as unknowns and rename them

$$x = y_1, \quad x' = y_2, \quad \text{and} \quad x'' = y_3.$$

Then

$$y_1' = x' = y_2, \quad y_2' = x'' = y_3, \quad \text{and} \quad y_3' = x'''.$$

Therefore $\{y_1, y_2, y_3\}$ satisfies the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= f(t, y_1, y_2, y_3). \end{aligned}$$

Since systems of differential equations involving higher derivatives can be rewritten as first order systems by the method used in Examples 10.1.5 -10.1.7 , we'll consider only first order systems.

Numerical Solution of Systems

The numerical methods that we studied in Chapter 3 can be extended to systems, and most differential equation software packages include programs to solve systems of equations. We will not go into detail on numerical methods for systems; however, for illustrative purposes we'll describe the Runge-Kutta method for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20} \end{aligned}$$

at equally spaced points t_0, t_1, …, $t_n = b$ in an interval $[t_0, b]$. Thus,

$$t_i = t_0 + ih, \quad i = 0, 1, \dots, n,$$

where

$$h = \frac{b - t_0}{n}.$$

We'll denote the approximate values of y_1 and y_2 at these points by $y_{10}, y_{11}, \dots, y_{1n}$ and $y_{20}, y_{21}, \dots, y_{2n}$. The Runge-Kutta method computes these approximate values as follows: given y_{1i} and y_{2i}, compute

$$\begin{aligned} I_{1i} &= g_1(t_i, y_{1i}, y_{2i}), \\ J_{1i} &= g_2(t_i, y_{1i}, y_{2i}), \\ I_{2i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{1i}, y_{2i} + \frac{h}{2} J_{1i}\right), \\ J_{2i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{1i}, y_{2i} + \frac{h}{2} J_{1i}\right), \\ I_{3i} &= g_1\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{2i}, y_{2i} + \frac{h}{2} J_{2i}\right), \\ J_{3i} &= g_2\left(t_i + \frac{h}{2}, y_{1i} + \frac{h}{2} I_{2i}, y_{2i} + \frac{h}{2} J_{2i}\right), \\ I_{4i} &= g_1(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}), \\ J_{4i} &= g_2(t_i + h, y_{1i} + hI_{3i}, y_{2i} + hJ_{3i}), \end{aligned}$$

and

$$\begin{aligned} y_{1,i+1} &= y_{1i} + \frac{h}{6}(I_{1i} + 2I_{2i} + 2I_{3i} + I_{4i}), \\ y_{2,i+1} &= y_{2i} + \frac{h}{6}(J_{1i} + 2J_{2i} + 2J_{3i} + J_{4i}) \end{aligned}$$

for $i = 0$, …, $n - 1$. Under appropriate conditions on g_1 and g_2, it can be shown that the global truncation error for the Runge-Kutta method is $O(h^4)$, as in the scalar case considered in Section 3.3.

This page titled 15.1: Introduction to Systems of Differential Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **10.1: Introduction to Systems of Differential Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

15.1.1: Introduction to Systems of Differential Equations (Exercises)

Q10.1.1

1. Tanks T_1 and T_2 contain 50 gallons and 100 gallons of salt solutions, respectively. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 1 gal/min, and a solution with 3 pounds of salt per gallon is pumped into T_2 from an external source at 2 gal/min. The solution from T_1 is pumped into T_2 at 3 gal/min, and the solution from T_2 is pumped into T_1 at 4 gal/min. T_1 is drained at 2 gal/min and T_2 is drained at 1 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2. Assume that both mixtures are well stirred.

2. Two 500 gallon tanks T_1 and T_2 initially contain 100 gallons each of salt solution. A solution with 2 pounds of salt per gallon is pumped into T_1 from an external source at 6 gal/min, and a solution with 1 pound of salt per gallon is pumped into T_2 from an external source at 5 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 1 gal/min. Both tanks are drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2, respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2 that's valid until a tank is about to overflow. Assume that both mixtures are well stirred.

3. A mass m_1 is suspended from a rigid support on a spring S_1 with spring constant k_1 and damping constant c_1. A second mass m_2 is suspended from the first on a spring S_2 with spring constant k_2 and damping constant c_2, and a third mass m_3 is suspended from the second on a spring S_3 with spring constant k_3 and damping constant c_3. Let $y_1 = y_1(t)$, $y_2 = y_2(t)$, and $y_3 = y_3(t)$ be the displacements of the three masses from their equilibrium positions at time t, measured positive upward. Derive a system of differential equations for y_1, y_2 and y_3, assuming that the masses of the springs are negligible and that vertical external forces F_1, F_2, and F_3 also act on the masses.

4. Let $\mathbf{X} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ be the position vector of an object with mass m, expressed in terms of a rectangular coordinate system with origin at Earth's center (Figure 10.1.3). Derive a system of differential equations for x, y, and z, assuming that the object moves under Earth's gravitational force (given by Newton's law of gravitation, as in Example 10.1.3) and a resistive force proportional to the speed of the object. Let α be the constant of proportionality.

5. Rewrite the given system as a first order system.

a.
$$x''' = f(t, x, y, y')$$
$$y'' = g(t, y, y')$$
$$u' = f(t, u, v, v', w')$$

b.
$$v'' = g(t, u, v, v', w)$$
$$w'' = h(t, u, v, v', w, w')$$

c. $y''' = f(t, y, y', y'')$

d. $y^{(4)} = f(t, y)$

e.
$$x'' = f(t, x, y)$$
$$y'' = g(t, x, y)$$

6. Rewrite the system Equation 10.1.14 of differential equations derived in Example 10.1.3 as a first order system.

7. Formulate a version of Euler's method (Section 3.1) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20}, \end{aligned}$$

on an interval $[t_0, b]$.

8. Formulate a version of the improved Euler method (Section 3.2) for the numerical solution of the initial value problem

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2), \quad y_1(t_0) = y_{10}, \\ y_2' &= g_2(t, y_1, y_2), \quad y_2(t_0) = y_{20}, \end{aligned}$$

on an interval $[t_0, b]$.

This page titled 15.1.1: Introduction to Systems of Differential Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **10.1E: Introduction to Systems of Differential Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

15.2: Linear Systems of Differential Equations

A first order system of differential equations that can be written in the form

$$\begin{aligned} y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \cdots + a_{1n}(t)y_n + f_1(t) \\ y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \cdots + a_{2n}(t)y_n + f_2(t) \\ &\vdots \\ y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \cdots + a_{nn}(t)y_n + f_n(t) \end{aligned} \tag{15.2.1}$$

is called a **linear system**.

The linear system Equation 15.2.1 can be written in matrix form as

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t)_{12}(t) & \cdots_{1n}(t) \\ a_{21}(t)_{22}(t) & \cdots_{2n}(t) \\ \vdots & \vdots \quad \ddots \quad \vdots \\ a_{n1}(t)_{n2}(t) & \cdots_{nn}(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix},$$

or more briefly as

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \tag{15.2.2}$$

where

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \mathbf{A(t)} = \begin{bmatrix} a_{11}(t)_{12}(t) & \cdots_{1n}(t) \\ a_{21}(t)_{22}(t) & \cdots_{2n}(t) \\ \vdots & \vdots \quad \ddots \quad \vdots \\ a_{n1}(t)_{n2}(t) & \cdots_{nn}(t) \end{bmatrix}, \quad \text{and} \quad \mathbf{f(t)} = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}.$$

We call A the **coefficient matrix** of Equation 15.2.2 and $\mathbf{f}$ the **forcing function**. We'll say that A and $\mathbf{f}$ are **continuous** if their entries are continuous. If $\mathbf{f} = \mathbf{0}$, then Equation 15.2.2 is **homogeneous**; otherwise, Equation 15.2.2 is **nonhomogeneous**.

An initial value problem for Equation 15.2.2 consists of finding a solution of Equation 15.2.2 that equals a given constant vector

$$\mathbf{k} = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}.$$

at some initial point t_0. We write this initial value problem as

$$\mathbf{y}' = \mathbf{A(t)y} + \mathbf{f(t)}, \quad \mathbf{y(t_0) = k}.$$

The next theorem gives sufficient conditions for the existence of solutions of initial value problems for Equation 15.2.2. We omit the proof.

Theorem 10.2.1 : Existence

Suppose the coefficient matrix A and the forcing function $\mathbf{f}$ are continuous on (a, b), let t_0 be in (a, b), and let $\mathbf{k}$ be an arbitrary constant n-vector. Then the initial value problem

$$\mathbf{y}' = \mathbf{A(t)y} + \mathbf{f(t)}, \quad \mathbf{y(t_0) = k}$$

has a unique solution on (a, b).

Example 10.2.1

a. Write the system

$$\begin{aligned} y_1' &= y_1 + 2y_2 + 2e^{4t} \\ y_2' &= 2y_1 + y_2 + e^{4t} \end{aligned} \tag{15.2.3}$$

in matrix form and conclude from Theorem 10.2.1 that every initial value problem for Equation 15.2.3 has a unique solution on $(-\infty, \infty)$.

b. Verify that

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \tag{15.2.4}$$

is a solution of Equation 15.2.3 for all values of the constants c_1 and c_2.

c. Find the solution of the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{y}(0) = \frac{1}{5}\begin{bmatrix} 3 \\ 22 \end{bmatrix}. \tag{15.2.5}$$

Solution a

The system Equation 15.2.3 can be written in matrix form as

$$\mathbf{y}' = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}.$$

An initial value problem for Equation 15.2.3 can be written as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad y(t_0) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

Since the coefficient matrix and the forcing function are both continuous on $(-\infty, \infty)$, Theorem 10.2.1 implies that this problem has a unique solution on $(-\infty, \infty)$.

Solution b

If $\mathbf{y}$ is given by Equation 15.2.4, then

$$\begin{aligned} A\mathbf{y} + \mathbf{f} &= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5}\begin{bmatrix} 22 \\ 23 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5}\begin{bmatrix} 32 \\ 28 \end{bmatrix} e^{4t} + 3c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \\ &= \mathbf{y}'. \end{aligned}$$

Solution c

We must choose c_1 and c_2 in Equation 15.2.4 so that

$$\frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 22 \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Solving this system yields $c_1 = 1$, $c_2 = -2$, so

$$\mathbf{y} = \frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

is the solution of Equation 15.2.5.

Note

The theory of $n \times n$ linear systems of differential equations is analogous to the theory of the scalar n-th order equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y = F(t) \tag{15.2.6}$$

as developed in Sections 9.1. For example by rewriting Equation 15.2.6 as an equivalent linear system it can be shown that Theorem 10.2.1 implies Theorem 9.1.1.

This page titled 15.2: Linear Systems of Differential Equations is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **10.2: Linear Systems of Differential Equations** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

15.2.1: Linear Systems of Differential Equations (Exercises)

Q10.2.1

1. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 and c_2.

a. $\begin{aligned} y_1' &= 2y_1 + 4y_2 \\ y_2' &= 4y_1 + 2y_2; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$

b. $\begin{aligned} y_1' &= -2y_1 - 2y_2 \\ y_2' &= -5y_1 + y_2; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} e^{3t}$

c. $\begin{aligned} y_1' &= -4y_1 - 10y_2 \\ y_2' &= 3y_1 + 7y_2; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} -5 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{t}$

d. $\begin{aligned} y_1' &= 2y_1 + y_2 \\ y_2' &= y_1 + 2y_2; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$

2. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1, c_2, and c_3.

a. $\begin{aligned} y_1' &= -y_1 + 2y_2 + 3y_3 \\ y_2' &= y_2 + 6y_3 \\ y_3' &= -2y_3; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-2t}$

b. $\begin{aligned} y_1' &= 2y_2 + 2y_3 \\ y_2' &= 2y_1 + 2y_3 \\ y_3' &= 2y_1 + 2y_2; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{4t}$

c. $\begin{aligned} y_1' &= -y_1 + 2y_2 + 2y_3 \\ y_2' &= 2y_1 - y_2 + 2y_3 \\ y_3' &= 2y_1 + 2y_2 - y_3; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t}$

d. $\begin{aligned} y_1' &= 3y_1 - y_2 - y_3 \\ y_2' &= -2y_1 + 3y_2 + 2y_3 \\ y_3' &= 4y_1 - y_2 - 2y_3; \end{aligned}$ $\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$

3. Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

a. $\begin{aligned} y_1' &= y_1 + y_2 \quad y_1(0) = 1 \\ y_2' &= -2y_1 + 4y_2, \quad y_2(0) = 0; \end{aligned}$ $\mathbf{y} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$

b. $\begin{aligned} y_1' &= 5y_1 + 3y_2 \quad y_1(0) = 12 \\ y_2' &= -y_1 + y_2, \quad y_2(0) = -6; \end{aligned}$ $\mathbf{y} = 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + 3 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{4t}$

4. Rewrite the initial value problem in matrix form and verify that the given vector function is a solution.

a.
$$\begin{aligned} y_1' &= 6y_1 + 4y_2 + 4y_3 & y_1(0) &= 3 \\ y_2' &= -7y_1 - 2y_2 - y_3, , & y_2(0) &= -6 \\ y_3' &= 7y_1 + 4y_2 + 3y_3 & y_3(0) &= 4 \end{aligned}$$
$$\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{6t} + 2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{-t}$$

b.
$$\begin{aligned} y_1' &= 8y_1 + 7y_2 + 7y_3 & y_1(0) &= 2 \\ y_2' &= -5y_1 - 6y_2 - 9y_3, & y_2(0) &= -4 \\ y_3' &= 5y_1 + 7y_2 + 10y_3, & y_3(0) &= 3 \end{aligned}$$
$$\mathbf{y} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{8t} + \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{t}$$

5. Rewrite the system in matrix form and verify that the given vector function satisfies the system for any choice of the constants c_1 and c_2.

a.
$$\begin{aligned} y_1' &= -3y_1 + 2y_2 + 3 - 2t \\ y_2' &= -5y_1 + 3y_2 + 6 - 3t \end{aligned}$$
$$\mathbf{y} = c_1 \begin{bmatrix} 2\cos t \\ 3\cos t - \sin t \end{bmatrix} + c_2 \begin{bmatrix} 2\sin t \\ 3\sin t + \cos t \end{bmatrix} + \begin{bmatrix} 1 \\ t \end{bmatrix}$$

b.
$$\begin{aligned} y_1' &= 3y_1 + y_2 - 5e^t \\ y_2' &= -y_1 + y_2 + e^t \end{aligned}$$
$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1+t \\ -t \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{t}$$

c.
$$\begin{aligned} y_1' &= -y_1 - 4y_2 + 4e^t + 8te^t \\ y_2' &= -y_1 - y_2 + e^{3t} + (4t+2)e^t \end{aligned}$$
$$\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} e^{3t} \\ 2te^t \end{bmatrix}$$

d.
$$\begin{aligned} y_1' &= -6y_1 - 3y_2 + 14e^{2t} + 12e^t \\ y_2' &= y_1 - 2y_2 + 7e^{2t} - 12e^t \end{aligned}$$
$$\mathbf{y} = c_1 \begin{bmatrix} -3 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + \begin{bmatrix} e^{2t} + 3e^t \\ 2e^{2t} - 3e^t \end{bmatrix}$$

6. Convert the linear scalar equation

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_n(t)y(t) = F(t) \tag{A}$$

into an equivalent $n \times n$ system

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t),$$

and show that A and $\mathbf{f}$ are continuous on an interval (a, b) if and only if (A) is normal on (a, b).

7. A matrix function

$$Q(t) = \begin{bmatrix} q_{11}(t) & q_{12}(t) & \cdots & q_{1s}(t) \\ q_{21}(t) & q_{22}(t) & \cdots & q_{2s}(t) \\ \vdots & \vdots & \ddots & \vdots \\ q_{r1}(t) & q_{r2}(t) & \cdots & q_{rs}(t) \end{bmatrix}$$

is said to be **differentiable** if its entries $\{q_{ij}\}$ are differentiable. Then the **derivative** Q' is defined by

$$Q(t) = \begin{bmatrix} q_{11}'(t) & q_{12}'(t) & \cdots & q_{1s}'(t) \\ q_{21}'(t) & q_{22}'(t) & \cdots & q_{2s}'(t) \\ \vdots & \vdots & \ddots & \vdots \\ q_{r1}'(t) & q_{r2}'(t) & \cdots & q_{rs}'(t) \end{bmatrix}$$

a. Prove: If P and Q are differentiable matrices such that $P+Q$ is defined and if c_1 and c_2 are constants, then

$$(c_1P+c_2Q)' = c_1P' + c_2Q'.$$

b. Prove: If P and Q are differentiable matrices such that PQ is defined, then

$$(PQ)' = P'Q + PQ'.$$

8. Verify that $Y' = AY$.

a. $Y = \begin{bmatrix} e^{6t} & e^{-2t} \\ e^{6t} & -e^{-2t} \end{bmatrix}$, $A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$

b. $Y = \begin{bmatrix} e^{-4t} & -2e^{3t} \\ e^{-4t} & 5e^{3t} \end{bmatrix}$, $A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}$

c. $Y = \begin{bmatrix} -5e^{2t} & 2e^{t} \\ 3e^{2t} & -e^{t} \end{bmatrix}$, $A = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix}$

d. $Y = \begin{bmatrix} e^{3t} & e^{t} \\ e^{3t} & -e^{t} \end{bmatrix}$, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

e. $Y = \begin{bmatrix} e^{t} & e^{-t} & e^{-2t} \\ e^{t} & 0 & -2e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}$, $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix}$

f. $Y = \begin{bmatrix} -e^{-2t} & -e^{-2t} & e^{4t} \\ 0 & e^{-2t} & e^{4t} \\ e^{-2t} & 0 & e^{4t} \end{bmatrix}$, $A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$

g. $Y = \begin{bmatrix} e^{3t} & e^{-3t} & 0 \\ e^{3t} & 0 & -e^{-3t} \\ e^{3t} & e^{-3t} & e^{-3t} \end{bmatrix}$, $A = \begin{bmatrix} -9 & 6 & 6 \\ -6 & 3 & 6 \\ -6 & 6 & 3 \end{bmatrix}$

h. $Y = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix}$, $A = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix}$

9. Suppose

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix}$$

are solutions of the homogeneous system

$$\mathbf{y}' = A(t)\mathbf{y}, \tag{A}$$

and define

$$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}.$$

a. Show that $Y' = AY$.

b. Show that if $\mathbf{c}$ is a constant vector then $\mathbf{y} = Y\mathbf{c}$ is a solution of (A).

c. State generalizations of (a) and (b) for $n \times n$ systems.

10. Suppose Y is a differentiable square matrix.

a. Find a formula for the derivative of Y^2.

b. Find a formula for the derivative of Y^n, where n is any positive integer.

c. State how the results obtained in (a) and (b) are analogous to results from calculus concerning scalar functions.

11. It can be shown that if Y is a differentiable and invertible square matrix function, then Y^{-1} is differentiable.

a. Show that $(Y^{-1})' = -Y^{-1}Y'Y^{-1}$. (Hint: Differentiate the identity $Y^{-1}Y = I$.)

b. Find the derivative of $Y^{-n} = (Y^{-1})^n$, where n is a positive integer.
c. State how the results obtained in (a) and (b) are analogous to results from calculus concerning scalar functions.

12. Show that Theorem 10.2.1 implies Theorem 9.1.1. HINT: **Write the scalar function**

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = F(x)$$

as an $n \times n$ system of linear equations.

13. Suppose $\mathbf{y}$ is a solution of the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), and that the $n \times n$ matrix P is invertible and differentiable on (a, b). Find a matrix B such that the function $\mathbf{x} = P\mathbf{y}$ is a solution of $\mathbf{x}' = B\mathbf{x}$ on (a, b).

This page titled 15.2.1: Linear Systems of Differential Equations (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **10.2E: Linear Systems of Differential Equations (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

15.3: Basic Theory of Homogeneous Linear Systems

In this section we consider homogeneous linear systems $\mathbf{y}' = A(t)\mathbf{y}$, where $A = A(t)$ is a continuous $n \times n$ matrix function on an interval (a, b). The theory of linear homogeneous systems has much in common with the theory of linear homogeneous scalar equations, which we considered in Sections 2.1, 5.1, and 9.1.

Whenever we refer to solutions of $\mathbf{y}' = A(t)\mathbf{y}$ we'll mean solutions on (a, b). Since $\mathbf{y} \equiv \mathbf{0}$ is obviously a solution of $\mathbf{y}' = A(t)\mathbf{y}$, we call it the **trivial** solution. Any other solution is **nontrivial**.

If $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ are vector functions defined on an interval (a, b) and $c_1, c_2, \ldots, c_n$ are constants, then

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n \tag{15.3.1}$$

is a **linear combination of** $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$. It's easy show that if $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), then so is any linear combination of $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$. We say that $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is a **fundamental set of solutions of** $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) on if every solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) can be written as a linear combination of $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$, as in Equation 15.3.1. In this case we say that Equation 15.3.1 is the **general solution of** $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

It can be shown that if A is continuous on (a, b) then $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on (a, b). The next definition will help to characterize fundamental sets of solutions of $\mathbf{y}' = A(t)\mathbf{y}$.

We say that a set $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of n-vector functions is **linearly independent** on (a, b) if the only constants $c_1, c_2, \ldots, c_n$ such that

$$c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) = 0, \quad a < t < b, \tag{15.3.2}$$

are $c_1 = c_2 = \cdots = c_n = 0$. If Equation 15.3.2 holds for some set of constants $c_1, c_2, \ldots, c_n$ that are not all zero, then $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is **linearly dependent** on (a, b)

The next theorem is analogous to Theorems 5.1.3 and 9.1.2.

Theorem 10.3.1

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b). Then a set $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ of n solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is a fundamental set if and only if it is linearly independent on (a, b).

Example 10.3.1

Show that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix}$$

are linearly independent on every interval (a, b).

Solution

Suppose

$$c_1\begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix} + c_2\begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix} + c_3\begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

We must show that $c_1 = c_2 = c_3 = 0$. Rewriting this equation in matrix form yields

$$\begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

Expanding the determinant of this system in cofactors of the entries of the first row yields

$$\begin{vmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} = e^t \begin{vmatrix} e^{3t} & e^{3t} \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 0 \end{vmatrix} + e^{2t} \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 1 \end{vmatrix}$$

$$= e^t(-e^{3t}) + e^{2t}(-e^{2t}) = -2e^{4t}.$$

Since this determinant is never zero, $c_1 = c_2 = c_3 = 0$.

We can use the method in Example 10.3.1 to test n solutions $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ of any $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$ for linear independence on an interval (a, b) on which A is continuous. To explain this (and for other purposes later), it is useful to write a linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ in a different way. We first write the vector functions in terms of their components as

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix}, \dots, \quad \mathbf{y}_n = \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}.$$

If

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$$

then

$$\mathbf{y} = c_1 \begin{bmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{bmatrix} + c_2 \begin{bmatrix} y_{12} \\ y_{22} \\ \vdots \\ y_{n2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} y_{1n} \\ y_{2n} \\ \vdots \\ y_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

This shows that

$$c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n = Y\mathbf{c}, \tag{15.3.3}$$

where

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

and

$$Y = [\mathbf{y}_1\ \mathbf{y}_2\ \cdots\ \mathbf{y}_n] = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}; \tag{15.3.4}$$

that is, the columns of Y are the vector functions $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$.

For reference below, note that

$$\begin{aligned} Y' &= [\mathbf{y}_1' \; \mathbf{y}_2' \; \cdots \; \mathbf{y}_n'] \\ &= [A\mathbf{y}_1 \; A\mathbf{y}_2 \; \cdots \; A\mathbf{y}_n] \\ &= A[\mathbf{y}_1 \; \mathbf{y}_2 \; \cdots \; \mathbf{y}_n] = AY; \end{aligned}$$

that is, Y satisfies the matrix differential equation

$$Y' = AY.$$

The determinant of Y,

$$W = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix} \tag{15.3.5}$$

is called the **Wronskian** of $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$. It can be shown (**Exercises 10.3.2** and **10.3.3**) that this definition is analogous to definitions of the Wronskian of scalar functions given in Sections 5.1 and 9.1. The next theorem is analogous to Theorems 5.1.4 and 9.1.3. The proof is sketched in **Exercise 10.3.4** for $n = 2$ and in **Exercise 10.3.5** for general n.

Theorem 10.3.2 : Abel's Formula

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b), let $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), and let t_0 be in (a, b). Then the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is given by

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} [a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)]\, ds\right), \quad a < t < b. \tag{15.3.6}$$

Therefore, either W has no zeros in (a, b) or $W \equiv 0$ on (a, b).

Note

The sum of the diagonal entries of a square matrix A is called the **trace** of A, denoted by $\mathrm{tr}(A)$. Thus, for an $n \times n$ matrix A,

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots a_{nn},$$

and Equation 15.3.6 can be written as

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \mathrm{tr}(A(s))\, ds\right), \quad a < t < b.$$

The next theorem is analogous to Theorems 5.1.6 and 9.1.4.

Theorem 10.3.3

Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) and let $\mathbf{y}_1$, $\mathbf{y}_2$, …,$\mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b). Then the following statements are equivalent; that is, they are either all true or all false:

a. The general solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \cdots + c_n\mathbf{y}_n$, where c_1, c_2, …, c_n are arbitrary constants.
b. $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).
c. $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is linearly independent on (a, b).
d. The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is nonzero at some point in (a, b).
e. The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n\}$ is nonzero at all points in (a, b).

We say that Y in Equation 15.3.4 is a **fundamental matrix** for $\mathbf{y}' = A(t)\mathbf{y}$ if any (and therefore all) of the statements (**a**)-(**e**) of Theorem 10.3.2 are true for the columns of Y. In this case, Equation 15.3.3 implies that the general solution of $\mathbf{y}' = A(t)\mathbf{y}$ can be written as $\mathbf{y} = Y\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant n-vector.

Example 10.3.2

The vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

are solutions of the constant coefficient system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y} \tag{15.3.7}$$

on $(-\infty, \infty)$. (Verify.)

a. Compute the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$ directly from the definition Equation 15.3.5
b. Verify Abel's formula Equation 15.3.6 for the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.
c. Find the general solution of Equation 15.3.7.
d. Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}. \tag{15.3.8}$$

Solution a

From Equation 15.3.5

$$W(t) = \begin{vmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{vmatrix} = e^{2t}e^{-t} \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = e^t. \tag{15.3.9}$$

Solution b

Here

$$A = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix},$$

so $\operatorname{tr}(A) = -4 + 5 = 1$. If t_0 is an arbitrary real number then Equation 15.3.6 implies that

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} 1\, ds\right) = \begin{vmatrix} -e^{2t_0} & -e^{-t_0} \\ 2e^{2t_0} & e^{-t_0} \end{vmatrix} e^{(t-t_0)} = e^{t_0}e^{t-t_0} = e^t,$$

which is consistent with Equation 15.3.9.

Solution c

Since $W(t) \neq 0$, Theorem 10.3.3 implies that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions of Equation 15.3.7 and

$$Y = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix for Equation 15.3.7. Therefore the general solution of Equation 15.3.7 is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1 \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{15.3.10}$$

Solution d

Setting $t = 0$ in Equation 15.3.10 and imposing the initial condition in Equation 15.3.8 yields

$$c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}.$$

Thus,

$$\begin{aligned} -c_1 - c_2 &= \quad 4 \\ 2c_1 + c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = -1$, $c_2 = -3$. Substituting these values into Equation 15.3.10 yields

$$\mathbf{y} = -\begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} - 3\begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} e^{2t} + 3e^{-t} \\ -2e^{2t} - 3e^{-t} \end{bmatrix}$$

as the solution of Equation 15.3.8.

This page titled 15.3: Basic Theory of Homogeneous Linear Systems is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **10.3: Basic Theory of Homogeneous Linear Systems** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

15.3.1: Basic Theory of Homogeneous Linear Systems (Exercises)

Q10.3.1

1. Prove: If $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ are solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b), then any linear combination of $\mathbf{y}_1$, $\mathbf{y}_2$, …, $\mathbf{y}_n$ is also a solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

2. In Section 5.1 the Wronskian of two solutions y_1 and y_2 of the scalar second order equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \tag{A}$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

a. Rewrite (A) as a system of first order equations and show that W is the Wronskian (as defined in this section) of two solutions of this system.

b. Apply Equation 10.3.6 to the system derived in (a), and show that

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x} \frac{P_1(s)}{P_0(s)}\,ds\right\},$$

which is the form of Abel's formula given in Theorem 9.1.3.

3. In Section 9.1 the Wronskian of n solutions y_1, y_2, …, y_n of the $n-$th order equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = 0 \tag{A}$$

was defined to be

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

a. Rewrite (A) as a system of first order equations and show that W is the Wronskian (as defined in this section) of n solutions of this system.

b. Apply Equation 10.3.6 to the system derived in (a), and show that

$$W(x) = W(x_0)\exp\left\{-\int_{x_0}^{x} \frac{P_1(s)}{P_0(s)}\,ds\right\},$$

which is the form of Abel's formula given in Theorem 9.1.3.

4. Suppose

$$\mathbf{y}_1 = \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix}$$

are solutions of the 2×2 system $\mathbf{y}' = A\mathbf{y}$ on (a, b), and let

$$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \quad \text{and} \quad W = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix}$$

thus, W is the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.

a. Deduce from the definition of determinant that

$$W' = \begin{vmatrix} y_{11}' & y_{12}' \\ y_{21} & y_{22} \end{vmatrix} + \begin{vmatrix} y_{11} & y_{12} \\ y_{21}' & y_{22}' \end{vmatrix}.$$

b. Use the equation $Y' = A(t)Y$ and the definition of matrix multiplication to show that

$$[y_{11}' \quad y_{12}'] = a_{11}[y_{11} \quad y_{12}] + a_{12}[y_{21} \quad y_{22}]$$

and

$$[y_{21}' \quad y_{22}'] = a_{21}[y_{11} \quad y_{12}] + a_{22}[y_{21} \quad y_{22}].$$

c. Use properties of determinants to deduce from (a) and (a) that

$$\begin{vmatrix} y_{11}' & y_{12}' \\ y_{21} & y_{22} \end{vmatrix} = a_{11}W \quad \text{and} \quad \begin{vmatrix} y_{11} & y_{12} \\ y_{21}' & y_{22}' \end{vmatrix} = a_{22}W.$$

d. Conclude from (c) that

$$W' = (a_{11} + a_{22})W,$$

and use this to show that if $a < t_0 < b$ then

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} [a_{11}(s) + a_{22}(s)]\, ds\right) \quad a < t < b.$$

5. Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b). Let

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix},$$

where the columns of Y are solutions of $\mathbf{y}' = A(t)\mathbf{y}$. Let

$$r_i = [y_{i1}\, y_{i2} \cdots y_{in}]$$

be the ith row of Y, and let W be the determinant of Y.

a. Deduce from the definition of determinant that

$$W' = W_1 + W_2 + \cdots + W_n,$$

where, for $1 \le m \le n$, the ith row of W_m is r_i if $i \ne m$, and r_m' if $i = m$.

b. Use the equation $Y' = AY$ and the definition of matrix multiplication to show that

$$r_m' = a_{m1}r_1 + a_{m2}r_2 + \cdots + a_{mn}r_n.$$

c. Use properties of determinants to deduce from (b) that

$$\det(W_m) = a_{mm}W.$$

d. Conclude from (a) and (c) that

$$W' = (a_{11} + a_{22} + \cdots + a_{nn})W,$$

and use this to show that if $a < t_0 < b$ then

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} [a_{11}(s) + a_{22}(s) + \cdots + a_{nn}(s)]\, ds\right), \quad a < t < b.$$

6. Suppose the $n \times n$ matrix A is continuous on (a, b) and t_0 is a point in (a, b). Let Y be a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).

a. Show that $Y(t_0)$ is invertible.

b. Show that if $\mathbf{k}$ is an arbitrary n-vector then the solution of the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y}, \quad \mathbf{y}(t_0) = \mathbf{k}$$

is

$$\mathbf{y} = Y(t)Y^{-1}(t_0)\mathbf{k}.$$

7. Let

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{6t} \\ e^{6t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-2t} \\ -e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}.$$

a. Verify that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$.
b. Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{k}. \tag{A}$$

c. Use the result of **Exercise 10.3.6** (b) to find a formula for the solution of (A) for an arbitrary initial vector $\mathbf{k}$.

8. Repeat **Exercise 10.3.7** with

$$A = \begin{bmatrix} -2 & -2 \\ -5 & 1 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{-4t} \\ e^{-4t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -2e^{3t} \\ 5e^{3t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 10 \\ -4 \end{bmatrix}.$$

9. Repeat **Exercise 10.3.7** with

$$A = \begin{bmatrix} -4 & -10 \\ 3 & 7 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} -5e^{2t} \\ 3e^{2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} -19 \\ 11 \end{bmatrix}.$$

10. Repeat **Exercise 10.3.7** with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{y}_1 = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}.$$

11. Let

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} e^{2t} \\ 0 \\ e^{2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{3t} \\ -e^{3t} \\ e^{3t} \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-t} \\ -3e^{-t} \\ 7e^{-t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 2 \\ -7 \\ 20 \end{bmatrix}.$$

a. Verify that $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions for $\mathbf{y}' = A\mathbf{y}$.
b. Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{k}. \tag{A}$$

c. Use the result of **Exercise 10.3.6** (b) to find a formula for the solution of (A) for an arbitrary initial vector $\mathbf{k}$.

12. Repeat **Exercise 10.3.11** with

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} -e^{-2t} \\ 0 \\ e^{-2t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-2t} \\ e^{-2t} \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{4t} \\ e^{4t} \\ e^{4t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 \\ -9 \\ 12 \end{bmatrix}.$$

13. Repeat **Exercise 10.3.11** with

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{bmatrix},$$

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_3 = \begin{bmatrix} e^{-2t} \\ -2e^{-2t} \\ e^{-2t} \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 5 \\ 5 \\ -1 \end{bmatrix}.$$

14. Suppose Y and Z are fundamental matrices for the $n \times n$ system $\mathbf{y}' = A(t)\mathbf{y}$. Then some of the four matrices YZ^{-1}, $Y^{-1}Z$, $Z^{-1}Y$, ZY^{-1} are necessarily constant. Identify them and prove that they are constant.

15. Suppose the columns of an $n \times n$ matrix Y are solutions of the $n \times n$ system $\mathbf{y}' = A\mathbf{y}$ and C is an $n \times n$ constant matrix.

a. Show that the matrix $Z = YC$ satisfies the differential equation $Z' = AZ$.
b. Show that Z is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$ if and only if C is invertible and Y is a fundamental matrix for $\mathbf{y}' = A(t)\mathbf{y}$.

16. Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) and t_0 is in (a, b). For $i = 1, 2, \ldots, n$, let $\mathbf{y}_i$ be the solution of the initial value problem $\mathbf{y}_i' = A(t)\mathbf{y}_i,\ \mathbf{y}_i(t_0) = \mathbf{e}_i$, where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

that is, the jth component of $\mathbf{e}_i$ is 1 if $j = i$, or 0 if $j \neq i$.

a. Show that $\{\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b).
b. Conclude from (a) and **Exercise 10.3.15** that $\mathbf{y}' = A(t)\mathbf{y}$ has infinitely many fundamental sets of solutions on (a, b).

17. Show that Y is a fundamental matrix for the system $\mathbf{y}' = A(t)\mathbf{y}$ if and only if Y^{-1} is a fundamental matrix for $\mathbf{y}' = -A^T(t)\mathbf{y}$, where A^T denotes the transpose of A. HINT: **See Exercise 10.3.11.**

18. Let Z be the fundamental matrix for the constant coefficient system $\mathbf{y}' = A\mathbf{y}$ such that $Z(0) = I$.

a. Show that $Z(t)Z(s) = Z(t + s)$ for all s and t. HINT: **For fixed s let $\Gamma_1(t) = Z(t)Z(s)$ and $\Gamma_2(t) = Z(t + s)$. Show that Γ_1 and Γ_2 are both solutions of the matrix initial value problem $\Gamma' = A\Gamma$, $\Gamma(0) = Z(s)$. Then conclude from Theorem 10.2.1 that $\Gamma_1 = \Gamma_2$.**
b. Show that $(Z(t))^{-1} = Z(-t)$.
c. The matrix Z defined above is sometimes denoted by e^{tA}. Discuss the motivation for this notation.

This page titled 15.3.1: Basic Theory of Homogeneous Linear Systems (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **10.3E: Basic Theory of Homogeneous Linear Systems (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

15.4: Constant Coefficient Homogeneous Systems I

We'll now begin our study of the homogeneous system

$$\mathbf{y}' = A\mathbf{y}, \tag{15.4.1}$$

where A is an $n \times n$ constant matrix. Since A is continuous on $(-\infty, \infty)$, Theorem 10.2.1 implies that all solutions of Equation 15.4.1 are defined on $(-\infty, \infty)$. Therefore, when we speak of solutions of $\mathbf{y}' = A\mathbf{y}$, we'll mean solutions on $(-\infty, \infty)$.

In this section we assume that all the eigenvalues of A are real and that A has a set of n linearly independent eigenvectors. In the next two sections we consider the cases where some of the eigenvalues of A are complex, or where A does not have n linearly independent eigenvectors.

In Example 10.3.2 we showed that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

form a fundamental set of solutions of the system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \tag{15.4.2}$$

but we did not show how we obtained $\mathbf{y}_1$ and $\mathbf{y}_2$ in the first place. To see how these solutions can be obtained we write Equation 15.4.2 as

$$\begin{aligned} y_1' &= -4y_1 - 3y_2 \\ y_2' &= 6y_1 + 5y_2 \end{aligned} \tag{15.4.3}$$

and look for solutions of the form

$$y_1 = x_1 e^{\lambda t} \quad \text{and} \quad y_2 = x_2 e^{\lambda t}, \tag{15.4.4}$$

where x_1, x_2, and λ are constants to be determined. Differentiating Equation 15.4.4 yields

$$y_1' = \lambda x_1 e^{\lambda t} \quad \text{and} \quad y_2' = \lambda x_2 e^{\lambda t}.$$

Substituting this and Equation 15.4.4 into Equation 15.4.3 and canceling the common factor $e^{\lambda t}$ yields

$$\begin{aligned} -4x_1 - 3x_2 &= \lambda x_1 \\ 6x_1 + 5x_2 &= \lambda x_2. \end{aligned}$$

For a given λ, this is a homogeneous algebraic system, since it can be rewritten as

$$\begin{aligned} (-4 - \lambda)x_1 - 3x_2 &= 0 \\ 6x_1 + (5 - \lambda)x_2 &= 0. \end{aligned} \tag{15.4.5}$$

The trivial solution $x_1 = x_2 = 0$ of this system isn't useful, since it corresponds to the trivial solution $y_1 \equiv y_2 \equiv 0$ of Equation 15.4.3, which can't be part of a fundamental set of solutions of Equation 15.4.2. Therefore we consider only those values of λ for which Equation 15.4.5 has nontrivial solutions. These are the values of λ for which the determinant of Equation 15.4.5 is zero; that is,

$$\begin{aligned} \begin{vmatrix} -4 - \lambda & -3 \\ 6 & 5 - \lambda \end{vmatrix} &= (-4 - \lambda)(5 - \lambda) + 18 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1) = 0, \end{aligned}$$

which has the solutions $\lambda_1 = 2$ and $\lambda_2 = -1$.

Taking $\lambda = 2$ in Equation 15.4.5 yields

$$-6x_1 - 3x_2 = 0$$
$$6x_1 + 3x_2 = 0,$$

which implies that $x_1 = -x_2/2$, where x_2 can be chosen arbitrarily. Choosing $x_2 = 2$ yields the solution $y_1 = -e^{2t}$, $y_2 = 2e^{2t}$ of Equation 15.4.3. We can write this solution in vector form as

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{2t}. \tag{15.4.6}$$

Taking $\lambda = -1$ in Equation 15.4.5 yields the system

$$-3x_1 - 3x_2 = 0$$
$$6x_1 + 6x_2 = 0,$$

so $x_1 = -x_2$. Taking $x_2 = 1$ here yields the solution $y_1 = -e^{-t}$, $y_2 = e^{-t}$ of Equation 15.4.3. We can write this solution in vector form as

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}. \tag{15.4.7}$$

In Equation 15.4.6 and Equation 15.4.7 the constant coefficients in the arguments of the exponential functions are the eigenvalues of the coefficient matrix in Equation 15.4.2, and the vector coefficients of the exponential functions are associated eigenvectors. This illustrates the next theorem.

Theorem 10.4.1

Suppose the $n \times n$ constant matrix A has n real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. Then the functions

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t},\ \mathbf{y}_2 = \mathbf{x}_2 e^{\lambda_2 t},\ \dots,\ \mathbf{y}_n = \mathbf{x}_n e^{\lambda_n t}$$

form a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$; that is, the general solution of this system is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} + \cdots + c_n\mathbf{x}_n e^{\lambda_n t}.$$

Proof

Differentiating $\mathbf{y}_i = \mathbf{x}_i e^{\lambda_i t}$ and recalling that $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$ yields

$$\mathbf{y}_i' = \lambda_i \mathbf{x}_i e^{\lambda_i t} = A\mathbf{x}_i e^{\lambda_i t} = A\mathbf{y}_i.$$

This shows that $\mathbf{y}_i$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is

$$\begin{vmatrix} x_{11}e^{\lambda_1 t} & x_{12}e^{\lambda_2 t} & \cdots & x_{1n}e^{\lambda_n t} \\ x_{21}e^{\lambda_1 t} & x_{22}e^{\lambda_2 t} & \cdots & x_{2n}e^{\lambda_n t} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1}e^{\lambda_1 t} & x_{n2}e^{\lambda_2 t} & \cdots & x_{nn}e^{\lambda x_n t} \end{vmatrix} = e^{\lambda_1 t}e^{\lambda_2 t}\cdots e^{\lambda_n t} \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}.$$

Since the columns of the determinant on the right are $\mathbf{x}_1$, $\mathbf{x}_2$, …, $\mathbf{x}_n$, which are assumed to be linearly independent, the determinant is nonzero. Therefore Theorem 10.3.3 implies that $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$.

Example 10.4.1

a. Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}. \tag{15.4.8}$$

b. Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}. \tag{15.4.9}$$

Solution a

The characteristic polynomial of the coefficient matrix A in Equation 15.4.8 is

$$\begin{aligned} \begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} &= (\lambda-2)^2 - 16 \\ &= (\lambda-2-4)(\lambda-2+4) \\ &= (\lambda-6)(\lambda+2). \end{aligned}$$

Hence, $\lambda_1 = 6$ and $\lambda_2 = -2$ are eigenvalues of A. To obtain the eigenvectors, we must solve the system

$$\begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{15.4.10}$$

with $\lambda = 6$ and $\lambda = -2$. Setting $\lambda = 6$ in Equation 15.4.10 yields

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

is a solution of Equation 15.4.8. Setting $\lambda = -2$ in Equation 15.4.10 yields

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = -x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

is a solution of Equation 15.4.8. From Theorem 10.4.1 , the general solution of Equation 15.4.8 is

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}. \tag{15.4.11}$$

Solution b

To satisfy the initial condition in Equation 15.4.9, we must choose c_1 and c_2 in Equation 15.4.11 so that

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

This is equivalent to the system

$$\begin{aligned} c_1 - c_2 &= \ \ 5 \\ c_1 + c_2 &= -1, \end{aligned}$$

so $c_1 = 2, c_2 = -3$. Therefore the solution of Equation 15.4.9 is

$$\mathbf{y} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} - 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

or, in terms of components,

$$y_1 = 2e^{6t} + 3e^{-2t}, \quad y_2 = 2e^{6t} - 3e^{-2t}.$$

Example 10.4.2

a. Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}. \tag{15.4.12}$$

b. Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}. \tag{15.4.13}$$

Solution a

The characteristic polynomial of the coefficient matrix A in Equation 15.4.12 is

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{vmatrix} = -(\lambda-2)(\lambda-3)(\lambda+1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -1$. To find the eigenvectors, we must solve the system

$$\begin{bmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{15.4.14}$$

with $\lambda = 2, 3, -1$. With $\lambda = 2$, the augmented matrix of Equation 15.4.14 is

$$\left[\begin{array}{ccc:c} 1 & -1 & -1 & 0 \\ -2 & 1 & 2 & 0 \\ 4 & -1 & -4 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc:c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

as a solution of Equation 15.4.12. With $\lambda = 3$, the augmented matrix of Equation 15.4.14 is

$$\left[\begin{array}{rrr:r} 0 & -1 & -1 & 0 \\ -2 & 0 & 2 & 0 \\ 4 & -1 & -5 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrr:r} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t}$$

as a solution of Equation 15.4.12. With $\lambda = -1$, the augmented matrix of Equation 15.4.14 is

$$\left[\begin{array}{rrr:r} 4 & -1 & -1 & 0 \\ -2 & 4 & 2 & 0 \\ 4 & -1 & -1 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{rrr:r} 1 & 0 & -\frac{1}{7} & 0 \\ 0 & 1 & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_3/7$ and $x_2 = -3x_3/7$. Taking $x_3 = 7$ yields

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$$

as a solution of Equation 15.4.12. By Theorem 10.4.1 , the general solution of Equation 15.4.12 is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t},$$

which can also be written as

$$\mathbf{y} = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{15.4.15}$$

Solution b

To satisfy the initial condition in Equation 15.4.13 we must choose c_1, c_2, c_3 in Equation 15.4.15 so that

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}.$$

Solving this system yields $c_1 = 3$, $c_2 = -2$, $c_3 = 1$. Hence, the solution of Equation 15.4.13 is

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} - 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}. \end{aligned}$$

Example 10.4.3

Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \mathbf{y}. \tag{15.4.16}$$

Solution

The characteristic polynomial of the coefficient matrix A in Equation 15.4.16 is

$$\begin{vmatrix} -3-\lambda & 2 & 2 \\ 2 & -3-\lambda & 2 \\ 2 & 2 & -3-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+5)^2.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 1, while $\lambda_2 = -5$ is an eigenvalue of multiplicity 2. Eigenvectors associated with $\lambda_1 = 1$ are solutions of the system with augmented matrix

$$\left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence, $x_1 = x_2 = x_3$, and we choose $x_3 = 1$ to obtain the solution

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t \tag{15.4.17}$$

of Equation 15.4.16. Eigenvectors associated with $\lambda_2 = -5$ are solutions of the system with augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right].$$

Hence, the components of these eigenvectors need only satisfy the single condition

$$x_1 + x_2 + x_3 = 0.$$

Since there's only one equation here, we can choose x_2 and x_3 arbitrarily. We obtain one eigenvector by choosing $x_2 = 0$ and $x_3 = 1$, and another by choosing $x_2 = 1$ and $x_3 = 0$. In both cases $x_1 = -1$. Therefore

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent eigenvectors associated with $\lambda_2 = -5$, and the corresponding solutions of Equation 15.4.16 are

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Because of this and Equation 15.4.17, Theorem 10.4.1 implies that the general solution of Equation 15.4.16 is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Geometric Properties of Solutions when n=2

We'll now consider the geometric properties of solutions of a 2×2 constant coefficient system

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \tag{15.4.18}$$

It is convenient to think of a "y_1-y_2 plane," where a point is identified by rectangular coordinates (y_1, y_2). If $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ is a non-constant solution of Equation 15.4.18, then the point $(y_1(t), y_2(t))$ moves along a curve C in the y_1-y_2 plane as t varies from $-\infty$ to ∞. We call C the **trajectory** of $\mathbf{y}$. (We also say that C is a trajectory of the system Equation 15.4.18.) I's important to note that C is the trajectory of infinitely many solutions of Equation 15.4.18, since if τ is any real number, then $\mathbf{y}(t - \tau)$ is a solution of Equation 15.4.18, and $(y_1(t - \tau), y_2(t - \tau))$ also moves along C as t varies from $-\infty$ to ∞. Moreover, **Exercise 10.4.28c** implies that distinct trajectories of Equation 15.4.18 can't intersect, and that two solutions $\mathbf{y}_1$ and $\mathbf{y}_2$ of Equation 15.4.18 have the same trajectory if and only if $\mathbf{y}_2(t) = \mathbf{y}_1(t - \tau)$ for some τ.

From **Exercise 10.4.28a**, a trajectory of a nontrivial solution of Equation 15.4.18 can't contain $(0, 0)$, which we define to be the trajectory of the trivial solution $\mathbf{y} \equiv 0$. More generally, if $\mathbf{y} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \neq \mathbf{0}$ is a constant solution of Equation 15.4.18 (which could occur if zero is an eigenvalue of the matrix of Equation 15.4.18), we define the trajectory of $\mathbf{y}$ to be the single point (k_1, k_2).

To be specific, this is the question: What do the trajectories look like, and how are they traversed? In this section we'll answer this question, assuming that the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

of Equation 15.4.18 has real eigenvalues λ_1 and λ_2 with associated linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then the general solution of Equation 15.4.18 is

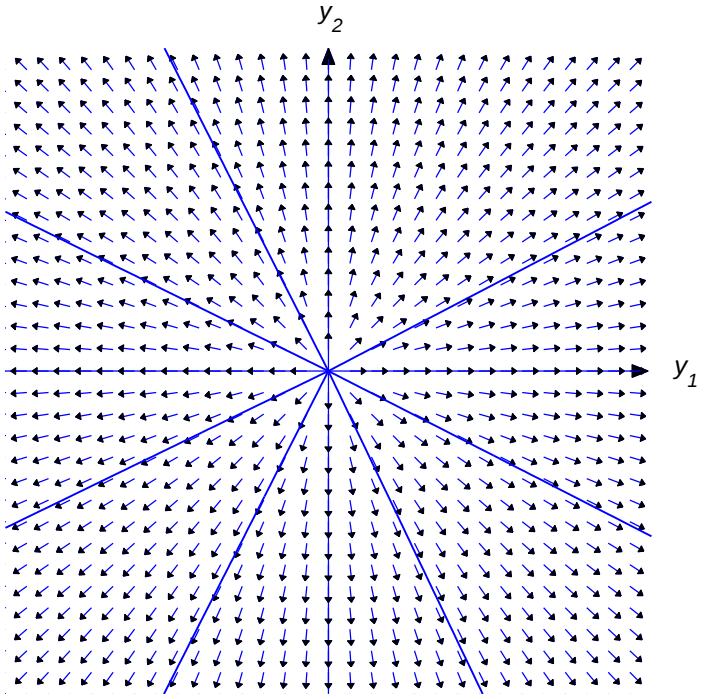
$$\mathbf{y} = c_1 \mathbf{x}_1 e^{\lambda_1 t} + c_2 \mathbf{x}_2 e^{\lambda_2 t}. \tag{15.4.19}$$

We'll consider other situations in the next two sections.

We leave it to you to classify the trajectories of Equation 15.4.18 if zero is an eigenvalue of A. We'll confine our attention here to the case where both eigenvalues are nonzero. In this case the simplest situation is where $\lambda_1 = \lambda_2 \neq 0$, so Equation 15.4.19 becomes

$$\mathbf{y} = (c_1\mathbf{x}_1 + c_2\mathbf{x}_2)e^{\lambda_1 t}.$$

Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent, an arbitrary vector $\mathbf{x}$ can be written as $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. Therefore the general solution of Equation 15.4.18 can be written as $\mathbf{y} = \mathbf{x}e^{\lambda_1 t}$ where $\mathbf{x}$ is an arbitrary 2-vector, and the trajectories of nontrivial solutions of Equation 15.4.18 are half-lines through (but not including) the origin. The direction of motion is away from the origin if $\lambda_1 > 0$ (Figure 10.4.1), toward it if $\lambda_1 < 0$ (Figure 10.4.2). (In these and the next figures an arrow through a point indicates the direction of motion along the trajectory through the point.)



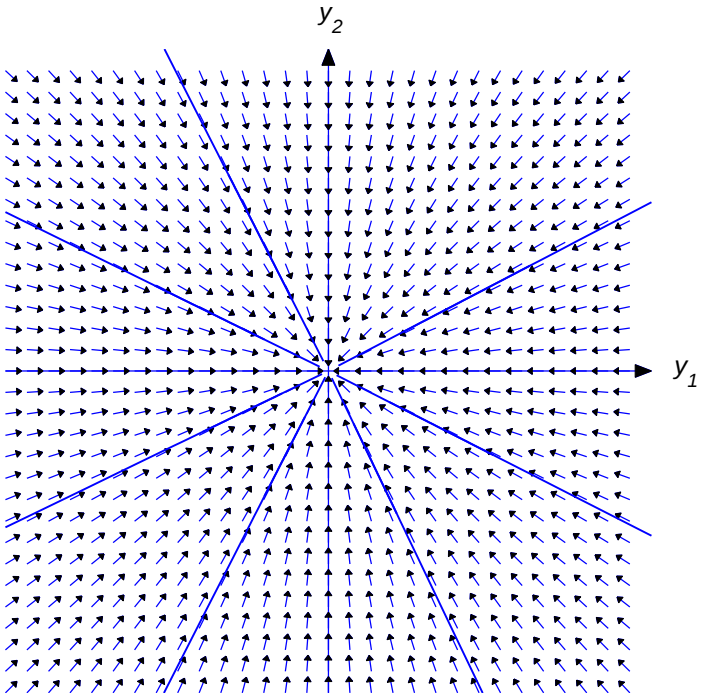


Figure 10.4.1 (left): Trajectories of a **2×2 system with a repeated positive eigenvalue** and Figure 10.4.2 (right): Trajectories of a **2×2 system with a repeated negative eigenvalue.**

Now suppose $\lambda_2 > \lambda_1$, and let L_1 and L_2 denote lines through the origin parallel to $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. By a half-line of L_1 (or L_2), we mean either of the rays obtained by removing the origin from L_1 (or L_2).

Letting $c_2 = 0$ in Equation 15.4.19 yields $\mathbf{y} = c_1\mathbf{x}_1e^{\lambda_1 t}$. If $c_1 \neq 0$, the trajectory defined by this solution is a half-line of L_1. The direction of motion is away from the origin if $\lambda_1 > 0$, toward the origin if $\lambda_1 < 0$. Similarly, the trajectory of $\mathbf{y} = c_2\mathbf{x}_2e^{\lambda_2 t}$ with $c_2 \neq 0$ is a half-line of L_2.

Henceforth, we assume that c_1 and c_2 in Equation 15.4.19 are both nonzero. In this case, the trajectory of Equation 15.4.19 can't intersect L_1 or L_2, since every point on these lines is on the trajectory of a solution for which either $c_1 = 0$ or $c_2 = 0$. (Remember: distinct trajectories can't intersect!). Therefore the trajectory of Equation 15.4.19 must lie entirely in one of the four open sectors bounded by L_1 and L_2, but do not any point on L_1 or L_2. Since the initial point $(y_1(0), y_2(0))$ defined by

$$\mathbf{y}(0) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

is on the trajectory, we can determine which sector contains the trajectory from the signs of c_1 and c_2, as shown in Figure 10.4.3 .

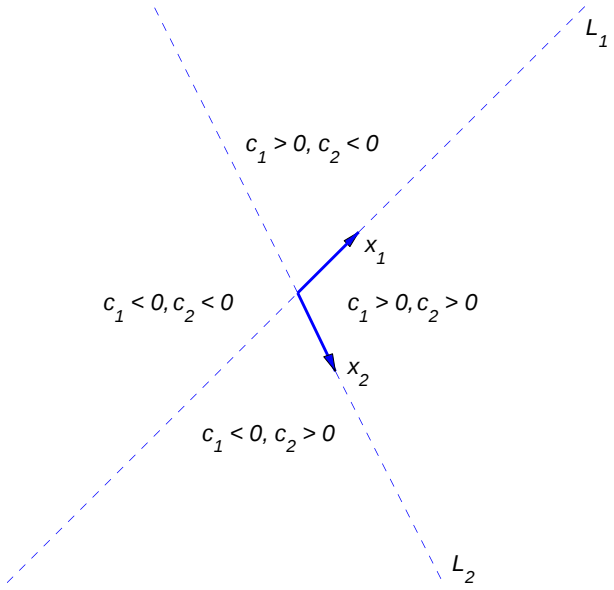


Figure 10.4.3 : Four open sectors bounded by L_1 and L_2.

The direction of $\mathbf{y}(t)$ in Equation 15.4.19 is the same as that of

$$e^{-\lambda_2 t}\mathbf{y}(t) = c_1\mathbf{x}_1e^{-(\lambda_2-\lambda_1)t} + c_2\mathbf{x}_2 \tag{15.4.20}$$

and of

$$e^{-\lambda_1 t}\mathbf{y}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2e^{(\lambda_2-\lambda_1)t}. \tag{15.4.21}$$

Since the right side of Equation 15.4.20 approaches $c_2\mathbf{x}_2$ as $t \to \infty$, the trajectory is asymptotically parallel to L_2 as $t \to \infty$. Since the right side of Equation 15.4.21 approaches $c_1\mathbf{x}_1$ as $t \to -\infty$, the trajectory is asymptotically parallel to L_1 as $t \to -\infty$.

The shape and direction of traversal of the trajectory of Equation 15.4.19 depend upon whether λ_1 and λ_2 are both positive, both negative, or of opposite signs. We'll now analyze these three cases.

Henceforth $\|\mathbf{u}\|$ denote the length of the vector $\mathbf{u}$.

Case 1: $\lambda_2 > \lambda_1$

Figure 10.4.4 shows some typical trajectories. In this case, $\lim_{t\to-\infty}\|\mathbf{y}(t)\| = 0$, so the trajectory is not only asymptotically parallel to L_1 as $t \to -\infty$, but is actually asymptotically tangent to L_1 at the origin. On the other hand, $\lim_{t\to\infty}\|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t\to\infty}\|\mathbf{y}(t) - c_2\mathbf{x}_2e^{\lambda_2 t}\| = \lim_{t\to\infty}\|c_1\mathbf{x}_1e^{\lambda_1 t}\| = \infty,$$

so, although the trajectory is asymptotically parallel to L_2 as $t \to \infty$, it is not asymptotically tangent to L_2. The direction of motion along each trajectory is away from the origin.

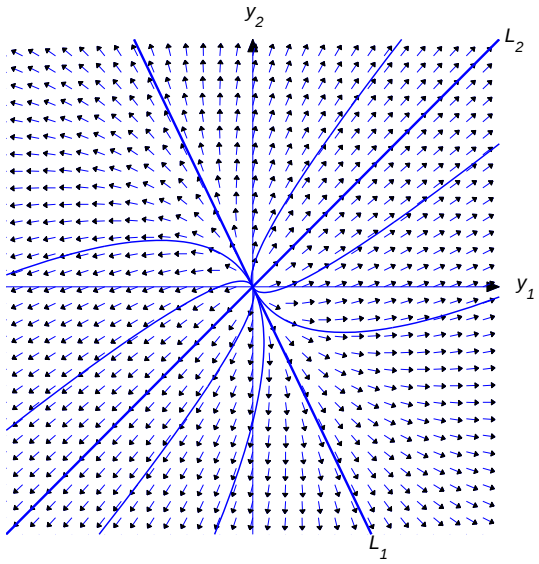


Figure 10.4.4 : Two positive eigenvalues; motion away from origin.

Case 2: 0 > λ₂ > λ₁

Figure 10.4.5 shows some typical trajectories. In this case, $\lim_{t\to\infty} \|\mathbf{y}(t)\| = 0$, so the trajectory is asymptotically tangent to $\boldsymbol{L_2}$ at the origin as $t \to \infty$. On the other hand, $\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty$ and

$$\lim_{t\to-\infty} \|\mathbf{y}(t) - c_1\mathbf{x}_1 e^{\lambda_1 t}\| = \lim_{t\to-\infty} \|c_2\mathbf{x}_2 e^{\lambda_2 t}\| = \infty,$$

so, although the trajectory is asymptotically parallel to $\boldsymbol{L_1}$ as $t \to -\infty$, it is not asymptotically tangent to it. The direction of motion along each trajectory is toward the origin.

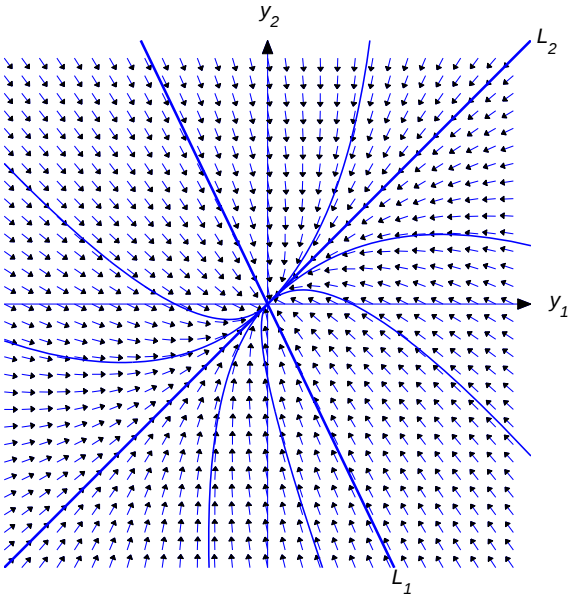


Figure 10.4.5 : Two negative eigenvalues; motion toward the origin.

Case 3: λ₂ > 0 > λ₁

Figure 10.4.6 shows some typical trajectories. In this case,

$$\lim_{t\to\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to\infty} \|\mathbf{y}(t) - c_2\mathbf{x}_2 e^{\lambda_2 t}\| = \lim_{t\to\infty} \|c_1\mathbf{x}_1 e^{\lambda_1 t}\| = 0,$$

so the trajectory is asymptotically tangent to $\boldsymbol{L_2}$ as $t \to \infty$. Similarly,

$$\lim_{t\to-\infty} \|\mathbf{y}(t)\| = \infty \quad \text{and} \quad \lim_{t\to-\infty} \|\mathbf{y}(t) - c_1\mathbf{x}_1 e^{\lambda_1 t}\| = \lim_{t\to-\infty} \|c_2\mathbf{x}_2 e^{\lambda_2 t}\| = 0,$$

so the trajectory is asymptotically tangent to L_1 as $t \to -\infty$. The direction of motion is toward the origin on L_1 and away from the origin on L_2. The direction of motion along any other trajectory is away from L_1, toward L_2.

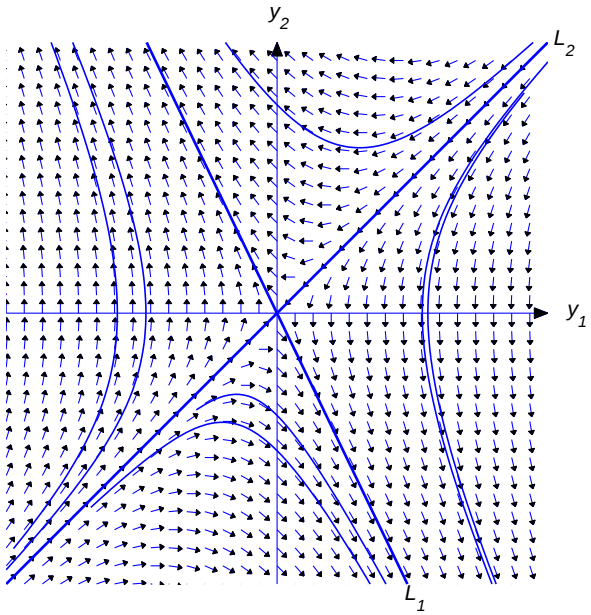


Figure 10.4.6 : Eigenvalues of different signs.

This page titled 15.4: Constant Coefficient Homogeneous Systems I is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **10.4: Constant Coefficient Homogeneous Systems I** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

15.4.1: Constant Coefficient Homogeneous Systems I (Exercises)

Q10.4.1

In **Exercises 10.4.1-10.4.15** find the general solution.

1. $\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y}$
2. $\mathbf{y}' = \frac{1}{4} \begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix} \mathbf{y}$
3. $\mathbf{y}' = \frac{1}{5} \begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y}$
4. $\mathbf{y}' = \begin{bmatrix} -1 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$
5. $\mathbf{y}' = \begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix} \mathbf{y}$
6. $\mathbf{y}' = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix} \mathbf{y}$
7. $\mathbf{y}' = \begin{bmatrix} -6 & -3 \\ 1 & -2 \end{bmatrix} \mathbf{y}$
8. $\mathbf{y}' = \begin{bmatrix} 1 & -1 & -2 \\ 1 & -2 & -3 \\ -4 & 1 & -1 \end{bmatrix} \mathbf{y}$
9. $\mathbf{y}' = \begin{bmatrix} -6 & -4 & -8 \\ -4 & 0 & -4 \\ -8 & -4 & -6 \end{bmatrix} \mathbf{y}$
10. $\mathbf{y}' = \begin{bmatrix} 3 & 5 & 8 \\ 1 & -1 & -2 \\ -1 & -1 & -1 \end{bmatrix} \mathbf{y}$
11. $\mathbf{y}' = \begin{bmatrix} 1 & -1 & 2 \\ 12 & -4 & 10 \\ -6 & 1 & -7 \end{bmatrix} \mathbf{y}$
12. $\mathbf{y}' = \begin{bmatrix} 4 & -1 & -4 \\ 4 & -3 & -2 \\ 1 & -1 & -1 \end{bmatrix} \mathbf{y}$
13. $\mathbf{y}' = \begin{bmatrix} -2 & 2 & -6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \mathbf{y}$
14. $\mathbf{y}' = \begin{bmatrix} 3 & 2 & -2 \\ -2 & 7 & -2 \\ -10 & 10 & -5 \end{bmatrix} \mathbf{y}$
15. $\mathbf{y}' = \begin{bmatrix} 3 & 1 & -1 \\ 3 & 5 & 1 \\ -6 & 2 & 4 \end{bmatrix} \mathbf{y}$

Q10.4.2

In **Exercises 10.4.16-10.4.27** solve the initial value problem.

16. $\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

17. $\mathbf{y}' = \frac{1}{6} \begin{bmatrix} 7 & 2 \\ -2 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

18. $\mathbf{y}' = \begin{bmatrix} 21 & -12 \\ 24 & -15 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

19. $\mathbf{y}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 7 \end{bmatrix}$

20. $\mathbf{y}' = \frac{1}{6} \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 7 \\ 1 \end{bmatrix}$

21. $\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 2 & -2 & 3 \\ -4 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

22. $\mathbf{y}' = \begin{bmatrix} 6 & -3 & -8 \\ 2 & 1 & -2 \\ 3 & -3 & -5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$

23. $\mathbf{y}' = \frac{1}{3} \begin{bmatrix} 2 & 4 & -7 \\ 1 & 5 & -5 \\ -4 & 4 & -1 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$

24. $\mathbf{y}' = \begin{bmatrix} 3 & 0 & 1 \\ 11 & -2 & 7 \\ 1 & 0 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 7 \\ 6 \end{bmatrix}$

25. $\mathbf{y}' = \begin{bmatrix} -2 & -5 & -1 \\ -4 & -1 & 1 \\ 4 & 5 & 3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 8 \\ -10 \\ -4 \end{bmatrix}$

26. $\mathbf{y}' = \begin{bmatrix} 3 & -1 & 0 \\ 4 & -2 & 0 \\ 4 & -4 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 7 \\ 10 \\ 2 \end{bmatrix}$

27. $\mathbf{y}' = \begin{bmatrix} -2 & 2 & 6 \\ 2 & 6 & 2 \\ -2 & -2 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 6 \\ -10 \\ 7 \end{bmatrix}$

Q10.4.3

28. Let A be an $n \times n$ constant matrix. Then Theorem 10.2.1 implies that the solutions of

$$\mathbf{y}' = A\mathbf{y} \tag{A}$$

are all defined on $(-\infty, \infty)$.

a. Use Theorem 10.2.1 to show that the only solution of (A) that can ever equal the zero vector is $\mathbf{y} \equiv \mathbf{0}$.

b. Suppose $\mathbf{y}_1$ is a solution of (A) and $\mathbf{y}_2$ is defined by $\mathbf{y}_2(t) = \mathbf{y}_1(t - \tau)$, where τ is an arbitrary real number. Show that $\mathbf{y}_2$ is also a solution of (A).

c. Suppose $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of (A) and there are real numbers t_1 and t_2 such that $\mathbf{y}_1(t_1) = \mathbf{y}_2(t_2)$. Show that $\mathbf{y}_2(t) = \mathbf{y}_1(t - \tau)$ for all t, where $\tau = t_2 - t_1$.

Q10.4.4

In **Exercises 10.4.29-10.4.34** describe and graph trajectories of the given system.

29. $\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{y}$

30. $\mathbf{y}' = \begin{bmatrix} -4 & 3 \\ -2 & -11 \end{bmatrix} \mathbf{y}$

31. $\mathbf{y}' = \begin{bmatrix} 9 & -3 \\ -1 & 11 \end{bmatrix} \mathbf{y}$

32. $\mathbf{y}' = \begin{bmatrix} -1 & -10 \\ -5 & 4 \end{bmatrix} \mathbf{y}$

33. $\mathbf{y}' = \begin{bmatrix} 5 & -4 \\ 1 & 10 \end{bmatrix} \mathbf{y}$

34. $\mathbf{y}' = \begin{bmatrix} -7 & 1 \\ 3 & -5 \end{bmatrix} \mathbf{y}$

Q10.4.5

35. Suppose the eigenvalues of the 2×2 matrix A are $\lambda = 0$ and $\mu \neq 0$, with corresponding eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Let L_1 be the line through the origin parallel to $\mathbf{x}_1$.

a. Show that every point on L_1 is the trajectory of a constant solution of $\mathbf{y}' = A\mathbf{y}$.
b. Show that the trajectories of nonconstant solutions of $\mathbf{y}' = A\mathbf{y}$ are half-lines parallel to $\mathbf{x}_2$ and on either side of L_1, and that the direction of motion along these trajectories is away from L_1 if $\mu > 0$, or toward L_1 if $\mu < 0$.

Q10.4.6

The matrices of the systems in **Exercises 10.4.36-10.4.41** are singular. Describe and graph the trajectories of nonconstant solutions of the given systems.

36. $\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{y}$

37. $\mathbf{y}' = \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix} \mathbf{y}$

38. $\mathbf{y}' = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix} \mathbf{y}$

39. $\mathbf{y}' = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix} \mathbf{y}$

40. $\mathbf{y}' = \begin{bmatrix} -4 & -4 \\ 1 & 1 \end{bmatrix} \mathbf{y}$

41. $\mathbf{y}' = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \mathbf{y}$

Q10.4.6

42. Let $P = P(t)$ and $Q = Q(t)$ be the populations of two species at time t, and assume that each population would grow exponentially if the other didn't exist; that is, in the absence of competition,

$$P' = aP \quad \text{and} \quad Q' = bQ, \tag{A}$$

where a and b are positive constants. One way to model the effect of competition is to assume that the growth rate per individual of each population is reduced by an amount proportional to the other population, so (A) is replaced by

$$P' = aP - \alpha Q$$
$$Q' = -\beta P + bQ,$$

where a and β are positive constants. (Since negative population doesn't make sense, this system holds only while P and Q are both positive.) Now suppose $P(0) = P_0 > 0$ and $Q(0) = Q_0 > 0$.

a. For several choices of a, b, α, and β, verify experimentally (by graphing trajectories of (A) in the P-Q plane) that there's a constant $\rho > 0$ (depending upon a, b, α, and β) with the following properties:
   i. If $Q_0 > \rho P_0$, then P decreases monotonically to zero in finite time, during which Q remains positive.
   ii. If $Q_0 < \rho P_0$, then Q decreases monotonically to zero in finite time, during which P remains positive.

b. Conclude from (a) that exactly one of the species becomes extinct in finite time if $Q_0 \neq \rho P_0$. Determine experimentally what happens if $Q_0 = \rho P_0$.

c. Confirm your experimental results and determine γ by expressing the eigenvalues and associated eigenvectors of

$$A = \begin{bmatrix} \alpha & -\alpha \\ -\beta & b \end{bmatrix}$$

in terms of a, b, α, and β, and applying the geometric arguments developed at the end of this section.

This page titled 15.4.1: Constant Coefficient Homogeneous Systems I (Exercises) is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by William F. Trench via source content that was edited to the style and standards of the LibreTexts platform.

- **10.4E: Constant Coefficient Homogeneous Systems I (Exercises)** by William F. Trench is licensed CC BY-NC-SA 3.0. Original source: https://digitalcommons.trinity.edu/mono/9.

CHAPTER OVERVIEW

16: Appendices

16.1: Trigonometric Identities
16.2: Review of Derivative Rules
16.3: Table of Derivatives
16.4: Table of Integrals

16: Appendices is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

16.1: Trigonometric Identities

Pythagorean Identities

$$\cos^2 x + \sin^2 x = 1$$

$$\sec^2 x - \tan^2 x = 1$$

Double-Angle Identities

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$$

Half-Angle Identities

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Angle Sum and Difference Identities

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

Angle Reflections and Shifts

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

$$\sin\left(x \pm \tfrac{\pi}{2}\right) = \pm \cos x$$

$$\cos\left(x \pm \tfrac{\pi}{2}\right) = \mp \sin x$$

Angle Supplement Identities

$$\sin(\pi - x) = \sin x$$

$$\cos(\pi - x) = -\cos x$$

$$\tan(\pi - x) = -\tan x$$

Periodicity Identities

$$\sin(x + 2\pi) = \sin x$$

$$\cos(x + 2\pi) = \cos x$$

$$\tan(x + \pi) = \tan x$$

16.1: Trigonometric Identities is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

16.2: Review of Derivative Rules

Here is a quick review of the basic derivative rules in words. See the Table of Derivatives for a list of Derivative Formulas.

- The derivative of a constant function is zero.
- The derivative of a power function is a function in which the power on x becomes the coefficient of the term and the power on x in the derivative decreases by 1.
- The derivative of a constant c multiplied by a function f is the same as the constant multiplied by the derivative.
- The derivative of the sum of a function f and a function g is the same as the sum of the derivative of f and the derivative of g.
- The derivative of the difference of a function f and a function g is the same as the difference of the derivative of f and the derivative of g.
- The derivative of a product of two functions is the derivative of the first function times the second function plus the derivative of the second function times the first function.
- The derivative of the quotient of two functions is the derivative of the first function times the second function minus the derivative of the second function times the first function, all divided by the square of the second function.
- We used the limit definition of the derivative to develop formulas that allow us to find derivatives without resorting to the definition of the derivative. These formulas can be used singly or in combination with each other.

16.2: Review of Derivative Rules is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

- **3.3: Differentiation Rules** by Edwin “Jed” Herman, Gilbert Strang is licensed CC BY-NC-SA 4.0. Original source: https://openstax.org/details/books/calculus-volume-1.

16.3: Table of Derivatives

General Formulas

1. $\frac{d}{dx}(c) = 0$
2. $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
3. $\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
4. $\frac{d}{dx}(x^n) = nx^{n-1}, \quad \text{for real numbers } n$
5. $\frac{d}{dx}(cf(x)) = cf'(x)$
6. $\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$
7. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
8. $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

Trigonometric Functions

9. $\frac{d}{dx}(\sin x) = \cos x$
10. $\frac{d}{dx}(\tan x) = \sec^2 x$
11. $\frac{d}{dx}(\sec x) = \sec x \tan x$
12. $\frac{d}{dx}(\cos x) = -\sin x$
13. $\frac{d}{dx}(\cot x) = -\csc^2 x$
14. $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Inverse Trigonometric Functions

15. $\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}$
16. $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$
17. $\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}}$
18. $\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$
19. $\frac{d}{dx}(\operatorname{arccot} x) = \frac{-1}{1+x^2}$
20. $\frac{d}{dx}(\operatorname{arccsc} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Exponential and Logarithmic Functions

21. $\frac{d}{dx}(e^x) = e^x$
22. $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$
23. $\frac{d}{dx}(b^x) = b^x \ln b$
24. $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$

Hyperbolic Functions

25. $\frac{d}{dx}(\sinh x) = \cosh x$
26. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
27. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
28. $\frac{d}{dx}(\cosh x) = \sinh x$
29. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
30. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

Inverse Hyperbolic Functions

31. $\frac{d}{dx}(\operatorname{arcsinh} x) = \frac{1}{\sqrt{x^2+1}}$
32. $\frac{d}{dx}(\operatorname{arctanh} x) = \frac{1}{1-x^2} \quad (|x| < 1)$
33. $\frac{d}{dx}(\operatorname{arcsech} x) = \frac{-1}{x\sqrt{1-x^2}} \quad (0 < x < 1)$
34. $\frac{d}{dx}(\operatorname{arccosh} x) = \frac{1}{\sqrt{x^2-1}} \quad (x > 1)$
35. $\frac{d}{dx}(\operatorname{arccoth} x) = \frac{1}{1-x^2} \quad (|x| > 1)$
36. $\frac{d}{dx}(\operatorname{arccsch} x) = \frac{-1}{|x|\sqrt{1+x^2}} \quad (x \neq 0)$

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Modified to change inverse trig notation by Paul Seeburger (Monroe Community College)

16.3: Table of Derivatives is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

16.4: Table of Integrals

For this course, all work must be shown to obtain most of these integral forms. **Of the integration formulas listed below, the only ones that can be applied without further work are #1 - 10, 15 - 17, and 49 and 50.** And even these will require work to be shown if a substitution is involved.

All others may be helpful for checking your final answers, but cannot be used to skip the necessary work to show you understand how to use the integration techniques taught in this course.

As you look through these formulas, you should be able to recognize which integration technique was needed to obtain the general formula. It may be very useful for you to try to obtain the general formula yourself using the techniques we learn in this course.

Basic Integrals

1. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$
2. $\int \frac{du}{u} = \ln|u| + C$
3. $\int e^u\,du = e^u + C$
4. $\int a^u\,du = \frac{a^u}{\ln a} + C$
5. $\int \sin u\,du = -\cos u + C$
6. $\int \cos u\,du = \sin u + C$
7. $\int \sec^2 u\,du = \tan u + C$
8. $\int \csc^2 u\,du = -\cot u + C$
9. $\int \sec u \tan u\,du = \sec u + C$
10. $\int \csc u \cot u\,du = -\csc u + C$
11. $\int \tan u\,du = \ln|\sec u| + C$
12. $\int \cot u\,du = \ln|\sin u| + C$
13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$
14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$
15. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$
16. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$
17. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

Trigonometric Integrals

18. $\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$
19. $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$
20. $\int \tan^2 u\,du = \tan u - u + C$
21. $\int \cot^2 u\,du = -\cot u - u + C$
22. $\int \sin^3 u\,du = -\frac{1}{3}(2 + \sin^2 u)\cos u + C$
23. $\int \cos^3 u\,du = \frac{1}{3}(2 + \cos^2 u)\sin u + C$
24. $\int \tan^3 u\,du = \frac{1}{2}\tan^2 u + \ln|\cos u| + C$
25. $\int \cot^3 u\,du = -\frac{1}{2}\cot^2 u - \ln|\sin u| + C$
26. $\int \sec^3 u\,du = \frac{1}{2}\sec u \tan u + \frac{1}{2}\ln|\sec u + \tan u| + C$
27. $\int \csc^3 u\,du = -\frac{1}{2}\csc u \cot u + \frac{1}{2}\ln|\csc u - \cot u| + C$

28. $\int \sin^n u\,du = \frac{-1}{n}\sin^{n-1} u\cos u + \frac{n-1}{n}\int \sin^{n-2} u\,du$
29. $\int \cos^n u\,du = \frac{1}{n}\cos^{n-1} u\sin u + \frac{n-1}{n}\int \cos^{n-2} u\,du$
30. $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du$
31. $\int \cot^n u\,du = \frac{-1}{n-1}\cot^{n-1} u - \int \cot^{n-2} u\,du$
32. $\int \sec^n u\,du = \frac{1}{n-1}\tan u\sec^{n-2} u + \frac{n-2}{n-1}\int \sec^{n-2} u\,du$
33. $\int \csc^n u\,du = \frac{-1}{n-1}\cot u\csc^{n-2} u + \frac{n-2}{n-1}\int \csc^{n-2} u\,du$
34. $\int \sin au\sin bu\,du = \frac{\sin(a-b)u}{2(a-b)} - \frac{\sin(a+b)u}{2(a+b)} + C$
35. $\int \cos au\cos bu\,du = \frac{\sin(a-b)u}{2(a-b)} + \frac{\sin(a+b)u}{2(a+b)} + C$
36. $\int \sin au\cos bu\,du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$
37. $\int u\sin u\,du = \sin u - u\cos u + C$
38. $\int u\cos u\,du = \cos u + u\sin u + C$
39. $\int u^n \sin u\,du = -u^n\cos u + n\int u^{n-1}\cos u\,du$
40. $\int u^n \cos u\,du = u^n\sin u - n\int u^{n-1}\sin u\,du$
41. $\int \sin^n u\cos^m u\,du =$ Use the methods for powers of sine and cosine

Exponential and Logarithmic Integrals

42. $\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C$
43. $\int u^n e^{au}\,du = \frac{1}{a}u^n e^{au} - \frac{n}{a}\int u^{n-1}e^{au}\,du$
44. $\int e^{au}\sin bu\,du = \frac{e^{au}}{a^2+b^2}(a\sin bu - b\cos bu) + C$
45. $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}(a\cos bu + b\sin bu) + C$
46. $\int \ln u\,du = u\ln u - u + C$
47. $\int u^n \ln u\,du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$
48. $\int \frac{1}{u\ln u}\,du = \ln|\ln u| + C$

Hyperbolic Integrals

49. $\int \sinh u\,du = \cosh u + C$
50. $\int \cosh u\,du = \sinh u + C$
51. $\int \tanh u\,du = \ln\cosh u + C$
52. $\int \coth u\,du = \ln|\sinh u| + C$
53. $\int \operatorname{sech} u\,du = \arctan|\sinh u| + C$
54. $\int \operatorname{csch} u\,du = \ln|\tanh\frac{1}{2}u| + C$
55. $\int \operatorname{sech}^2 u\,du = \tanh u + C$
56. $\int \operatorname{csch}^2 u\,du = -\coth u + C$
57. $\int \operatorname{sech} u\tanh u\,du = -\operatorname{sech} u + C$
58. $\int \operatorname{csch} u\coth u\,du = -\operatorname{csch} u + C$

Inverse Trigonometric Integrals

59. $\int \arcsin u\,du = u\arcsin u + \sqrt{1-u^2} + C$
60. $\int \arccos u\,du = u\arccos u - \sqrt{1-u^2} + C$

61. $\int \arctan u\,du = u\arctan u - \frac{1}{2}\ln(1+u^2) + C$

62. $\int u\arcsin u\,du = \frac{2u^2-1}{4}\arcsin u + \frac{u\sqrt{1-u^2}}{4} + C$

63. $\int u\arccos u\,du = \frac{2u^2-1}{4}\arccos u - \frac{u\sqrt{1-u^2}}{4} + C$

64. $\int u\arctan u\,du = \frac{u^2+1}{2}\arctan u - \frac{u}{2} + C$

65. $\int u^n \arcsin u\,du = \frac{1}{n+1}\left[u^{n+1}\arcsin u - \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

66. $\int u^n \arccos u\,du = \frac{1}{n+1}\left[u^{n+1}\arccos u + \int \frac{u^{n+1}\,du}{\sqrt{1-u^2}}\right], \quad n \neq -1$

67. $\int u^n \arctan u\,du = \frac{1}{n+1}\left[u^{n+1}\arctan u - \int \frac{u^{n+1}\,du}{1+u^2}\right], \quad n \neq -1$

Integrals Involving $a^2 + u^2$, $a > 0$

68. $\int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

69. $\int u^2\sqrt{a^2+u^2}\,du = \frac{u}{8}(a^2+2u^2)\sqrt{a^2+u^2} - \frac{a^4}{8}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

70. $\int \frac{\sqrt{a^2+u^2}}{u}\,du = \sqrt{a^2+u^2} - a\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right| + C$

71. $\int \frac{\sqrt{a^2+u^2}}{u^2}\,du = -\frac{\sqrt{a^2+u^2}}{u} + \ln\left(u+\sqrt{a^2+u^2}\right) + C$

72. $\int \frac{du}{\sqrt{a^2+u^2}} = \ln\left(u+\sqrt{a^2+u^2}\right) + C$

73. $\int \frac{u^2}{\sqrt{a^2+u^2}}\,du = \frac{u}{2}\left(\sqrt{a^2+u^2}\right) - \frac{a^2}{2}\ln\left(u+\sqrt{a^2+u^2}\right) + C$

74. $\int \frac{du}{u\sqrt{a^2+u^2}} = \frac{-1}{a}\ln\left|\frac{\sqrt{a^2+u^2}+a}{u}\right| + C$

75. $\int \frac{du}{u^2\sqrt{a^2+u^2}} = -\frac{\sqrt{a^2+u^2}}{a^2u} + C$

76. $\int \frac{du}{(a^2+u^2)^{3/2}} = \frac{u}{a^2\sqrt{a^2+u^2}} + C$

Integrals Involving $u^2 - a^2$, $a > 0$

77. $\int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

78. $\int u^2\sqrt{u^2-a^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{u^2-a^2} - \frac{a^4}{8}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

79. $\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\arccos\frac{a}{|u|} + C$

80. $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = -\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$

81. $\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C$

82. $\int \frac{u^2}{\sqrt{u^2-a^2}}\,du = \frac{u}{2}\sqrt{u^2-a^2} + \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + C$

83. $\int \frac{du}{u^2\sqrt{u^2-a^2}} = \frac{\sqrt{u^2-a^2}}{a^2u} + C$

84. $\int \frac{du}{(u^2-a^2)^{3/2}} = -\frac{u}{a^2\sqrt{u^2-a^2}} + C$

Integrals Involving $a^2 - u^2$, $a > 0$

85. $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\arcsin\frac{u}{a} + C$

86. $\int u^2\sqrt{a^2-u^2}\,du = \frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2} + \frac{a^4}{8}\arcsin\frac{u}{a} + C$

87. $\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

88. $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \frac{-1}{u}\sqrt{a^2-u^2} - \arcsin\frac{u}{a} + C$

89. $\int \frac{u^2}{\sqrt{a^2-u^2}}\,du = \frac{1}{2}\left(-u\sqrt{a^2-u^2} + a^2\arcsin\frac{u}{a}\right) + C$

90. $\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$

91. $\int \frac{du}{u^2\sqrt{a^2-u^2}} = -\frac{1}{a^2u}\sqrt{a^2-u^2} + C$

92. $\int (a^2-u^2)^{3/2}\,du = -\frac{u}{8}\left(2u^2-5a^2\right)\sqrt{a^2-u^2} + \frac{3a^4}{8}\arcsin\frac{u}{a} + C$

93. $\int \frac{du}{(a^2-u^2)^{3/2}} = -\frac{u}{a^2\sqrt{a^2-u^2}} + C$

Integrals Involving $2au - u^2$, $a > 0$

94. $\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\arccos\left(\frac{a-u}{a}\right) + C$

95. $\int \frac{du}{\sqrt{2au-u^2}} = \arccos\left(\frac{a-u}{a}\right) + C$

96. $\int u\sqrt{2au-u^2}\,du = \frac{2u^2-au-3a^2}{6}\sqrt{2au-u^2} + \frac{a^3}{2}\arccos\left(\frac{a-u}{a}\right) + C$

97. $\int \frac{du}{u\sqrt{2au-u^2}} = -\frac{\sqrt{2au-u^2}}{au} + C$

Integrals Involving $a + bu$, $a \neq 0$

98. $\int \frac{u}{a+bu}\,du = \frac{1}{b^2}(a+bu-a\ln|a+bu|) + C$

99. $\int \frac{u^2}{a+bu}\,du = \frac{1}{2b^3}\left[(a+bu)^2 - 4a(a+bu) + 2a^2\ln|a+bu|\right] + C$

100. $\int \frac{du}{u(a+bu)} = \frac{1}{a}\ln\left|\frac{u}{a+bu}\right| + C$

101. $\int \frac{du}{u^2(a+bu)} = -\frac{1}{au} + \frac{b}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

102. $\int \frac{u}{(a+bu)^2}\,du = \frac{a}{b^2(a+bu)} + \frac{1}{b^2}\ln|a+bu| + C$

103. $\int \frac{u}{u(a+bu)^2}\,du = \frac{1}{a(a+bu)} - \frac{1}{a^2}\ln\left|\frac{a+bu}{u}\right| + C$

104. $\int \frac{u^2}{(a+bu)^2}\,du = \frac{1}{b^3}\left(a+bu - \frac{a^2}{a+bu} - 2a\ln|a+bu|\right) + C$

105. $\int u\sqrt{a+bu}\,du = \frac{2}{15b^2}(3bu-2a)(a+bu)^{3/2} + C$

106. $\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu} + C$

107. $\int \frac{u^2}{\sqrt{a+bu}}\,du = \frac{2}{15b^3}(8a^2+3b^2u^2-4abu)\sqrt{a+bu} + C$

108. $\int \frac{du}{u\sqrt{a+bu}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bu}-\sqrt{a}}{\sqrt{a+bu}+\sqrt{a}}\right| + C, & \text{if } a > 0 \\ \frac{\sqrt{2}}{\sqrt{-a}}\arctan\sqrt{\frac{a+bu}{-a}} + C, & \text{if } a < 0 \end{cases}$

109. $\int \frac{\sqrt{a+bu}}{u}\,du = 2\sqrt{a+bu} + a\int \frac{du}{u\sqrt{a+bu}}$

110. $\int \frac{\sqrt{a+bu}}{u^2}\,du = -\frac{\sqrt{a+bu}}{u} + \frac{b}{2}\int \frac{du}{u\sqrt{a+bu}}$

111. $\int u^n\sqrt{a+bu}\,du = \frac{2}{b(2n+3)}\left[u^n(a+bu)^{3/2} - na\int u^{n-1}\sqrt{a+bu}\,du\right]$

112. $\int \frac{u^n}{\sqrt{a+bu}}\,du = \frac{2u^n\sqrt{a+bu}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int \frac{u^{n-1}}{\sqrt{a+bu}}\,du$

113. $\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$

Contributors

- Gilbert Strang (MIT) and Edwin “Jed” Herman (Harvey Mudd) with many contributing authors. This content by OpenStax is licensed with a CC-BY-SA-NC 4.0 license. Download for free at http://cnx.org.
- Introduction paragraphs and inverse trig notation changes by Paul Seeburger (Monroe Community College)

16.4: Table of Integrals is shared under a CC BY-NC-SA 4.0 license and was authored, remixed, and/or curated by LibreTexts.

Detailed Licensing

Overview

Title: Differential Equations and Linear Algebra (Zook)

Webpages: 144

Applicable Restrictions: Noncommercial

All licenses found:

- CC BY-NC-SA 4.0: 99.3% (143 pages)
- Undeclared: 0.7% (1 page)

By Page

- Differential Equations and Linear Algebra (Zook) - *CC BY-NC-SA 4.0*
  - Front Matter - *CC BY-NC-SA 4.0*
    - TitlePage - *CC BY-NC-SA 4.0*
    - InfoPage - *CC BY-NC-SA 4.0*
    - Table of Contents - *Undeclared*
    - Licensing - *CC BY-NC-SA 4.0*
  - 1: Systems of Equations - *CC BY-NC-SA 4.0*
    - 1.1: Systems of Equations, Geometry - *CC BY-NC-SA 4.0*
    - 1.2: Systems of Equations, Algebraic Procedures - *CC BY-NC-SA 4.0*
    - 1.3: Gaussian Elimination - *CC BY-NC-SA 4.0*
    - 1.4: Uniqueness of the Reduced Row-Echelon Form - *CC BY-NC-SA 4.0*
    - 1.5: Rank and Homogeneous Systems - *CC BY-NC-SA 4.0*
    - 1.6: Exercises - *CC BY-NC-SA 4.0*
  - 2: Matrices - *CC BY-NC-SA 4.0*
    - 2.1: Matrix Arithmetic - *CC BY-NC-SA 4.0*
    - 2.2: Multiplication of Matrices - *CC BY-NC-SA 4.0*
    - 2.3: The ijth Entry of a Product - *CC BY-NC-SA 4.0*
    - 2.4: Properties of Matrix Multiplication - *CC BY-NC-SA 4.0*
    - 2.5: The Transpose - *CC BY-NC-SA 4.0*
    - 2.6: The Identity and Inverses - *CC BY-NC-SA 4.0*
    - 2.7: Finding the Inverse of a Matrix - *CC BY-NC-SA 4.0*
    - 2.8: Elementary Matrices - *CC BY-NC-SA 4.0*
    - 2.9: More on Matrix Inverses - *CC BY-NC-SA 4.0*
    - 2.10: LU Factorization - *CC BY-NC-SA 4.0*
    - 2.11: Exercises - *CC BY-NC-SA 4.0*
  - 3: Determinants - *CC BY-NC-SA 4.0*
    - 3.1: Basic Techniques - *CC BY-NC-SA 4.0*
    - 3.2: Properties of Determinants - *CC BY-NC-SA 4.0*
    - 3.3: Finding Determinants using Row Operations - *CC BY-NC-SA 4.0*
    - 3.4: Applications of the Determinant - *CC BY-NC-SA 4.0*
    - 3.5: Exercises - *CC BY-NC-SA 4.0*
  - 4: R^n - *CC BY-NC-SA 4.0*
    - 4.1: Vectors in R^n - *CC BY-NC-SA 4.0*
    - 4.2: Vector Algebra - *CC BY-NC-SA 4.0*
    - 4.3: Geometric Meaning of Vector Addition - *CC BY-NC-SA 4.0*
    - 4.4: Length of a Vector - *CC BY-NC-SA 4.0*
    - 4.5: Geometric Meaning of Scalar Multiplication - *CC BY-NC-SA 4.0*
    - 4.6: Parametric Lines - *CC BY-NC-SA 4.0*
    - 4.7: Spanning, Linear Independence and Basis in R^n - *CC BY-NC-SA 4.0*
    - 4.8: Orthogonality - *CC BY-NC-SA 4.0*
    - 4.9: Applications - *CC BY-NC-SA 4.0*
    - 4.10: Exercises - *CC BY-NC-SA 4.0*
  - 5: Linear Transformations - *CC BY-NC-SA 4.0*
    - 5.1: Linear Transformations - *CC BY-NC-SA 4.0*
    - 5.2: The Matrix of a Linear Transformation I - *CC BY-NC-SA 4.0*
    - 5.3: Properties of Linear Transformations - *CC BY-NC-SA 4.0*
    - 5.4: Special Linear Transformations in R^2 - *CC BY-NC-SA 4.0*
    - 5.5: One-to-One and Onto Transformations - *CC BY-NC-SA 4.0*
    - 5.6: The Kernel and Image of A Linear Map - *CC BY-NC-SA 4.0*
    - 5.7: The Matrix of a Linear Transformation II - *CC BY-NC-SA 4.0*
    - 5.8: The General Solution of a Linear System - *CC BY-NC-SA 4.0*

- 5.9: Exercises - *CC BY-NC-SA 4.0*
- 6: Spectral Theory - *CC BY-NC-SA 4.0*
  - 6.1: Eigenvalues and Eigenvectors of a Matrix - *CC BY-NC-SA 4.0*
  - 6.2: Diagonalization - *CC BY-NC-SA 4.0*
  - 6.3: Applications of Spectral Theory - *CC BY-NC-SA 4.0*
  - 6.4: Orthogonality - *CC BY-NC-SA 4.0*
  - 6.5: Exercises - *CC BY-NC-SA 4.0*
- 7: What are Differential Equations? - *CC BY-NC-SA 4.0*
  - 7.1: Applications Leading to Differential Equations - *CC BY-NC-SA 4.0*
  - 7.2: Basic Concepts - *CC BY-NC-SA 4.0*
    - 7.2.1: Basic Concepts (Exercises) - *CC BY-NC-SA 4.0*
  - 7.3: Direction Fields for First Order Equations - *CC BY-NC-SA 4.0*
    - 7.3.1: Direction Fields for First Order Equations (Exercises) - *CC BY-NC-SA 4.0*
- 8: First Order Equations - *CC BY-NC-SA 4.0*
  - 8.1: Linear First Order Equations - *CC BY-NC-SA 4.0*
    - 8.1.1: Linear First Order Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 8.2: Separable Equations - *CC BY-NC-SA 4.0*
    - 8.2.1: Separable Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 8.3: Existence and Uniqueness of Solutions of Nonlinear Equations - *CC BY-NC-SA 4.0*
    - 8.3.1: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 8.4: Transformation of Nonlinear Equations into Separable Equations - *CC BY-NC-SA 4.0*
    - 8.4.1: Transformation of Nonlinear Equations into Separable Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 8.5: Exact Equations - *CC BY-NC-SA 4.0*
    - 8.5.1: Exact Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 8.6: Integrating Factors - *CC BY-NC-SA 4.0*
    - 8.6.1: Integrating Factors (Exercises) - *CC BY-NC-SA 4.0*
- 9: Applications of First Order Equations - *CC BY-NC-SA 4.0*
  - 9.1: Growth and Decay - *CC BY-NC-SA 4.0*
    - 9.1.1: Growth and Decay (Exercises) - *CC BY-NC-SA 4.0*
  - 9.2: Cooling and Mixing - *CC BY-NC-SA 4.0*
    - 9.2.1: Cooling and Mixing (Exercises) - *CC BY-NC-SA 4.0*
  - 9.3: Elementary Mechanics - *CC BY-NC-SA 4.0*
    - 9.3.1: Elementary Mechanics (Exercises) - *CC BY-NC-SA 4.0*
- 10: Linear Second Order Equations - *CC BY-NC-SA 4.0*
  - 10.1: Homogeneous Linear Equations - *CC BY-NC-SA 4.0*
    - 10.1.1: Homogeneous Linear Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 10.2: Constant Coefficient Homogeneous Equations - *CC BY-NC-SA 4.0*
    - 10.2.1: Constant Coefficient Homogeneous Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 10.3: Nonhomogeneous Linear Equations - *CC BY-NC-SA 4.0*
    - 10.3.1: Nonhomgeneous Linear Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 10.4: The Method of Undetermined Coefficients I - *CC BY-NC-SA 4.0*
    - 10.4.1: The Method of Undetermined Coefficients I (Exercises) - *CC BY-NC-SA 4.0*
  - 10.5: The Method of Undetermined Coefficients II - *CC BY-NC-SA 4.0*
    - 10.5.1: The Method of Undetermined Coefficients II (Exercises) - *CC BY-NC-SA 4.0*
  - 10.6: Reduction of Order - *CC BY-NC-SA 4.0*
    - 10.6.1: Reduction of Order (Exercises) - *CC BY-NC-SA 4.0*
- 11: Applications of Linear Second Order Equations - *CC BY-NC-SA 4.0*
  - 11.1: Spring Problems I - *CC BY-NC-SA 4.0*
    - 11.1.1: Spring Problems I (Exercises) - *CC BY-NC-SA 4.0*
  - 11.2: Spring Problems II - *CC BY-NC-SA 4.0*
    - 11.2.1: Spring Problems II (Exercises) - *CC BY-NC-SA 4.0*
- 12: Series Solutions of Linear Second Order Equations - *CC BY-NC-SA 4.0*
  - 12.1: Prelude to Series Solutions of Linear Second Order Equations - *CC BY-NC-SA 4.0*
  - 12.2: Review of Power Series - *CC BY-NC-SA 4.0*

- 12.2.1: Review of Power Series (Exercises) - *CC BY-NC-SA 4.0*
- 12.3: Series Solutions Near an Ordinary Point I - *CC BY-NC-SA 4.0*
  - 12.3.1: Series Solutions Near an Ordinary Point I (Exercises) - *CC BY-NC-SA 4.0*

13: Laplace Transforms - *CC BY-NC-SA 4.0*

- 13.1: Introduction to the Laplace Transform - *CC BY-NC-SA 4.0*
  - 13.1.1: Introduction to the Laplace Transform (Exercises) - *CC BY-NC-SA 4.0*
- 13.2: The Inverse Laplace Transform - *CC BY-NC-SA 4.0*
  - 13.2.1: The Inverse Laplace Transform (Exercises) - *CC BY-NC-SA 4.0*
- 13.3: Solution of Initial Value Problems - *CC BY-NC-SA 4.0*
  - 13.3.1: Solution of Initial Value Problems (Exercises) - *CC BY-NC-SA 4.0*
- 13.4: The Unit Step Function - *CC BY-NC-SA 4.0*
  - 13.4.1: The Unit Step Function (Exercises) - *CC BY-NC-SA 4.0*
- 13.5: Constant Coefficient Equations with Piecewise Continuous Forcing Functions - *CC BY-NC-SA 4.0*
  - 13.5.1: Constant Coefficient Equations with Piecewise Continuous Forcing Functions (Exercises) - *CC BY-NC-SA 4.0*
- 13.6: A Brief Table of Laplace Transforms - *CC BY-NC-SA 4.0*

14: Linear Higher Order Differential Equations - *CC BY-NC-SA 4.0*

- 14.1: Introduction to Linear Higher Order Equations - *CC BY-NC-SA 4.0*
  - 14.1.1: Introduction to Linear Higher Order Equations (Exercises) - *CC BY-NC-SA 4.0*
- 14.2: Higher Order Constant Coefficient Homogeneous Equations - *CC BY-NC-SA 4.0*
  - 14.2.1: Higher Order Constant Coefficient Homogeneous Equations (Exercises) - *CC BY-NC-SA 4.0*

15: Linear Systems of Differential Equations - *CC BY-NC-SA 4.0*

- 15.1: Introduction to Systems of Differential Equations - *CC BY-NC-SA 4.0*
  - 15.1.1: Introduction to Systems of Differential Equations (Exercises) - *CC BY-NC-SA 4.0*
- 15.2: Linear Systems of Differential Equations - *CC BY-NC-SA 4.0*
  - 15.2.1: Linear Systems of Differential Equations (Exercises) - *CC BY-NC-SA 4.0*
- 15.3: Basic Theory of Homogeneous Linear Systems - *CC BY-NC-SA 4.0*
  - 15.3.1: Basic Theory of Homogeneous Linear Systems (Exercises) - *CC BY-NC-SA 4.0*
- 15.4: Constant Coefficient Homogeneous Systems I - *CC BY-NC-SA 4.0*
  - 15.4.1: Constant Coefficient Homogeneous Systems I (Exercises) - *CC BY-NC-SA 4.0*

16: Appendices - *CC BY-NC-SA 4.0*

- 16.1: Trigonometric Identities - *CC BY-NC-SA 4.0*
- 16.2: Review of Derivative Rules - *CC BY-NC-SA 4.0*
- 16.3: Table of Derivatives - *CC BY-NC-SA 4.0*
- 16.4: Table of Integrals - *CC BY-NC-SA 4.0*

Back Matter - *CC BY-NC-SA 4.0*

- Detailed Licensing - *CC BY-NC-SA 4.0*
- Index - *CC BY-NC-SA 4.0*
- Glossary - *CC BY-NC-SA 4.0*
- Detailed Licensing - *CC BY-NC-SA 4.0*

Index

A

Abel's theorem
- 14.1: Introduction to Linear Higher Order Equations
- 15.3: Basic Theory of Homogeneous Linear Systems

augmented matrix
- 1.3: Gaussian Elimination

B

basic variables
- 1.4: Uniqueness of the Reduced Row-Echelon Form

Bernoulli equations
- 8.4: Transformation of Nonlinear Equations into Separable Equations

C

characteristic equation
- 10.2: Constant Coefficient Homogeneous Equations

characteristic polynomial
- 10.2: Constant Coefficient Homogeneous Equations
- 14.2: Higher Order Constant Coefficient Homogeneous Equations

Chebyshev's equation
- 12.3: Series Solutions Near an Ordinary Point I

coefficient matrix
- 15.2: Linear Systems of Differential Equations

cofactor
- 3.1: Basic Techniques

cofactor matrix
- 3.4: Applications of the Determinant

complementary equation
- 8.1: Linear First Order Equations
- 10.3: Nonhomogeneous Linear Equations

compound interest
- 9.1: Growth and Decay

constant coefficient equation
- 10.2: Constant Coefficient Homogeneous Equations
- 14.2: Higher Order Constant Coefficient Homogeneous Equations

constant matrix
- 1.3: Gaussian Elimination

Critically Damped Motion
- 11.2: Spring Problems II

D

determinant
- 3.1: Basic Techniques

determinants
- 3.2: Properties of Determinants

direction field
- 7.3: Direction Fields for First Order Equations

distance formula
- 4.4: Length of a Vector

E

elementary matrix
- 2.8: Elementary Matrices

Elementary operations
- 1.2: Systems of Equations, Algebraic Procedures

elementary row operations
- 1.3: Gaussian Elimination

equivalent matrices
- 1.4: Uniqueness of the Reduced Row-Echelon Form

escape velocity
- 9.3: Elementary Mechanics

Exact Differentials
- 8.5: Exact Equations

exact equations
- 8.5: Exact Equations

F

forcing function
- 10.1: Homogeneous Linear Equations
- 15.2: Linear Systems of Differential Equations

Fourier expansion
- 4.8: Orthogonality

free variables
- 1.4: Uniqueness of the Reduced Row-Echelon Form

G

Gaussian elimination
- 1.2: Systems of Equations, Algebraic Procedures
- 1.3: Gaussian Elimination

general solution
- 5.8: The General Solution of a Linear System

gravity
- 9.3: Elementary Mechanics

H

homogeneous
- 10.1: Homogeneous Linear Equations

Homogeneous Differential Equations with constant Coefficients
- 10.2: Constant Coefficient Homogeneous Equations

Hooke's law
- 11.1: Spring Problems I

I

identity matrix
- 2.6: The Identity and Inverses

implicit function theorem
- 8.2: Separable Equations

initial value problems
- 13.3: Solution of Initial Value Problems

integrating factor
- 8.6: Integrating Factors

interpolating polynomial
- 3.4: Applications of the Determinant

inverse Laplace transform
- 13.2: The Inverse Laplace Transform

K

kernel
- 5.6: The Kernel and Image of A Linear Map
- 5.8: The General Solution of a Linear System

Kronecker delta
- 2.6: The Identity and Inverses
- 4.8: Orthogonality

L

Laplace Expansion
- 3.1: Basic Techniques

Laplace transform
- 13: Laplace Transforms
- 13.1: Introduction to the Laplace Transform
- 13.3: Solution of Initial Value Problems

linear independence
- 4.8: Orthogonality

linear transformation
- 5.1: Linear Transformations
- 5.8: The General Solution of a Linear System

Lotka Volterra system
- 6.3: Applications of Spectral Theory

LU factorization
- 2.10: LU Factorization

M

Malthusian model
- 7.1: Applications Leading to Differential Equations

Markov matrices
- 6.3: Applications of Spectral Theory

matrix
- 2.1: Matrix Arithmetic

matrix exponential
- 6.3: Applications of Spectral Theory

matrix inverse
- 2.7: Finding the Inverse of a Matrix
- 2.9: More on Matrix Inverses

matrix multiplication
- 2.4: Properties of Matrix Multiplication

matrix transformations
- 5.1: Linear Transformations

mechanics
- 9.3: Elementary Mechanics

method of undetermined coefficients
- 10.4: The Method of Undetermined Coefficients I
- 10.5: The Method of Undetermined Coefficients II

migration matrix
- 6.3: Applications of Spectral Theory

minor of a matrix
- 3.1: Basic Techniques

minors or a matrix
- 3.1: Basic Techniques

multiplicity
- 14.2: Higher Order Constant Coefficient Homogeneous Equations

N

Newton's Law of Cooling
- 9.2: Cooling and Mixing

Nonhomgeneous Linear Equations
- 10.3: Nonhomogeneous Linear Equations

null space
- 5.8: The General Solution of a Linear System

O

open interval of convergence
- 12.2: Review of Power Series

open rectangle
- 8.3: Existence and Uniqueness of Solutions of Nonlinear Equations

ordinary point
- 12.3: Series Solutions Near an Ordinary Point I

orthogonal matrix
4.8: Orthogonality
orthogonality
4.8: Orthogonality
orthogonally diagonalizable
6.3: Applications of Spectral Theory
Overdamped Motion
11.2: Spring Problems II

P

Parametric Lines
4.6: Parametric Lines
particular solution
5.8: The General Solution of a Linear System
10.3: Nonhomogeneous Linear Equations
piecewise continuous function
13.1: Introduction to the Laplace Transform
polynomial operator
14.2: Higher Order Constant Coefficient Homogeneous Equations
Position Vector
4.1: Vectors in R^n
power series
12.2: Review of Power Series
power series solution
12.3: Series Solutions Near an Ordinary Point I
Principal Axis Theorem
6.3: Applications of Spectral Theory

R

radius of convergence
12.2: Review of Power Series
reduction of orders
10.6: Reduction of Order
Riccati equation
8.4.1: Transformation of Nonlinear Equations into Separable Equations (Exercises)
row operations
3.2: Properties of Determinants

S

Separable Differential Equations
8.2: Separable Equations
simple harmonic motion
11.1: Spring Problems I
skew lines
1.1: Systems of Equations, Geometry
skew symmetric matrix
2.5: The Transpose
spring constant
11.1: Spring Problems I
square matrix
2.1: Matrix Arithmetic
step function
13.4: The Unit Step Function
subspace
4.8: Orthogonality
superposition principle
10.3: Nonhomogeneous Linear Equations
system of linear equations
1.2: Systems of Equations, Algebraic Procedures

T

temperature decay constant
9.2: Cooling and Mixing
terminal velocity
9.3: Elementary Mechanics
transforms
5.1: Linear Transformations
transpose
2.5: The Transpose
triangular matrix
3.1: Basic Techniques
trivial solution
10.1: Homogeneous Linear Equations

U

Underdamped Motion
11.2: Spring Problems II

V

vector addition
4.3: Geometric Meaning of Vector Addition
Verhulst model
7.1: Applications Leading to Differential Equations

W

Wronskian
14.1: Introduction to Linear Higher Order Equations
15.3: Basic Theory of Homogeneous Linear Systems

Z

zero matrix
2.1: Matrix Arithmetic

Glossary

Sample Word 1 | Sample Definition 1

Detailed Licensing

Overview

Title: Differential Equations and Linear Algebra (Zook)

Webpages: 144

Applicable Restrictions: Noncommercial

All licenses found:

- CC BY-NC-SA 4.0: 99.3% (143 pages)
- Undeclared: 0.7% (1 page)

By Page

- Differential Equations and Linear Algebra (Zook) - *CC BY-NC-SA 4.0*
  - Front Matter - *CC BY-NC-SA 4.0*
    - TitlePage - *CC BY-NC-SA 4.0*
    - InfoPage - *CC BY-NC-SA 4.0*
    - Table of Contents - *Undeclared*
    - Licensing - *CC BY-NC-SA 4.0*
  - 1: Systems of Equations - *CC BY-NC-SA 4.0*
    - 1.1: Systems of Equations, Geometry - *CC BY-NC-SA 4.0*
    - 1.2: Systems of Equations, Algebraic Procedures - *CC BY-NC-SA 4.0*
    - 1.3: Gaussian Elimination - *CC BY-NC-SA 4.0*
    - 1.4: Uniqueness of the Reduced Row-Echelon Form - *CC BY-NC-SA 4.0*
    - 1.5: Rank and Homogeneous Systems - *CC BY-NC-SA 4.0*
    - 1.6: Exercises - *CC BY-NC-SA 4.0*
  - 2: Matrices - *CC BY-NC-SA 4.0*
    - 2.1: Matrix Arithmetic - *CC BY-NC-SA 4.0*
    - 2.2: Multiplication of Matrices - *CC BY-NC-SA 4.0*
    - 2.3: The ijth Entry of a Product - *CC BY-NC-SA 4.0*
    - 2.4: Properties of Matrix Multiplication - *CC BY-NC-SA 4.0*
    - 2.5: The Transpose - *CC BY-NC-SA 4.0*
    - 2.6: The Identity and Inverses - *CC BY-NC-SA 4.0*
    - 2.7: Finding the Inverse of a Matrix - *CC BY-NC-SA 4.0*
    - 2.8: Elementary Matrices - *CC BY-NC-SA 4.0*
    - 2.9: More on Matrix Inverses - *CC BY-NC-SA 4.0*
    - 2.10: LU Factorization - *CC BY-NC-SA 4.0*
    - 2.11: Exercises - *CC BY-NC-SA 4.0*
  - 3: Determinants - *CC BY-NC-SA 4.0*
    - 3.1: Basic Techniques - *CC BY-NC-SA 4.0*
    - 3.2: Properties of Determinants - *CC BY-NC-SA 4.0*
    - 3.3: Finding Determinants using Row Operations - *CC BY-NC-SA 4.0*
    - 3.4: Applications of the Determinant - *CC BY-NC-SA 4.0*
    - 3.5: Exercises - *CC BY-NC-SA 4.0*
  - 4: R^n - *CC BY-NC-SA 4.0*
    - 4.1: Vectors in R^n - *CC BY-NC-SA 4.0*
    - 4.2: Vector Algebra - *CC BY-NC-SA 4.0*
    - 4.3: Geometric Meaning of Vector Addition - *CC BY-NC-SA 4.0*
    - 4.4: Length of a Vector - *CC BY-NC-SA 4.0*
    - 4.5: Geometric Meaning of Scalar Multiplication - *CC BY-NC-SA 4.0*
    - 4.6: Parametric Lines - *CC BY-NC-SA 4.0*
    - 4.7: Spanning, Linear Independence and Basis in R^n - *CC BY-NC-SA 4.0*
    - 4.8: Orthogonality - *CC BY-NC-SA 4.0*
    - 4.9: Applications - *CC BY-NC-SA 4.0*
    - 4.10: Exercises - *CC BY-NC-SA 4.0*
  - 5: Linear Transformations - *CC BY-NC-SA 4.0*
    - 5.1: Linear Transformations - *CC BY-NC-SA 4.0*
    - 5.2: The Matrix of a Linear Transformation I - *CC BY-NC-SA 4.0*
    - 5.3: Properties of Linear Transformations - *CC BY-NC-SA 4.0*
    - 5.4: Special Linear Transformations in R^2 - *CC BY-NC-SA 4.0*
    - 5.5: One-to-One and Onto Transformations - *CC BY-NC-SA 4.0*
    - 5.6: The Kernel and Image of A Linear Map - *CC BY-NC-SA 4.0*
    - 5.7: The Matrix of a Linear Transformation II - *CC BY-NC-SA 4.0*
    - 5.8: The General Solution of a Linear System - *CC BY-NC-SA 4.0*

- 5.9: Exercises - *CC BY-NC-SA 4.0*
- 6: Spectral Theory - *CC BY-NC-SA 4.0*
  - 6.1: Eigenvalues and Eigenvectors of a Matrix - *CC BY-NC-SA 4.0*
  - 6.2: Diagonalization - *CC BY-NC-SA 4.0*
  - 6.3: Applications of Spectral Theory - *CC BY-NC-SA 4.0*
  - 6.4: Orthogonality - *CC BY-NC-SA 4.0*
  - 6.5: Exercises - *CC BY-NC-SA 4.0*
- 7: What are Differential Equations? - *CC BY-NC-SA 4.0*
  - 7.1: Applications Leading to Differential Equations - *CC BY-NC-SA 4.0*
  - 7.2: Basic Concepts - *CC BY-NC-SA 4.0*
    - 7.2.1: Basic Concepts (Exercises) - *CC BY-NC-SA 4.0*
  - 7.3: Direction Fields for First Order Equations - *CC BY-NC-SA 4.0*
    - 7.3.1: Direction Fields for First Order Equations (Exercises) - *CC BY-NC-SA 4.0*
- 8: First Order Equations - *CC BY-NC-SA 4.0*
  - 8.1: Linear First Order Equations - *CC BY-NC-SA 4.0*
    - 8.1.1: Linear First Order Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 8.2: Separable Equations - *CC BY-NC-SA 4.0*
    - 8.2.1: Separable Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 8.3: Existence and Uniqueness of Solutions of Nonlinear Equations - *CC BY-NC-SA 4.0*
    - 8.3.1: Existence and Uniqueness of Solutions of Nonlinear Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 8.4: Transformation of Nonlinear Equations into Separable Equations - *CC BY-NC-SA 4.0*
    - 8.4.1: Transformation of Nonlinear Equations into Separable Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 8.5: Exact Equations - *CC BY-NC-SA 4.0*
    - 8.5.1: Exact Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 8.6: Integrating Factors - *CC BY-NC-SA 4.0*
    - 8.6.1: Integrating Factors (Exercises) - *CC BY-NC-SA 4.0*
- 9: Applications of First Order Equations - *CC BY-NC-SA 4.0*
  - 9.1: Growth and Decay - *CC BY-NC-SA 4.0*
    - 9.1.1: Growth and Decay (Exercises) - *CC BY-NC-SA 4.0*
  - 9.2: Cooling and Mixing - *CC BY-NC-SA 4.0*
    - 9.2.1: Cooling and Mixing (Exercises) - *CC BY-NC-SA 4.0*
  - 9.3: Elementary Mechanics - *CC BY-NC-SA 4.0*
    - 9.3.1: Elementary Mechanics (Exercises) - *CC BY-NC-SA 4.0*
- 10: Linear Second Order Equations - *CC BY-NC-SA 4.0*
  - 10.1: Homogeneous Linear Equations - *CC BY-NC-SA 4.0*
    - 10.1.1: Homogeneous Linear Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 10.2: Constant Coefficient Homogeneous Equations - *CC BY-NC-SA 4.0*
    - 10.2.1: Constant Coefficient Homogeneous Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 10.3: Nonhomogeneous Linear Equations - *CC BY-NC-SA 4.0*
    - 10.3.1: Nonhomgeneous Linear Equations (Exercises) - *CC BY-NC-SA 4.0*
  - 10.4: The Method of Undetermined Coefficients I - *CC BY-NC-SA 4.0*
    - 10.4.1: The Method of Undetermined Coefficients I (Exercises) - *CC BY-NC-SA 4.0*
  - 10.5: The Method of Undetermined Coefficients II - *CC BY-NC-SA 4.0*
    - 10.5.1: The Method of Undetermined Coefficients II (Exercises) - *CC BY-NC-SA 4.0*
  - 10.6: Reduction of Order - *CC BY-NC-SA 4.0*
    - 10.6.1: Reduction of Order (Exercises) - *CC BY-NC-SA 4.0*
- 11: Applications of Linear Second Order Equations - *CC BY-NC-SA 4.0*
  - 11.1: Spring Problems I - *CC BY-NC-SA 4.0*
    - 11.1.1: Spring Problems I (Exercises) - *CC BY-NC-SA 4.0*
  - 11.2: Spring Problems II - *CC BY-NC-SA 4.0*
    - 11.2.1: Spring Problems II (Exercises) - *CC BY-NC-SA 4.0*
- 12: Series Solutions of Linear Second Order Equations - *CC BY-NC-SA 4.0*
  - 12.1: Prelude to Series Solutions of Linear Second Order Equations - *CC BY-NC-SA 4.0*
  - 12.2: Review of Power Series - *CC BY-NC-SA 4.0*

- 12.2.1: Review of Power Series (Exercises) - *CC BY-NC-SA 4.0*
- 12.3: Series Solutions Near an Ordinary Point I - *CC BY-NC-SA 4.0*
  - 12.3.1: Series Solutions Near an Ordinary Point I (Exercises) - *CC BY-NC-SA 4.0*

13: Laplace Transforms - *CC BY-NC-SA 4.0*

- 13.1: Introduction to the Laplace Transform - *CC BY-NC-SA 4.0*
  - 13.1.1: Introduction to the Laplace Transform (Exercises) - *CC BY-NC-SA 4.0*
- 13.2: The Inverse Laplace Transform - *CC BY-NC-SA 4.0*
  - 13.2.1: The Inverse Laplace Transform (Exercises) - *CC BY-NC-SA 4.0*
- 13.3: Solution of Initial Value Problems - *CC BY-NC-SA 4.0*
  - 13.3.1: Solution of Initial Value Problems (Exercises) - *CC BY-NC-SA 4.0*
- 13.4: The Unit Step Function - *CC BY-NC-SA 4.0*
  - 13.4.1: The Unit Step Function (Exercises) - *CC BY-NC-SA 4.0*
- 13.5: Constant Coefficient Equations with Piecewise Continuous Forcing Functions - *CC BY-NC-SA 4.0*
  - 13.5.1: Constant Coefficient Equations with Piecewise Continuous Forcing Functions (Exercises) - *CC BY-NC-SA 4.0*
- 13.6: A Brief Table of Laplace Transforms - *CC BY-NC-SA 4.0*

14: Linear Higher Order Differential Equations - *CC BY-NC-SA 4.0*

- 14.1: Introduction to Linear Higher Order Equations - *CC BY-NC-SA 4.0*
  - 14.1.1: Introduction to Linear Higher Order Equations (Exercises) - *CC BY-NC-SA 4.0*
- 14.2: Higher Order Constant Coefficient Homogeneous Equations - *CC BY-NC-SA 4.0*
  - 14.2.1: Higher Order Constant Coefficient Homogeneous Equations (Exercises) - *CC BY-NC-SA 4.0*

15: Linear Systems of Differential Equations - *CC BY-NC-SA 4.0*

- 15.1: Introduction to Systems of Differential Equations - *CC BY-NC-SA 4.0*
  - 15.1.1: Introduction to Systems of Differential Equations (Exercises) - *CC BY-NC-SA 4.0*
- 15.2: Linear Systems of Differential Equations - *CC BY-NC-SA 4.0*
  - 15.2.1: Linear Systems of Differential Equations (Exercises) - *CC BY-NC-SA 4.0*
- 15.3: Basic Theory of Homogeneous Linear Systems - *CC BY-NC-SA 4.0*
  - 15.3.1: Basic Theory of Homogeneous Linear Systems (Exercises) - *CC BY-NC-SA 4.0*
- 15.4: Constant Coefficient Homogeneous Systems I - *CC BY-NC-SA 4.0*
  - 15.4.1: Constant Coefficient Homogeneous Systems I (Exercises) - *CC BY-NC-SA 4.0*

16: Appendices - *CC BY-NC-SA 4.0*

- 16.1: Trigonometric Identities - *CC BY-NC-SA 4.0*
- 16.2: Review of Derivative Rules - *CC BY-NC-SA 4.0*
- 16.3: Table of Derivatives - *CC BY-NC-SA 4.0*
- 16.4: Table of Integrals - *CC BY-NC-SA 4.0*

Back Matter - *CC BY-NC-SA 4.0*

- Detailed Licensing - *CC BY-NC-SA 4.0*
- Index - *CC BY-NC-SA 4.0*
- Glossary - *CC BY-NC-SA 4.0*
- Detailed Licensing - *CC BY-NC-SA 4.0*