MAT186: PRE-CALCULUS - WALSH



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MAT186: Pre-calculus - Walsh

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Licensing

*A detailed breakdown of this resource's licensing can be found in **Back Matter/Detailed Licensing**.*

CHAPTER OVERVIEW

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1.1: Functions and Function Notation

Function

A **function** is a relation in which each possible input value leads to exactly one output value. We say “the output is a function of the input.”

The **input** values make up the **domain**, and the **output** values make up the **range**.

How To: Given a relationship between two quantities, determine whether the relationship is a function

1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, classify the relationship as a function. If any input value leads to two or more outputs, do not classify the relationship as a function.

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1.2: Domain and Range

Graphing Piecewise-Defined Functions

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function $f(x) = |x|$. With a domain of all real numbers and a range of values greater than or equal to 0, **absolute value** can be defined as the **magnitude**, or **modulus**, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0.

If we input 0, or a positive value, the output is the same as the input.

$$f(x) = x \text{ if } x \geq 0$$

If we input a negative value, the output is the opposite of the input.

$$f(x) = -x \text{ if } x < 0$$

Because this requires two different processes or pieces, the absolute value function is an example of a **piecewise function**. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%. The tax on a total income S would be $0.1S$ if $S \leq \$10,000$ and $\$1000 + 0.2(S - \$10,000)$ if $S > \$10,000$.

Piecewise Function

A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1} & \text{if x is in domain 1} \\ \text{formula 2} & \text{if x is in domain 2} \\ \text{formula 3} & \text{if x is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

How To: Given a piecewise function, write the formula and identify the domain for each interval.

1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

Example 1.2.8A: Writing a Piecewise Function

A museum charges \$5 per person for a guided tour with a group of 1 to 9 people or a fixed \$50 fee for a group of 10 or more people. Write a **function** relating the number of people, n, to the cost, C.

Solution

Two different formulas will be needed. For n-values under 10, $C = 5n$. For values of n that are 10 or greater, $C = 50$.

$$C(n) = \begin{cases} 5n & \text{if } n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

Analysis

The function is represented in Figure 1.2.20. The graph is a diagonal line from $n = 0$ to $n = 10$ and a constant after that. In this example, the two formulas agree at the meeting point where $n = 10$, but not all piecewise functions have this property.

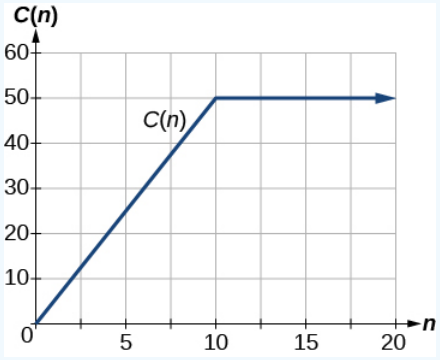


Figure 1.2.20

Example 1.2.8B: Working with a Piecewise Function

A cell phone company uses the function below to determine the cost, C, in dollars for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 25 + 10(g-2) & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

Soltuion

To find the cost of using 1.5 gigabytes of data, $C(1.5)$, we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data, C(4), we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 25 + 10(4-2) = \$45$$

Analysis

The function is represented in Figure 1.2.21. We can see where the function changes from a constant to a shifted and stretched identity at $g = 2$. We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.

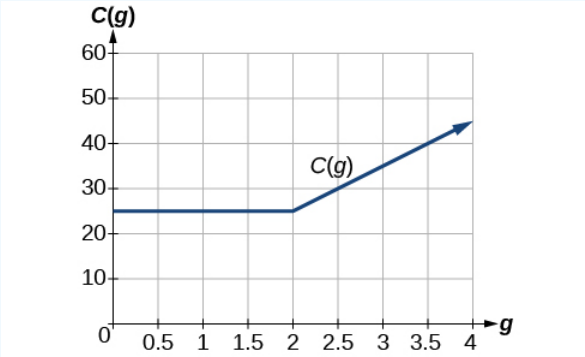


Figure 1.2.21

How To: Given a piecewise function, sketch a graph.

1. Indicate on the x-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

Example 1.2.8C: Graphing a Piecewise Function

Sketch a graph of the function.

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 3 & \text{if } 1 < x \leq 2 \\ x & \text{if } x > 2 \end{cases}$$

Solution

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.

Figure 1.2.20 shows the three components of the piecewise function graphed on separate coordinate systems.

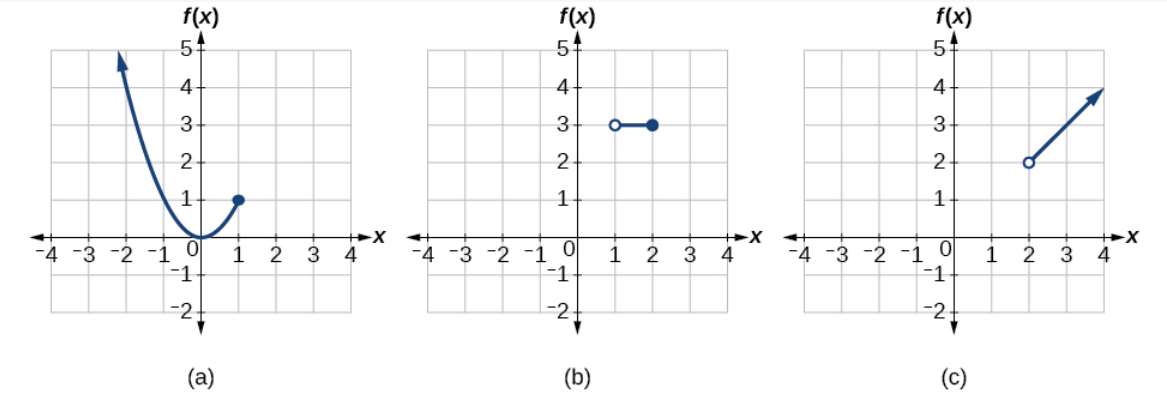


Figure 1.2.20: *Graph of each part of the piece-wise function f(x)*

(a) $f(x) = x^2$ if $x \leq 1$; (b) $f(x) = 3$ if $1 < x \leq 2$; (c) $f(x) = x$ if $x > 2$

Now that we have sketched each piece individually, we combine them in the same coordinate plane. See Figure 1.2.21.

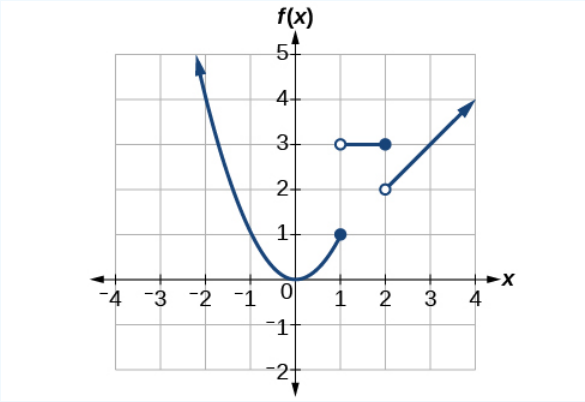


Figure 1.2.21: Graph of the entire function.

Analysis

Note that the graph does pass the vertical line test even at $x = 1$ and $x = 2$ because the points $(1, 3)$ and $(2, 2)$ are not part of the graph of the function, though $(1, 1)$ and $(2, 3)$ are.

Exercise 1.2.8

Graph the following piecewise function.

$$f(x) = \begin{cases} x^3 & \text{if } x < -1 \\ -2 & \text{if } -1 < x < 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$$

Answer

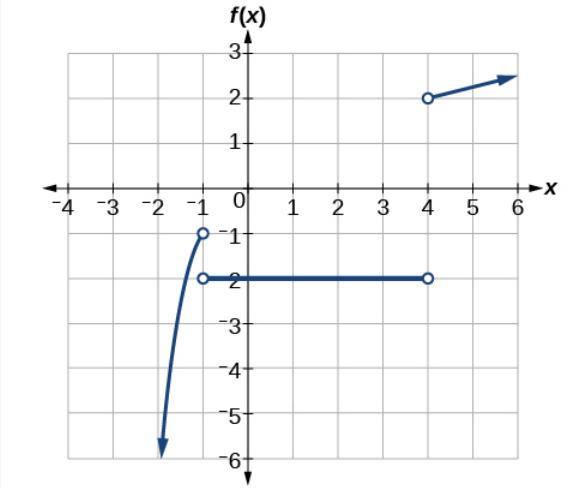


Figure 1.2.22

Can more than one formula from a piecewise function be applied to a value in the domain?

No. Each value corresponds to one equation in a piecewise formula.

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1.3: Rates of Change and Behavior of Graphs

Finding the Average Rate of Change of a Function

The price change per year is a **rate of change** because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in Table 1.3.1 did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the **average rate of change** over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.

$$\begin{aligned}\text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}\end{aligned}$$

The Greek letter Δ (delta) signifies the change in a quantity; we read the ratio as "delta-y over delta-x" or "the change in y divided by the change in x." Occasionally we write Δf instead of Δy, which still represents the change in the function's output value resulting from a change to its input value. It does not mean we are changing the function into some other function.

In our example, the gasoline price increased by $1.37 from 2005 to 2012. Over 7 years, the average rate of change was

$$\frac{\Delta y}{\Delta x} = \frac{\$1.37}{7\text{years}} \approx 0.196 \text{ dollars per year.} \tag{1.3.1}$$

On average, the price of gas increased by about 19.6¢ each year. Other examples of rates of change include:

- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by $4,000 per quarter

Definition: Rate of Change

A **rate of change** describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."

The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{1.3.2}$$

Figure 1.3.5: Definition of a local maximum

These observations lead us to a formal definition of local extrema.

Local Minima and Local Maxima

- A function f is an **increasing function** on an open interval if $f(b) > f(a)$ for every a, b interval where $b > a$.
- A function f is a **decreasing function** on an open interval if $f(b) < f(a)$ for every a, b interval where $b > a$.

A function f has a local maximum at a point b in an open interval (a, c) if $f(b)$ is greater than or equal to $f(x)$ for every point x (x does not equal b) in the interval. Likewise, f has a local minimum at a point b in (a, c) if $f(b)$ is less than or equal to

> $f(x)$ for every x (x does not equal b) in the interval.

Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in Figure 1.3.10, Figure 1.3.11, and Figure 1.3.12

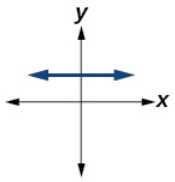
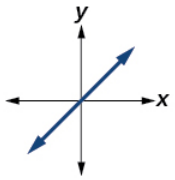
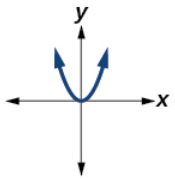
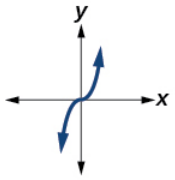
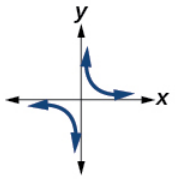
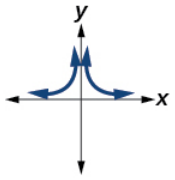
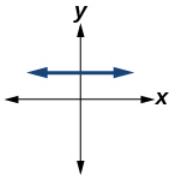
Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	

Figure 1.3.10

Function	Increasing/Decreasing	Example
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	

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Figure 1.3.11

Function	Increasing/Decreasing	Example
Constant Function $f(x) = c$	Neither increasing nor decreasing	

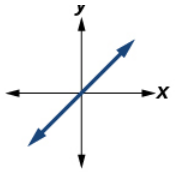
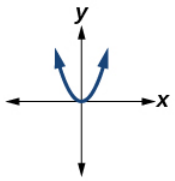
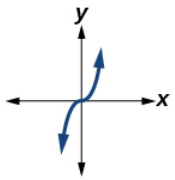
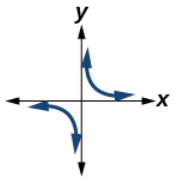
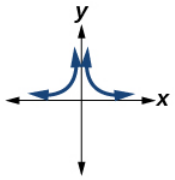
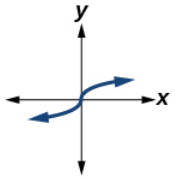
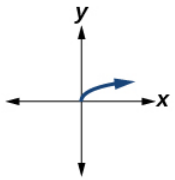
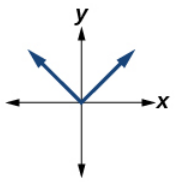
Identity Function $f(x) = x$	Increasing	
Quadratic Function $f(x) = x^2$	Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ Minimum at $x = 0$	
Cubic Function $f(x) = x^3$	Increasing	
Reciprocal $f(x) = \frac{1}{x}$	Decreasing $(-\infty, 0) \cup (0, \infty)$	
Reciprocal Squared $f(x) = \frac{1}{x^2}$	Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$	
Cube Root $f(x) = \sqrt[3]{x}$	Increasing	
Square Root $f(x) = \sqrt{x}$	Increasing on $(0, \infty)$	
Absolute Value $f(x) = \lvert x \rvert$	Increasing on $(0, \infty)$ Decreasing on $(\infty, 0)$	

***Figure* 1.3.12**

Use A Graph to Locate the Absolute Maximum and Absolute Minimum

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The y-coordinates (output) at the highest and lowest points are called the **absolute maximum** and **absolute minimum**, respectively. To locate absolute maxima and minima from a

graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function (Figure 1.3.13).

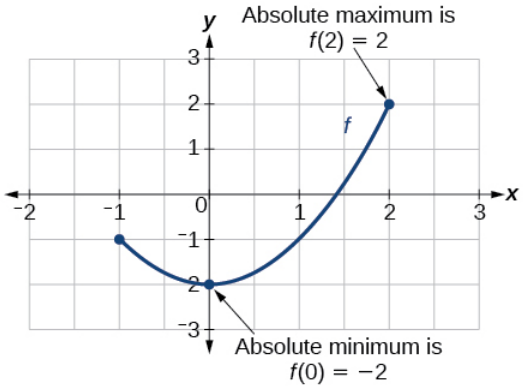


Figure 1.3.13: Graph of a segment of a parabola with an absolute minimum at (0, -2) and absolute maximum at (2, 2).

Not every function has an absolute maximum or minimum value. The toolkit function $f(x) = x^3$ is one such function.

Absolute Maxima and Minima

- The **absolute maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f.
- The **absolute minimum** of f at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all x in the domain of f.

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1.4: Composition of Functions

Create a Function by Composition of Functions

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year, we created a new function that takes a day as input and yields a cost as output. The process of **combining functions** so that the output of one function becomes the input of another is known as a **composition of functions**. The resulting function is known as a **composite function**. We represent this combination by the following notation:

$$f \circ g(x) = f(g(x)) \tag{1.4.1}$$

We read the left-hand side as"f composed with g at x," and the right-hand side as"f of g of x."The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol $\circ$ is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases $f(g(x)) \neq f(x)g(x)$.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the outside. In the equation above, the function g takes the input x first and yields an output $g(x)$. Then the function f takes $g(x)$ as an input and yields an output $f(g(x))$.

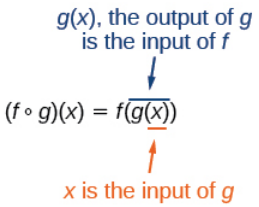


Figure 1.4.2: Explanation of the composite function.

In general, $f \circ g$ and $g \circ f$ are different functions. In other words, in many cases $f(g(x)) \neq g(f(x))$ for all x. We will also see that sometimes two functions can be composed only in one specific order.

For example, if $f(x) = x^2$ and $g(x) = x + 2$, then

$$\begin{aligned} f(g(x)) &= f(x+2) \\ &= (x+2)^2 \\ &= x^2 + 4x + 4 \end{aligned}$$

but

$$\begin{aligned} g(f(x)) &= g(x^2) \\ &= x^2 + 2 \end{aligned}$$

These expressions are not equal for all values of x, so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value $x = -\frac{1}{2}$.

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs.

Composition of Functions

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)) \tag{1.4.2}$$

The domain of the composite function $f \circ g$ is all x such that x is in the domain of g and $g(x)$ is in the domain of f.

It is important to realize that the product of functions fg is not the same as the function composition $f(g(x))$, because, in general, $f(x)g(x) \neq f(g(x))$.

Finding the Domain of a Composite Function

As we discussed previously, the **domain of a composite function** such as $f \circ g$ is dependent on the domain of g and the domain of f. It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as $f \circ g$. Let us assume we know the domains of the functions f and g separately. If we write the composite function for an input x as $f(g(x))$, we can see right away that x must be a member of the domain of g in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that $g(x)$ must be a member of the domain of f, otherwise the second function evaluation in $f(g(x))$ cannot be completed, and the expression is still undefined. Thus the domain of $f \circ g$ consists of only those inputs in the domain of g that produce outputs from g belonging to the domain of f. Note that the domain of f composed with g is the set of all x such that x is in the domain of g and g(x)\) is in the domain of f.

Definition: Domain of a Composite Function

The **domain of a composite function** $f(g(x))$ is the set of those inputs x in the domain of g for which $g(x)$ is in the domain of f.

How To...

Given a function composition $f(g(x))$, determine its domain.

1. Find the domain of g.
2. Find the domain of f.
3. Find those inputs x in the domain of g for which $g(x)$ is in the domain of f. That is, exclude those inputs x from the domain of g for which $g(x)$ is not in the domain of f. The resulting set is the domain of $f \circ g$.

Example 1.4.8A: Finding the Domain of a Composite Function

Find the domain of

$$(f \circ g)(x) \text{ where } f(x) = \frac{5}{x-1} \text{ and } g(x) = \frac{4}{3x-2}$$

Solution

The domain of $g(x)$ consists of all real numbers except $x = \frac{2}{3}$, since that input value would cause us to divide by 0. Likewise, the domain of f consists of all real numbers except 1. So we need to exclude from the domain of $g(x)$ that value of x for which $g(x) = 1$.

$$\begin{aligned} \frac{4}{3x-2} &= 1 \\ 4 &= 3x-2 \\ 6 &= 3x \\ x &= 2 \end{aligned}$$

So the domain of $f \circ g$ is the set of all real numbers except $\frac{2}{3}$ and 2. This means that

$$x \neq \frac{2}{3} \text{ or } x \neq 2$$

We can write this in interval notation as

$$\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 2\right) \cup (2, \infty)$$

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1.5: Transformation of Functions

Identifying Vertical Shifts

One simple kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function $g(x) = f(x) + k$, the function $f(x)$ is shifted vertically k units. See Figure 1.5.2 for an example.

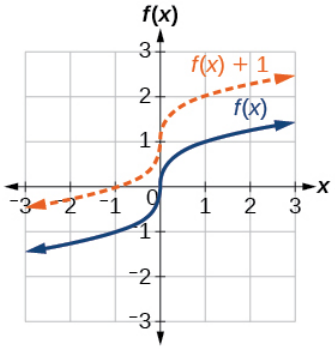


Figure 1.5.2: Vertical shift by $k = 1$ of the cube root function $f(x) = \sqrt[3]{x}$.

To help you visualize the concept of a vertical shift, consider that $y = f(x)$. Therefore, $f(x) + k$ is equivalent to $y + k$. Every unit of y is replaced by $y + k$, so the y-value increases or decreases depending on the value of k. The result is a shift upward or downward.

Definition: Vertical Shift

Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a **vertical shift** of the function $f(x)$. All the output values change by k units. If k is positive, the graph will shift up. If k is negative, the graph will shift down.

Example 1.5.1: Adding a Constant to a Function

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 1.5.3 shows the area of open vents V (in square feet) throughout the day in hours after midnight, t. During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.

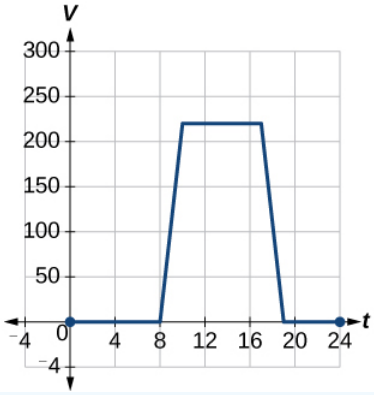


Figure 1.5.3

Solution

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up, as shown in Figure 1.5.4.

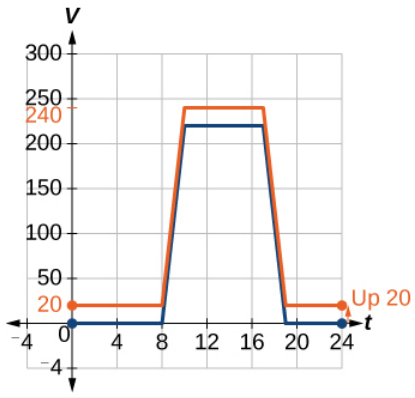


Figure 1.5.4

Graphing Functions Using Reflections about the Axes

Another transformation that can be applied to a function is a reflection over the x- or y-axis. A **vertical reflection** reflects a graph vertically across the x-axis, while a **horizontal reflection** reflects a graph horizontally across the y-axis. The reflections are shown in Figure 1.5.13.

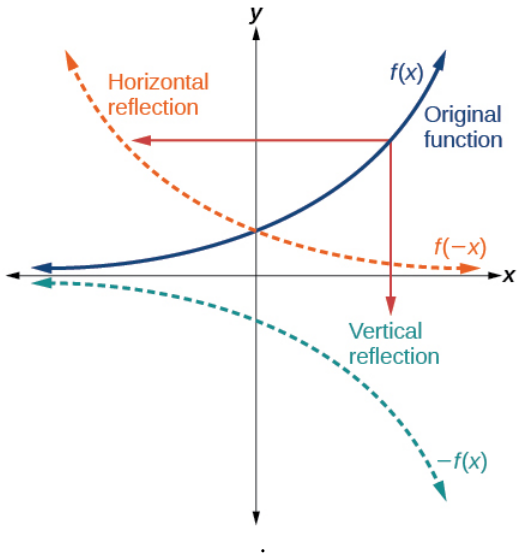


Figure 1.5.13: Graph of the vertical and horizontal reflection of a function.

Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the x-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the y-axis.

Definitions: Reflections

Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a reflection about (or over, or through) the x-axis.

Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a reflection about the y-axis.

How To...

Given a function, reflect the graph both vertically and horizontally.

1. Multiply all outputs by –1 for a vertical reflection. The new graph is a reflection of the original graph about the x-axis.
2. Multiply all inputs by –1 for a horizontal reflection. The new graph is a reflection of the original graph about the y-axis.

Example 1.5.10: Reflecting a Graph Horizontally and Vertically

Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.

Solution

a. Reflecting the graph vertically means that each output value will be reflected over the horizontal t-axis as shown in Figure 1.5.14

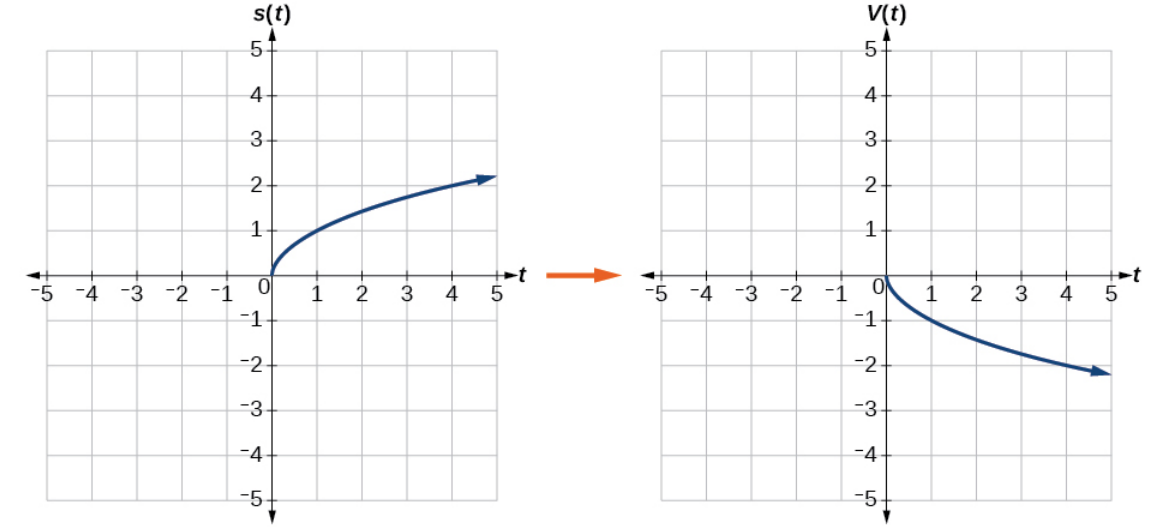


Figure 1.5.14: Graph of the vertical reflection of the square root function.

Because each output value is the opposite of the original output value, we can write

$$V(t) = -s(t) \text{ or } V(t) = -\sqrt{t}$$

Notice that this is an outside change, or vertical shift, that affects the output $s(t)$ values, so the negative sign belongs outside of the function.

b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown in Figure 1.5.15.

Determining Even and Odd Functions

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions $f(x) = x^2$ or $f(x) = |x|$ will result in the original graph. We say that these types of graphs are symmetric about the y-axis. Functions whose graphs are symmetric about the y-axis are called **even functions**.

If the graphs of $f(x) = x^3$ or $f(x) = \frac{1}{x}$ were reflected over both axes, the result would be the original graph, as shown in Figure 1.5.21.

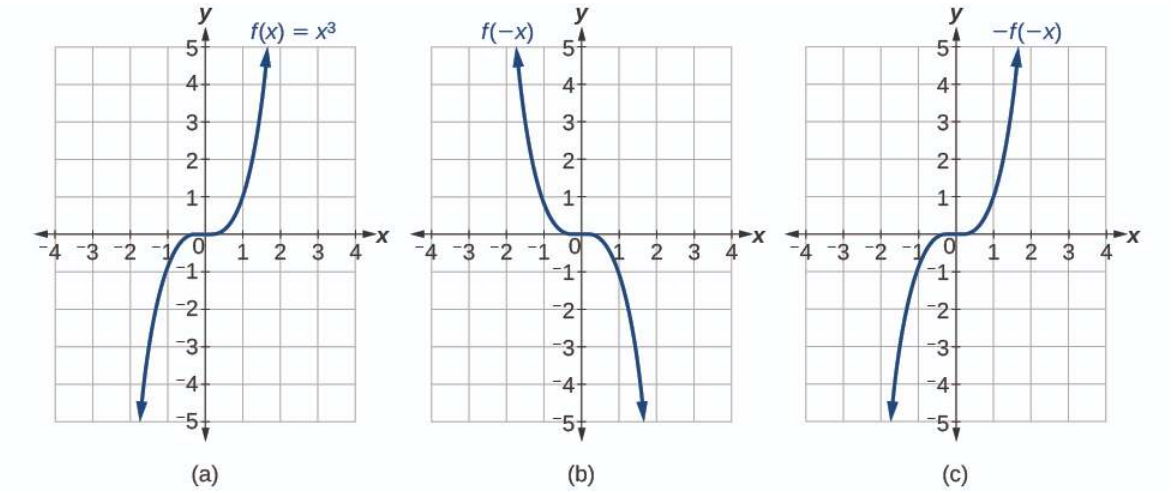


Figure 1.5.21: (a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an **odd function**.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, $f(x) = 2^x$ is neither even nor odd. Also, the only function that is both even and odd is the constant function $f(x) = 0$.

Definitions: Even and Odd Functions

A function is called an **even function** if for every input x

$f(x) = f(-x)$

The graph of an even function is symmetric about the y-axis.

A function is called an **odd function** if for every input x

$f(x) = -f(-x)$

The graph of an odd function is symmetric about the origin.

How To...

Given the formula for a function, determine if the function is even, odd, or neither.

1. Determine whether the function satisfies $f(x) = f(-x)$. If it does, it is even.
2. Determine whether the function satisfies $f(x) = -f(-x)$. If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

Example 1.5.13: Determining whether a Function Is Even, Odd, or Neither

Is the function $f(x) = x^3 + 2x$ even, odd, or neither?

Solution

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.

$$f(-x) = (-x)^3 + 2(-x) = -x^3 - 2x$$

This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.

$$-f(-x) = -(-x^3 - 2x) = x^3 + 2x$$

Because $-f(-x) = f(x)$, this is an odd function.

Analysis

Consider the graph of f in Figure 1.5.22 Notice that the graph is symmetric about the origin. For every point (x, y) on the graph, the corresponding point $(-x, -y)$ is also on the graph. For example, $(1, 3)$ is on the graph of f, and the corresponding point $(-1, -3)$ is also on the graph.

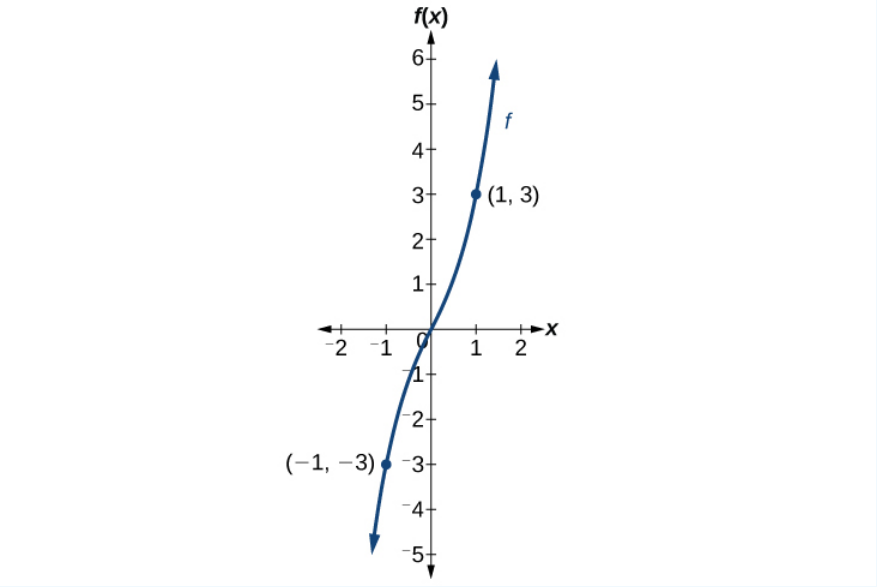


Figure 1.5.22: Graph of $f(x)$ with labeled points at $(1, 3)$ and $(-1, -3)$.

Graphing Functions Using Stretches and Compressions

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

Vertical Stretches and Compressions

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1, we get a **vertical stretch**; if the constant is between 0 and 1, we get a **vertical compression**. Figure 1.5.23 shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.

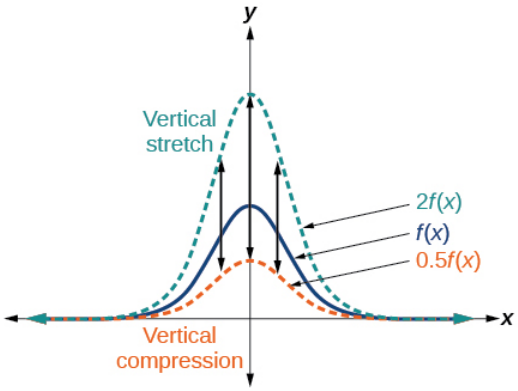


Figure 1.5.23: Vertical stretch and compression

Definitions: Vertical Stretches and Compressions

Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a **vertical stretch** or **vertical compression** of the function $f(x)$.

- If $a > 1$, then the graph will be stretched.
- If $0 < a < 1$, then the graph will be compressed.
- If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

Performing a Sequence of Transformations

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as $2f(x)+3$, which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of $f(x)$, we first multiply by 2, causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

Horizontal transformations are a little trickier to think about. When we write $g(x) = f(2x+3)$, for example, we have to think about how the inputs to the function g relate to the inputs to the function f. Suppose we know $f(7) = 12$. What input to g would produce that output? In other words, what value of x will allow $g(x) = f(2x+3) = 12$? We would need $2x+3 = 7$. To solve for x, we would first subtract 3, resulting in a horizontal shift, and then divide by 2, causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.

$$f(bx+p) = f(b(x+\frac{p}{b}))$$

Let's work through an example.

$$f(x) = (2x+4)^2$$

We can factor out a 2.

$$f(x) = (2(x+2))^2$$

Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression. Factoring in this way allows us to horizontally stretch first and then shift horizontally.

Combining Transformations

- When combining vertical transformations written in the form $af(x)+k$, first vertically stretch by a and then vertically shift by k.
- When combining horizontal transformations written in the form $f(bx+h)$, first horizontally shift by h and then horizontally stretch by $\frac{1}{b}$.
- When combining horizontal transformations written in the form $f(b(x+h))$, first horizontally stretch by $\frac{1}{b}$ and then horizontally shift by h.
- Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

Example 1.5.20: Finding a Triple Transformation of a Tabular Function

Given Table 1.5.18 for the function $f(x)$, create a table of values for the function $g(x) = 2f(3x)+1$.

Table 1.5.18

x	6	12	18	24

$f(x)$	10	14	15	17

Solution

There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, $f(3x)$ is a horizontal compression by $\frac{1}{3}$, which means we multiply each x-value by $\frac{1}{3}$.See Table 1.5.19

Table 1.5.19

x	2	4	6	8
$f(3x)$	10	14	15	17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation. See Table 1.5.20

Table 1.5.20

x	2	4	6	8
$2f(3x)$	20	28	30	34

Finally, we can apply the vertical shift, which will add 1 to all the output values. See Table 1.5.21.

Table 1.5.21

x	2	4	6	8
$g(x)=2f(3x)+1+1$	21	29	31	35

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1.6: Absolute Value Functions

Understanding Absolute Value

Recall that in its basic form $f(x) = |x|$, the **absolute value** function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance the number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

Absolute Value Function

The absolute value function can be defined as a piecewise function

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \tag{1.6.1}$$

Example 1.6.1: Determine a Number within a Prescribed Distance

Describe all values x within or including a distance of 4 from the number 5.

Solution

We want the distance between x and 5 to be less than or equal to 4. We can draw a number line, such as the one in , to represent the condition to be satisfied.

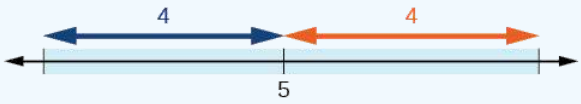


Figure 1.6.2: Number line describing the difference of the distance of 4 away from 5.

The distance from x to 5 can be represented using the absolute value as $|x-5|$. We want the values of x that satisfy the condition $|x-5| \leq 4$.

Analysis

Note that

$$\begin{aligned} -4 \leq x-5 \qquad & x-5 \leq 4 \\ 1 \leq x \qquad & x \leq 9 \end{aligned}$$

So $|x-5| \leq 4$ is equivalent to $1 \leq x \leq 9$.

However, mathematicians generally prefer absolute value notation.

Solving an Absolute Value Inequality

Absolute value equations may not always involve equalities. Instead, we may need to solve an equation within a range of values. We would use an **absolute value inequality** to solve such an equation. An absolute value inequality is an equation of the form

$$|A| < B, \ |A| \leq B, |A| > B,$$

or

$$|A| \geq B,$$

where an expression A (and possibly but not usually B) depends on a variable x. Solving the inequality means finding the set of all x that satisfy the inequality. Usually this set will be an interval or the union of two intervals.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two functions. The advantage of the algebraic approach is it yields solutions that may be difficult to read from the graph.

For example, we know that all numbers within 200 units of 0 may be expressed as

$$|x| < 200$$

or

$$-200 < x < 200$$

Suppose we want to know all possible returns on an investment if we could earn some amount of money within $200 of $600. We can solve algebraically for the set of values x such that the distance between x and 600 is less than 200. We represent the distance between x and 600 as $|x-600|$.

$$|x-600| < 200 \tag{1.6.2}$$

or

$$-200 < x-600 < 200 \tag{1.6.3}$$

$$-200+600 < x-600+600 < 200+600$$

$$400 < x < 800$$

This means our returns would be between $400 and $800.

Sometimes an absolute value inequality problem will be presented to us in terms of a shifted and/or stretched or compressed absolute value function, where we must determine for which values of the input the function's output will be negative or positive.

How To ...

Given an absolute value inequality of the form $|x-A| \leq B$ for real numbers a and b where b is positive, solve the absolute value inequality algebraically.

1. Find boundary points by solving $|x-A| = B$.
2. Test intervals created by the boundary points to determine where $|x-A| \leq B$.
3. Write the interval or union of intervals satisfying the inequality in interval, inequality, or set-builder notation.

Example 1.6.6: Solving an Absolute Value Inequality

Solve $|x-5| \leq 4$.

Solution

With both approaches, we will need to know first where the corresponding equality is true. In this case we first will find where $|x-5| = 4$. We do this because the absolute value is a function with no breaks, so the only way the function values can switch from being less than 4 to being greater than 4 is by passing through where the values equal 4. Solve $|x-5| = 4$.

$$\begin{aligned} x-5 &= 4 \quad \text{or} \quad x = 9 \\ x-5 &= -4 \quad\quad\;\; x = 1 \end{aligned}$$

After determining that the absolute value is equal to 4 at $x = 1$ and $x = 9$, we know the graph can change only from being less than 4 to greater than 4 at these values. This divides the number line up into three intervals:

$$x < 1, \; 1 < x < 9, \text{ and } x > 9.$$

To determine when the function is less than 4, we could choose a value in each interval and see if the output is less than or greater than 4, as shown in Table 1.6.1.

Table 1.6.1

Interval test x	$f(x)$	< 4 or > 4	
$x < 1$	0	$\|0-5\| = 5$	Greater than
$1 < x < 9$	6	$\|6-5\| = 1$	Less than
$x > 9$	11	$\|11-5\| = 6$	Greater than

Because $1 \leq x \leq 9$ is the only interval in which the output at the test value is less than 4, we can conclude that the solution to $|x-5| \leq 4$ is $1 \leq x \leq 9$, or $[1, 9]$.

To use a graph, we can sketch the function $f(x) = |x-5|$. To help us see where the outputs are 4, the line $g(x) = 4$ could also be sketched as in Figure 1.6.11.

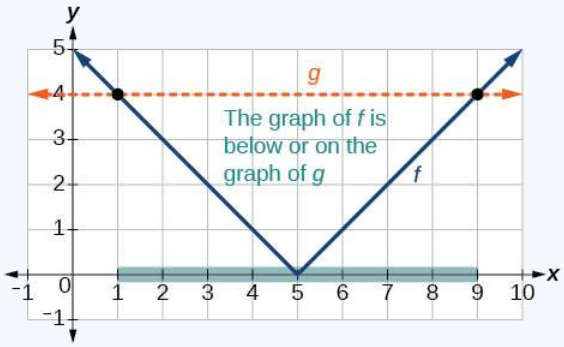


Figure 1.6.11: Graph to find the points satisfying an absolute value inequality.

We can see the following:

- The output values of the absolute value are equal to 4 at $x = 1$ and $x = 9$.
- The graph of f is below the graph of g on $1 < x < 9$. This means the output values of $f(x)$ are less than the output values of $g(x)$.
- The absolute value is less than or equal to 4 between these two points, when $1 \leq x \leq 9$. In interval notation, this would be the interval $[1, 9]$.

Analysis

For absolute value inequalities,

$$\begin{array}{ll} |x-A| < C, & |x-A| > C, \\ -C < x - A < C, & x - A < -C \text{ or } x - A > C. \end{array}$$

The $<$ or $>$ symbol may be replaced by $\leq$ or $\geq$.

So, for this example, we could use this alternative approach.

$$\begin{array}{ll} |x-5| \leq 4 & \\ -4 \leq x - 5 \leq 4 & \text{Rewrite by removing the absolute value bars.} \\ -4+5 \leq x - 5 + 5 \leq 4 + 5 & \text{Isolate the x.} \\ 1 \leq x \leq 9 & \end{array}$$

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1.7: Inverse Functions

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure 1.7.1 provides a visual representation of this question. In this section, we will consider the reverse nature of functions.

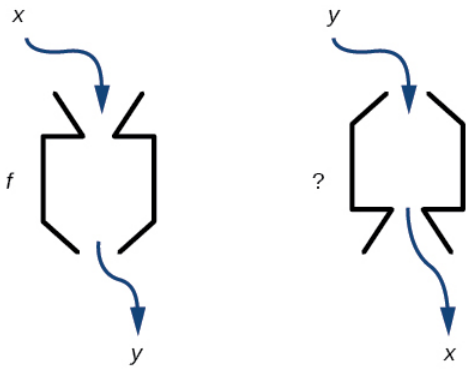


Figure* 1.7.1*: *Can a function "machine" operate in reverse?*

Verifying That Two Functions Are Inverse Functions

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the **Celsius** scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees **Fahrenheit** to degrees Celsius. She finds the formula

$$C = \frac{5}{9}(F - 32) \tag{1.7.1}$$

and substitutes 75 for F to calculate

$$\frac{5}{9}(75 - 32) \approx 24^{\circ} \tag{1.7.2}$$

Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast from Figure 1.7.2 for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

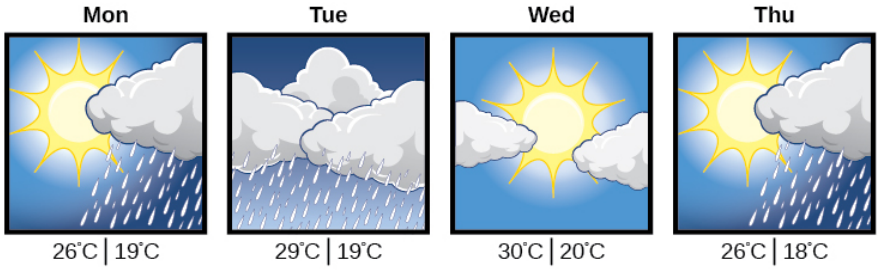


Figure 1.7.2: A forecast of Monday's through Thursday's weather.

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for F after substituting a value for C. For example, to convert 26 degrees Celsius, she could write

$$26 = \frac{5}{9}(F - 32) \tag{1.7.3}$$

$$26 \cdot \frac{9}{5} = F - 32 \tag{1.7.4}$$

$$F = 26 \cdot \frac{9}{5} + 32 \approx 79 \tag{1.7.5}$$

After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an **inverse function**, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function $f(x)$, we represent its inverse as $f^{-1}(x)$, read as "f inverse of x." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, $f^{-1}(x)$ does not mean $\frac{1}{f(x)}$ because $\frac{1}{f(x)}$ is the reciprocal of f and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as $a^{-1}a = 1$ (1 is the identity element for multiplication) for any nonzero number a, so $f^{-1} \circ f$ equals the identity function, that is,

$$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = f^{-1}(y) = x \tag{1.7.6}$$

This holds for all x in the domain of f. Informally, this means that inverse functions "undo" each other. However, just as zero does not have a **reciprocal**, some functions do not have inverses.

Given a function $f(x)$, we can verify whether some other function $g(x)$ is the inverse of $f(x)$ by checking whether either $g(f(x)) = x$ or $f(g(x)) = x$ is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, $y = 4x$ and $y = \frac{1}{4}x$ are inverse functions.

$$(f^{-1} \circ f)(x) = f^{-1}(4x) = \frac{1}{4}(4x) = x \tag{1.7.7}$$

and

$$(f \circ f^{-1})(x) = f\left(\frac{1}{4}x\right) = 4\left(\frac{1}{4}x\right) = x \tag{1.7.8}$$

A few coordinate pairs from the graph of the function $y = 4x$ are $(-2, -8)$, $(0, 0)$, and $(2, 8)$. A few coordinate pairs from the graph of the function $y = \frac{1}{4}x$ are $(-8, -2)$, $(0, 0)$, and $(8, 2)$. If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

Definition: Inverse Function

For any **one-to-one function** $f(x) = y$, a function $f^{-1}(x)$ is an **inverse function** of f if $f^{-1}(y) = x$. This can also be written as $f^{-1}(f(x)) = x$ for all x in the domain of f. It also follows that $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1} if f^{-1} is the inverse of f.

The notation f^{-1} is read "f inverse." Like any other function, we can use any variable name as the input for f^{-1}, so we will often write $f^{-1}(x)$, which we read as "f inverse of x." Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)} \tag{1.7.9}$$

and not all functions have inverses.

Example 1.7.1: Identifying an Inverse Function for a Given Input-Output Pair

If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

Solution

The inverse function reverses the input and output quantities, so if

$$\begin{aligned} f(2) = 4, \text{ then } f^{-1}(4) = 2; \\ f(5) = 12, \text{ then } f^{-1}(12) = 5 \end{aligned} \tag{1.7.10}$$

.

Alternatively, if we want to name the inverse function g, then $g(4) = 2$ and $g(12) = 5$.

Analysis

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed. See Table 1.7.1.

Table 1.7.1

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

Finding Domain and Range of Inverse Functions

The outputs of the function f are the inputs to f^{-1}, so the range of f is also the domain of f^{-1}. Likewise, because the inputs to f are the outputs of f^{-1}, the domain of f is the range of f^{-1}. We can visualize the situation as in Figure 1.7.3.

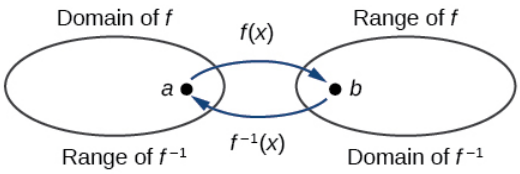


***Figure* 1.7.3:** *Domain and range of a function and its inverse.*

When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of $f(x) = \sqrt{x}$ is $f^{-1}(x) = x^2$, because a square "undoes" a square root; but the square is only the inverse of the square root on the domain $[0, \infty)$, since that is the range of $f(x) = \sqrt{x}$.

We can look at this problem from the other side, starting with the square (toolkit quadratic) function $f(x) = x^2$. If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0). For example, the output 9 from the quadratic function corresponds to the inputs 3 and –3. But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function $f(x) = x^2$ with its range limited to $[0, \infty)$, which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If $f(x) = (x-1)^2$ on $[1, \infty)$, then the inverse function is $f^{-1}(x) = \sqrt{x} + 1$.

- The domain of f = range of $f^{-1} = [1, \infty)$.
- The domain of f^{-1} = range of $f = [0, \infty)$.

Is it possible for a function to have more than one inverse?

No. If two supposedly different functions, say, g and h, both meet the definition of being inverses of another function f, then you can prove that $g = h$. We have just seen that some functions only have inverses if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverses. However, on any one domain, the original function still has only one unique inverse.

Note: Domain and Range of Inverse Functions

The range of a function $f(x)$ is the domain of the inverse function $f^{-1}(x)$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.

How To: Given a function, find the domain and range of its inverse.

1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

Example 1.7.4: Finding the Inverses of Toolkit Functions

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed in Table 1.7.2. We restrict the domain in such a fashion that the function assumes all y-values exactly once.

Table 1.7.2

Constant	Identity	Quadratic	Cubic	Reciprocal
$f(x) = c$	$f(x) = x$	$f(x) = x^2$	$f(x) = x^3$	$f(x) = \frac{1}{x}$
Reciprocal squared	Cube Root	Square Root	Absolute Value	
$f(x) = \frac{1}{x^2}$	$f(x) = \sqrt[3]{x}$	$f(x) = \sqrt{x}$	$f(x) = \lvert x \rvert$	

Solution

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.

The absolute value function can be restricted to the domain $[0, \infty)$,where it is equal to the identity function.

The reciprocal-squared function can be restricted to the domain $(0, \infty)$.

Analysis

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs, shown in Figure 1.7.4. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

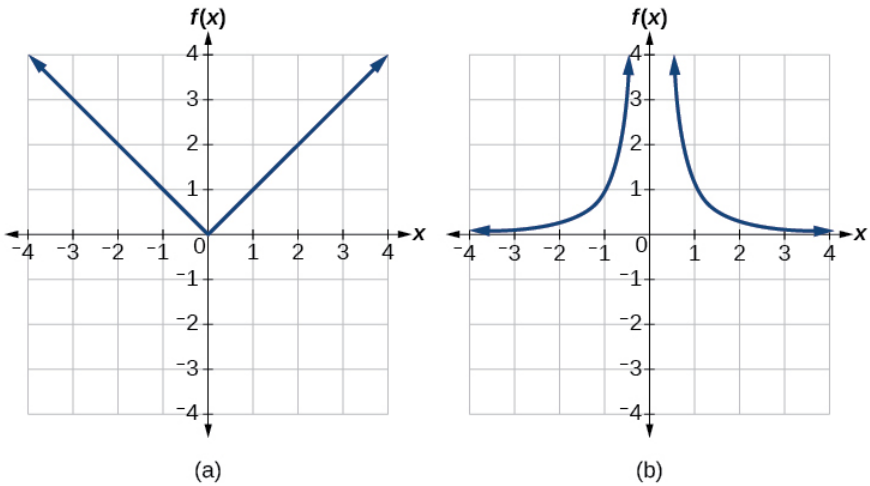


***Figure* 1.7.4:** *(a) Absolute value (b) Reciprocal squared*

1.7.4: The domain of function f is $(1, \infty)$ and the range of function f is $(-\infty, -2)$. Find the domain and range of the inverse function.

Solution

The domain of function f^{-1} is $(-\infty, -2)$ and the range of function f^{-1} is $(1, \infty)$.

Finding and Evaluating Inverse Functions

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

Inverting Tabular Functions

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

Example 1.7.5: Interpreting the Inverse of a Tabular Function

A function $f(t)$ is given in Table 1.7.3, showing distance in miles that a car has traveled in t minutes. Find and interpret $f^{-1}(70)$

Table 1.7.3

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

The inverse function takes an output of f and returns an input for f. So in the expression $f^{-1}(70)$, 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function f, 90 minutes, so $f^{-1}(70) = 90$. The interpretation of this is that, to drive 70 miles, it took 90 minutes.

Alternatively, recall that the definition of the inverse was that if $f(a) = b$, then $f^{-1}(b) = a$. By this definition, if we are given $f^{-1}(70) = a$, then we are looking for a value a so that $f(a) = 70$. In this case, we are looking for a t so that $f(t) = 70$, which is when $t = 90$.

Exercise 1.7.5

Using Table 1.7.4, find and interpret (a) $f(60)$,and (b) $f^{-1}(60)$.

Table 1.7.4

t (minutes)	30	50	60	70	90
$f(t)$ (miles)	20	40	50	60	70

Answer

$f(60) = 50$. In 60 minutes, 50 miles are traveled.
$f^{-1}(60) = 70$. To travel 60 miles, it will take 70 minutes.

Evaluating the Inverse of a Function, Given a Graph of the Original Function

We saw in **Functions and Function Notation** that the domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the *vertical* extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the *horizontal* extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

How To: Given the graph of a function, evaluate its inverse at specific points.

1. Find the desired input on the y-axis of the given graph.
2. Read the inverse function's output from the x-axis of the given graph.

Example 1.7.6: Evaluating a Function and Its Inverse from a Graph at Specific Points

A function $g(x)$ is given in Figure 1.7.5. Find $g(3)$ and $g^{-1}(3)$.

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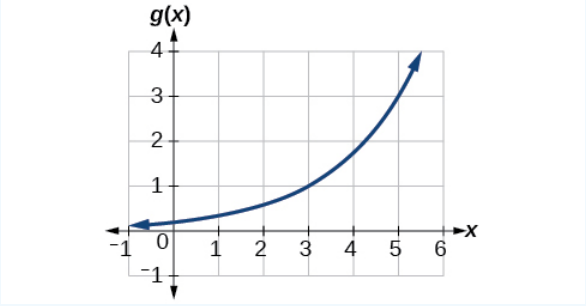


Figure 1.7.5: Graph of $g(x)$

Solution

To evaluate $g(3)$, we find 3 on the x-axis and find the corresponding output value on the y-axis. The point $(3, 1)$ tells us that $g(3) = 1$.

To evaluate $g^{-1}(3)$, recall that by definition $g^{-1}(3)$ means the value of x for which $g(x) = 3$. By looking for the output value 3 on the vertical axis, we find the point $(5, 3)$ on the graph, which means $g(5) = 3$, so by definition, $g^{-1}(3) = 5$. See Figure 1.7.6.

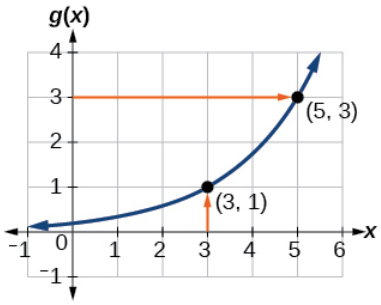


Figure 1.7.6: Graph of $g(x)$.

? Exercise 1.7.6

Using the graph in Figure 1.7.6, (a) find $g^{-1}(1)$,and (b) estimate $g^{-1}(4)$.

Answer a

3

Answer b

5.6

Finding Inverses of Functions Represented by Formulas

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula— for example, y as a function of x— we can often find the inverse function by solving to obtain x as a function of y.

How To: Given a function represented by a formula, find the inverse.

1. Make sure f is a one-to-one function.
2. Solve for x
3. Interchange x and y.

Example 1.7.7: Inverting the Fahrenheit-to-Celsius Function

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.

$$C = \frac{5}{9}(F - 32) \tag{1.7.11}$$

Solution

$$C = \frac{5}{9}(F - 32) \tag{1.7.12}$$

$$C \cdot \frac{9}{5} = F - 32 \tag{1.7.13}$$

$$F = \frac{9}{5}C + 32 \tag{1.7.14}$$

By solving in general, we have uncovered the inverse function. If

$$C = h(F) = \frac{5}{9}(F - 32) \tag{1.7.15}$$

,

then

$$F = h^{-1}(C) = \frac{9}{5}C + 32. \tag{1.7.16}$$

In this case, we introduced a function h to represent the conversion because the input and output variables are descriptive, and writing C^{-1} could get confusing.

Exercise 1.7.7

Solve for x in terms of y given $y = \frac{1}{3}(x-5)$

Answer

$x = 3y+5$

Example 1.7.8: Solving to Find an Inverse Function

Find the inverse of the function $f(x) = \frac{2}{x-3}+4$.

Solution

$$y = \frac{2}{x-3+4} \quad \text{Set up an equation.} \quad (1.7.17)$$

$$y-4 = \frac{2}{x-3} \quad \text{Subtract 4 from both sides.} \quad (1.7.18)$$

$$x-3 = \frac{2}{y-4} \quad \text{Multiply both sides by x−3 and divide by y−4.} \quad (1.7.19)$$

$$x = \frac{2}{y-4}+3 \quad \text{Add 3 to both sides.} \quad (1.7.20)$$

So $f^{-1}(y) = \frac{2}{y-4}+3$ or $f^{-1}(x) = \frac{2}{x-4}+3$.

Analysis

The domain and range of f exclude the values 3 and 4, respectively. f and f^{-1} are equal at two points but are not the same function, as we can see by creating Table 1.7.5.

Table 1.7.5

x	1	2	5	$f^{-1}(y)$
$f(x)$	3	2	5	y

Example 1.7.9: Solving to Find an Inverse with Radicals

Find the inverse of the function $f(x) = 2+\sqrt{x-4}$.

Solution

$$y = 2+\sqrt{x-4} \quad (1.7.21)$$

$$(y-2)^2 = x-4 \quad (1.7.22)$$

$$x = (y-2)^2+4 \quad (1.7.23)$$

So $f^{-1}(x) = (x-2)^2+4$.

The domain of f is $[4,\infty)$. Notice that the range of f is $[2,\infty)$, so this means that the domain of the inverse function f^{-1} is also $[2,\infty)$

Analysis

The formula we found for $f^{-1}(x)$ looks like it would be valid for all real x. However, f^{-1} itself must have an inverse (namely, f) so we have to restrict the domain of f^{-1} to $[2,\infty)$ in order to make f^{-1} a one-to-one function. This domain of f^{-1} is exactly the range of f.

Exercise 1.7.8

What is the inverse of the function $f(x) = 2 - \sqrt{x}$? State the domains of both the function and the inverse function.

Answer

$f^{-1}(x) = (2-x)^2$; domain of f: $[0, \infty)$; domain of f^{-1}: $(-\infty, 2]$

Finding Inverse Functions and Their Graphs

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function $f(x) = x^2$ restricted to the domain $[0, \infty)$, on which this function is one-to-one, and graph it as in Figure 1.7.7.

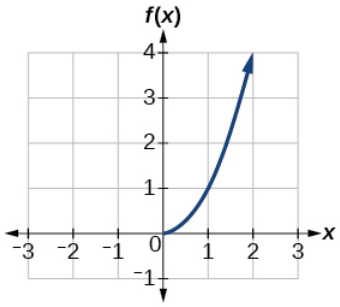


Figure* 1.7.7*: *Quadratic function with domain restricted to* $[0, \infty)$.

Restricting the domain to $[0, \infty)$ makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, $f^{-1}(x) = \sqrt{x}$. What happens if we graph both f and f^{-1} on the same set of axes, using the x-axis for the input to both f and f^{-1}?

We notice a distinct relationship: The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the identity line, shown in Figure 1.7.8.

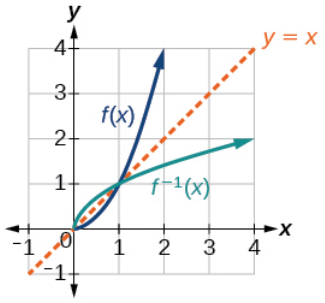


Figure* 1.7.8*: *Square and square-root functions on the non-negative domain*

This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

Example 1.7.10: Finding the Inverse of a Function Using Reflection about the Identity Line

Given the graph of $f(x)$ in Figure 1.7.9, sketch a graph of $f^{-1}(x)$.

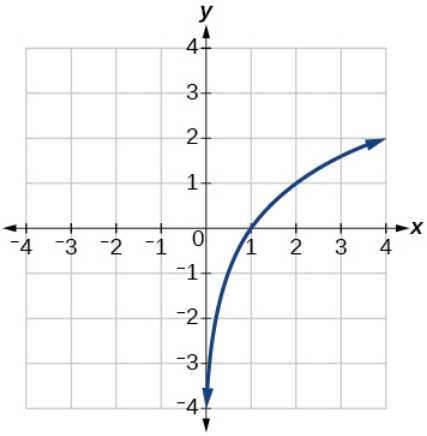


Figure 1.7.9: Graph of $f^(-1)(x)$.

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of $(0, \infty)$ and range of $(-\infty, \infty)$, so the inverse will have a domain of $(-\infty, \infty)$ and range of $(0, \infty)$.

If we reflect this graph over the line $y = x$, the point $(1, 0)$ reflects to $(0, 1)$ and the point $(4, 2)$ reflects to $(2, 4)$. Sketching the inverse on the same axes as the original graph gives Figure 1.7.10.

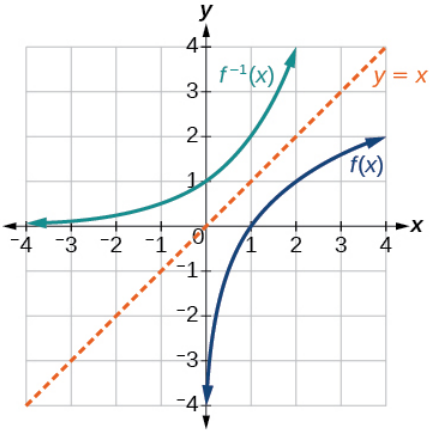


Figure 1.7.10: The function and its inverse, showing reflection about the identity line

Exercise 1.7.1

Draw graphs of the functions f and f^{-1} from Example 1.7.8.

Answer

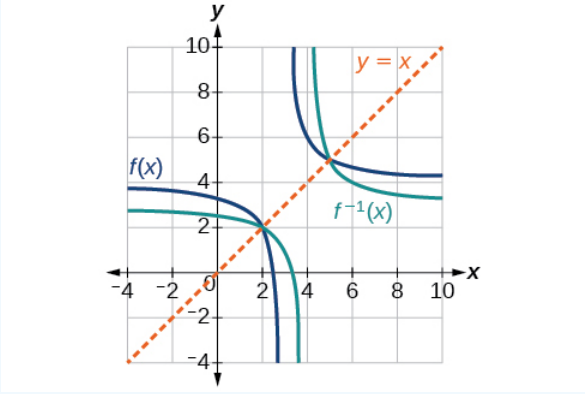


Figure 1.7.11: Graph of $f(x)$ and $f^(-1)(x)$.

Is there any function that is equal to its own inverse?

Yes. If $f = f^{-1}$, then $f(f(x)) = x$, and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because

$$\frac{1}{\frac{1}{x}} = x \tag{1.7.24}$$

Any function $f(x) = c - x$, where c is a constant, is also equal to its own inverse.

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1.R: Functions (Review)

This page is a draft and is under active development.

1.1: Functions and Function Notation

For the exercises 1-4, determine whether the relation is a function.

1) $\{(a,b),(c,d),(e,d)\}$

Answer

function

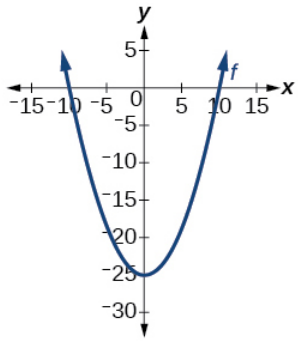
2) $\{(5,2),(6,1),(6,2),(4,8)\}$

3) $y^2+4=x$, for x the independent variable and y the dependent variable

Answer

not a function

4) Is the graph in the Figure below a function?



For the exercises 5-6, evaluate the function at the indicated values: $f(-3); f(2); f(-a); -f(a); f(a+h)$

5) $f(x)=-2x^2+3x$

Answer

$f(-3)=-27; f(2)=-2; f(-a)=-2a^2-3a; -f(a)=2a^2-3a; f(a+h)=-2a^2+3a-4ah+3h-2h^2$

6) $f(x)=2|3x-1|$

For the exercises 7-8, determine whether the functions are one-to-one.

7) $f(x)=-3x+5$

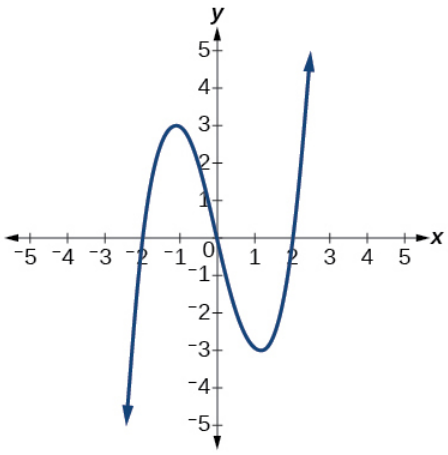
Answer

one-to-one

8) $f(x)=|x-3|$

For the exercises 9-11, use the vertical line test to determine if the relation whose graph is provided is a function.

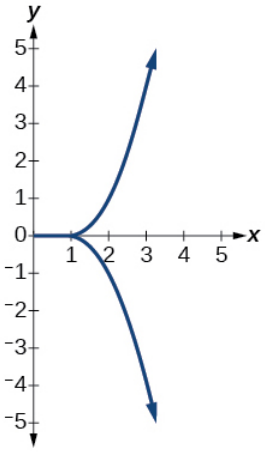
9)



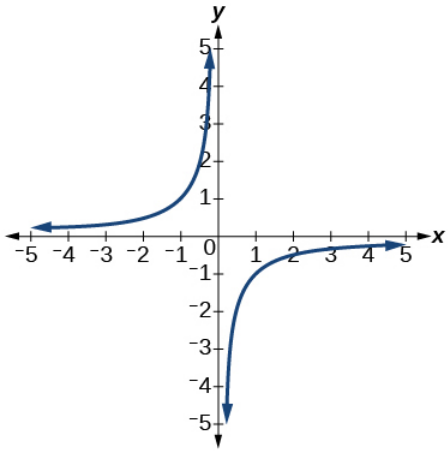
Answer

function

10)



11)



Answer

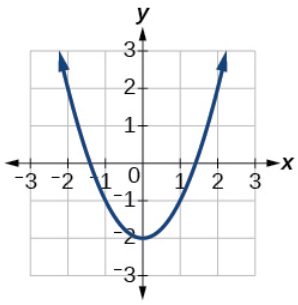
function

For the exercises 12-13, graph the functions.

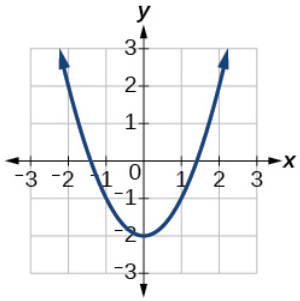
12) $f(x) = |x+1|$

13) $f(x) = x^2 - 2$

Answer



For the exercises 14-17, use the Figure below to approximate the values.



14) $f(2)$

15) $f(-2)$

Answer

2

16) If $f(x) = -2$, then solve for x

17) If $f(x) = 1$, then solve for x

Answer

$x = -1.8$ or $x = 1.8$

For the exercises 18-19, use the function $h(t) = -16t^2 + 80t$ to find the values.

18) $\dfrac{h(2) - h(1)}{2 - 1}$

19) $\dfrac{h(a) - h(1)}{a - 1}$

Answer

$$\frac{-64 + 80a - 16a^2}{-1 + a} = -16a + 64$$

1.2: Domain and Range

For the exercises 1-4, find the domain of each function, expressing answers using interval notation.

1) $f(x) = \dfrac{2}{3x + 2}$

2) $f(x) = \frac{x-3}{x^2-4x-12}$

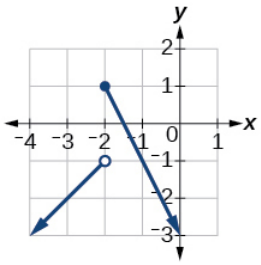
Answer

$(-\infty, -2) \cup (-2, 6) \cup (6, \infty)$

3)

4) Graph this piecewise function: $f(x) = \begin{cases} x+1 & x < -2 \\ -2x-3 & x \geq -2 \end{cases}$

Answer



1.3: Rates of Change and Behavior of Graphs

For the exercises 1-3, find the average rate of change of the functions from $x = 1$ to $x = 2$

1) $f(x) = 4x - 3$

2) $f(x) = 10x^2 + x$

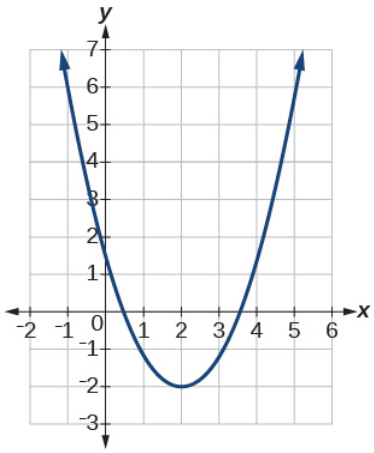
Answer

31

3) $f(x) = -\frac{2}{x^2}$

For the exercises 4-6, use the graphs to determine the intervals on which the functions are increasing, decreasing, or constant.

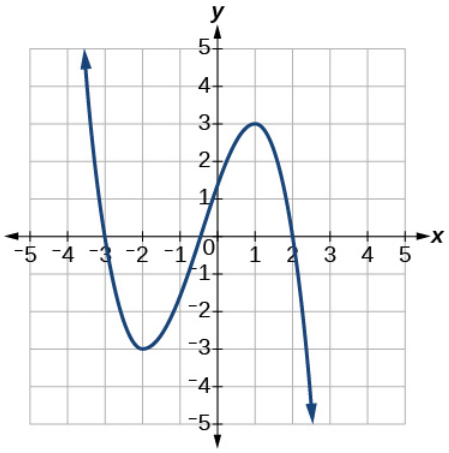
4)



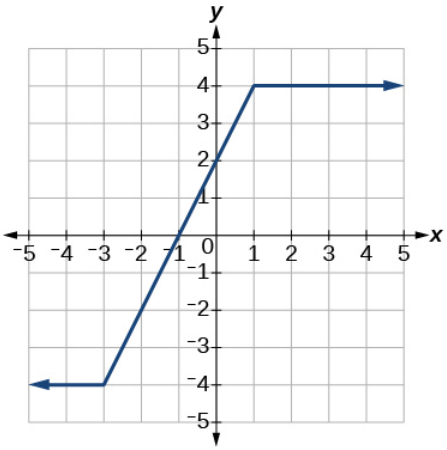
Answer

increasing $(2, \infty)$; decreasing $(-\infty, 2)$

5)



6)



Answer

increasing $(-3, 1)$; constant $(-\infty, -3) \cup (1, \infty)$

7) Find the local minimum of the function graphed in Exercise 4.

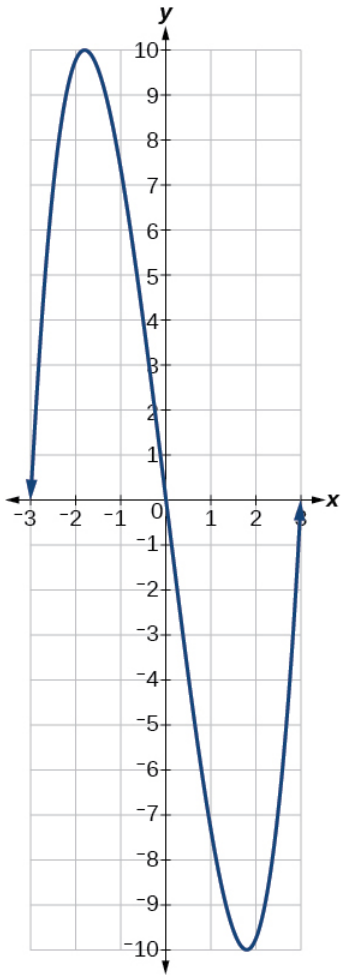
8) Find the local extrema for the function graphed in Exercise 5.

Answer

local minimum $(-2, -3)$; local maximum $(1, 3)$

9) For the graph in the Figure in Exercise 10, the domain of the function is $[-3, 3]$. The range is $[-10, 10]$. Find the absolute minimum of the function on this interval.

10) Find the absolute maximum of the function graphed in the Figure below.



Answer

$(-1.8, 10)$

1.4: Composition of Functions

For the exercises 1-5, find $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions.

1) $f(x) = 4 - x, g(x) = -4x$

2) $f(x) = 3x + 2, g(x) = 5 - 6x$

Answer

$(f \circ g)(x) = 17 - 18x; (g \circ f)(x) = -7 - 18x$

3) $f(x) = x^2 + 2x, g(x) = 5x + 1$

4) $f(x) = \sqrt{x+2}, g(x) = \dfrac{1}{x}$

Answer

$(f \circ g)(x) = \sqrt{\dfrac{1}{x} + 2}; (g \circ f)(x) = \dfrac{1}{\sqrt{x+2}}$

5) $f(x) = \dfrac{x+3}{2}, g(x) = \sqrt{1-x}$

For the exercises 6-9, find $(f \circ g)$ and the domain for $(f \circ g)(x)$ for each pair of functions.

6) $f(x)=\frac{x+1}{x+4}, g(x)=\frac{1}{x}$

Answer

$(f\circ g)(x)=\dfrac{1+x}{1+4x}, x\neq 0, x\neq -\dfrac{1}{4}$

7) $f(x)=\dfrac{1}{x+3}, g(x)=\dfrac{1}{x-9}$

8) $f(x)=\dfrac{1}{x}, g(x)=\sqrt{x}$

Answer

$(f\circ g)(x)=\frac{1}{\sqrt{x}}, x>0$

9) $f(x)=\frac{1}{x^2-1}, g(x)=\sqrt{x+1}$

For the exercises 10-11, express each function H as a composition of two functions f and g where $H(x)=(f\circ g)(x)$

10) $H(x)=\sqrt{\frac{2x-1}{3x+4}}$

Answer

sample: $g(x)=\dfrac{2x-1}{3x+4}; f(x)=\sqrt{x}$

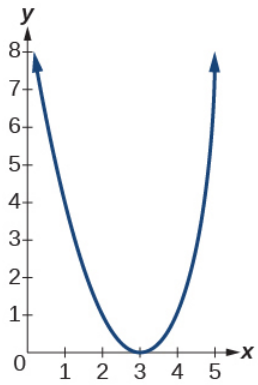
11) $H(x)=\dfrac{1}{(3x^2-4)^{-3}}$

1.5: Transformation of Functions

For the exercises 1-8, sketch a graph of the given function.

1) $f(x)=(x-3)^2$

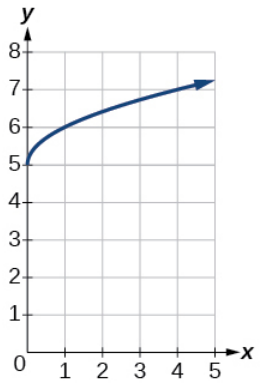
Answer



2) $f(x)=(x+4)^3$

3) $f(x)=\sqrt{x}+5$

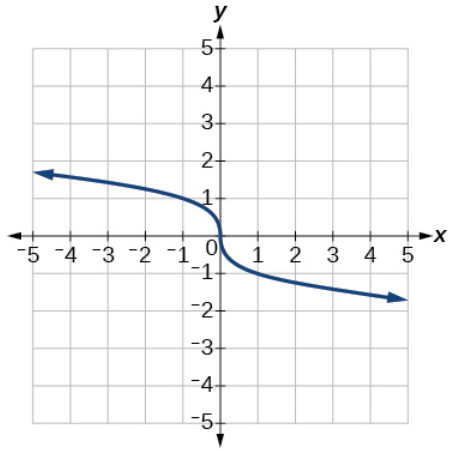
Answer



4) $f(x) = -x^3$

5) $f(x) = \sqrt[3]{-x}$

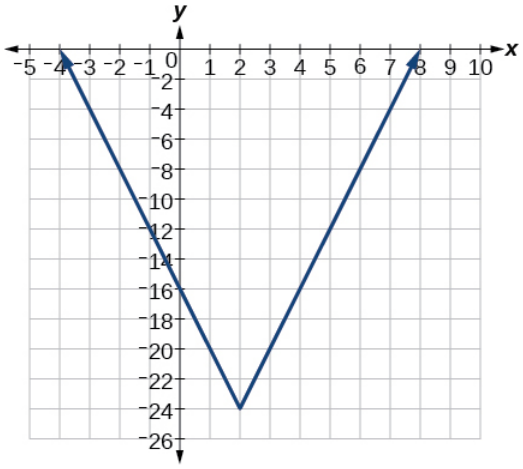
Answer



6) $f(x) = 5\sqrt{-x} - 4$

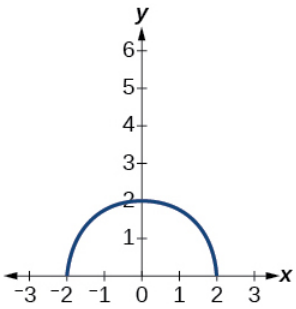
7) $f(x) = 4[|x-2| - 6]$

Answer



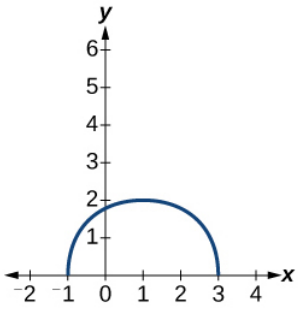
8) $f(x) = -(x+2)^2 - 1$

For the exercises 9-10, sketch the graph of the function g if the graph of the function f is shown in the Figure below.



9) $g(x) = f(x-1)$

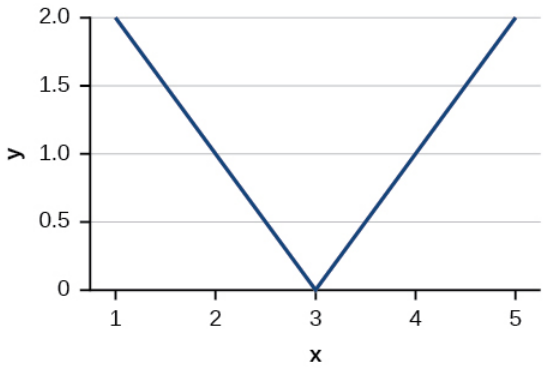
Answer



10) $g(x) = 3f(x)$

For the exercises 11-12, write the equation for the standard function represented by each of the graphs below.

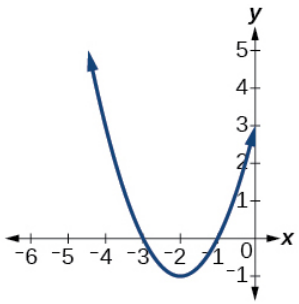
11)



Answer

$f(x) = |x-3|$

12)



For the exercises 13-15, determine whether each function below is even, odd, or neither.

13) $f(x) = 3x^4$

Answer

even

14) $g(x) = \sqrt{x}$

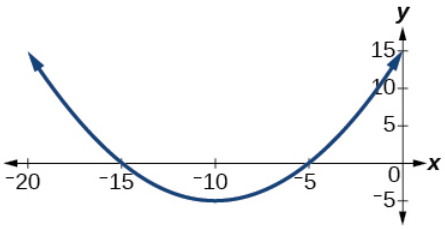
15) $h(x) = \frac{1}{x} + 3x$

Answer

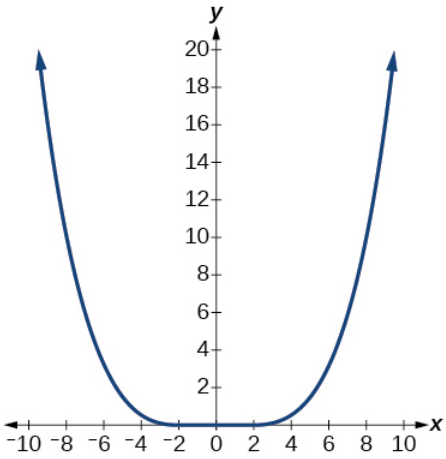
odd

For the exercises 16-18, analyze the graph and determine whether the graphed function is even, odd, or neither.

16)



17)

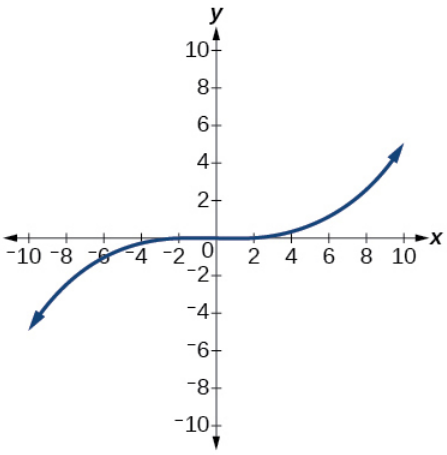


Answer

even

18)

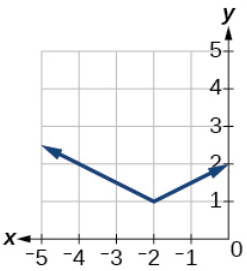
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1.6: Absolute Value Functions

For the exercises 1-3, write an equation for the transformation of $f(x) = |x|$.

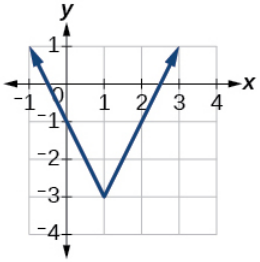
1)



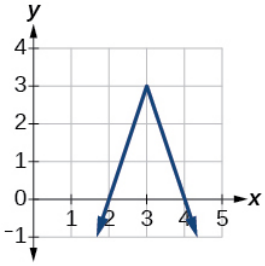
Answer

$$f(x) = \frac{1}{2}|x+2|+1$$

2)



3)



Answer

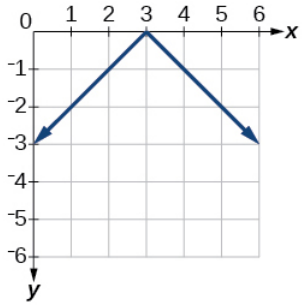
$f(x) = -3|x-3|+3$

For the exercises 4-6, graph the absolute value function.

4) $f(x) = |x-5|$

5) $f(x) = -|x-3|$

Answer



6) $f(x) = |2x-4|$

For the exercises 7-8, solve the absolute value equation.

7) $|x+4| = 18$

Answer

$x = -22, x = 14$

8) $\left|\frac{1}{3}x+5\right| = \left|\frac{3}{4}x-2\right|$

For the exercises 9-10, solve the inequality and express the solution using interval notation.

9) $|3x-2| < 7$

Answer

$\left(-\frac{5}{3}, 3\right)$

10) $\left|\frac{1}{3}x-2\right| \leq 7$

1.7: Inverse Functions

For the exercises 1-2, find $f^{-1}(x)$ for each function.

1) $f(x) = 9+10x$

2) $f(x) = \frac{x}{x+2}$

Answer

$f^{-1}(x) = \frac{-2x}{x-1}$

3) For the following exercise, find a domain on which the function f is one-to-one and non-decreasing. Write the domain in interval notation. Then find the inverse of f restricted to that domain.

$$f(x) = x^2+1 \tag{1.R.1}$$

4) Given $f(x) = x^3-5$ and $g(x) = \sqrt[3]{x+5}$:

a. Find $f(g(x))$ and $g(f(x))$.
b. What does the answer tell us about the relationship between $f(x)$ and $g(x)$?

Answer

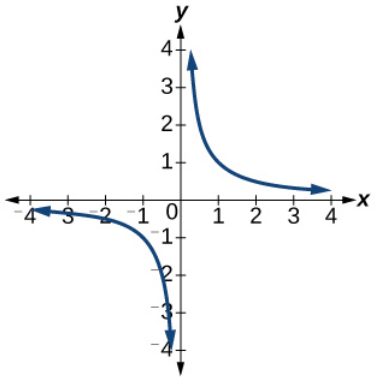
a. $f(g(x)) = x$ and $g(f(x)) = x$
b. This tells us that f and g are inverse functions

For the exercises 5-8, use a graphing utility to determine whether each function is one-to-one.

5) $f(x) = \dfrac{1}{x}$

Answer

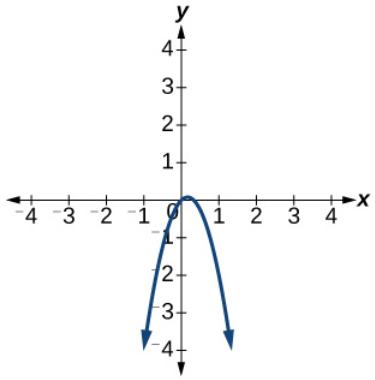
The function is one-to-one.



6) $f(x) = -3x^2 + x$

Answer

The function is not one-to-one.



7) If $f(5) = 2$, find $f^{-1}(2)$

Answer

5

8) If $f(1) = 4$, find $f^{-1}(4)$

Practice Test

For the exercises 1-2, determine whether each of the following relations is a function.

1) $y = 2x + 8$

Answer

The relation is a function.

2) $\{(2,1),(3,2),(-1,1),(0,-2)\}$

For the exercises 3-4, evaluate the function $f(x)=-3x^2+2x$ at the given input.

3) $f(-2)$

Answer

-16

4) $f(a)$

5) Show that the function $f(x)=-2(x-1)^2+3$ is not one-to-one.

Answer

The graph is a parabola and the graph fails the horizontal line test.

6) Write the domain of the function $f(x)=\sqrt{3-x}$ in interval notation.

7) Given $f(x)=2x^2-5x$, find $f(a+1)-f(1)$

Answer

$2a^2-a$

8) Graph the function $f(x)=\begin{cases} x+1 & \text{if} \quad -2<x<3 \\ -x & \text{if} \quad x\geq 3 \end{cases}$

9) Find the average rate of change of the function $f(x)=3-2x^2+x$ by finding $\dfrac{f(b)-f(a)}{b-a}$

Answer

$-2(a+b)+1$

For the exercises 10-11, use the functions $f(x)=3-2x^2+x$ and $g(x)=\sqrt{x}$ to find the composite functions.

10) $(g\circ f)(x)$

11) $(g\circ f)(1)$

Answer

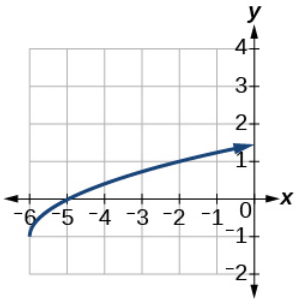
$\sqrt{2}$

12) Express $H(x)=\sqrt[3]{5x^2-3x}$ a composition of two functions, f and g, where $(f\circ g)(x)=H(x)$

For the exercises 13-14, graph the functions by translating, stretching, and/or compressing a toolkit function.

13) $f(x)=\sqrt{x+6}-1$

Answer



14) $f(x) = \dfrac{1}{x+2} - 1$

For the exercises 15-17, determine whether the functions are even, odd, or neither.

15) $f(x) = -\dfrac{5}{x^2} + 9x^6$

Answer

even

16) $f(x) = -\dfrac{5}{x^3} + 9x^5$

17) $f(x) = \dfrac{1}{x}$

Answer

odd

18) Graph the absolute value function $f(x) = -2|x-1| + 3$.

19) Solve $|2x-3| = 17$.

Answer

$x = -7$ and $x = 10$

20) Solve $-\left|\dfrac{1}{3}x - 3\right| \geq 17$. Express the solution in interval notation.

For the exercises 21-22, find the inverse of the function.

21) $f(x) = 3x - 5$

Answer

$f^{-1}(x) = \dfrac{x+5}{3}$

22) $f(x) = \dfrac{4}{x+7}$

For the exercises 23-26, use the graph of *g* shown in the Figure below.

23) On what intervals is the function increasing?

Answer

$(-\infty, -1.1)$ and $(1.1, \infty)$

24) On what intervals is the function decreasing?

25) Approximate the local minimum of the function. Express the answer as an ordered pair.

Answer

$(1.1, -0.9)$

26) Approximate the local maximum of the function. Express the answer as an ordered pair.

For the exercises 27-29, use the graph of the piecewise function shown in the Figure below.

27) Find $f(2)$.

Answer

$f(2) = 2$

28) Find $f(-2)$.

29) Write an equation for the piecewise function.

Answer

$$f(x) = \begin{cases} |x| & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$$

For the exercises 30-35, use the values listed in the Table below.

x	$F(x)$
0	1
1	3
2	5
3	7
4	9
5	11
6	13
7	15
8	17

30) Find $F(6)$.

31) Solve the equation $F(x) = 5$

Answer

$x = 2$

32) Is the graph increasing or decreasing on its domain?

33) Is the function represented by the graph one-to-one?

Answer

yes

34) Find $F^{-1}(15)$.

35) Given $f(x) = -2x + 11$, find $f^{-1}(x)$.

Answer

$$f^{-1}(x) = -\frac{x-11}{2}$$

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CHAPTER OVERVIEW

2: Linear Functions

Recall that a function is a relation that assigns to every element in the domain exactly one element in the range. Linear functions are a specific type of function that can be used to model many real-world applications, such as plant growth over time. In this chapter, we will explore linear functions, their graphs, and how to relate them to data.

2.1: Linear Functions
2.2: Graphs of Linear Functions
2.3: Modeling with Linear Functions
2.4: Fitting Linear Models to Data
2.R: Linear Functions(Review)

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2.1: Linear Functions

Representing Linear Functions

The function describing the train's motion is a **linear function**, which is defined as a function with a constant rate of change, that is, a polynomial of degree 1. There are several ways to represent a linear function, including word form, function notation, tabular form, and graphical form. We will describe the train's motion as a function using each method.

Representing a Linear Function in Word Form

Let's begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.

- *The train's distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at constant speed.*

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

Representing a Linear Function in Function Notation

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the form known as the slope-intercept form of a line, where xis the input value, m is the rate of change, and b is the initial value of the dependent variable.

$$\begin{aligned} &\text{Equation form} && y = mx + b \\ &\text{Equation notation} && f(x) = mx + b \end{aligned}$$

In the example of the train, we might use the notation $D(t)$ in which the total distance D is a function of the time t. The rate, m, is 83 meters per second. The initial value of the dependent variable b is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.

$$D(t) = 83t + 250 \tag{2.1.1}$$

Representing a Linear Function in Tabular Form

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in Figure 2.1.2. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.

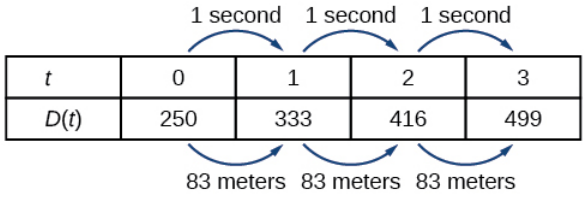


Figure 2.1.2: Tabular representation of the function D showing selected input and output values

Q/A? Can the input in the previous example be any real number?

No. The input represents time, so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of non-negative real numbers.

Representing a Linear Function in Graphical Form

Another way to represent linear functions is visually, using a graph. We can use the function relationship from above, $D(t) = 83t + 250$, to draw a graph, represented in Figure 2.1.3. Notice the graph is a line. When we plot a linear function, the graph is always a line.

The rate of change, which is constant, determines the slant, or **slope** of the line. The point at which the input value is zero is the vertical intercept, or **y-intercept**, of the line. We can see from the graph in Figure $2.1.3$ that the y-intercept in the train example we just saw is $(0, 250)$ and represents the distance of the train from the station when it began moving at a constant speed.

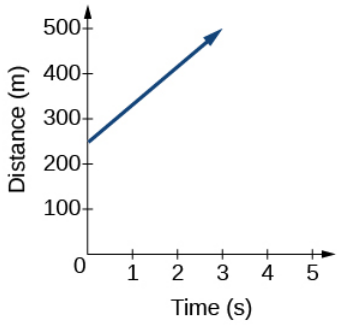


Figure 2.1.3: The graph of $D(t) = 83t + 250$. Graphs of linear functions are lines because the rate of change is constant.

Notice that the graph of the train example is restricted, but this is not always the case. Consider the graph of the line $f(x) = 2x + 1$. Ask yourself what numbers can be input to the function, that is, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled, and then have one added to the product.

Definition: Linear Function

A **linear function** is a function whose graph is a line. Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b \tag{2.1.2}$$

where b is the initial or starting value of the function (when input, $x = 0$), and m is the constant rate of change, or **slope** of the function. The **y-intercept** is at $(0, b)$.

Example : Using a Linear Function to Find the Pressure on a Diver

The pressure, P, in pounds per square inch (PSI) on the diver in Figure $2.1.4$ depends upon her depth below the water surface, d, in feet. This relationship may be modeled by the equation, $P(d) = 0.434d + 14.696$. Restate this function in words.



Figure 2.1.4: (credit: Ilse Reijs and Jan-Noud Hutten)Hutten)

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

Analysis

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.

Determining whether a Linear Function Is Increasing, Decreasing, or Constant

The linear functions we used in the two previous examples increased over time, but not every linear function does. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an **increasing function** has a positive slope. A line with a positive slope slants upward from left to right as in Figure 2.1.5(a). For a **decreasing function**, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in Figure 2.1.5(b). If the function is constant, the output values are the same for all input values so the slope is zero. A line with a slope of zero is horizontal as in Figure 2.1.5(c).

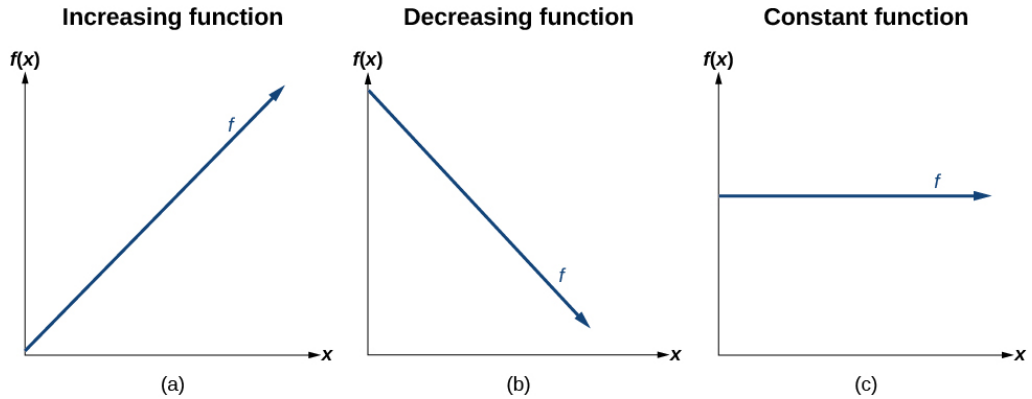


Figure 2.1.5: Three graphs depicting an increasing function, a decreasing function, and a constant function.]

Increasing and Decreasing Functions

Increasing and Decreasing Functions

The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a constant function.

- $f(x) = mx + b$ is an increasing function if $m > 0$.
- $f(x) = mx + b$ is an decreasing function if $m < 0$.
- $f(x) = mx + b$ is a constant function if $m = 0$.

Example : Deciding whether a Function Is Increasing, Decreasing, or Constant

Some recent studies suggest that a teenager sends an average of 60 texts per day. For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then, determine whether the graph of the function is increasing, decreasing, or constant.

a. The total number of texts a teen sends is considered a function of time in days. The input is the number of days, and output is the total number of texts sent.
b. A teen has a limit of 500 texts per month in his or her data plan. The input is the number of days, and output is the total number of texts remaining for the month.
c. A teen has an unlimited number of texts in his or her data plan for a cost of $50 per month. The input is the number of days, and output is the total cost of texting each month.

Solution

Analyze each function.

a. The function can be represented as $f(x) = 60x$ where x is the number of days. The slope, 60, is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
b. The function can be represented as $f(x) = 500 - 60x$ where x is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after x days.
c. The cost function can be represented as $f(x) = 50$ because the number of days does not affect the total cost. The slope is 0 so the function is constant.

Calculating and Interpreting Slope

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope given input and output values. Given two values for the input, x_1 and x_2, and two corresponding values for the output, y_1 and y_2—which can be represented by a set of points, (x_1, y_1) and (x_2, y_2)—we can calculate the slope m, as follows

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}}$$

$$= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where Δy is the vertical displacement and Δx is the horizontal displacement. Note in function notation two corresponding values for the output y_1 and y_2 for the function f, $y_1 = f(x_1)$ and $y_2 = f(x_2)$, so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Figure 2.1.6 indicates how the slope of the line between the points, (x_1, y_1) and (x_2, y_2), is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.

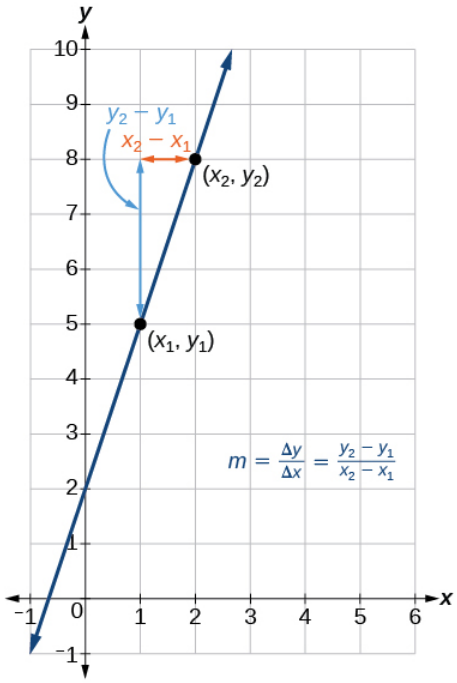


Figure 2.1.6: The slope of a function is calculated by the change in y divided by the change in x. It does not matter which coordinate is used as the (x_2, y_2) and which is the (x_1, y_1),as long as each calculation is started with the elements from the same coordinate pair.

Q & A

Are the units for slope always $\frac{\text{units for the output}}{\text{units for the input}}$?

Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

Calculate Slope

The slope, or rate of change, of a function m can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.1.3}$$

where x_1 and x_2 are input values, y_1 and y_2 are output values.

Writing the Point-Slope Form of a Linear Equation

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Here, we will learn another way to write a linear function, the **point-slope form**.

$$y - y_1 = m(x - x_1) \tag{2.1.4}$$

The point-slope form is derived from the slope formula.

$$\begin{aligned} m &= \frac{y - y_1}{x - x_1} && \text{assuming } x \neq x_1 \\ m(x - x_1) &= \frac{y - y_1}{x - x_1}(x - x_1) && \text{Multiply both sides by } (x - x_1). \\ m(x - x_1) &= y - y_1 && \text{Simplify} \\ y - y_1 &= m(x - x_1) && \text{Rearrange} \end{aligned}$$

Keep in mind that the slope-intercept form and the point-slope form can be used to describe the same function. We can move from one form to another using basic algebra. For example, suppose we are given an equation in point-slope form, $y - 4 = -\frac{1}{2}(x - 6)$. We can convert it to the slope-intercept form as shown.

$$\begin{aligned} y - 4 &= -\frac{1}{2}(x - 6) \\ y - 4 &= -\frac{1}{2}x + 3 && \text{Distribute the } -\frac{1}{2}. \\ y &= -\frac{1}{2}x + 7 && \text{Add 4 to each side.} \end{aligned}$$

Therefore, the same line can be described in slope-intercept form as $y = \frac{1}{2}x + 7$.

Point-Slope Form of a Linear Equation

The **point-slope** form of a linear equation takes the form

$$y - y_1 = m(x - x_1) \tag{2.1.5}$$

where m is the slope, x_1 and y_1 are the x and y coordinates of a specific point through which the line passes.

Writing the Equation of a Line Using a Point and the Slope

The point-slope form is particularly useful if we know one point and the slope of a line. Suppose, for example, we are told that a line has a slope of 2 and passes through the point $(4, 1)$. We know that $m = 2$ and that $x_1 = 4$ and $y_1 = 1$. We can substitute these values into the general point-slope equation.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 4) \end{aligned}$$

If we wanted to then rewrite the equation in slope-intercept form, we apply algebraic techniques.

$$\begin{aligned} y - 1 &= 2(x - 4) \\ y - 1 &= 2x - 8 && \text{Distribute the 2.} \\ y &= 2x - 7 && \text{Add 1 to each side.} \end{aligned}$$

Both equations, $y - 1 = 2(x - 4)$ and $y = 2x - 7$, describe the same line. See Figure 2.1.7.

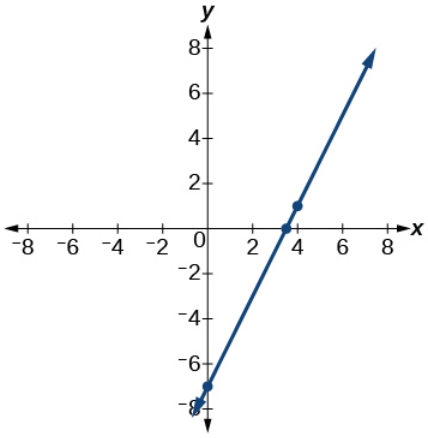


Figure 2.1.7

Writing the Equation of a Line Using Two Points

The point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points $(0, 1)$ and $(3, 2)$. We can use the coordinates of the two points to find the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2-1}{3-0} \\ &= \frac{1}{3} \end{aligned}$$

Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let use $(0, 1)$ for our point.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= \frac{1}{3}(x - 0) \end{aligned}$$

As before, we can use algebra to rewrite the equation in the slope-intercept form.

$$\begin{aligned} y - 1 &= \frac{1}{3}(x - 0) && \\ y - 1 &= \frac{1}{3}x && \text{Distribute the } \frac{1}{3}. \\ y &= \frac{1}{3}x + 1 && \text{Add 1 to each side.} \end{aligned}$$

Both equations describe the line shown in Figure 2.1.8.

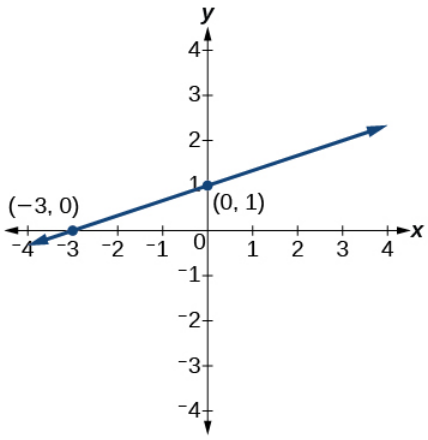


Figure 2.1.8

Example : Writing Linear Equations Using Two Points

Write the point-slope form of an equation of a line that passes through the points $(5, 1)$ and $(8, 7)$. Then rewrite it in the slope-intercept form.

Let's begin by finding the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{7-1}{8-5} \\ &= \frac{6}{3} \\ &= 2 \end{aligned}$$

So $m = 2$. Next, we substitute the slope and the coordinates for one of the points into the general point-slope equation. We can choose either point, but we will use $(5, 1)$.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 1 &= 2(x - 5) \end{aligned}$$

The point-slope equation of the line is $y_2 - 1 = 2(x_2 - 5)$. To rewrite the equation in slope-intercept form, we use algebra.

$$\begin{aligned} y - 1 &= 2(x - 5) \\ y - 1 &= 2x - 10 \\ y &= 2x - 9 \end{aligned}$$

The slope-intercept equation of the line is $y = 2x - 9$.

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2.2: Graphs of Linear Functions

Graphing Linear Functions

Previously, we saw that that the graph of a linear function is a straight line. We were also able to see the points of the function as well as the initial value from a graph. By graphing two functions, then, we can more easily compare their characteristics. There are three basic methods of graphing linear functions:

1. Plot the points and then drawing a line through the points.
2. Use the y-intercept and slope.
3. Use transformations of the identity function $f(x) = x$.

Graphing a Function by Plotting Points

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general, we should evaluate the function at a minimum of two inputs in order to find at least two points on the graph. For example, given the function, $f(x) = 2x$, we might use the input values 1 and 2. Evaluating the function for an input value of 1 yields an output value of 2, which is represented by the point $(1, 2)$. Evaluating the function for an input value of 2 yields an output value of 4, which is represented by the point $(2, 4)$. Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

How To: Given a linear function, graph by plotting points.

1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

Example 2.2.1: Graphing by Plotting Points

Graph $f(x) = -\frac{2}{3}x + 5$ by plotting points.

Solution

Begin by choosing input values. This function includes a fraction with a denominator of 3, so let's choose multiples of 3 as input values. We will choose 0, 3, and 6.

Evaluate the function at each input value, and use the output value to identify coordinate pairs.

$$\begin{aligned} x = 0 \qquad f(0) &= -\frac{2}{3}(0) + 5 = 5 \rightarrow (0, 5) \\ x = 3 \qquad f(3) &= -\frac{2}{3}(3) + 5 = 3 \rightarrow (3, 3) \\ x = 6 \qquad f(6) &= -\frac{2}{3}(6) + 5 = 1 \rightarrow (6, 1) \end{aligned}$$

Plot the coordinate pairs and draw a line through the points. Figure 2.2.1 represents the graph of the function $f(x) = -\frac{2}{3}x + 5$.

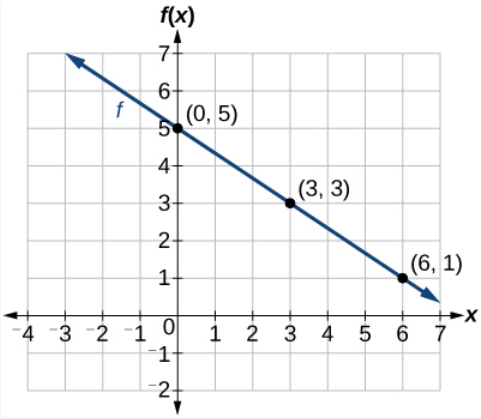


Figure 2.2.1: The graph of the linear function $f(x) = -\frac{2}{3}x + 5$.

Analysis

The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant, which indicates a negative slope. This is also expected from the negative constant rate of change in the equation for the function.

? Exercise 2.2.1

Graph $f(x) = -\frac{3}{4}x + 6$ by plotting points.

Answer

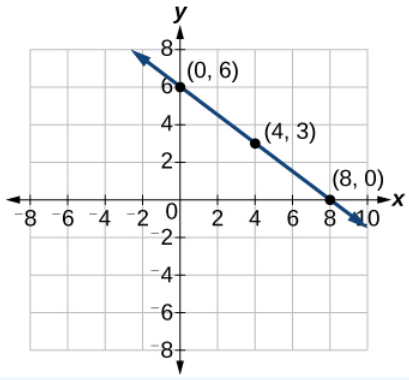


Figure 2.2.2

Graphing a Function Using y-intercept and Slope

Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its **y-intercept**, which is the point at which the input value is zero. To find the y-intercept, we can set $x = 0$ in the equation.

The other characteristic of the linear function is its slope m, which is a measure of its steepness. Recall that the slope is the rate of change of the function. The slope of a function is equal to the ratio of the change in outputs to the change in inputs. Another way to think about the slope is by dividing the vertical difference, or rise, by the horizontal difference, or run. We encountered both the y-intercept and the slope in Linear Functions.

Let's consider the following function.

$$f(x) = \frac{1}{2}x + 1 \tag{2.2.1}$$

The slope is $\frac{1}{2}$. Because the slope is positive, we know the graph will slant upward from left to right. The y-intercept is the point on the graph when $x = 0$. The graph crosses the y-axis at $(0, 1)$. Now we know the slope and the y-intercept. We can begin graphing by plotting the point $(0, 1)$ We know that the slope is rise over run, $m = \frac{\text{rise}}{\text{run}}$. From our example, we have $m = \frac{1}{2}$, which means

that the rise is 1 and the run is 2. So starting from our y-intercept $(0, 1)$, we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points, and then we draw a line through the points as shown in Figure $2.2.3$.

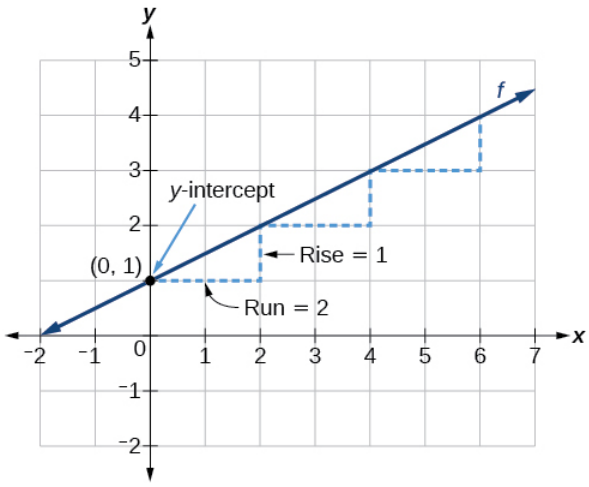


Figure $2.2.3$: Graph of $f(x)$ showing slope and y-intercept

Graphical Interpretation of a Linear Function

In the equation $f(x) = mx + b$

- b is the y-intercept of the graph and indicates the point $(0, b)$ at which the graph crosses the y-axis.
- m is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.2.2}$$

Graphing a Function Using Transformations

Another option for graphing is to use **transformations** of the identity function $f(x) = x$. A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

Vertical Stretch or Compression

In the equation $f(x) = mx$, the m is acting as the **vertical stretch** or **compression** of the identity function. When m is negative, there is also a vertical reflection of the graph. Notice in Figure $2.2.5$ that multiplying the equation of $f(x) = x$ by m stretches the graph of f by a factor of m units if $m > 1$ and compresses the graph of f by a factor of m units if \(0

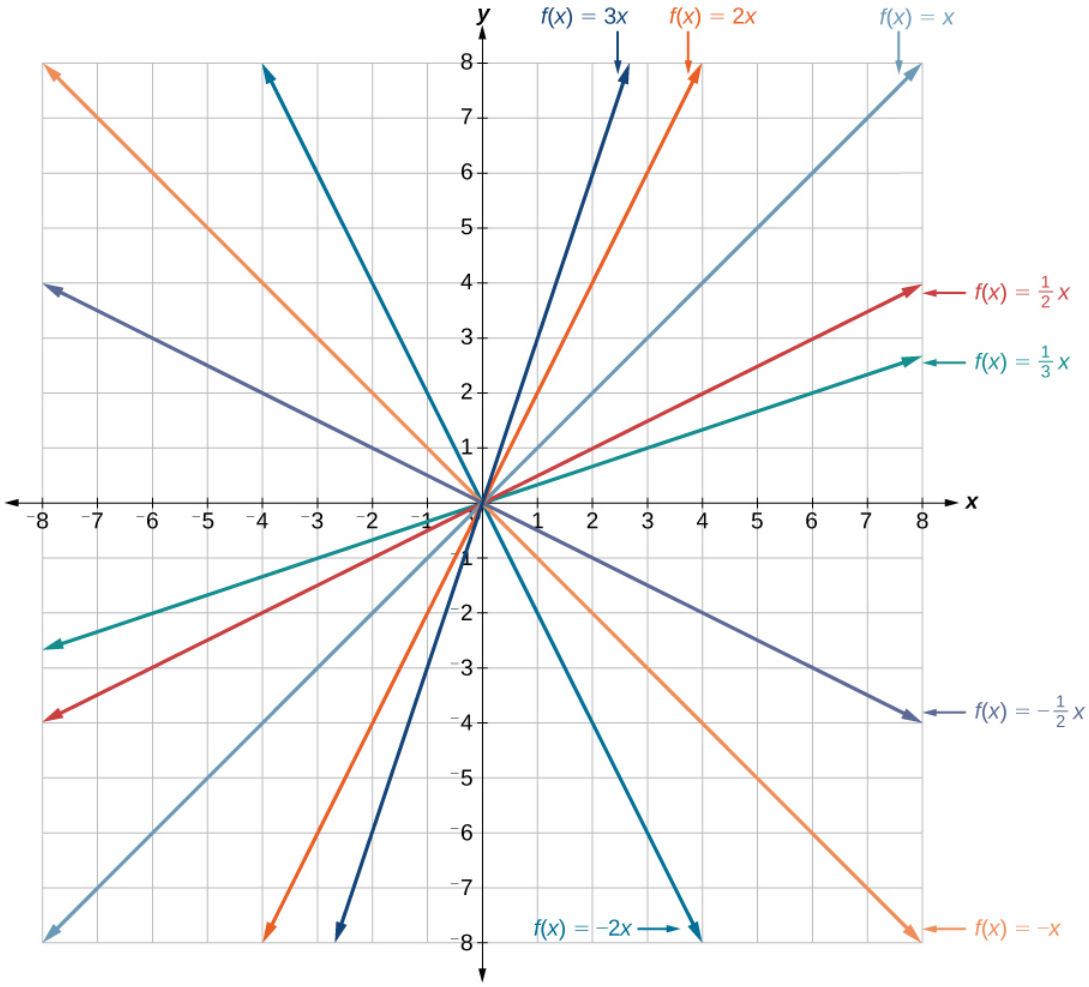


Figure 2.2.5: Vertical stretches and compressions and reflections on the function $f(x) = x$.

Vertical Shift

In $f(x) = mx + b$, the b acts as the **vertical shift**, moving the graph up and down without affecting the slope of the line. Notice in Figure 2.2.6 that adding a value of b to the equation of $f(x) = x$ shifts the graph of f a total of b units up if b is positive and $|b|$ units down if b is negative.

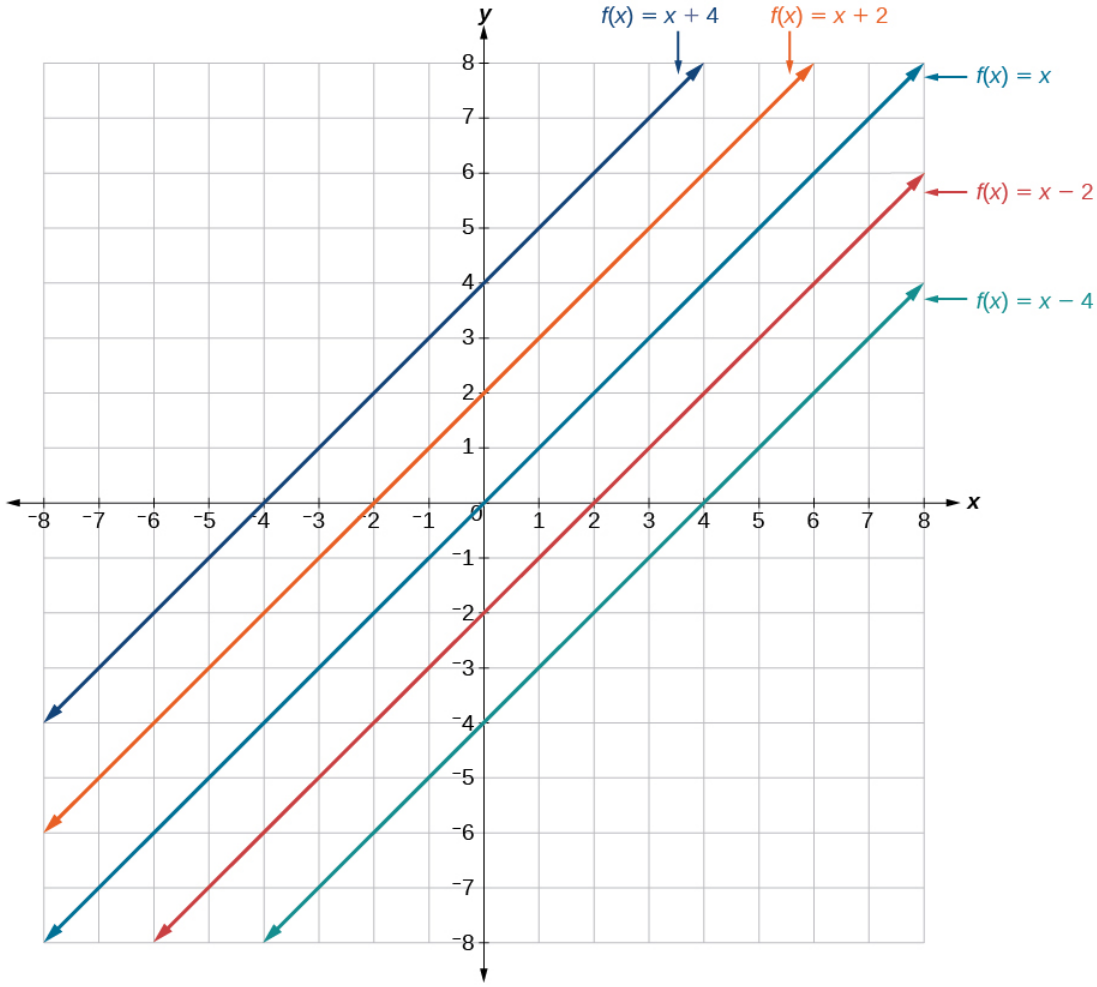
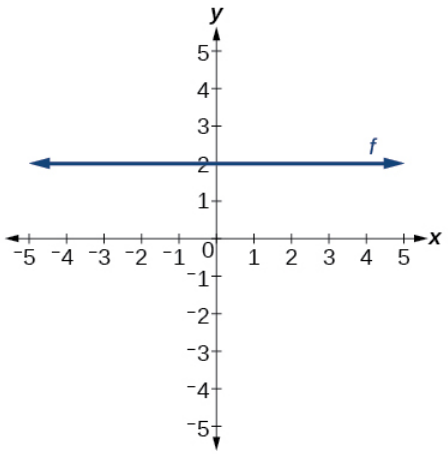


Figure 2.2.6: This graph illustrates vertical shifts of the function $f(x) = x$.

Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

Describing Horizontal and Vertical Lines

There are two special cases of lines on a graph—horizontal and vertical lines. A **horizontal line** indicates a constant output, or y-value. In Figure 2.2.15, we see that the output has a value of 2 for every input value. The change in outputs between any two points, therefore, is 0. In the slope formula, the numerator is 0, so the slope is 0. If we use $m = 0$ in the equation $f(x) = mx + b$, the equation simplifies to $f(x) = b$. In other words, the value of the function is a constant. This graph represents the function $f(x) = 2$.



x	-4	-2	0	2	4
y	2	2	2	2	2

Figure 2.2.15: A horizontal line representing the function $f(x) = 2$.

A **vertical line** indicates a constant input, or x-value. We can see that the input value for every point on the line is 2, but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.

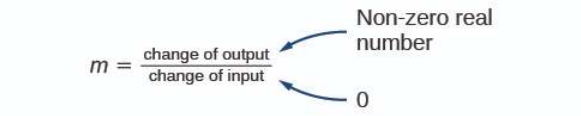
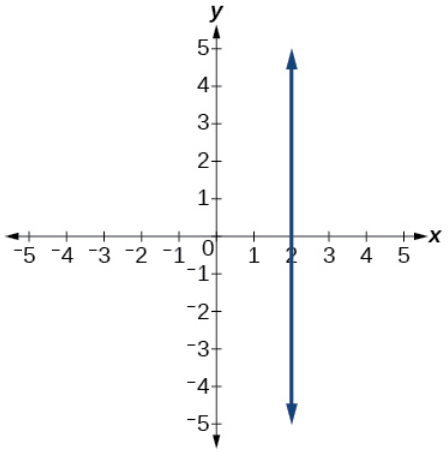


Figure 2.2.16

Notice that a vertical line, such as the one in Figure 2.2.17, has an x-intercept, but no y-intercept unless it's the line $x = 0$. This graph represents the line $x = 2$.



x	2	2	2	2	2
y	-4	-2	0	2	4

Figure 2.2.17: The vertical line, $x = 2$, which does not represent a function.

Definitions: Horizontal and Vertical Lines

Lines can be horizontal or vertical.

- A **horizontal line** is a line defined by an equation in the form $f(x) = b$.

- A **vertical line** is a line defined by an equation in the form $x = a$.

Writing the Equation of a Line Parallel or Perpendicular to a Given Line

If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

Writing Equations of Parallel Lines

Suppose for example, we are given the following equation.

$$f(x) = 3x + 1$$

We know that the slope of the line formed by the function is 3. We also know that the y-intercept is $(0, 1)$. Any other line with a slope of 3 will be parallel to $f(x)$. So the lines formed by all of the following functions will be parallel to $f(x)$.

$$\begin{aligned} g(x) &= 3x + 6 \\ h(x) &= 3x + 1 \\ p(x) &= 3x + \frac{2}{3} \end{aligned}$$

Suppose then we want to write the equation of a line that is parallel to f and passes through the point $(1, 7)$. We already know that the slope is 3. We just need to determine which value for b will give the correct line. We can begin with the point-slope form of an equation for a line, and then rewrite it in the slope-intercept form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 7 &= 3(x - 1) \\ y - 7 &= 3x - 3 \\ y &= 3x + 4 \end{aligned}$$

So $g(x) = 3x + 4$ is parallel to $f(x) = 3x + 1$ and passes through the point $(1, 7)$.

Solving a System of Linear Equations Using a Graph

A system of linear equations includes two or more linear equations. The graphs of two lines will intersect at a single point if they are not parallel. Two parallel lines can also intersect if they are coincident, which means they are the same line and they intersect at every point. For two lines that are not parallel, the single point of intersection will satisfy both equations and therefore represent the solution to the system.

To find this point when the equations are given as functions, we can solve for an input value so that $f(x) = g(x)$. In other words, we can set the formulas for the lines equal to one another, and solve for the input that satisfies the equation.

Example 2.2.12: Finding a Point of Intersection Algebraically

Find the point of intersection of the lines $h(t) = 3t - 4$ and $j(t) = 5 - t$.

Solution

Set $h(t) = j(t)$.

$$3t - 4 = 5 - t \tag{2.2.3}$$
$$4t = 9 \tag{2.2.4}$$
$$t = \frac{9}{4} \tag{2.2.5}$$

This tells us the lines intersect when the input is $\frac{9}{4}$.

We can then find the output value of the intersection point by evaluating either function at this input.

$$j\left(\frac{9}{4}\right) = 5 - \frac{9}{4} \tag{2.2.6}$$
$$= \frac{11}{4} \tag{2.2.7}$$

These lines intersect at the point $\left(\frac{9}{4}, \frac{11}{4}\right)$.

Analysis

Looking at Figure $2.2.25$, this result seems reasonable.

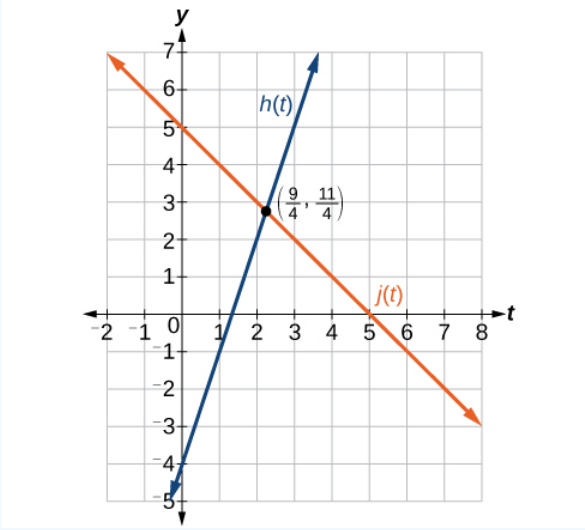


Figure 2.2.25: Graph of two functions $h(t) = 3t - 4$ and $j(t) = t + 5$ and their intersection at $(\frac{9}{4}, \frac{11}{4})$.

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2.3: Modeling with Linear Functions

Identifying Steps to Model and Solve Problems

When **modeling** scenarios with linear functions and solving problems involving quantities with a **constant rate of change**, we typically follow the same problem strategies that we would use for any type of function. Let's briefly review them:

Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
Carefully read the problem to identify important information. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
Carefully read the problem to determine what we are trying to find, identify, solve, or interpret.
Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
When needed, write a formula for the function.
Solve or evaluate the function using the formula.
Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
Clearly convey your result using appropriate units, and answer in full sentences when necessary.

Building Linear Models

Now let's take a look at the student in Seattle. In her situation, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can use this information to define our variables, including units.

- Output: M, money remaining, in dollars
- Input: t, time, in weeks

So, the amount of money remaining depends on the number of weeks: $M(t)$

We can also identify the initial value and the rate of change.

- Initial Value: She saved $3,500, so $3,500 is the initial value for M.
- Rate of Change: She anticipates spending $400 each week, so –$400 per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.

The **rate of change is constant**, so we can start with the linear model $M(t) = mt + b$. Then we can substitute the intercept and slope provided.

$$M(t) = mt + b$$

$$-400 \qquad 3500$$

$$M(t) = -400t + 3500$$

Figure 2.3.2

To find the x-intercept, we set the output to zero, and solve for the input.

$$\begin{aligned} 0 &= -400t + 3500 \\ t &= \frac{3500}{400} \\ &= 8.75 \end{aligned}$$

The x-intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Emily will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid—almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn't make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before she saved $3,500, but the

scenario discussed poses the question once she saved $3,500 because this is when her trip and subsequent spending starts. It is also likely that this model is not valid after the x-intercept, unless Emily will use a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is $0 \leq t \leq 8.75$.

In the above example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given, and use it appropriately to build a linear model.

Using a Given Intercept to Build a Model

Some real-world problems provide the y-intercept, which is the constant or initial value. Once the y-intercept is known, the x-intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is $1,000. She plans to pay $250 per month until her balance is $0. The y-intercept is the initial amount of her debt, or $1,000. The rate of change, or slope, is -$250 per month. We can then use the slope-intercept form and the given information to develop a linear model.

$$\begin{aligned} f(x) &= mx + b \\ &= -250x + 1000 \end{aligned}$$

Now we can set the function equal to 0, and solve for x to find the x-intercept.

$$\begin{aligned} 0 &= -250 + 1000 \\ 1000 &= 250x \\ 4 &= x \\ x &= 4 \end{aligned}$$

The x-intercept is the number of months it takes her to reach a balance of $0. The x-intercept is 4 months, so it will take Hannah four months to pay off her loan.

Using a Given Input and Output to Build a Model

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes instead be asked to evaluate the linear model at a given input or set the equation of the linear model equal to a specified output.

Building Systems of Linear Models

Real-world situations including two or more linear functions may be modeled with a **system of linear equations**. Remember, when solving a system of linear equations, we are looking for points the two lines have in common. Typically, there are three types of answers possible, as shown in Figure 2.3.6.

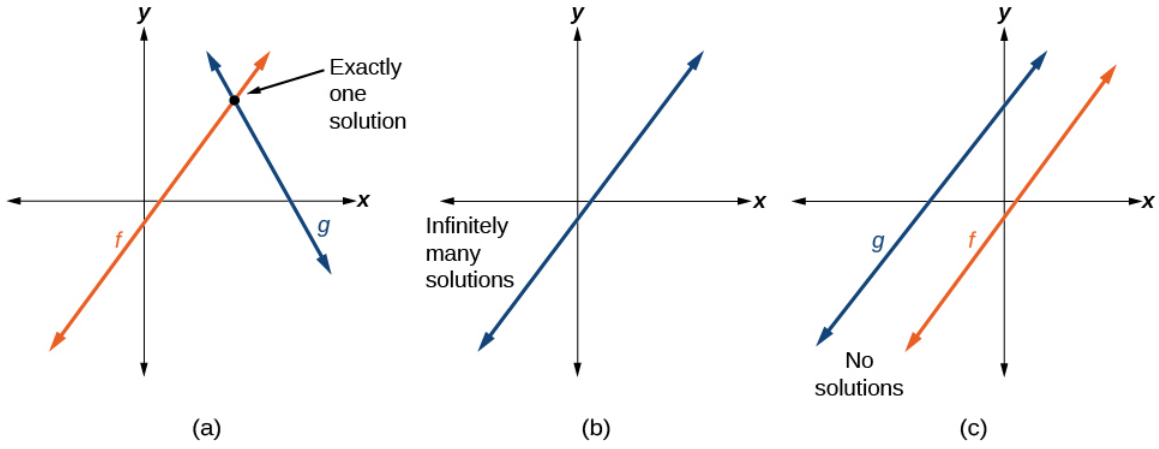


Figure 2.3.6

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2.4: Fitting Linear Models to Data

Drawing and Interpreting Scatter Plots

A scatter plot is a graph of plotted points that may show a relationship between two sets of data. If the relationship is from a **linear model**, or a model that is nearly linear, the professor can draw conclusions using his knowledge of linear functions. Figure 2.4.1 shows a sample scatter plot.

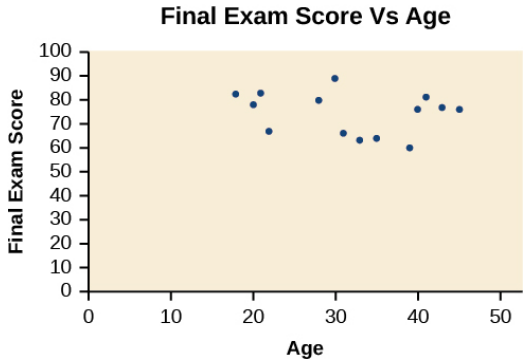


Figure 2.4.1: A scatter plot of age and final exam score variables

Notice this scatter plot does not indicate a **linear relationship**. The points do not appear to follow a trend. In other words, there does not appear to be a relationship between the age of the student and the score on the final exam.

Example 2.4.1: Using a Scatter Plot to Investigate Cricket Chirps

Table shows the number of cricket chirps in 15 seconds, for several different air temperatures, in degrees Fahrenheit[1]. Plot this data, and determine whether the data appears to be linearly related.

Table 2.4.1

Chirps	44	35	20.4	33	31	35	18.5	37	26
Temperature	80.5	70.5	57	66	68	72	52	73.5	53

Solution

Plotting this data, as depicted in Figure 2.4.2 suggests that there may be a trend. We can see from the trend in the data that the number of chirps increases as the temperature increases. The trend appears to be roughly linear, though certainly not perfectly so.

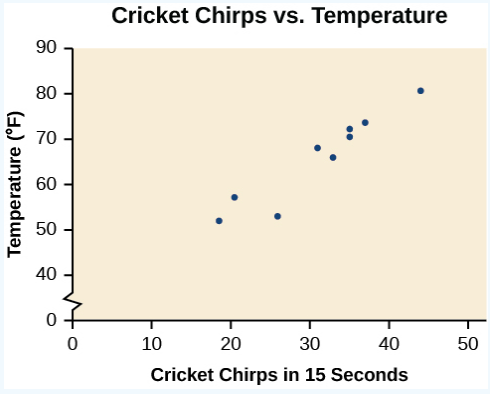


Figure 2.4.2: The line regression is generally positive.

Finding the Line of Best Fit

Once we recognize a need for a linear function to model that data, the natural follow-up question is "what is that linear function?" One way to approximate our linear function is to sketch the line that seems to best fit the data. Then we can extend the line until we can verify the y-intercept. We can approximate the slope of the line by extending it until we can estimate the $\frac{\text{rise}}{\text{run}}$.

Example 2.4.2: Finding a Line of Best Fit

Find a linear function that fits the data in Table 2.4.1 by "eyeballing" a line that seems to fit.

Solution

On a graph, we could try sketching a line.

Using the starting and ending points of our hand drawn line, points $(0, 30)$ and $(50, 90)$, this graph has a slope of

$$m = \frac{60}{50} = 1.2 \tag{2.4.1}$$

and a y-intercept at 30. This gives an equation of

$$T(c) = 1.2c + 30 \tag{2.4.2}$$

where c is the number of chirps in 15 seconds, and $T(c)$ is the temperature in degrees Fahrenheit. The resulting equation is represented in Figure 2.4.3.

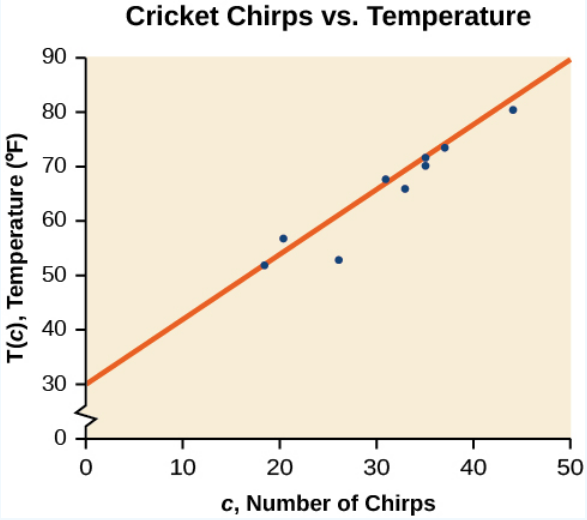


Figure 2.4.3: Scatter plot, showing the line of best fit.

Analysis

This linear equation can then be used to approximate answers to various questions we might ask about the trend.

Recognizing Interpolation or Extrapolation

While the data for most examples does not fall perfectly on the line, the equation is our best guess as to how the relationship will behave outside of the values for which we have data. We use a process known as **interpolation** when we predict a value inside the domain and range of the data. The process of **extrapolation** is used when we predict a value outside the domain and range of the data.

Figure 2.4.4 compares the two processes for the cricket-chirp data addressed in Example 2.4.2. We can see that interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44. Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

There is a difference between making predictions inside the domain and range of values for which we have data and outside that domain and range. Predicting a value outside of the domain and range has its limitations. When our model no longer applies after a certain point, it is sometimes called **model breakdown**. For example, predicting a cost function for a period of two years may

involve examining the data where the input is the time in years and the output is the cost. But if we try to extrapolate a cost when $x = 50$, that is in 50 years, the model would not apply because we could not account for factors fifty years in the future.

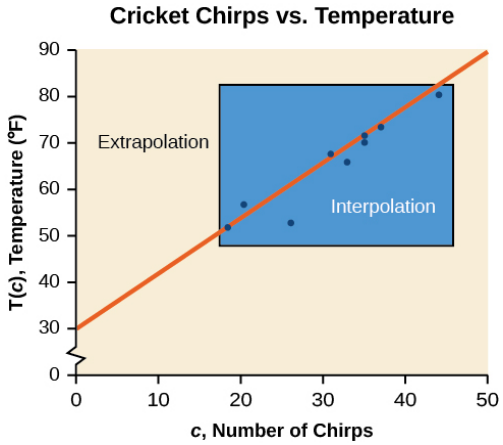


Figure 2.4.4: Interpolation occurs within the domain and range of the provided data whereas extrapolation occurs outside.

Interpolation and Extrapolation

Different methods of making predictions are used to analyze data.

- The method of **extrapolation** involves predicting a value outside the domain and/or range of the data.
- **Model breakdown** occurs at the point when the model no longer applies.

Example 2.4.3: Understanding Interpolation and Extrapolation

Use the cricket data from Table 2.4.1 to answer the following questions:

a. Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.
b. Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.

Solution

a. The number of chirps in the data provided varied from 18.5 to 44. A prediction at 30 chirps per 15 seconds is inside the domain of our data, so would be interpolation. Using our model:

$$T(30) = 30 + 1.2(30) \tag{2.4.3}$$
$$= 66 \text{ degrees} \tag{2.4.4}$$

Based on the data we have, this value seems reasonable.
b. The temperature values varied from 52 to 80.5. Predicting the number of chirps at 40 degrees is extrapolation because 40 is outside the range of our data. Using our model:

$$40 = 30 + 1.2c \tag{2.4.5}$$
$$10 = 1.2c \tag{2.4.6}$$
$$c \approx 8.33 \tag{2.4.7}$$

We can compare the regions of interpolation and extrapolation using Figure 2.4.5.

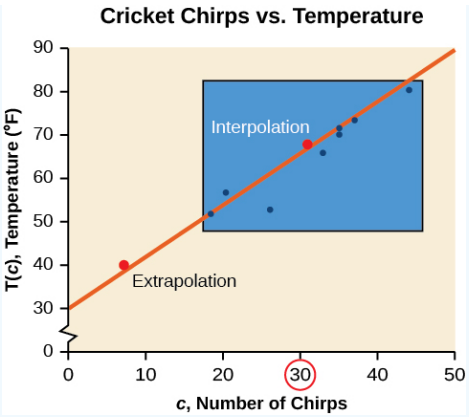


Figure 2.4.5: Scatter plot, showing the line of best fit and where interpolation and extrapolation occurs.

Analysis

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether below around 50 degrees.

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2.R: Linear Functions(Review)

2.1 Linear Functions

1) Determine whether the algebraic equation is linear: $2x + 3y = 7$

Answer

Yes

2) Determine whether the algebraic equation is linear: $6x^2 - y = 5$

3) Determine whether the function is increasing or decreasing: $f(x) = 7x - 2$

Answer

Increasing

4) Determine whether the function is increasing or decreasing: $g(x) = -x + 2$

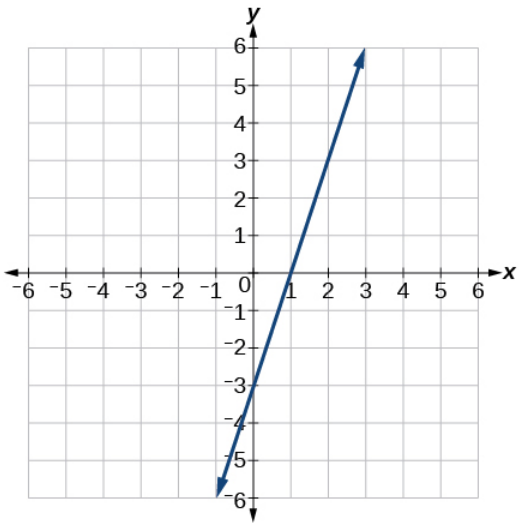
5) Given each set of information, find a linear equation that satisfies the given conditions, if possible. Passes through $(7, 5)$ and $(3, 17)$

Answer

$y = -3x + 26$

6) Given each set of information, find a linear equation that satisfies the given conditions, if possible. x -intercept at $(6, 0)$ and y -intercept at $(0, 10)$

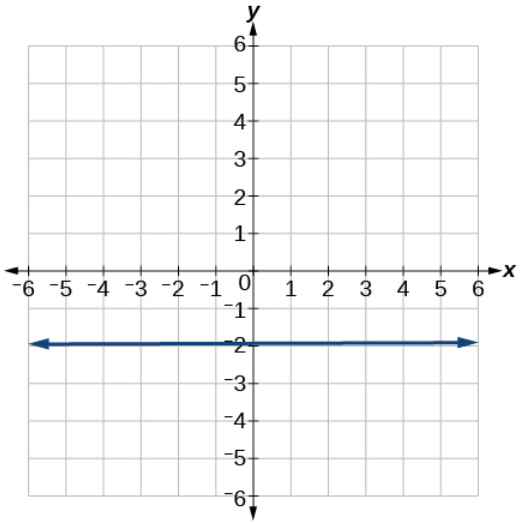
7) Find the slope of the line shown in the line graph.



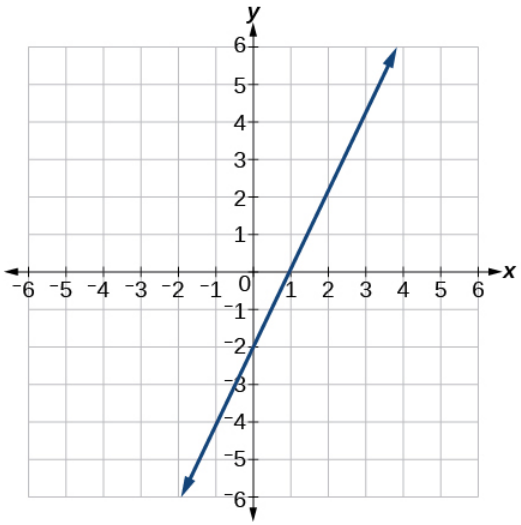
Answer

3

8) Find the slope of the line graphed.



9) Write an equation in slope-intercept form for the line shown.



Answer

$y = 2x - 2$

10) Does the following table represent a linear function? If so, find the linear equation that models the data.

x	–4	0	2	10
$g(x)$	18	–2	–12	–52

11) Does the following table represent a linear function? If so, find the linear equation that models the data.

x	6	8	12	26
$g(x)$	–8	–12	–18	–46

Answer

Not linear

12) On June ^{1st}, a company has $\$4,000,000$ profit. If the company then loses $150,000$ dollars per day thereafter in the month of June, what is the company's profit nth day after June ^{1st}?

2.2 Graphs of Linear Functions

For the exercises 1-2, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

1) $\begin{aligned} 2x-6y &= 12 \\ -x+3y &= 1 \end{aligned}$

Answer

parallel

2) $\begin{aligned} y &= \frac{1}{3}x-2 \\ 3x+y &= -9 \end{aligned}$

For the exercises 3-4, find the x- and y- intercepts of the given equation

3) $7x+9y=-63$

Answer

$(-9,0);(0,-7)$

4) $f(x)=2x-1$

For the exercises 5-11, use the descriptions of the pairs of lines to find the slopes of Line 1 and Line 2. Is each pair of lines parallel, perpendicular, or neither?

5)

Line 1: Passes through $(5,11)$ and $(10,1)$
Line 2 : Passes through $(-1,3)$ and $(-5,11)$

Answer

Line 1 : $m=-2$; Line 2 : $m=-2$; Parallel

6)

Line 1: Passes through $(8,-10)$ and $(0,-26)$
Line 2: Passes through $(2,5)$ and $(4,4)$

7) Write an equation for a line perpendicular to $f(x)=5x-1$ and passing through the point $(5,20)$.

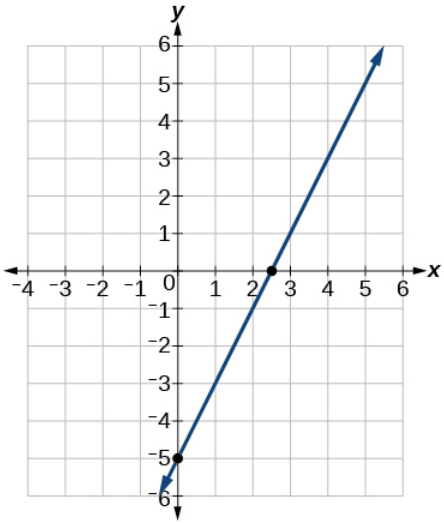
Answer

$y=-0.2x+21$

8) Find the equation of a line with a y -intercept of $(0,2)$ and slope $-\frac{1}{2}$.

9) Sketch a graph of the linear function $f(t)=2t-5$.

Answer



10) Find the point of intersection for the 2 linear functions: $\begin{aligned} x &= y+6 \\ 2x-y &= 13 \end{aligned}$

11) A car rental company offers two plans for renting a car.

Plan A: 25 dollars per day and 10 cents per mile

Plan B: 50 dollars per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

Answer

250

2.3 Modeling with Linear Functions

1) Find the area of a triangle bounded by the y axis, the line $f(x) = 10 - 2x$, and the line perpendicular to f that passes through the origin.

2) A town's population increases at a constant rate. In 2010 the population was $55,000$. By 2012 the population had increased to $76,000$. If this trend continues, predict the population in 2016.

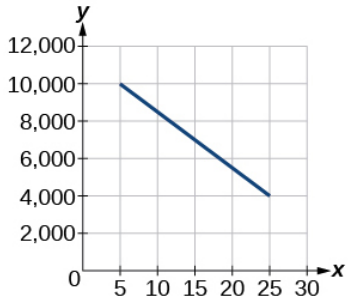
Answer

$118,000$

3) The number of people afflicted with the common cold in the winter months dropped steadily by 50 each year since 2004 until 2010. In 2004, 875 people were inflicted.

4) Find the linear function that models the number of people afflicted with the common cold C as a function of the year, t. When will no one be afflicted?

For the exercises 5-6, use the graph in the Figure below showing the profit, y, in thousands of dollars, of a company in a given year, x, where x represents years since 1980.



5) Find the linear function y, where y depends on x, the number of years since 1980.

Answer

$y = -300x + 11,500$

6) Find and interpret the y-intercept.

7) For the following exercise, consider this scenario: In 2004, a school population was $1,700$. By 2012 the population had grown to $2,500$. Assume the population is changing linearly.

a. How much did the population grow between the year 2004 and 2012?
b. What is the average population growth per year?
c. Find an equation for the population, P, of the school t years after 2004.

Answer

a. 800
b. 100 students per year
c. $P(t) = 100t + 1700$

For the exercises 8-9, consider this scenario: In 2000, the moose population in a park was measured to be $6,500$. By 2010, the population was measured to be $12,500$. Assume the population continues to change linearly.

8) Find a formula for the moose population, P.

9) What does your model predict the moose population to be in 2020?

Answer

$18,500$

For the exercises 10-, consider this scenario: The median home values in subdivisions Pima Central and East Valley (adjusted for inflation) are shown in the Table below. Assume that the house values are changing linearly.

Year	Pima Central	East Valley
1970	32,000	120,250
2010	85,000	150,000

10) In which subdivision have home values increased at a higher rate?

11) If these trends were to continue, what would be the median home value in Pima Central in 2015?

Answer

$\$91,625$

2.4 Fitting Linear Models to Data

1) Draw a scatter plot for the data in the Table below. Then determine whether the data appears to be linearly related.

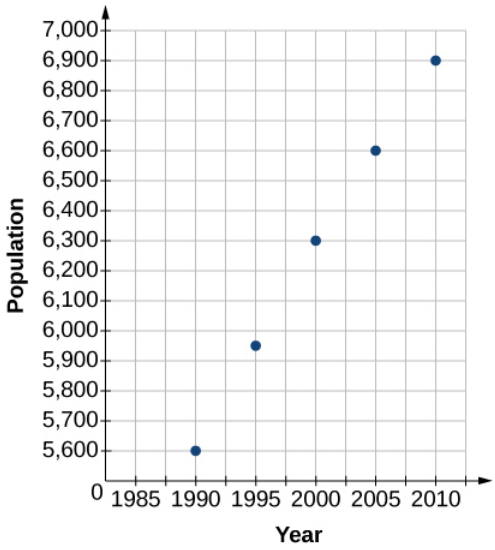
0	2	4	6	8	10
–105	–50	1	55	105	160

2) Draw a scatter plot for the data in the Table. If we wanted to know when the population would reach $15,000$, would the answer involve interpolation or extrapolation?

Year	Population
1990	5,600
1995	5,950
2000	6,300
2005	6,600
2010	6,900

Answer

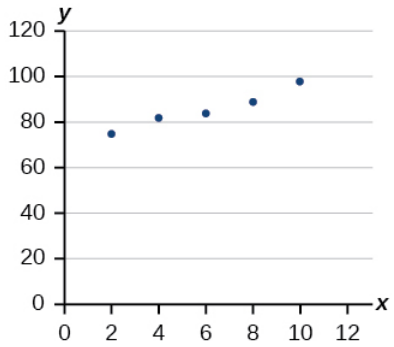
Extrapolation



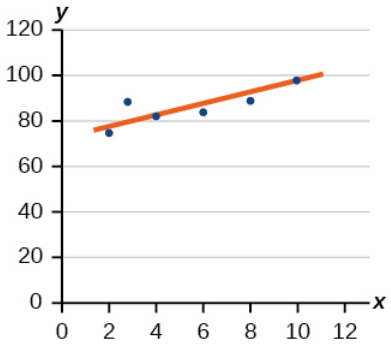
3) Eight students were asked to estimate their score on a 10-point quiz. Their estimated and actual scores are given in the Table. Plot the points, then sketch a line that fits the data.

Predicted	Actual
6	6
7	7
7	8
8	8
7	9
9	10
10	10
10	9

4) Draw a best-fit line for the plotted data.



Answer



For the exercises 5-6, consider the data in the Table below, which shows the percent of unemployed in a city of people 25 years or older who are college graduates is given below, by year.

Year	Percent Graduates
2000	6.5
2002	7.0
2005	7.4
2007	8.2
2010	9.0

5) Determine whether the trend appears to be linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

6) In what year will the percentage exceed 12%?

Answer

Midway through 2024

7) Based on the set of data given in the Table, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

x	17	20	23	26	29
y	15	25	31	37	40

8) Based on the set of data given in the Table, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient to three decimal places.

x	10	12	15	18	20
y	36	34	30	28	22

Answer

$y = -1.294x + 49.412; r = -0.974$

For the exercises 9-12, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs show the population and the year over the ten-year span (population, year) for specific recorded years: $(3,600, 2000); (4,000, 2001); (4,700, 2003); (6,000, 2006)$

9) Use linear regression to determine a function y, where the year depends on the population, to three decimal places of accuracy.

10) Predict when the population will hit $12,000$

Answer

Early in 2022

11) What is the correlation coefficient for this model to three decimal places of accuracy?

12) According to the model, what is the population in 2014?

Answer

$7,660$

Practice Test

1) Determine whether the following algebraic equation can be written as a linear function: $2x + 3y = 7$

Answer

Yes

2) Determine whether the following function is increasing or decreasing: $f(x) = -2x + 5$

3) Determine whether the following function is increasing or decreasing: $f(x) = 7x + 9$

Answer

Increasing

4) Given the following set of information, find a linear equation satisfying the conditions, if possible. Passes through $(5, 1)$ and $(3, -9)$

5) Given the following set of information, find a linear equation satisfying the conditions, if possible. x intercept at $(-4, 0)$ and y -intercept at $(0, -6)$

Answer

$y = -1.5x - 6$

6) Find the slope of the line in the Figure below.

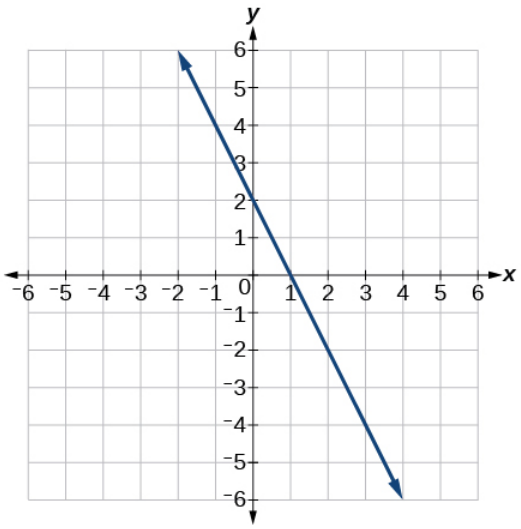
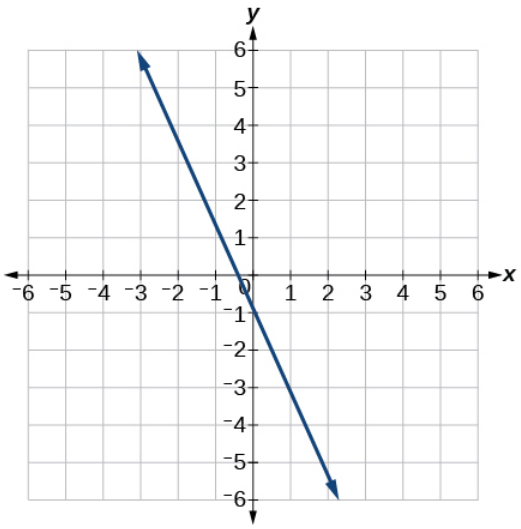


Figure below.



Answer

$y = -2x - 1$

8) Does the Table represent a linear function? If so, find a linear equation that models the data.

x	–6	0	2	4
$g(x)$	14	32	38	44

9) Does the Table represent a linear function? If so, find a linear equation that models the data.

x	1	3	7	11
$g(x)$	4	9	19	12

Answer

No

10) At 6 AM, an online company has sold 120 items that day. If the company sells an average of 30 items per hour for the remainder of the day, write an expression to represent the number of items that were sold n after 6 AM.

For the exercises 11-22, determine whether the lines given by the equations below are parallel, perpendicular, or neither parallel nor perpendicular:

11) $y = \frac{3}{4}x - 9$
$-4x - 3y = 8$

Answer

Perpendicular

12) $-2x + y = 3$
$3x + \frac{3}{2}y = 5$

13) Find the x - and y -intercepts of the equation $2x + 7y = -14$.

Answer

$(-7, 0); (0, -2)$

14) Given below are descriptions of two lines. Find the slopes of Line 1 and Line 2. Is the pair of lines parallel, perpendicular, or neither?

Line 1 : Passes through $(-2, -6)$ and $(3, 14)$
Line 2 : Passes through $(2, 6)$ and $(4, 14)$

15) Write an equation for a line perpendicular to $f(x) = 4x + 3$ and passing through the point $(8, 10)$.

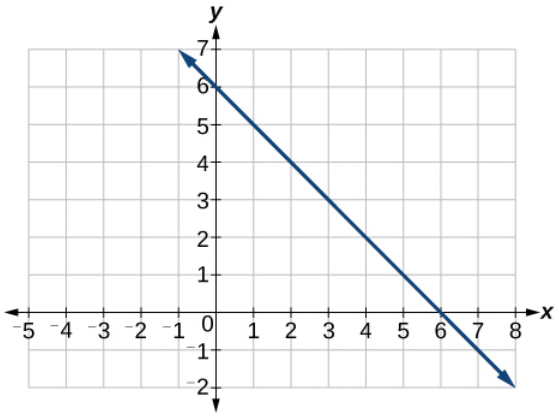
Answer

$y = -0.25x + 12$

16) Sketch a line with a y-intercept of $(0, 5)$ and slope $-\frac{5}{2}$.

17) Graph of the linear function $f(x) = -x + 6$.

Answer



18) For the two linear functions, find the point of intersection: $x = y + 2$
$2x - 3y = -1$

19) A car rental company offers two plans for renting a car.

Plan A: $\$25$ per day and $\$0.10$ per mile
Plan B: $\$40$ per day with free unlimited mileage

How many miles would you need to drive for plan B to save you money?

Answer

150

20) Find the area of a triangle bounded by the y axis, the line $f(x) = 12 - 4x$, and the line perpendicular to f that passes through the origin.

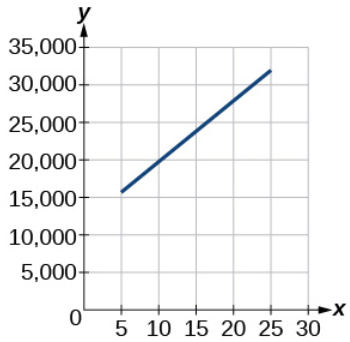
21) A town's population increases at a constant rate. In 2010 the population was $65,000$. By 2012 the population had increased to $90,000$. Assuming this trend continues, predict the population in 2018.

Answer

$165,000$

22) The number of people afflicted with the common cold in the winter months dropped steadily by 25 each year since 2002 until 2012. In 2002, $8,040$ people were inflicted. Find the linear function that models the number of people afflicted with the common cold C as a function of the year, t. When will less than $6,000$ people be afflicted?

For the exercises 23-24, use the graph in Figure, showing the profit, y, in thousands of dollars, of a company in a given year, x, where x represents years since 1980.



23) Find the linear function y, where y depends on x, the number of years since 1980.

Answer

$y = 875x + 10,675$

24) Find and interpret the y-intercept.

25) In 2004, a school population was 1250. By 2012 the population had dropped to 875. Assume the population is changing linearly.

a. How much did the population drop between the year 2004 and 2012?
b. What is the average population decline per year?
c. Find an equation for the population, P, of the school t years after 2004.

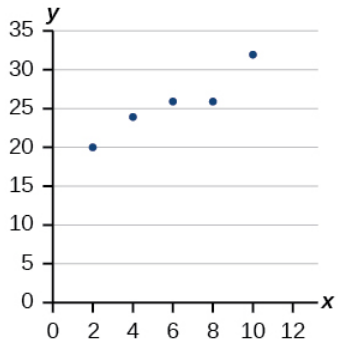
Answer

a. 375
b. dropped an average of 46.875,or about 47 people per year
c. $y = -46.875t + 1250$

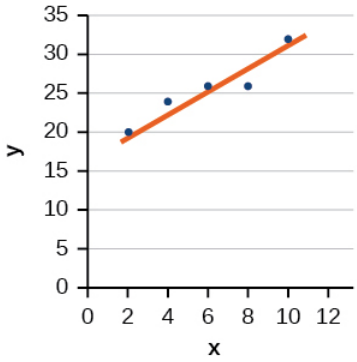
26) Draw a scatter plot for the data provided in the Table. Then determine whether the data appears to be linearly related.

0	2	4	6	8	10
–450	–200	10	265	500	755

27) Draw a best-fit line for the plotted data.



Answer



For the exercises 28-29, use the Table below, which shows the percent of unemployed persons 25 years or older who are college graduates in a particular city, by year.

Year	Percent Graduates
2000	8.5
2002	8.0
2005	7.2
2007	6.7
2010	6.4

28) Determine whether the trend appears linear. If so, and assuming the trend continues, find a linear regression model to predict the percent of unemployed in a given year to three decimal places.

29) In what year will the percentage drop below 4%?

Answer

Early in 2018

30) Based on the set of data given in the Table below, calculate the regression line using a calculator or other technology tool, and determine the correlation coefficient. Round to three decimal places of accuracy.

x	16	18	20	24	26
y	106	110	115	120	125

For the exercises 31-33, consider this scenario: The population of a city increased steadily over a ten-year span. The following ordered pairs shows the population (in hundreds) and the year over the ten-year span, (population, year) for specific recorded years: $(4,500,2000);(4,700,2001);(5,200,2003);(5,800,2006)$

31) Use linear regression to determine a function y, where the year depends on the population. Round to three decimal places of accuracy.

Answer

$y = 0.00455x + 1979.5$

32) Predict when the population will hit $20,000$

33) What is the correlation coefficient for this model?

Answer

$r = 0.999$

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CHAPTER OVERVIEW

3: Polynomial and Rational Functions

In this chapter, we will learn about these concepts and discover how mathematics can be used in such applications.

3.1: Complex Numbers
3.2: Quadratic Functions
3.3: Power Functions and Polynomial Functions
3.4: Graphs of Polynomial Functions
3.5: Dividing Polynomials
3.6: Zeros of Polynomial Functions
3.7: Rational Functions
3.8: Inverses and Radical Functions
3.9: Modeling Using Variation

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3.1: Complex Numbers

Expressing Square Roots of Negative Numbers as Multiples of *i*

We know how to find the square root of any positive real number. In a similar way, we can find the square root of a negative number. The difference is that the root is not real. If the value in the radicand is negative, the root is said to be an imaginary number. The imaginary number i is defined as the square root of negative 1.

$$\sqrt{-1} = i \tag{3.1.1}$$

So, using properties of radicals,

$$i^2 = (\sqrt{-1})^2 = -1 \tag{3.1.2}$$

We can write the square root of any negative number as a multiple of i. Consider the square root of –25.

$$\sqrt{-25} = \sqrt{25 \cdot (-1)} \tag{3.1.3}$$
$$= \sqrt{25}\sqrt{-1} \tag{3.1.4}$$
$$= 5i \tag{3.1.5}$$

We use $5i$ and not $-5i$ because the principal root of 25 is the positive root.

A **complex number** is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written $a + bi$ where a is the real part and bi is the imaginary part. For example, $5 + 2i$ is a complex number. So, too, is $3 + 4\sqrt{3}i$.

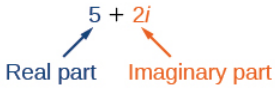


Figure 3.1.1

Imaginary numbers are distinguished from real numbers because a squared imaginary number produces a negative real number. Recall, when a positive real number is squared, the result is a positive real number and when a negative real number is squared, again, the result is a positive real number. Complex numbers are a combination of real and imaginary numbers.

Imaginary and Complex Numbers

A complex number is a number of the form $a + bi$ where

- a is the real part of the complex number.
- bi is the imaginary part of the complex number.

If $b = 0$, then $a + bi$ is a real number. If $a = 0$ and b is not equal to 0, the complex number is called an **imaginary number**. An imaginary number is an even root of a negative number.

Plotting a Complex Number on the Complex Plane

We cannot plot complex numbers on a number line as we might real numbers. However, we can still represent them graphically. To represent a complex number we need to address the two components of the number. We use the complex plane, which is a coordinate system in which the horizontal axis represents the real component and the vertical axis represents the imaginary component. Complex numbers are the points on the plane, expressed as ordered pairs (a, b), where a represents the coordinate for the horizontal axis and b represents the coordinate for the vertical axis.

Let's consider the number $-2 + 3i$. The real part of the complex number is-2 and the imaginary part is $3i$. We plot the ordered pair $(-2, 3)$ to represent the complex number $-2 + 3i$ as shown in Figure 3.1.2

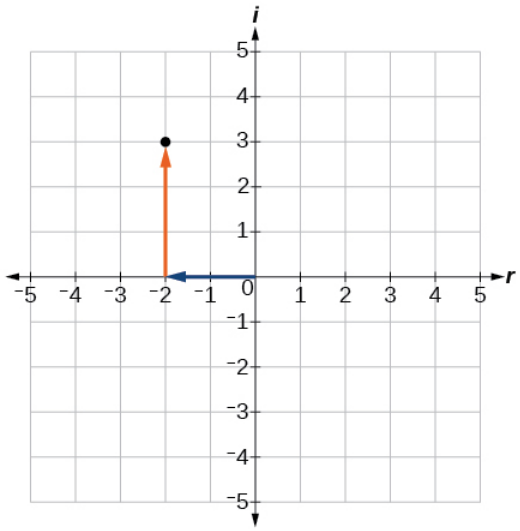


Figure 3.1.2: Plot of a complex number, $-2+3i$. Note that the real part (-2) is plotted on the x-axis and the imaginary part $(3i)$ is plotted on the y-axis.

Complex Plane

In the complex plane, the horizontal axis is the real axis, and the vertical axis is the imaginary axis as shown in Figure 3.1.3.

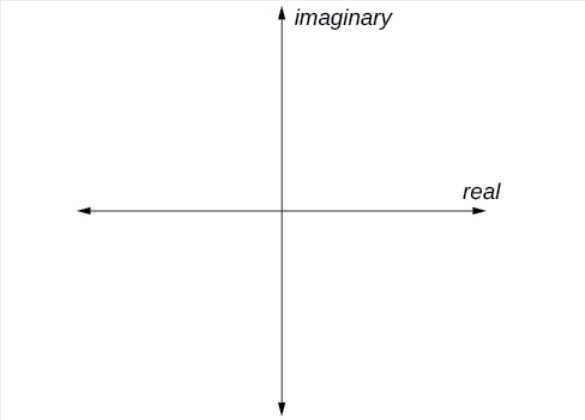


Figure 3.1.3: The complex plane showing that the horizontal axis (in the real plane, the x-axis) is known as the real axis and the vertical axis (in the real plane, the y-axis) is known as the imaginary axis.

Adding and Subtracting Complex Numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and combine the imaginary parts.

Complex Numbers: Addition and Subtraction

Adding complex numbers:

$$(a+bi)+(c+di)=(a+c)+(b+d)i \tag{3.1.6}$$

Subtracting complex numbers:

$$(a+bi)-(c+di)=(a-c)+(b-d)i \tag{3.1.7}$$

Dividing Complex Numbers

Division of two complex numbers is more complicated than addition, subtraction, and multiplication because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator. We need to find a term by which we can

multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with a real number as the denominator. This term is called the **complex conjugate** of the denominator, which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of $a+bi$ is $a-bi$.

Note that complex conjugates have a reciprocal relationship: The complex conjugate of $a+bi$ is $a-bi$, and the complex conjugate of $a-bi$ is $a+bi$. Further, when a quadratic equation with real coefficients has complex solutions, the solutions are always complex conjugates of one another.

Suppose we want to divide $c+di$ by $a+bi$, where neither a nor b equals zero. We first write the division as a fraction, then find the complex conjugate of the denominator, and multiply.

$$\frac{c+di}{a+bi} \text{ where } a\neq 0 \text{ and } b\neq 0$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(c+di)}{(a+bi)}\cdot\frac{(a-bi)}{(a-bi)}=\frac{(c+di)(a-bi)}{(a+bi)(a-bi)}$$

Apply the distributive property.

$$=\frac{ca-cbi+adi-bdi^2}{a^2-abi+abi-b^2i^2}$$

Simplify, remembering that $i^2=-1$.

$$=\frac{ca-cbi+adi-bd(-1)}{a^2-abi+abi-b^2(-1)}$$
$$=\frac{(ca+bd)+(ad-cb)i}{a^2+b^2}$$

Definition: The Complex Conjugate

The *complex conjugate* of a complex number $a+bi$ is $a-bi$. It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

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3.2: Quadratic Functions

Curved antennas, such as the ones shown in Figure 3.2.1, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

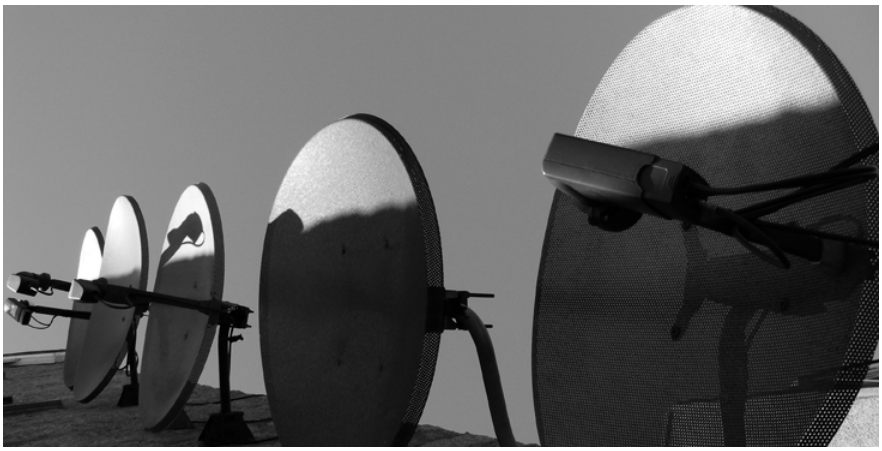


Figure 3.2.1: *An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)*

In this section, we will investigate quadratic functions, which frequently model problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree functions, so they provide a good opportunity for a detailed study of function behavior.

Recognizing Characteristics of Parabolas

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the **vertex**. If the parabola opens up, the vertex represents the lowest point on the graph, or the **minimum value** of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the **maximum value**. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the **axis of symmetry**. These features are illustrated in Figure 3.2.2.

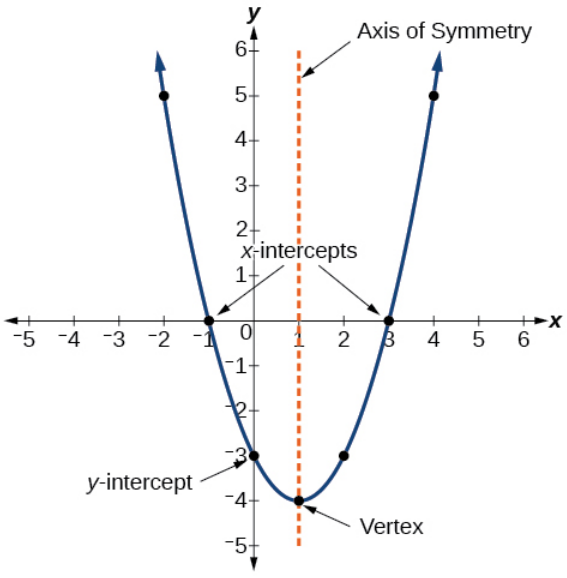


Figure 3.2.2: Graph of a parabola showing where the x and y intercepts, vertex, and axis of symmetry are.

The y-intercept is the point at which the parabola crosses the y-axis. The x-intercepts are the points at which the parabola crosses the x-axis. If they exist, the x-intercepts represent the **zeros**, or **roots**, of the quadratic function, the values of x at which $y = 0$.

Example 3.2.1: Identifying the Characteristics of a Parabola

Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in Figure 3.2.3.

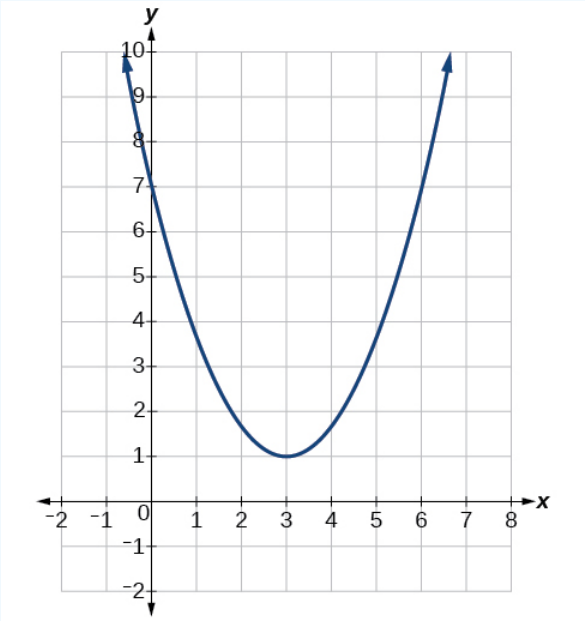


Figure 3.2.3.

Solution

The vertex is the turning point of the graph. We can see that the vertex is at $(3, 1)$. Because this parabola opens upward, the axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is $x = 3$. This parabola does not cross the x-axis, so it has no zeros. It crosses the y-axis at $(0, 7)$ so this is the y-intercept.

Definitions: Forms of Quadratic Functions

A quadratic function is a function of degree two. The graph of a *quadratic function* is a parabola.

- The *general form* of a quadratic function is $f(x) = ax^2 + bx + c$ where a, b, and c are real numbers and $a \neq 0$.
- The standard form of a quadratic function is $f(x) = a(x - h)^2 + k$.
- The vertex (h, k) is located at

$$h = -\frac{b}{2a},\ k = f(h) = f(\frac{-b}{2a}). \tag{3.2.1}$$

Finding the Domain and Range of a Quadratic Function

Any number can be the input value of a quadratic function. Therefore, the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum point, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all y-values greater than or equal to the y-coordinate at the turning point or less than or equal to the y-coordinate at the turning point, depending on whether the parabola opens up or down.

Definition: Domain and Range of a Quadratic Function

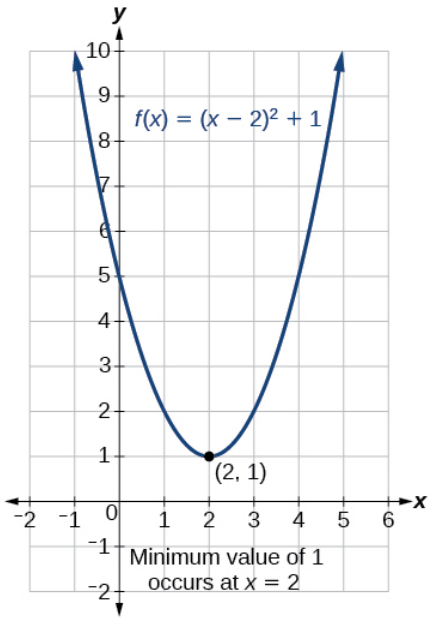
The domain of any **quadratic function** is all real numbers.

The range of a quadratic function written in general form $f(x) = ax^2 + bx + c$ with a positive a value is $f(x) \geq f(-\frac{b}{2a})$, or $[f(-\frac{b}{2a}), \infty)$; the range of a quadratic function written in general form with a negative a value is $f(x) \leq f(-\frac{b}{2a})$, or $(-\infty, f(-\frac{b}{2a})]$.

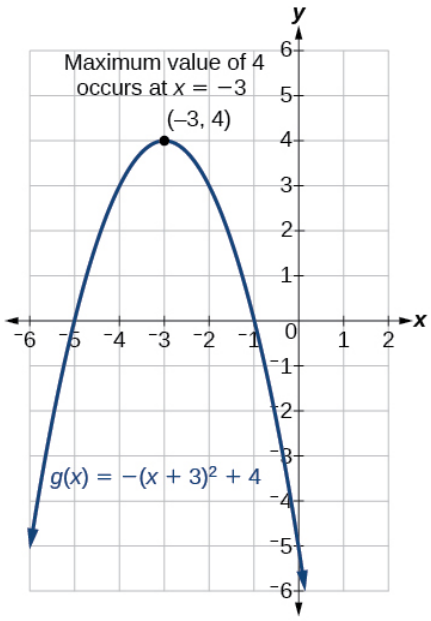
The range of a quadratic function written in standard form $f(x) = a(x-h)^2 + k$ with a positive a value is $f(x) \geq k$; the range of a quadratic function written in standard form with a negative a value is $f(x) \leq k$.

Determining the Maximum and Minimum Values of Quadratic Functions

The output of the quadratic function at the vertex is the maximum or minimum value of the function, depending on the orientation of the **parabola**. We can see the maximum and minimum values in Figure 3.2.9.



(a)



(b)

Figure 3.2.9: Minimum and maximum of two quadratic functions.

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue.

Finding the x- and y-Intercepts of a Quadratic Function

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the y-intercept of a quadratic by evaluating the function at an input of zero, and we find the x-intercepts at locations where the output is zero. Notice in Figure 3.2.13 that the number of x-intercepts can vary depending upon the location of the graph.

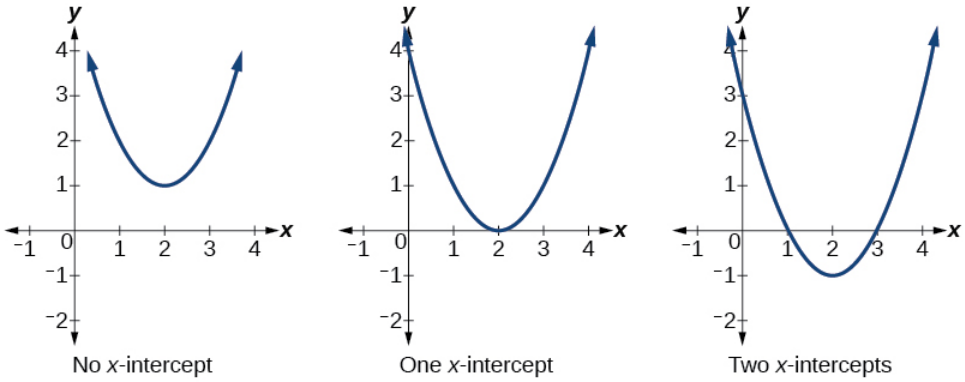


Figure 3.2.13: Number of x-intercepts of a parabola.

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3.3: Power Functions and Polynomial Functions

Identifying Power Functions

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, a coefficient, and a variable raised to a fixed real number. (A number that multiplies a variable raised to an exponent is known as a coefficient.)

As an example, consider functions for area or volume. The function for the area of a circle with radius r is

$$A(r) = \pi r^2$$

and the function for the volume of a sphere with radius r is

$$V(r) = \frac{4}{3}\pi r^3$$

Both of these are examples of power functions because they consist of a coefficient, π or $\frac{4}{3}\pi$, multiplied by a variable r raised to a power.

Definition: Power Function

A *power function* is a function that can be represented in the form

$$f(x) = kx^p \tag{3.3.1}$$

where k and p are real numbers, and k is known as the *coefficient*.

Identifying End Behavior of Power Functions

Figure 3.3.2 shows the graphs of $f(x) = x^2$, $g(x) = x^4$ and and $h(x) = x^6$, which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function in the toolkit. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

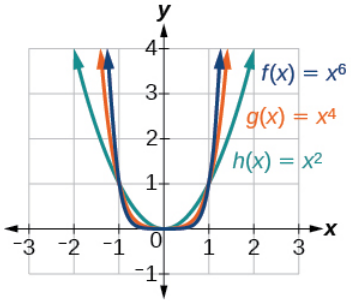


Figure 3.3.2: Even-power functions

To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol ∞ for positive infinity and $-\infty$ for negative infinity. When we say that "x approaches infinity," which can be symbolically written as $x \to \infty$, we are describing a behavior; we are saying that x is increasing without bound.

With the even-power function, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as x approaches positive or negative infinity, the $f(x)$ values increase without bound. In symbolic form, we could write

$$\text{as } x \to \pm\infty,\ f(x) \to \infty$$

Figure 3.3.3 shows the graphs of $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^7$, which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function in the toolkit. Again, as the power increases, the graphs flatten near the origin and become steeper away from the origin.

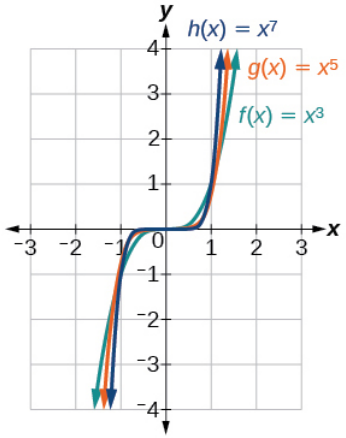


Figure 3.3.3: Odd-power function

These examples illustrate that functions of the form $f(x) = x^n$ reveal symmetry of one kind or another. First, in Figure 3.3.2 we see that even functions of the form $f(x) = x^n$, n even, are symmetric about the y-axis. In Figure 3.3.3 we see that odd functions of the form $f(x) = x^n$, n odd, are symmetric about the origin.

For these odd power functions, as x approaches negative infinity, $f(x)$ decreases without bound. As x approaches positive infinity, $f(x)$ increases without bound. In symbolic form we write

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$
$$\text{as } x \to \infty,\ f(x) \to \infty$$

The behavior of the graph of a function as the input values get very small $(x \to -\infty)$ and get very large $x \to \infty$ is referred to as the *end behavior* of the function. We can use words or symbols to describe end behavior.

Figure 3.3.4 shows the end behavior of power functions in the form $f(x) = kx^n$ where n is a non-negative integer depending on the power and the constant.

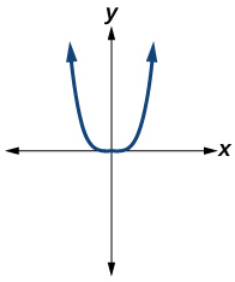
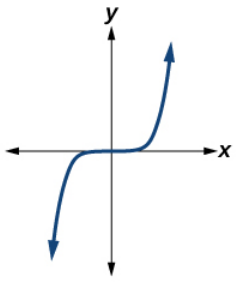
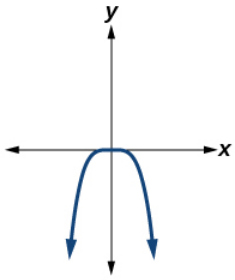
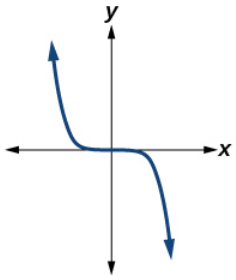
	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to \infty$	 $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to -\infty$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to -\infty$

Figure 3.3.4

Identifying Polynomial Functions

An oil pipeline bursts in the Gulf of Mexico, causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius r of the spill depends on the number of weeks w that have passed. This relationship is linear.

$$r(w) = 24 + 8w$$

We can combine this with the formula for the area A of a circle.

$$A(r) = \pi r^2$$

Composing these functions gives a formula for the area in terms of weeks.

$$\begin{aligned} A(w) &= A(r(w)) \\ &= A(24 + 8w) \\ &= \pi(24 + 8w)^2 \end{aligned}$$

Multiplying gives the formula.

$$A(w) = 576\pi + 384\pi w + 64\pi w^2$$

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

Definition: Polynomial Functions

Let n be a non-negative integer. A *polynomial function* is a function that can be written in the form

$$f(x) = a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0 \tag{3.3.2}$$

This is called the general form of a polynomial function. Each a_i is a coefficient and can be any real number. Each product $a_i x^i$ is a *term of a polynomial function.*

Example 3.3.4: Identifying Polynomial Functions

Which of the following are polynomial functions?

- $f(x) = 2x^3 \cdot 3x + 4$
- $g(x) = -x(x^2 - 4)$
- $h(x) = 5\sqrt{x} + 2$

Solution

The first two functions are examples of polynomial functions because they can be written in the form of Equation 3.3.2, where the powers are non-negative integers and the coefficients are real numbers.

- $f(x)$ can be written as $f(x) = 6x^4 + 4$.
- $g(x)$ can be written as $g(x) = -x^3 + 4x$.
- $h(x)$ cannot be written in this form and is therefore not a polynomial function.

Identifying the Degree and Leading Coefficient of a Polynomial Function

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order of power, or in general form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the highest power of the variable, or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

Terminology of Polynomial Functions

We often rearrange polynomials so that the powers are descending.

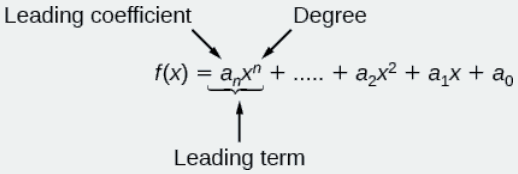


Figure 3.3.7

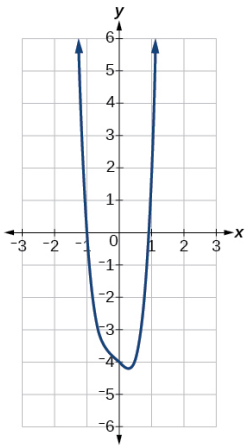
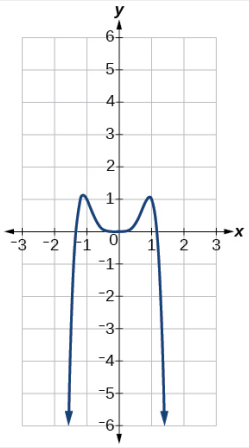
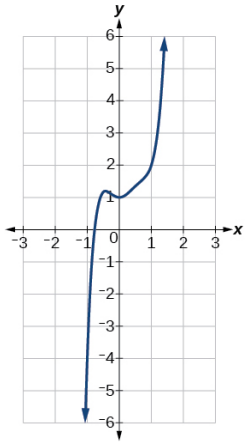
When a polynomial is written in this way, we say that it is in *general form*.

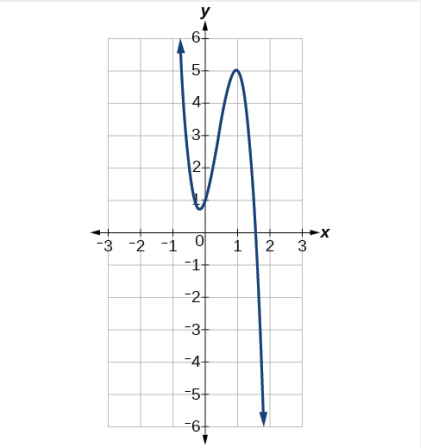
Identifying End Behavior of Polynomial Functions

Knowing the degree of a polynomial function is useful in helping us predict its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as x gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree (Table 3.3.3).

Table 3.3.3

Polynomial Function	Leading Term	Graph of Polynomial Function

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = 5x^4 + 2x^3 - x - 4$	$5x^4$	
$f(x) = -2x^6 - x^5 + 3x^4 + x^3$	$-2x^6$	
$f(x) = 3x^5 - 4x^4 + 2x^2 + 1$	$3x^5$	

Polynomial Function	Leading Term	Graph of Polynomial Function
$f(x) = -6x^3 + 7x^2 + 3x + 1$	$-6x^3$	

Example 3.3.6: Identifying End Behavior and Degree of a Polynomial Function

Describe the end behavior and determine a possible degree of the polynomial function in Figure 3.3.8.

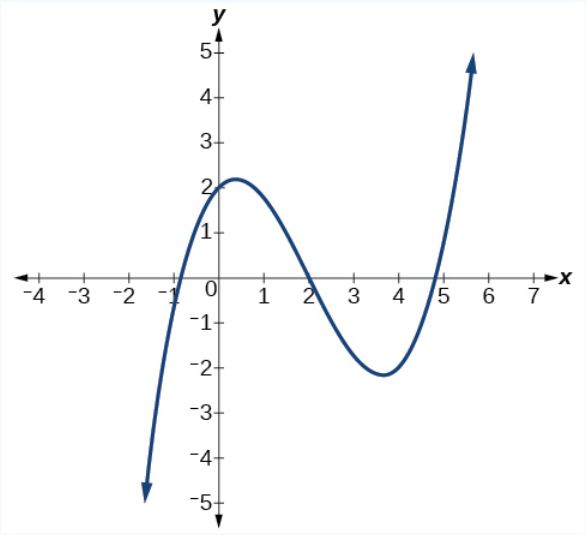


Figure 3.3.8.

Solution

As the input values x get very large, the output values $f(x)$ increase without bound. As the input values x get very small, the output values $f(x)$ decrease without bound. We can describe the end behavior symbolically by writing

$$\text{as } x \to \infty,\ f(x) \to \infty$$

$$\text{as } x \to -\infty,\ f(x) \to -\infty$$

In words, we could say that as x values approach infinity, the function values approach infinity, and as x values approach negative infinity, the function values approach negative infinity.

We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.

Identifying Local Behavior of Polynomial Functions

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A **turning point** is a point at which the function values

change from increasing to decreasing or decreasing to increasing.

We are also interested in the intercepts. As with all functions, the y-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one y-intercept $(0, a_0)$. The x-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one x-intercept. See Figure 3.3.10.

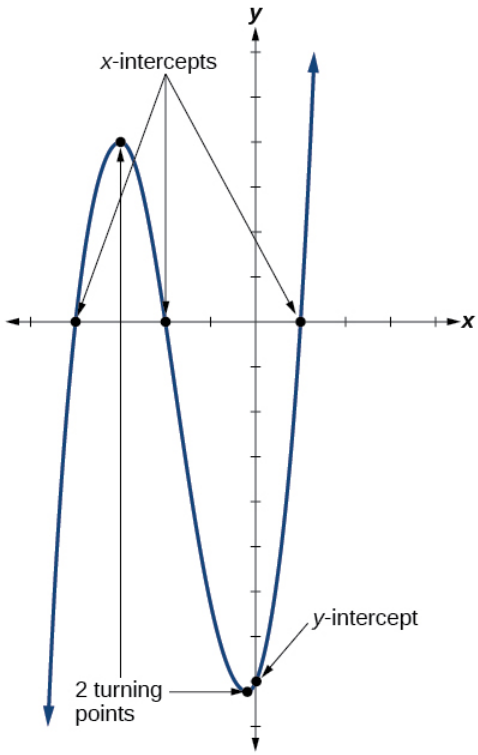


Figure 3.3.10

Defintion: Intercepts and Turning Points of Polynomial Functions

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The y-intercept is the point at which the function has an input value of zero. The x-intercepts are the points at which the output value is zero.

Comparing Smooth and Continuous Graphs

The degree of a polynomial function helps us to determine the number of x-intercepts and the number of turning points. A polynomial function of n^{th} degree is the product of n factors, so it will have at most n roots or zeros, or x-intercepts. The graph of the polynomial function of degree n must have at most $n-1$ turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

A **continuous function** has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A smooth curve is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

Intercepts and Turning Points of Polynomials

A polynomial of degree n will have, at most, n x-intercepts and $n-1$ turning points.

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3.4: Graphs of Polynomial Functions

Recognizing Characteristics of Graphs of Polynomial Functions

Polynomial functions of degree 2 or more have graphs that do not have sharp corners; recall that these types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. Figure 3.4.1 shows a graph that represents a **polynomial function** and a graph that represents a function that is not a polynomial.

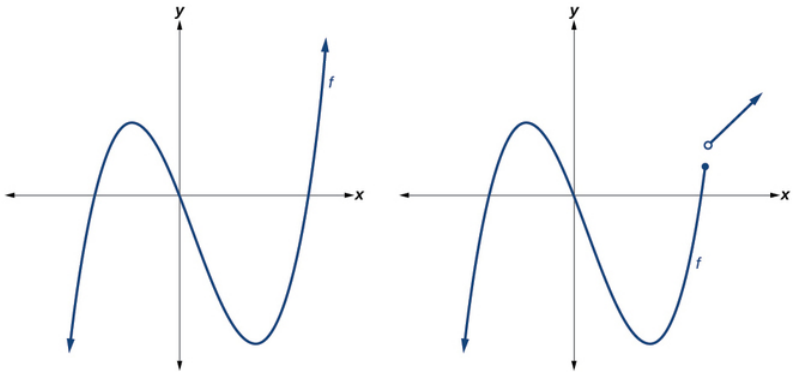


Figure 3.4.1: Graph of $f(x) = x^3 - 0.01x$.

Example 3.4.1: Recognizing Polynomial Functions

Which of the graphs in Figure 3.4.2 represents a polynomial function?

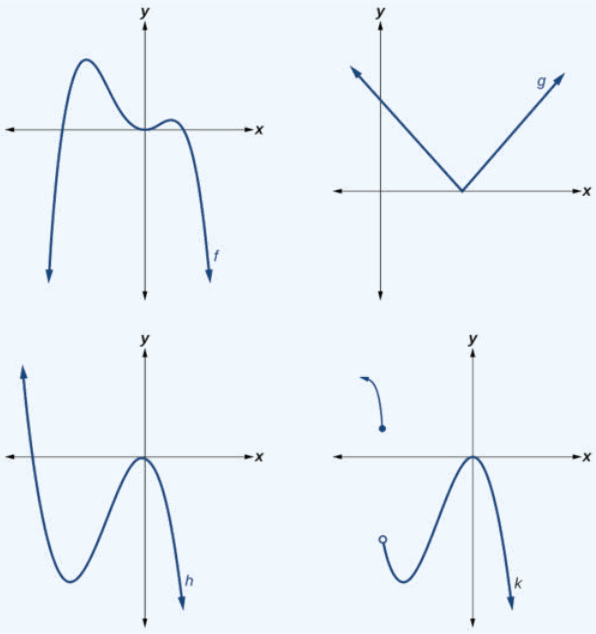


***Figure* 3.4.2**

Solution

- The graphs of f and h are graphs of polynomial functions. They are smooth and **continuous**.
- The graphs of g and k are graphs of functions that are not polynomials. The graph of function g has a sharp corner. The graph of function k is not continuous.

Using Factoring to Find Zeros of Polynomial Functions

Recall that if f is a polynomial function, the values of x for which $f(x) = 0$ are called **zeros** of f. If the equation of the polynomial function can be factored, we can set each factor equal to zero and solve for the zeros.

We can use this method to find x-intercepts because at the x-intercepts we find the input values when the output value is zero. For general polynomials, this can be a challenging prospect. While quadratics can be solved using the relatively simple quadratic formula, the corresponding formulas for cubic and fourth-degree polynomials are not simple enough to remember, and formulas do not exist for general higher-degree polynomials. Consequently, we will limit ourselves to three cases in this section:

The polynomial can be factored using known methods: greatest common factor and trinomial factoring.
The polynomial is given in factored form.
Technology is used to determine the intercepts.

Determining End Behavior

As we have already learned, the behavior of a graph of a polynomial function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \tag{3.4.1}$$

will either ultimately rise or fall as x increases without bound and will either rise or fall as x decreases without bound. This is because for very large inputs, say 100 or 1,000, the leading term dominates the size of the output. The same is true for very small inputs, say –100 or –1,000.

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, $a_n x^n$, is an even power function, as x increases or decreases without bound, $f(x)$ increases without bound. When the leading term is an odd power function, as x decreases without bound, $f(x)$ also decreases without bound; as x increases without bound, $f(x)$ also increases without bound. If the leading term is negative, it will change the direction of the end behavior. Figure $3.4.11$ summarizes all four cases.

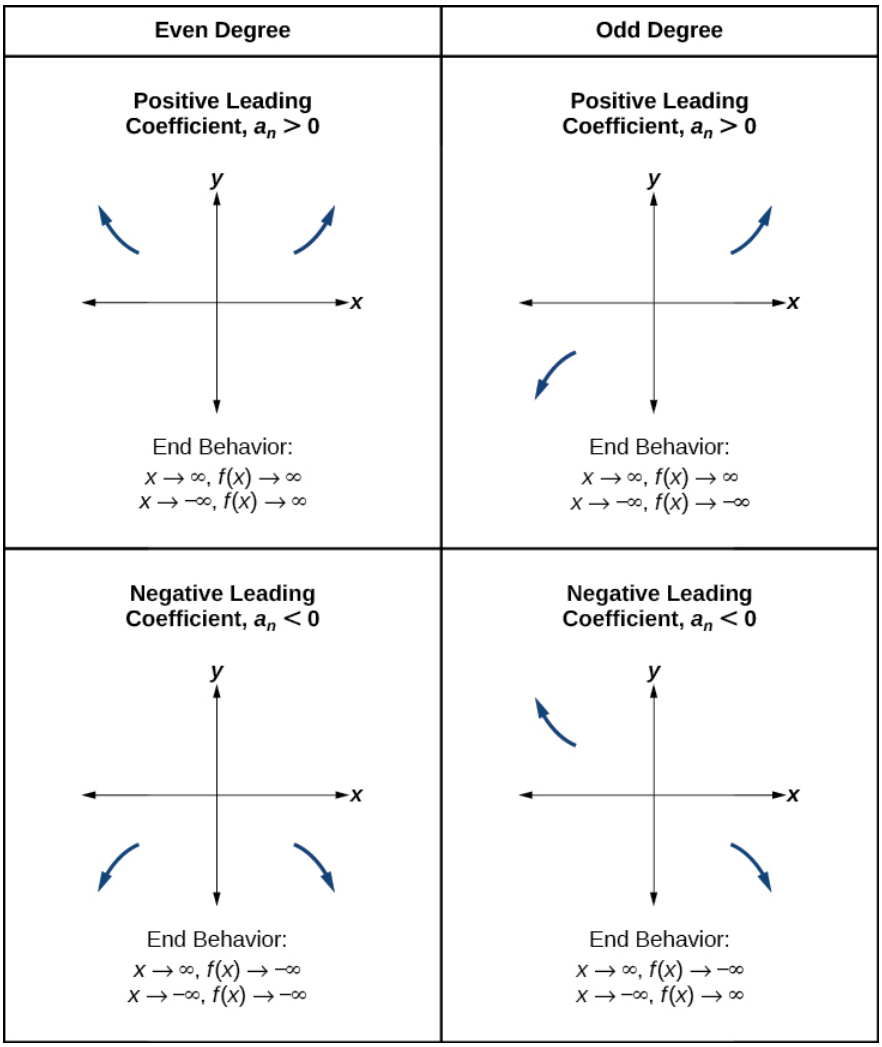


Figure 3.4.11.

Understanding the Relationship between Degree and Turning Points

In addition to the end behavior, recall that we can analyze a polynomial function's local behavior. It may have a turning point where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). Look at the graph of the polynomial function $f(x) = x^4 - x^3 - 4x^2 + 4x$ in Figure 3.4.12 The graph has three turning points.

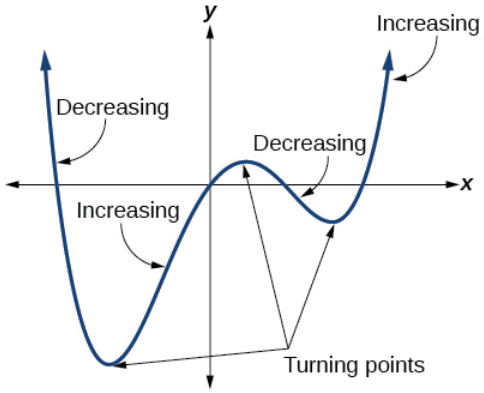


Figure 3.4.12: Graph of $f(x) = x^4 - x^3 - 4x^2 + 4x$

This function f is a 4th degree polynomial function and has 3 turning points. The maximum number of turning points of a polynomial function is always one less than the degree of the function.

Definition: Interpreting Turning Points

A turning point is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising). A polynomial of degree n will have at most $n-1$ turning points.

Graphing Polynomial Functions

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

Howto: Given a polynomial function, sketch the graph

1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetrical about the y-axis, that is, $f(-x) = f(x)$. If a function is an odd function, its graph is symmetrical about the origin, that is, $f(-x) = -f(x)$.
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch a graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

Example 3.4.8: Sketching the Graph of a Polynomial Function

Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

Solution

This graph has two x-intercepts. At $x = -3$, the factor is squared, indicating a multiplicity of 2. The graph will bounce at this x-intercept. At $x = 5$,the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.

The y-intercept is found by evaluating $f(0)$.

$$\begin{aligned} f(0) &= -2(0+3)^2(0-5) \\ &= -2 \cdot 9 \cdot (-5) \\ &= 90 \end{aligned}$$

The y-intercept is $(0, 90)$.

Additionally, we can see the leading term, if this polynomial were multiplied out, would be $-2x3$, so the end behavior is that of a vertically reflected cubic, with the outputs decreasing as the inputs approach infinity, and the outputs increasing as the inputs approach negative infinity. See Figure $3.4.13$

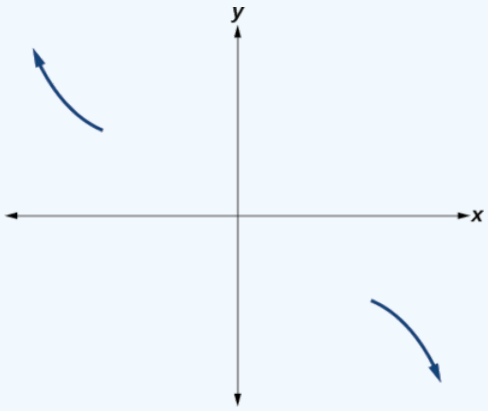


Figure 3.4.13: Showing the distribution for the leading term.

To sketch this, we consider that:

- As $x \to -\infty$ the function $f(x) \to \infty$,so we know the graph starts in the second quadrant and is decreasing toward the x-axis.
- Since $f(-x) = -2(-x+3)^2(-x-5)$ is not equal to $f(x)$, the graph does not display symmetry.
- At $(-3, 0)$, the graph bounces off of thex-axis, so the function must start increasing.
- At $(0, 90)$, the graph crosses the y-axis at the y-intercept. See Figure $3.4.14$

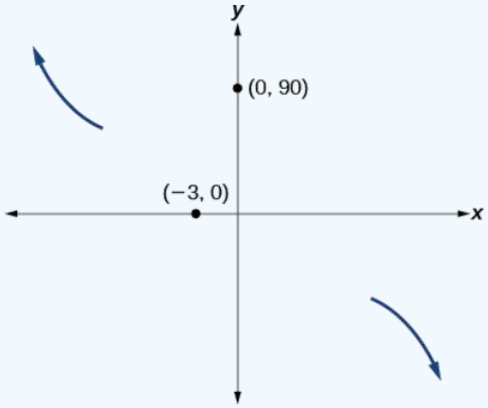


Figure 3.4.14: Graph of the end behavior and intercepts, $(-3, 0)$ and $(0, 90)$, for the function $f(x) = -2(x+3)^2(x-5)$.

Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at $(5, 0)$. See Figure $3.4.15$

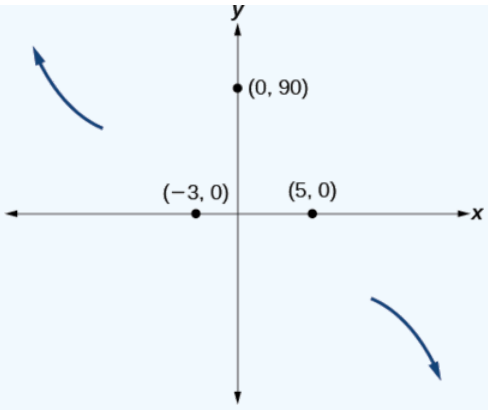


Figure 3.4.15: Graph of the end behavior and intercepts, $(-3,0)$, $(0,90)$ and $(5,0)$, for the function $f(x)=-2(x+3)^2(x-5)$.

As $x\to\infty$ the function $f(x)\to-\infty$,

so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.

Using technology, we can create the graph for the polynomial function, shown in Figure 3.4.16, and verify that the resulting graph looks like our sketch in Figure 3.4.15.

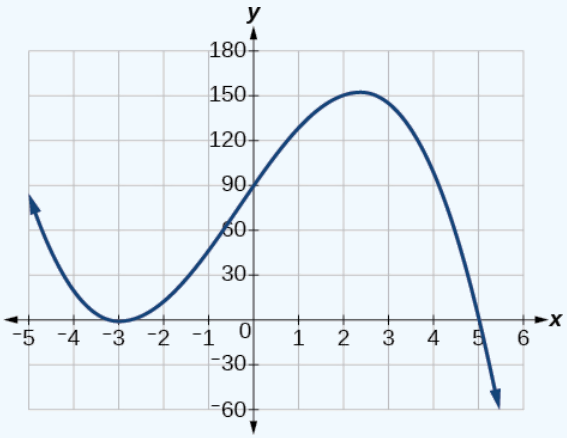


***Figure* 3.4.16:** *The complete graph of the polynomial function* $f(x)=-2(x+3)^2(x-5)$.

Using the Intermediate Value Theorem

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the x-axis, we can confirm that there is a zero between them. Consider a polynomial function f whose graph is smooth and continuous. The **Intermediate Value Theorem** states that for two numbers a and b in the domain of f,if $a<b$ and $f(a)\neq f(b)$,then the function f takes on every value between $f(a)$ and $f(b)$. We can apply this theorem to a special case that is useful in graphing polynomial functions. If a point on the graph of a continuous function f at $x=a$ lies above the x-axis and another point at $x=b$ lies below thex-axis, there must exist a third point between $x=a$ and $x=b$ where the graph crosses the x-axis. Call this point $(c,f(c))$.This means that we are assured there is a solution c where $f(c)=0$.

In other words, the **Intermediate Value Theorem** tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the x-axis. Figure 3.4.18shows that there is a zero between a and b.

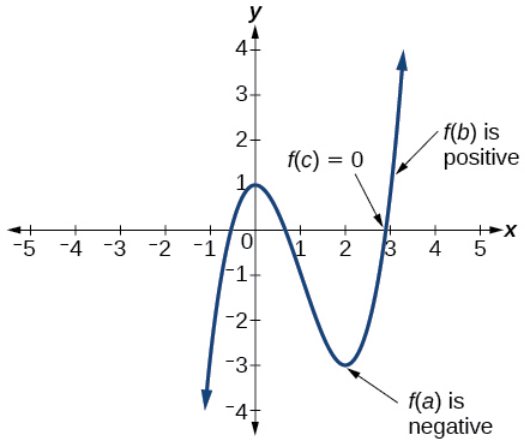


Figure 3.4.18: Using the Intermediate Value Theorem to show there exists a zero.

Definition: Intermediate Value Theorem

Let f be a polynomial function. The **Intermediate Value Theorem** states that if $f(a)$ and $f(b)$ have opposite signs, then there exists at least one value c between a and b for which $f(c) = 0$.

Using Local and Global Extrema

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a **global maximum** or a **global minimum**. These are also referred to as the absolute maximum and absolute minimum values of the function.

Note: Local and Global Extrema

A **local maximum** or **local minimum** at $x = a$ (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around $x = a$.If a function has a local maximum at a, then $f(a) \geq f(x)$for all x in an open interval around $x = a$. If a function has a local minimum at a, then $f(a) \leq f(x)$for all x in an open interval around $x = a$.

A **global maximum** or **global minimum** is the output at the highest or lowest point of the function. If a function has a global maximum at a, then $f(a) \geq f(x)$ for all x. If a function has a global minimum at a, then $f(a) \leq f(x)$ for all x.

We can see the difference between local and global extrema in Figure 3.4.22

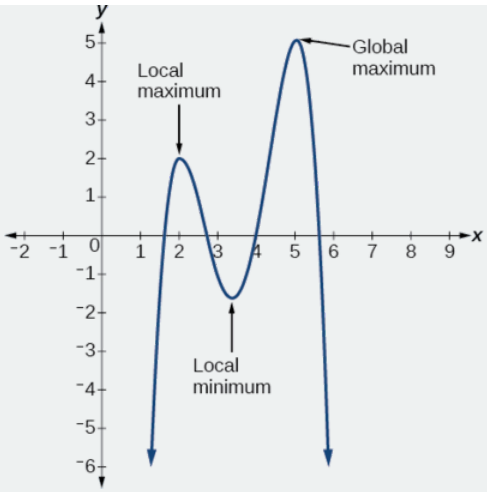


Figure 3.4.22: Graph of an even-degree polynomial that denotes the local maximum and minimum and the global maximum.

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3.5: Dividing Polynomials

Using Long Division to Divide Polynomials

We are familiar with the **long division** algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

Long Division

$$\begin{array}{r} 59 \\ 3\overline{)178} \\ -15\downarrow \\ \hline 28 \\ -27 \\ \hline 1 \end{array}$$

Step 1: $5 \times 3 = 15$ and $17 - 15 = 2$
Step 2: Bring down the 8
Step 3: $9 \times 3 = 27$ and $28 - 27 = 1$

Answer: $59\,R\,1$ or $59\frac{1}{3}$

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.

$$\begin{aligned} \text{dividend} &= (\text{divisor}\cdot\text{quotient}) + \text{remainder} \\ 178 &= (3\cdot 59) + 1 \\ &= 177 + 1 \\ &= 178 \end{aligned}$$

We call this the **Division Algorithm** and will discuss it more formally after looking at an example.

Division of polynomials that contain more than one term has similarities to long division of whole numbers. We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder. The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide $2x^3 - 3x^2 + 4x + 5$ by $x + 2$ using the long division algorithm, it would look like this:

$$x+2\overline{)2x^3-3x^2+4x+5}$$ Set up the division problem.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3-3x^2+4x+5} \end{array}$$ $2x^3$ divided by x is $2x^2$.

$$\begin{array}{r} 2x^2 \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \end{array}$$ Multiply $x+2$ by $2x^2$ and subtract.
Bring down the next term.

$$\begin{array}{r} 2x^2-7x \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \end{array}$$ $-7x^2$ divided by x is $-7x$.

$$\begin{array}{r} 2x^2-7x \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \\ \underline{-\left(-7x^2-14x\right)} \\ 18x+5 \end{array}$$ Multiply $x+2$ by $-7x$.
Subtract and bring down the next term.

$$\begin{array}{r} 2x^2-7x+18 \\ x+2\overline{)2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \\ \underline{-\left(-7x^2-14x\right)} \\ 18x+5 \\ \underline{-(18x+36)} \\ -31 \end{array}$$ $18x$ divided by x is 18.
Multiply $x+2$ by 18.
Subtract.

We have found

$$\frac{2x^3-3x^2+4x+5}{x+2}=2x^2-7x+18-\frac{31}{x+2}$$

or

$$2x^3-3x^2+4x+5=(x+2)(2x^2-7x+18)-31$$

We can identify the dividend, the **divisor**, the **quotient**, and the **remainder**.

Writing the result in this manner illustrates the Division Algorithm.

The Division Algorithm

The Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$, there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x)=d(x)q(x)+r(x) \tag{3.5.1}$$

$q(x)$ is the quotient and $r(x)$ is the remainder. The remainder is either equal to zero or has degree strictly less than $d(x)$.

If $r(x)=0$, then $d(x)$ divides evenly into $f(x)$. This means that, in this case, both $d(x)$ and $q(x)$ are factors of $f(x)$.

Using Synthetic Division to Divide Polynomials

As we've seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1.

To illustrate the process, recall the example at the beginning of the section.

Divide $2x^3-3x^2+4x+5$ by $x+2$ using the long division algorithm.

The final form of the process looked like this:

$$\begin{array}{r} 2x^2-7x+18 \\ x+2\overline{)\,2x^3-3x^2+4x+5} \\ \underline{-\left(2x^3+4x^2\right)} \\ -7x^2+4x \\ \underline{-\left(-7x^2-14x\right)} \\ 18x+5 \\ \underline{-\left(18x+36\right)} \\ -31 \end{array}$$

There is a lot of repetition in the table. If we don't write the variables but, instead, line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.

$$\begin{array}{r|rrrr} 2 & 2 & -3 & 4 & 5 \\ & -2 & -4 & & \\ \hline & & -7 & 14 & \\ & & & 18 & -36 \\ \hline & & & & -31 \end{array}$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the "divisor" to –2, multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & -36 \\ \hline & 2 & -7 & 18 & -31 \end{array}$$

We then multiply it by the "divisor" and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is $2x^2-7x+18$ and the remainder is -31. The process will be made more clear in Example $3.5.3$.

Synthetic Division

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form $x-k$. In *synthetic division*, only the coefficients are used in the division process.

Given two polynomials, use synthetic division to divide

1. Write k for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by k. Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by k. Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0, the next number from the right has degree 1, the next number from the right has degree 2, and so on.

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3.6: Zeros of Polynomial Functions

Evaluating a Polynomial Using the Remainder Theorem

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the **Remainder Theorem**. If the polynomial is divided by $x-k$, the remainder may be found quickly by evaluating the polynomial function at k, that is, $f(k)$. Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that, given a polynomial dividend $f(x)$ and a non-zero polynomial divisor $d(x)$ where the degree of $d(x)$ is less than or equal to the degree of $f(x)$,there exist unique polynomials $q(x)$ and $r(x)$ such that

$$f(x)=d(x)q(x)+r(x)$$

If the divisor, $d(x)$, is $x-k$, this takes the form

$$f(x)=(x-k)q(x)+r$$

Since the divisor $x-k$

is linear, the remainder will be a constant, r. And, if we evaluate this for $x=k$, we have

$$\begin{aligned} f(k) &= (k-k)q(k)+r \\ &= 0\cdot q(k)+r \\ &= r \end{aligned}$$

In other words, $f(k)$ is the remainder obtained by dividing $f(x)$by $x-k$.

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x-k$,then the remainder is the value $f(k)$.

Using the Factor Theorem to Solve a Polynomial Equation

The **Factor Theorem** is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm.

$$f(x)=(x-k)q(x)+r \tag{3.6.1}$$

If k is a zero, then the remainder r is $f(k)=0$ and $f(x)=(x-k)q(x)+0$ or $f(x)=(x-k)q(x)$.

Notice, written in this form, $x-k$ is a factor of $f(x)$. We can conclude if k is a zero of $f(x)$, then $x-k$ is a factor of $f(x)$.

Similarly, if $x-k$ is a factor of $f(x)$, then the remainder of the Division Algorithm $f(x)=(x-k)q(x)+r$ is 0. This tells us that k is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree n in the complex number system will have n zeros. We can use the Factor Theorem to completely factor a polynomial into the product of n factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

THE FACTOR THEOREM

According to the Factor Theorem, k is a zero of $f(x)$ if and only if $(x-k)$ is a factor of $f(x)$.

How to: Given a factor and a third-degree polynomial, use the Factor Theorem to factor the polynomial

1. Use synthetic division to divide the polynomial by $(x-k)$.
2. Confirm that the remainder is 0.
3. Write the polynomial as the product of $(x-k)$ and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

Using the Rational Zero Theorem to Find Rational Zeros

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial. But first we need a pool of rational numbers to test. The **Rational Zero Theorem** helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading **coefficient** of the polynomial

Consider a quadratic function with two zeros, $x = \frac{2}{5}$ and $x = \frac{3}{4}$. By the Factor Theorem, these zeros have factors associated with them. Let us set each factor equal to 0, and then construct the original quadratic function absent its stretching factor.

$x - \frac{2}{5} = 0$ or $x - \frac{3}{4} = 0$	Set each factor equal to 0.
$5x - 2 = 0$ or $4x - 3 = 0$	Multiply both sides of the equation to eliminate fractions.
$f(x) = (5x - 2)(4x - 3)$	Create the quadratic function, multiplying the factors.
$f(x) = 20x^2 - 23x + 6$	Expand the polynomial.
$f(x) = (5 \cdot 4)x^2 - 23x + (2 \cdot 3)$	

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3. Similarly, two of the factors from the leading coefficient, 20, are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

THE RATIONAL ZERO THEOREM

The Rational Zero Theorem states that, if the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ has integer coefficients, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

How to: Given a polynomial function $f(x)$, use the Rational Zero Theorem to find rational zeros.

1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of $f(\frac{p}{q})$.

Using the Fundamental Theorem of Algebra

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The **Fundamental Theorem of Algebra** tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose f is a polynomial function of degree four, and $f(x) = 0$. The Fundamental Theorem of Algebra states that there is at least one complex solution, call it c_1. By the Factor Theorem, we can write $f(x)$ as a product of $x - c_1$ and a polynomial quotient. Since $x - c_1$ is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it c_2. So we can write the polynomial quotient as a product of $x - c_2$ and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of $f(x)$.

THE FUNDAMENTAL THEOREM OF ALGEBRA

The *Fundamental Theorem of Algebra* states that, if $f(x)$ is a polynomial of degree $n > 0$, then $f(x)$ has at least one complex zero.

We can use this theorem to argue that, if $f(x)$ is a polynomial of degree $n > 0$, and a is a non-zero real number, then $f(x)$ has exactly n linear factors

$$f(x) = a(x - c_1)(x - c_2)\ldots(x - c_n) \tag{3.6.2}$$

where $c_1, c_2, ..., c_n$ are complex numbers. Therefore, $f(x)$ has n roots if we allow for multiplicities.

Using the Linear Factorization Theorem to Find Polynomials with Given Zeros

A vital implication of the **Fundamental Theorem of Algebra**, as we stated above, is that a polynomial function of degree n will have n zeros in the set of complex numbers, if we allow for multiplicities. This means that we can factor the polynomial function into n factors. The **Linear Factorization Theorem** tells us that a polynomial function will have the same number of factors as its degree, and that each factor will be in the form $(x - c)$, where c is a complex number.

Let f be a polynomial function with real coefficients, and suppose $a + bi$, $b \neq 0$, is a zero of $f(x)$. Then, by the Factor Theorem, $x - (a + bi)$ is a factor of $f(x)$. For f to have real coefficients, $x - (a - bi)$ must also be a factor of $f(x)$. This is true because any factor other than $x - (a - bi)$, when multiplied by $x - (a + bi)$, will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function f with real coefficients has a complex zero $a + bi$, then the complex conjugate $a - bi$ must also be a zero of $f(x)$. This is called the **Complex Conjugate Theorem**.

COMPLEX CONJUGATE THEOREM

According to the Linear Factorization Theorem, a polynomial function will have the same number of factors as its degree, and each factor will be in the form $(x - c)$, where c is a complex number.

If the polynomial function f has real coefficients and a complex zero in the form $a + bi$, then the complex conjugate of the zero, $a - bi$, is also a zero.

How to

Given the zeros of a polynomial function f and a point $(c, f(c))$ on the graph of f, use the Linear Factorization Theorem to find the polynomial function.

1. Use the zeros to construct the linear factors of the polynomial.
2. Multiply the linear factors to expand the polynomial.
3. Substitute $(c, f(c))$ into the function to determine the leading coefficient.
4. Simplify.

Using Descartes' Rule of Signs

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, **Descartes' Rule of Signs** tells us of a relationship between the number of sign changes in $f(x)$ and the number of positive real zeros. For example, the polynomial function below has one sign change.

$$f(x) = x^4 + x^3 + x^2 + x - 1$$

This tells us that the function must have 1 positive real zero.

There is a similar relationship between the number of sign changes in $f(-x)$ and the number of negative real zeros.

$$f(-x) = (-x)^4 + (-x)^3 + (-x)^2 + (-x) - 1$$
$$f(-x) = +x^4 - x^3 + x^2 - x - 1$$

In this case, $f(-x)$ has 3 sign changes. This tells us that $f(x)$ could have 3 or 1 negative real zeros.

DESCARTES' RULE OF SIGNS

According to *Descartes' Rule of Signs*, if we let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial function with real coefficients:

- The number of positive real zeros is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.

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3.7: Rational Functions

Using Arrow Notation

We have seen the graphs of the basic **reciprocal function** and the squared reciprocal function from our study of toolkit functions. Examine these graphs, as shown in Figure 3.7.1, and notice some of their features.

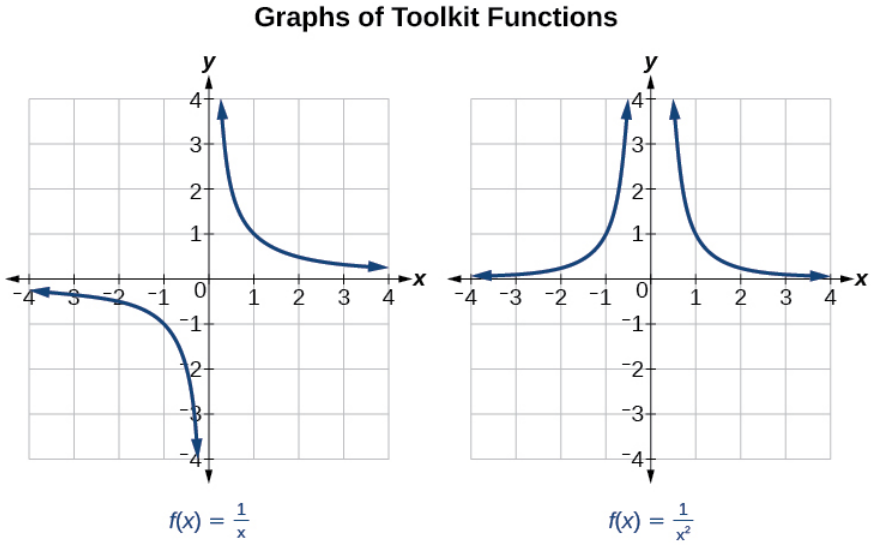


Figure 3.7.1

Several things are apparent if we examine the graph of $f(x) = \frac{1}{x}$.

- On the left branch of the graph, the curve approaches the x-axis $(y = 0)$ as $x \to -\infty$.
- As the graph approaches $x = 0$ from the left, the curve drops, but as we approach zero from the right, the curve rises.
- Finally, on the right branch of the graph, the curves approaches the x-axis $(y = 0)$ as $x \to \infty$.

To summarize, we use **arrow notation** to show that x or $f(x)$ is approaching a particular value (Table 3.7.1).

Table 3.7.1

Symbol	Meaning
$x \to a^-$	x approaches a from the left ($x < a$ but close to a)
$x \to a^+$	x approaches a from the right ($x > a$ but close to a)
$x \to \infty$	x approaches infinity (x increases without bound)
$x \to -\infty$	x approaches negative infinity (x decreases without bound)
$f(x) \to \infty$	the output approaches infinity (the output increases without bound)
$f(x) \to -\infty$	the output approaches negative infinity (the output decreases without bound)
$f(x) \to a$	the output approaches a

Local Behavior of $f(x) = \frac{1}{x}$

Let's begin by looking at the reciprocal function, $f(x) = \frac{1}{x}$. We cannot divide by zero, which means the function is undefined at $x = 0$; so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in Table 3.7.2.

Table 3.7.2

x	–0.1	–0.01	–0.001	–0.0001
$f(x)=\frac{1}{x}$	–10	–100	–1000	–10,000

We write in arrow notation

$$\text{as } x \to 0^-, f(x) \to -\infty$$

As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in Table 3.7.3.

Table 3.7.3

x	0.1	0.01	0.001	0.0001
$f(x)=\frac{1}{x}$	10	100	1000	10,000

We write in arrow notation

$$\text{As } x \to 0^+, f(x) \to \infty \, .$$

See Figure 3.7.2.

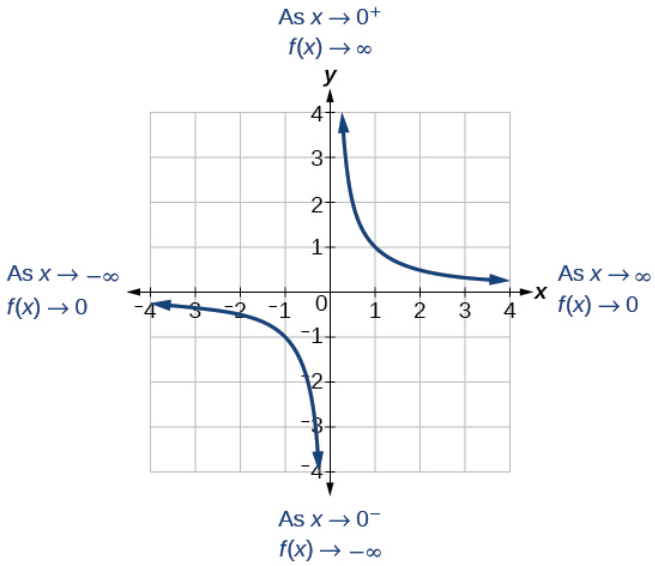


Figure 3.7.2.

This behavior creates a **vertical asymptote**, which is a vertical line that the graph approaches but never crosses. In this case, the graph is approaching the vertical line $x = 0$ as the input becomes close to zero (Figure 3.7.3).

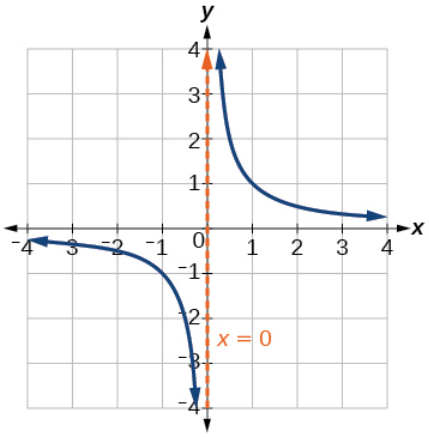


Figure 3.7.3.

Definition: VERTICAL ASYMPTOTE

A vertical asymptote of a graph is a vertical line $x = a$ where the graph tends toward positive or negative infinity as the inputs approach a. We write

As $x \to a$, $f(x) \to \infty$, or as $x \to a$, $f(x) \to -\infty$.

End Behavior of $f(x) = \frac{1}{x}$

As the values of x approach infinity, the function values approach 0. As the values of x approach negative infinity, the function values approach 0 (Figure $3.7.4$). Symbolically, using arrow notation

As $x \to \infty$, $f(x) \to 0$,and as $x \to -\infty$, $f(x) \to 0$.

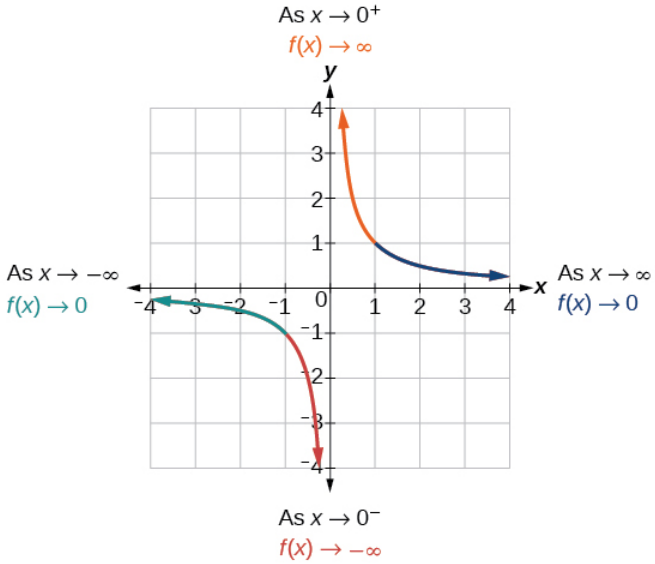


Figure $3.7.4$.

Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0; it seems to level off as the inputs become large. This behavior creates a **horizontal asymptote**, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line $y = 0$. See Figure $3.7.5$.

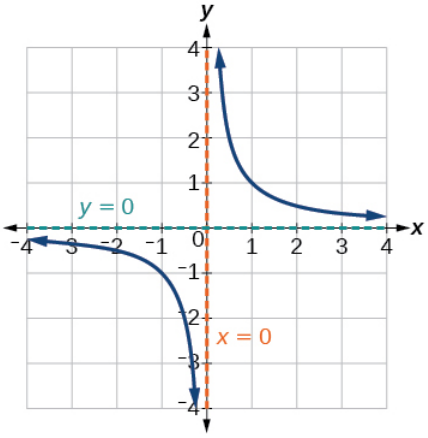


Figure $3.7.5$.

Definition: HORIZONTAL ASYMPTOTE

A *horizontal asymptote* of a graph is a horizontal line $y = b$ where the graph approaches the line as the inputs increase or decrease without bound. We write

$$\text{As } x \to \infty \text{ or } x \to -\infty \text{ , } f(x) \to b.$$

Solving Applied Problems Involving Rational Functions

In Example 3.7.2, we shifted a toolkit function in a way that resulted in the function $f(x)=\frac{3x+7}{x+2}$. This is an example of a rational function. A *rational function* is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

Definition: RATIONAL FUNCTION

A ***rational function*** is a function that can be written as the quotient of two polynomial functions $P(x)$ and $Q(x)$.

$$f(x)=\frac{P(x)}{Q(x)}=\frac{a_p x^p+a_{p-1}x^{p-1}+\ldots+a_1 x+a_0}{b_q x^q+b_{q-1}x^{q-1}+\ldots+b_1 x+b_0},\ Q(x)\neq 0 \tag{3.7.1}$$

Finding the Domains of Rational Functions

A *vertical asymptote* represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

Definition: DOMAIN OF A RATIONAL FUNCTION

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

How To: Given a rational function, find the domain.

1. Set the denominator equal to zero.
2. Solve to find the x-values that cause the denominator to equal zero.
3. The domain is all real numbers except those found in Step 2.

Identifying Vertical Asymptotes of Rational Functions

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Vertical Asymptotes

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

How To: Given a rational function, identify any vertical asymptotes of its graph

1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities, or “holes.”

Removable Discontinuities

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a *removable discontinuity*. For example, the function $f(x)=\dfrac{x^2-1}{x^2-2x-3}$ may be re-written by factoring the numerator and the denominator.

$$f(x)=\frac{(x+1)(x-1)}{(x+1)(x-3)}$$

Notice that $x+1$ is a common factor to the numerator and the denominator. The zero of this factor, $x=-1$, is the location of the removable discontinuity. Notice also that $(x-3)$ is not a factor in both the numerator and denominator. The zero of this factor, $x=3$, is the vertical asymptote. See Figure 3.7.11. [Note that removable discontinuities may not be visible when we use a graphing calculator, depending upon the window selected.]

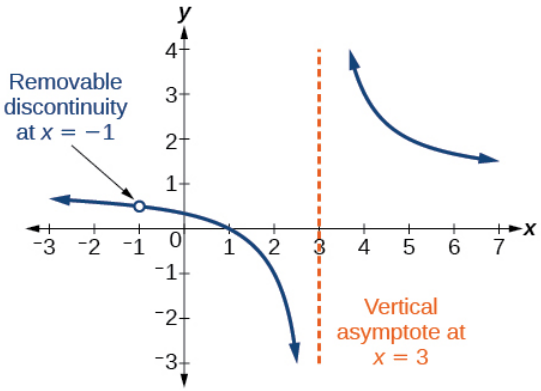


Figure 3.7.11.

REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS

A **removable discontinuity** occurs in the graph of a rational function at $x=a$ if a is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

Identifying Horizontal Asymptotes of Rational Functions

While vertical asymptotes describe the behavior of a graph as the *output* gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the *input* gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the function that is the ratio of the leading terms.

There are three distinct outcomes when checking for horizontal asymptotes:

Case 1: If the degree of the denominator > degree of the numerator, there is a **horizontal asymptote** at $y=0$.

Example: $f(x)=\dfrac{4x+2}{x^2+4x-5}$

In this case, the end behavior is $f(x)\approx\dfrac{4x}{x^2}=\dfrac{4}{x}$. This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x)=\dfrac{4}{x}$, and the outputs will approach zero, resulting in a horizontal asymptote at $y=0$. See **Figure** 3.7.13 Note that this graph crosses the horizontal asymptote.

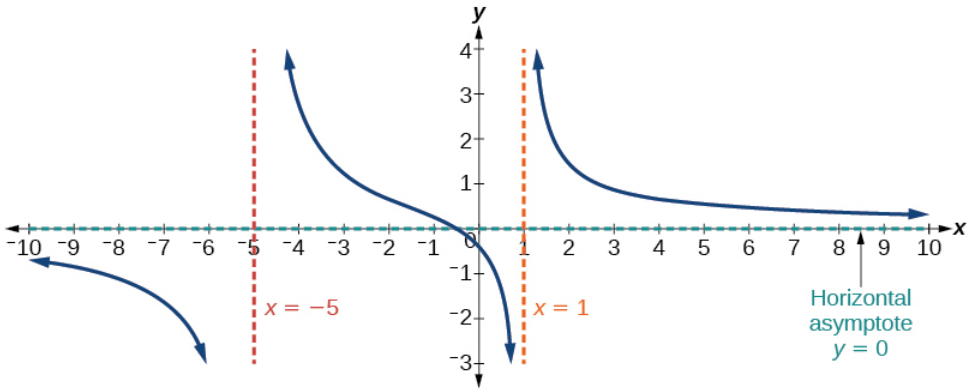


Figure 3.7.13: Horizontal asymptote $y = 0$ when $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of p < degree of q.

Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.

Example: $f(x) = \dfrac{3x^2 - 2x + 1}{x - 1}$

In this case, the end behavior is $f(x) \approx \dfrac{3x^2}{x} = 3x$. This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function $g(x) = 3x$. As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of $g(x) = 3x$ looks like a diagonal line, and since f will behave similarly to g, it will approach a line close to $y = 3x$. This line is a slant asymptote.

To find the equation of the slant asymptote, divide $\dfrac{3x^2 - 2x + 1}{x - 1}$. The quotient is $3x + 1$, and the remainder is 2. The slant asymptote is the graph of the line $g(x) = 3x + 1$. See Figure 3.7.14

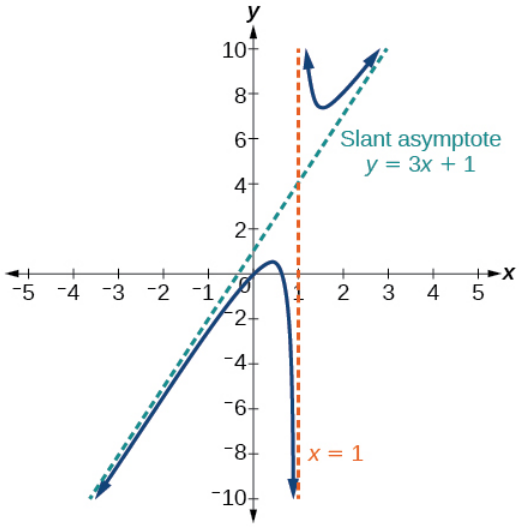


Figure 3.7.14: Slant asymptote when $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of p >degree of q by 1.

Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at $y = \dfrac{a_n}{b_n}$, where a_n and b_n are respectively the leading coefficients of $p(x)$ and $q(x)$ for $f(x) = \dfrac{p(x)}{q(x)}$, $q(x) \neq 0$.

Example: $f(x) = \dfrac{3x^2 + 2}{x^2 + 4x - 5}$

In this case, the end behavior is $f(x) \approx \frac{3x^2}{x^2} = 3$. This tells us that as the inputs grow large, this function will behave like the function $g(x) = 3$, which is a horizontal line. As $x \to \pm\infty$, $f(x) \to 3$, resulting in a horizontal asymptote at $y = 3$. See Figure 3.7.15. Note that this graph crosses the horizontal asymptote.

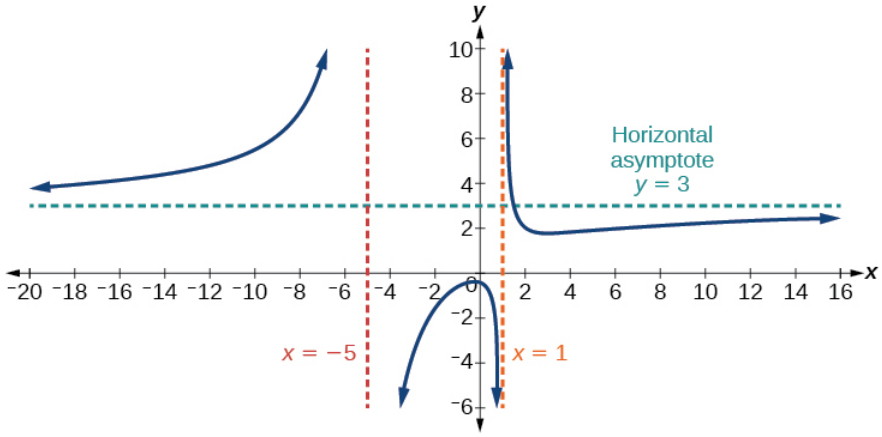


Figure 3.7.15: Horizontal asymptote when $f(x) = \frac{p(x)}{q(x)}$, $q(x) \neq 0$ where degree of p = degree of q.

Notice that, while the graph of a rational function will never cross a **vertical asymptote**, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the **end behavior** of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function

$$f(x) = \frac{3x^5 - x^2}{x + 3}$$

with end behavior

$$f(x) \approx \frac{3x^5}{x} = 3x^4$$

the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

$$x \to \pm\infty, f(x) \to \infty$$

HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS

The **horizontal asymptote** of a rational function can be determined by looking at the degrees of the numerator and denominator.

- Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
- Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

Example 3.7.7: Identifying Horizontal and Slant Asymptotes

For the functions listed, identify the horizontal or slant asymptote.

1. $g(x) = \frac{6x^3 - 10x}{2x^3 + 5x^2}$
2. $h(x) = \frac{x^2 - 4x + 1}{x + 2}$
3. $k(x) = \frac{x^2 + 4x}{x^3 - 8}$

Solution

For these solutions, we will use $f(x)=\dfrac{p(x)}{q(x)},\ q(x)\neq 0$.

1. $g(x)=\dfrac{6x^3-10x}{2x^3+5x^2}$: The degree of $p=$ degree of $q=3$, so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at $y=\frac{6}{2}$ or $y=3$.
2. $h(x)=\dfrac{x^2-4x+1}{x+2}$: The degree of $p=2$ and degree of $q=1$. Since $p>q$ by 1, there is a slant asymptote found at $\dfrac{x^2-4x+1}{x+2}$.

$$\begin{array}{r|rrr} -2 & 1 & -4 & 1 \\ & & -2 & 12 \\ \hline & 1 & -6 & 13 \end{array}$$

3. $k(x)=\dfrac{x^2+4x}{x^3-8}$: The degree of $p=2<$ degree of $q=3$, so there is a horizontal asymptote $y=0$.

Example 3.7.8 Identifying Horizontal Asymptotes

In the sugar concentration problem earlier, we created the equation $C(t)=\dfrac{5+t}{100+10t}$.

Find the horizontal asymptote and interpret it in context of the problem.

Solution

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is t, with coefficient 1. In the denominator, the leading term is 10t, with coefficient 10. The horizontal asymptote will be at the ratio of these values:

$$t\to\infty,\ C(t)\to\tfrac{1}{10}$$

This function will have a horizontal asymptote at $y=\frac{1}{10}$.

This tells us that as the values of t increase, the values of C will approach $\frac{1}{10}$. In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or $\frac{1}{10}$ pounds per gallon.

Example 3.7.9: Identifying Horizontal and Vertical Asymptotes

Find the horizontal and vertical asymptotes of the function $f(x)=\dfrac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}$

Solution

First, note that this function has no common factors, so there are no potential removable discontinuities.

The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at $x=1,-2,$and 5, indicating vertical asymptotes at these values.

The numerator has degree 2, while the denominator has degree 3. Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as $x\to\pm\infty$, $f(x)\to 0$. This function will have a horizontal asymptote at $y=0$. See Figure 3.7.16.

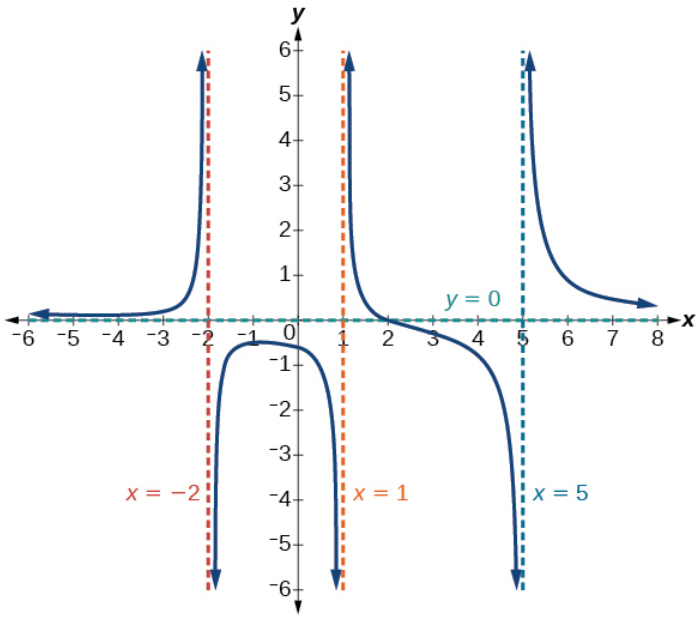


Figure 3.7.16.

Exercise 3.7.6

Find the vertical and horizontal asymptotes of the function:

$$f(x) = \frac{(2x-1)(2x+1)}{(x-2)(x+3)}$$

Answer

Vertical asymptotes at $x = 2$ and $x = -3$

horizontal asymptote at $y = 4$.

INTERCEPTS OF RATIONAL FUNCTIONS

A *rational function* will have a y-intercept at $f(0)$, if the function is defined at zero. A rational function will not have a y-intercept if the function is not defined at zero.

Likewise, a rational function will have x-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, x-intercepts can only occur when the numerator of the rational function is equal to zero.

Graphing Rational Functions

In Example 3.7.10, we see that the numerator of a rational function reveals the *x*-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity. See Figure 3.7.18

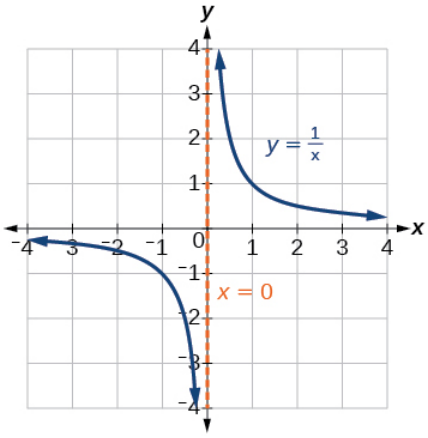


Figure 3.7.18.

When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides. See Figure 3.7.19

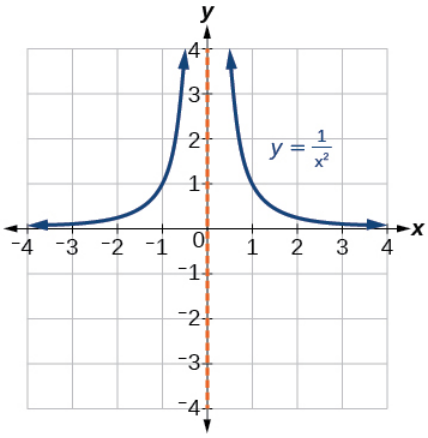


Figure 3.7.19.

For example, the graph of $f(x) = \dfrac{(x+1)^2(x-3)}{(x+3)^2(x-2)}$ is shown in Figure 3.7.20

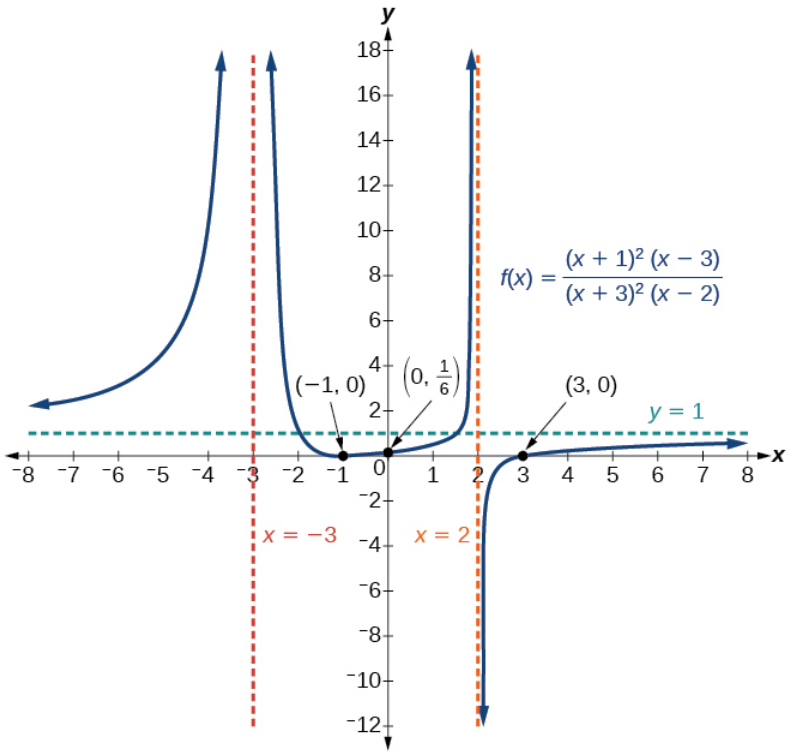


Figure 3.7.20.

- At the x-intercept $x = -1$ corresponding to the $(x+1)^2$ factor of the numerator, the graph "bounces", consistent with the quadratic nature of the factor.
- At the x-intercept $x = 3$ corresponding to the $(x-3)$ factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote $x = -3$ corresponding to the $(x+3)^2$ factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function $f(x) = \frac{1}{x^2}$.
- At the vertical asymptote $x = 2$, corresponding to the $(x-2)$ factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function $f(x) = \frac{1}{x}$.

Howto: Given a rational function, sketch a graph.

1. Evaluate the function at 0 to find the y-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the x-intercepts.
4. Find the multiplicities of the x-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

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3.8: Inverses and Radical Functions

Finding the Inverse of a Polynomial Function

Two functions f and g are inverse functions if for every coordinate pair in f, (a, b), there exists a corresponding coordinate pair in the inverse function, g, (b, a). In other words, the coordinate pairs of the inverse functions have the input and output interchanged. Only one-to-one functions have inverses. Recall that a one-to-one function has a unique output value for each input value and passes the horizontal line test.

For example, suppose a water runoff collector is built in the shape of a parabolic trough as shown in Figure 3.8.2. We can use the information in the figure to find the surface area of the water in the trough as a function of the depth of the water.

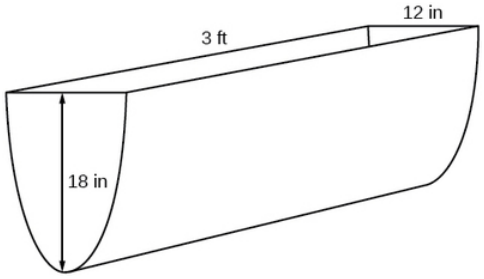


Figure 3.8.2

Because it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with x measured horizontally and y measured vertically, with the origin at the vertex of the parabola (Figure 3.8.3).

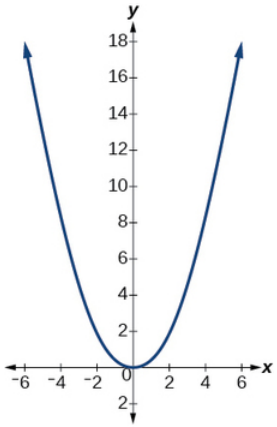


Figure 3.8.3

From this we find an equation for the parabolic shape. We placed the origin at the vertex of the parabola, so we know the equation will have form $y(x) = ax^2$. Our equation will need to pass through the point $(6, 18)$, from which we can solve for the stretch factor a.

$$18 = a6^2$$

$$a = \frac{18}{36}$$

$$= \frac{1}{2}$$

Our parabolic cross section has the equation

$$y(x) = \frac{1}{2}x^2$$

We are interested in the *surface area* of the water, so we must determine the width at the top of the water as a function of the water depth. For any depth y, the width will be given by $2x$, so we need to solve the equation above for x and find the inverse function.

However, notice that the original function is not one-to-one, and indeed, given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it *is* one-to-one. In this case, it makes sense to restrict ourselves to positive x values. On this domain, we can find an inverse by solving for the input variable:

$$\begin{aligned} y &= \frac{1}{2}x^2 \\ 2y &= x^2 \\ x &= \pm\sqrt{2y} \end{aligned}$$

This is not a function as written. We are limiting ourselves to positive x values, so we eliminate the negative solution, giving us the inverse function we're looking for.

$$y = \frac{x^2}{2}, x > 0$$

Because x is the distance from the center of the parabola to either side, the entire width of the water at the top will be $2x$. The trough is 3 feet (36 inches) long, so the surface area will then be:

$$\begin{aligned} \text{Area} &= l \cdot w \\ &= 36 \cdot 2x \\ &= 72x \\ &= 72\sqrt{2y} \end{aligned}$$

This example illustrates two important points:

1. When finding the inverse of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2. The inverse of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called **radical functions**. While it is not possible to find an inverse of most polynomial functions, some basic polynomials do have inverses. Such functions are called **invertible functions**, and we use the notation $f^{-1}(x)$.

Warning: $f^{-1}(x)$ is not the same as the reciprocal of the function $f(x)$. This use of "–1" is reserved to denote inverse functions. To denote the reciprocal of a function $f(x)$, we would need to write:

$$(f(x))^{-1} = \frac{1}{f(x)}. \tag{3.8.1}$$

An important relationship between inverse functions is that they "undo" each other. If f^{-1} is the inverse of a function f, then f is the inverse of the function f^{-1}. In other words, whatever the function f does to x, f^{-1} undoes it—and vice-versa.

$$f^{-1}(f(x)) = x, \text{ for all } x \text{ in the domain of } f$$

and

$$f(f^{-1}(x)) = x, \text{ for all } x \text{ in the domain of } f^{-1}$$

Note that the inverse switches the domain and range of the original function.

VERIFYING TWO FUNCTIONS ARE INVERSES OF ONE ANOTHER

Two functions, f and g, are inverses of one another if for all x in the domain of f and g,

$$g(f(x)) = f(g(x)) = x$$

Howto: Given a polynomial function, find the inverse of the function by restricting the domain in such a way that the new function is one-to-one

1. Replace $f(x)$ with y.
2. Interchange x and y.

3. Solve for y, and rename the function $f^{-1}(x)$.

Solving Applications of Radical Functions

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

Example 3.8.6: Solving an Application with a Cubic Function

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by

$$V = \frac{2}{3}\pi r^3$$

Find the inverse of the function $V = \frac{2}{3}\pi r^3$ that determines the volume V of a cone and is a function of the radius r. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi = 3.14$.

Solution

Start with the given function for V. Notice that the meaningful domain for the function is $r > 0$ since negative radii would not make sense in this context nor would a radius of 0. Also note the range of the function (hence, the domain of the inverse function) is $V > 0$. Solve for r in terms of V, using the method outlined previously. Note that in real-world applications, we do not swap the variables when finding inverses. Instead, we change which variable is considered to be the independent variable.

$$V = \frac{2}{3}\pi r^3$$

Solve for r^3.

$$r^3 = \frac{3V}{2\pi}$$

Solve for r.

$$r = \sqrt[3]{\frac{3V}{2\pi}}$$

This is the result stated in the section opener. Now evaluate this for $V = 100$ and $\pi = 3.14$.

$$\begin{aligned} r &= \sqrt[3]{\frac{3V}{2\pi}} \\ &= \sqrt[3]{\frac{3 \cdot 100}{2 \cdot 3.14}} \\ &\approx \sqrt[3]{47.7707} \\ &\approx 3.63 \end{aligned}$$

Therefore, the radius is about 3.63 ft.

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3.9: Modeling Using Variation

Solving Direct Variation Problems

In the example above, Nicole's earnings can be found by multiplying her sales by her commission. The formula $e = 0.16s$ tells us her earnings, e, come from the product of 0.16, her commission, and the sale price of the vehicle. If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive. See **Table 5.8.1**.

s, sales price	$e = 0.16s$	Interpretation
\$9,200	$e = 0.16(9,200) = 1,472$	A sale of a \$9,200 vehicle results in \$1472 earnings.
\$4,600	$e = 0.16(4,600) = 736$	A sale of a \$4,600 vehicle results in \$736 earnings.
\$18,400	$e = 0.16(18,400) = 2,944$	A sale of a \$18,400 vehicle results in \$2944 earnings.

Table 5.8.1

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from \$4,600 to \$9,200, and we double the earnings from \$736 to \$1,472. As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called **direct variation**. Each variable in this type of relationship **varies directly** with the other.

Figure 5.8.1 represents the data for Nicole's potential earnings. We say that earnings vary directly with the sales price of the car. The formula $y = kx^n$ is used for direct variation. The value k is a nonzero constant greater than zero and is called the **constant of variation**. In this case, $k = 0.16$ and $n = 1$. We saw functions like this one when we discussed power functions.

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Figure 5.8.1

A General Note: DIRECT VARIATION

If x and y are related by an equation of the form

$$y = kx^n$$

then we say that the relationship is **direct variation** and y **varies directly** with, or is proportional to, the nth power of x. In direct variation relationships, there is a nonzero constant ratio $k = \frac{y}{x^n}$, where k is called the **constant of variation**, which help defines the relationship between the variables.

Given a description of a direct variation problem, solve for an unknown.

1. Identify the input, x,and the output, y.
2. Determine the constant of variation. You may need to divide y by the specified power of x to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Solving Inverse Variation Problems

Water temperature in an ocean varies inversely to the water's depth. The formula $T = \frac{14,000}{d}$ gives us the temperature in degrees Fahrenheit at a depth in feet below Earth's surface. Consider the Atlantic Ocean, which covers 22% of Earth's surface. At a certain location, at the depth of 500 feet, the temperature may be 28°F.

If we create **Table 5.8.2**, we observe that, as the depth increases, the water temperature decreases.

d, depth	$T = \frac{14{,}000}{d}$	Interpretation
500 ft	$\frac{14{,}000}{500} = 28$	At a depth of 500 ft, the water temperature is 28° F.
1000 ft	$\frac{14{,}000}{1000} = 14$	At a depth of 1,000 ft, the water temperature is 14° F.
2000 ft	$\frac{14{,}000}{2000} = 7$	At a depth of 2,000 ft, the water temperature is 7° F.

Table 5.8.2

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be **inversely proportional** and each term **varies inversely** with the other. Inversely proportional relationships are also called **inverse variations**.

For our example, **Figure 5.8.3** depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula $y = \frac{k}{x}$ for inverse variation in this case uses $k = 14,000$.

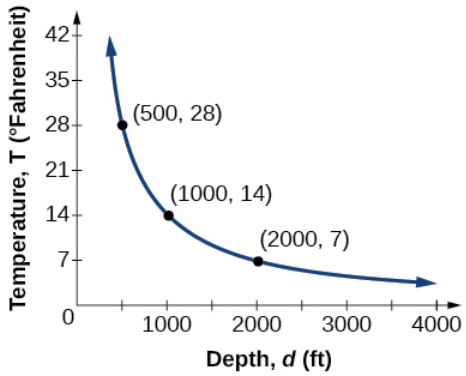


Figure 5.8.3

A General Note: INVERSE VARIATION

If x and y are related by an equation of the form

$$y = \frac{k}{x^n}$$

where k is a nonzero constant, then we say that y **varies inversely** with the nth power of x. In **inversely proportional** relationships, or **inverse variations**, there is a constant multiple $k = x^n y$.

Solving Problems Involving Joint Variation

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called **joint variation**. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable c,cost, varies jointly with the number of students, n,and the distance, d.

A General Note: JOINT VARIATION

Joint variation occurs when a variable varies directly or inversely with multiple variables.

For instance, if x varies directly with both y and z, we have $x = kyz$. If x varies directly with y and inversely with z,we have $x = \frac{ky}{z}$. Notice that we only use one constant in a joint variation equation.

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CHAPTER OVERVIEW

4: Exponential and Logarithmic Functions

In this chapter, we will explore exponential functions, which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions, which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

4.1: Exponential Functions
4.2: Graphs of Exponential Functions
4.3: Logarithmic Functions
4.4: Graphs of Logarithmic Functions
4.5: Logarithmic Properties
4.6: Exponential and Logarithmic Equations
4.7: Exponential and Logarithmic Models
4.8: Fitting Exponential Models to Data

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4.1: Exponential Functions

Defining an Exponential Function

A study found that the percent of the population who are vegans in the United States doubled from 2009 to 2011. In 2011, 2.5% of the population was vegan, adhering to a diet that does not include any animal products—no meat, poultry, fish, dairy, or eggs. If this rate continues, vegans will make up 10% of the U.S. population in 2015, 40% in 2019, and 80% in 2050.

What exactly does it mean to *grow exponentially*? What does the word *double* have in common with *percent increase*? People toss these words around errantly. Are these words used correctly? The words certainly appear frequently in the media.

- **Percent change** refers to a *change* based on a *percent* of the original amount.
- **Exponential growth** refers to an *increase* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* increase of the original amount over time.
- **Exponential decay** refers to a *decrease* based on a constant multiplicative rate of change over equal increments of time, that is, a *percent* decrease of the original amount over time.

For us to gain a clear understanding of **exponential growth**, let us contrast exponential growth with **linear growth**. We will construct two functions. The first function is exponential. We will start with an input of 0, and increase each input by 1. We will double the corresponding consecutive outputs. The second function is linear. We will start with an input of 0, and increase each input by 1. We will add 2 to the corresponding consecutive outputs (Table $4.1.1$).

From Table $4.1.1$ we can infer that for these two functions, exponential growth dwarfs linear growth.

- **Exponential growth** refers to the original value from the range increases by the *same percentage* over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increases by the *same amount* over equal increments found in the domain.

Table 4.1.1

x	$f(x)=2^x$	$g(x)=2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

Apparently, the difference between “the same percentage” and “the same amount” is quite significant. For exponential growth, over equal increments, the constant multiplicative rate of change resulted in doubling the output whenever the input increased by one. For linear growth, the constant additive rate of change over equal increments resulted in adding 2 to the output whenever the input was increased by one.

The general form of the **exponential function** is $f(x)=ab^x$, where a is any nonzero number, b is a positive real number not equal to 1.

- If $b>1$,the function grows at a rate proportional to its size.
- If $0<b<1$, the function decays at a rate proportional to its size.

Let’s look at the function $f(x)=2^x$ from our example. We will create a table (Table $4.1.2$) to determine the corresponding outputs over an interval in the domain from -3 to 3.

Table 4.1.2

x	-3	-2	-1	0	1	2	3
$f(x)=2^x$	$2^{-3}=\frac{1}{8}$	$2^{-2}=\frac{1}{4}$	$2^{-1}=\frac{1}{2}$	$2^0=1$	$2^1=2$	$2^2=4$	$2^3=8$

Let us examine the graph of f by plotting the ordered pairs from Table 4.1.2 and then make a few observations 4.1.1.

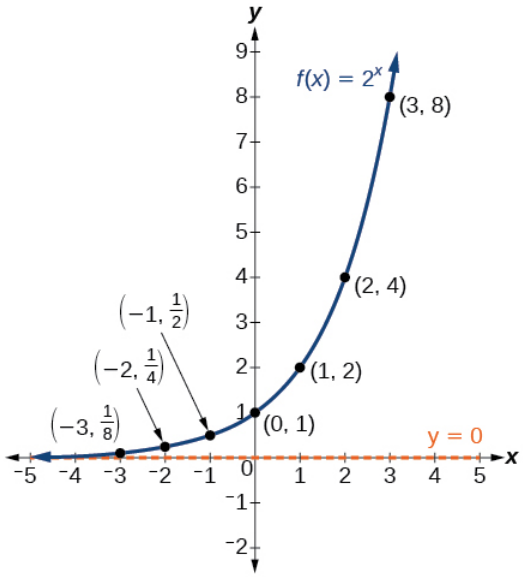


Figure 4.1.1

Let's define the behavior of the graph of the exponential function $f(x)=2^x$ and highlight some its key characteristics.

- the domain is $(-\infty,\infty)$,
- the range is $(0,\infty)$,
- as $x\to\infty$, $f(x)\to\infty$,
- as $x\to-\infty$, $f(x)\to 0$,
- $f(x)$ is always increasing,
- the graph of $f(x)$ will never touch the *x*-axis because base two raised to any exponent never has the result of zero.
- $y=0$ is the horizontal asymptote.
- the *y*-intercept is 1.

Definition: Exponential Functions

For any real number x,an exponential function is a function with the form

$$f(x)=ab^x \tag{4.1.1}$$

where

- a is a non-zero real number called the initial value and
- b is any positive real number such that $b\neq 1$.
- The domain of f is all real numbers.
- The range of f is all positive real numbers if $a>0$.
- The range of f is all negative real numbers if $a<0$.
- The *y*-intercept is $(0,a)$,and the horizontal asymptote is $y=0$.

Example 4.1.1: Identifying Exponential Functions

Which of the following equations are *not* exponential functions?

- $f(x)=4^{3(x-2)}$
- $g(x)=x^3$

- $h(x) = \left(\frac{1}{3}\right)^x$
- $j(x) = (-2)^x$

Solution

By definition, an exponential function has a constant as a base and an independent variable as an exponent. Thus, $g(x) = x^3$ does not represent an exponential function because the base is an independent variable. In fact, $g(x) = x^3$ is a power function.

Recall that the base b of an exponential function is always a positive constant, and $b \neq 1$. Thus, $j(x) = (-2)^x$ does not represent an exponential function because the base, -2, is less than 0.

Defining Exponential Growth

Because the output of exponential functions increases very rapidly, the term "exponential growth" is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

Definition: Exponential Growth

A function that models exponential growth grows by a rate proportional to the amount present. For any real number x and any positive real numbers a and b such that $b \neq 1$,an exponential growth function has the form

$$f(x) = ab^x \tag{4.1.2}$$

where

- a is the initial or starting value of the function.
- b is the growth factor or growth multiplier per unit x.

In more general terms, we have an *exponential function*, in which a constant base is raised to a variable exponent. To differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function $A(x) = 100 + 50x$. Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function $B(x) = 100(1 + 0.5)^x$.

A few years of growth for these companies are illustrated in Table $4.1.3$.

Table 4.1.3

Year, x	Stores, Company A	Stores, Company B
0	$100 + 50(0) = 100$	$100(1 + 0.5)^0 = 100$
1	$100 + 50(1) = 150$	$100(1 + 0.5)^1 = 150$
2	$100 + 50(2) = 200$	$100(1 + 0.5)^2 = 225$
3	$100 + 50(3) = 250$	$100(1 + 0.5)^3 = 337.5$
x	$A(x) = 100 + 50x$	$B(x) = 100(1 + 0.5)^x$

The graphs comparing the number of stores for each company over a five-year period are shown in Figure $4.1.2$. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.

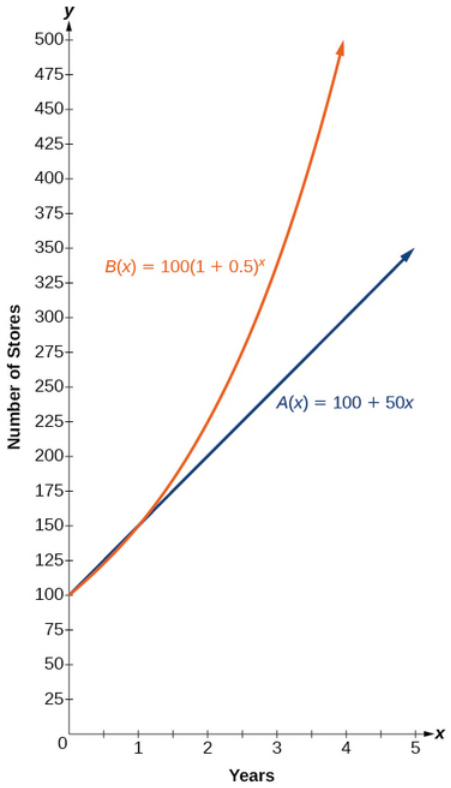


Figure 4.1.2: The graph shows the numbers of stores Companies A and B opened over a five-year period.

Notice that the domain for both functions is $[0, \infty)$,and the range for both functions is $[100, \infty)$. After year 1, Company B always has more stores than Company A.

Now we will turn our attention to the function representing the number of stores for Company B, $B(x) = 100(1 + 0.5)^x$. In this exponential function, 100 represents the initial number of stores, 0.50 represents the growth rate, and $1 + 0.5 = 1.5$ represents the growth factor. Generalizing further, we can write this function as $B(x) = 100(1.5)^x$,where 100 is the initial value, 1.5 is called the *base*, and x is called the *exponent*.

Finding Equations of Exponential Functions

In the previous examples, we were given an exponential function, which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, then determine the constants a, a and b, b,and evaluate the function.

How to: Given two data points, write an exponential model

1. If one of the data points has the form $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b.
2. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
3. Using the a and b found in the steps above, write the exponential function in the form $f(x) = a(b)^x$.

Example 4.1.4: Writing an Exponential Model When the Initial Value Is Known

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function $N(t)$ representing the population (N) of deer over time t.

Solution

We let our independent variable t be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: (0, 80) and (6, 180). Notice that by choosing our input variable to be measured as years after 2006, we

have given ourselves the initial value for the function, $a = 80$. We can now substitute the second point into the equation $N(t) = 80b^t$ to find b:

$$\begin{aligned} N(t) &= 80b^t \\ 180 &= 80b^6 && \text{Substitute using point } (6, 180) \\ \frac{9}{4} &= b^6 && \text{Divide and write in lowest terms} \\ b &= \left(\frac{9}{4}\right)^{\frac{1}{6}} && \text{Isolate b using properties of exponents} \\ b &\approx 1.1447 && \text{Round to 4 decimal places} \end{aligned}$$

> *Unless otherwise stated, do not round any intermediate calculations. Then round the final answer to four places for the remainder of this section.*

The exponential model for the population of deer is $N(t) = 80(1.1447)^t$. (Note that this exponential function models short-term growth. As the inputs gets large, the output will get increasingly larger, so much so that the model may not be useful in the long term.)

We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph in Figure 4.1.3 passes through the initial points given in the problem, $(0, 80)$ and $(6, 180)$. We can also see that the domain for the function is $[0, \infty)$,and the range for the function is $[80, \infty)$.

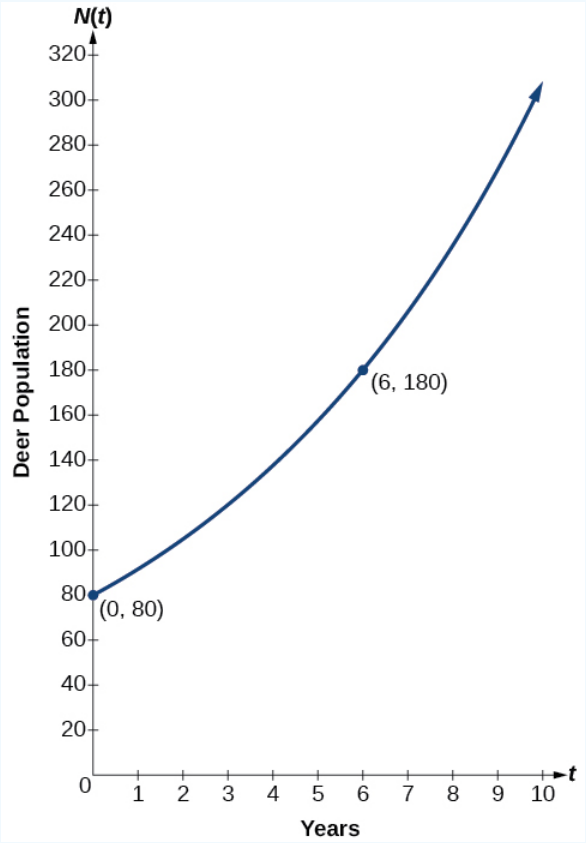


Figure 4.1.3: Graph showing the population of deer over time, $N(t) = 80(1.1447)^t$, t years after 2006

? Exercise 4.1.4

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013, the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population N of wolves over time t.

Answer

$(0, 129)$ and $(2, 236)$; $N(t) = 129(1.3526)^t$

Example 4.1.5: Writing an Exponential Model When the Initial Value is Not Known

Find an exponential function that passes through the points $(-2, 6)$ and $(2, 1)$.

Solution

Because we don't have the initial value, we substitute both points into an equation of the form $f(x) = ab^x$, and then solve the system for a and b.

- Substituting $(-2, 6)$ gives $6 = ab^{-2}$
- Substituting $(2, 1)$ gives $1 = ab^2$

Use the first equation to solve for a in terms of b:

$$\begin{aligned} 6 &= ab^{-2} \\ \frac{6}{b^{-2}} &= a \qquad \text{Divide} \\ a &= 6b^2 \qquad \text{Use properties of exponents to rewrite the denominator} \end{aligned}$$

Substitute a in the second equation, and solve for b:

$$\begin{aligned} 1 &= ab^2 \\ 1 &= 6b^2b^2 \\ &= 6b^4 \qquad \text{Substitute a} \\ b &= \left(\frac{1}{6}\right)^{\frac{1}{4}} \qquad \text{Round 4 decimal places rewrite the denominator} \\ b &\approx 0.6389 \end{aligned}$$

Use the value of b in the first equation to solve for the value of a:

$$\begin{aligned} a &= 6b^2 \\ &\approx 6(0.6389)^2 \\ &\approx 2.4492 \end{aligned}$$

Thus, the equation is $f(x) = 2.4492(0.6389)^x$.

We can graph our model to check our work. Notice that the graph in Figure 4.1.4 passes through the initial points given in the problem, $(-2, 6)$ and $(2, 1)$. The graph is an example of an exponential decay function.

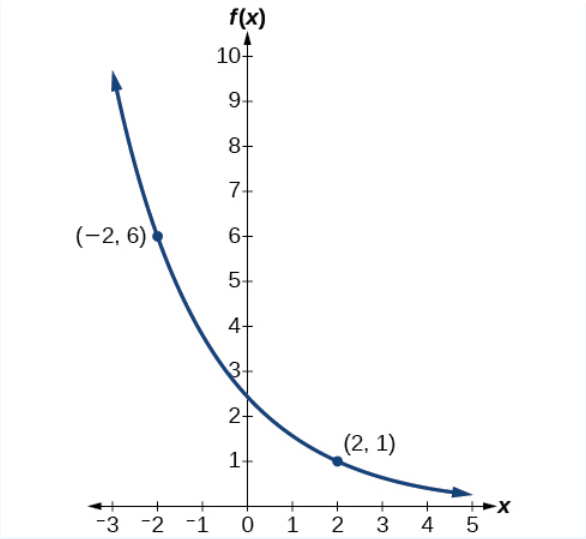


Figure 4.1.4: The graph of $f(x) = 2.4492(0.6389)^x$ models exponential decay.

How to: Given the graph of an exponential function, write its equation

1. First, identify two points on the graph. Choose the y-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the y-intercept $(0, a)$, then a is the initial value. Using a, substitute the second point into the equation $f(x) = a(b)^x$, and solve for b
3. If neither of the data points have the form $(0, a)$, substitute both points into two equations with the form $f(x) = a(b)^x$. Solve the resulting system of two equations in two unknowns to find a and b.
4. Write the exponential function, $f(x) = a(b)^x$.

Applying the Compound-Interest Formula

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use **compound interest**. The term *compounding* refers to interest earned not only on the original value, but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time t, principal P, APR r, and number of compounding periods in a year n:

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

For example, observe Table 4.1.4, which shows the result of investing $\$1,000$ at 10% for one year. Notice how the value of the account increases as the compounding frequency increases.

Table 4.1.4

Frequency	Value after 1 year
Annually	$1100
Semiannually	$1102.50
Quarterly	$1103.81
Monthly	$1104.71

Frequency	Value after 1 year
Daily	$1105.16

Definition: Compound Interest

Compound interest can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \tag{4.1.3}$$

where

- $A(t)$ is the account value,
- t is measured in years,
- P is the starting amount of the account, often called the principal, or more generally present value,
- r is the annual percentage rate (APR) expressed as a decimal, and
- n is the number of compounding periods in one year.

Example 4.1.8: Calculating Compound Interest

If we invest $\$3,000$ in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

Solution

Because we are starting with $\$3,000$, $P = 3000$. Our interest rate is 3%, so $r = 0.03$. Because we are compounding quarterly, we are compounding 4 times per year, so $n = 4$. We want to know the value of the account in 10 years, so we are looking for $A(10)$,the value when $t = 10$.

$$\begin{aligned} A(t) &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Use the compound interest formula} \\ A(10) &= 3000\left(1 + \frac{0.03}{4}\right)^{(4)\cdot(10)} && \text{Substitute using given values} \\ &\approx \$4045.05 && \text{Round to two decimal places} \end{aligned}$$

The account will be worth about $\$4,045.05$ in 10 years.

Evaluating Functions with Base e

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. Table 4.1.5 shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of $\$1$ invested at 100% interest for 1 year, compounded at various frequencies, listed in Table 4.1.5.

Table 4.1.5

Frequency	$A(t) = \left(1 + \frac{1}{n}\right)^{n}$	Value
Annually	$\left(1 + \frac{1}{1}\right)^{1}$	$2
Semiannually	$\left(1 + \frac{1}{2}\right)^{2}$	$2.25
Quarterly	$\left(1 + \frac{1}{4}\right)^{4}$	$2.441406

Frequency	$A(t)=\left(1+\frac{1}{n}\right)^n$	Value
Monthly	$\left(1+\frac{1}{12}\right)^{12}$	$2.613035
Daily	$\left(1+\frac{1}{365}\right)^{365}$	$2.714567
Hourly	$\left(1+\frac{1}{8760}\right)^{8760}$	$2.718127
Once per minute	$\left(1+\frac{1}{525600}\right)^{525600}$	$2.718279
Once per second	$\left(1+\frac{1}{31536000}\right)^{31536000}$	$2.718282

These values appear to be approaching a limit as n increases without bound. In fact, as n gets larger and larger, the expression $\left(1+\frac{1}{n}\right)^n$ approaches a number used so frequently in mathematics that it has its own name: the letter e. This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

Definition: The Number e

The letter e represents the irrational number

$$\left(1+\frac{1}{n}\right)^n \tag{4.1.4}$$

as n increases without bound

The letter e is used as a base for many real-world exponential models. To work with base e, we use the approximation, $e \approx 2.718282$. The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

Example 4.1.10: Using a Calculator to Find Powers of e

Calculate $e^{3.14}$. Round to five decimal places.

Solution

On a calculator, press the button labeled $[e^x]$. The window shows $[e^(]$. Type 3.14 and then close parenthesis, $[)]$. Press [ENTER]. Rounding to 5 decimal places, $e^{3.14} \approx 23.10387$. Caution: Many scientific calculators have an “Exp” button, which is used to enter numbers in scientific notation. It is not used to find powers of e.

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4.2: Graphs of Exponential Functions

Graphing Exponential Functions

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form $f(x) = b^x$ whose base is greater than one. We'll use the function $f(x) = 2^x$. Observe how the output values in Table 4.2.1 change as the input increases by 1.

Table 4.2.1

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function with the form $f(x) = ab^x$, b is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of a.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values increase without bound; and
- as x decreases, the output values grow smaller, approaching zero.

Figure 4.2.1 shows the exponential growth function $f(x) = 2^x$.

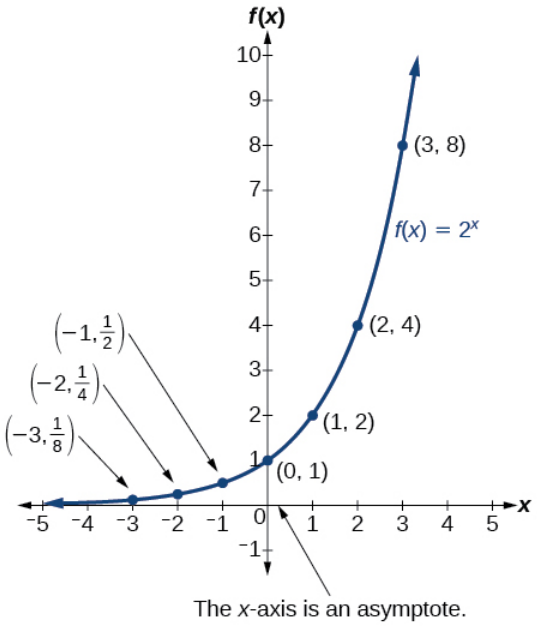


Figure 4.2.1: Notice that the graph gets close to the x-axis, but never touches it.

The domain of $f(x) = 2^x$ is all real numbers, the range is $(0, \infty)$, and the horizontal asymptote is $y = 0$.

To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form $f(x) = b^x$ whose base is between zero and one. We'll use the function $g(x) = \left(\frac{1}{2}\right)^x$. Observe how the output values in Table 4.2.2 change as the input increases by 1.

Table 4.2.2

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio $\frac{1}{2}$.

Notice from the table that

- the output values are positive for all values of x;
- as x increases, the output values grow smaller, approaching zero; and
- as x decreases, the output values grow without bound.

Figure 4.2.2 shows the exponential decay function, $g(x) = \left(\frac{1}{2}\right)^x$.

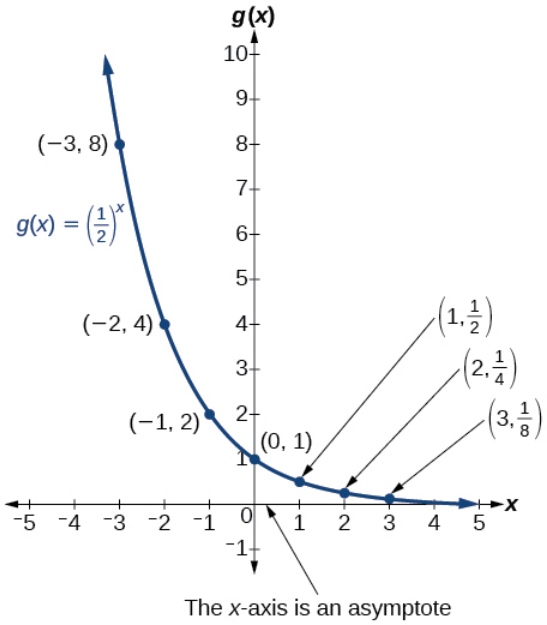


Figure 4.2.2

The domain of $g(x) = (\frac{1}{2})^x$ is all real numbers, the range is $(0, \infty)$,and the horizontal asymptote is $y = 0$.

CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION $F(X) = b^x$

An exponential function with the form $f(x) = b^x$, $b > 0$, $b \neq 1$,has these characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x-intercept: none
- y-intercept: $(0, 1)$
- increasing if $b > 1$
- decreasing if $b < 1$

Figure 4.2.3 compares the graphs of exponential growth and decay functions.

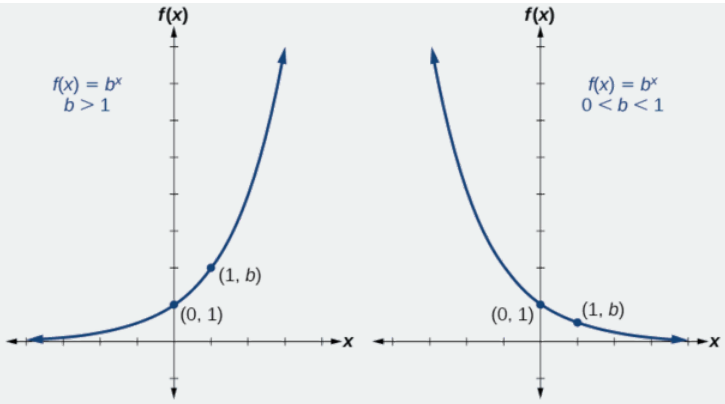


Figure 4.2.3

How to: Given an exponential function of the form $f(x) = b^x$,graph the function

1. Create a table of points.
2. Plot at least 3 point from the table, including the *y*-intercept $(0, 1)$.
3. Draw a smooth curve through the points.
4. State the domain, $(-\infty, \infty)$,the range, $(0, \infty)$,and the horizontal asymptote, $y = 0$.

Example 4.2.1: Sketching the Graph of an Exponential Function of the Form $f(x) = b^x$

Sketch a graph of $f(x) = 0.25^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and create a table of points for the graph.

Table 4.2.3

x	-3	-2	-1	0	1	2	3
$f(x) = 0.25^x$	64	16	4	1	0.25	\(0.0625\0	0.015625

- Since $b = 0.25$ is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote $y = 0$.
- Create a table of points as in Table 4.2.3.
- Plot the *y*-intercept, $(0, 1)$,along with two other points. We can use $(-1, 4)$ and $(1, 0.25)$.

Draw a smooth curve connecting the points as in Figure 4.2.4.

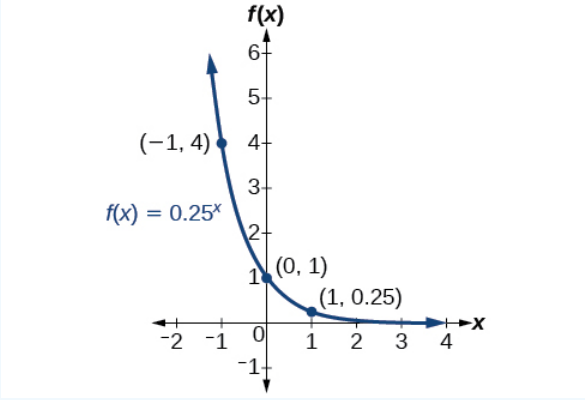


Figure 4.2.4

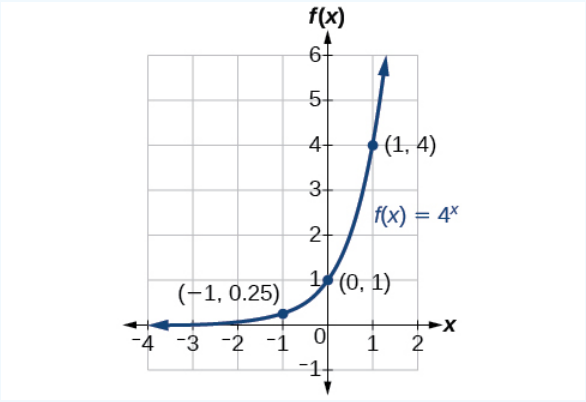
The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Exercise 4.2.1

Sketch the graph of $f(x) = 4^x$. State the domain, range, and asymptote.

Answer

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.



Graphing Transformations of Exponential Functions

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, reflections, stretches, and compressions—to the parent function $f(x) = b^x$ without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

Graphing a Vertical Shift

The first transformation occurs when we add a constant d to the parent function $f(x) = b^x$, giving us a vertical shiftd d units in the same direction as the sign. For example, if we begin by graphing a parent function, $f(x) = 2^x$, we can then graph two vertical shifts alongside it, using $d = 3$: the upward shift, $g(x) = 2^x + 3$ and the downward shift, $h(x) = 2^x - 3$. Both vertical shifts are shown in Figure $4.2.5$.

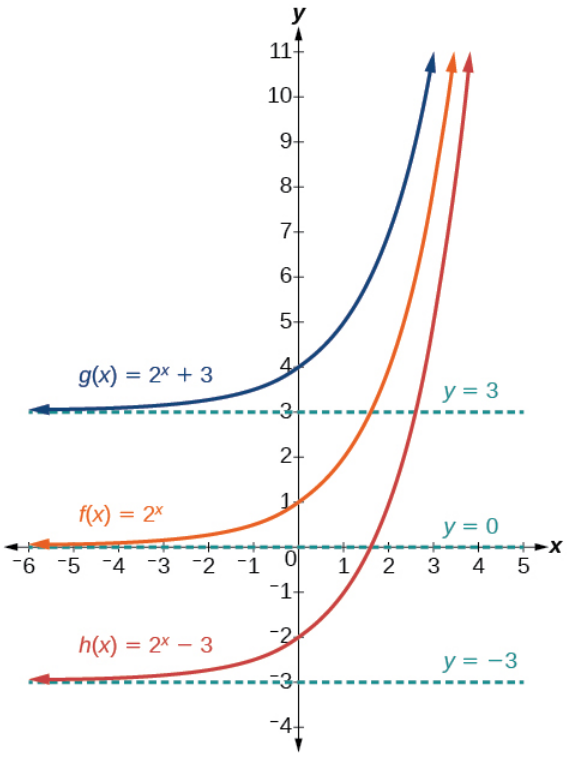


Figure 4.2.5

Observe the results of shifting $f(x) = 2^x$ vertically:

- The domain, $(-\infty, \infty)$ remains unchanged.
- When the function is shifted up 3 units to $g(x) = 2^x + 3$:
  - The *y*-intercept shifts up 3 units to $(0, 4)$.
  - The asymptote shifts up 3 units to $y = 3$.
  - The range becomes $(3, \infty)$.
- When the function is shifted down 3 units to $h(x) = 2^x - 3$:
  - The *y*-intercept shifts down 3 units to $(0, -2)$.
  - The asymptote also shifts down 3 units to $y = -3$.
  - The range becomes $(-3, \infty)$.

Graphing a Horizontal Shift

The next transformation occurs when we add a constant c to the input of the parent function $f(x) = b^x$, giving us a horizontal shift c units in the *opposite* direction of the sign. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph two horizontal shifts alongside it, using $c = 3$: the shift left, $g(x) = 2^{x+3}$, and the shift right, $h(x) = 2^{x-3}$. $h(x) = 2^{x-3}$. Both horizontal shifts are shown in Figure $4.2.6$.

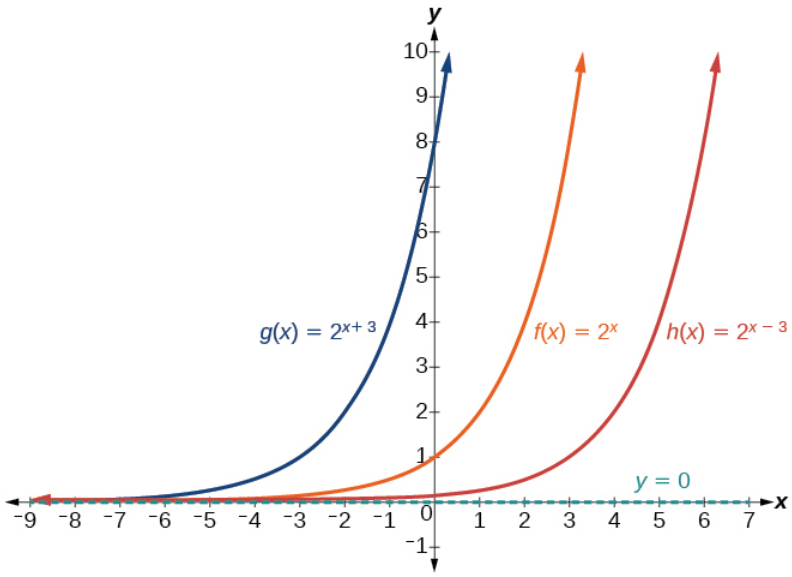


Figure 4.2.6

Observe the results of shifting $f(x) = 2^x$ horizontally:

- The domain, $(-\infty, \infty)$,remains unchanged.
- The asymptote, $y = 0$,remains unchanged.
- The *y*-intercept shifts such that:
  - When the function is shifted left 3 units to $g(x) = 2^{x+3}$,the *y*-intercept becomes $(0, 8)$. This is because $2^{x+3} = (8)2^x$,so the initial value of the function is 8.
  - When the function is shifted right 3 units to $h(x) = 2^{x-3}$,the *y*-intercept becomes $(0, \frac{1}{8})$. Again, see that $2^{x-3} = (\frac{1}{8})2^x$,so the initial value of the function is $\frac{1}{8}$.

SHIFTS OF THE PARENT FUNCTION $F(X) = b^x$

For any constants c and d,the function $f(x) = b^{x+c} + d$ shifts the parent function $f(x) = b^x$

- vertically d units, in the *same* direction of the sign of d.
- horizontally c units, in the *opposite* direction of the sign of c.
- The *y*-intercept becomes $(0, b^c + d)$.
- The horizontal asymptote becomes $y = d$.
- The range becomes (d, ∞).
- The domain, $(-\infty, \infty)$,remains unchanged.

How to: Given an exponential function with the form $f(x) = b^{x+c} + d$, graph the translation

1. Draw the horizontal asymptote $y = d$.
2. Identify the shift as $(-c, d)$. Shift the graph of $f(x) = b^x$ left c units if c is positive, and right c units if c is negative.
3. Shift the graph of $f(x) = b^x$ up d units if d is positive, and down d units if d is negative.
4. State the domain, $(-\infty, \infty)$,the range, (d, ∞),and the horizontal asymptote $y = d$.

Graphing a Stretch or Compression

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a **stretch** or **compression** occurs when we multiply the parent function $f(x) = b^x$ by a constant $|a| > 0$. For example, if we begin by graphing the parent function $f(x) = 2^x$,we can then graph the stretch, using $a = 3$,to get $g(x) = 3(2)^x$ as shown on the left in Figure 4.2.8, and the compression, using $a = \frac{1}{3}$,to get $h(x) = \frac{1}{3}(2)^x$ as shown on the right in Figure 4.2.8.

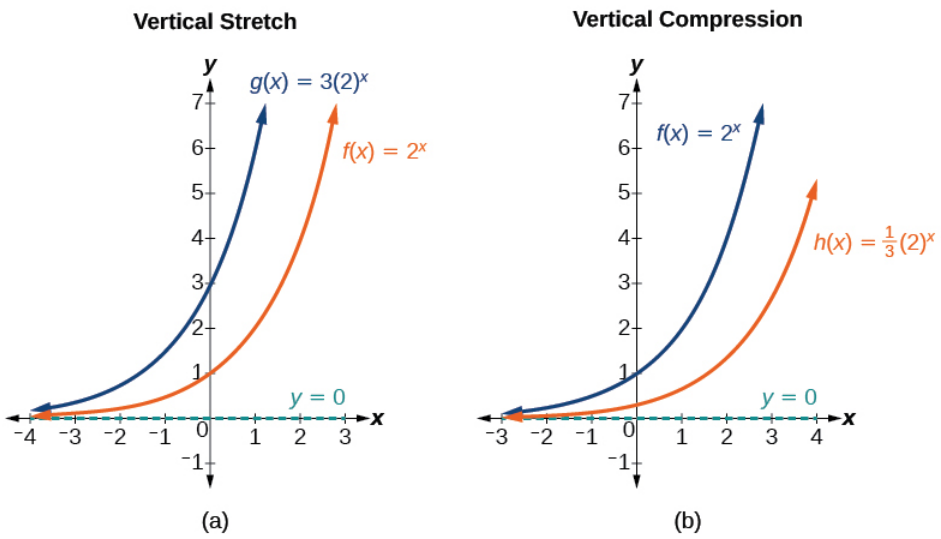


Figure 4.2.8: (a) $g(x) = 3(2)^x$ stretches the graph of $f(x) = 2^x$ vertically by a factor of 3. (b) $h(x) = \frac{1}{3}(2)^x$ compresses the graph of $f(x) = 2^x$ vertically by a factor of $\frac{1}{3}$.

STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION $F(X) = B^X$

For any factor $a > 0$,the function $f(x) = a(b)^x$

- is stretched vertically by a factor of a if $|a| > 1$.
- is compressed vertically by a factor of a if $|a| < 1$.
- has a y-intercept of $(0, a)$.
- has a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$,which are unchanged from the parent function.

Example 4.2.4: Graphing the Stretch of an Exponential Function

Sketch a graph of $f(x) = 4(\frac{1}{2})^x$. State the domain, range, and asymptote.

Solution

Before graphing, identify the behavior and key points on the graph.

- Since $b = \frac{1}{2}$ is between zero and one, the left tail of the graph will increase without bound as x decreases, and the right tail will approach the x-axis as x increases.
- Since $a = 4$,the graph of $f(x) = (\frac{1}{2})^x$ will be stretched by a factor of 4.
- Create a table of points as shown in Table 4.2.4.

Table 4.2.4

x	-3	-2	-1	0	1	2	3
$f(x) = 4(\frac{1}{2})^x$	32	16	8	4	2	1	0.5

- Plot the y-intercept, $(0, 4)$,along with two other points. We can use $(-1, 8)$ and $(1, 2)$.

Draw a smooth curve connecting the points, as shown in Figure 4.2.9.

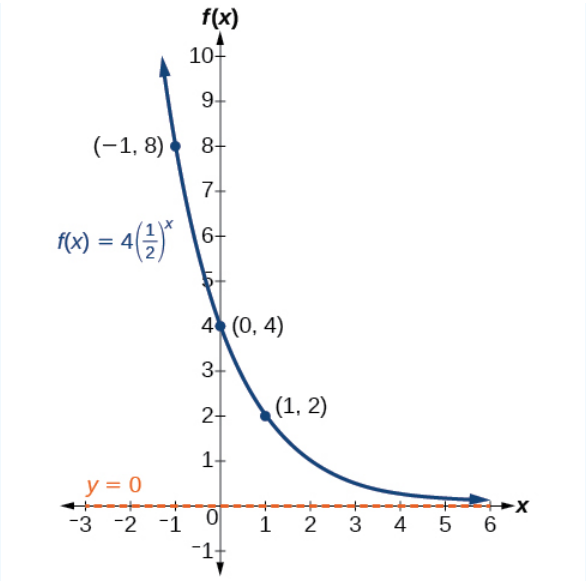


Figure 4.2.9

The domain is $(-\infty, \infty)$; the range is $(0, \infty)$; the horizontal asymptote is $y = 0$.

Graphing Reflections

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the *x*-axis or the *y*-axis. When we multiply the parent function $f(x) = b^x$ by -1,we get a reflection about the *x*-axis. When we multiply the input by -1,we get a reflection about the *y*-axis. For example, if we begin by graphing the parent function $f(x) = 2^x$, we can then graph the two reflections alongside it. The reflection about the *x*-axis, $g(x) = -2^x$,is shown on the left side of Figure 4.2.10, and the reflection about the y-axis $h(x) = 2^{-x}$, is shown on the right side of Figure 4.2.10.

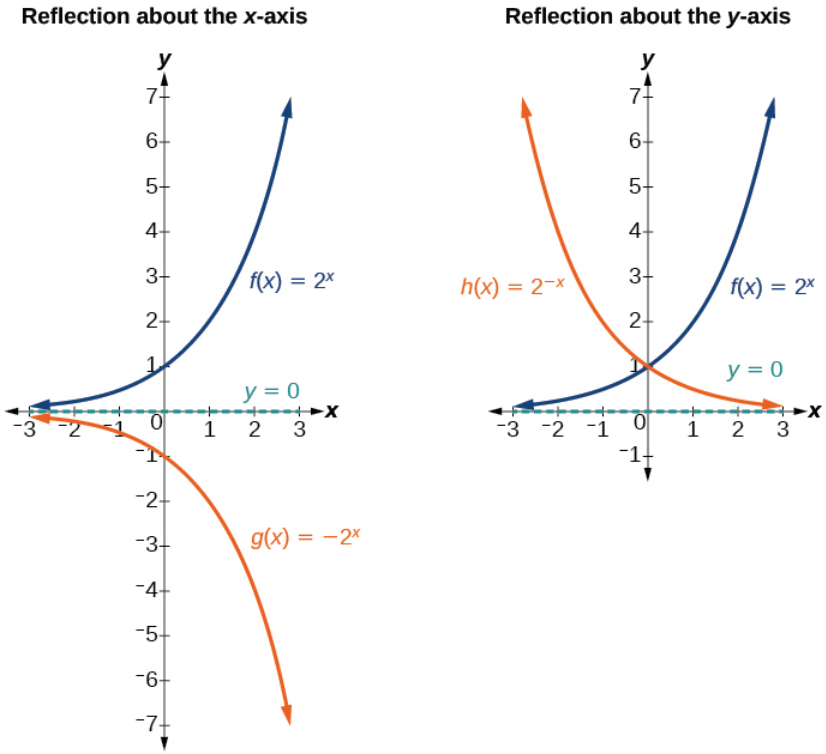


Figure 4.2.10: (a) $g(x) = -2^x$ reflects the graph of $f(x) = 2^x$ about the x-axis. (b) $g(x) = 2^{-x}$ reflects the graph of $f(x) = 2^x$ about the y-axis.

REFLECTIONS OF THE PARENT FUNCTION $F(X) = B^x$

The function $f(x) = -b^x$

- reflects the parent function $f(x) = b^x$ about the *x*-axis.
- has a *y*-intercept of $(0, -1)$.
- has a range of $(-\infty, 0)$
- has a horizontal asymptote at $y = 0$ and domain of $(-\infty, \infty)$,which are unchanged from the parent function.

The function $f(x) = b^{-x}$

- reflects the parent function $f(x) = b^x$ about the *y*-axis.
- has a *y*-intercept of $(0, 1)$, a horizontal asymptote at $y = 0$, a range of $(0, \infty)$, and a domain of $(-\infty, \infty)$, which are unchanged from the parent function.

Summarizing Translations of the Exponential Function

Now that we have worked with each type of translation for the exponential function, we can summarize them in **Table 4.2.6** to arrive at the general equation for translating exponential functions.

Table 4.2.6: Translations of the Parent Function $f(x) = b^x$

Translation	Form
Shift • Horizontally c units to the left • Vertically d units up	$f(x) = b^{x+c} + d$
Stretch and Compress • Stretch if $\lvert a\rvert > 1$ • Compression if $0 < \lvert a\rvert < 1$	$f(x) = ab^x$
Reflect about the x-axis	$f(x) = -b^x$
Reflect about the y-axis	$f(x) = b^{-x} = (\frac{1}{b})^x$
General equation for all translations	$f(x) = ab^{x+c} + d$

TRANSLATIONS OF EXPONENTIAL FUNCTIONS

A translation of an exponential function has the form

$$f(x) = ab^{x+c} + d$$

Where the parent function, $y = b^x$, $b > 1$,is

- shifted horizontally c units to the left.
- stretched vertically by a factor of $\lvert a\rvert$ if $\lvert a\rvert > 0$.
- compressed vertically by a factor of $\lvert a\rvert$ if $0 < \lvert a\rvert < 1$.
- shifted vertically d units.
- reflected about the *x*-axis when $a < 0$.

Note the order of the shifts, transformations, and reflections follow the order of operations.

Example 4.2.6: Writing a Function from a Description

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.

$f(x) = e^x$ is vertically stretched by a factor of 2 , reflected across the *y*-axis, and then shifted up 4 units.

Solution

We want to find an equation of the general form f(x)=abx+c+d. f(x)=abx+c+d. We use the description provided to finda, a, b,b, c,c, and d. d.

- We are given the parent function $f(x) = e^x$, so $b = e$.
- The function is stretched by a factor of 2, so $a = 2$.
- The function is reflected about the *y*-axis. We replace x with $-x$ to get: e^{-x}.
- The graph is shifted vertically 4 units, so $d = 4$.

Substituting in the general form we get,

$$\begin{aligned} f(x) &= ab^{x+c} + d \\ &= 2e^{-x+0} + 4 \\ &= 2e^{-x} + 4 \end{aligned}$$

The domain is $(-\infty, \infty)$; the range is $(4, \infty)$; the horizontal asymptote is $y = 4$.

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4.3: Logarithmic Functions

Converting from Logarithmic to Exponential Form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is $10^x = 500$, where x represents the difference in magnitudes on the Richter Scale. How would we solve for x?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve $10^x = 500$. We know that $10^2 = 100$ and $10^3 = 1000$, so it is clear that x must be some value between 2 and 3, since $y = 10^x$ is increasing. We can examine a graph, as in Figure 4.3.1, to better estimate the solution.

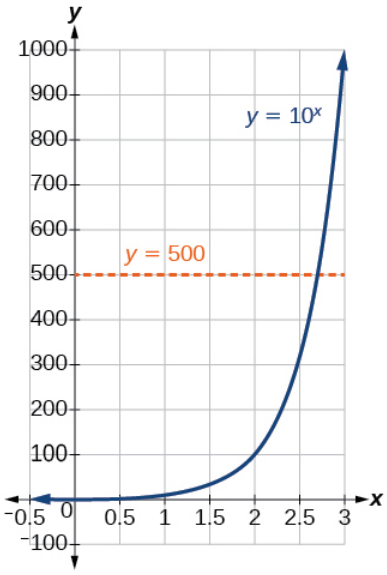


***Figure* 4.3.2**

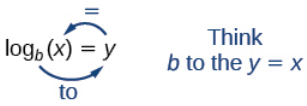
Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph in Figure 4.3.2 passes the horizontal line test. The exponential function $y = b^x$ is one-to-one, so its inverse, $x = b^y$ is also a function. As is the case with all inverse functions, we simply interchange x and y and solve for y to find the inverse function. To represent y as a function of x, we use a logarithmic function of the form $y = \log_b(x)$. The base b **logarithm** of a number is the exponent by which we must raise b to get that number.

We read a logarithmic expression as, "The logarithm with base b of x is equal to y," or, simplified, "log base b of x is y." We can also say, "b raised to the power of y is x," because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since $2^5 = 32$, we can write $\log_2 32 = 5$. We read this as "log base 2 of 32 is 5."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

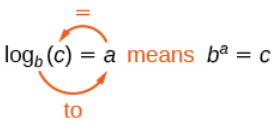
$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1 \tag{4.3.1}$$

Note that the base b is always positive.



Because logarithm is a function, it is most correctly written as $\log_b(x)$, using parentheses to denote function evaluation, just as we would with $f(x)$. However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as $\log_b x$. Note that many calculators require parentheses around the x.

We can illustrate the notation of logarithms as follows:



Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means $y = \log^b(x)$ and $y = b^x$ are inverse functions.

DEFINITION OF THE LOGARITHMIC FUNCTION

A logarithm base b of a positive number x satisfies the following definition.

For $x > 0$, $b > 0$, $b \neq 1$,

$$y = \log_b(x) \text{ is equivalent to } b^y = x \tag{4.3.2}$$

where,

- we read $\log_b(x)$ as, "the logarithm with base b of x" or the "log base b of x."
- the logarithm y is the exponent to which b must be raised to get x.

Also, since the logarithmic and exponential functions switch the x and y values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base b is $(0, \infty)$.
- the range of the logarithm function with base b is $(-\infty, \infty)$.

Using Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression $\log(x)$ means $\log_{10}(x)$. We call a base -10 logarithm a *common logarithm*. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

DEFINITION OF THE COMMON LOGARITHM

A common logarithm is a logarithm with base 10. We write $\log_{10}(x)$ simply as $\log(x)$. The common logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \log(x) \text{ is equivalent to } 10^y = x \tag{4.3.3}$$

We read $\log(x)$ as, "the logarithm with base 10 of x " or "log base 10 of x."

The logarithm y is the exponent to which 10 must be raised to get x.

How to: Given a common logarithm of the form $y = \log(x)$, evaluate it mentally

1. Rewrite the argument x as a power of 10: $10^y = x$.
2. Use previous knowledge of powers of 10 to identify y by asking, "To what exponent must 10 be raised in order to get x?"

Example 4.3.5: Finding the Value of a Common Logarithm Mentally

Evaluate $y = \log(1000)$ without using a calculator.

Solution

First we rewrite the logarithm in exponential form: $10^y = 1000$. Next, we ask, "To what exponent must 10 be raised in order to get 1000?" We know

$$10^3 = 1000$$

Therefore, $\log(1000) = 3$.

Exercise 4.3.5

Evaluate $y = \log(1,000,000)$

Answer

$\log(1,000,000) = 6$

How to: Given a common logarithm with the form $y = \log(x)$,evaluate it using a calculator

1. Press **[LOG]**.
2. Enter the value given for x,followed by **[ )]**.
3. Press **[ENTER]**.

Using Natural Logarithms

The most frequently used base for logarithms is e. Base e logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base e logarithm, $\log_e(x)$, has its own notation,$\ln(x)$. Most values of $\ln(x)$ can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, $\ln 1 = 0$. For other natural logarithms, we can use the $\ln$ key that can be found on most scientific calculators. We can also find the natural logarithm of any power of e using the inverse property of logarithms.

DEFINITION OF THE NATURAL LOGARITHM

A natural logarithm is a logarithm with base e. We write $\log_e(x)$ simply as $\ln(x)$. The natural logarithm of a positive number x satisfies the following definition.

For $x > 0$,

$$y = \ln(x) \text{ is equivalent to } e^y = x$$

We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x."

The logarithm y is the exponent to which e must be raised to get x.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = x$ for all x and $e^{\ln(x)} = x$ for $x > 0$.

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4.4: Graphs of Logarithmic Functions

Finding the Domain of a Logarithmic Function

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as $y = b^x$ for any real number x and constant $b > 0$, $b \neq 1$, where

- The domain of y is $(-\infty, \infty)$.
- The range of y is $(0, \infty)$.

In the last section we learned that the logarithmic function $y = \log_b(x)$ is the inverse of the exponential function $y = b^x$. So, as inverse functions:

- The domain of $y = \log_b(x)$ is the range of $y = b^x$: $(0, \infty)$.
- The range of $y = \log_b(x)$ is the domain of $y = b^x$: $(-\infty, \infty)$.

Transformations of the parent function $y = \log_b(x)$ behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations—shifts, stretches, compressions, and reflections—to the parent function without loss of shape.

In Graphs of Exponential Functions we saw that certain transformations can change the *range* of $y = b^x$. Similarly, applying transformations to the parent function $y = \log_b(x)$ can change the *domain*. When finding the domain of a logarithmic function, therefore, it is important to remember that the domain consists *only of positive real numbers*. That is, the argument of the logarithmic function must be greater than zero.

For example, consider $f(x) = \log_4(2x - 3)$. This function is defined for any values of x such that the argument, in this case $2x - 3$,is greater than zero. To find the domain, we set up an inequality and solve for x:

$$\begin{aligned} 2x - 3 &> 0 \quad \text{Show the argument greater than zero} \\ 2x &> 3 \quad \text{Add 3} \\ x &> 1.5 \quad \text{Divide by 2} \end{aligned}$$

In interval notation, the domain of $f(x) = \log_4(2x - 3)$ is $(1.5, \infty)$.

Given a logarithmic function, identify the domain

1. Set up an inequality showing the argument greater than zero.
2. Solve for x.
3. Write the domain in interval notation.

Graphing Logarithmic Functions

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function $y = \log_b(x)$ along with all its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function $y = \log_b(x)$. Because every logarithmic function of this form is the inverse of an exponential function with the form $y = b^x$, their graphs will be reflections of each other across the line $y = x$. To illustrate this, we can observe the relationship between the input and output values of $y = 2^x$ and its equivalent $x = \log_2(y)$ in Table 4.4.1.

Table 4.4.1

x	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from Table 4.4.1, we can build another table to observe the relationship between points on the graphs of the inverse functions $f(x) = 2^x$ and $g(x) = \log_2(x)$. See Table 4.4.2.

Table 4.4.2

$f(x) = 2^x$	$\left(-3, \frac{1}{8}\right)$	$\left(-2, \frac{1}{4}\right)$	$\left(-1, \frac{1}{2}\right)$	$(0, 1)$	$(1, 2)$	$(2, 4)$	$(3, 8)$
$g(x) = \log_2(x)$	$\left(\frac{1}{8}, -3\right)$	$\left(\frac{1}{4}, -2\right)$	$\left(\frac{1}{2}, -1\right)$	$(1, 0)$	$(2, 1)$	$(4, 2)$	$(8, 3)$

As we'd expect, the x- and y-coordinates are reversed for the inverse functions. Figure 4.4.2 shows the graph of f and g.

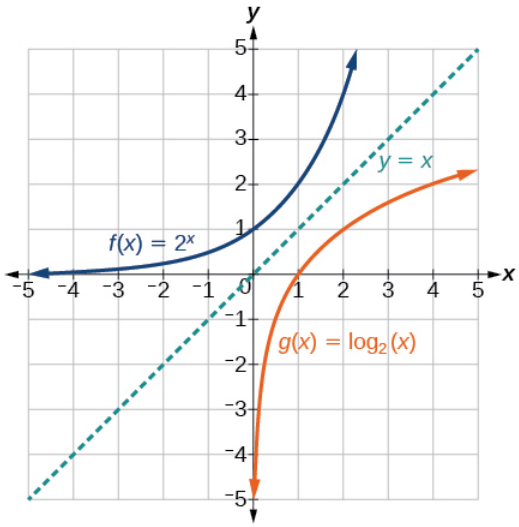


Figure 4.4.2: Notice that the graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are reflections about the line $y = x$.

Observe the following from the graph:

- $f(x) = 2^x$ has a y-intercept at $(0, 1)$ and $g(x) = \log_2(x)$ has an x- intercept at $(1, 0)$.
- The domain of $f(x) = 2^x$, $(-\infty, \infty)$, is the same as the range of $g(x) = \log_2(x)$.
- The range of $f(x) = 2^x$, $(0, \infty)$, is the same as the domain of $g(x) = \log_2(x)$.

CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION, $F(X) = LOG_B(X)$

For any real number x and constant $b > 0$, $b \neq 1$, we can see the following characteristics in the graph of $f(x) = \log_b(x)$:

- one-to-one function
- vertical asymptote: $x = 0$
- domain: $(0, \infty)$
- range: $(-\infty, \infty)$
- x-intercept: $(1, 0)$ and key point $(b, 1)$
- y-intercept: none
- increasing if $b > 1$
- decreasing if $0 < b < 1$

See Figure 4.4.3.

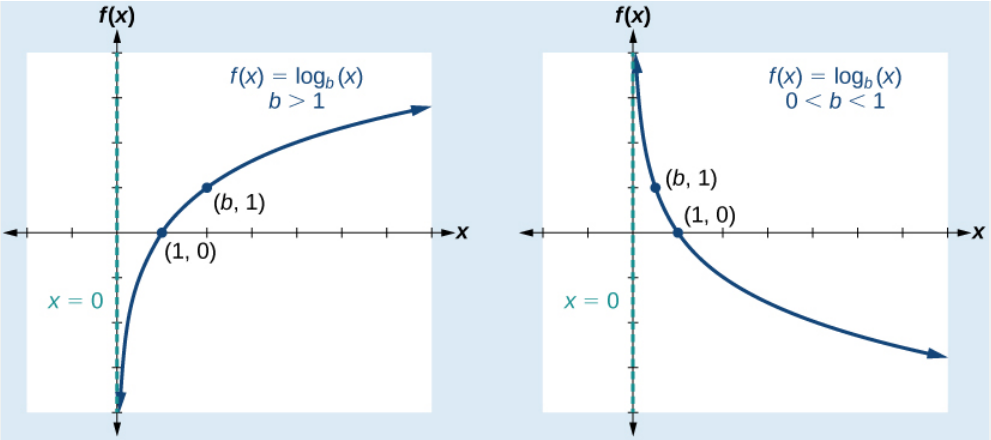


Figure 4.4.3

Figure 4.4.4 shows how changing the base b in $f(x) = \log_b(x)$ can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (Note: recall that the function $\ln(x)$ has base $e \approx 2.718$.)

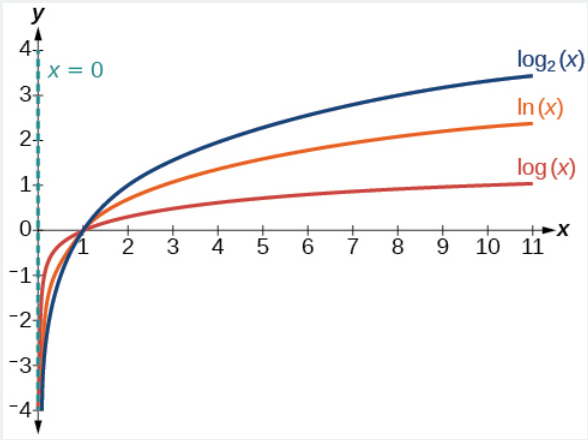


Figure 4.4.4: The graphs of three logarithmic functions with different bases, all greater than 1.

Given a logarithmic function with the form $f(x) = \log_b(x)$, graph the function.

1. Draw and label the vertical asymptote, $x = 0$.
2. Plot the *x*-intercept, $(1, 0)$.
3. Plot the key point $(b, 1)$.
4. Draw a smooth curve through the points.
5. State the domain, $(0, \infty)$,the range, $(-\infty, \infty)$, and the vertical asymptote, $x = 0$.

Graphing Transformations of Logarithmic Functions

As we mentioned in the beginning of the section, transformations of logarithmic graphs behave similarly to those of other parent functions. We can shift, stretch, compress, and reflect the parent function $y = \log_b(x)$ without loss of shape.

Graphing a Horizontal Shift of $f(x) = \log_b(x)$

When a constant c is added to the input of the parent function $f(x) = \log_b(x)$, the result is a horizontal shift c units in the *opposite* direction of the sign on c. To visualize horizontal shifts, we can observe the general graph of the parent function $f(x) = \log_b(x)$ and for $c > 0$ alongside the shift left, $g(x) = \log_b(x + c)$, and the shift right, $h(x) = \log_b(x - c)$. See Figure 4.4.7.

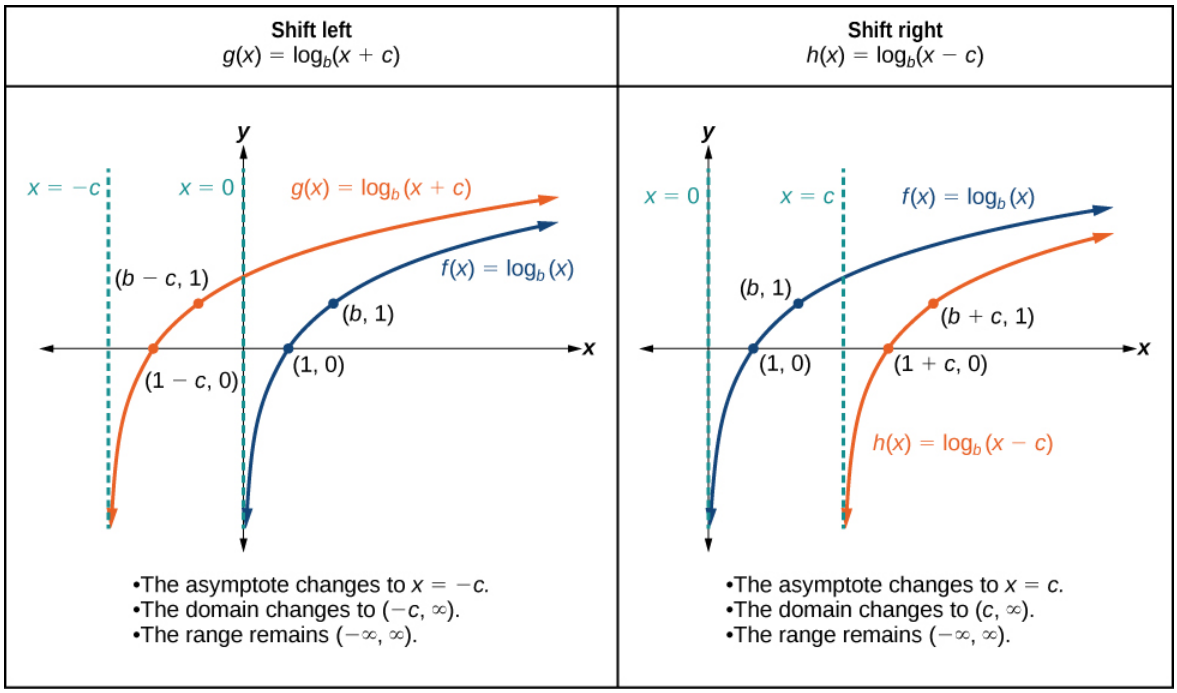


Figure 4.4.7

HORIZONTAL SHIFTS OF THE PARENT FUNCTION $Y = LOG_B(X)$

For any constant c,the function $f(x) = \log_b(x + c)$

- shifts the parent function $y = \log_b(x)$ left c units if $c > 0$.
- shifts the parent function $y = \log_b(x)$ right c units if $c < 0$.
- has the vertical asymptote $x = -c$.
- has domain $(-c, \infty)$.
- has range $(-\infty, \infty)$.

Given a logarithmic function with the form $f(x) = \log_b(x + c)$, graph the translation.

1. Identify the horizontal shift:
   - If $c > 0$,shift the graph of $f(x) = \log_b(x)$ left c units.
   - If $c < 0$,shift the graph of $f(x) = \log_b(x)$ right c units.
2. Draw the vertical asymptote $x = -c$.
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting c from the x coordinate.
4. Label the three points.
5. The Domain is $(-c, \infty)$,the range is $(-\infty, \infty)$, and the vertical asymptote is $x = -c$.

Graphing Reflections of $f(x) = log_b(x)$

When the parent function $f(x) = \log_b(x)$ is multiplied by -1,the result is a reflection about the x-axis. When the *input* is multiplied by -1,the result is a reflection about the y-axis. To visualize reflections, we restrict $b > 1$, and observe the general graph of the parent function $f(x) = \log_b(x)$ alongside the reflection about the x-axis, $g(x) = -\log_b(x)$ and the reflection about the y-axis, $h(x) = \log_b(-x)$.

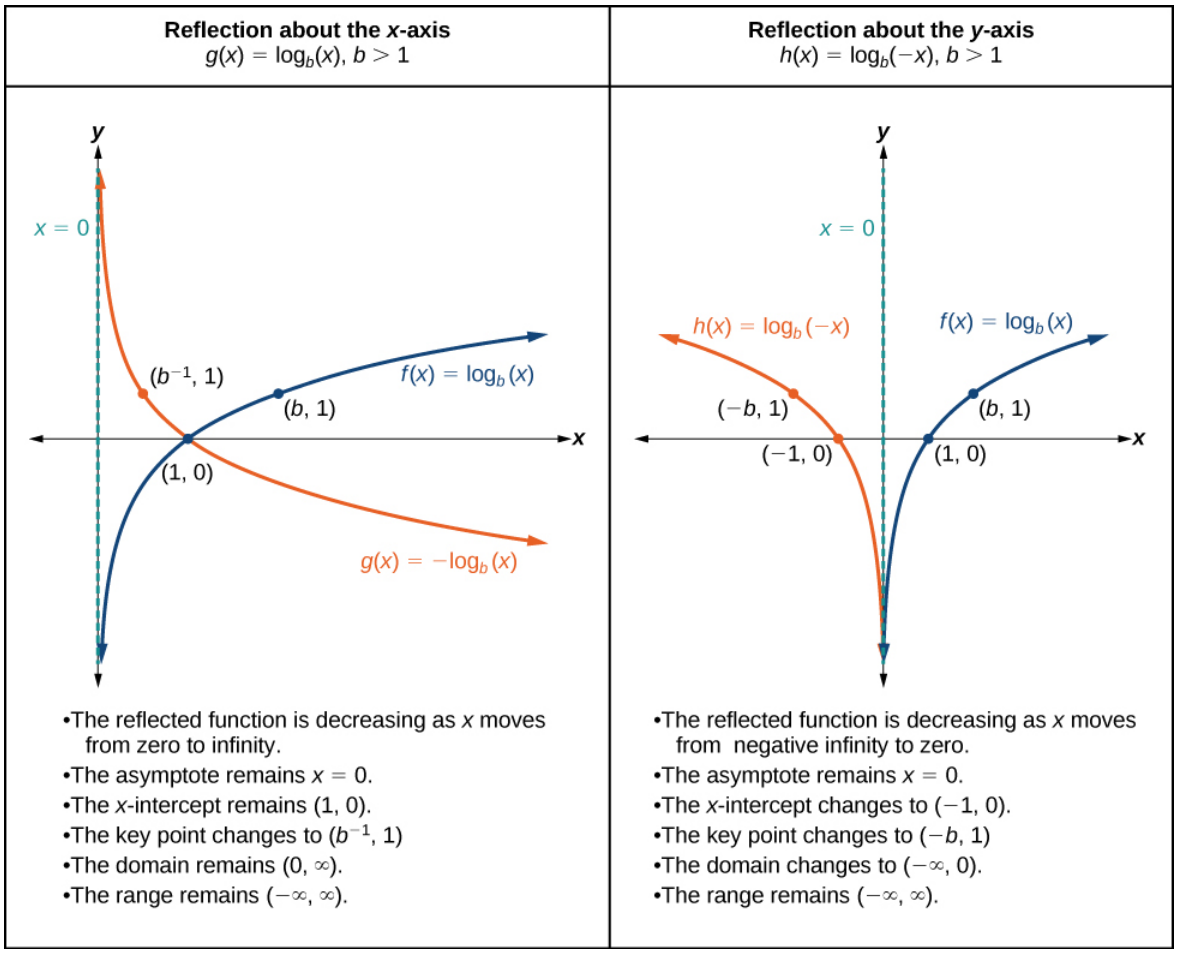


Figure 4.4.18

REFLECTIONS OF THE PARENT FUNCTION $Y = LOG_B(X)$

The function $f(x) = -\log_b(x)$

- reflects the parent function $y = \log_b(x)$ about the x-axis.
- has domain, $(0, \infty)$, range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

The function $f(x) = \log_b(-x)$

- reflects the parent function $y = \log_b(x)$ about the y-axis.
- has domain $(-\infty, 0)$.
- has range, $(-\infty, \infty)$, and vertical asymptote, $x = 0$, which are unchanged from the parent function.

Given a logarithmic function with the parent function $f(x) = \log_b(x)$, graph a translation.

Table 4.4.3

If $f(x) = -\log_b(x)$	If $f(x) = \log_b(-x)$
1. Draw the vertical asymptote, $x = 0$.	1. Draw the vertical asymptote, $x = 0$.
2. Plot the x-intercept, $(1, 0)$.	2. Plot the x-intercept, $(1, 0)$.
3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the x-axis.	3. Reflect the graph of the parent function $f(x) = \log_b(x)$ about the y-axis.
4. Draw a smooth curve through the points.	4. Draw a smooth curve through the points.

If $f(x) = -\log_b(x)$	If $f(x) = \log_b(-x)$
5. State the domain, $(0, \infty)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.	5. State the domain, $(-\infty, 0)$, the range, $(-\infty, \infty)$, and the vertical asymptote $x = 0$.

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4.5: Logarithmic Properties

Using the Product Rule for Logarithms

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.

$$\log_b 1 = 0$$

$$\log_b b = 1$$

For example, $\log_5 1 = 0$ since $5^0 = 1$. And $\log_5 5 = 1$ since $5^1 = 5$.

Next, we have the inverse property.

$$\log_b(b^x) = x$$

$$b^{\log_b x} = x,\ x > 0$$

For example, to evaluate $\log(100)$, we can rewrite the logarithm as $\log_{10}(10^2)$, and then apply the inverse property $\log_b(b^x) = x$ to get $\log_{10}(10^2) = 2$.

To evaluate $e^{\ln(7)}$, we can rewrite the logarithm as $e^{\log_e 7}$, and then apply the inverse property $b^{\log_b x} = x$ to get $e^{\log_e 7} = 7$.

Finally, we have the one-to-one property.

$$\log_b M = \log_b N \text{ if and only if } M = N \tag{4.5.1}$$

We can use the one-to-one property to solve the equation $\log_3(3x) = \log_3(2x+5)$ for x. Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for x:

$3x = 2x + 5$ Set the arguments equal.

$x = 5$ Subtract $2x$.

But what about the equation $\log_3(3x) + \log_3(2x+5) = 2$? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

Recall that we use the *product rule of exponents* to combine the product of exponents by adding: $x^a x^b = x^{a+b}$. We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number x and positive real numbers M, N, and b, where $b \neq 1$, we will show

$$\log_b(MN) = \log_b(M) + \log_b(N)\ .$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for M and N} \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents} \\ &= m+n && \text{Apply the inverse property of logs} \\ &= \log_b(M) + \log_b(N) && \text{Substitute for m and n} \end{aligned}$$

Note that repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. For example, consider $\log_b(wxyz)$. Using the product rule for logarithms, we can rewrite this logarithm of a product as the sum of logarithms of its factors:

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

The Product Rule for Logarithms

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0 \tag{4.5.2}$$

How to: Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

Using the Quotient Rule for Logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to combine the quotient of exponents by subtracting: $x^{\frac{a}{b}} = x^{a-b}$. The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms.

The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N \tag{4.5.3}$$

Just as with the product rule, we can use the inverse property to derive the quotient rule.

Proof

Given any real number x and positive real numbers M, N, and b, b, where $b \neq 1$, we will show

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)\ .$$

Let $m = \log_b M$ and $n = \log_b N$. In exponential form, these equations are $b^m = M$ and $b^n = N$. It follows that

$$\begin{aligned}
\log_b\left(\frac{M}{N}\right) &= \log_b\left(\frac{b^m}{b^n}\right) && \text{Substitute for M and N} \\
&= \log_b(b^{m-n}) && \text{Apply the quotient rule for exponents} \\
&= m - n && \text{Apply the inverse property of logs} \\
&= \log_b(M) - \log_b(N) && \text{Substitute for m and n}
\end{aligned}$$

For example, to expand $\log\left(\frac{2x^2+6x}{3x+9}\right)$, we must first express the quotient in lowest terms. Factoring and canceling we get,

$$\begin{aligned}
\log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{Factor the numerator and denominator} \\
&= \log\left(\frac{2x}{3}\right) && \text{Cancel the common factors}
\end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\log\left(\frac{2x}{3}\right) = \log(2x) - \log(3)$$
$$= \log(2) + \log(x) - \log(3)$$

How to: Given the logarithm of a quotient, use the quotient rule of logarithms to write an equivalent difference of logarithms

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

Using the Power Rule for Logarithms

We've explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as x^2? One method is as follows:

$$\log_b(x^2) = \log_b(x \cdot x)$$
$$= \log_b x + \log_b x$$
$$= 2\log_b x$$

Notice that we used the **product rule for logarithms** to find a solution for the example above. By doing so, we have derived the **power rule for logarithms**, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \quad \sqrt{3} = 3^{\frac{1}{2}} \quad \frac{1}{e} = e^{-1}$$

The Power Rule for Logarithms

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b M \tag{4.5.4}$$

How to: Given the logarithm of a power, use the power rule of logarithms to write an equivalent product of a factor and a logarithm

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

Expanding Logarithmic Expressions

Taken together, the product rule, quotient rule, and power rule are often called "laws of logs." Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\log_b\left(\frac{6x}{y}\right) = \log_b(6x) - \log_b y$$
$$= \log_b 6 + \log_b x - \log_b y$$

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:

$$\log_b\left(\frac{A}{C}\right) = \log_b(AC^{-1})$$
$$= \log_b(A) + \log_b(C^{-1})$$
$$= \log_b A + (-1)\log_b C$$
$$= \log_b A - \log_b C$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally, writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm.

Example 4.5.6: Expanding Logarithms Using Product, Quotient, and Power Rules

Rewrite $ln\left(\frac{x^4y}{7}\right)$ as a sum or difference of logs.

Solution

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln\left(\frac{x^4y}{7}\right) = \ln(x^4y) - \ln(7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln(x^4y) - \ln(7) = \ln(x^4) + \ln(y) - \ln(7)$$

Finally, we use the power rule on the first term:

$$\ln(x^4) + \ln(y) - \ln(7) = 4\ln(x) + \ln(y) - \ln(7)$$

Condensing Logarithmic Expressions

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

How to: Given a sum, difference, or product of logarithms with the same base, write an equivalent expression as a single logarithm

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

Using the Change-of-Base Formula for Logarithms

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 ore, e,we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the **one-to-one** property and **power rule for logarithms**.

Given any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,we show

$$\log_b M = \frac{\log_n M}{\log_n b}$$

Let $y = \log_b M$. By taking the log base n of both sides of the equation, we arrive at an exponential form, namely $b^y = M$. It follows that

$$\begin{aligned} \log_n(b^y) &= \log_n M && \text{Apply the one-to-one property} \\ y\log_n b &= \log_n M && \text{Apply the power rule for logarithms} \\ y &= \frac{\log_n M}{\log_n b} && \text{Isolate y} \\ \log_b M &= \frac{\log_n M}{\log_n b} && \text{Substitute for y} \end{aligned}$$

For example, to evaluate $\log_5 36$ using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\begin{aligned} \log_5 36 &= \frac{\log(36)}{\log(5)} && \text{Apply the change of base formula using base 10} \\ &\approx 2.2266 && \text{Use a calculator to evaluate to 4 decimal places} \end{aligned}$$

THE CHANGE-OF-BASE FORMULA

The change-of-base formula can be used to evaluate a logarithm with any base.

For any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,

$$\log_b M = \frac{\log_n M}{\log_n b} \tag{4.5.5}$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b} \tag{4.5.6}$$

and

$$\log_b M = \frac{\log M}{\log b} \tag{4.5.7}$$

How to: Given a logarithm with the form $\log_b M$, use the change-of-base formula to rewrite it as a quotient of logs with any positive base n, where $n \neq 1$

1. Determine the new base n, remembering that the common log, $\log(x)$, has base 10, and the natural log, $\ln(x)$,has base e.
2. Rewrite the log as a quotient using the change-of-base formula
   - The numerator of the quotient will be a logarithm with base n and argument M.
   - The denominator of the quotient will be a logarithm with base n and argument b.

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4.6: Exponential and Logarithmic Equations

Using Like Bases to Solve Exponential Equations

The first technique involves two functions with like bases. Recall that the one-to-one property of exponential functions tells us that, for any real numbers b, S, and T, where $b > 0$, $b \neq 1$, $b^S = b^T$ if and only if $S = T$.

In other words, when an **exponential equation** has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we use the fact that exponential functions are one-to-one to set the exponents equal to one another, and solve for the unknown.

For example, consider the equation $3^{4x-7} = \dfrac{3^{2x}}{3}$. To solve for x, we use the division property of exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for x:

$$
\begin{aligned}
3^{4x-7} &= \frac{3^{2x}}{3} && \\
3^{4x-7} &= \frac{3^{2x}}{3^1} && \text{Rewrite 3 as } 3^1 \\
3^{4x-7} &= 3^{2x-1} && \text{Use the division property of exponents} \\
4x-7 &= 2x-1 && \text{Apply the one-to-one property of exponents} \\
2x &= 6 && \text{Subtract 2x and add 7 to both sides} \\
x &= 3 && \text{Divide by 3}
\end{aligned}
$$

USING THE ONE-TO-ONE PROPERTY OF EXPONENTIAL FUNCTIONS TO SOLVE EXPONENTIAL EQUATIONS

For any algebraic expressions S and T, and any positive real number $b \neq 1$,

$$b^S = b^T \text{ if and only if } S = T \tag{4.6.1}$$

How to: Given an exponential equation with the form $b^S = b^T$, where S and T are algebraic expressions with an unknown, solve for the unknown.

1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form $b^S = b^T$.
2. Use the one-to-one property to set the exponents equal.
3. Solve the resulting equation, $S = T$, for the unknown.

Example 4.6.1: Solving an Exponential Equation with a Common Base

Solve $2^{x-1} = 2^{2x-4}$.

Solution

$$
\begin{aligned}
2^{x-1} &= 2^{2x-4} && \text{The common base is 2} \\
x-1 &= 2x-4 && \text{By the one-to-one property the exponents must be equal} \\
x &= 3 && \text{Solve for x}
\end{aligned}
$$

Solving Exponential Equations Using Logarithms

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since $\log(a) = \log(b)$ is equivalent to $a = b$, we may apply logarithms with the same base on both sides of an exponential equation.

How to: Given an exponential equation in which a common base cannot be found, solve for the unknown

1. Apply the logarithm of both sides of the equation.
   - If one of the terms in the equation has base 10, use the common logarithm.
   - If none of the terms in the equation has base 10, use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

Example 4.6.5: Solving an Equation Containing Powers of Different Bases

Solve $5^{x+2} = 4^x$.

Solution

$5^{x+2} = 4^x$	There is no easy way to get the powers to have the same base
$\ln 5^{x+2} = \ln 4^x$	Take ln of both sides
$(x+2)\ln 5 = x \ln 4$	Use laws of logs
$x \ln 5 + 2 \ln 5 = x \ln 4$	Use the distributive law
$x \ln 5 - x \ln 4 = -2 \ln 5$	Get terms containing x on one side, terms without x on the other
$x(\ln 5 - \ln 4) = -2 \ln 5$	On the left hand side, factor out an x
$x \ln\left(\frac{5}{4}\right) = \ln\left(\frac{1}{25}\right)$	Use the laws of logs
$x = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)}$	Divide by the coefficient of x

Exercise 4.6.5

Solve $2^x = 3^{x+1}$.

Answer

$$x = \frac{\ln 3}{\ln\left(\frac{2}{3}\right)}$$

Equations Containing e

One common type of exponential equations are those with base e. This constant occurs again and again in nature, in mathematics, in science, in engineering, and in finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

How to: Given an equation of the form $y = Ae^{kt}$, solve for t.

1. Divide both sides of the equation by A.
2. Apply the natural logarithm of both sides of the equation.
3. Divide both sides of the equation by k.

Example 4.6.6: Solve an Equation of the Form $y = Ae^{kt}$

Solve $100 = 20e^{2t}$.

Solution

$$
\begin{aligned}
100 &= 20e^{2t} \\
5 &= e^{2t} && \text{Divide by the coefficient of the power} \\
\ln 5 &= 2t && \text{Take ln of both sides. Use the fact that } ln(x) \text{ and } e^x \text{ are inverse functions} \\
t &= \frac{\ln 5}{2} && \text{Divide by the coefficient of t}
\end{aligned}
$$

Analysis

Using laws of logs, we can also write this answer in the form $t = \ln \sqrt{5}$. If we want a decimal approximation of the answer, we use a calculator.

Using the Definition of a Logarithm to Solve Logarithmic Equations

We have already seen that every logarithmic equation $\log_b(x) = y$ is equivalent to the exponential equation $b^y = x$. We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation $\log_2(2) + \log_2(3x - 5) = 3$. To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for x:

$$
\begin{aligned}
\log_2(2) + \log_2(3x - 5) &= 3 \\
\log_2(2(3x - 5)) &= 3 && \text{Apply the product rule of logarithms} \\
\log_2(6x - 10) &= 3 && \text{Distribute} \\
2^3 &= 6x - 10 && \text{Apply the definition of a logarithm} \\
8 &= 6x - 10 && \text{Calculate } 2^3 \\
18 &= 6x && \text{Add 10 to both sides} \\
x &= 3 && \text{Divide by 6}
\end{aligned}
$$

USING THE DEFINITION OF A LOGARITHM TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expression S and real numbers b and c,where $b > 0$, $b \neq 1$,

$$\log_b(S) = c \text{ if and only if } b^c = S \tag{4.6.2}$$

Example 4.6.9: Using Algebra to Solve a Logarithmic Equation

Solve $2 \ln x + 3 = 7$.

Solution

$$
\begin{aligned}
2 \ln x + 3 &= 7 \\
2 \ln x &= 4 && \text{Subtract 3} \\
\ln x &= 2 && \text{Divide by 2} \\
x &= e^2 && \text{Rewrite in exponential form}
\end{aligned}
$$

Using the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers $x > 0$, $S > 0$, $T > 0$ and any positive real number b, where $b \neq 1$,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x - 1) = \log_2(8), \text{ then } x - 1 = 8.$$

So, if $x - 1 = 8$, then we can solve for x,and we get $x = 9$. To check, we can substitute $x = 9$ into the original equation: $\log_2(9 - 1) = \log_2(8) = 3$. In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation $\log(3x-2)-\log(2)=\log(x+4)$. To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm, and then apply the one-to-one property to solve for x:

$$\begin{aligned}
\log(3x-2)-\log(2) &= \log(x+4) \\
\log\left(\frac{3x-2}{2}\right) &= \log(x+4) && \text{Apply the quotient rule of logarithms} \\
\frac{3x-2}{2} &= x+4 && \text{Apply the one to one property of a logarithm} \\
3x-2 &= 2x+8 && \text{Multiply both sides of the equation by 2} \\
x &= 10 && \text{Subtract 2x and add 2}
\end{aligned}$$

To check the result, substitute $x=10$ into $\log(3x-2)-\log(2)=\log(x+4)$.

$$\begin{aligned}
\log(3(10)-2)-\log(2) &= \log((10)+4) \\
\log(28)-\log(2) &= \log(14) \\
\log\left(\frac{28}{2}\right) &= \log(14) && \text{The solution checks}
\end{aligned}$$

USING THE ONE-TO-ONE PROPERTY OF LOGARITHMS TO SOLVE LOGARITHMIC EQUATIONS

For any algebraic expressions S and T and any positive real number b, where $b\neq 1$,

$$b^S=b^T \text{ if and only if } S=T \tag{4.6.3}$$

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

How to: Given an equation containing logarithms, solve it using the one-to-one property

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form $\log_b S=\log_b T$.
2. Use the one-to-one property to set the arguments equal.
3. Solve the resulting equation, $S=T$, for the unknown.

Example 4.6.12: Solving an Equation Using the One-to-One Property of Logarithms

Solve $\ln(x^2)=\ln(2x+3)$.

Solution

$$\begin{aligned}
\ln(x^2) &= \ln(2x+3) \\
x^2 &= 2x+3 && \text{Use the one-to-one property of the logarithm} \\
x^2-2x-3 &= 0 && \text{Get zero on one side before factoring} \\
(x-3)(x+1) &= 0 && \text{Factor using FOIL} \\
x-3 &= 0 && \text{or } x+1=0 \text{ If a product is zero, one of the factors must be zero} \\
x &= 3 \quad \text{or} \\
x &= -11 && \text{Solve for x}
\end{aligned}$$

Analysis

There are two solutions: 3 or -1. The solution -1 is negative, but it checks when substituted into the original equation because the argument of the logarithm functions is still positive.

Exercise 4.6.12

Solve $\ln(x^2)=\ln 1$.

Answer

$x = 1$ or $x = -1$

Solving Applied Problems Using Exponential and Logarithmic Equations

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its **half-life**. Table 4.6.1 lists the half-life for several of the more common radioactive substances.

Table 4.6.1

Substance	*Use*	*Half-life*
gallium-67	*nuclear medicine*	*80 hours*
cobalt-60	*manufacturing*	*5.3 years*
technetium-99m	*nuclear medicine*	*6 hours*
americium-241	*construction*	*432 years*
carbon-14	*archeological dating*	*5,715 years*
uranium-235	*atomic power*	*703,800,000 years*

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:

$$A(t) = A_0 e^{\frac{\ln(0.5)}{T}t} \tag{4.6.4}$$

$$A(t) = A_0 e^{\frac{\ln(0.5)t}{T}} \tag{4.6.5}$$

$$A(t) = A_0 (e^{\ln(0.5)})^{\frac{t}{T}} \tag{4.6.6}$$

$$A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}} \tag{4.6.7}$$

where

- A_0 is the amount initially present
- T is the half-life of the substance
- t is the time period over which the substance is studied
- y is the amount of the substance present after time t

Example 4.6.13: Using the Formula for Radioactive Decay to Find the Quantity of a Substance

How long will it take for ten percent of a 1000-gram sample of uranium-235 to decay?

Solution

$$
\begin{aligned}
y &= 1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} \\
900 &= 1000e^{\frac{\ln(0.5)}{703{,}800{,}000}t} && \text{After } 10\% \text{ decays, 900 grams are left} \\
0.9 &= e^{\frac{\ln(0.5)}{703{,}800{,}000}t} && \text{Divide by 1000} \\
\ln(0.9) &= \ln\left(e^{\frac{\ln(0.5)}{703{,}800{,}000}t}\right) && \text{Take ln of both sides} \\
\ln(0.9) &= \frac{\ln(0.5)}{703,800,000}t && \ln(e^M) = M \\
t &= 703,800,000 \times \frac{\ln(0.9)}{\ln(0.5)} && \text{years Solve for t} \\
t &\approx 106,979,777 && \text{years}
\end{aligned}
$$

Analysis

Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium-235 remaining is 900 grams.

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4.7: Exponential and Logarithmic Models

Modeling Exponential Growth and Decay

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function:

$$y = A_0 e^{kt} \tag{4.7.1}$$

where A_0 is equal to the value at time zero, e is Euler's constant, and k is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving **doubling time**, the time it takes for a quantity to double. Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth based on a doubling time. In some applications, however, as we will see when we discuss the logistic equation, the logistic model sometimes fits the data better than the exponential model.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the **exponential decay** model. Again, we have the form $y = A_0 e^{kt}$ where A_0 is the starting value, and e is Euler's constant. Now k is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating half-life, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in Figure 4.7.2 and Figure 4.7.3. It is important to remember that, although parts of each of the two graphs seem to lie on the x-axis, they are really a tiny distance above the x-axis.

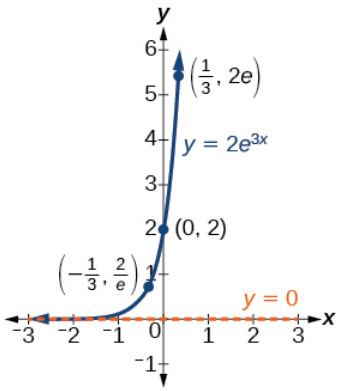


Figure 4.7.2: A graph showing exponential growth. The equation is $y = 2e^{3x}$.

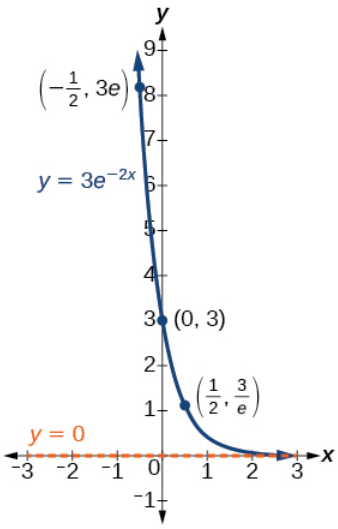


Figure 4.7.3: A graph showing exponential decay. The equation is $y = 3e^{-2x}$.

Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude. The **order of magnitude** is the power of ten, when the number is expressed in scientific notation, with one digit to the left of the decimal. For example, the distance to the nearest star, **Proxima Centauri**, measured in kilometers, is $40,113,497,200,000$kilometers. Expressed in scientific notation, this is 4.01134972×1013 So, we could describe this number as having order of magnitude 1013.

$y = a_0 e^{kt}$

An exponential function with the form $y = A_0 e^{kt}$ has the following characteristics:

- one-to-one function
- horizontal asymptote: $y = 0$
- domain: $(-\infty, \infty)$
- range: $(0, \infty)$
- x intercept: none
- y-intercept: $(0, A_0)$
- increasing if $k > 0$ (see Figure 4.7.4)
- decreasing if $k < 0$ (see Figure 4.7.4)

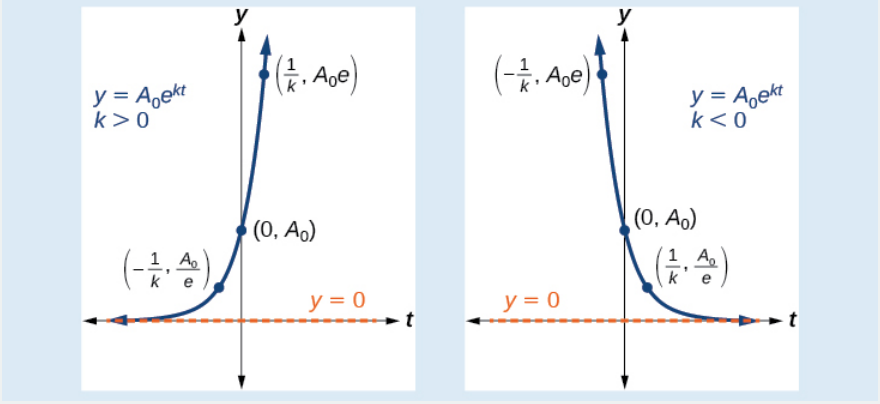


Figure 4.7.4: An exponential function models exponential growth when $k > 0$ and exponential decay when $k < 0$.

Using Newton's Law of Cooling

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless

the room temperature is zero, this will correspond to a **vertical shift** of the generic **exponential decay** function. This translation leads to **Newton's Law of Cooling**, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature

$$T(t) = ae^{kt} + T_s$$

This formula is derived as follows:

$$\begin{aligned} T(t) &= Ab^{ct} + T_s \\ T(t) &= Ae^{\ln(b^{ct})} + T_s \qquad \text{Laws of logarithms} \\ T(t) &= Ae^{ct\ln b} + T_s \qquad \text{Laws of logarithms} \\ T(t) &= Ae^{kt} + T_s \qquad \text{Rename the constant c } lnb\text{, calling it } k \end{aligned}$$

NEWTON'S LAW OF COOLING

The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$$T(t) = Ae^{kt} + T_s \tag{4.7.2}$$

where

- t is time
- A is the difference between the initial temperature of the object and the surroundings
- k is a constant, the continuous rate of cooling of the object

Given a set of conditions, apply Newton's Law of Cooling.

1. Set T_s equal to the y-coordinate of the horizontal asymptote (usually the ambient temperature).
2. Substitute the given values into the continuous growth formula $T(t) = Ae^{kt} + T_s$ to find the parameters A and k.
3. Substitute in the desired time to find the temperature or the desired temperature to find the time.

Using Logistic Growth Models

Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over 17 billion lines, or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually, an exponential model must begin to approach some limiting value, and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model, though the **exponential growth** model is still useful over a short term, before approaching the limiting value.

The **logistic growth model** is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound, called the **carrying capacity**. For constants a, b, and c, the logistic growth of a population over time x is represented by the model

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

The graph in Figure 4.7.6 shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate, at which point the rate of increase decreases.

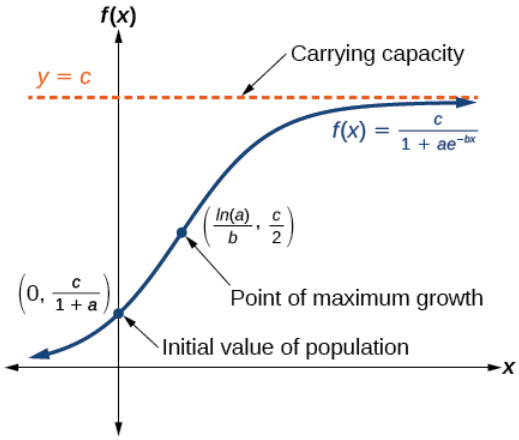


Figure 4.7.6

LOGISTIC GROWTH

The logistic growth model is

$$f(x) = \frac{c}{1 + ae^{-bx}} \tag{4.7.3}$$

where

- $\frac{c}{1+a}$ is the initial value
- c is the *carrying capacity*, or *limiting value*
- b is a constant determined by the rate of growth.

Example 4.7.6: Using the Logistic-Growth Model

An influenza epidemic spreads through a population rapidly, at a rate that depends on two factors: The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model a good one to study the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.

For example, at time $t = 0$ there is one person in a community of $1,000$ people who has the flu. So, in that community, at most $1,000$ people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is $b = 0.6030$. Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

Solution

We substitute the given data into the logistic growth model

$$f(x) = \frac{c}{1 + ae^{-bx}}$$

Because at most $1,000$ people, the entire population of the community, can get the flu, we know the limiting value is $c = 1000$. To find a,we use the formula that the number of cases at time $t = 0$ is $\frac{c}{1+a} = 1$, from which it follows that $a = 999$.This model predicts that, after ten days, the number of people who have had the flu is $f(x) = \frac{1000}{1 + 999e^{-0.6030x}} \approx 293.8$. Because the actual number must be a whole number (a person has either had the flu or not) we round to 294. In the long term, the number of people who will contract the flu is the limiting value, $c = 1000$.

Analysis

Remember that, because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values.

The graph in Figure 4.7.7 gives a good picture of how this model fits the data.

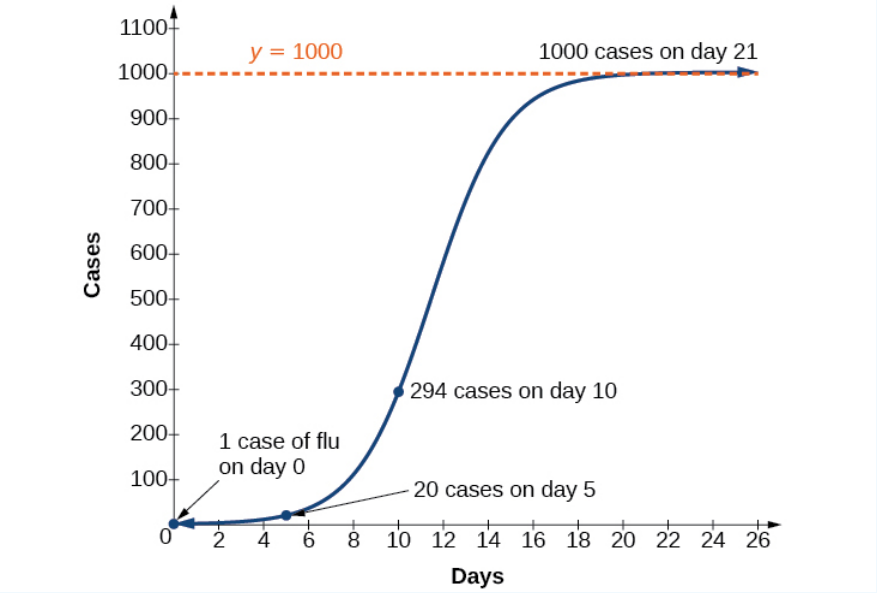


Figure 4.7.7: The graph of $f(x) = \dfrac{1000}{1 + 999e^{-0.6030x}}$

Expressing an Exponential Model in Base e

While powers and logarithms of any base can be used in modeling, the two most common bases are 10 and e. In science and mathematics, the base e is often preferred. We can use laws of exponents and laws of logarithms to change any base to base e.

Given a model with the form $y = ab^x$, change it to the form $y = A_0e^{kx}$.

1. Rewrite $y = ab^x$ as $y = ae^{\ln(b^x)}$.
2. Use the power rule of logarithms to rewrite y as $y = ae^{x\ln(b)} = ae^{\ln(b)^x}$.
3. Note that $a = A_0$ and $k = \ln(b)$ in the equation $y = A_0e^{kx}$.

Example 4.7.8: Changing to base e

Change the function $y = 2.5(3.1)^x$ so that this same function is written in the form $y = A_0e^{kx}$.

Solution

The formula is derived as follows

$$\begin{aligned} y &= 2.5(3.1)^x \\ &= 2.5e^{\ln(3.1^x)} && \text{Insert exponential and its inverse} \\ &= 2.5e^{x\ln 3.1} && \text{Laws of logs} \\ &= 2.5e^{(\ln 3.1)x} && \text{Commutative law of multiplication} \end{aligned}$$

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4.8: Fitting Exponential Models to Data

Building an Exponential Model from Data

As we've learned, there are a multitude of situations that can be modeled by exponential functions, such as investment growth, radioactive decay, atmospheric pressure changes, and temperatures of a cooling object. What do these phenomena have in common? For one thing, all the models either increase or decrease as time moves forward. But that's not the whole story. It's the *way* data increase or decrease that helps us determine whether it is best modeled by an exponential equation. Knowing the behavior of exponential functions in general allows us to recognize when to use exponential regression, so let's review exponential growth and decay.

Recall that exponential functions have the form $y = ab^x$ or $y = A_0e^{kx}$. When performing regression analysis, we use the form most commonly used on graphing utilities, $y = ab^x$. Take a moment to reflect on the characteristics we've already learned about the exponential function $y = ab^x$ (assume $a > 0$):

- b must be greater than zero and not equal to one.
- The initial value of the model is $y = a$.
  - If $b > 1$, the function models exponential growth. As x increases, the outputs of the model increase slowly at first, but then increase more and more rapidly, without bound.
  - If $0 < b < 1$, the function models **exponential decay**. As x increases, the outputs for the model decrease rapidly at first and then level off to become asymptotic to the *x*-axis. In other words, the outputs never become equal to or less than zero.

As part of the results, your calculator will display a number known as the *correlation coefficient*, labeled by the variable r, or r^2. (You may have to change the calculator's settings for these to be shown.) The values are an indication of the "goodness of fit" of the regression equation to the data. We more commonly use the value of r^2 instead of r, but the closer either value is to 1, the better the regression equation approximates the data.

EXPONENTIAL REGRESSION

Exponential regression is used to model situations in which growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero. We use the command "ExpReg" on a graphing utility to fit an exponential function to a set of data points. This returns an equation of the form,

$$y = ab^x \tag{4.8.1}$$

Note that:

- b must be non-negative.
- when $b > 1$, we have an exponential growth model.
- when $0 < b < 1$, we have an exponential decay model.

How to: Given a set of data, perform exponential regression using a graphing utility

1. Use the STAT then EDIT menu to enter given data.
   i. Clear any existing data from the lists.
   ii. List the input values in the L1 column.
   iii. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.
   i. Use ZOOM [9] to adjust axes to fit the data.
   ii. Verify the data follow an exponential pattern.
3. Find the equation that models the data.
   i. Select "ExpReg" from the STAT then CALC menu.
   ii. Use the values returned for *a* and *b* to record the model, $y = ab^x$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

Building a Logarithmic Model from Data

Just as with exponential functions, there are many real-world applications for logarithmic functions: intensity of sound, pH levels of solutions, yields of chemical reactions, production of goods, and growth of infants. As with exponential models, data modeled by logarithmic functions are either always increasing or always decreasing as time moves forward. Again, it is the *way* they increase or decrease that helps us determine whether a **logarithmic model** is best.

Recall that logarithmic functions increase or decrease rapidly at first, but then steadily slow as time moves on. By reflecting on the characteristics we've already learned about this function, we can better analyze real world situations that reflect this type of growth or decay. When performing logarithmic **regression analysis**, we use the form of the logarithmic function most commonly used on graphing utilities, $y = a + b\ln(x)$. For this function

- All input values, x,must be greater than zero.
- The point $(1, a)$ is on the graph of the model.
- If $b > 0$,the model is increasing. Growth increases rapidly at first and then steadily slows over time.
- If $b < 0$,the model is decreasing. Decay occurs rapidly at first and then steadily slows over time.

LOGARITHMIC REGRESSION

Logarithmic regression is used to model situations where growth or decay accelerates rapidly at first and then slows over time. We use the command "LnReg" on a graphing utility to fit a logarithmic function to a set of data points. This returns an equation of the form,

$$y = a + b\ln(x) \tag{4.8.2}$$

Note that

- all input values, x,must be non-negative.
- when $b > 0$, the model is increasing.
- when $b < 0$, the model is decreasing.

How to: Given a set of data, perform logarithmic regression using a graphing utility

1. Use the STAT then EDIT menu to enter given data.
   i. Clear any existing data from the lists.
   ii. List the input values in the L1 column.
   iii. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.
   i. Use ZOOM [9] to adjust axes to fit the data.
   ii. Verify the data follow a logarithmic pattern.
3. Find the equation that models the data.
   i. Select "LnReg" from the STAT then CALC menu.
   ii. Use the values returned for *a* and *b* to record the model, $y = a + b\ln(x)$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

Building a Logistic Model from Data

Like exponential and logarithmic growth, logistic growth increases over time. One of the most notable differences with logistic growth models is that, at a certain point, growth steadily slows and the function approaches an upper bound, or *limiting value*. Because of this, logistic regression is best for modeling phenomena where there are limits in expansion, such as availability of living space or nutrients.

It is worth pointing out that logistic functions actually model resource-limited exponential growth. There are many examples of this type of growth in real-world situations, including population growth and spread of disease, rumors, and even stains in fabric. When performing logistic regression analysis, we use the form most commonly used on graphing utilities:

$$y = \frac{c}{1 + ae^{-bx}}$$

Recall that:

- $\frac{c}{1+a}$ is the initial value of the model.
- when $b > 0$, the model increases rapidly at first until it reaches its point of maximum growth rate, $(\frac{\ln(a)}{b}, \frac{c}{2})$. At that point, growth steadily slows and the function becomes asymptotic to the upper bound $y = c$.
- c is the limiting value, sometimes called the *carrying capacity*, of the model.

LOGISTIC REGRESSION

Logistic regression is used to model situations where growth accelerates rapidly at first and then steadily slows to an upper limit. We use the command "Logistic" on a graphing utility to fit a logistic function to a set of data points. This returns an equation of the form

$$y = \frac{c}{1+ae^{-bx}} \tag{4.8.3}$$

Note that

- The initial value of the model is $\frac{c}{1+a}$.
- Output values for the model grow closer and closer to $y = c$ as time increases.

How to: Given a set of data, perform logistic regression using a graphing utility

1. Use the STAT then EDIT menu to enter given data.
   i. Clear any existing data from the lists.
   ii. List the input values in the L1 column.
   iii. List the output values in the L2 column.
2. Graph and observe a scatter plot of the data using the STATPLOT feature.
   i. Use ZOOM [9] to adjust axes to fit the data.
   ii. Verify the data follow a logistic pattern.
3. Find the equation that models the data.
   i. Select "Logistic" from the STAT then CALC menu.
   ii. Use the values returned for a, b, and c to record the model, $y = \frac{c}{1+ae^{-bx}}$.
4. Graph the model in the same window as the scatterplot to verify it is a good fit for the data.

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CHAPTER OVERVIEW

5: Trigonometric Functions

The trigonometric functions are functions of an angle. and relate the angles of a triangle to the lengths of its sides. They are important in the study of triangles and modeling periodic phenomena, among many other applications.

5.1: Angles

5.2: Unit Circle - Sine and Cosine Functions

5.3: The Other Trigonometric Functions

5.4: Right Triangle Trigonometry

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5.1: Angles

Drawing Angles in Standard Position

Properly defining an angle first requires that we define a *ray*. A ray consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its *endpoint* and any other point on it. The ray in Figure 5.1.1 can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

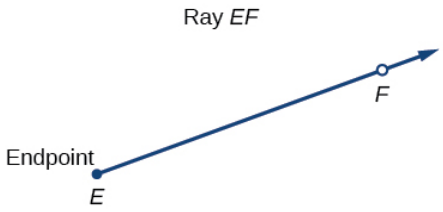


Figure 5.1.1

An *angle* is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in Figure 5.1.2 is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle *DEF*, or in symbol form $\angle DEF$.

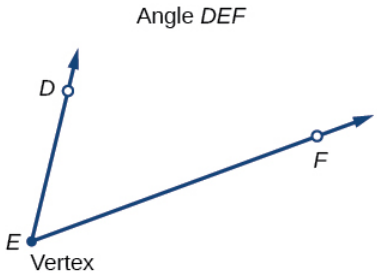


Figure 5.1.2

Greek letters are often used as variables for the measure of an angle. Table 5.1.1 is a list of Greek letters commonly used to represent angles, and a sample angle is shown in Figure 5.1.3.

Table 5.1.1

θ	φ or ϕ	α	β	γ
theta	phi	alpha	beta	gamma

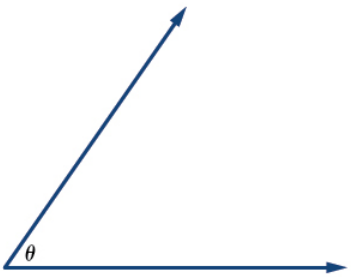


***Figure* 5.1.3**: *Angle theta, shown as* $\angle\theta$

Angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure 5.1.4.

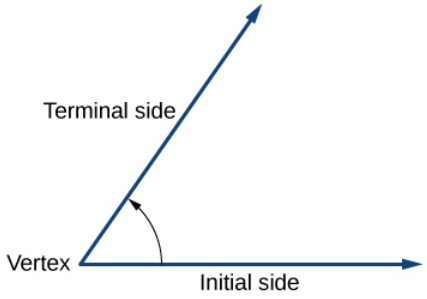


Figure 5.1.4

As we discussed at the beginning of the section, there are many applications for angles, but in order to use them correctly, we must be able to measure them. The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. One **degree** is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol °. For example, 90 degrees = 90°.

To formalize our work, we will begin by drawing angles on an *x*-*y* coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive *x*-axis. See Figure 5.1.5.

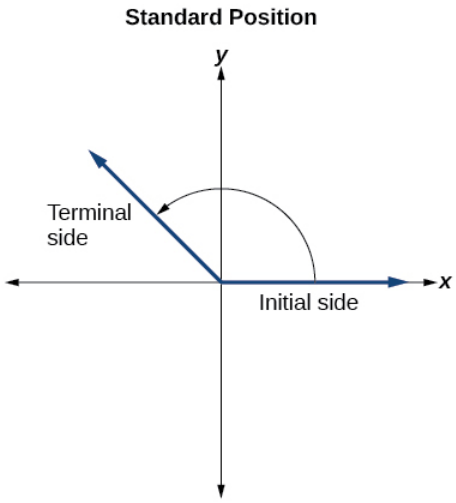


Figure 5.1.5

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Drawing an angle in standard position always starts the same way—draw the initial side along the positive *x*-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90^\circ}{360^\circ} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive *x*-axis. To draw a 360° angle, we calculate that $\frac{360^\circ}{360^\circ} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive *x*-axis. In this case, the initial side and the terminal side overlap. See Figure 5.1.6.

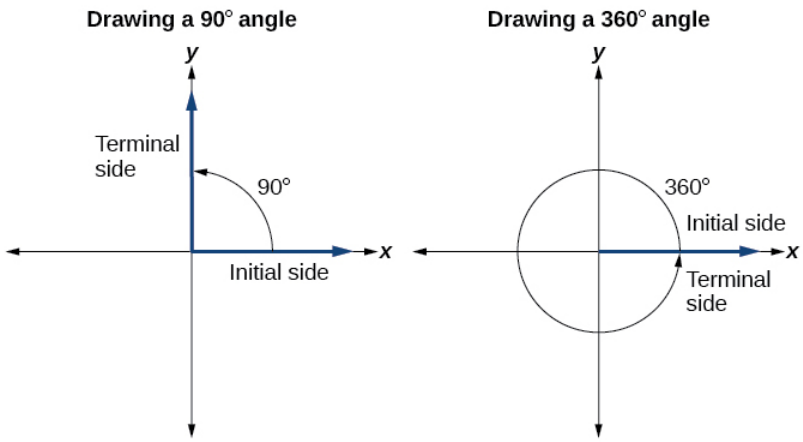


Figure 5.1.6

Since we define an angle in **standard position** by its initial side, we have a special type of angle whose terminal side lies on an axis, a **quadrantal angle**. This type of angle can have a measure of 0°, 90°, 180°, 270° or 360°. See Figure 5.1.7.

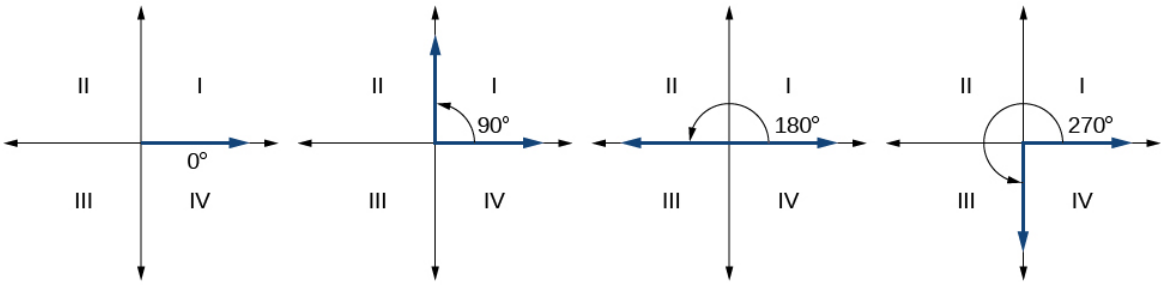


Figure 5.1.7: Quadrantal angles are angles in standard position whose terminal side lies along an axis. Examples are shown.

QUADRANTAL ANGLES

Quadrantal angles are angles in standard position whose terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.

Given an angle measure in degrees, draw the angle in standard position

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive *x*-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 5.1.1: Drawing an Angle in Standard Position Measured in Degrees

1. Sketch an angle of 30° in standard position.
2. Sketch an angle of −135° in standard position.

Solution

1. Divide the angle measure by 360°.

$$\frac{30^\circ}{360^\circ} = \frac{1}{12} \tag{5.1.1}$$

To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}\left(\frac{1}{4}\right) \tag{5.1.2}$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in Figure 5.1.8.

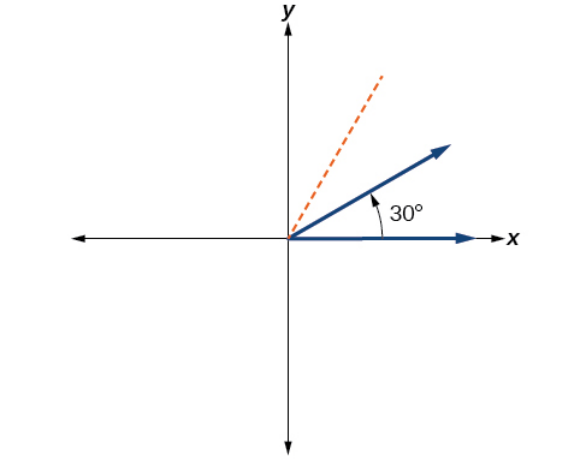


***Figure* 5.1.8**

2. Divide the angle measure by 360°.

$$\frac{-135^\circ}{360^\circ} = -\frac{3}{8} \tag{5.1.3}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}\left(\frac{1}{4}\right) \tag{5.1.4}$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in Figure 5.1.9.

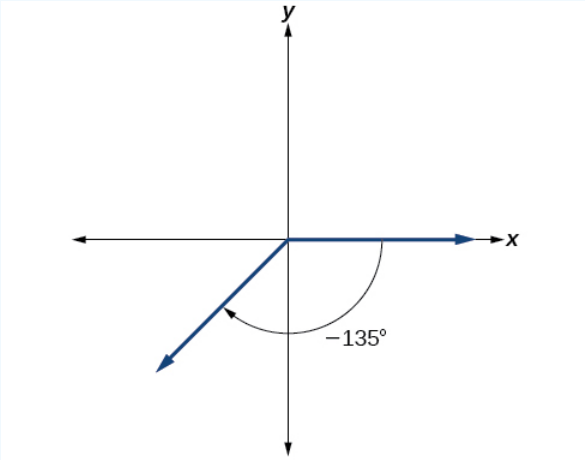


***Figure* 5.1.9**

? 5.1.1:

Show an angle of 240° on a circle in standard position.

Solution

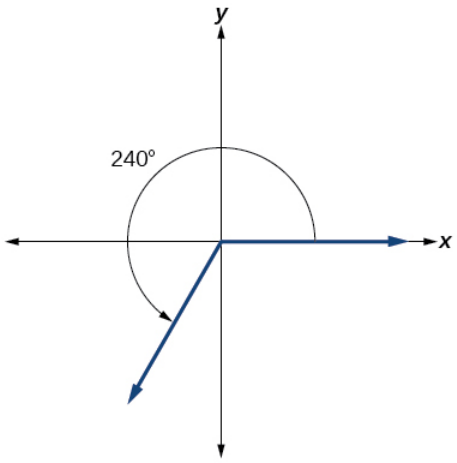


Figure 5.1.10

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An **arc** may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the **circumference** of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in Figure 5.1.11.

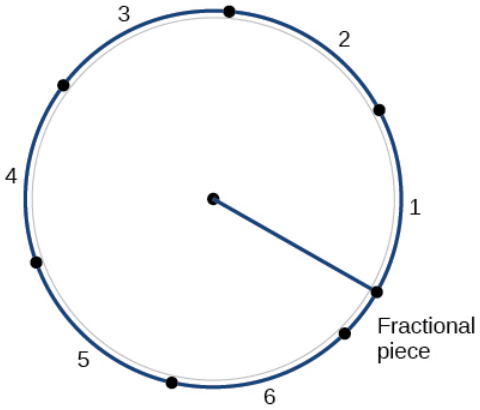


Figure 5.1.11

This brings us to our new angle measure. One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians } = 360^\circ \tag{5.1.5}$$

$$\pi \text{ radians } = \frac{360^\circ}{2} = 180^\circ \tag{5.1.6}$$

$$1 \text{ radian } = \frac{180^\circ}{\pi} \approx 57.3^\circ \tag{5.1.7}$$

See Figure 5.1.12. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

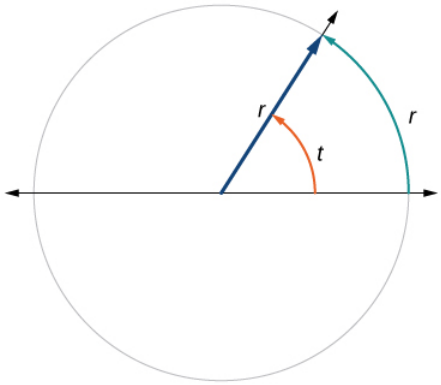


Figure 5.1.12: The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the **radian measure**, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s s to the radius r. See Figure 5.1.13.

$$s = r\theta \tag{5.1.8}$$

$$\theta = \frac{s}{r} \tag{5.1.9}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$ radian.

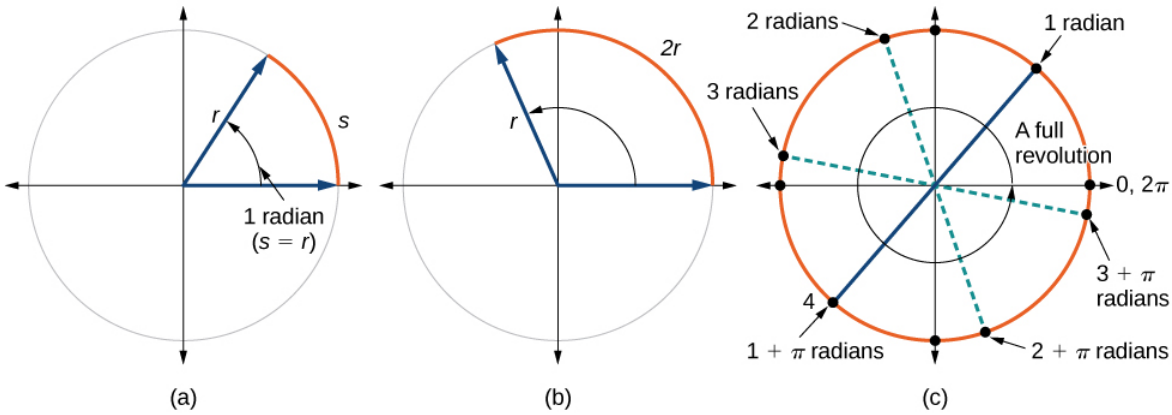


Figure 5.1.13: (a) In an angle of 1 radian, the arc length s equals the radius r. (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$,where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45° angle on the two circles, as in Figure 5.1.14.

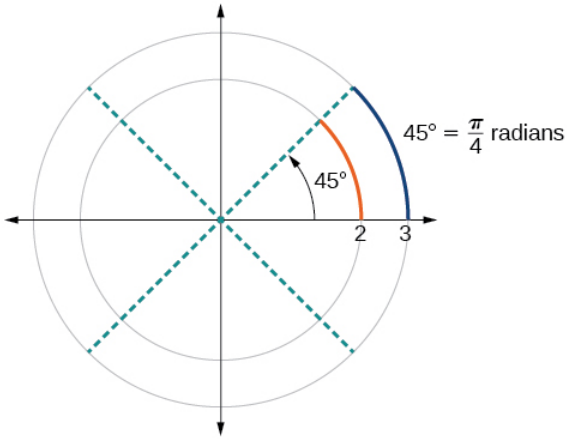


Figure 5.1.14: A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle: } \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi \tag{5.1.10}$$

$$\text{Larger circle: } \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi \tag{5.1.11}$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

RADIANS

One **radian** is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.

The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

Using Radians

Because **radian** measure is the ratio of two lengths, it is a unitless measure. For example, in Figure 5.1.14, suppose the radius were 2 inches and the distance along the arc were also 2 inches. When we calculate the radian measure of the angle, the “inches” cancel, and we have a result without units. Therefore, it is not necessary to write the label “radians” after a radian measure, and if we see an angle that is not labeled with “degrees” or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the **unit circle** (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$\begin{aligned} 1\text{rotation} &= 360^\circ &&= 2\pi \quad \text{radians} \\ \frac{1}{2}\text{rotation} &= 180^\circ &&= \pi \quad \text{radians} \\ \frac{1}{4}\text{rotation} &= 90^\circ &&= \frac{\pi}{2} \quad \text{radians} \end{aligned} \tag{5.1.12}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in Figure 5.1.15. Memorizing these angles will be very useful as we study the properties associated with angles.

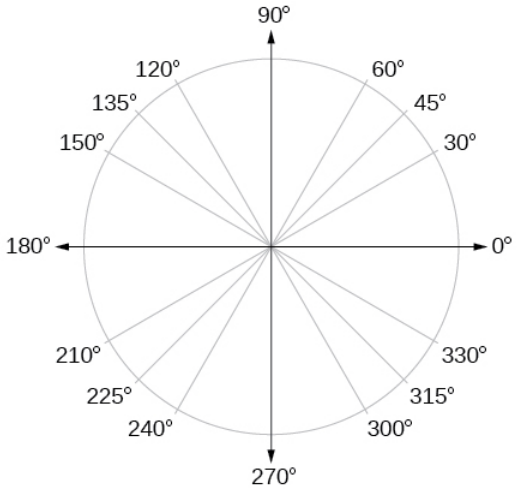


Figure 5.1.15: Commonly encountered angles measured in degrees

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in Figure 5.1.15, which are shown in Figure 5.1.16. Be sure you can verify each of these measures.

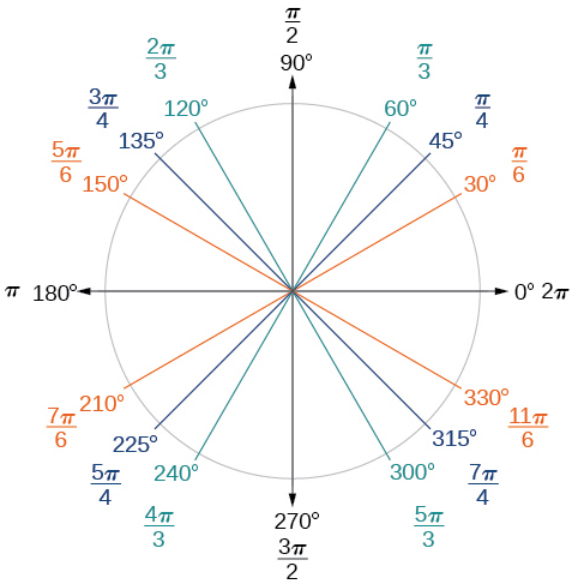


Figure 5.1.16: Commonly encountered angles measured in radians

Converting between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion.

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.13}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle θ in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{Radians}{\pi} \tag{5.1.14}$$

CONVERTING BETWEEN RADIANS AND DEGREES

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{\theta^R}{\pi} \tag{5.1.15}$$

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0° to 360°, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at Figure 5.1.17. The angle of 140° is a **positive angle**, measured counterclockwise. The angle of –220° is a **negative angle**, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are **coterminal** angles. Every angle greater than 360° or less than 0° is coterminal with an angle between 0° and 360°, and it is often more convenient to find the coterminal angle within the range of 0° to 360° than to work with an angle that is outside that range.

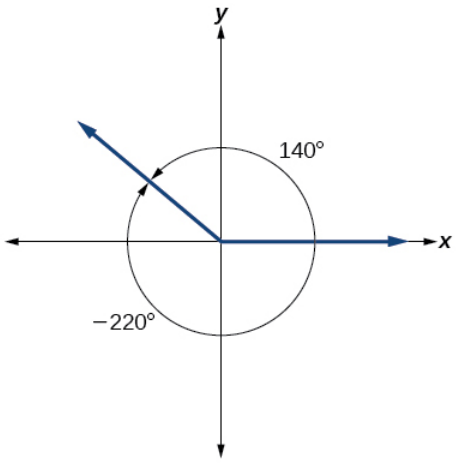


Figure 5.1.17: An angle of 140° and an angle of –220° are coterminal angles.

Any angle has infinitely many **coterminal angles** because each time we add 360° to that angle—or subtract 360° from it—the resulting value has a terminal side in the same location. For example, 100° and 460° are coterminal for this reason, as is −260°. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See Figure 5.1.18 for examples of reference angles for angles in different quadrants.

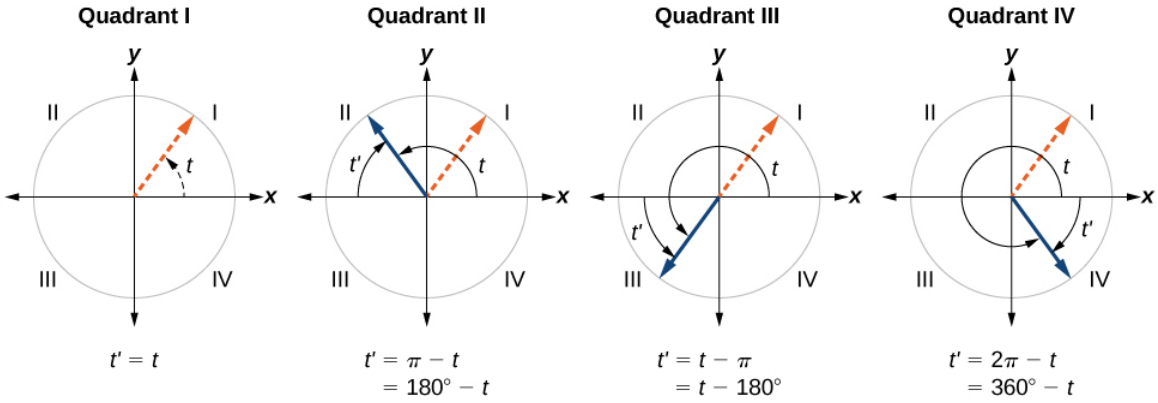


Figure 5.1.18

COTERMINAL AND REFERENCE ANGLES

- *Coterminal angles* are two angles in standard position that have the same terminal side.
- An angle's *reference angle* is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

Given an angle greater than 360°, find a coterminal angle between 0° and 360°

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Finding Coterminal Angles Measured in Radians

We can find **coterminal** angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Determining the Length of an Arc

Recall that the **radian measure** θ of an angle was defined as the ratio of the **arc length** s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

ARC LENGTH ON A CIRCLE

In a circle of radius *r*, the length of an arc s subtended by an angle with measure θ in radians, shown in Figure 5.1.22, is

$$s = r\theta \tag{5.1.16}$$

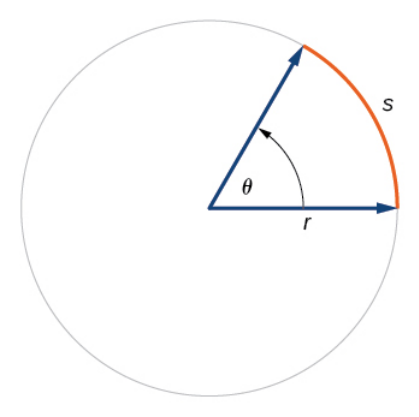


Figure 5.1.22

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of θ : $s = r\theta$.

Example 5.1.8: Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Solution

1. Let's begin by finding the circumference of Mercury's orbit.

$$C = 2\pi r \tag{5.1.17}$$
$$= 2\pi(36 \text{ million miles}) \tag{5.1.18}$$
$$\approx 226 \text{ million miles} \tag{5.1.19}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled:

$$(0.0114)226 \text{ million miles} = 2.58 \text{ million miles} \tag{5.1.20}$$

2. Now, we convert to radians:

$$\text{radian} = \frac{\text{arc length}}{\text{radius}} \tag{5.1.21}$$
$$= \frac{2.58 \text{ million miles}}{36 \text{ million miles}} \tag{5.1.22}$$
$$= 0.0717 \tag{5.1.23}$$

Finding the Area of a Sector of a Circle

In addition to arc length, we can also use angles to find the area of a **sector of a circle**. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A = \pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\text{Area of sector} = (\frac{\theta}{2\pi})\pi r^2 \tag{5.1.24}$$
$$= \frac{\theta \pi r^2}{2\pi} \tag{5.1.25}$$
$$= \frac{1}{2}\theta r^2 \tag{5.1.26}$$

AREA OF A SECTOR

The area of a **sector of a circle** with radius r subtended by an angle θ, measured in radians, is

$$A = \frac{1}{2}\theta r^2 \tag{5.1.27}$$

See Figure 5.1.23.

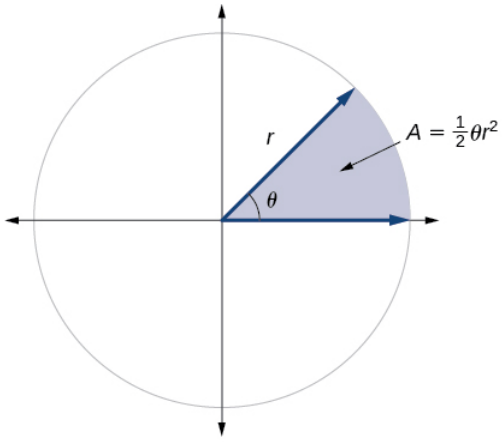


Figure 5.1.23: The area of the sector equals half the square of the radius times the central angle measured in radians.

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r : $A = \frac{1}{2}\theta r^2$.

Use Linear and Angular Speed to Describe Motion on a Circular Path

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its **displacement**) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10π inches, every second. So the linear speed of the point is 10π in./s. The equation for linear speed is as follows where v is linear speed, s is displacement, and t is time.

$$v = \frac{s}{t} \tag{5.1.28}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, angular speed is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}} = 90$ degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega = \frac{\theta}{t} \tag{5.1.29}$$

Combining the definition of angular speed with the arc length equation, $s = r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta = \omega t$. Substituting this into the arc length equation gives:

$$s = r\theta \tag{5.1.30}$$
$$= r\omega t \tag{5.1.31}$$

Substituting this into the linear speed equation gives:

$$v = \frac{s}{t} \qquad = \frac{r\omega t}{t} = r\omega \tag{5.1.32}$$

ANGULAR AND LINEAR SPEED

As a point moves along a circle of radius r, its **angular speed**, ω, is the angular rotation θ per unit time, t.

$$\omega = \frac{\theta}{t} \tag{5.1.33}$$

The **linear speed.** v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$$v = \frac{s}{t} \tag{5.1.34}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v = r\omega \tag{5.1.35}$$

This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

Given the amount of angle rotation and the time elapsed, calculate the angular speed

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

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5.2: Unit Circle - Sine and Cosine Functions

Finding Function Values for the Sine and Cosine

To define our trigonometric functions, we begin by drawing a unit circle, a circle centered at the origin with radius 1, as shown in Figure 5.2.2. The angle (in radians) that t intercepts forms an arc of length s. Using the formula $s = rt$, and knowing that $r = 1$, we see that for a **unit circle**, $s = t$.

Recall that the *x*- and *y*-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For any angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y). The coordinates x and y will be the outputs of the trigonometric functions $f(t) = \cos t$ and $f(t) = \sin t$, respectively. This means $x = \cos t$ and $y = \sin t$.

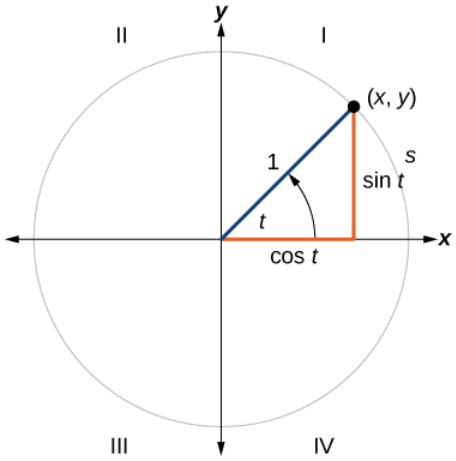


Figure 5.2.2: Unit circle where the central angle is t radians

UNIT CIRCLE

A unit circle has a center at $(0, 0)$ and radius 1. The length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x, y) be the endpoint on the unit circle of an arc of arc length s. The (x, y) coordinates of this point can be described as functions of the angle.

Finding Sines and Cosines of Angles on an Axis

For quadrantral angles, the corresponding point on the unit circle falls on the x- or y-axis. In that case, we can easily calculate cosine and sine from the values of x and y.

Example 5.2.2: Calculating Sines and Cosines along an Axis

Find $\cos(90^\circ)$ and $\sin(90^\circ)$.

Solution

Moving 90° counterclockwise around the unit circle from the positive x-axis brings us to the top of the circle, where the (x, y) coordinates are (0, 1), as shown in Figure 5.2.6.

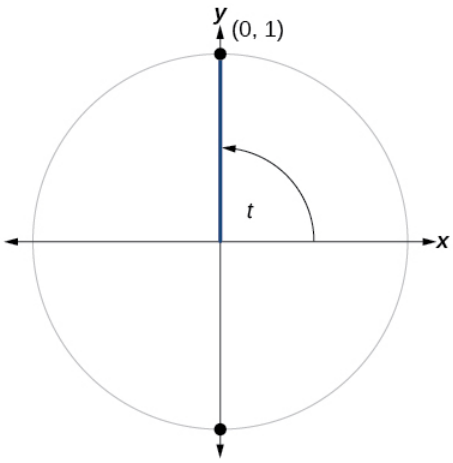


Figure 5.2.6

Using our definitions of cosine and sine,

$$x = \cos t = \cos(90^\circ) = 0$$
$$y = \sin t = \sin(90^\circ) = 1$$

The cosine of 90° is 0; the sine of 90° is 1.

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos t$ and $y = \sin t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See Figure 5.2.7.

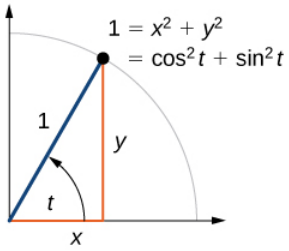


Figure 5.2.7

We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1 \tag{5.2.1}$$

How To: Given the sine of some angle t and its quadrant location, find the cosine of t

1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

Finding Sines and Cosines of Special Angles

We have already learned some properties of the special angles, such as the conversion from radians to degrees. We can also calculate sines and cosines of the special angles using the **Pythagorean Identity** and our knowledge of triangles.

Finding Sines and Cosines of 45° Angles

First, we will look at angles of 45° or $\frac{\pi}{4}$, as shown in Figure 5.2.9. A $45^\circ - 45^\circ - 90^\circ$ triangle is an isosceles triangle, so the x- and y-coordinates of the corresponding point on the circle are the same. Because the x- and y-values are the same, the sine and cosine values will also be equal.

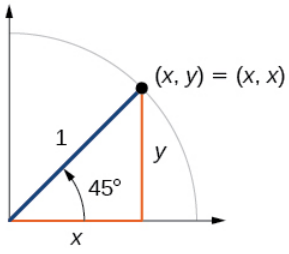


Figure 5.2.9

At $t = \frac{\pi}{4}$, which is 45 degrees, the radius of the unit circle bisects the first **quadrantal angle**. This means the radius lies along the line $y = x$. A unit circle has a radius equal to 1. So, the right triangle formed below the line $y = x$ has sides x and y (with $y = x$), and a radius = 1. See Figure 5.2.10.

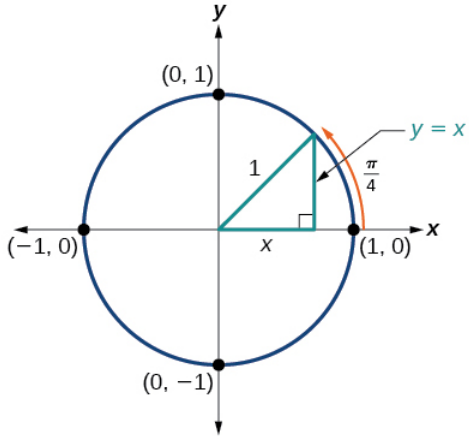


Figure 5.2.10

From the Pythagorean Theorem we get

$$x^2 + y^2 = 1$$

Substituting $y = x$, we get

$$x^2 + x^2 = 1$$

Combining like terms we get

$$2x^2 = 1$$

And solving for x, we get

$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$

In quadrant I, $x = \dfrac{1}{\sqrt{2}}$.

At $t = \frac{\pi}{4}$ or 45 degrees,

$$(x, y) = (x, x) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}}$$
$$\cos t = \frac{1}{\sqrt{2}},\ \sin t = \frac{1}{\sqrt{2}}$$

If we then rationalize the denominators, we get

$$\cos t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\sin t = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

Therefore, the (x, y) coordinates of a point on a circle of radius 1 at an angle of 45° are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

Finding Sines and Cosines of 30° and 60° Angles

Next, we will find the cosine and sine at an angle of 30°, or $\frac{\pi}{6}$. First, we will draw a triangle inside a circle with one side at an angle of 30°, and another at an angle of -30°, as shown in Figure 5.2.11. If the resulting two right triangles are combined into one large triangle, notice that all three angles of this larger triangle will be 60°, as shown in Figure 5.2.12

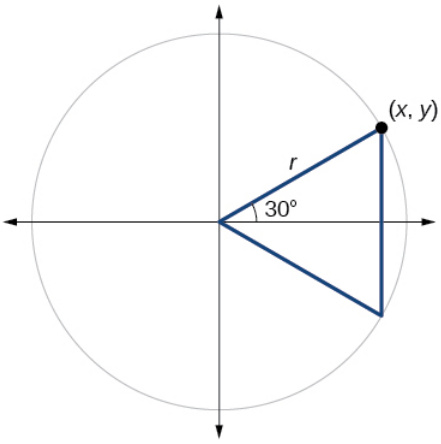


Figure 5.2.11

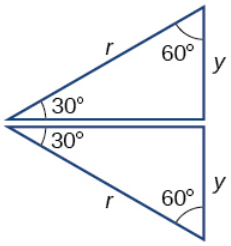


Figure 5.2.12

Because all the angles are equal, the sides are also equal. The vertical line has length $2y$, and since the sides are all equal, we can also conclude that $r = 2y$ or $y = \frac{1}{2}r$. Since $\sin t = y$,

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

And since $r = 1$ in our **unit circle**,

$$\begin{aligned}\sin\left(\frac{\pi}{6}\right) &= \frac{1}{2}(1)\\ &= \frac{1}{2}\end{aligned}$$

Using the Pythagorean Identity, we can find the cosine value.

$$\begin{aligned}\cos^2\left(\frac{\pi}{6}\right) + \sin^2\left(\frac{\pi}{6}\right) &= 1\\ \cos^2\left(\frac{\pi}{6}\right) + \left(\frac{1}{2}\right)^2 &= 1\\ \cos^2\left(\frac{\pi}{6}\right) &= \frac{3}{4} && \text{Use the square root property.}\\ \cos\left(\frac{\pi}{6}\right) &= \pm\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} && \text{Since } y \text{ is positive, choose the positive root.}\end{aligned}$$

The (x, y) coordinates for the point on a circle of radius 1 at an angle of 30° are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. At $t = \frac{\pi}{3}$ (60°), the radius of the unit circle, 1, serves as the hypotenuse of a 30-60-90 degree right triangle, BAD, as shown in Figure 5.2.13. Angle A has measure 60°. 60°. At point B, we draw an angle ABC with measure of 60°. We know the angles in a triangle sum to 180°, so the measure of angle C is also 60°. Now we have an equilateral triangle. Because each side of the equilateral triangle ABC is the same length, and we know one side is the radius of the unit circle, all sides must be of length 1.

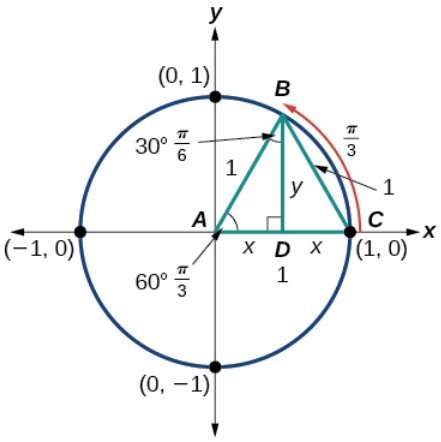


Figure 5.2.13

The measure of angle ABD is 30°. So, if double, angle ABC is 60°. BD is the perpendicular bisector of AC, so it cuts AC in half. This means that AD is $\frac{1}{2}$ the radius, or $\frac{1}{2}$. Notice that AD is the x-coordinate of point B, which is at the intersection of the 60° angle and the unit circle. This gives us a triangle BAD with hypotenuse of 1 and side x of length $\frac{1}{2}$.

From the Pythagorean Theorem, we get

$$x^2 + y^2 = 1$$

Substituting $x = \frac{1}{2}$, we get

$$\left(\frac{1}{2}\right)^2 + y^2 = 1$$

Solving for y, we get

$$\begin{aligned}\frac{1}{4}+y^2 &= 1\\ y^2 &= 1-\frac{1}{4}\\ y^2 &= \frac{3}{4}\\ y &= \pm\frac{\sqrt{3}}{2}\end{aligned}$$

Since $t=\frac{\pi}{3}$ has the terminal side in quadrant I where the y-coordinate is positive, we choose $y=\frac{\sqrt{3}}{2}$, the positive value.

At $t=\frac{\pi}{3}$ (60°), the (x,y) coordinates for the point on a circle of radius 1 at an angle of 60° are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, so we can find the sine and cosine.

$$(x,y)=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$$

$$x=\frac{1}{2},\ y=\frac{\sqrt{3}}{2}$$

$$\cos t=\frac{1}{2},\ \sin t=\frac{\sqrt{3}}{2}$$

We have now found the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle. Table 5.2.1 summarizes these values.

Table 5.2.1

Angle	0	$\frac{\pi}{6}$, or 30	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Figure 5.2.14 shows the common angles in the first quadrant of the unit circle.

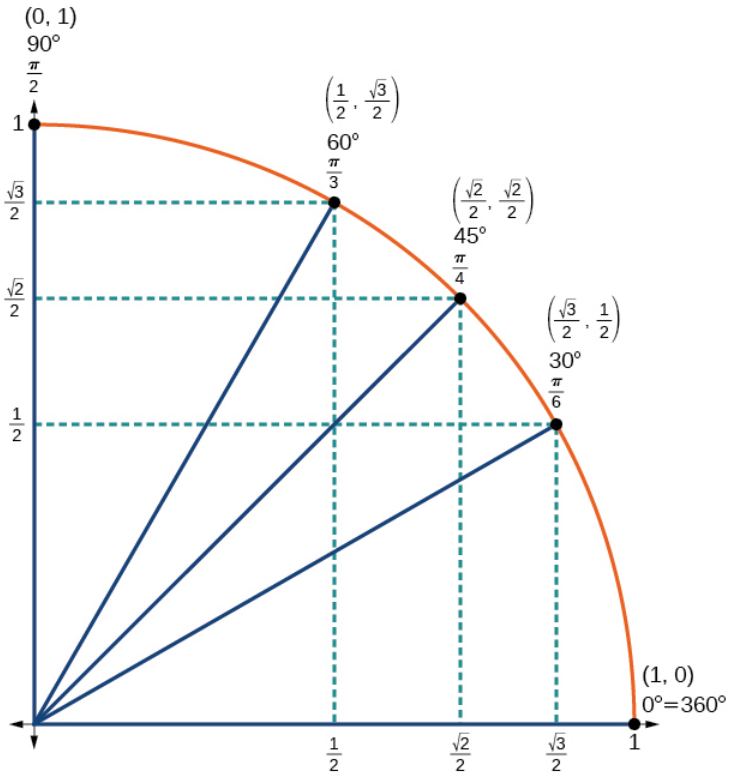


Figure 5.2.14

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the **special angles**, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate $\cos(30)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

How To: Given an angle in radians, use a graphing calculator to find the cosine

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Identifying the Domain and Range of Sine and Cosine Functions

Now that we can find the sine and cosine of an angle, we need to discuss their domains and ranges. What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of x, y, and r, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The input to the sine and cosine functions is the rotation from the positive x-axis, and that may be any real number.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the **unit circle**, as shown in Figure 5.2.15. The bounds of the x-coordinate are $[-1, 1]$. The bounds of the y-coordinate are also $[-1, 1]$. Therefore, the range of both the sine and cosine functions is $[-1, 1]$.

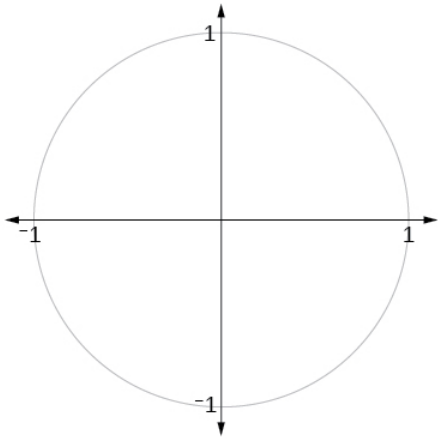


Figure 5.2.15

Finding Reference Angles

We have discussed finding the sine and cosine for angles in the first quadrant, but what if our angle is in another quadrant? For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the y-coordinate on the unit circle, the other angle with the same sine will share the same y-value, but have the opposite x-value. Therefore, its cosine value will be the opposite of the first angle's cosine value.

Likewise, there will be an angle in the fourth quadrant with the same cosine as the original angle. The angle with the same cosine will share the same x-value but will have the opposite y-value. Therefore, its sine value will be the opposite of the original angle's sine value.

As shown in Figure 5.2.16, angle α has the same sine value as angle t; the cosine values are opposites. Angle β has the same cosine value as angle t; the sine values are opposites.

$$\sin(t) = \sin(\alpha) \quad \text{and} \quad \cos(t) = -\cos(\alpha)$$
$$\sin(t) = -\sin(\beta) \quad \text{and} \quad \cos(t) = \cos(\beta)$$

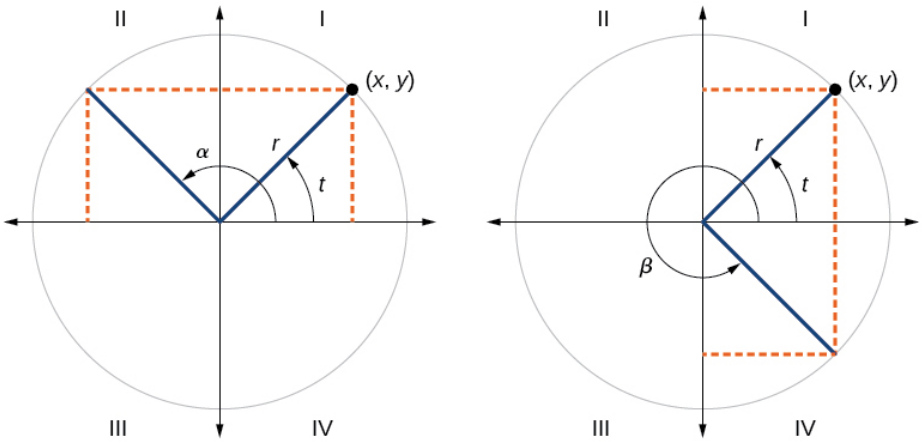


Figure 5.2.16

Recall that an angle's reference angle is the acute angle, t, formed by the terminal side of the angle t and the horizontal axis. A reference angle is always an angle between 0 and 90°, or 0 and $\frac{\pi}{2}$ radians. As we can see from Figure 5.2.17, for any angle in quadrants II, III, or IV, there is a reference angle in quadrant I.

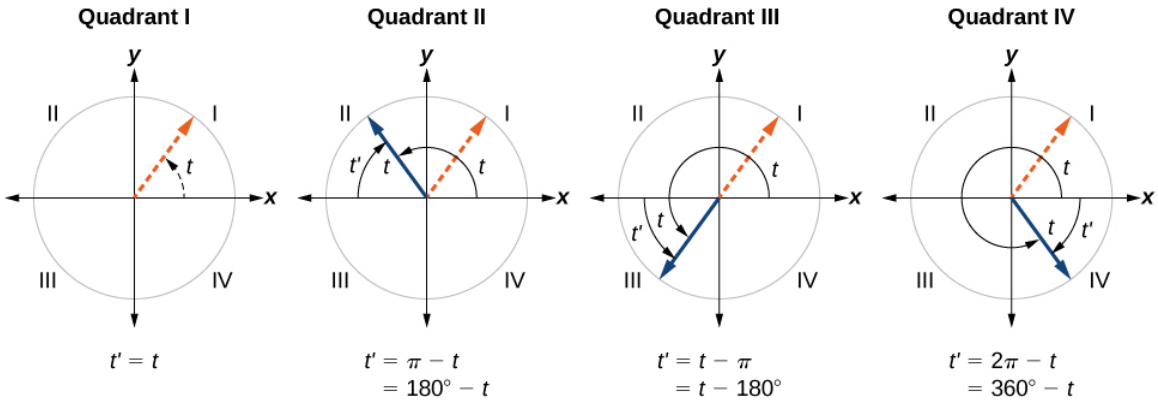


Figure 5.2.17

How To: Given an angle between 0 and 2π, find its reference angle

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180^\circ - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360^\circ - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Using Reference Angles

Now let's take a moment to reconsider the Ferris wheel introduced at the beginning of this section. Suppose a rider snaps a photograph while stopped twenty feet above ground level. The rider then rotates three-quarters of the way around the circle. What is the rider's new elevation? To answer questions such as this one, we need to evaluate the sine or cosine functions at angles that are greater than 90 degrees or at a **negative angle**. Reference angles make it possible to evaluate trigonometric functions for angles outside the first quadrant. They can also be used to find (x, y) coordinates for those angles. We will use the **reference angle** of the angle of rotation combined with the quadrant in which the terminal side of the angle lies.

Using Reference Angles to Evaluate Trigonometric Functions

We can find the cosine and sine of any angle in any quadrant if we know the cosine or sine of its reference angle. The absolute values of the cosine and sine of an angle are the same as those of the reference angle. The sign depends on the quadrant of the original angle. The cosine will be positive or negative depending on the sign of the x-values in that quadrant. The sine will be positive or negative depending on the sign of the y-values in that quadrant.

USING REFERENCE ANGLES TO FIND COSINE AND SINE

Angles have cosines and sines with the same absolute value as cosines and sines of their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To: Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Using Reference Angles to Find Coordinates

Now that we have learned how to find the cosine and sine values for special angles in the first quadrant, we can use symmetry and reference angles to fill in cosine and sine values for the rest of the special angles on the unit circle. They are shown in Figure 5.2.19. Take time to learn the (x, y) coordinates of all of the major angles in the first quadrant.

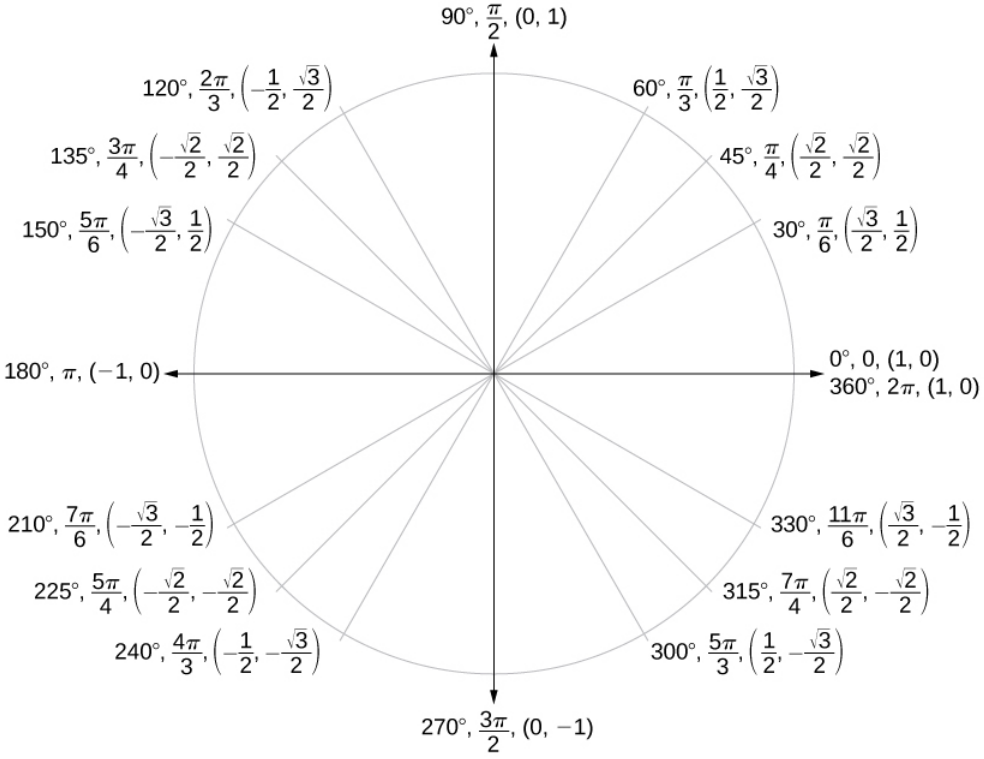


Figure 5.2.19: Special angles and coordinates of corresponding points on the unit circle

In addition to learning the values for special angles, we can use reference angles to find (x, y) coordinates of any point on the unit circle, using what we know of reference angles along with the **identities**

$$x = \cos t$$
$$y = \sin t$$

First we find the reference angle corresponding to the given angle. Then we take the sine and cosine values of the **reference angle**, and give them the signs corresponding to the y- and x-values of the quadrant.

How To: Given the angle of a point on a circle and the radius of the circle, find the (x, y) coordinates of the point

1. Find the reference angle by measuring the smallest angle to the x-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

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5.3: The Other Trigonometric Functions

Finding Exact Values of the Trigonometric Functions Secant, Cosecant, Tangent, and Cotangent

To define the remaining functions, we will once again draw a unit circle with a point (x, y) corresponding to an angle of t,as shown in Figure 5.3.1. As with the sine and cosine, we can use the (x, y) coordinates to find the other functions.

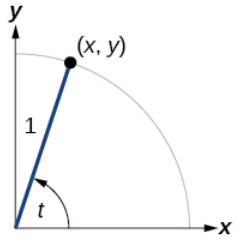


Figure 5.3.1

The first function we will define is the tangent. The **tangent** of an angle is the ratio of the *y*-value to the *x*-value of the corresponding point on the unit circle. In Figure 5.3.1, the tangent of angle t is equal to $\frac{y}{x}, x \neq 0$. Because the *y*-value is equal to the sine of t, and the *x*-value is equal to the cosine of t, the tangent of angle t can also be defined as $\frac{\sin t}{\cos t}, \cos t \neq 0$. The tangent function is abbreviated as tan. The remaining three functions can all be expressed as reciprocals of functions we have already defined.

- The **secant** function is the reciprocal of the cosine function. In Figure 5.3.1, the secant of angle t is equal to $\frac{1}{\cos t} = \frac{1}{x}, x \neq 0$. The secant function is abbreviated as sec.
- The **cotangent** function is the reciprocal of the tangent function. In Figure 5.3.1, the cotangent of angle t is equal to $\frac{\cos t}{\sin t} = \frac{x}{y}, y \neq 0$. The cotangent function is abbreviated as cot.
- The **cosecant** function is the reciprocal of the sine function. In Figure 5.3.1, the cosecant of angle t is equal to $\frac{1}{\sin t} = \frac{1}{y}, y \neq 0$. The cosecant function is abbreviated as csc.

TANGENT, SECANT, COSECANT, AND COTANGENT FUNCTIONS

If t is a real number and (x, y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

$$\tan t = \frac{y}{x}, x \neq 0 \tag{5.3.1}$$

$$\sec t = \frac{1}{x}, x \neq 0 \tag{5.3.2}$$

$$\csc t = \frac{1}{y}, y \neq 0 \tag{5.3.3}$$

$$\cot t = \frac{x}{y}, y \neq 0 \tag{5.3.4}$$

Example 5.3.1: Finding Trigonometric Functions from a Point on the Unit Circle

The point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the unit circle, as shown in Figure 5.3.2. Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.

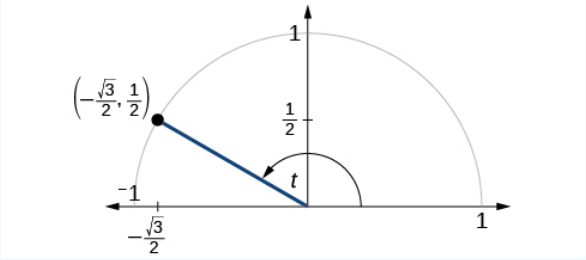


Figure 5.3.2

Solution

Because we know the (x, y) coordinates of the point on the unit circle indicated by angle t, we can use those coordinates to find the six functions:

$$\sin t = y = \frac{1}{2}$$
$$\cos t = x = -\frac{\sqrt{3}}{2}$$
$$\tan t = \frac{y}{x} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{2}\left(-\frac{2}{\sqrt{3}}\right) = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$
$$\sec t = \frac{1}{x} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$
$$\csc t = \frac{1}{y} = \frac{1}{\frac{1}{2}} = 2$$
$$\cot t = \frac{x}{y} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right) = -\sqrt{3}$$

? Exercise 5.3.1:

The point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is on the unit circle, as shown in Figure 5.3.3. Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$.

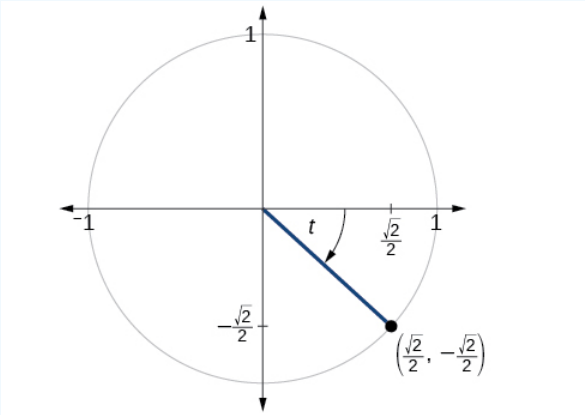


Figure 5.3.3

Solution

$\sin t = -\frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = -1, \sec t = \sqrt{2}, \csc t = -\sqrt{2}, \cot t = -1$

✓ Example 5.3.2: Finding the Trigonometric Functions of an Angle

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{6}$.

Solution

We have previously used the properties of equilateral triangles to demonstrate that $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$. We can use these values and the definitions of tangent, secant, cosecant, and cotangent as functions of sine and cosine to find the remaining function values.

$$\tan\frac{\pi}{6} = \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\sec\frac{\pi}{6} = \frac{1}{\cos\frac{\pi}{6}} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\csc\frac{\pi}{6} = \frac{1}{\sin\frac{\pi}{6}} = \frac{1}{\frac{1}{2}} = 2$$

$$\cot\frac{\pi}{6} = \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

? Exercise 5.3.2:

Find $\sin t, \cos t, \tan t, \sec t, \csc t$, and $\cot t$ when $t = \frac{\pi}{3}$.

Solution

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \tag{5.3.5}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} \tag{5.3.6}$$

$$\tan\frac{\pi}{3} = \sqrt{3} \tag{5.3.7}$$

$$\sec\frac{\pi}{3} = 2 \tag{5.3.8}$$

$$\csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3} \tag{5.3.9}$$

$$\cot\frac{\pi}{3} = \frac{\sqrt{3}}{3} \tag{5.3.10}$$

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well by setting x x equal to the cosine and y y equal to the sine and then using the definitions of tangent, secant, cosecant, and cotangent. The results are shown in Table 5.3.1.

Table 5.3.1

Angle	**0**	$\frac{\pi}{6}$, or **30**$^\circ$	$\frac{\pi}{4}$, or **45**$^\circ$	$\frac{\pi}{3}$, or **60**$^\circ$	$\frac{\pi}{2}$, or **90**$^\circ$
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Using Reference Angles to Evaluate Tangent, Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions. The procedure is the same: Find the **reference angle** formed by the terminal side of the given angle with the horizontal axis. The trigonometric function values for the original angle will be the same as those for the reference angle, except for the positive or negative sign, which is determined by *x*- and *y*-values in the original quadrant. Figure 5.3.4 shows which functions are positive in which quadrant.

To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "A Smart Trig Class." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise. In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive. In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive. In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive. Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.

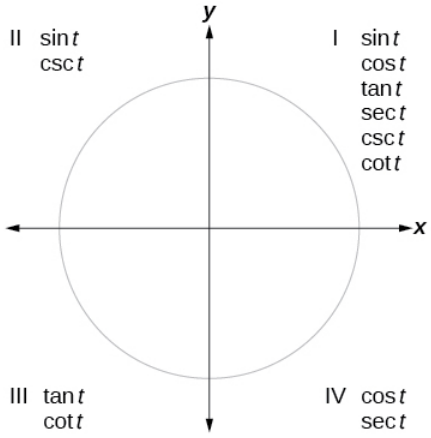


Figure 5.3.4

HOWTO: Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Using Even and Odd Trigonometric Functions

To be able to use our six trigonometric functions freely with both positive and negative angle inputs, we should examine how each function treats a negative input. As it turns out, there is an important difference among the functions in this regard.

Consider the function $f(x) = x^2$, shown in Figure 5.3.5. The graph of the function is symmetrical about the *y*-axis. All along the curve, any two points with opposite *x*-values have the same function value. This matches the result of calculation: $(4)^2 = (-4)^2$, $(-5)^2 = (5)^2$, and so on. So $f(x) = x^2$ is an **even function**, a function such that two inputs that are opposites have the same output. That means $f(-x) = f(x)$.

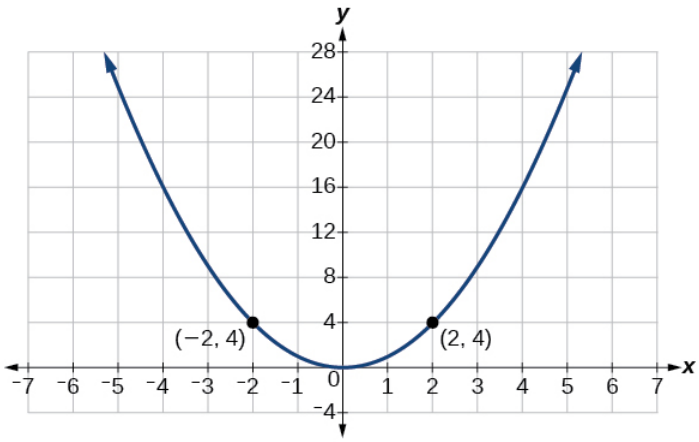


Figure 5.3.5: The function $f(x) = x^2$ is an even function.

Now consider the function $f(x) = x^3$, shown in Figure 5.3.6. The graph is not symmetrical about the *y*-axis. All along the graph, any two points with opposite *x*-values also have opposite *y*-values. So $f(x) = x^3$ is an **odd function**, one such that two inputs that are opposites have outputs that are also opposites. That means $f(-x) = -f(x)$.

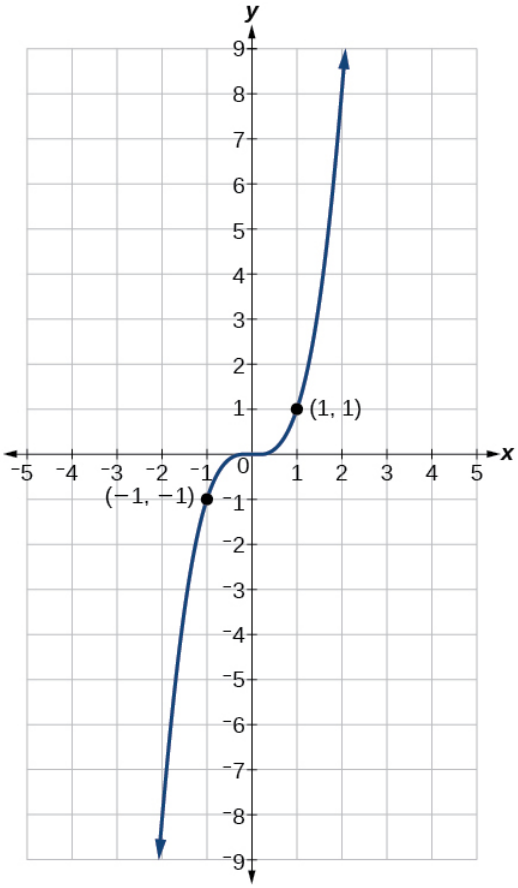


Figure 5.3.6: The function $f(x) = x^3$ is an odd function.

We can test whether a trigonometric function is even or odd by drawing a **unit circle** with a positive and a negative angle, as in Figure 5.3.7. The sine of the positive angle is y. The sine of the negative angle is $-y$. The **sine function**, then, is an odd function. We can test each of the six trigonometric functions in this fashion. The results are shown in Table 5.3.2.

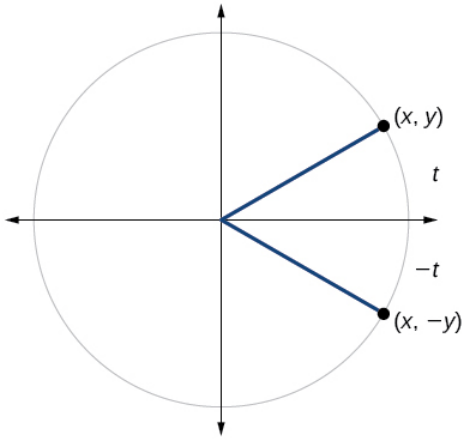


Figure 5.3.7

Table 5.3.2

$\sin t = y$ (5.3.11)	$\cos t = x$ (5.3.14)	$\tan(t) = \frac{y}{x}$ (5.3.17)
$\sin(-t) = -y$ (5.3.12)	$\cos(-t) = x$ (5.3.15)	$\tan(-t) = -\frac{y}{x}$ (5.3.18)
$\sin t \neq sin(-t)$ (5.3.13)	$\cos t = \cos(-t)$ (5.3.16)	$\tan t \neq \tan(-t)$ (5.3.19)
$\sec t = \frac{1}{x}$ (5.3.20)	$\csc t = \frac{1}{y}$ (5.3.23)	$\cot t = \frac{x}{y}$ (5.3.26)
$\sec(-t) = \frac{1}{x}$ (5.3.21)	$\csc(-t) = \frac{1}{-y}$ (5.3.24)	$\cot(-t) = \frac{x}{-y}$ (5.3.27)
$\sec t = \sec(-t)$ (5.3.22)	$\csc t \neq \csc(-t)$ (5.3.25)	$\cot t \neq \cot(-t)$ (5.3.28)

EVEN AND ODD TRIGONOMETRIC FUNCTIONS

- An **even function** is one in which $f(-x) = f(x)$.
- An **odd function** is one in which $f(-x) = -f(x)$.

Cosine and secant are even:

$$\cos(-t) = \cos t \quad (5.3.29)$$
$$\sec(-t) = \sec t \quad (5.3.30)$$

Sine, tangent, cosecant, and cotangent are odd:

$$\sin(-t) = -\sin t \quad (5.3.31)$$
$$\tan(-t) = -\tan t \quad (5.3.32)$$
$$\csc(-t) = -\csc t \quad (5.3.33)$$
$$\cot(-t) = -\cot t \quad (5.3.34)$$

Recognizing and Using Fundamental Identities

We have explored a number of properties of trigonometric functions. Now, we can take the relationships a step further, and derive some fundamental identities. Identities are statements that are true for all values of the input on which they are defined. Usually, identities can be derived from definitions and relationships we already know. For example, the Pythagorean Identity we learned earlier was derived from the Pythagorean Theorem and the definitions of sine and cosine.

FUNDAMENTAL IDENTITIES

We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t} \quad (5.3.35)$$

$$\sec t = \frac{1}{\cos t} \tag{5.3.36}$$

$$\csc t = \frac{1}{\sin t} \tag{5.3.37}$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t} \tag{5.3.38}$$

Example 5.3.5: Using Identities to Evaluate Trigonometric Functions

a. Given $\sin(45^\circ) = \frac{\sqrt{2}}{2}, \cos(45^\circ) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45^\circ)$.

b. Given $\sin(\frac{5\pi}{6}) = \frac{1}{2}, \cos(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$, evaluate $\sec(\frac{5\pi}{6})$.

Solution

Because we know the sine and cosine values for these angles, we can use identities to evaluate the other functions.

a.
$$\begin{aligned}\tan(45^\circ) &= \frac{\sin(45^\circ)}{\cos(45^\circ)} \\ &= \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} \\ &= 1\end{aligned}$$

b.
$$\begin{aligned}\sec(\frac{5\pi}{6}) &= \frac{1}{\cos(\frac{5\pi}{6})} \\ &= \frac{1}{-\frac{\sqrt{3}}{2}} \\ &= \frac{-2\sqrt{3}}{1} \\ &= \frac{-2}{\sqrt{3}} \\ &= -\frac{2\sqrt{3}}{3}\end{aligned}$$

Alternate Forms of the Pythagorean Identity

We can use these fundamental identities to derive alternative forms of the **Pythagorean Identity**, $\cos^2 t + \sin^2 t = 1$. One form is obtained by dividing both sides by $\cos^2 t$:

$$\frac{\cos^2 t}{\cos^2 t} + \frac{\sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} \tag{5.3.39}$$

$$1 + \tan^2 t = \sec^2 t \tag{5.3.40}$$

The other form is obtained by dividing both sides by $\sin^2 t$:

$$\frac{\cos^2 t}{\sin^2 t} + \frac{\sin^2 t}{\sin^2 t} = \frac{1}{\sin^2 t} \tag{5.3.41}$$

$$\cot^2 t + 1 = \csc^2 t \tag{5.3.42}$$

ALTERNATE FORMS OF THE PYTHAGOREAN IDENTITY

$$1 + \tan^2 t = \sec^2 t \tag{5.3.43}$$

$$\cot^2 t + 1 = \csc^2 t \tag{5.3.44}$$

Example 5.3.7: Using Identities to Relate Trigonometric Functions

If cos(t)=1213 cos(t)=1213 and t t is in quadrant IV, as shown in Figure 5.3.8, find the values of the other five trigonometric functions.

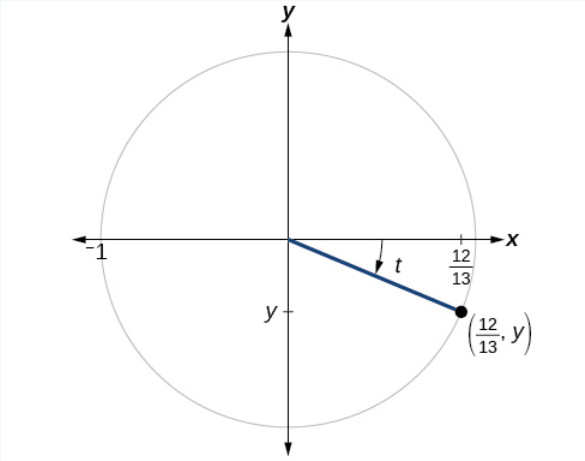


Figure 5.3.8

Solution

We can find the sine using the Pythagorean Identity, $\cos^2 t + \sin^2 t = 1$, and the remaining functions by relating them to sine and cosine.

$$(\frac{12}{13})^2 + \sin^2 t = 1 \tag{5.3.45}$$

$$\sin^2 t = 1 - (\frac{12}{13})^2 \tag{5.3.46}$$

$$\sin^2 t = 1 - \frac{144}{169} \tag{5.3.47}$$

$$\sin^2 t = \frac{25}{169} \tag{5.3.48}$$

$$\sin t = \pm\sqrt{\frac{25}{169}} \tag{5.3.49}$$

$$\sin t = \pm\frac{\sqrt{25}}{\sqrt{169}} \tag{5.3.50}$$

$$\sin t = \pm\frac{5}{13} \tag{5.3.51}$$

The sign of the sine depends on the *y*-values in the quadrant where the angle is located. Since the angle is in quadrant IV, where the *y*-values are negative, its sine is negative, $-\frac{5}{13}$.

The remaining functions can be calculated using identities relating them to sine and cosine.

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12} \tag{5.3.52}$$

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{12}{13}} = \frac{13}{12} \tag{5.3.53}$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5} \tag{5.3.54}$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5} \tag{5.3.55}$$

As we discussed in the chapter opening, a function that repeats its values in regular intervals is known as a **periodic function**. The trigonometric functions are periodic. For the four trigonometric functions, sine, cosine, cosecant and secant, a revolution of one circle, or 2π,will result in the same outputs for these functions. And for tangent and cotangent, only a half a revolution will result in the same outputs.

Other functions can also be periodic. For example, the lengths of months repeat every four years. If x x represents the length time, measured in years, and $f(x)$ represents the number of days in February, then $f(x+4)=f(x)$. This pattern repeats over and over through time. In other words, every four years, February is guaranteed to have the same number of days as it did 4 years earlier. The positive number 4 is the smallest positive number that satisfies this condition and is called the period. A **period** is the shortest interval over which a function completes one full cycle—in this example, the period is 4 and represents the time it takes for us to be certain February has the same number of days.

PERIOD OF A FUNCTION

The **period** P of a repeating function f f is the number representing the interval such that $f(x+P)=f(x)$ for any value of x.

The period of the cosine, sine, secant, and cosecant functions is 2π.

The period of the tangent and cotangent functions is π.

Evaluating Trigonometric Functions with a Calculator

We have learned how to evaluate the six trigonometric functions for the common first-quadrant angles and to use them as reference angles for angles in other quadrants. To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

Evaluating a tangent function with a scientific calculator as opposed to a graphing calculator or computer algebra system is like evaluating a sine or cosine: Enter the value and press the TAN key. For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

If we need to work with degrees and our calculator or software does not have a degree mode, we can enter the degrees multiplied by the conversion factor $\frac{\pi}{180}$ to convert the degrees to radians. To find the secant of 30°, we could press

$$(\text{for a scientific calculator}): \frac{1}{30\times\frac{\pi}{180}}\text{COS} \tag{5.3.56}$$

or

$$(\text{for a graphing calculator}): \frac{1}{\cos(\frac{30\pi}{180})} \tag{5.3.57}$$

how to: Given an angle measure in radians, use a scientific calculator to find the cosecant

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Enter the value of the angle inside parentheses.
4. Press the SIN key.
5. Press the = key.

how to: Given an angle measure in radians, use a graphing utility/calculator to find the cosecant

1. If the graphing utility has degree mode and radian mode, set it to radian mode.
2. Enter: 1 /
3. Press the SIN key.
4. Enter the value of the angle inside parentheses.
5. Press the ENTER key.

Example 5.3.10: Evaluating the Cosecant Using Technology

Evaluate the cosecant of $\frac{5\pi}{7}$.

Solution

For a scientific calculator, enter information as follows:

$$1/(5 \times \pi/7)\text{SIN} = \tag{5.3.58}$$

$$\csc(\frac{5\pi}{7}) \approx 1.279 \tag{5.3.59}$$

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5.4: Right Triangle Trigonometry

Using Right Triangles to Evaluate Trigonometric Functions

In earlier sections, we used a unit circle to define the *trigonometric functions*. In this section, we will extend those definitions so that we can apply them to right triangles. The value of the sine or cosine function of t is its value at t radians. First, we need to create our right triangle. Figure 5.4.1 shows a point on a unit circle of radius 1. If we drop a vertical line segment from the point (x, y) to the *x*-axis, we have a right triangle whose vertical side has length y and whose horizontal side has length x. We can use this right triangle to redefine sine, cosine, and the other trigonometric functions as ratios of the sides of a right triangle.

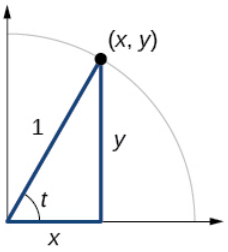


Figure 5.4.1

We know

$$\cos t = \frac{x}{1} = x \tag{5.4.1}$$

Likewise, we know

$$\sin t = \frac{y}{1} = y \tag{5.4.2}$$

These ratios still apply to the sides of a right triangle when no unit circle is involved and when the triangle is not in standard position and is not being graphed using (x, y) coordinates. To be able to use these ratios freely, we will give the sides more general names: Instead of x,we will call the side between the given angle and the right angle the **adjacent side** to angle t. (Adjacent means "next to.") Instead of y,we will call the side most distant from the given angle the **opposite side** from angle t. And instead of 1,we will call the side of a right triangle opposite the right angle the **hypotenuse**. These sides are labeled in Figure 5.4.2.

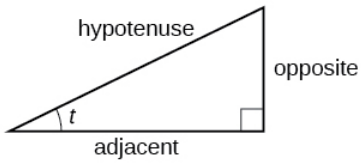


Figure 5.4.2: The sides of a right triangle in relation to angle t.

Understanding Right Triangle Relationships

Given a right triangle with an acute angle of t,

$$\sin(t) = \frac{\text{opposite}}{\text{hypotenuse}} \tag{5.4.3}$$

$$\cos(t) = \frac{\text{adjacent}}{\text{hypotenuse}} \tag{5.4.4}$$

$$\tan(t) = \frac{\text{opposite}}{\text{adjacent}} \tag{5.4.5}$$

A common mnemonic for remembering these relationships is SohCahToa, formed from the first letters of "Sine is opposite over hypotenuse, Cosine is adjacent over hypotenuse, Tangent is opposite over adjacent."

how to: Given the side lengths of a right triangle and one of the acute angles, find the sine, cosine, and tangent of that angle

1. Find the sine as the ratio of the opposite side to the hypotenuse.
2. Find the cosine as the ratio of the adjacent side to the hypotenuse.
3. Find the tangent is the ratio of the opposite side to the adjacent side.

Example 5.4.1: Evaluating a Trigonometric Function of a Right Triangle

Given the triangle shown in Figure 5.4.3, find the value of $\cos\alpha$.

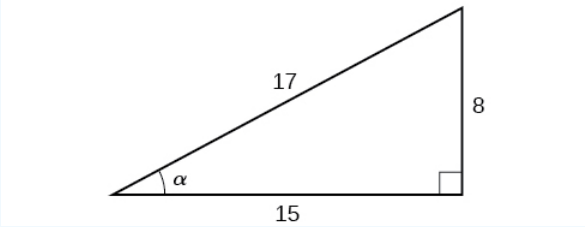


Figure 5.4.3

Solution

The side adjacent to the angle is 15, and the hypotenuse of the triangle is 17, so via Equation 5.4.4:

$$\cos(\alpha) = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$= \frac{15}{17}$$

Finding Trigonometric Functions of Special Angles Using Side Lengths

We have already discussed the trigonometric functions as they relate to the **special angles** on the unit circle. Now, we can use those relationships to evaluate triangles that contain those special angles. We do this because when we evaluate the special angles in trigonometric functions, they have relatively friendly values, values that contain either no or just one square root in the ratio. Therefore, these are the angles often used in math and science problems. We will use multiples of $30^\circ, 60^\circ,$ and 45°, however, remember that when dealing with right triangles, we are limited to angles between 0° and 90°.

Suppose we have a $30^\circ, 60^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ triangle. The sides have lengths in the relation $s, \sqrt{3}s, 2s$. The sides of a $45^\circ, 45^\circ, 90^\circ$ triangle, which can also be described as a $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ triangle, have lengths in the relation $s, s, \sqrt{2}s$. These relations are shown in Figure 5.4.8.

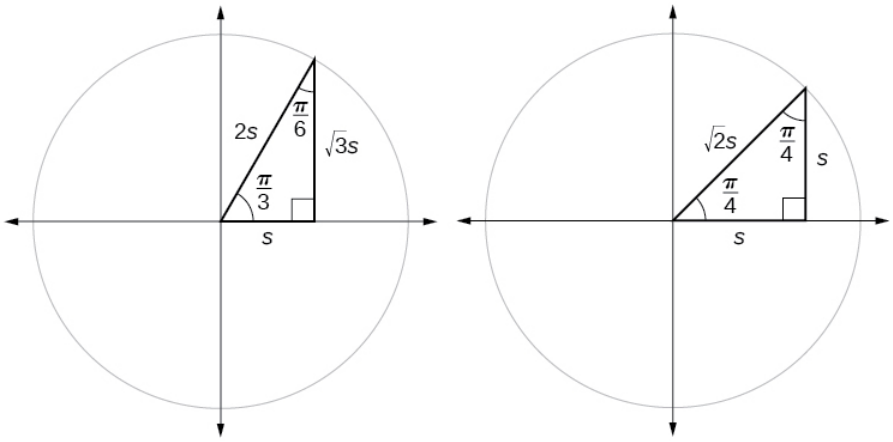


Figure 5.4.8: Side lengths of special triangles

We can then use the ratios of the side lengths to evaluate trigonometric functions of special angles.

Given trigonometric functions of a special angle, evaluate using side lengths.

1. Use the side lengths shown in Figure 5.4.8 for the special angle you wish to evaluate.
2. Use the ratio of side lengths appropriate to the function you wish to evaluate.

Example 5.4.3: Evaluating Trigonometric Functions of Special Angles Using Side Lengths

Find the exact value of the trigonometric functions of $\frac{\pi}{3}$, using side lengths.

Solution

$$\sin(\frac{\pi}{3}) = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\cos(\frac{\pi}{3}) = \frac{\text{adj}}{\text{hyp}} = \frac{s}{2s} = \frac{1}{2}$$
$$\tan(\frac{\pi}{3}) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{3}s}{s} = \sqrt{3}$$
$$\sec(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{adj}} = \frac{2s}{s} = 2$$
$$\csc(\frac{\pi}{3}) = \frac{\text{hyp}}{\text{opp}} = \frac{2s}{\sqrt{3}s} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$
$$\cot(\frac{\pi}{3}) = \frac{\text{adj}}{\text{opp}} = \frac{s}{\sqrt{3}s} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Using Equal Cofunction of Complements

If we look more closely at the relationship between the sine and cosine of the special angles relative to the unit circle, we will notice a pattern. In a right triangle with angles of $\frac{\pi}{6}$ and $\frac{\pi}{3}$, we see that the sine of $\frac{\pi}{3}$, namely $\frac{\sqrt{3}}{2}$, is also the cosine of $\frac{\pi}{6}$, while the sine of $\frac{\pi}{6}$, namely $\frac{1}{2}$, is also the cosine of $\frac{\pi}{3}$ (Figure 5.4.9).

$$\sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}$$
$$\sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}$$

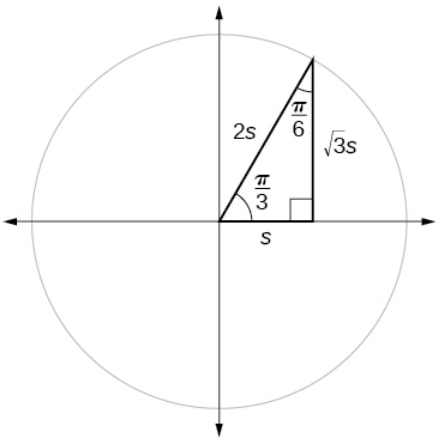


Figure 5.4.9: The sine of $\frac{\pi}{3}$ equals the cosine of $\frac{\pi}{6}$ and vice versa.

This result should not be surprising because, as we see from Figure 5.4.9, the side opposite the angle of $\frac{\pi}{3}$ is also the side adjacent to $\frac{\pi}{6}$, so $\sin(\frac{\pi}{3})$ and $\cos(\frac{\pi}{6})$ are exactly the same ratio of the same two sides, $\sqrt{3}s$ and $2s$. Similarly, $\cos(\frac{\pi}{3})$ and $\sin(\frac{\pi}{6})$ are also the same ratio using the same two sides, s and $2s$.

The interrelationship between the sines and cosines of $\frac{\pi}{6}$ and $\frac{\pi}{3}$ also holds for the two acute angles in any right triangle, since in every case, the ratio of the same two sides would constitute the sine of one angle and the cosine of the other. Since the three angles of a triangle add to π, and the right angle is $\frac{\pi}{2}$, the remaining two angles must also add up to $\frac{\pi}{2}$. That means that a right triangle can be formed with any two angles that add to $\frac{\pi}{2}$—in other words, any two complementary angles. So we may state a *cofunction identity*: If any two angles are complementary, the sine of one is the cosine of the other, and vice versa. This identity is illustrated in Figure 5.4.10.

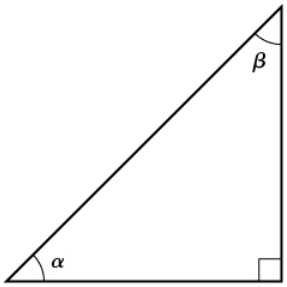


Figure 5.4.10: Cofunction identity of sine and cosine of complementary angles

Using this identity, we can state without calculating, for instance, that the sine of $\frac{\pi}{12}$ equals the cosine of $\frac{5\pi}{12}$, and that the sine of $\frac{5\pi}{12}$ equals the cosine of $\frac{\pi}{12}$. We can also state that if, for a certain angle t, $\cos t = \frac{5}{13}$, then $\sin(\frac{\pi}{2} - t) = \frac{5}{13}$ as well.

COFUNCTION IDENTITIES

The **cofunction identities** in radians are listed in Table 5.4.1.

Table 5.4.1

$\cos t = \sin(\frac{\pi}{2} - t)$	$\sin t = \cos(\frac{\pi}{2} - t)$
$\tan t = \cot(\frac{\pi}{2} - t)$	$\cot t = \tan(\frac{\pi}{2} - t)$
$\sec t = \csc(\frac{\pi}{2} - t)$	$\csc t = \sec(\frac{\pi}{2} - t)$

how to: Given the sine and cosine of an angle, find the sine or cosine of its complement.

1. To find the sine of the complementary angle, find the cosine of the original angle.
2. To find the cosine of the complementary angle, find the sine of the original angle.

Example 5.4.4: Using Cofunction Identities

If $\sin t = \frac{5}{12}$, find $(\cos\frac{\pi}{2} - t)$.

Solution

According to the cofunction identities for sine and cosine,

$$\sin t = \cos(\frac{\pi}{2} - t).$$

So

$$\cos(\frac{\pi}{2} - t) = \frac{5}{12}.$$

Using Right Triangle Trigonometry to Solve Applied Problems

Right-triangle trigonometry has many practical applications. For example, the ability to compute the lengths of sides of a triangle makes it possible to find the height of a tall object without climbing to the top or having to extend a tape measure along its height. We do so by measuring a distance from the base of the object to a point on the ground some distance away, where we can look up to the top of the tall object at an angle. The **angle of elevation** of an object above an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. The right triangle this position creates has sides that represent the unknown height, the measured distance from the base, and the angled line of sight from the ground to the top of the object. Knowing the measured distance to the base of the object and the angle of the line of sight, we can use trigonometric functions to calculate the unknown height. Similarly, we can form a triangle from the top of a tall object by looking downward. The **angle of depression** of an object below an observer relative to the observer is the angle between the horizontal and the line from the object to the observer's eye. See Figure 5.4.12

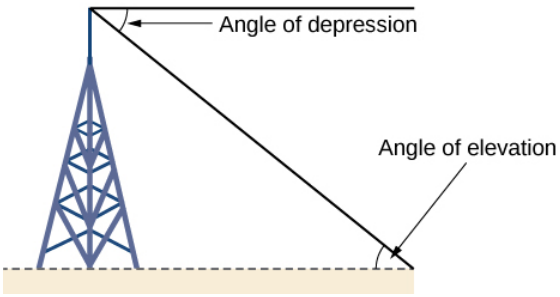


Figure 5.4.12

how to: Given a tall object, measure its height indirectly

1. Make a sketch of the problem situation to keep track of known and unknown information.
2. Lay out a measured distance from the base of the object to a point where the top of the object is clearly visible.
3. At the other end of the measured distance, look up to the top of the object. Measure the angle the line of sight makes with the horizontal.
4. Write an equation relating the unknown height, the measured distance, and the tangent of the angle of the line of sight.
5. Solve the equation for the unknown height.

Example 5.4.6: Measuring a Distance Indirectly

To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an angle of 57° 57° between a line of sight to the top of the tree and the ground, as shown in Figure 5.4.13. Find the height of the tree.

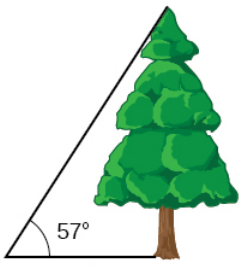


Figure 5.4.13

Solution

We know that the angle of elevation is 57° and the adjacent side is 30 ft long. The opposite side is the unknown height.

The trigonometric function relating the side opposite to an angle and the side adjacent to the angle is the tangent. So we will state our information in terms of the tangent of 57°, letting h be the unknown height.

$$\begin{aligned} \tan\theta &= \frac{\text{opposite}}{\text{adjacent}} \\ \tan(57^\circ) &= \frac{h}{30} && \text{Solve for } h. \\ h &= 30\tan(57^\circ) && \text{Multiply.} \\ h &\approx 46.2 && \text{Use a calculator.} \end{aligned} \tag{5.4.6}$$

The tree is approximately 46 feet tall.

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CHAPTER OVERVIEW

6: Periodic Functions

In this chapter, we will investigate graphs of sine, cosine, and other trigonometric functions.

6.1: Graphs of the Sine and Cosine Functions

6.2: Graphs of the Other Trigonometric Functions

6.3: Inverse Trigonometric Functions

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6.1: Graphs of the Sine and Cosine Functions

Graphing Sine and Cosine Functions

Recall that the sine and cosine functions relate real number values to the x- and y-coordinates of a point on the unit circle. So what do they look like on a graph on a coordinate plane? Let's start with the sine function. We can create a table of values and use them to sketch a graph. Table 6.1.1 lists some of the values for the sine function on a unit circle.

Table 6.1.1

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin(x)$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Plotting the points from the table and continuing along the *x*-axis gives the shape of the sine function. See Figure 6.1.2.

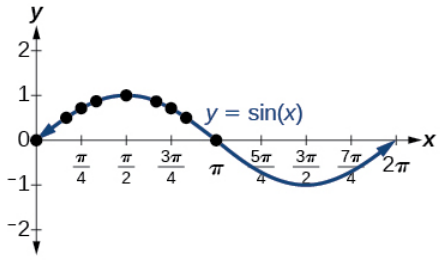


Figure 6.1.2: The sine function

Notice how the sine values are positive between 0 and π, which correspond to the values of the sine function in quadrants I and II on the unit circle, and the sine values are negative between π and 2π, which correspond to the values of the sine function in quadrants III and IV on the unit circle. See Figure 6.1.3.

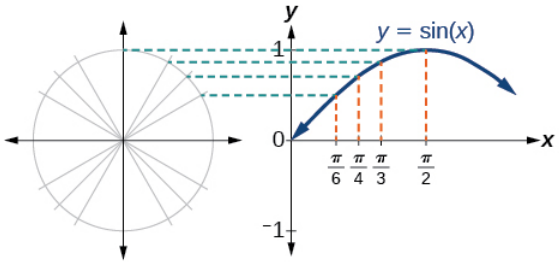


Figure 6.1.3: Plotting values of the sine function

Now let's take a similar look at the **cosine function**. Again, we can create a table of values and use them to sketch a graph. Table 6.1.2 lists some of the values for the cosine function on a unit circle.

Table 6.1.2

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(x)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

As with the sine function, we can plots points to create a graph of the cosine function as in Figure 6.1.4.

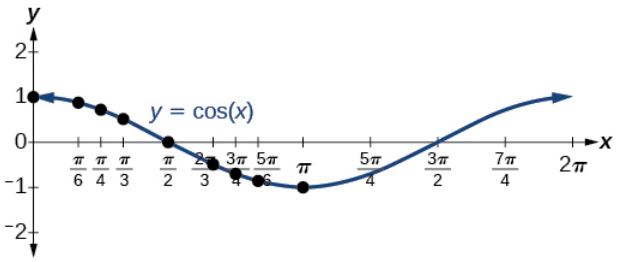


Figure 6.1.4: The cosine function

Because we can evaluate the sine and cosine of any real number, both of these functions are defined for all real numbers. By thinking of the sine and cosine values as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

In both graphs, the shape of the graph repeats after 2π, which means the functions are periodic with a period of 2π. A **periodic function** is a function for which a specific **horizontal shift**, P, results in a function equal to the original function: $f(x+P) = f(x)$ for all values of x in the domain of f. When this occurs, we call the smallest such horizontal shift with $P > 0$ the **period** of the function. Figure 6.1.5 shows several periods of the sine and cosine functions.

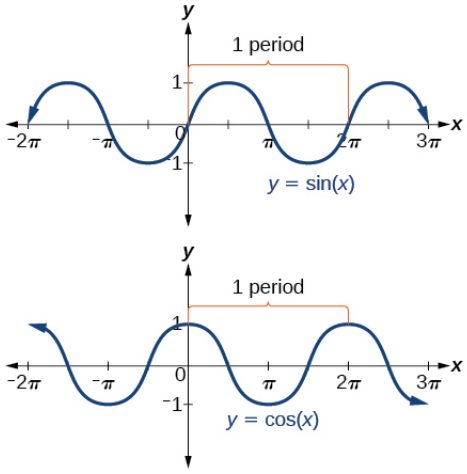


Figure 6.1.5

Looking again at the sine and cosine functions on a domain centered at the y-axis helps reveal symmetries. As we can see in Figure 6.1.6, the sine function is symmetric about the origin. Recall from The Other Trigonometric Functions that we determined from the unit circle that the sine function is an odd function because $\sin(-x) = -\sin\ x$. Now we can clearly see this property from the graph.

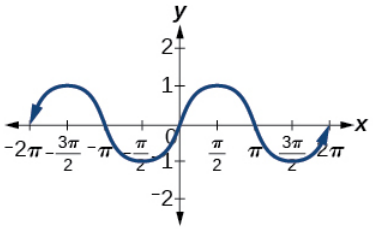


Figure 6.1.6: Odd symmetry of the sine function

Figure 6.1.7 shows that the cosine function is symmetric about the y-axis. Again, we determined that the cosine function is an even function. Now we can see from the graph that $\cos(-x) = \cos\ x$.

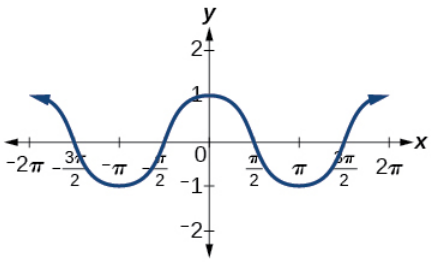


Figure 6.1.7: Even symmetry of the cosine function

CHARACTERISTICS OF SINE AND COSINE FUNCTIONS

The sine and cosine functions have several distinct characteristics:

- They are periodic functions with a period of 2π.
- The domain of each function is $(-\infty, \infty)$ and the range is $[-1, 1]$.
- The graph of $y = \sin\ x$ is symmetric about the origin, because it is an odd function.
- The graph of $y = \cos\ x$ is symmetric about they- y-axis, because it is an even function.

Investigating Sinusoidal Functions

As we can see, sine and cosine functions have a regular period and range. If we watch ocean waves or ripples on a pond, we will see that they resemble the sine or cosine functions. However, they are not necessarily identical. Some are taller or longer than others. A function that has the same general shape as a sine or **cosine function** is known as a **sinusoidal function**. The general forms of sinusoidal functions are

$$y = A\sin(Bx - C) + D \tag{6.1.1}$$

and

$$y = A\cos(Bx - C) + D \tag{6.1.2}$$

Determining the Period of Sinusoidal Functions

Looking at the forms of sinusoidal functions, we can see that they are transformations of the sine and cosine functions. We can use what we know about transformations to determine the period.

In the general formula, B is related to the period by $P = \dfrac{2\pi}{|B|}$. If $|B| > 1$, then the period is less than 2π and the function undergoes a horizontal compression, whereas if $|B| < 1$, then the period is greater than 2π and the function undergoes a horizontal stretch. For example, $f(x) = \sin(x)$, $B = 1$, so the period is 2π,which we knew. If $f(x) = \sin(2x)$, then $B = 2$, so the period is π and the graph is compressed. If $f(x) = \sin\left(\dfrac{x}{2}\right)$, then $B = \dfrac{1}{2}$, so the period is 4π and the graph is stretched. Notice in Figure 6.1.8 how the period is indirectly related to $|B|$.

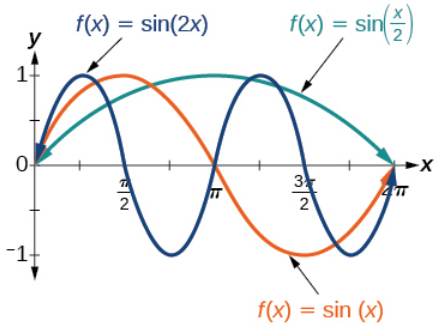


Figure 6.1.8

PERIOD OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

- $y = A\sin(Bx)$
- $y = A\cos(Bx)$

The period is $\dfrac{2\pi}{|B|}$.

Example 6.1.1: Identifying the Period of a Sine or Cosine Function

Determine the period of the function $f(x) = \sin\left(\dfrac{\pi}{6}x\right)$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx)$.

In the given equation, $B = \dfrac{\pi}{6}$, so the period will be

$$\begin{aligned} P &= \frac{2\pi}{|B|} \\ &= \frac{2\pi}{\frac{\pi}{6}} \\ &= 2\pi \cdot \frac{6}{\pi} \\ &= 12 \end{aligned}$$

Determining Amplitude

Returning to the general formula for a sinusoidal function, we have analyzed how the variable B relates to the period. Now let's turn to the variable A so we can analyze how it is related to the **amplitude**, or greatest distance from rest. A represents the vertical stretch factor, and its absolute value $|A|$ is the amplitude. The local maxima will be a distance $|A|$ above the vertical **midline** of the graph, which is the line $x = D$; because $D = 0$ in this case, the midline is the *x*-axis. The local minima will be the same distance below the midline. If $|A| > 1$, the function is stretched. For example, the amplitude of $f(x) = 4sinx$ is twice the amplitude of

$$f(x) = 2\sin x$$

If $|A| < 1$, the function is compressed. Figure 6.1.9 compares several sine functions with different amplitudes.

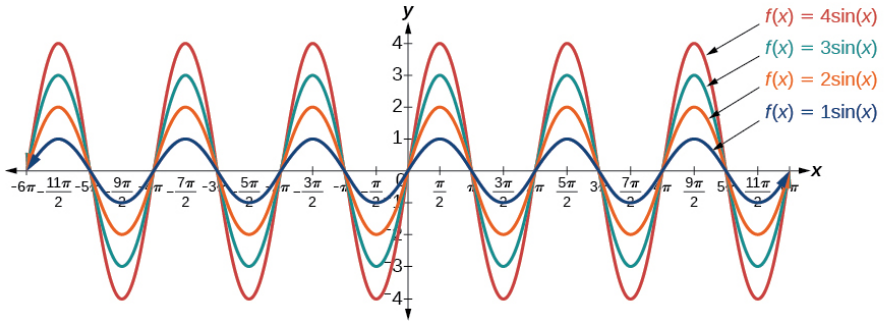


Figure 6.1.9

AMPLITUDE OF SINUSOIDAL FUNCTIONS

If we let $C = 0$ and $D = 0$ in the general form equations of the sine and cosine functions, we obtain the forms

$$y = A\sin(Bx) \text{ and } y = A\cos(Bx) \tag{6.1.3}$$

The amplitude is A, and the vertical height from the midline is $|A|$. In addition, notice in the example that

$$|A| = amplitude = \frac{1}{2}\,|\,maximum - minimum| \tag{6.1.4}$$

Example 6.1.2: Identifying the Amplitude of a Sine or Cosine Function

What is the amplitude of the sinusoidal function $f(x) = -4\sin(x)$? Is the function stretched or compressed vertically?

Solution

Let's begin by comparing the function to the simplified form $y = A\sin(Bx)$.

In the given function, $A = -4$, so the amplitude is $|A| = |-4| = 4$. The function is stretched.

Analysis

The negative value of A results in a reflection across the x-axis of the sine function, as shown in Figure 6.1.10.

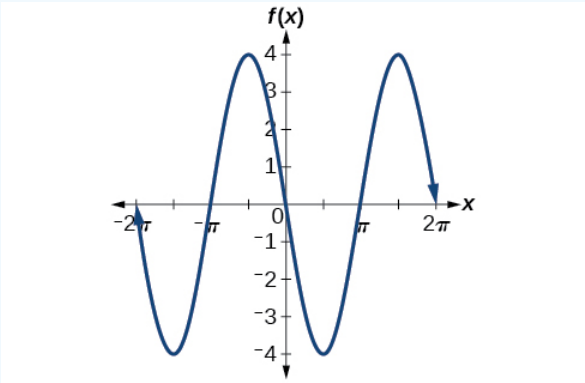


Figure 6.1.10

Analyzing Graphs of Variations of $y = \sin\ x$ and $y = \cos\ x$

Now that we understand how A and B relate to the general form equation for the sine and cosine functions, we will explore the variables C and D. Recall the general form:

$$y = A\sin(Bx - C) + D \qquad \text{and} \qquad y = A\cos(Bx - C) + D \tag{6.1.5}$$

or

$$y = A\sin\left(B\left(x - \frac{C}{B}\right)\right) + D \qquad \text{and} \qquad y = A\cos\left(B\left(x - \frac{C}{B}\right)\right) + D \tag{6.1.6}$$

The value $\frac{C}{B}$ for a sinusoidal function is called the **phase shift**, or the horizontal displacement of the basic sine or cosine function. If $C > 0$, the graph shifts to the right. If $C < 0$, the graph shifts to the left. The greater the value of $|C|$, the more the graph is shifted. Figure 6.1.11 shows that the graph of $f(x) = \sin(x - \pi)$ shifts to the right by π units, which is more than we see in the graph of $f(x) = \sin\left(x - \frac{\pi}{4}\right)$, which shifts to the right by $\frac{\pi}{4}$ units.

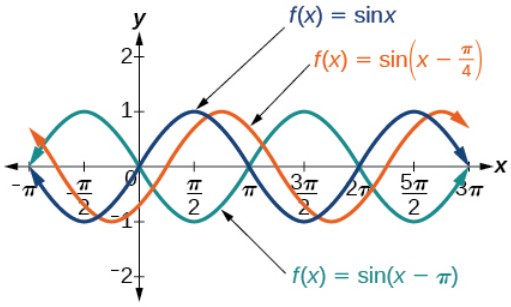


Figure 6.1.11

While C relates to the horizontal shift, D indicates the vertical shift from the midline in the general formula for a sinusoidal function. See Figure 6.1.12. The function $y = \cos(x) + D$ has its midline at $y = D$.

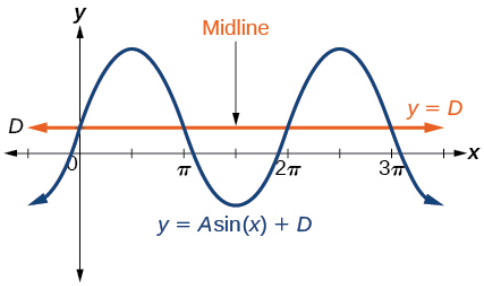


Figure 6.1.12

Any value of D other than zero shifts the graph up or down. Figure 6.1.13 compares $f(x) = \sin x$ with $f(x) = \sin x + 2$, which is shifted 2 units up on a graph.

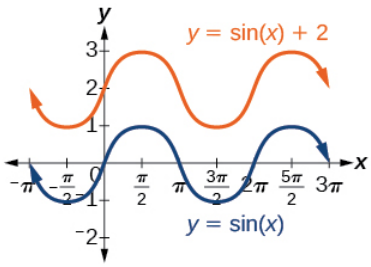


Figure 6.1.13

VARIATIONS OF SINE AND COSINE FUNCTIONS

Given an equation in the form $f(x) = A\sin(Bx - C) + D$ or $f(x) = A\cos(Bx - C) + D$, $\frac{C}{D}$ is the **phase shift** and D is the **vertical shift.**

Example 6.1.3: Identifying the Phase Shift of a Function

Determine the direction and magnitude of the phase shift for $f(x) = \sin\left(x + \frac{\pi}{6}\right) - 2$.

Solution

Let's begin by comparing the equation to the general form $y = A\sin(Bx - C) + D$.

In the given equation, notice that $B = 1$ and $C = -\frac{\pi}{6}$. So the phase shift is

$$\begin{aligned} \frac{C}{B} &= -\frac{\frac{\pi}{6}}{1} \\ &= -\frac{\pi}{6} \end{aligned}$$

or $\frac{\pi}{6}$ units to the left.

Analysis

We must pay attention to the sign in the equation for the general form of a sinusoidal function. The equation shows a minus sign before C. Therefore $f(x) = \sin(x + \frac{\pi}{6}) - 2$ can be rewritten as $f(x) = \sin\left(x - \left(-\frac{\pi}{6}\right)\right) - 2$. If the value of C is negative, the shift is to the left.

Exercise 6.1.3

Determine the direction and magnitude of the phase shift for $f(x) = 3\cos\left(x - \frac{\pi}{2}\right)$.

Answer

$\frac{\pi}{2}$; right

How to: Given a sinusoidal function in the form $f(x) = A\sin(Bx - C) + D$, identify the midline, amplitude, period, and phase shift

1. Determine the amplitude as $|A|$.
2. Determine the period as $P = \frac{2\pi}{|B|}$.
3. Determine the phase shift as $\frac{C}{B}$.
4. Determine the midline as $y = D$.

Example 6.1.7: Identifying the Equation for a Sinusoidal Function from a Graph

Determine the equation for the sinusoidal function in Figure 6.1.17.

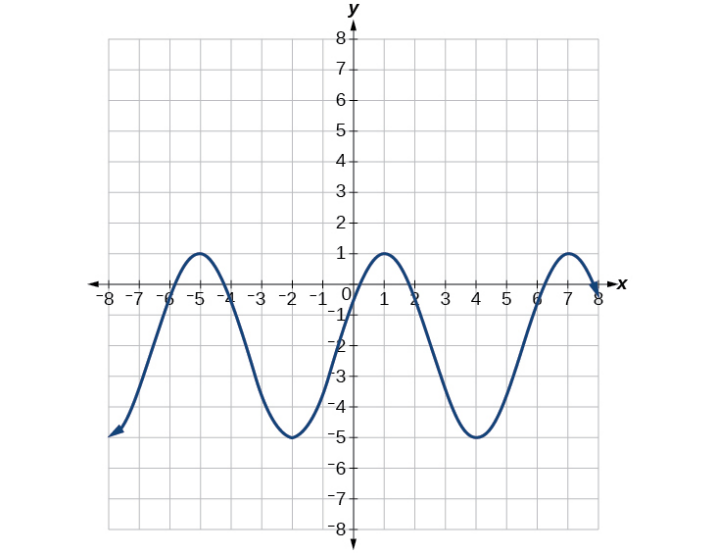


Figure 6.1.17

Solution

With the highest value at 1 and the lowest value at -5, the midline will be halfway between at -2. So $D = -2$.

The distance from the midline to the highest or lowest value gives an amplitude of $|A| = 3$.

The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$, or from the distance between the lowest points. Therefore, $P = \dfrac{2\pi}{|B|} = 6$. Using the positive value for B, we find that

$$\begin{aligned} B &= \frac{2\pi}{P} \\ &= \frac{2\pi}{6} \\ &= \frac{\pi}{3} \end{aligned}$$

So far, our equation is either $y = 3\sin\left(\frac{\pi}{3}x - C\right) - 2$ or $y = 3\cos\left(\frac{\pi}{3}x - C\right) - 2$. For the shape and shift, we have more than one option. We could write this as any one of the following:

- a cosine shifted to the right
- a negative cosine shifted to the left
- a sine shifted to the left
- a negative sine shifted to the right

While any of these would be correct, the cosine shifts are easier to work with than the sine shifts in this case because they involve integer values. So our function becomes

$$y = 3\cos\left(\frac{\pi}{3}x - \frac{\pi}{3}\right) - 2 \quad \text{or}$$
$$y = -3\cos\left(\frac{\pi}{3}x + \frac{2\pi}{3}\right) - 2$$

Again, these functions are equivalent, so both yield the same graph.

Graphing Variations of $y = \sin x$ and $y = \cos x$

Throughout this section, we have learned about types of variations of sine and cosine functions and used that information to write equations from graphs. Now we can use the same information to create graphs from equations.

Instead of focusing on the general form equations

$$y = A\sin(Bx - C) + D \text{ and } y = A\cos(Bx - C) + D$$

we will let $C = 0$ and $D = 0$ and work with a simplified form of the equations in the following examples.

Given the function $y = A\sin(Bx)$, sketch its graph.

1. Identify the amplitude, $|A|$.
2. Identify the period, $P = \dfrac{2\pi}{|B|}$.
3. Start at the origin, with the function increasing to the right if A is positive or decreasing if A is negative.
4. At $x = \dfrac{\pi}{2|B|}$ there is a local maximum for $A > 0$ or a minimum for $A < 0$, with $y = A$.
5. The curve returns to the *x*-axis at $x = \dfrac{\pi}{|B|}$.
6. There is a local minimum for $A > 0$ (maximum for $A < 0$) at $x = \dfrac{3\pi}{2|B|}$ with $y = -A$.
7. The curve returns again to the *x*-axis at $x = \dfrac{\pi}{2|B|}$.

Example 6.1.8: Graphing a Function and Identifying the Amplitude and Period

Sketch a graph of $f(x) = -2\sin\left(\dfrac{\pi x}{2}\right)$.

Solution

Let's begin by comparing the equation to the form $y = A\sin(Bx)$.

- *Step 1.* We can see from the equation that $A = -2$, so the amplitude is 2.
  $|A| = 2$
- *Step 2.* The equation shows that $B = \dfrac{\pi}{2}$, so the period is

$$\begin{aligned} P &= \frac{2\pi}{\frac{\pi}{2}} \\ &= 2\pi \cdot \frac{2}{\pi} \\ &= 4 \end{aligned}$$

- *Step 3.* Because A is negative, the graph descends as we move to the right of the origin.
- *Step 4.* The x-intercepts are at the beginning of one period, $x = 0$, the horizontal midpoints are at $x = 2$ and at the end of one period at $x = 4$.

The quarter points include the minimum at $x = 1$ and the maximum at $x = 3$. A local minimum will occur 2 units below the midline, at $x = 1$, and a local maximum will occur at 2 units above the midline, at $x = 3$. Figure 6.1.19 shows the graph of the function.

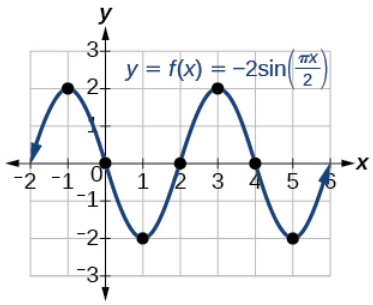


Figure 6.1.19

How to: Given a sinusoidal function with a phase shift and a vertical shift, sketch its graph

1. Express the function in the general form $y = A\sin(Bx - C) + D$ or $y = A\cos(Bx - C) + D$.
2. Identify the amplitude, $|A|$.
3. Identify the period, $P = \dfrac{2\pi}{|B|}$.
4. Identify the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $f(x) = A\sin(Bx)$ shifted to the right or left by $\dfrac{C}{B}$ and up or down by D.

Using Transformations of Sine and Cosine Functions

We can use the transformations of sine and cosine functions in numerous applications. As mentioned at the beginning of the chapter,**circular motion** can be modeled using either the sine or **cosine function**.

Example 6.1.11: Finding the Vertical Component of Circular Motion

A point rotates around a circle of radius 3 centered at the origin. Sketch a graph of the y-coordinate of the point as a function of the angle of rotation.

Solution

Recall that, for a point on a circle of radius r, the y-coordinate of the point is $y = r\sin(x)$, so in this case, we get the equation $y(x) = 3\sin(x)$. The constant 3 causes a vertical stretch of the y-values of the function by a factor of 3, which we can see in the graph in Figure $6.1.24$

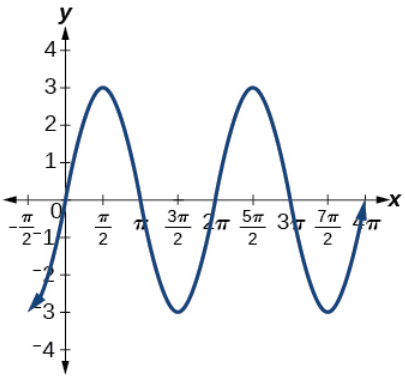


Figure 6.1.24

Analysis

Notice that the period of the function is still 2π; as we travel around the circle, we return to the point $(3, 0)$ for $x = 2\pi, 4\pi, 6\pi$,...Because the outputs of the graph will now oscillate between -3 and 3, the amplitude of the sine wave is 3.

Exercise 6.1.10

What is the amplitude of the function $f(x) = 7\cos(x)$? Sketch a graph of this function.

Answer

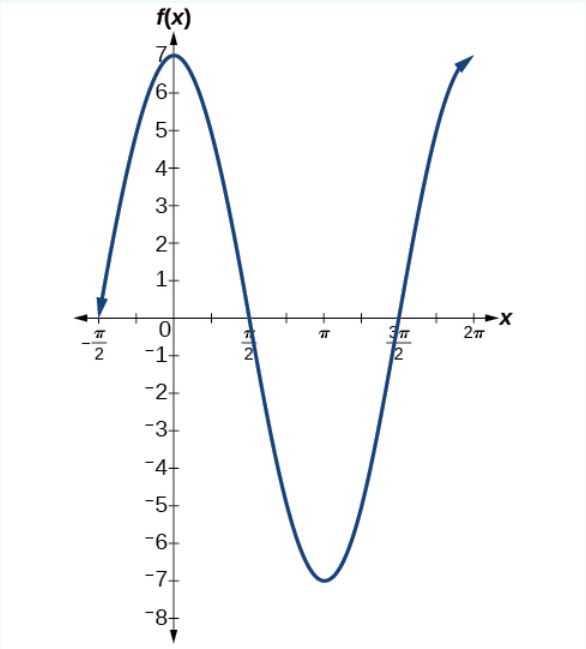


Figure 6.1.25

Example 6.1.12: Finding the Vertical Component of Circular Motion

A circle with radius 3 ft is mounted with its center 4 ft off the ground. The point closest to the ground is labeled P, as shown in Figure 6.1.26. Sketch a graph of the height above the ground of the point P as the circle is rotated; then find a function that gives the height in terms of the angle of rotation.

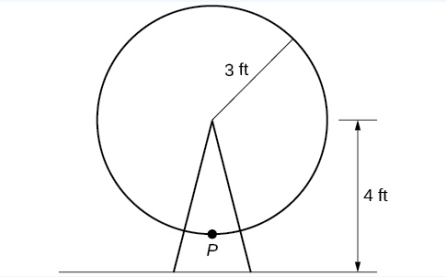


Figure 6.1.26

Solution

Sketching the height, we note that it will start 1 ft above the ground, then increase up to 7 ft above the ground, and continue to oscillate 3 ft above and below the center value of 4 ft, as shown in Figure 6.1.27.

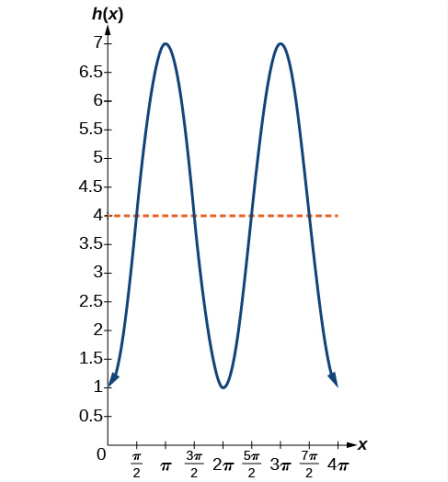


Figure 6.1.27

Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other. Let's use a cosine function because it starts at the highest or lowest value, while a **sine function** starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of 1, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together, we find that

$$y = -3\cos(x) + 4$$

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6.2: Graphs of the Other Trigonometric Functions

Analyzing the Graph of $y = \tan x$

We will begin with the graph of the tangent function, plotting points as we did for the sine and cosine functions. Recall that

$$\tan x = \frac{\sin x}{\cos x} \tag{6.2.1}$$

The period of the tangent function is π because the graph repeats itself on intervals of $k\pi$ where k is a constant. If we graph the tangent function on $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we can see the behavior of the graph on one complete cycle. If we look at any larger interval, we will see that the characteristics of the graph repeat.

We can determine whether tangent is an odd or even function by using the definition of tangent.

$$\begin{aligned} \tan(-x) &= \frac{\sin(-x)}{\cos(-x)} && \text{Definition of tangent} \\ &= \frac{-\sin x}{\cos x} && \text{Sine is an odd function, cosine is even} \\ &= -\frac{\sin x}{\cos x} && \text{The quotient of an odd and an even function is odd} \\ &= -\tan x && \text{Definition of tangent} \end{aligned}$$

Therefore, tangent is an odd function. We can further analyze the graphical behavior of the tangent function by looking at values for some of the special angles, as listed in Table 6.2.1.

Table 6.2.1

x	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

These points will help us draw our graph, but we need to determine how the graph behaves where it is undefined. If we look more closely at values when $\frac{\pi}{3} < x < \frac{\pi}{2}$, we can use a table to look for a trend. Because $\frac{\pi}{3} \approx 1.05$ and $\frac{\pi}{2} \approx 1.57$, we will evaluate x at radian measures $1.05 < x < 1.57$ as shown in Table 6.2.2.

Table 6.2.2

x	1.3	1.5	1.55	1.56
$\tan x$	3.6	14.1	48.1	92.6

As x approaches $\frac{\pi}{2}$, the outputs of the function get larger and larger. Because $y = \tan x$ is an odd function, we see the corresponding table of negative values in Table 6.2.3.

Table 6.2.3

x	−1.3	−1.5	−1.55	−1.56
$\tan x$	−3.6	−14.1	−48.1	−92.6

We can see that, as x approaches $-\frac{\pi}{2}$, the outputs get smaller and smaller. Remember that there are some values of x for which $\cos x = 0$. For example, $\cos\left(\frac{\pi}{2}\right) = 0$ and $\cos\left(\frac{3\pi}{2}\right) = 0$. At these values, the tangent function is undefined, so the graph of $y = \tan x$ has discontinuities at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. At these values, the graph of the tangent has vertical asymptotes. Figure 6.2.1 represents the graph of $y = \tan x$. The tangent is positive from 0 to $\frac{\pi}{2}$ and from π to $\frac{3\pi}{2}$, corresponding to quadrants I and III of the unit circle.

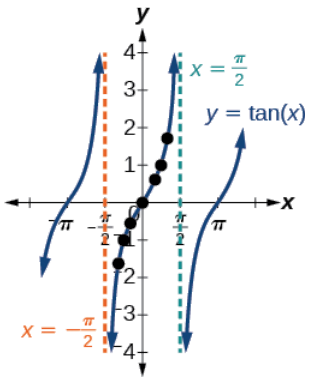


Figure* 6.2.1*: *Graph of the tangent function*

Graphing Variations of $y = \tan x$

As with the sine and cosine functions, the *tangent function* can be described by a general equation.

$$y = A\tan(Bx)$$

We can identify horizontal and vertical stretches and compressions using values of A and B. The horizontal stretch can typically be determined from the period of the graph. With tangent graphs, it is often necessary to determine a vertical stretch using a point on the graph.

Because there are no maximum or minimum values of a tangent function, the term *amplitude* cannot be interpreted as it is for the sine and cosine functions. Instead, we will use the phrase *stretching/compressing factor* when referring to the constant A.

FEATURES OF THE GRAPH OF $Y = A\tan(Bx)$

- The stretching factor is $|A|$.
- The period is $P = \dfrac{\pi}{|B|}$.
- The domain is all real numbers x,where $x \neq \dfrac{\pi}{2|B|} + \dfrac{\pi}{|B|}k$ such that k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{2|B|} + \dfrac{\pi}{|B|}k$ where k is an integer.
- $y = A\tan(Bx)$ is an odd function.

Graphing One Period of a Stretched or Compressed Tangent Function

We can use what we know about the properties of the **tangent function** to quickly sketch a graph of any stretched and/or compressed tangent function of the form $f(x) = A\tan(Bx)$. We focus on a single **period** of the function including the origin, because the periodic property enables us to extend the graph to the rest of the function's domain if we wish. Our limited domain is then the interval $\left(-\dfrac{P}{2}, \dfrac{P}{2}\right)$ and the graph has vertical asymptotes at $\pm\dfrac{P}{2}$ where $P = \dfrac{\pi}{B}$. On $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, the graph will come up from the left asymptote at $x = -\dfrac{\pi}{2}$, cross through the origin, and continue to increase as it approaches the right asymptote at $x = \dfrac{\pi}{2}$. To make the function approach the asymptotes at the correct rate, we also need to set the vertical scale by actually evaluating the function for at least one point that the graph will pass through. For example, we can use

$$f\left(\frac{P}{4}\right) = A\tan\left(B\frac{P}{4}\right) = A\tan\left(B\frac{\pi}{4B}\right) = A$$

because $\tan\left(\dfrac{\pi}{4}\right) = 1$.

Howto: Given the function $f(x) = A\tan(Bx)$, graph one period.

1. Identify the stretching factor, $|A|$.
2. Identify B and determine the period, $P = \dfrac{\pi}{|B|}$.
3. Draw vertical asymptotes at $x = -\dfrac{P}{2}$ and $x = \dfrac{P}{2}$.
4. For $A > 0$, the graph approaches the left asymptote at negative output values and the right asymptote at positive output values (reverse for $A < 0$).
5. Plot reference points at $\left(\dfrac{P}{4}, A\right)$, $(0, 0)$, and $\left(-\dfrac{P}{4}, -A\right)$, and draw the graph through these points.

Graphing One Period of a Shifted Tangent Function

Now that we can graph a **tangent function** that is stretched or compressed, we will add a vertical and/or horizontal (or phase) shift. In this case, we add C and D to the general form of the tangent function.

$$f(x) = A\tan(Bx - C) + D$$

The graph of a transformed tangent function is different from the basic tangent function $\tan x$ in several ways:

FEATURES OF THE GRAPH OF $Y = A\tan(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an odd integer.
- There is no amplitude.
- $y = A\tan(Bx)$ is and odd function because it is the qoutient of odd and even functions(sin and cosine perspectively).

Howto: Given the function $y = A\tan(Bx - C) + D$, sketch the graph of one period.

1. Express the function given in the form $y = A\tan(Bx - C) + D$.
2. Identify the **stretching/compressing** factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{\pi}{|B|}$.
4. Identify C and determine the phase shift, $\dfrac{C}{B}$.
5. Draw the graph of $y = A\tan(Bx)$ shifted to the right by $\dfrac{C}{B}$ and up by D.
6. Sketch the vertical asymptotes, which occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
7. Plot any three reference points and draw the graph through these points.

Analyzing the Graphs of $y = \sec x$ and $y = \csc x$

The secant was defined by the reciprocal identity $sec\, x = \dfrac{1}{\cos x}$. Notice that the function is undefined when the cosine is 0, leading to vertical asymptotes at $\dfrac{\pi}{2}$, $\dfrac{3\pi}{2}$ etc. Because the cosine is never more than 1 in absolute value, the secant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \sec x$ by observing the graph of the cosine function because these two functions are reciprocals of one another. See Figure 6.2.8. The graph of the cosine is shown as a dashed orange wave so we can see the relationship. Where the graph of the

cosine function decreases, the graph of the secant function increases. Where the graph of the cosine function increases, the graph of the secant function decreases. When the cosine function is zero, the secant is undefined.

The secant graph has vertical asymptotes at each value of x where the cosine graph crosses the x-axis; we show these in the graph below with dashed vertical lines, but will not show all the asymptotes explicitly on all later graphs involving the secant and cosecant.

Note that, because cosine is an even function, secant is also an even function. That is, $\sec(-x) = \sec x$.

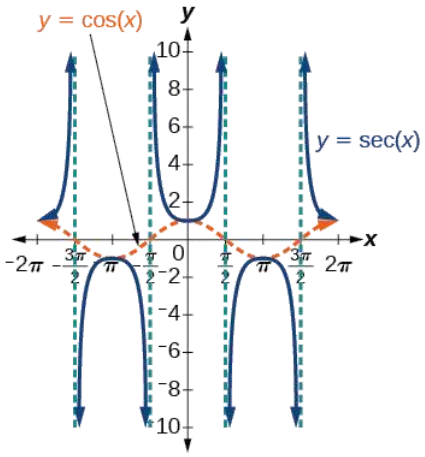


Figure* 6.2.8*: *Graph of the secant function,* $f(x) = \sec x = \dfrac{1}{\cos x}$

As we did for the tangent function, we will again refer to the constant $|A|$ as the stretching factor, not the amplitude.

FEATURES OF THE GRAPH OF $Y = A\sec(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{\pi}{2|B|}k$, where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

Similar to the secant, the **cosecant** is defined by the reciprocal identity $\csc x = \dfrac{1}{\sin x}$. Notice that the function is undefined when the sine is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the sine is never more than 1 in absolute value, the cosecant, being the reciprocal, will never be less than 1 in absolute value.

We can graph $y = \csc x$ by observing the graph of the sine function because these two functions are reciprocals of one another. See Figure $6.2.7$. The graph of sine is shown as a dashed orange wave so we can see the relationship. Where the graph of the sine function decreases, the graph of the cosecant function increases. Where the graph of the sine function increases, the graph of the **cosecant function** decreases.

The cosecant graph has vertical asymptotes at each value of x where the sine graph crosses the x-axis; we show these in the graph below with dashed vertical lines.

Note that, since sine is an odd function, the cosecant function is also an odd function. That is, $\csc(-x) = -\csc x$.

The graph of cosecant, which is shown in Figure $6.2.9$, is similar to the graph of secant.

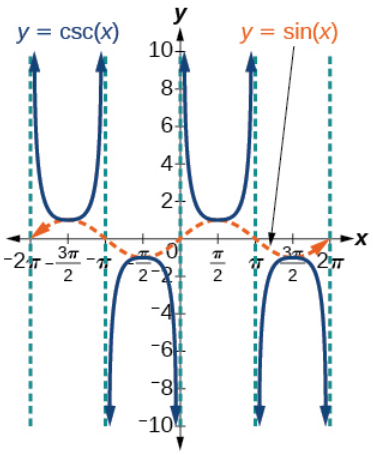


***Figure* 6.2.9:** *The graph of the cosecant function,* $f(x) = \csc x = \frac{1}{\sin x}$

FEATURES OF THE GRAPH OF $Y = A\csc(Bx)$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\csc(Bx)$ is an odd function because sine is an odd function.

Graphing Variations of $y = \sec x$ and $y = \csc x$

For shifted, compressed, and/or stretched versions of the secant and cosecant functions, we can follow similar methods to those we used for tangent and cotangent. That is, we locate the vertical asymptotes and also evaluate the functions for a few points (specifically the local extrema). If we want to graph only a single period, we can choose the interval for the period in more than one way. The procedure for secant is very similar, because the cofunction identity means that the secant graph is the same as the cosecant graph shifted half a period to the left. Vertical and phase shifts may be applied to the cosecant function in the same way as for the secant and other functions.The equations become the following.

$$y = A\sec(Bx - C) + D \tag{6.2.2}$$

$$y = A\csc(Bx - C) + D \tag{6.2.3}$$

FEATURES OF THE GRAPH OF $Y = A\sec(Bx - C) + D$

- The stretching factor is $|A|$.
- The period is $\dfrac{2\pi}{|B|}$.
- The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
- The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an odd integer.
- There is no amplitude.
- $y = A\sec(Bx)$ is an even function because cosine is an even function.

FEATURES OF THE GRAPH OF $Y = A\csc(Bx - C) + D$

1. The stretching factor is $|A|$.
2. The period is $\dfrac{2\pi}{|B|}$.
3. The domain is $x \neq \dfrac{C}{B} + \dfrac{\pi}{2|B|}k$,where k is an integer.
4. The range is $(-\infty, -|A|] \cup [|A|, \infty)$.
5. The vertical asymptotes occur at $x = \dfrac{C}{B} + \dfrac{\pi}{|B|}k$,where k is an integer.
6. There is no amplitude.
7. $y = A\csc(Bx)$ is an odd function because sine is an odd function.

HOWTO: Given a function of the form $y = A\sec(Bx)$, graph one period

1. Express the function given in the form $y = A\sec(Bx)$.
2. Identify the stretching/compressing factor, $|A|$.
3. Identify B and determine the period, $P = \dfrac{2\pi}{|B|}$.
4. Sketch the graph of $y = A\cos(Bx)$.
5. Use the reciprocal relationship between $y = \cos\ x$ and $y = \sec\ x$ to draw the graph of $y = A\sec(Bx)$.
6. Sketch the asymptotes.
7. Plot any two reference points and draw the graph through these points.

Analyzing the Graph of $y = \cot x$

The last trigonometric function we need to explore is cotangent. The cotangent is defined by the reciprocal identity $cot\ x = \dfrac{1}{\tan x}$. Notice that the function is undefined when the tangent function is 0, leading to a vertical asymptote in the graph at 0, π, etc. Since the output of the tangent function is all real numbers, the output of the cotangent function is also all real numbers.

We can graph $y = \cot x$ by observing the graph of the tangent function because these two functions are reciprocals of one another. See Figure 6.2.19. Where the graph of the tangent function decreases, the graph of the cotangent function increases. Where the graph of the tangent function increases, the graph of the cotangent function decreases.

The cotangent graph has vertical asymptotes at each value of x where $\tan x = 0$; we show these in the graph below with dashed lines. Since the cotangent is the reciprocal of the tangent, $\cot x$ has vertical asymptotes at all values of x where $\tan x = 0$, and $\cot x = 0$ at all values of x where $\tan x$ has its vertical asymptotes.

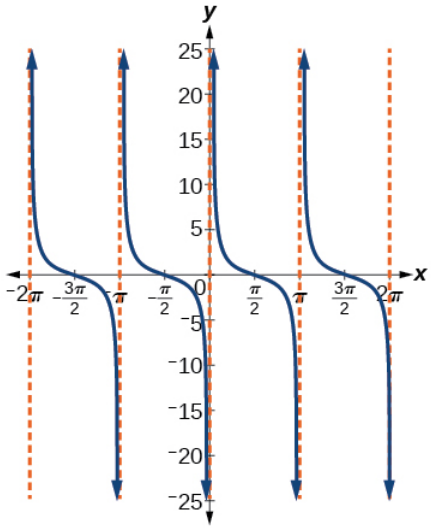


Figure* 6.2.19*: *The cotangent function*

FEATURES OF THE GRAPH OF $Y = A\cot(BX)$

- The stretching factor is $|A|$.
- The period is $P = \dfrac{\pi}{|B|}$.
- The domain is $x \neq \dfrac{\pi}{|B|}k$, where k is an integer.
- The range is $(-\infty, \infty)$.
- The asymptotes occur at $x = \dfrac{\pi}{|B|}k$, where k is an integer.
- $y = A\cot(Bx)$ is an odd function.

Using the Graphs of Trigonometric Functions to Solve Real-World Problems

Many real-world scenarios represent periodic functions and may be modeled by trigonometric functions. As an example, let's return to the scenario from the section opener. Have you ever observed the beam formed by the rotating light on a police car and wondered about the movement of the light beam itself across the wall? The periodic behavior of the distance the light shines as a function of time is obvious, but how do we determine the distance? We can use the **tangent function**.

Example 6.2.10: Using Trigonometric Functions to Solve Real-World Scenarios

Suppose the function $y = 5\tan(\dfrac{\pi}{4}t)$ marks the distance in the movement of a light beam from the top of a police car across a wall where t is the time in seconds and y is the distance in feet from a point on the wall directly across from the police car.

a. Find and interpret the stretching factor and period.
b. Graph on the interval $[0, 5]$.
c. Evaluate $f(1)$ and discuss the function's value at that input.

Solution

a. We know from the general form of $y = A\tan(Bt)$ that $|A|$ is the stretching factor and $\dfrac{\pi}{B}$ is the period.

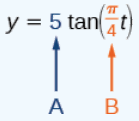


***Figure* 6.2.22**

We see that the stretching factor is 5. This means that the beam of light will have moved 5 ft after half the period.

The period is $\dfrac{\pi}{\frac{\pi}{4}} = \dfrac{\pi}{1}\cdot\dfrac{4}{\pi} = 4$. This means that every 4 seconds, the beam of light sweeps the wall. The distance from the spot across from the police car grows larger as the police car approaches.

b. To graph the function, we draw an asymptote at $t = 2$ and use the stretching factor and period. See Figure $6.2.23$

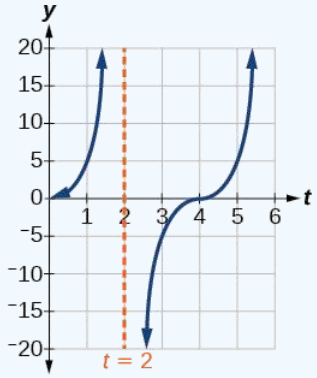


***Figure* 6.2.23**

c. period: $f(1) = 5\tan(\frac{\pi}{4}(1)) = 5(1) = 5$; after 1 second, the beam of has moved 5 ft from the spot across from the police car.

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6.3: Inverse Trigonometric Functions

Understanding and Using the Inverse Sine, Cosine, and Tangent Functions

In order to use inverse trigonometric functions, we need to understand that an inverse trigonometric function "undoes" what the original trigonometric function "does," as is the case with any other function and its inverse. In other words, the domain of the inverse function is the range of the original function, and vice versa, as summarized in Figure 6.3.1.

Trig Functions	Inverse Trig Functions
Domain: Measure of an angle	Domain: Ratio
Range: Ratio	Range: Measure of an angle

Figure 6.3.1

For example, if $f(x) = \sin\ x$, then we would write $f^{-1}(x) = \sin^{-1} x$. Be aware that $\sin^{-1} x$ does not mean $\dfrac{1}{\sin\ x}$. The following examples illustrate the inverse trigonometric functions:

- Since $\sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$, then $\dfrac{\pi}{6} = \sin^{-1}\left(\dfrac{1}{2}\right)$.
- Since $\cos(\pi) = -1$, then $\pi = \cos^{-1}(-1)$.
- Since $\tan\left(\dfrac{\pi}{4}\right) = 1$, then $\dfrac{\pi}{4} = \tan^{-1}(1)$.

In previous sections, we evaluated the trigonometric functions at various angles, but at times we need to know what angle would yield a specific sine, cosine, or tangent value. For this, we need inverse functions. Recall that, for a one-to-one function, if $f(a) = b$, then an inverse function would satisfy $f^{-1}(b) = a$.

Bear in mind that the sine, cosine, and tangent functions are not one-to-one functions. The graph of each function would fail the horizontal line test. In fact, no periodic function can be one-to-one because each output in its range corresponds to at least one input in every period, and there are an infinite number of periods. As with other functions that are not one-to-one, we will need to restrict the **domain** of each function to yield a new function that is one-to-one. We choose a domain for each function that includes the number 0. Figure 6.3.2 shows the graph of the sine function limited to $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ and the graph of the cosine function limited to $[0, \pi]$.

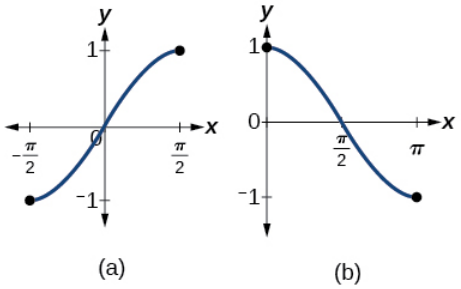


Figure 6.3.2: (a) Sine function on a restricted domain of $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$; (b) Cosine function on a restricted domain of $[0, \pi]$

Figure 6.3.3 shows the graph of the tangent function limited to $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

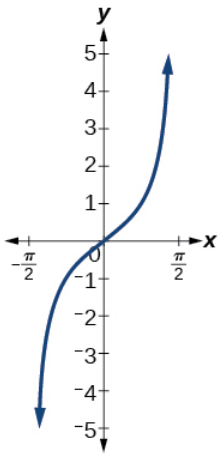


Figure 6.3.3: Tangent function on a restricted domain of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

These conventional choices for the restricted domain are somewhat arbitrary, but they have important, helpful characteristics. Each domain includes the origin and some positive values, and most importantly, each results in a one-to-one function that is invertible. The conventional choice for the restricted domain of the tangent function also has the useful property that it extends from one **vertical asymptote** to the next instead of being divided into two parts by an asymptote.

On these restricted domains, we can define the inverse trigonometric functions.

- The inverse sine function $y = \sin^{-1} x$ means $x = \sin\ y$. The inverse sine function is sometimes called the **arcsine** function, and notated $\arcsin\ x$.

$$y = \sin^{-1} x \text{ has domain } [-1, 1] \text{ and range } \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

- The inverse cosine function $y = \cos^{-1} x$ means $x = \cos\ y$. The inverse cosine function is sometimes called the **arccosine** function, and notated $\arccos\ x$.

$$y = \cos^{-1} x \text{ has domain } [-1, 1] \text{ and range } [0, \pi]$$

- The inverse tangent function $y = \tan^{-1} x$ means $x = \tan\ y$. The inverse tangent function is sometimes called the **arctangent** function, and notated $\arctan\ x$.

$$y = \tan^{-1} x \text{ has domain } (-\infty, \infty) \text{ and range } \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

The graphs of the inverse functions are shown in Figures 6.3.4 - 6.3.6. Notice that the output of each of these inverse functions is a *number*, an angle in radian measure. We see that $\sin^{-1} x$ has domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$, and $\tan^{-1} x$ has domain of all real numbers and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the domain and range of inverse trigonometric functions, switch the domain and range of the original functions. Each graph of the inverse trigonometric function is a reflection of the graph of the original function about the line $y = x$.

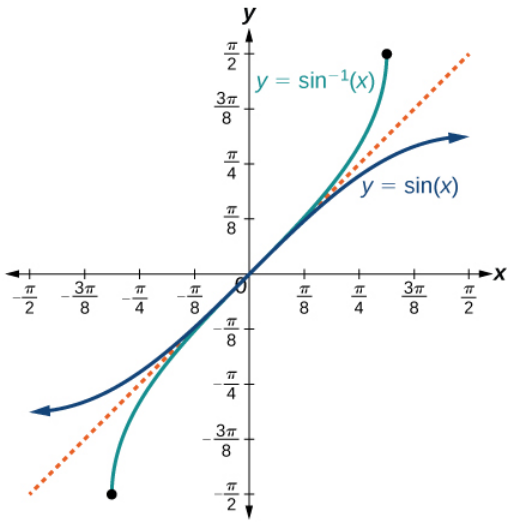


Figure 6.3.4: The sine function and inverse sine (or arcsine) function

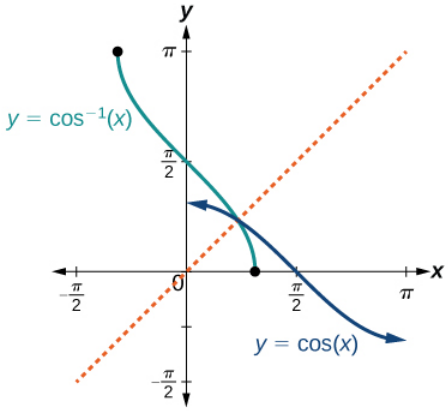


Figure 6.3.5: The cosine function and inverse cosine (or arccosine) function

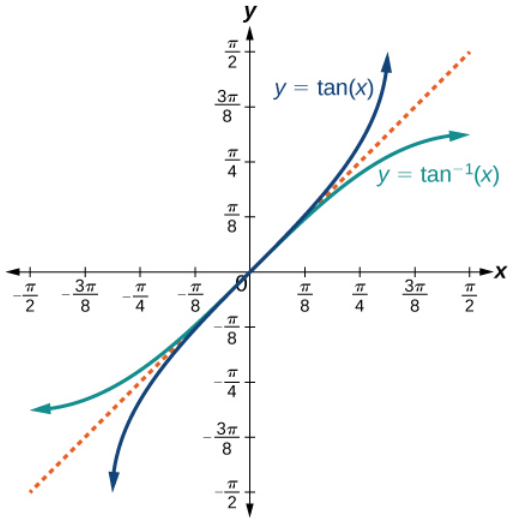


Figure 6.3.6: The tangent function and inverse tangent (or arctangent) function

RELATIONS FOR INVERSE SINE, COSINE, AND TANGENT FUNCTIONS

For angles in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, if $\sin y = x$, then $\sin^{-1} x = y$.

For angles in the interval $[0, \pi]$, if $\cos y = x$, then $\cos^{-1} x = y$.

For angles in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, if $\tan y = x$,then $\tan^{-1} x = y$.

Finding the Exact Value of Expressions Involving the Inverse Sine, Cosine, and Tangent Functions

Now that we can identify inverse functions, we will learn to evaluate them. For most values in their domains, we must evaluate the inverse trigonometric functions by using a calculator, interpolating from a table, or using some other numerical technique. Just as we did with the original trigonometric functions, we can give exact values for the inverse functions when we are using the special angles, specifically $\frac{\pi}{6}(30°)$, $\frac{\pi}{4}(45°)$, and $\frac{\pi}{3}(60°)$, and their reflections into other quadrants.

Given a "special" input value, evaluate an inverse trigonometric function.

1. Find angle x for which the original trigonometric function has an output equal to the given input for the inverse trigonometric function.
2. If x is not in the defined range of the inverse, find another angle y that is in the defined range and has the same sine, cosine, or tangent as x,depending on which corresponds to the given inverse function.

Finding Exact Values of Composite Functions with Inverse Trigonometric Functions

There are times when we need to compose a trigonometric function with an inverse trigonometric function. In these cases, we can usually find exact values for the resulting expressions without resorting to a calculator. Even when the input to the composite function is a variable or an expression, we can often find an expression for the output. To help sort out different cases, let $f(x)$ and $g(x)$ be two different trigonometric functions belonging to the set{ $\sin(x)$,$\cos(x)$,$\tan(x)$ } and let $f^{-1}(y)$ and $g^{-1}(y)$ be their inverses.

Evaluating Compositions of the Form $f(f^{-1}(y))$ and $f^{-1}(f(x))$

For any trigonometric function,$f(f^{-1}(y)) = y$ for all y in the proper domain for the given function. This follows from the definition of the inverse and from the fact that the range of f was defined to be identical to the domain of f^{-1}. However, we have to be a little more careful with expressions of the form $f^{-1}(f(x))$.

COMPOSITIONS OF A TRIGONOMETRIC FUNCTION AND ITS INVERSE

$$\begin{aligned}
\sin(\sin^{-1} x) &= x \quad && \text{for } -1 \le x \le 1 \\
\cos(\cos^{-1} x) &= x \quad && \text{for } -1 \le x \le 1 \\
\tan(\tan^{-1} x) &= x \quad && \text{for } -\infty < x < \infty \\
\sin^{-1}(\sin x) &= x \quad && \text{only for } -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\
\cos^{-1}(\cos x) &= x \quad && \text{only for } 0 \le x \le \pi \\
\tan^{-1}(\tan x) &= x \quad && \text{only for } -\frac{\pi}{2} < x < \frac{\pi}{2}
\end{aligned}$$

Given an expression of the form $f^{-1}(f(\theta))$ where $f(\theta) = \sin\theta$, $\cos\theta$, or $\tan\theta$, evaluate.

1. If θ is in the restricted domain of f, then $f^{-1}(f(\theta)) = \theta$.
2. If not, then find an angle ϕ within the restricted domain off f such that $f(\phi) = f(\theta)$. Then $f^{-1}(f(\theta)) = \phi$.

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CHAPTER OVERVIEW

7: Trigonometric Identities and Equations

In this chapter, we discuss how to manipulate trigonometric equations algebraically by applying various formulas and trigonometric identities. We will also investigate some of the ways that trigonometric equations are used to model real-life phenomena.

7.1: Solving Trigonometric Equations with Identities
7.2: Sum and Difference Identities
7.3: Double-Angle, Half-Angle, and Reduction Formulas
7.4: Sum-to-Product and Product-to-Sum Formulas
7.5: Solving Trigonometric Equations
7.6: Modeling with Trigonometric Equations

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7.1: Solving Trigonometric Equations with Identities

Verifying the Fundamental Trigonometric Identities

Identities enable us to simplify complicated expressions. They are the basic tools of trigonometry used in solving trigonometric equations, just as factoring, finding common denominators, and using special formulas are the basic tools of solving algebraic equations. In fact, we use algebraic techniques constantly to simplify trigonometric expressions. Basic properties and formulas of algebra, such as the difference of squares formula and the perfect squares formula, will simplify the work involved with trigonometric expressions and equations. We already know that all of the trigonometric functions are related because they all are defined in terms of the unit circle. Consequently, any trigonometric identity can be written in many ways.

To verify the trigonometric identities, we usually start with the more complicated side of the equation and essentially rewrite the expression until it has been transformed into the same expression as the other side of the equation. Sometimes we have to factor expressions, expand expressions, find common denominators, or use other algebraic strategies to obtain the desired result. In this first section, we will work with the fundamental identities: the **Pythagorean identities**, the even-odd identities, the reciprocal identities, and the quotient identities.

We will begin with the **Pythagorean identities** (Table 7.1.1), which are equations involving trigonometric functions based on the properties of a right triangle. We have already seen and used the first of these identifies, but now we will also use additional identities.

Table 7.1.1: Pythagorean Identities

$\sin^2\theta+\cos^2\theta=1$	$1+\cot^2\theta=\csc^2\theta$	$1+\tan^2\theta=\sec^2\theta$

The second and third identities can be obtained by manipulating the first. The identity $1+\cot^2\theta=\csc^2\theta$ is found by rewriting the left side of the equation in terms of sine and cosine.

Prove: $1+\cot^2\theta=\csc^2\theta$

$$\begin{aligned} 1+\cot^2\theta &= (1+\frac{\cos^2}{\sin^2}) && \text{Rewrite the left side} \\ &= \left(\frac{\sin^2}{\sin^2}\right)+\left(\frac{\cos^2}{\sin^2}\right) && \text{Write both terms with the common denominator} \\ &= \frac{\sin^2+\cos^2}{\sin^2} \\ &= \frac{1}{\sin^2} \\ &= \csc^2 \end{aligned}$$

Similarly,$1+\tan^2\theta=\sec^2\theta$ can be obtained by rewriting the left side of this identity in terms of sine and cosine. This gives

$$\begin{aligned} 1+\tan^2\theta &= 1+\left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Rewrite left side} \\ &= \left(\frac{\cos\theta}{\cos\theta}\right)^2+\left(\frac{\sin\theta}{\cos\theta}\right)^2 && \text{Write both terms with the common denominator} \\ &= \frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} \\ &= \frac{1}{\cos^2\theta} \\ &= \sec^2\theta \end{aligned}$$

Recall that we determined which trigonometric functions are odd and which are even. The next set of fundamental identities is the set of **even-odd identities.** The **even-odd identities** relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle (Table 7.1.2).

Table 7.1.2: Even-Odd Identities

$\tan(-\theta) = -\tan\theta$	$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$
$\cot(-\theta) = -\cot\theta$	$\csc(-\theta) = -\csc\theta$	$\sec(-\theta) = \sec\theta$

Recall that an **odd function** is one in which $f(-x) = -f(x)$ for all x in the domain off. f. The sine function is an odd function because $\sin(-\theta) = -\sin\theta$. The graph of an odd function is symmetric about the origin. For example, consider corresponding inputs of $\frac{\pi}{2}$ and $-\frac{\pi}{2}$. The output of $\sin\left(\frac{\pi}{2}\right)$ is opposite the output of $\sin\left(-\frac{\pi}{2}\right)$. Thus,

$$\sin\left(\frac{\pi}{2}\right) = 1$$
$$\sin\left(-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}\right)$$
$$= -1$$

This is shown in Figure 7.1.2.

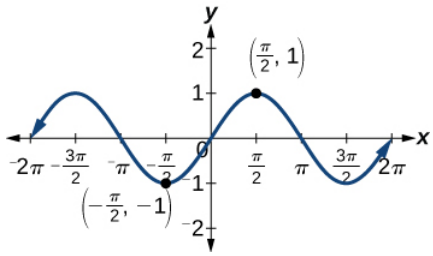


Figure 7.1.2: Graph of $y = \sin\theta$

Recall that an **even function** is one in which

$$f(-x) = f(x) \text{ for all } x \text{ in the domain of } f$$

The graph of an even function is symmetric about the *y*-axis. The cosine function is an even function because $\cos(-\theta) = \cos\theta$. For example, consider corresponding inputs $\frac{\pi}{4}$ and $-\frac{\pi}{4}$. The output of $\cos\left(\frac{\pi}{4}\right)$ is the same as the output of $\cos\left(-\frac{\pi}{4}\right)$. Thus,

$$\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$$
$$\approx 0.707$$

See Figure 7.1.3.

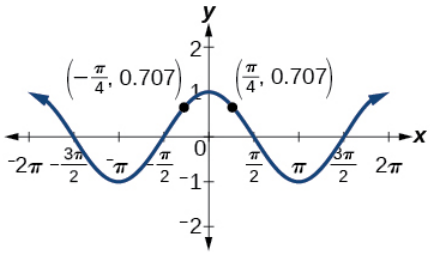


Figure 7.1.3: Graph of $y = \cos\theta$

For all θ in the domain of the sine and cosine functions, respectively, we can state the following:

- Since $\sin(-\theta) = -\sin\theta$,sine is an odd function.
- Since $\cos(-\theta) = \cos\theta$,cosine is an even function.

The other even-odd identities follow from the even and odd nature of the sine and cosine functions. For example, consider the tangent identity,$\tan(-\theta) = -\tan\theta$. We can interpret the tangent of a negative angle as

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta.$$

Tangent is therefore an odd function, which means that $\tan(-\theta) = -\tan(\theta)$ for all θ in the domain of the **tangent function.**

The cotangent identity, $\cot(-\theta) = -\cot\theta$,also follows from the sine and cosine identities. We can interpret the cotangent of a negative angle as

$$\cot(-\theta) = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta.$$

Cotangent is therefore an odd function, which means that $\cot(-\theta) = -\cot(\theta)$ for all θ in the domain of the **cotangent function.**

The **cosecant function** is the reciprocal of the sine function, which means that the cosecant of a negative angle will be interpreted as

$$\csc(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\csc\theta.$$

The cosecant function is therefore odd.

Finally, the secant function is the reciprocal of the cosine function, and the secant of a negative angle is interpreted as

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta.$$

The secant function is therefore even.

To sum up, only two of the trigonometric functions, cosine and secant, are even. The other four functions are odd, verifying the even-odd identities.

The next set of fundamental identities is the set of reciprocal identities, which, as their name implies, relate trigonometric functions that are reciprocals of each other. (Table 7.1.3). Recall that we first encountered these identities when defining trigonometric functions from right angles in Right Angle Trigonometry.

Table 7.1.3: Reciprocal Identities

$\sin\theta = \frac{1}{\csc\theta}$	$\csc\theta = \frac{1}{\sin\theta}$
$\cos\theta = \frac{1}{\sec\theta}$	$\sec\theta = \frac{1}{\cos\theta}$
$\tan\theta = \frac{1}{\cot\theta}$	$\cot\theta = \frac{1}{\tan\theta}$

The final set of identities is the set of **quotient identities,** which define relationships among certain trigonometric functions and can be very helpful in verifying other identities (Table 7.1.4).

Table 7.1.4: Quotient Identities

$\tan\theta = \frac{\sin\theta}{\cos\theta}$	$\cot\theta = \frac{\cos\theta}{\sin\theta}$

The reciprocal and quotient identities are derived from the definitions of the basic trigonometric functions.

SUMMARIZING TRIGONOMETRIC IDENTITIES

The Pythagorean identities are based on the properties of a right triangle.

$$\cos^2\theta + \sin^2\theta = 1 \tag{7.1.1}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{7.1.2}$$

$$1 + \tan^2\theta = \sec^2\theta \tag{7.1.3}$$

The even-odd identities relate the value of a trigonometric function at a given angle to the value of the function at the opposite angle.

$$\tan(-\theta) = -\tan\theta \tag{7.1.4}$$

$$\cot(-\theta) = -\cot\theta \tag{7.1.5}$$

$$\sin(-\theta) = -\sin\theta \tag{7.1.6}$$

$$\csc(-\theta) = -\csc\theta \tag{7.1.7}$$

$$\cos(-\theta) = \cos\theta \tag{7.1.8}$$

$$\sec(-\theta) = \sec\theta \tag{7.1.9}$$

The reciprocal identities define reciprocals of the trigonometric functions.

$$\sin\theta = \frac{1}{\csc\theta} \tag{7.1.10}$$

$$\cos\theta = \frac{1}{\sec\theta} \tag{7.1.11}$$

$$\tan\theta = \frac{1}{\cot\theta} \tag{7.1.12}$$

$$\csc\theta = \frac{1}{\sin\theta} \tag{7.1.13}$$

$$\sec\theta = \frac{1}{\cos\theta} \tag{7.1.14}$$

$$\cot\theta = \frac{1}{\tan\theta} \tag{7.1.15}$$

The **quotient identities** define the relationship among the trigonometric functions.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{7.1.16}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \tag{7.1.17}$$

Example 7.1.1: Graphing the Equations of an Identity

Graph both sides of the identity $\cot\theta = \frac{1}{\tan\theta}$. In other words, on the graphing calculator, graph $y = \cot\theta$ and $y = \frac{1}{\tan\theta}$.

Solution

See Figure 7.1.4.

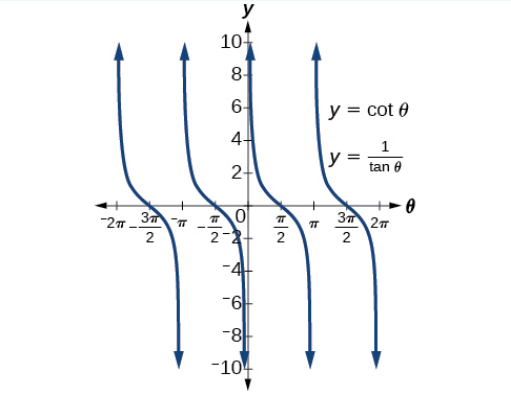


Figure 7.1.4

Analysis

We see only one graph because both expressions generate the same image. One is on top of the other. This is a good way to prove any identity. If both expressions give the same graph, then they must be identities.

How to: Given a trigonometric identity, verify that it is true.

1. Work on one side of the equation. It is usually better to start with the more complex side, as it is easier to simplify than to build.
2. Look for opportunities to factor expressions, square a binomial, or add fractions.
3. Noting which functions are in the final expression, look for opportunities to use the identities and make the proper substitutions.
4. If these steps do not yield the desired result, try converting all terms to sines and cosines.

Example 7.1.2: Verifying a Trigonometric Identity

Verify $\tan\theta\cos\theta = \sin\theta$.

Solution

We will start on the left side, as it is the more complicated side:

$$\begin{aligned}\tan\theta\cos\theta &= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta \\ &= \sin\theta\end{aligned}$$

Analysis

This identity was fairly simple to verify, as it only required writing $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$.

Example 7.1.3A: Verifying the Equivalency Using the Even-Odd Identities

Verify the following equivalency using the even-odd identities:

$$(1+\sin x)[1+\sin(-x)] = \cos^2 x$$

Solution

Working on the left side of the equation, we have

$$(1+\sin x)[1+\sin(-x)] = (1+\sin x)(1-\sin x)$$

Since

$$\begin{aligned}\sin(-x) &= -\sin x \\ [5pt] &= 1-\sin^2 x \qquad \text{Difference of squares} \\ [5pt] &= \cos^2 x \\ \cos^2 x &= 1-\sin^2 x\end{aligned}$$

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7.2: Sum and Difference Identities

Using the Sum and Difference Formulas for Cosine

Finding the exact value of the sine, cosine, or tangent of an angle is often easier if we can rewrite the given angle in terms of two angles that have known trigonometric values. We can use the special angles, which we can review in the unit circle shown in Figure 7.2.2.

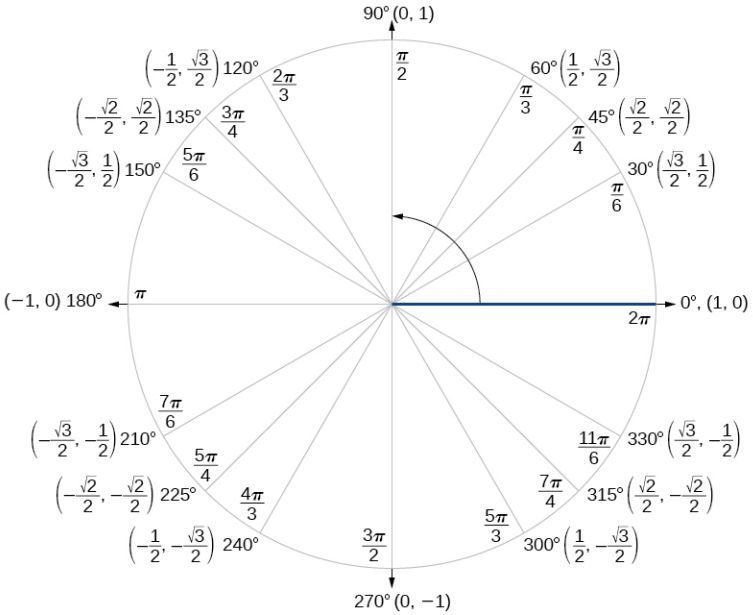


Figure 7.2.2: The Unit Circle

We will begin with the sum and difference formulas for cosine, so that we can find the cosine of a given angle if we can break it up into the sum or difference of two of the special angles (Table 7.2.1).

Table 7.2.1

Sum formula for cosine	$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
Difference formula for cosine	$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

First, we will prove the difference formula for cosines. Let's consider two points on the unit circle (Figure 7.2.3). Point P is at an angle α from the positive x-axis with coordinates $(\cos\alpha, \sin\alpha)$ and point Q is at an angle of β from the positive x-axis with coordinates $(\cos\beta, \sin\beta)$. Note the measure of angle POQ is $\alpha-\beta$.

Label two more points: A at an angle of $(\alpha-\beta)$ from the positive x-axis with coordinates $(\cos(\alpha-\beta), \sin(\alpha-\beta))$; and point B with coordinates $(1, 0)$. Triangle POQ is a rotation of triangle AOB and thus the distance from P to Q is the same as the distance from A to B.

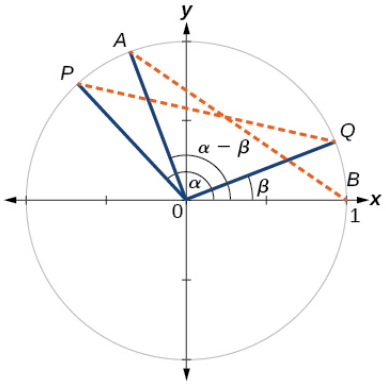


Figure 7.2.3

We can find the distance from P to Q using the distance formula.

$$
\begin{aligned}
d_{PQ} &= \sqrt{(\cos\alpha - \cos\beta)^2 + (sin\alpha - sin\beta)^2} \\
&= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} && \text{Apply Pythagorean identity and simplify.} \\
&= \sqrt{(\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\
&= \sqrt{1 + 1 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta} \\
&= \sqrt{2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta}
\end{aligned}
$$

Similarly, using the distance formula we can find the distance from A to B.

$$
\begin{aligned}
d_{AB} &= \sqrt{(\cos(\alpha-\beta) - 1)^2 + (\sin(\alpha-\beta) - 0)^2} \\
&= \sqrt{\cos^2(\alpha-\beta) - 2\cos(\alpha-\beta) + 1 + \sin^2(\alpha-\beta)} && \text{Apply Pythagorean identity and simplify} \\
&= \sqrt{(\cos^2(\alpha-\beta) + \sin^2(\alpha-\beta)) - 2\cos(\alpha-\beta) + 1} \\
&= \sqrt{1 - 2\cos(\alpha-\beta) + 1} \\
&= \sqrt{2 - 2\cos(\alpha-\beta)} && \text{Subtract 2 from both sides and divide both sides by } -2.
\end{aligned}
$$

$$\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$$

Thus, we have the difference formula for cosine. We can use similar methods to derive the cosine of the sum of two angles.

SUM AND DIFFERENCE FORMULAS FOR COSINE

These formulas can be used to calculate the cosine of sums and differences of angles.

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{7.2.1}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{7.2.2}$$

How to: Given two angles, find the cosine of the difference between the angles

1. Write the difference formula for cosine.
2. Substitute the values of the given angles into the formula.
3. Simplify.

Example 7.2.1: Finding the Exact Value Using the Formula for the Cosine of the Difference of Two Angles

Using the formula for the cosine of the difference of two angles, find the exact value of $\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right)$.

Solution

Begin by writing the formula for the cosine of the difference of two angles. Then substitute the given values.

$$
\begin{aligned}
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\cos\left(\frac{5\pi}{4} - \frac{\pi}{6}\right) &= \cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{5\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\
&= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{-\sqrt{6} - \sqrt{2}}{4}
\end{aligned}
$$

Keep in mind that we can always check the answer using a graphing calculator in radian mode.

Using the Sum and Difference Formulas for Sine

The sum and difference formulas for sine can be derived in the same manner as those for cosine, and they resemble the cosine formulas.

SUM AND DIFFERENCE FORMULAS FOR SINE

These formulas can be used to calculate the sines of sums and differences of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta \tag{7.2.3}$$

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta \tag{7.2.4}$$

How to: Given two angles, find the sine of the difference between the angles

1. Write the difference formula for sine.
2. Substitute the given angles into the formula.
3. Simplify.

Example 7.2.3: Using Sum and Difference Identities to Evaluate the Difference of Angles

Use the sum and difference identities to evaluate the difference of the angles and show that part *a* equals part *b*.

a. $\sin(45^\circ-30^\circ)$
b. $\sin(135^\circ-120^\circ)$

Solution

a. Let's begin by writing the formula and substitute the given angles.

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \sin(45^\circ-30^\circ) &= \sin(45^\circ)\cos(30^\circ)-\cos(45^\circ)\sin(30^\circ)\end{aligned}$$

Next, we need to find the values of the trigonometric expressions.

$$\sin(45^\circ)=\tfrac{\sqrt{2}}{2},\qquad \cos(30^\circ)=\tfrac{\sqrt{3}}{2},\qquad \cos(45^\circ)=\tfrac{\sqrt{2}}{2},\qquad \sin(30^\circ)=\tfrac{1}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(45^\circ-30^\circ) &= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)-\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

b. Again, we write the formula and substitute the given angles.

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \sin(135^\circ-120^\circ) &= \sin(135^\circ)\cos(120^\circ)-\cos(135^\circ)\sin(120^\circ)\end{aligned}$$

Next, we find the values of the trigonometric expressions.

$$\sin(135^\circ)=\tfrac{\sqrt{2}}{2},\qquad \cos(120^\circ)=-\tfrac{1}{2},\qquad \cos(135^\circ)=\tfrac{\sqrt{2}}{2},\qquad \sin(120^\circ)=\tfrac{\sqrt{3}}{2}$$

Now we can substitute these values into the equation and simplify.

$$\begin{aligned}\sin(135^\circ-120^\circ) &= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Using the Sum and Difference Formulas for Tangent

Finding exact values for the tangent of the sum or difference of two angles is a little more complicated, but again, it is a matter of recognizing the pattern.

Finding the sum of two angles formula for tangent involves taking quotient of the sum formulas for sine and cosine and simplifying. Recall, $\tan x=\dfrac{\sin x}{\cos x}$, when $\cos x\neq 0$.

Let's derive the sum formula for tangent.

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\ &= \frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{1-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} \\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\end{aligned}$$

We can derive the difference formula for tangent in a similar way.

SUM AND DIFFERENCE FORMULAS FOR TANGENT

The sum and difference formulas for tangent are:

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \tag{7.2.5}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \tag{7.2.6}$$

How to: Given two angles, find the tangent of the sum of the angles

1. Write the sum formula for tangent.
2. Substitute the given angles into the formula.
3. Simplify.

Example 7.2.5: Finding the Exact Value of an Expression Involving Tangent

Find the exact value of $\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$.

Solution

Let's first write the sum formula for tangent and then substitute the given angles into the formula.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\tan\left(\frac{\pi}{6}\right)+\tan\left(\frac{\pi}{4}\right)}{1-\left(\tan\left(\frac{\pi}{6}\right)\right)\left(\tan\left(\frac{\pi}{4}\right)\right)}$$

Next, we determine the individual function values within the formula:

$$\tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}},\quad \text{and}\quad \tan\left(\frac{\pi}{4}\right)=1 \tag{7.2.7}$$

So we have,

$$\tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\frac{1}{\sqrt{3}}+1}{1-\left(\frac{1}{\sqrt{3}}\right)(1)}$$
$$=\frac{\frac{1+\sqrt{3}}{\sqrt{3}}}{\frac{\sqrt{3}-1}{\sqrt{3}}}$$
$$=\frac{1+\sqrt{3}}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}-1}$$
$$=\frac{\sqrt{3}+1}{\sqrt{3}-1}$$

Using Sum and Difference Formulas for Cofunctions

Now that we can find the sine, cosine, and tangent functions for the sums and differences of angles, we can use them to do the same for their cofunctions. You may recall from Right Triangle Trigonometry that, if the sum of two positive angles is $\frac{\pi}{2}$, those two angles are complements, and the sum of the two acute angles in a right triangle is $\frac{\pi}{2}$, so they are also complements. In Figure 7.2.6, notice that if one of the acute angles is labeled as θ, then the other acute angle must be labeled $\frac{\pi}{2}-\theta$.

Notice also that $\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$,which is opposite over hypotenuse. Thus, when two angles are complimentary, we can say that the sine of θ equals the cofunction of the complement of θ. Similarly, tangent and cotangent are cofunctions, and secant and cosecant are cofunctions.

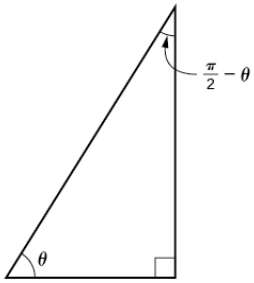


Figure 7.2.6

From these relationships, the cofunction identities are formed. Recall that you first encountered these identities in The Unit Circle: Sine and Cosine Functions.

COFUNCTION IDENTITIES

The cofunction identities are summarized in Table 7.2.2.

Table 7.2.2

$\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)$	$\cos\theta=\sin\left(\frac{\pi}{2}-\theta\right)$
$\tan\theta=\cot\left(\frac{\pi}{2}-\theta\right)$	$\cot\theta=\tan\left(\frac{\pi}{2}-\theta\right)$
$\sec\theta=\csc\left(\frac{\pi}{2}-\theta\right)$	$\csc\theta=\sec\left(\frac{\pi}{2}-\theta\right)$

Notice that the formulas in the table may also justified algebraically using the sum and difference formulas. For example, using

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

We can write

$$\cos\left(\frac{\pi}{2}-\theta\right)=\cos\frac{\pi}{2}\cos\theta+\sin\frac{\pi}{2}\sin\theta$$
$$=(0)\cos\theta+(1)\sin\theta$$
$$=\sin\theta$$

Example 7.2.7: Finding a Cofunction with the Same Value as the Given Expression

Write $\tan\frac{\pi}{9}$ in terms of its cofunction.

Solution

The cofunction of $\tan\theta = \cot\left(\frac{\pi}{2}-\theta\right)$. Thus,

$$\begin{aligned}\tan\left(\frac{\pi}{9}\right) &= \cot\left(\frac{\pi}{2}-\frac{\pi}{9}\right)\\ &= \cot\left(\frac{9\pi}{18}-\frac{2\pi}{18}\right)\\ &= \cot\left(\frac{7\pi}{18}\right)\end{aligned}$$

Using the Sum and Difference Formulas to Verify Identities

Verifying an identity means demonstrating that the equation holds for all values of the variable. It helps to be very familiar with the identities or to have a list of them accessible while working the problems. Reviewing the general rules presented earlier may help simplify the process of verifying an identity.

How to: Given an identity, verify using sum and difference formulas

1. Begin with the expression on the side of the equal sign that appears most complex. Rewrite that expression until it matches the other side of the equal sign. Occasionally, we might have to alter both sides, but working on only one side is the most efficient.
2. Look for opportunities to use the sum and difference formulas.
3. Rewrite sums or differences of quotients as single quotients.
4. If the process becomes cumbersome, rewrite the expression in terms of sines and cosines.

Example 7.2.8A: Verifying an Identity Involving Sine

Verify the identity $\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta$.

Solution

We see that the left side of the equation includes the sines of the sum and the difference of angles.

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

We can rewrite each using the sum and difference formulas.

$$\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta\\ &= 2\sin\alpha\cos\beta\end{aligned}$$

We see that the identity is verified.

Example 7.2.8B: Verifying an Identity Involving Tangent

Verify the following identity.

$$\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\tan\alpha-\tan\beta$$

Solution

We can begin by rewriting the numerator on the left side of the equation.

$$\begin{aligned}\frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta} &= \frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\\ &= \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta} && \text{Rewrite using a common denominator}\\ &= \frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta} && \text{Cancel}\\ &= \tan\alpha-\tan\beta && \text{Rewrite in terms of tangent}\end{aligned}$$

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7.3: Double-Angle, Half-Angle, and Reduction Formulas

Using Double-Angle Formulas to Find Exact Values

In the previous section, we used addition and subtraction formulas for trigonometric functions. Now, we take another look at those same formulas. The **double-angle formulas** are a special case of the sum formulas, where $\alpha = \beta$. Deriving the double-angle formula for sine begins with the sum formula,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{7.3.1}$$

If we let $\alpha = \beta = \theta$, then we have

$$\begin{aligned}\sin(\theta+\theta) &= \sin\theta\cos\theta + \cos\theta\sin\theta \\ \sin(2\theta) &= 2\sin\theta\cos\theta\end{aligned}$$

Deriving the double-angle for cosine gives us three options. First, starting from the sum formula, $\cos(\alpha+\beta) = \cos\,\alpha\,\cos\,\beta - \sin\,\alpha\,\sin\,\beta$,and letting $\alpha = \beta = \theta$, we have

$$\begin{aligned}\cos(\theta+\theta) &= \cos\theta\cos\theta - \sin\theta\sin\theta \\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Using the Pythagorean properties, we can expand this double-angle formula for cosine and get two more variations. The first variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= (1-\sin^2\theta) - \sin^2\theta\end{aligned}$$

The second variation is:

$$\begin{aligned}\cos(2\theta) &= \cos^2\theta - \sin^2\theta \\ &= \cos^2\theta - (1-\cos^2\theta) \\ &= 2\cos^2\theta - 1\end{aligned}$$

Similarly, to derive the double-angle formula for tangent, replacing $\alpha = \beta = \theta$ in the sum formula gives

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\ \tan(\theta+\theta) &= \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta} \\ \tan(2\theta) &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

DOUBLE-ANGLE FORMULAS

The *double-angle formulas* are summarized as follows:

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{7.3.2}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1 \tag{7.3.3}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \tag{7.3.4}$$

How to: Given the tangent of an angle and the quadrant in which it is located, use the double-angle formulas to find the exact value

1. Draw a triangle to reflect the given information.
2. Determine the correct double-angle formula.

3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 7.3.1: Using a Double-Angle Formula to Find the Exact Value Involving Tangent

Given that $\tan\theta = -\frac{3}{4}$ and θ is in quadrant II, find the following:

a. $\sin(2\theta)$
b. $\cos(2\theta)$
c. $\tan(2\theta)$

Solution

If we draw a triangle to reflect the information given, we can find the values needed to solve the problems on the image. We are given $\tan\theta = -\frac{3}{4}$,such that θ is in quadrant II. The tangent of an angle is equal to the opposite side over the adjacent side, and because θ is in the second quadrant, the adjacent side is on the *x*-axis and is negative. Use the **Pythagorean Theorem** to find the length of the hypotenuse:

$$\begin{aligned}(-4)^2+(3)^2 &= c^2\\ 16+9 &= c^2\\ 25 &= c^2\\ c &= 5\end{aligned}$$

Now we can draw a triangle similar to the one shown in Figure 7.3.2.

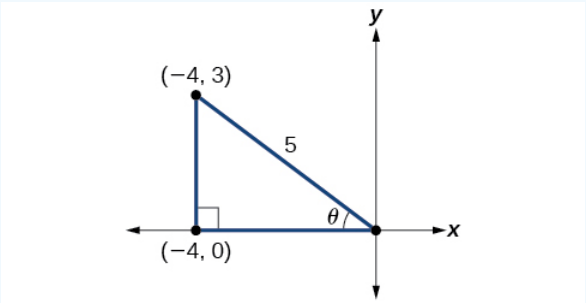


Figure 7.3.2

a. Let's begin by writing the double-angle formula for sine.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

We see that we to need to find $\sin\theta$ and $\cos\theta$. Based on Figure 7.3.2, we see that the hypotenuse equals 5, so $\sin\theta = 35$, $\sin\theta = 35$, and $\cos\theta = -45$. Substitute these values into the equation, and simplify.

Thus,

$$\begin{aligned}\sin(2\theta) &= 2\left(\frac{3}{5}\right)\left(-\frac{4}{5}\right)\\ &= -\frac{24}{25}\end{aligned}$$

b. Write the double-angle formula for cosine.

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Again, substitute the values of the sine and cosine into the equation, and simplify.

$$\begin{aligned}\cos(2\theta) &= \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 \\ &= \frac{16}{25} - \frac{9}{25} \\ &= \frac{7}{25}\end{aligned}$$

c. Write the double-angle formula for tangent.

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

In this formula, we need the tangent, which we were given as $\tan\theta = -\frac{3}{4}$. Substitute this value into the equation, and simplify.

$$\begin{aligned}\tan(2\theta) &= \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2} \\ &= \frac{-\frac{3}{2}}{1-\frac{9}{16}} \\ &= -\frac{3}{2}\left(\frac{16}{7}\right) \\ &= -\frac{24}{7}\end{aligned}$$

Use Reduction Formulas to Simplify an Expression

The double-angle formulas can be used to derive the **reduction formulas,** which are formulas we can use to reduce the power of a given expression involving even powers of sine or cosine. They allow us to rewrite the even powers of sine or cosine in terms of the first power of cosine. These formulas are especially important in higher-level math courses, calculus in particular. Also called the power-reducing formulas, three identities are included and are easily derived from the double-angle formulas.

We can use two of the three double-angle formulas for cosine to derive the reduction formulas for sine and cosine. Let's begin with $\cos(2\theta) = 1 - 2\sin^2\theta$. Solve for $\sin^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 1 - 2\sin^2\theta \\ 2\sin^2\theta &= 1 - \cos(2\theta) \\ \sin^2\theta &= \frac{1-\cos(2\theta)}{2}\end{aligned}$$

Next, we use the formula $\cos(2\theta) = 2\cos^2\theta - 1$. Solve for $\cos^2\theta$:

$$\begin{aligned}\cos(2\theta) &= 2\cos^2\theta - 1 \\ 1 + \cos(2\theta) &= 2\cos^2\theta \\ \frac{1+\cos(2\theta)}{2} &= \cos^2\theta\end{aligned}$$

The last reduction formula is derived by writing tangent in terms of sine and cosine:

$$\begin{aligned}\tan^2\theta &= \frac{\sin^2\theta}{\cos^2\theta} \\ &= \frac{\frac{1-\cos(2\theta)}{2}}{\frac{1+\cos(2\theta)}{2}} && \text{(Substitute the reduction formulas)} \\ &= \left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{2}{1+\cos(2\theta)}\right) \\ &= \frac{1-\cos(2\theta)}{1+\cos(2\theta)}\end{aligned}$$

REDUCTION FORMULAS

The reduction formulas are summarized as follows:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2} \tag{7.3.5}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2} \tag{7.3.6}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \tag{7.3.7}$$

Example 7.3.5: Writing an Equivalent Expression Not Containing Powers Greater Than 1

Write an equivalent expression for $\cos^4 x$ that does not involve any powers of sine or cosine greater than 1.

Solution

We will apply the reduction formula for cosine twice.

$$\begin{aligned}\cos^4 x &= \left(\cos^2 x\right)^2 \\ &= \left(\frac{1+\cos(2x)}{2}\right)^2 && \text{Substitute reduction formula} \\ &= \frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{4}\left(\frac{1+\cos^2(2x)}{2}\right) && \text{Substitute reduction formula for } \cos^2 x \\ &= \frac{1}{4}+\frac{1}{2}\cos(2x)+\frac{1}{8}+\frac{1}{8}\cos(4x) \\ &= \frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{8}\cos(4x)\end{aligned}$$

Analysis

The solution is found by using the reduction formula twice, as noted, and the perfect square formula from algebra.

Using Half-Angle Formulas to Find Exact Values

The next set of identities is the set of **half-angle formulas**, which can be derived from the reduction formulas and we can use when we have an angle that is half the size of a special angle. If we replace θ with $\frac{\alpha}{2}$,the half-angle formula for sine is found by simplifying the equation and solving for $\sin\left(\frac{\alpha}{2}\right)$. Note that the half-angle formulas are preceded by a $\pm$ sign. This does not mean that both the positive and negative expressions are valid. Rather, it depends on the quadrant in which $\frac{\alpha}{2}$ terminates.

The half-angle formula for sine is derived as follows:

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\left(\cos 2\cdot\frac{\alpha}{2}\right)}{2}$$

$$= \frac{1-\cos\alpha}{2}$$

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

To derive the half-angle formula for cosine, we have

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}{2}$$

$$= \frac{1+\cos\alpha}{2}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

For the tangent identity, we have

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

$$\tan^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\left(2\cdot\frac{\alpha}{2}\right)}{1+\cos\left(2\cdot\frac{\alpha}{2}\right)}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

HALF-ANGLE FORMULAS

The **half-angle formulas** are as follows:

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}} \tag{7.3.8}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \tag{7.3.9}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} \tag{7.3.10}$$

Example 7.3.7

Using a Half-Angle Formula to Find the Exact Value of a Sine Function. Find $\sin(15^\circ)$ using a half-angle formula.

Solution

Since $15^\circ = \frac{30^\circ}{2}$,we use the half-angle formula for sine (Equation 7.3.8):

$$\sin\frac{30^\circ}{2}=\sqrt{\frac{1-\cos 30^\circ}{2}}$$
$$=\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$$
$$=\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}}$$
$$=\sqrt{\frac{2-\sqrt{3}}{4}}$$
$$=\frac{\sqrt{2-\sqrt{3}}}{2}$$

Remember that we can check the answer with a graphing calculator.

Analysis

Notice that we used only the positive root because $\sin(15^\circ)$ is positive.

Howto: Given the tangent of an angle and the quadrant in which the angle lies, find the exact values of trigonometric functions of half of the angle.

1. Draw a triangle to represent the given information.
2. Determine the correct half-angle formula.
3. Substitute values into the formula based on the triangle.
4. Simplify.

Example 7.3.8: Finding Exact Values Using Half-Angle Identities

Given that $\tan\alpha=\frac{8}{15}$ and α lies in quadrant III, find the exact value of the following:

a. $\sin\left(\frac{\alpha}{2}\right)$
b. $\cos\left(\frac{\alpha}{2}\right)$
c. $\tan\left(\frac{\alpha}{2}\right)$

Solution

Using the given information, we can draw the triangle shown in Figure 7.3.3. Using the Pythagorean Theorem, we find the hypotenuse to be 17. Therefore, we can calculate $\sin\alpha=-\frac{8}{17}$ and $\cos\alpha=-\frac{15}{17}$.

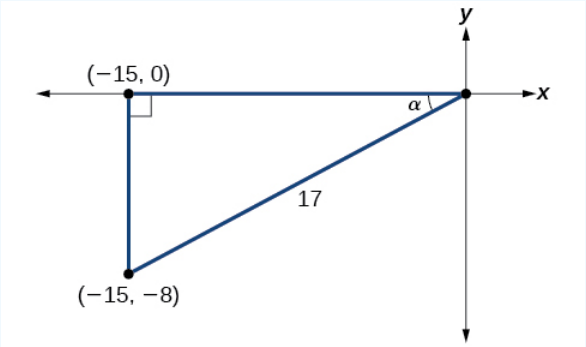


Figure 7.3.3

a. Before we start, we must remember that if α is in quadrant III, then $180^\circ < \alpha < 270^\circ$,so $\frac{180^\circ}{2} < \frac{\alpha}{2} < \frac{270^\circ}{2}$. This means that the terminal side of $\frac{\alpha}{2}$ is in quadrant II, since $90^\circ < \frac{\alpha}{2} < 135^\circ$. To find $\sin \frac{\alpha}{2}$,we begin by writing the half-angle formula for sine. Then we substitute the value of the cosine we found from the triangle in Figure $7.3.3$ and simplify.

$$\begin{aligned}
\sin \frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
&= \pm\sqrt{\frac{1-(-\frac{15}{17})}{2}} \\
&= \pm\sqrt{\frac{\frac{32}{17}}{2}} \\
&= \pm\sqrt{\frac{32}{17}\cdot\frac{1}{2}} \\
&= \pm\sqrt{\frac{16}{17}} \\
&= \pm\frac{4}{\sqrt{17}} \\
&= \frac{4\sqrt{17}}{17}
\end{aligned}$$

We choose the positive value of $\sin \frac{\alpha}{2}$ because the angle terminates in quadrant II and sine is positive in quadrant II.

b. To find $\cos \frac{\alpha}{2}$,we will write the half-angle formula for cosine, substitute the value of the cosine we found from the triangle in Figure $7.3.3$, and simplify.

$$\begin{aligned}
\cos \frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}} \\
&= \pm\sqrt{\frac{1+\left(-\frac{15}{17}\right)}{2}} \\
&= \pm\sqrt{\frac{\frac{2}{17}}{2}} \\
&= \pm\sqrt{\frac{2}{17}\cdot\frac{1}{2}} \\
&= \pm\sqrt{\frac{1}{17}} \\
&= -\frac{\sqrt{17}}{17}
\end{aligned}$$

We choose the negative value of $\cos \frac{\alpha}{2}$ because the angle is in quadrant II because cosine is negative in quadrant II.

c. To find $\tan \frac{\alpha}{2}$,we write the half-angle formula for tangent. Again, we substitute the value of the cosine we found from the triangle in Figure $7.3.3$ and simplify.

$$\begin{aligned}\tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\ &= \pm\sqrt{\frac{1-\left(-\frac{15}{17}\right)}{1+\left(-\frac{15}{17}\right)}} \\ &= \pm\sqrt{\frac{\frac{32}{17}}{\frac{2}{17}}} \\ &= \pm\sqrt{\frac{32}{2}} \\ &= -\sqrt{16} \\ &= -4\end{aligned}$$

We choose the negative value of $\tan\frac{\alpha}{2}$ because $\frac{\alpha}{2}$ lies in quadrant II, and tangent is negative in quadrant II.

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7.4: Sum-to-Product and Product-to-Sum Formulas

Expressing Products as Sums

We have already learned a number of formulas useful for expanding or simplifying trigonometric expressions, but sometimes we may need to express the product of cosine and sine as a sum. We can use the **product-to-sum formulas,** which express products of trigonometric functions as sums. Let's investigate the cosine identity first and then the sine identity.

Expressing Products as Sums for Cosine

We can derive the product-to-sum formula from the sum and difference identities for **cosine**. If we add the two equations, we get:

$$\begin{aligned} \cos\alpha\cos\beta+\sin\alpha\sin\beta &= \cos(\alpha-\beta) \\ +\cos\alpha\cos\beta-\sin\alpha\sin\beta &= \cos(\alpha+\beta) \\ \hline 2\cos\alpha\cos\beta &= \cos(\alpha-\beta)+\cos(\alpha+\beta) \end{aligned}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{7.4.1}$$

How to: Given a product of cosines, express as a sum

1. Write the formula for the product of cosines.
2. Substitute the given angles into the formula.
3. Simplify.

Example 7.4.1: Writing the Product as a Sum Using the Product-to-Sum Formula for Cosine

Write the following product of cosines as a sum: $2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right)$.

Solution

We begin by writing the formula for the product of cosines (Equation 7.4.1):

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

We can then substitute the given angles into the formula and simplify.

$$\begin{aligned} 2\cos\left(\frac{7x}{2}\right)\cos\left(\frac{3x}{2}\right) &= 2\left(\frac{1}{2}\right)\left[\cos\left(\frac{7x}{2}-\frac{3x}{2}\right)+\cos\left(\frac{7x}{2}+\frac{3x}{2}\right)\right] \\ &= \cos\left(\frac{4x}{2}\right)+\cos\left(\frac{10x}{2}\right) \\ &= \cos 2x+\cos 5x \end{aligned}$$

Exercise 7.4.1

Use the product-to-sum formula (Equation 7.4.1) to write the product as a sum or difference: $\cos(2\theta)\cos(4\theta)$.

Answer

$\frac{1}{2}(\cos 6\theta+\cos 2\theta)$

Expressing the Product of Sine and Cosine as a Sum

Next, we will derive the product-to-sum formula for sine and cosine from the sum and difference formulas for **sine**. If we add the sum and difference identities, we get:

$$\begin{aligned} \cos\alpha\cos\beta+\sin\alpha\sin\beta &= \cos(\alpha-\beta) \\ +\cos\alpha\cos\beta-\sin\alpha\sin\beta &= \cos(\alpha+\beta) \\ \hline 2\cos\alpha\cos\beta &= \cos(\alpha-\beta)+\cos(\alpha+\beta) \end{aligned}$$

Then, we divide by 2 to isolate the product of cosines:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$

Example 7.4.2: Writing the Product as a Sum Containing only Sine or Cosine

Express the following product as a sum containing only sine or cosine and no products: $\sin(4\theta)\cos(2\theta)$.

Solution

Write the formula for the product of sine and cosine. Then substitute the given values into the formula and simplify.

$$\begin{aligned} \sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \\ \sin(4\theta)\cos(2\theta) &= \frac{1}{2}[\sin(4\theta+2\theta)+\sin(4\theta-2\theta)] \\ &= \frac{1}{2}[\sin(6\theta)+\sin(2\theta)] \end{aligned}$$

Expressing Products of Sines in Terms of Cosine

Expressing the product of sines in terms of **cosine** is also derived from the sum and difference identities for cosine. In this case, we will first subtract the two cosine formulas:

$$\begin{aligned} \cos(\alpha-\beta) &= \cos\alpha\cos\beta+\sin\alpha\sin\beta \\ -\cos(\alpha+\beta) &= -(\cos\alpha\cos\beta-\sin\alpha\sin\beta) \\ \hline \cos(\alpha-\beta)-\cos(\alpha+\beta) &= 2\sin\alpha\sin\beta \end{aligned}$$

Then, we divide by 2 to isolate the product of sines:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

Similarly we could express the product of cosines in terms of sine or derive other product-to-sum formulas.

THE PRODUCT-TO-SUM FORMULAS

The **product-to-sum** formulas are as follows:

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{7.4.2}$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \tag{7.4.3}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \tag{7.4.4}$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)] \tag{7.4.5}$$

Expressing Sums as Products

Some problems require the reverse of the process we just used. The **sum-to-product formulas** allow us to express sums of sine or cosine as products. These formulas can be derived from the product-to-sum identities. For example, with a few substitutions, we can derive the sum-to-product identity for **sine**. Let $\frac{u+v}{2}=\alpha$ and $\frac{u-v}{2}=\beta$.

Then,

$$\begin{aligned}\alpha+\beta &= \frac{u+v}{2}+\frac{u-v}{2}\\ &= \frac{2u}{2}\\ &= u\\ \alpha-\beta &= \frac{u+v}{2}-\frac{u-v}{2}\\ &= \frac{2v}{2}\\ &= v\end{aligned}$$

Thus, replacing α and β in the product-to-sum formula with the substitute expressions, we have

$$\begin{aligned}\sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]\\ \sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) &= \frac{1}{2}[\sin u+\sin v] \qquad \text{Substitute for } (\alpha+\beta) \text{ and } (\alpha\beta)\\ 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) &= \sin u+\sin v\end{aligned}$$

The other sum-to-product identities are derived similarly.

SUM-TO-PRODUCT FORMULAS

The **sum-to-product formulas** are as follows:

$$\sin\alpha+\sin\beta=2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{7.4.6}$$

$$\sin\alpha-\sin\beta=2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{7.4.7}$$

$$\cos\alpha-\cos\beta=-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{7.4.8}$$

$$\cos\alpha+\cos\beta=2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{7.4.9}$$

Example 7.4.4: Writing the Difference of Sines as a Product

Write the following difference of sines expression as a product: $\sin(4\theta)-\sin(2\theta)$.

Solution

We begin by writing the formula for the difference of sines.

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

Substitute the values into the formula, and simplify.

$$\begin{aligned}\sin(4\theta) - \sin(2\theta) &= 2\sin\left(\frac{4\theta-2\theta}{2}\right)\cos\left(\frac{4\theta+2\theta}{2}\right)\\ &= 2\sin\left(\frac{2\theta}{2}\right)\cos\left(\frac{6\theta}{2}\right)\\ &= 2\sin\theta\cos(3\theta)\end{aligned}$$

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7.5: Solving Trigonometric Equations

Solving Linear Trigonometric Equations in Sine and Cosine

Trigonometric equations are, as the name implies, equations that involve trigonometric functions. Similar in many ways to solving polynomial equations or rational equations, only specific values of the variable will be solutions, if there are solutions at all. Often we will solve a trigonometric equation over a specified interval. However, just as often, we will be asked to find all possible solutions, and as trigonometric functions are periodic, solutions are repeated within each period. In other words, trigonometric equations may have an infinite number of solutions. Additionally, like rational equations, the domain of the function must be considered before we assume that any solution is valid. The **period** of both the sine function and the cosine function is 2π. In other words, every 2π units, the y-values repeat. If we need to find all possible solutions, then we must add $2\pi k$,where k is an integer, to the initial solution. Recall the rule that gives the format for stating all possible solutions for a function where the period is 2π:

$$\sin\theta = \sin(\theta \pm 2k\pi) \tag{7.5.1}$$

There are similar rules for indicating all possible solutions for the other trigonometric functions. Solving trigonometric equations requires the same techniques as solving algebraic equations. We read the equation from left to right, horizontally, like a sentence. We look for known patterns, factor, find common denominators, and substitute certain expressions with a variable to make solving a more straightforward process. However, with trigonometric equations, we also have the advantage of using the identities we developed in the previous sections.

Example 7.5.1A: Solving a Linear Trigonometric Equation Involving the Cosine Function

Find all possible exact solutions for the equation $\cos\theta = \dfrac{1}{2}$.

Solution

From the unit circle, we know that

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3},\ \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\theta = \frac{\pi}{3} \pm 2k\pi \quad \text{and} \quad \theta = \frac{5\pi}{3} \pm 2k\pi$$

where k is an integer.

Example 7.5.1B: Solving a Linear Equation Involving the Sine Function

Find all possible exact solutions for the equation $\sin t = \dfrac{1}{2}$.

Solution

Solving for all possible values of t means that solutions include angles beyond the period of 2π. From the section on Sum and Difference Identities, we can see that the solutions are $t = \dfrac{\pi}{6}$ and $t = \dfrac{5\pi}{6}$. But the problem is asking for all possible values that solve the equation. Therefore, the answer is

$$t = \frac{\pi}{6} \pm 2\pi k \quad \text{and} \quad t = \frac{5\pi}{6} \pm 2\pi k$$

where k is an integer.

Howto: Given a trigonometric equation, solve using algebra

1. Look for a pattern that suggests an algebraic property, such as the difference of squares or a factoring opportunity.
2. Substitute the trigonometric expression with a single variable, such as x or u.
3. Solve the equation the same way an algebraic equation would be solved.
4. Substitute the trigonometric expression back in for the variable in the resulting expressions.
5. Solve for the angle.

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7.6: Modeling with Trigonometric Equations

Determining the Amplitude and Period of a Sinusoidal Function

Any motion that repeats itself in a fixed time period is considered **periodic motion** and can be modeled by a **sinusoidal function**. The amplitude of a sinusoidal function is the distance from the midline to the maximum value, or from the midline to the minimum value. The midline is the average value. Sinusoidal functions oscillate above and below the midline, are periodic, and repeat values in set cycles. Recall from Graphs of the Sine and Cosine Functions that the **period** of the sine function and the cosine function is 2π. In other words, for any value of x,

$$\sin(x \pm 2\pi k) = \sin x \text{ and } \cos(x \pm 2\pi k) = \cos x \tag{7.6.1}$$

where k is an integer.

STANDARD FORM OF SINUSOIDAL EQUATIONS

The general forms of a sinusoidal equation are given as

$$y = A\sin(Bt - C) + D \tag{7.6.2}$$

or

$$y = A\cos(Bt - C) + D \tag{7.6.3}$$

where $\text{amplitude} = |A|$, B is related to period such that the $\text{period} = \frac{2\pi}{B}$, C is the phase shift such that $\frac{C}{B}$ denotes the horizontal shift, and D represents the vertical shift from the graph's parent graph.

Note that the models are sometimes written as

$$y = a\sin(\omega t \pm C) + D \tag{7.6.4}$$

or

$$y = a\cos(\omega t \pm C) + D, \tag{7.6.5}$$

with a period that is given as $\frac{2\pi}{\omega}$.

The difference between the sine and the cosine graphs is that the sine graph begins with the average value of the function and the cosine graph begins with the maximum or minimum value of the function.

Example 7.6.1: Showing How the Properties of a Trigonometric Function Can Transform a Graph

Show the transformation of the graph of $y = \sin x$ into the graph of $y = 2\sin(4x - \frac{\pi}{2}) + 2$.

Solution

Consider the series of graphs in Figure $7.6.1$ and the way each change to the equation changes the image.

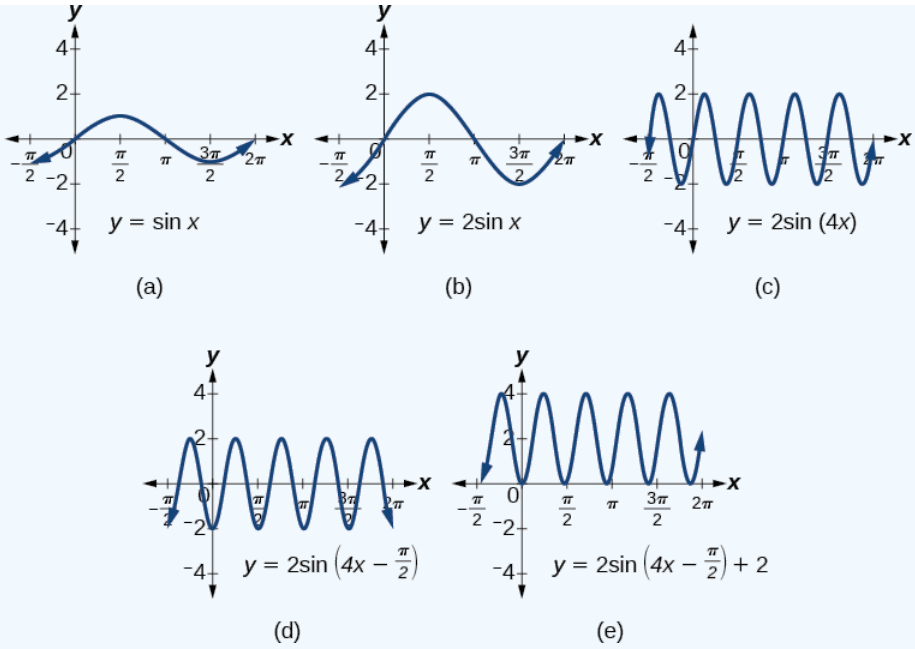


Figure 7.6.2:

a. The basic graph of $y = \sin x$
b. Changing the amplitude from 1 to 2 generates the graph of $y = 2\sin x$.
c. The period of the sine function changes with the value of B, such that period $= \frac{2\pi}{B}$. Here we have $B = 4$, which translates to a period of $\frac{\pi}{2}$. The graph completes one full cycle in $\frac{\pi}{2}$ units.
d. The graph displays a horizontal shift equal to $\frac{C}{B}$, or $\frac{\frac{\pi}{2}}{4} = \frac{\pi}{8}$.
e. Finally, the graph is shifted vertically by the value of D. In this case, the graph is shifted up by 2 units.

Modeling Periodic Behavior

We will now apply these ideas to problems involving periodic behavior.

Example 7.6.4: Modeling an Equation and Sketching a Sinusoidal Graph to Fit Criteria

The average monthly temperatures for a small town in Oregon are given in Table 7.6.1. Find a sinusoidal function of the form $y = A\sin(Bt - C) + D$ that fits the data (round to the nearest tenth) and sketch the graph.

Table 7.6.2

Month	Temperature, ^oF
January	42.5
February	44.5
March	48.5
April	52.5
May	58
June	63
July	68.5
August	69
September	64.5
October	55.5

Month	Temperature, ^oF
November	46.5
December	43.5

Solution

Recall that amplitude is found using the formula

$$A = \frac{\text{largest value} - \text{smallest value}}{2} \tag{7.6.6}$$

Thus, the amplitude is

$$\begin{aligned} |A| &= \frac{69 - 42.5}{2} \\ &= 13.25 \end{aligned}$$

The data covers a period of 12 months, so $\frac{2\pi}{B} = 12$ which gives $B = \frac{2\pi}{12} = \frac{\pi}{6}$.

The vertical shift is found using the following equation.

$$D = \frac{\text{highest value} + \text{lowest value}}{2} \tag{7.6.7}$$

Thus, the vertical shift is

$$D = \frac{69 + 42.5}{2} \qquad = 55.8$$

So far, we have the equation $y = 13.3\sin(\frac{\pi}{6}x - C) + 55.8$.

To find the horizontal shift, we input the x and y values for the first month and solve for C.

$$\begin{aligned} 42.5 &= 13.3\sin(\frac{\pi}{6}(1) - C) + 55.8 \\ -13.3 &= 13.3\sin(\frac{\pi}{6} - C) \\ -1 &= \sin(\frac{\pi}{6} - C) \qquad \sin\theta = -1 \to \theta = -\frac{\pi}{2} \\ \frac{\pi}{6} - C &= -\frac{\pi}{2} \\ \frac{\pi}{6} + \frac{\pi}{2} &= C \\ \frac{2\pi}{3} &= C \end{aligned}$$

We have the equation $y = 13.3\sin(\frac{\pi}{6}x - \frac{2\pi}{3}) + 55.8$. See the graph in Figure 7.6.9.

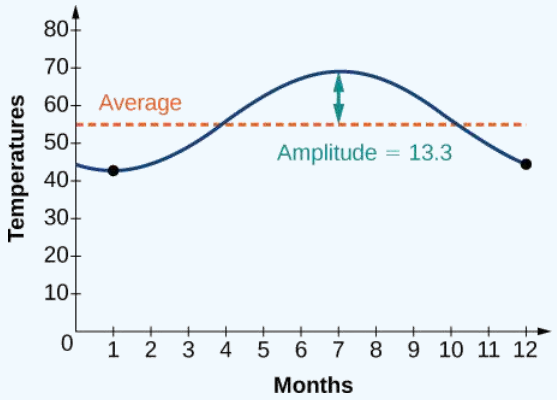


Figure 7.6.9

Modeling Harmonic Motion Functions

Harmonic motion is a form of periodic motion, but there are factors to consider that differentiate the two types. While general **periodic motion** applications cycle through their periods with no outside interference, **harmonic motion** requires a restoring force. Examples of harmonic motion include springs, gravitational force, and magnetic force.

Simple Harmonic Motion

A type of motion described as **simple harmonic motion** involves a restoring force but assumes that the motion will continue forever. Imagine a weighted object hanging on a spring, When that object is not disturbed, we say that the object is at rest, or in equilibrium. If the object is pulled down and then released, the force of the spring pulls the object back toward equilibrium and harmonic motion begins. The restoring force is directly proportional to the displacement of the object from its equilibrium point. When $t = 0, d = 0$.

SIMPLE HARMONIC MOTION

We see that **simple harmonic motion** equations are given in terms of displacement:

$$d = a\cos(\omega t) \text{ or } d = a\sin(\omega t) \tag{7.6.8}$$

where $|a|$ is the amplitude, $\frac{2\pi}{\omega}$ is the period, and $\frac{\omega}{2\pi}$ is the frequency, or the number of cycles per unit of time.

? Exercise 7.6.5: Finding the Displacement, Period, and Frequency, and Graphing a Function

For each the given functions:

a. $y = 5\sin(3t)$
b. $y = 6\cos(\pi t)$
c. $y = 5\cos(\frac{\pi}{2})t$

address the following questions:

1. Find the maximum displacement of an object.
2. Find the period or the time required for one vibration.
3. Find the frequency.
4. Sketch the graph.

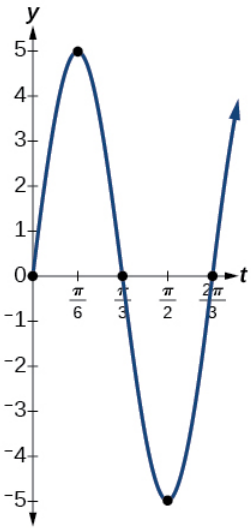


Figure 7.6.13

Answer a

1. The maximum displacement is equal to the amplitude,$|a|$, which is 5.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{3}$.

3. The frequency is given as $\frac{\omega}{2\pi} = \frac{3}{2\pi}$.
4. See Figure 7.6.13 The graph indicates the five key points.

Answer b

1. The maximum displacement is 6.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2$.
3. The frequency is $\frac{\omega}{2\pi} = \frac{\pi}{2\pi} = \frac{1}{2}$.
4. See Figure 7.6.14

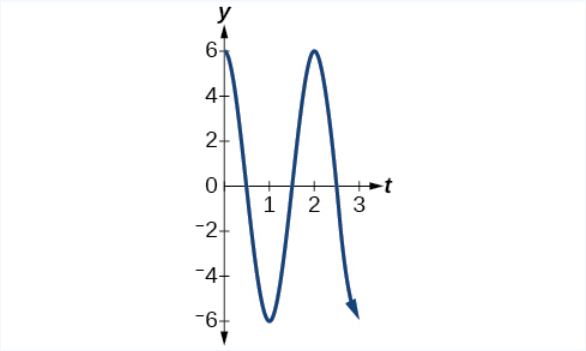


Figure 7.6.14

Answer c

1. The maximum displacement is 5.
2. The period is $\frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4$.
3. The frequency is $\frac{1}{4}$.
4. See Figure 7.6.15

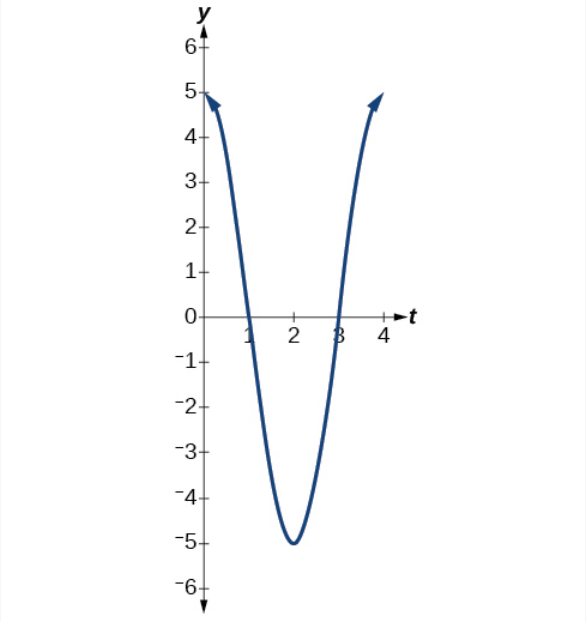


Figure 7.6.15

Damped Harmonic Motion

In reality, a pendulum does not swing back and forth forever, nor does an object on a spring bounce up and down forever. Eventually, the pendulum stops swinging and the object stops bouncing and both return to equilibrium. Periodic motion in which an energy-dissipating force, or damping factor, acts is known as **damped harmonic motion**. Friction is typically the damping factor.

In physics, various formulas are used to account for the damping factor on the moving object. Some of these are calculus-based formulas that involve derivatives. For our purposes, we will use formulas for basic damped harmonic motion models.

Definition: DAMPED HARMONIC MOTION

In *damped harmonic motion*, the displacement of an oscillating object from its rest position at time t is given as

$$f(t) = ae^{-ct}\sin(\omega t) \text{ or } f(t) = ae^{-ct}\cos(\omega t) \tag{7.6.9}$$

where c is a damping factor, $|a|$ is the initial displacement and $\frac{2\pi}{\omega}$ is the period.

Example 7.6.8: Modeling Damped Harmonic Motion

Model the equations that fit the two scenarios and use a graphing utility to graph the functions: Two mass-spring systems exhibit damped harmonic motion at a frequency of 0.5 cycles per second. Both have an initial displacement of 10 cm. The first has a damping factor of 0.5 and the second has a damping factor of 0.1.

Solution

At time $t = 0$, the displacement is the maximum of 10 cm, which calls for the cosine function. The cosine function will apply to both models.

We are given the frequency $f = \frac{\omega}{2\pi}$ of 0.5 cycles per second. Thus,

$$\begin{aligned} \frac{\omega}{2\pi} &= 0.5 \\ \omega &= (0.5)2\pi \\ &= \pi \end{aligned}$$

The first spring system has a damping factor of $c = 0.5$. Following the general model for damped harmonic motion, we have

$$f(t) = 10e^{-0.5t}\cos(\pi t)$$

Figure 7.6.16 models the motion of the first spring system.

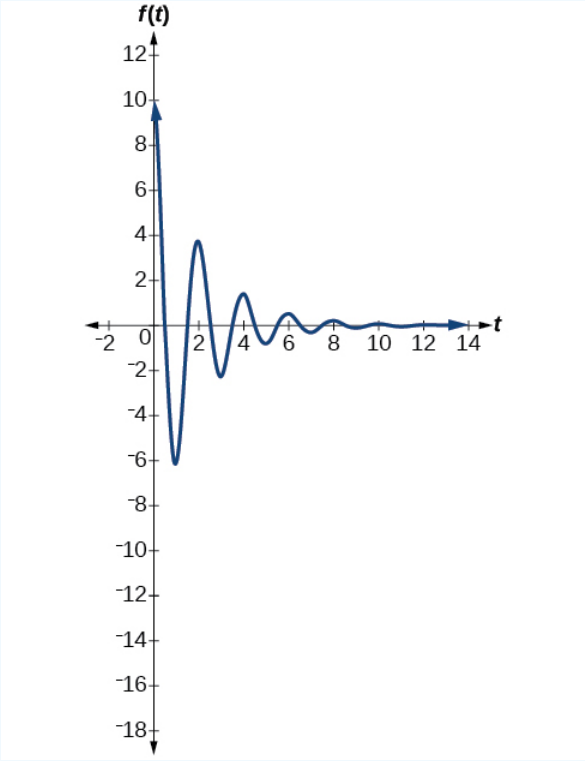


Figure 7.6.16

The second spring system has a damping factor of $c = 0.1$ and can be modeled as

$$f(t) = 10e^{-0.1t}\cos(\pi t) \tag{7.6.10}$$

Figure $7.6.17$models the motion of the second spring system.

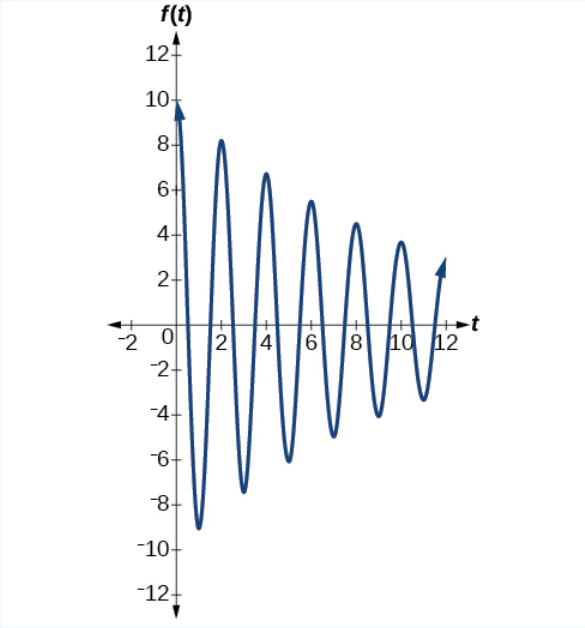


Figure 7.6.17

Analysis

Notice the differing effects of the damping constant. The local maximum and minimum values of the function with the damping factor $c = 0.5$ decreases much more rapidly than that of the function with $c = 0.1$.

Bounding Curves in Harmonic Motion

Harmonic motion graphs may be enclosed by bounding curves. When a function has a varying **amplitude**, such that the amplitude rises and falls multiple times within a period, we can determine the bounding curves from part of the function.

Example 7.6.12: Graphing an Oscillating Cosine Curve

Graph the function $f(x) = \cos(2\pi x)\cos(16\pi x)$

Solution

The graph produced by this function will be shown in two parts. The first graph will be the exact function $f(x)$ (see Figure $7.6.23; top$), and the second graph is the exact function $f(x)$ plus a bounding function (see Figure $7.6.23; bottom$). The graphs look quite different.



Figure_07_07_016.jpg" />

***Figure* 7.6.23**

Analysis

The curves $y = \cos(2\pi x)$ and $y = -\cos(2\pi x)$ are bounding curves: they bound the function from above and below, tracing out the high and low points. The harmonic motion graph sits inside the bounding curves. This is an example of a function

whose amplitude not only decreases with time, but actually increases and decreases multiple times within a period.

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CHAPTER OVERVIEW

8: Further Applications of Trigonometry

In this chapter, we will explore applications of trigonometry that will enable us to solve many different kinds of problems, including finding the height of a tree. We extend topics we introduced in Trigonometric Functions and investigate applications more deeply and meaningfully.

8.1: Non-right Triangles - Law of Sines
8.2: Non-right Triangles - Law of Cosines
8.3: Polar Coordinates
8.4: Polar Coordinates - Graphs
8.5: Polar Form of Complex Numbers
8.6: Parametric Equations

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8.1: Non-right Triangles - Law of Sines

Using the Law of Sines to Solve Oblique Triangles

In any triangle, we can draw an **altitude**, a perpendicular line from one vertex to the opposite side, forming two right triangles. It would be preferable, however, to have methods that we can apply directly to non-right triangles without first having to create right triangles.

Any triangle that is not a right triangle is an **oblique triangle**. Solving an oblique triangle means finding the measurements of all three angles and all three sides. To do so, we need to start with at least three of these values, including at least one of the sides. We will investigate three possible oblique triangle problem situations:

ASA (angle-side-angle) We know the measurements of two angles and the included side. See Figure 8.1.2.

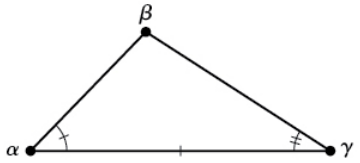


***Figure* 8.1.2**

AAS (angle-angle-side) We know the measurements of two angles and a side that is not between the known angles. See Figure 8.1.3.

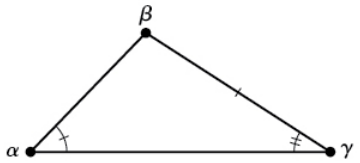


***Figure* 8.1.3**

SSA (side-side-angle) We know the measurements of two sides and an angle that is not between the known sides. See Figure 8.1.4.

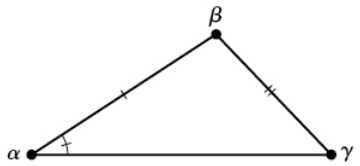


***Figure* 8.1.4**

Knowing how to approach each of these situations enables us to solve oblique triangles without having to drop a perpendicular to form two right triangles. Instead, we can use the fact that the ratio of the measurement of one of the angles to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. Let's see how this statement is derived by considering the triangle shown in Figure 8.1.5.

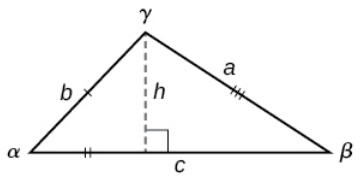


***Figure* 8.1.5**

Using the right triangle relationships, we know that $\sin\alpha = \frac{h}{b}$ and $\sin\beta = \frac{h}{a}$. Solving both equations for h gives two different expressions for h.

$$h = b\sin\alpha \text{ and } h = a\sin\beta$$

We then set the expressions equal to each other.

$$\begin{aligned} b\sin\alpha &= a\sin\beta \\ \left(\frac{1}{ab}\right)(b\sin\alpha) &= (a\sin\beta)\left(\frac{1}{ab}\right) \qquad \text{Multiply both sides by } \frac{1}{ab} \\ \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \end{aligned}$$

Similarly, we can compare the other ratios.

$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} \text{ and } \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Collectively, these relationships are called the **Law of Sines**.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

Note the standard way of labeling triangles: angle α (alpha) is opposite side a; angle β (beta) is opposite side b; and angle γ (gamma) is opposite side c. See Figure 8.1.6.

While calculating angles and sides, be sure to carry the exact values through to the final answer. Generally, final answers are rounded to the nearest tenth, unless otherwise specified.

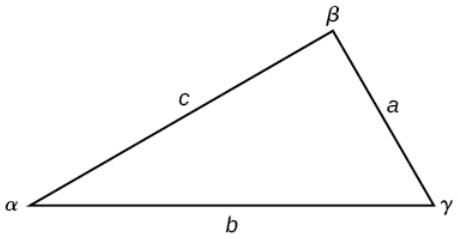


***Figure* 8.1.6**

Note: LAW OF SINES

Given a triangle with angles and opposite sides labeled as in Figure 8.1.6, the ratio of the measurement of an angle to the length of its opposite side will be equal to the other two ratios of angle measure to opposite side. All proportions will be equal. The **Law of Sines** is based on proportions and is presented symbolically two ways.

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \tag{8.1.1}$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \tag{8.1.2}$$

To solve an oblique triangle, use any pair of applicable ratios.

Example 8.1.1: Solving for Two Unknown Sides and Angle of an AAS Triangle

Solve the triangle shown in Figure 8.1.7 to the nearest tenth.

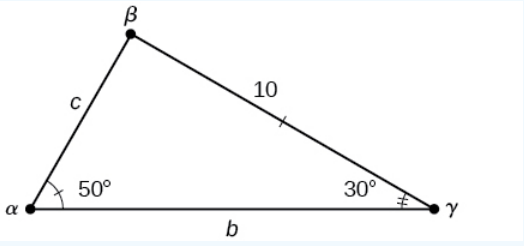


Figure **8.1.7**

Solution

The three angles must add up to 180 degrees. From this, we can determine that

$$\begin{aligned}\beta &= 180^\circ - 50^\circ - 30^\circ \\ &= 100^\circ\end{aligned}$$

To find an unknown side, we need to know the corresponding angle and a known ratio. We know that angle $\alpha = 50^\circ$ and its corresponding side $a = 10$. We can use the following proportion from the Law of Sines to find the length of c.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(30^\circ)}{c} \\ c\frac{\sin(50^\circ)}{10} &= \sin(30^\circ) \qquad \text{Multiply both sides by } c \\ c &= \sin(30^\circ)\frac{10}{\sin(50^\circ)} \qquad \text{Multiply by the reciprocal to isolate } c \\ c &\approx 6.5\end{aligned}$$

Similarly, to solve for b, we set up another proportion.

$$\begin{aligned}\frac{\sin(50^\circ)}{10} &= \frac{\sin(100^\circ)}{b} \\ b\sin(50^\circ) &= 10\sin(100^\circ) \qquad \text{Multiply both sides by } b \\ b &= \frac{10\sin(100^\circ)}{\sin(50^\circ)} \qquad \text{Multiply by the reciprocal to isolate } b \\ b &\approx 12.9\end{aligned}$$

Therefore, the complete set of angles and sides is

$$\begin{aligned}\alpha &= 50^\circ \quad & a &= 10 \\ \beta &= 100^\circ \quad & b &\approx 12.9 \\ \gamma &= 30^\circ \quad & c &\approx 6.5\end{aligned}$$

Finding the Area of an Oblique Triangle Using the Sine Function

Now that we can solve a triangle for missing values, we can use some of those values and the sine function to find the area of an oblique triangle. Recall that the area formula for a triangle is given as $Area = \frac{1}{2}bh$, where b is base and h is height. For oblique triangles, we must find h before we can use the area formula. Observing the two triangles in Figure 8.1.15, one acute and one obtuse, we can drop a perpendicular to represent the height and then apply the trigonometric property $\sin\alpha = \frac{opposite}{hypotenuse}$ to write an equation for area in oblique triangles. In the acute triangle, we have $\sin\alpha = \frac{h}{c}$ or $c\sin\alpha = h$. However, in the obtuse triangle, we drop the perpendicular outside the triangle and extend the base b to form a right triangle. The angle used in calculation is α', or $180 - \alpha$.

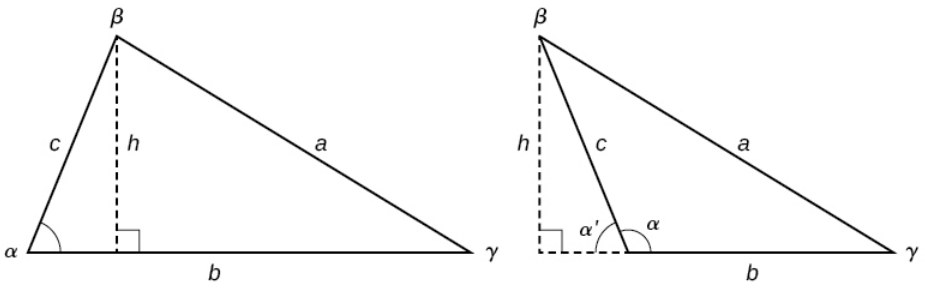


Figure **8.1.15**

Thus,

$$Area = \frac{1}{2}(base)(height) = \frac{1}{2}b(c\sin\alpha)$$

Similarly,

$$Area = \frac{1}{2}a(b\sin\gamma) = \frac{1}{2}a(c\sin\beta)$$

Note: AREA OF AN OBLIQUE TRIANGLE

The formula for the area of an oblique triangle is given by

$$Area = \frac{1}{2}bc\sin\alpha \tag{8.1.3}$$

$$Area = \frac{1}{2}ac\sin\beta \tag{8.1.4}$$

$$Area = \frac{1}{2}ab\sin\gamma \tag{8.1.5}$$

This is equivalent to one-half of the product of two sides and the sine of their included angle.

Example 8.1.5: Finding the Area of an Oblique Triangle

Find the area of a triangle with sides $a = 90$, $b = 52$, and angle $\gamma = 102^\circ$. Round the area to the nearest integer.

Solution

Using the formula, we have

$$Area = \frac{1}{2}ab\sin\gamma$$
$$Area = \frac{1}{2}(90)(52)\sin(102^\circ)$$
$$Area \approx 2289 \text{ square units}$$

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8.2: Non-right Triangles - Law of Cosines

Using the Law of Cosines to Solve Oblique Triangles

The tool we need to solve the problem of the boat's distance from the port is the *Law of Cosines*, which defines the relationship among angle measurements and side lengths in oblique triangles. Three formulas make up the Law of Cosines. At first glance, the formulas may appear complicated because they include many variables. However, once the pattern is understood, the Law of Cosines is easier to work with than most formulas at this mathematical level.

Understanding how the Law of Cosines is derived will be helpful in using the formulas. The derivation begins with the Generalized Pythagorean Theorem, which is an extension of the Pythagorean Theorem to non-right triangles. Here is how it works: An arbitrary non-right triangle ABC is placed in the coordinate plane with vertex A at the origin, side c drawn along the *x*-axis, and vertex C located at some point (x, y) in the plane, as illustrated in Figure 8.2.2. Generally, triangles exist anywhere in the plane, but for this explanation we will place the triangle as noted.

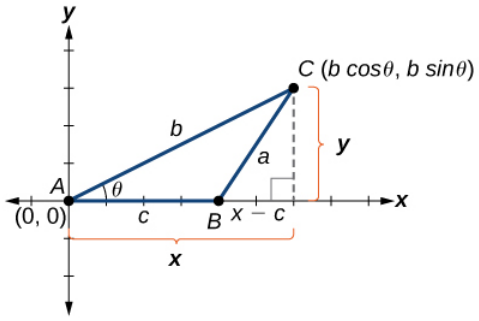


***Figure* 8.2.2**

We can drop a perpendicular from C to the *x*-axis (this is the altitude or height). Recalling the basic trigonometric identities, we know that

$$\cos\theta = \frac{x(adjacent)}{b(hypotenuse)} \text{ and } \sin\theta = \frac{y(opposite)}{b(hypotenuse)}$$

In terms of θ, $x = b\cos\theta$ and $y = b\sin\theta$. The (x, y) point located at C has coordinates $(b\cos\theta, b\sin\theta)$. Using the side $(x - c)$ as one leg of a right triangle and y as the second leg, we can find the length of hypotenuse a using the Pythagorean Theorem. Thus,

$$\begin{aligned} a^2 &= (x-c)^2 + y^2 \\ &= (b\cos\theta - c)^2 + (b\sin\theta)^2 && \text{Substitute } (b\cos\theta) \text{ for } x \text{ and } (b\sin\theta) \text{ for } y \\ &= (b^2\cos^2\theta - 2bc\cos\theta + c^2) + b^2\sin^2\theta && \text{Expand the perfect square.} \\ &= b^2\cos^2\theta + b^2\sin^2\theta + c^2 - 2bc\cos\theta && \text{Group terms noting that } \cos^2\theta + \sin^2\theta = 1 \\ &= b^2(\cos^2\theta + \sin^2\theta) + c^2 - 2bc\cos\theta && \text{Factor out } b^2 \\ a^2 &= b^2 + c^2 - 2bc\cos\theta \end{aligned}$$

The formula derived is one of the three equations of the Law of Cosines. The other equations are found in a similar fashion.

Keep in mind that it is always helpful to sketch the triangle when solving for angles or sides. In a real-world scenario, try to draw a diagram of the situation. As more information emerges, the diagram may have to be altered. Make those alterations to the diagram and, in the end, the problem will be easier to solve.

The LAW OF COSINES

The Law of Cosines states that the square of any side of a triangle is equal to the sum of the squares of the other two sides minus twice the product of the other two sides and the cosine of the included angle.

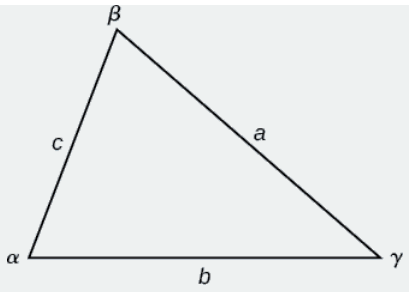


***Figure* 8.2.3**

For triangles labeled as in Figure 8.2.3, with angles α, β and γ, and opposite corresponding sides a, b, and c, respectively, the Law of Cosines is given as three equations.

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \tag{8.2.1}$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta \tag{8.2.2}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{8.2.3}$$

To solve for a missing side measurement, the corresponding opposite angle measure is needed.

When solving for an angle, the corresponding opposite side measure is needed. We can use another version of the Law of Cosines to solve for an angle.

$$\cos\alpha = \frac{b^2 + c^2 - a^2}{2bc} \tag{8.2.4}$$

$$\cos\beta = \frac{a^2 + c^2 - b^2}{2ac} \tag{8.2.5}$$

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} \tag{8.2.6}$$

How to: Given two sides and the angle between them (SAS), find the measures of the remaining side and angles of a triangle

1. Sketch the triangle. Identify the measures of the known sides and angles. Use variables to represent the measures of the unknown sides and angles.
2. Apply the Law of Cosines to find the length of the unknown side or angle.
3. Apply the Law of Sines or Cosines to find the measure of a second angle.
4. Compute the measure of the remaining angle.

Example 8.2.1: Finding the Unknown Side and Angles of a SAS Triangle

Find the unknown side and angles of the triangle in Figure 8.2.4.

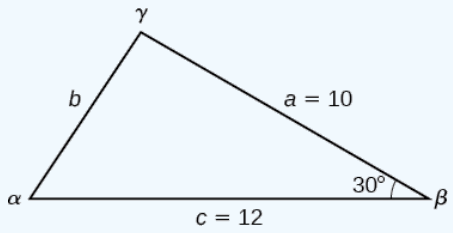


***Figure* 8.2.4**

Solution

First, make note of what is given: two sides and the angle between them. This arrangement is classified as SAS and supplies the data needed to apply the Law of Cosines.

Each one of the three laws of cosines begins with the square of an unknown side opposite a known angle. For this example, the first side to solve for is side b, as we know the measurement of the opposite angle β.

$$b^2 = a^2 + c^2 - 2ac\cos\beta$$

$$b^2 = 10^2 + 12^2 - 2(10)(12)\cos(30^\circ) \quad \text{Substitute the measurements for the known quantities.}$$

$$b^2 = 100 + 144 - 240\left(\frac{\sqrt{3}}{2}\right) \quad \text{Evaluate the cosine and begin to simplify.}$$

$$b^2 = 244 - 120\sqrt{3}$$

$$b = \sqrt{244 - 120\sqrt{3}} \quad \text{Use the square root property.}$$

$$b \approx 6.013$$

Because we are solving for a length, we use only the positive square root. Now that we know the length b, we can use the Law of Sines to fill in the remaining angles of the triangle. Solving for angle α, we have

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

$$\frac{\sin\alpha}{10} = \frac{\sin(30^\circ)}{6.013}$$

$$\sin\alpha = \frac{10\sin(30^\circ)}{6.013} \quad \text{Multiply both sides of the equation by 10.}$$

$$\alpha = \sin^{-1}\left(\frac{10\sin(30^\circ)}{6.013}\right) \quad \text{Find the inverse sine of } \frac{10\sin(30^\circ)}{6.013}.$$

$$\alpha \approx 56.3^\circ$$

The other possibility for α would be $\alpha = 180^\circ - 56.3^\circ \approx 123.7^\circ$. In the original diagram,α is adjacent to the longest side, so α is an acute angle and, therefore, 123.7° does not make sense. Notice that if we choose to apply the Law of Cosines, we arrive at a unique answer. We do not have to consider the other possibilities, as cosine is unique for angles between 0° and 180°. Proceeding with $\alpha \approx 56.3^\circ$, we can then find the third angle of the triangle.

$$\begin{aligned}\gamma &= 180^\circ - 30^\circ - 56.3^\circ \\ &\approx 93.7^\circ\end{aligned}$$

The complete set of angles and sides is

$$\alpha \approx 56.3^\circ \; a = 10$$

$$\beta = 30^\circ \; b \approx 6.013$$

$$\gamma \approx 93.7^\circ \; c = 12$$

Using Heron's Formula to Find the Area of a Triangle

We already learned how to find the area of an oblique triangle when we know two sides and an angle. We also know the formula to find the area of a triangle using the base and the height. When we know the three sides, however, we can use **Heron's formula** instead of finding the height. **Heron of Alexandria** was a geometer who lived during the first century A.D. He discovered a formula for finding the area of oblique triangles when three sides are known.

HERON'S FORMULA

Heron's formula finds the area of oblique triangles in which sides a, b,and c are known.

$$Area = \sqrt{s(s-a)(s-b)(s-c)} \tag{8.2.7}$$

where $s = \dfrac{(a+b+c)}{2}$ is one half of the perimeter of the triangle, sometimes called the semi-perimeter.

Example 8.2.4: Using Heron's Formula to Find the Area of a Given Triangle

Find the area of the triangle in Figure 8.2.9 using Heron's formula.

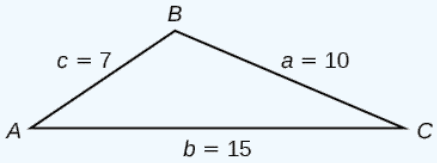


***Figure* 8.2.9**

Solution

First, we calculate s.

$$s = \frac{(a+b+c)}{2}$$
$$s = \frac{(10+15+7)}{2}$$
$$= 16$$

Then we apply the formula.

$$Area = \sqrt{s(s-a)(s-b)(s-c)}$$
$$Area = \sqrt{16(16-10)(16-15)(16-7)}$$
$$Area \approx 29.4$$

The area is approximately 29.4 square units.

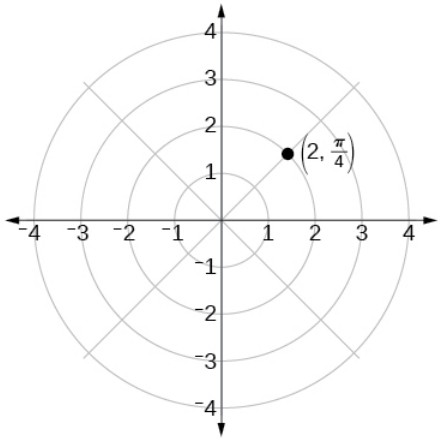
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8.3: Polar Coordinates

Plotting Points Using Polar Coordinates

When we think about plotting points in the plane, we usually think of *rectangular coordinates* (x, y) in the *Cartesian coordinate plane*. However, there are other ways of writing a coordinate pair and other types of grid systems. In this section, we introduce to polar coordinates, which are points labeled (r, θ) and plotted on a polar grid. The polar grid is represented as a series of concentric circles radiating out from the pole, or the origin of the coordinate plane.

The polar grid is scaled as the unit circle with the positive x-axis now viewed as the polar axis and the origin as the pole. The first coordinate r is the radius or length of the directed line segment from the pole. The angle θ, measured in radians, indicates the direction of r. We move counterclockwise from the polar axis by an angle of θ,and measure a directed line segment the length of r in the direction of θ. Even though we measure θ first and then r, the polar point is written with the r-coordinate first. For example, to plot the point $\left(2, \frac{\pi}{4}\right)$,we would move $\frac{\pi}{4}$ units in the counterclockwise direction and then a length of 2 from the pole. This point is plotted on the grid in Figure 8.3.2.



Polar Grid

***Figure* 8.3.2**

Example 8.3.1: Plotting a Point on the Polar Grid

Plot the point $\left(3, \frac{\pi}{2}\right)$ on the polar grid.

Solution

The angle $\frac{\pi}{2}$ is found by sweeping in a counterclockwise direction 90° from the polar axis. The point is located at a length of 3 units from the pole in the $\frac{\pi}{2}$ direction, as shown in Figure 8.3.3.

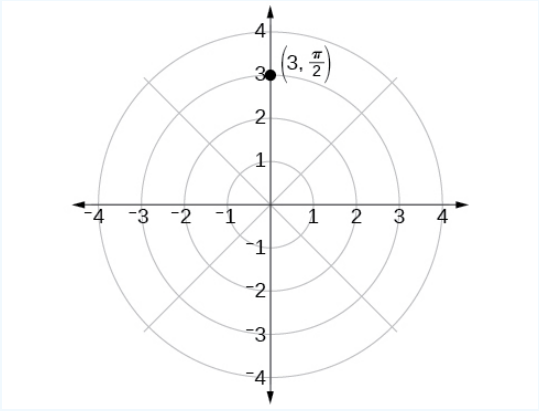


Figure **8.3.3**

Converting from Rectangular Coordinates to Polar Coordinates

To convert rectangular coordinates to polar coordinates, we will use two other familiar relationships. With this conversion, however, we need to be aware that a set of rectangular coordinates will yield more than one polar point.

CONVERTING FROM RECTANGULAR COORDINATES TO POLAR COORDINATES

Converting from rectangular coordinates to polar coordinates requires the use of one or more of the relationships illustrated in Figure 8.3.10.

$$\cos\theta = \frac{x}{r} \text{ or } x = r\cos\theta$$
$$\sin\theta = \frac{y}{r} \text{ or } y = r\sin\theta$$
$$r^2 = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$

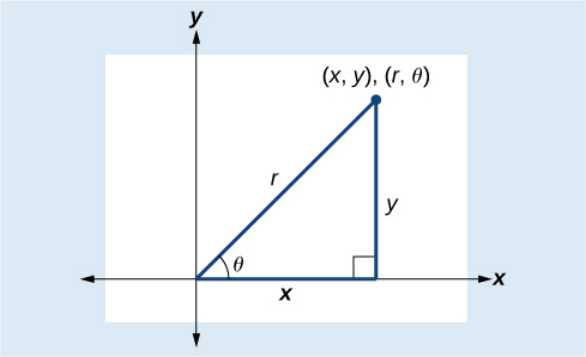


Figure **8.3.10**

Example 8.3.4: Writing Rectangular Coordinates as Polar Coordinates

Convert the rectangular coordinates $(3, 3)$ to polar coordinates.

Solution

We see that the original point $(3, 3)$ is in the first quadrant. To find θ, use the formula $\tan\theta = \frac{y}{x}$. This gives

$$\tan\theta = \frac{3}{3}$$
$$\tan\theta = 1$$
$$\tan^{-1}(1) = \frac{\pi}{4}$$

To find r, we substitute the values for x and y into the formula $r = \sqrt{x^2 + y^2}$. We know that r must be positive, as $\frac{\pi}{4}$ is in the first quadrant. Thus

$$r = \sqrt{3^2 + 3^2}$$
$$r = \sqrt{9 + 9}$$
$$r = \sqrt{18}$$
$$= 3\sqrt{2}$$

So, $r = 3\sqrt{2}$ and $\theta = \frac{\pi}{4}$, giving us the polar point $(3\sqrt{2}, \frac{\pi}{4})$. See Figure 8.3.11.

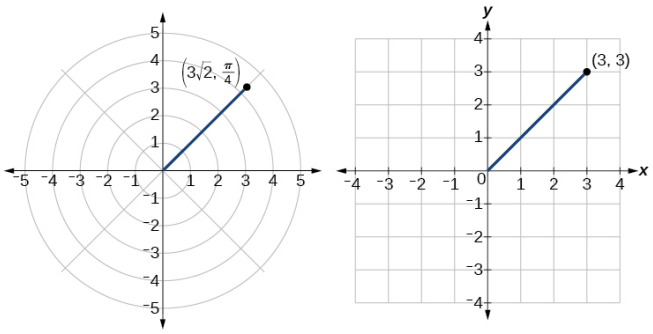


***Figure* 8.3.11**

Analysis

There are other sets of polar coordinates that will be the same as our first solution. For example, the points $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ and $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ will coincide with the original solution of $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. The point $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$ indicates a move further counterclockwise by π, which is directly opposite $\frac{\pi}{4}$. The radius is expressed as $-3\sqrt{2}$. However, the angle $\frac{5\pi}{4}$ is located in the third quadrant and, as r is negative, we extend the directed line segment in the opposite direction, into the first quadrant. This is the same point as $\left(3\sqrt{2}, \frac{\pi}{4}\right)$. The point $\left(3\sqrt{2}, -\frac{7\pi}{4}\right)$ is a move further clockwise by $-\frac{7\pi}{4}$, from $\frac{\pi}{4}$. The radius, $3\sqrt{2}$, is the same.

Transforming Equations between Polar and Rectangular Forms

We can now convert coordinates between polar and rectangular form. Converting equations can be more difficult, but it can be beneficial to be able to convert between the two forms. Since there are a number of polar equations that cannot be expressed clearly in Cartesian form, and vice versa, we can use the same procedures we used to convert points between the coordinate systems. We can then use a graphing calculator to graph either the rectangular form or the polar form of the equation.

How to: Given an equation in polar form, graph it using a graphing calculator

1. Change the **MODE** to **POL**, representing polar form.
2. Press the **Y=** button to bring up a screen allowing the input of six equations: r_1, r_2,..., r_6.
3. Enter the polar equation, set equal to r.
4. Press **GRAPH**.

Example 8.3.5*A*: Writing a Cartesian Equation in Polar Form

Write the Cartesian equation $x^2+y^2=9$ in polar form.

Solution

The goal is to eliminate x and y from the equation and introduce r and θ. Ideally, we would write the equation r as a function of θ. To obtain the polar form, we will use the relationships between (x, y) and (r, θ). Since $x = r\cos\theta$ and $y = r\sin\theta$, we can substitute and solve for r.

$$\begin{aligned} (r\cos\theta)^2+(r\sin\theta)^2 &= 9 \\ r^2\cos^2\theta+r^2\sin^2\theta &= 9 \\ r^2(\cos^2\theta+\sin^2\theta) &= 9 \\ r^2(1) &= 9 \qquad \text{Substitute } \cos^2\theta+\sin^2\theta=1 \\ r &= \pm 3 \qquad \text{Use the square root property.} \end{aligned}$$

Thus, $x^2+y^2=9$,$r=3$,and $r=-3$ should generate the same graph. See Figure 8.3.12

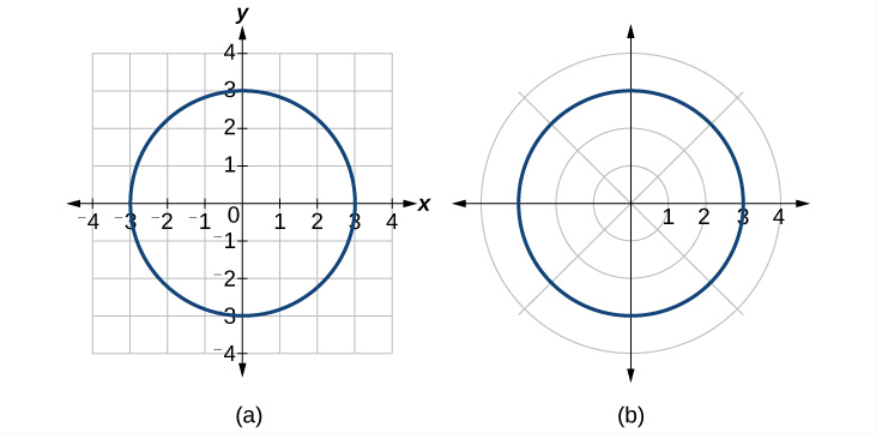


***Figure* 8.3.12:** *(a) Cartesian form $x^2+y^2=9$ (b) Polar form $r=3$*

To graph a circle in rectangular form, we must first solve for y.

$$\begin{aligned} x^2+y^2 &= 9 \\ y^2 &= 9-x^2 \\ y &= \pm\sqrt{9-x^2} \end{aligned}$$

Note that this is two separate functions, since a circle fails the vertical line test. Therefore, we need to enter the positive and negative square roots into the calculator separately, as two equations in the form $Y_1=\sqrt{9-x^2}$ and $Y_2=-\sqrt{9-x^2}$. Press **GRAPH.**

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8.4: Polar Coordinates - Graphs

Testing Polar Equations for Symmetry

Just as a rectangular equation such as $y = x^2$ describes the relationship between x and y on a Cartesian grid, a **polar equation** describes a relationship between r and θ on a polar grid. Recall that the coordinate pair (r, θ) indicates that we move counterclockwise from the polar axis (positive x-axis) by an angle of θ, and extend a ray from the pole (origin) r units in the direction of θ. All points that satisfy the polar equation are on the graph.

Symmetry is a property that helps us recognize and plot the graph of any equation. If an equation has a graph that is symmetric with respect to an axis, it means that if we folded the graph in half over that axis, the portion of the graph on one side would coincide with the portion on the other side. By performing three tests, we will see how to apply the properties of symmetry to polar equations. Further, we will use symmetry (in addition to plotting key points, zeros, and maximums of r) to determine the graph of a polar equation.

In the first test, we consider symmetry with respect to the line $\theta = \dfrac{\pi}{2}$ (y-axis). We replace (r, θ) with $(-r, -\theta)$ to determine if the new equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin\theta$;

$$\begin{aligned} r &= 2\sin\theta \\ -r &= 2\sin -\theta \qquad \text{Replace } (r, \theta) \text{ with } (-r, -\theta). \\ -r &= -2\sin\theta \qquad \text{Identity: } \sin(-\theta) = -\sin\theta. \\ r &= 2\sin\theta \qquad \text{Multiply both sides by } -1 \end{aligned}$$

This equation exhibits symmetry with respect to the line $\theta = \dfrac{\pi}{2}$.

In the second test, we consider symmetry with respect to the polar axis (x-axis). We replace (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ to determine equivalency between the tested equation and the original. For example, suppose we are given the equation $r = 1 - 2\cos\theta$.

$$\begin{aligned} r &= 1 - 2\cos\theta \\ r &= 1 - 2\cos(-\theta) \qquad \text{Replace } (r, \theta) \text{ with } (r, -\theta). \\ r &= 1 - 2\cos\theta \qquad \text{Even/Odd identity} \end{aligned}$$

The graph of this equation exhibits symmetry with respect to the polar axis.

In the third test, we consider symmetry with respect to the pole (origin). We replace (r, θ) with $(-r, \theta)$ to determine if the tested equation is equivalent to the original equation. For example, suppose we are given the equation $r = 2\sin(3\theta)$.

$$\begin{aligned} r &= 2\sin(3\theta) \\ -r &= 2\sin(3\theta) \end{aligned}$$

The equation has failed the symmetry test, but that does not mean that it is not symmetric with respect to the pole. Passing one or more of the symmetry tests verifies that symmetry will be exhibited in a graph. However, failing the symmetry tests does not necessarily indicate that a graph will not be symmetric about the line $\theta = \dfrac{\pi}{2}$, the polar axis, or the pole. In these instances, we can confirm that symmetry exists by plotting reflecting points across the apparent axis of symmetry or the pole. Testing for symmetry is a technique that simplifies the graphing of polar equations, but its application is not perfect.

Note: SYMMETRY TESTS

A polar equation describes a curve on the polar grid. The graph of a polar equation can be evaluated for three types of symmetry, as shown in Figure 8.4.2.

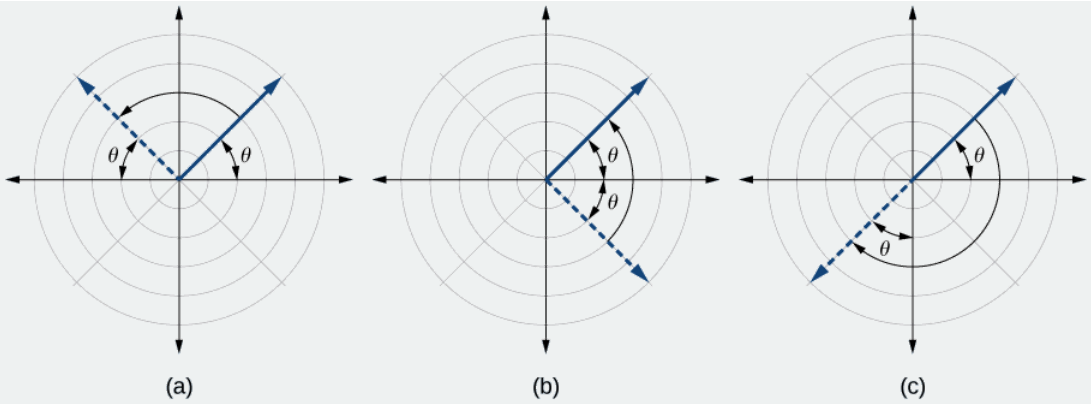


Figure 8.4.2: (a) A graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$ (y-axis) if replacing (r, θ) with $(-r, -\theta)$ yields an equivalent equation. (b) A graph is symmetric with respect to the polar axis (x-axis) if replacing (r, θ) with $(r, -\theta)$ or $(-r, \pi - \theta)$ yields an equivalent equation. (c) A graph is symmetric with respect to the pole (origin) if replacing (r, θ) with $(-r, \theta)$ yields an equivalent equation.

How to: Given a polar equation, test for symmetry.

1. Substitute the appropriate combination of components for (r, θ): $(-r, -\theta)$ for $\theta = \frac{\pi}{2}$ symmetry; $(r, -\theta)$ for polar axis symmetry; and $(-r, \theta)$ for symmetry with respect to the pole.
2. If the resulting equations are equivalent in one or more of the tests, the graph produces the expected symmetry.

Graphing Polar Equations by Plotting Points

To graph in the rectangular coordinate system we construct a table of x and y values. To graph in the polar coordinate system we construct a table of θ and r values. We enter values of θ into a polar equation and calculate r. However, using the properties of symmetry and finding key values of θ and r means fewer calculations will be needed.

Finding Zeros and Maxima

To find the zeros of a polar equation, we solve for the values of θ that result in $r = 0$. Recall that, to find the zeros of polynomial functions, we set the equation equal to zero and then solve for x. We use the same process for polar equations. Set $r = 0$, and solve for θ.

For many of the forms we will encounter, the maximum value of a polar equation is found by substituting those values of θ into the equation that result in the maximum value of the trigonometric functions. Consider $r = 5\cos\theta$; the maximum distance between the curve and the pole is 5 units. The maximum value of the cosine function is 1 when $\theta = 0$, so our polar equation is $5\cos\theta$, and the value $\theta = 0$ will yield the maximum $|r|$.

Similarly, the maximum value of the sine function is 1 when $\theta = \frac{\pi}{2}$, and if our polar equation is $r = 5\sin\theta$, the value $\theta = \frac{\pi}{2}$ will yield the maximum $|r|$. We may find additional information by calculating values of r when $\theta = 0$. These points would be polar axis intercepts, which may be helpful in drawing the graph and identifying the curve of a polar equation.

Example 8.4.2: Finding Zeros and Maximum Values for a Polar Equation

Using the equation in Example 8.4.1, find the zeros and maximum $|r|$ and, if necessary, the polar axis intercepts of $r = 2\sin\theta$.

Solution

To find the zeros, set r equal to zero and solve for θ.

$$\begin{aligned} 2\sin\theta &= 0 \\ \sin\theta &= 0 \\ \theta &= \sin^{-1} 0 \\ \theta &= n\pi \qquad \text{where n is an integer} \end{aligned}$$

Substitute any one of the θ values into the equation. We will use 0.

$$r = 2\sin(0)$$
$$r = 0$$

The points $(0, 0)$ and $(0, \pm n\pi)$ are the zeros of the equation. They all coincide, so only one point is visible on the graph. This point is also the only polar axis intercept.

To find the maximum value of the equation, look at the maximum value of the trigonometric function $\sin\theta$, which occurs when $\theta = \frac{\pi}{2} \pm 2k\pi$ resulting in $\sin\left(\frac{\pi}{2}\right) = 1$. Substitute $\frac{\pi}{2}$ for θ.

$$r = 2\sin\left(\frac{\pi}{2}\right)$$
$$r = 2(1)$$
$$r = 2$$

Analysis

The point $\left(2, \frac{\pi}{2}\right)$ will be the maximum value on the graph. Let's plot a few more points to verify the graph of a circle. See Table 8.4.1 and Figure 8.4.4.

Table 8.4.1

θ	$r = 2\sin\theta$	r
0	$r = 2\sin(0) = 0$	0
$\frac{\pi}{6}$	$r = 2\sin\left(\frac{\pi}{6}\right) = 1$	1
$\frac{\pi}{3}$	$r = 2\sin\left(\frac{\pi}{3}\right) \approx 1.73$	1.73
$\frac{pi}{2}$	$r = 2\sin\left(\frac{\pi}{2}\right) = 2$	2
$\frac{2\pi}{3}$	$r = 2\sin\left(\frac{2\pi}{3}\right) \approx 1.73$	1.73
$\frac{5\pi}{6}$	$r = 2\sin\left(\frac{5\pi}{6}\right) = 1$	1
π	$r = 2\sin(\pi) = 0$	0

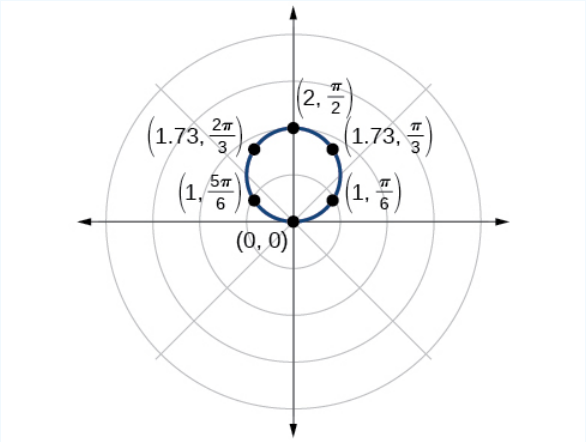


Figure 8.4.4

Investigating Circles

Now we have seen the equation of a circle in the polar coordinate system. In the last two examples, the same equation was used to illustrate the properties of symmetry and demonstrate how to find the zeros, maximum values, and plotted points that produced the graphs. However, the circle is only one of many shapes in the set of polar curves.

There are five classic polar curves**: cardioids**, **limaçons, lemniscates, rose curves**, and **Archimedes' spirals**. We will briefly touch on the polar formulas for the circle before moving on to the classic curves and their variations.

FORMULAS FOR THE EQUATION OF A CIRCLE

Some of the formulas that produce the graph of a circle in polar coordinates are given by $r = a\cos\theta$ and $r = a\sin\theta$, wherea a is the diameter of the circle or the distance from the pole to the farthest point on the circumference. The radius is $\frac{|a|}{2}$, or one-half the diameter. For $r = a\cos\theta$, the center is $\left(\frac{a}{2}, 0\right)$. For $r = a\sin\theta$, the center is $\left(\frac{a}{2}, \pi\right)$. Figure 8.4.5 shows the graphs of these four circles.

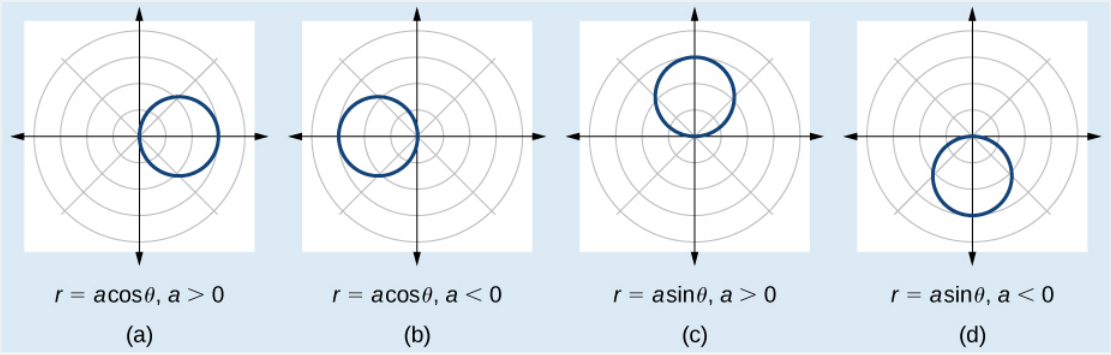


Figure 8.4.5

Example 8.4.3: Sketching the Graph of a Polar Equation for a Circle

Sketch the graph of $r = 4\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph is symmetric about the polar axis. Next, we find the zeros and maximum $|r|$ for $r = 4\cos\theta$. First, set $r = 0$, and solve for θ. Thus, a zero occurs at $\theta = \frac{\pi}{2} \pm k\pi$. A key point to plot is $\left(0, \frac{\pi}{2}\right)$.

To find the maximum value of r, note that the maximum value of the cosine function is 1 when $\theta = 0 \pm 2k\pi$. Substitute $\theta = 0$ into the equation:

$$\begin{aligned} r &= 4\cos\theta \\ r &= 4\cos(0) \\ r &= 4(1) \\ &= 4 \end{aligned}$$

The maximum value of the equation is 4. A key point to plot is $(4, 0)$.

As $r = 4\cos\theta$ is symmetric with respect to the polar axis, we only need to calculate r-values for θ over the interval $[0, \pi]$. Points in the upper quadrant can then be reflected to the lower quadrant. Make a table of values similar to Table 8.4.2. The graph is shown in Figure 8.4.6.

Table 8.4.2

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	3.46	2.83	2	0	-2	-2.83	-3.46	4

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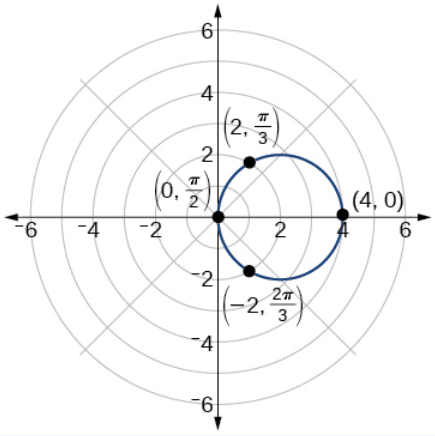


Figure 8.4.6

Investigating Cardioids

While translating from polar coordinates to Cartesian coordinates may seem simpler in some instances, graphing the classic curves is actually less complicated in the polar system. The next curve is called a cardioid, as it resembles a heart. This shape is often included with the family of curves called limaçons, but here we will discuss the cardioid on its own.

FORMULAS FOR A CARDIOID

The formulas that produce the graphs of a cardioid are given by $r = a \pm b\cos\theta$ and $r = a \pm b\sin\theta$ where $a > 0$, $b > 0$, and $\frac{a}{b} = 1$. The cardioid graph passes through the pole, as we can see in Figure 8.4.7.

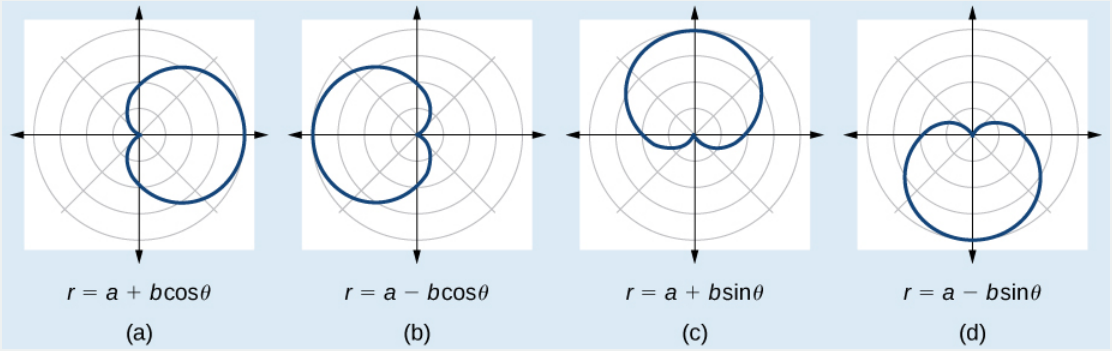


Figure 8.4.7

How to: Given the polar equation of a cardioid, sketch its graph

1. Check equation for the three types of symmetry.
2. Find the zeros. Set $r = 0$.
3. Find the maximum value of the equation according to the maximum value of the trigonometric expression.
4. Make a table of values for r and θ.
5. Plot the points and sketch the graph.

Example 8.4.4: Sketching the Graph of a Cardioid

Sketch the graph of $r = 2 + 2\cos\theta$.

Solution

First, testing the equation for symmetry, we find that the graph of this equation will be symmetric about the polar axis. Next, we find the zeros and maximums. Setting $r = 0$, we have $\theta = \pi + 2k\pi$. The zero of the equation is located at $(0, \pi)$. The graph passes through this point.

The maximum value of $r = 2 + 2\cos\theta$ occurs when $\cos\theta$ is a maximum, which is when $\cos\theta = 1$ or when $\theta = 0$. Substitute $\theta = 0$ into the equation, and solve for r.

$$\begin{aligned} r &= 2 + 2\cos(0) \\ r &= 2 + 2(1) \\ &= 4 \end{aligned}$$

The point $(4, 0)$ is the maximum value on the graph.

We found that the polar equation is symmetric with respect to the polar axis, but as it extends to all four quadrants, we need to plot values over the interval $[0, \pi]$. The upper portion of the graph is then reflected over the polar axis. Next, we make a table of values, as in Table 8.4.3, and then we plot the points and draw the graph. See Figure 8.4.8.

Table 8.4.3

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	4	3.41	2	1	0

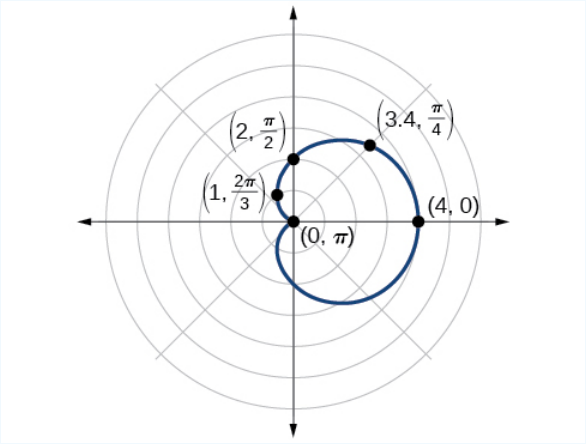


Figure 8.4.8

Investigating the Archimedes' Spiral

The final polar equation we will discuss is the Archimedes' spiral, named for its discoverer, the Greek mathematician Archimedes (c. 287 BCE - c. 212 BCE), who is credited with numerous discoveries in the fields of geometry and mechanics.

ARCHIMEDES' SPIRAL

The formula that generates the graph of the Archimedes' spiral is given by $r = \theta$ for $\theta \geq 0$. As θ increases, r increases at a constant rate in an ever-widening, never-ending, spiraling path. See Figure 8.4.21.

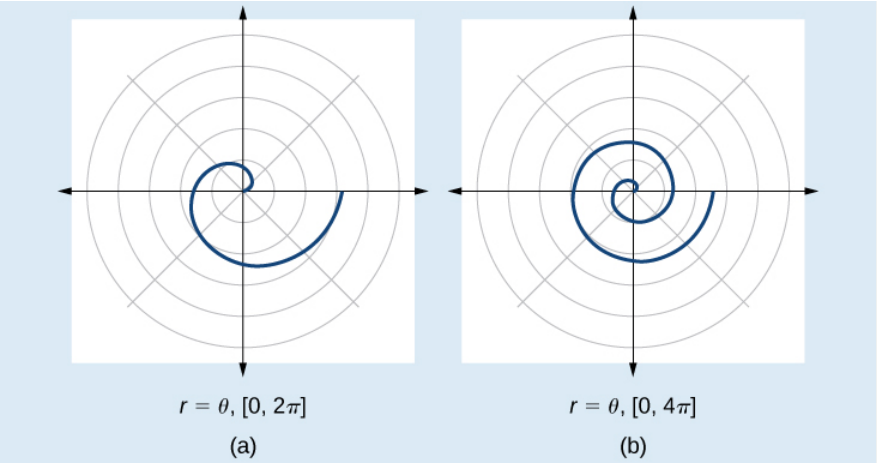


Figure 8.4.21

How to: Given an Archimedes' spiral over $[0, 2\pi]$,sketch the graph

1. Make a table of values for r and θ over the given domain.
2. Plot the points and sketch the graph.

Example 8.4.10: Sketching the Graph of an Archimedes' Spiral

Sketch the graph of $r = \theta$ over $[0, 2\pi]$.

Solution

As r is equal to θ, the plot of the Archimedes' spiral begins at the pole at the point $(0, 0)$. While the graph hints of symmetry, there is no formal symmetry with regard to passing the symmetry tests. Further, there is no maximum value, unless the domain is restricted.

Create a table such as Table 8.4.9.

Table 8.4.9

θ	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0.785	1.57	3.14	4.71	5.50	6.28

Notice that the r-values are just the decimal form of the angle measured in radians. We can see them on a graph in Figure 8.4.22

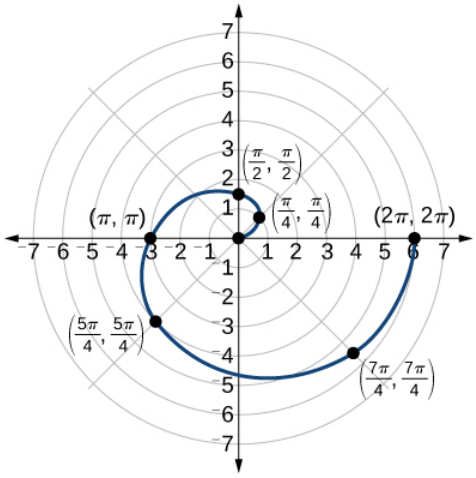


Figure 8.4.22: Archimedes' spiral

Analysis

The domain of this polar curve is $[0, 2\pi]$. In general, however, the domain of this function is $(-\infty, \infty)$. Graphing the equation of the Archimedes' spiral is rather simple, although the image makes it seem like it would be complex.

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8.5: Polar Form of Complex Numbers

Plotting Complex Numbers in the Complex Plane

Plotting a complex number $a+bi$ is similar to plotting a real number, except that the horizontal axis represents the real part of the number, a, and the vertical axis represents the imaginary part of the number, bi.

How to: Given a complex number $a+bi$, plot it in the complex plane.

1. Label the horizontal axis as the *real* axis and the vertical axis as the *imaginary axis.*
2. Plot the point in the complex plane by moving a units in the horizontal direction and b units in the vertical direction.

Example 8.5.1: Plotting a Complex Number in the Complex Plane

Plot the complex number $2-3i$ in the complex plane.

Solution

From the origin, move two units in the positive horizontal direction and three units in the negative vertical direction. See Figure 8.5.1.

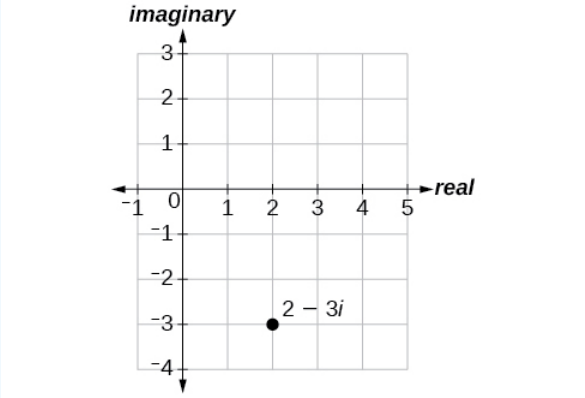


Figure 8.5.1

Finding the Absolute Value of a Complex Number

The first step toward working with a complex number in polar form is to find the absolute value. The absolute value of a complex number is the same as its magnitude, or $|z|$. It measures the distance from the origin to a point in the plane. For example, the graph of $z=2+4i$, in Figure 8.5.3, shows $|z|$.

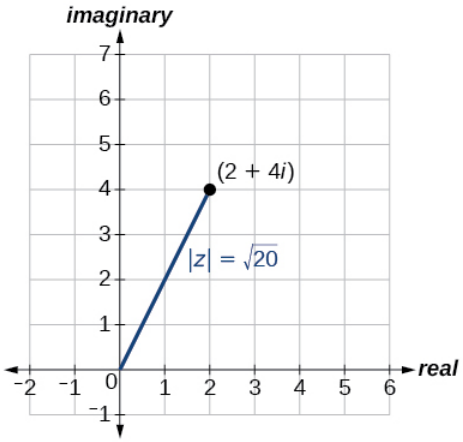


Figure 8.5.3

ABSOLUTE VALUE OF A COMPLEX NUMBER

Given $z = x + yi$, a complex number, the absolute value of z is defined as

$$|z| = \sqrt{x^2 + y^2} \tag{8.5.1}$$

It is the distance from the origin to the point (x, y).

Notice that the absolute value of a real number gives the distance of the number from 0, while the absolute value of a complex number gives the distance of the number from the origin, $(0, 0)$.

Example 8.5.2: Finding the Absolute Value of a Complex Number with a Radical

Find the absolute value of $z = \sqrt{5} - i$.

Solution

Using the formula, we have

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ |z| &= \sqrt{(\sqrt{5})^2 + (-1)^2} \\ |z| &= \sqrt{5+1} \\ |z| &= \sqrt{6} \end{aligned}$$

See Figure 8.5.4.

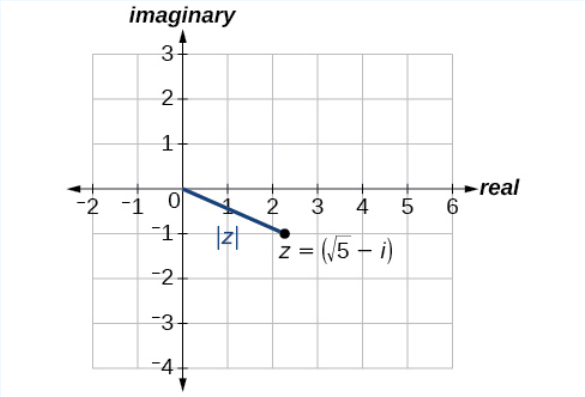


Figure 8.5.4

Writing Complex Numbers in Polar Form

The polar form of a complex number expresses a number in terms of an angle θ and its distance from the origin r. Given a complex number in rectangular form expressed as $z = x + yi$, we use the same conversion formulas as we do to write the number in trigonometric form:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r &= \sqrt{x^2 + y^2} \end{aligned}$$

We review these relationships in Figure 8.5.6.

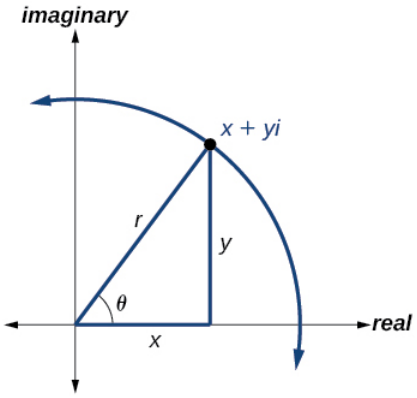


Figure 8.5.6

We use the term **modulus** to represent the absolute value of a complex number, or the distance from the origin to the point (x, y). The modulus, then, is the same as r, the radius in polar form. We use θ to indicate the angle of direction (just as with polar coordinates). Substituting, we have

$$\begin{aligned} z &= x + yi \\ z &= r\cos\theta + (r\sin\theta)i \\ z &= r(\cos\theta + i\sin\theta) \end{aligned}$$

POLAR FORM OF A COMPLEX NUMBER

Writing a complex number in polar form involves the following conversion formulas:

$$x = r\cos\theta \tag{8.5.2}$$
$$y = r\sin\theta \tag{8.5.3}$$
$$r = \sqrt{x^2 + y^2} \tag{8.5.4}$$

Making a direct substitution, we have

$$z = x + yi \tag{8.5.5}$$
$$z = (r\cos\theta) + i(r\sin\theta) \tag{8.5.6}$$
$$z = r(\cos\theta + i\sin\theta) \tag{8.5.7}$$

where r is the modulus and θ is the argument. We often use the abbreviation $r\ cis\theta$ to represent $r(\cos\theta + i\sin\theta)$.

Example 8.5.4: Expressing a Complex Number Using Polar Coordinates

Express the complex number $4i$ using polar coordinates.

Solution

On the complex plane, the number $z = 4i$ is the same as $z = 0 + 4i$. Writing it in polar form, we have to calculate r first.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ r &= \sqrt{0^2 + 4^2} \\ r &= \sqrt{16} \\ r &= 4 \end{aligned}$$

Next, we look at x. If $x = r\cos\theta$, and $x = 0$, then $\theta = \dfrac{\pi}{2}$. In polar coordinates, the complex number $z = 0 + 4i$ can be written as $z = 4\left(\cos\left(\dfrac{\pi}{2}\right) + i\sin\left(\dfrac{\pi}{2}\right)\right)$ or $4\ cis\left(\dfrac{\pi}{2}\right)$. See Figure 8.5.7.

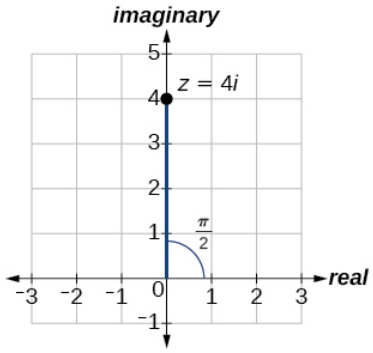


Figure 8.5.7

Finding Products of Complex Numbers in Polar Form

Now that we can convert complex numbers to polar form we will learn how to perform operations on complex numbers in polar form. For the rest of this section, we will work with formulas developed by French mathematician Abraham **de Moivre** (1667-1754). These formulas have made working with products, quotients, powers, and roots of complex numbers much simpler than they appear. The rules are based on multiplying the moduli and adding the arguments.

PRODUCTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the product of these numbers is given as:

$$z_1 z_2 = r_1 r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \tag{8.5.8}$$

$$z_1 z_2 = r_1 r_2\, cis(\theta_1+\theta_2) \tag{8.5.9}$$

Notice that the product calls for multiplying the moduli and adding the angles.

Example 8.5.7: Finding the Product of Two Complex Numbers in Polar Form

Find the product of $z_1 z_2$, given $z_1 = 4(\cos(80^\circ) + i\sin(80^\circ))$ and $z_2 = 2(\cos(145^\circ) + i\sin(145^\circ))$.

Solution

Follow the formula

$$\begin{aligned} z_1 z_2 &= 4\cdot 2[\cos(80^\circ+145^\circ) + i\sin(80^\circ+145^\circ)] \\ z_1 z_2 &= 8[\cos(225^\circ) + i\sin(225^\circ)] \\ z_1 z_2 &= 8\left[\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right] \\ z_1 z_2 &= 8\left[-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right] \\ z_1 z_2 &= -4\sqrt{2} - 4i\sqrt{2} \end{aligned}$$

Finding Quotients of Complex Numbers in Polar Form

The quotient of two complex numbers in polar form is the quotient of the two moduli and the difference of the two arguments.

QUOTIENTS OF COMPLEX NUMBERS IN POLAR FORM

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then the quotient of these numbers is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)],\ z_2 \neq 0 \tag{8.5.10}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\, cis(\theta_1-\theta_2),\ z_2 \neq 0 \tag{8.5.11}$$

Notice that the moduli are divided, and the angles are subtracted.

How to: Given two complex numbers in polar form, find the quotient

1. Divide $\frac{r_1}{r_2}$.
2. Find $\theta_1 - \theta_2$.
3. Substitute the results into the formula: $z = r(\cos\theta + i\sin\theta)$. Replace r with $\frac{r_1}{r_2}$, and replace θ with $\theta_1 - \theta_2$.
4. Calculate the new trigonometric expressions and multiply through by r.

Example 8.5.8: Finding the Quotient of Two Complex Numbers

Find the quotient of $z_1 = 2(\cos(213^\circ) + i\sin(213^\circ))$ and $z_2 = 4(\cos(33^\circ) + i\sin(33^\circ))$.

Solution

Using the formula, we have

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{2}{4}[\cos(213^\circ - 33^\circ) + i\sin(213^\circ - 33^\circ)]\\ \frac{z_1}{z_2} &= \frac{1}{2}[\cos(180^\circ) + i\sin(180^\circ)]\\ \frac{z_1}{z_2} &= \frac{1}{2}[-1 + 0i]\\ \frac{z_1}{z_2} &= -\frac{1}{2} + 0i\\ \frac{z_1}{z_2} &= -\frac{1}{2}\end{aligned}$$

Finding Powers of Complex Numbers in Polar Form

Finding powers of complex numbers is greatly simplified using **De Moivre's Theorem**. It states that, for a positive integer n, z^n is found by raising the modulus to the n^{th} power and multiplying the argument by n. It is the standard method used in modern mathematics.

DE MOIVRE'S THEOREM

If $z = r(\cos\theta + i\sin\theta)$ is a complex number, then

$$z^n = r^n[\cos(n\theta) + i\sin(n\theta)] \tag{8.5.12}$$
$$z^n = r^n\, cis(n\theta) \tag{8.5.13}$$

where n is a positive integer.

Example 8.5.9: Evaluating an Expression Using De Moivre's Theorem

Evaluate the expression $(1+i)^5$ using De Moivre's Theorem.

Solution

Since De Moivre's Theorem applies to complex numbers written in polar form, we must first write $(1+i)$ in polar form. Let us find r.

$$\begin{aligned}r &= \sqrt{x^2 + y^2}\\ r &= \sqrt{(1)^2 + (1)^2}\\ r &= \sqrt{2}\end{aligned}$$

Then we find θ. Using the formula $\tan\theta = \frac{y}{x}$ gives

$$\tan\theta = \frac{1}{1}$$
$$\tan\theta = 1$$
$$\theta = \frac{\pi}{4}$$

Use De Moivre's Theorem to evaluate the expression.

$$\begin{aligned}(a+bi)^n &= r^n[\cos(n\theta)+i\sin(n\theta)]\\ (1+i)^5 &= (\sqrt{2})^5\left[\cos\left(5\cdot\frac{\pi}{4}\right)+i\sin\left(5\cdot\frac{\pi}{4}\right)\right]\\ (1+i)^5 &= 4\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right]\\ (1+i)^5 &= 4\sqrt{2}\left[-\frac{\sqrt{2}}{2}+i\left(-\frac{\sqrt{2}}{2}\right)\right]\\ (1+i)^5 &= -4-4i\end{aligned}$$

Finding Roots of Complex Numbers in Polar Form

To find the n^{th} root of a complex number in polar form, we use the n^{th} Root Theorem or De Moivre's Theorem and raise the complex number to a power with a rational exponent. There are several ways to represent a formula for finding n^{th} roots of complex numbers in polar form.

THE N^{TH} ROOT THEOREM

To find the n^{th} root of a complex number in polar form, use the formula given as

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)+i\sin\left(\frac{\theta}{n}+\frac{2k\pi}{n}\right)\right] \tag{8.5.14}$$

where $k = 0, 1, 2, 3, \ldots, n-1$. We add $\frac{2k\pi}{n}$ to $\frac{\theta}{n}$ in order to obtain the periodic roots.

th Root of a Complex Number

Evaluate the cube roots of $z = 8\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right)$.

Solution

We have

$$\begin{aligned}z^{\frac{1}{3}} &= 8^{\frac{1}{3}}\left[\cos\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{\frac{2\pi}{3}}{3}+\frac{2k\pi}{3}\right)\right]\\ z^{\frac{1}{3}} &= 2\left[\cos\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)+i\sin\left(\frac{2\pi}{9}+\frac{2k\pi}{3}\right)\right]\end{aligned}$$

There will be three roots: $k = 0, 1, 2$. When $k = 0$, we have

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{2\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}\right)\right)$$

When $k = 1$, we have

$$\begin{aligned}z^{\frac{1}{3}} &= 2\left[\cos\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{6\pi}{9}\right)\right] \qquad \text{Add } \frac{2(1)\pi}{3} \text{ to each angle.}\\ z^{\frac{1}{3}} &= 2\left(\cos\left(\frac{8\pi}{9}\right)+i\sin\left(\frac{8\pi}{9}\right)\right)\end{aligned}$$

When $k = 2$, we have

$$z^{\frac{1}{3}} = 2\left[\cos\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)+i\sin\left(\frac{2\pi}{9}+\frac{12\pi}{9}\right)\right] \qquad \text{Add } \frac{2(2)\pi}{3} \text{ to each angle.}$$

$$z^{\frac{1}{3}} = 2\left(\cos\left(\frac{14\pi}{9}\right)+i\sin\left(\frac{14\pi}{9}\right)\right)$$

Remember to find the common denominator to simplify fractions in situations like this one. For $k=1$, the angle simplification is

$$\begin{aligned}\frac{\frac{2\pi}{3}}{3}+\frac{2(1)\pi}{3} &= \frac{2\pi}{3}\left(\frac{1}{3}\right)+\frac{2(1)\pi}{3}\left(\frac{3}{3}\right)\\ &= \frac{2\pi}{9}+\frac{6\pi}{9}\\ &= \frac{8\pi}{9}\end{aligned}$$

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8.6: Parametric Equations

Parameterizing a Curve

When an object moves along a curve—or curvilinear path—in a given direction and in a given amount of time, the position of the object in the plane is given by the x-coordinate and the y-coordinate. However, both x and y vary over time and so are functions of time. For this reason, we add another variable, the parameter, upon which both x and y are dependent functions. In the example in the section opener, the parameter is time, t. The x position of the moon at time, t, is represented as the function $x(t)$, and the y position of the moon at time, t, is represented as the function $y(t)$. Together, $x(t)$ and $y(t)$ are called parametric equations, and generate an ordered pair $(x(t), y(t))$. Parametric equations primarily describe motion and direction.

When we parameterize a curve, we are translating a single equation in two variables, such as x and y,into an equivalent pair of equations in three variables, x, y, and t. One of the reasons we parameterize a curve is because the parametric equations yield more information: specifically, the direction of the object's motion over time.

When we graph parametric equations, we can observe the individual behaviors of x and of y. There are a number of shapes that cannot be represented in the form $y = f(x)$, meaning that they are not functions. For example, consider the graph of a circle, given as $r^2 = x^2 + y^2$. Solving for y gives $y = \pm\sqrt{r^2 - x^2}$, or two equations: $y_1 = \sqrt{r^2 - x^2}$ and $y_2 = -\sqrt{r^2 - x^2}$. If we graph y_1 and y_2 together, the graph will not pass the vertical line test, as shown in Figure 8.6.2. Thus, the equation for the graph of a circle is not a function.

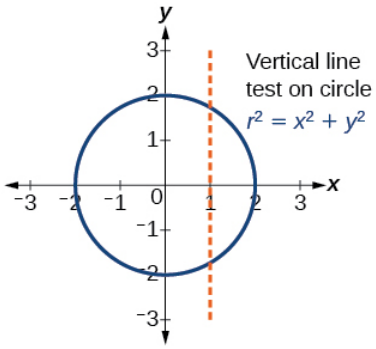


Figure 8.6.2

However, if we were to graph each equation on its own, each one would pass the vertical line test and therefore would represent a function. In some instances, the concept of breaking up the equation for a circle into two functions is similar to the concept of creating parametric equations, as we use two functions to produce a non-function. This will become clearer as we move forward.

PARAMETRIC EQUATIONS

Suppose t is a number on an interval, I. The set of ordered pairs, $(x(t), y(t))$, where $x = f(t)$ and $y = g(t)$,forms a plane curve based on the parameter t. The equations $x = f(t)$ and $y = g(t)$ are the parametric equations.

Example 8.6.1: Parameterizing a Curve

Parameterize the curve $y = x^2 - 1$ letting $x(t) = t$. Graph both equations.

Solution

If $x(t) = t$, then to find $y(t)$ we replace the variable x with the expression given in $x(t)$. In other words, $y(t) = t^2 - 1$ .Make a table of values similar to Table 8.6.1, and sketch the graph.

Table 8.6.1

t	$x(t)$	$y(t)$
-4	-4	$y(-4) = (-4)^2 - 1 = 15$
-3	-3	$y(-3) = (-3)^2 - 1 = 8$

t	$x(t)$	$y(t)$
-2	-2	$y(-2)=(-2)^2-1=3$
-1	-1	$y(-1)=(-1)^2-1=0$
0	0	$y(0)=(0)^2-1=-1$
1	1	$y(1)=(1)^2-1=0$
2	2	$y(2)=(2)^2-1=3$
3	3	$y(3)=(3)^2-1=8$
4	4	$y(4)=(4)^2-1=15$

See the graphs in Figure $8.6.3$. It may be helpful to use the TRACE feature of a graphing calculator to see how the points are generated as t increases.

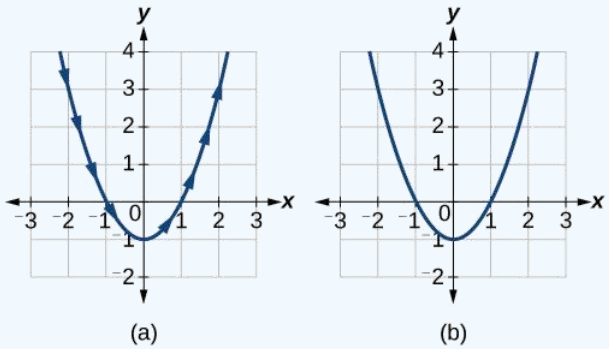


Figure 8.6.3: (a) Parametric $y(t)=t^2-1$ (b) Rectangular $y=x^2-1$

Analysis

The arrows indicate the direction in which the curve is generated. Notice the curve is identical to the curve of $y=x^2-1$.

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